15

Derived equivalence and non-vanishing loci II

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Abstract

We prove a few cases of a conjecture on the invariance of cohomological support loci under derived equivalence by establishing a concrete connection with the related problem of the invariance of Hodge numbers. We use the main case in order to study the derived behavior of fibrations over curves.

Dedicated to Rob Lazarsfeld on the occasion of his 60th birthday, with warmth and gratitude

1 Introduction

This paper is concerned with the following conjecture made in [11] on the behavior of the non-vanishing loci for the cohomology of deformations of the canonical bundle under derived equivalence. We recall that given a smooth projective X these loci, more commonly called cohomological support loci, are the closed algebraic subsets of the Picard variety defined as

$$V^i(\omega_X) := \{\alpha \mid H^i(X, \omega_X \otimes \alpha) \neq 0\} \subseteq \mathrm{Pic}^0(X).$$

All varieties we consider are defined over the complex numbers. We denote by $\mathbf{D}(X)$ the bounded derived category of coherent sheaves $\mathbf{D}^{\mathrm{b}}(\mathrm{Coh}(X))$.

Conjecture 1 ([11]) *Let X and Y be smooth projective varieties with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ as triangulated categories. Then*

$$V^i(\omega_X)_0 \simeq V^i(\omega_Y)_0 \quad \text{for all } i \geq 0,$$

where $V^i(\omega_X)_0$ denotes the union of the irreducible components of $V^i(\omega_X)$ passing through the origin, and similarly for Y.

We refer to [10] and [11] for a general discussion of this conjecture and its applications, and of the cases in which it has been known to hold (recovered below as well). The main point of this paper is to relate Conjecture 1 directly to part of the well-known problem of the invariance of Hodge numbers under derived equivalence; we state only the special case we need.

Conjecture 2   *Let X and Y be smooth projective varieties with* $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. *Then*

$$h^{0,i}(X) = h^{0,i}(Y) \quad \text{for all } i \geq 0.$$

Our main result is the following:

Theorem 3   *Conjecture 2 implies Conjecture 1. More precisely, Conjecture 1 for a given i is implied by Conjecture 2 for* $n - i$, *where* $n = \dim X$.

This leads to a verification of Conjecture 1 in a few important cases, corresponding to the values of i for which Conjecture 2 is already known to hold.

Corollary 4   *Let X and Y be smooth projective varieties of dimension n, with* $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. *Then*

$$V^i(\omega_X)_0 \simeq V^i(\omega_Y)_0 \quad \text{for} \quad i = 0, 1, n-1, n.$$

Proof   According to Theorem 3, we need to know that derived equivalence implies the invariance of $h^{0,n}$, $h^{0,n-1}$, and $h^{0,1}$. The first two are well-known consequences of the invariance of Hochschild homology, while the last is the main result of [12].   □

A stronger result than Theorem 3 and Corollary 4, involving the dimension of cohomology groups related via the isomorphism, is in fact proved in Section 3 (see Conjecture 11 and Theorem 12). For $i = 0, 1$ this was proved in [10] by means of a twisted version of Hochschild homology. We note also that the corollary above recovers a result first proved in [10, Section 4], namely that Conjecture 1 holds for varieties of dimension up to 3. We can also conclude that it holds for an important class of irregular fourfolds.

Corollary 5   *Conjecture 1 holds in dimension up to three, and for fourfolds of maximal Albanese dimension.*

Proof   The first part follows immediately from Corollary 4. For the second part, according to Corollary 4 and Theorem 3 it suffices to have $h^{0,2}(X) =$

$h^{0,2}(Y)$, which is proved for derived equivalent fourfolds of maximal Albanese dimension in [10, Corollary 1.8].[1] □

Overall, besides the unified approach, from the point of view of Conjecture 1 the key new result and applications here are in the case $i = n - 1$. By theorems of Beauville [2] and Green–Lazarsfeld [7], the cohomological support loci $V^{n-1}(\omega_X)$ are the most "geometric" among the V^i, corresponding in a quite precise way to fibrations of X over curves. This leads to the following structural application; note that while Fourier–Mukai equivalences between smooth projective surfaces are completely classified [4, 9], in higher dimension few results toward classification are available (see, e.g., [16]).

Theorem 6 *Let X and Y be smooth projective varieties with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, such that X admits a surjective morphism to a smooth projective curve C of genus $g \geq 2$. Then:*

(i) Y admits a surjective morphism to a curve of genus $\geq g$.

(ii) If X has a Fano fibration structure over C, then so does Y, and X and Y are K-equivalent.[2] In particular, if X is a Mori fiber space over C, then X and Y are isomorphic.

A slightly stronger statement is given in Theorem 14. We remark that it is known from results of Beauville and Siu that X admits a surjective morphism to a curve of genus $\geq g$ if and only if $\pi_1(X)$ has a surjective homomorphism onto Γ_g, the fundamental group of a Riemann surface of genus g (see the Appendix to [6]). On the other hand, it is also known that derived equivalent varieties do not necessarily have isomorphic fundamental groups (see [1, 14]), so this would not suffice in order to deduce Theorem 6 (*i*). A more precise version of (*i*) can be found in the Remark on p. 302; see also Question 13. The refinement we give in (*ii*) in the case of Fano fibrations answers a question posed to us by Y. Kawamata; for this, the method of proof is completely independent of the study of $V^i(\omega_X)$, relying instead on Kawamata's kernel technique [9] and on the structure of the Albanese map for varieties with nef anticanonical bundle [17]. The result, however, fits naturally in the present context.

Going back to the main results, the isomorphism between the V_0^i is realized, as in [10], via the Rouquier isomorphism associated to a Fourier–Mukai equivalence (see Section 2). To relate this to the behavior of Hodge numbers of type $h^{0,i}$ as in Theorem 3, the main new ingredients are Simpson's result describing

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[1] Note that the same holds for fourfolds of non-negative Kodaira dimension whose Albanese image has dimension 3, and for those with non-affine $\mathrm{Aut}^0(X)$.

[2] Recall that this means there exist a smooth projective Z and birational morphisms $f: Z \to X$ and $g: Z \to Y$ such that $f^*\omega_X \simeq g^*\omega_Y$.

the components of all $V^i(\omega_X)$ as torsion translates of abelian subvarieties of $\text{Pic}^0(X)$, used via a density argument involving torsion points of special prime order, and the comparison of the derived categories of cyclic covers associated to torsion line bundles mapped to each other via the Rouquier isomorphism, modeled after and slightly extending results of Bridgeland–Maciocia [3] on equivalences of canonical covers.

2 Derived equivalences of cyclic covers

Cyclic covers. Let X be a complex smooth projective variety and α be a d-torsion element of $\text{Pic}^0(X)$. We denote by

$$\pi_\alpha \colon X_\alpha \to X$$

the étale cyclic cover of order d associated to α (see, e.g., [8, Section 7.3]). Then

$$\pi_{\alpha*}O_{X_\alpha} \simeq \bigoplus_{i=0}^{d-1} \alpha^{-i} \tag{1}$$

and there is a free action of the group $G := \mathbf{Z}/d\mathbf{Z}$ on X_α such that $X_\alpha/G \simeq X$. The following lemma is analogous to [3, Proposition 2.5(b)]. We include a proof for completeness, entirely inspired by the approach in [3, Proposition 2.5(a)].

Lemma 7 *Let E be an object of $\mathbf{D}(X)$. There is an object E_α in $\mathbf{D}(X_\alpha)$ such that $\pi_{\alpha*}E_\alpha \simeq E$ if and only if $E \otimes \alpha \simeq E$.*

Proof For the nontrivial implication, let

$$s \colon E \xrightarrow{\simeq} E \otimes \alpha$$

be an isomorphism. We proceed by induction on the number r of nonzero cohomology sheaves of E. If E is a sheaf concentrated in degree zero, then the lemma is a standard fact. Indeed, it is well known that

$$\pi_{\alpha*} \colon \text{Coh}(X_\alpha) \to \text{Coh}(\mathcal{A})$$

is an equivalence between the category of coherent O_{X_α}-modules and the category of coherent $\mathcal{A} := (\bigoplus_{i=0}^{d-1} \alpha^i)$-algebras, while a coherent sheaf E on X belongs to $\text{Coh}(\mathcal{A})$ if and only if $E \otimes \alpha \simeq E$.

Suppose now that the lemma is true for all objects having at most $r-1$ nonzero cohomology sheaves, and consider an object E with r nonzero cohomology sheaves. By shifting E, we can assume that $\mathcal{H}^i(E) = 0$ for

$i \notin [-(r-1), 0]$. Since $E \otimes \alpha \simeq E$, we also have $\mathcal{H}^0(E) \otimes \alpha \simeq \mathcal{H}^0(E)$. Therefore, by the above, there exists a coherent sheaf M_α on X_α such that $\pi_{\alpha*} M_\alpha \simeq \mathcal{H}^0(E)$. Now the natural morphism $E \xrightarrow{j} \mathcal{H}^0(E)$ induces a distinguished triangle

$$E \xrightarrow{j} \mathcal{H}^0(E) \xrightarrow{f} F \to E[1]$$

such that the object F has $r-1$ nonzero cohomology sheaves. By the commutativity of the following diagram:

$$
\begin{array}{ccccccc}
E & \xrightarrow{\ j\ } & \mathcal{H}^0(E) & \xrightarrow{\ f\ } & F & \longrightarrow & E[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E \otimes \alpha & \xrightarrow{j \otimes \alpha} & \mathcal{H}^0(E) \otimes \alpha & \xrightarrow{f \otimes \alpha} & F \otimes \alpha & \longrightarrow & (E \otimes \alpha)[1]
\end{array}
$$

we obtain an isomorphism $F \simeq F \otimes \alpha$, and therefore by induction an object F_α in $\mathbf{D}(X_\alpha)$ such that $\pi_{\alpha*} F_\alpha \simeq F$.

To show the existence of an object E_α in $\mathbf{D}(X_\alpha)$ such that $\pi_{\alpha*} E_\alpha \simeq E$, we assume for a moment that there exists a morphism $f_\alpha \colon M_\alpha \to F_\alpha$ such that $\pi_{\alpha*} f_\alpha = f$. This is enough to conclude, since by completing f_α to a distinguished triangle

$$M_\alpha \xrightarrow{f_\alpha} F_\alpha \to E_\alpha[1] \to M_\alpha[1]$$

and applying $\pi_{\alpha*}$, we obtain $\pi_{\alpha*} E_\alpha \simeq E$.

We are left with showing the existence of f_α. Let $\lambda_\alpha \colon \pi_{\alpha*} M_\alpha \to \pi_{\alpha*} M_\alpha \otimes \alpha$ and $\mu_\alpha \colon \pi_{\alpha*} F_\alpha \to \pi_{\alpha*} F_\alpha \otimes \alpha$ be the isomorphisms determined by the diagram above. Note that

$$\mu_\alpha \circ f = (f \otimes \alpha) \circ \lambda_\alpha \quad \text{in} \quad \mathbf{D}(X). \tag{2}$$

We can replace F_α by an injective resolution

$$\cdots \to \mathcal{I}_\alpha^{-1} \xrightarrow{d^{-1}} \mathcal{I}_\alpha^0 \xrightarrow{d^0} \mathcal{I}_\alpha^1 \xrightarrow{d^1} \cdots,$$

so that f is represented (up to homotopy) by a morphism of $\mathcal{O}_X$-modules

$$u \colon \pi_{\alpha*} M_\alpha \to \pi_{\alpha*} \mathcal{I}_\alpha^0.$$

Let V be the image of the map

$$\mathrm{Hom}(\pi_{\alpha*} M_\alpha, -) \colon \mathrm{Hom}(\pi_{\alpha*} M_\alpha, \pi_{\alpha*} \mathcal{I}_\alpha^{-1}) \to \mathrm{Hom}(\pi_{\alpha*} M_\alpha, \pi_{\alpha*} \mathcal{I}_\alpha^0).$$

By (2), we have isomorphisms of $\mathcal{O}_X$-modules $a_1 \colon \pi_{\alpha*} M_\alpha \to \pi_{\alpha*} M_\alpha \otimes \alpha$ and $b_1 \colon \pi_{\alpha*} \mathcal{I}_\alpha^0 \to \pi_{\alpha*} \mathcal{I}_\alpha^0 \otimes \alpha$ such that

$$b_1 \circ u = (u \otimes \alpha) \circ a_1 \qquad \text{up to homotopy.} \tag{3}$$

By setting $a_i := (a_1 \otimes \alpha^{i-1}) \circ \cdots \circ (a_1 \otimes \alpha) \circ a_1$ (for $i \geq 2$) and similarly for b_i, we define an action of $G := \mathbf{Z}/d\mathbf{Z}$ on V as

$$g^i \cdot (-) := b_i^{-1} \circ (- \otimes \alpha^i) \circ a_i,$$

where g is a generator of G. Moreover, we define operators A and B on V as

$$A := \sum_{i=0}^{d-1} g^i \cdot (-), \qquad B := 1 - g \cdot (-).$$

Since $AB = 0$, we note that $\operatorname{Ker} A = \operatorname{Im} B$.

By (3), we have that $B(u) = u - b_1^{-1} \circ (u \otimes \alpha) \circ a_1$ is null-homotopic, and therefore $B(u) \in V$. Since $\operatorname{Ker} A = \operatorname{Im} B$, there exists a morphism $\eta \in V$ such that $B(\eta) = B(u)$. Consider the morphism $t := u - \eta \in \operatorname{Hom}(\pi_{\alpha*} M_\alpha, \pi_{\alpha*} \mathcal{I}_\alpha^0)$. It is easy to check that t is homotopic to u and therefore it represents f as well. But now $B(t) = 0$, so $t = \pi_{\alpha*}(v)$ for some morphism $v \colon M_\alpha \to \mathcal{I}_\alpha^0$, which concludes the proof. $\qquad \square$

Rouquier isomorphism. It is well known by Orlov's criterion that every equivalence $\Phi \colon \mathbf{D}(X) \to \mathbf{D}(Y)$ is of Fourier–Mukai type, i.e., induced by an object $\mathcal{E} \in \mathbf{D}(X \times Y)$, unique up to isomorphism, via

$$\Phi = \Phi_{\mathcal{E}} \colon \mathbf{D}(X) \to \mathbf{D}(Y), \quad \Phi_{\mathcal{E}}(-) = \mathbf{R}p_{Y*}(p_X^*(-) \overset{\mathbf{L}}{\otimes} \mathcal{E}).$$

For every such equivalence, Rouquier [13, Théorème 4.18] showed that there is an induced isomorphism of algebraic groups

$$F_{\mathcal{E}} \colon \operatorname{Aut}^0(X) \times \operatorname{Pic}^0(X) \to \operatorname{Aut}^0(Y) \times \operatorname{Pic}^0(Y)$$

which usually mixes the two factors. A concrete formula for $F_{\mathcal{E}}$ was worked out in [12, Lemma 3.1], namely

$$F_{\mathcal{E}}(\varphi, \alpha) = (\psi, \beta) \iff p_X^*\alpha \otimes (\varphi \times \operatorname{id}_Y)^*\mathcal{E} \simeq p_Y^*\beta \otimes (\operatorname{id}_X \times \psi)_*\mathcal{E}. \tag{4}$$

Derived equivalences of cyclic covers. Before stating the main theorem of this section, we recall two definitions from [3] (see also [8, Section 7.3]). Let $\widetilde{X}$ and $\widetilde{Y}$ be two smooth projective varieties on which the group $G := \mathbf{Z}/d\mathbf{Z}$ acts freely. Denote by $\pi_X \colon \widetilde{X} \to X$ and $\pi_Y \colon \widetilde{Y} \to Y$ the quotient maps of $\widetilde{X}$ and $\widetilde{Y}$ respectively.

Definition 8 A functor $\widetilde{\Phi} \colon \mathbf{D}(\widetilde{X}) \to \mathbf{D}(\widetilde{Y})$ is *equivariant* if there exist an automorphism μ of G and isomorphisms of functors

$$g^* \circ \widetilde{\Phi} \simeq \widetilde{\Phi} \circ \mu(g)^* \quad \text{for all} \quad g \in G.$$

Definition 9 Let $\Phi \colon \mathbf{D}(X) \to \mathbf{D}(Y)$ be a functor. A *lift* of Φ is a functor $\widetilde{\Phi} \colon \mathbf{D}(\widetilde{X}) \to \mathbf{D}(\widetilde{Y})$ inducing isomorphisms

$$\pi_{Y*} \circ \widetilde{\Phi} \simeq \Phi \circ \pi_{X*}, \tag{5}$$

$$\pi_Y^* \circ \Phi \simeq \widetilde{\Phi} \circ \pi_X^*. \tag{6}$$

Remark If $\Phi \colon \mathbf{D}(X) \to \mathbf{D}(Y)$ and $\widetilde{\Phi} \colon \mathbf{D}(\widetilde{X}) \to \mathbf{D}(\widetilde{Y})$ are equivalences, then by taking the adjoints (5) holds if and only if (6) holds.

Now we are ready to prove the main result of this section. It is a slight extension of the result of [3] on canonical covers, whose proof almost entirely follows the one given there, and which serves as a technical tool for our main theorem.

Theorem 10 *Let X and Y be smooth projective varieties, and $\alpha \in \mathrm{Pic}^0(X)$ and $\beta \in \mathrm{Pic}^0(Y)$ d-torsion elements. Denote by $\pi_\alpha \colon X_\alpha \to X$ and $\pi_\beta \colon Y_\beta \to Y$ the cyclic covers associated to α and β respectively.*

(i) *Suppose that $\Phi_{\mathcal{E}} \colon \mathbf{D}(X) \to \mathbf{D}(Y)$ is an equivalence, and that $F_{\mathcal{E}}(\mathrm{id}_X, \alpha) = (\mathrm{id}_Y, \beta)$. Then there exists an equivariant equivalence $\Phi_{\widetilde{\mathcal{E}}} \colon \mathbf{D}(X_\alpha) \to \mathbf{D}(Y_\beta)$ lifting $\Phi_{\mathcal{E}}$.*
(ii) *Suppose that $\Phi_{\widetilde{\mathcal{F}}} \colon \mathbf{D}(X_\alpha) \to \mathbf{D}(Y_\beta)$ is an equivariant equivalence. Then $\Phi_{\widetilde{\mathcal{F}}}$ is the lift of some equivalence $\Phi_{\mathcal{F}} \colon \mathbf{D}(X) \to \mathbf{D}(Y)$.*

Proof To see (*i*), consider the following commutative diagram, where p_1, p_2, r_1 and r_2 are projection maps:

$$
\begin{array}{ccccc}
X_\alpha & \xleftarrow{\ r_1\ } & X_\alpha \times Y & \xrightarrow{\ r_2\ } & Y \\
\downarrow{\scriptstyle \pi_\alpha} & & \downarrow{\scriptstyle \pi_\alpha \times \mathrm{id}_Y} & & \| \\
X & \xleftarrow{\ p_1\ } & X \times Y & \xrightarrow{\ p_2\ } & Y
\end{array}
$$

By (4), the condition $F_{\mathcal{E}}(\mathrm{id}_X, \alpha) = (\mathrm{id}_Y, \beta)$ is equivalent to the isomorphism in $\mathbf{D}(X \times Y)$:

$$p_1^* \alpha \otimes \mathcal{E} \simeq p_2^* \beta \otimes \mathcal{E}. \tag{7}$$

Pulling (7) back via the map $(\pi_\alpha \times \mathrm{id}_Y)$, we get an isomorphism

$$(\pi_\alpha \times \mathrm{id}_Y)^* \mathcal{E} \simeq r_2^* \beta \otimes (\pi_\alpha \times \mathrm{id}_Y)^* \mathcal{E}$$

as $\pi_\alpha^* \alpha \simeq \mathcal{O}_{X_\alpha}$. As the map $(\mathrm{id}_{X_\alpha} \times \pi_\beta) \colon X_\alpha \times Y_\beta \to X_\alpha \times Y$ is the étale cyclic cover associated to the line bundle $r_2^* \beta$, by Lemma 7 there exists an object $\widetilde{\mathcal{E}}$ such that

$$(\mathrm{id}_{X_\alpha} \times \pi_\beta)_* \widetilde{\mathcal{E}} \simeq (\pi_\alpha \times \mathrm{id}_Y)^* \mathcal{E}.$$

By [3, Lemma 4.4], there is an isomorphism

$$\pi_{\beta *} \circ \Phi_{\widetilde{\mathcal{E}}} \simeq \Phi_{\mathcal{E}} \circ \pi_{\alpha *}. \tag{8}$$

We now show that $\Phi_{\widetilde{\mathcal{E}}}$ is an equivalence. Let $\Psi_{\mathcal{E}'} : \mathbf{D}(Y) \to \mathbf{D}(X)$ be a quasi-inverse of $\Phi_{\mathcal{E}}$. Since $F_{\mathcal{E}'} = F_{\mathcal{E}}^{-1}$, we have that $F_{\mathcal{E}'}(\mathrm{id}_X, \beta) = (\mathrm{id}_Y, \alpha)$. By repeating the previous argument, one then sees that there exists an object $\widetilde{\mathcal{E}}'$ such that

$$(\pi_\alpha \times \mathrm{id}_{Y_\beta})_* \widetilde{\mathcal{E}}' \simeq (\mathrm{id}_X \times \pi_\beta)^* \mathcal{E}'$$

and an isomorphism of functors

$$\pi_{\alpha *} \circ \Psi_{\widetilde{\mathcal{E}}'} \simeq \Psi_{\mathcal{E}'} \circ \pi_{\beta *}. \tag{9}$$

Since $\Psi_{\mathcal{E}'} \circ \Phi_{\mathcal{E}} \simeq \mathrm{id}_{\mathbf{D}(X)}$, using (8) and (9) we get an isomorphism

$$\pi_{\alpha *} \circ \Psi_{\widetilde{\mathcal{E}}'} \circ \Phi_{\widetilde{\mathcal{E}}} \simeq \Psi_{\mathcal{E}'} \circ \pi_{\beta *} \circ \Phi_{\widetilde{\mathcal{E}}} \simeq \Psi_{\mathcal{E}'} \circ \Phi_{\mathcal{E}} \circ \pi_{\alpha *} \simeq \pi_{\alpha *}. \tag{10}$$

Hence, following the proof of [3, Lemma 4.3], we have that $\Psi_{\widetilde{\mathcal{E}}'} \circ \Phi_{\widetilde{\mathcal{E}}} \simeq g_*(L \otimes -)$ for some $g \in G$ and $L \in \mathrm{Pic}(X_\alpha)$. By taking left adjoints in (10), we obtain on the other hand that

$$(L^{-1} \otimes -) \circ g^* \circ \pi_\alpha^* \simeq \pi_\alpha^*,$$

which applied to O_X yields $L \simeq O_{X_\alpha}$. This gives $\Psi_{\widetilde{\mathcal{E}}'} \circ \Phi_{\widetilde{\mathcal{E}}} \simeq g_*$. Similarly, we can show that $\Phi_{\widetilde{\mathcal{E}}} \circ \Psi_{\widetilde{\mathcal{E}}'} \simeq h_*$ for some $h \in G$, and hence that $g^* \circ \Psi_{\widetilde{\mathcal{E}}'}$, or equivalently $\Psi_{\widetilde{\mathcal{E}}'} \circ h^*$, is a quasi-inverse of $\Phi_{\widetilde{\mathcal{E}}}$. Finally, Remark 2 implies that $\Phi_{\widetilde{\mathcal{E}}}$ is a lift of $\Phi_{\mathcal{E}}$.

The proofs of the fact that $\Phi_{\widetilde{\mathcal{E}}}$ is equivariant and of (*ii*) are now completely analogous to those of the corresponding statements in [3, Theorem 4.5]. $\qquad \square$

3 Comparison of cohomological support loci

A more precise statement than that of Conjecture 1 naturally involves the dimension of the cohomology groups of the line bundles mapped to each other via the Rouquier isomorphism. Such a statement was proved in [10] when $i = 0, 1$. The concrete statement, which also contains Conjecture 2 by specializing at the origin, is the following:

Conjecture 11 *Let X and Y be smooth projective varieties of dimension n, and let $\Phi_{\mathcal{E}}$ be a Fourier–Mukai equivalence between $\mathbf{D}(X)$ and $\mathbf{D}(Y)$. If $F = F_{\mathcal{E}}$ is the induced Rouquier isomorphism, then*

$$F\left(\mathrm{id}_X, V^i(\omega_X)_0\right) = \left(\mathrm{id}_Y, V^i(\omega_Y)_0\right)$$

for all i, so that $V^i(\omega_X)_0 \simeq V^i(\omega_Y)_0$. Moreover if $\alpha \in V^i(\omega_X)_0$ and $F(\mathrm{id}_X, \alpha) = (\mathrm{id}_Y, \beta)$, then

$$h^i(X, \omega_X \otimes \alpha) = h^i(Y, \omega_Y \otimes \beta).$$

In order to address this statement, we consider for each $m \geq 1$ the more refined cohomological support loci

$$V_m^i(\omega_X) := \{\alpha \in \mathrm{Pic}^0(X) \mid h^i(X, \omega_X \otimes \alpha) \geq m\}.$$

In this notation, $V^i(\omega_X)$ becomes $V_1^i(\omega_X)$. The following result is a strengthening of Theorem 3 in the Introduction.[3]

Theorem 12 *Conjecture 2 is equivalent to Conjecture 11. Specifically, if $n =$ dim X, Conjecture 2 for $n - i$ implies*

$$F\left(\mathrm{id}_X, V_m^i(\omega_X)_0\right) = \left(\mathrm{id}_Y, V_m^i(\omega_Y)_0\right)$$

for all $m \geq 1$.

Proof Note first that F induces an isomorphism on the locus of line bundles $\alpha \in \mathrm{Pic}^0(X)$ with the property that $F(\mathrm{id}_X, \alpha) = (\mathrm{id}_Y, \beta)$ for some $\beta \in \mathrm{Pic}^0(Y)$, so indeed the first assertion in the statement follows from the second one, which we prove in a few steps.
Step 1. We first show that if $\alpha \in V^i(\omega_X)_0$, then it does satisfy the property above, namely there exists $\beta \in \mathrm{Pic}^0(Y)$ such that

$$F(\mathrm{id}_X, \alpha) = (\mathrm{id}_Y, \beta).$$

A more general statement has already been proved in [10, Theorem 3.2]. We extract the argument we need here in order to keep the proof self-contained, following [12, Section 3] as well. The Rouquier isomorphism F induces a morphism

$$\pi \colon \mathrm{Pic}^0(Y) \to \mathrm{Aut}^0(X), \quad \pi(\beta) = p_1(F^{-1}(\mathrm{id}_Y, \beta)),$$

whose image is an abelian variety A and where p_1 is the projection from $\mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X)$ onto the first factor. If A is trivial there is nothing to prove, so we can assume that A is positive-dimensional.

As A is an abelian variety of automorphisms of X, according to [5, Section 3] there exists a finite subgroup $H \subset A$ and an étale locally trivial fibration

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[3] We are grateful to the referee, who suggested that our original argument for $m = 1$ applies in fact to all m. This was crucial for deducing this more refined statement, as opposed to just that of Conjecture 1, from Conjecture 2.

$p\colon X \to A/H$ which is trivialized by base change to A. In other words, there is a cartesian diagram

$$
\begin{array}{ccc}
A \times Z & \xrightarrow{\ g\ } & X \\
\downarrow & & \downarrow{\scriptstyle p} \\
A & \longrightarrow & A/H
\end{array}
$$

where $Z = p^{-1}(0)$. Restricting g to the fiber $A \times \{z_0\}$, where z_0 is an arbitrary point in Z, we obtain a morphism $f\colon A \to X$, which is in fact an orbit of the action of A on X. It is shown in the proof of [12, Theorem A] that $\mathrm{Ker}(f^*)^0 \simeq \mathrm{Ker}(\pi)^0$, where $(\cdot)^0$ denotes the connected component of the identity; this is based on a theorem of Matsumura–Nishi, essentially saying that the induced $f^*\colon \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(A)$ is surjective. Consequently, we only need to show that $\alpha \in \mathrm{Ker}(f^*)$; it will then automatically be in $\mathrm{Ker}(f^*)^0$, since it lives in $V^i(\omega_X)_0$, which is a union of abelian subvarieties.

To this end, note that $\alpha \in V^i(\omega_X)_0$ implies that

$$
H^i\left(A \times Z, g^*(\omega_X \otimes \alpha)\right) \simeq H^i\left(A \times Z, f^*\alpha \boxtimes (\omega_Z \otimes \alpha_{|Z})\right) \neq 0.
$$

Applying the Künneth formula, we conclude that we must have

$$
H^k(A, f^*\alpha) \neq 0 \quad \text{for some } 0 \leq k \leq i,
$$

which implies that $f^*\alpha \simeq O_A$.

Step 2. Since one can repeat the argument in Step 1 for F^{-1}, it is enough to show that if $\alpha \in V_m^i(\omega_X)_0$ and $F(\mathrm{id}_X, \alpha) = (\mathrm{id}_Y, \beta)$, then $\beta \in V_m^i(\omega_Y)_0$ as well. In this step we show that it is enough to prove this assertion in the case when $\alpha \in \mathrm{Pic}^0(X)$ is a torsion point of (special) prime order. First, since F is a group isomorphism, α is torsion of some order if and only if β is torsion of the same order.

According to a well-known theorem of Simpson [15], every irreducible component Z of $V_m^i(\omega_Y)$ is a torsion translate $\tau_Z + A_Z$ of an abelian subvariety of $\mathrm{Pic}^0(Y)$. We consider the set P_i of all prime numbers that do not divide $\mathrm{ord}(\tau_Z)$ for any such component Z. As $V_m^i(\omega_Y)$ is an algebraic set by the semicontinuity theorem, we are only throwing away a finite set of primes. We will show that it is enough to prove the assertion above when α is torsion with order in P_i. First note that it is a standard fact that torsion points of prime order are Zariski dense in a complex abelian variety.[4] Consequently, torsion points with order in the set P_i are dense as well.

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[4] This follows for instance from the fact that real numbers can be approximated by rational numbers with prime denominators.

Now let W be a component of $V_m^i(\omega_X)_0$. It suffices to show that

$$Z := p_2(F(\mathrm{id}_X, W)) \subset V_m^i(\omega_Y)_0,$$

where p_2 is the projection onto the second component of $\mathrm{Aut}^0(Y) \times \mathrm{Pic}^0(Y)$. Indeed, since one can repeat the same argument for the inverse homomorphism F^{-1}, this implies that Z has to be a component of $V_m^i(\omega_Y)_0$, isomorphic to W via F. Now Z is an abelian variety, and therefore by the discussion above torsion points β of order in P_i are dense in Z. By semicontinuity, it suffices to show that $\beta \in V_m^i(\omega_Y)_0$. These βs are precisely the images of $\alpha \in W$ of order in P_i, which concludes our reduction step.

Step 3. Now let $\alpha \in V_m^i(\omega_X)_0$ be a torsion point of order belonging to the set P_i, and $F(\mathrm{id}_X, \alpha) = (\mathrm{id}_Y, \beta)$. Denote

$$p = \mathrm{ord}(\alpha) = \mathrm{ord}(\beta).$$

Consider the cyclic covers $\pi_\alpha \colon X_\alpha \to X$ and $\pi_\beta \colon Y_\beta \to Y$ associated to α and β respectively. We can apply Theorem 10 to conclude that there exists a Fourier–Mukai equivalence

$$\Phi_{\widetilde{\mathcal{E}}} \colon \mathbf{D}(X_\alpha) \to \mathbf{D}(Y_\beta)$$

lifting $\Phi_{\mathcal{E}}$. Assuming Conjecture 2, we have in particular that

$$h^{0,n-i}(X) = h^{0,n-i}(Y) \quad \text{and} \quad h^{0,n-i}(X_\alpha) = h^{0,n-i}(Y_\beta).$$

On the other hand, using (1), we have

$$H^{n-i}(X_\alpha, O_{X_\alpha}) \simeq \bigoplus_{j=0}^{p-1} H^{n-i}(X, \alpha^{-j}) \quad \text{and}$$

$$H^{n-i}(Y_\beta, O_{Y_\beta}) \simeq \bigoplus_{j=0}^{p-1} H^{n-i}(Y, \beta^{-j}).$$

The terms on the left hand side and the terms corresponding to $j = 0$ on the right hand side have the same dimension. On the other hand, since every component of $V_m^i(\omega_X)_0$ is an abelian subvariety of $\mathrm{Pic}^0(X)$, we have that $\alpha^j \in V_m^i(\omega_X)_0$ for all j, so

$$h^{n-i}(X, \alpha^{-j}) \geq m \quad \text{for all } j.$$

We conclude that

$$h^{n-i}(Y, \beta^{-k}) \geq m \quad \text{for some} \quad 1 \leq k \leq p-1.$$

This says that $\beta^k \in V_m^i(\omega_Y)$. We claim that in fact $\beta^k \in V_m^i(\omega_Y)_0$. Assuming that this is the case, we can conclude the argument. Indeed, pick a component

$T \subset V_m^i(\omega_Y)_0$ such that $\beta^k \in T$. But β^k generates the cyclic group of prime order $\{1, \beta, \ldots, \beta^{p-1}\}$, so $\beta \in T$ as well, since T is an abelian variety.

We are left with proving that $\beta^k \in V_m^i(\omega_Y)_0$. Pick any component S in $V_m^i(\omega_Y)$ containing β^k. By the Simpson theorem mentioned above, we have that $S = \tau + B$, where τ is a torsion point and B is an abelian subvariety of $\mathrm{Pic}^0(Y)$. We claim that we must have $\tau \in B$, so that $S = B$, confirming our statement.[5] To this end, switching abusively to additive notation, say $k\beta = \tau + b$ with $b \in B$, and denote the torsion order of τ by r. Since the order p of β is assumed to be in the set P_i, we have that r and p are coprime. Now on the one hand $r\tau = 0 \in B$, while on the other hand $p\tau + pb = kp\beta = 0$, so $p\tau \in B$ as well. Since r and p are coprime, one easily concludes that $\tau \in B$. $\qquad\square$

4 Fibrations over curves

Fibration structure via derived equivalence. We now apply the derived invariance of $V^{n-1}(\omega_X)_0$ to deduce Theorem 6 (*i*) in the Introduction.

Proof of Theorem 6 (i) Let $f : X \to C$ be a surjective morphism onto a smooth projective curve of genus $g \geq 2$. Using Stein factorization, we can assume that f has connected fibers. We have that $f^* \mathrm{Pic}^0(C) \subset V^{n-1}(\omega_X)_0$. Since by Corollary 4 we have $V^{n-1}(\omega_X)_0 \simeq V^{n-1}(\omega_Y)_0$, there exists a component T of $V^{n-1}(\omega_Y)_0$ of dimension at least g. By [2, Corollaire 2.3], there exists a smooth projective curve D and a surjective morphism with connected fibers $g : Y \to D$ such that $T = g^* \mathrm{Pic}^0(D)$. Note that $g(D) = \dim T \geq g$. $\qquad\square$

Remark The discussion above shows in fact the following more refined statement. For a smooth projective variety Z, define

$$A_Z := \{g \in \mathbf{N} \mid g = \dim T \text{ for some irreducible component } T \subset V^{n-1}(\omega_Z)_0\}.$$

Then if $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, we have $A_X = A_Y$. Denoting this set by A, for each $g \in A$ both X and Y have surjective maps onto curves of genus g. The maximal genus of a curve admitting a surjective map from X (or Y) is $\max(A)$.

Question 13 If $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, is the set of curves of genus at least 2 admitting non-constant maps from X the same as that for Y? Or at least the set of curves corresponding to irreducible components of $V^{n-1}(\omega_X)_0$?

Fano fibrations. The following is a slightly more precise version of Theorem 6 (*ii*) in the Introduction.

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[5] Note that in fact we are proving something stronger: β^k belongs *only* to components of $V_m^i(\omega_Y)$ passing through the origin.

Theorem 14 *Let X and Y be smooth projective complex varieties such that* $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. *Assume that there is an algebraic fiber space* $f : X \to C$ *such that C is a smooth projective curve of genus at least* 2 *and the general fiber of f is Fano. Then:*

(i) *X and Y are K-equivalent.*

(ii) *There is an algebraic fiber space* $g : Y \to C$ *such that for* $c \in C$ *where the fibers* X_c *and* Y_c *are smooth, with* X_c *Fano, one has* $Y_c \simeq X_c$.

(iii) *If* ω_X^{-1} *is f-ample (e.g., if f is a Mori fiber space), then* $X \simeq Y$.

Proof Let p and q be the projections of $X \times Y$ onto the first and second factor respectively. Consider the unique-up-to-isomorphism $\mathcal{E} \in \mathbf{D}(X \times Y)$ such that the given equivalence is the Fourier–Mukai functor $\Phi_{\mathcal{E}}$. Then by [8, Corollary 6.5], there exists a component Z of $\mathrm{Supp}(\mathcal{E})$ such that $p_{|Z} : Z \to X$ is surjective. We first claim that $\dim Z = \dim X$.

Assuming by contradiction that $\dim Z > \dim X$, we show that ω_X^{-1} is nef. We denote by F the general fiber of f, which is Fano. We also define $Z_F := p_Z^{-1}(F) \subset Z$, while $q_F : Z_F \to Y$ is the projection obtained by restricting q to Z_F. Since ω_F^{-1} is ample, we obtain that q_F is finite onto its image; see [8, Corollary 6.8]. On the other hand, the assumption that $\dim Z > \dim X$ implies that $\dim Z_F \geq \dim X = \dim Y$, so q_F must be surjective (and consequently $\dim Z_F = \dim X$).

By passing to its normalization if necessary, we can assume without loss of generality that Z_F is normal. Denoting by p_F the projection of Z_F to X, by [8, Corollary 6.9] we have that there exists $r > 0$ such that

$$p_F^* \, \omega_X^{-r} \simeq q_F^* \, \omega_Y^{-r}.$$

Now since p_F factors through F and ω_F^{-1} is ample, we have that $p_F^* \omega_X^{-1}$ is nef, hence by the isomorphism above so is $q_F^* \, \omega_Y^{-1}$. Finally, since q_F is finite and surjective, we obtain that ω_Y^{-1} is nef, so by [9, Theorem 1.4], ω_X^{-1} is nef as well.

We can now conclude the proof of the claim using the main result of Zhang [17] (part of a conjecture of Demailly–Peternell–Schneider), saying that a smooth projective variety with nef anticanonical bundle has surjective Albanese map. In our case, since the general fiber of f is Fano, the Albanese map of X is obtained by composing f with the Abel–Jacobi embedding of C. But this implies that C has genus at most 1, a contradiction. The claim is proved, so

$$\dim Z = \dim X = \dim Y.$$

At this stage, the *K*-equivalence statement follows from Lemma 15 below.

For statements (*ii*) and (*iii*) we emphasize that, once we know that X and Y are K-equivalent, the argument is standard and independent of derived equivalence.[6] Note first that smooth birational varieties have the same Albanese variety and Albanese image. Since f is the Albanese map of X, it follows that the Albanese map of Y is a surjective morphism $g\colon Y \to C$. Furthermore, C is the Albanese image of any other birational model as well, hence any smooth model Z inducing a K-equivalence between X and Y sits in a commutative diagram
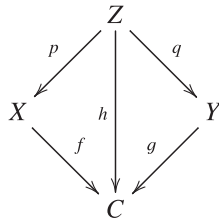
Note that in particular g has connected fibers since f does.

For a point $c \in C$, denote by X_c, Y_c, Z_c the fibers of f, g, h over c. By adjunction, Z_c realizes a K-equivalence between X_c and Y_c. First, assuming that c is chosen such that X_c and Y_c are smooth, with X_c Fano, we show that $X_c \simeq Y_c$.

To this end, if we assume that the induced rational map $\varphi_c\colon Y_c \to X_c$ is not a morphism, there must be a curve $B \subset Z_c$ which is contracted by q_c but not by p_c. Then $q_c^* \omega_{Y_c} \cdot B = 0$, and so $p_c^* \omega_{X_c} \cdot B = 0$ as well. On the other hand $\omega_{X_c}^{-1} \cdot p_c(B) < 0$, which is a contradiction. Therefore we obtain that φ_c is a birational morphism with the property that $\varphi_c^* \omega_{X_c} \simeq \omega_{Y_c}$, which implies that φ_c is an isomorphism.

If in fact ω_X^{-1} is f-ample, this argument can be globalized: indeed, assuming that the rational map $\varphi\colon Y \to X$ is not a morphism, there exists a curve $B \subset Z$ which is contracted by q and hence h, but not by p. Since B lives in a fiber of f (by the commutativity of the diagram), we again obtain a contradiction. Once we know that φ is a morphism, the same argument as above implies that it is an isomorphism. □

The following lemma used in the proof above is due to Kawamata, and can be extracted from his argument leading to the fact that derived equivalent varieties of general type are K-equivalent [9]; we sketch the argument for convenience.

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[6] We thank Alessio Corti for pointing this out to us.

Lemma 15 *Let* $\Phi_{\mathcal{E}}\colon \mathbf{D}(X) \to \mathbf{D}(Y)$ *be a derived equivalence, and assume that there exists a component Z of the support of $\mathcal{E}$ such that* $\dim Z = \dim X$ *and Z dominates X. Then X and Y are K-equivalent.*

Proof   Denote by p and q the projections of Z to X and Y. Since p is surjective, [8, Corollary 6.12] tells us that p is birational, and Z is the unique component of Supp($\mathcal{E}$) dominating X. We claim that q is also surjective, in which case by the same reasoning q is birational as well. Since (on the normalization of Z) we have $p^*\omega_X^r \simeq q^*\omega_Y^r$ for some $r \geq 1$, this suffices to conclude that X and Y are K-equivalent as in [9, Theorem 2.3] (see also [8, p. 149]).

Assuming that q is not surjective, we can find general points x_1 and x_2 in X such that $p^{-1}(x_1)$ and $p^{-1}(x_2)$ consist of one point, and $q(p^{-1}(x_1)) = q(p^{-1}(x_2)) = y$ for some $y \in Y$. One then sees that

$$\operatorname{Supp} \Phi_{\mathcal{E}}(O_{x_1}) = \operatorname{Supp} \Phi_{\mathcal{E}}(O_{x_2}) = \{y\}.$$

This implies in standard fashion that

$$\operatorname{Hom}^{\bullet}_{\mathbf{D}(X)}(O_{x_1}, O_{x_2}) \simeq \operatorname{Hom}^{\bullet}_{\mathbf{D}(Y)}(\Phi_{\mathcal{E}}(O_{x_1}), \Phi_{\mathcal{E}}(O_{x_2})) \neq 0,$$

a contradiction. $\qquad\square$

References

[1] Bak, A. 2009. The spectral construction for a $(1, 8)$-polarized family of abelian varieties. arXiv:0903.5488.

[2] Beauville, A. 1992. Annulation du H^1 pour les fibrés en droites plats. In *Complex Algebraic Varieties (Bayreuth, 1990)*, pp. 1–15. Lectures Notes in Mathematics, Vol. 1507. Berlin: Springer-Verlag.

[3] Bridgeland, T. and Maciocia, A. 1998. Fourier–Mukai transforms for quotient varieties. arXiv:9811101.

[4] Bridgeland, T. and Maciocia, A. 2001. Complex surfaces with equivalent derived categories. *Math. Z.*, **236**(4), 677–697.

[5] Brion, M. 2009. Some basic results on actions of non-affine algebraic groups. In *Symmetry and Spaces: In honor of Gerry Schwarz*, pp. 1–20. Progress in Mathematics, Vol. 278, Boston: Birkhäuser.

[6] Catanese, F. 1991. Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations. *Invent. Math.*, **104**(2), 389–407.

[7] Green, M. and Lazarsfeld, R. 1991. Higher obstructions to deforming cohomology groups of line bundles. *J. Amer. Math. Soc.*, **1**(4), 87–103.

[8] Huybrechts, H. 2006. *Fourier–Mukai Transforms in Algebraic Geometry*. Oxford Mathematical Monographs. Oxford: The Clarendon Press.

[9] Kawamata, Y. 2002. *D*-equivalence and *K*-equivalence. *J. Diff. Geom.*, **61**(1), 147–171.

[10] Lombardi, L. 2014. Derived invariants of irregular varieties and Hochschild homology. *Algebra and Number Theory*, **8**(3), 513–542.

[11] Popa, M. 2013. Derived equivalence and non-vanishing loci. "A celebration of Algebraic Geometry", *Clay Math. Proc*. 18, 567–575, Amer. Math. Soc., Providence, RI.

[12] Popa, M. and Schnell, C. 2011. Derived invariance of the number of holomorphic 1-forms and vector fields. *Ann. Sci. ENS*, **44**(3), 527–536.

[13] Rouquier, R. 2011. Automorphismes, graduations et catégories triangulées. *J. Inst. Math. Jussieu*, **10**(3), 713–751.

[14] Schnell, C. 2012. The fundamental group is not a derived invariant. In *Derived Categories in Algebraic Geometry*, pp. 279–285. EMS Series of Congress Reports. Zürich: European Mathematical Society.

[15] Simpson, C. 1993. Subspaces of moduli spaces of rank one local systems. *Ann. Sci. ENS*, **26**(3), 361–401.

[16] Toda, T. 2006. Fourier-Mukai transforms and canonical divisors. *Compos. Math.*, **142**(4), 962–982.

[17] Zhang, Q. 1996. On projective manifolds with nef anticanonical bundles. *J. Reine Angew. Math.*, **478**, 57–60.