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Line arrangements modeling curves of high degree: Equations, syzygies, and secants

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Abstract

We study curves consisting of unions of projective lines whose intersections are given by graphs. Under suitable hypotheses on the graph, these so-called *graph curves* can be embedded in projective space as line arrangements. We discuss property N_p for these embeddings and are able to obtain products of linear forms that generate the ideal in certain cases. We also briefly discuss questions regarding the higher-dimensional subspace arrangements obtained by taking the secant varieties of graph curves.

1 Introduction

An arrangement of linear subspaces, or subspace arrangement, is the union of a finite collection of linear subspaces of projective space. In this paper we study arrangements of lines called graph curves with high degree relative to genus. We are particularly interested in the defining equations and syzygies of these subspace arrangements. We will assume an algebraically closed ground field of characteristic zero throughout.

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Let $G = (V, E)$ be a simple, connected graph with vertex set V and edge set E. Following [9], we assume that G is *subtrivalent*, meaning that each vertex has degree at most three. The (abstract) *graph curve* C_G associated with G is constructed by taking the union of $\{L_v \mid v \in V\}$, where each L_v is a copy of $\mathbb{P}^1$ and lines L_u and L_v intersect in a node if and only if there is an edge between u and v in G. (Note that if we think of the nodes of C_G as vertices and the lines L_v as edges, then C_G is the graph dual to G.) Since we are assuming that each vertex has degree less than or equal to three, C_G is specified by purely combinatorial data; we may assume that on each component of C_G the nodes are at 0, 1 or ∞. Note that if each vertex of G is trivalent, then each copy of $\mathbb{P}^1$ in C_G contains three nodes, and C_G is stable (see [4, 9]).

The motivation for the work presented here was to see if the syzygies of a high-degree graph curve and its secant varieties would behave as they are expected to when the curve is smooth. The kth secant variety, Σ_k, of a smooth curve in $\mathbb{P}^r$ has expected dimension $\min\{2k + 1, r\}$. Thus, we expect the kth secant variety of C_G to be an arrangement of subspaces of dimension $2k + 1$.

Many authors [10, 16, 22, 25] have given generalizations of the results for smooth curves to higher-dimensional varieties, showing that embeddings via line bundles satisfying various positivity conditions will also satisfy property N_p. However, recent work of Ein and Lazarsfeld [10] shows that these results describe only a small portion of the minimal free resolution of a higher-dimensional variety, and what happens in the remaining piece is quite complicated, contrary to the belief that positivity of an embedding simplifies syzygies.

One can view the conjecture of [29], which says that we should expect property $N_{k+2,p}$ (ideal generators of degree $k + 2$ and linear syzygies through stage p) for the kth secant variety of a smooth curve of genus g embedded via a complete linear series of degree at least $2g + 2k + 1 + p$, as an alternate way of generalizing property N_p for curves to higher-dimensional varieties. Some progress was made for first secant varieties, using geometric methods in [27], but the recursive nature of these methods makes generalizing those techniques to higher secant varieties daunting. If a similar result were true for the secant varieties of graph curves, the proof methods would necessarily be very different and the hope is that they would shed new light on understanding secant varieties of smooth curves.

Based on many examples computed with Macaulay2 [18], the situation looks promising. However, when $g > 2$, the combinatorics can be intricate even if we only consider curves and not secant varieties. We will focus on curves in Sections 2 and 3 and turn to a discussion of the syzygies of secant varieties of graph curves in Section 4.

We begin by setting some assumptions and notation. Let d be the number of vertices in G. The topology of G determines the arithmetic genus of C_G as we may view G as a 1-dimensional simplicial complex, from which it follows that $p_a(C_G) = h^1(G, k)$ if G is connected (see Proposition 1.1 in [4]). We refer to this quantity as the genus g of G, and $|E| = d + g - 1$. Note that g is not the genus of G in the usual graph-theoretic sense.

The story that we wish to generalize to the setting of graph curves began with Green and Lazarsfeld [19] in the early 1980s, who showed that if C is a smooth and irreducible curve of genus g embedded in projective space via a complete linear series of degree $d \geq 2g+1+p$, then C satisfies property N_p. In other words, its ideal is generated by quadrics with syzygy modules generated by linear forms through the pth stage of the resolution.

We conjecture that if G satisfies Assumption 1.1, then property N_p will hold for C_G embedded as a line arrangement in $\mathbb{P}^{d-g}$.

Assumption 1.1 Fix $p \geq 0$ and let G be a simple, connected, subtrivalent graph with $d \geq 2g + 1 + p$. Assume that if G' is a connected subgraph induced on $V' \subset V$, $d' = |V'|$, and g' is the genus of G', then $d' \geq 2g' + 1 + p$ if $g' \geq 1$.

To see that the recursive hypotheses are necessary, note that a graph may satisfy $d \geq 2g + 2$, but if it contains a triangle, then the ideal of C_G cannot be generated by quadrics.

If Assumption 1.1 is satisfied for some $p \geq 0$, then C_G embeds in $\mathbb{P}^{d-g}$ as a line arrangement via [7] and is arithmetically Cohen–Macaulay by [15]. If C_G is arithmetically Cohen–Macaulay, we may proceed as in [19] and property N_p for C_G will follow if property N_p holds for a general hyperplane section. In [19], Green and Lazarsfeld deduce property N_p for points in linearly general position, and conjecture that the failure of property N_p for a set of $2r + 1 + p$ points implies the existence of a subset of $2k + 2 - p$ points on a $\mathbb{P}^k$. As shown in [14, 20], this conjecture for point sets is a consequence of the linear syzygy conjecture of Eisenbud, Koh, and Stillman [13]. Green proved the linear syzygy conjecture in [20], and for graph curves of degree $g \leq 2$ we can show that an embedding of C_G as a line arrangement via a complete linear series must satisfy N_p if Assumption 1.1 is satisfied.

Graph curves associated with graphs in which every vertex is trivalent are canonical curves, and have been studied in several different contexts. For example, Ciliberto, Harris, and Miranda [8] used graph curves to understand the surjectivity of the Wahl map, Ciliberto and Miranda [9] related graph curves to graph colorings, and Bayer and Eisenbud [4] studied graph curves in connection with Green's conjecture. In fact, Proposition 3.1 in [4] gives an explicit description of generators of the ideal of a canonical graph curve using the combinatorics of G. More recently, Ballico has written several papers about graph curves [1, 2].

We present an explicit embedding of C_G into projective space in Section 3. If the ideal of C_G is generated by quadrics, this allows us to show that I_{C_G} may be generated by products of linear forms (Theorem 3.7).

Although a subspace arrangement may always be cut from products of linear forms set-theoretically, we do not generally expect the ideal of a subspace arrangement to be generated by products of linear forms, cf. Propositions 5.4 and 5.7 in [5]. The most interesting examples of subspace arrangements with ideals generated by products of linear forms occur when the intersections among the subspaces have a rich combinatorial structure [5, 23, 24]. If G is a path or a cycle, then C_G can be embedded in projective space so that its ideal is generated by square-free monomials. In both cases, the ideals of the nontrivial secant varieties of these curves are also generated by square-free monomials and are examples of "combinatorial secant varieties" [28].

In addition to viewing graph curves and their secant varieties as combinatorial models of smooth curves and their secant varieties, we can also think of them as a new way of generating arrangements of linear subspaces with interesting interactions between the combinatorics of the arrangements, the geometry of the embeddings, and their defining equations. We present conjectures and questions for further work in this direction in Section 5.

2 Regularity and property N_p

In this section we will show that if $g \leq 2$, then the ideal of a linearly normal embedding of C_G as a line arrangement satisfies property N_p if G satisfies Assumption 1.1 for some $p \geq 0$, following the idea of the "quick" proof that a smooth and irreducible curve of degree $d \geq 2g+1+p$ satisfies N_p given in [19].

A key assumption in [19] is that a hyperplane section of a smooth curve of degree $d \geq 2g + 1 + p$ will consist of points in linearly general position. This fact is used to show that the points in a hyperplane section of the curve impose independent conditions on quadrics.

Using Lemma 2.1 we can show that this is not the case for a graph curve if G contains a cycle as a proper subgraph.

Lemma 2.1 *If G is a cycle on d vertices, then a hyperplane section of C_G has a 1-dimensional space of linear dependence relations and all of the points are contained in the support of the relation.*

Proof A cycle of length d embeds into $\mathbb{P}^{d-1}$, so the hyperplane section consists of d points in $\mathbb{P}^{d-2}$. A set of d points spanning $\mathbb{P}^{d-2}$ must satisfy a unique relation up to scalar. □

Therefore, if we have a cycle as a proper subset of a graph G, the points of a hyperplane section must fail to be in linearly general position. Because N_p fails if G contains a cycle of length $p+2$, it will often be impossible to reproduce the graded Betti diagrams of a smooth curve with the graded Betti diagrams of a graph curve. For instance, for genus $g = 2$ and degree d, the length of the smallest cycle has an upper bound of $\lfloor \frac{2d-1}{3} \rfloor + 1$.

Nevertheless, we will show that if G satisfies Assumption 1.1 for $g \leq 2$, then a general hyperplane section of C_G imposes independent conditions on quadrics. This follows from the weaker assumption that no $2k+2$ of the points lie on a $\mathbb{P}^k$ using ideas from [14].

Theorem 2.2 *Suppose that G satisfies Assumption 1.1 for some $p \geq 0$, $g \leq 2$, and C_G is embedded in $\mathbb{P}^{d-g}$ as a line arrangement via a complete linear series. If H is a general hyperplane and $X = H \cap C_G$, then there is no set of $2k+2-p$ dependent points of X lying on a $\mathbb{P}^k$.*

Proof Let $Y \subset X$. Suppose for contradiction that $|Y| = 2k+2-p$ and Y spans a $\mathbb{P}^k$. This means that there is a $2k+2-p-(k+1) = k+1-p = m$-dimensional space of dependence relations on Y. Since $g \leq 2$, we know that $m \leq 2$. If $m = 0$, then the points are independent, which contradicts our hypotheses.

If $m = 1$, then $k = p$. Either the support of the unique dependence relation on Y contains a cycle of points, or the relation is a linear combination of dependence relations on two cycles in which at least one point has been eliminated from their support. If $\{\gamma_i\}$ form a basis for $H_1(G; \mathbb{R})$, the corresponding dependence relations $\{R_i\}$ form a basis for the space of linear relations on X, and Assumption 1.1 implies that $\gamma_1 \cup \gamma_2$ contains at least $5+p$ points. The cycles γ_1 and γ_2 can be combined in $H_1(G; \mathbb{R})$ to form a distinct cycle γ_3, which also supports a unique linear dependence. Therefore, if we fix the coefficient of R_1 there is a unique multiple of R_2 that eliminates the shared points in the interior of their common path to create a dependence relation with support on γ_3. Consequently, we see that we cannot simultaneously eliminate the endpoints of this path and the points between them from the support. Therefore, if a linear combination of R_1 and R_2 is not supported on a full cycle, it contains at least $2 \cdot 2 + 1 + p - 2 = 2 + 1 + p = 3 + p$ points, implying that Y spans a projective space of dimension at least $p+1$, which is a contradiction as $k = p$.

If $m = 2$, then $k = p + 1$. In this case $g = 2$, and Y must contain the support of both cycles of G, in which case $2k + 2 - p \geq 2 \cdot 2 + 1 + p$, or $2k \geq 2p + 3$, which contradicts $k = p + 1$. □

Remark We conjecture that if G satisfies Assumption 1.1 and C_G is embedded via a complete linear series, then Theorem 2.2 holds for all g. The idea is that if there is an m-dimensional space of dependence relations on Y, then we need at least m independent cycles of G to span this space. The support of m cycles contains at least $2m+1+p$ points. If more than m cycles are needed to span the space of dependence relations of Y, then we may have eliminated some points from the support, but we will always have at least $2m + 1 + p$ points remaining.

Theorem 2.3 *If G satisfies Assumption 1.1 for some $p \geq 0$, and no $2k+2-p$ points of X lie on a $\mathbb{P}^k$, then a general hyperplane section of C_G has a 3-regular ideal and satisfies property N_p.*

Proof The proposition on p. 169 of [14] states that X imposes independent conditions on quadrics if X does not contain a subset of $2k + 2$ points on a projective k-plane. This implies that the ideal of X is 3-regular by Lemma 2 of [14]. The ideal of X satisfies N_p as a consequence of Theorem 2.1 in [20]. □

Theorem 2.4 *Suppose that G satisfies Assumption 1.1 for some $p \geq 0$, $d \geq 2g + 1 + p$, and C_G is embedded in $\mathbb{P}^{d-g}$ as a line arrangement via a complete linear series. If no $2k + 2 - p$ points of a general hyperplane section lie on a $\mathbb{P}^k$, then this embedding is arithmetically Cohen–Macualay, 3-regular, and satisfies N_p.*

Proof For 3-regularity we need $H^1(\mathcal{I}_{C_G}(2)) = H^2(\mathcal{I}_{C_G}(1)) = 0$. We know that $H^1(O_{C_G}(1)) = 0$ by Serre duality and our hypothesis that $d \geq 2g + 2$. This implies that $H^2(\mathcal{I}_{C_G}(1)) = 0$. To see the vanishing of $H^1(\mathcal{I}_{C_G}(2))$, note via Theorem 2.3 the regularity of the ideal of a general hyperplane section X of C_G is 3, which implies that $H^1(\mathcal{I}_X(2)) = 0$. Since C_G is embedded via a complete linear series, $H^1(\mathcal{I}_{C_G}(1)) = 0$, and we conclude that $H^1(\mathcal{I}_{C_G}(2)) = 0$.

The curve $C_G \subset \mathbb{P}^{d-g}$ is arithmetically Cohen–Macaulay if its homogeneous coordinate ring is Cohen–Macaulay. Equivalently, the hypersurfaces of degree m are a complete linear series, which holds if and only if $H^1(\mathcal{I}_{C_G}(m)) = 0$ for all $m \geq 0$ (see Section 8A of [11]). When $m = 0$, this follows because C_G is connected. We know that $H^1(\mathcal{I}_{C_G}(1)) = 0$ from the linear normality of the embedding, and $H^1(\mathcal{I}_{C_G}(k)) = 0$ for all $k \geq 2$ by the 3-regularity of the ideal. □

Corollary 2.5 *If G satisfies Assumption 1.1 for some $p \geq 0$, and $g \leq 2$, then an embedding of C_G as a line arrangement via a complete linear series is arithmetically Cohen–Macualay, 3-regular, and satisfies N_p.*

Proof Theorem 2.2 implies that the hypotheses of Theorems 2.3 and 2.4 are satisfied. □

Note that by Theorem 4.2 of [15], we know that if Assumption 1.1 holds for some $p \geq 0$, then an embedding of C_G as a line arrangement is always Cohen–Macaulay, as our singularities are planar. Moreover, Ballico and Franciosi [3] proved that a line bundle L on a reduced curve C satisfies property N_p under certain numerical conditions on the positivity of L with respect to subcurves constructed from an ordering of the irreducible components of C. Their hypothesis on the degree of L restricted to an irreducible component fails if G contains a cycle or if $p > 0$, and L has degree 1 on each line. However, if G is a tree, then Assumption 1.1 is automatically satisfied, so we expect that the ideal of C_G is 2-regular in this case. In fact, this follows from [12] because the lines in C_G can be ordered in such a way that the ith line intersects the span of the previous lines in a single point.

3 Line arrangements generated by products of linear forms

In this section we present an embedding of C_G into projective space if its edges can be labeled according to certain rules described below. If the ideal of C_G is generated by quadrics, then we identify conditions on the labeling that guarantee the existence of generators of the ideal of C_G that are monomial and binomial products of linear forms.

Given a graph G satisfying Assumption 1.1, construct $\tilde{G}$ from G by adding a loop to each vertex of degree 1 so that vertices of degree 1 in G are incident to two edges in $\tilde{G}$. For the induction in Theorem 3.7 we also need to allow the possibility of the addition of a loop at vertices with degree 2 in G. We describe the embedding of $C_G \subset \mathbb{P}^{d-g}$ by labeling the edges of $\tilde{G}$ with monomial and binomial linear forms in $S[x_0, \ldots, x_{d-g}]$ that indicate how coordinates of $\mathbb{P}^{d-g}$ parameterize each line L_v.

Label each edge of $\tilde{G}$ with a monomial x_i or a binomial $x_i - x_j$ subject to the following rules:

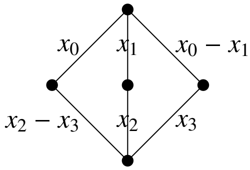
1. We require that each variable x_i appears as a monomial edge label exactly once.
2. Binomials only appear on non-loop edges.

3. Each edge labeled with a binomial is incident to a vertex with three incident edges.
4. If v has three incident edges, then they are labeled x_j, x_k, and $x_j - x_k$, where $j \neq k \in \{0, \dots, d - g\}$.

For a fixed graph G, it may be the case that some $\tilde{G}$ can be labeled according to these rules and others can not.

To define the ideal of L_v, let Ω_v be the set defined by deleting all of the variables appearing on the edges incident to v from the set of variables of S and then adding in the binomial edge label incident to v if v has only two incident edges in $\tilde{G}$. We let $I_v = \langle \Omega_v \rangle$ be the ideal of L_v. Thus, the line L_v is parameterized by the coordinates on the incident edges, with coordinates i and j equal if $x_i - x_j$ appears at v but x_i and x_j do not.

Example 3.1 The graph G below has $g = 2$ and $d = 5$.



The ideals of the five lines are:

$$\begin{aligned} &\langle x_2, x_3 \rangle \\ &\langle x_1, x_2 - x_3 \rangle \\ &\langle x_0, x_3 \rangle \\ &\langle x_0 - x_1, x_2 \rangle \\ &\langle x_0, x_1 \rangle. \end{aligned}$$

Via Macaulay2 [18], the ideal of the arrangement is

$$\langle x_0x_2 - x_0x_3 + x_1x_3, x_1x_2x_3, x_0x_1x_3 - x_1^2x_3 \rangle.$$

The labeling gives rise to an embedding of C_G, but the ideal of this embedding is not generated by products of linear forms and is not generated by quadrics.

Theorem 3.2 *If G satisfies Assumption 1.1 for $p \geq 0$ and $\tilde{G}$ is labeled as described above, then $I = \cap_{v \in V} I_v$ is the ideal of an embedding of C_G into $\mathbb{P}^{d-g}$.*

Proof If u and v are connected by an edge in G and ℓ is the linear form on the edge that joins them, then the lines L_u and L_v intersect at the point of $\mathbb{P}^{d-g}$ that has coordinates appearing in ℓ set to 1 and all other coordinates set to 0.

To see that a labeling defines an embedding of C_G we must show that if u and v are not connected by an edge, then L_u and L_v do not intersect. If they intersect then a variable appearing on an edge incident to u must also appear on an edge incident to v. If x_i is an edge label at u, and v is incident to an edge with a label containing x_i, we must have the configuration on the left in Figure 1. But then the only coordinates of L_v with x_i nonzero also have x_j nonzero, and L_u does not contain any points with x_j nonzero unless w is trivalent and

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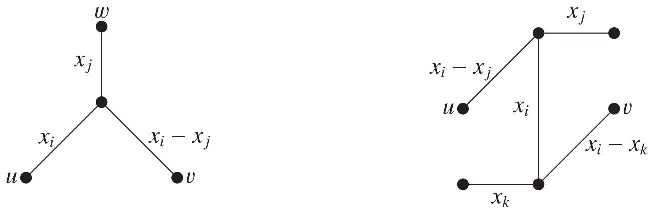


Figure 1

there is an edge labeled $x_j - x_k$ incident to u. However, this is not possible, because G does not contain any triangles. Hence, the lines cannot intersect.

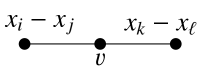
The only other possibility is that x_i appears in a binomial at u and at v as in the diagram on the right in Figure 1. But then if the lines L_u and L_v intersect, the three coordinates x_i, x_j, x_k must all be nonzero and equal. This means that the edge labeled x_k must be incident to u and the edge labeled x_j must be incident to v. But this is forbidden because $d \geq 2g + 1$ for all subgraphs of genus g. □

Our method of labeling edges with linear forms is similar in spirit to the description of the generators of the ideal of a canonical graph curve (corresponding to a trivalent graph) in [4]. Bayer and Eisenbud label the edges in G with a basis for the space of 1-cochains of G and intersect an ideal generated by monomials in this basis with the ring generated by the 1-cocycles.

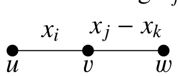
In order to describe the generators of I_{C_G} explicitly, we must make some further assumptions on the relative placement of labels.

Assumption 3.3 The labeling on $\tilde{G}$ satisfies the following conditions:

1. Incident edges never both have binomial labels. In other words, the labeling below never appears:



2. If v is a vertex of degree 2 as depicted below (with i, j, k distinct), then there are no other edges with labels containing x_i that are incident to edges with labels containing x_j or x_k:



3. The vertices of G are ordered $v_1, \ldots, v_d$, G_i is the graph induced on $v_1, \ldots, v_i$, and $\tilde{G}_{i-1}$ is obtained from $\tilde{G}_i$ by removing v_i and replacing any non-loop edge uv_i labeled with a monomial by a loop at u labeled with the same monomial:

(a) G_i is connected;
(b) v_i has at most two incident edges in $\tilde{G}_i$;
(c) if v_i is connected to a vertex u in G_{i-1} via an edge labeled with a binomial, then u is incident to three edges in G_i. (i.e., L_u has a monomial ideal).

In what follows, let $G_{\hat{v}}$ denote the subgraph of G obtained by removing v and all of its incident edges. If C_G is embedded in $\mathbb{P}^{d-g}$, we let $C_{G_{\hat{v}}}$ be the corresponding subcurve. Note that if $\deg v = 1$ in G and we remove the line L_v from C_G embedded as above, then $C_{G_{\hat{v}}}$ is embedded as a line arrangement via a complete linear series in a hyperplane. If $\deg v = 2$ in G and v is contained in a cycle, then $G_{\hat{v}}$ is still connected, the genus drops by 1, and the remaining subcurve is embedded via a complete linear series. We do not allow the removal of vertices of degree 3 because if $\deg v = 3$, and $G_{\hat{v}}$ is connected, then the genus drops by 2 and $C_{G_{\hat{v}}}$ is not embedded via a complete linear series.

Lemma 3.4 *Suppose that Assumption 1.1 holds for some $p \geq 1$ and Assumption 3.3 also holds. If the configuration in part 2 of Assumption 3.3 appears in a labeling of $\tilde{G}$, then $x_i(x_j - x_k)$ is in the ideal of C_G and $x_i x_j$, $x_i x_k$ are in the ideal of $C_{G_{\hat{v}}}$.*

Proof To see that $x_i(x_j - x_k)$ is in the ideal of C_G, note that $x_j - x_k$ is in the ideal of L_v. Our hypotheses imply that for any vertex $v' \neq v$, if x_i appears on an incident edge, neither x_j nor x_k do, so $x_j - x_k$ is in the ideal of $L_{v'}$. Otherwise, x_i is in the ideal of $L_{v'}$. Hence, $x_i(x_j - x_k)$ is in the ideal of each line.

It is easy to see that neither $x_i x_j$ nor $x_i x_k$ vanish on L_v. Since this is the only line where the coordinate x_i is paired with x_j or x_k, it follows that these two monomials are contained in the ideal of $C_{G_{\hat{v}}}$. □

Example 3.5 If $g = 2$, G has precisely two trivalent vertices, and it satisfies Assumption 1.1 for some $p \geq 1$, then it can be labeled according to Assumption 3.3. If the cycles are disjoint, then G must consists of two cycles and a bridge between them. Putting one binomial label in each cycle satisfies Assumption 3.3 because each cycle has length at least 4.

If the cycles overlap, then we have three paths between trivalent vertices u and v. Label the shortest path with monomials and put one binomial label on each of the remaining paths. For example, the graph in Figure 2 satisfies Assumption 3.3 and has defining ideal $\langle x_3x_4, x_0x_4 - x_2x_4, x_0x_3 - x_1x_3, x_1x_2 \rangle$.

Theorem 3.6 *Suppose that G satisfies Assumption 1.1 with $p = 1$. Fix a $\tilde{G}$ and a labeling that gives an embedding of C_G into $\mathbb{P}^{d-g}$ as a line arrangement. If Assumption 3.3 is satisfied, and $I_{C_{G_i}}$ is generated by quadrics for all*

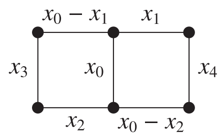


Figure 2

$i \geq 2$, then I_{C_G} is generated by elements of the form $x_i x_j$, $x_i(x_j - x_k)$, where the variables in each product are distinct.

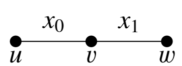
Proof of Theorem 3.7 We proceed by induction on d. The result is easy to check when $d = 2$. Assume the result for all graphs on $d-1$ vertices satisfying our hypotheses. Our hypotheses hold for G_i and $\tilde{G}_i$ for all $i \geq 2$. Let $v = v_{i+1}$. We may assume that $G = G_{i+1}$ and $G_i = G_{\hat{v}}$.

Case 1 *v has degree 1 in G.* The vertex v is incident to exactly one vertex $u \in G_{\hat{v}}$ with $u \neq v$. We may assume that L_v is spanned by a point p in $C_{G_{\hat{v}}}$ and the point $[0 : \cdots : 0 : 1]$. (By Assumption 3.3 part 3, all loops are labeled by monomials.) Then $I_{C_{G_{\hat{v}}}} = Q + \langle x_{d-g} \rangle$, where Q is generated by elements of the form $x_i x_j$ and $x_i(x_j - x_k)$ in which no term is divisible by x_{d-g}.

We argue that $Q \subset I_{L_v}$. Let $q = fh$ be one of the generators of Q fixed above where f and h are linear forms. Since q must vanish at p, without loss of generality we may assume that f vanishes at p. Since x_{d-g} does not appear in f, then f must also vanish on $[0 : \cdots : 1]$. Thus, f is a linear form vanishing at two points of L_v; hence it must vanish on all of L_v. Therefore, $Q \subset I_{L_v}$. Thus, we see that $I_{C_G} = Q + \langle x_{d-g} \rangle \cdot I_{L_v}$. Moreover, we see that I_{C_G} is generated by the generators of Q and elements of the form $x_{d-g} x_i$ and $x_{d-g}(x_j - x_k)$.

Case 2 *v has degree 2 in G.* By Assumption 3.3 part 3, there cannot be a loop at v. Then there are two cases: without loss of generality, either the labels on the edges incident to v have the form x_0 and x_1 or they have the form x_0 and $x_1 - x_2$.

In the first case, we claim that $x_0 x_1 \in I_{C_{G_{\hat{v}}}}$. Indeed, we have the configuration



If z is a vertex in $G_{\hat{v}}$ such that x_0 does not vanish on L_z, then x_0 must appear in a label on an edge incident at z. If $z = u$, then x_1 cannot appear in a binomial on any edge incident at z via Assumption 3.3 part 2, and so x_1 vanishes on L_z. If $z \neq u$, then $x_0 - x_j$ must appear on an edge incident at z. Again, if x_1 does not vanish on L_z, then it must appear on an edge incident at z. It cannot appear in a binomial by Assumption 3.3 part 1, in which case z must be equal to w, which creates a triangle. We conclude that either x_0 or x_1 vanishes on every irreducible component in $C_{G_{\hat{v}}}$, and hence that $x_0x_1 \in I_{C_{G_{\hat{v}}}}$.

Define a binomial minimal generator of $I_{C_{G_{\hat{v}}}}$ to be a binomial quadric in the ideal such that neither of its monomials is in $I_{C_{G_{\hat{v}}}}$. If $x_0(x_1 - x_i)$ is in $I_{C_{G_{\hat{v}}}}$, then so is x_0x_i. Hence we may assume that we have no minimal binomial generators of the form $x_0(x_1 - x_i)$. Similarly, we may assume that we have no generators of the form $x_1(x_0 - x_i)$.

The ideal of I_{C_G} is the intersection of $I_{C_{G_{\hat{v}}}}$ with $I_v = \langle x_2, \dots, x_{d-g}\rangle$, and it is generated by quadrics. The only monomial quadrics not contained in $\langle x_2, \dots, x_{d-g}\rangle$ are x_0^2, x_1^2, x_0x_1. The monomials x_0^2, x_1^2 do not appear in any minimal generator of $I_{C_{G_{\hat{v}}}}$. Since $x_0(x_1 - x_i)$ and $x_1(x_0 - x_i)$ are not generators of $I_{C_{G_{\hat{v}}}}$, every generator of the form x_ix_j and $x_i(x_j - x_k)$ must be in I_v, except for x_0x_1. Therefore, since I_{C_G} is generated by a space of quadrics whose dimension must be less than the dimension of the space of quadrics generating $I_{C_{G_{\hat{v}}}}$, we conclude that I_{C_G} is generated by the generators of $I_{C_{G_{\hat{v}}}}$ minus x_0x_1.

In the second case, x_0x_1 and x_0x_2 are in $I_{C_{G_{\hat{v}}}}$ but not I_{C_G} by Lemma 3.4 and I_{C_G} is the intersection of $I_{C_{G_{\hat{v}}}}$ with $I_v = \langle x_1 - x_2, \dots, x_{d-g}\rangle$. We can find generators of $I_{C_{G_{\hat{v}}}}$ that have the form x_ix_j and $x_i(x_j - x_k)$. Note that all square-free monomials x_ix_j are in I_v except for x_0x_1, x_0x_2, and x_1x_2. The monomial x_1x_2 cannot be in the ideal of $C_{G_{\hat{v}}}$ because it contains the line parameterized by x_1 and x_2.

We claim that all of the binomial minimal generators of $I_{C_{G_{\hat{v}}}}$ are contained in I_v. If $x_i(x_j - x_k)$ is not contained in I_v, then i must be 0, 1, or 2. If it is 0, then exactly one of j and k is in the set $\{1, 2\}$. But then x_0x_1 and x_0x_2 are already in $I_{C_{G_{\hat{v}}}}$, so $x_0(x_j - x_k)$ is not a binomial minimal generator.

So, without loss of generality, assume $i = 1$. Let w be the trivalent vertex with L_w parameterized by x_1 and x_2, and note that $w \in G_{\hat{v}}$. If neither of j or k is in the set $\{0, 2\}$, then $x_i(x_j - x_k)$ is in I_v. If one of them is equal to 0, then $x_1(x_0 - x_k)$ is not a binomial minimal generator because $x_0x_1 \in I_{C_{G_{\hat{v}}}}$. So assume that we have $x_1(x_2 - x_k)$ with $k \neq 0, 1, 2$. Then x_1x_k is in the ideal of L_w. If $x_1(x_2 - x_k)$ were in $I_{C_{G_{\hat{v}}}}$ it would also have to be in the ideal of L_w. But $x_1x_k, x_1(x_2 - x_k)$ in the ideal of L_w would imply that x_1x_2 would also be in the ideal of L_w, which is a contradiction. Therefore, we have no generators of the form $x_1(x_2 - x_k)$.

We conclude that all of the monomial and binomial minimal generators of $I_{C_{G_{\hat{v}}}}$ are in I_{C_G} except for x_0x_1 and x_0x_2. But $x_0(x_1 - x_2) \in I_{C_G}$, and the conclusion follows as in the first case since we have identified a space of quadrics in I_{C_G} of dimension exactly one less than the dimension of the space of quadrics in $I_{C_{G_{\hat{v}}}}$. □

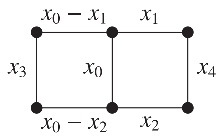


Figure 3

Via Corollary 2.5, if $g \leq 2$, we know that each $I_{C_{G_i}}$ is generated by quadrics and we obtain the following:

Corollary 3.7 *Suppose that G satisfies Assumption 1.1 for $p = 1$. Fix a $\tilde{G}$ and a labeling that gives an embedding of C_G into $\mathbb{P}^{d-g}$ as a line arrangement. If Assumption 3.3 is satisfied, and $g \leq 2$, then I_{C_G} is generated by elements of the form $x_i x_j, x_i(x_j - x_k)$, where the variables in each product are distinct.*

The result in Corollary 3.7 is sharp, as witnessed by the following example.

Example 3.8 Let G be the graph in Figure 3, where $d = 6$ and $g = 2$. Both of the vertices on the left fail part 1 of Assumption 3.3.

The ideal of the embedding corresponding to this labeling is

$$(x_3x_4, x_0x_4, x_0x_3 - x_1x_3 - x_2x_3, x_1x_2).$$

The terms in the trinomial do not appear in any other generators of the ideal. Therefore, it is impossible to find a set of minimal generators that does not contain an element with at least three terms.

Corollary 3.9 *If G satisfies Assumption 1.1 for $p = 1$, $g \leq 2$, and G has at most two trivalent vertices, then there exists an embedding of $C_G \subset \mathbb{P}^{d-g}$ so that I_{C_G} is generated by elements of the form $x_i x_j$ and $x_i(x_j - x_k)$.*

Proof Let the vertices of degree 2 with an incident edge labeled by a binomial be the last vertices in the order (so the first to get stripped off in the induction). Combine Corollary 2.5 with Example 3.5 and Corollary 3.7. □

4 Secant varieties and property $N_{k,p}$

In this section we show when $N_{3,p}$ must fail for the secant line variety Σ_1. The key idea of the proof comes from [12], whose authors state that their Theorem 1.1 has a natural generalization for higher-degree forms. We give a precise statement of a special case below:

Theorem 4.1 *Suppose that $X \subset \mathbb{P}^n$ is a variety that satisfies $N_{k,p}$. Let W be a linear subspace of dimension p with $Z = X \cap W$. If* $\dim Z = 0$, *then Z contains at most $\binom{p+k-1}{p}$ points.*

Proof It follows from the proof of Theorem 1.1 from [12] that the ideal of Z in the homogeneous coordinate ring of W is k-regular. Via Theorem 4.2 in [11], the degree in which the Hilbert function and the Hilbert polynomial of S_Z agree is the regularity of S_Z. We know that the Hilbert polynomial of S_Z is constant, equal to the number of points in Z. If $I(Z)$ is k-regular then S_Z is $k-1$-regular.

If $\dim(S_Z)_{k-1}$ is equal to the size of Z, then $\dim S_{k-1}$ must be at least the size of Z. Hence, $|Z| \leq \binom{p+k-1}{p}$. □

Corollary 4.2 *If C_G contains a cycle of m lines, then $N_{3,m-4}$ fails for the secant variety of C_G.*

Proof Since the m lines in the cycle are contained in a $\mathbb{P}^{m-1}$, so is the span of any subset of these lines. Thus, each 3-plane obtained by taking the span of non-adjacent lines in the cycle is contained in this $\mathbb{P}^{m-1}$. There are $\binom{m}{2} - m = \frac{1}{2}m(m-3)$ such 3-planes.

A general plane of dimension $m-4$ intersects a 3-plane in $\mathbb{P}^{m-1}$ in a point. Therefore, a general $(m-4)$-plane in this $\mathbb{P}^{m-1}$ intersects the secant variety of X in $\frac{1}{2}m(m-3)$ points. However, $\binom{m-4+3-1}{m-4} = \binom{m-2}{2} = \frac{1}{2}(m-2)(m-3)$. Thus, $N_{3,m-4}$ fails. □

5 Questions and conjectures

Computations with Macaulay 2 [18] were essential in all of our computations of embeddings of graph curves. In addition to the results proved in this paper we have several questions and conjectures regarding the defining equations and syzygies of graph curves and their secant varieties motivated by the examples that we have seen.

In Section 3 we saw that under certain hypotheses I_{C_G} is generated by products of linear forms that can be described explicitly in terms of the combinatorics of the graph G. The combinatorics of the kth secant variety of C_G is encoded in an *intersection lattice* whose elements are constructed by intersecting subsets of the subspaces. From the intersection lattice of an arrangement, we get a partially ordered set ordered by reverse inclusion of subspaces.

Question 1 Does the partially ordered set associated with the kth secant variety have any interesting combinatorial features? We conjecture that Σ_k is Cohen–Macaulay, so will the corresponding poset be shellable?

It is also natural to ask if there is an analogue of Theorem 3.7 for secant varieties, perhaps requiring additional hypotheses on the intersection lattice of the secant varieties of C_G.

Question 2 Are the secant varieties of C_G defined by products of linear forms?

Finding generators of I_{Σ_k} that are products of linear forms is equivalent to finding an explicit and special basis for the ideal that may have combinatorial interest. Of course, a module does not typically have a unique generating set or a unique minimal free graded resolution. However, the number of minimal generators of degree j of the ith syzygy module is invariant under a change of basis. Given a finitely generated graded module M, the graded Betti number $\beta_{i,j}$ is the number of minimal generators of degree j required at the ith stage of a minimal free graded resolution of M. A standard way of displaying the graded Betti numbers of a module is with a graded Betti diagram organized as follows:

$$\begin{array}{c|cccc} & 0 & 1 & 2 & \\ \hline 0 & \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \cdots \\ 1 & \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \cdots \end{array}$$

Bounds on the number of rows and columns of the graded Betti diagram of a module give a rough sense of how complicated it is. Specifically, recall that the *regularity* of a finitely generated graded module M is equal to $\sup\{j - i \mid \beta_{i,j} \neq 0 \text{ for some } i\}$, and thus regularity gives a bound on the number of rows of the graded Betti diagram of M. Additionally, by the Auslander–Buchsbaum formula, a variety $X \subset \mathbb{P}^n$ is arithmetically Cohen–Macaulay if $\sup\{i \mid \beta_{i,j} \neq 0 \text{ for some } i\} = \operatorname{codim} X$, which bounds the number of columns of the graded Betti diagram off M. The following conjecture is the graph curve analogue of Conjecture 1.4 in [27], which refines conjectures from [29].

Conjecture 1 If Assumption 1.1 holds for some $p \geq 2k$, then the kth secant variety of C_G has regularity equal to $2k + 1$ and is arithmetically Cohen–Macaulay.

Note that as the secant varieties of C_G are not normal, we cannot expect projective normality.

In addition to bounding the length and width of the graded Betti diagram, we conjecture that under certain conditions one particular graded Betti number counts the number of cycles of minimal length in the graph. Recall that the *girth* of a graph is the length of its smallest cycle.

Conjecture 2 Let G be a graph on d vertices, embedded as in Theorem 1.3. Let n denote the girth of G. Assume that $d = 2g + 1 + p$ and $n - 2 \leq p$. Then property N_p fails and $\beta_{n-2,n}$ is equal to the number of cycles of length n in G.

Example 5.1 gives an illustration of the properties discussed in Conjectures 1, 2.

Example 5.1 ($g = 2$, $d = 10$) Let G be as given in Figure 4.
The ideal of C_G corresponding to this labeling is given below:

$$I_{C_G} = \begin{array}{lllllll} (x_5x_8, & x_4x_8, & x_3x_8, & x_2x_8, & x_1x_8, & x_6x_7, & x_5x_7, \\ x_2x_7, & x_1x_7, & x_0x_7, & x_4x_6, & x_3x_6, & x_2x_6, & x_1x_6, \\ x_3x_5, & x_2x_5, & x_1x_5, & x_0x_5, & x_2x_4, & x_1x_4, & x_0x_4, \\ x_1x_3, & x_0x_3, & x_0x_2, & x_3x_7 - x_4x_7, & x_0x_8 - x_6x_8). & & \end{array}$$

The graded Betti diagram of S/I_{C_G} shows that $N_{2,5}$ fails as $\beta_{5,7} = 2$. As Conjecture 2 predicts, the girth of G is 7, and G contains precisely two cycles of length 7:

	0	1	2	3	4	5	6	7
total:	1	26	98	168	154	72	15	2
0:	1	.	.	.	.	.	.	..
1:	.	26	98	168	154	70	8	.
2:	.	.	.	.	.	2	7	2

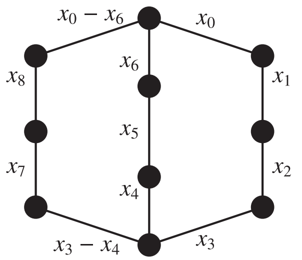


Figure 4

We can also compute the ideal of Σ:

$$I(\Sigma) = \begin{array}{llll} (x_3x_5x_8, & x_2x_5x_8, & x_1x_5x_8, & x_0x_4x_8 - x_4x_6x_8, \\ x_2x_4x_8, & x_1x_4x_8, & x_1x_3x_8, & x_0x_3x_8 - x_3x_6x_8, \\ x_2x_6x_7, & x_1x_6x_7, & x_2x_5x_7, & x_0x_2x_8 - x_2x_6x_8, \\ x_1x_5x_7, & x_0x_5x_7, & x_0x_2x_7, & x_3x_6x_7 - x_4x_6x_7, \\ x_2x_4x_6, & x_1x_4x_6, & x_1x_3x_6, & x_1x_3x_7 - x_1x_4x_7, \\ x_1x_3x_5, & x_0x_3x_5, & x_0x_2x_5, & x_0x_3x_7 - x_0x_4x_7, \\ x_0x_2x_4) & & & \end{array}$$

and its graded Betti diagram

	0	1	2	3	4	5
total:	1	25	58	43	12	3
0:	1	.	.	.	.	.
1:	.	.	.	.	.	..
2:	.	25	58	41	.	.
3:	.	.	.	.	7	.
4:	.	.	.	2	5	3

We see that $N_{3,3}$ fails for Σ and that $\beta_{3,7} = 2$, which is the number of cycles of length equal to the girth of G. We can also see from the graded Betti diagram that Σ is arithmetically Cohen–Macaulay and that $I(\Sigma)$ has regularity 5.

It is natural to ask if combinatorics can be used to compute other values of the $\beta_{i,j}$. One result that gives the flavor of what might be possible is due to Gasharov, Peeva, and Welker [17], who used the lcm lattice of a monomial ideal to compute graded Betti numbers of monomial ideals.

Question 3 Is there an analogue of the lcm lattice for graph curves and their secant varieties that would allow us to compute (or estimate) the graded Betti numbers of graph curves?

Further work on understanding the graded Betti numbers of graph curves has been done in [6].

It is also interesting to consider C_G as a deformation of a smooth curve. In Example 5.1, C_G has a 7-secant $\mathbb{P}^5$ while a smooth curve of genus 2 in $\mathbb{P}^8$ has no such $\mathbb{P}^5$. As any strictly subtrivalent graph curve $C_G \subset \mathbb{P}^n$ is smoothable in $\mathbb{P}^n$ [21, 29.9], it is our expectation that we have a family of seven 6-secant $\mathbb{P}^5$s to smooth curves that collapse to the 7-secant $\mathbb{P}^5$ in the singular limit C_G. It also seems reasonable to believe that the secant varieties to embedded curves in a flat family themselves form a flat family, and so the secant varieties to C_G should, in particular, have the same dimension and degree as those to smooth curves. In fact, since each pair of disjoint lines in C_G spans a $\mathbb{P}^3$, we have a

3-dimensional secant plane for each edge in the complement of the graph G. If C_G has degree d and genus g, then G has d vertices and $d+g-1$ edges. Thus, the number of edges in the complement of G is $\binom{d}{2}-d-g+1=\binom{d-1}{2}-g$, which is the degree of the secant variety of a smooth curve of degree d and genus g.

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References

[1] Ballico, E. On the gonality of graph curves. *Manuscripta Math.*, **129**(2): 169–180, 2009.

[2] Ballico, E. Graph curves with gonality one. *Rend. Semin. Mat. Univ. Politec. Torino*, **68**(1): 17–28, 2010.

[3] Ballico, E. and Franciosi, M. On property N_p for algebraic curves. *Kodai Math. J.*, **23**(3): 432–441, 2000.

[4] Bayer, D. and Eisenbud, D. Graph curves. *Adv. Math.*, **86**(1): 1–40, 1991. With an appendix by Sung Won Park.

[5] Björner, A., Peeva, I., and Sidman, J. Subspace arrangements defined by products of linear forms. *J. London Math. Soc. (2)*, **71**(2): 273–288, 2005.

[6] Bruce, D., Kao, P.-H., Nash, E., and Vermeire, P. Betti tables of reducible algebraic curves. *Proc. Amer. Math. Soc.*, in press.

[7] Catanese, F., Franciosi, M., Hulek, K., and Reid, M. Embeddings of curves and surfaces. *Nagoya Math. J.*, **154**: 185–220, 1999.

[8] Ciliberto, C. Harris, J., and Miranda, R. On the surjectivity of the Wahl map. *Duke Math. J.*, **57**(3): 829–858, 1988.

[9] Ciliberto, C. and Miranda, R. Graph curves, colorings, and matroids. In *Zero-Dimensional Schemes (Ravello, 1992)*, pp. 89–111. Berlin: de Gruyter, 1994.

[10] Ein, L. and Lazarsfeld, R. Asymptotic syzygies of algebraic varieties. *Invent. Math.*, **190**(3): 603–646, 2012.

[1] Readers of [26] should note that references there refer to the original version of this paper, and results are not in the same place or form here.

[11] Eisenbud, D. *The Geometry of Syzygies*: Volume 229 of *Graduate Texts in Mathematics*. New York: Springer-Verlag, 2005. A second course in commutative algebra and algebraic geometry.

[12] Eisenbud, D., Green, M., Hulek, K., and Popescu, S. Restricting linear syzygies: algebra and geometry. *Compos. Math.*, **141**(6): 1460–1478, 2005.

[13] Eisenbud, D., Koh, J., and Stillman, M. Determinantal equations for curves of high degree. *Amer. J. Math.*, **110**(3): 513–539, 1988.

[14] Eisenbud, D. and Koh, J.-H. Remarks on points in a projective space. In *Commutative Algebra (Berkeley, CA, 1987)*. Vol. 15 of Mathematical Sciences Research Institute Publications, pp. 157–172. New York: Springer-Verlag, 1989.

[15] Franciosi, M. and Elisa, T. The canonical ring of a 3-connected curve*. arXiv:1107.5535v2.

[16] Gallego, F. J. and Purnaprajna, B. P. Projective normality and syzygies of algebraic surfaces. *J. Reine Angew. Math.*, **506**: 145–180, 1999.

[17] Gasharov, V., Peeva, I., and Welker, V. The lcm-lattice in monomial resolutions. *Math. Res. Lett.*, **6**(5 & 6): 521–532, 1999.

[18] Grayson, D. R. and Stillman, M. E. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[19] Green, M. and Lazarsfeld, R. Some results on the syzygies of finite sets and algebraic curves. *Compos. Math.*, **67**(3): 301–314, 1988.

[20] Green, M. L. The Eisenbud–Koh–Stillman conjecture on linear syzygies. *Invent. Math.*, **136**(2): 411–418, 1999.

[21] Hartshorne, R. *Deformation Theory*. Vol. 257 of *Graduate Texts in Mathematics*. New York: Springer-Verlag, 2010.

[22] Hering, M., Schenck, H., and Smith, G. G. Syzygies, multigraded regularity and toric varieties. *Compos. Math.*, **142**(6): 1499–1506, 2006.

[23] Li, W. C. W. and Li, S.-Y. R. On generators of ideals associated with unions of linear varieties. *Bull. London Math. Soc.*, **13**(1): 59–65, 1981.

[24] Lovász, L. Stable sets and polynomials. *Discrete Math.*, **124**(1–3): 137–153, 1994. *Graphs and Combinatorics (Qawra, 1990)*.

[25] Ottaviani, G. and Paoletti, R. Syzygies of Veronese embeddings. *Compositio Math.*, **125**(1): 31–37, 2001.

[26] Schenck, H. and Sidman, J. Commutative algebra of subspace and hyperplane arrangements. In *Commutative Algebra*, pp. 639–665. New York: Springer-Verlag, 2013.

[27] Sidman, J. and Vermeire, P. Syzygies of the secant variety of a curve. *Algebra Number Theory*, **3**(4): 445–465, 2009.

[28] Sturmfels, B. and Sullivant, S. Combinatorial secant varieties. *Pure Appl. Math. Q.*, **2**(3, part 1): 867–891, 2006.

[29] Vermeire, P. Regularity and normality of the secant variety to a projective curve. *J. Algebra*, **319**(3): 1264–1270, 2008.