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Threefold divisorial contractions to singularities of cE type

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Abstract

We survey some recent progress in the classification of three-dimensional divisorial contractions to cE points. In particular, we introduce a new structure of three-dimensional cE singularity and use this structure to explain the work of Hayakawa. We also provide some new examples.

Dedicated to Rob Lazarsfeld on the occasion of his sixtieth birthday

1 Introduction

The minimal model program has been one of the main tools in the study of birational algebraic geometry. After some recent advances in the study of the geometry of complex 3-folds, one might hope to build up an explicit classification theory for 3-folds similar to the theory of surfaces by using the minimal model program.

In the minimal model program, divisorial contractions, flips, and flops are considered to be elementary maps. Any birational map obtained from the

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minimal model program consists of a combination of the above-mentioned maps. Let us briefly recall some known results about three-dimensional birational maps. First of all, Mori and then Cutkosky classified birational maps from nonsingular 3-folds and Gorenstein 3-folds respectively [4, 20]. Tziolas has produced a series of work on divisorial contractions to curves passing through Gorenstein singularities (cf. [23–25]). The recent project of Mori and Prokhorov (cf. [21, 22]) on extremal contractions provides a treatment which is valid for divisorial contractions to curves and for conic bundles. They classified completely divisorial contractions to curves of type IA, IC, and IIB. Flops are studied in Kollár's article [16]. Flips are still quite mysterious, except for some examples in [2, 18].

Divisorial contractions to points are probably the best understood, due mainly to the works of Kawamata, Hayakawa, Markushevich, and Kawakita (cf. [5–7, 10–15, 19]). Divisorial contractions to points of index > 1 are now completely classified and realized as weighted blow-ups. Therefore, it remains to consider contractions to points of index 1, i.e., terminal Gorenstein singularities. The description of contractions to index 1 points can be found in [13]. In fact, contractions to cA points are classified completely in [13]. Recently, Hayakawa started a project to classify contractions to cD and cE points [8, 9]. The project is not yet complete. Especially, the existence of divisorial contractions with discrepancy > 1 listed in [13, Table 3, e2, e3, e7] is unknown.

The purpose of this paper is to analyze the known examples in [9] and give the structure of various weighted blow-ups. We introduce a new structure of three-dimensional cE singularities and use this structure to explain the work of Hayakawa. We also provide some new examples in the last section.

2 Normal form of cE singularities

For any $F \in \mathbb{C}\{x_1, \ldots, x_n\}$ the set $(F = 0)$ is a germ of a complex analytic set. For $F \in \mathbb{C}[[x_1, \ldots x_n]]$, by the singularity $(F = 0)$, we mean the scheme $\operatorname{Spec}\mathbb{C}[[x_1, \ldots, x_n]]/(F)$.

$F, G \in \mathbb{C}[[x_1, \ldots, x_n]]$ (resp. $\mathbb{C}\{x_1, \ldots, x_n\}$) are called equivalent if there is an automorphism of $\mathbb{C}[[x_1, \ldots, x_n]]$ (resp. $\mathbb{C}\{x_1, \ldots, x_n\}$) given by $x_i \mapsto \phi_i(x_1, \ldots, x_n)$ and a unit $u(x_1, \ldots, x_n)$ such that

$$u(x_1, \ldots, x_n)G(x_1, \ldots, x_n) = F(\phi_1, \ldots, \phi_n).$$

Note that if $F, G \in \mathbb{C}\{x_1, \ldots, x_n\}$ have isolated singularities at the origin, then F and G are equivalent in $\mathbb{C}\{x_1, \ldots, x_n\}$ if and only if they are equivalent in $\mathbb{C}[[x_1, \ldots, x_n]]$ (see, e.g., [1, 17]).

For a power series F, F_d denotes the degree-d homogeneous part and $F_{\geq d}$ (resp. $F_{>d}$) denotes the part of degree $\geq d$ (resp. $> d$).

Theorem 2.1 ([17]) *Assume that $F(x, y, z, u)$ defines a terminal singularity of type cE. Then F is equivalent to one of the following:*

- cE_6 : $x^2 + y^3 + yg_{\geq 3}(z, u) + h_{\geq 4}(z, u)$, *where* $h_4 \neq 0$.
- cE_7 : $x^2 + y^3 + yg_{\geq 3}(z, u) + h_{\geq 5}(z, u)$, *where* $g_3 \neq 0$.
- cE_8 : $x^2 + y^3 + yg_{\geq 4}(z, u) + h_{\geq 5}(z, u)$, *where* $h_5 \neq 0$.

We call such a form a normal form of F.

The following consequence is immediate:

Lemma 2.2 *Normal forms are equivalent if and only if there exists an automorphism of the form*

$$\begin{cases} x \mapsto x; \\ y \mapsto y; \\ z \mapsto \phi_3(z, u); \\ u \mapsto \phi_4(z, u). \end{cases}$$

Therefore, changing z, u by a linear transformation and up to a constant, we may and do assume that in the case of cE_6,

$$h_4 \in \{z^4, z^3(z + u), (z^2 + zu)^2, z^2(z^2 + u^2), z(z^3 + u^3)\}.$$

In particular, $z^4 \in h_4$.

In the case of cE_7, $g_3 \neq 0$, we may and do assume that

$$g_3 \in \{z^3, z^2(z + u), z(z^2 + u^2)\}.$$

In particular, $z^3 \in g_3$.

In the case of cE_8, $h_5 \neq 0$, we may and do assume that

$$h_5 \in \left\{ \begin{array}{l} z^5, z^4(z + u), z^3(z + u)^2, z^3(z^2 + u^2), \\ z^2(z + u)^2(z - u), z^2(z^3 + u^3), z(z^4 + u^4) \end{array} \right\}.$$

In particular, $z^5 \in h_5$.

Remark 2.3 Consider

$$F = x^2 + y^3 + yg(z, u) + h(z, u),$$

which possibly contains lower-degree terms in g or h. An isolated singularity with the above description is called a cE-like singularity.

An isolated cE-like singularity is at worst of type cD (resp. cE_6, cE_7, cE_8) if $g_m \neq 0$ for some $m \leq 2$ or $h_m \neq 0$ for some $m \leq 3$ (resp. $h_m \neq 0$ for some $m \leq 4$, $g_m \neq 0$ for some $m \leq 3$, $h_m \neq 0$ for some $m \leq 5$).

3 Admissible weights and canonical form

Given a three-dimensional terminal Gorenstein singularity $(P \in X)$, it is known that there exists a divisorial contraction to $(P \in X)$ with discrepancy 1, which is realized as a weighted blow-up (cf. [19]). In this section we consider weights that might be admissible for a weighted blow-up with discrepancy 1.

Given a terminal Gorenstein singularity $(P \in X)$, we always identify it with $(F = 0) \subset \mathbb{C}^4$ for some F of normal form. We consider a weighted blow-up $wBl_w \colon \mathcal{Y} \to \mathcal{X} = \mathbb{C}^4$ with weight $w = (a, b, c, d)$. Let $\mathcal{E}$ be the exceptional divisor and write $\mathcal{Y} = \cup_{i=1}^4 U_i$. Let Q_i denote the origin of U_i. There is an induced map $wBl_w \colon Y \to X$, where Y is realized as the proper transform of X in the weighted blow-up of $\mathbb{C}^4$. Let $E := \mathcal{E} \cap Y$.

In considering weighted blow-ups $wBl_w \colon Y \to X$, we have the following questions:

1. Is E irreducible and reduced? If so, then E gives rise to a valuation with discrepancy $a + b + c + d - wt_w(F) - 1$. In other words, $K_Y = wBl_w^* K_X + \alpha E$ with $\alpha = a + b + c + d - wt_w(F) - 1$, whenever it makes sense. We thus call wBl_w a *divisorial blow-up* in this situation.
2. Does Y have only isolated singularities? If not, then clearly Y is not terminal.
3. Does Y have terminal singularities? If so, then $wBl_w \colon Y \to X$ is indeed a divisorial contraction.

Since divisorial contractions with minimal discrepancy 1 play a pivotal role in the study of geometry over terminal Gorenstein points, we would like to first consider all possible weights such that $\alpha = a+b+c+d-wt_w(F)-1 = 1$, which we call *admissible weights* of F. However, we exclude the weights $(2, 1, 1, 1)$ and $(1, 1, 1, 1)$ because $\mathcal{E} \cap Y$ is never reduced, even though $\alpha = 1$ is satisfied.

Proposition 3.1 *The admissible weights for cE_6 are*

$$(6, 4, 3, 1), (4, 3, 2, 1), (3, 2, 2, 1), (2, 2, 1, 1).$$

The admissible weights for cE_7 are

$$(9, 6, 4, 1), (7, 5, 3, 1), (6, 4, 3, 1), (5, 4, 2, 1), (5, 3, 2, 1),$$

$$(4, 3, 2, 1), (3, 2, 2, 1), (3, 3, 1, 1), (3, 2, 1, 1), (2, 2, 1, 1).$$

The admissible weights for cE_8 are

$$(15, 10, 6, 1), (12, 8, 5, 1), (10, 7, 4, 1), (9, 6, 4, 1), (8, 5, 3, 1), (7, 5, 3, 1),$$

$$(6, 4, 3, 1), (5, 4, 2, 1), (5, 3, 2, 1), (4, 3, 2, 1), (3, 2, 2, 1), (3, 2, 1, 1), (2, 2, 1, 1).$$

Proof We first consider the cE_6 case. We have

$$\begin{cases} wt(x^2), wt(y^3), wt(z^4) \geq wt(F), \\ wt(xyzu) - wt(F) - 2 = 0. \end{cases}$$

It follows that

$$wt(xyz) \geq \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) wt(F)$$

and therefore

$$2 \geq \frac{1}{12} wt(F) + wt(u).$$

It is then straightforward to solve for the admissible weights.

We next consider the cE_7 case. We have

$$\begin{cases} wt(x^2), wt(y^3), wt(yz^3) \geq wt(F), \\ wt(xyzu) - wt(F) - 2 = 0. \end{cases}$$

It follows that

$$wt(x^3y^3z^3) = wt(x^3 \cdot y^2 \cdot yz^3) \geq \left(\frac{3}{2} + \frac{2}{3} + 1\right) wt(F)$$

and therefore

$$2 \geq \frac{1}{18} wt(F) + wt(u).$$

It is then straightforward to solve for the admissible weights.

The case of cE_8 is similar to the case of cE_6. We have

$$2 \geq \frac{1}{30} wt(F) + wt(u).$$

□

Definition 3.2 Given an admissible weight w of the form $(a, b, k, 1)$ with $a \geq b \geq k \geq 1$, we define the following notions:

- The level of w, denoted $lev(w)$, is k.
- $\sigma(w) := a + b + k - 1$.
- $\sigma_y(w) := a + k - 1$.
- A weight denoted by $w_{\sigma(w)}$, e.g., $w_{12} = (6, 4, 3, 1)$. There are two weights with $\sigma = 6$: $w_6 = (3, 2, 2, 1)$ and $w'_6 = (3, 3, 1, 1)$.
- w is in the main series, or of stage 0, if $w = (3k - 3, 2k - 2, k, 1)$. This consists of $wt_{30} = (15, 10, 6, 1), w_{24} = (12, 8, 5, 1), w_{18} = (9, 6, 4, 1), w_{12} = (6, 4, 3, 1),\ w_6 = (3, 2, 2, 1)$.

  Note that $\sigma = 6k - 6, \sigma_y = 4k - 4$ in this situation.

- w is in series I, or of stage 1, if $w = (3k-2, 2k-1, k, 1)$.
  This consists of $w_{20} = (10, 7, 4, 1), w_{14} = (7, 5, 3, 1), w_8 = (4, 3, 2, 1)$.
  Note that $\sigma = 6k - 4, \sigma_y = 4k - 3$ in this situation.
- w is in series II, or of stage 2, if $w = (3k-1, 2k-1, k, 1)$.
  This consists of $w_{15} = (8, 5, 3, 1), w_9 = (5, 3, 2, 1)$.
  Note that $\sigma = 6k - 3, \sigma_y = 4k - 2$ in this situation.
- w is in series III, or of stage 3, if $w = (3k-1, 2k, k, 1)$.
  This consists of $w_{10} = (5, 4, 2, 1), w_4 = (2, 2, 1, 1)$.
  Note that $\sigma = 6k - 2, \sigma_y = 4k - 2$ in this situation.
- w is in series IV, or of stage 4, if $w = (3k, 2k, k, 1)$.
  This consists of $w_5 = (3, 2, 1, 1)$.
  Note that $\sigma = 6k - 1, \sigma_y = 4k - 1$ in this situation.
- w is in series V, or of stage 5, if $w = (3k, 2k+1, k, 1)$.
  This consists of $w'_6 = (3, 3, 1, 1)$.
  Note that $\sigma = 6k, \sigma_y = 4k - 1$ in this situation and this only happens in the case of cE_7.

Definition 3.3 Given a normal form F, we say that F is of level $\geq k$ if $wt_w(F) \geq \sigma(w)$ for weight w of level $k \geq 2$ in the main series. Otherwise, we say that F is of level 1.

Lemma 3.4 *If F is of level $\geq k_1$, then F is of level $\geq k_2$ for any $k_1 \geq k_2 \geq 1$.*

Proof We write $g = \sum a_{ij} z^i u^j$, $h = \sum b_{ij} z^i u^j$ and let $I_g := \{(i, j) | a_{ij} \neq 0\}$, $I_h := \{(i, j) | b_{ij} \neq 0\}$.

If F is of level $\geq k_1$, then for $(i, j) \in I_h$ (resp. I_g), $k_1 i + j \geq 6(k_1 - 1)$ (resp. $4(k_1 - 1)$). It follows that for $(i, j) \in I_h$

$$k_2 i + j \geq \frac{k_2}{k_1}(k_1 i + j) \geq \frac{k_2}{k_1} 6(k_1 - 1) \geq 6(k_2 - 1).$$

Similarly, for $(i, j) \in I_g$, one has $k_2 i + j \geq 4(k_2 - 1)$. Therefore, F is of level $\geq k_2$. □

Definition 3.5 Fix a normal form F of level k, we say that F is of stage $\geq m$ if $wt_w(F) \geq \sigma(w)$ for the admissible weight w of level k of stage $m \geq 1$. Otherwise, we say that F is of level k of stage 0.

Lemma 3.6 *If F is of stage $\geq m_1$, then F is of stage $\geq m_2$ for any $m_1 \geq m_2$.*

Proof Notice that weights of level k and stage 0 consist of $v_0 = (3k-3, 2k-2, k, 1)$ with $\sigma = 6k - 6$. Hence F is of level k stage 0 if and only if $wt_{v_0}(g) \geq 4k - 4$ and $wt_{v_0}(h) \geq 6k - 6$.

Weights of stage 1 consist of $v_1 = (3k-2, 2k-1, k, 1)$ with $\sigma = 6k-4$. Hence F is of level k stage ≥ 1 if and only if $wt_{v_1}(g) \geq 4k-3$ and $wt_{v_1}(h) \geq 6k-4$. Since $wt_{v_0}(g) = wt_{v_1}(g)$ and $wt_{v_0}(h) = wt_{v_1}(h)$, it follows immediately that if F is of stage ≥ 1 then F is of stage 0.

Similarly, weights of stage 2 consist of $v_2 = (3k-1, 2k-1, k, 1)$ with $\sigma = 6k-3$. Hence F is of level k stage ≥ 2 if and only if $wt_{v_2}(g) \geq 4k-2$ and $wt_{v_2}(h) \geq 6k-3$. It follows immediately that if F is of stage ≥ 2 then F is of stage ≥ 1.

The comparisons of other stages are similar. □

Definition 3.7 Given F, we say that F is of level k if it is of level $\geq k$ but not of level $\geq k+1$. We say that F is of stage 0 if it is not of stage ≥ 1 and of stage m if it is of stage $\geq m$ but not of stage $\geq m+1$.

A normal form F is said to be a *canonical form* if it admits the highest level and then the highest stage among all equivalent normal forms.

Given a cE singularity $P \in X$, we associate a canonical form F so that $(P \in X) \cong o \in (F = 0) \subset \mathbb{C}^4$. We define the level and stage of $P \in X$ to be the level and stage of its canonical form F.

Therefore, we may classify isolated cE points $P \in X \cong o \in (F = 0) \subset \mathbb{C}^4$ according to their level and stage.

4 Admissible weighted blow-ups

In this section we study weighted blow-ups of cE singularities by admissible weights. Let us first fix some notation. Fix an admissible weight $w = (a, b, k, 1)$ such that $wt_w(F) \geq \sigma(w)$. We may write

$$F = F^w_\sigma + F^w_{>\sigma} \quad \text{or simply } F_\sigma + F_{>\sigma},$$

where F_σ (resp. $F_{>\sigma}$) denotes the homogeneous part of weighted degree = $\sigma(w)$ (resp. the part of weighted degree $> \sigma(w)$). More explicitly, we may also similarly write

$$F = x^2 + y^3 + yg_{\sigma_y} + yg_{>\sigma_y} + h_\sigma + h_{>\sigma}$$

or

$$F = x^2 + y^3 + y \sum_{ki+j \geq \sigma_y} a_{ij} z^i u^j + \sum_{ki+j \geq \sigma} b_{ij} z^i u^j.$$

We set $I_g := \{(i, j) | a_{ij} \neq 0\}$ and $I_h := \{(i, j) | b_{ij} \neq 0\}$.

Lemma 4.1 *Let F be of level k_1. Let $wBl_w : Y \to X$ be a weighted blow-up of weight w of level k_2 such that $k_1 > k_2 \geq 2$ and $wt_w(F) \geq \sigma(w)$.*

Suppose that

- *w is of stage $0, 1, 2$, or*
- *w is of stage 3 (hence $k_2 = 2$ or 1) and $k_1 \geq k_2 + 2$, or*
- *w is of stage 4 (hence $k_2 = 1$) and $k_1 \geq 3$.*

Then Y is not terminal.

Proof We shall prove that $Q_4 \in Y \cap U_4$ is not terminal. Since F is of level k_1, we have $k_1 i + j \geq 6k_1 - 6$ (resp. $4k_1 - 4$) for $(i, j) \in I_h$ (resp. I_g).

Let w be a weight of level k_2, then $Y \cap U_4$ is given by

$$\tilde{F} = x^2 u^{wt_w(x^2)-\sigma(w)} + y^3 u^{wt_w(y^3)-\sigma(w)} + y \sum a_{ij} z^i u^{j'} + \sum b_{ij} z^i u^{j''},$$

where

$$\begin{cases} j' = k_2 i + j - \sigma_y(w); \\ j'' = k_2 i + j - \sigma(w). \end{cases}$$

Therefore, if $(i, j) \in I_g$, then

$$i + j' + \sigma_y(w) = (k_2 + 1)i + j \geq \frac{k_2 + 1}{k_1}(k_1 i + j) \geq \frac{k_2 + 1}{k_1} 4(k_1 - 1) \geq 4k_2. \quad (\dagger_g)$$

Similarly, if $(i, j) \in I_h$, then

$$i + j'' + \sigma(w) \geq 6k_2. \quad (\dagger_h)$$

Case 1 w is of stage 0.
Note that we have $\sigma(w) = 6k_2 - 6, \sigma_y(w) = 4k_2 - 4$. By $\dagger_g, \dagger_h$, $\tilde{F}$ is of the form

$$x^2 + y^3 + y \sum a_{ij} z^i u^{j'} + \sum b_{ij} z^i u^{j''},$$

with $i + j' \geq 4$ and $i + j'' \geq 6$. Hence Q_4 is not terminal.

Case 2 w is of stage 1.
Now we have $\sigma(w) = 6k_2 - 4, \sigma_y(w) = 4k_2 - 3$.

By $\dagger_g, \dagger_h$, $Y \cap U_4$ is given by

$$\tilde{F} = x^2 + y^3 u + y \sum a_{ij} z^i u^{j'} + \sum b_{ij} z^i u^{j''},$$

with $i + j' \geq 3$ and $i + j'' \geq 4$. Hence Q_4 is not terminal.

Case 3 w is of stage 2.
Now we have $\sigma(w) = 6k_2 - 3, \sigma_y(w) = 4k_2 - 2$. Note that $Y \cap U_4$ is given by

$$\tilde{F} = x^2 u + y^3 + y \sum a_{ij} z^i u^{j'} + \sum b_{ij} z^i u^{j''}.$$

By $\dagger_g, \dagger_h$, $i + j' \geq 2$ and $i + j'' \geq 3$. Hence Q_4 is not terminal.

Case 4 w is of stage 3 and $k_2 = 2$.
Now $w = (5, 4, 2, 1)$ with $\sigma = 10, \sigma_y = 6$ and

$$\tilde{F} = x^2 + y^3u^2 + y\sum a_{ij}z^iu^{j'} + \sum b_{ij}z^iu^{j''}.$$

Note that $\dagger_g$ shows that $i + j' + \sigma_y(w) \geq 12\frac{k_1-1}{k_1}$ and hence $i + j' \geq 3$ if $k_1 \geq 4$. Similarly, $\dagger_h$ shows that $i + j'' + \sigma(w) \geq 18\frac{k_1-1}{k_1}$ and hence $i + j'' \geq 4$ if $k_1 \geq 4$. Hence Q_4 is not terminal if $k_1 \geq 4$.

Case 5 w is of stage 3 and $k_2 = 1$.
Now $w = (2, 2, 1, 1)$ with $\sigma = 4, \sigma_y = 2$ and

$$\tilde{F} = x^2 + y^3u^2 + y\sum a_{ij}z^iu^{j'} + \sum b_{ij}z^iu^{j''}.$$

Note that $\dagger_g$ shows that $i + j' + \sigma_y(w) \geq 8\frac{k_1-1}{k_1}$ and hence $i + j' \geq 3$ if $k_1 \geq 3$. Similarly, $\dagger_h$ shows that $i + j'' + \sigma(w) \geq 12\frac{k_1-1}{k_1}$ and hence $i + j'' \geq 4$ if $k_1 \geq 3$. Hence Q_4 is not terminal if $k_1 \geq 3$.

Case 6 w is of stage 4 and $k_2 = 1$.
Now $w = (3, 2, 1, 1)$ with $\sigma = 5, \sigma_y = 3$ and

$$\tilde{F} = x^2u + y^3u + y\sum a_{ij}z^iu^{j'} + \sum b_{ij}z^iu^{j''}.$$

Note that $\dagger_g$ shows that $i + j' + \sigma_y(w) \geq 8\frac{k_1-1}{k_1}$ and hence $i + j' \geq 2$ if $k_1 \geq 3$. Similarly, $\dagger_h$ shows that $i + j'' + \sigma(w) \geq 12\frac{k_1-1}{k_1}$ and hence $i + j'' \geq 3$ if $k_1 \geq 3$. Hence Q_4 is not terminal if $k_1 \geq 3$. □

We will need the following useful criterion, due to Hayakawa (cf. [7]), to determine whether a weighted blow-up is a divisorial contraction or not. For the reader's convenience, we reproduce the proof.

Theorem 4.2 *Given $P \in X \cong o \in (F = 0) \subset \mathbb{C}^4$ the germ of a terminal singularity of cE type, where F is a normal form, let $f = wBl_v : Y \to X$ be the weighted blow-up with weight $v = (a, b, k, 1)$ and exceptional divisor E. Suppose that*

- *E is irreducible;*
- *$a + b + k - wt_v(F) = 1$;*
- *$Y \cap U_4$ is terminal.*

Then $Y \to X$ is a divisorial contraction.

Proof Suppose that E is irreducible, then $K_Y = f^*K_X + a(E,X)E$ with $a(E,X) = a + b + k - wt_v(F) - 1$. Let $D = (u = 0) \subset \mathrm{Div}(X)$ and D_Y be its proper transform in Y. One has $f^*D = D_Y + E$. Hence

$$f^*(K_X + D) = K_Y + D_Y$$

and $D_Y \sim_X -K_Y$.

Let $g : Z \to Y$ be a resolution of Y. For any exceptional divisor F in Z such that $g(Z) \subset D_Y$, one has $g^*D_Y = D_Z + mF + \ldots$ for some $m > 0$. It follows that $a(F,Y) = m > 0$. Therefore, it remains to consider exceptional divisors whose center in Y is not contained in D_Y. Hence Y is terminal if $Y - D_Y = Y \cap U_4$ is terminal. Once Y is terminal, it is then easy to see that $Y \to X$ is a divisorial contraction. □

Lemma 4.3 *Let $Y \to X \ni P$ be a divisorial contraction with discrepancy 1 to a terminal Gorenstein singularity $P \in X$. Let E be its exceptional divisor. Then for any divisor F with discrepancy 1 over P, its center in Y is a singular point of index > 1 and contained in E.*

Therefore, to search for exceptional divisors over P with discrepancy 1, it suffices to search for exceptional divisors over singular points of index > 1 in E. We denote by $\mathrm{Sing}(Y)_{>1}$ the set of singular points of index > 1 on Y (which is contained in E).

4.1 Weights of stage 0

Proposition 4.4 *Let $(F = 0)$ be a cE singularity of level $\geq k$. We consider a weighted blow-up with weight $w = (3k-3, 2k-2, k, 1)$ of level k of stage 0. Then the exceptional divisor is irreducible.*

Proof This is clear since the exceptional divisor is defined by $(F^w_\sigma = 0)$ and F^w_σ contains x^2, y^3. □

Now suppose that F is of level k. Fix a weight $w = (3k-3, 2k-2, k, 1)$. We may write

$$F = x^2 + y^3 + yg_{\sigma_y} + yg_{\sigma_y+1} + yg_{>\sigma_y+1} + h_\sigma + h_{\sigma+1} + h_{>\sigma+1}.$$

Lemma 4.5 *Suppose that F is of level k. Consider $Y \to X$ the weighted blow-up with weight $w = (3k-3, 2k-2, k, 1)$. Then $\mathrm{Sing}(Y) \cap U_4$ is isolated unless:*

$$\exists\, s(z,u) \text{ s.t.} \begin{cases} g_{\sigma_y} = -3s(z,u)^2, \\ h_\sigma = 2s(z,u)^3, \\ h_{\sigma+1} = -s(z,u)g_{\sigma_y+1}. \end{cases} \qquad ♮$$

Proof It is clear that $\mathrm{Sing}(Y) \cap U_1 = \emptyset$. Hence we have $\mathrm{Sing}(Y) \cap U_4 \subset (x = 0)$. Moreover, $\mathrm{Sing}(Y) \subset E$. Hence we have $\mathrm{Sing}(Y) \cap U_4 \subset (u = 0)$.

Therefore, let $\tilde{F}$ be the defining equation of $Y \cap U_4$. We have

$$\begin{aligned} \mathrm{Sing}(Y) \cap U_4 &\subset (x = u = 0) \cap (\tilde{F} = \tilde{F}_y = \tilde{F}_u = 0) \\ &\subset (x = u = 0) \cap \Sigma, \end{aligned}$$

where Σ is defined as

$$\begin{cases} y^3 + y g_{\sigma_y} + h_\sigma = 0, \\ 3y^2 + g_{\sigma_y} = 0, \\ y g_{\sigma_y+1} + h_{\sigma+1} = 0. \end{cases}$$

If g_{σ_y} is not a perfect square, then $3y^2 + g_{\sigma_y}$ is irreducible and hence Σ is finite. If g_{σ_y} is a perfect square, then we write it as $g_{\sigma_y} = -3s^2$. One sees that Σ is finite unless $y - s$ or $y + s$ divides the above three polynomials. The statement now follows. □

Once we know that $\mathrm{Sing}(Y) \cap U_4$ is isolated, then it is easy to check whether the singularities in $Y \cap U_4$ are terminal or not. Notice that $\tilde{F}$ is of the form $x^2 + y^3 + y\tilde{g} + \tilde{h}$. It is straightforward to see that $\tilde{F}$ is at worst of cE type at $Q_4 \in U_4$ if and only if F is not of level $k + 1$.

Next we study $(0, 0, \gamma, 0) \in U_4$. We rewrite

$$F = x^2 + y^3 + y \sum a'_{ij}(z - \gamma u^k)^i u^j + \sum b'_{ij}(z - \gamma u^k)^i u^j$$

and correspondingly

$$\tilde{F} = x^2 + y^3 + y \sum a'_{ij}(z - \gamma)^i u^{j'} + \sum b'_{ij}(z - \gamma)^i u^{j''},$$

with $j' = ki + j - \sigma_y$ and $j'' = ki + j - \sigma$.

If $\tilde{F}$ has a non-terminal singularity at $(0, 0, \gamma, 0) \in U_4$, then $i + j' \geq 4$ and $i + j'' \geq 6$. It follows that F has level $\geq k + 1$, a contradiction. Notice also that singularity the at $(\alpha, \beta, \gamma, \delta) \in U_4$ with $(\alpha, \beta) \neq (0, 0)$ is at worst of cA type. We thus conclude the following:

Theorem 4.6 *Given a canonical form F, suppose that F is of level k stage* 0*. Then the weighted blow-up with weight $w = (3k - 3, 2k - 2, k, 1)$ is a divisorial contraction if ♮ does not hold.*

Suppose that ♮ holds. Then by considering a change of coordinate $\bar{y} = y - s(z, u)$, one sees that F is equivalent to

$$G := x^2 + y^3 + s(z, u)y^2 + y g_{>\sigma_y+1} + h_{>\sigma+1}.$$

Note that $wt_w(s(z,u)) = 2k-2$. Hence consider $w_1 = (3k-2, 2k-1, k, 1)$ the weight of level k of stage 1. One sees that $wt_{w_1}(G) \geq 6k-4 = \sigma(w_1)$, that is, G is of level k of stage ≥ 1. Therefore, we may say that G is a *stage lifting* of F.

5 Divisorial contractions to cE points with discrepancy 1

We classify divisorial contractions to cE points with discrepancy 1 of higher level in this section. Moreover, we determine the number of exceptional divisors with discrepancy 1 over cE_6 points.

5.1 cE Singularity of level 6

Proposition 5.1 *Suppose that* $(P \in X)$ *is of level* 6*, then there are eight different valuations with discrepancy* 1 *corresponding to weighted blow-ups of different weights. Among them, there is only one divisorial contraction with discrepancy* 1 *over* $(P \in X)$*, which is given by* $wBl_{w_{30}}$.

Proof Let F be a canonical form of $(P \in X)$. If F is of level 6, then $wt_{w_{30}}(F) \geq 30$. This only happens in cE_8 singularity and $h_5 = z^5$.

1. Let $f = wBl_{w_{30}} : Y \to X$, where $w_{30} = (15, 10, 6, 1)$. It follows from Theorem 4.6 that Y is terminal.
2. $\mathrm{Sing}(Y)_{>1} = \{R_{23}, R_{13}, R_{12}\}$, where R_{ij} is a quotient singularity in $(x_i = x_j = 0) \subset E$ of index $gcd(a_i, a_j)$.
3. Take an economic resolution $Z \to Y$. We have exceptional divisors F_1, G_1, G_2, $H_1, \ldots, H_4$ with discrepancy 1, where F_i (resp. G_i, H_i) denotes the exceptional divisor over R_{23} (resp. R_{13}, R_{12}). Computation shows that all these divisors are of discrepancy 1 over X. Together with E, there are eight divisors with discrepancy 1.

Take F_1 for example, obtained by a Kawamata blow-up of weights $\frac{1}{2}(1,1,1)$ over R_{23}. In fact, this can be realized as a weighted blow-up with vector w_{15}. More precisely, let $\mathcal{X}_1 \to \mathcal{X}_0$ be the weighted blow-up with vector w_{30} and let $g : \mathcal{X}_2 \to \mathcal{X}_1$ be the weighted blow-up with vector w_{15}. Since $w_{15} = \frac{1}{2}w_{30} + \frac{1}{2}e_1 + \frac{1}{2}e_4$, the map g is obtained by subdivision of $\sigma_2 = \langle e_1, w_{30}, e_3, e_4\rangle$ and $\sigma_3 = \langle e_1, e_2, w_{30}, e_4\rangle$ along w_{15}. One sees that g is the weighted blow-up along the curve $(y = z = 0) \subset \mathcal{E}$, which is singular of type $\frac{1}{2}(1,1,1) \times \mathbb{P}^1$. The proper transform of Y in $\mathcal{X}_2$, denoted Z, and its induced map $g : Z \to Y$ then gives the Kawamata blow-up.

One can consider the exceptional divisors $G_1, \ldots, H_4$ similarly. The computation can be summarized as follows:

Div	*center*$_Y$	$a(Z/Y)$	wt_Y	$a(Z/X)$	wt_X
F_1	R_{23}	1/2	$\frac{1}{2}(1,1,1)$	1	$(8,5,3,1) = w_{15}$
G_1	R_{13}	1/3	$\frac{1}{3}(1,1,2)$	1	$(10,7,4,1) = w_{20}$
G_2	R_{13}	2/3	$\frac{1}{3}(2,2,1)$	1	$(5,4,2,1) = w_{10}$
H_1	R_{12}	1/5	$\frac{1}{5}(1,1,4)$	1	$(12,8,5,1) = w_{24}$
H_2	R_{12}	2/5	$\frac{1}{5}(2,2,3)$	1	$(9,6,4,1) = w_{18}$
H_3	R_{12}	3/5	$\frac{1}{5}(3,3,2)$	1	$(6,4,3,1) = w_{12}$
H_4	R_{12}	4/5	$\frac{1}{5}(4,4,1)$	1	$(3,2,2,1) = w_6$

where *center*$_Y$ denotes the center in Y, $a(Z/Y)$ (resp. $a(Z/X)$) denotes the discrepancy over Y (resp. over X), and wt_Y (resp. wt_X) denotes the weights over Y (resp. over X).

4. $wBl_v(X)$ is not terminal if $v = w_{24}, w_{18}, w_{12}, w_6, w_{20}, w_{10}, w_{15}$. By Lemma 4.1, it is straightforward to see that any of these blow-ups is a divisorial blow-up, i.e., its exceptional divisor is irreducible.

5. $wBl_{w_{30}}$ is the unique divisorial contraction with discrepancy 1. Suppose that $f' : Y' \to X$ is another divisorial contraction to P with discrepancy 1. Let E' be its exceptional divisor. Then $E' \in \{F_1, G_1, G_2, H_1, \ldots, H_4\}$ as valuations. If $E' = F_1$, then we consider $f'' = wBl_{w_{24}} : Y'' \to X$ for the corresponding vector $w_{24} = (12, 8, 5, 1)$. One sees that E'' is irreducible and hence by [10, Lemma 4.3], $f'' \cong f'$ and $Y'' \cong Y'$. However, this is absurd since Y'' is not terminal.

As we have seen, all exceptional divisors with discrepancy 1 (other than E) correspond to a weighted blow-up with the weights given above and none of these weighted blow-ups is a divisorial contraction. We thus conclude that there is no divisorial contraction other than $wBl_{w_{30}}$. □

5.2 cE Singularity of level 5

Proposition 5.2 *Suppose that $P \in X$ is of level 5, then there are seven different valuations with discrepancy 1 corresponding to weighted blow-ups of different weights. Among them, there is only one divisorial contraction over $P \in X$ which is given by $wBl_{w_{24}}$.*

Proof Let F be the canonical form. If F is of level 5, then it can only happen for a cE_8 singularity and $h_5 = z^5$.

1. Let $f = wBl_{w_{24}} : Y \to X$, where $w_{24} = (12, 8, 5, 1)$. Since $z^5 \in h$, one sees that ♮ does not hold. It follows from Theorem 4.6 that $Y \to X$ is a divisorial contraction.

2. $\mathrm{Sing}(Y)_{>1} = \{R_{12}, Q_3\}$, where R_{12} is a point in $(x = y = 0) \subset E$ of index 4 and Q_3 is quotient of index 5.
3. Take an economic resolution $Z \to Y$. We have exceptional divisors $\{F_i\}_{i\le 3}$ and $\{G_j\}_{i\le 4}$ over R_{12} and Q_3 respectively. Computation yields the following:

Div	*center$_Y$*	$a(Z/Y)$	wt_Y	$a(Z/X)$	wt_X
F_1	R_{12}	1/4	$\frac{1}{4}(3,1,1)$	1	$(9,6,4,1) = w_{18}$
F_2	R_{12}	2/4	$\frac{1}{4}(2,2,2)$	1	$(6,4,3,1) = w_{12}$
F_3	R_{12}	3/4	$\frac{1}{4}(1,3,3)$	1	$(3,2,2,1) = w_6$
G_1	Q_3	1/5	$\frac{1}{5}(2,3,4,1)$	1	$(10,7,4,1) = w_{20}$
G_2	Q_3	2/5	$\frac{1}{5}(4,1,3,2)$	1	$(8,5,3,1) = w_{15}$
G_3	Q_3	3/5	$\frac{1}{5}(1,4,2,3)$	1	$(5,4,2,1) = w_{10}$
G_4	Q_3	4/5	$\frac{1}{5}(3,2,6,4)$	2	

4. $wBl_v(X)$ is a divisorial blow-up but not terminal if v is one of w_{18}, w_{12}, w_6, w_{20}, w_{15}, w_{10} by Lemma 4.1.

We conclude that $wBl_{w_{24}}$ is the unique divisorial contraction with discrepancy 1. □

5.3 cE Singularity of level 4

We need to consider different stages.

Stage 1 Weights $w_{20} = (10, 7, 4, 1)$.

Proposition 5.3 *Suppose that F is of level 4 of stage 1, then there are six different valuations with discrepancy 1 corresponding to weighted blow-ups of different weights. Among them, there is only one divisorial contraction over $P \in X$ which is given by $wBl_{w_{20}}$.*

Proof This only happens in cE_8 singularity.
1. Let $f = wBl_{w_{20}} : Y \to X$, where $w_{20} = (10, 7, 4, 1)$. Since $z^5 \in h$, one sees that the exceptional divisor of f is irreducible. Since F is not of level 5, it is straightforward to check that $Y \to X$ is a divisorial contraction. $\mathrm{Sing}(Y)_{>1} = \{R_{13}, Q_2\}$, where R_{13} is a point of index 2 and Q_2 is a quotient of index 7.
2. Take an economic resolution $Z \to Y$. We have exceptional divisors F_1 and $\{G_j\}_{i\le 6}$ over R_{13} and Q_2 respectively. Computation shows that F_1, G_1, G_2, G_3, G_5 are with discrepancy 1 and G_4, G_6 are with discrepancy 2. Hence there are six divisors with discrepancy 1.

3. Similarly, the exceptional divisors F_1, G_1, G_2, G_3, G_5 correspond to vectors $w_{10}, w_{18}, w_{15}, w_{12}, w_6$ respectively. Clearly, a weighted blow-up with any of these weights has irreducible exceptional divisor. Moreover, $wBl_v(X)$ is not terminal if v is one of $w_{10}, w_{15}, w_{12}, w_6$ by Lemma 4.1.
4. We consider $wBl_{w_{18}}$. Since $\natural$ holds, one sees that $wBl_{w_{18}}(X)$ is not terminal for it contains non-isolated singularities.

We conclude that $wBl_{w_{20}}$ is the unique divisorial contraction with discrepancy 1 and there are six divisors with discrepancy 1, realized by weighted blow-ups. □

Stage 0 $w_{18} = (9, 6, 4, 1)$.

Case 1 F is of type cE_7.
Since $z^3 \in g$, $\natural$ does not hold. By Theorem 4.6, the weighted blow-up with weight w_{18} is a divisorial contraction. There are singularities R_{12}, R_{23}, Q_3 of quotient type of index $3, 2, 4$ respectively. An economic resolution produces exceptional divisors $F_1, F_2, G_1, H_1, H_2, H_3$. All of these are of discrepancy 1 over X. They correspond to weights w_{12}, w_6, w_9, w_{14}, w_{10}, w_6' respectively. Clearly, $w_{12}, w_6, w_{14}, w_{10}$ give rise to divisorial blow-ups which are not terminal by Lemma 4.1.

It is clear that $wBl_{w_6'}$ has an irreducible exceptional divisor. Note that $Y \cap U_4$ is given by

$$\tilde{F} = x^2 + y^3u^3 + y \sum a_{ij} z^i u^{j'} + \sum b_{ij} z^i u^{j''}.$$

One has

$$\begin{aligned} i + j'' &= 2i + j - 6 \geq \tfrac{1}{2}(4i + j) - 6 \geq 3, \quad && \text{if } (i, j) \in I_h; \\ i + j' &= 2i + j - 3 \geq \tfrac{1}{2}(4i + j) - 3 \geq 3, \quad && \text{if } (i, j) \in I_g. \end{aligned}$$

In fact $4i + 2j > 18$ if $4i + j = 18$. Hence $i + j'' > 3$ if $(i, j) \in I_h$ and therefore Q_4 is not terminal.

We now consider $w_9 = (5, 3, 2, 1)$. The weighted blow-up with weight w_9 has irreducible exceptional divisor if there is some $z^i u^j \in h$ with $2i + j = 9$. However, this can not happen since $4i + j \geq 18$. Hence the exceptional divisor defined by $y^3 + yz^3$ is reducible. We need to consider an isomorphism $P' \in X' \subset \mathbb{C}^5$ such that

$$P' \in X' = \left(\begin{array}{l} x^2 + yt + yg_{>\sigma_y(w_9)} + h_{\geq\sigma(w_9)} = 0, \\ t = y^2 + z^3 \end{array} \right) \subset \mathbb{C}^5.$$

We take $wBl_v : Y' \to X$ with $v = (5, 3, 2, 1, 7)$. One sees that wBl_v has irreducible exceptional divisor. Note that Y' is non-terminal, since $Y' \cap U_4$ is not a hypersurface singularity.

Hence there are seven divisors with discrepancy 1 realized by weighted blow-ups and there is exactly one divisorial contraction with discrepancy 1 among these weighted blow-ups.

Case 2 F is of type cE_8 and ♮ holds.
Then we need to consider its stage lifting G. Now G is of level 4 of stage 1. The same argument in stage 1 holds for G. We conclude that there is a unique divisorial contraction and six divisors with discrepancy 1. All of them are realized by weighted blow-ups.

Case 3 F is of type cE_8 and ♮ does not hold.
Then one sees that the weighted blow-up of weight w_{18} is a divisorial contraction. There are singularities R_{12}, Q_3, where R_{12} is of quotient type of index 3 and Q_3 is of type $cAx/4$ with axial weight 2. By resolving R_{12} as in Case 1 of cE_7, there are two exceptional divisors with discrepancy 1 corresponding to w_{12} and w_6 respectively.

It remains to consider Q_3, whose nature varies depending on the appearance of z^4u^2.

If $z^4u^2 \notin F$, then $Y \cap U_3$ is given by

$$(\tilde{F} = x^2 + y^3 + z^2 + \ldots = 0) \subset \mathbb{C}^4/\frac{1}{4}(1, 2, 3, 1).$$

We thus have a resolution over Q_3 given by the following vectors:

$a(Z/Y)$	wt_Y	$a(Z/X)$	wt_X
1/4	$\frac{1}{4}(5, 2, 3, 1)$	1	$(8, 5, 3, 1) = w_{15}$
1/2	$\frac{1}{2}(3, 2, 3, 1)$	2	
1/2	$\frac{1}{2}(1, 2, 1, 1)$	1	$(5, 4, 2, 1) = w_{10}$
3/4	$\frac{1}{4}(3, 2, 5, 3)$	2	
1	$(1, 1, 1, 1)$	2	

Clearly, w_{15}, w_{10} give rise to divisorial blow-ups which are not terminal by Lemma 4.1.

If $z^4u^2 \in F$, then we change coordinates by $\bar{x} = x + z^2u$ to get

$$G = \bar{x}^2 - 2z^2u\bar{x} + y^3 + \ldots$$

Then $Y \cap U_3$ is given by

$$(\tilde{G} = \bar{x}^2 - 2u\bar{x} + y^3 + \ldots = 0) \subset \mathbb{C}^4/\frac{1}{4}(1, 2, 3, 1).$$

We thus have resolution over Q_3 given by the following vectors:

$a(Z/Y)$	wt_Y	$a(Z/X)$	wt_X
1/4	$\frac{1}{4}(5,2,3,1)$	1	$(8,5,3,1)=w_{15}$
1/4	$\frac{1}{4}(1,2,3,1)$	1	$(7,5,3,1)=w_{14}$
1/2	$\frac{1}{2}(1,2,1,1)$	1	$(5,4,2,1)=w_{10}$
3/4	$\frac{1}{4}(3,2,5,3)$	2	
1	$(1,1,1,1)$	2	

Clearly, w_{15}, w_{10} give rise to divisorial blow-ups which are not terminal by Lemma 4.1. Indeed,

$$G = \bar{x}^2 - 2z^2u\bar{x} + y^3 + yg_{>\sigma_y(w_{14})} + h_{>\sigma(w_{14})}.$$

We need to consider an isomorphism $P' \in X' \subset \mathbb{C}^5$:

$$P' \in X' = \left(\begin{array}{l} \bar{x}t + y^3 + yg_{>\sigma_y(w_{14})} + h_{>\sigma(w_{14})} = 0, \\ t = \bar{x} - 2z^2u \end{array} \right) \subset \mathbb{C}^5.$$

The weighted blow-up with weight $(7,5,3,1,8)$, which is clearly a divisorial blow-up but not a divisorial contraction, realizes the exceptional divisor corresponding to w_{14}.

We thus conclude that there are six (resp. five) exceptional divisors with discrepancy 1 if $z^4u^2 \in F$ (resp. $z^4u^2 \notin F$). In any event, there is exactly one divisorial contraction with discrepancy 1.

5.4 cE_6 Singularities

We consider cE_6 singularities in this subsection. Instead of providing a detailed classification, we are interested in determining the number of divisorial contractions and exceptional divisors with discrepancy 1 over a given singularity.

Level 3 It is clear that $w_{12} = (6,4,3,1)$ is the only admissible weight of level 3. Since $h_4 = z^4$, $\natural$ does not hold and therefore the weighted blow-up $wBl_{w_{12}}\colon Y \to X$ with weight w_{12} is a divisorial contraction. There are two singularities of type $\frac{1}{3}(1,2,1)$ and one singularity of type $\frac{1}{2}(1,1,1)$. Let $Z \to Y$ be the economic resolution over these singular points. One sees that there are six exceptional divisors with discrepancy 1, say $F_{11}, F_{12}, F_{21}, F_{22}, G_1$, and E.

Then G_1 corresponds to the divisorial weighted blow-up of weight $(3,2,2,1)$, which is not a divisorial contraction by Lemma 4.1. Next, we consider the coordinate change $\bar{x} = x \pm z^2$ to get

$$G = \bar{x}^2 \mp 2\bar{x}z^2 + y^3 + yg_{\geq 3} + h_{\geq 5}.$$

Take weighted blow-ups with weight $(5, 3, 2, 1)$ so that the exceptional set is irreducible. This realizes two more divisors.

Since $wt_{(3,1)}g_3 \geq 8$ (resp. $wt_{(3,1)}h_5 \geq 12$), it is clear that $g_3 = z^2 g'$ (resp. $h_5 = z^2 h'$) for some g' (resp. h'). The singularity is also isomorphic to

$$\left(\begin{array}{l} \bar{x}^2 + y^3 + z^2 t + y g_{\geq 4} + h_{\geq 6} = 0 \\ t = \mp 2\bar{x} + y g' + h' \end{array} \right).$$

Take weighted blow-ups with weight $(3, 2, 1, 1, 4)$ so that the exceptional set is irreducible. Then one realizes two more divisors with discrepancy 1. These five additional weighted blow-ups do not produce divisorial contractions.

Level 2 and Stage 1 Now we consider $w_8 = (4, 3, 2, 1)$. Since F is not of level 3, one sees that $wBl_{w_8} : Y \to X$ is a divisorial contraction if its exceptional divisor is irreducible. There are two singularities of type $\frac{1}{2}(1, 1, 1)$ and one singularity of type $\frac{1}{3}(1, 2, 1)$. Let $Z \to Y$ be the economic resolution over these singular points. One sees that there are four exceptional divisors with discrepancy 1, say F_1, F_2, G_1, and E. G_1 corresponds to the weighted blow-up of weight $(3, 2, 2, 1)$, which is clearly not a divisorial contraction by Lemma 4.1. To realize the other two divisors, a similar argument as in Level 3 shows that weighted blow-ups with weight $(3, 2, 1, 1, 4)$ realize the other two divisors.

Level 2 and Stage 0 Suppose that $\natural$ holds for F, then we can consider its stage lifting G. The situation is then exactly the same as in the above level 2 and stage 1 case. That is, there are four exceptional divisors with discrepancy 1.

Suppose that $\natural$ doesn't hold. Then wBl_{w_6} is a divisorial contraction, where Q_3 is the only singularity of index > 1.

If h_4 is not a square, then Q_3 is of type $cA/2$ with axial weight 3 and τ-$wt = 1$ (cf. [7, Section 8]), hence there is only one exceptional divisor F_1 with discrepancy $\frac{1}{2}$ over Y by weighted blow-up with weight $\frac{1}{2}(1, 2, 1, 1)$. This is a divisor with discrepancy 1 over X corresponding to a weighted blow-up with weight $(2, 2, 1, 1)$.

If $h_4 = -q(z, u)^2$ is a square, then we change coordinates by $\bar{x} = x - q(z, u)$. Now F is equivalent to

$$G = \bar{x}^2 \mp 2\bar{x}q(z, u) + y^3 + y g_{\geq 3} + h_{\geq 5}.$$

By considering $wBl_{w_6} : Y \to X$, one sees that Q_3 is of type $cA/2$ with axial weight 3 and τ-$wt = 2$ or 3. Computation shows that there are two or three exceptional divisors with discrepancy $\frac{1}{2}$ over Q_3, however there is only one divisor with discrepancy 1 over $P \in X$. This divisor corresponds to divisorial weighted blow-up of G with weights $(3, 2, 1, 1)$ and $(4, 2, 1, 1)$ respectively.

In any case, there are two divisors with discrepancy 1.

Level 1 If $h_4 = -q(z,u)^2$ is a square, then we consider another stage lifting by $\bar{x} = x \pm q(z,u)$,

$$G = \bar{x}^2 \mp 2\bar{x}q(z,u) + y^3 + yg_{\geq 3} + h_{\geq 5}.$$

Now G has level 1 of stage 4, and hence we consider the admissible weighted blow-up $wBl_{w_5}\colon Y \to X$ with weight $w_5 = (3,2,1,1)$. Since F is of level 1, there exists yu^3 or u^5 in F. One can thus check that Y is terminal by considering $Y \cap U_4$. There are singularities Q_1, Q_2 of type $\frac{1}{3}(2,1,1)$ and $\frac{1}{2}(1,1,1)$ respectively. Let $Z \to Y$ be the economic resolution. We have exceptional divisors F_1, F_2, G_1. Computation shows that $a(F_1, X) = 1$ and $a(F_2, X) = a(G_1, X) = 2$.

Indeed, F_1 corresponds to the weight $(2,2,1,1)$. To realize this valuation as an exceptional divisor of a weighted blow-up, we proceed as in the level 3 case. The singularity is isomorphic to

$$\left(\begin{array}{l} \bar{x}t + y^3 + yg_{\geq 3} + h_{\geq 5} \\ t = \bar{x} \mp 2q(z,u) \end{array} \right).$$

Take weighted blow-ups with weight $(2,2,1,1,3)$. We thus conclude that there are exactly two divisors with discrepancy 1 in this case.

If h_4 is not a square, then we consider $w_4 = (2,2,1,1)$. One sees that $wBl_{w_4}\colon Y \to X$ is a divisorial contraction. The higher-index singular point is Q_2 of type $cAx/2$, given by

$$x^2 + y^2 + h_4 + \text{ other terms} = 0 \subset \mathbb{C}^4/\frac{1}{2}(2,1,1,1).$$

By [5, Theorem 8.4], the only exceptional divisor of discrepancy $\frac{1}{2}$ is given by a weighted blow-up of weight $\frac{1}{2}(2,3,1,1)$. Hence its discrepancy over X is 2. We thus conclude that there is a unique divisorial contraction and unique exceptional divisor of discrepancy 1 in this case.

We thus have the following observation:

Corollary 5.4 *Let F be a canonical form of cE_6. Suppose that F has exactly one exceptional divisor with discrepancy* 1. *Then F is of level* 1 *and h_4 is not a square.*

6 Divisorial contractions with higher discrepancies

By the classification of Kawakita [13, Theorem 1.2.ii], divisorial contractions to a cE point with discrepancy > 1 are not of ordinary type and a brief description was given in [13, Table 3]. The purpose of this section is to give some more examples which were not previously known and also to provide some characterization of cE singularities admitting divisorial contractions with higher discrepancy.

The only previously known example with higher discrepancy is the following:

Example 6.1 ([13, Example 8.9]) Let $P \in X$ be the germ

$$o \in (x^2 + y^3 + yz^3 + u^7 = 0) \subset \mathbb{C}^4.$$

P is of type cE_7. Take the weighted blow-up with weight $(7, 5, 3, 2)$. Then it is a divisorial contraction with discrepancy 2.

Let $P \in X$ be the germ

$$o \in (x^2 + y^3 + z^5 + u^7 = 0) \subset \mathbb{C}^4.$$

P is of type cE_8. Take the weighted blow-up with weight $(7, 5, 3, 2)$. Then it is a divisorial contraction with discrepancy 2.

We provide another example:

Example 6.2 Let $P \in X$ be the germ

$$o \in \left(\begin{array}{l} x^2 + yt + u^5 = 0 \\ t = y^2 + z^3 \end{array} \right) \subset \mathbb{C}^5.$$

P is of type cE_7. Take the weighted blow-up with weight $(5, 3, 2, 2, 7)$. Then it is a divisorial contraction with discrepancy 2.

In fact, by the studies in [3, Case IIc], it is known that there is only one exceptional divisor with discrepancy 1 over $P \in X$. The unique divisorial contraction with discrepancy 1 is given by a weighted blow-up with weight $(3, 2, 1, 1, 3)$.

Example 6.3 If $P \in X$ is a germ of cE_6 such that it admits a divisorial contraction $Y \to X \ni P$ of discrepancy 3, then a singularity of higher index in Y is a $cAx/4$ point of axial weight 2. By [3, Case IId], there is exactly one exceptional divisor with discrepancy 1 over $P \in X$. By Corollary 5.4, one see that its canonical form must be of level 1 and h_4 is not a square. However, we do not know if such an example exists or not.

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