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Special prime Fano fourfolds of degree 10 and index 2

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Abstract

We analyze (complex) prime Fano fourfolds of degree 10 and index 2. Mukai gave in [M1] a complete geometric description; in particular, most of them are contained in a Grassmannian $G(2,5)$. As in the case of cubic fourfolds, they are unirational and some are rational, as already remarked by Roth in 1949.

We show that their middle cohomology is of K3 type and that their period map is dominant, with smooth 4-dimensional fibers, onto a 20-dimensional bounded symmetric period domain of type IV. Following Hassett, we say that such a fourfold is *special* if it contains a surface whose cohomology class does not come from the Grassmannian $G(2,5)$. Special fourfolds correspond to a countable union of hypersurfaces (the Noether–Lefschetz locus) in the period domain, labelled by a positive integer d. We describe special fourfolds for some low values of d. We also characterize those integers d for which special fourfolds do exist.

Dedicated to Robert Lazarsfeld on the occasion of his sixtieth birthday

1 Introduction

One of the most vexing classical questions in complex algebraic geometry is whether there exist irrational smooth cubic hypersurfaces in $\mathbf{P}^5$. They are all

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unirational, and rational examples are easy to construct (such as Pfaffian cubic fourfolds) but no smooth cubic fourfold has yet been proven to be irrational. The general feeling seems to be that the question should have an affirmative answer but, despite numerous attempts, it is still open.

In a couple of very interesting articles on cubic fourfolds ([H1], [H2]), Hassett adopted a Hodge-theoretic approach and, using the period map (proven to be injective by Voisin in [V]) and the geometry of the period domain, a 20-dimensional bounded symmetric domain of type IV, he related geometric properties of a cubic fourfold to arithmetical properties of its period point.

We do not solve the rationality question in this paper, but investigate instead similar questions for another family of Fano fourfolds (see Section 2 for their definition). Again, they are all unirational (see Section 3) and rational examples were found by Roth (see [R]; also [P] and Section 7), but no irrational examples are known.

We prove in Section 4 that the moduli stack $\mathscr{X}_{10}$ associated with these fourfolds is smooth of dimension 24 (Proposition 4.1) and that the period map is smooth and dominant onto, again, a 20-dimensional bounded symmetric domain of type IV (Theorem 4.3). We identify the underlying lattice in Section 5. Then, following [H1], we define in Section 6.1 hypersurfaces in the period domain which parametrize "special" fourfolds X, whose period point satisfies a nontrivial arithmetical property depending on a positive integer d, the *discriminant.* As in [H1], we characterize in Proposition 6.6 those integers d for which the nonspecial cohomology of a special X of discriminant d is essentially the primitive cohomology of a K3 surface; we say that this K3 surface is *associated* with X. Similarly, we characterize in Proposition 6.7 those d for which the nonspecial cohomology of a special X of discriminant d is the nonspecial cohomology of a cubic fourfold in the sense of [H1].

In Section 7, we give geometric constructions of special fourfolds X for $d \in \{8, 10, 12\}$; in particular, we discuss some rational examples (already present in [R] and [P]). When $d = 10$, the associated K3 surface (in the sense of Proposition 6.6) does appear in the construction of X; when $d = 12$, so does the associated cubic fourfold (in the sense of Proposition 6.7) and it is birationally isomorphic to X.

In Section 8, we characterize the positive integers d for which there exist (smooth) special fourfolds of discriminant d. As in [H1], our construction relies on the surjectivity of the period map for K3 surfaces. Finally, in Section 9, we ask a question about the image of the period map.

So in some sense, the picture is very similar to what we have for cubic fourfolds, with one big difference: the Torelli theorem does not hold. In a forthcoming article, building on the link between our fourfolds and EPW sextics discovered in [IM], we will analyze the (4-dimensional) fibers of the period map.

2 Prime Fano fourfolds of degree 10 and index 2

Let X be a (smooth) prime Fano fourfold of degree 10 (i.e., of "genus" 6) and index 2; this means that $\mathrm{Pic}(X)$ is generated by the class of an ample divisor H such that $H^4 = 10$ and $-K_X \underset{\mathrm{lin}}{\equiv} 2H$. Then H is very ample and embeds X in $\mathbf{P}^8$ as follows ([M2]; [IP], Theorem 5.2.3).

Let V_5 be a 5-dimensional vector space (our running notation is V_k for any k-dimensional vector space). Let $G(2, V_5) \subset \mathbf{P}(\wedge^2 V_5)$ be the Grassmannian in its Plücker embedding and let $CG \subset \mathbf{P}(\mathbf{C} \oplus \wedge^2 V_5) \simeq \mathbf{P}^{10}$ be the cone, with vertex $v = \mathbf{P}(\mathbf{C})$, over $G(2, V_5)$. Then

$$X = CG \cap \mathbf{P}^8 \cap Q,$$

where Q is a quadric. There are two cases:

- either $v \notin \mathbf{P}^8$, in which case X is isomorphic to the intersection of $G(2, V_5) \subset \mathbf{P}(\wedge^2 V_5)$ with a hyperplane (the projection of $\mathbf{P}^8$ to $\mathbf{P}(\wedge^2 V_5)$) and a quadric;
- or $v \in \mathbf{P}^8$, in which case $\mathbf{P}^8$ is a cone over a $\mathbf{P}^7 \subset \mathbf{P}(\wedge^2 V_5)$ and X is a double cover of $G(2, V_5) \cap \mathbf{P}^7$ branched along its intersection with a quadric.

The varieties obtained by the second construction will be called "of Gushel type" (after Gushel, who studied the 3-dimensional case in [G]). They are specializations of varieties obtained by the first construction.

Let $\mathscr{X}_{10}$ be the irreducible moduli stack for (smooth) prime Fano fourfolds of degree 10 and index 2, let $\mathscr{X}_{10}^G$ be the (irreducible closed) substack of those which are of Gushel type, and let $\mathscr{X}_{10}^0 := \mathscr{X}_{10} - \mathscr{X}_{10}^G$. We have

$$\dim(\mathscr{X}_{10}) = 24, \quad \dim(\mathscr{X}_{10}^G) = 22.$$

3 Unirationality

Let $G := G(2, V_5)$ and let $X := G \cap \mathbf{P}^8 \cap Q$ be a fourfold of type $\mathscr{X}_{10}^0$. We prove in this section that X is unirational.

The hyperplane $\mathbf{P}^8$ is defined by a nonzero skew-symmetric form ω on V_5, and the singular locus of $G^\omega := G \cap \mathbf{P}^8$ is isomorphic to $G(2, \mathrm{Ker}(\omega))$. Since X is smooth, this singular locus must be finite, hence ω must be of maximal rank and G^ω is also smooth. The variety G^ω is the unique del Pezzo fivefold

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of degree 5 ([IP], Theorem 3.3.1); it parameterizes isotropic 2-planes for the form ω.

If $V_1^\omega \subset V_5$ is the kernel of ω, the 3-plane $\mathbf{P}_0^3 := \mathbf{P}(V_1^\omega \wedge V_5)$ of lines passing through $[V_1^\omega] \in \mathbf{P}(V_5)$ is contained in G^ω, hence X contains $\Sigma_0 := \mathbf{P}_0^3 \cap Q$, a "$\sigma$-quadric" surface,[1] possibly reducible. Any irreducible σ-quadric contained in X is Σ_0.

Proposition 3.1 *Any fourfold X of type $\mathscr{X}_{10}^0$ is unirational. More precisely, there is a rational double cover $\mathbf{P}^4 \dashrightarrow X$.*

Proof If $p \in \Sigma_0$, the associated $V_{2,p} \subset V_5$ contains V_1^ω, hence its ω-orthogonal complement is a hyperplane $V_{2,p}^\perp \subset V_5$ containing $V_{2,p}$. Then

$$Y := \bigcup_{p \in \Sigma_0} \mathbf{P}(V_{2,p}) \times \mathbf{P}(V_{2,p}^\perp / V_{2,p})$$

is the fiber product of the projectivizations of two vector bundles over Σ_0, hence is rational.

A general point of Y defines a flag $V_1^\omega \subset V_{2,p} \subset V_3 \subset V_{2,p}^\perp \subset V_5$, hence a line in $G(2, V_5)$ passing through p and contained in $\mathbf{P}^8$. This line meets $X - \Sigma_0$ at a unique point, and this defines a rational map $Y \dashrightarrow X$.

This map has degree 2: if x is general in X, lines in $G(2, V_5)$ through x meet $\mathbf{P}(V_1^\omega \wedge V_5)$ in points p such that $V_{2,p} = V_1^\omega \oplus V_1$, with $V_1 \subset V_{2,x}$, hence the intersection is $\mathbf{P}(V_1^\omega \wedge V_{2,x})$. This is a line, therefore it meets Σ_0 in two points. □

4 Cohomology and the local period map

This section contains more or less standard computations of various cohomology groups of fourfolds of type $\mathscr{X}_{10}$.

As in Section 3, we set $G := G(2, V_5)$ and let $G^\omega := G \cap \mathbf{P}^8$ be a smooth hyperplane section of G.

4.1 The Hodge diamond of X

The inclusion $G^\omega \subset G$ induces isomorphisms

$$H^k(G, \mathbf{Z}) \xrightarrow{\sim} H^k(G^\omega, \mathbf{Z}) \qquad \text{for all } k \in \{0, \dots, 5\}. \tag{1}$$

[1] This means that the lines in $\mathbf{P}(V_5)$ parameterized by Σ_0 all pass through a fixed point. Since X is smooth, it contains no 3-planes by the Lefschetz theorem, hence Σ_0 is indeed a surface.

The Hodge diamond for a fourfold $X := G^\omega \cap Q$ of type $\mathscr{X}_{10}^0$ was computed in [IM], Lemma 4.1; its upper half is as follows:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 1 & & 22 & & 1 & & 0 \end{array} \tag{2}$$

When X is of Gushel type, the Hodge diamond remains the same. In all cases, the rank-2 lattice $H^4(G, \mathbf{Z})$ embeds into $H^4(X, \mathbf{Z})$ and *we define the vanishing cohomology* $H^4(X, \mathbf{Z})_{\mathrm{van}}$ *as the orthogonal complement* (for the intersection form) *of the image of* $H^4(G, \mathbf{Z})$ *in* $H^4(X, \mathbf{Z})$. It is a lattice of rank 22.

4.2 The local deformation space

We compute the cohomology groups of the tangent sheaf T_X of a fourfold X of type $\mathscr{X}_{10}$.

Proposition 4.1 *For any fourfold X of type $\mathscr{X}_{10}$, we have*

$$H^p(X, T_X) = 0 \qquad \textit{for} \quad p \neq 1$$

and $h^1(X, T_X) = 24$. *In particular, the group of automorphisms of X is finite and the local deformation space* $\mathrm{Def}(X)$ *is smooth of dimension 24.*

Proof For $p \geq 2$, the conclusion follows from the Kodaira–Akizuki–Nakano theorem since $T_X \simeq \Omega_X^3(2)$. We assume that $X = G^\omega \cap Q$ is not of Gushel type (the proof in the case where X is of Gushel type is similar, and left to the reader).

Let us prove $H^0(X, T_X) = 0$. We have inclusions $X \subset G^\omega \subset G$. The conormal exact sequence $0 \to \mathscr{O}_X(-2) \to \Omega^1_{G^\omega}|_X \to \Omega^1_X \to 0$ induces an exact sequence

$$0 \to \Omega_X^2 \to \Omega^3_{G^\omega}(2)|_X \to T_X \to 0.$$

Since $H^1(X, \Omega_X^2)$ vanishes, it is enough to show $H^0(X, \Omega^3_{G^\omega}(2)|_X) = 0$. Since $H^1(G^\omega, \Omega^3_{G^\omega}) = 0$, it is enough to show that $H^0(G^\omega, \Omega^3_{G^\omega}(2))$, or equivalently its Serre dual $H^5(G^\omega, \Omega^2_{G^\omega}(-2))$, vanishes.

The conormal exact sequence of G^ω in G induces an exact sequence

$$0 \to \Omega^1_{G^\omega}(-3) \to \Omega^2_G(-2)|_{G^\omega} \to \Omega^2_{G^\omega}(-2) \to 0.$$

The desired vanishing follows since $H^5(G, \Omega^2_G(-2)) = H^6(G, \Omega^2_G(-3)) = 0$ by Bott's theorem. Since X is (anti)canonically polarized, this vanishing implies that its group of automorphisms is a discrete subgroup of $\mathrm{PGL}(9, \mathbf{C})$, hence is finite.

We also leave the computation of $h^1(X, T_X) = -\chi(X, T_X) = -\chi(X, \Omega^1_X(-2))$ to the reader. □

4.3 The local period map

Let X be a fourfold of type $\mathscr{X}_{10}$ and let Λ be a fixed lattice isomorphic to $H^4(X, \mathbf{Z})_{\mathrm{van}}$. By Proposition 4.1, X has a smooth (simply connected) local deformation space $\mathrm{Def}(X)$ of dimension 24. By (2), the Hodge structure of $H^4(X)_{\mathrm{van}}$ is of K3 type, hence we can define a morphism

$$\mathrm{Def}(X) \to \mathbf{P}(\Lambda \otimes \mathbf{C})$$

with values in the smooth 20-dimensional quadric

$$\mathscr{Q} := \{\omega \in \mathbf{P}(\Lambda \otimes \mathbf{C}) \mid (\omega \cdot \omega) = 0\}.$$

We show below (Theorem 4.3) that the restriction $p\colon \mathrm{Def}(X) \to \mathscr{Q}$, the *local period map,* is a submersion.

Recall from Section 3 that the hyperplane $\mathbf{P}^8$ is defined by a skew-symmetric form on V_5 whose kernel is a 1-dimensional subspace V_1^ω of V_5.

Lemma 4.2 *There is an isomorphism* $H^1(G^\omega, \Omega^3_{G^\omega}(2)) \simeq V_5/V_1^\omega$.

Proof From the normal exact sequence of the embedding $G^\omega \subset G$, we deduce the exact sequences

$$0 \to \Omega^1_{G^\omega} \to \Omega^2_G(1)|_{G^\omega} \to \Omega^2_{G^\omega}(1) \to 0, \tag{3}$$

$$0 \to \Omega^2_{G^\omega}(1) \to \Omega^3_G(2)|_{G^\omega} \to \Omega^3_{G^\omega}(2) \to 0. \tag{4}$$

By Bott's theorem, $\Omega^2_G(1)$ is acyclic, so we have

$$H^q(G^\omega, \Omega^2_G(1)|_{G^\omega}) \simeq H^{q+1}(G, \Omega^2_G) \simeq \delta_{q,1}\mathbf{C}^2.$$

On the other hand, by (1), we have $H^q(G^\omega, \Omega^1_{G^\omega}) \simeq \delta_{q,1}\mathbf{C}$. Therefore, we also get, by (3), $H^q(G^\omega, \Omega^2_{G^\omega}(1)) \simeq \delta_{q,1} V_1^\omega$.

By Bott's theorem again, $\Omega^3_G(1)$ is acyclic, hence using (4) we obtain

$$H^q(G^\omega, \Omega^3_G(2)|_{G^\omega}) \simeq H^q(G, \Omega^3_G(2)) \simeq \delta_{q,1} V_5.$$

This finishes the proof of the lemma. □

Theorem 4.3 *For any fourfold X of type $\mathscr{X}_{10}$, the local period map $p\colon \mathrm{Def}(X) \to \mathscr{Q}$ is a submersion.*

Proof The tangent map to p at the point $[X]$ defined by X has the same kernel as the morphism

$$T\colon H^1(X,T_X) \to \mathrm{Hom}(H^{3,1}(X), H^{3,1}(X)^\perp/H^{3,1}(X))$$
$$\simeq \mathrm{Hom}(H^1(X,\Omega_X^3), H^2(X,\Omega_X^2))$$

defined by the natural pairing $H^1(X,T_X)\otimes H^1(X,\Omega_X^3) \to H^2(X,\Omega_X^2)$ (by (2), $H^1(X,\Omega_X^3)$ is 1-dimensional).

Again, we will only explain the proof when X is not of Gushel type, i.e. , when it is a smooth quadratic section of G^ω, leaving the Gushel case to the reader. Recall the isomorphism $T_X \simeq \Omega_X^3(2)$. The normal exact sequence of the embedding $X\subset G^\omega$ yields the exact sequence $0\to\Omega_X^2\to\Omega_{G^\omega}^3(2)|_X\to T_X\to 0$.

Moreover, the induced coboundary map

$$H^1(X,T_X)\to H^2(X,\Omega_X^2)$$

coincides with the cup product by a generator of $H^1(X,\Omega_X^3)\simeq \mathbf{C}$, hence is the morphism T. Since $H^{2,1}(X) = 0$ (see (2)), its kernel K is isomorphic to $H^1(X,\Omega_{G^\omega}^3(2)|_X)$.

In order to compute this cohomology group, we consider the exact sequence $0\to\Omega_{G^\omega}^3\to\Omega_{G^\omega}^3(2)\to\Omega_{G^\omega}^3(2)|_X\to 0$. Since, by (1), we have $H^1(G^\omega,\Omega_{G^\omega}^3) = H^2(G^\omega,\Omega_{G^\omega}^3) = 0$, we get

$$K\simeq H^1(X,\Omega_{G^\omega}^3(2)|_X)\simeq H^1(G^\omega,\Omega_{G^\omega}^3(2))\simeq V_5/V_1^\omega$$

by Lemma 4.2. Since $\mathrm{Def}(X)$ is smooth of dimension 24 and Q is smooth of dimension 20, this concludes the proof of the theorem in this case. □

The fact that the period map is dominant implies a Noether–Lefschetz-type result.

Corollary 4.4 *If X is a very general fourfold of type $\mathscr{X}_{10}$, we have $H^{2,2}(X)\cap H^4(X,\mathbf{Q}) = H^4(G,\mathbf{Q})$ and the Hodge structure $H^4(X,\mathbf{Q})_{\mathrm{van}}$ is simple.*

Proof For $H^{2,2}(X)\cap H^4(X,\mathbf{Q})_{\mathrm{van}}$ to be nonzero, the corresponding period must be in one of the (countably many) hypersurfaces $\alpha^\perp\cap\mathscr{Q}$, for some $\alpha\in\mathbf{P}(\Lambda\otimes\mathbf{Q})$. Since the local period map is dominant, this does not happen for X very general.

For any X, a standard argument (see, e.g., [Z], Theorem 1.4.1) shows that the transcendental lattice $(H^4(X,\mathbf{Z})_{\mathrm{van}}\cap H^{2,2}(X))^\perp$ inherits a simple rational Hodge structure. For X very general, the transcendental lattice is $H^4(X,\mathbf{Z})_{\mathrm{van}}$. □

Remark 4.5 If X is of Gushel type, we may consider, inside $\mathrm{Def}(X)$, the locus $\mathrm{Def}^G(X)$ where the deformation of X remains of Gushel type and the restriction $p^G\colon \operatorname{Def}^G(X) \to \mathscr{Q}$ of the local period map. One can show that the kernel of $T_{p^G,[X]}$ is 2-dimensional. In particular, p^G is a submersion at $[X]$.

Also, the conclusion of Corollary 4.4 remains valid for very general fourfolds of Gushel type.

5 The period domain and the period map

5.1 The vanishing cohomology lattice

Let $(L,\cdot)$ be a lattice; we denote by $L^\vee$ its dual $\operatorname{Hom}_{\mathbf{Z}}(L,\mathbf{Z})$. The symmetric bilinear form on L defines an embedding $L \subset L^\vee$. The *discriminant group* is the finite abelian group $D(L) := L^\vee/L$; it is endowed with the symmetric bilinear form $b_L\colon D(L)\times D(L) \to \mathbf{Q}/\mathbf{Z}$ defined by $b_L([w],[w']) := w \cdot_{\mathbf{Q}} w' \pmod{\mathbf{Z}}$ ([N], Section 1, 3°). We define the *divisibility* $\operatorname{div}(w)$ of a nonzero element w of L as the positive generator of the ideal $w\cdot L \subset \mathbf{Z}$, so that $w/\operatorname{div}(w)$ is primitive in $L^\vee$. We set $w_* := [w/\operatorname{div}(w)] \in D(L)$. If w is primitive, $\operatorname{div}(w)$ is the order of w_* in $D(L)$.

Proposition 5.1 *Let X be a fourfold of type $\mathscr{X}_{10}$. The* vanishing cohomology lattice $H^4(X,\mathbf{Z})_{\mathrm{van}}$ *is even and has signature* $(20,2)$ *and discriminant group* $(\mathbf{Z}/2\mathbf{Z})^2$. *It is isometric to*

$$\Lambda := 2E_8 \oplus 2U \oplus 2A_1. \tag{5}$$

Proof By (2), the Hodge structure on $H^4(X)$ has weight 2 and the unimodular lattice $\Lambda_X := H^4(X,\mathbf{Z})$, endowed with the intersection form, has signature $(22,2)$. Since $22-2$ is not divisible by 8, this lattice must be odd, hence of type $22\langle 1\rangle \oplus 2\langle -1\rangle$, often denoted by $I_{22,2}$ ([S], Chapitre V, Section 2, Corollaire 1 of Théorème 2 and Théorème 4).

The intersection form on the lattice $\Lambda_G := H^4(G(2,V_5),\mathbf{Z})|_X$ has matrix $\begin{pmatrix}2&2\\2&4\end{pmatrix}$ in the basis $(\sigma_{1,1}|_X,\sigma_2|_X)$. It is of type $2\langle 1\rangle$ and embeds as a primitive sublattice in $H^4(X,\mathbf{Z})$. The vanishing cohomology lattice $\Lambda_X^0 := H^4(X,\mathbf{Z})_{\mathrm{van}} = \Lambda_G^\perp$ therefore has signature $(20,2)$ and $D(\Lambda_X^0) \simeq D(\Lambda_G) \simeq (\mathbf{Z}/2\mathbf{Z})^2$ ([N], Proposition 1.6.1).

An element x of $I_{22,2}$ is *characteristic* if

$$\forall y \in I_{22,2} \quad x\cdot y \equiv y^2 \pmod 2.$$

The lattice $x^\perp$ is then even. One has from [BH], Section 16.2,

$$\begin{aligned} c_1(T_X) &= 2\sigma_1|_X, \\ c_2(T_X) &= 4\sigma_1^2|_X - \sigma_2|_X. \end{aligned} \tag{6}$$

Wu's formula (see [W]) then gives

$$\forall y \in \Lambda_X \quad y^2 \equiv y \cdot (c_1^2 + c_2) \equiv y \cdot \sigma_2|_X \pmod 2. \tag{7}$$

In other words, $\sigma_2|_X$ is characteristic, hence Λ_X^0 is an even lattice. As one can see from Table 15.4 in [CS], there is only one genus of even lattices with signature $(20, 2)$ and discriminant group $(\mathbf{Z}/2\mathbf{Z})^2$ (it is denoted by $II_{20,2}(2_I^2)$ in that table); moreover, there is only one isometry class in that genus ([CS], Theorem 21). In other words, any lattice with these characteristics, such as the one defined in (5), is isometric to Λ_X^0. □

One can also check that Λ is the orthogonal complement in $I_{22,2}$ of the lattice generated by the vectors

$$u := e_1 + e_2 \quad \text{and} \quad v' := e_1 + \cdots + e_{22} - 3f_1 - 3f_2$$

in the canonical basis $(e_1, \dots, e_{22}, f_1, f_2)$ for $I_{22,2}$. Putting everything together, we see that there is an isometry $\gamma\colon \Lambda_X \xrightarrow{\sim} I_{22,2}$ such that

$$\gamma(\sigma_{1,1}|_X) = u, \quad \gamma(\sigma_2|_X) = v', \quad \gamma(\Lambda_X^0) \simeq \Lambda. \tag{8}$$

We let $\Lambda_2 \subset I_{22,2}$ be the rank-2 sublattice $\langle u, v' \rangle = \langle u, v \rangle$, where $v := v' - u$. Then u and v both have divisibility 2, $D(\Lambda_2) = \langle u_*, v_* \rangle$, and the matrix of b_{Λ_2} associated with these generators is $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

5.2 Lattice automorphisms

One can construct $I_{20,2}$ as an overlattice of Λ as follows. Let e and f be respective generators for the last two A_1-factors of Λ (see (5)). They both have divisibility 2 and $D(\Lambda) \simeq (\mathbf{Z}/2\mathbf{Z})^2$, with generators e_* and f_*; the form b_Λ has matrix $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$. In particular, $e_* + f_*$ is the only isotropic nonzero element in $D(\Lambda)$. By [N], Proposition 1.4.1, this implies that there is a unique unimodular overlattice of Λ. Since there is just one isometry class of unimodular lattices of signature $(20, 2)$, this is $I_{20,2}$.

Note that Λ is an even sublattice of index 2 of $I_{20,2}$, so it is the maximal even sublattice $\{x \in I_{20,2} \mid x^2 \text{ even}\}$ (it is contained in that sublattice, and has the same index in $I_{20,2}$).

Every automorphism of $I_{20,2}$ will preserve the maximal even sublattice, so $O(I_{20,2})$ is a subgroup of $O(\Lambda)$. On the other hand, the group $O(D(\Lambda))$ has order 2 and fixes $e_* + f_*$. It follows that every automorphism of Λ fixes $I_{20,2}$, and we obtain $O(I_{20,2}) \simeq O(\Lambda)$.

Now let us try to extend to $I_{22,2}$ an automorphism $\mathrm{Id} \oplus h$ of $\Lambda_2 \oplus \Lambda$. Again, this automorphism permutes the overlattices of $\Lambda_2 \oplus \Lambda$, such as $I_{22,2}$, according to its action on $D(\Lambda_2) \oplus D(\Lambda)$. By [N], overlattices correspond to isotropic subgroups of $D(\Lambda_2) \oplus D(\Lambda)$ that map injectively to both factors. Among them is $I_{22,2}$; after perhaps permuting e and f, it corresponds to the (maximal isotropic) subgroup

$$\{0, u_* + e_*, v_* + f_*, u_* + v_* + e_* + f_*\}.$$

Any automorphism of Λ leaves $e_* + f_*$ fixed. So either h acts trivially on $D(\Lambda)$, in which case $\mathrm{Id} \oplus h$ leaves $I_{22,2}$ fixed, hence extends to an automorphism of $I_{22,2}$; or h switches the other two nonzero elements, in which case $\mathrm{Id} \oplus h$ does not extend to $I_{22,2}$.

In other words, the image of the restriction map

$$\{g \in O(I_{22,2}) \mid g|_{\Lambda_2} = \mathrm{Id}\} \hookrightarrow O(\Lambda)$$

is the *stable orthogonal group*

$$\widetilde{O}(\Lambda) := \mathrm{Ker}(O(\Lambda) \to O(D(\Lambda))). \tag{9}$$

It has index 2 in $O(\Lambda)$ and a generator for the quotient is the involution $r \in O(\Lambda)$ that exchanges e and f and is the identity on $\langle e, f\rangle^\perp$. Let r_2 be the involution of Λ_2 that exchanges u and v. It follows from the discussion above that the involution $r_2 \oplus r$ of $\Lambda_2 \oplus \Lambda$ extends to an involution r_I of $I_{22,2}$.

5.3 The period domain and the period map

Fix a lattice Λ as in (5); it has signature $(20, 2)$. The manifold

$$\Omega := \{\omega \in \mathbf{P}(\Lambda \otimes \mathbf{C}) \mid (\omega \cdot \omega) = 0\,, (\omega \cdot \bar{\omega}) < 0\}$$

is a homogeneous space for the real Lie group $\mathrm{SO}(\Lambda \otimes \mathbf{R}) \simeq \mathrm{SO}(20, 2)$. This group has two components, and one of them reverses the orientation on the negative definite part of $\Lambda \otimes \mathbf{R}$. It follows that Ω has two components, Ω^+ and Ω^-, both isomorphic to the 20-dimensional open complex manifold $\mathrm{SO}_0(20, 2)/\,\mathrm{SO}(20) \times \mathrm{SO}(2)$, a bounded symmetric domain of type IV.

Let $\mathscr{U}$ be a smooth (irreducible) quasi-projective variety parameterizing all fourfolds of type $\mathscr{X}_{10}$. Let u be a general point of $\mathscr{U}$ and let X be the corresponding fourfold. The group $\pi_1(\mathscr{U}, u)$ acts on the lattice $\Lambda_X := H^4(X, \mathbf{Z})$ by isometries and the image Γ_X of the morphism $\pi_1(\mathscr{U}, u) \to O(\Lambda_X)$ is called the monodromy group. The group Γ_X is contained in the subgroup (see (9))

$$\widetilde{O}(\Lambda_X) := \{g \in O(\Lambda_X) \mid g|_{\Lambda_G} = \mathrm{Id}\}.$$

Choose an isometry $\gamma\colon \Lambda_X \xrightarrow{\sim} I_{22,2}$ satisfying (8). It induces an isomorphism $\widetilde{O}(\Lambda_X) \simeq \widetilde{O}(\Lambda)$. The group $\widetilde{O}(\Lambda)$ acts on the manifold Ω defined above and, by a theorem of Baily and Borel, the quotient $\mathscr{D} := \widetilde{O}(\Lambda)\backslash\Omega$ has the structure of an irreducible quasi-projective variety. One defines as usual a period map $\mathscr{U} \to \mathscr{D}$ by sending a variety to its period; it is an algebraic morphism. It descends to "the" period map

$$\wp\colon \mathscr{X}_{10} \to \mathscr{D}.$$

By Theorem 4.3 (and Remark 4.5), $\wp$ is dominant with 4-dimensional smooth fibers *as a map of stacks.*

Remark 5.2 As in the 3-dimensional case ([DIM1]), we do not know whether our fourfolds have a coarse moduli space, even in the category of algebraic spaces (the main unresolved issue is whether the corresponding moduli functor is separated). If such a space $\mathbf{X}_{10}$ exists, note however that it is *singular along the Gushel locus:* any fourfold X of Gushel type has a canonical involution; if X has no other nontrivial automorphisms, $\mathbf{X}_{10}$ is then locally around $[X]$ the product of a smooth 22-dimensional germ and the germ of a surface node. The fiber of the period map $\mathbf{X}_{10} \to \mathscr{D}$ then has multiplicity 2 along the surface corresponding to Gushel fourfolds (see Remark 4.5).

6 Special fourfolds

Following [H1], Section 3, we say that a fourfold X of type $\mathscr{X}_{10}$ is *special* if it contains a surface whose cohomology class "does not come" from $G(2, V_5)$. Since the Hodge conjecture is true (over $\mathbf{Q}$) for Fano fourfolds (more generally, by [CM], for all uniruled fourfolds), this is equivalent to saying that the rank of the (positive definite) lattice $H^{2,2}(X) \cap H^4(X, \mathbf{Z})$ is at least 3, hence by Corollary 4.4, a very general X is not special. The set of special fourfolds is sometimes called the Noether–Lefschetz locus.

6.1 Special loci

For each primitive, positive definite, rank-3 sublattice $K \subset I_{22,2}$ containing the lattice Λ_2 defined at the end of Section 5.1, we define an irreducible hypersurface of Ω^+ by setting

$$\Omega_K := \{\omega \in \Omega^+ \mid K \subset \omega^\perp\}.$$

A fourfold X is *special* if and only if its period is in one of these (countably many) hypersurfaces. We now investigate these lattices K.

Lemma 6.1 *The discriminant d of K is positive and $d \equiv 0, 2, \text{or } 4 \pmod 8$.*

Proof Since K is positive definite, d must be positive. Completing the basis (u, v) of Λ_2 from Section 5.1 to a basis of K, we see that the matrix of the intersection form in that basis is $\begin{pmatrix} 2 & 0 & a \\ 0 & 2 & b \\ a & b & c \end{pmatrix}$, whose determinant is $d = 4c - 2(a^2 + b^2)$. By Wu's formula (7) (or equivalently, since v is characteristic), we have $c \equiv a + b \pmod 2$, hence $d \equiv 2(a^2 + b^2) \pmod 8$. This proves the lemma. □

We keep the notation of Section 5.

Proposition 6.2 *Let d be a positive integer such that $d \equiv 0, 2, \text{or } 4 \pmod 8$ and let $\mathscr{O}_d$ be the set of orbits for the action of the group*

$$\widetilde{O}(\Lambda) = \{g \in O(I_{22,2}) \mid g|_{\Lambda_2} = \mathrm{Id}\} \subset O(\Lambda)$$

on the set of primitive, positive definite, rank-3, discriminant-d, sublattices $K \subset I_{22,2}$ containing Λ_2. Then:

(a) if $d \equiv 0 \pmod 8$, $\mathscr{O}_d$ has one element and $K \simeq \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & d/4 \end{pmatrix}$;

(b) if $d \equiv 2 \pmod 8$, $\mathscr{O}_d$ has two elements, which are interchanged by the involution r_I of $I_{22,2}$, and $K \simeq \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & (d+2)/4 \end{pmatrix}$;

(c) if $d \equiv 4 \pmod 8$, $\mathscr{O}_d$ has one element and $K \simeq \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & (d+4)/4 \end{pmatrix}$.

In case (b), one orbit is characterized by the properties $K \cdot u = \mathbf{Z}$ and $K \cdot v = 2\mathbf{Z}$, and the other by $K \cdot u = 2\mathbf{Z}$ and $K \cdot v = \mathbf{Z}$.

Proof By a theorem of Eichler (see, e.g., [GHS], Lemma 3.5), the $\widetilde{O}(\Lambda)$-orbit of a primitive vector w in the even lattice Λ is determined by its length w^2 and its class $w_* \in D(\Lambda)$.

If $\mathrm{div}(w) = 1$, we have $w_* = 0$ and the orbit is determined by w^2. The lattice $\Lambda_2 \oplus \mathbf{Z}w$ is primitive: if $\alpha u + \beta v + \gamma w = mw'$, and if $w \cdot w'' = 1$, we obtain $\gamma = mw' \cdot w''$, hence $\alpha u + \beta v = m((w' \cdot w'')w - w')$ and m divides α, β, and γ. Its discriminant is $4w^2 \equiv 0 \pmod 8$.

If $\mathrm{div}(w) = 2$, we have $w_* \in \{e_*, f_*, e_* + f_*\}$. Recall from Section 5.2 that $\frac{1}{2}(u + e)$, $\frac{1}{2}(v + f)$, and $\frac{1}{2}(u + v + e + f)$ are all in $I_{22,2}$. It follows that exactly one of $\frac{1}{2}(u+w)$, $\frac{1}{2}(v+w)$, and $\frac{1}{2}(u+v+w)$ is in $I_{22,2}$, and $\Lambda_2 \oplus \mathbf{Z}w$ has index 2 in its saturation K in $I_{22,2}$. In particular, K has discriminant w^2. If $w_* \in \{e_*, f_*\}$, this is $\equiv 2 \pmod 8$; if $w_* = e_* + f_*$, this is $\equiv 4 \pmod 8$.

Now if K is a lattice as in the statement of the proposition, we let $K^\perp$ be its orthogonal complement in $I_{22,2}$, so that the rank-1 lattice $K^0 := K \cap \Lambda$ is the orthogonal complement of $K^\perp$ in Λ. From $K^0 \subset \Lambda$, we can therefore recover $K^\perp$, then $K \supset \Lambda_2$. The preceding discussion applied to a generator w of K^0 gives the statement, except that we still have to prove that there are indeed elements w of the various types for all d, i.e., we need to construct elements in each orbit to show they are not empty.

Let u_1 and u_2 be standard generators for a hyperbolic factor U of Λ. For any integer m, set $w_m := u_1 + mu_2$. We have $w_m^2 = 2m$ and $\operatorname{div}(w_m) = 1$. The lattice $\Lambda_2 \oplus \mathbf{Z}w_m$ is saturated with discriminant $8m$.

We have $(e + 2w_m)^2 = 8m + 2$ and $\operatorname{div}(e + 2w_m) = 2$. The saturation of the lattice $\Lambda_2 \oplus \mathbf{Z}(e + 2w_m)$ has discriminant $d = 8m + 2$, and similarly upon replacing e with f (same d) or $e + f$ ($d = 8m + 4$). □

Let K be a lattice as above. The image in $\mathscr{D} = \widetilde{O}(\Lambda)\backslash\Omega$ of the hypersurface $\Omega_K \subset \Omega^+$ depends only on the $\widetilde{O}(\Lambda)$-orbit of K. Also, the involution $r \in O(\Lambda)$ induces a nontrivial involution $r_{\mathscr{D}}$ of $\mathscr{D}$.

Corollary 6.3 *The periods of the special fourfolds of discriminant d are contained in*

(a) if $d \equiv 0 \pmod 4$*, an irreducible hypersurface* $\mathscr{D}_d \subset \mathscr{D}$*;*
(b) if $d \equiv 2 \pmod 8$*, the union of two irreducible hypersurfaces* $\mathscr{D}'_d$ *and* $\mathscr{D}''_d$*, which are interchanged by the involution* $r_{\mathscr{D}}$*.*

Assume $d \equiv 2 \pmod 8$ (case (b)). Then, $\mathscr{D}'_d$ (resp. $\mathscr{D}''_d$) corresponds to lattices K with $K \cdot u = \mathbf{Z}$ (resp. $K \cdot v = \mathbf{Z}$). In other words, given a fourfold X of type $\mathscr{X}_{10}$ whose period point is in $\mathscr{D}_d = \mathscr{D}'_d \cup \mathscr{D}''_d$, it is in $\mathscr{D}'_d$ if $K \cdot \sigma_1^2 \subset 2\mathbf{Z}$, and it is in $\mathscr{D}''_d$ if $K \cdot \sigma_{1,1} \subset 2\mathbf{Z}$.

Remark 6.4 The divisors $\mathscr{D}_d$ appear in the theory of modular forms under the name of *Heegner divisors*. In the notation of [B]:

- when $d \equiv 0 \pmod 8$, we have $\mathscr{D}_d = h_{d/8,0}$;
- when $d \equiv 2 \pmod 8$, we have $\mathscr{D}'_d = h_{d/2,e_*}$ and $\mathscr{D}''_d = h_{d/2,f_*}$;
- when $d \equiv 4 \pmod 8$, we have $\mathscr{D}_d = h_{d/2,e_*+f_*}$.

Remark 6.5 Zarhin's argument, already used in the proof of Corollary 4.4, proves that if X is a fourfold whose period is very general in any given $\mathscr{D}_d$, the lattice $K = H^4(X, \mathbf{Z}) \cap H^{2,2}(X)$ has rank exactly 3 and the rational Hodge structure $K^\perp \otimes \mathbf{Q}$ is simple.

6.2 Associated K3 surface

As we will see in the next section, K3 surfaces sometimes occur in the geometric description of special fourfolds X of type $\mathscr{X}_{10}$. This is related to the fact that, for some values of d, the nonspecial cohomology of X looks like the primitive cohomology of a K3 surface.

Following [H1], we determine, in each case of Proposition 6.2, the discriminant group of the *nonspecial lattice* $K^\perp$ and the symmetric form $b_{K^\perp} = -b_K$. We then find all cases when the nonspecial lattice of X is isomorphic (with a change of sign) to the primitive cohomology lattice of a (pseudo-polarized, degree-d) K3 surface. Although this property is only lattice-theoretic, the surjectivity of the period map for K3 surfaces then produces an actual K3 surface, which is said to be "associated with X." For $d = 10$, we will see in Sections 7.1 and 7.3 geometric constructions of the associated K3 surface.

Finally, there are other cases where geometry provides an "associated" K3 surface S (see Section 7.6), but not in the sense considered here: the Hodge structure of S is only isogeneous to that of the fourfold. So there might be integers d not in the list provided by the proposition below, for which special fourfolds of discriminant d are still related in some way to K3 surfaces (of degree different from d).

Proposition 6.6 *Let d be a positive integer such that $d \equiv 0, 2,$ or $4 \pmod 8$ and let (X, K) be a special fourfold of type $\mathscr{X}_{10}$ with discriminant d. Then:*

(a) if $d \equiv 0 \pmod 8$, we have $D(K^\perp) \simeq (\mathbf{Z}/2\mathbf{Z})^2 \times (\mathbf{Z}/(d/4)\mathbf{Z})$;
(b) if $d \equiv 2 \pmod 8$, we have $D(K^\perp) \simeq \mathbf{Z}/d\mathbf{Z}$ and we may choose this isomorphism so that $b_{K^\perp}(1,1) = -\frac{d+8}{2d} \pmod{\mathbf{Z}}$;
(c) if $d \equiv 4 \pmod 8$, we have $D(K^\perp) \simeq \mathbf{Z}/d\mathbf{Z}$ and we may choose this isomorphism so that $b_{K^\perp}(1,1) = -\frac{d+2}{2d} \pmod{\mathbf{Z}}$.

The lattice $K^\perp$ is isomorphic to the opposite of the primitive cohomology lattice of a pseudo-polarized K3 surface (necessarily of degree d) if and only if we are in case (b) or (c) and the only odd primes that divide d are $\equiv 1 \pmod 4$.

In these cases, there exists a pseudo-polarized, degree-d, K3 surface S such that the Hodge structure $H^2(S, \mathbf{Z})^0(-1)$ is isomorphic to $K^\perp$. Moreover, if the period point of X is not in $\mathscr{D}_8$, the pseudo-polarization is a polarization.

The first values of d that satisfy the conditions for the existence of an associated K3 surface are: 2, 4, 10, 20, 26, 34, 50, 52, 58, 68, 74, 82, 100, . . .

Proof Since $I_{22,2}$ is unimodular, we have $(D(K^\perp), b_{K^\perp}) \simeq (D(K), -b_K)$ ([N], Proposition 1.6.1). Case (a) follows from Proposition 6.2.

Let e, f, and g be the generators of K corresponding to the matrix given in Proposition 6.2. The matrix of $b_{K^\perp}$ in the dual basis $(e^\vee, f^\vee, g^\vee)$ of $K^\perp$ is the inverse of that matrix.

In case (b), one checks that $e^\vee + g^\vee$ generates $D(K)$, which is isomorphic to $\mathbf{Z}/d\mathbf{Z}$. Its square is $\frac{1}{2} + \frac{4}{d} = \frac{d+8}{2d}$.

In case (c), one checks that $e^\vee$ generates $D(K)$, which is isomorphic to $\mathbf{Z}/d\mathbf{Z}$. Its square is $\frac{d+2}{2d}$.

The opposite of the primitive cohomology lattice of a pseudo-polarized K3 surface of degree d has discriminant group $\mathbf{Z}/d\mathbf{Z}$ and the square of a generator is $\frac{1}{d}$. So case (a) is impossible.

In case (b), the forms are conjugate if and only if $-\frac{d+8}{2d} \equiv \frac{n^2}{d} \pmod{\mathbf{Z}}$ for some integer n prime to d, or $-\frac{d+8}{2} \equiv n^2 \pmod{d}$. Set $d = 2d'$ (so that $d' \equiv 1 \pmod 4$); then this is equivalent to saying that $d' - 4$ is a square in the ring $\mathbf{Z}/d\mathbf{Z}$. Since d' is odd, this ring is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/d'\mathbf{Z}$, hence this is equivalent to asking that -4, or equivalently -1, is a square in $\mathbf{Z}/d'\mathbf{Z}$. This happens if and only if the only odd primes that divide d' (or d) are $\equiv 1 \pmod 4$.

In case (c), the reasoning is similar: we need $-\frac{d+2}{2d} \equiv \frac{n^2}{d} \pmod{\mathbf{Z}}$ for some integer n prime to d. Set $d = 4d'$, with d' odd. This is equivalent to $-2 \equiv 2n^2 \pmod{d'}$, and we conclude as above.

As already explained, the existence of the polarized K3 surface (S, f) follows from the surjectivity of the period map for K3 surfaces. Finally, if $\wp([X])$ is not in $\mathscr{D}_8$, there are no classes of type $(2,2)$ with square 2 in $H^4(X,Z)_{\mathrm{van}}$, hence no (-2)-curves on S orthogonal to f, so f is a polarization. □

6.3 Associated cubic fourfold

Cubic fourfolds also sometimes occur in the geometric description of special fourfolds X of type $\mathscr{X}_{10}$ (see Section 7.2). We determine for which values of d the nonspecial cohomology of X is isomorphic to the nonspecial cohomology of a special cubic fourfold. Again, this is only a lattice-theoretic association, but the surjectivity of the period map for cubic fourfolds then produces a (possibly singular) actual cubic. We will see in Section 7.2 that some special fourfolds X of discriminant 12 are actually birationally isomorphic to their associated special cubic fourfold.

Proposition 6.7 *Let d be a positive integer such that $d \equiv 0, 2, \text{or } 4 \pmod 8$ and let (X, K) be a special fourfold of type $\mathscr{X}_{10}$ with discriminant d. The lattice*

$K^\perp$ is isomorphic to the nonspecial cohomology lattice of a (possibly singular) special cubic fourfold (necessarily of discriminant d) if and only if:

(a) either $d \equiv 2$ *or* $20 \pmod{24}$*, and the only odd primes that divide d are* $\equiv \pm 1 \pmod{12}$*;*
(b) or $d \equiv 12$ *or* $66 \pmod{72}$*, and the only primes* ≥ 5 *that divide d are* $\equiv \pm 1 \pmod{12}$.

In these cases, if moreover the period point of X is general in $\mathscr{D}_d$ *and* $d \neq 2$*, there exists a* smooth *special cubic fourfold whose nonspecial Hodge structure is isomorphic to* $K^\perp$.

The first values of d that satisfy the conditions for the existence of an associated cubic fourfold are: 2, 12, 26, 44, 66, 74, 92, 122, 138, 146, 156, 194, ...

Proof Recall from [H1], Section 4.3, that (possibly singular) special cubic fourfolds of positive discriminant d exist for $d \equiv 0$ or $2 \pmod 6$ (for $d = 2$, the associated cubic fourfold is the (singular) determinantal cubic; for $d = 6$, it is nodal). Combining that condition with that of Lemma 6.1, we obtain the necessary condition $d \equiv 0, 2, 8, 12, 18, 20 \pmod{24}$. Write $d = 24d' + e$, with $e \in \{0, 2, 8, 12, 18, 20\}$.

Then, one needs to check whether the discriminant forms are isomorphic. Recall from [H1], Proposition 3.2.5, that the discriminant group of the nonspecial lattice of a special cubic fourfold of discriminant d is isomorphic to $(\mathbf{Z}/3\mathbf{Z}) \times (\mathbf{Z}/(d/3)\mathbf{Z})$ if $d \equiv 0 \pmod 6$, and to $\mathbf{Z}/d\mathbf{Z}$ if $d \equiv 2 \pmod 6$. This excludes $e = 0$ or 8; for $e = 12$, we need $d' \not\equiv 1 \pmod 3$ and for $e = 18$, we need $d' \not\equiv 0 \pmod 3$. In all these cases, the discriminant group is cyclic.

When $e = 2$, the discriminant forms are conjugate if and only if $-\frac{d+8}{2d} \equiv n^2 \frac{2d-1}{3d} \pmod{\mathbf{Z}}$ for some integer n prime to d (Proposition 6.6 and [H1], Proposition 3.2.5), or equivalently, since 3 is invertible modulo d, if and only if $\frac{d}{2} + 12 \equiv 3\frac{d+8}{2} \equiv n^2 \pmod d$. This is equivalent to saying that $12d' + 13$ is a square in $\mathbf{Z}/d\mathbf{Z} \simeq (\mathbf{Z}/(12d' + 1)\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z})$, or that 3 is a square in $\mathbf{Z}/(12d' + 1)\mathbf{Z}$. Using quadratic reciprocity, we see that this is equivalent to saying that the only odd primes that divide d are $\equiv \pm 1 \pmod{12}$.

When $e = 20$, we need $-\frac{d+2}{2d} \equiv n^2 \frac{2d-1}{3d} \pmod{\mathbf{Z}}$ for some integer n prime to d, or equivalently, $\frac{d}{2} + 3 \equiv n^2 \pmod d$. Again, we get the same condition.

When $e = 12$, we need $9 \nmid d$ and $-\frac{d+2}{2d} \equiv n^2 \left(\frac{2}{3} - \frac{3}{d}\right) \pmod{\mathbf{Z}}$ for some integer n prime to d, or equivalently, $-12d' - 7 \equiv n^2(16d' + 5) \pmod d$. Modulo 3, we get that $1 - d'$ must be a nonzero square, hence $3 \mid d'$. Modulo 4, there are no conditions. Then we need $1 \equiv 3n^2 \pmod{2d' + 1}$ and we conclude as above.

Finally, when $e = 18$, we need $9 \nmid d$ and $-\frac{d+8}{2d} \equiv n^2\left(\frac{2}{3} - \frac{3}{d}\right) \pmod{\mathbf{Z}}$ for some n prime to d, or equivalently, $-12d' - 13 \equiv n^2(16d' + 9) \pmod{d}$. Modulo 3, we get $d' \equiv 2 \pmod 3$, and then $4 \equiv 3n^2 \pmod{4d' + 3}$ and we conclude as above.

At this point, we have a Hodge structure on $K^\perp$ which is, as a lattice, isomorphic to the nonspecial cohomology of a special cubic fourfold. It corresponds to a point in the period domain $\mathscr{C}$ of cubic fourfolds. To make sure that it corresponds to a (then unique) smooth cubic fourfold, we need to check that it is not in the special loci $\mathscr{C}_2 \cup \mathscr{C}_6$ ([La], Theorem 1.1). If the period point of X is general in $\mathscr{D}_d$, the period point in $\mathscr{C}$ is general in $\mathscr{C}_d$, hence is not in $\mathscr{C}_2 \cup \mathscr{C}_6$ if $d \notin \{2, 6\}$. □

Remark 6.8 One can be more precise and figure out explicit conditions on $\wp([X])$ for the associated cubic fourfold to be smooth (but calculations are complicated). For example, when $d = 12$, we find that it is enough to assume $\wp([X]) \notin \mathscr{D}_2 \cup \mathscr{D}_4 \cup \mathscr{D}_8 \cup \mathscr{D}_{16} \cup \mathscr{D}_{28} \cup \mathscr{D}_{60} \cup \mathscr{D}_{112} \cup \mathscr{D}_{240}$.

7 Examples of special fourfolds

Assume that a fourfold X of type $\mathscr{X}_{10}$ contains a smooth surface S. Then, by (6),

$$c(T_X)|_S = 1 + 2\sigma_1|_S + (4\sigma_1^2|_S - \sigma_2|_S) = c(T_S)c(N_{S/X}).$$

This implies $c_1(T_S) + c_1(N_{S/X}) = 2\sigma_1|_S$ and

$$4\sigma_1^2|_S - \sigma_2|_S = c_1(T_S)c_1(N_{S/X}) + c_2(T_S) + c_2(N_{S/X}).$$

We obtain

$$(S)_X^2 = c_2(N_{S/X}) = 4\sigma_1^2|_S - \sigma_2|_S - c_1(T_S)(2\sigma_1|_S - c_1(T_S)) - c_2(T_S).$$

Write $[S] = a\sigma_{3,1} + b\sigma_{2,2}$ in $G(2, V_5)$. Using Noether's formula, we obtain

$$(S)_X^2 = 3a + 4b + 2K_S \cdot \sigma_1|_S + 2K_S^2 - 12\chi(\mathscr{O}_S). \tag{10}$$

The determinant of the intersection matrix in the basis $(\sigma_{1,1}|_X, \sigma_2|_X - \sigma_{1,1}|_X, [S])$ is then

$$d = 4(S)_X^2 - 2(b^2 + (a - b)^2). \tag{11}$$

We remark that $\sigma_2|_X - \sigma_{1,1}|_X$ is the class of the unique σ-quadric surface Σ_0 contained in X (see Section 3).

The results of this section are summarized in Section 7.7.

7.1 Fourfolds containing a σ-plane (divisor $\mathscr{D}''_{10}$)

A σ-plane is a 2-plane in $G(2, V_5)$ of the form $\mathbf{P}(V_1 \wedge V_4)$; its class in $G(2, V_5)$ is $\sigma_{3,1}$. Fourfolds of type $\mathscr{X}^0_{10}$ containing such a 2-plane were already studied by Roth ([R], Section 4) and Prokhorov ([P], Section 3).

Proposition 7.1 *Inside $\mathscr{X}_{10}$, the family $\mathscr{X}_{\sigma\text{-plane}}$ of fourfolds containing a σ-plane is irreducible of codimension 2. The period map induces a dominant map $\mathscr{X}_{\sigma\text{-plane}} \to \mathscr{D}''_{10}$ whose general fiber has dimension 3 and is rationally dominated by a $\mathbf{P}^1$-bundle over a degree-10 K3 surface.*

A general member of $\mathscr{X}_{\sigma\text{-plane}}$ is rational.

During the proof, we present an explicit geometric construction of a general member X of $\mathscr{X}_{\sigma\text{-plane}}$, starting from a general degree-10 K3 surface $S \subset \mathbf{P}^6$, a general point p on S, and a smooth quadric Y containing the projection $\widetilde{S} \subset \mathbf{P}^5$ from p. The birational isomorphism $Y \dashrightarrow X$ is given by the linear system of cubics containing $\widetilde{S}$.

Proof A parameter count ([IM], Lemma 3.6) shows that $\mathscr{X}_{\sigma\text{-plane}}$ is irreducible of codimension 2 in $\mathscr{X}_{10}$. Let $P \subset X$ be a σ-plane. From (10), we obtain $(P)^2_X = 3$ and from (11), $d = 10$. Since $\sigma_1^2 \cdot P$ is odd, we are in $\mathscr{D}''_{10}$.

For X general in $\mathscr{X}_{\sigma\text{-plane}}$ (see [P], Section 3, for the precise condition), the image of the projection $\pi_P \colon X \dashrightarrow \mathbf{P}^5$ from P is a smooth quadric $Y \subset \mathbf{P}^5$ and, if $\widetilde{X} \to X$ is the blow-up of P, the projection π_P induces a birational morphism $\widetilde{X} \to Y$ which is the blow-up of a smooth degree-9 surface $\widetilde{S}$, itself the blow-up of a smooth degree-10 K3 surface S at one point ([P], Proposition 2).

Conversely, let $S \subset \mathbf{P}^6$ be a degree-10 K3 surface. When S is general, the projection from a general point p on S induces an embedding $\widetilde{S} \subset \mathbf{P}^5$ of the blow-up of S at p. Given any *smooth* quadric Y containing $\widetilde{S}$, one can reverse the construction above and produce a fourfold X containing a σ-plane (we will give more details about this construction and explicit genericity assumptions during the proof of Theorem 8.1).

There are isomorphisms of polarized integral Hodge structures

$$\begin{aligned} H^4(\widetilde{X}, \mathbf{Z}) &\simeq H^4(X, \mathbf{Z}) \oplus H^2(P, \mathbf{Z})(-1) \\ &\simeq H^4(Y, \mathbf{Z}) \oplus H^2(\widetilde{S}, \mathbf{Z})(-1) \\ &\simeq H^4(Y, \mathbf{Z}) \oplus H^2(S, \mathbf{Z})(-1) \oplus \mathbf{Z}(-2). \end{aligned}$$

For S very general, the Hodge structure $H^2(S, \mathbf{Q})_0$ is simple, hence it is isomorphic to the nonspecial cohomology $K^\perp \otimes \mathbf{Q}$ (where K is the lattice spanned by $H^4(G(2, V_5), \mathbf{Z})$ and $[P]$ in $H^4(X, \mathbf{Z})$). Moreover, the lattice $H^2(S, \mathbf{Z})_0(-1)$

embeds isometrically into $K^\perp$. Since they both have rank 21 and discriminant 10, they are isomorphic. The surface S is thus the (polarized) K3 surface associated with X as in Proposition 6.6.

Since the period map for polarized degree-10 K3 surfaces is dominant onto their period domain, the period map for $\mathscr{X}_{\sigma\text{-plane}}$ is dominant onto $\mathscr{D}''_{10}$ as well. Since the Torelli theorem for K3 surfaces holds, S is determined by the period point of X, hence the fiber $\wp^{-1}([X])$ is rationally dominated by the family of pairs (p, Y), where $p \in S$ and Y belongs to the pencil of quadrics in $\mathbf{P}^5$ containing $\widetilde{S}$. □

With the notation above, the inverse image of the quadric $Y \subset \mathbf{P}^5$ by the projection $\mathbf{P}^8 \dashrightarrow \mathbf{P}^5$ from P is a rank-6 non-Plücker quadric in $\mathbf{P}^8$ containing X, with vertex P. We show in Section 7.5 that $\mathscr{X}_{\sigma\text{-plane}}$ is contained in the irreducible hypersurface of $\mathscr{X}_{10}$ parameterizing the fourfolds X contained in such a quadric.

7.2 Fourfolds containing a ρ-plane (divisor $\mathscr{D}_{12}$)

A ρ-plane is a 2-plane in $G(2, V_5)$ of the form $\mathbf{P}(\wedge^2 V_3)$; its class in $G(2, V_5)$ is $\sigma_{2,2}$. Fourfolds of type $\mathscr{X}_{10}$ containing such a 2-plane were already studied by Roth ([R], Section 4).

Proposition 7.2 *Inside $\mathscr{X}_{10}$, the family $\mathscr{X}_{\rho\text{-plane}}$ of fourfolds containing a ρ-plane is irreducible of codimension 3. The period map induces a dominant map $\mathscr{X}_{\rho\text{-plane}} \to \mathscr{D}_{12}$ whose general fiber is the union of two rational surfaces.*

A general member of $\mathscr{X}_{\rho\text{-plane}}$ is birationally isomorphic to a cubic fourfold containing a smooth cubic surface scroll.

The proof presents a geometric construction of a general member of $\mathscr{X}_{\rho\text{-plane}}$, starting from any smooth cubic fourfold $Y \subset \mathbf{P}^5$ containing a smooth cubic surface scroll T. The birational isomorphism $Y \dashrightarrow X$ is given by the linear system of quadrics containing T.

Proof A parameter count ([IM], Lemma 3.6) shows that $\mathscr{X}_{\rho\text{-plane}}$ is irreducible of codimension 3 in $\mathscr{X}_{10}$. Let $P = \mathbf{P}(\wedge^2 V_3) \subset X$ be a ρ-plane. From (10), we obtain $(P)^2_X = 4$. From (11), we obtain $d = 12$ and we are in $\mathscr{D}_{12}$.

As shown in [R], Section 4, the image of the projection $\pi_P : X \dashrightarrow \mathbf{P}^5$ from P is a cubic hypersurface Y and the image of the intersection of X with the Schubert hypersurface

$$\Sigma_P = \{V_2 \subset V_5 \mid V_2 \cap V_3 \neq 0\} \subset G(2, V_5)$$

is a cubic surface scroll T (contained in Y). If $\widetilde{X} \to X$ is the blow-up of P, with exceptional divisor E_P, the projection π_P induces a birational morphism $\widetilde{\pi}_P\colon \widetilde{X} \to Y$. One checks (with the same arguments as in [P], Section 3) that all fibers have dimension ≤ 1 and hence that $\widetilde{\pi}_P$ is the blow-up of the smooth surface T. The image $\widetilde{\pi}_P(E_P)$ is the (singular) hyperplane section $Y_0 := Y \cap \langle T\rangle$.

Conversely, a general cubic fourfold Y containing a smooth cubic scroll contains two families (each parameterized by $\mathbf{P}^2$) of such surfaces (see [HT1] and [HT2], Example 7.12). For each such smooth cubic scroll, one can reverse the construction above and produce a smooth fourfold X containing a ρ-plane.

As in Section 7.1, there are isomorphisms of polarized integral Hodge structures

$$H^4(\widetilde{X},\mathbf{Z}) \simeq H^4(X,\mathbf{Z}) \oplus H^2(P,\mathbf{Z})(-1) \simeq H^4(Y,\mathbf{Z}) \oplus H^2(T,\mathbf{Z})(-1).$$

Let K be the lattice spanned by $H^4(G(2,V_5),\mathbf{Z})$ and $[P]$ in $H^4(X,\mathbf{Z})$. For X very general in $\mathscr{X}_{\rho\text{-plane}}$, the Hodge structure $K^\perp \otimes \mathbf{Q}$ is simple (Remark 6.5), hence it is isomorphic to the Hodge structure $\langle h^2,[T]\rangle^\perp \subset H^4(Y,\mathbf{Q})$. Moreover, the lattices $K^\perp$ and $\langle h^2,[T]\rangle^\perp \subset H^4(Y,\mathbf{Z})$, which both have rank 21 and discriminant 12 (see [H1], Section 4.1.1), are isomorphic. This case fits into the setting of Proposition 6.7: the special cubic fourfold Y is associated with X.

Finally, since the period map for cubic fourfolds containing a cubic scroll surface is dominant onto the corresponding hypersurface in their period domain, the period map for $\mathscr{X}_{\rho\text{-plane}}$ is dominant onto $\mathscr{D}_{12}$ as well. Since the Torelli theorem holds for cubic fourfolds ([V]), Y is determined by the period point of X, hence the fiber $\wp^{-1}([X])$ is rationally dominated by the family of smooth cubic scrolls contained in Y. It is therefore the union of two rational surfaces. □

With the notation above, let $V_4 \subset V_5$ be a general hyperplane containing V_3. Then $G(2,V_4) \cap X$ is the union of P and a cubic scroll surface.

7.3 Fourfolds containing a τ-quadric surface (divisor $\mathscr{D}'_{10}$)

A τ-quadric surface in $G(2,V_5)$ is a linear section of $G(2,V_4)$; its class in $G(2,V_5)$ is $\sigma_1^2 \cdot \sigma_{1,1} = \sigma_{3,1} + \sigma_{2,2}$.

Proposition 7.3 *The closure $\overline{\mathscr{X}}_{\tau\text{-quadric}} \subset \mathscr{X}_{10}$ of the family of fourfolds containing a τ-quadric surface is an irreducible component of $\wp^{-1}(\mathscr{D}'_{10})$. The period map induces a dominant map $\mathscr{X}_{\tau\text{-quadric}} \to \mathscr{D}'_{10}$ whose general fiber* *is birationally isomorphic to the quotient by an involution of the symmetric square of a degree-10 K3 surface.*

A general member of $\mathscr{X}_{\tau\text{-quadric}}$ is rational.

During the proof, we present a geometric construction of a general member of $\mathscr{X}_{\tau\text{-quadric}}$, starting from a general degree-10 K3 surface $S \subset \mathbf{P}^6$ and two general points on S: if $S_0 \subset \mathbf{P}^4$ is the (singular) projection of S from these two points, the birational isomorphism $\mathbf{P}^4 \dashrightarrow X$ is given by the linear system of quartics containing S_0.

Proof A parameter count shows that $\mathscr{X}_{\tau\text{-quadric}}$ is irreducible of codimension 1 in $\mathscr{X}_{10}$ (one can also use the parameter count at the end of the proof). Let $\Sigma \subset X$ be a smooth τ-quadric surface. From (10), we obtain $(\Sigma)^2_X = 3$ and from (11), $d = 10$. Since $\sigma_1^2 \cdot \Sigma$ is even, we are in $\mathscr{D}'_{10}$. The family $\mathscr{X}_{\tau\text{-quadric}}$ is therefore a component of the divisor $\wp^{-1}(\mathscr{D}'_{10})$.

The projection from the 3-plane $\langle \Sigma \rangle$ induces a birational map $X \dashrightarrow \mathbf{P}^4$ (in particular, X is rational!). If $\varepsilon \colon \widetilde{X} \to X$ is the blow-up of Σ, one checks that it induces a birational *morphism* $\pi \colon \widetilde{X} \to \mathbf{P}^4$ which is more complicated than just the blow-up of a smooth surface (compare with Section 7.1).

In the first part of the proof, we analyze the birational structure of π by factorizing it into a composition of blow-ups with smooth centers and their inverses (see diagram (12)). This gives an explicit construction of X, and in the second part of the proof we prove that any such construction does give an X containing a τ-quadric surface.

Since Σ is contained in a $G(2, V_4)$, the quartic surface $X \cap G(2, V_4)$ is the union of Σ and another τ-quadric surface $\Sigma^\star$. The two 3-planes $\langle \Sigma \rangle$ and $\langle \Sigma^\star \rangle$ meet along a 2-plane, hence (the strict transform of) $\Sigma^\star$ is contracted by π to a point. Generically, the only quadric surfaces contained in X are the σ-quadric surface Σ_0 (defined in Section 3) and the τ-quadric surfaces Σ and $\Sigma^\star$. Using the fact that X is an intersection of quadrics, one checks that $\Sigma^\star$ is the only surface contracted (to a point) by π.

Let $\ell' \subset \widetilde{X}$ be a line contracted by ε. If $\ell \subset \widetilde{X}$ is (the strict transform of) a line contained in $\Sigma^\star$, it meets Σ and is contracted by π. Since $\widetilde{X}$ has Picard number 2, the rays $\mathbf{R}^+[\ell]$ and $\mathbf{R}^+[\ell']$ are extremal, hence span the cone of curves of $\widetilde{X}$. These two classes have $(-K_{\widetilde{X}})$-degree 1, hence $\widetilde{X}$ is a Fano fourfold. Extremal contractions on smooth fourfolds have been classified ([AM], Theorem 4.1.3). In our case, we have:

- π is a divisorial contraction, its (irreducible) exceptional divisor D contains $\Sigma^\star$, and $D \underset{\text{lin}}{\equiv} 3H - 4E$;

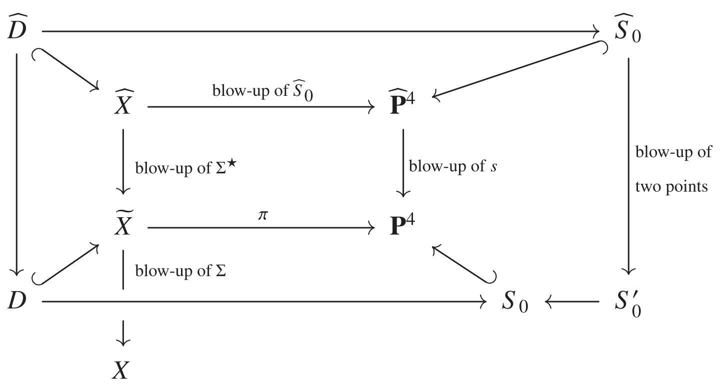
- $S_0 := \pi(D)$ is a surface with a single singular point $s := \pi(\Sigma^\star)$, where it is locally the union of two smooth 2-dimensional germs meeting transversely;
- outside of s, the map π is the blow-up of S_0 in $\mathbf{P}^4$.

Let $\widehat{X} \to \widetilde{X}$ be the blow-up of $\Sigma^\star$, with exceptional divisor $\widehat{E}$, and let $\widehat{\mathbf{P}^4} \to \mathbf{P}^4$ be the blow-up of s, with exceptional divisor $\mathbf{P}^3_s$. The strict transform $\widehat{S}_0 \subset \widehat{\mathbf{P}^4}$ of S_0 is the blow-up of its (smooth) normalization S'_0 at the two points lying over s and meets $\mathbf{P}^3_s$ along the disjoint union of the two exceptional curves L_1 and L_2. There is an induced morphism $\widehat{X} \to \widehat{\mathbf{P}^4}$ which is an extremal contraction ([AM], Theorem 4.1.3), hence is the blow-up of the smooth surface $\widehat{S}_0$, with exceptional divisor the strict transform $\widehat{D} \subset \widehat{X}$ of D; it induces by restriction a morphism $\widehat{E} \to \mathbf{P}^3_s$ which is the blow-up of $L_1 \sqcup L_2$.

It follows that we have isomorphisms of polarized Hodge structures

$$\begin{aligned} H^4(\widehat{X}, \mathbf{Z}) &\simeq H^4(X, \mathbf{Z}) \oplus H^2(\Sigma, \mathbf{Z})(-1) \oplus H^2(\Sigma^\star, \mathbf{Z})(-1) \\ &\simeq H^4(\mathbf{P}^4, \mathbf{Z}) \oplus H^2(\mathbf{P}^3_s, \mathbf{Z})(-1) \oplus \mathbf{Z}[L_1] \oplus \mathbf{Z}[L_2] \oplus H^2(S'_0, \mathbf{Z})(-1). \end{aligned}$$

In particular, we have $b_2(S'_0) = 24 + 2 + 2 - 1 - 1 - 1 - 1 = 24$ and $h^{2,0}(S'_0) = h^{3,1}(\widehat{X}) = 1$; moreover, the Picard number of S'_0 is 3 for X general. The situation is as follows:



(12)

To compute the degree d of S_0, we consider the (smooth) inverse image $P \subset \widetilde{X}$ of a 2-plane in $\mathbf{P}^4$. It is isomorphic to the blow-up of $\mathbf{P}^2$ at d points, hence $K_P^2 = 9 - d$. On the other hand, we have by adjunction

$$K_P \underset{\text{lin}}{\equiv} (K_{\widetilde{X}} + 2(H - E))|_P \underset{\text{lin}}{\equiv} (-2H + E + 2(H - E))|_P = -E|_P,$$

hence $K_P^2 = E^2 \cdot (H - E)^2 = 1$ and $d = 8$.

Consider now a general hyperplane $h \subset \mathbf{P}^4$. Its intersection with S_0 is a smooth connected curve C of degree 8, and its inverse image in $\widetilde{X}$ is the blow-up of h along C, with exceptional divisor its intersection with D. From [IP], Lemma 2.2.14, we obtain

$$D^3 \cdot (H - E) = -2g(C) + 2 + K_h \cdot C = -2g(C) + 2 - 4\deg(C) = -2g(C) - 30,$$

from which we get $g(C) = 6$. In particular, $c_1(S'_0) \cdot h = 2$. On the other hand, using a variant of the formula for smooth surfaces in $\mathbf{P}^4$, we obtain

$$d^2 - 2 = 10d + c_1^2(S'_0) - c_2(S'_0) + 5c_1(S'_0) \cdot h,$$

hence $c_1^2(S'_0) - c_2(S'_0) = -28$. We then use a formula from [P], Lemma 2:

$$\begin{aligned}\widehat{D}^4 &= (c_2(\widehat{\mathbf{P}}^4) - c_1^2(\widehat{\mathbf{P}}^4)) \cdot \widehat{S}_0 + c_1(\widehat{\mathbf{P}}^4)|_{\widehat{S}_0} \cdot c_1(\widehat{S}_0) - c_2(\widehat{S}_0)\\ &= (-15h^2 - 7[\mathbf{P}^3_s]^2) \cdot \widehat{S}_0 + (-5h^2 + 3[\mathbf{P}^3_s])|_{\widehat{S}_0} \cdot c_1(\widehat{S}_0) - c_2(\widehat{S}_0)\\ &= (-15h^2 - 7[\mathbf{P}^3_s]^2) \cdot \widehat{S}_0 + (-5h^2 + 3[\mathbf{P}^3_s])|_{\widehat{S}_0} \cdot c_1(\widehat{S}_0) - c_2(\widehat{S}_0)\\ &= -120 + 14 - 10 - 6 - c_2(\widehat{S}_0).\end{aligned}$$

Since $\widehat{D}^4 = D^4 = (3H - 4E)^4 = -150$, we obtain $c_2(\widehat{S}_0) = 28$, hence $c_2(S'_0) = 26$ and $c_1^2(S'_0) = -2$. Noether's formula implies $\chi(S'_0, \mathscr{O}_{S'_0}) = 2$, hence $h^1(S'_0, \mathscr{O}_{S'_0}) = 0$. The classification of surfaces implies that S'_0 is the blow-up at two points of a K3 surface S of degree 10. By the simplicity argument used before, the integral polarized Hodge structures $H^2(S, \mathbf{Z})_0(-1)$ and $K^\perp$ are isomorphic: S is the (polarized) K3 surface associated with X via Proposition 6.6.

What happens if we start from the τ-quadric $\Sigma^\star$ instead of Σ? Blowing up Σ and then the strict transform of $\Sigma^\star$ is not the same as doing it in the reverse order, but the end products have a common open subset $\widetilde{X}^0$ (whose complements have codimension 2). The morphisms $\widetilde{X}^0 \to \widetilde{\mathbf{P}}^4 \to \mathbf{P}^3$ (where the second morphism is induced by projection from s) are then the same, because they are induced by the projection of X from the 4-plane $\langle \Sigma, \Sigma^\star \rangle$, and the locus where they are not smooth is the common projection S_1 in $\mathbf{P}^3$ of the surfaces $S_0 \subset \mathbf{P}^4$ and $S_0^\star \subset \mathbf{P}^4$ from their singular points.

This surface S_1 is also the projection of the K3 surface $S \subset \mathbf{P}^6$ from the 2-plane spanned by p, p', q, q'. The end result is therefore the same K3 surface S (as it should be, because its period is determined by that of X), but the pair of points is now q, q'. We let ι_S denote the birational involution on $S^{[2]}$ defined by $p + p' \mapsto q + q'$ (in [O], Proposition 5.20, O'Grady proves that for S general, the involution ι_S is biregular on the complement of a 2-plane).

Conversely, let $S = G(2, V_5) \cap Q' \cap \mathbf{P}^6$ be a general K3 surface of degree 10 and let p (corresponding to $V_2 \subset V_5$) and p' (corresponding to $V'_2 \subset V_5$)

be two general points on S. If $V_4 := V_2 \oplus V_2'$, the intersection $S \cap G(2, V_4)$ is a set of four points p, p', q, q' in the 2-plane $\mathbf{P}(\wedge^2 V_4) \cap \mathbf{P}^6$. Projecting S from the line pp' gives a nonnormal degree-8 surface $S_0 := S_{pp'} \subset \mathbf{P}^4$, where q and q' have been identified. Its normalization S_0' is the blow-up of S at p and p'. Now let $\widehat{\mathbf{P}}^4 \to \mathbf{P}^4$ be the blow-up of the singular point of S_0, and let $\widehat{X} \to \widehat{\mathbf{P}}^4$ be the blow-up of the strict transform of S_0 in $\widehat{\mathbf{P}}^4$. The strict transform in $\widehat{X}$ of the exceptional divisor $\mathbf{P}^3_s \subset \widehat{\mathbf{P}}^4$ can be blown down by $\widehat{X} \to \widetilde{X}$.

The resulting smooth fourfold $\widetilde{X}$ is a Fano variety with Picard number 2. One extremal contraction is $\pi\colon \widetilde{X} \to \mathbf{P}^4$. The other extremal contraction gives the desired X. This construction depends on 23 parameters (19 for the surface S and 4 for $p, p' \in S$).

All this implies (as in the proofs of Propositions 7.1 and 7.2) that the period map for $\mathscr{X}_{\rho\text{-plane}}$ is dominant onto $\mathscr{D}_{10}'$, with fiber birationally isomorphic to $S^{[2]}/\iota_S$. □

7.4 Fourfolds containing a cubic scroll (divisor $\mathscr{D}_{12}$)

We consider rational cubic scroll surfaces obtained as smooth hyperplane sections of the image of a morphism $\mathbf{P}(V_2) \times \mathbf{P}(V_3) \to G(2, V_5)$, where $V_5 = V_2 \oplus V_3$; their class in $G(2, V_5)$ is $\sigma_1^2 \cdot \sigma_2 = 2\sigma_{3,1} + \sigma_{2,2}$.

Proposition 7.4 *The closure $\overline{\mathscr{X}}_{\text{cubic scroll}} \subset \mathscr{X}_{10}$ of the family of fourfolds containing a cubic scroll surface is the irreducible component of $\wp^{-1}(\mathscr{D}_{12})$ that contains the family $\mathscr{X}_{\rho\text{-plane}}$.*

Proof Let us count parameters. We have $6 + 6 = 12$ parameters for the choice of V_2 and V_3, hence a priori 12 parameters for cubic scroll surfaces in the isotropic Grassmannian $G_\omega(2, V_5)$. However, one checks that there is a 1-dimensional family of V_3 which all give the same cubic scroll, so there are actually only 11 parameters. Then, for X to contain a given cubic scroll F represents $h^0(F, \mathscr{O}_F(2,2)) = 12$ conditions. It follows that $\mathscr{X}_{\text{cubic scroll}}$ is irreducible of codimension $12 - 11 = 1$ in $\mathscr{X}_{10}$.

Let $F \subset X$ be a cubic scroll. Since K_F has type $(-1, -2)$, we obtain $(F)^2_X = 4$ from (10). From (11), we obtain $d = 12$ and we are in $\mathscr{D}_{12}$. The family $\overline{\mathscr{X}}_{\text{cubic scroll}}$ is therefore a component of the hypersurface $\wp^{-1}(\mathscr{D}_{12})$.

In the degenerate situation where $V_4 = V_2 + V_3$ is a hyperplane, the associated rational cubic scroll is contained in $G(2, V_4)$ and is a cubic scroll surface as in the comment at the end of Section 7.2. It follows that $\mathscr{X}_{\rho\text{-plane}}$ is contained in $\overline{\mathscr{X}}_{\text{cubic scroll}}$. □

7.5 Fourfolds containing a quintic del Pezzo surface (divisor $\mathscr{D}''_{10}$)

We consider quintic del Pezzo surfaces obtained as the intersection of $G(2, V_5)$ with a $\mathbf{P}^5$; their class is $\sigma_1^4 = 3\sigma_{3,1} + 2\sigma_{2,2}$ in $G(2, V_5)$. Fourfolds of type $\mathscr{X}_{10}$ containing such a surface were already studied by Roth ([R], Section 4).

Proposition 7.5 *The closure $\overline{\mathscr{X}}_{\text{quintic}} \subset \mathscr{X}_{10}$ of the family of fourfolds containing a quintic del Pezzo surface is the irreducible component of $\wp^{-1}(\mathscr{D}''_{10})$ that contains $\mathscr{X}_{\sigma\text{-plane}}$.*

A general member of $\mathscr{X}_{\text{quintic}}$ is rational.

Proof Let us count parameters. We have $\dim G(5, \mathbf{P}^8) = 18$ parameters for the choice of the $\mathbf{P}^5$ that defines a del Pezzo surface T. Then, for X to contain a given quintic del Pezzo surface T represents $h^0(\mathbf{P}^5, \mathscr{O}(2)) - h^0(\mathbf{P}^5, \mathscr{I}_T(2)) = 21 - 5 = 16$ conditions.

Since $h^0(\mathbf{P}^8, \mathscr{I}_X(2)) = 6 = h^0(\mathbf{P}^5, \mathscr{I}_T(2)) + 1$, there exists a unique (non-Plücker) quadric $Q \subset \mathbf{P}^8$ containing X and $\mathbf{P}^5$. This quadric has rank ≤ 6, hence it is a cone with vertex a 2-plane over a (in general) smooth quadric in $\mathbf{P}^5$. Such a quadric contains two 3-dimensional families of 5-planes. The intersection of such a 5-plane with X is, in general, a quintic del Pezzo surface, hence X contains (two) 3-dimensional families of quintic del Pezzo surfaces. It follows that $\mathscr{X}_{\text{quintic}}$ has codimension $16 - 18 + 3 = 1$ in $\mathscr{X}_{10}$.

Let $T \subset X$ be a quintic del Pezzo surface. From (10), we obtain $(T)^2_X = 5$ and from (11), $d = 10$. Since $\sigma_{1,1} \cdot T$ is odd, we are in $\mathscr{D}''_{10}$. The family $\overline{\mathscr{X}}_{\text{quintic}}$ is therefore a component of the divisor $\wp^{-1}(\mathscr{D}''_{10})$.

The lattice spanned by $H^4(G(2, V_5), \mathbf{Z})$ and $[T]$ in $H^4(X, \mathbf{Z})$ is the same as for fourfolds containing a σ-plane P, and $[T] = \sigma_2|_X - [P]$. We will now explain this fact geometrically.

If X contains a quintic del Pezzo surface, we saw that X is contained in a (non-Plücker) quadric $Q \subset \mathbf{P}^8$ of rank ≤ 6. Conversely, if X is contained in such a quadric, this quadric contains 5-planes and the intersection of such a 5-plane with X is, in general, a quintic del Pezzo surface.

If follows that $\mathscr{X}_{\text{quintic}}$ has the same closure in $\mathscr{X}_{10}$ as the set of X contained in a non-Plücker rank-6 quadric Q. When the vertex of Q is contained in X, it is a σ-plane, hence $\overline{\mathscr{X}}_{\text{quintic}}$ contains $\mathscr{X}_{\sigma\text{-plane}}$.

Finally, note after [R], Section 5.(5), that the general fibers of the projection $X \dashrightarrow \mathbf{P}^2$ from $\langle T \rangle$ are again degree-5 del Pezzo surfaces (they are residual surfaces to T in the intersection of X with a 6-plane $\langle T, x \rangle$, and this intersection is contained in $\langle T, x \rangle \cap Q$, which is the union of two hyperplanes). It follows from a theorem of Enriques that X is rational ([E], [SB]). □

7.6 Nodal fourfolds (divisor $\mathscr{D}_8$)

Let X be a general prime *nodal* Fano fourfold of index 2 and degree 10. As in the 3-dimensional case ([DIM2], Lemma 4.1), X is the intersection of a smooth $G^\omega := G(2,V_5)\cap\mathbf{P}^8$ with a nodal quadric Q, singular at a point O general in G^ω.

One checks that, as in the case of cubic fourfolds (see [V], Section 4; [H1], Proposition 4.2.1), the limiting Hodge structure is pure, and the period map extends to the moduli stack $\overline{\mathscr{X}}_{10}$ of our fourfolds with at most one node as

$$\overline{\wp}\colon \overline{\mathscr{X}}_{10} \to \mathscr{D}.$$

Proposition 7.6 *The closure $\overline{\mathscr{X}}_{\text{nodal}} \subset \overline{\mathscr{X}}_{10}$ of the family of nodal fourfolds is an irreducible component of $\overline{\wp}^{-1}(\mathscr{D}_8)$.*

Proof Let X be a prime nodal Fano fourfold with a node at O, obtained as above. If $\widetilde{X}\to X$ is the blow-up of O, the (pure) limiting Hodge structure is the direct sum of $\langle\delta\rangle$, where δ is the vanishing cycle, with self-intersection 2, and $H^4(\widetilde{X},\mathbf{Z})$. In the basis $(\sigma_{1,1}|_X, \sigma_2|_X-\sigma_{1,1}|_X, \delta)$, the corresponding lattice K has intersection matrix $\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}$, hence we are in $\mathscr{D}_8$.

The point O defines a pencil of Plücker quadrics, singular at O, and the image G^ω_O of G^ω by the projection $p_O\colon \mathbf{P}^8 \dashrightarrow \mathbf{P}^7_O$ is the base-locus of a pencil of rank-6 quadrics (see [DIM2], Section 3). One checks that G^ω_O contains the 4-plane $\mathbf{P}^4_O := p_O(\mathbf{T}_{G^\omega,O})$ and that G^ω_O is singular along a cubic surface contained in $\mathbf{P}^4_O$. If $\widetilde{\mathbf{P}}^7_O \to \mathbf{P}^7_O$ is the blow-up of $\mathbf{P}^4_O$, the strict transform $\widetilde{G}^\omega_O \subset \widetilde{\mathbf{P}}^7_O \subset \mathbf{P}^7_O\times\mathbf{P}^2$ of G^ω_O is smooth and the projection $\widetilde{G}^\omega_O \to \mathbf{P}^2$ is a $\mathbf{P}^3$-bundle (this can be checked by explicit computations as in [DIM2], Section 9.2).

The image $X_O := p_O(X)$ is thus the base locus in $\mathbf{P}^7_O$ of a net of quadrics $\mathbf{P}$, containing a special line of rank-6 Plücker quadrics. The strict transform $\widetilde{X}_O \subset \widetilde{G}^\omega_O$ of X_O is smooth. The induced projection $\widetilde{X}_O \to \mathbf{P}^2$ is a quadric bundle, with discriminant a smooth sextic curve $\Gamma^\star_6 \subset \mathbf{P}^2$ (compare with [DIM2], Proposition 4.2) and associated double cover $S \to \mathbf{P}^2$ ramified along $\Gamma^\star_6$. It follows that S is a (smooth) K3 surface with a degree-2 polarization. By [L], Theorem II.3.1, there is an exact sequence

$$0 \longrightarrow H^4(\widetilde{X}_O,\mathbf{Z})_0 \xrightarrow{\Phi} H^2(S,\mathbf{Z})_0(-1) \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow 0.$$

Both desingularizations $\widetilde{X} \to X_O$ and $\widetilde{X}_O \to X_O$ are small and their fibers all have dimension ≤ 1; by [FW], Proposition 3.1, the graph of the rational map $\widetilde{X} \dashrightarrow \widetilde{X}_O$ induces an isomorphism $H^4(\widetilde{X}_O,\mathbf{Z}) \xrightarrow{\sim} H^4(\widetilde{X},\mathbf{Z})$ of polarized Hodge structures. The usual simplicity argument implies that this isomorphism sends

$H^4(\widetilde{X}_O, \mathbf{Z})_0$ onto the nonspecial cohomology $K^\perp$, which therefore has index 2 in $H^2(S, \mathbf{Z})_0(-1)$.

When X is general, so is S among degree-2 K3 surfaces, hence the image $\overline{\wp}(\mathscr{X}_{\text{nodal}})$ has dimension 19. It follows that $\mathscr{X}_{\text{nodal}}$ is an irreducible component of $\overline{\wp}^{-1}(\mathscr{D}_8)$. □

7.7 Summary of results

We summarize the results of this section in the table below. Please refer to the corresponding subsections for exact statements.

X contains a	Dimension of family	Image in period domain	Fiber of period map	General X birational to
σ-plane	22	$\mathscr{D}''_{10}$	$\mathbf{P}^1$-bundle over K3	$\mathbf{P}^4$
quintic del Pezzo	23	$\mathscr{D}''_{10}$	?	$\mathbf{P}^4$
τ-quadric	23	$\mathscr{D}'_{10}$	$K3^{[2]}$/inv.	$\mathbf{P}^4$
ρ-plane	21	$\mathscr{D}_{12}$	2 rational surfaces	cubic
cubic scroll	23	$\mathscr{D}_{12}$	?	
node	23	$\mathscr{D}_8$	?	int. of 3 quadrics

8 Construction of special fourfolds

Again following Hassett (particularly [H1], Section 4.3), we construct special fourfolds with given discriminant. Hassett's idea was to construct, using the surjectivity of the (extended) period map for K3 surfaces, nodal cubic fourfolds whose Picard group also contains a rank-2 lattice with discriminant d and to smooth them using the fact that the period map remains a submersion on the nodal locus ([V], p. 597). This method should work in our case, but would require first making the construction of Section 7.6 of a nodal fourfold X of type $\mathscr{X}_{10}$ from a given degree-2 K3 surface more explicit, and second proving that the extended period map remains submersive at any point of the nodal locus.

We prefer here to use the simpler construction of Section 7.1 to prove the following:

Theorem 8.1 *The image of the period map* $\wp\colon \mathscr{X}^0_{10} \to \mathscr{D}$ *meets all divisors* $\mathscr{D}_d$*, for* $d \equiv 0 \pmod 4$ *and* $d \geq 12$*, and all divisors* $\mathscr{D}'_d$ *and* $\mathscr{D}''_d$*, for* $d \equiv 2 \pmod 8$ *and* $d \geq 10$*, except possibly* $\mathscr{D}''_{18}$*.*

Actually, the divisor $\mathscr{D}''_{18}$ also meets the image of the period map: in a forthcoming article, we construct birational transformations that take elements of $\wp^{-1}(\mathscr{D}'_d)$ to elements of $\wp^{-1}(\mathscr{D}''_d)$.

Proof Our starting point is Lemma 4.3.3 of [H1]: let Γ be a rank-2 indefinite even lattice containing a primitive element h with $h^2 = 10$, and assume there is no $c \in \Gamma$ with

- either $c^2 = -2$ and $c \cdot h = 0$;
- or $c^2 = 0$ and $c \cdot h = 1$;
- or $c^2 = 0$ and $c \cdot h = 2$.

Then there exists a K3 surface S with $\mathrm{Pic}(S) = \Gamma$ and h is very ample on S, hence embeds it in $\mathbf{P}^6$. Assuming moreover that S is not trigonal, e.g., that there are no classes $c \in \Gamma$ with $c^2 = 0$ and $c \cdot h = 3$, it has Clifford index 2 and is therefore obtained as the intersection of a Fano threefold $Z := G(2, V_5) \cap \mathbf{P}^6$ with a quadric ([M3], (3.9); [JK], Theorem 10.3 and Proposition 10.5).

In particular, S is an intersection of quadrics, and since a general point p of S is not on a line contained in S, the projection from p of S is a (degree-9) smooth surface $\widetilde{S}_p \subset \mathbf{P}^5$.

On the other hand, if $\Pi \subset \mathbf{P}(\wedge^2 V_5^\vee)$ is the 2-plane of hyperplanes that cut out $\mathbf{P}^6$ in $\mathbf{P}(\wedge^2 V_5)$, one has ([PV], Corollary 1.6)

$$\mathrm{Sing}(Z) = \Pi^\perp \cap \bigcup_{[\omega]\in\Pi} G(2, \mathrm{Ker}(\omega)).$$

Since S is smooth, $\mathrm{Sing}(Z)$ is finite. If $\mathrm{Sing}(Z) \neq \varnothing$, some $[\omega_0] \in \Pi$ must have rank 2 and one checks that there exists $V_2 \subset \mathrm{Ker}(\omega_0)$ such that Z contains a family of lines through $[V_2]$, parameterized by a rational cubic curve. The intersection of the cone swept out by these lines and the quadric that defines S in Z is a sextic curve of genus 2 in S. Its class c' thus satisfies $c'^2 = 2$ and $c' \cdot h = 6$, hence $(h - c')^2 = 0$ and $(h - c') \cdot h = 4$.

So if we assume finally that there are no classes $c \in \Gamma$ with $c^2 = 0$ and $c \cdot h = 4$, the threefold Z is smooth. It is then known ([PV], Theorem 7.5 and Proposition 7.6) that $\mathrm{Aut}(Z)$ is isomorphic to $\mathrm{PGL}(2, \mathbf{C})$ and acts on Z with three (rational) orbits of respective dimensions 3, 2, and 1. Since S is not rational, it must meet the open orbit. Take $p \in S$ in that orbit.

It is then classical ([PV], Section 7) that Z contains three lines passing through p, so that the projection $\widetilde{Z}_p \subset \mathbf{P}^5$ from p of Z has exactly three singular points, which are also on the smooth surface $\widetilde{S}_p$. One then checks on explicit equations of $\widetilde{Z}_p$ ([I], Section 3.1) that $\widetilde{Z}_p$ is contained in a smooth quadric $Y \subset \mathbf{P}^5$. Consider the blow-up $\widetilde{Y} \to Y$ of $\widetilde{S}_p$. The inverse image

$E \subset \widetilde{Y}$ of $\widetilde{Z}_p$ is then a small resolution, which is isomorphic to the blow-up of p in Z.

The morphism $E \to \widetilde{Z}_p$ is in particular independent of the choice of S, p, and Y and it follows from the description of the general case in the proof of Proposition 7.1 that E is a $\mathbf{P}^1$-bundle over $\mathbf{P}^2$. More precisely, the linear system $|H|$ on $\widetilde{Y}$ given by cubics containing $\widetilde{S}_p$ induces on E a morphism $E \to \mathbf{P}^2$ with $\mathbf{P}^1$-fibers (in the notation of that proof, E is the exceptional divisor of the blow-up $\widetilde{X} \to X$ of the plane P).

The linear system $|H|$ is base-point-free and injective outside of E (because $\widetilde{Z}_p$ is a quadratic section of Y which contains $\widetilde{S}_p$) and base-point-free on E as we just saw. Since $H^4 = 10$ and $h^0(\widetilde{Y}, H) = 9$, it defines a birational morphism $\widetilde{Y} \twoheadrightarrow X \subset \mathbf{P}^8$ which maps E onto a 2-plane $P \subset X$. This morphism is one of the two $K_{\widetilde{Y}}$-negative extremal contractions of $\widetilde{Y}$ (the other one being the blow-up $\widetilde{Y} \to Y$); its fibers all have dimension ≤ 1, hence X is *smooth* and the contraction is the blow-up of P ([AM], Theorem 4.1.3).

It is then easy to check that X is a (special) Fano fourfold of type $\mathscr{X}_{10}^0$ containing P as a σ-plane. As explained in the proof of Proposition 7.1, its nonspecial cohomology is isomorphic to the primitive cohomology of S.

We will now apply this construction with various lattices Γ to produce examples of smooth fourfolds X which will all be in $\mathscr{X}_{\sigma\text{-plane}}$, hence with period point in $\mathscr{D}''_{10}$, but whose lattice $H^{2,2}(X) \cap H^4(X, \mathbf{Z})$ will contain other sublattices of rank 3 with various discriminants.

Apply first Hassett's lemma with the rank-2 lattice Γ with matrix $\begin{pmatrix} 10 & 0 \\ 0 & -2e \end{pmatrix}$ in a basis (h, w). When $e > 1$, the conditions we need on Γ are satisfied and we obtain a K3 surface S and a smooth fourfold X containing a σ-plane P, such that $H^4(X, \mathbf{Z}) \cap H^{2,2}(X)$ contains a lattice $K_{10} = \langle u, v, w''_{10} \rangle$ with matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ and discriminant 10 (here $w''_{10} = [P]$; see proof of Proposition 7.1). Moreover, $H^2(S, \mathbf{Z})_0(-1) \simeq K_{10}^{\perp}$ as polarized integral Hodge structures. The element $w \in \Gamma \cap H^2(S, \mathbf{Z})_0$ corresponds to $w_X \in K_{10}^{\perp} \cap H^{2,2}(X)$, and $w_X^2 = -w^2 = 2e$. Therefore, $H^4(X, \mathbf{Z}) \cap H^{2,2}(X)$ is the lattice $\langle u, v, w''_{10}, w_X \rangle$, with matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 2e \end{pmatrix}.$$

It contains the lattice $\langle u, v, w_X\rangle$, with matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2e \end{pmatrix}$. Therefore, the period point of X belongs to $\mathscr{D}_{8e}$, and this proves the theorem when $d \equiv 0 \pmod 8$.

It also contains the lattice $\langle u, v, w''_{10} + w_X\rangle$, with matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2e+3 \end{pmatrix}$ and discriminant $8e + 10$, hence we are also in $\mathscr{D}''_{8e+10}$.

Now let $e \geq 0$ and apply Hassett's lemma with the lattice Γ with matrix $\begin{pmatrix} 10 & 5 \\ 5 & -2e \end{pmatrix}$ in a basis (h, g). The orthogonal complement of h is spanned by $w := h - 2g$. One checks that primitive classes $c \in \Gamma$ such that $c^2 = 0$ satisfy $c \cdot h \equiv 0 \pmod 5$. All the conditions we need are thus satisfied and we obtain a K3 surface S and a smooth fourfold X such that $H^4(X, \mathbf{Z}) \cap H^{2,2}(X)$ contains a lattice K_{10} of discriminant 10 and $H^2(S, \mathbf{Z})_0(-1) \simeq K_{10}^\perp$ as polarized Hodge structures. Again, w corresponds to $w_X \in K_{10}^\perp \cap H^{2,2}(X)$ with $w_X^2 = -w^2 = 8e + 10$. Set

$$K := (\Lambda_2 \oplus \mathbf{Z}w_X)^{\mathrm{sat}}.$$

To compute the discriminant of K, we need to know the ideal $w_X \cdot \Lambda$. As in the proof of Proposition 6.2, let w_{10} be a generator of $K_{10} \cap \Lambda$; it satisfies $w_{10}^2 = 10$. Then $K_{10}^\perp \oplus \mathbf{Z}w_{10}$ is a sublattice of Λ and, taking discriminants, we find that the index is 5. Let u be an element of Λ whose class generates the quotient. We have

$$w_X \cdot \Lambda = \mathbf{Z}w_X \cdot u + w_X \cdot (K_{10}^\perp \oplus \mathbf{Z}w_{10}) = \mathbf{Z}w_X \cdot u + w_X \cdot K_{10}^\perp = \mathbf{Z}w_X \cdot u + w \cdot H^2(S, \mathbf{Z})_0.$$

One checks directly on the K3 lattice that $w \cdot H^2(S, \mathbf{Z})_0 = 2\mathbf{Z}$. Since $5u \in K_{10}^\perp \oplus \mathbf{Z}w_{10}$, we have $5w_X \cdot u \in 2\mathbf{Z}$, hence $w_X \cdot u \in 2\mathbf{Z}$. All in all, we have proved $w_X \cdot \Lambda = 2\mathbf{Z}$, hence the proof of Proposition 6.2 implies that the discriminant of K is $w_X^2 = 8e + 10$. Therefore, the period point of X belongs to $\mathscr{D}_{8e+10}$.

Since the period point of X is in $\mathscr{D}''_{10}$, we saw in the proof of Proposition 6.2 that $w''_{10} := \frac{1}{2}(v + w_{10})$ is in $H^4(X, \mathbf{Z})$. Similarly, either $w'_X := \frac{1}{2}(u + w_X)$ or $w''_X := \frac{1}{2}(v + w_X)$ is in K. Taking intersections with w''_{10} (and recalling $w_X \cdot w_{10} = 0$ and $v \cdot w_{10} = 1$), we see that we are in the first case, hence the period point of X is actually in $\mathscr{D}'_{8e+10}$.

More precisely, $H^4(X, \mathbf{Z}) \cap H^{2,2}(X)$ is the lattice $\langle u, v, w''_{10}, w'_X\rangle$, with matrix

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & 0 & 2e+3 \end{pmatrix}.$$

This lattice also contains the lattice $\langle u, v, w''_{10}+w'_X\rangle$, with matrix $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2e+6 \end{pmatrix}$ and discriminant $8e + 20$, hence we are also in $\mathscr{D}_{8e+20}$.

Since we know from Section 7.2 that the period points of some smooth fourfolds X of type $\mathscr{X}^0_{10}$ lie in $\mathscr{D}_{12}$, this proves the theorem when $d \equiv 4 \pmod 8$. □

9 A question

It would be very interesting, as Laza did for cubic fourfolds ([La], Theorem 1.1), to determine the exact image in the period domain $\mathscr{D}$ of the period map for our fourfolds.

Question 9.1 Is the image of the period map equal to $\mathscr{D} - \mathscr{D}_2 - \mathscr{D}_4 - \mathscr{D}_8$?

Answering this question seems far from our present possibilities; to start with, inspired by the results of [H1], one could ask (see Theorem 8.1) whether the image of the period map is disjoint from the hypersurfaces $\mathscr{D}_2$, $\mathscr{D}_4$, and $\mathscr{D}_8$.

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