13

Generic vanishing fails for singular varieties and in characteristic $p > 0$

C. D. Hacon[a]
University of Utah

S. J. Kovács[b]
University of Washington

To Rob Lazarsfeld on the occasion of his 60th birthday

1 Introduction

In recent years there has been considerable interest in understanding the geometry of irregular varieties, i.e., varieties admitting a nontrivial morphism to an abelian variety. One of the central results in the area is the following, conjectured by M. Green and R. Lazarsfeld (cf. [GL91, 6.2]) and proven in [Hac04] and [PP09].

Theorem 1.1 *Let $\lambda : X \to A$ be a generically finite (onto its image) morphism from a compact Kähler manifold to a complex torus. If $\mathscr{L} \to X \times \mathrm{Pic}^0(A)$ is the universal family of topologically trivial line bundles, then*

$$R^i \pi_{\mathrm{Pic}^0(A)*} \mathscr{L} = 0 \qquad \text{for } i < n.$$

At first sight the above result appears to be quite technical, however it has many concrete applications (see, e.g., [CH11], [JLT11], and [PP09]). In this paper we will show that Theorem 1.1 does not generalize to characteristic $p > 0$ or to singular varieties in characteristic 0.

[a] Partially supported by NSF research grants DMS-0757897 and DMS-1300750 and a grant from the Simons Foundation.
[b] Partially supported by NSF grants DMS-0856185, DMS-1301888, and the Craig McKibben and Sarah Merner Endowed Professorship in Mathematics at the University of Washington.

From *Recent Advances in Algebraic Geometry*, edited by Christopher Hacon, Mircea Mustaţă and Mihnea Popa

Notation 1.2 Let A be an abelian variety over an algebraically closed field k, $\widehat{A}$ its dual abelian variety, $\mathscr{P}$ the normalized Poincaré bundle on $A \times \widehat{A}$, and $p_{\widehat{A}} \colon A \times \widehat{A} \to \widehat{A}$ the projection. Let $\lambda \colon X \to A$ be a projective morphism, $\pi_{\widehat{A}} \colon X \times \widehat{A} \to \widehat{A}$ the projection, and $\mathscr{L} := (\lambda \times \mathrm{id}_{\widehat{A}})^* \mathscr{P}$ where $(\lambda \times \mathrm{id}_{\widehat{A}}) \colon X \times \widehat{A} \to A \times \widehat{A}$ is the product morphism.

Theorem 1.3 *Let k be an algebraically closed field. Then, using Notation 1.2, there exists a projective variety X over k such that*

- *if* $\operatorname{char} k = p > 0$, *then X is smooth and*
- *if* $\operatorname{char} k = 0$, *then X has isolated Gorenstein log canonical singularities*

and a separated projective morphism to an abelian variety $\lambda \colon X \to A$ which is generically finite onto its image such that

$$R^i \pi_{\widehat{A}*} \mathscr{L} \neq 0 \qquad \text{for some } 0 \leq i < n.$$

Remark 1.4 Owing to the birational nature of the statement, Theorem 1.1 generalizes trivially to the case of X having only rational singularities. Arguably, Gorenstein log canonical singularities are the simplest examples of singularities that are not rational. Therefore, the characteristic 0 part of Theorem 1.3 may be interpreted as saying that generic vanishing does not extend to singular varieties in a nontrivial way.

Remark 1.5 Note that Theorem 1.3 seems to contradict the main result of [Par03].

2 Preliminaries

Let A be a g-dimensional abelian variety over an algebraically closed field k, $\widehat{A}$ its dual abelian variety, p_A and $p_{\widehat{A}}$ the projections of $A \times \widehat{A}$ onto A and $\widehat{A}$, and $\mathscr{P}$ the normalized Poincaré bundle on $A \times \widehat{A}$. We denote by $\mathbf{R}\widehat{S} \colon \mathbf{D}(A) \to \mathbf{D}(\widehat{A})$ the usual Fourier–Mukai functor given by $\mathbf{R}\widehat{S}(\mathscr{F}) = \mathbf{R}p_{\widehat{A}*}(p_A^* \mathscr{F} \otimes \mathscr{P})$ (cf. [Muk81]). There is a corresponding functor $\mathbf{R}S \colon \mathbf{D}(\widehat{A}) \to \mathbf{D}(A)$ such that

$$\mathbf{R}S \circ \mathbf{R}\widehat{S} = (-1_A)^*[-g] \qquad \text{and} \qquad \mathbf{R}\widehat{S} \circ \mathbf{R}S = (-1_{\widehat{A}})^*[-g].$$

Definition 2.1 An object $F \in \mathbf{D}(A)$ is called *WIT-i* if $R^j\widehat{S}(F) = 0$ for all $j \neq i$. In this case we use the notation $\widehat{F} = R^i\widehat{S}(F)$.

Notice that if F is a WIT-i coherent sheaf (in degree 0), then $\widehat{F}$ is a WIT-$(g-i)$ coherent sheaf (in degree i) and $F \simeq (-1_A)^* R^{g-i} S(\widehat{F})$.

One easily sees that if F and G are arbitrary objects, then

$$\operatorname{Hom}_{\mathbf{D}(A)}(F, G) = \operatorname{Hom}_{\mathbf{D}(\widehat{A})}(\mathbf{R}\widehat{S}F, \mathbf{R}\widehat{S}G).$$

An easy consequence (cf. [Muk81, 2.5]) is that if F is a WIT-i sheaf and G is a WIT-j sheaf (or if F is a WIT-i locally free sheaf and G is a WIT-j object – not necessarily a sheaf), then

$$\begin{aligned}\operatorname{Ext}^k_{\mathscr{O}_A}(F, G) &\simeq \operatorname{Hom}_{\mathbf{D}(A)}(F, G[k]) \\ &\simeq \operatorname{Hom}_{\mathbf{D}(\widehat{A})}(\mathbf{R}\widehat{S}F, \mathbf{R}\widehat{S}G[k]) \\ &= \operatorname{Hom}_{\mathbf{D}(\widehat{A})}(\widehat{F}[-i], \widehat{G}[k-j]) \simeq \operatorname{Ext}^{k+i-j}_{\mathscr{O}_{\widehat{A}}}(\widehat{F}, \widehat{G}). \quad (1)\end{aligned}$$

Let L be any ample line bundle on $\widehat{A}$, then $\mathbf{R}S(L) = R^0S(L) = \widehat{L}$ is a vector bundle on A of rank $h^0(L)$. For any $x \in A$, let $t_x\colon A \to A$ be the translation by x and let $\phi_L\colon \widehat{A} \to A$ be the isogeny determined by $\phi_L(\widehat{x}) = t^*_{\widehat{x}}L \otimes L^\vee$, then $\phi^*_L(\widehat{L}) = \bigoplus_{h^0(L)} L^\vee$.

Let $\lambda\colon X \to A$ be a projective morphism of normal varieties, and $\mathscr{L} = (\lambda \times \mathrm{id}_{\widehat{A}})^*\mathscr{P}$. We let $\mathbf{R}\Phi\colon \mathbf{D}(X) \to \mathbf{D}(\widehat{A})$ be the functor defined by $\mathbf{R}\Phi(F) = \mathbf{R}\pi_{\widehat{A}*}(\pi^*_X F \otimes \mathscr{L})$, where π_X and $\pi_{\widehat{A}}$ denote the projections of $X \times \widehat{A}$ onto the first and second factor. Note that

$$\begin{aligned}\mathbf{R}\Phi(F) &= \mathbf{R}\pi_{\widehat{A}*}(\pi^*_X F \otimes \mathscr{L}) \\ &\simeq^1 \mathbf{R}p_{\widehat{A}*}\mathbf{R}(\lambda \times \mathrm{id}_{\widehat{A}})_*(\pi^*_X F \otimes (\lambda \times \mathrm{id}_{\widehat{A}})^*\mathscr{P}) \\ &\simeq^2 \mathbf{R}p_{\widehat{A}*}\left(\mathbf{R}(\lambda \times \mathrm{id}_{\widehat{A}})_*(\pi^*_X F) \otimes \mathscr{P}\right) \\ &\simeq^3 \mathbf{R}p_{\widehat{A}*}(p^*_A \mathbf{R}\lambda_* F \otimes \mathscr{P}) \simeq \mathbf{R}\widehat{S}(\mathbf{R}\lambda_* F), \quad (2)\end{aligned}$$

where $\simeq^1$ follows by composition of derived functors [Har66, II.5.1], $\simeq^2$ follows by the projection formula [Har66, II.5.6], and $\simeq^3$ follows by flat base change [Har66, II.5.12].

We also define $\mathbf{R}\Psi\colon \mathbf{D}(\widehat{A}) \to \mathbf{D}(X)$ by $\mathbf{R}\Psi(F) = \mathbf{R}\pi_{X*}(\pi^*_{\widehat{A}}F \otimes \mathscr{L})$. Notice that if F is a locally free sheaf, then $\pi^*_{\widehat{A}}F \otimes \mathscr{L}$ is also a locally free sheaf. In particular, for any $i \in \mathbb{Z}$, we have that

$$R^i\Psi(F) \simeq R^i\pi_{X*}(\pi^*_{\widehat{A}}F \otimes \mathscr{L}). \quad (3)$$

We will need the following fact (which is also proven during the proof of Theorem B of [PP11]):

Lemma 2.2 *Let L be an ample line bundle on $\widehat{A}$, then*

$$\mathbf{R}\Psi(L^\vee) = R^g\Psi(L^\vee) = \lambda^*\widehat{L^\vee}.$$

Proof Since L is ample, $H^i(\widehat{A}, L^\vee \otimes \mathscr{L}_x) = H^i(\widehat{A}, L^\vee \otimes \mathscr{P}_{\lambda(x)}) = 0$ for $i \neq g$, where $\mathscr{P}_{\lambda(x)} = \mathscr{P}|_{\lambda(x)\times\widehat{A}}$ and $\mathscr{L}_x = \mathscr{L}|_{x\times\widehat{A}}$ are isomorphic. By cohomology and base change, $\mathbf{R}\Psi(L^\vee) = R^g\Psi(L^\vee)$ (resp. $\widehat{L^\vee}$) is a vector bundle of rank $h^g(\widehat{A}, L^\vee)$ on X (resp. on A).

The natural transformation $\mathrm{id}_{A\times\widehat{A}} \to (\lambda\times\mathrm{id}_{\widehat{A}})_*(\lambda\times\mathrm{id}_{\widehat{A}})^*$ induces a natural morphism

$$\widehat{L^\vee} = R^g p_{A*}(p_{\widehat{A}}^* L^\vee \otimes \mathscr{P}) \to R^g p_{A*}(\lambda\times\mathrm{id}_{\widehat{A}})_*(\pi_{\widehat{A}}^* L^\vee \otimes \mathscr{L}).$$

Let $\sigma = p_A \circ (\lambda\times\mathrm{id}_{\widehat{A}}) = \lambda\circ\pi_X$. By the Grothendieck spectral sequence associated with $p_{A*}\circ(\lambda\times\mathrm{id}_{\widehat{A}})_*$ there exists a natural morphism

$$R^g p_{A*}(\lambda\times\mathrm{id}_{\widehat{A}})_*(\pi_{\widehat{A}}^* L^\vee \otimes \mathscr{L}) \to R^g\sigma_*(\pi_{\widehat{A}}^* L^\vee \otimes \mathscr{L}),$$

and similarly by the Grothendieck spectral sequence associated with $\lambda_*\circ\pi_{X*}$ there exists a natural morphism

$$R^g\sigma_*(\pi_{\widehat{A}}^* L^\vee \otimes \mathscr{L}) \to \lambda_* R^g\pi_{X*}(\pi_{\widehat{A}}^* L^\vee \otimes \mathscr{L}).$$

Combining the above three morphisms gives a natural morphism

$$\widehat{L^\vee} \to \lambda_* R^g\pi_{X*}(\pi_{\widehat{A}}^* L^\vee \otimes \mathscr{L}) = \lambda_* R^g\Psi(L^\vee),$$

and hence by adjointness a natural morphism

$$\eta\colon \lambda^*\widehat{L^\vee} \to R^g\Psi(L^\vee).$$

For any point $x\in X$, by cohomology and base change, the induced morphism on the fiber over x is an isomorphism:

$$\begin{aligned}\eta_x\colon \lambda^*\widehat{L^\vee}\otimes\kappa(x) &\simeq H^g(\lambda(x)\times\widehat{A}, L^\vee\otimes\mathscr{P}_{\lambda(x)})\\ &\xrightarrow{\simeq} H^g(x\times\widehat{A}, L^\vee\otimes\mathscr{L}_x) \simeq R^g\Psi(L^\vee)\otimes\kappa(x).\end{aligned}$$

Therefore η_x is an isomorphism for all $x \in X$ and hence η is an isomorphism. □

3 Examples

Notation 3.1 Let $T \subseteq \mathbb{P}^n$ be a projective variety. The cone over T in $\mathbb{A}^{n+1}$ will be denoted by $C(T)$. In other words, if $T \simeq \operatorname{Proj} S$, then $C(T) \simeq \operatorname{Spec} S$.

Linear equivalence between (Weil) divisors is denoted by $\sim$ and strict transform of a subvariety T by the inverse of a birational morphism σ is denoted by $\sigma_*^{-1}T$.

Example 3.2 Let k be an algebraically closed field, $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$ two smooth projective varieties over k, and $p \in V$ a closed point. Let $x_0, \dots, x_n$ and $y_0, \dots, y_m$ be homogeneous coordinates on $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively.

Consider the embedding $V \times W \subset \mathbb{P}^N$ induced by the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$. We may choose homogeneous coordinates z_{ij} for $i = 0, \dots, n$ and $j = 0, \dots, m$ on $\mathbb{P}^N$, and in these coordinates $\mathbb{P}^n \times \mathbb{P}^m$ is defined by the equations $z_{\alpha\gamma} z_{\beta\delta} - z_{\alpha\delta} z_{\beta\gamma}$ for all $0 \le \alpha, \beta \le n$ and $0 \le \gamma, \delta \le m$.

Next let $H \subset W$ such that $\{p\} \times H \subset \{p\} \times W$ is a hyperplane section of $\{p\} \times W$ in $\mathbb{P}^N$. Let $Y = C(V \times W) \subset \mathbb{A}^{N+1}$ and $Z = C(V \times H) \subset Y$, and let $v \in Z \subset Y$ denote the common vertex of Y and Z. If $\dim W = 0$, then $H = \emptyset$. In this case let $Z = \{v\}$, the vertex of Y. Finally, let $\mathfrak{m}_v$ denote the ideal of v in the affine coordinate ring of Y. It is generated by all the variables z_{ij}.

Proposition 3.3 *Let $f \colon X \to Y$ be the blowing up of Y along Z. Then f is an isomorphism over $Y \setminus \{v\}$ and the scheme-theoretic pre-image of v (whose support is the exceptional locus) is isomorphic to V:*

$$f^{-1}(v) \simeq V.$$

Proof As Z is of codimension 1 in Y and $Y \setminus \{v\}$ is smooth, it follows that $Z \setminus \{v\}$ is a Cartier divisor in $Y \setminus \{v\}$ and hence f is indeed an isomorphism over $Y \setminus \{v\}$.

To prove the statement about the exceptional locus of f, first assume that $V = \mathbb{P}^n$, $W = \mathbb{P}^m$, $p = [1 : 0 : \cdots : 0]$, and $\{p\} \times H = (z_{0m} = 0) \cap (\{p\} \times W)$. Then $H = (y_m = 0) \subseteq W$ and hence $I = I(Z)$, the ideal of Z in the affine coordinate ring of Y, is generated by $\{z_{im} | i = 0, \dots, n\}$. Then by the definition of blowing up, $X = \operatorname{Proj} \oplus_{d \ge 0} I^d$ and $f^{-1}v \simeq \operatorname{Proj} \oplus_{d \ge 0} I^d / I^d \mathfrak{m}_v$.

Notice that $I^d / I^d \mathfrak{m}_v$ is a k-vector space generated by the degree-d monomials in the variables $\{z_{im} | i = 0, \dots, n\}$. It follows that the graded ring $\oplus_{d \ge 0} I^d / I^d \mathfrak{m}_v$ is nothing else but $k[z_{im} | i = 0, \dots, n]$ and hence $f^{-1}v \simeq \mathbb{P}^n = V$, so the claim is proved in this case.

Next consider the case when $V \subseteq \mathbb{P}^n$ is arbitrary, but $W = \mathbb{P}^m$. In this case the calculation is similar, except that we have to account for the defining equations of V. They show up in the definition of the coordinate ring of Y in the following way. If a homogeneous polynomial $g \in k[x_0, \dots, x_n]$ vanishes on V (i.e., $g \in I(V)_h$), then define $g_\gamma \in k[z_{ij}]$ for any $0 \le \gamma \le m$ by replacing x_α with $z_{\alpha\gamma}$ for each $0 \le \alpha \le n$. Then $\{g_\gamma | 0 \le \gamma \le m, g \in I(V)_h\}$ generates the ideal of Y in the affine coordinate ring of $C(\mathbb{P}^n \times \mathbb{P}^m)$. It follows that the above computation goes through in the same way, except that the variables $\{z_{im} | i = 0, \dots, n\}$ on the exceptional $\mathbb{P}^n$ are subject to the equations $\{g_m | g \in I(V)_h\}$. However, this simply means that the exceptional locus of f, i.e., $f^{-1}v$, is cut out from $\mathbb{P}^n$ by these equations and hence it is isomorphic to V.

Finally, consider the general case. The way W changes the setup is the same as what we described for V. If a homogeneous polynomial $h \in k[y_0, \dots, y_m]$ vanishes on W (i.e., $h \in I(W)_h$), then define $h_\alpha \in k[z_{ij}]$ for any $0 \le \alpha \le n$ by replacing y_γ with $z_{\alpha\gamma}$ for each $0 \le \gamma \le m$. Then $\{h_\alpha | 0 \le \alpha \le n, h \in I(W)_h\}$ generates the ideal of Y in the affine coordinate ring of $C(V \times \mathbb{P}^m)$.

However, in this case, differently from the case of V, we do not get any additional equations. Indeed, we chose the coordinates so that $H = (y_m = 0)$ and hence $y_m \notin I(W)$, which means that we may choose the rest of the coordinates such that $[0 : \cdots : 0 : 1] \in W$. This implies that no polynomial in the ideal of W may have a monomial term that is a constant multiple of a power of y_m. It follows that, since $I = I(Z)$ is generated by the elements $\{z_{im} | i = 0, \dots, n\}$, any monomial term of any polynomial in the ideal of Y in the affine coordinate ring of $C(V \times \mathbb{P}^m)$ that lies in I^d for some $d > 0$ also lies in $I^d \mathfrak{m}_v$. Therefore, these new equations do not change the ring $\oplus I^d / I^d \mathfrak{m}_v$ and so $f^{-1}v$ is still isomorphic to V. □

Notation 3.4 We will use the notation introduced in Proposition 3.3 for X, Y, Z, and f. We will also use $X_{\mathbb{P}}$, $Y_{\mathbb{P}}$, $Z_{\mathbb{P}}$, and $f_{\mathbb{P}} \colon X_{\mathbb{P}} \to Y_{\mathbb{P}}$ to denote the same objects in the case $W = \mathbb{P}^m$, i.e., $Y_{\mathbb{P}} = C(V \times \mathbb{P}^m)$, $Z_{\mathbb{P}} = C(V \times H)$, where $H \subset \mathbb{P}^m$ is such that $\{p\} \times H \subset \{p\} \times \mathbb{P}^m$ is a hyperplane section of $\{p\} \times \mathbb{P}^m$ in $\mathbb{P}^N$.

Corollary 3.5 *$f_{\mathbb{P}}$ is an isomorphism over $Y_{\mathbb{P}} \setminus \{v\}$ and the scheme-theoretic pre-image of v (whose support is the exceptional locus) via $f_{\mathbb{P}}$ is isomorphic to V:*

$$f_{\mathbb{P}}^{-1} v \simeq V.$$

Proof This was proven as an intermediate step in Proposition 3.3, and is also straightforward by taking $W = \mathbb{P}^m$. □

Proposition 3.6 *Assume that V and W are both positive dimensional, $W \subseteq \mathbb{P}^m$ is a complete intersection, and the embedding $V \times \mathbb{P}^r \subset \mathbb{P}^N$ for any linear subvariety $\mathbb{P}^r \subseteq \mathbb{P}^m$ induced by the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ is projectively normal. Then X is Gorenstein.*

Proof First note that the projective normality assumption implies that $Y_{\mathbb{P}} = C(V \times \mathbb{P}^m)$ is normal and hence we may consider divisors and their linear equivalences.

Let $H' \subset \mathbb{P}^m$ be an arbitrary hypersurface (different from H and not necessarily linear). Observe that $H' \sim d \cdot H$ with $d = \deg H'$, so $V \times H' \sim d \cdot (V \times H)$, and hence $C(V \times H') \sim d \cdot C(V \times H)$ as divisors on $Y_{\mathbb{P}}$.

Since $f_{\mathbb{P}}$ is a small morphism it follows that the strict transforms of these divisors on $X_{\mathbb{P}}$ are also linearly equivalent: $f_*^{-1}C(V \times H') \sim d \cdot f_*^{-1}C(V \times H)$ (where by abuse of notation we let $f = f_{\mathbb{P}}$). By the basic properties of blowing up, the (scheme-theoretic) pre-image of $C(V \times H)$ is a Cartier divisor on X which coincides with $f_*^{-1}C(V \times H)$ (as f is small). However, then $f_*^{-1}C(V \times H')$ is also a Cartier divisor and hence it is Gorenstein if and only if $X_{\mathbb{P}}$ is. Note that $f_*^{-1}C(V \times H')$ is nothing else but the blow-up of $C(V \times H')$ along $C(V \times (H' \cap H))$.

By assumption W is a complete intersection, so applying the above argument for the intersection of the hypersurfaces cutting out W shows that X is Gorenstein if and only if $X_{\mathbb{P}}$ is Gorenstein. In other words, it is enough to prove the statement with the additional assumption that $W = \mathbb{P}^m$. In particular, we have $X = X_{\mathbb{P}}$, etc.

In this case the same argument as above shows that the statement holds for m if and only if it holds for $m - 1$, so we only need to prove it for $m = 1$. In that case $H \in \mathbb{P}^1$ is a single point. Choose another point $H' \in \mathbb{P}^1$. As above, $f_*^{-1}C(V \times H')$ is a Cartier divisor in X and it is the blow-up of $C(V \times H')$ along the intersection $C(V \times H') \cap C(V \times H)$.

We claim that this intersection is just the vertex of $C(V)$. To see this, view $Y = Y_{\mathbb{P}} = C(V \times \mathbb{P}^1)$ as a subscheme of $C(\mathbb{P}^n \times \mathbb{P}^1)$. Inside $C(\mathbb{P}^n \times \mathbb{P}^1)$ the cones $C(\mathbb{P}^n \times H)$ and $C(\mathbb{P}^n \times H')$ are just linear subspaces of dimension $n + 1$ whose scheme-theoretic intersection is the single reduced point v. Therefore we have that

$$C(V \times H') \cap C(V \times H) \subseteq C(\mathbb{P}^m \times H') \cap C(\mathbb{P}^m \times H) = \{v\},$$

proving the same for this intersection.

Finally then $f_*^{-1}C(V \times H')$, the blow-up of $C(V \times H')$ along the intersection $C(V \times H') \cap C(V \times H)$, is just the blow-up of $C(V)$ at its vertex and hence it is smooth and in particular Gorenstein. This completes the proof. □

Lemma 3.7 *Let $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$ be two normal complete intersection varieties of positive dimension. Assume that either* $\dim V + \dim W > 2$ *or if* $\dim V = \dim W = 1$*, then $n = m = 2$. The embedding $V \times W \subset \mathbb{P}^N$ induced by the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ is then projectively normal.*

Proof It follows easily from the definition of the Segre embedding that it is itself projectively normal and hence it is enough to prove that

$$H^0(\mathbb{P}^n \times \mathbb{P}^m, \mathscr{O}_{\mathbb{P}^N}(d)|_{\mathbb{P}^n \times \mathbb{P}^m}) \to H^0(V \times W, \mathscr{O}_{\mathbb{P}^N}(d)|_{V \times W}) \tag{1}$$

is surjective for all $d \in \mathbb{N}$.

We prove this by induction on the combined number of hypersurfaces cutting out V and W. When this number is 0, then $V = \mathbb{P}^n$ and $W = \mathbb{P}^m$ so we are done.

Otherwise, assume that $\dim V \leq \dim W$ and if $\dim V = \dim W = 1$ then $\deg V = e \geq \deg W$. Let $V' \subseteq \mathbb{P}^n$ be a complete intersection variety of dimension $\dim V + 1$ such that $V = V' \cap H'$, where $H' \subset \mathbb{P}^n$ is a hypersurface of degree e. Then $V \times W \subset V' \times W$ is a Cartier divisor with ideal sheaf $\mathscr{I} \simeq \pi_1^* \mathscr{O}_{V'}(-e)$, where $\pi_1 \colon V' \times W \to V'$ is the projection to the first factor. It follows that for every $d \in \mathbb{N}$ there exists a short exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^N}(d)|_{V' \times W} \otimes \pi_1^* \mathscr{O}_{V'}(-e) \to \mathscr{O}_{\mathbb{P}^N}(d)|_{V' \times W} \to \mathscr{O}_{\mathbb{P}^N}(d)|_{V \times W} \to 0,$$

and hence an induced exact sequence of cohomology

$$H^0(V' \times W, \mathscr{O}_{\mathbb{P}^N}(d)|_{V' \times W}) \to H^0(V \times W, \mathscr{O}_{\mathbb{P}^N}(d)|_{V \times W}) \to H^1(V' \times W, \pi_1^* \mathscr{O}_{V'}(d-e) \otimes \pi_2^* \mathscr{O}_W(d)),$$

where $\pi_2 \colon V' \times W \to W$ is the projection to the second factor.

Since by assumption V' is a complete intersection variety of dimension at least 2, it follows that $H^1(V', \mathscr{O}_{V'}(d-e)) = 0$.

If $\dim W > 1$, then it follows similarly that $H^1(W, \mathscr{O}_W(d)) = 0$.

If $\dim W = 1$, then since $0 < \dim V \leq \dim W$ we also have $\dim V = 1$. By assumption V and W are normal and hence regular, and in this case we assumed earlier that $\deg V = e \geq \deg W$. It follows that as long as $e > d$, then $H^0(V', \mathscr{O}_{V'}(d-e)) = 0$ and if $e \leq d$, then $d \geq \deg W$ and hence $H^1(W, \mathscr{O}_W(d)) = 0$.

In both cases we obtain that by the Künneth formula (cf. [EGAIII$_2$, (6.7.8)], [Kem93, 9.2.4]),

$$H^1(V' \times W, \pi_1^* \mathscr{O}_{V'}(d-e) \otimes \pi_2^* \mathscr{O}_W(d)) = 0$$

and hence

$$H^0(V' \times W, \mathscr{O}_{\mathbb{P}^N}(d)|_{V' \times W}) \to H^0(V \times W, \mathscr{O}_{\mathbb{P}^N}(d)|_{V \times W})$$

is surjective. By induction we may assume that

$$H^0(\mathbb{P}^n \times \mathbb{P}^m, \mathscr{O}_{\mathbb{P}^N}(d)|_{\mathbb{P}^n \times \mathbb{P}^m}) \to H^0(V' \times W, \mathscr{O}_{\mathbb{P}^N}(d)|_{V' \times W})$$

is surjective, so it follows that the desired map in (1) is surjective as well and the statement is proven. □

Corollary 3.8 *Let $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$ be two positive-dimensional normal complete intersection varieties and assume that if* $\dim V = 1$*, then* $n = 2$*. X is then Gorenstein.*

Proof Follows by combining Proposition 3.6 and Lemma 3.7. Note that in Proposition 3.6 the embedding $V \times W \hookrightarrow \mathbb{P}^N$ does not need to be projectively normal, only $V \times \mathbb{P}^r \hookrightarrow \mathbb{P}^N$ does, which indeed follows from Lemma 3.7. □

Example 3.9 Let k be an algebraically closed field. We will construct a birational projective morphism $f\colon X \to Y$ such that X is Gorenstein (and log canonical) and $R^1 f_*\omega_X \neq 0$.

Let $E_1, E_2 \subseteq \mathbb{P}^2$ be two smooth projective cubic curves. Consider the construction in Example 3.2 with $V = E_1$, $W = E_2$. As in that construction let $f\colon X \to Y$ be the blow-up of $Y = C(E_1 \times E_2)$ along $Z = C(E_1 \times H)$, where $H \subseteq E_2$ is a hyperplane section. The common vertex of Y and Z will still be denoted by $v \in Z \subset Y$. The map f is an isomorphism over $Y \setminus \{v\}$ and $f^{-1}v \simeq E_1$ by Proposition 3.3.

Proposition 3.10 *Both X and Y are smooth in codimension* 1 *with trivial canonical divisor and X is Gorenstein and hence Cohen–Macaulay.*

Proof By construction $Y \setminus \{v\} \simeq X \setminus f^{-1}v$ is smooth, so the first statement follows. Furthermore, $Y \setminus \{v\} \simeq X \setminus f^{-1}v$ is an affine bundle over $E_1 \times E_2$, so by the choice of E_1 and E_2, the canonical divisor of $Y \setminus \{v\} \simeq X \setminus f^{-1}v$ is trivial. However, the complement of this set has codimension at least 2 in both X and Y and hence their canonical divisors are trivial as well. Since $E_1, E_2 \subset \mathbb{P}^2$ are hypersurfaces, X is Gorenstein by Corollary 3.8. □

Let E denote $f^{-1}v$, so we have that $E \simeq E_1$ and there is a short exact sequence

$$0 \to \mathscr{I}_E \to \mathscr{O}_X \to \mathscr{O}_E \to 0.$$

Pushing this forward via f we obtain a homomorphism $\phi\colon R^1 f_*\mathscr{O}_X \to R^1 f_*\mathscr{O}_E$. Since the maximum dimension of any fiber of f is 1, we have $R^2 f_*\mathscr{I}_E = 0$. It follows that $R^1 f_*\omega_X = R^1 f_*\mathscr{O}_X \neq 0$, because $R^1 f_*\mathscr{O}_E \neq 0$ (it is a sheaf supported on v of length $h^1(\mathscr{O}_E) = 1$).

Example 3.11 Let k be an algebraically closed field of characteristic $p \neq 0$. Then there exists a birational morphism $f\colon X \to Y$ of varieties (defined over k) such that X is smooth of dimension 7 and $R^i f_*\omega_X \neq 0$, for some $i \in \{1, 2, 3, 4, 5\}$.

Let Z be a smooth 6-dimensional variety and L a very ample line bundle such that $H^1(Z, \omega_Z \otimes L) \neq 0$ (such varieties exist by [LR97]). By Serre vanishing, $H^i(Z, \omega_Z \otimes L^j) = 0$ for all $i > 0$ and $j \gg 0$. Let m be the largest positive integer such that $H^i(Z, \omega_Z \otimes L^m) \neq 0$ for some $i > 0$.

After replacing L by L^m we may assume that there exists a $q > 0$ such that $H^q(Z, \omega_Z \otimes L) \neq 0$, but $H^i(Z, \omega_Z \otimes L^j) = 0$ for all $i > 0$ and $j \geq 2$. Note that $q < 6$, because $H^6(Z, \omega_Z \otimes L)$ is dual to $H^0(Z, L^{-1}) = 0$.

Let Y be the cone over the embedding of Z given by L, $f\colon X \to Y$ the blow-up of the vertex $v \in Y$, and $E = f^{-1}v$ the exceptional divisor of f. Note that $E \simeq Z$ and $\omega_E(-jE) \simeq \omega_Z \otimes L^j$ for any j.

For $j \geq 1$ consider the short exact sequence

$$0 \to \omega_X(-jE) \to \omega_X(-(j-1)E) \to \omega_E(-jE) \to 0.$$

Claim $R^i f_* \omega_X(-E) = 0$ for all $i > 0$ and $R^i f_* \omega_X = 0$ for all $i > 0$, such that $H^i(Z, \omega_Z \otimes L) = 0$.

Proof of claim As $-E$ is f-ample we have, by Serre vanishing again, that $R^i f_* \omega_X(-jE) = 0$ for all $i > 0$ and some $j > 0$. If either $j > 1$ or $j = 1$ and $H^i(Z, \omega_Z \otimes L) = 0$, then $R^i f_* \omega_E(-jE) = H^i(Z, \omega_Z \otimes L^j) = 0$ by the choice of L. Therefore, the exact sequence

$$0 = R^i f_* \omega_X(-jE) \to R^i f_* \omega_X(-(j-1)E) \to R^i f_* \omega_E(-jE) = 0$$

gives that $R^i f_* \omega_X(-(j-1)E) = 0$. The claim follows by induction. □

From the above claim it follows that

$$0 = R^q f_* \omega_X(-E) \to R^q f_* \omega_X \to R^q f_* \omega_E(-E) \to R^{q+1} f_* \omega_X(-E) = 0.$$

Since $R^q f_* \omega_E(-E) = H^q(Z, \omega_Z \otimes L) \neq 0$, we obtain that $R^q f_* \omega_X \neq 0$ as claimed. □

Remark 3.12 The above example is certainly well known (see, e.g., [CR11b, 4.7.2]) and one can easily construct examples in dimension ≥ 3 (using, e.g., the results of [Ray78] and [Muk79]). We have chosen to include the above example because of its elementary nature.

Proposition 3.13 *There exists a variety T and a generically finite projective separable morphism to an abelian variety $\lambda\colon T \to A$ defined over an algebraically closed field k such that:*

- *if* $\operatorname{char} k = 0$*, then T is Gorenstein (and hence Cohen–Macaulay) with a single isolated log canonical singularity and $R^1 \lambda_* \omega_T \neq 0$;*
- *if* $\operatorname{char} k = p > 0$*, then T is smooth and $R^i \lambda_* \omega_T \neq 0$ for some $i > 0$.*

Proof First assume that $\operatorname{char} k = 0$ and let $f\colon X \to Y$ be as in Example 3.9. We may assume that X and Y are projective. Let $X' \to X$ and $Y' \to Y$ be birational morphisms that are isomorphisms near $f^{-1}(v)$ and v respectively such that there is a birational morphism $f'\colon X' \to Y'$ and a generically finite morphism $g\colon Y' \to \mathbb{P}^n$. Let $v' \in Y'$ be the inverse image of $v \in Y$ and $p \in \mathbb{P}^n$

its image. We may assume that there is an open subset $\mathbb{P}^n_0 \subset \mathbb{P}^n$ such that $g|_{Y'_0}$ is finite, where $Y'_0 = g^{-1}(\mathbb{P}^n_0)$. Note that if we let X'_0 be the inverse image of Y'_0 and $g' = g \circ f'$, then we have $R^i g'_* \omega_{X'_0} = g_* R^i f'_* \omega_{X'_0}$.

Let A be an n-dimensional abelian variety, $A' \to A$ a birational morphism of smooth varieties, and $A' \to \mathbb{P}^n$ a generically finite morphism. We may assume that there are points $a' \in A'$ and $a \in A$ such that $(A', a') \to (A, a)$ is locally an isomorphism and $(A', a') \to (\mathbb{P}^n, p)$ is locally étale.

Let U be the normalization of the main component of $X' \times_{\mathbb{P}^n} A'$ and $h\colon U \to X'$ the corresponding morphism. We let $E \subset (f' \circ h)^{-1}(v') \subset U$ be the component corresponding to $(v', a') \in Y' \times_{\mathbb{P}^n} A'$. Then, the morphism $(U, E) \to (Y' \times_{\mathbb{P}^n} A', (v', a')) \to (A, a)$ is étale locally (on the base) isomorphic to $(X, f^{-1}(v)) \to (Y, v) \to (\mathbb{P}^n, p)$.

Let $\nu\colon T \to U$ be a birational morphism such that ν is an isomorphism over a neighborhood of $E \subset U$ and $T \setminus \nu^{-1}(E)$ is smooth. Let $\lambda\colon T \to A$ be the induced morphism. It is clear from what we have observed above that $\lambda(E)$ is one of the components of the support of $R^1\lambda_*\omega_T \neq 0$ and T has the required singularities.

Assume now that $\operatorname{char} k = p > 0$ and let $f\colon X \to Y$ be a birational morphism of varieties such that X is smooth and $R^i f_* \omega_X \neq 0$ for some $i > 0$. This i will be fixed for the rest of the proof. The existence of such morphisms is well known (see Remark 3.12), and Example 3.11 is explicit in dimension 7. Further, let A be an abelian variety of the same dimension as X and Y and set $n = \dim A = \dim X = \dim Y$. There are embeddings $Y \subset \mathbb{P}^{m_1}$, $A \subset \mathbb{P}^{m_2}$, and $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2} \subset \mathbb{P}^M$. Let H be a very ample divisor on $\mathbb{P}^M$ and $U \subset Y \times A$ the intersection of n general members $H_1, \dots, H_n \in |H|$ with $Y \times A$. By choice, the induced maps $h\colon U \to Y$ and $a\colon U \to A$ are generically finite, U intersects $v \times A$ transversely so that $V = U \cap (v \times A)$ is a finite set of reduced points, and $U \setminus V$ is smooth by Bertini's theorem (cf. [Har77, II.8.18] and its proof). It follows that any singular point $u \in U$ is a point in V and (U, u) is locally isomorphic to (Y, v). We claim that a is finite in a neighborhood of $u \in U$. Consider any contracted curve, i.e., any curve $C \subset U \cap (Y \times a(u))$. We must show that $u \notin C$. Let $\nu\colon T \to U$ be the blow-up of U along V and $\tilde{C}$ the strict transform of C on T. We let $\mu\colon \mathrm{Bl}_V \mathbb{P}^M \to \mathbb{P}^M$, $E = \mu^{-1}(u) \cong \mathbb{P}^{M-1}$ and we denote by $h_i = \mu_*^{-1} H_i|_E$ the corresponding hyperplanes. To verify the claim it suffices to check that $\nu^{-1}(u) \cap \tilde{C} = \emptyset$. But this is now clear, as $\nu^{-1}(u) \cong Z \subset \mathbb{P}^{M-1}$ and the h_i are general hyperplanes so that $Z \cap h_1 \cap \ldots \cap h_n = \emptyset$ as Z is $(n-1)$-dimensional.

Let $\lambda = a \circ \nu\colon T \to A$ be the induced morphism. By construction, the support of the sheaf $R^i \nu_* \omega_T$ is V. Since a is finite on a neighborhood of $u \in U$, it follows that $0 \neq a_* R^i \nu_* \omega_T \subset R^i \lambda_* \omega_T$ and hence $R^i \lambda_* \omega_T \neq 0$ for the same $i > 0$. □

4 Main result

Proposition 4.1 *Assume that $\lambda\colon X \to A$ is generically finite onto its image, where X is a projective Cohen–Macaulay variety and A is an abelian variety. If* $\mathrm{char}(k) = p > 0$, *then we assume that there is an ample line bundle L on A whose degree is not divisible by p. If $R^i\pi_{\widehat{A}*}\mathscr{L} = 0$ for all $i < n$, then $R^i\lambda_*\omega_X = 0$ for all $i > 0$.*

Proof By Theorem A of [PP11], $R^i\Phi(\mathscr{O}_X) = R^i\pi_{\widehat{A}*}\mathscr{L} = 0$ for all $i < n$ is equivalent to

$$H^i(X, \omega_X \otimes R^g\Psi(L^\vee)) = 0 \qquad \forall\, i > 0,$$

where L is sufficiently ample on $\widehat{A}$ and $R^g\Psi(L^\vee) = \lambda^*\widehat{L^\vee}$ (cf. Lemma 2.2). It is easy to see that this in turn is equivalent to

$$H^i(X, \omega_X \otimes \lambda^*(\widehat{t_{\widehat{a}}^* L^\vee})) = 0 \qquad \forall\, i > 0,\ \forall\, \widehat{a} \in \widehat{A},$$

where L is sufficiently ample on $\widehat{A}$. By [Muk81, 3.1], we have $\widehat{t_{\widehat{a}}^*L^\vee} = \widehat{L^\vee} \otimes P_{-\widehat{a}}$ and hence $H^i(X, \omega_X \otimes \lambda^*(\widehat{L^\vee} \otimes P_{-\widehat{a}})) = 0$. Thus, by cohomology and base change, we have that

$$\mathbf{R}\widehat{S}(\mathbf{R}\lambda_*\omega_X \otimes \widehat{L^\vee}) =^{(2)} \mathbf{R}\Phi(\omega_X \otimes \lambda^*\widehat{L^\vee}) = R^0\Phi(\omega_X \otimes \lambda^*\widehat{L^\vee}).$$

In particular, $\mathbf{R}\lambda_*\omega_X \otimes \widehat{L^\vee}$ is WIT-0. □

Claim 4.2 For any ample line bundle M on A, we have that

$$H^i(X, \omega_X \otimes \lambda^*(\widehat{L^\vee} \otimes M \otimes P_{-\widehat{a}})) = 0 \qquad \forall\, i > 0,\ \forall\, \widehat{a} \in \widehat{A}.$$

Proof We follow the argument in [PP03, 2.9]. For any $P = P_{-\widehat{a}}$,

$$\begin{aligned} H^i(X, \omega_X \otimes \lambda^*(\widehat{L^\vee} \otimes M \otimes P)) &= R^i\Gamma(X, \omega_X \otimes \lambda^*(\widehat{L^\vee} \otimes M \otimes P)) \\ =^{\text{P.F.}}\ R^i\Gamma(A, \mathbf{R}\lambda_*\omega_X \otimes \widehat{L^\vee} \otimes M \otimes P) &= \mathrm{Ext}^i_{D(A)}((M \otimes P)^\vee, \mathbf{R}\lambda_*\omega_X \otimes \widehat{L^\vee}) \\ &=^{(1)} \mathrm{Ext}^{i+g}_{D(\widehat{A})}(R^g\widehat{S}((M \otimes P)^\vee), R^0\Phi(\omega_X \otimes \lambda^*\widehat{L^\vee})) \\ &= H^{i+g}(\widehat{A}, R^0\Phi(\omega_X \otimes \lambda^*\widehat{L^\vee}) \otimes R^g\widehat{S}((M \otimes P)^\vee)^\vee) = 0 \qquad i > 0. \end{aligned}$$

(The third equality follows as $M \otimes P$ is free, the fifth follows since $R^g\widehat{S}(M \otimes P)^\vee$ is free, and the last one since $i + g > g = \dim \widehat{A}$.) □

Let $\phi_L\colon \widehat{A} \to A$ be the isogeny induced by $\phi_L(\widehat{x}) = t_{\widehat{x}}^*L \otimes L^\vee$, then $\phi_L^*\widehat{L^\vee} = L^{\oplus h^0(L)}$. We may assume that the characteristic does not divide the degree of L, so that ϕ_L is separable. Let $X' = X \times_A \widehat{A}$, $\phi\colon X' \to X$, and $\lambda'\colon X' \to \widehat{A}$ be the induced morphisms. Note that $\phi_*\mathscr{O}_{X'} = \lambda^*(\phi_{L*}\mathscr{O}_{\widehat{A}}) = \lambda^*(\oplus P_{\alpha_i})$, where the α_i are the elements in $K \subset \widehat{A}$, the kernel of the induced homomorphism $\phi_L : \widehat{A} \to A$. By the above equation and a flat base change,

$$H^i(X', \omega_{X'} \otimes \lambda'^* \phi_L^* (\widehat{L^\vee} \otimes M)) = \bigoplus_{\alpha \in K} H^i(X, \omega_X \otimes \lambda^* (\widehat{L^\vee} \otimes M \otimes P_\alpha)) = 0$$

for all $i > 0$. But then $H^i(X', \omega_{X'} \otimes \lambda'^*(L \otimes \phi_L^* M)) = 0$ for all $i > 0$. Note that if M is sufficiently ample on A then so is $L \otimes \phi_L^* M$ on $\widehat{A}$. It follows by an easy (and standard) spectral sequence argument that $R^i \lambda'_* \omega_{X'} = 0$ for $i > 0$. Since ω_X is a summand of $\phi_* \omega_{X'} = \mathbf{R}\phi_* \omega_{X'}$, and $\mathbf{R}\lambda_* \mathbf{R}\phi_* \omega_{X'} = \mathbf{R}\phi_{L*} \mathbf{R}\lambda'_* \omega_{X'}$, it follows that $R^i \lambda_* \omega_X$ is a summand of $R^i \lambda_* \phi_* \omega_{X'} = \phi_{L*} R^i \lambda'_* \omega_{X'}$ and hence $R^i \lambda_* \omega_X = 0$ for all $i > 0$. □

Proof of Theorem 1.3 Immediate from Propositions 3.13 and 4.1. □

Acknowledgments The authors would like to thank G. Pareschi, A. Langer, and M. Popa for useful discussions and comments.

References

[CH11] Chen, J. A. and Hacon, C. D. Kodaira dimension of irregular varieties. *Invent. Math.* **186**(3) (2011) 481–500.

[CR11b] Chatzistamatiou, A. and Rülling, K. Hodge Witt cohomology and Witt-rational singularities. arXiv:1104.2145v1.

[GL91] Green, M. and Lazarsfeld, R. Higher obstructions to deforming cohomology groups of line bundles. *J. Amer. Math. Soc.* **4** (1991) 87–103.

[EGAIII$_2$] Grothendieck, A. Éléments de géométrie algébrique, III (seconde partie). *Inst. Hautes Études Sci. Publ. Math.* **17** (1963) 91.

[Hac04] Hacon, C. A derived category approach to generic vanishing. *J. Reine Angew. Math.* **575** (2004) 173–187.

[Har66] Hartshorne, R. Residues and Duality. Lecture notes of a seminar on the work of A. Grothendieck given at Harvard University 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Berlin: Springer-Verlag, 1966.

[Har77] Hartshorne, R. Algebraic Geometry. Graduate Texts in Mathematics, No. 52. New York: Springer-Verlag, 1977.

[JLT11] Jiang, Z., Lahoz, M., and Tirabassi, S. On the Iitaka fibration of varieties of maximal Albanese dimension. arXiv:1111.6279.

[Kem93] Kempf, G. R. Algebraic Varieties. London Mathematical Society Lecture Note Series, Vol. 172. Cambridge: Cambridge University Press, 1993.

[LR97] Lauritzen, N. and Rao, A. P. Elementary counterexamples to Kodaira vanishing in prime characteristic. *Proc. Indian Acad. Sci.* **107** (1997) 21–25.

[Muk79] Mukai, S. On counterexamples for the Kodaira vanishing theorem and the Yau inequality in positive characteristics. In Symposium on Algebraic Geometry, Kinosaki, 1979, p. 923 (in Japanese).

[Muk81] Mukai, S. Duality between $D(X)$ and $D(\widehat{X})$ with its application to Picard sheaves. *Nagoya Math. J.* **81** (1981) 153–175.

[Par03] Pareschi, G.: Generic vanishing, Gaussian maps, and Fourier–Mukai transform. math.arXiv:0310026v1.

[PP03] Pareschi, G. and Popa, M.: Regularity on abelian varieties I. *J. Amer. Math. Soc.* **16**(2) (2003) 285–302.

[PP09] Pareschi, G. and Popa, M. Strong generic vanishing and a higher dimensional Castelnuovo–de Franchis inequality. *Duke Math. J.* **150**(2) (2009) 269–285.

[PP11] Pareschi, G. and Popa, M.: GV-sheaves, Fourier–Mukai transform, and generic vanishing. *Amer. J. Math.* **133** (2011) 235–271.

[Ray78] Raynaud, M. Contre-exemple au "vanishing theorem" en caractéristique $p > 0$. C. P. Ramanujam—A Tribute. Tata Institute of Fundamental Research Studies in Mathematics, Vol. 8. Berlin: Springer-Verlag, 1978, pp. 273–278.