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# Lower-order asymptotics for Szegö and Toeplitz kernels under Hamiltonian circle actions

R. Paoletti
*Università degli Studi di Milano Bicocca*

## Abstract

We consider a natural variant of Berezin–Toeplitz quantization of compact Kähler manifolds, in the presence of a Hamiltonian circle action lifting to the quantizing line bundle. Assuming that the moment map is positive, we study the diagonal asymptotics of the associated Szegö and Toeplitz operators, and specifically their relation to the moment map and to the geometry of a certain symplectic quotient. When the underlying action is trivial and the moment map is taken to be identically equal to one, this scheme coincides with the usual Berezin–Toeplitz quantization. This continues previous work on near-diagonal scaling asymptotics of equivariant Szegö kernels in the presence of Hamiltonian torus actions.

*Dedicated to Rob Lazarsfeld on the occasion of his 60th birthday*

## 1 Introduction

The object of this paper are the asymptotics of Szegö and Toeplitz operators in a non-standard version of the Berezin–Toeplitz quantization of a complex projective Kähler manifold $(M, J, \omega)$.

In Berezin–Toeplitz quantization, one typically adopts as "quantum spaces" the Hermitian spaces $H^0\left(M, A^{\otimes k}\right)$ of global holomorphic sections of higher powers of the polarizing line bundle $(A, h)$; here $(A, h)$ is a positive, hence ample, Hermitian holomorphic line bundle on $M$. Quantum observables, on the contrary, correspond to Toeplitz operators associated with real $C^\infty$ functions on $M$.

Here we assume given a Hamiltonian action $\mu^M$ of the circle group $U(1) = \mathbf{T}^1$ on $M$, with positive moment map $\Phi$, and admitting a metric-preserving linearization to $A$. It is then natural to replace the spaces $H^0\left(M, A^{\otimes k}\right)$ with certain new "quantum spaces" which arise by decomposing the Hardy space associated with $A$ into isotypes for the action; these are generally not spaces of sections of powers of $A$. One is thus led to consider analogues of the usual constructs of Berezin–Toeplitz quantization. In particular, it is interesting to investigate how the symplectic geometry of the underlying action, encapsulated in $\Phi$, influences the semiclassical asymptotics in this quantization scheme.

This picture generalizes the usual Berezin–Toeplitz quantization of $(M, J, \omega)$, for one falls back on the standard case by considering the trivial action of $\mathbf{T}^1$ on $M$ with moment map $\Phi = 1$. Then the lifted action is essentially fiberwise scalar multiplication, and the corresponding equivariant spaces are the usual spaces of global holomorphic sections.

This theme was considered in [28] for general Hamiltonian torus actions; the focus there was on near-diagonal scaling asymptotics of the associated equivariant Szegö kernels. Here we shall restrict our analysis to circle actions, and investigate the lower-order terms of these asymptotic expansions, as well as their analogues for Toeplitz operators.

In the usual standard setting of Berezin–Toeplitz quantization, a huge amount of work has been devoted to these themes, involving a variety of approaches and techniques; see for example (obviously with no pretense of being exhaustive) [1, 2, 4, 5, 7, 8, 10–12, 14, 22, 25, 26, 29–31, 34, 35] and references cited therein.

In the present paper, we follow the general approach to quantization based on the microlocal analysis of the Szegö kernel on the circle bundle X of $A^\vee$; this train of thought was first introduced in the grounding work [5], and afterwards explored by many authors. We shall also specifically build on ideas and results from [13, 17, 19]; in fact, the present paper was inspired considerably by the concise approach in [19] to the derivation of the lower-order terms in the TYZ expansion for real-analytic metrics.

Now let us make our discussion more precise. Let $(M, J)$ be a connected complex $d$-dimensional projective manifold, and let $A$ be an ample holomorphic line bundle on $M$, with dual line bundle $A^\vee$ and projection $\widehat{\pi} \colon A^\vee \to M$.

There is an Hermitian metric $\ell_A$ on $A$ such that the unique covariant derivative $\nabla_A$ on $A$ compatible with $\ell_A$ and the holomorphic structure has curvature $\Theta_A = -2i\,\omega$, where $\omega$ is a Kähler form on $M$. Then $dV_M =: \omega^{\wedge d}/d!$ is a volume form on $M$.

Let $X \subseteq A^{\vee}$ be the unit circle bundle, with projection $\pi = \widehat{\pi}\big|_X : X \to M$. Then $\nabla$ corresponds to a connection contact form $\alpha \in \Omega^1(X)$, such that $d\alpha = 2\pi^*(\omega)$ and $dV_X =: (1/2\pi)\,\alpha \wedge \pi^*(dV_M)$ is a volume form on $X$. Let $L^2(X) =: L^2(X, dV_X)$, and identify functions with densities and half-densities on $X$. Also, let $H(X) =: \ker(\overline{\partial}_b) \cap L^2(X)$ be the Hardy space of $X$, where $\overline{\partial}_b$ is the Cauchy–Riemann operator on $X$.

Suppose that the action $\mu^M : \mathbf{T}^1 \times M \to M$ is holomorphic with respect to $J$ and Hamiltonian with respect to $2\omega$, with moment map $\Phi : M \to \mathbb{R}$; suppose furthermore that $(\mu^M, \Phi)$ can be linearized to a holomorphic action $\mu^A$ on $A$ leaving $\ell_A$ invariant. Then $\mathbf{T}^1$ acts on $X$ as a group of contactomorphisms under the naturally induced action $\mu^X : \mathbf{T}^1 \times X \to X$ lifting $\mu^M$.

Infinitesimally, the relation between $\mu^M$ and $\mu^X$ is as follows. Let $\partial/\partial\theta \in \mathfrak{X}(X)$ be the infinitesimal generator of the action of $\mathbf{T}^1$ on $X$ given by fiber-wise scalar multiplication, mult: $(e^{i\theta}, x) \mapsto e^{i\theta} \cdot x$; also, let $\xi_M \in \mathfrak{X}(M)$ be the infinitesimal generator of $\mu^M$, with horizontal lift $\xi_M^{\sharp}$ to $X$. Then the infinitesimal generator $\xi_X \in \mathfrak{X}(X)$ of $\mu^X$ is given by

$$\xi_X = \xi_M^{\sharp} - \Phi\,\frac{\partial}{\partial\theta}, \tag{1}$$

where we write $\Phi$ for $\Phi \circ \pi$. Thus $\mu^X$ depends crucially on the choice of $\Phi$; for example, when $\mu^M$ is trivial, choosing $\Phi = 0$ yields the trivial action on $X$, while if $\Phi = 1$ we obtain the action

$$\nu^X : \mathbf{T}^1 \times X \to X, \quad (e^{i\theta}, x) \mapsto e^{-i\theta} \cdot x. \tag{2}$$

Since $\mu^X$ preserves $\alpha$ and is a lifting of the holomorphic action $\mu^M$, it leaves $H(X)$ invariant; therefore, it determines a unitary action of $\mathbf{T}^1$ on $H(X)$. Thus $H(X)$ equivariantly and unitarily decomposes into the Hilbert direct sum of its isotypes,

$$H_k^{\mu}(X) =: \left\{ f \in H(X) \; : \; f\left(\mu_{g^{-1}}^X(x)\right) = g^k\,f(x) \; \forall\,(g, x) \in \mathbf{T}^1 \times X \right\}, \tag{3}$$

where $k \in \mathbb{Z}$. If $\mu^M$ is trivial and $\Phi = 1$, (3) is the standard $k$th equivariant Szegö kernel $H_k(X)$, which is unitarily isomorphic to $H^0\left(M, A^{\otimes k}\right)$ in a natural manner. However, in general $H_k^{\mu}(X)$ is not a space of sections of powers of $A$, and may even be infinite-dimensional. For example, if $\mu^M$ is trivial and $\Phi = 0$ then $H_0^{\mu}(X) = H(X)$, while $H_k^{\mu}(X)$ is the null space for $k \neq 0$.

Nonetheless, if $\Phi > 0$ then $H_k^{\mu}(X)$ is finite-dimensional for any $k \in \mathbb{Z}$, and is the null space if $k < 0$ [28]; in particular, the orthogonal projector

$\Pi_k^\mu \colon L^2(X) \to H_k^\mu(X)$ is a smoothing operator, with Schwartz kernel $\Pi_k^\mu(\cdot, \cdot) \in C^\infty(X \times X)$ given by

$$\Pi_k^\mu(x, y) = \sum_j s_j^{(k)}(x) \cdot \overline{s_j^{(k)}(y)} \qquad (x, y \in X) \qquad (4)$$

for any choice of an orthonormal basis $\left(s_j^{(k)}\right)$ of $H_k^\mu(X)$. The diagonal restriction $x \mapsto \Pi_k^\mu(x, x)$ descends to a well-defined $C^\infty$ function on $M$.

Also, if $\Phi > 0$ then $\xi_X(x) \neq 0$ for every $x \in X$ by (1); hence $\mu^X$ is locally free, and every $x \in X$ has finite stabilizer $T_x \subseteq \mathbf{T}^1$. As $\mu^X$ commutes with scalar multiplication, $T_x$ only depends on $m = \pi(x) \in M$; we shall emphasize this by denoting $T_x$ by $T_m$. For instance, $T_m = \{1\}$ for every $m \in M$ if $\mu^M$ is trivial and $\Phi = 1$. While $T_m$ is generally not constant on $M$, it equals a fixed finite subgroup $T_{\mathrm{gen}} \subseteq \mathbf{T}^1$ on a dense open subset $M' \subseteq M$. Then $T_{\mathrm{gen}}$ stabilizes every $x \in X$; after passing to the quotient, we may reduce to the case $T_{\mathrm{gen}} = \{1\}$. By Corollary 1.1 of [28], at a point $x \in X$ where $T_m$ is trivial, $\Pi_k^\mu(x, x)$ satisfies an asymptotic expansion as $k \to +\infty$ of the form

$$\Pi_k^\mu(x, x) \sim \left(\frac{k}{\pi}\right)^d \sum_{j \geq 0} k^{-j} S_j^\mu(m), \qquad (5)$$

where $S_0^\mu(m) = \Phi(m)^{-(d+1)}$. Here we shall focus on the lower-order terms $S_j^\mu$.

More generally, given a real $f \in C^\infty(M)$, one can consider the associated Toeplitz operators $T_k^\mu[f] =: \Pi_k^\mu \circ M_f \circ \Pi_k^\mu$, viewed as self-adjoint endomorphisms of $H_k^\mu(X)$; here $M_f \colon L^2(X) \to L^2(X)$ is multiplication by $f \circ \pi$. Assuming $\Phi > 0$, this is also a smoothing operator, whose distributional kernel may be expressed as

$$
\begin{aligned}
T_k^\mu[f](x, x') &= \int_X \Pi_k^\mu(x, y)\, f(y)\, \Pi_k^\mu(y, x')\, dV_X(y) \\
&= \sum_j T_k^\mu[f](s_j^{(k)})(x) \cdot \overline{s_j^{(k)}(y)} \qquad (x, y \in X),
\end{aligned} \qquad (6)
$$

where we write $f(y)$ for $f(\pi(y))$. The diagonal restriction $x \mapsto T_k^\mu[f](x, x)$ also descends to $M$. We shall see that $T_k^\mu[f](x, x')$ has near diagonal scaling asymptotics (that is, for $x \to x'$) analogous to those of $\Pi_k^\mu$ in Theorem 1 of [28], and investigate the lower-order terms in the asymptotics of the diagonal restriction $T_k^\mu[f](x, x)$. We shall then derive from this an asymptotic expansion for an "equivariant Berezin transform," and consider the relation between commutators of Toeplitz operators and Poisson brackets of the corresponding Hamiltonians. Before describing our results in detail, we need to specify the geometric setting somewhat.

We shall assume without loss that $T_{\text{gen}}$ is trivial; then $\mu^X$ is free on a dense $\nu^X \times \mu^X$-invariant open subset $X' \subseteq X$ (since $\nu^X$ – given by (2) – and $\mu^X$ commute, we may consider the product action). Thus $M' =: \pi(X') \subseteq M$ is also open and dense.

The quotient $N = X/\mathbf{T}^1$ is an orbifold, and the dense open subset $N' =: X'/\mathbf{T}^1 \subseteq N$ is a manifold; the restricted projection $\kappa: X' \to N'$ is a circle bundle, and passing from $\pi$ to $\kappa$ the roles of $\mu^X$ and $\nu^X$ get interchanged.

More precisely, $\beta =: \alpha/\Phi$ is a connection 1-form for $\kappa$, defining the same horizontal distribution as $\alpha$, and there is on $N'$ a naturally induced Kähler structure $(N', I, \eta)$ with $d\beta = 2\kappa^*(\eta)$, and if $\omega$ is real-analytic then so is $\eta$. Furthermore, $\nu^X$ descends to an action $\nu^N: \mathbf{T}^1 \times N' \to N'$, which turns out to be holomorphic with respect to $I$ and Hamiltonian with respect to $2\eta$. If as generating Hamiltonian for $\nu^N$ we choose $\Phi^{-1}$, descended to a function on $N'$, $\nu^X$ is the corresponding contact lift of $\nu^N$ to $(X', \beta)$ in the sense of (1).

Every $\mu^M$-invariant $C^\infty$ function $f = f(m)$ on $M$ lifts to a $\nu^X \times \mu^X$-invariant function $f = f(x)$ on $X$, and then descends to a $\nu^N$-invariant $C^\infty$ function $f = f(n)$ on $N'$. In the reverse direction, a $C^\infty$ $\nu^N$-invariant function $f = f(n)$ on $N'$ yields a $\mu^M$-invariant $C^\infty$ function $f = f(m)$ on $M'$. We thus have a natural algebraic isomorphism between spaces of invariant smooth functions:

$$C^\infty(M')^\mu \cong C^\infty(N')^\nu.$$

If $\omega$ is real-analytic, this restricts to an isomorphism between the corresponding subspaces of invariant real-analytic functions:

$$C^\varpi(M')^\mu \cong C^\varpi(N')^\nu.$$

With this understanding, we shall think of $\Phi$ as being defined on $M$, $X$, or $N$ according to the context, and drop the symbols of pull-back or push-forward. Similarly, let $\varrho_N$ be the scalar curvature of the Kähler structure $(N', I, 2\eta)$; then $\varrho_N$ is $\nu^N$-invariant, and may be viewed as a $\mu^M$-invariant function on $M'$. By the same principle, the Laplace–Beltrami operator $\Delta_N$ of $(N', I, 2\eta)$ acts on $\mu^M$-invariant functions on $M'$ (see Section 2.1 for precise definitions).

An important ingredient of the present analysis is the study by Engliš of the asymptotics of Laplace integrals on a real-analytic Kähler manifold. Namely, let $(g_{k\bar{l}})$ be a real-analytic Kähler metric on an open subset $U \subseteq \mathbf{C}^d$, and suppose that $\Xi$ is a Kähler potential for $(g_{k\bar{l}})$ on $U$. Let $\widetilde{\Xi}$ be a sesquiholomorphic extension of $\Xi$ to some open neighborhood $\widehat{U} \subseteq U \times U$ of the diagonal. Calabi's *diastasis function* is given by

$$\mathcal{D}(z, w) =: \Xi(z) + \Xi(w) - \widetilde{\Xi}(z, w) - \widetilde{\Xi}(w, z) \qquad \big((z, w) \in \widehat{U}\big); \qquad (7)$$

it is an intrinsic attribute of $(g_{k\bar{l}})$, that is, it does not depend on the choice of $\Xi$, and it satisfies $\mathcal{D}(z, z) = 0$ and $\mathcal{D}(z, w) > 0$ if $z \ne w$ [9] (see also the discussions in [8] and [20]).

In [13], Engliš considers the asymptotics as $\lambda \to +\infty$ of integrals of the form

$$I(\lambda, y) =: \int_U e^{-\lambda \mathcal{D}(x,y)} f(x) g(x) \, dx, \tag{8}$$

where $g =: \det[g_{k\bar{l}}]$ and $dx$ denotes the Lebesgue measure on $\mathbb{C}^d$. By Theorem 3 of [13], there is an asymptotic expansion of the form

$$I(\lambda, y) \sim \left(\frac{\pi}{\lambda}\right)^d \sum_{j \ge 0} \lambda^{-j} \left. R_j^U(f) \right|_y, \tag{9}$$

where the $R_j^U$s are covariant differential operators, that may be expressed in a universal manner in terms of the metric, the curvature tensor, and their covariant derivatives; in particular, $R_0 = \mathrm{id}$ and $R_1 = \Delta_N - \varrho_N/2$ (the opposite sign convention is used in [13] for the curvature tensor and for $\varrho_N$). Engliš also provided an explicit description of $R_j^U$ for $j \le 3$; the higher $R_j^U$s and their differential geometric significance were investigated further in [20], and a graph-theoretic formula for them was given in [33]. Because $\mathcal{D}$ and the $R_l^U$s are intrinsically defined, the expansion (9) holds globally on any real-analytic Kähler manifold $(S, g)$, in which case we shall denote the covariant operators by $R_j^S$.

**Theorem 1**   *With the notation above, suppose that $\omega$ is real-analytic, $\Phi > 0$, and $T_{\text{gen}}$ is trivial. Then the invariant functions $S_j^\mu: M' \to \mathbb{R}$ in (5) are determined as follows. First, $S_0^\mu = \Phi^{-(d+1)}$. Next, for some $j \ge 0$ suppose inductively that*

$$S_0^\mu, \dots, S_j^\mu \in C^{\varpi}(M')^\mu \cong C^{\varpi}(N')^\nu$$

*have been constructed, and let $\widetilde{S_0^\mu}, \dots, \widetilde{S_j^\mu}$ be their respective sesquiholomorphic extensions as elements of $C^{\varpi}(N')^\nu$. Define*

$$Z_j(n_0, n) =: \Phi(n)^{d+1} \sum_{a+b=j} \widetilde{S_a^\mu}(n_0, n) \, \widetilde{S_b^\mu}(n, n_0). \tag{10}$$

*Then, thinking of the $R_r^N$s as acting on the variable n and of $n_0$ as a parameter,*

$$S_{j+1}^\mu(n_0) = -\Phi(n_0)^{d+1} \sum_{l=1}^{j} S_l^\mu(n_0) \, S_{j+1-l}^\mu(n_0)$$

$$- \sum_{r=1}^{j+1} R_r^N \left( Z_{j+1-r}(n_0, \cdot) \right) |_{n=n_0}. \tag{11}$$

Since the $R_r^N$s are universal intrinsic attributes of the Kähler manifold $(N, K, \eta)$, (11) expresses the $S_j^\mu$s as a universal intrinsic attribute of the Hamiltonian action, through the geometry of its quotient. As mentioned, the $R_r^N$s were computed in Section 4 of [13], in [20] and [33]; thus, in principle, (11) determines $S_l^\mu$ explicitly in terms of the geometry of the quotient $N'$. Let us consider $S_1^\mu$:

**Corollary 2** *Under the assumptions of Theorem 1, we have*

$$S_1^\mu = \frac{1}{2} \varrho_N \, \Phi^{-(d+1)}$$

$$+(d+1) \, \Phi(n_0)^{-(d+2)} \left[ \frac{1}{2 \, \Phi} \left\| \mathrm{grad}_N(\Phi) \right\|^2 - \Delta_N(\Phi) \right].$$

Here $\varrho_N$, the gradient $\mathrm{grad}_N \Phi$ of $\Phi$ as a function on $N'$, and the Laplacian $\Delta_N(\Phi)$ taken with respect to the Kähler structure $(N, I, 2\eta)$, and $\| \cdot \|_N$ is the norm in the same metric. Their relation to the corresponding objects on $M$ is explained in Sections 2.2 and 2.8 (see (34) and (71)). If $\Phi = 1$, we recover Lu's subprincipal term [22].

*Remark 3* A notational remark is in order. If, working in a system of local holomorphic coordinates, $\gamma_{a\bar{b}}$ is a Kähler form, the corresponding Kähler metric here is $\rho_{a\bar{b}} = -i \gamma_{a\bar{b}}$ (see the discussion in Section 2.1 and (17)). In the literature, often a factor $1/2$ (or $1/(2\pi)$) is included on the LHS of the previous relation; with this convention, the previous invariants would be associated with $(N, I, \eta)$ [22, 32].

Next let us dwell on the local asymptotics of the Toeplitz kernels $T_k^\mu[f](\cdot, \cdot)$. Firstly, by Theorem 1 of [28] we have $\Pi_k^\mu(x', x'') = O(k^{-\infty})$ uniformly for $\mathrm{dist}_X \left( \mathbf{T}^1 \cdot x', x'' \right) \geq C \, k^{\epsilon - 1/2}$, for any given $\epsilon > 0$. In view of (6), the same holds of $T_k^\mu[f]$. We can then focus on the local asymptotics of $T_k^\mu[f](x', x'')$ for $x'' \to \mathbf{T}^1 \cdot x'$. In view of (3) and (4), for any $e^{i\vartheta} \in \mathbf{T}^1$ we have

$$T_k^\mu[f]\left( \mu_{e^{-i\vartheta}}^X(x'), x'' \right) = e^{ik\vartheta} \, T_k^\mu[f]\left( x', x'' \right) = T_k^\mu[f]\left( x', \mu_{e^{i\vartheta}}^X(x'') \right). \tag{12}$$

Therefore, we need only consider the asymptotics of $T_k^\mu[f](x', x'')$ for $x'' \to x'$. Predictably, these exhibit the same kind of scaling behavior as the asymptotics of $\Pi_k^\mu(x', x'')$ for $x' \to x''$ (Theorem 2 of [28]).

This is best expressed in terms of Heisenberg local coordinates (in the following: HLC for short) $x + (\theta, \mathbf{v})$ centered at a given $x \in X$; here $(\theta, \mathbf{v}) \in (-\pi, \pi) \times B_{2d}(\mathbf{0}, \delta)$, where $B_{2d}(\mathbf{0}, \delta) \subseteq \mathbb{C}^d$ is the open ball centered at the origin and of radius $\delta > 0$. It is in these coordinates that the near-diagonal scaling asymptotics of the standard equivariant Szegö kernels $\Pi_k$ exhibit their universal nature [3, 30], and by [28] the same holds of the $\Pi_k^\mu$s. While we refer to [30]

for a precise definition, let us recall that Heisenberg local coordinates enjoy the following properties.

Firstly, the parameterized submanifold $\gamma_x : \mathbf{v} \mapsto x + (0, \mathbf{v})$ is horizontal, that is, tangent to $\ker(\alpha) \subseteq TX$, at $\mathbf{v} = \mathbf{0}$. In view of (1), and given that $\Phi > 0$, $\gamma_x$ is transverse to the $\mu^X$-orbit $\mathbf{T}^1 \cdot x$; hence for $\mathbf{v} \sim \mathbf{0}$ we have

$$D_2 \,\|\mathbf{v}\| \geq \operatorname{dist}_X \left( \mathbf{T}^1 \cdot x, x + \mathbf{v} \right) \geq D_1 \,\|\mathbf{v}\|, \tag{13}$$

for some fixed $D_1, D_2 > 0$.

Since HLC centered at $x \in X$ come with a built-in unitary isomorphism $T_m M \cong \mathbb{C}^d$, where $m = \pi(x) \in X$, we may use the expression $x + (\theta, \mathbf{v})$ when $\mathbf{v} \in T_m M$ has sufficiently small norm.

Finally, scalar multiplication by $e^{i\vartheta} \in \mathbf{T}^1$ is expressed in HLC by a translation in the angular coordinate: where both terms are defined, we have

$$e^{i\vartheta} \cdot (x + (\theta, \mathbf{v})) = x + (\vartheta + \theta, \mathbf{v}). \tag{14}$$

We shall set $x + \mathbf{v} =: x + (0, \mathbf{v})$.

Given (12) and the previous transversality argument, we need only consider the asymptotics of $T_k^\mu[f](x + \mathbf{v}, x + \mathbf{w})$ for $\mathbf{v}, \mathbf{w} \to 0$. Following [30], let us define, for $\mathbf{v}, \mathbf{w} \in T_m M$,

$$\psi_2(\mathbf{v}, \mathbf{w}) =: -i\,\omega_m(\mathbf{v}, \mathbf{w}) - \frac{1}{2}\,\|\mathbf{v} - \mathbf{w}\|_m^2, \tag{15}$$

where $\|\cdot\|_m$ is the Euclidean norm on the unitary vector space $(T_m M, \omega_m, J_m)$.

**Theorem 4**  *Assume as above that $\mathbf{\Phi} > 0$. Then for any $f \in C^\infty(M)^\mu$ we have*

*(1) $T_k^\mu[f] = 0$ for any $k \leq 0$.*
*(2) For any $C, \epsilon > 0$, we have $T_k^\mu[f](x', x'') = O\left(k^{-\infty}\right)$ as $k \to +\infty$, uniformly for $\operatorname{dist}_X\left(\mathbf{T}^1 \cdot x', x''\right) \geq C\,k^{\epsilon - 1/2}$.*
*(3) Suppose $x \in X$ and fix a system of HLC on $X$ centered at $x$. Set $m =: \pi(x)$. Then uniformly for $\mathbf{v}, \mathbf{w} \in T_m M$ with $\|\mathbf{v}\|, \|\mathbf{w}\| \leq C\,k^{1/9}$, as $k \to +\infty$ we have an asymptotic expansion of the form*

$$T_k^\mu[f]\left(x + \frac{\mathbf{v}}{\sqrt{k}}, x + \frac{\mathbf{w}}{\sqrt{k}}\right)$$
$$= \left(\frac{k}{\pi}\right)^d \sum_{t \in T_m} t^k \, e^{\psi_2\left(d_m \mu_{t^{-1}}^M(\mathbf{v}), \mathbf{w}\right)/\Phi(m)} \cdot A_t(m, \mathbf{v}, \mathbf{w}),$$

*with*

$$A_t(m, \mathbf{v}, \mathbf{w}, f) \sim \sum_{j \geq 0} k^{-j/2}\, R_j\left(m, d_m \mu_{t^{-1}}^M(\mathbf{v}), \mathbf{w}, f\right),$$

*where the $R_j(\cdot, \cdot, \cdot, \cdot)$'s are polynomial in $\mathbf{v}$ and $\mathbf{w}$ and differential operators in $f$. In particular,*

$$R_0\left(m, d_m\mu_{t^{-1}}^M(\mathbf{v}), \mathbf{w}, f\right) = \Phi(m)^{-(d+1)}\, f(m).$$

*(4) The previous asymptotic expansion goes down by integer steps when $\mathbf{v} = \mathbf{w} = 0$ (that is, only powers of $k^{-1}$ appear in the diagonal asymptotics).*

Theorem 4 might be proven by a microlocal argument along the lines of that used for Theorem 1 of [28]; to avoid introducing too much machinery, we shall instead deduce it as a consequence of Theorem 1 of [28], by inserting in (6) the near-diagonal scaling asymptotics for $\Pi_k^\mu$.

**Corollary 5** *In the situation of Theorem 4, suppose in addition that $T_{\text{gen}}$ is trivial. If $x \in X'$, then as $k \to +\infty$ there is an asymptotic expansion*

$$T_k^\mu[f](x,x) \sim \left(\frac{k}{\pi}\right)^d \sum_{j \geq 0} k^{-j}\, S_j^\mu[f](m),$$

*where $m = \pi(x)$ and every $S_j^\mu[f] \in C^\infty(M')^\mu$. In particular, we have*

$$S_0^\mu[f] = \Phi^{-(d+1)} \cdot f.$$

When $\Phi = 1$, corresponding results were obtained in Lemma 4.6 of [24] and Lemma 7.2.4 of [23], covering the case of symplectic manifolds in the presence of a twisting vector bundle.

Let us consider the lower-order $S_j^\mu[f]$s.

**Theorem 6** *Under the assumptions of Corollary 5, assume also that $\omega$ is real-analytic. Then for every $j = 0, 1, 2, \ldots$ we have $S_j^\mu[f] = P_j^\mu(f)$, where each $P_j^\mu$ is a differential operator of degree $\leq 2j$. More precisely, viewed as a $\nu^N$-invariant function on $N$, $S_j^\mu[f]$ is given by*

$$S_j^\mu[f](n_0) = P_j^\mu(f)(n_0) = \sum_{r+s=j} R_r^N(f(\cdot)\, Z_s(n_0, \cdot))\Big|_{n=n_0}.$$

*Remark 7* Clearly, $S_j^\mu = S_j^\mu[1]$ for every $j \geq 0$.

**Corollary 8** *In the situation of Theorem 6,*

$$S_1^\mu[f] = \Phi^{-(d+1)}\Delta_N(f) + S_1^\mu \cdot f.$$

For $\Phi = 1$, the corresponding result to Corollary 8 was obtained in (0.13) of [26].

For a general discussion of the Berezin transform in the Kähler context, we refer, say, to [1, 7, 13, 21, 29]. Here we adopt the following natural adjustment:

**Definition 9**   If $f \in C^\infty(M)$ and $k = 0, 1, 2, \ldots$, let the $k$th $\mu$-*equivariant Berezin transform* of $f$ be given by

$$\mathrm{Ber}_k^\mu[f](m) =: \frac{T_k^\mu[f](x, x)}{\Pi_k^\mu(x, x)} \qquad (m \in M)$$

for any choice of $x \in \pi^{-1}(m)$.

**Corollary 10**   *Assume that $\omega$ is real-analytic, $\Phi > 0$, and $T_{\mathrm{gen}} = \{1\}$. If $f \in C^\infty(M)^\mu$, then as $k \to +\infty$ on $M'$, uniformly on compact subsets of $M'$, there is an asymptotic expansion of the form*

$$\mathrm{Ber}_k^\mu[f] \sim \sum_{j \geq 0} k^{-j} B_j^\mu(f),$$

*where every $B_j^\mu$ is a differential operator of degree $2j$. In particular, $B_0^\mu = \mathrm{id}$ and $B_1^\mu = \Delta_N$.*

A corresponding result for $\Phi = 1$ was given in [13].

The following analogue of the Heisenberg correspondence relates the commutator of two equivariant Toeplitz operators to the Poisson brackets of the corresponding Hamiltonians. Let $\{\cdot, \cdot\}_M$ and $\{\cdot, \cdot\}_N$ denote, respectively, Poisson brackets on $(M, 2\,\omega)$ and $(N', 2\,\eta)$. By restriction, they yield maps

$$\{\cdot, \cdot\}_M, \ \{\cdot, \cdot\}_N \colon C^\infty(M')^\mu \times C^\infty(M')^\mu \to C^\infty(M')^\mu.$$

**Theorem 11**   *Assume that $\omega$ is real-analytic, $\Phi > 0$, and $T_{\mathrm{gen}} = \{1\}$. Let $f, g \in C^\infty(M)^\mu$ be real-valued, and denote by $E_k^\mu[f, g](\cdot, \cdot) \in C^\infty(X \times X)$ the Schwartz kernel of the composition $T_k^\mu[f] \circ T_k^\mu[g]$. Then uniformly on compact subsets of $M'$ as $k \to +\infty$ we have*

$$\begin{aligned}
& E_k^\mu[f, g](x, x) - E_k^\mu[f, g](x, x) \\
&= \left(\frac{k}{\pi}\right)^d \left[-\frac{i}{k}\, \Phi(m)^{-(d+1)} \{f, g\}_N(m) + O\left(k^{-2}\right)\right] \\
&= \left(\frac{k}{\pi}\right)^d \left[-\frac{i}{k}\, \Phi(m)^{-d} \{f, g\}_M(m) + O\left(k^{-2}\right)\right],
\end{aligned}$$

*for any $x \in \pi^{-1}(m)$.*

In the course of the proof, one actually establishes an asymptotic expansion for $E_k^\mu[f, g](x, x)$ (see (118)):

$$E_k^\mu[f, g](x, x) \sim \left(\frac{k}{\pi}\right)^d \sum_j k^{-j} A_j[f, g](x), \qquad (16)$$

where $A_0[f,g] = \Phi^{-(d+1)} \cdot f\,g$ and

$$A_1[f,g] = \Phi^{-(d+1)} \left[ f\,\Delta_N g + g\,\Delta_N f + \left\langle \operatorname{grad}_N(f)^{(0,1)}, \operatorname{grad}_N(g)^{(1,0)} \right\rangle \right]$$
$$+ S_1^\mu \cdot f\,g$$

(we leave the explicit computation to the reader). When $\Phi = 1$, the formula for $A_1[f,g]$ was obtained in (0.16) of [26].

As explained in the references above for the standard case, this expansion can be used to define in a natural manner a $*$-product on $C^\infty(M')^\mu$ (depending on $\Phi$), but we won't discuss this here.

## 2  Preliminaries

### 2.1  Some notation and recalls from Kähler geometry

Let $(P, K)$ be a $d$-dimensional complex manifold and let $(P, K, \gamma)$ be a Kähler structure on it, with associated covariant metric tensor $\rho(\cdot, \cdot) =: \gamma(\cdot, K(\cdot))$; also, let $\ell =: \rho - i\gamma$ be the associated Hermitian metric. Given holomorphic local coordinates $(z_a)$ on $P$, we shall let $\partial_a =: \partial/\partial z_a$ and $\partial_{\bar a} =: \partial/\partial \bar z_a$, $\rho_{a\bar b} =: \rho(\partial_a, \partial_{\bar b})$, $\gamma_{a\bar b} =: \gamma(\partial_a, \partial_{\bar b})$, $\ell_{a\bar b} =: \ell(\partial_a, \partial_{\bar b})$. Then locally

$$\gamma = \sum_{a,b} \gamma_{a\bar b}\, dz_a \wedge d\bar z_b = i \sum_{a,b} \rho_{a\bar b}\, dz_a \wedge d\bar z_b = \frac{i}{2} \sum_{a,b} \ell_{a\bar b}\, dz_a \wedge d\bar z_b. \tag{17}$$

Consider the real local frame $\mathcal{B} = (\partial/\partial x_1, \ldots, \partial/\partial x_d, \partial/\partial y_1, \ldots, \partial/\partial y_d)$, where $z_j = x_j + iy_j$ is the decomposition in real and imaginary parts, and denote by $M_{\mathcal{B}}(\rho)$ the matrix representing $\rho$ in this frame. Then

$$\det M_{\mathcal{B}}(\rho) = 4^d \, \det\left([\rho_{a\bar b}]\right)^2.$$

Therefore, the Riemannian volume form of $(P, \rho)$ is

$$dV_P = \frac{1}{d!}\,\gamma^{\wedge d} = \sqrt{\det(M_{\mathcal{B}}(\rho))} \cdot dx_1 \wedge \cdots dx_d \wedge dy_1 \cdots \wedge dy_d$$
$$= 2^d \, \det\left([\rho_{k\bar l}]\right) \cdot dx_1 \wedge \cdots dx_d \wedge dy_1 \cdots \wedge dy_d$$
$$= \det\left([2\rho_{k\bar l}]\right) \cdot dx_1 \wedge \cdots dx_d \wedge dy_1 \cdots \wedge dy_d. \tag{18}$$

Let $R$ be the covariant curvature tensor of the Riemannian manifold $(P, \rho)$, with components $R_{a\bar b c\bar d} = R(\partial_a, \partial_{\bar b}, \partial_c, \partial_{\bar d})$ [32].

We shall set (leaving the metric understood and adopting Einstein notation)

$$\varrho_P =: \rho^{\bar b a}\,\rho^{\bar d c}\,R_{a\bar b c\bar d}; \tag{19}$$

this is $1/2$ of the ordinary Riemannian scalar curvature $\operatorname{scal}_P$.

Similarly, for $f \in C^\infty$, we shall let

$$\Delta_P(f) =: \rho^{\bar{b}a} \, \partial_a \partial_{\bar{b}} f, \tag{20}$$

which is $1/2$ times the ordinary Riemannian Laplace–Beltrami operator.

The gradient of $f$ is locally given by

$$\text{grad}_P(f) = \rho^{\bar{b}a} \, (\partial_{\bar{b}} f) \, \partial_a + \rho^{\bar{b}a} \, (\partial_a f) \, \partial_{\bar{b}}, \tag{21}$$

and its square norm is given by

$$\left\| \text{grad}_P(f) \right\|^2 = 2 \rho^{\bar{b}a} \, (\partial_a f) \, (\partial_{\bar{b}} f). \tag{22}$$

Since $\Delta_P$ here is $1/2$ times the ordinary Laplace–Beltrami operator, we have for any $f_1, \, f_2 \in C^\infty(P)$:

$$\Delta_P(f_1 \cdot f_2) = f_1 \, \Delta_P(f_2) + \rho(\text{grad}_P(f_1), \text{grad}_P(f_2)) + f_2 \, \Delta_P(f_1).$$

It follows inductively that for any $f \in C^\infty(P)$ and $l \geq 0$, we have

$$\Delta_P \left( f^l \right) = l \, f^{l-1} \, \Delta_P(f) + \frac{(l-1) \, l}{2} \, f^{l-2} \left\| \text{grad}_P(f) \right\|^2. \tag{23}$$

Let us now consider the Poisson brackets $\{f, g\}_P = \gamma(H_f, H_g)$ of two real functions $f, \, g \in C^\infty(P)$ in the symplectic structure $(P, \gamma)$; here $H_f$ is the Hamiltonian vector field of $f$ with respect to $\gamma$. We have $H_f = -K(\text{grad}_P(f))$, hence given (21)

$$\begin{aligned}
\{f, g\}_P &= \gamma\big( K(\text{grad}_P(f)), K(\text{grad}_P(g)) \big) = \gamma(\text{grad}_P(f), \text{grad}_P(g)) \\
&= \rho\big( K(\text{grad}_P(f)), \text{grad}_P(g) \big) \\
&= i \, \rho\big( \rho^{\bar{b}a} \, (\partial_{\bar{b}} f) \, \partial_a - \rho^{\bar{b}a} \, (\partial_a f) \, \partial_{\bar{b}}, \rho^{\bar{d}c} \, (\partial_{\bar{d}} g) \, \partial_c + \rho^{\bar{d}c} \, (\partial_c g) \, \partial_{\bar{d}} \big) \\
&= \frac{1}{i} \, \rho^{\bar{d}c} \left[ (\partial_c f) \, (\partial_{\bar{d}} g) - (\partial_c g) \, (\partial_{\bar{d}} f) \right]. \tag{24}
\end{aligned}$$

**Lemma 12** *Let $(P, K, \gamma)$ be a Kähler manifold, with $\gamma$ real-analytic. Let $\Phi : P \to \mathbb{R}$ be a real $C^\infty(M)$ function whose Hamiltonian flow with respect to $\gamma$ is holomorphic with respect to $K$. Then $\Phi$ is real-analytic.*

*Proof* Let $T^c P = TP \otimes \mathbb{C}$ be the complexified tangent bundle of $P$, and $T^c P = T'P \oplus T''P$ its decomposition into $\pm i$-eigenbundles of $K$. Let $v_\Phi \in \mathfrak{X}(P)$ be the Hamiltonian vector field of $\Phi$ with respect to $\gamma$. If $v_\Phi = v'_\Phi \oplus v''_\Phi$, with $v'_\Phi \in T'P$ and $v''_\Phi = \overline{v'_\Phi} \in T''P$, then $v'_\Phi$ is holomorphic, whence real-analytic. Then clearly $v_\Phi$ is real-analytic as well, and therefore so is its differential

$d\upsilon_\Phi = \iota(\upsilon_\Phi)\gamma$. This forces $\Phi$ itself to be real-analytic (say by Proposition 2.2.10 of [18]). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

## The Laplacian and sesquiholomorphic extensions

We give here a couple of technical lemmas that will be handy in the proof of Corollary 2.

**Lemma 13** *Let $(P, K, \gamma)$ be a Kähler manifold, and consider $f \in C^\varpi(P)$ with $f > 0$. Let $\widetilde{f}(\cdot, \cdot)$ be the sesquiholomorphic extension of $f$ to an open neighborhood $\widetilde{P} \subseteq P \times P$ of the diagonal (thus $\widetilde{f}(\cdot, \cdot)$ is holomorphic in the first entry and antiholomorphic in the second, and $\widetilde{f}(p, p) = f(p)$ for all $p \in P$). Given $p_0 \in P$, let $P' \subseteq P$ be an open neighborhood of $p_0$ so small that $P' \times P' \subseteq \widetilde{P}$ and $\widetilde{f}(p_0, p) \neq 0$ for all $p \in P'$. Define $f_1, f_2, F_f \in C^\varpi(P')$ by setting for $p \in P'$:*

$$f_1(p) =: f(p_0, p), \quad f_2(p) =: f(p, p_0) = \overline{f_1(p)}, \quad F_f(p) = \frac{f(p)}{f_1(p)\, f_2(p)}.$$

*Thus $f_1$ is antiholomorphic, $f_2$ is holomorphic, and $F_f > 0$. Then*

$$\Delta_P(F_f)(p_0) = \frac{1}{f(p_0)^2}\left[\Delta_P(f)(p_0) - \frac{1}{2\,f(p_0)}\left\|\mathrm{grad}_P(f)(p_0)\right\|^2\right], \qquad (25)$$

*where the terms involved are given by (20) and (22).*

*Remark 14* To be precise, we should really write $F_{f,p_0}$ for $F_f$, since the latter also depends on the reference point.

*Proof* As above, let $\rho$ be the metric tensor. In a local holomorphic chart $(z_a)$ for $P$ centered at $p_0$, given that $\partial_a \partial_{\bar{b}} f_j = 0$ we have

$$\Delta_P(F) = \rho^{\bar{b}a}\, \partial_a \partial_{\bar{b}}\left(\frac{f}{f_1\, f_2}\right)$$

$$= \rho^{\bar{b}a}\, \partial_a\left(\frac{1}{f_1\, f_2}\, \partial_{\bar{b}} f - \frac{f}{f_1^2\, f_2}\, \partial_{\bar{b}} f_1\right)$$

$$= \rho^{\bar{b}a}\left(-\frac{1}{f_1\, f_2^2}\, \partial_a f_2\, \partial_{\bar{b}} f + \frac{1}{f_1\, f_2}\, \partial_a \partial_{\bar{b}} f - \frac{1}{f_1^2\, f_2}\, \partial_a f\, \partial_{\bar{b}} f_1\right.$$

$$\left. + \frac{f}{f_1^2\, f_2^2}\, \partial_a f_2\, \partial_{\bar{b}} f_1\right). \qquad (26)$$

At $p_0$, $\partial_a f_2(p_0) = \partial_a f(p_0)$, $\partial_{\bar{b}} f_1(p_0) = \partial_{\bar{b}} f(p_0)$, and $f_1(p_0) = f_2(p_0) = f(p_0)$.

Thus, (26) yields

$$
\begin{aligned}
\Delta_P(F)(p_0) &= \rho^{\bar{b}a}(p_0) \left( -\frac{1}{f(p_0)^3} \, \partial_a f(p_0) \, \partial_{\bar{b}} f(p_0) + \frac{1}{f(p_0)^2} \, \partial_a \, \partial_{\bar{b}} f(p_0) \right. \\
&\qquad \left. -\frac{1}{f(p_0)^3} \, \partial_a f(p_0) \partial_{\bar{b}} f(p_0) + \frac{1}{f(p_0)^3} \, \partial_a f(p_0) \partial_{\bar{b}} f(p_0) \right) \\
&= \frac{1}{f(p_0)^2} \left[ \rho^{\bar{b}a}(p_0) \, \partial_a \, \partial_{\bar{b}} f(p_0) - \frac{1}{f(p_0)} \rho^{\bar{b}a}(p_0) \, \partial_a f(p_0) \, \partial_{\bar{b}} f(p_0) \right] \\
&= \frac{1}{f(p_0)^2} \left[ \Delta_P(f)(p_0) - \frac{1}{2 \, f(p_0)} \, \big\| \mathrm{grad}_P(f)(p_0) \big\|_P^2 \right].
\end{aligned}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

**Lemma 15**  *With the hypothesis and notation of Lemma 13, we have*

$$
\mathrm{grad}_P(F_f)(p_0) = 0.
$$

*Proof*  Let again $(z_a)$ be a local holomorphic coordinate chart for $P$ centered at $p_0$. Then for every $a$ we have

$$
\begin{aligned}
\partial_a(F_f)(p_0) &= \frac{1}{f_1(p_0)} \, \partial_a \left( \frac{f}{f_2} \right)(p_0) \\
&= \frac{f_2(p_0) \, \partial_a f(p_0) - f(p_0) \, \partial_a f_2(p_0)}{f_1(p_0) \, f_2(p_0)^2} = \frac{f(p_0) \, \partial_a f(p_0) - f(p_0) \, \partial_a f(p_0)}{f(p_0)^3} = 0.
\end{aligned}
$$

Similarly, $\partial_{\bar{a}} F_f(p_0) = 0$ for every $a$.                                        $\square$

## 2.2  The Kähler structure on $N'$

We are assuming $\Phi > 0$ and $T_{\mathrm{gen}}$ trivial. Then the two projections

$$
M' \xleftarrow{\ \pi\ } X' \xrightarrow{\ \kappa\ } N'
$$

are circle bundle structures; the fibers of $\pi$ are the orbits in $X'$ of $\nu^X$ and those of $\kappa$ are the orbits in $X'$ of $\mu^X$.

Let $\mathcal{H} = \ker(\alpha) \subseteq TX$ be the horizontal distribution for $\pi$. Since $\alpha$ is $\mu^X$-invariant, so is $\mathcal{H}$. In addition, by (1) $\mathcal{H}$ is transverse to every $\mu^X$-orbit. Therefore, it may be viewed as an invariant horizontal distribution for $\kappa$ as well.

Let $J_{\mathcal{H}}$ be the complex structure that $\mathcal{H}$ inherits from $J$ by the isomorphism $d\pi|_{\mathcal{H}} : \mathcal{H} \cong \pi^*(TM)$. Since $\mu^M$ is holomorphic, $J_{\mathcal{H}}$ is $\mu^X$-invariant. Therefore, given the isomorphism $d\kappa|_{\mathcal{H}} : \mathcal{H} \cong \kappa^*(TN')$, it descends to an almost complex structure $I$ on $N'$.

**Proposition 16** *I is a complex structure.*

*Proof* Let $\mathcal{J}$ be the complex distribution on $M$ associated with $J$. Thus

$$\mathcal{J} = \{\mathbf{v} - i\,J(\mathbf{v}) \,:\, \mathbf{v} \in TM\} = \ker(J - i\,\mathrm{id}) \subseteq TM \otimes \mathbb{C}.$$

As $J$ is integrable, $\mathcal{J}$ is involutive.

Similarly, let

$$\mathcal{J}_{\mathcal{H}} = \{\mathbf{h} - i\,J_{\mathcal{H}}(\mathbf{h}) \,:\, \mathbf{h} \in \mathcal{H}\} = \ker(J_{\mathcal{H}} - i\,\mathrm{id}) \subseteq \mathcal{H} \otimes \mathbb{C}.$$

Evidently, $\mathcal{J}_{\mathcal{H}}$ is the horizontal lift of $\mathcal{J}_M$.

**Lemma 17** $\mathcal{J}_{\mathcal{H}}$ *is involutive.*

*Proof* If $V \in \mathfrak{X}(M)$ is a real vector field on $M$, then $U =: V - i\,J(V)$ is a complex vector field on $X$ tangent to $\mathcal{J}_{\mathcal{H}}$, and its horizontal lift

$$U^{\sharp} = V^{\sharp} - i\,J(V)^{\sharp} = V^{\sharp} - i\,J_{\mathcal{H}}\left(V^{\sharp}\right)$$

is a complex vector field on $X$ tangent to $\mathcal{J}_{\mathcal{H}}$. It is clear that $\mathcal{J}_{\mathcal{H}}$ is locally spanned by vector fields of this form, so it suffices to show that $\left[U_1^{\sharp}, U_2^{\sharp}\right]$ is tangent to $\mathcal{J}_{\mathcal{H}}$, for any pair of complex vector fields $U_1$, $U_2$ on $M$ tangent to $\mathcal{J}_M$.

Since $\mathcal{J}_M$ is involutive, $[U_1, U_2]$ is tangent to $\mathcal{J}_M$. Given that $\left[U_1^{\sharp}, U_2^{\sharp}\right]$ is $\pi$-correlated to $[U_1, U_2]$, to show that $\left[U_1^{\sharp}, U_2^{\sharp}\right]$ is tangent to $\mathcal{J}_{\mathcal{H}}$ it suffices to show that it is horizontal.

On the one hand, by compatibility of $\omega$ and $J$ and because by construction $J(U_l) = i\,U_l$, we have

$$\omega(U_1, U_2) = \omega(J(U_1), J(U_2)) = i^2\,\omega(U_1, U_2),$$

so that $\omega(U_1, U_2) = 0$. On the other hand, since $U_l^{\sharp}$ is horizontal we have $\alpha\left(U_l^{\sharp}\right) = 0$; therefore, given that $d\alpha = 2\,\pi^*(\omega)$, we get

$$
\begin{aligned}
0 = 2\,\omega(U_1, U_2) &= 2\,\pi^*(\omega)\left(U_1^{\sharp}, U_2^{\sharp}\right) = d\alpha\left(U_1^{\sharp}, U_2^{\sharp}\right) \\
&= U_1^{\sharp} \cdot \alpha\left(U_2^{\sharp}\right) - U_2^{\sharp} \cdot \alpha\left(U_1^{\sharp}\right) - \alpha\left(\left[U_1^{\sharp}, U_2^{\sharp}\right]\right) \\
&= -\alpha\left(\left[U_1^{\sharp}, U_2^{\sharp}\right]\right).
\end{aligned}
$$

$\square$

Finally, let us set

$$\mathcal{I} = \{\mathbf{s} - i\,I(\mathbf{s}) \,:\, \mathbf{s} \in TN'\} = \ker(I - i\,\mathrm{id}) \subseteq TN' \otimes \mathbb{C}.$$

We need to prove that $\mathcal{I}$ is an involutive complex distribution. Let $S_1, S_2 \in \mathfrak{X}(N') \otimes \mathbb{C}$ be complex vector fields on $N'$ tangent to $\mathcal{I}$, and let $\widehat{S}_1, \widehat{S}_2$ be their

horizontal lifts to $X$. By definition of $I$, it follows that the restriction of $\mathcal{J}_{\mathcal{H}}$ to $X'$ is the horizontal lift of $\mathcal{I}$ under $\kappa$. Therefore, $\widehat{S}_l$ is tangent to $\mathcal{J}_{\mathcal{H}}$ and $\mu^X$-invariant. Then the same holds of their commutator $\left[\widehat{S}_1, \widehat{S}_2\right]$ because $\mathcal{J}_{\mathcal{H}}$ is involutive and $\mu^X$-invariant. Since $\left[\widehat{S}_1, \widehat{S}_2\right]$ is $\kappa$-correlated to $[S_1, S_2]$, we conclude that $[S_1, S_2]$ is tangent to $\mathcal{I}$. $\qquad\square$

Let us define

$$\beta =: \frac{1}{\Phi}\,\alpha. \tag{27}$$

**Lemma 18**  *$\beta$ is a connection form for $\kappa \colon X' \to N'$, with respect to which the horizontal tangent bundle is $\mathcal{H}$ (the horizontal tangent bundle of $\pi$).*

*Proof*  Since $\mu^X$ preserves $\alpha$ and lifts $\mu^M$, which is a Hamiltonian action with moment map $\Phi$, $\beta$ is $\mu^X$-invariant. Furthermore, we see from (1) and (27) that $\beta(\xi_X) = -1$. $\qquad\square$

Thus $\mathcal{H} \subseteq TX'$ is the horizontal tangent space for both $\pi$ and $\kappa$. If $V$ is a vector field on $M$, we shall denote by $V^\sharp$ its horizontal lift to $X$ under $\pi$; it is a $\nu^X$-invariant section of $\mathcal{H}$ on $X$. Similarly, if $U$ is a vector field on $N'$, we shall denote by $\widehat{U}$ its horizontal lift to $X'$ under $\kappa$; it is a $\mu^X$-invariant section of $\mathcal{H}$ on $X'$. Clearly, vector fields on $M$ are the same as $\nu^X$-invariant sections of $\mathcal{H}$ on $X$, and vector fields on $N'$ are the same as $\mu^X$-invariant sections of $\mathcal{H}$ on $X$.

**Lemma 19**  *There exists a unique Kähler form $\eta$ on $N'$ such that $d\beta = 2\,\kappa^*(\eta)$.*

*Proof*  We have

$$d\beta = \frac{1}{\Phi}\,d\alpha - \frac{1}{\Phi^2}\,d\Phi \wedge \alpha = \frac{2}{\Phi}\,\pi^*(\omega) - \frac{1}{\Phi^2}\,d\Phi \wedge \alpha, \tag{28}$$

and direct inspection using (1) shows that $\iota(\xi_X^\sharp)d\beta = 0$. Since $d\beta$ is $\mu^X$-invariant, it follows that there exists a necessarily unique 2-form $\eta$ on $N'$ such that $d\beta = \kappa^*(2\,\eta)$.

Thus, $\eta$ is a closed 2-form on $N'$. To see that it is in fact a Kähler form, we need to check that it is compatible with the complex structure and non-degenerate. To this end, we fix an arbitrary $n \in M'$, choose an arbitrary $x \in \kappa^{-1}(n)$, and set $m = \pi(x)$. Our construction then yields natural complex-linear isomorphisms $(T_m M, J_m) \cong (\mathcal{H}_x, J_{\mathcal{H}_x}) \cong (T_n N', I_n)$. To see that $\eta_n$ is non-degenerate on $T_n N'$ and compatible with $I_n$, it then suffices to see that the restriction of $d\beta$ is non-degenerate on $\mathcal{H}_x$, and compatible with $J_{\mathcal{H}_x}$.

By (28), under the complex-linear isomorphism $(T_m M, J_m) \cong (\mathcal{H}_x, J_{\mathcal{H}_x})$, the restriction of $d\beta$ on $\mathcal{H}_x$ may be identified with $2\omega_m/\Phi(m)$ on $T_m M$. Since $\omega$ is Kähler on $(M, J)$, it is non-degenerate on $T_m M$ and compatible with $J_m$, and this completes the proof. $\qquad\square$

Suppose $f \in C^\infty(M')^\mu \cong C^\infty(N')^\nu$, and let $H_f$ be its Hamiltonian vector field on $(M', 2\omega)$. Since $f$ is $\mu^M$-invariant, so is $H_f$. Let $H_f^\sharp$ be the horizontal lift of $H_f$ to $X'$. Then $H_f^\sharp$ is a $\mu^X \times \nu^X$-invariant horizontal vector field on $X'$, and therefore it descends to a $\nu^N$-invariant vector field $\overline{H}_f$, respectively.

**Lemma 20** *Let $K_f$ be the Hamiltonian vector field of $f \in C^\infty(M')^\mu \cong C^\infty(N')^\nu$ on $(N, 2\eta)$. Then $K_f = \Phi \overline{H}_f$.*

*Proof* We need to show that for any $n \in N'$ and $\mathbf{u} \in T_n N'$ we have

$$2\,\Phi(n) \cdot \eta_n(\overline{H}_f(n), \mathbf{u}) = d_n^N f(\mathbf{u}), \tag{29}$$

where $d^N f$ is the differential of $f$ when $f$ is viewed as a function on $N$.

Choose as before $x \in \kappa^{-1}(n)$ and let $m =: \pi(x) \in M'$. Let $\widehat{\mathbf{u}} \in \mathcal{H}_x$ be the horizontal lift of $\mathbf{u}$ under $\kappa$, and set $\mathbf{v} = d_x\pi(\widehat{\mathbf{u}})$. Thus $\widehat{\mathbf{u}} = \mathbf{v}^\sharp$. Since $f$ is invariant, on $X$ we have $f \circ \pi = f \circ \kappa$; thus,

$$d_m^M f(\mathbf{v}) = d_x^X f\left(\mathbf{v}^\sharp\right) = d_x^X f\left(\widehat{\mathbf{u}}\right) = d_n^N f(\mathbf{u}). \tag{30}$$

On the contrary, since $H_f^\sharp = \widehat{\overline{H}_f}$, we have

$$2\,\Phi(n) \cdot \eta_n\left(\overline{H}_f(n), \mathbf{u}\right) \tag{31}$$

$$= \Phi(m) \cdot d_x\beta\left(H_f(m)^\sharp, \mathbf{v}^\sharp\right) = \Phi(m) \cdot \frac{1}{\Phi(m)}\, d_x\alpha\left(H_f(m)^\sharp, \mathbf{v}^\sharp\right)$$

$$= 2\,\omega_m(H_f(m), \mathbf{v}) = d_m^M f(\mathbf{v}).$$

(29) follows from (30) and (31). □

Suppose $f, g \in C^\infty(M)^\mu$. Since $C^\infty(M')^\mu \cong C^\infty(N')^\nu$, we have Poisson brackets $\{f, g\}_M \in C^\infty(M')^\mu$ and $\{f, g\}_N \in C^\infty(N')^\nu$ on $(M', 2\omega)$ and $(N', 2\eta)$, respectively. The relation between them under the previous isomorphism is as follows:

**Corollary 21** *For any $f, g \in C^\infty(M')^\mu \cong C^\infty(N')^\nu$, we have $\{f, g\}_N = \Phi\{f, g\}_M$.*

*Proof* We have, omitting symbols of pull-back,

$$\{f, g\}_N = 2\,\eta(K_f, K_g) = \Phi^2\, d\beta\left(H_f^\sharp . H_g^\sharp\right)$$

$$= \Phi^2\, \frac{1}{\Phi}\, d\alpha\left(H_f^\sharp . H_g^\sharp\right) = \Phi \cdot 2\,\omega(H_f, H_g)$$

$$= \Phi \cdot \{f, g\}_M. \tag{32}$$

□

We can similarly relate the gradients $\text{grad}_M(f)$ and $\text{grad}_N(f)$ of an invariant $f$ on $(M', 2\,g)$ and $(N', 2\,h)$, where $g(\cdot, \cdot) = \omega(\cdot, J(\cdot))$ and $h = \eta(\cdot, I(\cdot))$ are the Riemannian metrics on $M$ and $N$, respectively. We have

$$\text{grad}_N(f) = I(K_f) = \Phi\,I\left(\overline{H_f}\right) = \Phi\,\overline{J(H_f)} = \Phi\,\overline{\text{grad}_M(f)}. \tag{33}$$

Passing to square norms, we get

$$
\begin{aligned}
\left\|\text{grad}_N(f)\right\|_N^2 &= 2\,h(\text{grad}_N(f), \text{grad}_N(f)) = 2\,\eta\Big(\text{grad}_N(f), I(\text{grad}_N(f))\Big) \\
&= 2\,\Phi^2\,\eta\left(\overline{\text{grad}_M(f)}, \overline{J(\text{grad}_M(f))}\right) \\
&= 2\,\Phi^2\,d\beta\left(\text{grad}_M(f)^\sharp, J(\text{grad}_M(f)^\sharp)\right) \\
&= 2\,\Phi\,d\alpha\left(\text{grad}_M(f)^\sharp, J(\text{grad}_M(f)^\sharp)\right) \\
&= \Phi \cdot 2\,\omega(\text{grad}_M(f), J(\text{grad}_M(f))) \\
&= \Phi\left\|\text{grad}_M(f)\right\|_M^2. 
\end{aligned}
\tag{34}
$$

### 2.3 The descended action on $N$

Let us dwell on the Hamiltonian nature of the descended action $\nu^N$. Recall that the action $\nu^X$ given by (2), that is, scalar multiplication composed with inversion, commutes with $\mu^X$, hence it descends to an action $\nu^N \colon \mathbf{T}^1 \times N \to N$.

**Lemma 22** $\nu^N$ *is an holomorphic action on* $(N', I)$.

*Proof* Choose $n \in N'$ and $x \in \kappa^{-1}(n)$, and let $m =: \pi(x)$. Fix $t = e^{i\theta} \in \mathbf{T}^1$. By construction, we have complex-linear isomorphisms $T_n N' \cong \mathcal{H}_x \cong T_m M$ that inter-twine $d_n \nu_t^N \colon T_n N' \to T_{\nu_t^N(n)} N'$ with $d_x \nu_t^X \colon \mathcal{H}_x \to \mathcal{H}_{\nu_t^X(x)} = \mathcal{H}_{e^{-i\theta} \cdot x}$, hence with the identity map of $T_m M$. The statement follows. $\square$

**Lemma 23** $\nu^N$ *is a symplectic action on* $(N', \eta)$.

*Proof* This follows as for Lemma 22, since in view of (28) under the previous isomorphism $\eta_n$ corresponds to $\omega_m / \Phi(m)$. $\square$

Thus $\nu_t^N$ is an automorphism of the Kähler manifold $(N', I, \eta)$, for each $t \in \mathbf{T}^1$.

**Lemma 24** $\nu^N$ *is a Hamiltonian action on* $(N', 2\,\eta)$*, with moment map* $1/\Phi$ *(viewed as a function on $N$).*

*Proof* The vector field $-\partial/\partial\theta$ on $X$ is $\mu^X$-invariant, hence it descends to a vector field $\upsilon$ on $N$, which is the infinitesimal generator of $\nu^X$. We need to show that $2\,\iota(\upsilon)\,\eta = d^N(1/\Phi)$, that is, for any $n \in N'$ and any $\mathbf{u} \in T_n N'$ we have

$$2\,\eta_n(\upsilon, \mathbf{u}) = -\Phi(n)^{-2}\,d_n^N \Phi(\mathbf{u}). \tag{35}$$

As before, let $\widehat{\mathbf{u}}$ be the horizontal lift of $\mathbf{u}$ with respect to $\kappa$, and set $\mathbf{v} =: d_x\pi(\widehat{\mathbf{u}})$, so that $\widehat{\mathbf{u}} = \mathbf{v}^\sharp$. Thus

$$
\begin{aligned}
-\Phi(n)^{-2}\, d_n^N\Phi(\mathbf{u}) &= -\Phi(x)^{-2}\, d_x^X\Phi(\widehat{\mathbf{u}}) \\
&= -\Phi(x)^{-2}\, d_x^X\Phi(\mathbf{v}^\sharp) = -\Phi(m)^{-2}\, d_m^M\Phi(\mathbf{v}).
\end{aligned}
\tag{36}
$$

On the contrary, since $\kappa^*(2\,\eta) = d\beta$, we have

$$
d_x\beta\left(-\frac{\partial}{\partial\theta}, \mathbf{v}^\sharp\right) = d_x\beta\left(-\frac{\partial}{\partial\theta}, \widehat{\mathbf{u}}\right) = 2\,\eta_n(\upsilon, \mathbf{u}).
\tag{37}
$$

Then (35) is equivalent to the equality

$$
d_x\beta\left(\frac{\partial}{\partial\theta}, \mathbf{v}^\sharp\right) = \frac{1}{\Phi(m)^2}\, d_m^M\Phi(\mathbf{v}),
\tag{38}
$$

for any $m \in M$, $\mathbf{v} \in T_mM$, and $x \in \pi^{-1}(x)$. The latter is an immediate consequence of (28). $\qquad\square$

Now $\beta$ is a connection 1-form for the circle bundle $\kappa\colon X' \to N'$ and is preserved by $\upsilon^X$; therefore, for an appropriate constant $c$, $\upsilon^X$ is a contact lift to $(X', \beta)$ of $\upsilon^N$, with respect to the Hamiltonian $c + 1/\Phi$.

**Lemma 25** *The correct choice is $c = 0$. Furthermore, the horizontal lift of $\upsilon$ with respect to $\kappa$ is*

$$
\widehat{\upsilon} = -\frac{1}{\Phi}\,\xi_M^\sharp.
$$

*Proof* We want to give a decomposition of $-\partial/\partial\theta$ analogous to (1), but referred to the circle bundle structure $\kappa\colon X' \to N'$. To this end, let $\beta$ be a locally defined angular coordinate on $X'$ referred to $\kappa$, so that $\xi_X = -\partial/\partial\beta$. Since the horizontal component of $-\partial/\partial\theta$ with respect to $\kappa$ is $\widehat{\upsilon}$, the analogue of (1) is

$$
\begin{aligned}
-\frac{\partial}{\partial\theta} &= \widehat{\upsilon} - \left(c + \frac{1}{\Phi}\right)\frac{\partial}{\partial\beta} \\
&= \widehat{\upsilon} + \left(c + \frac{1}{\Phi}\right)\left(\xi_M^\sharp - \Phi\frac{\partial}{\partial\theta}\right) \\
&= \left[\widehat{\upsilon} + \left(c + \frac{1}{\Phi}\right)\xi_M^\sharp\right] - (1 + c\,\Phi)\frac{\partial}{\partial\theta},
\end{aligned}
\tag{39}
$$

where the latter is a decomposition into horizontal and vertical components with respect to $\pi$. The latter equality is equivalent to the claimed statement. $\qquad\square$

## 2.4 The complexified action on $A_0^\vee$

The action $\mu^M \colon \mathbf{T}^1 \times M \to M$ complexifies to a holomorphic action $\widetilde{\mu}^M \colon \mathbb{T}^1 \times M \to M$, where $\mathbb{T}^1 = \mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^*$ (see, for instance, the discussion in Section 4 of [16]). Let $(\rho, \vartheta)$ be polar coordinates on $\mathbb{C}^*$, and let $\xi =: \partial/\partial\vartheta$, $\eta =: -\rho\,\partial/\partial\rho$; then $\eta = J_0(\xi)$ ($J_0$ being the complex structure on $\mathbb{C}^*$). By holomorphicity, if $\xi_M$ and $\eta_M$ are the induced vector fields on $M$, then $\eta_M = J_M(\xi_M)$.

On the contrary, the contact lift $\mu^X \colon \mathbf{T}^1 \times X \to X$ of $\mu^M$ extends to a linearized action $\mu^{A^\vee} \colon \mathbf{T}^1 \times A_0^\vee \to A_0^\vee$. There is a natural diffeomorphism $X \times \mathbb{R}_+ \cong A_0^\vee$, given by $(x, r) \mapsto r \cdot x$; as a function on $A_0^\vee$, $r$ is simply the norm for the given Hermitian structure. If $\theta$ is a locally defined angular coordinate on $X$, depending on the choice of a local unitary frame of $A^\vee$, then $(r, \theta)$ restrict to polar coordinates along the fibers of $A_0^\vee$. Thus, if $J_{A^\vee}$ is the complex structure of $A^\vee$, then the globally defined vertical vector fields $\partial/\partial\theta$ and $\partial/\partial r$ on $A_0^\vee$ are related by $J_{A^\vee}(\partial/\partial\theta) = -r\,\partial/\partial r$. By (1), the infinitesimal generator of $\mu^{A^\vee}$ is

$$\xi_{A^\vee} = \xi_M^\sharp - \Phi\,\frac{\partial}{\partial\theta}, \tag{40}$$

where the horizontal lift is now taken in the tangent bundle of $A^\vee$, with respect to the extended connection.

The action $\mu^{A^\vee} \colon \mathbf{T}^1 \times A_0^\vee \to A_0^\vee$ again extends to a holomorphic action $\widetilde{\mu}^{A^\vee} \colon \mathbb{T}^1 \times A_0^\vee \to A_0^\vee$, which is of course a linearization of $\widetilde{\mu}^M$ (see the discussion in Section 5 of [16]). By holomorphicity, the induced vector fields $\xi_{A^\vee}$ and $\eta_{A^\vee}$, with $\xi_{A^\vee}$ given by (40), satisfy

$$\eta_{A^\vee} = J_{A^\vee}(\xi_{A^\vee}) = \eta_M^\sharp + \Phi\,r\,\frac{\partial}{\partial r}. \tag{41}$$

Let $\mathcal{N}_A \colon A_0^\vee \to \mathbb{R}$ be the square norm function; thus $\mathcal{N}_A = r^2$ under the previous diffeomorphism $A_0^\vee \cong X \times \mathbb{R}_+$. Then

$$\xi_{A^\vee}(\mathcal{N}_A) = 0, \qquad \eta_{A^\vee}(\mathcal{N}_A) = 2\,\Phi\,\mathcal{N}_A > 0. \tag{42}$$

**Lemma 26** *Let $a =: \min|\Phi|$, $A =: \max|\Phi|$. Then, for every $\lambda \in A_0^\vee$, we have*

$$e^{2at}\,\mathcal{N}_A(\lambda) \le \mathcal{N}_A\left(\widetilde{\mu}_{e^{-t}}^{A^\vee}(\lambda)\right) \le e^{2At}\,\mathcal{N}_A(\lambda)$$

*if $t \ge 0$, and*

$$e^{2At}\,\mathcal{N}_A(\lambda) \le \mathcal{N}_A\left(\widetilde{\mu}_{e^{-t}}^{A^\vee}(\lambda)\right) \le e^{2at}\,\mathcal{N}_A(\lambda)$$

*if $t < 0$.*

*Proof*  The invariant vector field $\eta = -\rho\,\partial/\partial\rho$ on $\mathbb{C}^*$ is associated with the 1-parameter subgroup $t \mapsto e^{-t}$. Therefore, if given $\lambda \in A_0^\vee$ we define

$$\mathcal{N}_A^\lambda(t) =: \mathcal{N}_A\left(\widetilde{\mu}^{A^\vee}\left(e^{-t},\lambda\right)\right) \qquad (t \in \mathbb{R})$$

then by (42)

$$\frac{d}{dt}\mathcal{N}_A^\lambda(t) = \eta_{A^\vee}(\mathcal{N}_A)\left(\widetilde{\mu}^{A^\vee}\left(e^{-t},\lambda\right)\right) = 2\,\Phi\left(\widetilde{\mu}_{e^{-t}}^{A^\vee}(\lambda)\right)\mathcal{N}_A^\lambda(t),$$

which can be rewritten

$$\frac{d}{dt}\ln(\mathcal{N}_A^\lambda)\bigg|_{t=t_0} = 2\,\Phi\left(\widetilde{\mu}_{e^{-t_0}}^{A^\vee}(\lambda)\right) \tag{43}$$

for any $t_0 \in \mathbb{R}$. We deduce from (43) that

$$2\,a \le \frac{d}{dt}\ln(\mathcal{N}_A^\lambda) \le 2\,A, \tag{44}$$

which easily implies the claim. $\qquad\qquad\square$

Let us set, for $(z,\lambda) \in \mathbb{C}^* \times A_0^\vee$:

$$z \bullet \lambda =: \widetilde{\mu}^{A^\vee}\left(z^{-1},\lambda\right) = \widetilde{\mu}_{z^{-1}}^{A^\vee}(\lambda). \tag{45}$$

**Corollary 27**  *If $|z| \ge 1$, then*

$$|z|^{2a}\,\mathcal{N}_A(\lambda) \le \mathcal{N}_A\,(z \bullet \lambda) \le |z|^{2A}\,\mathcal{N}_A(\lambda).$$

*If $0 < |z| < 1$, then*

$$|z|^{2A}\,\mathcal{N}_A(\lambda) \le \mathcal{N}_A\,(z \bullet \lambda) \le |z|^{2a}\,\mathcal{N}_A(\lambda).$$

*Proof*  If $z = e^{t+is}$, with $t, s \in \mathbb{R}$ and $|z| = e^t$, then because the action of $e^{is} \in \mathbf{T}^1$ is metric-preserving we have

$$\mathcal{N}_A\,(z \bullet \lambda) = \mathcal{N}_A\left(e^t \bullet \lambda\right) = \mathcal{N}_A\left(\widetilde{\mu}_{e^{-t}}^{A^\vee}(\lambda)\right).$$

Thus the corollary is just a restatement of Lemma 26. $\qquad\qquad\square$

**Corollary 28**  *The $C^\infty$ map $\Upsilon \colon \mathbb{R}_+ \times X' \to A_0^{\vee\prime}$ given by $(t,x) \mapsto t \bullet x$ is a bijection.*

*Proof*  By Corollary 27, $\mathcal{N}_A(t \bullet \lambda) \to +\infty$ as $t \to +\infty$, and $\mathcal{N}_A(t \bullet \lambda) \to 0^+$ as $t \to 0^+$; thus for any $\lambda \in A_0^{\vee\prime}$ there exists $t_\lambda \in \mathbb{R}_+$ such that $t_\lambda^{-1} \bullet \lambda \in X'$. Corollary 27 also implies that $\mathcal{N}_A\,(t \bullet x) > \mathcal{N}_A\,(x)$ for any $t > 1$ and $x \in X'$. Therefore, $t \mapsto \mathcal{N}_A\,(t \bullet \lambda)$ is a strictly increasing function, since if $t_1 < t_2$ then

$$\mathcal{N}_A(t_2 \bullet \lambda) = \mathcal{N}_A\left(\left(\frac{t_2}{t_1}\,t_1\right) \bullet \lambda\right)$$

$$= \mathcal{N}_A\left(\left(\frac{t_2}{t_1}\right) \bullet (t_1 \bullet \lambda)\right) > \mathcal{N}_A(t_1 \bullet \lambda). \tag{46}$$

Hence $t_\lambda$ is in fact unique. □

*Remark 29* More is true. Since $X' \subseteq A_0^\vee$ is a real-analytic submanifold (see Corollary 33), $\mathbb{R}_+ \times X'$ is a real-analytic submanifold of $\mathbb{C}^* \times A_0^\vee$. Being the restriction of the holomorphic map (45), $\Upsilon$ is then a real-analytic bijection of real-analytic manifolds. It is in fact also a local diffeomorphism, for its differential has everywhere maximal rank; by the real-analytic inverse function theorem (Theorem 2.5.1 of [18]), $\Upsilon$ is a real-analytic equivalence between $\mathbb{R}_+ \times X'$ and $A_0^{\vee\prime}$

Recall that a Lie group action on a manifold $P$ is called *proper* if the associated *action map* $G \times P \to P \times P$, $(g, p) \mapsto (g \cdot p, p)$, is proper (Definition B2 of [15]).

Let $A_0^{\vee\prime} \subseteq A_0^\vee$ be the inverse image of $M'$; in other words, in terms of the diffeomorphism $A_0^\vee \cong X \times \mathbb{R}_+$, we have $A_0^{\vee\prime} \cong X' \times \mathbb{R}_+$.

**Corollary 30** *The complexified action $\widetilde{\mu}^{A^\vee} : \mathbb{T}^1 \times A_0^\vee \to A_0^\vee$ is proper. In addition, its restriction to $A_0^{\vee\prime}$ is free.*

*Proof* Let $\Upsilon : \mathbb{T}^1 \times A_0^\vee \to A_0^\vee \times A_0^\vee$ be the action map of $\widetilde{\mu}^{A^\vee}$, and let $R \subset A_0^\vee \times A_0^\vee$ be a compact subset. If $\pi_j : A_0^\vee \times A_0^\vee \to A_0^\vee$ is the projection onto the $j$th factor, let $R_j =: \pi_j(R)$. Then $R_j$ is compact and $R \subset R_1 \times R_2$. Therefore, to prove that $\Upsilon$ is proper, it suffices to show that $\Upsilon^{-1}(R_1 \times R_2)$ is compact, for any pair of compact subsets $R_1, R_2 \subset A_0^\vee$. Clearly, $\Upsilon^{-1}(R_1 \times R_2) \subseteq \mathbb{T}^1 \times R_2$. For $j = 1, 2$ let $\ell_j =: \min_{R_j} \mathcal{N}_A$ and $L_j =: \max_{R_j} \mathcal{N}_A$.

Suppose $(w, \lambda) \in \Upsilon^{-1}(R_1 \times R_2)$, and set $z = w^{-1}$, so that $\widetilde{\mu}^{A^\vee}(w, \lambda) = z \bullet \lambda$. If $|w| \le 1$, then $|z| \ge 1$ and so by Corollary 27

$$|z|^{2a}\,\mathcal{N}_A(\lambda) \le \mathcal{N}_A(z \bullet \lambda) \le |z|^{2A}\,\mathcal{N}_A(\lambda).$$

Since $\lambda \in R_2$, we have $l_2 \le \mathcal{N}_A(\lambda)$, and since $z \bullet \lambda \in R_1$, we have $\mathcal{N}_A(z \bullet \lambda) \le L_1$. Therefore, if $|z| \ge 1$ then

$$\ell_2\,|z|^{2a} \le L_1 \quad \Longrightarrow \quad |z| \le \left(\frac{L_1}{l_2}\right)^{1/2a}.$$

In other words, if $|w| \le 1$ then $(\ell_2/L_1)^{1/2a} \le |w|$. Similarly, one sees that if $|w| \ge 1$ then $|w| \le (L_2/\ell_1)^{1/2a}$.

Therefore, if $p\colon \mathbf{T}^1 \times A_0^{\vee} \to A_0^{\vee}$ is the projection onto the first factor, then $S\left(\Upsilon^{-1}(R_1 \times R_2)\right)$ is compact. Hence, $\Upsilon^{-1}(R_1 \times R_2) \subseteq S \times R_2$ is also compact, and this completes the proof that the action is proper.

The statement about the freeness of the action follows immediately from (44) and the definition of $X'$. $\qquad\square$

## 2.5 The circle bundle structures

Let us view $\kappa\colon X' \to N'$ as a circle bundle over $N'$, with the action of $\mathbf{T}^1$ on $X'$ given by $\left(e^{i\theta}, x\right) \mapsto e^{i\theta} \bullet x$; the latter is defined in (45). On $N'$, associated with the Kähler structure we have the volume form $dV_N = \eta^{\wedge d}/d!$; thus on $X'$, viewed as a circle bundle over $N'$, we have the natural choice of a volume form $dW_X = (1/2\pi)\beta \wedge \kappa^*(dV_N)$. Algebraically, $L^2(X, dV_X) = L^2(X', dV_X) = L^2(X', dW_X)$, although the metrics are different. Explicitly,

$$
\begin{aligned}
dW_X &= \frac{1}{2\pi}\,\beta \wedge \kappa^*(dV_N) \\
&= \frac{1}{2\pi}\,\frac{\alpha}{\Phi} \wedge \frac{1}{d!}\left(\frac{1}{\Phi}\,\pi^*(\omega) - \frac{1}{2\Phi^2}\,d\Phi \wedge \alpha\right)^{\wedge d} \\
&= \Phi^{-(d+1)}\left[\frac{1}{2\pi}\,\alpha \wedge \pi^*(\omega)^{\wedge d}\right] \\
&= \Phi^{-(d+1)}\,dV_X.
\end{aligned}
\tag{47}
$$

Furthermore, the two circle bundles $\pi$ and $\kappa$ have different CR structures, because they do not have the same vertical tangent bundle. However, by construction they share the same horizontal distribution, and the same horizontal complex structure $J_{\mathcal{H}}$. Let $\mathcal{H}^{(0,1)} \subseteq \mathcal{H} \otimes \mathbb{C}$ be the $-i$-eigenbundle of $J_{\mathcal{H}}$; then the boundary CR operator of either $X$ or $X'$ is defined by setting $\bar{\partial}_b f = df|_{\mathcal{H}^{(0,1)}}$, for any $C^\infty$ function $f$ on $X$ or $X'$, respectively. Therefore, the boundary CR operator of $X$,

$$
\bar{\partial}_b\colon C^\infty(X) \to C^\infty\left(X, \mathcal{H}^{(0,1)\,\vee}\right),
$$

restricts to the corresponding operator of $X'$. It follows that there is a natural algebraic (non-isometric) inclusion of corresponding Hardy spaces, $H(X) \hookrightarrow H(X')$. The latter is an algebraic isomorphism if $\mathrm{codim}_{\mathbb{C}}(M \setminus M', M) \geq 2$.

The action $\mu^X$ plays the role of the structure circle action of $\mathbf{T}^1$ with respect to $\kappa$. Let $\widetilde{H}_k(X')$ be the $k$th isotype for the latter action. Condition (3) for $s \in L^2(X)$ to belong to $H_k^\mu(X)$ may be rewritten $s\left(e^{i\theta} \bullet x\right) = e^{ik\theta}\,s(x)$, for any $e^{i\theta} \in \mathbf{T}^1$ and $x \in X$. Therefore, the previous inclusion of Hardy spaces yields for every $k = 0, 1, 2, \ldots$ an algebraic inclusion $H_k^\mu(X) \hookrightarrow \widetilde{H}_k(X')$.

## 2.6 The line bundle on $N'$

Let $B$ be the complex line bundle on $N'$ associated with $\kappa$ and the tautological action of $\mathbf{T}^1 = U(1)$ on $\mathbb{C}$, and let $B^\vee$ be its dual; thus, $B$ (resp. $B^\vee$) is the quotient of $X' \times \mathbb{C}$ by the equivalence relation $(x, w) \sim \left(e^{i\theta} \bullet x, e^{i\theta} w\right)$ (resp. $(x, w) \sim \left(e^{i\theta} \bullet x, e^{-i\theta} w\right)$). We can embed $\jmath \colon X' \hookrightarrow B^\vee$ by $x \mapsto [x, 1]$. Then $B$ and $B^\vee$ inherit natural Hermitian structures, that we shall denote by $\ell_B$, uniquely determined by imposing $\jmath \colon X' \hookrightarrow B^\vee$ to embed as the unit circle bundle. We shall denote by $\widehat{\kappa} \colon B^\vee \to N'$ the projection, so that $\kappa = \widehat{\kappa} \circ \jmath$.

The connection form $\beta$ on $X$ determines a unique metric covariant derivative $\nabla_B$ on $B$, with curvature $\Theta_B = -2i\,\kappa^*(\eta)$. Since $\eta$ is a Kähler form on $N'$, there is a uniquely determined holomorphic structure on $B$, such that $\nabla_B$ is the only covariant derivative on $B$ compatible with both the metric and the latter holomorphic structure. A local section $\sigma$ of $B$ is holomorphic for this structure if and only if the connection matrix with respect to $\sigma$ is of type $(1,0)$.

**Lemma 31** *There is a natural biholomorphism $\overline{\Gamma} \colon B_0^\vee \cong A_0^{\vee\prime}$ of bundles over $N'$; when we view $X'$ as a submanifold of $A_0^{\vee\prime}$ and $B_0^\vee$ in the natural manner, $\overline{\Gamma}$ restricts to the identity $X' \to X'$ (that is, $\overline{\Gamma}(\jmath(x)) = x$ for any $x \in X'$). Furthermore, $\overline{\Gamma}$ preserves the horizontal distributions, and maps biholomorphically the fibers of the bundle projection $\widehat{\kappa} \colon B^\vee \to N'$ onto the orbits of the action $\widetilde{\mu}^{A^\vee} \colon \mathbb{T}^1 \times A_0^\vee \to A_0^\vee$.*

*Proof* We have $B_0^\vee = X' \times \mathbb{C}^* / \sim$, where $(x, w) \sim \left(e^{i\theta} \bullet x, e^{-i\theta} w\right)$, for any $e^{i\theta} \in \mathbf{T}^1$. If $\Psi \colon X' \times \mathbb{C}^* \to B_0^\vee$ is the quotient map, $\Psi(x, w) = [x, w]$, for any $(x, w) \in X' \times \mathbb{C}^*$ the differential $d_{(x,w)}\Psi$ induces a $\mathbb{C}$-linear isomorphism

$$\mathcal{H}_x \oplus \mathbb{C} \subset T_{(x,w)}\left(X' \times \mathbb{C}^*\right) \cong T_{\Psi(x,w)} B_0^\vee,$$

which maps $\mathcal{H}_x \oplus (\mathbf{0})$ and $(\mathbf{0}) \oplus \mathbb{C}$, respectively, onto the horizontal and vertical tangent spaces of $B^\vee$ at $\Psi(x, w)$.

Let us consider the map

$$\Gamma \colon X' \times \mathbb{C}^* \to A_0^{\vee\prime}, \qquad (x, w) \mapsto w \bullet x.$$

Holomorphicity of the complexified action $\widetilde{\mu}$ implies that the differential $d_{(x,w)}\Gamma$ induces a $\mathbb{C}$-linear isomorphism

$$\mathcal{H}_x \oplus \mathbb{C} \subset T_{(x,w)}\left(X' \times \mathbb{C}^*\right) \cong T_{\Gamma(x,w)} A_0^\vee, \tag{48}$$

under which $\mathcal{H}_x \oplus \mathbb{C}$ maps onto the horizontal tangent space of $A^\vee$ and $(\mathbf{0}) \oplus \mathbb{C}$ onto the tangent space to the complex orbit $\widetilde{\mu}^{A^\vee}$ at $\Gamma(x, w)$.

On the other hand, for any $(x, w) \in X' \times \mathbb{C}^*$ and $e^{i\theta} \in \mathbf{T}^1$, we have

$$\Gamma(x, w) = w \bullet x = \left(w\, e^{-i\theta}\right) \bullet \left(e^{i\theta} x\right) = \Gamma\left(e^{i\theta} x, e^{-i\theta} w\right). \tag{49}$$

Therefore, $\Gamma$ passes to the quotient under $\Psi$, that is, there exists a $C^\infty$ map $\overline{\Gamma}: B_0^\vee \to A_0^{\vee'}$ such that $\Gamma = \overline{\Gamma} \circ \Psi$:

$$\overline{\Gamma}([x,w]) = \Gamma(x,w) \qquad ((x,w) \in X' \times \mathbb{C}^*);$$

the previous discussion implies that $\overline{\Gamma}$ is holomorphic.

Corollary 28 evidently implies that $\Gamma$ is surjective, and therefore so is $\overline{\Gamma}$. To see that $\overline{\Gamma}$ is also injective, suppose that $\lambda_j = \Psi(x_j, w_j)$, $j = 1, 2$, satisfy $\overline{\Gamma}(\lambda_1) = \overline{\Gamma}(\lambda_2)$. Thus $w_1 \bullet x_1 = \Gamma(x_1, w_1) = \Gamma_2(x_2, w_2) = w_2 \bullet x_2$, whence $\left(w_2^{-1} w_1\right) \bullet x_1 = x_2$. This evidently implies $\mathcal{N}_A\left(\left(w_2^{-1} w_1\right) \bullet x_1\right) = \mathcal{N}_A(x_1) = \mathcal{N}_A(x_2) = 1$. Since by Corollary 27 the map $t \mapsto \mathcal{N}_A(t \bullet x_1)$ is strictly increasing, this forces $|w_1| = |w_2|$. If $w_2^{-1} w_1 = e^{i\theta}$, we then have $w_2 = e^{-i\theta} w_1$, $x_2 = e^{i\theta} \bullet x_1$; hence $(x_2, w_2) = \left(e^{i\theta} \bullet x_1, e^{-i\theta} \bullet w_1\right) \sim (x_1, w_1)$. Therefore, $\lambda_1 = \lambda_2$.

Finally, any $x \in X' \subset A_0^{\vee'}$ corresponds to $[x, 1] \in B_0^\vee$, and $\overline{\Gamma}([x, 1]) = 1 \bullet x = x$. Therefore, with the previous identification $\overline{\Gamma}$ induces the identity map on $X'$. $\square$

*Remark 32* $\overline{\Gamma}$ interwines fiberwise scalar multiplication $\cdot_B$ on $B_0^\vee$ and the map (45). In fact, if $b = [x, w] \in B_0^\vee$ and $z \in \mathbb{C}^*$, then $z \cdot_B b = [x, zw]$. Therefore,

$$\overline{\Gamma}(z \cdot_B b) = \overline{\Gamma}([x, zw]) = (zw) \bullet x = z \bullet (w \bullet x) = z \bullet \overline{\Gamma}(x).$$

Let $\mathcal{N}_B: A_0^\vee \to \mathbb{R}$ be the norm function associated with the Hermitian structure of $B^\vee$, viewed as a function on $A_0^\vee$ by means of the biholomorphism of Lemma 31. Then $\mathcal{N}_B(z \bullet \lambda) = |z|^2 \|\lambda\|^2$.

**Corollary 33** *$X' \subseteq A_0^\vee$ is a real-analytic submanifold, and the projection $\kappa: X' \to N'$ is real-analytic.*

*Proof* Since the Hermitian metric $h$ on $A^\vee$ is real-analytic by assumption, the norm function $\mathcal{N}_A: A_0^\vee \to \mathbb{R}$ is a positive real-analytic function. Therefore, $X' = \mathcal{N}_A^{-1}(1) \cap A_0^{\vee'}$ is a real-analytic submanifold of $A_0^{\vee'}$ (see Section 2.7 of [18]). On the contrary, we have

$$\kappa = \widehat{\kappa} \circ \jmath = \widehat{\kappa} \circ \overline{\Gamma}^{-1}\Big|_X.$$

Thus $\kappa$ is the restriction of a holomorphic map to a real-analytic submanifold, hence it is real-analytic. $\square$

**Proposition 34** *The Kähler form $\eta$ on $N'$ is real-analytic.*

*Proof* It suffices to prove that for any $n \in N'$ there is a real-analytic chart for $N'$, defined on an open neighborhood $V \subseteq N'$ of $n$, such that the local expression of $\eta$ in that chart is real-analytic. To this end, choose $x \in \kappa^{-1}(n)$ and let $m =: \pi(x) \in M'$. On some open neighborhood $U \subseteq M$ of $m$, we can find

a local holomorphic frame $\varphi$ on $A^\vee$ such that $\varphi(m) = x$. We can also suppose, without loss of generality, that $\varphi$ is horizontal at $m$, that it, its differential at $m$ maps isomorphically $T_m M$ to the horizontal tangent space $\mathcal{H}_x$. If we assume, as we may, that $U$ is the domain of a holomorphic local coordinate chart $(z_j)$ centered at $m$, this means that $\mathcal{N}_A \circ \varphi = h(\varphi, \varphi) = 1 + O\left(\|z\|^2\right)$.

Let us write $\|\varphi\| =: \sqrt{h(\varphi, \varphi)}$, so that $\zeta =: \varphi/\|\varphi\| : U \to X'$ is a local unitary frame. Since $h$ is real-analytic, $\zeta$ is real-analytic. The previous remark shows, in addition, that

$$d_m \zeta = d_m \varphi : T_m M \longrightarrow T_x X \subseteq T_x A^\vee;$$

therefore $\zeta$ is also horizontal at $m$, whence it is transverse at $m$ to the $\mu^X$-orbit through $x$, in view of (1) and the positivity of $\Phi$. Since the latter orbit is the fiber through $x$ of the projection $\kappa : X' \to N'$, this implies that the composition $\kappa \circ \zeta : U \to N'$ is a real-analytic local diffeomorphism at $m$; we have $\kappa \circ \zeta(m) = \kappa(x) = n$. Therefore, perhaps after replacing $U$ with a smaller open neighborhood of $m$, we may assume that $\kappa \circ \zeta$ induces a real-analytic equivalence $U \cong V$, where $V =: \kappa \circ \zeta(U)$ is an open neighborhood of $n$ (see Theorem 2.5.1 of [18]). Given the holomorphic chart on $U$, we may then interpret $\kappa \circ \zeta$ as a real-analytic chart for $N'$ in the neighborhood of $n$.

Let $\theta_\zeta^\vee = i \zeta^*(\alpha)$ be the connection form of $A^\vee$ in the local frame $\zeta$. Then under our assumptions, $\theta_\zeta^\vee$ is a real-analytic imaginary 1-form. The local expression of $2\eta$ in this chart, by (28), is

$$\begin{aligned}
(\kappa \circ \zeta)^*(2\eta) &= \zeta^* \left( \kappa^*(2\eta) \right) \\
&= \zeta^* \left( \frac{2}{\Phi} \pi^*(\omega) - \frac{1}{\Phi^2} d\Phi \wedge \alpha \right) \\
&= \frac{2}{\Phi} \omega + \frac{i}{\Phi^2} d\Phi \wedge \theta_\zeta^\vee.
\end{aligned} \tag{50}$$

In view of Lemma 12, we conclude that (50) is real-analytic, and this completes the proof.  $\square$

This also follows from

**Lemma 35**  *The Hermitian metric $h$ on $B^\vee$ is real-analytic.*

*Proof*  It suffices to show that the norm function $\mathcal{N}_B : B_0^\vee \to \mathbb{R}_+$ is real-analytic. To this end, it is equivalent to show that the composition $\mathcal{N}_B \circ \overline{\Gamma}^{-1} : A_0^\vee \to \mathbb{R}_+$ is real-analytic. Again, let us simplify our discussion by biholomorphically identifying $A_0^\vee$ with $B_0^\vee$, and leaving $\overline{\Gamma}^{-1}$ implicit. Then fiberwise scalar multiplication on $B_0^\vee$ corresponds to the map (45). Thus if $(t, x) \in \mathbb{R}_+ \times X'$, then $\mathcal{N}_B(t \bullet x) = t^2$.

Let $\Upsilon \colon \mathbb{R}_+ \times X' \to A_0^{\vee\,\prime}$ be as in Corollary 28; then $\Upsilon$ is a real-analytic equivalence by Remark 29, and the previous remark implies that

$$\mathcal{N}_B \circ \Upsilon \colon \mathbb{R}_+ \times X' \to \mathbb{R}_+$$

is real-analytic. Therefore, so is $\mathcal{N}_B = (\mathcal{N}_B \circ \Upsilon) \circ \Upsilon^{-1}$.                    $\square$

We can consider the *equivariant distortion function* $K_k^\mu \colon M \to \mathbb{R}$ defined by setting

$$K_k^\mu(m) =: \Pi_k^\mu(x, x) = \sum_j \left| s_j^{(k)}(x) \right|^2, \tag{51}$$

for $m \in M$ and any choice of $x \in \pi^{-1}(m) \subseteq X$, where the $s_j^{(k)}$s are an orthonormal basis of $H_k^\mu(X)$ (see (4)). That $K_k^\mu$ is well-defined follows from the fact that $\mu^X$ and $\nu^X$ commute (Lemma 2.1 of [28]). For any $m \in M$ and $t \in \mathbf{T}^1$, given $x \in \pi^{-1}(m)$ by (3) we have

$$K_k^\mu\left(\mu_{t^{-1}}^M(m)\right) = \Pi_k^\mu\left(\mu_{t^{-1}}^X(x), \mu_{t^{-1}}^X(x)\right) = \sum_j \left| s_j^{(k)}\left(\mu_{t^{-1}}^X(x)\right) \right|^2$$

$$= \sum_j \left| s_j^{(k)}(x) \right|^2 = \Pi_k^\mu(x, x) = K_k^\mu(m).$$

Therefore, $K_k^\mu \in C^\infty(M)^\mu$; it may thus be regarded as a function on $N'$ in a natural manner. We have in fact:

**Lemma 36**    $K_k^\mu \in C^\varpi(M)^\mu$. *As a function on $N'$, $K_k^\mu \in C^\varpi(N')^\nu$.*

*Proof*   By its very definition, $\Pi_k^\mu \in C^\infty(X \times X)$ restricts to a sesquiholomorphic complex function on $A_0^\vee \times A_0^\vee$, which is then *a fortiori* real-analytic. Since $X \times X$ is a real-analytic submanifold of $A_0^\vee \times A_0^\vee$ by Corollary 33, we have $\Pi_k^\mu \in C^\varpi(X \times X)$. If now $\varphi$ is a local holomorphic frame on an open subset $U \subset M$, the unitarization $\varphi_u = \varphi/\|\varphi\|_A \colon U \to X$ is real-analytic, where $\|\varphi\|_A =: (\mathcal{N}_A \circ \varphi)^{1/2}$. Therefore,

$$K_k^\mu(m) = \Pi_k^\mu(\varphi_u(m), \varphi_u(m)) \qquad (m \in U)$$

is real-analytic on $U$. The second statement is proved similarly (Lemma 35).
                    $\square$

## 2.7  Asymptotics of sesquiholomorphic extensions

Every $s \in H_k^\mu(X)$ extends uniquely to a holomorphic function $\widetilde{s} \colon A_0^\vee \to \mathbb{C}$. Holomorphicity of the extended action $\widetilde{\mu}^{A^\vee}$ implies, in view of (3) and (45), that for every $(z, \lambda) \in \mathbb{C}^* \times A_0^\vee$ we have

$$\widetilde{s}(z \bullet \lambda) = z^k \, \widetilde{s}(\lambda). \tag{52}$$

Given this and (4), we see that $\Pi_k^\mu \colon X \times X \to \mathbb{C}$ extends uniquely to a sesquiholomorphic function $\mathcal{P}_k^\mu \colon A_0^\vee \times A_0^\vee \to \mathbb{C}$, given by

$$\mathcal{P}_k^\mu(\lambda, \lambda') = \sum_j \widetilde{s}_j^{(k)}(\lambda) \cdot \overline{\widetilde{s}_j^{(k)}(\lambda')} \qquad \left(\lambda, \lambda' \in A_0^\vee\right), \tag{53}$$

and satisfying, by (52),

$$\mathcal{P}_k^\mu(z \bullet \lambda, w \bullet \lambda') = z^k \, \overline{w}^k \, \mathcal{P}_k^\mu(\lambda, \lambda'), \tag{54}$$

for every $z, w \in \mathbb{C}^*$.

Let $\sigma$ be a local holomorphic frame of $B^\vee$ on an open subset $V \subseteq N'$. Then

$$\mathcal{P}_k^\mu \circ (\sigma \times \sigma) \colon V \times V \to \mathbb{C}, \quad (n, n') \mapsto \mathcal{P}_k^\mu(\sigma(n), \sigma(n'))$$

is sesquiholomorphic. The unitarization $\sigma_u =: (1/\|\sigma\|_B) \bullet \sigma \colon V \to X'$ (see Remark 32) is a real-analytic section. Given (54), we have

$$\begin{aligned}
\Pi_k^\mu(\sigma_u(n), \sigma_u(n')) &= \mathcal{P}_k^\mu \left( \frac{1}{\|\sigma(n)\|_B} \bullet \sigma(n), \frac{1}{\|\sigma(n')\|_B} \bullet \sigma(n') \right) \\
&= \frac{1}{\|\sigma(n)\|_B^k} \frac{1}{\|\sigma(n')\|_B^k} \mathcal{P}_k^\mu(\sigma(n), \sigma(n')) \\
&= e^{-\frac{k}{2}\left(\Xi(n) + \Xi(n')\right)} \mathcal{P}_k^\mu(\sigma(n), \sigma(n')),
\end{aligned} \tag{55}$$

where we have set, for $n \in V$,

$$\Xi(n) = \ln\left(\|\sigma(n)\|_B^2\right) = \ln\left(\ell_B(\sigma(n), \sigma(n))\right). \tag{56}$$

Then $\Xi$ is real-analytic by Lemma 35, and furthermore $\partial_N \overline{\partial}_N \Xi = \Theta_B$, where $\Theta_B = -2i\,\eta \in \Omega^2(N')$ is the curvature form of $B$. In any given local coordinate chart $(z_k)$ for $N'$ this means that

$$\frac{\partial^2 \Xi}{\partial z_k \, \partial z_{\bar{l}}} = \Theta_{B k\bar{l}} = -2i\,\eta_{k\bar{l}} = 2\,h_{k\bar{l}},$$

where $h$ is the Riemannian metric of $(N', I, \eta)$. In other words, $\Xi$ is a Kähler potential for $2\,h$.

Being real-analytic, $\Xi$ has a unique sesquiholomorphic extension $\widetilde{\Xi}$ to an open neighborhood of the diagonal $\widetilde{V} \subseteq V \times V$. Similarly, by Lemma 36, $K_k^\mu$ also has a unique sesquiholomorphic extension $\widetilde{K_k^\mu}$ to an open neighborhood of the diagonal in $N' \times N'$.

**Lemma 37** *Let $\widetilde{V} \subseteq V \times V$ be an appropriate open neighborhood of the diagonal. Then for every $(n, n') \in \widetilde{V}$, we have*

$$\mathcal{P}_k^\mu(\sigma(n), \sigma(n')) = e^{k\,\overline{\Xi}(n,n')}\, \widetilde{K}_k^\mu(n, n'). \tag{57}$$

*Proof* Both sides being sesquiholomorphic, it suffices to show that they have equal restrictions on the diagonal. If $n = n'$, by (55) we have

$$\mathcal{P}_k^\mu(\sigma(n), \sigma(n)) = e^{k\,\Xi(n)}\,\Pi_k^\mu(\sigma_u(n), \sigma_u(n)) \tag{58}$$
$$= e^{k\,\overline{\Xi}(n,n)}\, K_k^\mu(n) = e^{k\,\overline{\Xi}(n,n)}\, \widetilde{K}_k^\mu(n, n).$$

$\square$

Inserting (57) in (55), we obtain for $(n, n') \in \widetilde{V}$:

$$\Pi_k^\mu(\sigma_u(n), \sigma_u(n')) = e^{k\left[\overline{\Xi}(n,n') - \frac{1}{2}\Xi(n) - \frac{1}{2}\Xi(n')\right]}\, \widetilde{K}_k^\mu(n, n'). \tag{59}$$

As discussed in the Introduction, by [28] if $m \in M'$ and $x \in \pi^{-1}(m)$ there is an asymptotic expansion (5), smoothly varying on $M'$ and uniform on compact subsets of $M'$, with leading coefficient $S_0^\mu = \Phi^{-(d+1)}$. Since $\Pi_k^\mu(x, x)$ is $\mu^M$-invariant, so is every $S_j^\mu$. Therefore, viewing $K_k^\mu$ as defined on $N'$, the expansion may naturally be interpreted as holding on $N'$ (see (51) and Lemma 36)):

$$K_k^\mu(n) \sim \left(\frac{k}{\pi}\right)^d \sum_{j \geq 0} k^{-j}\, S_j^\mu(n), \tag{60}$$

where $S_0^\mu = \Phi^{-(d+1)}$. This suggest, heuristically, that $\widetilde{K}_k^\mu(n, n')$ should satisfy a similar expansion, with coefficients the sesquiholomorphic extensions of the $S_j^\mu$s. This is indeed the case.

To see this, let us consider first the asymptotics of $\Pi_k^\mu(\sigma_u(n), \sigma_u(n'))$ for $(n, n') \in \widetilde{V}$. Let

$$\Pi(x, y) = \int_0^{+\infty} e^{it\,\psi(x,y)}\, s(x, y, t)\, dt \tag{61}$$

be the usual Fourier integral representation of the Szegö kernel of $X$ determined in [6]; here we think of $X$ as the unit circle bundle of $A^\vee$, with volume form $dV_X$. In particular, $\Im(\psi) \geq 0$, and $s$ is a semiclassical symbol admitting an asymptotic expansion of the form

$$s(x, y, t) \sim \sum_{j \geq 0} t^{d-j}\, s_j(x, y) \tag{62}$$

(see also the discussion in [30] and [35]). For some $\epsilon > 0$, let $\varrho_1 \in C_0^\infty(-2\epsilon, 2\epsilon)$ be a bump function identically equal to 1 on $(-\epsilon, \epsilon)$. For some $C > 0$, let

$\varrho_2 \in C_0^\infty(1/(2C), 2C)$ be a bump function identically equal to 1 on $(1/C, C)$. Let us write $\mu_{-\vartheta}^X$ for $\mu_{e^{-i\vartheta}}^X$. Then, arguing as in the proof of Theorem 1 of [28],

$$
\begin{aligned}
\Pi_k^\mu(\sigma_u(n), \sigma_u(n')) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\vartheta} \, \Pi\left(\mu_{-\vartheta}^X(\sigma_u(n)), \sigma_u(n')\right) d\vartheta \qquad (63) \\
&\sim \frac{1}{2\pi} \int_0^{+\infty} \int_{-\pi}^{\pi} e^{-ik\vartheta + it\,\psi\left(\mu_{-\vartheta}^X(\sigma_u(n)), \sigma_u(n')\right)} s\left(\mu_{-\vartheta}^X(\sigma_u(n)), \sigma_u(n'), t\right) \varrho_1(\vartheta) \, dt \, d\vartheta \\
&\sim \frac{k}{2\pi} \int_0^{+\infty} \int_{-\pi}^{\pi} e^{ik\,\Psi(n,n',t,\vartheta)} s\left(\mu_{-\vartheta}^X(\sigma_u(n)), \sigma_u(n'), kt\right) \varrho_1(\vartheta) \, \varrho_2(t) \, dt \, d\vartheta,
\end{aligned}
$$

where

$$
\Psi(n, n', t, \vartheta) =: t\,\psi\left(\mu_{-\vartheta}^X(\sigma_u(n)), \sigma_u(n')\right) - \vartheta.
$$

The last line of (63) is an oscillatory integral with phase $\Psi(n, n', t, \vartheta)$, and $\mathfrak{I}(\psi) \geq 0$ implies $\mathfrak{I}(\Psi) \geq 0$.

Suppose first $n = n'$. Then one can see by (a slight adaptation of) the argument in the proof of Theorem 1 of [28] that the phase $\Psi(n, n, t, \vartheta)$ has a unique stationary point $P(n, n) = (t_0, \vartheta_0) = (1/\Phi(n), 0)$, where as usual we think of the invariant function $\Phi$ as descended on $N$. Since $\psi(x, x) = 0$ identically, we have $\Psi(n, n, t_0, \vartheta_0) = 0$. Furthermore, the Hessian matrix at $P_0$ is

$$
H_{P_0}(\Psi) = \begin{pmatrix} 0 & \Phi(n) \\ \Phi(n) & \partial_{\vartheta\vartheta}^2 \Psi(P_0) \end{pmatrix}.
$$

Therefore, $P_0$ is a non-degenerate critical point, and by applying the stationary phase lemma to it we obtain the asymptotic expansion (60).

By the theory of [27], the stationary point and the asymptotic expansion will deform smoothly with $(n, n') \in \widetilde{V}$, although the stationary point may cease to be real when $n \neq n'$ (and should then be regarded as the stationary point of an almost analytic extension of $\Psi$). More precisely, if $\widetilde{\Psi}\left(\widetilde{n}, \widetilde{n}', \widetilde{t}, \vartheta\right)$ denotes an almost analytic extension of $\Psi(n, n', t, \vartheta)$, then the condition that $P(\widetilde{n}, \widetilde{n}') = \left(\widetilde{t}(\widetilde{n}, \widetilde{n}'), \widetilde{\vartheta}(\widetilde{n}, \widetilde{n}')\right)$ be a stationary point of $\widetilde{\Psi}(\widetilde{n}, \widetilde{n}', \cdot, \cdot)$ defines an almost analytic manifold $\left(\widetilde{t}, \widetilde{\vartheta}\right) = \left(\widetilde{t}(\widetilde{n}, \widetilde{n}'), \widetilde{\vartheta}(\widetilde{n}, \widetilde{n}')\right)$.

Applying to (63) the stationary phase lemma for complex phase functions from Section 2 of [27] for $(n, n') \in \widetilde{V}$ we obtain a smoothly varying asymptotic expansion

$$
\Pi_k^\mu(\sigma_u(n), \sigma_u(n')) \sim \left(\frac{k}{\pi}\right)^d e^{ik\,\widetilde{\Psi}(n,n',P(n,n'))} \sum_{j \geq 0} k^{-j} \, S_j(n, n'), \qquad (64)
$$

for appropriate smooth functions $S_j(\cdot, \cdot)$ on $\widetilde{V} \subseteq V \times V$.

Given (55) and (64), we get

$$\mathcal{P}_k^\mu\left(\sigma(n),\sigma(n')\right) \tag{65}$$

$$\sim e^{k\left[\frac{1}{2}\left(\Xi(n)+\Xi(n')\right)+i\,\widetilde{\Psi}\left(n,n',P(n,n')\right)\right]}\left(\frac{k}{\pi}\right)^d\sum_{j\geq 0}k^{-j}\,S_j(n,n').$$

Since the expansion holds in $C^j$-norm for every $j$ and the LHS is sesqui-holomorphic, so is every term on the RHS. Therefore, each term

$$e^{k\left[\frac{1}{2}\left(\Xi(n)+\Xi(n')\right)+i\,\widetilde{\Psi}\left(n,n',P(n,n')\right)\right]}\,S_j(n,n')$$

is the sesquiholomorphic extension of its diagonal restriction.

On the contrary, on the diagonal (65) restricts to the uniquely determined asymptotic expansion for (58), and so we need to have $S_j(n,n) = S_j^\mu(n)$, whence $S_j(n,n') = \widetilde{S_j^\mu}(n,n')$. Furthermore, we see that

$$\frac{1}{2}\left(\Xi(n) + \Xi(n')\right) + i\,\widetilde{\Psi}(n,n',P(n,n')) = \widetilde{\Xi}(n,n').$$

Inserting this in (64), we obtain

$$\Pi_k^\mu(\sigma_u(n),\sigma_u(n')) \sim \left(\frac{k}{\pi}\right)^d e^{k\left[\widetilde{\Xi}(n,n')-\frac{1}{2}\left(\Xi(n)+\Xi(n')\right)\right]}$$

$$\cdot\sum_{j\geq 0}k^{-j}\,\widetilde{S_j^\mu}(n,n'). \tag{66}$$

Now (59) and (55) imply

$$\widetilde{K_k^\mu}(n,n') \sim \left(\frac{k}{\pi}\right)^d\sum_{j\geq 0}k^{-j}\,\widetilde{S_j^\mu}(n,n') \tag{67}$$

(see [17] and [35] for analogues in the standard case $\Phi = 1$).

Analogous considerations hold for Toeplitz operators; see Section 9.

## 2.8  The Laplacian on invariant functions

Let us now dwell on the relation between the Laplacian operators $\Delta_N$ and $\Delta_M$ of $(M,J,\omega)$ and $(N',I,\eta)$ acting on invariant functions. Thus let $f \in C^\infty(M)^\mu$, so that $f$ determines in a natural manner functions on $X$ and $N'$, respectively. It is convenient in the present argument to explicitly distinguish the domain of definition of the function in point, so we shall write $f = f_M$, and $f_X$ and $f_N$ to denote the induced functions on $X$ and $N'$, respectively. It is also notationally convenient to leave $\overline{\Gamma}$ implicit, and to identify $B_0^\vee$ with $A_0^{\vee\prime}$ (see Lemma 31). Thus we have holomorphic line bundle structures $\widehat{\pi}\colon A^\vee \to M$

and $\widehat{\kappa} \colon A^{\vee} \to N'$, where we write $\widehat{\kappa}$ for $\widehat{\kappa} \circ \overline{\Gamma}^{-1}$. The fibers of the latter are the orbits of the complexified action $\widetilde{\mu}^{A^{\vee}}$.

Suppose $m \in M'$, $x \in \pi^{-1}(m) \in X \subset A_0^{\vee}$, and set $n =: \kappa(x)$. Choose a local holomorphic frame $\varphi$ for $A^{\vee}$ on an open neighborhood $U \subset M'$ of $m$, such that $\varphi(m) = x$ and which is horizontal at $m$, in the sense of the proof of Proposition 34. Then, as remarked in the same proof, $\varphi \colon U \to A^{\vee}$ is transverse at $m$ to the orbit of $\mu^X$ through $x$. In fact, in view of (41), $\varphi \colon U \to A^{\vee}$ is transverse at $m$ to the full orbit of $\widetilde{\mu}^{A^{\vee}}$ through $x$. Thus, the composition $\widehat{\kappa} \circ \varphi \colon U \to N'$ is holomorphic, has maximal rank at $m$, and satisfies $\widehat{\kappa} \circ \varphi(m) = n$. Therefore, there exists an open neighborhood $U \subseteq M'$ of $m$ such that $V =: \widehat{\kappa} \circ \varphi(U)$ is open, and the induced map $\widehat{\kappa} \circ \varphi \colon U \to N'$ is a biholomorphism.

Let us set $Z =: \varphi(U) \subseteq A_0^{\vee}$. Then $Z$ is a complex submanifold of $A_0^{\vee}$, and the restrictions of $\widehat{\pi}$ and $\widehat{\kappa}$ to $Z$ determine biholomorphic maps $\pi_Z \colon Z \to U$ and $\kappa_Z \colon Z \to V$. The invariance hypothesis on $f$ implies that $f_M \circ \pi_Z = f_N \circ \kappa_Z$; let us write $f_Z$ for this function.

Furthermore, if $K$ is the complex structure on $Z$ then by holomorphicity we can pull back the Kähler structures $(M, J, \omega)$ and $(N, I, \eta)$ under $\pi_Z$ and $\kappa_Z$, respectively, to Kähler structures $(Z, K, \omega')$ and $(Z, K, \eta')$. Clearly

$$\Delta_M(f_M) \circ \pi_Z = \Delta_1(f_Z), \qquad \Delta_N(f_N) \circ \kappa_Z = \Delta_2(f_Z),$$

where $\Delta_1$ and $\Delta_2$ are the Laplacian operators in the Kähler structures $(Z, K, 2\omega')$ and $(Z, K, 2\eta')$, respectively. Therefore,

$$\Delta_M(f_M)(m) = \Delta_1(f_Z)(x), \qquad \Delta_N(f_N)(n) = \Delta_2(f_Z)(x). \tag{68}$$

Recall that $g(\cdot, \cdot) = \omega(\cdot, J(\cdot))$ and $h = \eta(\cdot, I(\cdot))$ are the Riemannian metrics on $(M, J, \omega)$ and $(N, I, \eta)$, and by pull-back we view them as the Riemannian metrics of $(Z, K, \omega')$ and $(Z, K, \eta')$, respectively. Perhaps after restricting $U$ to a smaller open neighborhood of $m$ in $M'$, we may assume without loss that on $Z$ there is a global holomorphic coordinate chart $(z_j)$. Let $g_{a\overline{b}} = g(\partial_a, \partial_{\overline{b}})$ and $h_{a\overline{b}} = h(\partial_a, \partial_{\overline{b}})$ be the respective covariant metric tensors, with associated contravariant tensors $(g^{\overline{b}a})$ and $(h^{\overline{b}a})$.

In particular, $(T_x Z, K_x, \omega_x') = (\mathcal{H}_x, J_{\mathcal{H},x}, \omega_x)$, where $\omega_x$ is $\omega_m$ pulled back to $\mathcal{H}_x$ under $d_x\pi$. Similarly, with the same abuse of language, $(T_x Z, K_x, \eta_x') = (\mathcal{H}_x, J_{\mathcal{H},x}, \eta_x)$. By horizontality, expression (28) for $\kappa^*(2\eta)$ implies that $\eta_x = \omega_x / \Phi(m)$. Hence $h_{a\overline{b}}(x) = g_{a\overline{b}}(x)/\Phi(m)$, and so $h^{\overline{b}a}(x) = \Phi(m)\, g^{\overline{b}a}(x)$. Thus we conclude that

$$
\begin{aligned}
\Delta_2(f_Z)(x) &= \frac{1}{2}\, h^{\overline{b}a}(x)\, \partial_a\, \partial_{\overline{b}} f_Z(x) \\
&= \frac{1}{2}\, \Phi(m)\, g^{\overline{b}a}(x)\, \partial_a\, \partial_{\overline{b}} f_Z(x) = \Phi(m)\, \Delta_1(f_Z)(x).
\end{aligned}
\tag{69}
$$

Given (68) and (69), we conclude that

$$\Delta_N(f_N)(n) = \Phi(m)\,\Delta_M(f_M)(m). \tag{70}$$

Interpreting $\Delta_M$ and $\Delta_N$ as endomorphisms of $C^\infty(M')^\mu$, we can restate (70) by writing

$$\Delta_N = \Phi \cdot \Delta_M. \tag{71}$$

## 2.9 $\mu$-Adapted Heisenberg local coordinates

As mentioned in the Introduction, HLC for $X$ centered at some $x \in X$ were defined in [30]; it is in these local coordinates that near-diagonal Szegö kernel scaling asymptotics exhibit their universal nature. While we refer to [30] for a detailed discussion, let us recall that they consist of the choice of an adapted local coordinate chart for $M$ centered at $m = \pi(x)$, intertwining the unitary structure on $T_mM$ with the standard one on $\mathbb{C}^d$, and a preferred local frame of $A^\vee$ on a neighborhood of $m$, having a prescribed second-order jet at $m$.

Let $\mathfrak{r}\colon (-\pi,\pi)\times B_{2d}(\mathbf{0},\epsilon) \to X$, $\mathfrak{r}(\theta,\mathbf{v}) = x + (\theta,\mathbf{v})$, be a system of HLC centered at $x$. Then $\mathfrak{r}^*(dV_X)(\theta,\mathbf{0}) = (2\pi)^{-1}\,|d\theta|\,d\mathcal{L}(\mathbf{v})$, where $d\mathcal{L}(\mathbf{v})$ is the Lebesgue measure on $\mathbf{R}^{2d}$. For $\mathbf{v} \in B_{2d}(\mathbf{0},\epsilon)$, let us set $x + \mathbf{v} =: \mathfrak{r}(0,\mathbf{v})$.

It is natural here to modify the previous prescription so as to incorporate $\mu^X$ into an "equivariant" HLC system. Namely, let us define $\mathfrak{y}'\colon \mathbf{T}^1\times B_{2d}(\mathbf{0},\epsilon) \to X$ by letting

$$\mathfrak{y}'\left(e^{i\vartheta},\mathbf{w}\right) =: e^{i\vartheta} \bullet (x + \mathbf{w}). \tag{72}$$

Working in coordinates on $\mathbf{T}^1$, this yields a map $\mathfrak{y}\colon (-\pi,\pi)\times B_{2d}(\mathbf{0},\epsilon) \to X$ by setting

$$\mathfrak{y}(\vartheta,\mathbf{w}) =: e^{i\vartheta} \bullet (x + \mathbf{w}). \tag{73}$$

If $\mathbf{H}(m) \in \mathbf{R}^{2d}$ is the local coordinate expression of $\xi_M(m) \in T_mM$ (viewed as a column vector) then the local HLC expression of $\xi_X(x)$ is $(\mathbf{H}(m),-\Phi(m)) \in \mathbb{R}^{2d}\times\mathbb{R}$. If $(\theta,\mathbf{v}) \sim (0,\mathbf{0})$, then by (1)

$$\mathfrak{y}(\vartheta,\mathbf{w}) = x + (\mathbf{w} + \vartheta\,\mathbf{H}(m),-\vartheta\,\Phi(m)) + O\left(\|(\mathbf{w},\vartheta)\|^2\right). \tag{74}$$

The Jacobian matrix at the origin of $\mathfrak{r}^{-1} \circ \mathfrak{y}$ is then

$$\mathrm{Jac}_{(0,\mathbf{0})}\left(\mathfrak{r}^{-1}\circ\mathfrak{y}\right) = \begin{pmatrix} I_{2d} & \mathbf{H}_m \\ \mathbf{0}^t & -\Phi(m) \end{pmatrix}. \tag{75}$$

Since $\Phi > 0$, $\mathfrak{y}'$ is a local diffeomorphism at $(1,\mathbf{0}) \in \mathbf{T}^1 \times B_{2d}(\mathbf{0},\epsilon)$. Therefore, if $T_\delta =: \left\{e^{i\vartheta}\colon\, -\delta < \vartheta < \delta\right\} \subseteq \mathbf{T}^1$ then for all sufficiently small $\delta,\,\epsilon > 0$, the restriction of $\mathfrak{y}'$ to $T_\delta \times B_{2d}(\mathbf{0},\epsilon)$ is a diffeomorphism onto its image.

**Lemma 38**  *Suppose $x \in X'$. Then, for all sufficiently small $\epsilon > 0$, the restriction $\mathfrak{y}' : \mathbf{T}^1 \times B_{2d}(\mathbf{0}, \epsilon) \to X$ is injective. Its image is a $\mu^X$-invariant tubular neighborhood of the $\mu^X$-orbit of $x$.*

*Proof*  If not, there exists $\epsilon_j \to 0^+$ and for every $j$ a choice of distinct pairs

$$(e^{i\vartheta_j}, \mathbf{w}_j), \quad (e^{i\vartheta'_j}, \mathbf{w}'_j) \in \mathbf{T}^1 \times B_{2d}(\mathbf{0}, \epsilon_j),$$

and such that, if $\lambda_j =: \vartheta'_j - \vartheta_j$,

$$e^{i\vartheta_j} \bullet (x + \mathbf{w}_j) = e^{i\vartheta'_j} \bullet (x + \mathbf{w}'_j) \quad \Longrightarrow \quad x + \mathbf{w}_j = e^{i\lambda_j} \bullet (x + \mathbf{w}'_j). \tag{76}$$

If $e^{i\lambda_j} \in T_\delta$, the previous considerations imply that $e^{i\lambda_j} = 1$, whence $e^{i\vartheta_j} = e^{i\vartheta'_j}$, and $\mathbf{w}_j = \mathbf{w}'_j$, against the assumptions. Therefore, it follows from (76) that $e^{i\lambda_j} \in \mathbf{T}^1 \setminus T_\delta$, a compact subset of $\mathbf{T}^1$. Perhaps after passing to a subsequence, we may therefore assume without loss that $e^{i\lambda_j} \to e^{i\lambda_\infty} \in \mathbf{T}^1 \setminus T_\delta$ as $j \to +\infty$. Since obviously $x + \mathbf{w}_j, \, x + \mathbf{w}'_j \to x$ as $j \to +\infty$, passing to the limit in (76) we obtain $e^{i\lambda_\infty} \bullet x = x$. But this is absurd by definition of $X'$, given that $e^{i\lambda_\infty} \neq 1$ and $x \in X'$. $\qquad \square$

It follows easily that if $x \in X'$ then $\mathfrak{y}' : \mathbf{T}^1 \times B_{2d}(\mathbf{0}, \epsilon) \to X$ is a diffeomorphism onto its image for all sufficiently small $\epsilon > 0$, and therefore that $\mathfrak{y} : (-\pi, \pi) \times B_{2d}(\mathbf{0}, \epsilon) \to X$ is a local coordinate chart. We shall say that $\eta$ is a system of $\mu$-adapted HLC.

In general, $\mathfrak{y}' : \mathbf{T}^1 \times B_{2d}(\mathbf{0}, \epsilon) \to X$ is an $l : 1$-covering, where $l = |T_m|$. To see this, let us consider the following generalization of Lemma 38:

**Lemma 39**  *Suppose $l = |T_m|$, where $m = \pi(x)$. Then, for all sufficiently small $\epsilon > 0$, the restriction $\mathfrak{y}' : \mathbf{T}^1 \times B_{2d}(\mathbf{0}, \epsilon) \to X$ is an $l : 1$-covering. Its image is a $\mu^X$-invariant tubular neighborhood of the $\mu^X$-orbit of $x$.*

*Proof*  Suppose $x' = e^{i\vartheta_0} \bullet x \in \mathbf{T}^1 \cdot x$. Then, for any $g \in T_m$, we have that $\mathfrak{y}((e^{i\vartheta_0} g, \mathbf{0})) = x'$. Therefore, the inverse image $\mathfrak{y}'^{-1}(x')$ contains $l$ distinct elements $(e^{i\vartheta_0} g, \mathbf{0})$ $(g \in T_m)$, and at each of these $\mathfrak{y}'$ is a local diffeomorphism. It follows that any $x''$ sufficiently close to the orbit $\mathbf{T}^1 \cdot x$ has at least $l$ inverse images under $\mathfrak{y}'$, and that at each of these the latter is a local diffeomorphism.

I claim that in fact any $x''$ sufficiently close to the orbit $\mathbf{T}^1 \cdot x$ has exactly $l$ inverse images under $\mathfrak{y}'$. If not, there exist $\epsilon_j \to 0^+$ and for every $j$ distinct pairs

$$(g_j^{(a)}, \mathbf{v}_j^{(a)}) \in \mathbf{T}^1 \times B_{2d}(\mathbf{0}, \epsilon), \quad 1 \leq a \leq l + 1,$$

such that $g_j^{(a)} \bullet \mathbf{v}_j^{(a)} = g_j^{(b)} \bullet \mathbf{v}_j^{(b)}$, for every $1 \leq a, b \leq l+1$. Arguing as in the proof of Lemma 38, we conclude that $g_j^{(a)} g_j^{(b)^{-1}} \notin T_\delta$ for any $1 \leq b < a \leq l + 1$ and

$j \gg 0$. In particular, perhaps after passing to a subsequence, for $a = 2, \dots, l+1$ we have $g_j^{(a)} g_j^{(1)-1} \to \lambda_\infty^{(a)} \in T_m \setminus T_\delta$.

Suppose $\lambda_\infty^{(a)} = \lambda_\infty^{(b)}$ for $2 \le a < b \le l+1$. Then $g_j^{(a)} g_j^{(b)-1} \to 1 \in T_\delta$ as $j \to +\infty$, absurd. Therefore, $T_m$ contains the $l+1$ distinct elements $\{1, \lambda_\infty^{(2)}, \dots, \lambda_\infty^{(l+1)}\}$, a contradiction. $\qquad\square$

**Lemma 40** *For any $\vartheta \in (-\pi, \pi)$, we have*

$$\mathfrak{y}^*(dV_X)(\vartheta, \mathbf{0}) = \frac{1}{2\pi} \Phi(m) |d\vartheta| \, d\mathcal{L}(\mathbf{w}),$$

*where $d\mathcal{L}(\mathbf{w})$ is the Lebesgue measure on $\mathbf{R}^{2d}$.*

*Proof* Let us write $\mathfrak{r}^*(dV_X) = \mathcal{V}(\theta, \mathbf{v}) |d\theta| \, d\mathcal{L}(\mathbf{v})$, so that $\mathcal{V}(\theta, \mathbf{0}) = (2\pi)^{-1}$. Then

$$
\begin{aligned}
\mathfrak{y}^*(dV_X) &= \left( \mathfrak{r} \circ \mathfrak{r}^{-1} \circ \mathfrak{y} \right)^* (dV_X) = \left( \mathfrak{r}^{-1} \circ \mathfrak{y} \right)^* \left( \mathfrak{r}^*(dV_X) \right) \\
&= \left( \mathfrak{r}^{-1} \circ \mathfrak{y} \right)^* \left( \mathcal{V}(\theta, \mathbf{v}) \, d\theta \, d\mathcal{L}(\mathbf{v}) \right) \\
&= \left( \mathcal{V} \circ \left( \mathfrak{r}^{-1} \circ \mathfrak{y} \right) \right) \cdot \left| \det \left( \mathrm{Jac} \left( \mathfrak{r}^{-1} \circ \mathfrak{y} \right) \right) \right| \, |d\vartheta| \, d\mathcal{L}(\mathbf{w}).
\end{aligned}
$$

At $(0, \mathbf{0})$, in view of (75) and since $\Phi > 0$, we get

$$\mathfrak{y}^*(dV_X)(0, \mathbf{0}) = \frac{1}{2\pi} \Phi(m) |d\vartheta| \, d\mathcal{L}(\mathbf{w}). \tag{77}$$

This proves the claim at $(0, \mathbf{0})$. To prove it at $(\vartheta_0, \mathbf{0})$, we replace $\vartheta \sim \vartheta_0$ by $\vartheta + \vartheta_0$ with $\vartheta \sim 0$ and note that $e^{i(\vartheta + \vartheta_0)} \bullet (x + \mathbf{v}) = e^{i\vartheta} \bullet (e^{i\vartheta_0} \bullet (x + \mathbf{v}))$. Since $\mathfrak{r}_{\vartheta_0}(\theta, \mathbf{v}) = e^{i\theta} \cdot (e^{i\vartheta_0} \bullet (x + \mathbf{v}))$ is a system of HLC centered at $e^{i\vartheta_0} \bullet x$, one can argue as in the previous case. $\qquad\square$

**Corollary 41** *Under the assumptions of Lemma 39, if $\epsilon > 0$ is sufficiently small let $V_\epsilon = \mathfrak{y}'(\mathbf{T}^1 \times B_{2d}(\mathbf{0}, \epsilon))$. Then for any continuous function on $X$, we have*

$$\int_{V_\epsilon} f \, dV_X = \frac{1}{2\pi |T_m|} \int_{-\pi}^\pi \int_{B_{2d}(\mathbf{0}, \epsilon)} f \circ \mathfrak{y} \cdot (\Phi(m) + A(\mathbf{w})) \, |d\vartheta| \, d\mathcal{L}(\mathbf{w}),$$

*where $A(\mathbf{w}) = O(\|\mathbf{w}\|)$.*

# 3 Proof of Theorem 1

*Proof* As the orthogonal projector $\Pi_k^\mu : L^2(X, dV_X) \to H_k^\mu(X)$, $\Pi_k^\mu$ is idempotent, then for every $x \in X$ the Schwartz kernel $\Pi_k^\mu \in C^\infty(X \times X)$ satisfies

$$\Pi_k^\mu(x, x) = \int_X \Pi_k^\mu(x, y) \, \Pi_k^\mu(y, x) \, dV_X(y). \tag{78}$$

Let us fix $x_0 \in X'$ and set $m_0 =: \pi(x_0) \in M'$, $n_0 =: \kappa(x_0) \in N'$, and apply (78) with $x = x_0$. Let $\sigma$ be a local holomorphic frame of $B^\vee$ on an open neighborhood $V \subseteq N'$ of $n_0$; as usual we implicitly identify $B_0^\vee$ with $A_0^\vee$ by means of $\overline{\Gamma}$ (Lemma 31). We may assume without loss that $\sigma(n_0) = x_0$. Let $\|\sigma\|_B =: (\mathcal{N}_B \circ \sigma)^{1/2}$. Then $\|\sigma\|_B$ is a positive real-analytic function on $V$ by Lemma 35. Therefore, the unitarization $\sigma_u =: (1/\|\sigma\|_B) \bullet \sigma : V \to X'$ (see Remark 32) is a real-analytic section and $\sigma_u(n_0) = x_0$.

There exists $\epsilon > 0$ such that $\mathrm{dist}_X(x_0, \mathbf{T}^1 \cdot y) \ge \delta$ for every $y \in X \setminus \kappa^{-1}(V)$. Therefore, by Theorem 1 of [28] we have $\Pi_k^\mu(x_0, \cdot) = O(k^{-\infty})$ uniformly on $X \setminus \kappa^{-1}(V)$. If $\sim$ stands for "has the same asymptotics as," we see from this and (78) for $x = x_0$ that

$$\Pi_k^\mu(x_0, x_0) \sim \int_{\kappa^{-1}(V)} \Pi_k^\mu(x_0, y) \, \Pi_k^\mu(y, x_0) \, dV_X(y). \tag{79}$$

We can parameterize the invariant open neighborhood $\kappa^{-1}(V) \subseteq X'$ by setting

$$\varrho : \mathbf{T}^1 \times V \to \kappa^{-1}(V), \quad \left(e^{i\vartheta}, n\right) \mapsto e^{i\vartheta} \bullet \sigma_u(n). \tag{80}$$

Then

$$\varrho^*(dW_X) = \frac{1}{2\pi} \, d\vartheta \wedge dV_N \tag{81}$$

where $dV_N = (1/d!) \eta^{\wedge d}$ (see Section 2.5). Now (3) means that $s(e^{i\vartheta} \bullet x) = e^{ik\vartheta} s(x)$, for every $e^{i\vartheta} \in \mathbf{T}^1$ and $x \in X$. Therefore, given (4), we have

$$\Pi_k^\mu \left(x_0, e^{i\vartheta} \bullet \sigma_u(n)\right) \Pi_k^\mu \left(e^{i\vartheta} \bullet \sigma_u(n), x\right)$$
$$= \left[e^{-ik\vartheta} \, \Pi_k^\mu(x_0, \sigma_u(n))\right] \left[e^{ik\vartheta} \, \Pi_k^\mu(\sigma_u(n), x_0)\right]$$
$$= \Pi_k^\mu(x_0, \sigma_u(n)) \, \Pi_k^\mu(\sigma_u(n), x_0).$$

Inserting this and (47) into (79), we obtain

$$\Pi_k^\mu(x_0, x_0)$$
$$\sim \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_V \Pi_k^\mu(x_0, \sigma_u(n)) \, \Pi_k^\mu(\sigma_u(n), x_0) \, \Phi(n)^{d+1} \, d\vartheta \, dV_N(n)$$
$$= \int_V \Pi_k^\mu(\sigma_u(n_0), \sigma_u(n)) \, \Pi_k^\mu(\sigma_u(n), \sigma_u(n_0)) \, \Phi(n)^{d+1} \, dV_N(n). \tag{82}$$

If we use (59) in (82) we get

$$\Pi_k^\mu(x_0, x_0)$$
$$\sim \int_V e^{-k\,\mathcal{D}_N(n_0,n)}\,\widetilde{K_k^\mu}(n_0, n)\,\widetilde{K_k^\mu}(n, n_0)\,\Phi(n)^{d+1}\,dV_N(n), \tag{83}$$

where $\mathcal{D}$ is Calabi's diastasis function of $(N', I, 2\eta)$, defined in (7).

Let us set, for simplicity, $\eta' = 2\,\eta$. Also, suppose without loss that $V$ is the domain of a holomorphic local coordinate chart $(z_a)$ for $N'$. If $z_a + i\,y_a$, with $x_a$, $y_a$ real-valued, then by (18) we have

$$dV_N = \det\left([2\,\eta_{k\bar{l}}]\right) \cdot dx_1 \wedge \cdots dx_d \wedge dy_1 \cdots \wedge dy_d$$
$$= \det\left([\eta'_{k\bar{l}}]\right) \cdot dx_1 \wedge \cdots dx_d \wedge dy_1 \cdots \wedge dy_d. \tag{84}$$

In view of (67), we can thus rewrite (83) as follows:

$$\Pi_k^\mu(x_0, x_0) \tag{85}$$
$$\sim \left(\frac{k}{\pi}\right)^{2d} \sum_{j \geq 0} k^{-j} \int_B e^{-k\,\mathcal{D}_N(n_0,n)}\,Z_j(n_0, n)\,\det\left([\eta'_{k\bar{l}}]\right)dx\,dy,$$

where now $B \subseteq \mathbb{C}^d$ is some open ball centered at the origin, and for every $j \geq 0$ we have

$$Z_j(n, n') =: \Phi(n')^{d+1} \sum_{a+b=j} \widetilde{S_a^\mu}(n, n')\,\widetilde{S_b^\mu}(n', n) \quad ((n, n') \in V \times V). \tag{86}$$

In particular, since $S_0^\mu = \Phi^{-(d+1)}$, for $j = 0$ we get from (86):

$$Z_0(n, n') = \Phi(n')^{d+1}\,\widetilde{\Phi}(n, n')^{-(d+1)}\,\widetilde{\Phi}(n', n)^{-(d+1)} \tag{87}$$
$$= \left(\frac{\Phi(n')}{\widetilde{\Phi}(n, n')\,\widetilde{\Phi}(n', n)}\right)^{d+1} = F_\Phi(n')^{d+1},$$

with the notation of Lemma 13, taking $p_0 = n$, and where $\widetilde{\Phi}$ is the sesquiholomorphic extension of $\Phi$ (as a function on $N'$) to some open neighborhood $\widetilde{N}$ of the diagonal (and we assume $V \times V \subseteq \widetilde{N}$). On the diagonal, $Z_0(n, n) = \Phi(n)^{-(d+1)}$.

On the contrary, for $j \geq 1$ we get

$$Z_j(n, n') = \Phi(n')^{d+1}\left[\widetilde{\Phi}(n, n')^{-(d+1)}\,\widetilde{S_j^\mu}(n', n) + \widetilde{S_j^\mu}(n, n')\,\widetilde{\Phi}(n', n)^{-(d+1)}\right]$$
$$+\Phi(n')^{d+1} \sum_{0<a<j} \widetilde{S_a^\mu}(n, n')\,\widetilde{S_{j-a}^\mu}(n', n). \tag{88}$$

On the diagonal,

$$Z_j(n, n) = 2\,S_j^\mu(n) + \Phi(n)^{d+1} \sum_{0<a<j} S_a^\mu(n)\,S_{j-a}^\mu(n). \tag{89}$$

Let us now consider the asymptotics of the $j$th summand in (85). Because $\mathcal{D}_N$ is the diastasis function of $\eta'$, we can apply Theorem 3 of [13], and obtain an asymptotic expansion of the form

$$\int_B e^{-k\,\mathcal{D}_N(n_0,n)}\, Z_j(n_0,n)\, \det\left([\eta'_{k\bar{l}}]\right) dx\, dy$$
$$\sim \left(\frac{\pi}{k}\right)^d \sum_{l\geq 0} k^{-l}\, R_l^N(Z_j(n_0,\cdot))\Big|_{n=n_0}, \tag{90}$$

where the $R_j^N$s are Engliš operators for the Kähler manifold $(N, I, \eta')$.

Using (90) within (85), we get

$$\Pi_k^\mu(x_0,x_0) \sim \left(\frac{k}{\pi}\right)^d \sum_{j,l\geq 0} k^{-j-l}\, R_l^N(Z_j(n_0,\cdot))\Big|_{n=n_0}$$
$$= \left(\frac{k}{\pi}\right)^d \sum_{j\geq 0} k^{-j} \sum_{a+b=j} R_a^N(Z_b(n_0,\cdot))\Big|_{n=n_0}. \tag{91}$$

It follows from (5) and (91) that

$$S_j^\mu(n_0) = \sum_{a+b=j} R_a^N(Z_b(n_0,\cdot))\Big|_{n=n_0}$$
$$= Z_j(n_0,n_0) + \sum_{a=1}^{j} R_a^N(Z_{j-a}(n_0,\cdot))\Big|_{n=n_0}. \tag{92}$$

Given (89), the latter relation may be rewritten

$$S_j^\mu(n_0) = 2\, S_j^\mu(n_0) + \Phi(n_0)^{d+1} \sum_{0<a<j} S_a^\mu(n_0)\, S_{j-a}^\mu(n_0) \tag{93}$$
$$+ \sum_{a=1}^{j} R_a^N(Z_{j-a}(n_0,\cdot))\Big|_{n=n_0}.$$

It follows that

$$S_j^\mu(n_0) = -\Phi(n_0)^{d+1} \sum_{0<a<j} S_a^\mu(n_0)\, S_{j-a}^\mu(n_0)$$
$$- \sum_{a=1}^{j} R_a^N(Z_{j-a}(n_0,\cdot))\Big|_{n=n_0}, \tag{94}$$

which determines $S_j^\mu$ for any $j \geq 1$ in terms of the $S_k^\mu$s with $0 \leq k < j$ and their sesquiholomorphic extensions. The proof is complete, for (94) is (11), with $j$ in place of $j+1$. $\qquad\square$

# 4 Proof of Corollary 2

*Proof* Let us apply (94) with $j = 1$. We get

$$
\begin{aligned}
S_1^\mu(n_0) &= - R_1^N(Z_0(n_0, \cdot))\big|_{n=n_0} \\
&= -\left(\Delta_N - \frac{1}{2}\,\varrho_N\right)(Z_0(n_0, \cdot))\bigg|_{n=n_0},
\end{aligned}
\tag{95}
$$

where $Z_0$ is defined by (87), and $\Delta_N$ and $\varrho_N$ are defined by (20) and (19), respectively, with reference to the Kähler manifold $(P, K, \gamma) = (N, I, \eta')$, where $\eta' = 2\,\eta$.

We have, by (87),

$$
Z_0(n_0, n) = \left(\frac{\Phi(n)}{\widetilde{\Phi}(n_0, n)\,\widetilde{\Phi}(n, n_0)}\right)^{d+1} = F_\Phi(n)^{d+1},
\tag{96}
$$

where $F_\Phi$ is defined as in Lemma 13, with $f = \Phi$ and $n_0 = p_0$. Applying (23) with $l = d + 1$ and $f = F_\Phi$, we get

$$
\begin{aligned}
\Delta_N(Z_0(n_0, \cdot))\big|_{n=n_0} &= \Delta_N\left(F_\Phi^{d+1}\right)(n_0) \\
&= (d + 1)\,F_\Phi(n_0)^d \cdot \Delta_N(F_\Phi)(n_0) + \frac{d(d + 1)}{2}\,F_\Phi(n_0)^{d-1}\left\|\mathrm{grad}_N(F_\Phi)(n_0)\right\|^2 \\
&= (d + 1)\,\Phi(n_0)^{-d} \cdot \Delta_N(F_\Phi)(n_0),
\end{aligned}
\tag{97}
$$

where the gradient and the norm are taken with respect to the Riemannian metric $h' = 2\,h$, and in the last equation we have made use of Lemma 15.

Let us apply Lemma 13 with $(P, K, \gamma) = (N', I, \eta')$, $f = \Phi^{d+1} \in C^\varpi(N')$, and $p_0 = n_0$, so that in the statement we have $F = Z_0(n_0, \cdot)$. We obtain

$$
\Delta_N(Z_0(n_0, \cdot))\big|_{n=n_0}
\tag{98}
$$

$$
= (d + 1)\,\Phi(n_0)^{-(d+2)}\left[\Delta_P(\Phi)(n_0) - \frac{1}{2\,\Phi(n_0)}\left\|\mathrm{grad}_N(\Phi)(n_0)\right\|^2\right].
$$

Inserting (98) into (95),

$$
\begin{aligned}
S_1^\mu(n_0) = \frac{1}{2}\,\varrho_N(n_0)\,\Phi(n_0)^{-(d+1)} \\
+ (d + 1)\,\Phi(n_0)^{-(d+2)}\left[\frac{1}{2\,\Phi(n_0)}\left\|\mathrm{grad}_N(\Phi)(n_0)\right\|^2 - \Delta_P(\Phi)(n_0)\right].
\end{aligned}
$$

$\square$

# 5  Proof of Theorem 4

*Proof*  Statements (1) and (2) follow quite straightforwardly by using the corresponding properties of $\Pi_k^\mu$ in Theorem 1 of [28] in the first line of (6).

To prove (3), we start from the relation

$$
T_k^\mu(f)\left(x + \frac{\mathbf{v}}{\sqrt{k}}, x + \frac{\mathbf{w}}{\sqrt{k}}\right) \tag{99}
$$
$$
= \int_X \Pi_k^\mu\left(x + \frac{\mathbf{v}}{\sqrt{k}}, y\right) f(y)\, \Pi_k^\mu\left(y, x + \frac{\mathbf{w}}{\sqrt{k}}\right) dV_X(y).
$$

If integration in $dV_X(y)$ in (99) is restricted to a given invariant tubular neighborhood $V$ of the orbit $\mathbf{T}^1 \cdot x$, only a negligible contribution to the asymptotics is lost. On the contrary, on $V$ we can introduce $\mu$-adapted HLC as in Section 2.9, so as to write $y = e^{i\theta} \bullet (x + \mathbf{u})$. Applying Corollary 41 (with $V = V_\epsilon$), we get

$$
T_k^\mu(f)\left(x + \frac{\mathbf{v}}{\sqrt{k}}, x + \frac{\mathbf{w}}{\sqrt{k}}\right) \tag{100}
$$
$$
\sim \frac{1}{2\pi\,|T_m|} \int_{-\pi}^{\pi} \int_{B_{2d}(\mathbf{0},\epsilon)} (\Phi(m) + A(\mathbf{u}))
$$
$$
\cdot \Pi_k^\mu\left(x + \frac{\mathbf{v}}{\sqrt{k}}, e^{i\theta} \bullet (x + \mathbf{u})\right) f(m + \mathbf{u})\, \Pi_k^\mu\left(e^{i\theta} \bullet (x + \mathbf{u}), x + \frac{\mathbf{w}}{\sqrt{k}}\right)
$$
$$
\cdot |d\vartheta|\, d\mathcal{L}(\mathbf{u}),
$$

where we used the fact that $f \in C^\infty(M)^\mu$.

Let $D_1, D_2 > 0$ be as in (13). Since $\|\mathbf{v}\|, \|\mathbf{w}\| \le C\, k^{1/9}$, we have

$$
\mathrm{dist}_X\left(\mathbf{T}^1 \cdot x, x + \frac{\mathbf{v}}{\sqrt{k}}\right) \le D_2\, C\, k^{-7/18}. \tag{101}
$$

If $\mathrm{dist}_X\left(\mathbf{T}^1 \cdot x, y\right) \ge 2\, D_2\, C\, k^{-7/18}$, then by (101) we have

$$
\mathrm{dist}_X\left(\mathbf{T}^1 \cdot y, x + \frac{\mathbf{v}}{\sqrt{k}}\right) \ge D_2\, C\, k^{-7/18},
$$

and similarly for $\mathbf{w}$. It follows from this and statement (2). (with $\epsilon = 1/9$) that the contribution to (99) and (100) coming from the locus where $\mathrm{dist}_X(\mathbf{T}^1 \cdot x, y) \ge 2\, D_2\, C\, k^{-7/18}$ is rapidly decreasing. By (13), this means that in (100) the contribution of the locus where $\|\mathbf{u}\| \ge (2D_2/D_1)\, C\, k^{-7/18}$ is rapidly decreasing. Therefore, only a negligible contribution is lost in (100) if the integrand is multiplied by $\varrho(k^{7/18}\, \mathbf{w})$, where $\varrho$ is an appropriate radial bump function, identically equal to 1 near the origin.

Furthermore, using (3) and (45), for any $x, x', x'' \in X$ and $e^{i\theta} \in \mathbf{T}^1$ we have

$$\Pi_k^\mu \left( x', e^{i\theta} \bullet x'' \right) = e^{-ik\theta} \Pi_k^\mu (x', x'') = \overline{\Pi_k^\mu (e^{i\theta} \bullet x'', x')}.$$

Inserting this in (100), and applying the rescaling $\mathbf{u} \mapsto \mathbf{u}/\sqrt{k}$, we obtain

$$T_k^\mu(f) \left( x + \frac{\mathbf{v}}{\sqrt{k}}, x + \frac{\mathbf{w}}{\sqrt{k}} \right) \tag{102}$$

$$\sim \frac{k^{-d}}{|T_m|} \int_{\mathbb{C}^d} \left( \Phi(m) + A \left( \frac{\mathbf{u}}{\sqrt{k}} \right) \right)$$

$$\cdot \Pi_k^\mu \left( x + \frac{\mathbf{v}}{\sqrt{k}}, x + \frac{\mathbf{u}}{\sqrt{k}} \right) f \left( m + \frac{\mathbf{u}}{\sqrt{k}} \right) \Pi_k^\mu \left( x + \frac{\mathbf{u}}{\sqrt{k}}, x + \frac{\mathbf{w}}{\sqrt{k}} \right)$$

$$\cdot \varrho \left( k^{-1/9} \mathbf{u} \right) d\mathcal{L}(\mathbf{u});$$

integration in $d\mathbf{u}$ is really over an expanding ball of radius $O(k^{1/9})$ in $\mathbb{C}^d$.

Now by (3) of Theorem 1 of [28] (and the remark immediately following the statement of that theorem) with $\upsilon_1 = (0, \mathbf{v})$ and $\upsilon_2 = (0, \mathbf{w})$, the sought expansion holds for $\Pi_k^\mu$ (that is, for $f = 1$). Thus

$$\Pi_k^\mu \left( x + \frac{\mathbf{v}}{\sqrt{k}}, x + \frac{\mathbf{u}}{\sqrt{k}} \right) \sim \left( \frac{k}{\pi} \right)^d \cdot \sum_{t \in T_m} t^k \, e^{\psi_2 \left( d_m \mu_{t^{-1}}^M (\mathbf{v}), \mathbf{u} \right)/\Phi(m)} \tag{103}$$

$$\cdot \left( \Phi(m)^{-(d+1)} + \sum_{j \geq 1} k^{-j/2} \, R_j \left( m, d_m \mu_{t^{-1}}^M (\mathbf{v}), \mathbf{u} \right) \right),$$

where $\psi_2$ is as in (15), and $R_j(m, \mathbf{v}, \mathbf{u})$ is a polynomial function of $\mathbf{v}$ and $\mathbf{u}$. Clearly,

$$\frac{1}{\Phi(m)} \psi_2(\mathbf{v}, \mathbf{u}) = \psi_2 \left( \frac{1}{\sqrt{\Phi(m)}} \mathbf{v}, \frac{1}{\sqrt{\Phi(m)}} \mathbf{u} \right) = \psi_2 (\mathbf{v}', \mathbf{u}'),$$

where for any $\mathbf{p} \in \mathbb{C}^d$ we set $\mathbf{p}' = \mathbf{p}/\sqrt{\Phi(m)}$.

Using this and the Taylor expansion for $f(m + \mathbf{u}/\sqrt{k})$ at $m$, we get for (102) an asymptotic expansion in descending powers of $k^{1/2}$, whose leading term is given by

$$\frac{k^{-d}}{|T_m|} \Phi(m)^{-2d-1} f(m) \left( \frac{k}{\pi} \right)^{2d} \tag{104}$$

$$\cdot \sum_{t,s \in T_m} (s\,t)^k \int_{\mathbb{C}^d} e^{\psi_2 \left( d_m \mu_{t^{-1}}^M (\mathbf{v}'), \mathbf{u}' \right) + \psi_2 \left( \mathbf{u}', d_m \mu_s^M (\mathbf{w}') \right)} d\mathcal{L}(\mathbf{u}).$$

Applying the change of variable $\mathbf{u} = \sqrt{\Phi(m)}\,\mathbf{s}$, (104) becomes

$$\frac{1}{|T_m|}\,\Phi(m)^{-(d+1)}\,f(m)\,\frac{k^d}{\pi^{2d}} \tag{105}$$

$$\cdot \sum_{t,s\in T_m}(s\,t)^k \int_{\mathbb{C}^d} e^{\psi_2\left(d_m\mu_{t^{-1}}^M(\mathbf{v}'),\mathbf{s}\right)+\psi_2\left(\mathbf{s},d_m\mu_s^M(\mathbf{w}')\right)}\,d\mathcal{L}(\mathbf{s}).$$

$$= \frac{1}{|T_m|}\,\Phi(m)^{-(d+1)}\,f(m)\,\frac{k^d}{\pi^{2d}}\,\pi^d \sum_{t,s\in T_m}(s\,t)^k\,e^{\psi_2\left(d_m\mu_{t^{-1}}^M(\mathbf{v}'),d_m\mu_s^M(\mathbf{w}')\right)}$$

$$= \frac{1}{|T_m|}\,\Phi(m)^{-(d+1)}\,f(m)\left(\frac{k}{\pi}\right)^d \sum_{t,s\in T_m}(s\,t)^k\,e^{\psi_2\left(d_m\mu_{(st)^{-1}}^M(\mathbf{v}'),\mathbf{w}'\right)}$$

$$= \Phi(m)^{-(d+1)}\,f(m)\left(\frac{k}{\pi}\right)^d \sum_{t\in T_m}t^k\,e^{\psi_2\left(d_m\mu_{t^{-1}}^M(\mathbf{v}'),\mathbf{w}'\right)}$$

$$= \Phi(m)^{-(d+1)}\,f(m)\left(\frac{k}{\pi}\right)^d \sum_{t\in T_m}t^k\,e^{\psi_2\left(d_m\mu_{t^{-1}}^M(\mathbf{v}),\mathbf{w}\right)/\Phi(m)}.$$

We have used the fact that if $A\colon \mathbb{C}^d \to \mathbb{C}^d$ is unitary, then

$$\psi_2(\mathbf{u}, A\mathbf{t}) = \psi_2\left(A^{-1}\mathbf{u}, \mathbf{t}\right)$$

for any $\mathbf{u},\mathbf{t}\in\mathbb{C}^d$, and the relation

$$\int_{\mathbb{C}^d} e^{\psi_2(\mathbf{v},\mathbf{u})+\psi_2(\mathbf{u},\mathbf{w})}\,d\mathcal{L}(\mathbf{u}) = \pi^d\,e^{\psi_2(\mathbf{v},\mathbf{w})}.$$

Finally, when $\mathbf{v} = \mathbf{w} = \mathbf{0}$ the appearance of descending powers of $k$ in the asymptotic expansion for (102) originates from Taylor expanding the integrand in $\mathbf{u}/\sqrt{k}$; half-integer powers of $k$ are thus associated with odd homogeneous polynomials in $\mathbf{u}$, and therefore the corresponding contributions to the integral vanish by parity considerations.                                           $\square$

## 6  Proof of Theorem 6

*Proof*   The proof of Theorem 6 is an adaptation of the proof of Theorem 1, so we'll be very sketchy. Adopting the same setup, rather than (78), (79), and (83) we now have

$$T_k^\mu[f](x_0, x_0) = \int_X \Pi_k^\mu(x_0, y)\,f(y)\,\Pi_k^\mu(y, x_0)\,dV_X(y) \tag{106}$$

$$\sim \int_{\kappa^{-1}(V)} \Pi_k^\mu(x_0, y)\,f(y)\,\Pi_k^\mu(y, x_0)\,dV_X(y)$$

$$= \int_V e^{-k\,\mathcal{D}_N(n_0, n)}\,\widetilde{K_k^\mu}(n_0, n)\,\widetilde{K_k^\mu}(n, n_0)\,f(n)\,\Phi(n)^{d+1}\,dV_N(n).$$

Therefore, we get in place of (85) and (91):

$$
\begin{aligned}
& T_k^\mu[f](x_0, x_0) \\
& \sim \left(\frac{k}{\pi}\right)^{2d} \sum_{j \geq 0} k^{-j} \int_B e^{-k\,\mathcal{D}_N(n_0,n)}\, Z_j(n_0, n)\, f(n)\, \det\left([\eta'_{k\bar{l}}]\right) dx\, dy \\
& \sim \left(\frac{k}{\pi}\right)^{d} \sum_{j \geq 0} k^{-j} \sum_{a+b=j} R_a^N\big(Z_b(n_0, \cdot)\, f(\cdot)\big)\Big|_{n=n_0},
\end{aligned}
\tag{107}
$$

which proves the claim (and reproves Corollary 5). □

## 7 Proof of Corollary 8

*Proof*  Let us simplify notation in the following arguments by setting $f_j^\mu :=$ $S_j^\mu[f]$. To begin with, we have from (88) that $Z_1(n_0) = 2\, S_1^\mu(n_0)$. We see from (107) that

$$
\begin{aligned}
f_1^\mu(n_0) &= R_0^N\big(Z_1(n_0, \cdot)\, f(\cdot)\big)\Big|_{n=n_0} + R_1^N\big(Z_0(n_0, \cdot)\, f(\cdot)\big)\Big|_{n=n_0} \\
&= Z_1(n_0, n_0)\, f(n_0) + \left(\Delta_N - \frac{1}{2}\, \varrho_N\right)\big(Z_0(n_0, \cdot)\, f(\cdot)\big)\Big|_{n=n_0} \\
&= \left[2\, S_1^\mu(n_0) - \frac{1}{2}\, \varrho_N(n_0)\, \Phi(n_0)^{-(d+1)}\right] f(n_0) \\
&\quad + \Delta_N\big(Z_0(n_0, \cdot)\, f(\cdot)\big)\Big|_{n=n_0}.
\end{aligned}
\tag{108}
$$

Now in view of Lemma 15 we have

$$
\begin{aligned}
& \Delta_N\big(Z_0(n_0, \cdot)\, f(\cdot)\big)\Big|_{n=n_0} \\
& = \Delta_N\big(Z_0(n_0, \cdot)\big)\Big|_{n=n_0} f(n_0) + \Phi(n_0)^{-(d+1)}\, \Delta_N\big(f(\cdot)\big)\Big|_{n=n_0}.
\end{aligned}
$$

Inserting this in (108), and recalling (95), we obtain

$$
\begin{aligned}
f_1^\mu(n_0) &= \Phi(n_0)^{-(d+1)}\, \Delta_N\big(f(\cdot)\big)\Big|_{n=n_0} + 2\, S_1^\mu(n_0)\, f(n_0) \\
&\quad + \left(-\frac{1}{2}\, \varrho_N(n_0)\, \Phi(n_0)^{-(d+1)} + \Delta_N\big(Z_0(n_0, \cdot)\big)\Big|_{n=n_0}\right) f(n_0) \\
&= \Phi(n_0)^{-(d+1)}\, \Delta_N\big(f(\cdot)\big)\Big|_{n=n_0} + S_1^\mu(n_0)\, f(n_0).
\end{aligned}
\tag{109}
$$

□

## 8 Proof of Corollary 10

*Proof* Notation being as in Definition 9 and the proof of Corollary 8, by Corollaries 2 and 8 we have on $M'$:

$$\mathrm{Ber}_k^\mu[f] = \frac{f_0^\mu + k^{-1} f_1^\mu + O\left(k^{-2}\right)}{S_0^\mu + k^{-1} S_1^\mu + O\left(k^{-2}\right)} = \frac{f_0^\mu}{S_0^\mu} \cdot \frac{1 + k^{-1} \left(f_1^\mu/f_0^\mu\right) + O\left(k^{-2}\right)}{1 + k^{-1} \left(S_1^\mu/S_0^\mu\right) + O\left(k^{-2}\right)}$$

$$= f + k^{-1} f \cdot \left(\frac{f_1^\mu}{f_0^\mu} - \frac{S_1^\mu}{S_0^\mu}\right) + O\left(k^{-2}\right).$$

Thus $B_0^\mu(f) = f$; furthermore, by Corollary 8 we have

$$B_1^\mu(f) = f \cdot \left(\frac{f_1^\mu}{f_0^\mu} - \frac{S_1^\mu}{S_0^\mu}\right) = \Phi^{d+1} f_1^\mu - f \, \Phi^{d+1} S_1^\mu$$

$$= \Delta_N(f) + \Phi^{d+1} S_1^\mu \cdot f - f \cdot \Phi^{d+1} S_1^\mu = \Delta_N(f).$$

$\square$

## 9 Proof of Theorem 11

Before tackling the proof, let us remark that considerations similar to those in Section 2.7 hold for Toeplitz operators. Namely, if $f \in C^\infty(M)^\mu$ let $T_k^\mu[f] \colon H_k^\mu(X) \to H_k^\mu(X)$ and $T_k^\mu[f] \in C^\infty(X \times X)$ denote both the induced operator and its Schwartz kernel, given by (6). The latter extends uniquely to a sesquiholomorphic function $\mathcal{T}_k^\mu[f] \colon A_0^\vee \times A_0^\vee \to \mathbb{C}$, which is the Toeplitz analogue of (53); explicitly, it is given by

$$\mathcal{T}_k^\mu[f](\lambda, \lambda') = \sum_j T_k^\mu \widetilde{[f]}\left(s_j^{(k)}\right)(\lambda) \cdot \overline{\tilde{s}_j^{(k)}(\lambda')} \qquad \left(\lambda, \lambda' \in A_0^\vee\right), \tag{110}$$

and satisfies the equivariance law (54). Corresponding to (55) we now have

$$T_k^\mu[f](\sigma_u(n), \sigma_u(n')) = e^{-\frac{k}{2}\left(\Xi(n) + \Xi(n')\right)} \mathcal{T}_k^\mu[f](\sigma(n), \sigma(n')). \tag{111}$$

Let us define $K_k^\mu[f] \colon X \to \mathbb{R}$, the Toeplitz analogue of (51), by setting

$$K_k^\mu[f](x) =: T_k^\mu[f](x, x) = \sum_j T_k^\mu[f]\left(s_j^{(k)}\right)(x) \, \overline{s_j^{(k)}(x)} \qquad (x \in X); \tag{112}$$

since $f$ is real, $T_k^\mu[f] \colon H_k^\mu(X) \to H_k^\mu(X)$ is self-adjoint, and so

$$T_k^\mu[f](x, x) \in \mathbb{R}.$$

Then $K_k^\mu[f]$ descends to a $\nu$-invariant $C^\varpi$ function on $N$, by an obvious analogue of Lemma 36, and so we can consider its unique sesquiholomorphic

extension $\widetilde{K_k^\mu[f]}$ to a neighborhood of the diagonal in $N \times N$. In place of (57) we now have that

$$\mathcal{T}_k^\mu[f](\sigma(n), \sigma(n')) = e^{k \overline{\Xi}(n,n')} \widetilde{K_k^\mu[f]}(n, n'). \tag{113}$$

Finally, a Toeplitz operator $T[f] = \Pi \circ M_f \circ \Pi$ is a zeroth-order FIO associated with the same almost complex Lagrangian relation as $\Pi$, and therefore also has a microlocal structure of the form (61), with an amplitude having an asymptotic expansion as in (62). Repeating the arguments following (63), therefore, leads to the Toeplitz generalization of the asymptotic expansion (66) and (67):

$$T_k^\mu[f](\sigma_u(n), \sigma_u(n')) \sim \left(\frac{k}{\pi}\right)^d e^{k\left[\overline{\Xi}(n,n') - \frac{1}{2}\left(\Xi(n) + \Xi(n')\right)\right]}$$
$$\cdot \sum_{j \geq 0} k^{-j} \widetilde{S_j^\mu[f]}(n, n'); \tag{114}$$

$$\widetilde{K_k^\mu[f]}(n, n') \sim \left(\frac{k}{\pi}\right)^d \sum_{j \geq 0} k^{-j} \widetilde{S_j^\mu[f]}(n, n'). \tag{115}$$

Let us prove Theorem 11.

*Proof* We adopt the notation and setting of the proof of Theorem 1. Given (114), arguing as in the derivation of (85) we now obtain

$$E_k^\mu[f, g](x_0, x_0) = \left(T_k^\mu[f] \circ T_k^\mu[g]\right)(x_0, x_0) \tag{116}$$
$$\sim \left(\frac{k}{\pi}\right)^{2d} \sum_{j \geq 0} k^{-j} \int_B e^{-k \mathcal{D}_N(n_0, n)} Z_j[f, g](n_0, n) \det\left([\eta_{kl}']\right) dx \, dy,$$

where

$$Z_j[f, g](n, n') \tag{117}$$
$$=: \Phi(n')^{d+1} \sum_{a+b=j} \widetilde{S_a^\mu[f]}(n, n') \widetilde{S_b^\mu[g]}(n', n) \qquad ((n, n') \in V \times V).$$

Corresponding to (91), we have

$$E_k^\mu[f, g](x_0, x_0) \sim \left(\frac{k}{\pi}\right)^d \sum_{j,l \geq 0} k^{-j-l} \left. R_l^N\left(Z_j[f, g](n_0, \cdot)\right)\right|_{n=n_0} \tag{118}$$
$$= \left(\frac{k}{\pi}\right)^d \sum_{j \geq 0} k^{-j} \sum_{a+b=j} \left. R_a^N\left(Z_j[f, g](n_0, \cdot)\right)\right|_{n=n_0}$$
$$= \left(\frac{k}{\pi}\right)^d \left\{Z_0[f, g](n_0, n_0) + k^{-1} A_1[f, g](n_0) + O\left(k^{-2}\right)\right\},$$

where

$$A_1[f,g](n_0) =: R_1^N\big(Z_0[f,g](n_0,\cdot)\big)\Big|_{n=n_0} + Z_1[f,g](n_0,n_0). \qquad (119)$$

By (117), we have

$$
\begin{aligned}
Z_0[f,g](n_0,n_0) &= \Phi(n_0)^{d+1}\, S_a^\mu[f](n_0)\, S_b^\mu[g](n_0)\\
&= \Phi(n_0)^{-(d+1)}\, f(n_0)\, g(n_0)\\
&= Z_0[g,f](n_0,n_0);
\end{aligned}
\qquad (120)
$$

therefore,

$$
\begin{aligned}
&E_k^\mu[f,g](x_0,x_0) - E_k^\mu[g,f](x_0,x_0)\\
&= \left(\frac{k}{\pi}\right)^d\left[k^{-1}\big(A_1[f,g](n_0) - A_1[g,f](n_0)\big) + O\big(k^{-2}\big)\right].
\end{aligned}
\qquad (121)
$$

Furthermore, by (117) we have

$$
\begin{aligned}
Z_1[f,g](n_0,n_0) &= \Phi(n_0)^{d+1}\left[S_0^\mu[f](n_0)\, S_1^\mu[g](n_0) + S_1^\mu[f](n_0)\, S_0^\mu[g](n_0)\right]\\
&= Z_1[g,f](n_0,n_0).
\end{aligned}
\qquad (122)
$$

We see from (119) and (122) that

$$
\begin{aligned}
& A_1[f,g](n_0) - A_1[g,f](n_0) && (123)\\
&= R_1^N\big(Z_0[f,g](n_0,\cdot)\big)\Big|_{n=n_0} - R_1^N\big(Z_0[g,f](n_0,\cdot)\big)\Big|_{n=n_0}\\
&= \Delta_N\big(Z_0[f,g](n_0,\cdot)\big)\Big|_{n=n_0} - \Delta_N\big(Z_0[g,f](n_0,\cdot)\big)\Big|_{n=n_0};
\end{aligned}
$$

in the latter equality, we have used the fact that $R_1^N = \Delta_N - \varrho_N/2$ and (120).

To compute the latter commutator, let us remark that

$$
\begin{aligned}
Z_0[f,g](n_0,n) &= \Phi(n)^{d+1}\, \widetilde{S_0^\mu[f]}(n_0,n)\, \widetilde{S_0^\mu[g]}(n,n_0)\\
&= Z_0(n_0,n)\, \widetilde{f}(n_0,n)\, \widetilde{g}(n,n_0),
\end{aligned}
\qquad (124)
$$

where $Z_0(n_0,n)$ is as in (87).

It follows from (124) and Lemma 15 that

$$
\begin{aligned}
\Delta_N\big(Z_0[f,g](n_0,\cdot)\big)\Big|_{n=n_0} &= \Delta_N\big(Z_0(n_0,\cdot)\big)\Big|_{n=n_0}\cdot f(n_0)\, g(n_0) && (125)\\
&\quad + Z_0(n_0,n_0)\,\Delta_N\big(\widetilde{f}(n_0,\cdot)\,\widetilde{g}(\cdot,n_0)\big)\Big|_{n=n_0}.
\end{aligned}
$$

Let $(h'^{\bar{r}s})$ be the contravariant metric tensor of $(N', I, \eta')$, where $\eta' = 2\eta$ (thus $h'^{\bar{r}s} = h^{\bar{r}s}/2$). Since the former summand on the RHS of (125) is symmetric in $f$ and $g$, we have

$$A_1[f,g](n_0) - A_1[g,f](n_0)$$

$$= \Phi(n_0)^{-(d+1)} \left[ \Delta_N\big(\widetilde{f}(n_0,\cdot)\,\widetilde{g}(\cdot,n_0)\big)\Big|_{n=n_0} - \Delta_N\big(\widetilde{g}(n_0,\cdot)\,\widetilde{f}(\cdot,n_0)\big)\Big|_{n=n_0} \right]$$

$$= \Phi(n_0)^{-(d+1)}\, h'^{\bar{r}s} \left( \partial_{\bar{r}}f(n_0)\,\partial_s g(n_0) - \partial_{\bar{r}}g(n_0)\,\partial_s f(n_0) \right)$$

$$= -i\,\Phi(n_0)^{-(d+1)}\,\{f,g\}_N, \tag{126}$$

where in the latter step we have used (24). The last equality in the statement now follows from (126) and Corollary 21.   □

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