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Some remarks on surface moduli and determinants

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For Rob Lazarsfeld

1 Introduction

The slope of a vector bundle on a smooth projective curve C defines the families of stable and semi-stable bundles (of fixed degree and rank) that are coarsely represented by projective moduli spaces [17]. On a smooth projective polarized surface S, Mumford's H-slope defines H-stable and semi-stable torsion-free sheaves, but this is only the infinitessimal tip of the stability iceberg for S, in which slopes and compatible t-structures on the bounded **derived category** of coherent sheaves on S are the points of a **complex manifold** $\mathrm{Stab}(S)$ of stability conditions [14]. This gives rise to variations of determinant line bundles on moduli spaces of stable objects as the notion of stability varies. The stability conditions that are "closest" to the geometry of coherent sheaves on S are traditionally given in terms of a central charge (see e.g. [15]). Here I want to describe them in terms of a positive cohomology class α on S, my excuse being that the relationship with the determinant line bundles on moduli becomes *linear* in this coordinate system, and therefore the connection with moduli of (Gieseker) H-stable sheaves (and the classical moduli constructed by geometric invariant theory [16]) becomes more transparent.

The Chern character of a vector bundle E on C is the cohomology class $\mathrm{ch}(E) = \mathrm{rk}(E) + c_1(E) \in \mathrm{H}^0(C,\mathbb{Z}) \oplus \mathrm{H}^2(C,\mathbb{Z})$, and the slope of E can therefore be written in terms of the Chern character as

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)} = \frac{\langle \mathrm{ch}(E), 1\rangle}{\langle \mathrm{ch}(E), H\rangle}$$

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where $H \in \mathrm{H}^2(C, \mathbb{Z})$ is the positive generator and $\langle \cdot, \cdot \rangle$ is the standard pairing on cohomology.

Mumford's H-slope on a surface S may similarly be defined as

$$\mu_H(E) := \frac{\langle \mathrm{ch}(E), H \rangle}{\langle \mathrm{ch}(E), H^2 \rangle}$$

where H is an *ample* divisor class on S. The slopes we wish to consider here, however, have the form

$$\mu_\alpha(E) := \frac{\langle \mathrm{ch}(E), \alpha \rangle}{\langle \mathrm{ch}(E), \alpha \cdot H \rangle}$$

where $\alpha = \alpha_0 + \alpha_1 + \alpha_2 \in \mathrm{H}^{2*}(S, \mathbb{Q})$ is positive in a different sense:

$$\alpha_0 > 0 \text{ and } \langle \alpha_1, \alpha_1 \rangle \geq 2\langle \alpha_0, \alpha_2 \rangle.$$

By a theorem of Bogomolov, these are the inequalities that hold for the Chern characters of an H-stable vector bundle on S. The ample divisor H itself is not, of course, positive in this sense since it lacks the inequality $\alpha_0 > 0$. It is, however, a *limit* of positive classes $\epsilon \cdot \mathrm{td}(S) + H'$ as $\epsilon \to 0$ if $\mathrm{td}(S)$ is taken to be the Todd class. This is roughly the difference between Mumford and Gieseker stability.

Expressing the slope in terms of α amounts to a change of coordinates from Bridgeland's description of the slope in terms of a central charge defined via a divisor D and an ample class tH (as modified in [4]):

$$Z_{D,tH}(E) = -\int_S e^{-(D+itH)}\, \mathrm{ch}(E) \text{ and } \mu_Z(E) := -Re(Z(E))/Im(Z(E))$$

since with this definition

$$\mu_Z(E) = \frac{\langle \mathrm{ch}(E), 1 - D + \frac{1}{2}\left(D^2 - t^2H^2\right)\rangle}{\langle \mathrm{ch}(E), t(H - D \cdot H)\rangle} = \frac{\langle \mathrm{ch}(E), \alpha \rangle}{\langle \mathrm{ch}(E), \alpha \cdot (tH) \rangle}.$$

In the α coordinates, the orthogonal complement $c^\perp$ of a Chern class maps linearly via the determinant line bundle to $\mathrm{H}^2(M(c), \mathbb{Q})$, for moduli $M(c)$ of stable objects of class c. The recent positivity result of Bayer and Macrí [7] relates critical stability conditions (for which there exist strictly semi-stable objects) with critical values in the variation of the determinant line bundles. This relates the "wall and chamber structures" for the birational geometry of $M(c)$ with wall and chambers defined on the stability manifold [12]. The choice of a base point in $c^\perp \cap \{\alpha_0 = 1\}$ will allow us to identify rays in the ample cone of S with rays in the stability manifold that "point toward Gieseker stability." We will consider Chern classes of torsion-free sheaves, and then revisit the Serre map for surfaces in the context of torsion sheaves. This evokes fond memories for me, since Rob gave me my start with a question about the Serre map for curves.

2 Slopes and stabilities

A slope function on an abelian category $\mathcal{A}$, expressed as a ratio

$$\mu(A) = \frac{\lambda_d(A)}{\lambda_r(A)};\ \lambda_d, \lambda_r : K(\mathcal{A})_{\mathbb{R}} \to \mathbb{R}$$

of $\mathbb{R}$-linear functions defines a pre-stability condition on $\mathcal{A}$ if

- $\lambda_r(A) \geq 0$ for all objects A of $\mathcal{A}$, and
- $\lambda_d(A) > 0$ for all objects (other than zero) for which $\lambda_r(A) = 0$.

If $\mathcal{A}$ is the heart of a bounded t-structure on a triangulated category $\mathcal{D}$, then the pre-stability condition extends to $\mathcal{D}$ in the obvious way.

Definition An object A of $\mathcal{A}$ is μ-stable if $\mu(B) < \mu(A)$ for all $B \subset A$.

It follows immediately from the two bullet points that:

Schur's Lemma 1 *Let A and B be μ-stable objects of $\mathcal{A}$. Then*

(i) $Hom(A, B) = 0$ if $\mu(A) > \mu(B)$.
(ii) Each nonzero $\phi \in \mathrm{Hom}(A, B)$ is an isomorphism if $\mu(A) = \mu(B)$.

If the abelian category is the heart of a bounded t-structure on a triangulated category $\mathcal{D}$, then the pre-slope extends to a Bridgeland pre-stability condition on $\mathcal{D}$. A different sort of boundedness is also required for the promotion of a pre-stability condition to a stability condition.

Definition Given a slope function μ on an abelian category $\mathcal{A}$, a bounded Harder–Narasimhan filtration on an object A has the form

$$0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$$

where $\mu_i := \mu(A_i/A_{i-1})$ are strictly decreasing, and each $B_i = A_i/A_{i-1}$ admits a finite Jordan–Holder filtration

$$0 = B_i^0 \subset B_i^1 \subset \cdots \subset B_i^{m_i} = B_i$$

where each $C_i^j = B_i^j/B_i^{j-1}$ is stable of the *same* slope $\mu_i = \mu(B_i)$.

Objects B that admit Jordan–Holder filtrations are called *semi-stable*. It is a consequence of Schur's Lemma 1 that the Harder–Narasimhan filtration of each A is unique but that the Jordan–Holder filtrations, while not necessarily unique, do have associated graded objects $\oplus C_i^j$ that are unique up to isomorphism and reordering of the summands. Two semi-stable objects B and B' with isomorphic associated grades are said to be *s-equivalent*.

Definition A pre-stability condition is a *stability condition* if bounded Harder–Narasimhan filtrations exist for all nonzero objects of $\mathcal{A}$.

Our main example Let X be a smooth complex projective variety and $\mathcal{D}(X)$ be the bounded derived category of coherent sheaves on X. We will only consider stability conditions that factor through the Chern character, i.e., slope functions or central charges of the form

$$\mu(E) = \frac{\langle \mathrm{ch}(E), \alpha \rangle}{\langle \mathrm{ch}(E), \beta \rangle} \text{ or equivalently } Z(E) = -\langle \mathrm{ch}(E), \alpha \rangle + i\langle \mathrm{ch}(E), \beta \rangle$$

on the hearts $\mathcal{A}$ of compatible t-structures. Bridgeland proved that the locus of stability conditions has the structure of a complex manifold Stab(X) locally homeomorphic to a finite-dimensional complex vector space via the map

$$\text{stability condition } \sigma = (\mu_\sigma, \mathcal{A}_\sigma) \mapsto Z \in \mathrm{Hom}(K(X)/\equiv, \mathbb{C})$$

where $\equiv$ is *numerical equivalence*.

In this main example, we also have:

Schur's Lemma 2 *If E is stable, then $Hom(E, E) = \mathbb{C} \cdot id_E$.*

Consider the case $X = C$. Evidently, the slope function

$$\mu(E) = \frac{\langle \mathrm{ch}(E), 1 \rangle}{\langle \mathrm{ch}(E), H \rangle} = \frac{\deg(E)}{\mathrm{rk}(E)}$$

on the category $\mathcal{A}$ of coherent sheaves satisfies the bullet points, since

- the rank of a coherent sheaf on C is non-negative, and
- the length of a rank-zero coherent sheaf is its length

and the only sheaf of rank zero **and** length zero is the zero sheaf.

The stable coherent sheaves on a curve are either skyscraper sheaves (infinite slope) or stable vector bundles (finite slope). Geometric invariant theory can be used to show that s-equivalence classes of semi-stable bundles (of fixed rank and degree) have projective coarse moduli. The s-equivalence classes of semi-stable sheaves of rank zero also have projective moduli since they are points of the symmetric powers of C. Projectivity of moduli is certainly not a direct consequence of the definition of stability and begs the following:

Question Given a stability condition $\sigma = (\mu_\sigma, \mathcal{A}_\sigma)$ on $\mathcal{D}(X)$, when are the coarse moduli spaces of semi-stable objects of $\mathcal{A}_\sigma$ projective?

Definition A torsion-free sheaf E on a polarized smooth projective surface S is *H-stable* if $\mu_H(F) < \mu_H(E)$ for all subsheaves $F \subset E$ such that the support of E/F has codimension ≤ 1.

The problem with H-stability is that it violates the second bullet:

$$\mu_H(\mathbb{C}_x) = \frac{0}{0}$$

for skyscraper (and finite-length) sheaves.

In spite of this problem, Harder–Narasimhan filtrations exist, commencing with the torsion subsheaf $E_{tor} \subset E$ and continuing with a filtration on E/E_{tor} with H-semi-stable quotients B_i of strictly decreasing H-slope. As in the curve case the s-equivalence classes of H-semi-stable torsion-free sheaves B on S have projective coarse moduli (although the Gieseker slope is better suited for the contruction of moduli spaces by geometric invariant theory [16]).

There is one important new feature of H-stability:

Theorem (Bogomolov [13]) *The Chern classes of an H-stable sheaf E satisfy the following inequality:*

$$\langle c_1(E), c_1(E)\rangle \geq 2\langle \mathrm{rk}(E), ch_2(E)\rangle.$$

This inequality allows us to implement Bridgeland's tilting construction to produce t-structures that are compatible with slope functions of the form

$$\mu_\alpha(E) = \frac{\langle \mathrm{ch}(E), \alpha\rangle}{\langle \mathrm{ch}(E), \alpha \cdot H\rangle}$$

for $\alpha = \alpha_0 + \alpha_1 + \alpha_2 \in \mathrm{H}^*(S, \mathbb{Q})$ satisfying $\langle \alpha_1, \alpha_1\rangle > 2\langle \alpha_0, \alpha_2\rangle$.

Lemma 1 *If E is an H-stable sheaf and $\langle ch(E), \alpha \cdot H\rangle = 0$, then*

$$\langle ch(E), \alpha\rangle < 0.$$

Proof The inequalities at our disposal are

$$\alpha_0 > 0, \;\; \mathrm{rk}(E) > 0, \; \langle \alpha_1, \alpha_1\rangle > 2\alpha_0\alpha_2, \; \langle c_1(E), c_1(E)\rangle \geq 2\,\mathrm{rk}(E)\,\mathrm{ch}_2(E).$$

The vanishing assumption $\langle \mathrm{ch}(E), \alpha H\rangle = \langle \alpha_0 c_1(E) + \alpha_1 \mathrm{rk}(E), H\rangle = 0$ (i.e., the class $D = \alpha_0 c_1(E) + \alpha_1 \mathrm{rk}(E)$ is perpendicular to H), together with the inequalities, gives

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$$\langle \mathrm{ch}(E), \alpha\rangle = \alpha_0 \,\mathrm{ch}_2(E) + \langle \alpha_1, \mathrm{ch}_1(E)\rangle + \alpha_2 \,\mathrm{rk}(E) < \frac{1}{2\alpha_0 \,\mathrm{rk}(E)}\langle D, D\rangle \le 0$$

by the Hodge index theorem. □

Bridgeland's tilting construction requires the data of a *torsion pair.* This consists of a pair of full subcategories $(\mathcal{F}, \mathcal{T})$ of the category of coherent sheaves on S with the property that

(i) $\mathrm{Hom}(T, F) = 0$ for all objects T of $\mathcal{T}$ and F of $\mathcal{F}$.
(ii) Each coherent sheaf E fits into an exact sequence

$$0 \to T \to E \to F \to 0.$$

These produce a t-structure on the derived category $\mathcal{D}(S)$ whose heart consists of the complexes that have cohomologies only in two degrees: -1 (and belonging to $\mathcal{F}$) and 0 (and belonging to $\mathcal{T}$). That is, the tilted abelian category consists of objects $E^{\cdot}$ of the derived category that admit a cohomology sequence

$$0 \to F[1] \to E^{\cdot} \to T \to 0; \quad F \in \mathcal{F}, T \in \mathcal{T}.$$

These are determined by T and F and a *second* extension class $\epsilon \in \mathrm{Ext}^2(T, F)$ in the abelian category of coherent sheaves which is a first extension class $\epsilon \in \mathrm{Ext}^1(T, F[1])$ in the new tilted abelian category $\mathcal{A}^{\#}$.

The particular torsion pair associated with α is defined as follows:

- The objects of $\mathcal{F}_\alpha$ are all torsion-free sheaves F such that

$$\langle \mathrm{ch}(F'), \alpha \cdot H\rangle \le 0 \text{ for all subsheaves } F' \subseteq F.$$

- The objects of $\mathcal{T}_\alpha$ are all coherent sheaves T such that

$$\langle \mathrm{ch}(T''), \alpha \cdot H\rangle > 0 \text{ or } \mathrm{len}(T'') < \infty \text{ for all quotients } T \to T'' \to 0.$$

Observation The pair $(\mathcal{F}_\alpha, \mathcal{T}_\alpha)$ satisfies (i) by definition, and (ii) by the Harder–Narasimhan filtrations defined above for a H-slope. The tilted abelian category with respect to this pair will be denoted by $\mathcal{A}_\alpha$.

Corollary 2 *The slope function μ_α satisfies the bullet points for a pre-stability condition on the tilted abelian category $\mathcal{A}_\alpha$.*

Proof The objects of $\mathcal{A}_\alpha$ are complexes of the form

$$0 \to F[1] \to E^{\cdot} \to T \to 0.$$

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By definition, objects of T all satisfy $\langle \mathrm{ch}(T), \alpha \cdot H \rangle > 0$ *with the exception of torsion sheaves of finite length*. But these sheaves satisfy

$$\langle \mathrm{ch}(T), \alpha \rangle = \alpha_0 \,\mathrm{len}(T) > 0.$$

Objects of $F[1]$ satisfy $\langle \mathrm{ch}(F[1]), \alpha \cdot H \rangle = -\langle \mathrm{ch}(F), \alpha \cdot H \rangle > 0$ *with the exception of H-semi-stable torsion-free sheaves that pair to* 0. But by the key lemma (and linearity), such sheaves satisfy

$$\langle \mathrm{ch}(F[1]), \alpha \rangle = -\langle \mathrm{ch}(F), \alpha \rangle > 0$$

and hence the bullet points are satisfied for objects of $\mathcal{A}_\alpha$. □

Remark The finiteness of Harder–Narasimhan filtrations is not difficult to prove since we are restricting to rational coefficients [6]. It is curious that it is much more difficult to prove when the coefficients are real, even though the rational stability conditions are dense. Generalizations of the Bogomolov inequality to third Chern classes of stable complexes on threefolds have had some success [2, 9, 10, 18–21], although a useful such inequality for any projective Calabi–Yau threefold has yet to be found.

3 Determinants and moduli

In this section, B is always a "reasonable" base scheme (e.g., of finite type and quasi-projective over $\mathbb{C}$) and $S \times B$ is equipped with projections

$$p : S \times B \to S \text{ and } \pi : S \times B \to B.$$

A family of derived objects on S is a (not necessarily flat) coherent sheaf E_B on $S \times B$, or more generally an object of the derived category of $S \times B$. Families can be pushed forward to B (in the derived category), and the associated line bundle on the base

$$\Delta(E_B) := c_1(R\pi_* E_B)$$

is the determinant of the family. This will not, in general, define a line bundle on coarse moduli spaces of *isomorphism classes* of sheaves or derived objects. However, in the case where the Euler characteristic of the (derived) fibers over closed points vanishes:

$$\chi(S_b, E_b) := \chi(S \times \{b\}, Li_b^* E_B) = 0$$

then the determinant satisfies $\Delta(E_B) = \Delta(E_B \otimes \pi^* L)$ for any line bundle L on B and therefore descends to isomorphism classes of simple objects (i.e., objects with minimal automorphism group $\mathbb{C} \cdot id$).

In other words, in light of the Schur lemmas, the $\chi = 0$ condition on families is the precise condition that ought to result in a line bundle that descends to the coarse moduli spaces of σ-stable objects for any given stability condition σ.

A family of vector bundles E_B of rank r and degree d on a curve C of genus g can be transformed into a family of $\chi = 0$ vector bundles by choosing a vector F bundle on C with the property that

$$\mu(E_b) + \mu(F) = g - 1$$

and replacing E_B with the family $E_B \otimes p^*F$.

This not only gives a determinant line bundle, but also a pseudo-divisor on B associated with each family, with support

$$\Theta_F(E_B) := \{b \in B \mid \mathrm{H}^1(C_b, E_b \otimes F) \neq 0\}.$$

That is, the locus where the cohomology (in either degree) is nonzero.

The coarse moduli spaces $\mathcal{M}_C(s, L)$ of semi-stable vector bundles of rank s and fixed determinant $\wedge^s F = L$ are unirational and their Picard groups are generated by a single line bundle. It follows that the determinant line bundles Δ_F on $M_C(r, d)$ are independent of the choice of semi-stable bundle $F \in \mathcal{M}_C(s, L)$ (in each rank) and therefore that the pseudo-divisors Θ_F are linearly equivalent. Moreover, the line bundles Δ_F are ample, which can be proved directly or else by appealing to the fact that the Δ_F coincide with the ample line bundles (up to scaling) arising from the geometric invariant theory construction of moduli.

When we look for analogous polarizations on the moduli spaces of σ-stable objects on a surface S, we immediately run into the following:

Observation Orthogonal classes to $\mathrm{ch}(E)$ are not unique. Let

$$\widetilde{\mathrm{NS}}(S)_{\mathbb{Q}} = \mathrm{H}^0(S, \mathbb{Q}) \oplus \mathrm{NS}(S)_{\mathbb{Q}} \oplus \mathrm{H}^4(S, \mathbb{Q})$$

be the extended Néron–Severi vector space, of rank $\rho + 2$ over $\mathbb{Q}$, with the induced inner product from cohomology and let

$$c^\perp = \{\alpha \in \widetilde{\mathrm{NS}}(S)_{\mathbb{Q}} \mid \langle c, \alpha \rangle = 0\}$$

be the orthogonal complement of a Chern class $c = c_0 + c_1 + c_2$.

Now suppose $\alpha \in c^\perp$ and that F is an H-stable vector bundle with

$$\mathrm{ch}(F) \cdot \mathrm{td}(S) = \alpha \in \widetilde{\mathrm{NS}}(S)_{\mathbb{Q}}.$$

By the Hirzebruch–Riemann–Roch theorem, $\chi(S, E^\cdot \otimes F) = 0$ for any object $E^\cdot$ with $\mathrm{ch}(E^\cdot) = c$. Thus any *family* $E^\cdot_B$ of such objects gives rise to a determinant line bundle $\Delta_F(E^\cdot_B)$ on the base of the family with the desired invariance under tensoring by a line bundle from B. This is the candidate for the line bundle on coarse moduli of stable objects of class c, but it is important to notice that **both** the coarse moduli space (of α-stable objects of $\mathcal{A}_\alpha$) **and** the determinant line bundle on that moduli space *depend upon the choice of* $\alpha \in c^\perp$.

The dependence of the line bundles Δ_F on the class $\alpha \in c^\perp$ is linear. Indeed, the Grothendieck–Riemann–Roch theorem gives

$$c_1(R\pi_*(E_B^\cdot \otimes p^*F)) = \pi_*\left(\mathrm{ch}_1(E_B^\cdot)p^*\alpha_2 + \mathrm{ch}_2(E_B^\cdot)p^*\alpha_1 + \mathrm{ch}_3(E_B^\cdot)p^*\alpha_0\right)_1$$

where π_* is the push-forward on Chern classes.

Consider, for example, the **Hilbert schemes** $S^{[n]}$ with universal ideal sheaf $\mathcal{I}_{\mathcal{Z}} \subset O_{S\times S^{[n]}}$. This is a deceptively nice example. Since Hilbert schemes are smooth and projective of the right dimension $2n$, they are defined independently of the choice of ample class H and have a preferred universal family $\mathcal{I}_{\mathcal{Z}}$.

The Chern character of the ideal sheaf $\mathcal{I}_Z$ of n points is $c = (1, 0, -n)$. Since $c^\perp = \{\alpha \mid n\alpha_0 = \alpha_2\}$, we have

$$c^\perp \cap \{\alpha_0 = 1\} = \{\alpha = (1, D, n) \mid D \in \mathrm{NS}(S)\}$$

and the positivity condition on α means that α determines a stability condition on S when

$$\langle D, D\rangle > 2n.$$

Let $D_t = -\frac{1}{2}K_S + tH'$ and $\alpha_t = (1, D_t, n)$ where H' is a second choice of ample divisor class (this will be important in later examples). For integer values of t, the resulting α is of the form

$$\mathrm{ch}(\mathcal{I}_W(tH'))\mathrm{td}(S) = \alpha_t$$

for a subscheme $W \subset S$ of the appropriate length. Thus, in this case, we may let $F = \mathcal{I}_W(tH')$ in the computation of the determinant line bundle (keeping in mind that this is torsion free and not locally free). Notice that for sufficiently large and small values of t, the class α_t defines stability conditions. When $t >> 0$, the second cohomology vanishes:

$$\mathrm{H}^2(S, \mathcal{I}_Z \otimes F) = \mathrm{H}^2(S, \mathcal{I}_Z \otimes \mathcal{I}_W \otimes O_S(tH')) = 0$$

and it follows that for such values of t, the pseudo-divisors

$$\Theta_F = \{Z \mid \mathrm{H}^1(S, \mathcal{I}_Z \otimes F) \neq 0\}$$

represent the line bundle Δ_F. Its class is easily computed [3]:

$$\Delta_F = \pi_*\left(\mathrm{ch}_2(\mathcal{I}_{\mathcal{Z}})p^*D_t + \mathrm{ch}_3(\mathcal{I}_{\mathcal{Z}})\right) = -q^*(D_t^{(n)}) + \frac{1}{2}E$$

where $q : S^{[n]} \to S^n/\Sigma_n$ is the Hilbert–Chow morphism from the Hilbert scheme to the symmetric product, E is the exceptional divisor, and $D^{(n)}$ is the symmetric divisor on S^n/Σ_n associated with D.

Notice that this class is anti-ample for $t >> 0$. This is a consequence of the fact that the determinant line bundle is the divisor class given by the support of the sheaf $\mathrm{R}^1\pi_*(\mathcal{I}_{\mathcal{Z}} \otimes F)$, which has odd degree.

When $t << 0$, there is a similar result. The family of derived objects to consider in this case is the shift of the derived dual $\mathcal{I}_{\mathcal{Z}}^{\vee}[1]$ in the derived category of $S \times S^{[n]}$. This is a family of objects of the derived category of S with Chern character invariants $-c = (-1, 0, n)$. In this case, the determinant line bundle is the same as above but with opposite sign, reflecting the fact that in this case the pseudo-divisor gives the support of the sheaf $\mathrm{R}^2\pi_*(\mathcal{I}_{\mathcal{Z}}^{\vee}[1] \otimes F)$, which has even degree.

Turning to a more general example, suppose $c = (c_0, c_1, c_2) = \mathrm{ch}(E)$ where E is a Gieseker H-stable torsion-free sheaf on S. Gieseker stability is given in terms of the Hilbert polynomial, but can be defined for polarized surfaces as follows:

Definition A torsion-free sheaf E on S is *Gieseker H-unstable* if there is a subsheaf $F \subset E$ such that either

(i) $\mu_H(F) > \mu_H(E)$ (i.e., E is Mumford H-unstable), or else
(ii) $\mu_H(F) = \mu_H(E)$ and $\frac{\chi(S,F)}{\mathrm{rk}(F)} > \frac{\chi(S,E)}{\mathrm{rk}(E)}$.

It is *Gieseker H-stable* if for every subsheaf $F \subset E$, either

(i) $\mu_H(F) < \mu_H(E)$, or else
(ii) $\mu_H(F) = \mu_H(E)$ and $\frac{\chi(S,F)}{\mathrm{rk}(F)} < \frac{\chi(S,E)}{\mathrm{rk}(E)}$

and Gieseker H-semi-stable is defined in the usual way.

The moduli of Gieseker semi-stable equivalence classes have a natural construction via geometric invariant theory. This is also reflected in their naturalness from the point of view of α-stability conditions. As with the Hilbert scheme, consider the one-parameter family of elements of $c^{\perp}$, defined by another ample divisor class H' via

$$\alpha_t = (1, D_t, d_t), \text{ where } D_t = -\frac{K_S}{2} + tH', d_t = -\frac{1}{c_0}(D_t \cdot c_1 + c_2).$$

Remark For large t, we have the required positivity

$$\langle D_t, D_t \rangle > 2d_t$$

since the left side grows quadratically with t and the right grows linearly.

Lemma 3 *Suppose $ch(E^{\cdot}) = c$ and $E^{\cdot}$ is α_t-stable for all $t > t_0$. Then*

(i) $\mathrm{H}^{-1}(E^{\cdot}) = 0$, *i.e.,* $E^{\cdot} = E = \mathrm{H}^0(E^{\cdot})$ *is a coherent sheaf.*

(ii) *E is Mumford H'-semi-stable.*

(iii) *E is Gieseker H'-semi-stable.*

Proof Let $F = H^{-1}(E^{\cdot})$. For $F[1]$ to belong to the tilted category $\mathcal{A}_{\alpha_t}$, it is required that F be a torsion-free sheaf and that

$$\langle F, \alpha_t \cdot H\rangle = \langle c_1(F), H\rangle + \left\langle \mathrm{ch}_0(F), \left(-\frac{K_S}{2} + tH'\right) \cdot H\right\rangle \leq 0.$$

But this is positive when t is sufficiently large, proving (i).

Next, suppose E is H'-unstable, i.e., that there is an $F \subset E$ such that

$$\frac{\langle c_1(F), H'\rangle}{\mathrm{ch}_0(F)} > \frac{\langle c_1(E), H'\rangle}{\mathrm{ch}_0(E)}.$$

Then

$$\langle \mathrm{ch}(F), \alpha_t\rangle = t\left[-\frac{\mathrm{ch}_0(F)\langle H', c_1(E)\rangle}{\mathrm{ch}_0(E)} + \langle c_1(F), H'\rangle\right] + \text{constant}$$

and this is positive when t is large. But $\langle \mathrm{ch}(E), \alpha_t\rangle = 0$ by construction, so $\mu_{\alpha_t}(F) > 0 = \mu_{\alpha_t}(F)$ when t is large. Similarly, if E is Gieseker H'-unstable, then either it is Mumford unstable (already done) or else there is an $F \subset E$ with $\mu_H(F) = \mu_H(E)$ and $\frac{\chi(S,F)}{ch_0(F)} > \frac{\chi(S,E)}{ch_0(E)}$. In this case the linear term in $\langle \mathrm{ch}(F), \alpha_t\rangle$ vanishes, but the constant term is

$$-\frac{\mathrm{ch}_0(F)}{\mathrm{ch}_0(E)}\langle \mathrm{ch}(E), \mathrm{td}(S)\rangle + \langle \mathrm{ch}(F), \mathrm{td}(S)\rangle > 0$$

by the Riemann–Roch theorem. □

Remark There is little dependence here upon the choice of H. In fact, because $\langle \mathrm{ch}(E^{\cdot}), \alpha_t\rangle = 0$ for all t, it follows that $E^{\cdot}$ is α_t-stable if and only if $\langle \mathrm{ch}(F^{\cdot}), \alpha_t\rangle < 0$ for all $F^{\cdot} \subset E^{\cdot}$. The **only** dependence upon H is in the categories $\mathcal{A}_{\alpha_t}$ in which the inclusions $F^{\cdot} \subset E^{\cdot}$ take place. Since these categories $\mathcal{A}_{\alpha_t}$ eventually include any given coherent sheaf and exclude the shift of any given coherent sheaf, it is unsurprising that H disappears in the limit.

A converse to the lemma is also easy:

Lemma 4 *Suppose E is a Gieseker H'-semi-stable torsion-free sheaf with* $\mathrm{ch}(E) = c$ *and* $E \in \mathcal{A}_{\alpha_{t_0}}$. *Any α_{t_0}-destabilizing subobject $F^{\cdot} \subset E$ will fail to destabilize E for large t.*

Proof From the long exact sequence in cohomology, it follows that a subobject $F^{\cdot} \subset E$ is a coherent sheaf (though it may not be a subsheaf). If $Q^{\cdot} = E/F$ is the quotient in $\mathcal{A}_{t_0}$, then

$$0 \to \mathrm{H}^{-1}(Q^{\cdot}) \to F \to E \to \mathrm{H}^0(Q^{\cdot}) \to 0$$

is a long exact sequence of coherent sheaves. Now, as in the previous lemma, if $\mathrm{H}^{-1}(Q^{\cdot}) \neq 0$ then it is eventually not in the category $\mathcal{A}_{\alpha_t}$, and the inclusion $F \subset E$ is destroyed (even though both coherent sheaves are in the category $\mathcal{A}_{\alpha_t}$). On the contrary, if $\mathrm{H}^{-1}(Q^{\cdot}) = 0$ then $F \subset E$ is a subsheaf, and it follows that the Gieseker slope of F is either smaller than that of E, in which case the α_t-slope is also eventually smaller, or else they are equal, in which case $\langle \mathrm{ch}(F), \alpha_t \rangle = 0$ for **all** t and $F \subset E$ was not a destabilizing subobject at $t = t_0$. □

This is evidence that the coarse moduli spaces $\mathcal{M}_{H'}(c)$ of Gieseker semi-stable sheaves should be the moduli of α_t-semi-stable objects of $\mathcal{A}_{\alpha_t}$ for large t. More evidence is also available in the positivity of the determinant line bundle on the moduli space $\mathcal{M}_{H'}(c)$:

Theorem ([1]) *The determinant line bundles Δ_F for large t and F semi-stable with* $\mathrm{ch}(F) \cdot td(S) = \lambda\alpha_t$ *are positive on families of Gieseker H'-stable sheaves, nef on families of semi-stable sheaves and descend to a positive line bundle on the coarse moduli space $\mathcal{M}_{H'}(c)$.*

This is, at least, consistent with the theorem of Bayer–Macrí [7], which comes to the same conclusion for the determinant line bundle on the moduli of α_t-stable objects. It is hard to see how to make this into a proof, however, without uniform versions of the lemmas. That is, we are faced with the following:

Problems (a) How to show that there is a uniform bound T for all H'-semi-stable sheaves E such that for $t > T$, all "objections" to α_t-stability (in the form of nontrivial Harder–Narasimhan filtrations) disappear?

(b) How to show that there is a uniform bound T such that for all $t > T$, **all** objects other than semi-stable sheaves fail to be α_t-stable?

In special cases, these problems have been solved (e.g., [3, 4, 11] and notably [8], in which essentially everything is done for K3 surfaces), but there is not (to my knowledge) a one-size-fits-all-surfaces solution.

To sum up, we have an attractive picture in these α coordinates.

The picture For each Chern class $c = \mathrm{ch}(E)$, there is a base point

$$\alpha_0 = \left(1, -\frac{1}{2}K_S, d_0\right) \in c^{\perp} \cap \{\alpha_0 = 1\}$$

in the affine space, from which rays emanate in ample directions

$$\alpha_t := \left(1, -\frac{1}{2}K_S + tH', d_0 - t\left(\frac{c_1 \cdot H'}{c_0}\right)\right);\ t \geq 0$$

with the following properties:

(i) Each ray eventually enters the stability manifold.
(ii) Gieseker H'-semi-stable sheaves are eventually α_t-semi-stable.
(iii) Everything else is eventually α_t-unstable.

And it is expected that:

(iv) The moduli of α_t-stable sheaves for $t > T$ coincide with $\mathcal{M}_{H'}(c)$.

In any case, Gieseker H'-semi-stable sheaves are identified with rays in the affine space $c^{\perp} \cap \{\alpha_0 = 1\}$ emanating from the base point α_0. The choice of base point is actually quite important. If, for example, H' is a critical value for Gieseker moduli, in which there are strictly semi-stable sheaves although the invariants $c = (c_0, c_1, c_2)$ are primitive (meaning that for "nearby" ample classes H, semi-stability and stability coincide), then we expect to detect the intermediate variations in moduli, discovered by Matsuki and Wentworth [22], by varying the base point while continuing to point the ray in the direction H'.

4 The Serre map

One case remains, namely that of Gieseker-stable torsion sheaves on a surface S. Several examples have been studied recently in the context of Bridgeland stability, but I want to focus on one case, namely that of sheaves $\mathcal{E}$ with the following invariants:

$$c_1(\mathcal{E}) = 2H - K_S,\ \ \chi(S, \mathcal{E}) = 0$$

where H is an ample divisor class (for which $2H - K_S$ is effective).

One example of such sheaves are the quotients of maps

$$0 \to L^{-1} \otimes \omega_S \to L \to \mathcal{E} \to 0$$

when $L = \mathcal{O}_S(H)$ and $\omega_S = \mathcal{O}_S(K_S)$.

As before, we consider stability conditions α_t along a ray:

$$\alpha_t = \left(1, -\frac{1}{2}K_S, -t\right) \in c^\perp \cap \{\alpha_0 = 1\}$$

although in this case the variation is entirely concentrated in $H^4(S, \mathbb{Q})$. There are several interesting properties of these stability conditions α_t:

(i) The ray enters the stability manifold for some $t \leq \frac{1}{8}K_S^2$.
(ii) The tilted categories $\mathcal{A}_{\alpha_t}$ are constant, since

$$\langle \mathrm{ch}(E), \alpha_t \cdot H\rangle = \mathrm{ch}_0(E)\left(-\frac{1}{2}K_S \cdot H\right) + \mathrm{ch}_1(E) \cdot H$$

is independent of t. Recall that extensions $0 \to F[1] \to E^{\cdot} \to T \to 0$ give the elements of $\mathcal{A}_{\alpha_t}$, where the semi-stable pieces of the Harder–Narasimhan filtration of F satisfy $\frac{c_1(E)\cdot H}{rk(E)} \leq \frac{1}{2}K_S \cdot H$ and those of the sheaf T (other than the torsion) satisfy $\frac{c_1(E)\cdot H}{rk(F)} > \frac{1}{2}K_S \cdot H$.

In particular, there is a Serre map for surfaces, analogous to the family of vector bundles on a curve parametrized by

$$\mathbb{P}(\mathrm{Ext}^1(L, \omega_C)) = \mathbb{P}(\mathrm{H}^0(C, L)^*)$$

via the "lines" $\{\lambda\epsilon : 0 \to \omega_C \to E \to L \to 0; \lambda \in \mathbb{C}^*\}$ of extensions. Recall that the generic such extension gives a **stable** vector bundle of rank two provided that $\deg(L) > \deg(K_C)$.

On a surface, a family of objects of $\mathcal{A}_{\alpha_t}$ is parametrized by

$$\mathbb{P}(\mathrm{Ext}^2(L, L^{-1} \otimes \omega_S)) = \mathbb{P}(\mathrm{H}^0(S, L^{\otimes 2}))$$

via the "lines" $\{\lambda\epsilon : 0 \to L^{-1} \otimes \omega_S[1] \to E^{\cdot} \to L \to 0; \lambda \in \mathbb{C}^*\}$ of extensions provided that $\langle H, H\rangle > \langle \frac{1}{2}K_S, H\rangle$. Each value of t gives a *different* stability criterion for the objects $E^{\cdot}$. By the computation $\mu_{\alpha_t}(L) = 0 = \mu_{\alpha_t}(K_S - L) \Leftrightarrow t_1 = \langle H - K_S, H\rangle$ it follows that if L and $L^{-1} \otimes \omega_S[1]$ are both stable at t_1 (a nontrivial assertion! [5]), then

- each of the $E^{\cdot}$ parametrized by $\epsilon \in \mathbb{P}(\mathrm{Ext}^2(L, L^{-1} \otimes \omega_S))$ and
- each of the sheaves $\mathcal{E}$ parametrized by $\delta \in \mathbb{P}(\mathrm{Hom}(L^{-1} \otimes \omega_S, L))$

are strictly semi-stable objects, with Jordan–Hölder filtrations

$$0 \to L^{-1} \otimes \omega_S[1] \to E^{\cdot} \to L \to 0 \text{ and } 0 \to L \to \mathcal{E} \to L^{-1} \otimes \omega_S[1] \to 0$$

respectively. This suggests that for $t_2 < t < t_1$, a general extension ϵ will parametrize an α_t-stable object of $\mathcal{A}_{\alpha_t}$, while for $t > t_1$ one suspects (and it is proved in many cases) that already the moduli of Gieseker-stable sheaves coincides with the moduli of α_t-stable objects, i.e., that T is the uniform bound sought in the previous section. My student, Christian Martinez, has recently proven that the closure of the image of the Serre map has a particularly nice property:

Theorem 5 ([21]) *The involution on the derived category*

$$E^{\cdot} \mapsto (E^{\cdot})^{\vee} \otimes \omega_S[1]$$

induces an involution on the moduli of α_t-stable objects of Chern class $c = \mathrm{ch}(L) - \mathrm{ch}(L^{-1} \otimes \omega_S)$ and this involution fixes objects in the image of the Serre map.

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