16

The automorphism groups of Enriques surfaces covered by symmetric quartic surfaces

S. Mukai[a]
Research Institute for Mathematical Sciences, Kyoto University

H. Ohashi[a]
Tokyo University of Science

Abstract

Let S be the (minimal) Enriques surface obtained from the symmetric quartic surface $(\sum_{i<j} x_i x_j)^2 = kx_1x_2x_3x_4$ in $\mathbb{P}^3$ with $k \neq 0, 4, 36$ by taking a quotient of the Cremona action $(x_i) \mapsto (1/x_i)$. The automorphism group of S is a semi-direct product of a free product $\mathcal{F}$ of four involutions and the symmetric group $\mathfrak{S}_4$. Up to action of $\mathcal{F}$, there are exactly 29 elliptic pencils on S.

Dedicated to Prof. Robert Lazarsfeld on his 60th birthday

The automorphism groups of very general Enriques surfaces, namely those corresponding to very general points in moduli, were computed by Barth and Peters [1] as an explicitly described infinite arithmetic group. Also, many authors [1, 3, 5, 9] have studied Enriques surfaces with only finitely many automorphisms. The article [1] also includes an example whose automorphism group is infinite but still virtually abelian. In this paper we give a concrete example of an Enriques surface whose automorphism group is not virtually abelian. Moreover, the automorphism group is described explicitly in terms of generators and relations. See also Remark 5.

We work over any algebraically closed field whose characteristic is not 2. Let us introduce the quartic surface with parameters k and l,

$$\overline{X}\colon \{s_2^2 = ks_4 + ls_1s_3\} \subset \mathbb{P}^3, \tag{1}$$

[a] Supported in part by the JSPS Grant-in-Aid for Scientific Research (B) 22340007, (S) 19104001, (S) 22224001, (S) 25220701, (A) 22244003, for Exploratory Research 20654004 and for Young Scientists (B) 23740010.

From *Recent Advances in Algebraic Geometry*, edited by Christopher Hacon, Mircea Mustaţă and Mihnea Popa

where s_d are the fundamental symmetric polynomials of degree d in the homogeneous coordinates $x_1, \dots, x_4$. It is singular at the four coordinate points $(1:0:0:0), \dots, (0:0:0:1)$ and has an action of the symmetric group $\mathfrak{S}_4$. It also admits the action of the standard Cremona transformation

$$\varepsilon\colon (x_1 : \cdots : x_4) \mapsto \left(\frac{1}{x_1} : \cdots : \frac{1}{x_4}\right)$$

which commutes with $\mathfrak{S}_4$. After taking the minimal resolution X, the quotient surface $S = X/\varepsilon$ becomes an Enriques surface, whenever $\overline{X}$ avoids the eight fixed points $(\pm 1 : \pm 1 : \pm 1 : 1)$ of ε. This condition is equivalent to $k + 16l \neq 36, k \neq 4$, and $4l + k \neq 0$.

The projection from one of four coordinate points exhibits X as a double cover of the projective plane $\mathbb{P}^2$. The associated covering involution commutes with ε and defines an involution of the Enriques surface S. In this way we obtain four involutions σ_i $(i = 1, \dots, 4)$. The action of $\mathfrak{S}_4$ also descends to S. Therefore, by mapping the generators of C_2^{*4} to σ_i, we obtain a group homomorphism

$$\mathfrak{S}_4 \ltimes (C_2^{*4}) \to \mathrm{Aut}(S), \tag{2}$$

where $\mathfrak{S}_4$ acts on the free product as a permutation of the four factors.

In this paper we study the automorphism group and elliptic fibrations of S in the case $l = 0$. Our main result is as follows:

Theorem 1 (= Theorem 3.7) *In equation* (1), *let* $l = 0$ *and* $k \neq 0, 4, 36$. *Then* (2) *is an isomorphism. Namely,* $\mathrm{Aut}(S)$ *is isomorphic to the semi-direct product of the free product* $\mathcal{F}$ *of four involutions* σ_i $(i = 1, \dots, 4)$ *and the symmetric group* $\mathfrak{S}_4$.

In the proof of this theorem, we also obtain the following results on elliptic pencils and smooth rational curves. Let S be as in Theorem 1.

Theorem 2 (= Theorem 3.4) *Up to the action of the free product* $\mathcal{F} \simeq C_2^{*4}$, *there are exactly* 29 *elliptic pencils on* S. *They are classified into five types and the main properties are as follows:*

	Singular fibers	*Mordell–Weil rank*	*Number*
(1)	$\tilde{E}_7 + \tilde{A}_1$	0	12
(2)	$\tilde{E}_6 + \tilde{A}_2$	0	4
(3)	$\tilde{D}_6 + \tilde{A}_1$	1	6
(4)	$\tilde{A}_7 + \tilde{A}_1$	0	3
(5)	$2\tilde{A}_5 + \tilde{A}_2 + \tilde{A}_1$	0	4

Here $2\tilde{A}_n$ denotes the multiple singular fiber of type $\tilde{A}_n$ and the Mordell–Weil rank stands for that of its Jacobian fibration.

Theorem 3 (= Theorem 3.3) *Up to the action of the free product $\mathcal{F} \simeq C_2^{*4}$, there are exactly 16 smooth rational curves on S. They are represented by the curves in the configuration $10A + 6B$ (see below).*

The proof of Theorem 1 uses some 16 smooth rational curves on S and the fact that four involutions $\sigma_1, \ldots, \sigma_4$ are numerically reflective. First using the four singularities of type D_4 and four tropes on $\overline{X}$, we find 10 smooth rational curves on S with the dual graph as in Figure 1 (Section 1). We call this the $10A$ configuration.

Also, by looking at some other plane sections, we find six further smooth rational curves on S with the dual graph as in Figure 2. This is called the $6B$ configuration.

We denote by $NS(S)_f$ the Néron–Severi lattice of S modulo torsion. The action of involutions $\sigma_1, \ldots, \sigma_4$ on $NS(S)_f$ is the reflection in (-2) classes $G_1, \ldots, G_4 \in NS(S)_f$, respectively. For instance, the class G_1 is $E_2 + E_{23} + E_3 + E_{34} + E_4 + E_{24} - E_1$ in terms of Figure 1 (Proposition 2.1). The dual graph of these four (-2) classes is the complete graph in four vertices with doubled edges. This is called the $4C$ configuration.

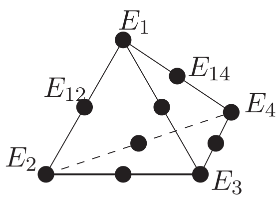

Figure 1 The $10A$ configuration.

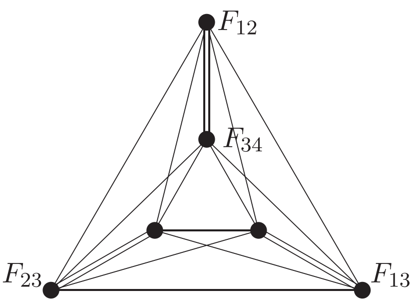

Figure 2 The $6B$ configuration.

We can check that the 20 (-2) classes E_i, E_{ij}, F_{ij}, G_i define a convex polyhedron whose Coxeter diagram satisfies Vinberg's condition [12]. Namely, the subgroup $W(10A + 6B + 4C)$ generated by reflections in these 20 classes has finite index in the orthogonal group $O(NS(S)_f)$. In fact, the limit of our Enriques surfaces as $k \to \infty$ is of type V in Kondo [5] (see Remark 2.3), and our diagram coincides with Kondo's. Although in his case the classes $G_1, \dots, G_4$ were also represented by smooth rational curves, in our case they appear just as the *center* of the reflective involutions $\sigma_1, \dots, \sigma_4$ and are not effective (Corollary 2.2).

To prove our Theorem 1, we divide the generators of $W(10A + 6B + 4C)$ into two parts, those coming from $10A + 6B$ and those from $4C$. By a lemma of Vinberg [11], $W(10A+6B+4C)$ is the semi-direct product $W(4C) \ltimes \overline{N}(W(10A+6B))$, where $\overline{N}$ denotes the normal closure. Since the whole $10A + 6B + 4C$ configuration has only $\mathfrak{S}_4$-symmetry, we obtain our Theorem 1 and the others (Section 3).

Remark 4 There are some interesting cases in $l \neq 0$ too.

(1) When $(k - 4)(l - 4) = 16$, the surface $\overline{X}$ is Kummer's quartic surface $\mathrm{Km}(J(C))$ written in Hutchinson's form. It has 16 nodes. Our four involutions σ_i are called *projections*. As is shown in [7], S is an Enriques surface of Hutchinson–Göpel type and the four involutions are numerically reflective. Especially in the case $(k, l) = (-4, 2)$, the hyperelliptic curve C branches over the vertices of a regular octahedron and the equation of $\overline{X}$ becomes

$$(x_1^2x_2^2 + x_3^2x_4^2) + (x_1^2x_3^2 + x_2^2x_4^2) + (x_1^2x_4^2 + x_2^2x_3^2) + 2x_1x_2x_3x_4 = 0.$$

This is the case of the octahedral Enriques surface [8] and S is isomorphic to the normalization of the singular sextic surface

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + \sqrt{-1}\left(\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \frac{1}{x_4^2}\right)x_1x_2x_3x_4 = 0.$$

In these cases we know that there exist automorphisms on S induced from X other than projections, namely some switches and correlations. The automorphism group of the octahedral Enriques surface will be discussed elsewhere.

(2) The quartic surface $\overline{X} : ks_4 + ls_1s_3 = 0$ is the Hessian of the cubic surface

$$k(x_1^3 + x_2^3 + x_3^3 + x_4^3) + l(x_1 + x_2 + x_3 + x_4)^3 = 0.$$

The case $(k : l) = (1 : -1)$ is most symmetric among this 1-parameter family. In this special case, the Enriques surface $S = X/\varepsilon$ is of type VI in Kondo [5] and the automorphism group is isomorphic to $\mathfrak{S}_5$. In particular, the homomorphism (2) is neither injective nor surjective.

Remark 5 In terms of virtual cohomological dimensions of discrete groups [10], our example can be located in the following way. The virtual cohomological dimension is equal to 0 for finite groups. At the other extreme, the discrete group $\mathrm{Aut}(S)$ for very general Enriques surfaces S has the virtual cohomological dimension 8. See [2]. In our case, the automorphism group has virtual cohomological dimension 1.

1 Smooth rational curves

Under the condition $l = 0$, equation (1) becomes

$$\overline{X}\colon (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)^2 = kx_1x_2x_3x_4. \tag{3}$$

This surface has four rational double points of type D_4 at the four coordinate points $(1 : 0 : 0 : 0), \ldots, (0 : 0 : 0 : 1)$ and by taking the quotient of the minimal resolution X by the standard Cremona involution

$$\varepsilon\colon (x_1 : \cdots : x_4) \mapsto \left(\frac{1}{x_1} : \cdots : \frac{1}{x_4}\right),$$

we obtain an Enriques surface $S = X/\varepsilon$. We begin with a study of the configuration of smooth rational curves on the surfaces.

The desingularization X has 16 smooth rational curves as the exceptional curves of the four D_4 singularities. Also, each coordinate plane cuts the quartic doubly along a conic, which defines a smooth rational curve on X. These are called *tropes*. The configuration of these 20 curves is shown in Figure 3, which depicts the dual graph. Black vertices come from the singularities and white ones are tropes.

The standard Cremona involution ε acts on Figure 3 by point symmetry. Therefore, the Enriques surface S has 10 smooth rational curves whose dual

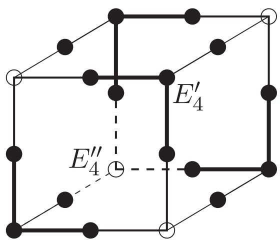

Figure 3 The quartic surface with four D_4 singularities.

graph is that in Figure 1. In what follows, we call these 10 curves on S the $10A$ configuration. The indexing is given as follows. Since a vertex of the tetrahedron corresponds to two curves on X, namely the trope $\{x_i = 0\}$ and the central component of the exceptional curves at $(0 : \cdots : 1 : \cdots : 0)$ (the ith coordinate is 1), we denote the curve at the vertex by E_i $(i = 1, \ldots, 4)$. Also, if a vertex at the middle of an edge is connected to two vertices, say E_i and E_j, then we denote the curve by E_{ij}. This is the first configuration of smooth rational curves on S of interest. It is convenient to note that the 10 curves $\{E_i, E_{ij}\}$ generate $NS(S)_f$ over the rationals; the Gram matrix of these curves has determinant -64.

Next let us consider the six plane sections $\{x_i + x_j = 0\}$ $(i = 1, \ldots, 4)$. In equation (3), we see that each plane section decomposes into two conics which are disjoint on X and exchanged by ε. Thus we obtain six further smooth rational curves on S, naturally indexed as F_{ij}. The intersection relation between these curves is shown in Figure 2. We call it the $6B$ configuration. Moreover, we can clarify the intersection relations between the configurations as follows:

$$(E_k, F_{ij}) = 0; \qquad (E_{kl}, F_{ij}) = \begin{cases} 2 & \text{if } \{k, l\} = \{i, j\}, \\ 0 & \text{otherwise.} \end{cases}$$

The configuration of 16 curves thus obtained is denoted by $10A + 6B$.

2 Numerically reflective involutions

The quartic surface (3) can be exhibited as a double cover of $\mathbb{P}^2$ by the projection from one of the coordinate points, say $(0 : 0 : 0 : 1)$. The branch $B \subset \mathbb{P}^2$ is the sextic plane curve defined by

$$x_1 x_2 x_3 \left\{ 4(x_1 + x_2 + x_3)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) x_1 x_2 x_3 - k x_1 x_2 x_3 \right\} = 0. \qquad (4)$$

It is the union of the coordinate triangle $\{x_1 x_2 x_3 = 0\}$ and the cubic curve

$$C\colon 4(x_1 + x_2 + x_3)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) - k = 0, \qquad (5)$$

which is invariant under the Cremona transformation $(x_i) \mapsto (1/x_i)$ of $\mathbb{P}^2$. See Figure 4. In this double-plane picture, the 20 rational curves in Figure 3 can be seen as the 12 rational curves above the three triple points of B, three rational curves above the three nodes of B, three tropes as the inverse image of the coordinate triangle, and some components of inverse images of the curves

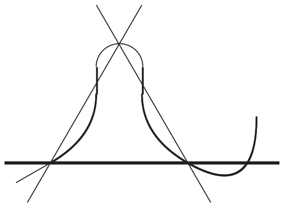

Figure 4 The branch sextic B.

L: $\{x_1 + x_2 + x_3 = 0\}$ and Q: $\{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 0\}$. (We note that the line L must pass through the three simple intersection points of C with the triangle in Figure 4, although it is not visible.)

The covering transformation of this double cover $X \to \mathbb{P}^2$ is called the *projection*. It is an antisymplectic involution acting on X. It stabilizes all the curves above the branch curve B (including those above the singularities of B). In particular, in Figure 3, if E_4'' is the trope $\{x_4 = 0\}$, then the projection stabilizes all the curves except for E_4'' and its antipodal E_4' (coming from the singularity at $(0 : 0 : 0 : 1)$). It is easy to determine the fixed curves of the projection, consisting of six rational curves (vertices of the cube except for E_4' and E_4'') and the inverse image of the elliptic curve C.

Since the projection commutes with the Cremona involution of $\mathbb{P}^3$, we obtain an involution of the Enriques surface S. It is denoted by σ_4, where the index is in accordance with the center of the projection $(0 : 0 : 0 : 1)$.

Proposition 2.1 *The involution $\sigma_4 \in \mathrm{Aut}(S)$ is numerically reflective. Moreover, its action on the Néron–Severi lattice $NS(S)_f$ is the reflection in the divisor $G_4 = E_1 + E_{12} + E_2 + E_{23} + E_3 + E_{13} - E_4$ of self-intersection (-2). In Figure 1, the six positive components in G_4 are just the cycle of curves disjoint from E_4.*

Proof From our description of fixed curves of the projection as above, we see that σ_4 preserves all the curves E_i and E_{ij} except for E_4. Compare Figures 1 and 3.

Consider the elliptic fibration $f: S \to \mathbb{P}^1$ defined by the divisor $2D_f = 2(E_1 + E_{12} + E_2 + E_{23} + E_3 + E_{13})$. It gives the multiple fiber of type ${}_2\mathrm{I}_6$ in Kodaira's notation. From Figure 1, we see that the curve E_4 sits inside a reducible fiber which we denote by D'. In comparison with Figure 4, f corresponds to the pencil $\mathcal{L}$ of cubics on $\mathbb{P}^2$ spanned by the triangle $\{x_1x_2x_3 = 0\}$ and the cubic curve C of (5). Thus we see that the multiple fibers of f are exactly the transform of the triangle, which is nothing but the divisor $2D_f$ of type ${}_2\mathrm{I}_6$, and the transform of C, namely some irreducible fiber of type ${}_2\mathrm{I}_0$. On the other hand, the cubic

 http://dx.doi.org/10.1017/CBO9781107416000.017

$$C_\infty := L + Q \in \mathcal{L} \tag{6}$$

corresponds to the reducible fiber of f which contains E_4. Therefore, the fiber $D' = E_4 + B$ is of Dynkin type $\tilde{A}_1$ and is not multiple. (More precisely, it is of type III in characteristic 3 and otherwise I_2.) Since the Cremona involution of $\mathbb{P}^2$ interchanges L and Q, we see that σ_4 interchanges E_4 and B.

From the linear equivalence $E_4 + B \sim 2(E_1 + E_{12} + E_2 + E_{23} + E_3 + E_{13})$, we see that the action is

$$\sigma_4 \colon E_4 \mapsto B = 2(E_1 + E_{12} + E_2 + E_{23} + E_3 + E_{13}) - E_4.$$

By taking the first paragraph into account, we see that σ_4 is numerically reflective and acts on $NS(S)_f$ by the reflection in the divisor

$$G_4 = E_1 + E_{12} + E_2 + E_{23} + E_3 + E_{13} - E_4.$$

□

By symmetry, we obtain divisors G_i $(i = 1, \dots, 4)$ which describe the numerically reflective involutions σ_i in a similar manner. We see that $(G_i, G_j) = 2$ for $i \neq j$, so that the intersection diagram associated with divisors $G_1, \dots, G_4$ is the complete graph in four vertices with all edges doubled. In what follows we denote this configuration by $4C$.

We note that the automorphism σ_i sends G_i to its negative. It implies the following corollary:

Corollary 2.2 *The numerical classes of G_i are not effective.*

We can compute the intersections of G_i and the $10A + 6B$ configuration. We have the following:

$$(G_i, E_j) = \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \qquad (G_i, E_{kl}) = 0, \quad (G_i, F_{kl}) = \begin{cases} 2 & \text{if } i \notin \{k, l\}, \\ 0 & \text{if } i \in \{k, l\}. \end{cases}$$

Remark 2.3 The limit of our quartic surface $\overline{X}$ in (3) as $k \to \infty$ is the double $\mathbb{P}^2$ with branch the union of the coordinate triangle and the reducible cubic C_∞ in (6). Hence the limit of our Enriques surfaces is of type V in Kondo [5]. (See [6] also.) In this limit our divisor class G_4 becomes effective and corresponds to the new singular point coming from the intersection $L \cap Q$ in (6). Thus G_4 can be regarded as the vanishing cycle of this specialization $k \to \infty$. Furthermore, the numerically reflective involution σ_4 becomes numerically trivial in this limit. More precisely, the limit of its graph as $k \to \infty$ is the union of that of the limit involution and the product $C_4 \times C_4$, where C_4 is the unique (-2) curve representing G_4 in the limiting Enriques surface.

3 Proof of the theorems

In the previous two sections, we obtained 16 smooth rational curves with the configuration $10A + 6B$ and four numerically reflective involutions σ_i whose centers G_i have the configuration $4C$. We begin with consideration of the natural representation $r\colon \mathrm{Aut}(S) \to O(NS(S)_f)$.

Proposition 3.1 *The homomorphism r is injective, namely there are no nontrivial numerically trivial automorphisms on S.*

Proof Let g be a numerically trivial automorphism of S, which is tame by virtue of Dolgachev [4]. It preserves each (-2) curve, in particular in the $10A + 6B$ configuration. The curves $E_1, \dots, E_4$ in Figure 1 must be pointwise fixed, since $\mathrm{Aut}(\mathbb{P}^1)$ is sharply triply transitive and since each E_i has three distinct intersections with its neighbors.

We again focus on the elliptic fibration $f\colon S \to \mathbb{P}^1$ defined by $D_f = E_1 + E_{12} + E_2 + E_{23} + E_3 + E_{13}$ as in Proposition 2.1. We saw that $E_4 + B = E_4 + \sigma_4(E_4)$ is a non-multiple fiber of f. Therefore the bisections E_{14}, E_{24}, E_{34} of f must intersect B. By a suitable choice of a bisection $C_f \in \{E_{14}, E_{24}, E_{34}\}$, we can assume that C_f does not pass through the intersection $E_4 \cap B$. Then, since g preserves all (-2) curves, the curve C_f has three distinct fixed points $E_4 \cap C_f$, $B \cap C_f$, and $E_i \cap C_f$, where E_i is another vertex of the edge containing C_f in Figure 1. It follows that g fixes C_f pointwise, hence the singular curve $E_4 + C_f$ too. It follows that $g = \mathrm{id}_S$, since g is tame and of finite order. □

In what follows we denote the hyperbolic lattice $NS(S)_f$ by L. Let us denote by $O'(L)$ the group of integral isometries whose $\mathbb{R}$-extensions preserve the positive cone of $L \otimes \mathbb{R}$. We denote by Λ the 9-dimensional Lobachevsky space associated with the positive cone. Then $O'(L)$ acts on Λ as a discrete group of motions. We refer the reader to [12] for the theory of discrete groups generated by reflections acting on Lobachevsky spaces.

We let

$$P^c = \{\mathbb{R}_+ x \in PS(L) \mid (x, E) \geq 0 \text{ for all } E \in \{E_i, E_{ij}, F_{ij}, G_i\}\}$$

be the convex polyhedron defined by the 20 roots from the $10A + 6B + 4C$ configuration in the projective sphere $PS(L) = (L - \{0\})/\mathbb{R}_+$ (see [12, Section 2]). We have seen that every intersection number of two distinct divisors in $10A + 6B + 4C$ is between 0 and 2, hence the Coxeter diagram associated with these 20 roots has no dotted lines or Lanner's subdiagrams. Also, by an easy check of the $10A + 6B + 4C$ configuration, we have the following:

 http://dx.doi.org/10.1017/CBO9781107416000.017

Lemma 3.2 *The Coxeter diagram of the polyhedron $P = P^c \cap \Lambda$ has exactly* 29 *parabolic subdiagrams of maximal rank* 8*. They are as follows:*

	Subdiagram	*Number*	10A	6B	4C
(1)	$\tilde{E}_7 + \tilde{A}_1$	12	8	1	1
(2)	$\tilde{E}_6 + \tilde{A}_2$	4	7	3	0
(3)	$\tilde{D}_6 + \tilde{A}_1 + \tilde{A}_1$	6	8	1	2
(4)	$\tilde{A}_7 + \tilde{A}_1$	3	8	2	0
(5)	$\tilde{A}_5 + \tilde{A}_2 + \tilde{A}_1$	4	7	3	1

Here, each column $10A, 6B, 4C$ *shows the number of vertices used from the configuration.*

It is easy to check that every connected parabolic subdiagram is a connected component of some parabolic subdiagram of rank 8, using the previous table. By Theorem 2.6 of [12], we see that P has finite volume and we obtain $P^c \subset \overline{\Lambda}$. This polyhedron gives the fundamental domain of the associated discrete reflection group generated by 20 reflections in the 20 roots $\{E_i, E_{ij}, F_{ij}, G_i\}$, which we denote by $W = W(10A + 6B + 4C)$. Algebro-geometrically, 16 of the generators are the Picard–Lefschetz transformations in (-2) curves in the $10A + 6B$ configuration and the remaining four are the involutions σ_i ($i = 1, \ldots, 4$) corresponding to $4C$. As an abstract group, we see that W has the structure of a Coxeter group whose fundamental relations are given by the Coxeter diagram (see [12]) of P. We note that the quasi-polarization (namely a nef and big divisor)

$$H = \sum_i E_i + \sum_{i<j} E_{ij}$$

defines an element $\mathbb{R}_+ H$ in P.

Now let $W(4C)$ be the subgroup of W generated by four reflections in G_i. Via the homomorphism $r\colon \mathrm{Aut}(S) \to O(NS(S)_f)$, the subgroup $\mathcal{F} \subset \mathrm{Aut}(S)$ generated by the four numerically reflective involutions σ_i is mapped onto this Coxeter subgroup $W(4C) \simeq C_2^{*4}$. It follows that $\mathcal{F} \simeq W(4C)$. Let $W(10A + 6B)$ be the subgroup generated by 16 reflections in E_i, E_{ij} and F_{ij}, and let $\overline{N}(W(10A + 6B))$ be the minimal normal subgroup of W which contains $W(10A + 6B)$. Since the intersection numbers between elements of $4C$ and $10A + 6B$ are all even, by [11, Proposition] we have the exact sequence

$$1 \longrightarrow \overline{N}(W(10A + 6B)) \longrightarrow W \longrightarrow W(4C) \longrightarrow 1. \qquad (7)$$

The kernel is exactly the subgroup generated by the conjugates

$$\{\sigma g \sigma^{-1} \mid \sigma \in W(4C), g \text{ a generator of } W(10A + 6B)\}.$$

We have the corresponding geometric consequence as follows:

Theorem 3.3 *There are exactly 16 smooth rational curves on S up to the action of $\mathcal{F}$:*

Proof Let E be a smooth rational curve on S. We consider the orbit $\mathcal{F}.E$. Since the divisor H above is nef, we can choose $E_0 \in \mathcal{F}.E$ such that the degree (E_0, H) is minimal. We show that E_0 is one of 16 curves in $10A + 6B$.

In fact, by the automorphism σ_i, we have

$$(E_0, H) \leq (\sigma_i(E_0), H) = (E_0, H) + (E_0, G_i)(G_i, H),$$

and $0 \leq (E_0, G_i)$ for all i. Suppose that E_0 intersects non-negatively all 16 curves in $10A + 6B$. Then we have $\mathbb{R}_+ E_0 \in P^c$. But from $P^c \subset \overline{\Lambda}$ we obtain $(E_0^2) \geq 0$, which is a contradiction. Hence E_0 is negative on some curve in $10A + 6B$ and we see that E_0 is one of them.

Next let us show that two distinct curves E, E' in the $10A + 6B$ configuration are inequivalent under $\mathcal{F}$. For the six curves E_{ij} from $10A$, we have $(E_{ij}, G_k) = 0$ for all $i < j$ and k. Therefore, by an easy induction, we see that the sextuple $((E_{ij}, E))_{1 \leq i < j \leq 4}$ consisting of intersection numbers is an invariant of the orbit $\mathcal{F}.E$. Suppose that $(E_{ij}, E) = (E_{ij}, E')$ for all $i < j$. Since E and E' are both in the $10A + 6B$ configuration, we see easily that $E = E'$. This shows that the orbits $\mathcal{F}.E$ and $\mathcal{F}.E'$ are disjoint. □

In other words, the group $\overline{N}(W(10A + 6B))$ is nothing but the Weyl group of S generated by Picard–Lefschetz reflections in all (-2) curves. We can proceed to elliptic pencils.

Theorem 3.4 *There are exactly* 29 *elliptic pencils on S up to the action of $\mathcal{F}$. Their properties are as in the table of Theorem 2.*

Proof Let $2f$ be a fiber class of an elliptic pencil on S. As before, we choose an element $f_0 \in \mathcal{F}.f$ such that the degree (f_0, H) is minimal. We have

$$(f_0, H) \leq (\sigma_i(f_0), H) = (f_0, H) + (f_0, G_i)(G_i, H),$$

hence $(f_0, G_i) \geq 0$ for all i. Moreover, since f_0 is nef we have $(f_0, E) \geq 0$ for all E in the $10A + 6B$ configuration. Therefore, $f_0 \in P^c$. This shows that f_0 corresponds to one of the maximal parabolic subdiagrams classified in Lemma 3.2.

Conversely, we can construct 29 elliptic pencils from the 29 subdiagrams in Lemma 3.2 as follows. The two types (2) and (4) in the lemma are easiest since they do not contain a class in $4C$. The elliptic pencils of types (2) and (4) have singular fibers of type $\tilde{E}_6 + \tilde{A}_2$ and $\tilde{A}_7 + \tilde{A}_1$, respectively as in the case of [5, Table 2].

In the case of type (1) (resp. (5)), one component of the parabolic subdiagram is $\tilde{A}_1$ consisting of a (-2) curve E in $6B$ (resp. $10A$) and (-2) class G in $4C$. Moreover, the sum $E + G$ is half of $\tilde{E}_7$ (resp. $\tilde{A}_2$). Hence $E + \sigma(E)$ is a non-multiple fiber of type $\tilde{A}_1$ since it is linearly equivalent to $2(E + G)$, where σ is the reflection in G.

In the case of type (3), one component is $\tilde{A}_1$ consisting of two classes G and G' in $4C$. But the other two components consist of (-2) curves. Therefore, the Mordell–Weil group is of rank 1 since neither G nor G' is effective. (The composite $\sigma\sigma'$ of two reflections in G and G' is the translation by a generator of the Mordell–Weil group.)

That these 29 pencils are inequivalent under $\mathcal{F}$ follows from the previous result for rational curves. □

To study the image of the representation $r\colon \mathrm{Aut}(S) \to O(NS(S)_f)$, we need some lemma. We denote by $4A'$ the set of four roots $\{E_i\}$ and by $6A''$ the set $\{E_{ij}\}$. Recall that by Theorem 3.3, all (-2) curves on S are in the $\mathcal{F}$-orbit of the three sets $4A', 6A''$, and $6B$.

Lemma 3.5 *Let τ be any automorphism of S. Then τ preserves each of the three orbits of rational curves $\mathcal{F}.(4A')$, $\mathcal{F}.(6A'')$, and $\mathcal{F}.(6B)$.*

Proof Any automorphism τ permutes smooth rational curves on S, and hence induces a symmetry of the dual graph of the set of rational curves. Thus, for the proof, it suffices to give a characterization of each orbit in terms of this infinite graph. We use the (full) subgraphs which are isomorphic to the dual graph of reducible fibers of elliptic fibrations.

Consider a vertex v in $\mathcal{F}.(6B)$. Then there exists a subgraph of fiber type I_3 passing through v. Conversely, if for a vertex v there is a subgraph of fiber type I_3, by Theorem 3.4, it is equivalent to a vertex in $6B$ under $\mathcal{F}$. Thus the vertices in $\mathcal{F}.(6B)$ are characterized by the property that there exists a subgraph of fiber type I_3 passing through them.

Similarly, vertices in $\mathcal{F}.(10A)$ are characterized by subgraphs of type I_8. Moreover, the vertices v in $\mathcal{F}.(4A')$ are characterized by the property that there exists a subgraph of type IV^* which has v as its end. In the opposite way, vertices in $\mathcal{F}.(6A'')$ are those which do not have such IV^* subgraphs. Thus the three orbits are all characterized and τ preserves these orbits. □

 http://dx.doi.org/10.1017/CBO9781107416000.017

Corollary 3.6 *The set of six curves* $\{E_{ij}\}$ *is preserved under any automorphism.*

Proof In fact, for any E_{kl} and any σ_i we have $\sigma_i(E_{kl}) = E_{kl}$. Hence $\mathcal{F}.(6A'') = \{E_{ij}\}$. □

Recall that S has action by $\mathfrak{S}_4$ from the symmetry of the defining equation of $\overline{X}$. Explicitly, it acts on the curves in the $10A + 6B$ configuration by the permutation of indices. For involutions σ_i, the same holds true if we regard the action as taking conjugates. It is easy to see that this group $\mathfrak{S}_4$ can be identified with the symmetry group $\mathrm{Sym}(P)$ of the polyhedron $P \subset \Lambda$ via r. We can also regard this group as acting on the reflection group W and the exact sequence (7) is preserved under this action. In particular, $W(4C)$ and $\mathrm{Sym}(P)$ generate a group isomorphic to $\mathfrak{S}_4 \ltimes C_2^{*4}$.

Theorem 3.7 *The representation r induces an isomorphism of* $\mathrm{Aut}(S)$ *onto the group generated by* $W(4C)$ *and* $\mathrm{Sym}(P)$*, hence we obtain* $\mathrm{Aut}(S) \simeq \mathfrak{S}_4 \ltimes C_2^{*4}$.

Proof Since r maps $\mathcal{F}$ onto $W(4C)$, the image of r includes the groups $W(4C)$ and $\mathrm{Sym}(P)$.

Conversely, let us pick up an arbitrary automorphism τ. We consider the image $\tau(H)$ of H. We use the elliptic fibration defined by the divisor $f = H - E_{12} - E_{34}$ of type I_8. By Theorem 3.4, the image $\tau(f)$ is equivalent to one of three elliptic pencils described in item (4) under $\mathcal{F}$. Moreover, since $\mathfrak{S}_4$ acts transitively on these three pencils, we can assume that $\tau(f) = f$ by composing τ with some elements of $\mathcal{F}$ and $\mathfrak{S}_4$. Thus we have $\tau(H) = f + \tau(E_{12}) + \tau(E_{34})$. By the previous corollary, $\tau(E_{12})$ and $\tau(E_{34})$ are in the set $\{E_{ij}\}$. By an easy check of intersection numbers, we see that $\tau(E_{12}) + \tau(E_{34}) = E_{12} + E_{34}$. In particular we obtain $\tau(H) = H$ as divisors. Since any permutation of the $10A$ configuration can be induced from the automorphism group $\mathfrak{S}_4$, this shows that the image of r is contained in the group generated by $W(4C)$ and $\mathrm{Sym}(P)$. □

References

[1] Barth, W. and Peters, C. Automorphisms of Enriques surfaces. *Invent. Math.* **73** (1983) 383–411.

[2] Borel, A. and Serre, J. P. Corners and arithmetic groups. *Comment. Math. Helv.* **48** (1973) 436–491.

[3] Dolgachev, I. On automorphisms of Enriques surfaces. *Invent. Math.* **76** (1984) 163–177.

[4] Dolgachev, I. Numerical trivial automorphisms of Enriques surfaces in arbitrary characteristic. In *Arithmetic and Geometry of K3 Surfaces and Calabi–Yau Threefolds*. Fields Institute Communication No. 67, 2013, pp. 267–283.
[5] Kondo, S. Enriques surfaces with finite automorphism groups. *Japan. J. Math.* **12** (1986) 191–282.
[6] Mukai, S. Numerically trivial involutions of Kummer type of an Enriques surface. *Kyoto J. Math.* **50** (2010) 889–902.
[7] Mukai, S. Kummer's quartics and numerically reflective involutions of Enriques surfaces. *J. Math. Soc. Japan* **64** (2012) 231–246.
[8] Mukai, S. and Ohashi, H. Enriques surfaces of Hutchinson–Göpel type and Mathieu automorphisms. In *Arithmetic and Geometry of K3 Surfaces and Calabi–Yau Threefolds*. Fields Institute Communication No. 67, 2013, pp. 429–454.
[9] Nikulin, V. V. On a description of the automorphism groups of Enriques surfaces. *Soviet Math. Dokl.* **30** (1984) 282–285.
[10] Serre, J. P. Cohomologie des groupes discrets. In *Prospects in Mathematics* (Proceedings of Symposium, Princeton University, Princeton, NJ, 1970), pp. 77–169. Annals of Mathematical Studies, No. 70. Princeton, NJ: Princeton University Press, 1971.
[11] Vinberg, E. B. The two most algebraic *K*3 surfaces. *Math. Ann.* **265** (1983) 1–21.
[12] Vinberg, E. B. Some arithmetical discrete groups in Lobaĉevskiî spaces. In *Discrete Subgroups of Lie Groups and Applications to Moduli (Bombay 1973)*. Oxford: Oxford University Press, 1975, pp. 323–348.