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Subcanonical graded rings which are not Cohen–Macaulay

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Abstract

We answer a question by Jonathan Wahl, giving examples of regular surfaces (so that the canonical ring is Gorenstein) with the following properties:

(1) the canonical divisor $K_S \equiv rL$ is a positive multiple of an ample divisor L;
(2) the graded ring $\mathcal{R} := \mathcal{R}(X, L)$ associated to L is not Cohen–Macaulay.

In the Appendix, Wahl shows how these examples lead to the existence of Cohen–Macaulay singularities with K_X $\mathbb{Q}$-Cartier which are not $\mathbb{Q}$-Gorenstein, since their index one cover is not Cohen–Macaulay.

Dedicated to Rob Lazarsfeld on the occasion of his 60th birthday

1 Introduction

The situation that we consider in this paper is the following: L is an ample divisor on a complex projective manifold X of complex dimension n, and we assume that L is subcanonical, i.e., there exists an integer h such that we have the linear equivalence $K_X \equiv hL$, where $h \neq 0$. There are then two cases: $h < 0$ and X is a Fano manifold, or $h > 0$ and X is a manifold with ample canonical divisor (in particular X is of general type). Assume that X is a Fano manifold and that $-K_X = rL$, with $r > 0$: then, by Kodaira vanishing,

$$H^j(mL) := H^j(\mathcal{O}_X(mL)) = 0, \qquad \forall m \in \mathbb{Z}, \forall 1 \leq j \leq n-1.$$

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For $m < 0$ this follows from Kodaira vanishing (and holds for $j \geq 1$), while for $m \geq 0$ Serre duality gives $h^j(mL) = h^{n-j}(K - mL) = h^{n-j}((-r - m)L) = 0$. At the other extreme, if K_X is ample and $K_X \equiv rL$ (thus $r > 0$), by the same argument we get vanishing outside of the interval

$$0 \leq m \leq r.$$

We associate to L, as usual, the finitely generated graded $\mathbb{C}$-algebra

$$\mathcal{R}(X, L) := \oplus_{m \geq 0} H^0(X, \mathcal{O}_X(mL)).$$

Therefore in the Fano case, the divisor L is arithmetically Cohen–Macaulay (see [Hart77]) and the above graded ring is a Gorenstein ring. The question is whether, in the case where K_X is ample, one may also hope for such a good property.

The above graded ring is integral over the canonical ring $\mathcal{A} := \mathcal{R}(X, K_X)$, which is a Gorenstein ring if and only if we have *pluri-regularity*, i.e., vanishing

$$H^j(\mathcal{O}_X) = 0, \qquad \forall\ 1 \leq j \leq n - 1.$$

Jonathan Wahl asked the following question (which makes sense only for $n \geq 2$):

Question 1 (J. Wahl) *Are there examples of subcanonical pluri-regular varieties X such that the graded ring $\mathcal{R}(X, L)$ is not Cohen–Macaulay?*

We show that the answer is positive, also in the case of regular subcanonical surfaces with K_X ample, where by the assumption we have the vanishing

$$H^1(mL) = 0, \qquad \forall m \leq 0, \text{ or } r \leq m$$

and the question boils down to requiring the vanishing also for $1 \leq m \leq r - 1$.

The following theorem answers the question by J. Wahl:

Theorem 2 *For each $r = n - 3$, where $n \geq 7$ is relatively prime to* 30*, and for each m, $1 \leq m \leq r - 1$, there are Beauville-type surfaces S with $q(S) = 0$ ($q(S) := \dim H^1(S, \mathcal{O}_S)$) such that $K_S = rL$ and $H^1(mL) \neq 0$.*

We therefore get examples of the following situation: $\mathcal{A} := \mathcal{R}(S, K_S)$ is a Gorenstein graded ring, and a subring of the ring $\mathcal{R} := \mathcal{R}(S, L)$, which is not arithmetically Cohen–Maculay. Hence, we have constructed examples of non-Cohen–Macaulay singularities (Spec($\mathcal{R}$)) with K_Y Cartier which are cyclic quasi-étale covers of a Gorenstein singularity (Spec($\mathcal{A}$)). In the Appendix, J. Wahl uses these to construct Cohen–Macaulay singularities with K_X $\mathbb{Q}$-Cartier whose index one cover is not Cohen–Macaulay.

In fact, we can consider three graded rings, two of which are subrings of the third, and which are cones associated to line bundles on the surface S:

- $Y := \mathrm{Spec}(\mathcal{R})$, the cone associated to L, which is not Cohen–Macaulay, while K_Y is Cartier;
- $Z := \mathrm{Spec}(\mathcal{A})$, the cone associated to K_S, which is Gorenstein;
- $X := \mathrm{Spec}(\mathcal{B})$, the cone associated to $K_S + L$ (for instance), which is Cohen–Macaulay with K_X $\mathbb{Q}$-Cartier, but whose index 1 (or canonical) cover $Y = \mathrm{Spec}(\mathcal{R})$ is not Cohen–Macaulay.

2 The special case of even surfaces

Recall: a smooth projective surface S is said to be **even** if there is a divisor L such that $K_S \equiv 2L$. This is a topological condition; it means that the second Stiefel Whitney class $w_2(S) = 0$, or, equivalently, the intersection form

$$H^2(S, \mathbb{Z}) \to \mathbb{Z}$$

is even (takes only even values). In particular, an even surface is a minimal surface. In particular, if S is of general type and even, the self-intersection

$$K_S^2 = 4L^2 = 8k$$

for some integer $k \geq 1$. The first numerical case is therefore the case $K_S^2 = 8$.

Proposition 3 *Assume that S is an even surface of general type with $K_S^2 = 8$ and $p_g(S) = h^0(K_S) = 0$. Then, if $K_S \equiv 2L$, we have $H^1(L) = 0$.*

Proof We have made the assumption that S is even, $K \equiv 2L$, and $p_g = 0$. Since the intersection form is even, and $K^2 \leq 9$ by the Bogomolov–Miyaoka–Yau inequality, we obtain that $L^2 = 2$. The Riemann–Roch theorem tells us that $\chi(L) = 1 + \frac{1}{2}L(L-K) = 1 + \frac{1}{2}L(-L) = 0$. On the other hand, by Serre duality $\chi(L) = 2h^0(L) - h^1(L)$, so if $H^1(L)$ is different from zero, then $H^0(L) \neq 0$, contradicting $p_g = 0$. □

Our construction for $n = 5$ shows in particular that the "Beauville surface," constructed in [Bea78] (see also [BPHV]) is an even surface with $K_S^2 = 8, q(S) = p_g(S) = 0$, but with $H^1(L) = 0$.

3 Canonical linearization on Fermat curves

Fix a positive integer $n \geq 5$, and let C be the degree-n Fermat curve

$$C := \{(x, y, z) \in \mathbb{P}^2 \mid f(x, y, z) := x^n + y^n + z^n = 0\}.$$

Let, as usual, μ_n be the group of n-roots of unity. Then the group

$$G := \mu_n^2 = \mu_n^3/\mu_n$$

acts on C, and we obtain a natural linearization of $\mathcal{O}_C(1)$ by letting $(\zeta, \eta) \in \mu_n^2$ act as follows:

$$z \mapsto z, x \mapsto \zeta x, y \mapsto \eta y.$$

In other words, $H^0(\mathcal{O}_C(1))$ splits as a direct sum of 1-dimensional eigenspaces (respectively generated by x, y, z) corresponding to the characters $(1,0), (0,1), (0,0) \in (\mathbb{Z}/n)^2 \cong \mathrm{Hom}(G, \mathbb{C}^*)$. Similarly, for $m \leq n-1$, the monomial $x^a y^b z^{m-a-b} \in H^0(\mathcal{O}_C(m))$ generates the unique eigenspace for the character (a, b) (we identify here $\mathbb{Z}/n \cong \{0, 1, \ldots, n-1\}$ and we obviously require $a + b \leq m$). However, any two linearizations differ (see [Mum70]) by a character of the group.

Definition 4 Assume that n is not divisible by 3. We call the **canonical** linearization on $H^0(\mathcal{O}_C(1))$ the one obtained from the natural one by twisting with the character $(n-3)^{-1}(1,1)$. Thus x corresponds to the character $v_1 := (1,0) + (n-3)^{-1}(1,1) = (-3)^{-1}(-2,1)$, y corresponds to the character $v_2 := (0,1) + (n-3)^{-1}(1,1) = (-3)^{-1}(1,-2)$, and z corresponds to the character $v_3 := (-3)^{-1}(1,1)$.

Remark 5 (I) Observe that v_1, v_2 are a basis of $(\mathbb{Z}/n)^2$ as soon as n is not divisible by 3. Indeed, $v_1 + v_2 = \frac{1}{3}(1,1) = 3^{-1}(1,1)$, hence

$$(1,0) = v_1 + 3^{-1}(1,1) = 2v_1 + v_2, \quad (0,1) = 2v_2 + v_1.$$

(II) Observe that the above linearization induces a linearization on all multiples of L, and, in the case where $m = (n-3)$, we obtain the natural linearization on the canonical divisor of C, $\mathcal{O}_C(n-3) \cong \Omega^1_C$. Since, if we take affine coordinates where $z = 1$, and we let f be the equation of C, we have

$$H^0(\Omega^1_C) = \left\{ P(x,y)\frac{dx}{f_y} = -P(x,y)\frac{dy}{f_x} \right\}$$

and the monomial $P = x^a y^b$ corresponds under this isomorphism to the character $(a+1, b+1)$.

(III) In particular, Serre duality

$$H^0(\mathcal{O}_C(m)) \times H^1(\Omega^1_C(-m)) \to H^1(\Omega^1_C) \cong \mathbb{C},$$

where $\mathbb{C}$ is the trivial G-representation, is G-invariant.

From the previous discussion it follows that:

Lemma 6 *The monomial* $x^a y^b z^c \in H^0(\mathcal{O}_C(m))$ *(here* $a, b, c \geq 0,\ a+b+c = m$*) corresponds to the character* χ*, equal to*

$$(a, b) + (-3)^{-1}(m, m) = (a - c)v_1 + (b - c)v_2.$$

Proof $v_1 + v_2 = \frac{1}{3}(1, 1)$, hence $(a, b) + (-3)^{-1}(m, m) = av_1 + bv_2 + (-m + a + b)(3)^{-1}(1, 1) = (a - c)v_1 + (b - c)v_2$. □

4 Abelian Beauville surfaces and their subcanonical divisors

We recall now the construction (see also [Cat00], [BCG05], or [Cat08]) of a Beauville surface with Abelian group $G \cong (\mathbb{Z}/n)^2$, where n is not divisible by 2 and by 3.

Definition 7 (1) Let $\Sigma \subset G$ be the union of the three respective subgroups generated by $(1, 0), (0, 1), (1, 1)$.

(2) Let $\psi : G \to G$ be a homomorphism such that, setting $\Sigma^* := \Sigma \setminus \{(0, 0)\}$, $\psi(\Sigma^*) \cap \Sigma^* = \emptyset$ (equivalently, $\psi(\Sigma) \cap \Sigma = \{(0, 0)\}$).

(3) Let C be the degree-n Fermat curve and let

$$S = (C \times C)/(\mathrm{Id} \times \psi)(G),$$

i.e., the quotient of $C \times C$ by the action of G s.t. $g(P_1, P_2) = (g(P_1), \psi(g)(P_2))$.

Remark 8 (i) By property (2), G acts freely and S is a projective smooth surface with ample canonical divisor.

(ii) The line bundle $\mathcal{O}_{C\times C}(1, 1)$ is $G \times G$ linearized, in particular it is $G \cong (\mathrm{Id} \times \psi)(G)$-linearized, therefore it descends to S, and we get a divisor L on S such that the pull-back of $\mathcal{O}_S(L)$ is the above G-linearized bundle.

(iii) By the previous remarks, we have a linear equivalence

$$K_S \equiv (n - 3)L.$$

5 Cohomology of multiples of the subcanonical divisor L

We consider now an integer m with

$$1 \leq m \leq n - 4$$

and determine the space $H^1(\mathcal{O}_S(mL))$.

Observe first that $H^1(\mathcal{O}_S(mL)) \cong H^1(\mathcal{O}_{C\times C}(m,m))^G$. By the Künneth formula

$$(9)\quad H^1(\mathcal{O}_{C\times C}(m,m)) \cong [H^0(\mathcal{O}_C(m)) \otimes H^1(\mathcal{O}_C(m))] \bigoplus [H^1(\mathcal{O}_C(m)) \otimes H^0(\mathcal{O}_C(m))].$$

We want to decompose the right hand side as a representation of $G \cong (\mathrm{Id}\times\psi)(G)$.

Explicitly, $H^0(\mathcal{O}_C(m)) = \oplus_\chi V_\chi$, where if we write the character $\chi = (a,b) + (-3)^{-1}(m,m)$ ($\chi = (a-(m-a-b))v_1 + (b-(m-a-b))v_2$ as we saw) then V_χ has dimension equal to one and corresponds to the monomial $x^a y^b z^{m-a-b}$, where $a,b \geq 0,\ a+b \leq m$. By Serre duality, $H^1(\mathcal{O}_C(m)) = \oplus_{\chi'} V_{-\chi'}$, where if we write as above $\chi' = (a',b') + (-3)^{-1}(m',m')$, then $V_{-\chi'}$ is the dual of $V_{\chi'}$, corresponding to the monomial $x^{a'} y^{b'} z^{m'-a'-b'}$, where $m' = n-3-m$, so $1 \leq m' \leq n-4$ also, and where $a',b' \geq 0,\ a'+b' \leq m'$.

Now, the homomorphism $\psi\colon G \to G$ induces a dual homomorphism $\phi := \psi^\vee\colon G^\vee \to G^\vee$, therefore we can finally write $H^1(\mathcal{O}_{C\times C}(m,m))$ as a representation of $G \cong (\mathrm{Id}\times\psi)(G)$:

$$H^1(\mathcal{O}_{C\times C}(m,m)) = \bigoplus_{\chi,\chi'} [(V_\chi \otimes V_{-\phi(\chi')}) \oplus (V_{-\chi'} \otimes V_{\phi(\chi)})].$$

We have proven therefore:

Lemma 10 *$H^1(\mathcal{O}_S(mL)) \neq 0$ if and only if there are characters $\chi = (a-c)v_1 + (b-c)v_2$ and $\chi' = (a'-c')v_1 + (b'-c')v_2$ with $a,b \geq 0,\ a+b \leq m$, $a',b' \geq 0,\ a'+b' \leq m' = n-3-m$ such that*

$$\chi = \phi(\chi') \quad \text{or} \quad \chi' = \phi(\chi).$$

Proof of Theorem 2 We now take ϕ to be given by a diagonal matrix in the basis v_1, v_2, i.e., such that

$$\phi(v_j) = \lambda_j v_j,\ j = 1,2,\ \lambda_j \in (\mathbb{Z}/n)^*.$$

For further use we also set $\lambda := \lambda_1, \mu := \lambda_2$.

Given n relatively prime to 30 and $1 \leq m \leq n-4$, we want to find λ_1 and μ such that the equations

$$(a-c) = \lambda(a'-c')$$

$$(b-c) = \mu(b'-c')$$

have solutions with $a,b,c \geq 0, a+b+c = m$, and $a',b',c' \geq 0, a'+b'+c' = m'$.

The first idea is simply to take $b = c$ and $b' = c'$, so that μ can be taken arbitrarily. For the first equation some care is needed, since we want that λ be a unit: for this it suffices that $(a - c), (a' - c')$ are both units, for instance they could be chosen to be equal to one of the three numbers $1, 2, 3$, according to the congruence class of m, respectively m', modulo 3. With this proviso we have to verify that we have a free action on the product.

Lemma 11 *If $n \geq 7$, given λ a unit, there exists a unit μ such that $\psi = \phi^\vee$ satisfies the condition $\psi(\Sigma) \cap \Sigma = \{(0, 0)\}$.*

Proof Since $(1, 0) = 2v_1 + v_2$ and $(0, 1) = v_1 + 2v_2$, the matrix of ϕ in the standard basis is the matrix

$$\phi = \frac{1}{3}\begin{pmatrix} 4\lambda - \mu & 2(\lambda - \mu) \\ 2(\mu - \lambda) & 4\mu - \lambda \end{pmatrix}$$

while the matrix of ψ is the matrix

$$\psi = \frac{1}{3}\begin{pmatrix} A := 4\lambda - \mu & B := 2(\mu - \lambda) \\ C := 2(\lambda - \mu) & D := 4\mu - \lambda \end{pmatrix}.$$

The conditions for a free action boil down to

$$A, B, C, D, A + B, C + D \text{ are units in } \mathbb{Z}/n$$

and moreover $A \neq B$, $C \neq D, A + B \neq C + D$. These are in turn equivalent to the condition that

$$\lambda, \mu, \lambda - 4\mu, \lambda - \mu, \mu - 4\lambda, \lambda + 2\mu, 2\lambda + \mu \in (\mathbb{Z}/n)^*.$$

Given $\lambda \in (\mathbb{Z}/n)^*$, consider its direct-sum decomposition given by the Chinese remainder theorem and the primary factorization of n. For each prime p dividing n, the residue classes modulo p which are excluded by the above condition are at most five values inside $(\mathbb{Z}/p)^*$, hence we are done if $(\mathbb{Z}/p)^*$ has at least six elements.

Now, since n is relatively prime to 30, each prime number dividing it is greater than or equal to $p = 7$. □

Proposition 12 *Consider the Beauville surface S constructed in [Bea78], corresponding to the case $n = 5$. Then S is an even surface and $K_S \equiv 2L$, where $H^1(L) = 0$.*

Proof We observe that L is unique, because the torsion group of S is of exponent 5 (see [BC04]). The existence of L follows exactly as in the proof of the main theorem, where the condition $n \geq 7$ was not used. That $H^1(L) = 0$ follows directly from Proposition 3. □

Acknowledgments I would like to thank Jonathan Wahl for asking the above question. In the Appendix below he describes a construction based on our main result.

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Appendix by J. Wahl: A non-$\mathbb{Q}$-Gorenstein Cohen–Macaulay cone X with K_X $\mathbb{Q}$-Cartier

A germ $(X, 0)$ of an isolated normal complex singularity of dimension $n \geq 2$ is called $\mathbb{Q}$-*Gorenstein* if:

1. $(X, 0)$ is Cohen–Macaulay.
2. The dualizing sheaf K_X is $\mathbb{Q}$-Cartier (i.e., the invertible sheaf $\omega_{X-\{0\}}$ has finite order r).

3. The corresponding cyclic *index one* (or *canonical*) cover $(Y, 0) \to (X, 0)$ is Cohen–Macaulay, hence Gorenstein.

Alternatively, $(X, 0)$ is the quotient of a Gorenstein singularity by a cyclic group acting freely off the singular point. Some early definitions did not require the third condition, which is of course automatic for $n = 2$.

If $(X, 0)$ is $\mathbb{Q}$-Gorenstein, a 1-parameter deformation $(\mathcal{X}, 0) \to (\mathbb{C}, 0)$ is called $\mathbb{Q}$-Gorenstein if it is the quotient of a deformation of the index one cover of $(X, 0)$; this is exactly the condition that $(\mathcal{X}, 0)$ is itself $\mathbb{Q}$-Gorenstein. These notions were introduced by Kollár and Shepherd-Barron [1], who made extensive use of the author's explicit smoothings of certain cyclic quotient surface singularities in [2] (5.9); these deformations were patently $\mathbb{Q}$-Gorenstein, and it was important to name this property.

Recently, the author and others considered rational surface singularities admitting a rational homology disk smoothing (i.e., with Milnor number 0). The 3-dimensional total space of the smoothing had a rational singularity with K $\mathbb{Q}$-Cartier, but it was not initially clear whether the smoothings were $\mathbb{Q}$-Gorenstein. (This was later established [4] by proving the stronger result that the total spaces were log-terminal.) In fact, one needs to be careful because of the example of A. Singh:

Example ([3]) There is a 3-dimensional isolated rational (hence Cohen–Macaulay) complex singularity $(X, 0)$ with K_X $\mathbb{Q}$-Cartier which, however, is not $\mathbb{Q}$-Gorenstein.

The purpose of this appendix is to use F. Catanese's result to provide other examples; they are not rational, but are cones over a smooth projective variety, which could for instance be assumed to be projectively normal with ideal generated by quadrics.

Proposition 13 *Let S be a surface as in Theorem 2, with $h^1(S, \mathcal{O}_S) = 0$, L ample, $K_S = rL$ (for some $r > 1$), and $h^1(mL) \neq 0$ for some $m > 0$. Let t be greater than r and relatively prime to it. Then*

1. *The cone $R = \mathcal{R}(S, tL) := \oplus_{m \geq 0} H^0(S, \mathcal{O}_S(mtL))$ is Cohen–Macaulay.*
2. *The dualizing sheaf of R is torsion, of order t.*
3. *The index 1 cover is $\mathcal{R}(S, L) := \oplus_{m \geq 0} H^0(S, \mathcal{O}_S(mL))$, and is not Cohen–Macaulay.*

In particular, R is not $\mathbb{Q}$-Gorenstein.

Proof The Cohen–Macaulayness for R follows because $h^1(itL) = 0$, for all i, thanks to Kodaira vanishing. Let $\pi\colon V \to S$ be the geometric line bundle corresponding to $-tL$; then $H^0(V, \mathcal{O}_V) \equiv R$. Since $K_V \equiv \pi^*(K_S + tL)$, one has that $jK_R \equiv \oplus_{n \in \mathbf{Z}} H^0(S, j(K_S + tL) + nL)$; since $tK_S = r(tL)$ with r and t relatively prime, K_R has order t. Making a cyclic t-fold cover and normalizing gives that $\mathcal{R}(S, L)$ is the index 1 cover, which as Catanese noted is not Cohen–Macaulay. □

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