

Partial positivity: Geometry and cohomology of q -ample line bundles

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Abstract

We give an overview of partial positivity conditions for line bundles, mostly from a cohomological point of view. Although the current work is to a large extent expository in nature, we present some minor improvements on the existing literature and a new result: a Kodaira-type vanishing theorem for effective q -ample Du Bois divisors and log canonical pairs.

To Rob Lazarsfeld on the occasion of his 60th birthday

1 Introduction

Ampleness is one of the central notions of algebraic geometry, possessing the extremely useful feature that it has geometric, numerical, and cohomological characterizations. Here we will concentrate on its cohomological side. The fundamental result in this direction is the theorem of Cartan–Serre–Grothendieck (see [Laz04, Theorem 1.2.6]): for a complete projective scheme X , and a line bundle \mathcal{L} on X , the following are equivalent to \mathcal{L} being ample:

1. There exists a positive integer $m_0 = m_0(X, \mathcal{L})$ such that $\mathcal{L}^{\otimes m}$ is very ample for all $m \geq m_0$.
2. For every coherent sheaf \mathcal{F} on X , there exists a positive integer $m_1 = m_1(X, \mathcal{F}, \mathcal{L})$ for which $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is globally generated for all $m \geq m_1$.
3. For every coherent sheaf \mathcal{F} on X , there exists a positive integer $m_2 = m_2(X, \mathcal{F}, \mathcal{L})$ such that

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$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = \{0\}$$

for all $i \geq 1$ and all $m \geq m_2$.

We will focus on the direction pointed out by Serre’s vanishing theorem, part 3 above, and concentrate on line bundles with vanishing cohomology above a certain degree.

Historically, the first result in this direction is due to Andreotti and Grauert [AG62]. They prove that given a compact complex manifold X of dimension n , and a holomorphic line bundle \mathcal{L} on X equipped with a Hermitian metric whose curvature is a $(1, 1)$ -form with at least $n - q$ positive eigenvalues at every point of X , then for every coherent sheaf \mathcal{F} on X , there exists a natural number $m_0(\mathcal{L}, \mathcal{F})$ such that

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = \{0\} \quad \text{for all } m \geq m_0(\mathcal{L}, \mathcal{F}) \text{ and for all } i > q. \quad (1)$$

In [DPS96], Demailly, Peternell, and Schneider posed the question of under what circumstances the converse does hold. That is, they asked: assume that for every coherent sheaf \mathcal{F} there exists $m = m_0(\mathcal{L}, \mathcal{F})$ such that the vanishing (1) holds. Does \mathcal{L} admit a Hermitian metric with the expected number of positive eigenvalues? In dimension two, Demailly [Dem11] proved an asymptotic version of this converse to the Andreotti–Grauert theorem using tools related to asymptotic cohomology; subsequently, Matsumura [Mat13] gave a positive answer to the question for surfaces. However, there exist higher-dimensional counterexamples to the converse Andreotti–Grauert problem in the range $\frac{\dim X}{2} - 1 < q < \dim X - 2$, constructed by Ottem [Ott12].

We study line bundles with the property of the conclusion of the Andreotti–Grauert theorem; let X be a complete scheme of dimension n , $0 \leq q \leq n$ an integer. A line bundle \mathcal{L} is called *naively q -ample*, or simply *q -ample* if for every coherent sheaf \mathcal{F} on X there exists an integer $m_0 = m_0(\mathcal{L}, \mathcal{F})$ for which

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = \{0\}$$

for $m \geq m_0$ and for all $i > q$.

It is a consequence of the Cartan–Grothendieck–Serre result discussed above that 0-ampleness reduces to the usual notion of ampleness. In [DPS96], the authors studied naive q -ampleness along with various other notions of partial cohomological positivity. Part of their approach is to look at positivity of restrictions to elements of a complete flag, and they use it to give a partial vanishing theorem similar to that of Andreotti–Grauert.

In a beautiful paper [Tot13], Totaro proves that the competing partial positivity concepts are in fact all equivalent in characteristic 0, thus laying down the foundations for a very satisfactory theory. His result goes as follows: let X

be a projective scheme of dimension n , \mathcal{A} a sufficiently ample line bundle on X , $0 \leq q \leq n$ a natural number. Then for all line bundles \mathcal{L} on X , the following are equivalent:

1. \mathcal{L} is naively q -ample.
2. \mathcal{L} is uniformly q -ample, that is, there exists a constant $\lambda > 0$ such that

$$H^i(X, \mathcal{L}^{\otimes m} \otimes \mathcal{A}^{\otimes -j}) = \{0\}$$

for all $i > q$, $j > 0$, and $m \geq \lambda j$.

3. There exists a positive integer $m_1 = m_1(\mathcal{L}, \mathcal{A})$ such that

$$H^{q+i}(X, \mathcal{L}^{\otimes m_1} \otimes \mathcal{A}^{\otimes -(n+i)}) = \{0\}$$

for all $1 \leq i \leq n - q$.

As an outcome, Totaro can prove that on the one hand, for a given q , the set of q -ample line bundles forms an open cone in the Néron–Severi space, and on the other hand, q -ampleness is an open property in families as well.

In his recent article [Ott12], Ottem works out other basic properties of q -ample divisors and employs them to study subvarieties of higher codimension. We will give an overview of his results in Section 2.4.

Interestingly enough, prior to [DPS96], Sommese [Som78] defined a geometric version of partial ampleness by studying the dimensions of the fibers of the morphism associated with a given line bundle. In the case of semi-ample line bundles, where Sommese’s notion is defined, he proves his condition to be equivalent to naive q -ampleness. Although more limited in scope, Sommese’s geometric notion extends naturally to vector bundles as well.

One of the major technical vanishing theorems for ample divisors that does not follow directly from the definition is Kodaira’s vanishing theorem if X is a smooth projective variety, defined over an algebraically closed field of characteristic 0, and \mathcal{L} is an ample line bundle on X , then

$$H^j(X, \omega_X \otimes \mathcal{L}) = \{0\} \quad \text{for all } j > 0.$$

We refer the reader to [Kod53] for the original analytic proof, to [DI87] for a subsequent algebraic proof, to [Ray78] for a counterexample in characteristic p , and to [Kol95, Chapter 9] as well as to [EV92] for a general discussion of vanishing theorems. It is a natural question to ask whether an analogous vanishing holds for q -ample divisors. While one of the ingredients of classical proofs of Kodaira vanishing, namely Lefschetz’ hyperplane section theorem, has been generalized to the q -ample setup by Ottem [Ott12, Corollary 5.2], at the same time he gives a counterexample to the Kodaira vanishing theorem for q -ample divisors, which we recall in Section 2.4. In Ottem’s example, the

chosen q -ample divisor is not pseudo-effective. In Section 3 we show that there is a good reason for this: we prove that q -Kodaira vanishing holds for reduced effective divisors which are not too singular; more precisely, we prove two versions of q -Kodaira vanishing which are related to log canonicity and the Du Bois condition.

Theorem (= Theorem 3.4) *Let X be a normal proper variety, $D \subset X$ a reduced effective (Cartier) divisor such that*

1. $\mathcal{O}_X(D)$ is q -ample;
2. $X \setminus D$ is smooth, and D (with its reduced subscheme structure) is Du Bois.

Then, we have

$$H^j(X, \omega_X \otimes \mathcal{O}_X(D)) = \{0\} \quad \text{for all } j > q$$

as well as

$$H^j(X, \mathcal{O}_X(-D)) = \{0\} \quad \text{for all } j < \dim X - q.$$

Theorem (= Theorem 3.8) *Let X be a proper Cohen–Macaulay variety, \mathcal{L} a q -ample line bundle on X , and D_i different irreducible Weil divisors on X . Assume that $\mathcal{L}^m \cong \mathcal{O}_X(\sum d_j D_j)$ for some integers $1 \leq d_j < m$. Set $m_j := m/\gcd(m, d_j)$. Assume furthermore that the pair $(X, \sum(1 - \frac{1}{m_j})D_j)$ is log canonical. Then, we have*

$$H^j(X, \omega_X \otimes \mathcal{L}) = \{0\} \quad \text{for all } j > q$$

as well as

$$H^j(X, \mathcal{L}^{-1}) = \{0\} \quad \text{for all } j < \dim X - q.$$

Global conventions If not mentioned otherwise, we work over the complex numbers, and all divisors are assumed to be Cartier.

2 Overview of the theory of q -ample line bundles

2.1 Vanishing of cohomology groups and partial ampleness

Starting with the pioneering work [DPS96] of Demailly, Peternell, and Schneider related to the Andreotti–Grauert problem, there has been a certain interest in studying line bundles with vanishing cohomology above a given degree. Just as big line bundles are a generalization of ample ones along its geometric side, these so-called q -ample bundles focus on a weakening of the cohomological

characterization of ampleness. In general, there exist competing definitions of various flavors, which were shown to be equivalent in characteristic 0 in [Tot13].

Definition 2.1 (Definitions of partial ampleness) Let X be a complete scheme of dimension n over an algebraically closed field of arbitrary characteristic, \mathcal{L} an invertible sheaf on X , q a natural number.

1. The invertible sheaf \mathcal{L} is called *naively q -ample* if for every coherent sheaf \mathcal{F} on X there exists a natural number $m_0 = m_0(\mathcal{L}, \mathcal{F})$ having the property that

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = \{0\} \quad \text{for all } i > q \text{ and } m \geq m_0.$$

2. Fix a very ample invertible sheaf \mathcal{A} on X . We call \mathcal{L} *uniformly q -ample* if there exists a constant $\lambda = \lambda(\mathcal{A}, \mathcal{L})$ such that

$$H^i(X, \mathcal{L}^{\otimes m} \otimes \mathcal{A}^{\otimes -j}) = \{0\} \quad \text{for all } i > q, j > 0, \text{ and } m \geq \lambda \cdot j.$$

3. Fix a Koszul-ample invertible sheaf \mathcal{A} on X . We say that \mathcal{L} is *q - T -ample* if there exists a positive integer $m_1 = m_1(\mathcal{A}, \mathcal{L})$ satisfying

$$\begin{aligned} H^{q+1}(X, \mathcal{L}^{\otimes m_1} \otimes \mathcal{A}^{\otimes -(n+1)}) &= H^{q+2}(X, \mathcal{L}^{\otimes m_1} \otimes \mathcal{A}^{\otimes -(n+2)}) \\ &\dots = H^n(X, \mathcal{L}^{\otimes m_1} \otimes \mathcal{A}^{\otimes -2n+q}) = \{0\}. \end{aligned}$$

An integral Cartier divisor is called *naively q -ample/uniformly q -ample/ q - T -ample* if the invertible sheaf $\mathcal{O}_X(D)$ has the appropriate property.

Remark 2.2 (Koszul-ampleness) We recall that a connected locally finite¹ graded ring $R_\bullet = \bigoplus_{i=0}^\infty R_i$ is called N -Koszul for a positive integer N if the field $k = R_0$ has a resolution

$$\dots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow k \longrightarrow 0$$

as a graded R_\bullet -module, where for all $i \geq N$ the module M_i is free and generated in degree i .

In turn, a very ample line bundle \mathcal{A} on a projective scheme X (taken to be connected and reduced to arrange that the ring of its regular functions $k \stackrel{\text{def}}{=} \mathcal{O}_X(X)$ is a field) is called N -Koszul if the section ring $R(X, \mathcal{A})$ is N -Koszul. The line bundle \mathcal{A} is said to be *Koszul-ample* if it is N -Koszul with $N = 2 \dim X$. It is important to point out that Castelnuovo–Mumford regularity with respect to a Koszul-ample line bundle has favorable properties.

¹ In this context locally finite means that $\dim R_i < \infty$ for all graded pieces.

If \mathcal{A} is an arbitrary ample line bundle on X , then Backelin [Bac86] showed that there exists a positive integer $k_0 \in \mathbb{N}$ having the property that $\mathcal{A}^{\otimes k}$ is Koszul-ample for all $k \geq k_0$.

Remark 2.3 The idea of naive q -ampleness is the immediate extension of the Grothendieck–Cartan–Serre vanishing criterion for ampleness. Uniform q -ampleness first appeared in [DPS96]; the term q - T -ampleness was coined by Totaro in [Tot13, Section 7], extending the idea of Castelnuovo–Mumford regularity.

A line bundle is ample if and only if it is naively 0-ample, while all line bundles are $n = \dim X$ -ample.

Example 2.4 One source of examples of q -ample divisors comes from ample vector bundles: according to [Ott12, Proposition 4.5], if \mathcal{E} is an ample vector bundle of rank $r \leq \dim X$ on a scheme X , $s \in H^0(X, \mathcal{E})$, then $Y \stackrel{\text{def}}{=} Z(s) \subseteq X$ is an ample subvariety. By definition, this means that $\mathcal{O}_{X'}(E)$ is $(r - 1)$ -ample, where $\pi: X' \rightarrow X$ is the blow-up of X along Y with exceptional divisor E , cf. Definition 2.48.

Remark 2.5 A straightforward sufficient condition for (naive) q -ampleness can be obtained by studying restrictions of \mathcal{L} to general complete intersection subvarieties. More specifically, the following claim is shown in [Kür10, Theorem A]: let X be a projective variety over the complex numbers, L a Cartier divisor, A_1, \dots, A_q very ample Cartier divisors on X such that $L|_{E_1 \cap \dots \cap E_q}$ is ample for general $E_j \in |A_j|$. Then, for any coherent sheaf \mathcal{F} on X there exists an integer $m(L, A_1, \dots, A_q, \mathcal{F})$ such that

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mL + N + \sum_{j=1}^q k_j A_j)) = \{0\}$$

for all $i > q$, $m \geq m(L, A_1, \dots, A_q, \mathcal{F})$, $k_j \geq 0$, and all nef divisors N . In particular, the conditions of the above claim ensure that L is q -ample.

Remark 2.6 There are various implications among the three definitions over an arbitrary field. As was verified in [DPS96, Proposition 1.2] via an argument resolving coherent sheaves by direct sums of ample line bundles,² a uniformly q -ample line bundle is necessarily naively q -ample. On the other hand, naive q -ampleness implies q - T -ampleness by definition.

² Note that the resulting resolution is not guaranteed to be finite; see [Laz04, Example 1.2.21] for a discussion of this possibility.

Remark 2.7 The idea behind the definition of q - T -ampleness is to reduce the question of q -ampleness to the vanishing of finitely many cohomology groups. It is not known whether the three definitions coincide in positive characteristic.

The following result of Totaro compares the three different approaches to q -ampleness in characteristic 0.

Theorem 2.8 (Totaro; [Tot13, Theorem 8.1]) *Over a field of characteristic 0, the three definitions of partial ampleness are equivalent.*

The proof uses methods from positive characteristics to generalize a vanishing result of Arapura [Ara06, Theorem 5.4], at the same time it relies on earlier work of Orlov and Kawamata on resolutions of the diagonal via Koszul-ample line bundles.

Remark 2.9 (Resolution of the diagonal) One of the main building blocks of [Tot13] is an explicit resolution of the diagonal as a sheaf on $X \times X$ depending on an ample line bundle \mathcal{A} on X . This is used as a tool to prove an important result on the regularity of tensor products of sheaves (see Theorem 2.10 below, itself an improvement over a statement of Arapura [Ara06, Corollary 1.12]), which in turn is instrumental in showing that q - T -ampleness implies uniform q -ampleness, the nontrivial part of Totaro’s theorem on the equivalence of the various definitions of partial vanishing.

The resolution in question – which exists over an arbitrary field – had first been constructed by Orlov [Orl97, Proposition A.1] under the assumption that \mathcal{A} is sufficiently ample, and subsequently improved by Kawamata [Kaw04] by making the more precise assumption that the coordinate ring $R(X, \mathcal{A})$ is a Koszul algebra.

Totaro reproves Kawamata’s result under the weaker hypothesis that for \mathcal{A} , the section ring $R(X, \mathcal{A})$ is N -Koszul for some positive integer N (for the most of [Tot13] one will set $N = 2 \dim X$).

To construct the Kawamata–Orlov–Totaro resolution of the diagonal, we will proceed as follows. First, we define a sequence of k -vector spaces B_i by setting

$$B_i \stackrel{\text{def}}{=} \begin{cases} k & \text{if } i = 0, \\ H^0(X, \mathcal{A}) & \text{if } i = 1, \\ \ker(B_{i-1} \otimes H^0(X, \mathcal{A}) \rightarrow B_{i-2} \otimes H^0(X, \mathcal{A}^2)) & \text{if } i \geq 2. \end{cases}$$

Note that \mathcal{A} is N -Koszul precisely if the following sequence of graded $R(X, \mathcal{A})$ -modules cooked up from the B_i s is exact:

$$B_N \otimes R(X, \mathcal{A})(-N) \rightarrow \dots \rightarrow B_1 \otimes R(X, \mathcal{A})(-1) \rightarrow R(X, \mathcal{A}) \rightarrow k \rightarrow 0.$$

Next, set

$$\mathcal{R}_i \stackrel{\text{def}}{=} \begin{cases} \mathcal{O}_X & \text{if } i = 0, \\ \ker(B_i \otimes \mathcal{O}_X \rightarrow B_{i-1} \otimes \mathcal{A}) & \text{if } i > 0. \end{cases}$$

Totaro’s claim [Tot13, Theorem 2.1] goes as follows: if \mathcal{A} is an N -Koszul line bundle on X , then there exists an exact sequence

$$\mathcal{R}_{N-1} \boxtimes \mathcal{A}^{-N+1} \rightarrow \dots \rightarrow \mathcal{R}_1 \boxtimes \mathcal{A}^{-1} \rightarrow \mathcal{R}_0 \boxtimes \mathcal{O}_X \rightarrow \mathcal{O}_\Delta \rightarrow 0,$$

where \boxtimes denotes the external tensor product on $X \times X$, and $\Delta \subset X \times X$ stands for the diagonal.

The above construction leads to a statement of independent interest about the regularity of tensor products of sheaves, which had already appeared in some form in [Ara06].

Theorem 2.10 (Totaro; [Tot13, Theorem 3.4]) *Let X be a connected and reduced projective scheme of dimension n , \mathcal{A} a $2n$ -Koszul line bundle, \mathcal{E} a vector bundle, \mathcal{F} a coherent sheaf on X . Then,*

$$\text{reg}(\mathcal{E} \otimes \mathcal{F}) \leq \text{reg}(\mathcal{E}) + \text{reg}(\mathcal{F}).$$

Remark 2.11 In the case of $X = \mathbb{P}^n_{\mathbb{C}}$ the above theorem is a simple application of Koszul complexes (see [Laz04, Proposition 1.8.9], for instance).

Remark 2.12 (Positive characteristic methods) Another crucial point in the proof of the equivalence of the various definitions of q -ampleness is a vanishing result in positive characteristic originating in the work of Arapura in the smooth case [Ara06, Theorem 5.3] that was extended to possibly singular schemes by Totaro [Tot13, Theorem 5.1] exploiting a flatness property of the Frobenius over arbitrary schemes over fields of prime cardinality.

The statement is essentially as follows: let X be a connected and reduced projective scheme of dimension n over a field of positive characteristic p , \mathcal{A} a Koszul-ample line bundle on X , q a natural number. Let \mathcal{L} be a line bundle on X satisfying

$$H^{q+1}(X, \mathcal{L} \otimes \mathcal{A}^{\otimes(-n-1)}) = H^{q+2}(X, \mathcal{L} \otimes \mathcal{A}^{\otimes(-n-2)}) = \dots = \{0\}.$$

Then, for any coherent sheaf \mathcal{F} on X one has

$$H^i(X, \mathcal{L}^{\otimes p^m} \otimes \mathcal{F}) = \{0\} \text{ for all } i > q \text{ and } p^m \geq \text{reg}_{\mathcal{A}}(\mathcal{F}).$$

2.2 Basic properties of q -ampleness

From now on we return to our blanket assumption and work over the complex number field; we will immediately see that the equivalence of the various definitions brings all sorts of perks.

First, we point out that q -ampleness enjoys many formal properties analogous to ampleness. The following statements have been part of the folklore; for precise proofs we refer the reader to [Ott12, Proposition 2.3] and [DPS96, Lemma 1.5].

Lemma 2.13 *Let X be a projective scheme, \mathcal{L} a line bundle on X , q a natural number. Then, the following hold:*

1. \mathcal{L} is q -ample if and only if $\mathcal{L}|_{X_{\text{red}}}$ is q -ample on X_{red} .
2. \mathcal{L} is q -ample precisely if $\mathcal{L}|_{X_i}$ is q -ample on X_i for every irreducible component X_i of X .
3. For a finite morphism $f: Y \rightarrow X$, if \mathcal{L} on X is q -ample then so is $f^*\mathcal{L}$. Conversely, if f is surjective as well, then the q -ampleness of $f^*\mathcal{L}$ implies the q -ampleness of \mathcal{L} .

The respective proofs of the ample case go through with minimal modifications. Another feature surviving in an unchanged form is the fact that to check (naive) q -ampleness we can restrict our attention to line bundles.

Lemma 2.14 *Let X be a projective scheme, \mathcal{L} a line bundle, \mathcal{A} an arbitrary ample line bundle on X . Then, \mathcal{L} is q -ample precisely if there exists a natural number $m_0 = m_0(\mathcal{A}, \mathcal{L})$ having the property that*

$$H^i(X, \mathcal{L}^{\otimes m} \otimes \mathcal{A}^{\otimes -k}) = \{0\}$$

for all $i > q$, $k \geq 0$, and $m \geq m_0k$.

Proof Follows immediately by decreasing induction on q from the fact that every coherent sheaf \mathcal{F} on X has a possibly infinite resolution by finite direct sums of non-positive powers of the ample line bundle \mathcal{A} ([Laz04, Example 1.2.21]). □

Ample line bundles are good to work with for many reasons, but the fact that they are open both in families and in the Néron–Severi space contributes considerably. As it turns out, the same properties are valid for q -ample line bundles as well.

Theorem 2.15 (Totaro; [Tot13, Theorem 9.1]) *Let $\pi: X \rightarrow B$ be a flat projective morphism of schemes (over \mathbb{Z}) with connected fibers, \mathcal{L} a line bundle*

on X , q a natural number. Then, the subset of points b of B having the property that $\mathcal{L}|_{X_b}$ is q -ample is Zariski open.

Sketch of proof This is one point where q - T -ampleness plays a role, since in that formulation one only needs to check vanishing for a finite number of cohomology groups.

Assume that $\mathcal{L}|_{X_b}$ is q - T -ample for a given point $b \in B$; let U be an affine open neighborhood on $b \in B$, and \mathcal{A} a line bundle on $\pi^{-1}(U) \subseteq X$, whose restriction to X_b is Koszul-ample. Since Koszul-ampleness is a Zariski-open property, $\mathcal{A}|_{X_{b'}}$ is again Koszul-ample for an open subset of points $b' \in U$; without loss of generality we can assume that this holds on the whole of U .

We will use the line bundle $\mathcal{A}|_{X_{b'}}$ to check q - T -ampleness of $\mathcal{L}_{X_{b'}}$ in an open neighborhood on $b \in U$. By the q - T -ampleness of $\mathcal{L}|_{X_b}$ there exists a positive integer m_0 satisfying

$$H^{q+1}(X_b, \mathcal{L}^{\otimes m_0} \otimes \mathcal{A}^{\otimes -n-1}) = \dots = H^n(X_b, \mathcal{L}^{\otimes m_0} \otimes \mathcal{A}^{\otimes -2n+q}) = \{0\}.$$

It follows from the semicontinuity theorem that the same vanishing holds true for points in an open neighborhood of b . □

In a different direction, Demailly, Peternell, and Schneider proved that uniform q -ampleness is open in the Néron–Severi space. To make this precise we need the fact that uniform q -ampleness is a numerical property; once this is behind us, we can define q -ampleness for numerical equivalence classes of \mathbb{R} -divisors.

Remark 2.16 Note that a line bundle \mathcal{L} is q -ample if and only if $\mathcal{L}^{\otimes m}$ is q -ample for some positive integer m . Therefore it makes sense to talk about q -ampleness of \mathbb{Q} -Cartier divisors; a \mathbb{Q} -divisor D is said to be q -ample if it has a multiple mD that is integral and $\mathcal{O}_X(mD)$ is q -ample.

Theorem 2.17 (q -ampleness is a numerical property) *Let D and D' be numerically equivalent integral Cartier divisors on an irreducible complex projective variety X , q a natural number. Then,*

$$D \text{ is } q\text{-ample} \Leftrightarrow D' \text{ is } q\text{-ample}.$$

Demailly, Peternell, and Schneider [DPS96, Proposition 1.4] only prove this claim for smooth projective varieties. The proof in [DPS96] cites the completeness of $\text{Pic}^0(X)$, hence it is far from obvious how to extend it. Here we give a proof that is valid under the more general given hypothesis. Instead of dealing with uniform q -ampleness, we use the naive formulation.

Proof Let \mathcal{N}' be a numerically trivial line bundle, \mathcal{L} a q -ample line bundle on X . This means that for a given coherent sheaf \mathcal{F} , we have

$$H^i(X, \mathcal{L}^{\otimes m} \otimes \mathcal{F}) = \{0\} \text{ for } i > q \text{ and } m_0 = m_0(\mathcal{L}, \mathcal{F}).$$

We need to prove that

$$H^i(X, (\mathcal{L} \otimes \mathcal{N}')^{\otimes m} \otimes \mathcal{F}) = \{0\}$$

holds for all $i > q$, and for suitable $m \geq m_1(\mathcal{L}, \mathcal{N}', \mathcal{F})$. To this end, we will study the function

$$f_i^{(m)} : \mathcal{N} \mapsto h^i(X, \mathcal{L}^{\otimes m} \otimes \mathcal{N} \otimes \mathcal{F})$$

as a function on the closed points of the subscheme \mathcal{X} of the Picard scheme that parameterizes numerically trivial line bundles on X , which is a scheme of finite type by the boundedness of numerically trivial line bundles [Laz04, Theorem 1.4.37]. We know that

$$f_i^{(m)}(\mathcal{O}_X) = h^i(X, \mathcal{L}^{\otimes m} \otimes \mathcal{F}) = \{0\}$$

for $i > q$, and $m \geq m_0$. By the semicontinuity theorem, $f_i^{(m)}$ attains the same value on a dense open subset of \mathcal{X} .

By applying Noetherian induction and semicontinuity on the irreducible components of the complement (on each of which we apply the q -ampleness of \mathcal{L} for coherent sheaves of the shape $\mathcal{F} \otimes \mathcal{N}$, \mathcal{N} numerically trivial) we will eventually find a value $m'_0 = m'_0(\mathcal{L}, \mathcal{F})$ such that

$$f_i^{(m)} \equiv 0$$

for all $i > q$ and $m \geq m'_0$.

But this implies that

$$H^i(X, (\mathcal{L} \otimes \mathcal{N}')^{\otimes m} \otimes \mathcal{F}) = H^i(X, \mathcal{L}^{\otimes m} \otimes ((\mathcal{N}')^{\otimes m}) \otimes \mathcal{F}) = \{0\}$$

for $m \geq m'_0$, since the required vanishing holds for an arbitrary numerically trivial divisor in place of $(\mathcal{N}')^{\otimes m}$. □

Remark 2.18 As a result, we are in a position to extend the definition of q -ampleness elements of $N^1(X)_{\mathbb{Q}}$: if α is a numerical equivalence class of \mathbb{Q} -divisors, then we will call it q -ample if one (equivalently all) of its representatives are q -ample.

Remark 2.19 Since Fujita's vanishing theorem also holds over algebraically closed fields of positive characteristic (see [Fuj83] or [Laz04, Remark 1.4.36]), the boundedness of numerically trivial line bundles holds again in this case by

[Laz04, Proposition 1.4.37] and we can conclude that naive q -ampleness is invariant with respect to numerical equivalence in that situation as well.

Definition 2.20 (q -ampleness for \mathbb{R} -divisors) An \mathbb{R} -divisor D on a complex projective variety is q -ample if

$$D = D' + A,$$

where D' is a q -ample \mathbb{Q} -divisor, A an ample \mathbb{R} -divisor.

The result that q -ample \mathbb{R} -divisors form an open cone in $N^1(X)_{\mathbb{R}}$ was proved in [DPS96]. Here we face the same issue as with Theorem 2.17: in the article [DPS96] only the smooth case is considered, and the proof given there does not seem to generalize to general varieties. Here we present a proof of the general case.

Definition 2.21 Given $\alpha \in N^1(X)_{\mathbb{R}}$, we set

$$q(\alpha) \stackrel{\text{def}}{=} \min \{q \in \mathbb{N} \mid \alpha \text{ is } q\text{-ample}\} .$$

Theorem 2.22 (Upper-semicontinuity of q -ampleness) *Let X be an irreducible projective variety over the complex numbers. Then, the function*

$$q : N^1(X)_{\mathbb{R}} \longrightarrow \mathbb{N}$$

is upper-semicontinuous. In particular, for a given $q \in \mathbb{N}$, the set of q -ample classes forms an open cone.

In order to be able to prove this result, we need some auxiliary statements. To this end, Demailly, Peternell, and Schneider introduce the concept of height of coherent sheaves with respect to a given ample divisor. Roughly speaking, the height of a coherent sheaf tells us, what multiples of the given ample divisor we need to obtain a linear resolution.

Definition 2.23 (Height) Let X be an irreducible projective variety, \mathcal{F} a coherent sheaf, \mathcal{A} an ample line bundle. Consider the set \mathcal{R} of all resolutions

$$\dots \rightarrow \bigoplus_{1 \leq l \leq m_k} \mathcal{A}^{\otimes -d_{k,l}} \rightarrow \dots \rightarrow \bigoplus_{1 \leq l \leq m_0} \mathcal{A}^{\otimes -d_{0,l}} \rightarrow \mathcal{F} \rightarrow 0$$

of \mathcal{F} by non-positive powers of \mathcal{A} (that is, $d_{k,l} \geq 0$). Then,

$$\text{ht}_{\mathcal{A}}(\mathcal{F}) \stackrel{\text{def}}{=} \min_{\mathcal{R}} \max_{0 \leq k \leq \dim X, 1 \leq l \leq m_k} d_{k,l} .$$

Remark 2.24 One could define the height by looking at resolutions without truncating, that is, by

$$\widetilde{\text{ht}}_{\mathcal{A}}(\mathcal{F}) \stackrel{\text{def}}{=} \min_{\mathcal{R}} \max_{0 \leq k, 1 \leq l \leq m_k} d_{k,l} .$$

On a general projective variety there might be sheaves that do not possess finite locally free resolutions at all, and it can happen that the height of a sheaf is infinite if we do not truncate resolutions.

A result of Arapura [Ara04, Corollary 3.2] gives effective estimates on the height of a coherent sheaf in terms of its Castelnuovo–Mumford regularity.

Lemma 2.25 *Let X be an irreducible projective variety, \mathcal{A} an ample and globally generated line bundle, \mathcal{F} a coherent sheaf on X . Given a natural number k , there exist vector spaces V_i for $1 \leq i \leq k$ and a resolution*

$$V_k \otimes \mathcal{A}^{\otimes -r_{\mathcal{F}} - kr_X} \rightarrow \dots \rightarrow V_1 \otimes \mathcal{A}^{\otimes -r_{\mathcal{F}} - r_X} \rightarrow V_0 \otimes \mathcal{A}^{\otimes -r_{\mathcal{F}}} \rightarrow \mathcal{F} \rightarrow 0,$$

where

$$r_{\mathcal{F}} \stackrel{\text{def}}{=} \text{reg}_{\mathcal{A}}(\mathcal{F}) \quad \text{and} \quad r_X \stackrel{\text{def}}{=} \max\{1, \text{reg}_{\mathcal{A}}(\mathcal{O}_X)\}.$$

Proof Without loss of generality we can assume that \mathcal{F} is 0-regular by replacing \mathcal{F} by $\mathcal{F} \otimes \mathcal{A}^{r_{\mathcal{F}}}$, thus we can assume $r_{\mathcal{F}} = 0$. Consequently, \mathcal{F} is globally generated by Mumford’s theorem; set $V_0 \stackrel{\text{def}}{=} H^0(X, \mathcal{F})$, and

$$\mathcal{K}_0 \stackrel{\text{def}}{=} \mathcal{F} \otimes \mathcal{A}^{\otimes -r_X}, \quad \mathcal{K}_1 \stackrel{\text{def}}{=} \ker(V_0 \otimes \mathcal{O}_X \rightarrow \mathcal{K}_0 \otimes \mathcal{A}^{\otimes r_X}).$$

A quick cohomology computation [Ara06, Lemma 3.1] shows that \mathcal{K}_1 is r_X -regular, hence we can repeat the above process for \mathcal{K}_1 in place of \mathcal{F} . This leads to a sequence of vector spaces V_i , and sheaves \mathcal{K}_i which fit into the exact sequences

$$0 \rightarrow \mathcal{K}_{i+1} \rightarrow V_i \otimes \mathcal{O}_X \rightarrow \mathcal{K}_i \otimes \mathcal{A}^{\otimes r_X} \rightarrow 0$$

or, equivalently,

$$0 \rightarrow \mathcal{K}_{i+1} \otimes \mathcal{A}^{\otimes -ir_X} \rightarrow V_i \otimes \mathcal{A}^{\otimes -ir_X} \rightarrow \mathcal{K}_i \otimes \mathcal{A}^{\otimes (1-i)r_X} \rightarrow 0.$$

We obtain the statement of the lemma by combining these sequences into the required resolution. □

Corollary 2.26 *With notation as above, the height of an $r_{\mathcal{F}}$ -regular coherent sheaf \mathcal{F} is*

$$\text{ht}_{\mathcal{A}}(\mathcal{F}) \leq r_{\mathcal{F}} + r_X \cdot \dim X.$$

Proposition 2.27 (Properties of height) *Let X be an irreducible projective variety of dimension n , \mathcal{A} an ample line bundle. Then, the following hold:*

1. *For coherent sheaves \mathcal{F}_1 and \mathcal{F}_2 we have*

$$\text{ht}_{\mathcal{A}}(\mathcal{F}_1 \otimes \mathcal{F}_2) \leq \text{ht}_{\mathcal{A}}(\mathcal{F}_1) + \text{ht}_{\mathcal{A}}(\mathcal{F}_2).$$

2. There exists a positive constant $M = M(X, \mathcal{A})$ having the property that

$$\text{ht}_{\mathcal{A}}(\mathcal{N}) \leq M$$

for all numerically trivial line bundles \mathcal{N} on X .

Proof The first statement is an immediate consequence of the fact that the tensor product of appropriate resolutions of \mathcal{F}_1 and \mathcal{F}_2 is a resolution of $\mathcal{F}_1 \otimes \mathcal{F}_2$ of the required type.

The second claim is a consequence of the fact that numerically trivial divisors on a projective variety are parameterized by a quasi-projective variety. Indeed, it follows by Lemma 2.28 and the Noetherian property of the Zariski topology that there exists a constant M' satisfying

$$\text{reg}_{\mathcal{A}}(\mathcal{N}) \leq M'$$

for all numerically trivial line bundles \mathcal{N} . By the Corollary of Lemma 2.25,

$$\text{ht}_{\mathcal{A}}(\mathcal{N}) \leq M \stackrel{\text{def}}{=} M' + r_X \cdot \dim X,$$

as required. □

Lemma 2.28 (Upper-semicontinuity of Castelnuovo–Mumford regularity) *Let X be an irreducible projective variety, \mathcal{A} an ample and globally generated line bundle on X . Given a flat family of line bundles \mathcal{L} on X parameterized by a quasi-projective variety T , the function*

$$T \ni t \mapsto \text{reg}_{\mathcal{A}}(\mathcal{L}_t)$$

is upper-semicontinuous.

Proof Since Castelnuovo–Mumford regularity is checked by the vanishing of finitely many cohomology groups of line bundles, the statement follows from the semicontinuity theorem for cohomology. □

Example 2.29 (Height and regularity on projective spaces) Here we discuss the relationship between height and regularity with respect to $\mathcal{O}(1)$ on an n -dimensional projective space \mathbb{P} . We claim that

$$\text{ht}(\mathcal{F}) = \text{reg}(\mathcal{F}) + n$$

holds for an arbitrary coherent sheaf \mathcal{F} on \mathbb{P} .

For the inequality

$$\text{ht}(\mathcal{F}) \leq \text{reg}(\mathcal{F}) + n$$

observe that by [Laz04, Proposition 1.8.8] there must exist a long exact sequence

$$\dots \rightarrow \bigoplus \mathcal{O}_{\mathbb{P}}(-\text{reg}(\mathcal{F}) - 1) \rightarrow \bigoplus \mathcal{O}_{\mathbb{P}}(-\text{reg}(\mathcal{F})) \rightarrow \mathcal{F} \rightarrow 0,$$

hence we are done by the definition of height.

To see the reverse inequality, let

$$\dots \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F} \rightarrow 0$$

be a resolution of \mathcal{F} with

$$\mathcal{F}_k = \bigoplus_{i=1}^k \mathcal{O}_{\mathbb{P}}(-d_{k,i}).$$

Since $d_{k,i} \leq \text{ht}(\mathcal{F})$ by definition, we obtain that \mathcal{F}_k is $\text{ht}(\mathcal{F}) - k$ regular for all $n \geq k \geq 0$, therefore

$$\text{reg}(\mathcal{F}) \leq \text{ht}(\mathcal{F}) - n$$

according to [Laz04, Example 1.8.7].

Remark 2.30 (Height and Serre vanishing with estimates) The introduction of the height of a coherent sheaf leads to an effective version of Serre’s vanishing theorem. With the notation of the Introduction, we have

$$m_0(\mathcal{A}, \mathcal{F}) \leq m_0(\mathcal{A}, \mathcal{A}) + \text{ht}_{\mathcal{A}}(\mathcal{F}).$$

Lemma 2.31 (Demailly–Peternell–Schneider; [DPS96, Proposition 1.2]) *Let \mathcal{L} be a uniformly q -ample line bundle on X with respect to an ample line bundle \mathcal{A} for a given constant $\lambda = \lambda(\mathcal{A}, \mathcal{L})$. Given a coherent sheaf \mathcal{F} on X ,*

$$H^i(X, \mathcal{L}^{\otimes m} \otimes \mathcal{F}) = \{0\}$$

for all $i > q$ and $m \geq \lambda \cdot (\text{ht}_{\mathcal{A}}(\mathcal{F}) + 1)$.

Proof Let

$$\dots \rightarrow \bigoplus_{1 \leq l \leq m_k} \mathcal{A}^{\otimes -d_{k,l}} \rightarrow \dots \rightarrow \bigoplus_{1 \leq l \leq m_0} \mathcal{A}^{\otimes -d_{0,l}} \rightarrow \mathcal{F} \rightarrow 0$$

be a resolution where the value of $\text{ht}_{\mathcal{A}}(\mathcal{F})$ is attained, and write \mathcal{F}_k for the image sheaf of the k th differential in the above sequence (note that $\mathcal{F}_0 = \mathcal{F}$). Chopping up the resolution of \mathcal{F} into short exact sequences yields

$$0 \longrightarrow \mathcal{F}_{k+1} \longrightarrow \bigoplus_{1 \leq l \leq m_k} \mathcal{A}^{\otimes -d_{k,l}} \longrightarrow \mathcal{F}_k \longrightarrow 0$$

for all $0 \leq k \leq \dim X$. By the uniform q -ampleness assumption on \mathcal{L} we obtain that

$$H^i(X, \mathcal{L}^{\otimes m} \otimes \mathcal{A}^{\otimes -d_{k,l}}) = \{0\} \quad \text{for all } i > q \text{ and } m \geq \lambda(d_{k,l} + 1).$$

By induction on k we arrive at

$$H^i(X, \mathcal{L}^{\otimes m} \otimes \mathcal{F}) \simeq \dots \simeq H^{i+k}(X, \mathcal{L}^{\otimes m} \otimes \mathcal{F}_k) \simeq H^{i+k+1}(X, \mathcal{L}^{\otimes m} \otimes \mathcal{F}_{k+1})$$

for all $i > q$ and $m \geq \lambda(\text{ht}_{\mathcal{A}}(\mathcal{F}) + 1)$. The statement of the lemma follows by taking $k = \dim X$. □

The idea for the following modification of the proof of [DPS96, Proposition 1.4] was suggested to us by Burt Totaro.

Proof of Theorem 2.22 Fix an integral ample divisor A , as well as integral Cartier divisors B_1, \dots, B_ρ whose numerical equivalence classes form a basis of the rational Néron–Severi space. Let D be an integral uniformly q -ample divisor (for a constant $\lambda = \lambda(D, A)$), D' a \mathbb{Q} -Cartier divisor, and write

$$D' \equiv D + \sum_{i=1}^{\rho} \lambda_i B_i$$

for rational numbers λ_i . Let k be a positive integer clearing all denominators, then

$$kD' = kD + \sum_{i=1}^{\rho} k\lambda_i B_i + N$$

for a numerically trivial (integral) divisor N . We want to show that

$$H^i(X, mkD' - pA) = \{0\}$$

whenever $m \geq \lambda(D', A) \cdot p$ for a suitable positive constant λ . By Lemma 2.31 applied with

$$\mathcal{L} = \mathcal{O}_X(D), \mathcal{A} = \mathcal{O}_X(A), \text{ and } \mathcal{F} = \mathcal{O}_X\left(\sum_{i=1}^{\rho} mk\lambda_i B_i + mN - pA\right),$$

this will happen whenever

$$m \geq \lambda(D, A) \cdot \text{ht}_A\left(\sum_{i=1}^{\rho} mk\lambda_i B_i + mN - pA\right).$$

Observe that

$$\begin{aligned} \text{ht}_A\left(\sum_{i=1}^{\rho} mk\lambda_i B_i + mN - pA\right) &= \sum_{i=1}^{\rho} \text{ht}_A(mk\lambda_i B_i) + \text{ht}_A(mN) + \text{ht}_A(-pA) \\ &\leq \sum_{i=1}^{\rho} mk|\lambda_i| \cdot \max\{\text{ht}_A(B_i), \text{ht}_A(-B_i)\} + M + p, \end{aligned}$$

where M is the constant from Proposition 2.27; note that $\text{ht}_A(-pA) = p$ for $p \geq 0$. Therefore, if the λ_i s are close enough to zero so that

$$\lambda \cdot \sum_{i=1}^{\rho} k|\lambda_i| \cdot \max\{\text{ht}_A(B_i), \text{ht}_A(-B_i)\} < \frac{1}{2},$$

then it suffices to require

$$m \geq 2\lambda(M + p),$$

and D' will be q -ample. This shows the upper-semicontinuity of uniform q -ampleness. \square

Remark 2.32 If D_1 is a q -ample and D' is an r -ample divisor, then their sum $D + D'$ can only be guaranteed to be $q + r$ -ample; this bound is sharp, as shown in [Tot13, Section 8]. As a consequence, the cone of q -ample \mathbb{R} -divisor classes is not necessarily convex. We denote this cone by $\text{Amp}^q(X)$.

Interestingly enough, if we restrict our attention to semi-ample divisors, then Sommese proves in [Som78] (see also Corollary 2.45 below) that the sum of q -ample divisors retains this property.

It is an interesting question how to characterize the cone of q -ample divisors for a given integer q . If $q = 0$, then the Cartan–Serre–Grothendieck theorem implies that $\text{Amp}^0(X)$ equals the ample cone.

Totaro describes the $(n - 1)$ -ample cone with the help of duality theory.

Theorem 2.33 (Totaro; [Tot13, Theorem 10.1]) *For an irreducible projective variety X we have*

$$\text{Amp}^{n-1}(X) = N^1(X)_{\mathbb{R}} \setminus \overline{(-\text{Eff}(X))}.$$

Corollary 2.34 (1-ampleness on surfaces) *If X is a surface, then a divisor D on X is 1-ample if and only if $(D \cdot A) > 0$ for some ample divisor A on X .*

Remark 2.35 The cone of q -ample divisors on a \mathbb{Q} -factorial projective toric variety has been shown to be polyhedral (more precisely, to be the interior of the union of finitely many rational polyhedral cones) by Broomhead and

Predergast-Smith [BPS12, Theorem 3.3]. Nevertheless, an explicit combinatorial description in terms of the fan of the underlying toric variety along the lines of [HKP06] is not yet known.

Totaro links partial positivity to the vanishing of higher asymptotic cohomology. Generalizing the main result of [dFKL07] (see also [Kür06] for terminology), he asks the following question:

Question 2.36 (Totaro) Let D be an \mathbb{R} -divisor class on a complex projective variety, $0 \leq q \leq n$ an integer. Assume that $\widehat{h}^i(X, D') = 0$ for all $i > q$ and all $D' \in N^1(X)_{\mathbb{R}}$ in a neighborhood of D . Is it true that D is q -ample?

Remark 2.37 Broomhead and Predergast-Smith [BPS12, Theorem 5.1] answered Totaro’s question positively for toric varieties.

It is expected the q -ampleness should have more significance in the case of big line bundles. A first move in this direction comes from the following Fujita-type vanishing statement (see [Fuj83] or [Laz04, Theorem 1.4.35] for Fujita’s original statement).

Theorem 2.38 ([Kür10], Theorem C) *Let X be a complex projective scheme, L a big Cartier divisor, \mathcal{F} a coherent sheaf on X . Then there exists a positive integer $m_0(L, \mathcal{F})$ such that*

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mL + D)) = \{0\}$$

for all $i > \dim \mathbf{B}_+(L)$, $m \geq m_0(L, \mathcal{F})$, and all nef divisors D on X .

Remark 2.39 (Augmented base loci on schemes) The augmented base locus of a \mathbb{Q} -Cartier divisor L is defined in [ELM⁺06] via

$$\mathbf{B}_+(L) \stackrel{\text{def}}{=} \bigcap_A \mathbf{B}(L - A),$$

where A runs through all ample \mathbb{Q} -Cartier divisors. As opposed to the stable base locus of a divisor, the augmented base locus is invariant with respect to numerical equivalence of divisors. The augmented base locus of a \mathbb{Q} -divisor L is empty precisely if L is ample.

Although it is customary to define the stable base locus and the augmented base locus of a divisor in the setting of projective varieties, as pointed out in [Kür10, Section 3], these notions make perfect sense on more general schemes.

For an invertible sheaf \mathcal{L} on an arbitrary scheme X , let us denote by $\mathcal{F}_{\mathcal{L}}$ the quasi-coherent subsheaf of \mathcal{L} generated by $H^0(X, \mathcal{L})$. Then we can set

$$\mathfrak{b}(\mathcal{L}) \stackrel{\text{def}}{=} \text{ann}_{\mathcal{O}_X}(\mathcal{L}/\mathcal{F}_{\mathcal{L}}),$$

and define $\text{Bs}(\mathcal{L})$ to be the closed subscheme corresponding to $\mathfrak{b}(\mathcal{L})$. Furthermore, we define

$$\mathbf{B}(\mathcal{L}) \stackrel{\text{def}}{=} \bigcap_{m=1}^{\infty} \text{Bs}(\mathcal{L}^{\otimes m})_{\text{red}} \subseteq X$$

as a closed subset of the topological space associated with X . All basic properties of the stable base locus are retained (see again [Kür10, Section 3]), in particular, if X is complete and algebraic over \mathbb{C} (by which we mean separated, and of finite type over \mathbb{C}), then we recover the original definition of stable base loci.

Assuming X to be projective and algebraic over \mathbb{C} , we define the augmented base locus of a \mathbb{Q} -Cartier divisor L via

$$\mathbf{B}_+(L) \stackrel{\text{def}}{=} \bigcap_A \mathbf{B}(L - A),$$

where A runs through all ample \mathbb{Q} -Cartier divisors. Again, basic properties are preserved, and in the case of projective varieties we recover the original definition.

An interesting further step in this direction is provided by Brown’s work, where he connects q -ampleness of a big line bundle to its behavior when restricted to its augmented base locus.

Theorem 2.40 (Brown; [Bro12, Theorem 1.1]) *Let \mathcal{L} be a big line bundle on a complex projective scheme X , denote by $\mathbf{B}_+(\mathcal{L})$ the augmented base locus of \mathcal{L} . For a given integer $0 \leq q \leq n$, \mathcal{L} is q -ample if and only if $\mathcal{L}|_{\mathbf{B}_+(\mathcal{L})}$ is q -ample.*

We give a very rough outline of the proof of Brown’s result. First, if \mathcal{L} is q -ample on X and $Y \subseteq X$ denotes $\mathbf{B}_+(\mathcal{L})$ with the reduced induced scheme structure, then the projection formula and the preservation of cohomology groups under push-forward by closed immersions imply that $\mathcal{L}|_{\mathbf{B}_+(\mathcal{L})}$ is q -ample as well.

The other implication comes from the following useful observation from [Bro12], a restriction theorem for line bundles that are not q -ample [Bro12, Theorem 2.1]: let \mathcal{L} be a line bundle on a reduced projective scheme X , which is not q -ample, and let \mathcal{L}' be a line bundle on X with a nonzero section s having the property that $\mathcal{L}^{\otimes a} \otimes \mathcal{L}'^{\otimes -b}$ is ample for some positive integers a, b . Then, $\mathcal{L}|_{Z(s)}$ is not q -ample.

2.3 Sommesse’s geometric q -ampleness

In this section, we will discuss Sommesse’s geometric notion of q -ampleness, and relate it to the more cohomologically oriented discussion in the previous sections.

Definition 2.41 (Sommese; [Som78, Definition 1.3]) Let X be a projective variety, \mathcal{L} a line bundle on X . We say that \mathcal{L} is *geometrically q -ample* for a natural number q if

1. \mathcal{L} is semi-ample, i.e., $\mathcal{L}^{\otimes m}$ is globally generated for some natural number $m \geq 1$;
2. the maximal fiber dimension of $\phi_{|\mathcal{L}^{\otimes m}|}$ is at most q .

More generally, Sommesse defines a vector bundle \mathcal{E} over X to be geometrically q -ample if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is geometrically q -ample, and goes on to prove many interesting results for vector bundles (see [Som78, Proposition 1.7 or 1.12], for instance). In this paper we will only treat the line bundle case.

Remark 2.42 (Iitaka fibration) We briefly recall the semi-ample or Iitaka fibration associated with a semi-ample line bundle \mathcal{L} on a normal projective variety X [Laz04, Theorem 2.1.27]: there exists an algebraic fiber space (a surjective projective morphism with connected fibers) $\phi: X \rightarrow Y$ with the property that for any sufficiently large and divisible $m \in \mathbb{N}$ one has

$$\phi_{|\mathcal{L}^{\otimes m}|} = \phi \text{ and } Y_m = Y,$$

where Y_m denotes the image of X under $\phi_{|\mathcal{L}^{\otimes m}|}$.

In addition there exists an ample line bundle \mathcal{A} on X such that

$$\phi^* \mathcal{A} = \mathcal{L}^{\otimes k}$$

for a suitable positive integer k .

Remark 2.43 If X is a normal variety then Sommesse’s conditions are equivalent to requiring that \mathcal{L} is semi-ample and its semi-ample fibration has fiber dimension at most q . In the case when X is not normal, it is a priori not clear if the set of integers q for which \mathcal{L} is q -ample depends on the choice of m ; a posteriori this follows from Sommesse’s theorem. Nevertheless, for this reason the definition via the Iitaka fibration is cleaner in the case of normal varieties.

We summarize Sommesse’s results in this direction.

Theorem 2.44 (Sommese; [Som78, Proposition 1.7]) *Let X be a projective variety, \mathcal{L} a semi-ample line bundle over X with Iitaka fibration $\phi_{\mathcal{L}}$. For a natural number q , the following are equivalent:*

- (i) The line bundle \mathcal{L} is geometrically q -ample.
- (ii) The maximal dimension of an irreducible subvariety $Z \subseteq X$ with the property that $\mathcal{L}|_Z$ is trivial is at most q .
- (iii) If $\psi: Z \rightarrow X$ is a morphism from a projective variety Z such that $\phi^*\mathcal{L}$ is trivial, then $\dim Z \leq q$.
- (iv) The line bundle \mathcal{L} is naively q -ample, that is, for every coherent sheaf \mathcal{F} on X there exists a natural number $m_0 = m_0(\mathcal{L}, \mathcal{F})$ with the property that

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0 \text{ for all } i > q \text{ and } m \geq m_0.$$

Proof All equivalences are treated in [Som78]. Here we describe the equivalence between (i) and (iv), that is, we prove that for semi-ample line bundles geometric and cohomological q -ampleness agree.

Let $m_0 \geq 1$ be an integer for which $\mathcal{L}^{\otimes m_0}$ is globally generated, and let $\phi: X \rightarrow Y \subseteq \mathbb{P}$ denote the associated morphism. Then there exists an ample line bundle \mathcal{A} on Y and a positive integer k such that $\mathcal{L}^{\otimes k} = \phi^*\mathcal{A}$. Fix a coherent sheaf \mathcal{F} on X , and consider the Leray spectral sequence

$$H^p(Y, R^r\phi_*(\mathcal{F} \otimes \mathcal{L}^{\otimes m})) \implies H^{p+r}(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}).$$

Let us write $m = sk + t$ with $0 \leq t < k$ and $s \geq 0$ integers. For the cohomology groups on the LHS

$$\begin{aligned} H^p(Y, R^r\phi_*(\mathcal{F} \otimes \mathcal{L}^{\otimes m})) &= H^p(Y, R^r\phi_*(\mathcal{F} \otimes (\mathcal{L}^{\otimes k})^{\otimes s} \otimes \mathcal{L}^{\otimes t})) \\ &= H^p(Y, R^r\phi_*(\mathcal{F} \otimes \mathcal{L}^{\otimes t}) \otimes \mathcal{A}^{\otimes s}) \end{aligned}$$

by the projection formula. Serre’s vanishing theorem yields

$$H^p(Y, R^r\phi_*(\mathcal{F} \otimes \mathcal{L}^{\otimes t}) \otimes \mathcal{A}^{\otimes s}) = 0 \text{ for all } p \geq 1, s \gg 0 \text{ and all } 0 \leq t < k,$$

hence

$$H^0(Y, R^r\phi_*(\mathcal{F} \otimes \mathcal{L}^{\otimes m})) \simeq H^r(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m})$$

for $m \gg 0$. On the other hand, if the maximal fiber dimension of ϕ is q , then $R^r\phi_*(\mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$ for all $r > q$ and therefore

$$H^r(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0 \text{ for all } m \gg 0 \text{ and } r > q.$$

Consequently, \mathcal{L} is naively q -ample as claimed.

For the other implication assume that \mathcal{L} is not geometrically q -ample, hence has a fiber $F \subseteq X$ of dimension $f > q$. Starting from here one constructs a coherent sheaf \mathcal{F} on F having the property that $R^f\phi_*\mathcal{F}$ is a skyscraper sheaf, which, by the Leray spectral sequence above, would imply that

$$H^f(X, \mathcal{F} \otimes \mathcal{L}^{\otimes ks}) \neq 0 \text{ for all } s \geq 1.$$

Without loss of generality we can assume that F is irreducible, otherwise we replace it by one of its top-dimensional irreducible components. Let $\pi: \widetilde{F} \rightarrow F$ denote a resolution of singularities of F , and consider the Grauert–Riemenschneider canonical sheaf

$$\mathcal{F} \stackrel{\text{def}}{=} \mathcal{K}_{\widetilde{F}/F} \stackrel{\text{def}}{=} \pi_* \omega_{\widetilde{F}}.$$

By Grauert–Riemenschneider [GR70]

$$R^f(\phi|_F)_* \mathcal{F} \simeq H^f(F, \mathcal{F}) \neq 0$$

as $\phi|_F$ maps F to a point. □

Corollary 2.45 *Let D_1 and D_2 be geometrically q -ample divisors on a smooth variety. Then so is $D_1 + D_2$.*

Proof This is [Som78, Corollary 1.10.2]. □

Additionally, Kodaira–Akizuki–Nakano vanishing continues to hold for the expected range of cohomology groups and degrees of differential forms.

Theorem 2.46 (Kodaira–Akizuki–Nakano for geometrically q -ample bundles) *Let \mathcal{L} be a geometrically q -ample line bundle on a smooth projective variety X . Then,*

$$H^i(X, \wedge^j \Omega_X \otimes \mathcal{L}) = \{0\} \text{ for all } i + j > \dim X + q.$$

In particular, the following q -Kodaira vanishing holds:

$$H^i(X, \omega_X \otimes \mathcal{L}) = \{0\} \text{ for all } i > q.$$

Proof This is proven in [Som78, Proposition 1.12]. □

Remark 2.47 Sommesse’s version of the Kodaira–Akizuki–Nakano vanishing was later shown by Esnault and Viehweg to hold for an even larger range of values for i and j , see [EV89] and [EV92, Corollary 6.6].

In Example 2.54 and Section 3 below we discuss the question of whether Kodaira vanishing still continues to hold when one drops the semi-ampleness condition, i.e., for general q -ample line bundles.

2.4 Ample subschemes and a Lefschetz hyperplane theorem for q -ample divisors

Building upon the theory of q -ample line bundles and Hartshorne’s classical work [Har70], Ottem [Ott12] defines the notion of an ample subvariety (or

subscheme) and goes on to verify that ample subvarieties share many of the significant algebro-geometric and topological properties of their codimension-1 counterparts. One of the highlights of his work is a Lefschetz-type hyperplane theorem, which we will use for a proof of Kodaira’s vanishing theorem for effective q -ample Du Bois divisors in Section 3 below. Here we briefly recall the theory obtained in [Ott12].

Definition 2.48 (Ample subschemes) Let X be a projective scheme, $Y \subseteq X$ a closed subscheme of codimension r , $\pi: \widetilde{X} \rightarrow X$ the blow-up of Y with exceptional divisor E . Then, Y is called *ample* if E is $(r - 1)$ -ample on \widetilde{X} .

The idea behind this notion is classical: it has been known for a long time that positivity properties of Y can often be read off from the geometry of the complement $X - Y \simeq \widetilde{X} - E$. In spite of this, the concept has not been defined until recently.

Example 2.49 As can be expected, linear subspaces of projective spaces are ample.

Remark 2.50 (Cohomological dimension of the complement of an ample subscheme) An important geometric feature of ample divisors is that their complement is affine. In terms of cohomology, this is equivalent to requiring

$$H^i(X, \mathcal{F}) = \{0\} \text{ for all } i > 0, \mathcal{F} \text{ coherent sheaf on } X.$$

If we denote as customary the cohomological dimension of a subset $Y \subseteq X$ by $\text{cd}(Y)$, then we can phrase Ottem’s generalization [Ott12, Proposition 5.1] to the q -ample case as follows: if $U \subseteq X$ is an open subset of a projective scheme X having the property that $X \setminus U$ is the support of a q -ample divisor, then

$$\text{cd}(U) \leq q.$$

The observation on cohomological dimensions of complements leads to the following Lefschetz-type statement:

Theorem 2.51 (Generalized Lefschetz hyperplane theorem, Ottem; [Ott12, Corollary 5.2]) *Let D be an effective q -ample divisor on a projective variety X with smooth complement. Then, the restriction morphism*

$$H^i(X, \mathbb{Q}) \longrightarrow H^i(D, \mathbb{Q}) \text{ is } \begin{cases} \text{an isomorphism for } 0 \leq i \leq n - q - 1, \\ \text{injective for } i = n - q - 1. \end{cases}$$

We give Ottem’s proof to show the principles at work.

Proof Via the long exact sequence for relative cohomology and Lefschetz duality, the statement reduces to the claim that

$$H^i(X \setminus D, \mathbb{C}) = \{0\} \text{ for } i > n + q.$$

This latter follows from the Frölicher spectral sequence

$$E_1^{st} = H^s(X \setminus D, \Omega_{X \setminus D}^t) \implies H^{s+t}(X \setminus D, \mathbb{C}),$$

as

$$H^s(X \setminus D, \Omega_{X \setminus D}^t) = \{0\} \text{ for all } s + t > n + q$$

by Remark 2.50. □

Corollary 2.52 (Lefschetz hyperplane theorem for ample subvarieties; [Ott12, Corollary 5.3]) *Let Y be an ample local complete intersection subscheme in a smooth complex projective variety X . Then, the restriction morphism*

$$H^i(X, \mathbb{Q}) \longrightarrow H^i(Y, \mathbb{Q}) \text{ is } \begin{cases} \text{an isomorphism} & \text{if } i < \dim Y, \\ \text{injective} & \text{if } i = \dim Y. \end{cases}$$

The following is a summary of properties of smooth ample subschemes:

Theorem 2.53 (Properties of smooth ample subschemes; [Ott12, Corollary 5.6 and Theorem 6.6]) *Let X be a smooth projective variety, $Y \subseteq X$ a non-singular ample subscheme of dimension $d \geq 1$. Then, the following hold:*

1. *The normal bundle $N_{Y|X}$ of Y is an ample vector bundle.*
2. *For every irreducible $(\dim X - d)$ -dimensional subvariety $Z \subseteq X$ one has $(Y \cdot Z) > 0$. In particular, Y meets every divisor.*
3. *The Lefschetz hyperplane theorem holds for rational cohomology on Y :*

$$H^i(X, \mathbb{Q}) \longrightarrow H^i(Y, \mathbb{Q}) \text{ is } \begin{cases} \text{an isomorphism} & \text{if } i < \dim Y, \\ \text{injective} & \text{if } i = \dim Y. \end{cases}$$

4. *Let \widehat{X} denote the completion of X with respect to Y . For any coherent sheaf \mathcal{F} on X one has*

$$H^i(X, \mathcal{F}) \longrightarrow H^i(\widehat{X}, \mathcal{F}) \text{ is } \begin{cases} \text{an isomorphism} & \text{if } i < \dim Y, \\ \text{injective} & \text{if } i = \dim Y. \end{cases}$$

5. *The inclusion $Y \hookrightarrow X$ induces a surjection*

$$\pi_1(Y) \twoheadrightarrow \pi_1(X)$$

on the level of fundamental groups.

Proof We will only discuss point 5, since it is the only statement that is slightly different from its original source. Because Y is smooth over a reduced base, it is automatically reduced and by smoothness again, it is irreducible exactly when it is connected. But point 3 in the case of $i = 0$ implies that Y is connected. The rest follows from Ottem’s proof. \square

Based on the fact that a Lefschetz-type theorem holds for q -ample divisors, as seen in Theorem 2.51, and that this forms one of the ingredients of the proof of the Kodaira vanishing theorem in the classical setup, cf. [Laz04, Section 4.2], as well as on the fact that q -Kodaira vanishing continues to hold for *geometrically* q -ample divisors, Theorem 2.46, one might hope that there is a q -Kodaira vanishing theorem for q -ample divisors. The following example shows that this is not true in general.

Example 2.54 (Counterexample to Kodaira vanishing for non-pseudo-effective q -ample divisors, Ottem; [Ott12, Section 9]) Let $G = SL_3(\mathbb{C})$, $B \leq G$ be the Borel subgroup consisting of upper triangular matrices, and consider the homogeneous space G/B . By the Bott–Borel–Weil theorem and a brief computation, Ottem shows the existence of a non-pseudo-effective line bundle \mathcal{L} on G/B , which is 1-ample, but for which the cohomology group $H^2(G/B, \omega_{G/B} \otimes \mathcal{L})$ does not vanish.

It turns out, however, that by putting geometric restrictions on the q -ample divisor in question, one can in fact prove Kodaira-style vanishing theorems. We will do this in the next section.

3 q -Kodaira vanishing for Du Bois divisors and log canonical pairs

This section contains the proofs of various versions of Kodaira’s vanishing theorem for q -ample divisors. First we present the argument in the smooth case, where the reasoning is particularly transparent and simple.

Theorem 3.1 *Let X be a smooth projective variety, D a smooth reduced effective q -ample divisor on X . Then,*

$$H^i(X, \mathcal{O}_X(K_X + D)) = \{0\} \text{ for all } i > q.$$

Proof By Serre duality, it suffices to show

$$H^i(X, \mathcal{O}_X(-D)) = \{0\} \text{ for all } i \leq n - q - 1.$$

To this end, follow the proof of Kodaira vanishing in [Laz04, Section 4.2] with minor modifications. Ottem’s generalized Lefschetz hyperplane theorem for effective q -ample divisors, Theorem 2.51, asserts that

$$H^i(X, \mathbb{C}) \longrightarrow H^i(D, \mathbb{C})$$

is an isomorphism for $0 \leq i < n - q - 1$ and an injection for $i = n - q - 1$.

The Hodge decomposition then gives rise to homomorphisms

$$r_{k,l}: H^l(X, \Omega_X^k) \longrightarrow H^l(D, \Omega_D^k) \tag{1}$$

for which $r_{k,l}$ is an isomorphism for $0 \leq k + l < n - q - 1$, and an injection for $k + l = n - q - 1$.

Consequently, one has that $r_{o,l}: H^l(X, \mathcal{O}_X) \rightarrow H^l(D, \mathcal{O}_D)$ is an isomorphism for $l < n - q - 1$, and is injective for $l = n - q - 1$.

Finally, consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0.$$

The properties of $r_{o,l}$ applied to the associated long exact sequence imply that

$$H^i(X, \mathcal{O}_X(-D)) = \{0\} \text{ for all } i \leq n - q - 1$$

as we wished. □

Remark 3.2 Our proof here should also be compared with the discussion in [EV92, Section 4], where bounds on the cohomological dimension of the complement of a smooth divisor are also used to derive vanishing theorems of the type considered here.

Remark 3.3 Note that we have used only part of the information provided by the homomorphisms (1). The remaining instances lead to an Akizuki–Nakano-type vanishing result for effective reduced smooth q -ample divisors; cf. the discussion in [Laz04, Section 4.2]. Note however that a generalization to the singular setup cannot be expected, as already for ample divisors on Kawamata log terminal varieties the natural generalization of Kodaira–Akizuki–Nakano vanishing does not hold in general; we refer the reader to [GKP12, Section 4] for a discussion and for explicit counterexamples.

One of the original contributions of our work is the observation that the argument in the proof of Theorem 3.1 can be modified to go through in the Du Bois case, in particular, for log canonical pairs on smooth projective varieties. Our discussion here is very much influenced by “Kollár’s principle,” cf. [Rei97, Section 3.12], that vanishing occurs when a coherent cohomology group has a

topological interpretation, see [Kol86, Section 5]. For this, we depend heavily on the discussion of vanishing theorems in [Kol95].

Theorem 3.4 (q -Kodaira vanishing for reduced effective Du Bois divisors)
 Let X be a normal proper variety, $D \subset X$ a reduced effective (Cartier) divisor such that

1. $\mathcal{O}_X(D)$ is q -ample,
2. $X \setminus D$ is smooth,
3. D (with its reduced subscheme structure) is Du Bois.

Then, we have

$$H^j(X, \omega_X \otimes \mathcal{O}_X(D)) = \{0\} \quad \text{for all } j > q \tag{2}$$

as well as

$$H^j(X, \mathcal{O}_X(-D)) = \{0\} \quad \text{for all } j < \dim X - q. \tag{3}$$

We refer the reader to [KS11] and [Kol95, Chapter 12] for introductions to the theory of Du Bois singularities, as well as to [Kov12] for a simple characterization of the Du Bois property for projective varieties, related to the properties of Du Bois singularities used here.

Proof Since D is effective and q -ample, and $X \setminus D$ is smooth, Theorem 2.51 states that the restriction morphism

$$H^j(X, \mathbb{C}) \xrightarrow{\Phi_j} H^j(D, \mathbb{C})$$

is an isomorphism for $0 \leq j < n - q - 1$, and injective for $j = n - q - 1$. If F^\bullet denotes Deligne’s Hodge filtration on $H^j(D, \mathbb{C})$ with associated graded pieces $Gr_{F^\bullet}^j$, then without any assumption on D , the natural map $\alpha_j: H^j(D, \mathbb{C}) \rightarrow Gr_F^0(H^j(D, \mathbb{C}))$ factors as

$$\begin{array}{ccccc}
 & & \alpha_j & & \\
 & \curvearrowright & & \curvearrowleft & \\
 H^j(D, \mathbb{C}) & \xrightarrow{\beta_j} & H^j(D, \mathcal{O}_D) & \xrightarrow{\gamma_j} & Gr_F^0(H^j(D, \mathbb{C}))
 \end{array} \tag{4}$$

for each j .

Moreover, since D is assumed to be Du Bois, the map γ_j is an isomorphism, see [Kov12, Section 1]. By abuse of notation, the Hodge filtration on $H^j(X, \mathbb{C})$ will likewise be denoted by F^\bullet . By standard results of Hodge theory (e.g., see [PS08, Theorem 5.33.iii]), the map Φ_j is a morphism of Hodge structures; in

particular it is compatible with the filtrations F^\bullet on $H^j(X, \mathbb{C})$ and $H^j(D, \mathbb{C})$. Hence, for each j , we obtain a natural commutative diagram

$$\begin{array}{ccc}
 H^j(X, \mathbb{C}) & \xrightarrow{\Phi_j} & H^j(D, \mathbb{C}) \\
 \downarrow & & \downarrow \beta_j \\
 H^j(X, \mathcal{O}_X) & \xrightarrow{\phi_j} & H^j(D, \mathcal{O}_D) \\
 \downarrow \cong & & \downarrow \cong \gamma_j \\
 Gr_F^0(H^j(X, \mathbb{C})) & \longrightarrow & Gr_F^0(H^j(D, \mathbb{C}))
 \end{array}$$

Since Φ_j is an isomorphism for $0 \leq j < n - q - 1$, and injective for $j = n - q - 1$, Hodge theory [PS08, Corollaries 3.6 and 3.7] implies that the same is true for ϕ_j ; i.e., ϕ_j is an isomorphism for $0 \leq j < n - q - 1$, and injective for $j = n - q - 1$. Looking at the long exact cohomology sequence associated with

$$0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$$

we conclude that

$$H^j(X, \mathcal{O}_X(-D)) = \{0\} \quad \text{for } 0 \leq j \leq n - q - 1, \tag{5}$$

as claimed in equation (3).

Now, as $X \setminus D$ is smooth and D is Du Bois, X itself has rational singularities by a result of Schwede [Sch07, Theorem 5.1], see also [KS11, Section 12]. In particular, X is Cohen–Macaulay, [KM98, Theorem 5.10], and hence we may apply Serre duality [KM98, Theorem 5.71] to equation (5) to obtain

$$H^j(X, \omega_X \otimes \mathcal{O}_X(D)) = \{0\} \text{ for all } j > q,$$

as claimed in equation (2). □

Corollary 3.5 (*q*-Kodaira vanishing for reduced log canonical pairs) *Let X be a smooth projective variety, $D \subset X$ a reduced effective (Cartier) divisor such that*

1. $\mathcal{O}_X(D)$ is *q*-ample,
2. the pair (X, D) is log canonical.

Then, we have

$$H^j(X, \omega_X \otimes \mathcal{O}_X(D)) = \{0\} \quad \text{for all } j > q \tag{6}$$

as well as

$$H^j(X, \mathcal{O}_X(-D)) = \{0\} \quad \text{for all } j < \dim X - q. \tag{7}$$

Proof Since D is a union of log canonical centers of the pair (X, D) , it is Du Bois by [KK10, Theorem 1.4]. Hence, the claim follows immediately from Theorem 3.4. \square

Example 3.6 To give an example of a q -ample divisor that satisfies the assumptions of Theorems 3.4 and 3.1 above, and for which the desired vanishing does not follow directly from Sommese’s results, let $Z \subset X$ be a smooth ample subscheme of pure codimension r in a projective manifold X , and $\mathcal{O}_{\hat{X}}(E)$ the corresponding $(r - 1)$ -ample line bundle on the blow-up \hat{X} of X along Z , cf. Example 2.4. Then, $\mathcal{O}_{\hat{X}}(E)$ is clearly not semi-ample, and hence not geometrically q -ample in the sense of Definition 2.41. However, E is effective, reduced, and smooth, and hence fulfills the assumptions of Theorems 3.4 and 3.1. In this special case, the desired vanishing can also be derived from the results presented in [EV92, Section 4].

Finally, we will prove a version of the above for line bundles that are only \mathbb{Q} -effective. We start with the following slight generalization of [Kol95, Theorem 12.10]:

Proposition 3.7 *Let X be a normal and proper variety, \mathcal{L} a rank 1 reflexive sheaf on X , and D_i different irreducible Weil divisors on X . Assume that $\mathcal{L}^{[m]} := (\mathcal{L}^{\otimes m})^{**} \cong \mathcal{O}_X(\sum d_j D_j)$ for some integers $1 \leq d_j < m$. Set $m_j := m/\text{gcd}(m, d_j)$. Assume furthermore that the pair $(X, \sum(1 - \frac{1}{m_j})D_j)$ is log canonical. Then, for every $i \geq 0$ and $n_j \geq 0$, the natural map*

$$H^i(X, \mathcal{L}^{[-1]}(-\sum n_j D_j)) \rightarrow H^i(X, \mathcal{L}^{[-1]})$$

is surjective.

Proof Let $p: Y \rightarrow X$ be the normalization of the cyclic cover corresponding to the isomorphism $\mathcal{L}^{[m]} \cong \mathcal{O}_X(\sum d_j D_j)$. By [Kol92, Proposition 20.2], we have

$$K_Y = p^*(K_X + \sum(1 - \frac{1}{m_j})D_j),$$

and therefore Y is log canonical by [Kol92, Proposition 20.3]. Hence, Y is Du Bois by [KK10, Theorem 1.4], and the natural map $H^j(Y, \mathbb{C}) \rightarrow H^j(Y, \mathcal{O}_Y)$ is surjective, cf. the discussion in the proof of Theorem 3.4. Consequently, the assumptions of [Kol95, Theorem 9.12] are fulfilled. This implies the claim. \square

We are now in a position to prove a version of q -Kodaira vanishing that works for \mathbb{Q} -effective line bundles.

Theorem 3.8 (*q*-Kodaira vanishing for effective log canonical \mathbb{Q} -divisors) *Let X be a proper, normal, Cohen–Macaulay variety, \mathcal{L} a q -ample line bundle on X , and D_i different irreducible Weil divisors on X . Assume that $\mathcal{L}^m \cong \mathcal{O}_X(\sum d_j D_j)$ for some integers $1 \leq d_j < m$. Set $m_j := m/\text{gcd}(m, d_j)$. Assume furthermore that the pair $(X, \sum(1 - \frac{1}{m_j})D_j)$ is log canonical. Then, we have*

$$H^j(X, \omega_X \otimes \mathcal{O}_X(D)) = \{0\} \quad \text{for all } j > q \tag{8}$$

as well as

$$H^j(X, \mathcal{O}_X(-D)) = \{0\} \quad \text{for all } j < \dim X - q. \tag{9}$$

Proof With Proposition 3.7 at hand, the proof is the same as in the klt case, cf. [Kol95, Chapter 10]. By Proposition 3.7, for all $i \geq 0$ and for all $k \in \mathbb{N}^{>0}$ such that $k - 1$ is divisible by m , we obtain a surjection

$$H^i(X, \mathcal{L}^{-k}) \rightarrow H^i(X, \mathcal{L}^{-1}).$$

Since X is Cohen–Macaulay, by Serre duality [KM98, Theorem 5.71], this surjection is dual to an injection

$$H^{n-i}(X, \omega_X \otimes \mathcal{L}) \hookrightarrow H^{n-i}(X, \omega_X \otimes \mathcal{L}^k) \quad \text{for all } k \text{ as above.} \tag{10}$$

As \mathcal{L} is q -ample, there exists a $k \gg 0$ such that

$$H^{n-i}(X, \omega_X \otimes \mathcal{L}^k) = \{0\} \quad \text{for all } n - i > q.$$

Hence, owing to the injection (10), we obtain

$$H^j(X, \omega_X \otimes \mathcal{L}) = \{0\} \quad \text{for all } j > q,$$

as claimed in equation (8). The dual vanishing (9) then follows from a further application of Serre duality. □

Remark 3.9 For related work discussing ample divisors on (semi)-log canonical varieties, the reader is referred to [KSS10].

Remark 3.10 (Necessity of assumptions on the singularities) To see that some assumption on the singularities of the pair (X, D) is necessary in Theorems 3.4 and 3.8, we note that Kodaira vanishing may fail already for ample line bundles on Gorenstein varieties with worse than log canonical singularities, see [BS95, Example 2.2.10]. Moreover, we note that for the dual form (9) of Kodaira vanishing the Cohen–Macaulay condition is strictly necessary. If X is a projective variety with ample (Cartier) divisor D for which (9) holds (with $q = 0$), then X is Cohen–Macaulay by [KM98, Corollary 5.72].

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