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Deformations of elliptic Calabi–Yau manifolds

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Abstract

We investigate deformations and characterizations of elliptic Calabi–Yau varieties, building on earlier works of Wilson and Oguiso. We show that if the second cohomology of the structure sheaf vanishes, then every deformation is again elliptic. More generally, all non-elliptic deformations derive from abelian varieties or K3 surfaces. We also give a numerical characterization of elliptic Calabi–Yau varieties under some positivity assumptions on the second Todd class. These results lead to a series of conjectures on fibered Calabi–Yau varieties.

To Robert Lazarsfeld on the occasion of his sixtieth birthday

The aim of this paper is to answer some questions about Calabi–Yau manifolds that were raised during the workshop *String Theory for Mathematicians,* which was held at the Simons Center for Geometry and Physics.

F-theory posits that the "hidden dimensions" constitute a Calabi–Yau 4-fold X that has an elliptic structure with a section. That is, there are morphisms $g\colon X \to B$ whose general fibers are elliptic curves and $\sigma\colon B \to X$ such that $g \circ \sigma = 1_B$ (see [Vaf96, Don98]). In his lecture, Donagi asked the following:

Question 1 Is every small deformation of an elliptic Calabi–Yau manifold also an elliptic Calabi–Yau manifold?

Question 2 Is there a good numerical characterization of elliptic Calabi–Yau manifolds?

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Clearly, an answer to Question 2 should give a solution of Question 1. The answers to these problems are quite sensitive to which variant of the definition of Calabi–Yau manifolds one uses. For instance, a general deformation of the product of an Abelian variety and of an elliptic curve has no elliptic fiber space structure and every elliptic K3 surface has non-elliptic deformations. We prove in Section 5 that these are essentially the only such examples, even for singular Calabi–Yau varieties (Theorem 31). In the smooth case, the answer is especially simple.

Theorem 3 *Let X be an elliptic Calabi–Yau manifold such that $H^2(X, O_X) = 0$. Then every small deformation of X is also an elliptic Calabi–Yau manifold.*

In dimension 3 this was proved in [Wil94, Wil98].

Our results on Question 2 are less complete. Let $L_B \in H^2(B, \mathbb{Q})$ be an ample cohomology class and set $L := g^* L_B$. We interpret Question 2 to mean: *Characterize pairs (X, L) that are elliptic fiber spaces.* Following [Wil89, Ogu93], one is led to the following:

Conjecture 4 *A Calabi–Yau manifold X is elliptic iff there is a $(1,1)$-class $L \in H^2(X, \mathbb{Q})$ such that $(L \cdot C) \geq 0$ for every algebraic curve $C \subset X$, $(L^{\dim X}) = 0$ and $(L^{\dim X - 1}) \neq 0$.*

For threefolds, the more general results of [Ogu93, Wil94] imply Conjecture 4 if L is effective or $(L \cdot c_2(X)) \neq 0$. As in the earlier works, in higher dimensions we study the interrelation of L and of the second Chern class $c_2(X)$. By a result of [Miy88], $(L^{n-2} \cdot c_2(X)) \geq 0$ and we distinguish two cases.

- (Main case) If $(L^{n-2} \cdot c_2(X)) > 0$ then Conjecture 4 is solved in Corollary 11. We also check that all elliptic Calabi–Yau manifolds with a section belong to this class (46).
- (Isotrivial case) If $(L^{n-2} \cdot c_2(X)) = 0$ then there is an elliptic curve E, a finite subgroup $G \subset \mathrm{Aut}(E)$, and a Calabi–Yau manifold Y with a G-action such that $X \cong (E \times Y)/G$ (see Theorem 43). If, in addition, $H^2(X, O_X) = 0$ then by Theorem 39 every deformation of X is obtained by deforming E and Y.

  However, I have not been able to prove that the numerical conditions of Conjecture 4 guarantee the existence of an elliptic structure.

Following [Ogu93] and [MP97, Lecture 10], the plan is to put both questions in the more general framework of the abundance conjecture [Rei83, 4.6]; see Conjectures 50 and 51 for the precise formulation.

This approach suggests that the key is to understand the rate of growth of $h^0(X, L^m)$. If (X, L) is elliptic, then $h^0(X, L^m)$ grows like $m^{\dim X - 1}$. Given a pair

(X, L), the most important deformation-invariant quantity is the holomorphic Euler characteristic

$$\chi(X, L^m) = h^0(X, L^m) - h^1(X, L^m) + h^2(X, L^m) \cdots$$

The difficulty is that in our case $h^0(X, L^m)$ and $h^1(X, L^m)$ both grow like $m^{\dim X - 1}$ and they cancel each other out. That is,

$$\chi(X, L^m) = O(m^{\dim X - 2}).$$

For the main series $\chi(X, L^m)$ does grow like $m^{\dim X - 2}$, which implies that $h^0(X, L^m)$ grows at least like $m^{\dim X - 2}$.

For the isotrivial series the order of growth of $\chi(X, L^m)$ is even smaller; in fact, $\chi(X, L^m)$ can be identically zero.

Several of the ideas of this paper can be traced back to other sources. Sections 2–4 owe a lot to [Kaw85a, Ogu93, Wil94, Fuj11]; Sections 5 and 6 to [Hor76, KL09]; Sections 7 and 8 to [Kol93, Nak99] and to some old results of Matsusaka. Ultimately the origin of many of these methods is the work of Kodaira on elliptic surfaces [Kod63, Section 12]. (See [BPV84, Sections V.7–13] for a more modern treatment.)

1 Calabi–Yau fiber spaces

For many reasons it is of interest to study proper morphisms with connected fibers $g' : X' \to B$ whose general fibers are birational to Calabi–Yau varieties. A special case of the minimal model conjecture, proved by [Lai11, HX13], implies that every such fiber space is birational to a projective morphism with connected fibers $g: X \to B$ where X has terminal singularities and its canonical class K_X is relatively trivial, at least rationally. That is, there is a Cartier divisor F on B such that $mK_X \sim g^*F$ for some $m > 0$.

We will work with varieties with log terminal singularities, or later even with klt pairs (X, Δ), but I will state the main results for smooth varieties as well. See [KM98, Section 2.3] for the definitions and basic properties of the singularities we use. Note also that, even if one is primarily interested in smooth Calabi–Yau varieties X, the natural setting is to allow at least canonical singularities on X and at least log terminal singularities on the base B of the elliptic fibration.

Definition 5 In this paper a *Calabi–Yau variety* is a projective variety X with log terminal singularities such that $K_X \sim_{\mathbb{Q}} 0$, that is, mK_X is linearly equivalent to 0 for some $m > 0$. By [Kaw85b] this is equivalent to assuming that $(K_X \cdot C) = 0$ for every curve $C \subset X$.

Note that we allow a rather broad definition of Calabi–Yau varieties. This is very natural for algebraic geometry but less so for physical considerations.

A *Calabi–Yau fiber space* is a proper morphism with connected fibers $g\colon X \to B$ onto a normal variety where X has log terminal singularities and $K_{X_g} \sim_{\mathbb{Q}} 0$ where $X_g \subset X$ is a general fiber.

We say that $g\colon X \to B$ is an *elliptic* (or *Abelian*, etc.) fiber space if in addition general fibers are elliptic curves (or Abelian varieties, etc.). Our main interest is in the elliptic case, but in Sections 7 and 8 we also study the general setting.

Let X be a projective, log terminal variety and L a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor (or divisor class) on X. We say that (X, L) is a Calabi–Yau fiber space if there is a Calabi–Yau fiber space $g\colon X \to B$ and an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L_B on B such that $L \sim_{\mathbb{Q}} g^*L_B$.

In general, a divisor L is called *semi-ample* if it is the pull-back of an ample divisor by a morphism and *nef* if $(L \cdot C) \geq 0$ for every irreducible curve $C \subset X$. Every semi-ample divisor is nef, but the converse usually fails. However, the hope is that for Calabi–Yau varieties nef and semi-ample are essentially equivalent; see Conjectures 50 and 51.

We say that a Calabi–Yau fiber space $g\colon X \to B$ is *relatively minimal* if $K_X \sim_{\mathbb{Q}} g^*F$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor F on B. This condition is automatic if X itself is Calabi–Yau. (These are called crepant log structures in [Kol13b].)

If $g\colon X \to B$ is a relatively minimal Calabi–Yau fiber space and X has canonical (resp. log terminal) singularities, then every other relatively minimal Calabi–Yau fiber space $g'\colon X' \to B$ that is birational to $g\colon X \to B$ also has canonical (resp. log terminal) singularities.

By [Nak88], if X has log terminal singularities then B has rational singularities; more precisely, there is an effective divisor D_B such that (B, D_B) is klt.

6 (Elliptic threefolds) Elliptic threefolds have been studied in detail. The papers [Wil89, Gra91, Nak91, Gra93, DG94, Gra94, Gro94, Wil94, Gro97, Nak02a, Nak02b, CL10, HK11, Klo13] give rather complete descriptions of their local and global structure. However, neither Question 1 nor 2 was fully answered for threefolds.

By contrast, not even the local structure of elliptic fourfolds is understood. Double covers of the $\mathbb{P}^1$-contractions described in [AW98] give some rather surprising examples; there are probably much more complicated ones as well.

Definition 7 Let $g\colon X \to B$ be a morphism between normal varieties. A divisor $D \subset X$ is called *horizontal* if $g(D) = B$, *vertical* if $g(D) \subset B$ has codimension ≥ 1, and *exceptional* if $g(D)$ has codimension ≥ 2 in B.

If g is birational the latter coincides with the usual notion of exceptional divisors, but the above version makes sense even if $\dim X > \dim B$. (If g is birational then there are no horizontal divisors.)

8 (Birational models of Calabi–Yau fiber spaces) We see in Lemma 18 that if X is smooth (or $\mathbb{Q}$-factorial), g is a Calabi–Yau fiber space, and $D \subset X$ is exceptional then D is not g-nef. Thus, by [Lai11, HX13] the $(X, \epsilon D)$ minimal model program over B (cf. [KM98, Section 3.7]) contracts D. Thus every Calabi–Yau fiber space $g_2 \colon X_2 \to B_2$ is birational to a relatively minimal Calabi–Yau fiber space $g_1 \colon X_1 \to B_1 = B_2$ that has no exceptional divisors. Furthermore, let $B \to B_1$ be a small $\mathbb{Q}$-factorialization. Let L_B be an effective, ample divisor and L_1 its pull-back to X_1 by the rational map $X_1 \dashrightarrow B$. Applying [Lai11, HX13] and Theorem 14 to $(X_1, \epsilon L_1)$, we get a birational model $g \colon X \to B$ where B is also $\mathbb{Q}$-factorial. (In general, such a model is not unique.) Thus, in birational geometry, it is reasonable to focus on the study of relatively minimal Calabi–Yau fiber spaces $g \colon X \to B$ without exceptional divisors, where X and B are $\mathbb{Q}$-factorial and X is log terminal.

Let $\phi \colon X_1 \dashrightarrow X_2$ be a birational equivalence of two relatively minimal Calabi–Yau fiber spaces $g_i \colon X_i \to B$. Thus ϕ is an isomorphism between dense open sets $\phi \colon X_1^0 \cong X_2^0$. If the X_i are smooth (or they have terminal singularities) then we can choose these sets such that their complements $X_i \setminus X_i^0$ have codimension ≥ 2 (cf. [KM98, 3.52.2]). More generally, this holds if there are no exceptional divisors E_i of discrepancy 0 over X_i such that the center of E_i on X_i is disjoint from X_i^0.

Even in the smooth case, ϕ can be a rather complicated composite of flops.

From the point of view of F-theory it is especially interesting to study the examples $g' \colon X' \to B$ with a section $\sigma' \colon B \to X'$, where X' itself is Calabi–Yau. In this case the so-called *Weierstrass model* is a relatively minimal model without exceptional divisors that can be constructed explicitly as follows.

Let L_B be an ample divisor on B. Then $\sigma'(B) + mg'^* L_B$ is nef and big on X' for $m \gg 1$, hence a large multiple of it is base point free (cf. [KM98, Section 3.2]). This gives a morphism $h \colon X' \to X$, where X is still Calabi–Yau (usually with canonical singularities) and $g \colon X \to B$ has a section $\sigma \colon B \to X$ whose image is g-ample. Thus every fiber of g has dimension 1 and so $g \colon X \to B$ has no exceptional divisors.

Furthermore, $R^1 h_* \mathcal{O}_{X'} = 0$ which implies that every deformation of X' comes from a deformation of X (see 53).

The next result says that once $g \colon X \to B$ looks like a relatively minimal Calabi–Yau fiber space outside a subset of codimension ≥ 2, then it is a relatively minimal Calabi–Yau fiber space.

Proposition 9 *Let $g\colon X \to B$ be a projective fiber space with X log terminal. Assume the following:*

(1) There are no g-exceptional divisors (Definition 7).
(2) There is a closed subset $Z \subset B$ of codimension ≥ 2 such that K_X is numerically trivial on the fibers over $B \setminus Z$.
(3) B is $\mathbb{Q}$-factorial.

Then $g\colon X \to B$ is a relatively minimal Calabi–Yau fiber space.

Proof First note that, as a very special case of Theorem 14, there is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor F_1 on $B \setminus Z$ such that

$$K_X|_{X \setminus g^{-1}(Z)} \sim_{\mathbb{Q}} g^* F_1.$$

Since B is $\mathbb{Q}$-factorial, F_1 extends to a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor F on B.

Thus every point $b \in B$ has an open neighborhood $b \in U_b \subset B$ and an integer $m_b > 0$ such that

$$\mathcal{O}_X(m_b K_X)|_{g^{-1}(U_b \setminus Z)} \cong g^* \mathcal{O}_{U_b}(m_b F|_{U_b}) \cong g^* \mathcal{O}_{U_b} \cong \mathcal{O}_{g^{-1}(U_b)}.$$

By (1), $g^{-1}(Z)$ has codimension ≥ 2 in $g^{-1}(U_b)$ and hence the constant 1 section of $\mathcal{O}_{g^{-1}(U_b \setminus Z)}$ extends to a global section of $\mathcal{O}_X(m_b K_X)|_{g^{-1}(U_b)}$ that has neither poles nor zeros. Thus

$$\mathcal{O}_X(m_b K_X)|_{g^{-1}(U_b)} \cong \mathcal{O}_{g^{-1}(U_b)}.$$

Since this holds for every $b \in B$, we conclude that $K_X \sim_{\mathbb{Q}} g^* F$. □

2 The main case

The next theorem gives a characterization of the main series of elliptic Calabi–Yau fiber spaces. (For the log version, see 54.) The proof is quite short but it relies on auxiliary results that are proved in the next two sections.

Theorem 10 *Let X be a projective variety of dimension n with log terminal singularities and L a Cartier divisor on X. Assume that K_X is nef and $(L^{n-2} \cdot \mathrm{td}_2(X)) > 0$, where $\mathrm{td}_2(X)$ is the second Todd class of X (24). Then (X, L) is a relatively minimal, elliptic fiber space iff*

(1) L is nef,
(2) $L - \epsilon K_X$ is nef for $0 \leq \epsilon \ll 1$,
(3) $(L^n) = 0$, and
(4) (L^{n-1}) is nonzero in $H^{2n-2}(X, \mathbb{Q})$.

Note that if (X, L) is a relatively minimal elliptic fiber space then L is semi-ample (Definition 5) and, as we see in 13 below, the only hard part of Theorem 10 is to show that conditions (1)–(4) imply L is semi-ample. In particular, Theorem 10 also holds over fields that are not algebraicaly closed.

This immediately yields the following partial answer to Question 1:

Corollary 11 *Let X be a smooth, projective variety of dimension n and L a Cartier divisor on X. Assume that $K_X \sim_{\mathbb{Q}} 0$ and $(L^{n-2} \cdot c_2(X)) > 0$. Then (X, L) is an elliptic fiber space iff*

(1) L is nef,
(2) $(L^n) = 0$, and
(3) (L^{n-1}) is nonzero in $H^{2n-2}(X, \mathbb{Q})$. □

Definition 12 Let Y be a projective variety and D a Cartier divisor on X. If $m > 0$ is sufficiently divisible, then, up to birational equivalence, the map given by global sections of $\mathcal{O}_Y(mD)$

$$Y \dashrightarrow I(Y, D) \overset{bir}{\sim} I_m(Y, D) \hookrightarrow \mathbb{P}(H^0(Y, \mathcal{O}_Y(mD))) \quad \text{is independent of } m.$$

It is called the *Iitaka fibration* of (Y, D). The *Kodaira dimension* of D (or of (Y, D)) is $\kappa(D) = \kappa(Y, D) := \dim I(Y, D)$.

If D is nef, the *numerical dimension* of D (or of (Y, D)), denoted by $\nu(D)$ or $\nu(Y, D)$, is the largest natural number r such that the self-intersection $(D^r) \in H^{2r}(Y, \mathbb{Q})$ is nonzero. Equivalently, $(D^r \cdot H^{n-r}) > 0$ for some (or every) ample divisor H.

It is easy to see that $\kappa(D) \leq \nu(D)$. This was probably first observed by Matsusaka as a corollary of his theory of variable intersection cycles (see [Mat72] or [LM75, p. 515]).

13 (Proof of Theorem 10) First note that $\kappa(L) \geq n - 2$ by Lemma 25. We will also need this for some perturbations of L.

By Theorem 10, (2) and (3) we have $0 = (L^n) \geq \epsilon(L^{n-1} \cdot K_X) \geq 0$, thus $(L^{n-1} \cdot K_X) = 0$.

Set $L_m := L - \frac{1}{m} K_X$. For $m \gg 1$ we see that L_m is nef, $(L_m^{n-2} \cdot \mathrm{td}_2(X)) > 0$, and (L_m^{n-1}) is nonzero in $H^{2n-2}(X, \mathbb{Q})$. Note that $mL = K_X + mL_m$, hence

$$m^n (L^n) = \textstyle\sum_{i=0}^n m^{n-i} (K_X^i \cdot L_m^{n-i}).$$

Since K_X and L_m are both nef, all the terms on the RHS are ≥ 0. Their sum is zero by assumption, hence $(K_X^i \cdot L_m^{n-i}) = 0$ for every i. Thus Lemma 25 also applies to L_m and we get that $\kappa(L_m) \geq n - 2$.

We can now apply Proposition 15 with $\Delta = 0$, $D := 2mL_m$, and $K_X + 2mL_m = 2mL_{2m}$ to conclude that $\nu(L_m) \le \kappa(L_{2m})$. Since we know that $\nu(L_m) = \dim X - 1$, we conclude that $\kappa(L_{2m}) = \dim X - 1$.

Finally, use Theorem 14 with S = (point), $2mL$ instead of L and $a = 1$ to obtain that some multiple of L is semi-ample. That is, there is a morphism with connected fibers $g: X \to B$ and an ample $\mathbb{Q}$-divisor L_B such that $L \sim_{\mathbb{Q}} g^* L_B$. Note that $(L^{\dim B}) \neq 0$ but $(L^{\dim B+1}) = 0$ so, comparing with Theorem 10, (3) and (4), we see that $\dim B = \dim X - 1$. By the adjunction formula, the canonical class of the general fiber is proportional to $(L^{n-1} \cdot K_X) = 0$, thus $g: X \to B$ is an elliptic fiber space. □

We have used the following theorem due to [Kaw85a] and [Fuj11]:

Theorem 14 *Let (X, Δ) be an irreducible, projective, klt pair and $g: X \to S$ a morphism with generic fiber X_g. Let L be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Assume that*

(1) L and $L - K_X - \Delta$ are g-nef and
(2) $\nu((L - K_X - \Delta)|_{X_g}) = \kappa((L - K_X - \Delta)|_{X_g}) = \nu(((1 + a)L - K_X - \Delta)|_{X_g}) = \kappa(((1 + a)L - K_X - \Delta)|_{X_g})$ for some $a > 0$.

Then there is a factorization $g: X \stackrel{h}{\to} B \stackrel{\pi}{\to} S$ and a π-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L_B on B such that $L \sim_{\mathbb{Q}} h^* L_B$. □

3 Adjoint systems of large Kodaira dimension

The following is modeled on [Ogu93, 2.4]:

Proposition 15 *Let (X, Δ) be a projective, klt pair such that $K_X + \Delta$ is pseudo-effective, that is, its cohomology class is a limit of effective classes. Let D be an effective, nef, $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $\kappa(K_X + \Delta + D) \ge \dim X - 2$.*

Then $\nu(D) \le \kappa(K_X + \Delta + D)$.

Proof There is nothing to prove if $\kappa(K_X + \Delta + D) = \dim X$. Thus assume that $\kappa(K_X + \Delta + D) \le \dim X - 1$ and let $g: X \dashrightarrow B$ be the Iitaka fibration (cf. [Laz04, 2.1.33]). After some blow-ups we may assume in addition that g is a morphism and X, B are smooth.

The generic fiber of g is a smooth curve or surface (S, Δ_S) such that $K_S + \Delta_S$ is pseudo-effective. Since abundance holds for curves and surfaces [Kol92, Section 11], this implies that $\kappa(K_S + \Delta_S) \ge 0$. Furthermore, by Iitaka's theorem (cf. [Laz04, 2.1.33]), $\kappa(K_S + \Delta_S + D|_S) = 0$.

If D is disjoint from S then, by Lemma 17, (2), $\nu(D) \le \dim B = \kappa(K_X+\Delta+D)$ and we are done. Otherwise $D|_S$ is an effective, nonzero, nef divisor on S. We obtain a contradiction by proving that $\kappa(K_S + \Delta_S + D|_S) \ge 1$.

If S is a curve, then $\deg D|_S > 0$ and hence $\kappa(K_S + \Delta_S + D|_S) \ge \kappa(D|_S) = 1$. If S is a surface, then $\kappa(K_S + \Delta_S + D|_S) \ge 1$ is proved in Lemma 16. □

Lemma 16 *Let (S, Δ_S) be a projective, klt surface such that $\kappa(K_S + \Delta_S) \ge 0$. Let D be a nonzero, effective, nef $\mathbb{Q}$-divisor. Then $\kappa(K_S + \Delta_S + D) \ge 1$.*

Proof Since $\kappa(K_S + \Delta_S + D) \ge \kappa(K_S + \Delta_S)$ we only need to consider the case when $\kappa(K_S + \Delta_S) = 0$. Let $\pi\colon (S, \Delta_S) \to (S^m, \Delta_S^m)$ be the minimal model. It is obtained by repeatedly contracting curves that have negative intersection number with $K_S + \Delta_S$. These curves also have negative intersection number with $K_S + \Delta_S + \epsilon D$ for $0 < \epsilon \ll 1$. Thus

$$\pi\colon (S, \Delta_S + \epsilon D) \to (S^m, \Delta_S^m + \epsilon D^m)$$

is also the minimal model and $(S^m, \Delta^m + \epsilon D^m)$ is klt for $0 < \epsilon \ll 1$. By the Hodge index theorem, every effective divisor contracted by π has negative self-intersection, thus D cannot be π-exceptional. So D^m is again a nonzero, effective, nef $\mathbb{Q}$-divisor.

Since abundance holds for klt surface pairs (cf. [Kol92, Section 11]), we see that $K_{S^m} + \Delta^m \sim_{\mathbb{Q}} 0$ and $\kappa\big(K_{S^m} + \Delta^m + \epsilon D^m\big) \ge 1$. Since D is effective, we obtain that $\kappa(K_S + \Delta_S + D) \ge \kappa(K_S + \Delta_S + \epsilon D) = \kappa\big(K_{S^m} + \Delta^m + \epsilon D^m\big) \ge 1$. □

Lemma 17 *Let $g\colon X \to B$ be a proper morphism with connected general fiber X_g. Let D be an effective, nef, $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then*

(1) either $D|_{X_g}$ is a nonzero nef divisor,
(2) or D is disjoint from X_g and $(D^{\dim B+1}) = 0$. Thus $\nu(D) \le \dim B$.

Proof We are done if $D|_{X_g}$ is nonzero. If it is zero then D is vertical, hence there is an ample divisor L_B such that $g^*L_B \sim D + E$ where E is effective. Then

$$(g^*L_B^r) - (D^r) = \textstyle\sum_{i=0}^{r-1} (E \cdot g^*L_B^i \cdot D^{r-1-i})$$

shows that $(D^r) \le (g^*L_B^r)$. Since $((g^*L_B)^{\dim B+1}) = g^*(L_B^{\dim B+1}) = 0$, we conclude that $(D^{\dim B+1}) = 0$. □

Lemma 18 *Let $g\colon X \to B$ be a proper morphism with connected fibers and D an effective, exceptional, $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then D is not g-nef.*

Proof Let $|H|$ be a very ample linear system on X and $S \subset X$ the intersection of $\dim X - 2$ general members of $|H|$. Then $g|_S : S \to B$ is generically finite over its image and $D \cap S$ is $g|_S$-exceptional. By the Hodge index theorem we conclude that $(D^2 \cdot H^{\dim X-2}) < 0$, a contradiction. □

4 Asymptotic estimates for cohomology groups

19 Let X be a smooth variety and $g\colon X \to B$ a Calabi–Yau fiber space of relative dimension m over a smooth curve B. Assume that $K_{X_g} \sim 0$, where X_g denotes a general fiber. It is easy to see that the sheaves $R^m g_* \mathcal{O}_X$ and $g_* \omega_{X/B}$ are line bundles and dual to each other. For elliptic surfaces these sheaves were computed by Kodaira. His results were clarified and extended to higher dimensions by [Fuj78]. We will need the following consequences of their results.

The degree of $g_* \omega_{X/B}$ is ≥ 0 and can be written as a sum of two terms. One is a global term (determined by the j-invariant of the fibers in the elliptic case) which is zero iff $g\colon X \to B$ is *generically isotrivial*, that is, g is an analytically locally trivial fiber bundle over a dense open set $B^0 \subset B$. The other is a local term, supported at the points where the local monodromy of the local system $R^m g_* \mathbb{Q}_{X^0}$ is nontrivial. There is a precise formula for the local term, but we only need to understand what happens with generically isotrivial families. For these the local term is positive iff the local monodromy has eigenvalue $\neq 1$ on $g_* \omega_{X^0/B^0} \subset \mathcal{O}_{B^0} \otimes_{\mathbb{Q}} R^m g_* \mathbb{Q}_{X^0}$.

Over higher-dimensional bases, $R^m g_* \mathcal{O}_X$ and $g_* \omega_{X/B}$ are rank 1 sheaves, and the above considerations describe their codimension 1 behavior. In particular, we see the following:

(1) $c_1(g_* \omega_{X/B})$ is linearly equivalent to a sum of effective $\mathbb{Q}$-divisors. It is zero only if $g\colon X \to B$ is isotrivial over a dense open set B^0 and the local monodromy around each irreducible component of $B \setminus B^0$ has eigenvalue $= 1$ on $g_* \omega_{X^0/B^0} \subset \mathcal{O}_{B^0} \otimes_{\mathbb{Q}} R^m g_* \mathbb{Q}_{X^0}$.

(2) $c_1(R^m g_* \mathcal{O}_X) = -c_1(g_* \omega_{X/B})$.

Frequently $c_1(g_* \omega_{X/B})$ is denoted by $\Delta_{X/B}$.

Corollary 20 *Let $g\colon X \to B$ be an elliptic fiber space of dimension n and L a line bundle on B. Then*

$$\begin{aligned} \chi(X, g^* L^m) &= \tfrac{(L^{n-2} \cdot \Delta_{X/B})}{(n-2)!} m^{n-2} + O(m^{n-3}) \quad \textit{and} \\ h^i(X, g^* L^m) &= O(m^{n-3}) \quad \textit{for } i \geq 2. \end{aligned}$$

Proof By the Leray spectral sequence,

$$\chi(X, g^* L^m) = \textstyle\sum_i (-1)^i \chi(B, L^m \otimes R^i g_* \mathcal{O}_X).$$

For $i \geq 2$ the support of $R^i g_* \mathcal{O}_X$ has codimension ≥ 2 in B, hence its cohomologies contribute only to the $O(m^{n-3})$ term.

Since g has connected fibers, $g_* \mathcal{O}_X \cong \mathcal{O}_B$ and $c_1(R^1 g_* \mathcal{O}_X) \sim_{\mathbb{Q}} -\Delta_{X/B}$ by 19, item (2). We conclude by applying Lemma 23 to both terms. □

21 Similar formulas apply to arbitrary Calabi–Yau fiber spaces $g\colon X \to B$ with general fiber F. If L is ample on B then, for $m \gg 1$, we have

$$H^i(X, g^*L^m) = H^0(B, L^m \otimes R^i g_*\mathcal{O}_X) = \chi(B, L^m \otimes R^i g_*\mathcal{O}_X). \tag{21.1}$$

Setting $k = \dim B$, Lemma 23 computes $H^i(X, g^*L^m)$ as

$$\frac{m^k}{k!} h^i(F, \mathcal{O}_F)(L^k) + \frac{m^{k-1}}{(k-1)!}\Big(L^{k-1} \cdot \big(c_1(R^i g_*\mathcal{O}_X) - \tfrac{h^i(F,\mathcal{O}_F)}{2} K_B\big)\Big) + O(m^{k-2}).$$

These imply that

$$\chi(X, g^*L^m) = \chi(F, \mathcal{O}_F) \cdot \frac{m^k}{k!}(L^k) + O(m^{k-1}). \tag{21.2}$$

If $\chi(F, \mathcal{O}_F) \neq 0$ then this describes the asymptotic behavior of $\chi(X, g^*L^m)$. However, if $\chi(F, \mathcal{O}_F) = 0$, which happens for Abelian fibers, then we have to look at the next term, which gives that

$$\chi(X, g^*L^m) = \frac{m^{k-1}}{(k-1)!}\Big(L^{k-1} \cdot \textstyle\sum_{i=1}^{\dim F}(-1)^i c_1(R^i g_*\mathcal{O}_X)\Big) + O(m^{k-2}). \tag{21.3}$$

If F is an elliptic curve then the sum on the RHS has only one nonzero term. For higher-dimensional Abelian fibers there are usually several nonzero terms and sometimes they cancel each other.

This is one reason why elliptic fibers are easier to study than higher-dimensional Abelian fibers. The other difficulty with higher-dimensional fibers is that the Euler characteristic only tells us that $h^0 + h^2 + h^4 + \cdots$ grows as expected. Proving that $h^0 \neq 0$ would need additional arguments.

The next result, while stated in all dimensions, is truly equivalent to Kodaira's formula [BPV84, V.12.2].

Corollary 22 *Let $g\colon X \to B$ be a relatively minimal elliptic fiber space of dimension n and L a line bundle on B. Then* $(L^{n-2} \cdot \Delta_{X/B}) = (g^*L^{n-2} \cdot \mathrm{td}_2(X))$.

Proof Expanding the Riemann–Roch formula $\chi(X, g^*L) = \int_X \mathrm{ch}(g^*L) \cdot \mathrm{td}(X)$ and taking into account that $(g^*L^n) = (g^*L^{n-1} \cdot K_X) = 0$ gives

$$\chi(X, g^*L^m) = \frac{(g^*L^{n-2} \cdot \mathrm{td}_2(X))}{(n-2)!} \cdot m^{n-2} + O(m^{n-3}).$$

Comparing this with Corollary 20 yields the claim. □

We used several versions of the asymptotic Riemann–Roch formula.

Lemma 23 *Let Y be a normal, projective variety of dimension n, L a line bundle on X, and F a coherent sheaf of rank r that is locally free in codimension 1. Then*

$$\chi(Y, L^m \otimes F) = \frac{(L^n)\cdot r}{n!}m^n + \frac{(L^{n-1}\cdot(c_1(F) - \frac{r}{2}K_Y))}{(n-1)!}m^{n-1} + O(m^{n-2}). \quad \square$$

24 (Riemann–Roch with rational singularities) The Todd classes of a singular variety X are not always easy to compute, but if X has rational singularities then there is a straightforward formula in terms of the Chern classes of any resolution $h\colon X' \to X$.

By definition, rational singularity means that $R^i h_* \mathcal{O}_{X'} = 0$ for $i > 0$. Thus $\chi(X, L) = \chi(X', h^*L)$ for any line bundle L on X. By the projection formula this implies that $\chi(X, L) = \int_X \mathrm{ch}(L)\cdot h_*\,\mathrm{td}(X')$ and in fact $\mathrm{td}(X) = h_*\,\mathrm{td}(X')$ (cf. [Ful98, Theorem 18.2].) In particular, we see that the second Todd class of X is

$$\mathrm{td}_2(X) = h_*\Big(\frac{c_1(X')^2 + c_2(X')}{12}\Big).$$

The following numerical version of Corollary 20 was used in the proof of Theorem 10.

Lemma 25 *Let X be a normal, projective variety of dimension n. Let L be a nef line bundle on X such that* $(L^n) = (L^{n-1}\cdot K_X) = 0$ *but* $(L^{n-1}) \neq 0$. *Then*

$$h^0(X, L^m) - h^1(X, L^m) = \frac{(L^{n-2}\cdot \mathrm{td}_2(X))}{(n-2)!}\cdot m^{n-2} + O(m^{n-3}).$$

Proof The assumptions $(L^n) = (L^{n-1}\cdot K_X) = 0$ imply that the RHS equals $\chi(X, L^m)$. Thus the equality follows if $h^i(X, L^m) = O(m^{n-3})$ for $i \geq 2$. The latter is a special case of Lemma 26. $\square$

Lemma 26 *Let X be a projective variety of dimension n and F a torsion-free coherent sheaf on X. Let L be a nef line bundle on X and set* $d = \nu(X, L)$. *Then*

$$\begin{aligned} h^i(X, F\otimes L^m) &= O(m^d) \quad \text{for } i = 0, \dots, n-d \text{ and} \\ h^{n-j}(X, F\otimes L^m) &= O(m^{j-1}) \quad \text{for } j = 0, \dots, d-1. \end{aligned}$$

Note the key feature of the estimate: the order of growth of H^i is m^d for $i \leq n-d$, then for $i = n-d+1$ it drops by 2 to m^{d-2}, and then it drops by 1 for each increase of i. This strengthens [Laz04, 1.4.40] but the proof is essentially the same.

Proof We use induction on $\dim X$. By Fujita's theorem (cf. [Laz04, 1.4.35]) we can choose a general very ample divisor A on X such that

$$h^i(X, F\otimes \mathcal{O}_X(A)\otimes L^m) = 0 \quad \text{for all } i \geq 1 \text{ and } m \geq 1.$$

We get an exact sequence

$$0 \to F \otimes L^m \to F \otimes \mathcal{O}_X(A) \otimes L^m \to G \otimes L^m \to 0,$$

where G is a torsion-free coherent sheaf on A. For $i \geq 1$ its long cohomology sequence gives surjections (even isomorphisms for $i \geq 2$)

$$H^{i-1}(A, G \otimes L^m) \twoheadrightarrow H^i(X, F \otimes L^m).$$

By induction this shows the claim except for $i = 0$.

One can realize F as a subsheaf of a sum of line bundles, thus it remains to prove that $H^0(X, F \otimes L^m) = O(m^d)$ when $F \cong \mathcal{O}_X(H)$ is a very ample line bundle. The exact sequence

$$0 \to L^m \to \mathcal{O}_X(H) \otimes L^m \to \mathcal{O}_H(H|_H) \otimes L^m \to 0$$

finally reduces the problem to $\kappa(L) \leq \nu(L)$, which was discussed in Definition 12. □

5 Deforming morphisms

Here we answer Question 1, but first two technical issues need to be discussed: the distinction between étale and quasi-étale covers and the existence of non-Calabi–Yau deformations. Both appear only for singular Calabi–Yau varieties.

Definition 27 Following [Cat07], a finite morphism $\pi\colon U \to V$ is called *quasi-étale* if there is a closed subvariety $Z \subset V$ of codimension ≥ 2 such that π is étale over $V \setminus Z$.

If V is a normal variety, then there is a one-to-one correspondence between quasi-étale covers of V and finite, étale covers of $V \setminus \operatorname{sing} V$.

In particular, if X is a Calabi–Yau variety then there is a quasi-étale morphism $X_1 \to X$ such that $K_{X_1} \sim 0$.

Among all such covers $X_1 \to X$ there is a unique smallest one, called the *index 1 cover* of X, which is Galois with cyclic Galois group. We denote it by $X^{\text{ind}} \to X$.

28 (Deformation theory) For a general introduction, see [Har10]. By a deformation of a proper scheme (or analytic space) X we mean a flat, proper morphism $g\colon \mathbf{X} \to (0 \in S)$ to a pointed scheme (or analytic space) together with a fixed isomorphism $X_0 \cong X$.

By a deformation of a morphism of proper schemes (or analytic spaces) $f\colon X \to Y$ we mean a morphism $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ where $\mathbf{X}$ is a deformation of X, $\mathbf{Y}$ is a deformation of Y, and $\mathbf{f}|_{X_0} = f$.

When we say that an assertion holds for all *small deformations* of X, this means that for every deformation $g\colon \mathbf{X} \to (0 \in S)$ there is an étale (or analytic) neighborhood $(0 \in S') \to (0 \in S)$ such that the assertion holds for $g'\colon \mathbf{X} \times_S S' \to (0 \in S')$.

29 (Deformations of Calabi–Yau varieties) Let X be a Calabi–Yau variety. If X is smooth (or has canonical singularities) then every small deformation of X is again a Calabi–Yau variety. This, however, fails in general; see Example 47, where X is a surface with quotient singularities.

Dealing with such unexpected deformations is a basic problem in the moduli theory of higher-dimensional varieties; see [Kol13a, Section 4], [HK10, Section 14B], or [AH11] for a discussion and solutions. For Calabi–Yau varieties one can use a global trivialization of the canonical bundle to get a much simpler answer.

We say that a deformation $g\colon \mathbf{X} \to (0 \in S)$ of X over a reduced, local space S is a *Calabi–Yau deformation* if the following equivalent conditions hold:

(1) Every fiber of g is a Calabi–Yau variety.
(2) The deformation can be lifted to a deformation $g^{\text{ind}}\colon \mathbf{X}^{\text{ind}} \to (0 \in S)$ of X^{ind}, the index 1 cover of X.

Thus, studying Calabi–Yau deformations of Calabi–Yau varieties is equivalent to studying deformations of Calabi–Yau varieties whose canonical class is Cartier. As we noted, for the latter every deformation is automatically a Calabi–Yau deformation. Thus we do not have to deal with this issue at all.

Theorem 30 *Let X be a Calabi–Yau variety and $g\colon X \to B$ an elliptic fiber space. Then at least one of the following holds:*

(1) The morphism g extends to every small Calabi–Yau deformation of X.
(2) There is a quasi-étale cover $\tilde{X} \to X$ such that the Stein factorization $\tilde{g}\colon \tilde{X} \to \tilde{B}$ of $\tilde{X} \to B$ is one of the following:
 (a) $(\tilde{g}\colon \tilde{X} \to \tilde{B}) \cong (p_1\colon \tilde{B} \times (\text{elliptic curve}) \to \tilde{B})$ where p_1 is the first projection or
 (b) $(\tilde{g}\colon \tilde{X} \to \tilde{B}) \cong (p_1\colon \tilde{Z} \times (\text{elliptic K3}) \to \tilde{Z} \times \mathbb{P}^1)$ where $\tilde{Z}$ is a Calabi–Yau variety of dimension $\dim X - 2$ and p_1 is the product of the first projection with the elliptic pencil map of the K3 surface.

Proof As noted in 29, we may assume that $K_X \sim 0$.

By [KMM92] there is a unique map (up to birational equivalence) $h\colon B \dashrightarrow Z$ whose general fiber F is rationally connected and whose target Z is not uniruled by [GHS03]. (See [Kol96, Chapter IV] for a detailed treatment or [AK03] for an introduction.) Next apply [KL09, Theorem 14] to $X \dashrightarrow Z$ to conclude that there is a finite, quasi-étale cover $\tilde{X} \to X$, a product decomposition $\tilde{X} \cong Y \times \tilde{Z}$, and a generically finite map $\tilde{Z} \dashrightarrow Z$ that factors $\tilde{X} \dashrightarrow Z$.

If $\dim Z = \dim B$ then we are in case (2a). If $\dim Z = \dim B - 1$ then the generic fiber of $\tilde{B} \to \tilde{Z}$ is $\mathbb{P}^1$. Furthermore, $\dim Y = 2$, hence either Y is an elliptic K3 surface and we are in case (2b) or Y is an Abelian surface that has an elliptic pencil and after a further cover we are again in case (2a).

It remains to prove that if $\dim F \geq 2$ then the assertion of (1) holds. By Theorem 35 it is sufficient to check that

$$\mathrm{Hom}_B(\Omega_B, R^1 g_* \mathcal{O}_X) = 0.$$

Note that 19 and $\omega_X \sim \mathcal{O}_X$ imply that $R^1 g_* \mathcal{O}_X \cong (g_* \omega_{X/B})^{-1} \cong \omega_B$, at least over the smooth locus of B. Since $R^1 g_* \mathcal{O}_X$ is reflexive by [Kol86a, 7.8], the isomorphism holds everywhere. Thus

$$\mathcal{H}om_B(\Omega_B, R^1 g_* \mathcal{O}_X) \cong \mathcal{H}om_B(\Omega_B, \omega_B) \cong (\Omega_B^{\dim B - 1})^{**},$$

where $(\ )^{**}$ denotes the double dual or reflexive hull. By taking global sections we get that

$$\mathrm{Hom}_B(\Omega_B, R^1 g_* \mathcal{O}_X) = H^0(B, (\Omega_B^{\dim B - 1})^{**}).$$

Let $B' \to B$ be a resolution of singularities such that $B' \to Z$ is a morphism and $F' \subset B'$ a general fiber. Since F' is rationally connected, it is covered by rational curves $C \subset F'$ such that

$$T_{F'}|_C \cong \sum \mathcal{O}_C(a_i) \quad \text{where } a_i > 0 \ \forall\, i;$$

see [Kol96, IV.3.9]. Thus $T_{B'}|_C$ is a sum of line bundles $\mathcal{O}_C(a_i)$ where $a_i > 0$ for $\dim F$ summands and $a_i = 0$ for the rest. Since $\dim F \geq 2$ we conclude that

$$\wedge^{\dim B - 1} T_{B'}|_C \cong \sum \mathcal{O}_C(b_i) \quad \text{where } b_i > 0 \text{ for every } i.$$

By duality this gives that $H^0(B', \Omega_{B'}^{\dim B - 1}) = 0$. Finally we use that B has log terminal singularities by [Nak88] and so [GKKP11] shows that

$$\mathrm{Hom}_B(\Omega_B, R^1 g_* \mathcal{O}_X) = H^0(B, (\Omega_B^{\dim B - 1})^{**}) = H^0(B', \Omega_{B'}^{\dim B - 1}) = 0.$$

□

We are now ready to answer Question 1.

Theorem 31 *Let X be an elliptic Calabi–Yau variety such that $H^2(X, \mathcal{O}_X) = 0$. Then every small Calabi–Yau deformation of X is also an elliptic Calabi–Yau variety.*

Proof Let $g\colon X \to B$ be an elliptic Calabi–Yau variety. By Theorem 30 every small Calabi–Yau deformation of X is also an elliptic Calabi–Yau variety except possibly when there is a quasi-étale cover $\tilde{X} \to X$ such that

(1) either $\tilde{X} \cong \tilde{Z} \times$ (elliptic curve)
(2) or $\tilde{X} \cong \tilde{Z} \times$ (elliptic K3).

In both cases, $\tilde{X}$ can have non-elliptic deformations but we show that these do not correspond to a deformation of X. Here we use that $H^2(X, \mathcal{O}_X) = 0$.

Let $\pi\colon \mathbf{X} \to (0 \in S)$ be a flat deformation of X over a local scheme S. Let L be the pull-back of an ample line bundle from B to X. Since $H^2(X, \mathcal{O}_X) = 0$, L lifts to a line bundle $\mathbf{L}$ on $\mathbf{X}$ (cf. [Gro62, p. 236–16]) thus we get a line bundle $\tilde{\mathbf{L}}$ on $\tilde{\mathbf{X}}$. We need to show that a large multiple of $\mathbf{L}$ is base-point-free over S; then it gives the required morphism $\mathbf{g}\colon \mathbf{X} \to \mathbf{B}$. One can check base-point-freeness of some multiple after a finite surjection, thus it is enough to show that some multiple of $\tilde{\mathbf{L}}$ is base-point-free over S.

The first case (more generally, deformations of products with Abelian varieties) is treated in Lemma 40.

In the K3 case note first that every small deformation of $\tilde{X}$ is of the form $\tilde{\mathbf{Z}} \times_S \tilde{\mathbf{F}}$, where $\tilde{\mathbf{F}} \to S$ is a flat family of K3 surfaces. This is a trivial case of Theorem 35; see 53 for an elementary argument. Hence we only need to show that the restriction of $\tilde{\mathbf{L}}$ to $\tilde{\mathbf{F}}$ is base-point-free over S. Equivalently, that the elliptic structure of the central K3 surface $\tilde{F}$ is preserved by our deformation. The restriction of $\tilde{\mathbf{L}}$ to every fiber of $\tilde{\mathbf{F}} \to S$ gives a nonzero, nef line bundle with self-intersection 0, hence an elliptic pencil. □

32 (Deformation of sections) Let $g\colon X \to B$ be an elliptic Calabi–Yau fiber space with a section $S \subset X$. Let us assume first that S is a Cartier divisor in X. (This is automatic if X is smooth.) Then S is g-nef, g-big, and $S \sim_{\mathbb{Q},g} K_X + S$ hence $R^i g_* \mathcal{O}_X(S) = 0$ for $i > 0$ (cf. [KM98, Section 2.5]). Thus $H^i(X, \mathcal{O}_X(S)) = H^i(B, g_*\mathcal{O}_X(S))$ for every i. In order to compute $g_*\mathcal{O}_X(S)$ we use the exact sequence

$$0 \to \mathcal{O}_B = g_*\mathcal{O}_X \xrightarrow{\alpha} g_*\mathcal{O}_X(S) \to g_*\mathcal{O}_S(S|_S).$$

A degree-1 line bundle over an elliptic curve has only one section, thus α is an isomorphism over an open set where the fiber is a smooth elliptic curve. Since $g_*\mathcal{O}_S(S|_S) \cong \mathcal{O}_S(S|_S)$ is torsion-free we conclude that $g_*\mathcal{O}_X(S) \cong \mathcal{O}_B$. Thus

$$H^1(X, \mathcal{O}_X(S)) = H^1(B, \mathcal{O}_B) \subset H^1(X, \mathcal{O}_X).$$

If $H^2(X, \mathcal{O}_X) = 0$ then the line bundle $\mathcal{O}_X(S)$ lifts to every small deformation of X and if $H^1(X, \mathcal{O}_X) = 0$ then the unique section of $\mathcal{O}_X(S)$ also lifts.

The situation is quite different if the section is not assumed Cartier. For instance, let $X_0 \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a general hypersurface of multidegree $(3,3)$ containing $S := \mathbb{P}^2 \times \{p\}$ for some point p. Then X_0 is a Calabi–Yau variety and the first projection shows that it is elliptic with a section. Note that X_0 is singular, it has nine ordinary nodes along S.

By contrast, if $X_t \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a smooth hypersurface of multidegree $(3,3)$ then the restriction map $\mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^2) \to \mathrm{Pic}(X_t)$ is an isomorphism by the Lefschetz hyperplane theorem. Thus the degree of every divisor $D \subset X_t$ on the general fiber of the first projection $X_t \to \mathbb{P}^2$ is a multiple of 3. Therefore $X_t \to \mathbb{P}^2$ does not even have a rational section.

As an aside, we consider the general question of deforming morphisms $g: X \to Y$ whose target is not uniruled.

There are some obvious examples when not every deformation of X gives a deformation of $g: X \to Y$. For example, let A_1, A_2 be positive-dimensional Abelian varieties and $g: A_1 \times A_2 \to A_2$ the second projection. A general deformation of $A_1 \times A_2$ is a simple Abelian variety which has no maps to lower-dimensional Abelian varieties. One can now get more complicated examples by replacing $A_1 \times A_2$ by say a complete intersection subvariety or by a cyclic cover. The next result says that this essentially gives all examples.

Theorem 33 *Let X be a projective variety with rational singularities, Y a normal variety, and $g: X \to Y$ a surjective morphism with connected fibers. Assume that Y is not uniruled. Then at least one of the following holds:*

(1) Every small deformation of X gives a deformation of $(g: X \to Y)$.
(2) There is a quasi-étale cover $\tilde{Y} \to Y$, a normal variety Z, and positive-dimensional Abelian varieties A_1, A_2 such that the lifted morphism $\tilde{g}: \tilde{X} := X \times_Y \tilde{Y} \to \tilde{Y}$ factors as

$$\begin{array}{ccc} \tilde{X} & \to & Z \times A_2 \times A_1 \\ \tilde{g} \downarrow & & \downarrow \\ \tilde{Y} & \cong & Z \times A_2 \end{array}$$

Proof By Theorem 35 every deformation of X gives a deformation of $g: X \to Y$ if

$$\mathrm{Hom}_Y(\Omega_Y, R^1 g_* O_X) = 0. \tag{3}$$

Thus we need to show that if Theorem 33, (3) fails then we get a structural description as in (2).

Assuming that there is a nonzero map $\phi: \Omega_Y \to R^1 g_* O_X$, let $E \subset R^1 g_* O_X$ denote its image. Our first aim is to prove that E becomes trivial after a quasi-étale base change.

Let $C \subset Y$ be a high-degree, general, complete intersection curve. First we show that $E|_C$ is stable and has degree 0.

Since Y is not uniruled, $\Omega_Y|_C$ is semi-positive by [Miy88] (see also [Kol92, Section 9]). Thus $E|_C$ is also semi-positive.

Let $Y^0 \subset Y$ be a dense open set and $X^0 := g^{-1}(Y^0)$ such that $g^0 \colon X^0 \to Y^0$ is smooth. Set $C^0 := Y^0 \cap C$. By [Ste76], $(R^1 g_* \mathcal{O}_X)|_C$ is the (lower) canonical extension of the top quotient of the variation of Hodge structures $R^1 g^0_* \mathbb{Q}_{X^0}|_{C^0}$. (Note that [Ste76] works with ω_{X^0/Y^0} but the proof is essentially the same; see [Kol86b, pp. 177–179].) Thus $(R^1 g_* \mathcal{O}_X)|_C$ is semi-negative by [Ste76] and so is $E|_C$. Thus $E|_C$ is stable of degree 0, hence it corresponds to a unitary representation ρ of $\pi_1(C)$.

By [Gri70, Section 5], ρ is a subrepresentation of the monodromy representation on $R^1 g^0_* \mathbb{Q}_{X^0}|_{C^0}$ and by [Del71, Theorem 4.2.6], it is even a direct summand $\mathbb{E}$. Since we have a polarized variation of Hodge structures, the monodromy representation on $\mathbb{E}$ has finite image. Thus E becomes trivial after a quasi-étale base change and then it corresponds to a direct factor of the relative Albanese variety of $X_1 := Y_1 \times_Y X$, giving the Abelian variety A_1.

Furthermore, in this case $T_{Y_1}|_C = \mathcal{H}om_{Y_1}(\Omega_{Y_1}, \mathcal{O}_{Y_1})|_C$ has a global section. Since T_{Y_1} is reflexive, the Enriques–Severi–Zariski lemma (as proved, though not as claimed in [Har77, III.7.8]) implies that $H^0(Y_1, T_{Y_1}) \neq 0$. Therefore $\dim \operatorname{Aut}(Y_1) > 0$. Since Y_1 is not uniruled, $\operatorname{Aut}^0(Y_1)$ has no linear algebraic subgroups, thus the connected component $\operatorname{Aut}^0(Y_1)$ is an Abelian variety A_2. By Proposition 34, A_2 becomes a direct factor after a suitable étale cover $\tilde{Y} \to Y_1 \to Y$. □

The following result was known in [Ser01, Ses63]; see [Bri10] for the general theory.

Proposition 34 *Let $W \to S$ be a flat, projective morphism with normal fibers over a field of characteristic 0 and $A \to S$ an Abelian scheme acting faithfully on W. Then there is a flat, projective morphism with normal fibers $Z \to S$ and an A-equivariant étale morphism $A \times_S Z \to W$.* □

The following is a combination of [Hor76, Theorem 8.1] and the method of [Hor76, Theorem 8.2] in the smooth case and [BHPS12, Proposition 3.10] in general.

Theorem 35 *Let $f \colon X \to Y$ be a morphism of proper schemes over a field such that $f_* \mathcal{O}_X = \mathcal{O}_Y$ and $\operatorname{Hom}_Y(\Omega_Y, R^1 f_* \mathcal{O}_X) = 0$.*

Then for every small deformation $\mathbf{X}$ of X there is a small deformation $\mathbf{Y}$ of Y such that f lifts to $\mathbf{f} \colon \mathbf{X} \to \mathbf{Y}$. □

6 Smoothings of very singular varieties

One can frequently construct smooth varieties by first exhibiting some very singular, even reducible schemes with suitable numerical invariants and then smoothing them. For such Calabi–Yau examples, see [KN94]. Thus it is of interest to know when an elliptic fiber space structure is preserved by a smoothing. In some cases, when Theorem 31 does not apply, the following result, relying on Corollary 11, provides a quite satisfactory answer.

Proposition 36 *Let X be a projective, reduced, Gorenstein scheme of pure dimension n such that ω_X is numerically trivial and $H^2(X, O_X) = 0$. Let $g\colon X \to B$ be a morphism whose general fibers (over every irreducible component of B) are curves of arithmetic genus 1. Assume also that every irreducible component of X dominates an irreducible component of B.*

Let L_B be an ample line bundle on B and assume that $\chi(X, g^*L_B^m)$ is a polynomial of degree $\dim X - 2$. Then every smoothing (and every log terminal deformation) of X is an elliptic fiber space.

Warning Note that we do not claim that g lifts to every deformation of X. In Example 49 X has smoothings, which are elliptic, and also other singular deformations that are not elliptic.

Proof As before, $H^2(X, O_X) = 0$ implies that g^*L_B lifts to every small deformation [Gro62, p. 236–16]. Thus we have a deformation $h\colon (\mathbf{X}, \mathbf{L}) \to (0 \in S)$ of $(X_0, L_0) \cong (X, L = g^*L_B)$.

We claim that $\mathbf{L}$ is h-nef and $K_{\mathbf{X}}$ is trivial on the fibers of h. This is a somewhat delicate point since being nef is not an open condition in general [Les12]. We get around this problem as follows.

Let (X_{gen}, L_{gen}) be a generic fiber. (Note the difference between generic and general.) First we show that L_{gen} is nef and $K_{X_{gen}} \equiv 0$. Indeed, assume that $(L_{gen} \cdot C_{gen}) < 0$ for some curve C_{gen}. Let $C_0 \subset X_0$ be a specialization of C_{gen}. Then $(L_0 \cdot C_0) = (L_{gen} \cdot C_{gen}) < 0$ gives a contradiction. A similar argument shows that $(K_{X_{gen}} \cdot C_{gen}) = 0$ for every curve C_{gen}.

Next, the deformation invariance of $\chi(X, g^*L_B^m)$ and Riemann–Roch (cf. Corollary 22 and 24) show that

$$(L_{gen}^{n-2} \cdot c_2(X_{gen})) = (n-2)! \cdot (\text{coefficient of } m^{n-2} \text{ in } \chi(X, g^*L_B^m)).$$

Therefore $(L_{gen}^{n-2} \cdot c_2(X_{gen})) > 0$ and, as we noted after Theorem 10, this implies that $|mL_{gen}|$ is base-point-free for some $m > 0$.

Thus there is a dense Zariski open subset $S^0 \subset S$ such that $|mL_s|$ is base-point-free for $s \in S^0$, hence (X_s, L_s) is an elliptic fiber space for $s \in S^0$. We repeat the argument for the generic points of $S \setminus S^0$ and conclude by Noetherian induction. □

It may be useful to see how to modify the above proof to work in the analytic case when there are no generic points.

The (Barlet or Douady) space of curves in $h\colon \mathbf{X} \to (0 \in S)$ has only countably many irreducible components, thus there are countably many closed subspaces $S_i \subsetneq S$ such that every curve $C_s \subset X_s$ is deformation equivalent to a curve $C_0 \subset X_0$ for $s \notin \cup S_i$. In particular, L_s is nef and $K_{X_s} \equiv 0$ whenever $s \notin \cup S_i$. Thus (X_s, L_s) is an elliptic fiber space for $s \notin \cup S_i$.

By semicontinuity, there are closed subvarieties $T_m \subsetneq S$ such that

$$h_*\mathcal{O}_{\mathbf{X}}(m\mathbf{L}) \otimes \mathbb{C}_s = H^0(X_s, \mathcal{O}_{X_s}(mL_s)) \quad \text{for } s \notin T_m.$$

Thus if $s \notin \cup_i S_i \bigcup \cup_m T_m$ and $\mathcal{O}_{X_s}(m_0 L_s)$ is generated by global sections then

$$\phi_{m_0}\colon h^*(h_*\mathcal{O}_{\mathbf{X}}(m_0\mathbf{L})) \to \mathcal{O}_{\mathbf{X}}(m_0\mathbf{L})$$

is surjective along X_s. Thus there is a dense Zariski open subset $S^0 \subset S$ such that ϕ_{m_0} is surjective for all $s \in S^0$. Now we can finish by Noetherian induction as before.

7 Calabi–Yau orbibundles

The techniques of this section are mostly taken from [Kol93, Section 6] and [Nak99].

Definition 37 A Calabi–Yau fiber space $g\colon X \to B$ is called an *orbibundle* if it can be obtained by the following construction.

Let $\tilde{B}$ be a normal variety, F a Calabi–Yau variety, and $\tilde{X} := \tilde{B} \times F$. Let G be a finite group, $\rho_B\colon G \to \mathrm{Aut}(\tilde{B})$ and $\rho_F\colon G \to \mathrm{Aut}(F)$ two faithful representations. Set

$$(g\colon X \to B) := (\tilde{X}/G \to \tilde{B}/G);$$

it is a generically isotrivial Calabi–Yau fiber space with general fiber F.

(It would seem more natural to require the above property only locally on B. We see in Theorem 43 that in the algebraic case the two versions are equivalent. However, for complex manifolds, the local and global versions are different.)

For any normal variety Z with non-negative Kodaira dimension, the connected component $\mathrm{Aut}^0(Z)$ of $\mathrm{Aut}(Z)$ is an Abelian variety; we call its elements translations. The quotient $\operatorname{Aut}(Z)/\operatorname{Aut}^0(Z)$ is the discrete part of the automorphism group.

For G acting on F, let $G_t := \rho_F^{-1}\operatorname{Aut}^0(F) \subset G$ be the normal subgroup of translations and set $X^d := \tilde{X}/G_t$. Then $G_d := G/G_t$ acts on X^d and $X = X^d/G_d$. Thus every orbibundle comes with two covers:

$$\begin{array}{ccccc} X & \xleftarrow{\tau_X} & X^d & \xleftarrow{\pi_X} & \tilde{X} \\ g\downarrow & & g^d\downarrow & & \tilde{g}\downarrow \\ B & \xleftarrow{\tau_B} & B^d & \xleftarrow{\pi_B} & \tilde{B} \end{array} \tag{37.1}$$

We see during the proof of Theorem 43 that the cover $X \leftarrow X^d$ corresponding to the discrete part of the monodromy representation is uniquely determined by $g\colon X \to B$. By contrast, the $X^d \leftarrow \tilde{X}$ part is not unique. Its group of deck transformations is $G_t \subset \operatorname{Aut}^0(F)$, hence Abelian. It is not even clear that there is a natural "smallest" choice of $X^d \leftarrow \tilde{X}$.

If $F = A$ is an Abelian variety, then $g^d\colon X^d \to B^d$ is a Seifert bundle where an orbibundle $g^s\colon X^s \to B^s$ is called a *Seifert bundle* if $F = A$ is an Abelian variety and G acts on A by translation. Note that in this case the A-action on $\tilde{B}\times A$ descends to an A-action on X^s and $B^s = X^s/A$. Thus the reduced structure of every fiber is a smooth Abelian variety isogenous to A.

Lemma 38 *Notation as above. Then*

(1) π_X and τ_X are étale in codimension 1 (that is, quasi-étale),
(2) π_X and τ_X are étale in codimension 2 if one of the following holds:

  (a) G acts freely on F outside a codimension ≥ 2 subset or
  (b) $K_F \sim 0$ and $\Delta_{X/B} = 0$.

Proof The first claim is clear since both ρ_F, ρ_B are faithful.

Since ρ_F, ρ_B are faithful, τ_X fails to be étale in codimension 2 iff some $1 \neq g \in G$ fixes a divisor $\tilde{D}_B \subset \tilde{B}$ and also a divisor $D_F \subset F$. This is excluded by (2a).

Next we check that (2b) implies (2a). At a general point $p \in D_F$ choose local g-equivariant coordinates $x_1, \dots, x_m$ such that $D_F = (x_1 = 0)$. Thus $\rho_F(g)^*$ acts on x_1 nontrivially but it fixes $x_2, \dots, x_m$. Let ω_0 be a nonzero section of ω_F. Locally near p we can write

$$\omega_0 = f \cdot dx_1 \wedge \cdots \wedge dx_m,$$

thus $\rho_F(g)^*$ acts on $H^0(F, \omega_F)$ with the same eigenvalue as on x_1.

Thus, by 19, (1), the image of $\tilde{D}_X$ gives a positive contribution to $\Delta_{X/B}$. This contradicts $\Delta_{X/B} = 0$. □

There are some obvious deformations of X obtained by deforming $\tilde{B}$ and F in a family $\{(\tilde{B}_t, F_t)\}$ such that the representations ρ_B, ρ_F lift to $\rho_{B,t}\colon G \to \mathrm{Aut}(\tilde{B}_t)$ and $\rho_{F,t}\colon G \to \mathrm{Aut}(F_t)$.

In general, not every deformation of X arises this way. For instance, let $\tilde{B}$ and $F = A$ be elliptic curves and X the Kummer surface of $\tilde{B} \times A$. The obvious deformations of X form a 2-dimensional family obtained by deforming $\tilde{B}$ and A. Thus a general deformation of X is not obtained this way and it is not even elliptic. Even worse, a general elliptic deformation of X is also not Kummer; thus not every deformation of the morphism $(g\colon X \to B)$ is obtained by the quotient construction.

Theorem 39 *Let $g\colon X \to B$ be a Calabi–Yau orbibundle with general fiber F. Assume that X has log terminal singularities, $H^2(X, \mathcal{O}_X) = 0$, $\kappa(X) \geq 0$, $K_F \sim 0$, and $\Delta_{X/B} = 0$. Then every flat deformation of X arises from a flat deformation of $(\tilde{B}, F, \rho_B, \rho_F)$.*

Proof Let L_B be an ample line bundle on B and set $L := g^* L_B$.

Let $h\colon \mathbf{X} \to (0 \in S)$ be a deformation of $X_0 \cong X$. In the sequel we will repeatedly replace S by a smaller analytic (or étale) neighborhood of 0 if necessary.

Since $H^2(X, \mathcal{O}_X) = 0$, L lifts to a line bundle $\mathbf{L}$ on $\mathbf{X}$ by [Gro62, p. 236–16].

Since $K_F \sim 0$ and $\Delta_{X/B} = 0$, Lemma 38 implies that $\pi\colon \tilde{X} \to X$ is étale in codimension 2. Since X is log terminal, so is $\tilde{X}$, hence it is Cohen–Macaulay (see, e.g., [KM98, 5.10 and 5.22]). Thus, by [Kol95, Corollary 12.7], the cover π lifts to a cover $\Pi\colon \tilde{\mathbf{X}} \to \mathbf{X}$.

Finally we show that the product decomposition $\tilde{X} \cong \tilde{B} \times F$ lifts to a product decomposition

$$\tilde{\mathbf{X}} \cong \tilde{\mathbf{B}} \times_S \mathbf{F}$$

where $\tilde{\mathbf{B}} \to S$ is a flat deformation of $\tilde{B}$ and $\mathbf{F} \to S$ is a family of Calabi–Yau varieties over S. After a further étale cover of $\tilde{F} \to F$ we may assume that $\tilde{F} \cong Z \times A$, where $H^1(Z, \mathcal{O}_Z) = 0$ and A is an Abelian variety. Set $\hat{X} := \tilde{B} \times Z \times A$; then $\hat{X} \to \tilde{X}$ lifts to a deformation $\widehat{\mathbf{X}} \to \tilde{\mathbf{X}} \to S$.

First we use Lemma 40 and Proposition 34 to show that the product decomposition $\hat{X} \cong (\tilde{B} \times Z) \times A$ lifts to a product decomposition

$$\widehat{\mathbf{X}} \cong \widehat{\mathbf{BZ}} \times_S \mathbf{A}$$

where $\widehat{\mathbf{BZ}} \to S$ is a flat deformation of $\tilde{B} \times Z$ and $\mathbf{A} \to S$ is a family of Abelian varieties over S. The deformation of the product $\tilde{B} \times Z$ is much easier to understand; we discuss it in 53. □

Lemma 40 *Let $Y \to S$ be a flat, proper morphism whose fibers are normal and L a line bundle on Y. Let $0 \in S$ be a point such that*

(1) Y_0 is not birationally ruled,
(2) an Abelian variety $A_0 \subset \mathrm{Aut}^0(Y_0)$ acts faithfully on Y_0,
(3) L_0 is nef, L_0 is numerically trivial on the A_0-orbits but not numerically trivial on general A_0'-orbits for any $A_0 \subsetneq A_0' \subset \mathrm{Aut}^0(Y_0)$.

Then, possibly after shrinking S, there is an Abelian scheme $A \to S$ extending A_0 such that A acts faithfully on Y.

Proof By [Mat68, p. 217] (see also [Kol85, p. 392]), possibly after shrinking S, $g^a\colon \mathrm{Aut}^0(Y/S) \to S$ is a smooth Abelian scheme, where $\mathrm{Aut}^0(Y/S)$ denotes the identity component of the automorphism scheme $\mathrm{Aut}(Y/S)$. The fibers are normal, hence $Y \to S$ is smooth over a dense subset of every fiber. Since a smooth morphism has sections étale locally, we may assume after an étale base change that there is a section $Z \subset Y$. Acting on Z gives a morphism $\rho_Z\colon \mathrm{Aut}^0(Y/S) \to Y$. Then $\rho_Z^* L$ is a nef line bundle on $\mathrm{Aut}^0(Y/S)$. The kernel of the cup-product map

$$c_1(\rho_Z^* L)\colon R^1 g^a_* \mathbb{Q} \to R^3 g^a_* \mathbb{Q}$$

is a variation of sub-Hodge structures, hence it corresponds to a smooth Abelian subfamily $A \subset \mathrm{Aut}^0(Y/S)$. By (3), this is the required extension of A_0.

The quotient then exists by [Ses63]. □

We will also need to understand the class group of an orbibundle.

41 (Divisors on orbibundles) We use the notation of Definitions 37 and 42.

By [BGS11, 5.3] (see also [HK11, CL10] for the elliptic case), the class group of the product $\tilde{B} \times F$ is

$$\mathrm{Cl}(\tilde{B} \times F) = \mathrm{Cl}(\tilde{B}) + \mathrm{Cl}(F) + \mathrm{Hom}(\mathrm{Alb}^{\mathrm{rat}}(\tilde{B}), \mathrm{Pic}^0(F)). \tag{41.1}$$

This comes with a natural G-action and, up to torsion, the class group of the quotient is

$$\mathrm{Cl}(B) + \mathrm{Cl}(F)^G + \mathrm{Hom}(\mathrm{Alb}^{\mathrm{rat}}(\tilde{B}), \mathrm{Pic}^0(F))^G. \tag{41.2}$$

Here $\mathrm{Cl}(F)^G + \mathrm{Hom}(\mathrm{Alb}^{\mathrm{rat}}(\tilde{B}), \mathrm{Pic}^0(F))^G$ can be identified with the class group of the generic fiber of g. If $\tilde{B}$ has rational singularities, then $\mathrm{Alb}^{\mathrm{rat}}(\tilde{B}) = \mathrm{Alb}(\tilde{B})$. Thus the extra component $\mathrm{Hom}(\mathrm{Alb}(\tilde{B}), \mathrm{Pic}^0(F))$ corresponds to divisors that are pulled back from $\mathrm{Alb}(\tilde{B}) \times F$, hence they are Cartier.

We will use the following variant of these observations:

Claim 41.3 Let $g\colon X \to B$ be an orbibundle such that X has log terminal singularities. Then the natural map

$$\mathrm{Cl}(B)/\operatorname{Pic}(B) + (\mathrm{Cl}(F)/\operatorname{Pic}(F))^G \to \mathrm{Cl}(X)/\operatorname{Pic}(X)$$

is an isomorphism modulo torsion. In particular, if B and the generic fiber of g are $\mathbb{Q}$-factorial, then so is X.

Proof By Lemma 38, $\tau_X\colon X^d \to X$ is étale in codimension 1, hence X^d also has log terminal singularities. As noted in Definition 5, this implies that B^d has rational singularities.

Let us now study more carefully the RHS of (41.2). Let $G_t \subset G$ denote the subgroup of translations. Then

$$\operatorname{Hom}(\mathrm{Alb}^{\mathrm{rat}}(\tilde{B}), \operatorname{Pic}^0(F))^G \subset \operatorname{Hom}(\mathrm{Alb}^{\mathrm{rat}}(\tilde{B}), \operatorname{Pic}^0(F))^{G_t}.$$

Since translations act trivially on $\operatorname{Pic}^0(F)$, the latter can be identified (up to torsion) as

$$\begin{aligned}\operatorname{Hom}(\mathrm{Alb}^{\mathrm{rat}}(\tilde{B}), \operatorname{Pic}^0(F))^{G_t} \otimes \mathbb{Q} &\cong \operatorname{Hom}(\mathrm{Alb}^{\mathrm{rat}}(\tilde{B})^{G_t}, \operatorname{Pic}^0(F)) \otimes \mathbb{Q}\\ &\cong \operatorname{Hom}(\mathrm{Alb}^{\mathrm{rat}}(B^d), \operatorname{Pic}^0(F)) \otimes \mathbb{Q}\\ &\cong \operatorname{Hom}(\mathrm{Alb}(B^d), \operatorname{Pic}^0(F)) \otimes \mathbb{Q}.\end{aligned}$$

Thus this extra term gives only $\mathbb{Q}$-Cartier divisors on X^d and hence also on X. □

The following local example shows that it is not enough to assume that B has rational singularities. Set $\tilde{B} = (u^3+v^3+w^3 = 0) \subset \mathbb{A}^3$ and $E = (x^3+y^3+z^3 = 0) \subset \mathbb{P}^2$. On both factors, $\mathbb{Z}/3$ acts by weights $(0,0,1)$. Then $B = \tilde{B}/\frac{1}{3}(0,0,1) \cong \mathbb{A}^2$ is even smooth, but

$$X = \tilde{B} \times E/\tfrac{1}{3}(0,0,1) \times (0,0,1)$$

is not $\mathbb{Q}$-factorial. For instance, the closure of the graph of the natural projection $\tilde{B} \dashrightarrow E$ gives a non-$\mathbb{Q}$-Cartier divisor on X.

Definition 42 (Albanese varieties) For a smooth projective variety V let $\mathrm{Alb}(V)$ denote the Albanese variety, that is, the target of the universal morphism from V to an Abelian variety. (See [BPV84, Section I.13] or [Gro62, p. 236–16] for introductions.)

There are two ways to generalize this concept to normal varieties.

The above definition yields what we again call the *Albanese variety* $\mathrm{Alb}(V)$. Alternatively, the *rational Albanese* variety $\mathrm{Alb}^{\mathrm{rat}}(V)$ is defined as the target of the universal rational map from V to an Abelian variety. One can identify $\mathrm{Alb}^{\mathrm{rat}}(V) = \mathrm{Alb}(V')$, where $V' \to V$ is any resolution of singularities.

It is easy to see that if V has log terminal (more generally rational) singularities, then $\mathrm{Alb}^{\mathrm{rat}}(V) = \mathrm{Alb}(V)$.

8 Generically isotrivial Calabi–Yau fiber spaces

In this section we prove that all generically isotrivial Calabi–Yau fiber spaces are essentially Calabi–Yau orbibundles.

Theorem 43 *Let $g\colon X \to B$ be a projective, generically isotrivial, Calabi–Yau fiber space with general fiber F. Then*

(1) $g\colon X \to B$ is birational to an orbibundle $(g^{\mathrm{orb}}\colon X^{\mathrm{orb}} \to B)$.
(2) $g\colon X \to B$ is isomorphic to $(g^{\mathrm{orb}}\colon X^{\mathrm{orb}} \to B)$ if

- *(a) X is $\mathbb{Q}$-factorial, log terminal,*
- *(b) $g\colon X \to B$ is relatively minimal, without exceptional divisors,*
- *(c) B is $\mathbb{Q}$-factorial, and*
- *(d) one of the following holds:*
  - *(i) $K_F \sim 0$ and $\Delta_{X/B} = 0$, or*
  - *(ii) there is a closed subset $Z_B \subset B$ of codimension ≥ 2 such that $g\colon X \to B$ is locally an orbibundle over $B \setminus Z_B$.*

Proof Let $B^0 \subset B$ be a Zariski open subset over which $X^0 \to B^0$ is isotrivial with general fiber F. This gives a well-defined representation

$$\rho\colon \pi_1(B^0) \to \mathrm{Aut}(F)/\,\mathrm{Aut}^0(F).$$

Let $B^{(d,0)} \to B^0$ be the corresponding étale, Galois cover with group G_d and $B^d \to B$ its extension to a (usually ramified) Galois cover of B with group G_d. This gives the well-defined cover in (37.1).

The trivialization of the translation part is more subtle and it depends on additional choices.

A general $\mathrm{Aut}^0(F)$-orbit $A_F \subset F$ defines an isotrivial Abelian family $X^{(d,0)} \supset A_X^{(d,0)} \to B^{(d,0)}$. By assumption there is a g-ample line bundle L on X. It pulls back to a relatively ample line bundle L_A on $A_X^{(d,0)}$. We may assume that its degree on the general fiber is at least 3. Let $T^{(d,0)} \subset A_X^{(d,0)}$ be the subscheme as in 44. Since L_A is G_d-invariant, $T^{(d,0)}$ is G_d-equivariant hence it defines a monodromy representation of $\pi_1(B^{(d,0)}) \to \mathrm{Aut}^0(F)$. Let $\Gamma \subset \pi_1(B^{(d,0)})$ be a finite-index subgroup that is normal in $\pi_1(B^0)$ and $\tilde{B}^0 \to B^0$ the corresponding étale, Galois cover with group $G = \pi_1(B^0)/\Gamma$. Let $\tilde{B} \to B$ denote its extension to a (usually ramified) Galois cover of B with group G.

By pull-back we obtain an isotrivial, Abelian fiber space $\tilde{A}^0_X \to \tilde{B}^0$ with a trivialization of the m-torsion points. For $m \geq 3$ this implies that $\tilde{A}^0_X \cong \tilde{B}^0 \times A$. (This is quite elementary, cf. [ACG11, p. 513].) Thus the same pull-back also trivializes $X^0 \to B^0$. We can compactify $\tilde{X}^0$ as $\tilde{X} := \tilde{B} \times F$.

The G-action on $\tilde{X}$ can be given as

$$\gamma\colon (\tilde{b}, c) \mapsto (\rho_B(\gamma) \cdot \tilde{b}, \rho_{F,\tilde{b}}(\gamma) \cdot c).$$

Note that $\rho_{F,\tilde{b}}$ preserves the m-torsion points and the automorphisms of an Abelian torsor that preserve any finite non-empty set form a discrete group. Thus in fact $\rho_{F,\tilde{b}}$ is independent of $\tilde{b}$ and hence the G-action on $\tilde{X}$ is given by

$$\gamma\colon (\tilde{b}, c) \mapsto (\rho_B(\gamma) \cdot \tilde{b}, \rho_F(\gamma) \cdot c)$$

for some isomorphism $\rho_B\colon G \cong \mathrm{Gal}(\tilde{B}/B)$ and homomorphism $\rho_F\colon G \to \mathrm{Aut}(F)$. We can replace $\tilde{B}$ by $\tilde{B}/\ker \rho_F$, hence we may assume that ρ_F is faithful.

Thus we have $g^{\mathrm{orb}} : X^{\mathrm{orb}} := \tilde{X}/G \to B$ and a birational map

$$\phi\colon X \dashrightarrow X^{\mathrm{orb}} \quad \text{such that} \quad g = g^{\mathrm{orb}} \circ \phi.$$

Assume next that conditions (2a–d) hold. First we use (2d) to prove that ϕ extends to an isomorphism over codimension 1 points of B. Then we use conditions (2a–c) to show that ϕ is an isomorphism everywhere.

In order to understand the codimension 1 behavior, we can take a transversal curve section (or localize at a codimension 1 point). Thus we may assume that $B = (0 \in D)$ is a unit disc (or the spectrum of a DVR) and $g\colon X \to D$ is isotrivial on $D \setminus \{0\}$. Thus X^{orb} is of the form

$$X^{\mathrm{orb}} \cong (F \times D)/(\rho, e^{2\pi i/m}),$$

where ρ is an automorphism of order m of F.

In case (2d(i)) ρ acts trivially on $H^0(F, \omega_F)$ by 19, (1), thus the canonical class of $F_0 := F/\langle\rho\rangle$ is trivial and so F_0 has canonical singularities. By inversion of adjunction [Kol13b, Theorem 4.9], the pair (X^{orb}, F_0) is also canonical. It is clear that $a(E, X^{\mathrm{orb}}) > a(E, X^{\mathrm{orb}}, F_0) \geq 0$ for every divisor over X^{orb} dominating $0 \in D$; cf. [KM98, 2.27]. Thus, as we noted in 8, ϕ restricts to a birational map of the central fibers $\phi_0\colon X_0 \dashrightarrow F_0$.

Now let H be a relatively ample divisor on $X \to D$. Then $\phi_* H$ is a divisor on X^{orb} that is Cartier and ample outside F_0. Since every curve on F_0 is the specialization of a curve in F, we see that $\phi_* H$ is $\mathbb{Q}$-Cartier and ample everywhere. Thus ϕ is an isomorphism by Lemma 45.

In case (2d(ii)), let $g_i\colon X_i \to D$ be two orbibundles and $\phi\colon X_1 \dashrightarrow X_2$ a birational map that is an isomorphism over $D^0 := D \setminus \{0\}$. Let $\Gamma \subset X_1 \times_D X_2$

denote the closure of the graph of ϕ. We need to prove that the coordinate projections $\Gamma \to X_i$ are finite. It is enough to check this after a finite base change. Thus we may assume that $X_i \cong F \times D$. Then ϕ can be identified with a map $D^0 \to \mathrm{Aut}(F)$ and this extends to $D \to \mathrm{Aut}(F)$ since every connected component of $\mathrm{Aut}(F)$ is proper. Thus ϕ is an isomorphism.

Now we return to the general case. We have shown that there is a subset $Z_B \subset B$ of codimension ≥ 2 such that $\phi\colon X \dashrightarrow X^{\mathrm{orb}}$ is an isomorphism over $B \setminus Z_B$. By assumption (2b), the pre-image $g^{-1}(Z_B)$ has codimension ≥ 2 and the pre-image $(g^{\mathrm{orb}})^{-1}(Z_B)$ has codimension ≥ 2 by construction. Since X is $\mathbb{Q}$-factorial, so is the generic fiber of g, hence X^{orb} is $\mathbb{Q}$-factorial by Claim 41.3. Thus the assumptions of Lemma 45, part (2) are satisfied and hence ϕ is an isomorphism. □

44 (Multisections of Abelian families) Let E be a smooth projective curve of genus 1 and L a line bundle of degree m on E. If $m = 1$ then L has a unique section, thus we can associate a point $p \in E$ to L. If $m \geq 2$, then sections define a linear equivalence class $|L|$ of m points. If we fix a point $0 \in E$ to be the origin, then we can add these m points together and get a well-defined point of E associated with L. This, however, depends on the choice of the origin.

To get something invariant, let us look at the points $p \in E$ such that $m \cdot p \in |L|$. There are m^2 such points, together forming a translate of the subgroup of m-torsion points. This construction also works in families.

Let $g\colon X \to B$ be a smooth, projective morphism whose fibers E_b are curves of genus 1. Let L be a line bundle on X that has degree m on each fiber. Then there is a closed subscheme $T \subset X$ such that $g|_T\colon T \to B$ is étale of degree m^2 and every fiber $T_b \subset E_b$ is a translate of the subgroup of m-torsion points.

There is a similar construction for higher-dimensional Abelian varieties. For clarity, I say *Abelian torsor* when talking about an Abelian variety without a specified origin.

Thus let A be an Abelian torsor of dimension d and L an ample line bundle on A. It has a first Chern class $\tilde{c}_1(L)$ in the Chow group and we get $\tilde{c}_1(L)^d$ as an element of the Chow group of 0-cycles. (It is important to use the Chow group, the Chern class in cohomology is not sufficient.) Let its degree be m.

Fix a base point $0 \in A$. This defines a map from the Chow group of 0-cycles to $(A, 0)$; let $\alpha(\tilde{c}_1(L)^d)$ denote the image.

Finally let $T \subset A$ be the set of points $t \in A$ such that $m \cdot t = \alpha(\tilde{c}_1(L)^d)$. This T is a translate of the subgroup of m-torsion points. As before, the key point is that T is independent of the choice of the base point $0 \in A$. Indeed, if we change 0 by a translation by $c \in A$ then $\alpha(\tilde{c}_1(L)^d)$ is changed by translation by $m \cdot c$ so T is changed by translation by c.

Furthermore, if (A_b, L_b) is a family of polarized Abelian torsors that varies analytically (or algebraically) with b then $T_b \subset A_b$ is a family of subschemes that also vary analytically (or algebraically) with b. Thus we obtain the following.

Let $g\colon X \to B$ be a smooth, projective morphism whose fibers are Abelian torsors. Then there is a closed subscheme $T \subset X$ such that $g|_T\colon T \to B$ is étale and every fiber $T_b \subset A_b$ is a translate of the subgroup of m-torsion points (where $\deg T/B = m^{2d}$).

Lemma 45 *Let $g_i\colon X_i \to B$ be projective fiber spaces, the X_i normal and $\phi\colon X_1 \dashrightarrow X_2$ a rational map. Assume that there are closed subsets $Z_i \subset X_i$ such that $\operatorname{codim}_{X_i} Z_i \geq 2$ and ϕ induces an isomorphism $X_1 \setminus Z_1 \cong X_2 \setminus Z_2$. Let H_1 be a g_1-ample divisor on X_1 and set $H_2 := \phi_* H_1$.*

(1) ϕ is an isomorphism iff H_2 is g_2-ample.

(2) If ϕ induces an isomorphism of the generic fibers, X_2 is $\mathbb{Q}$-factorial and every curve $C \subset X_2$ contracted by g_2 is $\mathbb{Q}$-homologous to a curve in a general fiber, then H_2 is g_2-ample.

Proof The first claim is a lemma of Matsusaka and Mumford [MM64]; see [KSC04, 5.6] or [Kol10, Exercise 75] for the variant used here.

It follows from assumption (2) that H_2 is $\mathbb{Q}$-Cartier and strictly positive on the cone of curves, hence it is g_2-ample. □

46 (F-theory examples) Let X be a smooth, projective variety and $g\colon X \to B$ a relatively minimal elliptic fiber space with a section $\sigma\colon B \to X$. Since X is smooth, so is B.

Assume that $\Delta_{X/S} = 0$. Then, by Lemma 38, it can have only multiple smooth fibers at codimension-1 points, but then the section shows that there are no multiple fibers. Thus there is an open subset $B^0 \subset B$ such that $\operatorname{codim}_B(B \setminus B^0) \geq 2$ and $X^0 \to X$ is a fiber bundle with fiber a pointed elliptic curve $(E, 0)$. Thus X^0 is given by the data

$$(B^0, E, \rho\colon \pi_1(B^0) \to \operatorname{Aut}(E, 0)).$$

Note that $\pi_1(B^0) = \pi_1(B)$ since B is smooth and $\operatorname{codim}_B(B \setminus B^0) \geq 2$. Thus X is birational to a fiber bundle $g'\colon X' \to B$ given by the data

$$(B, E, \rho\colon \pi_1(B) \to \operatorname{Aut}(E, 0)).$$

All the fibers of g' are elliptic curves, but the exceptional locus of a flip or a flop is always covered by rational curves (cf. [Kol96, VI.1.10]). Thus in fact

$X \cong X'$, hence $g\colon X \to B$ is a locally trivial fiber bundle. The image of the monodromy representation $\rho\colon \pi_1(B) \to \operatorname{Aut}(E, 0)$ is usually $\mathbb{Z}/2$, but for elliptic curves with extra automorphisms it can also be $\mathbb{Z}/3, \mathbb{Z}/4$, or $\mathbb{Z}/6$.

It is easy to write down examples where $K_X \sim 0$ and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$. However, $\pi_1(X)$ is always infinite, so such an X cannot be a "true" Calabi–Yau manifold.

By Theorem 39, if $H^2(X, \mathcal{O}_X) = 0$ then every small deformation of X is obtained by deforming B and, if the image of ρ is $\mathbb{Z}/2$, also deforming E.

9 Examples

The first example is an elliptic Calabi–Yau surface with quotient singularities that has a flat smoothing which is neither Calabi–Yau nor elliptic.

Example 47 We start with a surface S_F^* which is the quotient of the square of the Fermat cubic curve by $\mathbb{Z}/3$:

$$S_F^* \cong (u_1^3 = v_1^3 + w_1^3) \times (u_2^3 = v_2^3 + w_2^3)/\tfrac{1}{3}(1,0,0;1,0,0).$$

To describe the deformation, we need a different representation of it.

In $\mathbb{P}^3$ consider two lines $L_1 = (x_0 = x_1 = 0)$ and $L_2 = (x_2 = x_3 = 0)$. The linear system $\left|\mathcal{O}_{\mathbb{P}^2}(2)(-L_1 - L_2)\right|$ is spanned by the four reducible quadrics $x_i x_j$ for $i \in \{0, 1\}$ and $j \in \{2, 3\}$. They satisfy a relation $(x_0x_2)(x_1x_3) = (x_0x_3)(x_1x_2)$. Thus we get a morphism

$$\pi\colon B_{L_1+L_2}\mathbb{P}^3 \to \mathbb{P}^1 \times \mathbb{P}^1,$$

which is a $\mathbb{P}^1$-bundle whose fibers are the birational transforms of lines that intersect both of the L_i.

Let $S \subset \mathbb{P}^3$ be a cubic surface such that $\mathbf{p} := S \cap (L_1 + L_2)$ is six distinct points. Then we get $\pi_S : B_{\mathbf{p}}S \to \mathbb{P}^1 \times \mathbb{P}^1$.

In general, none of the lines connecting two points of $\mathbf{p}$ is contained in S. Thus in this case π_S is a finite triple cover.

Both of the lines L_i determine an elliptic pencil on $B_{\mathbf{p}}S$ but if we move the six points $\mathbf{p}$ into general position, we lose both elliptic pencils.

At the other extreme we have the Fermat-type surface

$$S_F := (x_0^3 + x_1^3 = x_2^3 + x_3^3) \subset \mathbb{P}^3.$$

We can factor both sides and write its equation as $m_1m_2m_3 = n_1n_2n_3$. The nine lines $L_{ij} := (m_i = n_j = 0)$ are all contained in S_F. Let $L'_{ij} \subset B_{\mathbf{p}}S_F$ denote their birational transforms. Then the self-intersections $(L'_{ij} \cdot L'_{ij})$ equal -3 and

π_{S_F} contracts these nine curves L'_{ij}. Thus the Stein factorization of π_{S_F} gives a triple cover $S_F^* \to \mathbb{P}^1 \times \mathbb{P}^1$ and S_F^* has nine singular points of type $\mathbb{A}^2/\frac{1}{3}(1,1)$. We see furthermore that

$$-3K_{S_F} \sim \sum_{ij} L_{ij} \quad \text{and} \quad -3K_{B_{\mathbf{P}} S_F} \sim \sum_{ij} L'_{ij}.$$

Thus $-3K_{S_F^*} \sim 0$.

To see that this is the same S_F^*, note that the morphism of the original S_F^* to $\mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$(u_1{:}v_1{:}w_1) \times (u_2{:}v_2{:}w_2) \mapsto (v_1{:}w_1) \times (v_2{:}w_2)$$

and the rational map to the cubic surface is given by

$$(u_1{:}v_1{:}w_1) \times (u_2{:}v_2{:}w_2) \mapsto (v_2 u_1 u_2^2{:}u_1 u_2^2{:}v_1 u_2^3{:}u_2^3).$$

Varying S gives a flat deformation whose central fiber is S_F^*, a surface with quotient singularities and torsion canonical class and whose general fiber is a cubic surface blown up at six general points, hence rational and without elliptic pencils.

The next example gives local models of generically isotrivial elliptic orbibundles that have a crepant resolution.

Example 48 Let $Z \subset \mathbb{P}^N$ be an anticanonically embedded Fano variety and $X \subset \mathbb{A}_{\mathbf{x}}^{N+1}$ the cone over Z. Let $0 \in E$ be an elliptic curve with a marked point. Consider the elliptic fiber space

$$Y := X \times E/(-1,-1) \to X/(-1).$$

We claim that Y has a crepant resolution.

First we blow up the vertex of X. We get $B_0 X \to X$ with exceptional divisor $F \cong Z$. Note further that $B_0 X \to X$ is crepant. The involution lifts to $B_0 X \times E/(-1,-1)$. The fixed point set of this action is $F \times \{0\}$; a smooth subvariety of codimension 2. Thus $B_0 X \times E/(-1,-1)$ is resolved by blowing up the singular locus.

The next example shows that for surfaces with normal crossing singularities, a deformation may lose the elliptic structure.

Example 49 Let $S \subset \mathbb{P}^1 \times \mathbb{P}^2$ be a smooth surface of bi-degree $(1,3)$. The first projection $\pi\colon S \to \mathbb{P}^1$ is an elliptic fiber space. The other projection $\tau\colon S \to \mathbb{P}^2$ exhibits it as the blow-up of $\mathbb{P}^2$ at nine base points of an elliptic pencil. Let $F_1, \dots, F_9 \subset S$ denote the nine exceptional curves. Thus S is an elliptic dP_9. In particular, specifying $\pi\colon S \to \mathbb{P}^1$ plus a fiber of π is equivalent to a pair

$(E \subset \mathbb{P}^2)$ plus nine points $P_1, \ldots, P_9 \in E$ such that $P_1 + \cdots + P_9 \sim \mathcal{O}_{\mathbb{P}^2}(3)|_E$. The elliptic pencils are given by $\pi^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \tau^*\mathcal{O}_{\mathbb{P}^2}(3)(-F_1 - \cdots - F_9)$.

Let us now vary the points on E in a family $P_i(t) : t \in \mathbb{C}$. The line bundle giving the elliptic pencil deforms as $\tau^*\mathcal{O}_{\mathbb{P}^2}(3)(-F_1(t) - \cdots - F_9(t))$ but the elliptic pencil deforms only if $P_1(t) + \cdots + P_9(t) \sim \mathcal{O}_{\mathbb{P}^2}(3)|_E$ holds for every t.

Let $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a smooth threefold of bi-degree $(1, 3)$. The first projection $\pi\colon X \to \mathbb{P}^2$ is an elliptic fiber space.

If $C \subset \mathbb{P}^2$ is a conic, its pre-image $X_C \to C$ is an elliptic K3 surface. If C is general then X_C is smooth.

If $C = L_1 \cup L_2$ is a pair of general lines then $X_C = S_1 \cup S_2$ is a singular K3 surface which is a union of two smooth dP_9 that intersect along a smooth elliptic curve E.

We can thus think of X_C as obtained from two pairs $(E^i \subset \mathbb{P}^2)$ $(i = 1, 2)$ with an isomorphism $\phi\colon E^1 \to E^2$ by blowing up nine points $P^i_j \subset E^i$ $(j = 1, \ldots, 9)$ and gluing the resulting surfaces along the birational transforms of E^1 and E^2.

Let us now vary the points on both curves $P^1_i(t)$ and $P^2_i(t)$. We get two families $S_1(t), S_2(t)$ and this induces a deformation $X_C(t) = S_1(t) \cup S_2(t)$.

Although the line bundle $\pi^*\mathcal{O}_C(1)$ giving the elliptic pencil $X_C \to C$ deforms on both of the $S_i(t)$, in general we do not get a line bundle on $X_C(t)$ unless

$$P^1_1(t) + \cdots + P^1_9(t) \sim \phi^*(P^2_1(t) + \cdots + P^2_9(t))$$

holds for every t. We can thus arrange that $\pi^*\mathcal{O}_C(1)$ deforms along $X_C(t)$ but we lose the elliptic pencil.

10 General conjectures

A straightforward generalization of Conjecture 4 is the following (cf. [Ogu93] and [MP97, Lecture 10]):

Conjecture 50 (Strong abundance for Calabi–Yau manifolds) *Let X be a Calabi–Yau manifold and $L \in H^2(X, \mathbb{Q})$ a $(1, 1)$-class such that $(L \cdot C) \geq 0$ for every algebraic curve $C \subset X$. Then there is a unique morphism with connected fibers $g\colon X \to B$ onto a normal variety B and an ample $L_B \in H^2(B, \mathbb{Q})$ such that $L = g^*L_B$.*

The usual abundance conjecture assumes that L is effective, but this may not be necessary.

One expects Conjecture 50 to get harder as $\dim X - \dim B$ increases. The easiest case, when $\dim X - \dim B = 1$, corresponds to Questions 1 and 2.

From the point of view of higher-dimensional birational geometry, it is natural to consider a more general setting.

A *log Calabi–Yau fiber space* is a proper morphism with connected fibers $g\colon (X, \Delta) \to B$ onto a normal variety where (X, Δ) is klt (or possibly lc) and $(K_X + \Delta)|_{X_g} \sim_{\mathbb{Q}} 0$, where $X_g \subset X$ is a general fiber.

Let (X, Δ) be a proper klt pair such that $K_X + \Delta$ is nef and $g\colon (X, \Delta) \to B$ a relatively minimal Calabi–Yau fiber space. Let L_B be an ample $\mathbb{Q}$-divisor on B and set $L := g^* L_B$. Then $L - \epsilon(K_X + \Delta)$ is nef for $0 \le \epsilon \ll 1$. The converse fails in some rather simple cases, for instance when $X = B \times E$ for an elliptic curve E and we twist L by a degree-0 non-torsion line bundle on E.

It is natural to expect that the above are essentially the only exceptions.

Conjecture 51 *Let (X, Δ) be a proper klt pair such that $K_X + \Delta$ is nef and $H^1(X, \mathcal{O}_X) = 0$. Let L be a Cartier divisor on X such that $L - \epsilon(K_X + \Delta)$ is nef for $0 \le \epsilon \ll 1$.*

Then there is a relatively minimal log Calabi–Yau fiber space structure $g\colon (X, \Delta) \to B$ and an ample $\mathbb{Q}$-divisor L_B on B such that $L \sim_{\mathbb{Q}} g^* L_B$.

If $L - \epsilon(K_X + \Delta)$ is effective then Conjecture 51 is implied by the abundance conjecture. Note also that Example 49 shows that Conjecture 51 fails if (X, Δ) is log canonical.

Conjecture 52 *Let $g_0\colon (X_0, \Delta_0) \to B_0$ be a relatively minimal log Calabi–Yau fiber space where (X_0, Δ_0) is a proper klt pair and $H^2(X_0, \mathcal{O}_{X_0}) = 0$.*

Let (X, Δ) be a klt pair and $h\colon (X, \Delta) \to (0 \in S)$ a flat proper morphism whose central fiber is (X_0, Δ_0).

Then, after passing to an analytic or étale neighborhood of $0 \in S$, there is a proper, flat morphism $B \to (0 \in S)$ whose central fiber is B_0 such that g_0 extends to a log Calabi–Yau fiber space $g\colon (X, \Delta) \to B$.[1]

53 Although Conjecture 52 looks much more general than Theorem 31, it seems that Abelian fibrations comprise the only unknown case.

Indeed, let X_0, B_0 be projective varieties with rational singularities and $g_0\colon X_0 \to B_0$ a morphism with connected general fiber F_0. Assume that $H^1(F_0, \mathcal{O}_{F_0}) = 0$. Then $R^1(g_0)_* \mathcal{O}_{X_0}$ is a torsion sheaf. On the contrary, it is reflexive by [Kol86a, 7.8]. Thus $R^1(g_0)_* \mathcal{O}_{X_0} = 0$.

We could use Theorem 35, but there is an even simpler argument. Let L_{B_0} be a sufficiently ample line bundle on B_0 and set $L_0 := g_0^* L_{B_0}$. Then $H^1(X_0, L_0) = 0$ by (21.1). Thus, if $h\colon X \to (0 \in S)$ is a deformation of X_0 such that L_0 lifts to a line bundle L on X then every section of L_0 lifts to a

[1] Recent work of Katzarkov, Kontsevich, and Pantev establishes this in case X_0 is smooth.

section of L (after passing to an analytic or étale neighborhood of $0 \in S$). Thus Conjecture 52 holds in this case.

Furthermore, the method of Theorem 30 suggests that the most difficult case is Abelian pencils over $\mathbb{P}^1$.

Note also that it is easy to write down examples of Abelian Calabi–Yau fiber spaces $f: X \to B = \mathbb{P}^1$ such that $\mathrm{Hom}_B(\Omega_B, R^1 f_* O_X) \neq 0$, thus Theorem 35 does not seem to be sufficient to prove Conjecture 52.

54 (Log elliptic fiber spaces) As before, $g: (X, \Delta) \to B$ is a log elliptic fiber space iff $(L^{\dim X}) = 0$ but $(L^{\dim X-1}) \neq 0$. There are three cases to consider.

(1) If $(L^{\dim X-1} \cdot \Delta) > 0$ then Riemann–Roch shows that $h^0(X, L^m)$ grows like $m^{\dim X-1}$ and we get Conjecture 51 as in Theorem 10. In this case the general fiber of g is $F \cong \mathbb{P}^1$ and $(F \cdot \Delta) = 2$.

(2) If $(L^{\dim X-1} \cdot \Delta) = 0$ but $(L^{n-2} \cdot \mathrm{td}_2(X)) > 0$ then the proof of Theorem 10 works with minor changes.

(3) The hard and unresolved case is again when $(L^{\dim X-1} \cdot \Delta) = 0$ and $(L^{n-2} \cdot \mathrm{td}_2(X)) = 0$, so $\chi(X, L^m) = O(m^{\dim X-3})$.

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References

[ACG11] Arbarello, E., Cornalba, M., and Griffiths, P. A. Geometry of Algebraic Curves, Vol. II. Grundlehren der Mathematischen Wissenschaften, Vol. 268. Heidelberg: Springer-Verlag, 2011. With a contribution by J. D. Harris.

[AH11] Abramovich, D. and Hassett, B. Stable varieties with a twist. In *Classification of Algebraic Varieties*. EMS Series of Congress Reports, European Mathematical Society, Zürich, 2011, pp. 1–38.

[AK03] Araujo, C. and Kollár, J. Rational curves on varieties. *Higher Dimensional Varieties and Rational Points (Budapest, 2001)*. Bolyai Society Mathematics Studies, Vol. 12. Berlin: Springer-Verlag, 2003, pp. 13–68.

[AW98] Andreatta, M. and Wiśniewski, J. A. On contractions of smooth varieties. *J. Alg. Geom.* **7**(2) (1998) 253–312.

[BGS11] Boissière, S. Gabber, O. and Serman, O. Sur le produit de variétés localement factorielles ou Q-factorielles. ArXiv:1104.1861.

[BHPS12] Bhatt, B. Ho, W. Patakfalvi, Z. and Schnell, C. Moduli of products of stable varieties. ArXiv:1206.0438.

[BPV84] Barth, W. Peters, C. and Van de Ven, A. *Compact Complex Surfaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 4. Berlin: Springer-Verlag, 1984.

[Bri10] Brion, M. *Some Basic Results on Actions of Nonaffine Algebraic Groups*. *Symmetry and spaces*. Progress in Mathematics, Vol. 278. Boston: Inc. Birkhäuser, 2010, pp. 1–20.

[Cat07] Catanese, F. Q.E.D. for algebraic varieties. *J. Diff. Geom.* **77** (2007) 43–75. With an appendix by S. Rollenske.

[CL10] Cogolludo-Agustin, J. I. and Libgober, A. Mordell-Weil groups of elliptic threefolds and the Alexander module of plane curves. ArXiv:1008.2018.

[Del71] Deligne, P. Théorie de Hodge, II. *Inst. Hautes Études Sci. Publ. Math.* (1971) 5–57.

[DG94] Dolgachev, I. and Gross, M. Elliptic threefolds. I. Ogg-Shafarevich theory. *J. Alg. Geom.* **3** (1994) 39–80.

[Don98] Donagi, R. Y. ICMP lecture on heterotic/F-theory duality. ArXiv:hep-th/9802093.

[Fuj78] Fujita, T. On Kähler fiber spaces over curves. *J. Math. Soc. Japan* **30**(4) (1978) 779–794.

[Fuj11] Fujino, O. On Kawamata's theorem. In *Classification of Algebraic Varieties*. EMS Series of Congress Reports, European Mathematical Society, Zürich, 2011, pp. 305–315.

[Ful98] Fulton, W. Intersection Theory, 2nd edn. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, Vol. 2. Berlin: Springer-Verlag, 1998.

[GHS03] Graber, T., Harris, J., and Starr, J. Families of rationally connected varieties. *J. Amer. Math. Soc.* **16** (2003) 57–67.

[GKKP11] Greb, D., Kebekus, S., Kovács, S. J., and Peternell, T. Differential forms on log canonical spaces, *Publ. Math. Inst. Hautes Études Sci.* **114** (2011) 87–169.

[Gra91] Grassi, A. On minimal models of elliptic threefolds. *Math. Ann.* **290**(2) (1991) 287–301.

[Gra93] Grassi, A. The singularities of the parameter surface of a minimal elliptic threefold. *Int. J. Math.* **4**(2) (1993) 203–230.

[Gra94] Grassi, A. On a question of J. Kollár. In *Classification of Algebraic Varieties (L'Aquila, 1992)*. Contemporary Mathematics, Vol. 162. Providence, RI: American Mathematical Society, 1994, pp. 209–214.

[Gri70] Griffiths, P. A. Periods of integrals on algebraic manifolds, III. Some global differential-geometric properties of the period mapping. *Inst. Hautes Études Sci. Publ. Math.* **38** (1970) 125–180.

[Gro62] Grothendieck, A. *Fondements de la géométrie algébrique. [Extraits du Séminaire Bourbaki, 1957–1962*.] Paris: Secrétariat mathématique, 1962.

[Gro94] Gross, M. A finiteness theorem for elliptic Calabi–Yau threefolds. *Duke Math. J.* **74**(2) (1994) 271–299.

[Gro97] Gross, M. Elliptic three-folds, II. Multiple fibres. *Trans. Amer. Math. Soc.* **349**(9) (1997) 3409–3468.

[Har77] Hartshorne, R. *Algebraic Geometry*. Graduate Texts in Mathematics, No. 52. New York: Springer-Verlag, 1977.

[Har10] Hartshorne, R. *Deformation Theory*. Graduate Texts in Mathematics, No. 257. New York: Springer-Verlag, 2010.

[HK10] Hacon, C.D. and Kovács, S.J. *Classification of Higher dimensional Algebraic Varieties*. Oberwolfach Seminars, Vol. 41. Basel: Birkhäuser, 2010.

[HK11] Hulek, K. and Kloosterman, R. Calculating the Mordell–Weil rank of elliptic threefolds and the cohomology of singular hypersurfaces. *Ann. Inst. Fourier (Grenoble)* **61**(3) (2011) 1133–1179.

[Hor76] Horikawa, E. On deformations of holomorphic maps, III. *Math. Ann.* **222**(3) (1976) 275–282.

[HX13] Hacon, C. D. and Xu, C. Existence of log canonical closures. *Invent. Math.* **192** (2013) 161–195.

[Kaw85a] Kawamata, Y. Pluricanonical systems on minimal algebraic varieties. *Invent. Math.* **79**(3) (1985) 567–588.

[Kaw85b] Kawamata, Y. Minimal models and the Kodaira dimension of algebraic fiber spaces. *J. Reine Angew. Math.* **363** (1985) 1–46.

[KL09] Kollár, J. and Larsen, M. *Quotients of Calabi–Yau Varieties*. Algebra, Arithmetic, and Geometry: In honor of Yu. I. Manin, Vol. II. Progress in Mathematics, Vol. 270. Boston: Birkhäuser, 2009, pp. 179–211.

[Klo13] Kloosterman, R. Cuspidal plane curves, syzygies and a bound on the MW-rank. *J. Algebra* **375** (2013) 216–234.

[KM98] Kollár, J. and Mori, S. *Birational Geometry of Algebraic Varieties*. Cambridge Tracts in Mathematics, Vol. 134. Cambridge: Cambridge University Press, 1998. With the collaboration of C. H. Clemens and A. Corti. Translated from the 1998 Japanese original.

[KMM92] Kollár, J. Miyaoka, Y. and Mori, S. Rationally connected varieties. *J. Alg. Geom.* **1**(3) (1992) 429–448.

[KN94] Kawamata, Y. and Namikawa, Y. Logarithmic deformations of normal crossing varieties and smoothing of degenerate Calabi–Yau varieties. *Invent. Math.* **118**(3) (1994) 395–409.

[Kod63] Kodaira, K. *On compact analytic surfaces, II, III*. *Ann. Math. (2)* **77** (1963) 563–626; **78** (1963) 1–40.

[Kol85] Kollár, J. Toward moduli of singular varieties. *Compos. Math.* **56**(3) (1985) 369–398.

[Kol86a] Kollár, J. Higher direct images of dualizing sheaves, I. *Ann. Math. (2)* **123** (1986) 11–42.

[Kol86b] Kollár, J. Higher direct images of dualizing sheaves, II. *Ann. Math. (2)* **124** (1986) 171–202.

[Kol92] Kollár, J. (ed.). Flips and abundance for algebraic threefolds. Société Mathématique de France, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, UT, August 1991. Astérisque No. 211, 1992.

[Kol93] Kollár, J. Shafarevich maps and plurigenera of algebraic varieties. *Invent. Math.* **113** (1993) 177–215.

[Kol95] Kollár, J. Flatness criteria. *J. Algebra* **175**(2) (1995) 715–727.

[Kol96] Kollár, J. Rational Curves on Algebraic Varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Vol. 32. Berlin: Springer-Verlag, 1996.

[Kol10] Kollár, J. Exercises in the Birational Geometry of Algebraic Varieties. *Analytic and algebraic geometry*. IAS/Park City Mathematics Series, vol. 17. Providence, RI: American Mathematical Society, 2010, pp. 495–524.

[Kol13a] Kollár, J. Moduli of Varieties of General Type. Handbook of Moduli (G. Farkas and I. Morrison, eds), Advanced Lectures in Mathematics. Somerville, MA: International Press, 2013, pp. 131–158.

[Kol13b] Kollár, J. Singularities of the Minimal Model Program. Cambridge Tracts in Mathematics, Vol. 200, Cambridge: Cambridge University Press, 2013. With the collaboration of S. Kovács.

[KSC04] Kollár, J., Smith, K. E., and Corti, A. Rational and Nearly Rational Varieties. Cambridge Studies in Advanced Mathematics, Vol. 92. Cambridge: Cambridge University Press, 2004.

[Lai11] Lai, C.-J. Varieties fibered by good minimal models. *Math. Ann.* **350**(3) (2011) 533–547.

[Laz04] Lazarsfeld, R. Positivity in Algebraic Geometry, I–II. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Vol. 48–49. Berlin: Springer-Verlag, 2004.

[Les12] Lesieutre, J. The diminished base locus is not always closed. ArXiv e-prints (2012).

[LM75] Lieberman, D. and Mumford, D. Matsusaka's big theorem. In *Algebraic Geometry* (Proceedings of Symposium on Pure Mathematics, Vol. 29, Humboldt State University, Arcata, CA, 1974), Providence, RI: American Mathematical Society, 1975, pp. 513–530.

[Mat68] Matsusaka, T. Algebraic deformations of polarized varieties. *Nagoya Math. J.* **31** (1968) 185–245. Corrections: **33** (1968) 137; **36** (1968) 119.

[Mat72] Matsusaka, T. Polarized varieties with a given Hilbert polynomial. *Amer. J. Math.* **94** (1972) 1027–1077.

[Miy88] Miyaoka, Y. On the Kodaira dimension of minimal threefolds. *Math. Ann.* **281**(2) (1988) 325–332.

[MM64] Matsusaka, T. and Mumford, D. Two fundamental theorems on deformations of polarized varieties. *Amer. J. Math.* **86** (1964) 668–684.

[MP97] Miyaoka, Y. and Peternell, T. Geometry of Higher-Dimensional Algebraic Varieties. DMV Seminar, Vol. 26, Basel: Birkhäuser, 1997.

[Nak88] Nakayama, N. The singularity of the canonical model of campact Kähler manifolds. *Math. Ann.* **280** (1988) 509–512.

[Nak91] Nakayama, N. Elliptic fibrations over surfaces, I. In *Algebraic Geometry and Analytic Geometry (Tokyo, 1990)*. ICM-90 Satellite Conference Proceedings. Tokyo: Springer-Verlag, 1991, pp. 126–137.

[Nak99] Nakayama, N. Projective algebraic varieties whose universal covering spaces are biholomorphic to $\mathbf{C}^n$. *J. Math. Soc. Japan* **51**(3) (1999) 643–654.

[Nak02a] Nakayama, N. Global structure of an elliptic fibration. *Publ. Res. Inst. Math. Sci.* **38**(3) (2002) 451–649.

[Nak02b] Nakayama, N. Local structure of an elliptic fibration. In *Higher Dimensional Birational Geometry (Kyoto, 1997)*. Advanced Studies in Pure Mathematics, Vol. 35. Mathematical Society of Japan, Tokyo, 2002, pp. 185–295.

[Ogu93] Oguiso, K. On algebraic fiber space structures on a Calabi–Yau 3-fold. *Int. J. Math.* **4**(3) (1993) 439–465. With an appendix by N. Nakayama.

[Rei83] Reid, M. Minimal models of canonical 3-folds. In *Algebraic Varieties and Analytic Varieties (Tokyo, 1981)*. Advanced Studies in Pure Mathematics, Vol. 1. Amsterdam: North-Holland, 1983, pp. 131–180.

[Ser01] Serre, J.-P. Espaces fibrés algèbriques. *Séminaire C. Chevalley (1958)*. Documents Mathématiques, Vol. 1. Paris: Société Mathématique de France, 2001.

[Ses63] Seshadri, C. S. Quotient space by an abelian variety. *Math. Ann.* **152** (1963) 185–194.

[Ste76] Steenbrink, J. Limits of Hodge structures. *Invent. Math.* **31**(3) (1975/76) 229–257.

[Vaf96] Vafa, C. Evidence for F-theory. *Nucl. Phys. B* **469** (1996) 403–415.

[Wil89] Wilson, P. M. H. Calabi–Yau manifolds with large Picard number. *Invent. Math.* **98** (1989) 139–155.

[Wil94] Wilson, P. M. H. The existence of elliptic fibre space structures on Calabi-Yau threefolds. *Math. Ann.* **300**(4) (1994) 693–703.

[Wil98] Wilson, P. M. H. The existence of elliptic fibre space structures on Calabi–Yau threefolds, II. *Math. Proc. Cambridge Philos. Soc.* **123**(2) (1998) 259–262.