5

Rationally connected manifolds and semipositivity of the Ricci curvature

F. Campana
Université de Lorraine

J.-P. Demailly
Université de Grenoble I

Th. Peternell
Universität Bayreuth

Abstract

This paper establishes a structure theorem for compact Kähler manifolds with semipositive anticanonical bundle. Up to finite étale cover, it is proved that such manifolds split holomorphically and isometrically as a product of Ricci flat varieties and of rationally connected manifolds. The proof is based on a characterization of rationally connected manifolds through the non-existence of certain twisted contravariant tensor products of the tangent bundle, along with a generalized holonomy principle for pseudoeffective line bundles. A crucial ingredient for this is the characterization of uniruledness by the property that the anticanonical bundle is not pseudoeffective.

Dedicated to Rob Lazarsfeld on the occasion of this 60th birthday

1 Main results

The goal of this work is to understand the geometry of compact Kähler manifolds with semipositive Ricci curvature, and especially to study the relations that tie Ricci semipositivity with rational connectedness. Many of the ideas are borrowed from [DPS96] and [BDPP]. Recall that a compact complex manifold X is said to be rationally connected if any two points of X can be joined by a chain of rational curves. A line bundle L is said to be

From *Recent Advances in Algebraic Geometry*, edited by Christopher Hacon, Mircea Mustaţă and Mihnea Popa

hermitian semipositive if it can be equipped with a smooth hermitian metric of semipositive curvature form. A sufficient condition for hermitian semipositivity is that some multiple of L is spanned by global sections; on the other hand, the hermitian semipositivity condition implies that L is numerically effective (nef) in the sense of [DPS94], which, for X projective algebraic, is equivalent to saying that $L \cdot C \geq 0$ for every curve C in X. Examples contained in [DPS94] show that all three conditions are different (even for X projective algebraic). The Ricci curvature is the curvature of the anticanonical bundle $K_X^{-1} = \det(T_X)$, and by Yau's solution of the Calabi conjecture (see [Aub76], [Yau78]), a compact Kähler manifold X has a hermitian semipositive anticanonical bundle K_X^{-1} if and only if X admits a Kähler metric ω with $\mathrm{Ricci}(\omega) \geq 0$. A classical example of a projective surface with K_X^{-1} nef is the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ blown-up in nine points, no three of which are collinear and no six of which lie on a conic; in that case Brunella [Bru10] showed that there are configurations of the nine points for which K_X^{-1} admits a smooth (but non-real analytic) metric with semipositive Ricci curvature; depending on some diophantine condition introduced in [Ued82], there are also configurations for which some multiple K_X^{-m} of K_X^{-1} is generated by sections and others for which K_X^{-1} is nef without any smooth metric. Finally, let us recall that a line bundle $L \to X$ is said to be pseudoeffective if there exists a singular hermitian metric h on L such that the Chern curvature current $T = i\Theta_{L,h} = -i\partial\overline{\partial} \log h$ is non-negative; equivalently, if X is projective algebraic, this means that the first Chern class $c_1(L)$ belongs to the closure of the cone of effective $\mathbb{Q}$-divisors.

We first give a criterion characterizing rationally connected manifolds by the non-existence of sections in certain twisted tensor powers of the cotangent bundle; this is only a minor variation of Theorem 5.2 in [Pet06], cf. also Remark 5.3 therein.

1.1 Criterion for rational connectedness *Let X be a projective algebraic n-dimensional manifold. The following properties are equivalent:*

(a) X is rationally connected.
(b) For every invertible subsheaf $\mathcal{F} \subset \Omega^p_X := \mathcal{O}(\Lambda^p T^*_X)$, $1 \leq p \leq n$, $\mathcal{F}$ is not pseudoeffective.
(c) For every invertible subsheaf $\mathcal{F} \subset \mathcal{O}((T^*_X)^{\otimes p})$, $p \geq 1$, $\mathcal{F}$ is not pseudoeffective.
(d) For some (resp. for any) ample line bundle A on X, there exists a constant $C_A > 0$ such that

$$H^0(X, (T^*_X)^{\otimes m} \otimes A^{\otimes k}) = 0 \qquad \text{for all } m, k \in \mathbb{N}^* \text{ with } m \geq C_A k.$$

1.2 Remark The proof follows easily from the uniruledness criterion established in [BDPP]: a non-singular projective variety X is uniruled if and only if

K_X is not pseudoeffective. A conjecture attributed to Mumford asserts that the weaker assumption

(d′) $H^0(X, (T^*_X)^{\otimes m}) = 0$

for all $m \geq 1$ should be sufficient to imply rational connectedness. Mumford's conjecture can actually be proved by essentially the same argument if one uses the abundance conjecture in place of the more demanding uniruledness criterion from [BDPP] – more specifically that $H^0(X, K_X^{\otimes m}) = 0$ for all $m \geq 1$ would imply uniruledness.

1.3 Remark By [DPS94], Criteria 1.1 (b) and (c) make sense on an arbitrary compact complex manifold and imply that $H^0(X, \Omega^2_X) = 0$. If X is assumed to be compact Kähler, then X is automatically projective algebraic by Kodaira [Kod54], therefore, 1.1 (b) or (c) also characterizes rationally connected manifolds among all compact Kähler ones. □

The following structure theorem generalizes the Bogomolov–Kobayashi–Beauville structure theorem for Ricci-flat manifolds ([Bog74a], [Bog74b], [Kob81], [Bea83]) to the Ricci semipositive case. Recall that a *holomorphic symplectic manifold X* is a compact Kähler manifold admitting a holomorphic symplectic 2-form ω (of maximal rank everywhere); in particular, $K_X = \mathcal{O}_X$. A *Calabi–Yau* manifold is a simply connected projective manifold with $K_X = \mathcal{O}_X$ and $H^0(X, \Omega^p_X) = 0$ for $0 < p < n = \dim X$ (or a finite étale quotient of such a manifold).

1.4 Structure theorem *Let X be a compact Kähler manifold with K_X^{-1} hermitian semipositive. Then*

(a) The universal cover $\widetilde{X}$ admits a holomorphic and isometric splitting

$$\widetilde{X} \simeq \mathbb{C}^q \times \prod Y_j \times \prod S_k \times \prod Z_\ell$$

where Y_j, S_k, Z_ℓ are compact simply connected Kähler manifolds of respective dimensions n_j, n'_k, n''_ℓ with irreducible holonomy, Y_j being Calabi–Yau manifolds (holonomy $\mathrm{SU}(n_j)$*), S_k holomorphic symplectic manifolds (holonomy* $\mathrm{Sp}(n'_k/2)$*), and Z_ℓ rationally connected manifolds with $K_{Z_\ell}^{-1}$ semipositive (holonomy* $\mathrm{U}(n''_\ell)$*).*

(b) There exists a finite étale Galois cover $\widehat{X} \to X$ such that the Albanese variety $\mathrm{Alb}(\widehat{X})$ *is a q-dimensional torus and the Albanese map $\alpha\colon \widehat{X} \to \mathrm{Alb}(\widehat{X})$ is an (isometrically) locally trivial holomorphic fiber bundle whose fibers are products $\prod Y_j \times \prod S_k \times \prod Z_\ell$ of the type described in (a). Even more holds after possibly another finite étale cover: $\hat{X}$ is a fiber bundle with fiber $\prod Z_\ell$ on $\prod Y_j \times \prod S_k \times \mathrm{Alb}(\widehat{X})$.*

(c) *We have* $\pi_1(\widehat{X}) \simeq \mathbb{Z}^{2q}$ *and* $\pi_1(X)$ *is an extension of a finite group* Γ *by the normal subgroup* $\pi_1(\widehat{X})$*. In particular, there is an exact sequence*

$$0 \to \mathbb{Z}^{2q} \to \pi_1(X) \to \Gamma \to 0,$$

and the fundamental group $\pi_1(X)$ *is almost abelian.*

The proof relies on the holonomy principle, De Rham's splitting theorem [DR52], and Berger's classification [Ber55]. Foundational background can be found in papers by Lichnerowicz [Lic67], [Lic71] and Cheeger–Gromoll [CG71], [CG72]. The restricted holonomy group of a hermitian vector bundle (E, h) of rank r is by definition the subgroup $H \subset \mathrm{U}(r) \simeq U(E_{z_0})$ generated by parallel transport operators with respect to the Chern connection ∇ of (E, h), along loops based at z_0 that are contractible (up to conjugation, H does not depend on the base point z_0). We need here a generalized "pseudoeffective" version of the holonomy principle, which can be stated as follows.

1.5 Generalized holonomy principle *Let E be a holomorphic vector bundle of rank r over a compact complex manifold X. Assume that E is equipped with a smooth hermitian structure h and X with a hermitian metric* ω*, viewed as a smooth positive* $(1, 1)$*-form* $\omega = i \sum \omega_{jk}(z) dz_j \wedge d\overline{z}_k$*. Finally, suppose that the* ω*-trace of the Chern curvature tensor* $\Theta_{E,h}$ *is semipositive, that is*

$$i\Theta_{E,h} \wedge \frac{\omega^{n-1}}{(n-1)!} = B\,\frac{\omega^n}{n!}, \qquad B \in \mathrm{Herm}(E, E), \quad B \geq 0 \text{ on } X,$$

and denote by H the restricted holonomy group of (E, h)*.*

(a) *If there exists an invertible sheaf* $\mathcal{L} \subset \mathcal{O}((E^*)^{\otimes m})$ *which is pseudoeffective as a line bundle, then* $\mathcal{L}$ *is flat and* $\mathcal{L}$ *is invariant under parallel transport by the connection of* $(E^*)^{\otimes m}$ *induced by the Chern connection* ∇ *of* (E, h)*; in fact, H acts trivially on* $\mathcal{L}$*.*

(b) *If H satisfies* $H = \mathrm{U}(r)$*, then none of the invertible subsheaves* $\mathcal{L}$ *of* $\mathcal{O}((E^*)^{\otimes m})$ *can be pseudoeffective for* $m \geq 1$*.*

The generalized holonomy principle is based on an extension of the Bochner formula as found in [BY53], [Kob83]: for (X, ω) Kähler, every section u in $H^0(X, (T^*_X)^{\otimes m})$ satisfies

$$\Delta(\|u\|^2) = \|\nabla u\|^2 + Q(u), \tag{1.6}$$

where $Q(u) \geq m\lambda_1 \|u\|^2$ is bounded from below by the smallest eigenvalue λ_1 of the Ricci curvature tensor of ω. If $\lambda_1 \geq 0$, the equality $\int_X \Delta(\|u\|^2)\omega^n = 0$ implies $\nabla u = 0$ and $Q(u) = 0$. The generalized principle consists essentially of considering a general vector bundle E rather than $E = T^*_X$, and replacing

$\|u\|^2_\omega$ with $\|u\|^2_\omega e^\varphi$ where u is a local trivializing section of $\mathcal{L}$, where φ is the corresponding local plurisubharmonic weight representing the metric of $\mathcal{L}$ and ω a Gauduchon metric, cf. (3.2).

1.7 Remark If one makes the weaker assumption that K_X^{-1} is nef, then Qi Zhang [Zha96, Zha05] proved that the Albanese mapping $\alpha\colon X \to \mathrm{Alb}(X)$ is surjective in the case where X is projective, and Păun [Pau12] recently extended this result to the general Kähler case (cf. also [CPZ03]). One may wonder whether there still exists a holomorphic splitting

$$\widetilde{X} \simeq \mathbb{C}^q \times \prod Y_j \times \prod S_k \times \prod Z_\ell$$

of the universal covering as above. However, the example where $X = \mathbb{P}(E)$ is the ruled surface over an elliptic curve $C = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ associated with a nontrivial rank 2 bundle $E \to C$ with

$$0 \to \mathcal{O}_C \to E \to \mathcal{O}_C \to 0$$

shows that $\widetilde{X} = \mathbb{C} \times \mathbb{P}^1$ cannot be an isometric product for a Kähler metric ω on X. Actually, such a situation would imply that $K_X^{-1} = \mathcal{O}_{\mathbb{P}(E)}(1)$ is semipositive, but we know by [DPS94] that $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef and non-semipositive. Under the mere assumption that K_X^{-1} is nef, it is unknown whether the Albanese map $\alpha\colon X \to \mathrm{Alb}(X)$ is a submersion, unless X is a projective threefold [PS98], and even if it is supposed to be so, it seems to be unknown whether the fibers of α may exhibit nontrivial variation of the complex structure (and whether they are actually products of Ricci flat manifolds by rationally connected manifolds). The main difficulty is that, a priori, the holonomy argument used here breaks down – a possibility would be to consider some sort of "asymptotic holonomy" for a sequence of Kähler metrics satisfying $\mathrm{Ricci}(\omega_\varepsilon) \geq -\varepsilon\omega_\varepsilon$, and dealing with the Gromov–Hausdorff limit of the variety. □

2 Proof of the criterion for rational connectedness

In this section we prove Criterion 1.1. Observe first that if X is rationally connected, then there exists an immersion $f\colon \mathbb{P}^1 \subset X$ passing through any given finite subset of X such that f^*T_X is ample, see e.g. [Kol96, Theorem 3.9, p. 203]. In other words $f^*T_X = \bigoplus \mathcal{O}_{\mathbb{P}^1}(a_j)$, $a_j > 0$, while $f^*A = \mathcal{O}_{\mathbb{P}^1}(b)$, $b > 0$. Hence

$$H^0(\mathbb{P}^1, f^*((T_X^*)^{\otimes m} \otimes A^{\otimes k})) = 0 \qquad \text{for } m > kb/\min(a_i).$$

As the immersion f moves freely in X, we immediately see from this that 1.1 (a) implies 1.1 (d) with any constant value $C_A > b/\min(a_j)$.

To see that 1.1 (d) implies 1.1 (c), assume that $\mathcal{F} \subset (T_X^*)^{\otimes p}$ is a pseudoeffective line bundle. Then there exists $k_0 \gg 1$ such that

$$H^0(X, \mathcal{F}^{\otimes m} \otimes A^{k_0}) \neq 0$$

for all $m \geq 0$ (for this, it is sufficient to take k_0 such that $A^{k_0} \otimes (K_X \otimes G^{n+1})^{-1} > 0$ for some very ample line bundle G). This implies $H^0(X, (T_X^*)^{\otimes mp} \otimes A^{k_0}) \neq 0$ for all m, contradicting assumption 1.1 (d).

The implication 1.1 (c) $\Rightarrow$ 1.1 (b) is trivial.

It remains to show that 1.1 (b) implies 1.1 (a). First note that K_X is not pseudoeffective, as one sees by applying the assumption 1.1 (b) with $p = n$. Hence X is uniruled by [BDPP]. We consider the quotient with maximal rationally connected fibers (rational quotient or MRC fibration, see [Cam92], [KMM92])

$$f : X \dashrightarrow W$$

to a smooth projective variety W. By [GHS01], W is not uniruled, otherwise we could lift the ruling to X and the fibers of f would not be maximal. We may further assume that f is holomorphic. In fact, assumption 1.1 (b) is invariant under blow-ups. To see this, let $\pi : \hat{X} \to X$ be a birational morphism from a projective manifold $\hat{X}$ and consider a line bundle $\hat{\mathcal{F}} \subset \Omega^p_{\hat{X}}$. Then $\pi_*(\hat{\mathcal{F}}) \subset \pi_*(\Omega^p_{\hat{X}}) = \Omega^p_X$, hence we introduce the line bundle

$$\mathcal{F} := (\pi_*(\hat{\mathcal{F}}))^{**} \subset \Omega^p_X.$$

Now, if $\hat{\mathcal{F}}$ were pseudoeffective, so would $\mathcal{F}$ be. Thus 1.1 (b) is invariant under π and we may suppose f holomorphic. In order to show that X is rationally connected, we need to prove that $p := \dim W = 0$. Otherwise, $K_W = \Omega^p_W$ is pseudoeffective by [BDPP] and we obtain a pseudoeffective invertible subsheaf $\mathcal{F} := f^*(\Omega^p_W) \subset \Omega^p_X$, in contradiction to 1.1 (b). □

3 Bochner formula and generalized holonomy principle

Let (E, h) be a hermitian holomorphic vector bundle over an n-dimensional compact complex manifold X. The semipositivity hypothesis on $B = \mathrm{Tr}_\omega \Theta_{E,h}$ is invariant by a conformal change of metric ω. Without loss of generality we can assume that ω is a Gauduchon metric, i.e., that $\partial\bar{\partial}\omega^{n-1} = 0$ (cf. [Gau77]). We consider the Chern connection ∇ on (E, h) and the corresponding parallel transport operators. At every point $z_0 \in X$, there exists a local coordinate system $(z_1, \ldots, z_n)$ centered at z_0 (i.e., $z_0 = 0$ in coordinates), and a holomorphic frame $(e_\lambda(z))_{1\le\lambda\le r}$ such that

$$\langle e_\lambda(z), e_\mu(z)\rangle_h = \delta_{\lambda\mu} - \sum_{1\le j,k\le n} c_{jk\lambda\mu} z_j \bar{z}_k + O(|z|^3),\ 1 \le \lambda, \mu \le r, \tag{3.1}$$

$$\Theta_{E,h}(z_0) = \sum_{1\le j,k,\lambda,\mu\le n} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e^*_\lambda \otimes e_\mu,\ c_{kj\mu\lambda} = \overline{c_{jk\lambda\mu}}, \tag{3.1$'$}$$

where $\delta_{\lambda\mu}$ is the Kronecker symbol and $\Theta_{E,h}(z_0)$ is the curvature tensor of the Chern connection ∇ of (E, h) at z_0.

Assume that we have an invertible sheaf $\mathcal{L} \subset O((E^*)^{\otimes m})$ that is pseudoeffective. There exist a covering U_j by coordinate balls and holomorphic sections f_j of $\mathcal{L}_{|U_j}$ generating $\mathcal{L}$ over U_j. Then $\mathcal{L}$ is associated with the Čech cocycle g_{jk} in O_X^* such that $f_k = g_{jk} f_j$, and the singular hermitian metric $e^{-\varphi}$ of $\mathcal{L}$ is defined by a collection of plurisubharmonic functions $\varphi_j \in \mathrm{PSH}(U_j)$ such that $e^{-\varphi_k} = |g_{jk}|^2 e^{-\varphi_j}$. It follows that we have a globally defined bounded measurable function

$$\psi = e^{\varphi_j} \|f_j\|^2 = e^{\varphi_j} \|f_j\|^2_{h^{*m}}$$

over X, which can be viewed also as the hermitian metric ratio $(h^*)^m / e^{-\varphi}$ along $\mathcal{L}$, i.e., $\psi = (h^*)^m_{|\mathcal{L}} e^{\varphi}$. We are going to compute the Laplacian $\Delta_\omega \psi$. For simplicity of notation, we omit the index j and consider a local holomorphic section f of $\mathcal{L}$ and a local weight $\varphi \in \mathrm{PSH}(U)$ on some open subset U of X. In a neighborhood of an arbitrary point $z_0 \in U$, we write

$$f = \sum_{\alpha\in\mathbb{N}^m} f_\alpha\, e^*_{\alpha_1} \otimes \ldots \otimes e^*_{\alpha_m}, \qquad f_\alpha \in O(U),$$

where (e^*_λ) is the dual holomorphic frame of (e_λ) in $O(E^*)$. The hermitian matrix of (E^*, h^*) is the transpose of the inverse of the hermitian matrix of (E, h), hence (3.1) implies

$$\langle e^*_\lambda(z), e^*_\mu(z)\rangle_h = \delta_{\lambda\mu} + \sum_{1\le j,k\le n} c_{jk\mu\lambda} z_j \bar{z}_k + O(|z|^3), \qquad 1 \le \lambda, \mu \le r.$$

On the open set U the function $\psi = (h^*)^m_{|\mathcal{L}} e^{\varphi}$ is given by

$$\psi = \Big(\sum_{\alpha\in\mathbb{N}^m} |f_\alpha|^2 + \sum_{\alpha,\beta\in\mathbb{N}^m,\ 1\le j,k\le n,\ 1\le\ell\le m} f_\alpha \overline{f_\beta}\, c_{jk\beta_\ell\alpha_\ell} z_j \bar{z}_k + O(|z|^3)|f|^2\Big) e^{\varphi(z)}.$$

By taking $i\partial\bar\partial(\ldots)$ of this at $z = z_0$ in the sense of distributions (that is, for almost every $z_0 \in X$), we find

$$i\partial\bar\partial\psi = e^{\varphi}\Big(|f|^2 i\partial\bar\partial\varphi + i\langle \partial f + f\partial\varphi, \partial f + f\partial\varphi\rangle + \sum_{\alpha,\beta,j,k,1\le\ell\le m} f_\alpha \overline{f_\beta}\, c_{jk\beta_\ell\alpha_\ell}\, i dz_j \wedge d\bar{z}_k\Big).$$

Since $i\partial\overline{\partial}\psi \wedge \frac{\omega^{n-1}}{(n-1)!} = \Delta_\omega\psi \frac{\omega^n}{n!}$ (we actually take this as a definition of Δ_ω), a multiplication by ω^{n-1} yields the fundamental inequality

$$\Delta_\omega\psi \geq |f|^2 e^\varphi(\Delta_\omega\varphi + m\lambda_1) + |\nabla_h^{1,0} f + f\partial\varphi|^2_{\omega,h^{*m}}\, e^\varphi, \tag{3.2}$$

where $\lambda_1(z) \geq 0$ is the lowest eigenvalue of the hermitian endomorphism $B = \mathrm{Tr}_\omega\,\Theta_{E,h}$ at an arbitrary point $z \in X$. As $\partial\overline{\partial}\omega^{n-1} = 0$, we have

$$\int_X \Delta\psi \frac{\omega^n}{n!} = \int_X i\partial\overline{\partial}\psi \wedge \frac{\omega^{n-1}}{(n-1)!} = \int_X \psi \wedge \frac{i\partial\overline{\partial}(\omega^{n-1})}{(n-1)!} = 0$$

by Stokes' formula. Since $i\partial\overline{\partial}\varphi \geq 0$, (3.2) implies $\Delta_\omega\varphi = 0$, i.e., $i\partial\overline{\partial}\varphi = 0$, and $\nabla_h^{1,0} f + f\partial\varphi = 0$ almost everywhere. This means in particular that the line bundle $(\mathcal{L}, e^{-\varphi})$ is flat. In each coordinate ball U_j the pluriharmonic function φ_j can be written $\varphi_j = w_j + \overline{w}_j$ for some holomorphic function $w_j \in \mathcal{O}(U_j)$, hence $\partial\varphi_j = dw_j$ and the condition $\nabla_h^{1,0} f_j + f_j\partial\varphi_j = 0$ can be rewritten $\nabla_h^{1,0}(e^{w_j} f_j) = 0$ where $e^{w_j} f_j$ is a local holomorphic section. This shows that $\mathcal{L}$ must be invariant by parallel transport and that the local holonomy of the Chern connection of (E, h) acts trivially on $\mathcal{L}$. Statement 1.5 (a) follows.

Finally, if we assume that the restricted holonomy group H of (E, h) is equal to $\mathrm{U}(r)$, there cannot exist any holonomy-invariant invertible subsheaf $\mathcal{L} \subset \mathcal{O}((E^*)^{\otimes m})$, $m \geq 1$, on which H acts trivially, since the natural representation of $\mathrm{U}(r)$ on $(\mathbb{C}^r)^{\otimes m}$ has no invariant line on which $\mathrm{U}(r)$ induces a trivial action. Property 1.5 (b) is proved. □

4 Proof of the structure theorem

We suppose here that X is equipped with a Kähler metric ω such that $\mathrm{Ricci}(\omega) \geq 0$, and we set $n = \dim_{\mathbb{C}} X$. We consider the holonomy representation of the tangent bundle $E = T_X$ equipped with the hermitian metric $h = \omega$. Here

$$B = \mathrm{Tr}_\omega\,\Theta_{E,h} = \mathrm{Tr}_\omega\,\Theta_{T_X,\omega} \geq 0$$

is nothing but the Ricci operator.

Proof of 1.4 (a) Let

$$(\widetilde{X}, \omega) \simeq \prod (X_i, \omega_i)$$

be the De Rham decomposition of $(\widetilde{X}, \omega)$, induced by a decomposition of the holonomy representation in irreducible representations. Since the holonomy is contained in $\mathrm{U}(n)$, all factors (X_i, ω_i) are Kähler manifolds with irreducible holonomy and holonomy group $H_i \subset \mathrm{U}(n_i)$, $n_i = \dim X_i$. By Cheeger–Gromoll [CG71], there is possibly a flat factor $X_0 = \mathbb{C}^q$ and the other factors X_i, $i \geq 1$, are compact and simply connected. Also, the product structure shows that each $K_{X_i}^{-1}$ is hermitian semipositive. By Berger's classification of holonomy groups [Ber55] there are only three possibilities, namely $H_i = \mathrm{U}(n_i)$, $H_i = \mathrm{SU}(n_i)$, or $H_i = \mathrm{Sp}(n_i/2)$. The case $H_i = \mathrm{SU}(n_i)$ leads to X_i being a Calabi–Yau manifold, and the case $H_i = \mathrm{Sp}(n_i/2)$ implies that X_i is holomorphic symplectic (see, e.g., [Bea83]). Now, if $H_i = \mathrm{U}(n_i)$, the generalized holonomy principle 1.5 shows that none of the invertible subsheaves $\mathcal{L} \subset \mathcal{O}((T_{X_i}^*)^{\otimes m})$ can be pseudoeffective for $m \geq 1$. Therefore, X_i is rationally connected by Criterion 1.1. □

Proof of 1.4 (b) Set $X' = \prod_{i\geq 1} X_i$. The group of covering transformations acts on the product $\widetilde{X} = \mathbb{C}^q \times X'$ by holomorphic isometries of the form $x = (z, x') \mapsto (u(z), v(x'))$. At this point, the argument is slightly more involved than in Beauville's paper [Bea83], because the group G' of holomorphic isometries of X' need not be finite (e.g., X' may be a projective space); instead, we imitate the proof of ([CG72], Theorem 9.2) and use the fact that X' and $G' = \mathrm{Isom}(X')$ are compact. Let $E_q = \mathbb{C}^q \rtimes U(q)$ be the group of unitary motions of $\mathbb{C}^q$. Then $\pi_1(X)$ can be seen as a discrete subgroup of $E_q \times G'$. As G' is compact, the kernel of the projection map $\pi_1(X) \to E_q$ is finite and the image of $\pi_1(X)$ in E_q is still discrete with compact quotient. This shows that there is a subgroup Γ of finite index in $\pi_1(X)$ which is isomorphic to a crystallographic subgroup of $\mathbb{C}^q$. By Bieberbach's theorem, the subgroup $\Gamma_0 \subset \Gamma$ of elements which are translations is a subgroup of finite index. Taking the intersection of all conjugates of Γ_0 in $\pi_1(X)$, we find a normal subgroup $\Gamma_1 \subset \pi_1(X)$ of finite index, acting by translations on $\mathbb{C}^q$. Then $\widehat{X} = \widetilde{X}/\Gamma_1$ is a fiber bundle over the torus $\mathbb{C}^q/\Gamma_1$, with X' as fiber and $\pi_1(X') = 1$. Therefore, $\widehat{X}$ is the desired finite étale covering of X. □

For the second assertion we consider fiberwise the rational quotient and obtain a factorization

$$\widehat{X} \xrightarrow{\beta} W \xrightarrow{\gamma} \mathrm{Alb}(\widehat{X})$$

with fiber bundles β (fiber $\prod Z_\ell$) and γ (fiber $\prod Y_j \times \prod S_k$). Since clearly $K_W \equiv 0$, the claim follows from the Beauville–Bogomolov decomposition theorem.

Proof of 1.4 (c) The statement is an immediate consequence of 1.4 (b), using the homotopy exact sequence of a fibration. □

5 Further remarks

We finally point out two direct consequences of Theorem 1.4. Since the property

$$H^0(X,(T^*_X)^{\otimes m}) = 0 \qquad (m \geq 1)$$

is invariant under finite étale covers, we obtain immediately from Theorem 1.4:

5.1 Corollary *Let X be a compact Kähler manifold with K_X^{-1} hermitian semipositive. Assume that $H^0(X,(T^*_X)^{\otimes m}) = 0$ for all positive m. Then X is rationally connected.*

This establishes Mumford's conjecture in case X has semipositive Ricci curvature.

Theorem 1.4 also gives strong implications for small deformations of a manifold with semipositive Ricci curvature:

5.2 Corollary *Let X be a compact Kähler manifold with K_X^{-1} hermitian semipositive. Let $\pi\colon \mathcal{X} \to \Delta$ be a proper submersion from a Kähler manifold $\mathcal{X}$ to the unit disk $\Delta \subset \mathbb{C}$. Assume that $X_0 = \pi^{-1}(0) \simeq X$. Then there exists a finite étale cover $\widehat{\mathcal{X}} \to \mathcal{X}$ with projection $\hat{\pi}\colon \widehat{\mathcal{X}} \to \Delta$ such that – after possibly shrinking Δ – the following holds:*

(a) The relative Albanese map $\alpha\colon \widehat{\mathcal{X}} \to \mathrm{Alb}(\mathcal{X}/\Delta)$ is a surjective submersion, thus the Albanese map $\alpha_t\colon \widehat{X}_t = \widehat{\pi}^{-1}(t) \to \mathrm{Alb}(X_t)$ is a surjective submersion for all t.

(b) Every fiber of α_t is a product of Calabi–Yau manifolds, irreducible symplectic manifolds, and irreducible rationally connected manifolds.

(c) There exists a factorization of α,

$$\widehat{\mathcal{X}} \xrightarrow{\beta} \mathcal{Y} \xrightarrow{\gamma} \mathrm{Alb}(\mathcal{X}/\Delta),$$

such that $\beta_t = \beta_{|\widehat{X}_t}$ is a submersion and a rational quotient of $\widehat{X}_t$ for all t, and $\gamma_t = \gamma_{|Y_t}$ is a trivial fiber bundle.

Corollary 5.2 is an immediate consequence of Theorem 1.4 and the following proposition:

5.3 Proposition *Let $\pi\colon \mathcal{Y} \to \Delta$ be a proper Kähler submersion over the unit disk. Assume that $Y_0 \simeq \prod X_i \times \prod Y_j \times \prod Z_k$ with X_i Calabi–Yau, Y_j irreducible symplectic, and Z_k irreducible rationally connected. Then (possibly after shrinking Δ) every Y_t has a decomposition*

$$Y_t \simeq \prod X_{i,t} \times \prod Y_{j,t} \times \prod Z_{k,t}$$

with factors of the same type as above, and the factors form families $\mathcal{X}_i$, $\mathcal{Y}_j$, and $\mathcal{Z}_k$.

Proof It suffices to treat the case of two factors, say $Y_0 = A_1 \times A_2$ where the A_i are Calabi–Yau, irreducible symplectic, or rationally connected. Since $H^1(A_j, \mathcal{O}_{A_j}) = 0$, the factors A_j deform to the neighboring Y_t. By the properness of the relative cycle space, we obtain families $q_i \colon U_i \to S_i$ over Δ with projections $p_i \colon U_i \to \mathcal{Y}$. Possibly after shrinking Δ, this yields holomorphic maps $f_i \colon \mathcal{Y} \to S_i$. Then the map

$$f_1 \times f_2 \colon \mathcal{Y} \to S_1 \times S_2$$

is an isomorphism, since $A_t \cdot B_t = A_0 \cdot B_0 = 1$. This gives the families $(A_i)_t$ we are looking for. □

A Appendix (by Jean-Pierre Demailly) A flag variety version of the holonomy principle

Our goal here is to derive a related version of the holonomy principle over flag varieties, based on a modified Bochner formula which we hope to be useful in other contexts (especially since no assumption on the base manifold is needed). If E is as before a holomorphic vector bundle of rank r over an n-dimensional complex manifold, we denote by $F(E)$ the flag manifold of E, namely the bundle $F(E) \to X$ whose fibers consist of flags

$$\xi : \; E_x = V_0 \supset V_1 \supset \ldots \supset V_r = \{0\}, \quad \dim E_x = r, \quad \operatorname{codim} V_\lambda = \lambda$$

in the fibers of E, along with the natural projection $\pi \colon F(E) \to X$, $(x, \xi) \mapsto x$. We let Q_λ, $1 \le \lambda \le r$ be the tautological line bundles over $F(E)$ such that

$$Q_{\lambda,\xi} = V_{\lambda-1}/V_\lambda,$$

and for a weight $a = (a_1, \ldots, a_r) \in \mathbb{Z}^r$ we set

$$Q^a = Q_1^{a_1} \otimes \ldots \otimes Q_r^{a_r}.$$

In additive notation, viewing the Q_j as divisors, we also denote by

$$a_1 Q_1 + \ldots + a_r Q_r$$

any real linear combination ($a_j \in \mathbb{R}$). Our goal is to compute explicitly the curvature tensor of the line bundles Q^a with respect to the tautological metric induced by h. For convenience of notation, we prefer to work on the dual flag manifold $F(E^*)$, although there is a biholomorphism $F(E) \simeq F(E^*)$ given by

$$(E_x = W_0 \supset W_1 \supset \ldots \supset W_r = \{0\}) \mapsto (E_x^* = V_0 \supset V_1 \supset \ldots \supset V_r = \{0\}),$$

where $V_\lambda = W_{r-\lambda}^\dagger$ is the orthogonal subspace of $W_{r-\lambda}$ in E_x^*. In this context, we have an isomorphism

$$V_{\lambda-1}/V_\lambda = W_{r-\lambda+1}^\dagger / W_{r-\lambda}^\dagger \simeq (W_{r-\lambda}/W_{r-\lambda+1})^*.$$

This shows that $Q^a \to F(E^*)$ is isomorphic to $Q^b \to F(E)$ where $b_\lambda = -a_{r-\lambda+1}$, that is

$$(b_1, b_2, \ldots, b_{r-1}, b_r) = (-a_r, -a_{r-1}, \ldots, -a_2, -a_1).$$

We now proceed to compute the curvature of $Q^a \to F(E^*)$, using the same notation as in Section 3. In a neighborhood of every point $z_0 \in X$, we can find a local coordinate system $(z_1, \ldots, z_n)$ centered at z_0 and a holomorphic frame $(e_\lambda)_{1\le\lambda\le r}$ such that

(A.1) $$\langle e_\lambda(z), e_\mu(z)\rangle = \mathbf{1}_{\{\lambda=\mu\}} - \sum_{1\le j,k\le n} c_{jk\lambda\mu} z_j \bar z_k + O(|z|^3),\ 1 \le \lambda, \mu \le r,$$

(A.1′) $$\Theta_{E,h}(z_0) = \sum_{1\le j,k,\lambda,\mu\le n} c_{jk\lambda\mu} dz_j \wedge d\bar z_k \otimes e_\lambda^* \otimes e_\mu,\ c_{kj\mu\lambda} = \overline{c_{jk\lambda\mu}},$$

where $\mathbf{1}_S$ denotes the characteristic function of the set S. For a given point $\xi_0 \in F(E_{z_0}^*)$ in the flag variety, one can always adjust the frame (e_λ) in such a way that the flag corresponding to ξ_0 is given by

(A.2) $$V_{\lambda,0} = \mathrm{Vect}(e_1, \ldots, e_\lambda)^\dagger \subset E_{z_0}^*.$$

A point (z, ξ) in a neighborhood of (z_0, ξ_0) is likewise represented by the flag associated with the holomorphic tangent frame $(\widetilde e_\lambda(z,\xi))_{1\le\lambda\le r}$ defined by

(A.3) $$\widetilde e_\lambda(z,\xi) = e_\lambda(z) + \sum_{\lambda<\mu\le r} \xi_{\lambda\mu} e_\mu(z), \qquad (\xi_{\lambda\mu})_{1\le\lambda<\mu\le r} \in \mathbb{C}^{r(r-1)/2}.$$

We obtain in this way a local coordinate system $(z_j, \xi_{\lambda\mu})$ near (z_0, ξ_0) on the total space of $F(E^*)$, where the $(\xi_{\lambda\mu})$ are the fiber coordinates. The frame $\widetilde e(z,\xi)$ is not orthonormal, but by the Gram–Schmidt orthogonalization process the flag ξ is also induced by the (non-holomorphic) orthonormal frame $(\widehat e_\lambda(z,\xi))$ obtained inductively by putting $\widehat e_1 = \widetilde e_1/|\widetilde e_1|$ and

$$\widehat e_\lambda = \Big(\widetilde e_\lambda - \sum_{1\le\mu<\lambda} \langle \widetilde e_\lambda, \widehat e_\mu\rangle \widehat e_\mu\Big)/(\text{norm of numerator}).$$

Straightforward calculations imply that the hermitian inner products involved are $\mathcal{O}(|\xi| + |z|^2)$ and the norms equal to $1 + \mathcal{O}((|\xi| + |z|)^2)$, hence we get

$$\widehat{e_\lambda}(z,\xi) = e_\lambda(z,\xi) + \sum_{\lambda<\mu\leq r} \xi_{\lambda\mu} e_\mu(z) - \sum_{1\leq\mu<\lambda} \overline{\xi}_{\mu\lambda} e_\mu(z) + O((|\xi| + |z|)^2)$$

and more precisely (omitting variables for simplicity of notation)

$$\begin{aligned}\widehat{e_\lambda} = \Big(1 - \frac{1}{2}\sum_{1\leq\mu<\lambda} |\xi_{\mu\lambda}|^2 - \frac{1}{2}\sum_{\lambda<\mu\leq r} |\xi_{\lambda\mu}|^2 + \frac{1}{2}\sum_{1\leq j,k\leq n} c_{jk\lambda\lambda} z_j \overline{z}_k\Big) e_\lambda \\ + \sum_{\lambda<\mu\leq r} \xi_{\lambda\mu} e_\mu \\ - \sum_{1\leq\mu<\lambda} \Big(\overline{\xi}_{\mu\lambda} + \sum_{\lambda<\nu\leq r} \xi_{\lambda\nu}\overline{\xi}_{\mu\nu} - \sum_{1\leq j,k\leq n} c_{jk\lambda\mu} z_j \overline{z}_k\Big) e_\mu \\ + \mathcal{O}((|\xi| + |z|)^3).\end{aligned} \tag{A.4}$$

The curvature of the tautological line bundle $Q_\lambda = V_{\lambda-1}/V_\lambda$ can be evaluated by observing that the dual line bundle

$$Q_\lambda^* = V_\lambda^\dagger / V_{\lambda-1}^\dagger = \mathrm{Vect}(\widetilde{e}_1, \ldots, \widetilde{e}_\lambda)/\,\mathrm{Vect}(\widetilde{e}_1, \ldots, \widetilde{e}_{\lambda-1})$$

admits a holomorphic section given by

$$v_\lambda(z,\xi) = \widetilde{e}_\lambda(z,\xi) \bmod \mathrm{Vect}(\widetilde{e}_1, \ldots, \widetilde{e}_{\lambda-1}).$$

The tautological norm of this section is

$$\begin{aligned}|v_\lambda|^2 &= |\widetilde{e}_\lambda|^2 - \sum_{1\leq\mu<\lambda} |\langle \widetilde{e}_\lambda, \widehat{e}_\mu\rangle|^2 \\ &= 1 - \sum_{1\leq j,k\leq n} c_{jk\lambda\lambda} z_j \overline{z}_k \\ &\quad + \sum_{\lambda<\mu\leq r} |\xi_{\lambda\mu}|^2 - \sum_{1\leq\mu<\lambda} |\xi_{\mu\lambda}|^2 + \mathcal{O}((|z| + |\xi|)^3).\end{aligned}$$

Therefore we obtain the formula

$$\begin{aligned}\Theta_{Q_\lambda}(z_0,\xi_0) &= \partial\overline{\partial} \log |v_\lambda|^2_{|(z_0,\xi_0)} \\ &= -\sum_{1\leq j,k\leq n} c_{jk\lambda\lambda} dz_j \wedge d\overline{z}_k \\ &\quad + \sum_{\lambda<\mu\leq r} d\xi_{\lambda\mu} \wedge d\overline{\xi}_{\lambda\mu} - \sum_{1\leq\mu<\lambda} d\xi_{\mu\lambda} \wedge d\overline{\xi}_{\mu\lambda},\end{aligned}$$

$$\begin{aligned}\Theta_{Q^a}(z_0,\xi_0) &= \sum_{1\le\lambda\le r} a_\lambda \Theta_{Q_\lambda}(z_0,\xi_0)\\ &= -\sum_{1\le j,k\le n,\,1\le\lambda\le r} a_\lambda c_{jk\lambda\lambda} dz_j\wedge d\overline{z}_k\\ &\quad+\sum_{1\le\lambda<\mu\le r}(a_\lambda-a_\mu)d\xi_{\lambda\mu}\wedge d\overline{\xi}_{\lambda\mu}.\end{aligned}$$

This calculation holds true only at (z_0,ξ_0), but it shows that we have at every point a decomposition of Θ_{Q^a} in horizontal and vertical parts given by

$$\Theta_{Q^a} = \theta_a^H + \theta_a^V, \tag{A.5}$$

$$\begin{cases}\theta_a^H(z_0,\xi_0) &= -\sum_{j,k,\lambda} a_\lambda c_{jk\lambda\lambda} dz_j\wedge d\overline{z}_k\\ &= -\sum_{1\le\lambda\le r} a_\lambda\,\pi^*\langle\Theta_{T_X,\omega}(e_\lambda),e_\lambda\rangle,\end{cases} \tag{A.6^H}$$

$$\theta_a^V(z_0,\xi_0) = \sum_{1\le\lambda<\mu\le r}(a_\lambda-a_\mu)d\xi_{\lambda\mu}\wedge d\overline{\xi}_{\lambda\mu}. \tag{A.6^V}$$

The decomposition is taken here with respect to the C^∞ splitting of the exact sequence

$$0\to T_{Y/X}\to T_Y\to \pi^*T_X\to 0,\qquad Y:=F(E^*) \tag{A.7}$$

provided by the Chern connection ∇ of (E,h); horizontal directions are those coming from flags associated with ∇-parallel frames. In order to express (A.6^H) in a more intrinsic way at an arbitrary point $(z,\xi)\in Y$, we have to replace $(e_\lambda(z))$ by the orthonormal frame $(\widehat{e}_\lambda(z,\xi))$ associated with the flag ξ. Such frames are not unique, actually they are defined up to the action of $(S^1)^r$, but such a change does not affect the expression of θ_a^H. We then get the intrinsic formula

$$\begin{aligned}\theta_a^H(z,\xi) &= -\sum_{1\le\lambda\le r} a_\lambda\,\pi^*\langle\Theta_{T_X,\omega}(\widehat{e}_\lambda(z,\xi)),\widehat{e}_\lambda(z,\xi)\rangle\\ &= -\sum_{1\le\lambda\le r} a_\lambda \sum_{1\le j,k\le n,\,1\le\sigma,\tau\le r} c_{jk\sigma\tau}(z)\,\widehat{e}_{\lambda\sigma}(z,\xi)\,\overline{\widehat{e}_{\lambda\tau}(z,\xi)}\,dz_j\wedge d\overline{z}_k,\end{aligned} \tag{A.8}$$

where we put

$$\widehat{e}_\lambda(z,\xi) = \sum_{1\le\sigma\le r}\widehat{e}_{\lambda\sigma}(z,\xi)\,e_\sigma(z)$$

(the coefficients $\widehat{e}_{\lambda\sigma}(z,\xi)$ can be computed from (A.4)). Moreover, since θ_a^V and Θ_{Q^a} have the same restriction to the fibers of $Y\to X$, we conclude that θ_a^V is in fact unitary invariant along the fibers (the tautological metric of Q^a clearly has this property). Let us consider the vertical and normalized unitary invariant relative volume form η of $Y \to X$ given by

$$\text{(A.9)} \qquad \eta(z_0, \xi_0) = \bigwedge_{1 \le \lambda < \mu \le r} i\, d\xi_{\lambda\mu} \wedge d\overline{\xi}_{\lambda\mu} \qquad \text{at } (z_0, \xi_0).$$

Let $N = r(r-1)/2$ be the fiber dimension. For a strictly dominant weight a, i.e., $a_1 > a_2 > \ldots > a_r$, the line bundle Q^a is relatively ample with respect to the projection $\pi \colon Y = F(E^*) \to X$, and $i\theta_a^V$ induces a Kähler form on the fibers. Formula (A.6^V) shows that the corresponding volume form is

$$(i\theta_a^V)^N = N! \prod_{1 \le \lambda < \mu \le r} (a_\lambda - a_\mu)\, \eta.$$

A.10 Curvature formulas *Consider as above $Q^a \to Y := F(E^*)$. Then*

(a) The curvature form of Q^a is given by $\Theta_{Q^a} = \theta_a^H + \theta_a^V$, where the horizontal part is given by

$$\theta_a^H = -\sum_{1 \le \lambda \le r} a_\lambda\, \pi^* \langle \Theta_{T_X,\omega}(\widehat{e}_\lambda), \widehat{e}_\lambda \rangle$$

and the vertical part by

$$\theta_a^V(z_0, \xi_0) = \sum_{1 \le \lambda < \mu \le r} (a_\lambda - a_\mu) d\xi_{\lambda\mu} \wedge d\overline{\xi}_{\lambda\mu}$$

in normal coordinates at any point (z_0, ξ_0).

(b) The relative canonical bundle $K_{Y/X}$ is isomorphic with Q^ρ for the (anti-dominant) canonical weight $\rho_\lambda = 2\lambda - r - 1$, $1 \le \lambda \le r$. For any positive definite $(1,1)$-form ω on X we have

$$\begin{aligned} i\partial\overline{\partial}\eta \wedge \pi^*\omega^{n-1} &= -i\theta_\rho^H \wedge \eta \wedge \pi^*\omega^{n-1} \\ &= \sum_{1 \le \lambda \le r} \rho_\lambda\, \pi^* \langle i\Theta_{T_X,\omega}(\widehat{e}_\lambda), \widehat{e}_\lambda \rangle \wedge \eta \wedge \pi^*\omega^{n-1}. \end{aligned}$$

Proof (a) follows entirely from the previous discussion.
(b) The formula for the canonical weight is a classical result in the theory of flag varieties. As $(i\theta_a^V)^N$ and η are proportional for a strictly dominant, we compute instead

$$\partial\overline{\partial}(\theta_a^V)^N = N\,(\theta_a^V)^{N-1} \wedge \partial\overline{\partial}\theta_a^V + N(N-1)\,(\theta_a^V)^{N-2} \wedge \partial\theta_a^V \wedge \overline{\partial}\theta_a^V,$$

and for this, we use a Taylor expansion of order 2 at (z_0, ξ_0). Since Θ_{Q^a} is closed, we have $\partial\overline{\partial}\theta_a^V = -\partial\overline{\partial}\theta_a^H$, hence

$$\partial\overline{\partial}\theta_a^V = \partial\overline{\partial} \sum_{1 \le \lambda \le r} a_\lambda \sum_{1 \le j,k \le n,\, 1 \le \sigma,\tau \le r} c_{jk\sigma\tau}(z)\, \widehat{e}_{\lambda\sigma}(z,\xi)\, \overline{\widehat{e}_{\lambda\tau}(z,\xi)}\, dz_j \wedge d\overline{z}_k,$$

and we have similar formulas for $\partial(\theta_a^V)$ and $\overline{\partial}(\theta_a^V)$. When taking ∂, $\overline{\partial}$ and $\partial\overline{\partial}$ we need only consider the differentials in ξ, otherwise we get terms $\Lambda^{\geq 3}(dz, d\overline{z})$ of degree at least three in the dz_j or $d\overline{z}_k$ and the wedge product of these with $\pi^*\omega^{n-1}$ is zero. For the same reason, $\partial\theta_a^V \wedge \overline{\partial}\theta_a^V$ will not contribute to the result since it produces terms of degree four or more in dz_j, $d\overline{z}_k$. Formula (A.4) gives

$$\begin{aligned}\widehat{e}_{\lambda\sigma} = \mathbf{1}_{\{\lambda=\sigma\}}\Big(1 - \frac{1}{2}\sum_{1\leq\mu<\lambda}|\xi_{\mu\lambda}|^2 - \frac{1}{2}\sum_{\lambda<\mu\leq r}|\xi_{\lambda\mu}|^2\Big)\\ + \mathbf{1}_{\{\lambda<\sigma\}}\xi_{\lambda\sigma} - \mathbf{1}_{\{\sigma<\lambda\}}\Big(\overline{\xi}_{\sigma\lambda} + \sum_{\mu>\lambda}\xi_{\lambda\mu}\overline{\xi}_{\sigma\mu}\Big) + O(|z|^2 + |\xi|^3).\end{aligned}$$

Notice that we do not need to look at the terms $O(|z|)$, $O(|z|^2)$ as they will produce no contribution at (z_0, ξ_0). From this we infer

$$\begin{aligned}\widehat{e}_{\lambda\sigma}\overline{\widehat{e}_{\lambda\tau}} =& \mathbf{1}_{\{\lambda=\sigma=\tau\}}\Big(1 - \sum_{1\leq\mu<\lambda}|\xi_{\mu\lambda}|^2 - \sum_{\lambda<\mu\leq r}|\xi_{\lambda\mu}|^2\Big)\\ &+ \mathbf{1}_{\{\lambda=\tau<\sigma\}}\xi_{\lambda\sigma} - \mathbf{1}_{\{\sigma<\lambda=\tau\}}\overline{\xi}_{\sigma\lambda} + \mathbf{1}_{\{\lambda=\sigma<\tau\}}\overline{\xi}_{\lambda\tau} - \mathbf{1}_{\{\tau<\lambda=\sigma\}}\xi_{\tau\lambda}\\ &+ \mathbf{1}_{\{\sigma,\tau>\lambda\}}\xi_{\lambda\sigma}\overline{\xi}_{\lambda\tau} + \mathbf{1}_{\{\sigma,\tau<\lambda\}}\xi_{\tau\lambda}\overline{\xi}_{\sigma\lambda}\\ &+ \mathbf{1}_{\{\tau<\lambda<\sigma\}}\xi_{\lambda\sigma}\xi_{\tau\lambda} + \mathbf{1}_{\{\sigma<\lambda<\tau\}}\overline{\xi}_{\sigma\lambda}\overline{\xi}_{\lambda\tau}\\ &- \sum_{1\leq\mu\leq r}\mathbf{1}_{\{\sigma<\lambda=\tau<\mu\}}\xi_{\lambda\mu}\overline{\xi}_{\sigma\mu} + \mathbf{1}_{\{\tau<\lambda=\sigma<\mu\}}\xi_{\tau\mu}\overline{\xi}_{\lambda\mu} \mod(|z|^2, |\xi|^3).\end{aligned}$$

By virtue of (A.7), only "diagonal terms" of the form $d\xi_{\lambda\mu} \wedge d\overline{\xi}_{\lambda\mu}$ in the $\partial\overline{\partial}$ of this expression can contribute to $(\theta_a^V)^{N-1} \wedge \partial\overline{\partial}\theta_a^V$, all others vanish at $z = \xi = 0$. The useful terms are thus

$$\begin{aligned}\partial\overline{\partial}\Big(\sum_{1\leq\lambda\leq r}a_\lambda\sum_{1\leq\sigma,\tau\leq r}c_{jk\sigma\tau}\widehat{e}_{\lambda\sigma}\overline{\widehat{e}_{\lambda\tau}}dz_j \wedge d\overline{z}_k\Big) = (\text{unneeded terms})\\ + \sum_{1\leq\lambda<\mu\leq r}(-a_\mu c_{jk\mu\mu} - a_\lambda c_{jk\lambda\lambda} + a_\lambda c_{jk\mu\mu} + a_\mu c_{jk\lambda\lambda})\, d\xi_{\lambda\mu}\wedge d\overline{\xi}_{\lambda\mu}\wedge dz_j\wedge d\overline{z}_k\\ = \sum_{1\leq\lambda<\mu\leq r}(a_\lambda - a_\mu)(c_{jk\mu\mu} - c_{jk\lambda\lambda})\, d\xi_{\lambda\mu} \wedge d\overline{\xi}_{\lambda\mu} \wedge dz_j \wedge d\overline{z}_k + (\text{unneeded}).\end{aligned}$$

From this we infer

$$\partial\overline{\partial}(\theta_a^V)^N \wedge \pi^*\omega^{n-1} = (\theta_a^V)^N \wedge \sum_{1\leq j,k\leq n,\, 1\leq\lambda\leq r}(2\lambda - 1 - r)\, c_{jk\lambda\lambda}\, dz_j \wedge d\overline{z}_k \wedge \pi^*\omega^{n-1}.$$

In fact, the coefficient of $c_{jk\lambda\lambda}$ is the number $(\lambda - 1)$ of indices $< \lambda$ (coming from the term $(a_\lambda - a_\mu)c_{jk\mu\mu}$ above) minus the number $r - \lambda$ of indices $> \lambda$ (coming from the term $-(a_\lambda - a_\mu)c_{jk\lambda\lambda}$). Formula A.10 (b) follows. □

A.11 Bochner formula *Assume that X is a compact complex manifold possessing a balanced metric. That is, a positive smooth $(1,1)$-form $\omega = i\sum_{1\le j,k\le n}\omega_{jk}(z)\,dz_j\wedge d\overline{z}_k$ such that $d\omega^{n-1}=0$. Assume also that for some dominant weight a $(a_1\ge\ldots\ge a_r\ge 0)$, the $\mathbb{R}$-line bundle Q^a is pseudoeffective on $Y := F(E^*)$,* i.e., *that there exists a quasi-plurisubharmonic function φ such that $i(\Theta_{Q^a}+\partial\overline{\partial}\varphi)\ge 0$ on Y. Then*

$$\int_Y (i\partial\varphi\wedge\overline{\partial}\varphi - i\theta^H_{a-\rho})e^\varphi\,\eta\wedge\pi^*\omega^{n-1}\le 0,$$

or equivalently

$$\int_Y \Big(i\partial\varphi\wedge\overline{\partial}\varphi + \sum_{1\le\lambda\le r}(a_\lambda-\rho_\lambda)\langle i\Theta_{TX,\omega}(\widehat{e}_\lambda),\widehat{e}_\lambda\rangle\Big)e^\varphi\,\eta\wedge\pi^*\omega^{n-1}\le 0.$$

Proof The idea is to use the $\partial\overline{\partial}$-formula

$$\int_Y i\partial\overline{\partial}(e^\varphi)\wedge\eta\wedge\pi^*\omega^{n-1} - e^\varphi\wedge i\partial\overline{\partial}\eta\wedge\pi^*\omega^{n-1}$$
$$= \int_Y d\Big(i\overline{\partial}(e^\varphi)\wedge\eta\wedge\pi^*\omega^{n-1} + e^\varphi\, i\partial\eta\wedge\pi^*\omega^{n-1}\Big) = 0,$$

which follows immediately from Stokes. We get

$$\text{(A.12)}\qquad \int_Y (i\partial\overline{\partial}\varphi + i\partial\varphi\wedge\overline{\partial}\varphi)e^\varphi\wedge\eta\wedge\pi^*\omega^{n-1} - e^\varphi\, i\partial\overline{\partial}\eta\wedge\pi^*\omega^{n-1} = 0.$$

Now, $i\partial\overline{\partial}\varphi\ge -i\Theta_{Q^a}$ in the sense of currents, and therefore by A.10 (a) and (b) we obtain

$$\text{(A.13)}\quad i\partial\overline{\partial}\varphi\wedge\eta\wedge\pi^*\omega^{n-1} - i\partial\overline{\partial}\eta\wedge\pi^*\omega^{n-1}\ge(-i\theta^H_a + i\theta^H_\rho)\wedge\eta\wedge\pi^*\omega^{n-1}.$$

The combination of (A.12) and (A.13) yields the inequality of Corollary A.11. □

The parallel transport operators of (E,h) can be considered to operate on the global flag variety $Y=F(E^*)$ as follows. For any piecewise smooth path $\gamma\colon[0,1]\to X$, we get a (unitary) hermitian isomorphism $\tau_\gamma\colon E_p\to E_q$ where $p=\gamma(0)$, $q=\gamma(1)$. Therefore τ_γ induces an isomorphism $\widetilde{\tau}_\gamma\colon F(E^*_p)\to F(E^*_q)$ of the corresponding flag varieties, and an isomorphism over $\widetilde{\tau}_\gamma$ of the tautological line bundles Q^a. Given a local C^∞ vector field v on an open set $U\subset X$, there is a unique horizontal lifting $\widetilde{v}$ of v to a C^∞ vector field on $\pi^{-1}(U)\subset Y$, where horizontality refers again to $\nabla=\nabla_{E,h}$. Now, the flow of $\widetilde{v}$ consists of parallel transport operators along the trajectories of v. By definition, h is invariant by parallel transport, therefore the associated hermitian metric h_a on each line bundle Q^a is also invariant. Another metric $h_{a,\varphi} = h_a e^{-\varphi}$ is invariant if and only if the weight function φ is invariant by the flows of all such vector fields $\widetilde{v}$ on Y, that is if $d\varphi(\zeta) = 0$ for all horizontal vector fields $\zeta \in T_Y$.

A.14 Theorem *Let $E \to X$ be a holomorphic vector bundle of rank r over a compact complex manifold X. Assume that X is equipped with a hermitian metric ω and E with a hermitian structure h such that $B := \mathrm{Tr}_\omega(i\Theta_{E,h}) \geq 0$. At each point $z \in X$, let*

$$0 \leq b_1(z) \leq \ldots \leq b_r(z)$$

be the eigenvalues of $B(z)$ with respect to $h(z)$. Finally, let Q^a be a pseudo-effective $\mathbb{R}$-line bundle on $Y := F(E^*)$ associated with a dominant weight $a_1 \geq \ldots \geq a_r \geq 0$, and let φ be a quasi-plurisubharmonic function on Y such that $i(\Theta_{Q^a} + \partial\overline{\partial}\varphi) \geq 0$. Then

(a) The function $\psi(z) = \sup_{\xi \in F(E_z^*)} \varphi(z, \xi)$ is constant and $b_\lambda \equiv 0$ as soon as $a_\lambda > 0$, and in particular $B \equiv 0$ if $a_r > 0$.

(b) Assume that $B \equiv 0$. Then the function φ must be invariant by parallel transport on Y.

Proof Since our hypotheses are invariant by a conformal change on the metric ω, we can assume by Gauduchon [Gau77] that $\partial\overline{\partial}\omega^{n-1} = 0$.

(a) Notice that if a is integral and φ is given by a holomorphic section of Q^a, then e^φ is the square of the norm of that section with respect to h, and e^ψ is the sup of that norm on the fibers of $Y \to X$. In general, formula A.10 (a) shows that

$$i\partial\overline{\partial}\varphi(z, \xi) \geq -i\theta_a^H(z, \xi) - i\theta_a^V(z, \xi),$$

hence

$$i\partial\overline{\partial}^H\varphi(z, \xi) \wedge \omega^{n-1}(z) \geq \sum_{1 \leq \lambda \leq r} a_\lambda \, \langle i\Theta_{TX,\omega}(\widehat{e_\lambda}), \widehat{e_\lambda}\rangle(z, \xi) \wedge \omega^{n-1}(z), \tag{A.15}$$

where $i\partial\overline{\partial}^H\varphi$ means the restriction of $i\partial\overline{\partial}\varphi$ to the horizontal directions in T_Y. By taking the supremum in ξ, we conclude from standard arguments of subharmonic function theory that

$$\Delta_\omega \psi(z) \geq \sum_{1 \leq \lambda \leq r} a_\lambda b_\lambda(z),$$

since the RHS is the minimum of the coefficient of the (n, n)-form occurring in the RHS of (A.15). Therefore, ψ is ω-subharmonic and so must be constant on

X by Aronszajn [Aro57]. It follows that $b_\lambda \equiv 0$ whenever $a_\lambda > 0$, in particular $B \equiv 0$ if $a_r > 0$.

(b) Under the assumption $B = \mathrm{Tr}_\omega \Theta_{E,h} \equiv 0$, the calculations made in the course of the proof of A.10 (b) imply that

$$\partial\eta \wedge \pi^*\omega^{n-1} = 0, \qquad \partial\overline{\partial}\eta \wedge \pi^*\omega^{n-1} = 0.$$

By the proof of the Bochner formula A.11 (the fact that $\partial\overline{\partial}\omega^{n-1} = 0$ is enough here), we get

$$0 \le \int_Y i\partial\varphi \wedge \overline{\partial}\varphi \wedge \eta \wedge \pi^*\omega^{n-1} \le 0,$$

and we conclude from this that the horizontal derivatives $\partial^H\varphi$ vanish. Therefore, φ is invariant by parallel transport. □

In the vein of Criterion 1.1, we have the following additional statement:

A.16 Proposition *Let X be a compact Kähler manifold. Then X is projective and rationally connected if and only if none of the $\mathbb{R}$-line bundles Q^a over $Y = F(T_X^*)$ is pseudoeffective for weights $a \neq 0$ with $a_1 \geq \ldots \geq a_r \geq 0$.*

Proof If X is projective rationally connected and some Q^a, $a \neq 0$, is pseudoeffective, we obtain a contradiction to Theorem A.14 by pulling back T_X and Q^a via a map $f\colon \mathbb{P}^1 \to X$ such that $E = f^*T_X$ is ample on $\mathbb{P}^1$ (as $B > 0$ in this circumstance).

Conversely, if the $\mathbb{R}$-line bundles Q^a, $a \neq 0$, are not pseudoeffective on $Y = F(T_X^*)$, we obtain by taking $a_1 = \ldots = a_p = 1$, $a_{p+1} = \ldots = a_n = 0$ that $\pi_* Q^a = \Omega_X^p$. Therefore, $H^0(X, \Omega_X^p) = 0$ and all invertible subsheaves $\mathcal{F} \subset \Omega_X^p$ are not pseudoeffective for $p \geq 1$. Hence X is projective (take $p = 2$ and apply Kodaira [Kod54]) and rationally connected by Criterion 1.1 (b). □

Acknowledgments This work was completed while the three authors were visiting the Mathematisches Forschungsinstitut Oberwolfach in September 2012. They wish to thank the Institute for its hospitality and the exceptional quality of the environment.

References

[Aro57] Aronszajn, N. A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. *J. Math. Pures Appl.* **36**, 235–249 (1957).

[Aub76] Aubin, T. Equations du type Monge–Ampère sur les variétés kähleriennes compactes. *C. R. Acad. Sci. Paris Ser. A* **283**, 119–121 (1976); *Bull. Sci. Math.* **102**, 63–95 (1978).

[Bea83] Beauville, A. Variétés kähleriennes dont la première classe de Chern est nulle. *J. Diff. Geom.* **18**, 775–782 (1983).

[Ber55] Berger, M. Sur les groupes d'holonomie des variétés à connexion affine des variétés riemanniennes. *Bull. Soc. Math. France* **83**, 279–330 (1955).

[BY53] Bochner, S. and Yano, K. Curvature and Betti numbers. *Annals of Mathematics Studies*, No. 32. Princeton, NJ: Princeton University Press, 1953.

[Bog74a] Bogomolov, F.A. On the decomposition of Kähler manifolds with trivial canonical class. *Math. USSR Sbornik* **22**, 580–583 (1974).

[Bog74b] Bogomolov, F.A. Kähler manifolds with trivial canonical class. *Izvestija Akad. Nauk* **38**, 11–21 (1974).

[BDPP] Boucksom, S., Demailly, J.-P., Paun, M., and Peternell, T. The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. *J. Alg. Geom.* **22**, 201–248 (2013).

[Bru10] Brunella, M. On Kähler surfaces with semipositive Ricci curvature. *Riv. Math. Univ. Parma (N.S.)*, **1**, 441–450 (2010).

[Cam92] Campana, F. Connexité rationnelle des variétés de Fano. *Ann. Sci. Ec. Norm. Sup.* **25**, 539–545 (1992).

[CPZ03] Campana, F., Peternell, Th., and Zhang, Q. On the Albanese maps of compact Kähler manifolds. *Proc. Amer. Math. Soc.* **131**, 549–553 (2003).

[CG71] Cheeger, J. and Gromoll, D. The splitting theorem for manifolds of nonnegative Ricci curvature. *J. Diff. Geom.* **6**, 119–128 (1971).

[CG72] Cheeger, J. and Gromoll, D. On the structure of complete manifolds of nonnegative curvature. *Ann. Math.* **96**, 413–443 (1972).

[DPS94] Demailly, J.-P., Peternell, T., and Schneider, M. Compact complex manifolds with numerically effective tangent bundles. *J. Alg. Geom.* **3**, 295–345 (1994).

[DPS96] Demailly, J.-P., Peternell, T., and Schneider, M. Compact Kähler manifolds with hermitian semipositive anticanonical bundle. *Compos. Math.* **101**, 217–224 (1996).

[DR52] de Rham, G. Sur la reductibilité d'un espace de Riemann. *Comment. Math. Helv.* **26**, 328–344 (1952).

[Gau77] Gauduchon, P. Le théorème de l'excentricité nulle. *C. R. Acad. Sci. Paris* **285**, 387–390 (1977).

[GHS01] Graber, T., Harris, J., and Starr, J. Families of rationally connected varieties. *J. Amer. Math. Soc.* **16**, 57–67 (2003).

[KMM92] Kollár, J., Miyaoka, Y., and Mori, S. Rationally connected varieties. *J. Alg.* **1**, 429–448 (1992).

[Kob81] Kobayashi, S. Recent results in complex differential geometry. *Jber. dt. Math.-Verein.* **83**, 147–158 (1981).

[Kob83] Kobayashi, S. Topics in complex differential geometry. In *DMV Seminar*, Vol. 3. Berlin: Birkhäuser, 1983.

[Kod54] Kodaira, K. On Kähler varieties of restricted type. *Ann. Math.* **60**, 28–48 (1954).

[Kol96] Kollár, J. *Rational Curves on Algebraic Varieties*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 32. Berlin: Springer-Verlag, 1996.

[Lic67] Lichnerowicz, A. Variétés kähleriennes et première classe de Chern. *J. Diff. Geom.* **1**, 195–224 (1967).

[Lic71] Lichnerowicz, A. Variétés Kählériennes à première classe de Chern non négative et variétés riemanniennes à courbure de Ricci généralisée non négative. *J. Diff. Geom.* **6**, 47–94 (1971).

[Pau12] Păun, M. Relative adjoint transcendental classes and the Albanese map of compact Kähler manifolds with nef Ricci classes. arXiv: 1209.2195.

[Pet06] Peternell, Th. Kodaira dimension of subvarieties II. *Int. J. Math.* **17**, 619–631 (2006).

[PS98] Peternell, Th. and Serrano, F. Threefolds with anti canonical bundles. *Coll. Math.* **49**, 465–517 (1998).

[Ued82] Ueda, T. On the neighborhood of a compact complex curve with topologically trivial normal bundle. *J. Math. Kyoto Univ.* **22**, 583–607 (1982/83).

[Yau78] Yau, S. T. On the Ricci curvature of a complex Kähler manifold and the complex Monge–Ampère equation I. *Comm. Pure Appl. Math.* **31**, 339–411 (1978).

[Zha96] Zhang, Q. On projective manifolds with nef anticanonical bundles. *J. Reine Angew. Math.* **478**, 57–60 (1996).

[Zha05] Zhang, Q. On projective varieties with nef anticanonical divisors. *Math. Ann.* **332**, 697–703 (2005).