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Gaussian maps and generic vanishing I: Subvarieties of abelian varieties

G. Pareschi

Università di Roma, Tor Vergata

Abstract

The aim of this paper is to present an approach to Green–Lazarsfeld's generic vanishing combining Gaussian maps and the Fourier–Mukai transform associated with the Poincaré line bundle. As an application, we prove the generic vanishing theorem for all normal Cohen–Macaulay subvarieties of abelian varieties over an algebraically closed field.

Dedicated to my teacher, Rob Lazarsfeld, on the occasion of his 60th birthday

1 Introduction

We work with irreducible projective varieties on an algebraically closed field of any characteristic, henceforth called *varieties*. The contents of this paper are:

(1) A general criterion expressing the vanishing of the higher cohomology of a line bundle on a Cohen–Macaulay variety in terms of certain first-order conditions on hyperplane sections (Theorem 2). Such conditions involve *Gaussian maps* and the criterion is a generalization of well-known results on hyperplane sections of K3 and abelian surfaces.

(2) Using a relative version of the above, we prove the vanishing of higher direct images of Poincaré line bundles of normal Cohen–Macaulay subvarieties of abelian varieties[1] (Theorem 5). As is well known, this is equivalent to Green–Lazarsfeld's *generic vanishing*, a condition satisfied by all irregular compact Kahler manifolds [5]. This implies in turn a Kodaira-type vanishing

[1] By the Poincaré line bundle of a subvariety X of an abelian variety A we mean the pull-back to $X \times \mathrm{Pic}^0 A$ of a Poincaré line bundle on $A \times \mathrm{Pic}^0 A$.

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for line bundles which are restrictions to normal Cohen–Macaulay subvarieties of abelian varieties, of ample line bundles on the abelian variety (Corollary 6).

Concerning point (2), it should be mentioned that at present we are not able to extend this approach efficiently to the general generic vanishing theorem (GVT), i.e., for varieties *mapping to* abelian varieties, even for smooth projective varieties over the complex numbers (where it is well known by the work of Green and Lazarsfeld). This will be the object of further research. However, concerning possible extensions of the general GVT to singular varieties and/or to positive characteristic, one should keep in mind the work of Hacon and Kovacs [8] where – by exploiting the relation between GVT and the Grauert–Riemenschneider vanishing theorem – they show examples of failure of the GVT for mildly singular varieties (over $\mathbb{C}$) and even smooth varieties (in characteristic $p > 0$) of dimension ≥ 3, with a (separable) generically finite map to an abelian variety. This disproved an erroneous theorem of a previous preprint of the author.

Now we turn to a more detailed presentation of the above topics.

1.1 Motivation: Gaussian maps on curves and vanishing of the H^1 of line bundles on surfaces

We introduce part (1) starting from a particular case, where the essence of the story becomes apparent: the vanishing of the H^1 of a line bundle on a surface in terms of Gaussian maps on a sufficiently positive hyperplane section (Theorem 1 below).

To begin with, let us recall what Gaussian maps are. Given a curve C and a line bundle A on C, denote by M_A the kernel of the evaluation map of global sections of A:

$$0 \to M_A \to H^0(C, A) \otimes \mathcal{O}_C \to A.$$

This comes equipped with a natural $\mathcal{O}_C$-linear differentiation map

$$M_A \to \Omega^1_C \otimes A$$

defined as

$$M_A = p_*(\mathcal{I}_\Delta \otimes q^* A) \to p_*((\mathcal{I}_\Delta \otimes A)_{|\Delta}) = \Omega^1_C \otimes A,$$

where p, q, and Δ are the projections and the diagonal of the product $C \times C$. Twisting with another line bundle B and taking global sections, one gets the *Gaussian map* (or *Wahl map* [21]) of A and B:

$$\gamma_{A,B} : Rel(A,B) := H^0(C, M_A \otimes B) \to H^0(C, \Omega^1_C \otimes A \otimes B)\,.^2$$

In our treatment it is more natural to set $A = N \otimes P$ and $B = \omega_C \otimes P^\vee$ for suitable line bundles N and P on the curve C, and to consider the dual map

$$g_{N,P} : \mathrm{Ext}^1_C(\Omega^1_C \otimes N, \mathcal{O}_C) \to \mathrm{Ext}^1_C(M_{N\otimes P}, P). \tag{1}$$

Note that $g_{N,P}$ can be defined directly (even if ω_C is not a line bundle) as $\mathrm{Ext}^1_C(\,\cdot\,, P)$ of the differentiation map of $M_{N\otimes P}$.

The relation with the vanishing of the H^1 of line bundles on surfaces lies in the following result, whose proof follows closely arguments contained in the papers of Beaville and Mérindol [2] and Colombo *et al.* [3]. Let X be a Cohen–Macaulay surface and Q a line bundle on X. Let L be a base-point-free line bundle on X such that $L \otimes Q$ is also base-point-free, and let C be a (reduced and irreducible) Cartier divisor in $|L|$, not contained in the singular locus of X. Let $N_C = L_{|C}$ be the normal bundle of C. We have the extension class

$$e \in \mathrm{Ext}^1_C(\Omega^1_C \otimes N_C, \mathcal{O}_C)$$

of the normal sequence

$$0 \to N_C^\vee \to (\Omega^1_X)_{|C} \to \Omega^1_C \to 0.$$

We consider the (dual) Gaussian map

$$g_{N_C,Q_{|C}} : \mathrm{Ext}^1_C(\Omega^1_C \otimes N_C, \mathcal{O}_C) \to \mathrm{Ext}^1_C(M_{N_C\otimes Q}, Q_{|C}). \tag{2}$$

Theorem 1 *(a) If $H^1(X,Q) = 0$, then $e \in \ker(g_{N_C,Q_{|C}})$.*

(b) If L is sufficiently positive,[3] then the converse also holds: if $e \in \ker(g_{N_C,Q_{|C}})$, then $H^1(X,Q) = 0$.

(Note that e is nonzero if L is sufficiently positive.) For example, if X is a smooth surface with trivial canonical bundle and $Q = \mathcal{O}_X$, then (a) says that if X is a K3 then $e \in \ker(g_{K_C,\mathcal{O}_C})$. This is a result of [2]. Conversely, (b) says that if X is abelian and C is sufficiently positive then $e \notin \ker(g_{K_C,\mathcal{O}_C})$. This is a result of [3].

[2] The source is denoted $Rel(A,B)$, as it is the kernel of the multiplication of global sections of A and B.

[3] By this we mean that L is a sufficiently high multiple of a fixed ample line bundle on X.

1.2

The proof is a calculation with extension classes whose geometric motivation is as follows. Suppose that C is a curve in a surface X and that C is embedded in an ambient variety Z. From the cotangent sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to (\Omega^1_Z)_{|C} \to \Omega^1_C \to 0$$

(where $\mathcal{I}$ is the ideal of C in Z), one gets the long cohomology sequence

$$\cdots \to \mathrm{Hom}_C(\mathcal{I}/\mathcal{I}^2, N_C^\vee) \stackrel{H_Z}{\to} \mathrm{Ext}^1_C(\Omega^1_C, N_C^\vee) \stackrel{G_Z}{\to} \mathrm{Ext}^1_C((\Omega^1_Z)_{|C}, N_C^\vee) \to \cdots \quad (3)$$

The problem of extending the embedding $C \hookrightarrow Z$ to the surface X has a natural first-order obstruction, namely the class e must belong to $\ker G_Z = \mathrm{I}m\, H_Z$. Indeed, as is well known, if the divisor $2C$ on X, seen as a scheme, is embedded in Z, then it lives (as an embedded first-order deformation) in the Hom on the left.[4] The forgetful map H_Z, disregarding the embedding, takes it to the class of the normal sequence $e \in \mathrm{Ext}^1_C(\Omega^1_C, N_C^\vee)$.

Now we specialize this to the case where the ambient variety is a projective space, specifically:

$$Z = \mathbb{P}(H^0(C, N_C \otimes Q)^\vee) := \mathbb{P}_Q$$

(in this informal discussion we are assuming, for simplicity, that the line bundle $L \otimes Q$ is very ample). By the Euler sequence, the map $G_{\mathbb{P}_Q}$ is the (dual) Gaussian map $g_{N_C, Q_{|C}}$ of (2). Notice that in this case there is a special feature that our extension problem can be relaxed to the problem of extending the embedding of C in $\mathbb{P}_Q$ to an embedding of the surface X in a possibly bigger projective space $\mathbb{P}$, containing $\mathbb{P}_Q$ as a linear subspace. However, since the restriction to $\mathbb{P}_Q$ of the conormal sheaf of C in $\mathbb{P}$ splits, this has the same first-order obstruction, namely $e \in \ker(g_{N_C, Q_{|C}})$.

The relation of all that with the vanishing of the H^1 is classical: the embedding of C in $\mathbb{P}_Q$ can be extended (in the above relaxed sense) to an embedding of X if and only if the restriction map $\rho_X : H^0(X, L \otimes Q) \to H^0(C, N_C \otimes Q)$ is surjective. This is implied by the vanishing of $H^1(X, Q)$, so we get (a). The converse is a bit more complicated: by Serre vanishing, if L is sufficiently positive then the vanishing of $H^1(X, Q)$ is *equivalent* to the surjectivity of the restriction map ρ_X, and also to the surjectivity of the restriction map $\rho_{2C} : H^0(2C, (L \otimes Q)_{|2C}) \to H^0(C, N_C \otimes Q)$, hence to the fact that $2C$ "lives" in $\mathrm{Hom}_C(\mathcal{I}/\mathcal{I}^2, N_C^\vee)$. Now if e is in the kernel of $g_{N_C, Q_{|C}} = G_{\mathbb{P}_Q}$, then e comes from some embedded deformations in $\mathrm{Hom}_C(\mathcal{I}/\mathcal{I}^2, N_C^\vee)$. However,

[4] More precisely, the ideal of $2C$ in Z induces the morphism of $\mathcal{O}_Z$-modules $\mathcal{I}/\mathcal{I}^2 \to N_C$ whose kernel is $\mathcal{I}_{2C/Z}/\mathcal{I}^2$ (see, e.g., [1] or [4]).

these do not necessarily include $2C$. A more refined analysis proves that this is indeed the case as soon as L is sufficiently positive.

1.3 Gaussian maps on hyperplane sections and vanishing

The criterion of part (1) above is a generalization of the previous theorem to a higher dimension and to a relative (flat) setting. The relevant case deals with the vanishing of the H^n of a line bundle on a variety of dimension $n+1$.[5,6] To this end, we consider "hybrid" Gaussian maps as follows: let C be a curve in an n-dimensional variety Y and let A_C be a line bundle on C. The *Lazarsfeld sheaf* (see [7]), denoted $F^Y_{A_C}$, is the kernel of the evaluation map of A_C, *seen as a sheaf on* Y:

$$0 \to F^Y_{A_C} \to H^0(A_C) \otimes \mathcal{O}_Y \to A_C$$

(note that $F^Y_{A_C}$ is never locally free if $\dim Y \geq 3$). As above, it comes equipped with a $\mathcal{O}_Y$-linear differentiation map

$$F^Y_{A_C} \to \Omega^1_Y \otimes A_C.$$

If B is a line bundle on Y, we define the *Gaussian map of* A_C *and* B as

$$\gamma^Y_{A_C,B} : Rel(A_C, B) = H^0(Y, F_{A_C} \otimes B) \to H^0(Y, \Omega^1_Y \otimes A_C \otimes B).$$

As above, we rather use the dual map

$$g^Y_{M_C,R} : \mathrm{Ext}^n_Y(\Omega^1_Y \otimes M_C, \mathcal{O}_Y) \to \mathrm{Ext}^n_Y(F_{M_C \otimes R}, R),$$

where M_C and R are line bundles respectively on C and Y such that $A_C = M_C \otimes R$ and $B = \omega_Y \otimes R^\vee$. Again, this map can be defined directly (even if ω_Y is not a line bundle) as $\mathrm{Ext}^1_C(\cdot\,, R)$ of the differentiation map of $F^Y_{M_C \otimes R}$. The case $n = 1$ is recovered by taking $Y = C$.

These maps can be extended to a relative flat setting. In this paper we consider only the simplest case, namely a family of line bundles on a fixed variety Y, as this is the only case needed in subsequent applications. In the notation above, let T be another projective CM variety (or scheme), and let $\mathcal{R}$ be a line bundle on $Y \times T$. Let ν and π denote the two projections, respectively

[5] In fact, for all positive k, with $k < n$, the vanishing of H^k can be reduced to this case, as it is equivalent (by Serre vanishing) to the vanishing of H^k of the restriction of the given line bundle to a sufficiently positive $(k+1)$-dimensional hyperplane section.

[6] *Note*: One could think of using the equality $h^n(X, Q) = h^1(X, \omega_X \otimes Q^\vee)$ and then reducing, as in the previous footnote, to a surface. However, this is not possible in the relative case, since in general there is no Serre duality isomorphism of the direct images. Even in the non-relative case, the resulting criterion is usually more difficult to apply.

on Y and T. We can consider the *relative Lazarsfeld sheaf* $\mathcal{F}^Y_{M_C,\mathcal{R}}$, kernel of the relative evaluation map

$$0 \to \mathcal{F}^Y_{M_C,\mathcal{R}} \to \pi^*\pi_*(\mathcal{R} \otimes \nu^* M_C) \to \mathcal{R} \otimes \nu^* M_C$$

where, as above, we see M_C as a sheaf on Y. The $\mathcal{O}_{Y\times T}$-module $\mathcal{F}^Y_{M_C,\mathcal{R}}$ is equipped with its $\mathcal{O}_{Y\times T}$-linear differentiation map (see Section 2.1 below)

$$\mathcal{F}^Y_{M_C,\mathcal{R}} \to \nu^*(\Omega^1_Y \otimes M_C) \otimes \mathcal{R}. \tag{4}$$

Applying $\mathrm{Ext}^n_{Y\times T}(\,\cdot\,,\mathcal{R})$ and restricting to the direct summand $\mathrm{Ext}^n_Y(\Omega^1_Y \otimes M_C, \mathcal{O}_Y)$, we get the (dual) Gaussian map

$$g^Y_{M_C,\mathcal{R}} : \mathrm{Ext}^n_Y(\Omega^1_Y \otimes M_C, \mathcal{O}_Y) \to \mathrm{Ext}^n_{Y\times T}(\mathcal{F}^Y_{M_C,\mathcal{R}}, \mathcal{R}).$$

The announced generalization of Theorem 1 is as follows. Let X be an $(n+1)$-dimensional Cohen–Macaulay variety, let T be a CM variety, and let $\mathcal{Q}$ be a line bundle on $X\times T$. In order to avoid heavy notation, we still denote by ν and π the two projections of $X\times T$ (however, see Notation 1 in Section 2.1). Let L be a line bundle on X, with n irreducible effective divisors $Y_1, \ldots, Y_n \in |L|$ such that their intersection is an integral curve C not contained in the singular locus of X. We assume also that the line bundle $\mathcal{Q} \otimes \nu^* L^{\otimes n}$ is relatively base-point-free, namely the relative evaluation map $\pi^*\pi_*(\mathcal{Q} \otimes \nu^* L^{\otimes n}) \to \mathcal{Q} \otimes \nu^* L^{\otimes n}$ is surjective. We choose a divisor among $Y_1, \ldots, Y_n$, say $Y = Y_1$, such that C is not contained in the singular locus of Y. Let N_C denote the line bundle $L_{|C}$. We consider the "restricted normal sequence"

$$0 \to N_C^\vee \to (\Omega^1_X)_{|C} \to (\Omega^1_Y)_{|C} \to 0. \tag{5}$$

Via the canonical isomorphism

$$\mathrm{Ext}^1_C(\Omega^1_Y \otimes N_C, \mathcal{O}_C) \cong \mathrm{Ext}^n_Y(\Omega^1_Y \otimes N_C^{\otimes n}, \mathcal{O}_Y) \tag{6}$$

(see Section 2.1 below), we see that the class e of (5) belongs to $\mathrm{Ext}^n_Y(\Omega^1_Y \otimes N_C^{\otimes n}, \mathcal{O}_Y)$. Finally, we consider the (dual) Gaussian map

$$g^Y_{N_C^{\otimes n},\mathcal{Q}_{|Y\times T}} :\ \mathrm{Ext}^n_Y\left(\Omega^1_Y \otimes N_C^{\otimes n}, \mathcal{O}_Y\right) \to \mathrm{Ext}^n_{Y\times T}\left(\mathcal{F}^Y_{N_C^{\otimes n},\mathcal{Q}_{|Y\times T}}, \mathcal{Q}_{|Y\times T}\right). \tag{7}$$

Then we have the following result, recovering part (b) of Theorem 1 as the case $n = 1$ and $T = \{point\}$:

Theorem 2 *If L is sufficiently positive and e is an element of* $\ker(g^Y_{N_C^{\otimes n},\mathcal{Q}_{|Y\times T}})$, *then $R^n\pi_*\mathcal{Q} = 0$.*

The following version is technically easier to apply:

Corollary 3 *Keeping the notation of Theorem 2, if the line bundle L is sufficiently positive, then the kernel of the map* $g^Y_{N_C^{\otimes n},Q_{|Y\times T}}$ *is at most 1-dimensional (spanned by e). Therefore, if* $g^Y_{N_C^{\otimes n},Q_{|Y\times T}}$ *is non-injective, then* $R^n\pi_*Q = 0$.

Concerning the other implication, what we can prove is:

Proposition 4 *(a) Assume that* $T = \{point\}$*. If* $H^n(X,Q) = 0$*, then* $e \in \ker(g_{N_C^{\otimes n},Q_{|Y}})$.

(b) In general, assume that $R^i\pi_*(Q_{|Y\times T}) = 0$ *for* $i < n$*. If also* $R^n\pi_*Q = 0$*, then* $e \in \ker(g^Y_{N_C^{\otimes n},Q_{|Y\times T}})$.

1.4

To motivate these statements, let us go back to the informal discussion of Section 1.2. We assume for simplicity that $T = \{point\}$. Let X be an $(n+1)$-dimensional variety and C a curve in X as above. It is easily seen, using the Koszul resolution of the ideal of C and Serre vanishing, that the vanishing of $H^n(X,Q)$ implies the surjectivity of the restriction map $\rho_X : H^0(X, L^{\otimes n}\otimes Q) \to H^0(C, N_C^{\otimes n}\otimes Q)$, and in fact the two conditions are equivalent as soon as L is sufficiently positive. Hence it is natural to look for first-order obstructions to extend to X an embedding of the curve C (a 1-dimensional complete intersection of linearly equivalent divisors of X) into

$$\mathbb{P}_Q := \mathbb{P}(H^0(C, N_C^{\otimes n}\otimes Q)^\vee)\ .$$

More generally, we can consider the same problem for any given ambient variety Z, rather than projective space.

To find a first-order obstruction one can no longer replace X by the first-order neighborhood of C in X. We rather have to pick a divisor in $|L|$ containing C, say $Y = Y_1$, and replace X by the scheme $2Y\cap Y_2\cap\cdots\cap Y_n$. In analogy with the case of curves on surfaces, it is natural to consider the long cohomology sequence

$$\cdots\to \mathrm{Hom}_C\,\mathcal{I}_Y/\mathcal{I}_Y^2, N_C^\vee) \stackrel{H^Y_Z}{\to} \mathrm{Ext}^1_C((\Omega^1_Y)_{|C}, N_C^\vee) \stackrel{G^Y_Z}{\to} \mathrm{Ext}^1_C((\Omega^1_Z)_{|C}, N_C^\vee)\to\cdots \tag{8}$$

(where $\mathcal{I}_Y$ is the ideal of Y in Z). As above, a necessary condition for the lifting to X of the embedding of $C \hookrightarrow Z$ is that the "restricted normal class" e of (5) belongs to $\ker(G^Y_Z)$.

However, looking for *sufficient* conditions for lifting (in the relaxed sense, as in Section 1.2) the embedding $C \hookrightarrow \mathbb{P}_Q$ to X, one cannot assume that the divisor Y is already embedded in $\mathbb{P}_Q$. This is the reason why, differently from the case when X is a surface, the map $g_{N_C^{\otimes n}, Q_{|Y}}$ appearing in the statement of Theorem 2 and Corollary 3 is not the map G_Z^Y with $Z = \mathbb{P}_Q$, but rather a slightly more complicated "hybrid" version of a (dual) Gaussian map. After this modification, the geometric motivation for Theorem 2 is similar to that of Section 1.2.

1.5 Generic vanishing for subvarieties of abelian varieties

Although difficult – if not impossible – to use in most cases, the above results can be applied in some very special circumstances. For example, in analogy with the literature on curves sitting on K3 surfaces and Fano threefolds, Proposition 4 can supply nontrivial necessary conditions for an n-dimensional variety to sit in some very special $(n+1)$-dimensional varieties.

However, in this paper we rather focus on the sufficient condition for vanishing provided by Theorem 2 and Corollary 3, as it provides an approach to *generic vanishing*, a far-reaching concept introduced by Green and Lazarsfeld [5, 6]. Namely, we consider a variety X with a map to an abelian variety, generically finite onto its image

$$a : X \to A. \tag{9}$$

Denoting by $\mathrm{Pic}^0 A = \widehat{A}$ the dual variety, we consider the pull-back to $X \times \widehat{A}$ of a Poincaré line bundle $\mathcal{P}$ on $A \times \widehat{A}$:

$$Q = (a \times \mathrm{id}_{\widehat{A}})^* \mathcal{P}. \tag{10}$$

We keep the notation of the previous section. In particular, we denote by ν and π the projections of $X \times \widehat{A}$. A way of expressing generic vanishing is the vanishing of higher direct images

$$R^i \pi_* Q = 0 \quad \text{for} \quad i < \dim X. \tag{11}$$

For smooth varieties over the complex numbers, (11) was proved (as a particular case of a more general statement) by Hacon [7], settling a conjecture of Green and Lazarsfeld. Another way of expressing the generic vanishing condition involves the *cohomological support loci*

$$V_a^i(X) = \{\alpha \in \mathrm{Pic}^0 A \mid h^i(X, a^*\alpha) > 0\}.$$

Green and Lazarsfeld's theorem [5, 6] is that, if the map a is generically finite, then

$$\mathrm{codim}_{\widehat{A}} V_a^i(X) \geq \dim X - i.^7 \tag{12}$$

[7] In general, if the map a is not generically finite, Hacon's and Green and Lazarsfeld's theorems are respectively $R^i \pi_* Q = 0$ for $i < \dim a(X)$ and $\mathrm{codim}_{\mathrm{Pic}^0 A} V_a^i(X) \geq \dim a(X) - i$. However,

It is easy to see that (11) implies (12). Subsequently, it has been observed in [16, 17] that (11) is in fact *equivalent* to (12).[8] The heart of Hacon's proof of (11) consists of a clever reduction to Kodaira–Kawamata–Viehweg vanishing, while the argument of Green and Lazarsfeld for (12) uses Hodge theory. Both need characteristic 0, and that the variety X is smooth (or with rational singularities).

On the contrary, a characteristic-free example of both (11) and (12) is given by abelian varieties themselves [15, p. 127]. Here we extend this by proving that (11) (and, therefore, (12)) holds for normal Cohen–Macaulay subvarieties of abelian varieties on an algebraically closed field of any characteristic.

Theorem 5 *In the above notation, assume that X is normal Cohen–Macaulay and the morphism a is an embedding. Then $R^i\pi_*Q = 0$ for all $i < \dim X$.*

The strategy of the proof consists of applying Theorem 2 to the Poincaré line bundle Q. In order to do so we take a general complete intersection $C = Y \cap Y_2 \cap \cdots \cap Y_n$ of X, with $Y_i \in |L|$, where, as above, L is a sufficiently positive line bundle on X and $n + 1 = \dim X$. The main issue of the argument consists of comparing two spaces of first-order deformations: the first is the kernel of the (dual) Gaussian map $g_{N_C^{\otimes n}, Q_{Y\times\widehat{A}}}$. The second is the kernel of the map G^Y_Z of (8) with Z equal to the ambient abelian variety A[9] (by (6), the two maps have the same source). As in the discussion of Section 1.4, the variety $X \subset A$ induces naturally, via the restricted normal extension class e, a nontrivial element of $\ker G^Y_A$. Hence, to get the vanishing of $R^n\pi_*Q$, it would be enough to prove that $\ker G^Y_A$ is contained in $\ker g_{N_C^{\otimes n}, Q_{Y\times\widehat{A}}}$, or at least – in view of Corollary 3 – that the intersection of $\ker G^Y_A$ and $\ker g_{N_C^{\otimes n}, Q_{|Y\times\widehat{A}}}$ is nonzero. This analysis is accomplished by means of the Fourier–Mukai transform associated with the Poincaré line bundle.[10] In doing this we were inspired by the classical papers [10, 13] where the conceptually related problem of comparing the first-order embedded deformations of a curve in its Jacobian and the first-order deformations of the Picard bundle on the dual was solved.

The vanishing of $R^i\pi_*Q$ for $i < n$ follows from this step, after reducing to a sufficiently positive $(i + 1)$-dimensional hyperplane section.

Note that conditions (12) can be expressed dually as

$$\operatorname{codim}_{\operatorname{Pic}^0 A}\{\alpha \in \operatorname{Pic}^0 A \mid h^i(\omega_X \otimes \alpha) > 0\} \geq i \qquad \text{for all} \quad i > 0.$$

they can be reduced to the case of generically finite a by taking sufficiently positive hyperplane sections of dimension equal to the rank of a.

8 In [17] this is stated only in the smooth case, but this hypothesis is unnecessary.

9 This is simply the dual of the multiplication map $V \otimes H^0(N_C \otimes \omega_C) \to H^0(\Omega^1_Y \otimes N_C \otimes \omega_C)$, where V is the cotangent space of A at the origin.

10 We remark, incidentally, that (11) for abelian varieties (Mumford's theorem) is the key point assuring that the Fourier–Mukai transform is an equivalence of categories.

According to the terminology of [17], this is stated by saying that the dualizing sheaf ω_X is a GV-sheaf. As a first application of Theorem 2 we note that, combined with Proposition 3.1 of [18] ("GV tensor $IT_0 \Rightarrow IT_0$"), we get the following Kodaira-type vanishing:

Corollary 6 *Let X be a normal Cohen–Macaulay subvariety of an abelian variety A, and let L be an ample line bundle on A. Then $H^i(X, \omega_X \otimes L) = 0$ for all $i > 0$.*

The rest of this paper is organized as follows: in Section 2 we prove Theorem 2 (and Proposition 4). Section 3 contains the proof of Corollary 3. In Sections 4 and 5 we establish the setup of the argument for Theorem 5. In particular, we interpret Gaussian maps in terms of the Fourier–Mukai transform. The conclusion of the proof of Theorem 5 takes up Section 6.

It seems possible that these methods can find application in wider generality.

2 Proof of Theorem 2 and Proposition 4

2.1 Preliminaries

The argument consists of a computation with extension classes. The geometric motivation is outlined in the Introduction (Sections 1.2 and 1.4). To get a first idea of the argument, it could be helpful to have a look at the proof of Lemma 3.1 of [3].

Notation 1 In the first place, some warning about the notation. We have the three varieties $C \subset Y \subset X$ (respectively of dimension 1, n, and $n + 1$). The projections of $X \times T$ onto X and T are denoted respectively by ν and π. It will be different to consider the relative evaluation maps of a sheaf $\mathcal{A}$ on $C \times T$ seen as a sheaf on $Y \times T$, or on $X \times T$, or on $C \times T$ itself: their kernels are the various relative Lazarsfeld sheaves attached to $\mathcal{A}$ in different ambient varieties (see Section 1.3). Therefore, we denote

$$\pi_Y = \pi_{|Y\times T}, \qquad \pi_C = \pi_{|C\times T}.$$

For example, on $Y \times T$ we have

$$0 \to \mathcal{F}^Y_{A,Q_{|Y\times T}} \to \pi_Y^*\pi_*(Q \otimes \nu^*A) \to Q \otimes \nu^*A \tag{13}$$

while on $X \times T$

$$0 \to \mathcal{F}^X_{A,Q} \to \pi^*\pi_*(Q \otimes \nu^*A) \to Q \otimes \nu^*A. \tag{14}$$

Next, we clarify a few points appearing in the Introduction.

The differentiation map (4). We describe explicitly the differentiation map (4). We keep the notation there: M_C is a line bundle on the curve C while $\mathcal{R}$ is a line bundle on $Y \times T$. Now let p, q, and $\widetilde{\Delta}$ denote the two projections and the diagonal of the fibered product $(Y \times T) \times_T (Y \times T)$. Concerning the Lazarsfeld sheaf $\mathcal{F}^Y_{M_C,\mathcal{R}}$, we claim that there is a canonical isomorphism

$$\mathcal{F}^Y_{M_C,\mathcal{R}} \cong p_*(\mathcal{I}_{\widetilde{\Delta}} \otimes q^*(\mathcal{R} \otimes \nu^* M_C)). \tag{15}$$

Admitting the claim, the differentiation map (4) is defined as usual, as p_* of the restriction to $\widetilde{\Delta}$. The isomorphism (15): in the first place $p_*(\mathcal{I}_{\widetilde{\Delta}} \otimes q^*(\mathcal{R} \otimes \nu^* M_C))$ is the kernel of the map (p_* of the restriction map)

$$p_* q^*(\mathcal{R} \otimes \nu^* M_C) \to p_* q^*((\mathcal{R} \otimes \nu^* M_C)_{|\widetilde{\Delta}}) \cong \mathcal{R} \otimes \nu^* M_C \tag{16}$$

(it is easily seen that the sequence $0 \to \mathcal{I}_{\widetilde{\Delta}} \to \mathcal{O}_{Y \times_T Y} \to \mathcal{O}_{\widetilde{\Delta}} \to 0$ remains exact when restricted to $(Y \times T) \times_T (C \times T)$). To prove (15) we note that, by a flat base change,

$$\pi_Y^* \pi_*(\mathcal{R} \otimes \nu^* M_C) \cong p_* q^*(\mathcal{R} \otimes \nu^* M_C)$$

and, via such an isomorphism, the map (16) is identified with the relative evaluation map.

The isomorphism (6). This follows from the spectral sequence

$$\mathrm{Ext}^i_C(\Omega^1_Y \otimes N_C^{\otimes n}, \mathcal{E}xt^j_Y(\mathcal{O}_C, \mathcal{O}_Y)) \Rightarrow \mathrm{Ext}^{i+j}_Y(\Omega^1_Y \otimes N_C^{\otimes n}, \mathcal{O}_Y)$$

using the fact that, C being the complete intersection of $n-1$ divisors in the linear system $|L_{|Y}|$, we have $\mathcal{E}xt^j_Y(\mathcal{O}_C, \mathcal{O}_Y) = N_C^{\otimes n-1}$ if $j = n-1$ and zero otherwise. Seeing the elements of Ext-groups as higher extension classes with their natural multiplicative structure (Yoneda Exts; see, e.g., [12, Chapter III]), we denote by

$$\kappa \in \mathrm{Ext}^{n-1}_Y(\mathcal{O}_C, L_{|Y}^{\otimes -(n-1)}) \tag{17}$$

the extension class of the Koszul resolution of $\mathcal{O}_C$ as a $\mathcal{O}_Y$-module:

$$0 \to L_{|Y}^{\otimes -(n-1)} \to \cdots \to (L_{|Y}^{\otimes -1})^{\oplus n-1} \to \mathcal{O}_Y \to \mathcal{O}_C \to 0\ . \tag{18}$$

Then the multiplication by κ,

$$\mathrm{Ext}^1_C(\Omega^1_Y \otimes N_C, \mathcal{O}_C) \xrightarrow{\cdot\kappa} \mathrm{Ext}^n_Y(\Omega^1_Y \otimes N_C, L_{|Y}^{\otimes -(n-1)}) \cong \mathrm{Ext}^n_Y(\Omega^1_Y \otimes N_C^{\otimes n}, \mathcal{O}_Y),$$

is an isomorphism coinciding, up to scalar, with (6).

2.2 First step (statement)

Notation 2 From this point on we will adopt the hypotheses and notation of Theorem 2. We also adopt the following typographical abbreviations:

$$\mathcal{F}^Y = \mathcal{F}^Y_{N_C^{\otimes n}, Q_{|Y\times T}} \qquad \mathcal{F}^X = \mathcal{F}^X_{N_C^{\otimes n}, Q} \qquad g = g_{N_C^{\otimes n}, Q_{|Y\times T}}$$

The first, and most important, step of the proof of Theorem 2 and Proposition 4 consists of an explicit calculation of the class $g(e)$. This is the content of Lemma 7 below. The strategy is as follows. Applying $\mathrm{Ext}^n_{Y\times Y}(\,\cdot\,, Q_{|Y\times T})$ to the basic sequence

$$0 \to \mathcal{F}^Y \to \pi_Y^*\pi_*(Q\otimes \nu^* N_C^{\otimes n}) \to Q\otimes \nu^* N_C^{\otimes n} \to 0$$

(namely (13) for $A = N_C^{\otimes n}$ and $\mathcal{R} = Q_{|Y\times T}$ [11]), we get the following diagram with exact (in the middle) column

$$\begin{array}{ccc}
 & & \mathrm{Ext}^n_{Y\times T}(Q\otimes \nu^* N_C^{\otimes n}, Q_{|Y\times T}) \\
 & & \downarrow{\scriptstyle h} \\
 & & \mathrm{Ext}^n_{Y\times T}(\pi_Y^*\pi_*(Q\otimes \nu^* N_C^{\otimes n}), Q_{|Y\times T}) \\
 & & \downarrow{\scriptstyle f} \\
\mathrm{Ext}^n_Y(\Omega^1_Y\otimes N_C^{\otimes n}, \mathcal{O}_Y) & \xrightarrow{\ g\ } & \mathrm{Ext}^n_{Y\times T}(\mathcal{F}^Y, Q_{|Y\times T})
\end{array} \tag{19}$$

In Definition 8 below we produce a certain class b in the source of f, namely

$$b \in \mathrm{Ext}^n_{Y\times T}(\pi_Y^*\pi_*(Q\otimes \nu^* N_C^{\otimes n}), Q_{|Y\times T}) \tag{20}$$

such that its coboundary map

$$\delta_b : \pi_*(Q\otimes \nu^* \mathcal{N}_C^{\otimes n}) \to R^n\pi_*(Q_{|Y\times T})$$

is the composition

$$\begin{array}{ccc}
\pi_*(Q\otimes \nu^* \mathcal{N}_C^{\otimes n}) & \xrightarrow{\ \alpha\ } & R^n\pi_*(Q) \\
 & \searrow^{\delta_b} & \downarrow{\scriptstyle \beta} \\
 & & R^n\pi_*(Q_{|Y\times T})
\end{array} \tag{21}$$

where the horizontal map α is the coboundary map of the natural extension of $\mathcal{O}_{X\times T}$-modules

$$0 \to Q \to \cdots \to (Q\otimes \nu^* L^{\otimes n-1})^{\oplus n} \to Q\otimes \nu^* L^{\otimes n} \to Q\otimes \nu^* N_C^{\otimes n} \to 0 \tag{22}$$

(ν^* of the Koszul resolution of $\mathcal{O}_C$ as an $\mathcal{O}_X$-module, twisted by $Q\otimes\nu^* L^{\otimes n}$) and the vertical map β is simply $R^n\pi_*$ of the restriction map $Q \to Q_{|Y\times T}$. The main lemma is:

[11] The surjectivity on the right follows from the hypotheses of Theorem 2.

Lemma 7 $f(b) = g(e)$.

Note that this will already prove Proposition 4. Indeed, if $T = \{point\}$ then the vector space of (20) is

$$\mathrm{Ext}^n_Y(H^0(Q \otimes \nu^* N_C^{\otimes n}) \otimes O_Y, Q_{|Y}) \cong \mathrm{Hom}_k(H^0(Q \otimes \nu^* N_C^{\otimes n}), H^n(Y, Q_{|Y})). \quad (23)$$

Hence the class b coincides, up to scalar, with its coboundary map δ_b. From the description of δ_b we have that $\delta_b = 0$ if $H^n(X, Q) = 0$. If this is the case, then Lemma 7 says that $g(e) = 0$, proving Proposition 4 in this case. If $\dim T > 0$, we consider the spectral sequence

$$\mathrm{Ext}^i_T(\pi_*(Q \otimes N_C^{\otimes n}), R^j\pi_*(Q_{|Y\times T})) \Rightarrow \mathrm{Ext}^{i+j}_{Y\times T}(\pi_Y^*\pi_*(p^*(Q \otimes \nu^* N_C^{\otimes n})), Q_{|Y\times T})$$

coming from the isomorphism

$$\mathbf{R}\,\mathrm{Hom}_T(\pi_*(Q \otimes N_C^{\otimes n}), \mathbf{R}\pi_*(Q_{|Y\times T})) \cong \mathbf{R}\,\mathrm{Hom}_{Y\times T}(\pi_Y^*\pi_*(p^*(Q \otimes \nu^* N_C^{\otimes n})), Q_{|Y\times T}).$$

Since we are assuming that $R^i\pi_*(Q_{|Y\times T}) = 0$ for $i < n$, the spectral sequence degenerates, providing an isomorphism as (23), and Proposition 4 follows in the same way. □

Next, we give a definition of the class b of (20). In order to do so, we introduce some additional notation.

Notation 3 We denote by $\mathbf{K}^\bullet_{C,X}$ (resp. $\mathbf{H}^\bullet_{C,Y}$) the ν^* of the Koszul resolution of the ideal of C in X tensored with $Q \otimes \nu^* L^{\otimes n}$ (resp. the ν^* of the Koszul resolution of O_C as an O_Y-module, tensored with $Q \otimes \nu^* L_{|Y}^{\otimes n-1}$):

$$\begin{array}{llllll} \mathbf{K}^\bullet_{C,X} & 0 \to & Q & \to \cdots \to (Q \otimes \nu^* L^{\otimes n-2})^{\oplus\binom{n}{2}} \to & (Q \otimes \nu^* L^{\otimes n-1})^{\oplus n} \\ \mathbf{H}^\bullet_{C,Y} & 0 \to & Q_{|Y\times T} & \to \cdots \to (Q \otimes \nu^* L_{|Y}^{\otimes n-2})^{\oplus n-1} \to & Q \otimes \nu^* L_{|Y}^{\otimes n-1} \end{array}$$

(note that they have the same length). For example, with this notation the exact complex of $O_{X\times T}$-modules (22) is written as

$$\mathbf{K}^\bullet_{C,X} \to Q \otimes L^{\otimes n} \to Q \otimes \nu^* N_C^{\otimes n} \to 0. \quad (24)$$

Definition 8 (The class b of (20)) Composing (24) with the relative evaluation map of $Q \otimes \nu^* N_C^{\otimes n}$ (seen as a sheaf on $X \times T$),

$$\pi^*\pi_*(Q \otimes \nu^* N_C^{\otimes n}) \to Q \otimes \nu^* N_C^{\otimes n},$$

we get the commutative exact diagram

$$\begin{array}{ccccccc} \mathbf{K}^{\bullet}_{C,X} & \to & \mathcal{E} & \to & \pi^*\pi_*(Q\otimes\nu^*N_C^{\otimes n}) & \to 0 \\ \| & & \downarrow & & \downarrow & \\ \mathbf{K}^{\bullet}_{C,X} & \to & Q\otimes L^{\otimes n} & \to & Q\otimes\nu^*N_C^{\otimes n} & \to 0 \end{array} \tag{25}$$

where $\mathcal{E}$ is an $\mathcal{O}_{X\times T}$-module. Since $tor^i_{X\times T}(\pi^*\pi_*(Q\otimes\nu^*N_C^{\otimes n}),\nu^*\mathcal{O}_Y) = 0$ for $i > 0$, restricting the top row of (25) to $Y\times T$ we get an *exact* complex of $\mathcal{O}_{Y\times T}$-modules

$$(\mathbf{K}^{\bullet}_{C,X})_{|Y\times T} \to \mathcal{E}_{|Y\times T} \to \pi_Y^*\pi_*(Q\otimes\nu^*N_C^{\otimes n}) \to 0. \tag{26}$$

We define the class $b \in \mathrm{Ext}^n_{Y\times T}(\pi_Y^*\pi_*(Q\otimes\nu^*N_C^{\otimes n}), Q_{|Y\times T})$ of (20) as the extension class of the exact complex (26). The assertion about its coboundary map follows from its definition.

We will need the following:

Lemma 9 *The row of the following diagram*

$$\begin{array}{ccccc} \mathbf{H}^{\bullet}_{C,Y} & \longrightarrow & & \longrightarrow & \mathcal{E}_{|Y\times T} \to \pi_Y^*\pi_*(Q\otimes\nu^*N_C^{\otimes n}) \to 0 \\ & \searrow & & \nearrow & \\ & & Q\otimes\nu^*N_C^{\otimes n-1} & & \\ & \nearrow & & \searrow & \\ 0 & & & & 0 \end{array} \tag{27}$$

is an exact complex having the same extension class as (26), namely $b \in \mathrm{Ext}^n_{Y\times T}(\pi_Y^*\pi_*(Q\otimes\nu^*N_C^{\otimes n}), Q_{|Y\times T})$.

Proof For $n = 1$, i.e., $C = Y$, there is nothing to prove. For $n > 1$, recall that, by its definition, the top row of (25) is

$$\begin{array}{ccccc} \mathbf{K}^{\bullet}_{C,X} & \longrightarrow & & \longrightarrow & \mathcal{E} \to \pi^*\pi_*(Q\otimes\nu^*N_C^{\otimes n}) \to 0 \\ & \searrow & & \nearrow & \\ & & \mathcal{I}_{C/X}\otimes Q\otimes\nu^*L^{\otimes n} & & \\ & \nearrow & & \searrow & \\ 0 & & & & 0 \end{array}$$

Recalling that the curve C is the complete intersection $Y_1\cap\cdots\cap Y_n$, with $Y_i \in |L|$, and that $Y = Y_1$, restricting the ideal sheaf $\mathcal{I}_{C/X}$ to Y one gets $\mathcal{I}_{C/Y}\oplus N_C^{-1}$. Accordingly the Koszul resolution of $\mathcal{I}_{C/X}$, restricted to Y, splits as the direct sum of the Koszul resolution of $\mathcal{I}_{C/Y}$ and the Koszul resolution of $\mathcal{O}_C$, as an $\mathcal{O}_Y$-module, tensored with $L^{-1}_{|Y}$:

$$0\to \begin{array}{c} 0 \\ \oplus \\ L^{-n}_{|Y} \end{array} \to\cdots\to \begin{array}{c} (L^{-2}_{|Y})^{\oplus\binom{n-1}{2}} \\ \oplus \\ (L^{-2}_{|Y})^{\oplus n-1} \end{array} \to \begin{array}{c} (L^{-1}_{|Y})^{\oplus n-1} \\ \oplus \\ L^{-1}_{|Y} \end{array} \left(\to \begin{array}{c} \mathcal{I}_{C/Y} \\ \oplus \\ N_C^{-1} \end{array} \to 0 \right).$$

Now, restricting the exact complex (25) to Y one gets the exact complex (26) whose "tail," namely the exact complex $(\mathbf{K}^\bullet_{C,X})_{|Y\times T}$, splits as above. Therefore, deleting the exact complex corresponding to the above upper row one gets the equivalent – as an extension – exact complex (27). This proves the claim. $\square$

2.3 First step (proof)

In this section we prove Lemma 7. We first compute $g(e)$.[12] The exact sequences defining $\mathcal{F}^X$ and $\mathcal{F}^Y$ (see Notation 2) fit into the commutative diagram

$$\begin{array}{ccccccc} 0\to & \mathcal{F}^X & \to & \pi^*\pi_*(\mathcal{Q}\otimes\nu^*N_C^{\otimes n}) & \to & \mathcal{Q}\otimes\nu^*N_C^{\otimes n} & \to 0 \\ & \downarrow & & \downarrow & & \| & \\ 0\to & \mathcal{F}^Y & \to & \pi_Y^*\pi_*(\mathcal{Q}\otimes\nu^*N_C^{\otimes n}) & \to & \mathcal{Q}\otimes\nu^*N_C^{\otimes n} & \to 0 \end{array}$$

yielding, after restricting the top row to $Y\times T$, the exact sequence

$$0\to\mathcal{Q}\otimes\nu^*N_C^{\otimes n-1}\to(\mathcal{F}^X)_{|Y\times T}\to\mathcal{F}^Y\to 0 \tag{28}$$

where the sheaf on the left is $tor_1^{\mathcal{O}_{X\times T}}(\mathcal{Q}\otimes\nu^*N_C^{\otimes n},\mathcal{O}_{Y\times T})$.

This sequence in turn fits into the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0\to & \mathcal{Q}\otimes\nu^*N_C^{\otimes n-1} & \to & (\mathcal{F}^X)_{|Y\times T} & \to & \mathcal{F}^Y & \to 0 \\ & \| & & \downarrow & & \downarrow & \\ 0\to & \mathcal{Q}\otimes\nu^*N_C^{\otimes n-1} & \to & \mathcal{Q}\otimes\nu^*(\Omega^1_X\otimes N_C^{\otimes n}) & \to & \mathcal{Q}\otimes\nu^*(\Omega^1_Y\otimes N_C^{\otimes n}) & \to 0 \end{array}$$

where the class of the bottow row is $\nu^*(e)\in\nu^*\operatorname{Ext}^1_C(\Omega^1_Y\otimes N_C,\mathcal{O}_C)$. It follows that $g(e)$ (where now e is seen in $\operatorname{Ext}^n_Y(\Omega^1_Y\otimes N_C^{\otimes n},\mathcal{O}_Y)$, see (6) and Section 2.1)) is the class of the sequence (28) with $\mathbf{H}^\bullet_{C,Y}$ attached on the left:

$$\mathbf{H}^\bullet_{C,Y}\to(\mathcal{F}^X)_{|Y\times T}\to\mathcal{F}^Y\to 0\ . \tag{29}$$

Next, we compute $f(b)$. The exact complex (25) is the middle row of the commutative exact diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{F}^X & = & \mathcal{F}^X & & \\ & & \downarrow & & \downarrow & & \\ \mathbf{K}^\bullet_{C,X} & \to & \mathcal{E} & \to & \pi^*\pi_*(\mathcal{Q}\otimes\nu^*N_C^{\otimes n}) & \to 0 & \\ \| & & \downarrow & & \downarrow & & \\ \mathbf{K}^\bullet_{C,X} & \to & \mathcal{Q}\otimes\nu^*L^{\otimes n} & \to & \mathcal{Q}\otimes\nu^*N_C^{\otimes n} & \to 0 & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \tag{30}$$

[12] This argument follows [20, p. 252].

This provides us with the commutative exact diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ \mathbf{H}^{\bullet}_{C,Y} & \to & (\mathcal{F}^X)_{|Y\times T} & \to & \mathcal{F}^Y & \to 0 \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{E}_{|Y\times T} & \to & \pi_Y^*\pi_*(Q\otimes \nu^* N_C^{\otimes n}) & \to 0 \end{array}$$

where the long row is (29), whose class is $g(e)$. By Lemma 9, we can complete the above diagram as follows:

$$\begin{array}{ccccccc} \mathbf{H}^{\bullet}_{C,Y} & \to & (\mathcal{F}^X)_{|Y\times T} & \to & \mathcal{F}^Y & \to 0 \\ \| & & \downarrow & & \downarrow & & \\ \mathbf{H}^{\bullet}_{C,Y} & \to & \mathcal{E}_{|Y\times T} & \to & \pi_Y^*\pi_*(Q\otimes \nu^* N_C^{\otimes n}) & \to 0 \end{array}$$

and the class of the bottow row is b. By definition, the class of the top row is $f(b)$, and it is equal to $g(e)$. This proves Lemma 7. □

2.4 Conclusion of the proof of Theorem 2

The last step is:

Lemma 10 *We keep the notation and setting of Lemma 7. Assume that the line bundle L on X is sufficiently positive. If $f(b) = 0$ then $b = 0$.*

Assuming this, Theorem 2 follows: if $g(e) = 0$ then, by Lemmas 7 and 10, it follows that $b = 0$, hence its coboundary map $\delta_b = \beta\circ\alpha$ is zero (see (21)). Taking L sufficiently positive, Serre vanishing yields that α is surjective and β is injective. Therefore the target of δ_b, namely $R^n\pi_*Q$, is zero. □

Proof (of Lemma 10) The proof is a somewhat tedious repeated application of Serre vanishing. Going back to diagram (19), we have that if $f(b) = 0$ then there is a $c \in \mathrm{Ext}^n_{Y\times T}(Q\otimes \nu^* N_{|C}^{\otimes n}, Q_{|Y\times T})$ such that

$$h(c) = b \ . \tag{31}$$

Now we consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^n_{Y\times T}(Q\otimes \nu^* N_C^{\otimes n}, Q_{|Y\times T}) & \xrightarrow{r} & \mathrm{Ext}^n_{X\times T}(Q\otimes \nu^* L^{\otimes n}, Q_{|Y\times T}) \\ \downarrow h & & \downarrow h' \\ \mathrm{Ext}^n_{Y\times T}(\pi_Y^*\pi_*(Q\otimes \nu^* N_C^{\otimes n}), Q_{|Y\times T}) & \xrightarrow{s} & \mathrm{Ext}^n_{X\times T}(\pi^*\pi_*(Q\otimes \nu^* L^{\otimes n}), Q_{|Y\times T}) \\ \downarrow \mu & & \downarrow \mu' \\ \mathrm{Hom}(\pi_*(Q\otimes \nu^* N_C^{\otimes n}), R^n\pi_*Q_{|Y\times T}) & \xrightarrow{t} & \mathrm{Hom}(\pi_*(Q\otimes \nu^* L^{\otimes n}), R^n\pi_*Q_{|Y\times T}) \end{array} \tag{32}$$

where:
(a) h is as above and h' is the analogous map $\mathrm{Ext}^n_X(ev_X, Q_{|Y\times T})$, where ev_X is the relative evaluation map on $X \times T$: $\pi^*\pi_*(Q \otimes \nu^* L^{\otimes n}) \to Q \otimes \nu^* L^{\otimes n}$.
(b) μ is the map taking an extension to its coboundary map. Consequently, the map $\mu \circ h$ takes an extension class $e \in \mathrm{Ext}^n_{Y\times T}(Q \otimes \nu^* N_C^{\otimes n}, Q_{|Y\times T})$ to its coboundary map

$$\pi_*(Q \otimes \nu^* N_C^{\otimes n}) \to R^n\pi_*(Q_{|Y\times T}).$$

The map $\mu' \circ h'$ operates in the same way.
(c) Notice that the target of r is simply $H^n(\nu^* L^{\otimes -n}_{|Y\times T})$, i.e., $\mathrm{Ext}^n_{Y\times T}(\nu^* L^{\otimes n}_{|Y}, \mathcal{O}_{Y\times T})$. Via this identification, the map r is defined as the natural map

$$\mathrm{Ext}^n_{Y\times T}(\nu^* L^{\otimes n}_{|C}, \mathcal{O}_{Y\times T}) \to \mathrm{Ext}^n_{Y\times T}(\nu^* L^{\otimes n}_{|Y}, \mathcal{O}_{Y\times T}).$$

(d) s and t are the natural maps.

We know that the coboundary map of the extension class b factorizes through the natural coboundary map $\alpha : \pi_*(Q \otimes \nu^* N_C^{\otimes n}) \to R^n\pi_*(Q)$. This implies that $(t \circ \mu)(b) = 0$. Therefore, by (31) and (32), we have that $(\mu' \circ h' \circ r)(c) = 0$. The lemma will follow from the fact that both r and $\mu' \circ h'$ are injective.

Injectivity of r: In the case $n = 1$, i.e. $Y = C$, the map r is just the identity (cf. (c) above). Assume that $n > 1$. Chasing in the Koszul resolution of $\mathcal{O}_C$ as an $\mathcal{O}_Y$-module one finds that the injectivity of r holds as soon as $\mathrm{Ext}^{n-i}_{Y\times T}(\nu^* L^{\otimes n-i}_{|Y}, \mathcal{O}_{Y\times T}) = 0$ for $i = 1, \dots, n-1$. But these are simply $H^{n-i}(Y \times T, L^{\otimes i-n}_{|Y} \boxtimes \mathcal{O}_T)$ and the result follows easily from Künneth decomposition, Serre vanishing, and Serre duality.

Injectivity of $\mu' \circ h'$: We have that $\mathrm{Ext}^n_{X\times T}(Q \otimes \nu^* L^{\otimes n}, Q_{|Y\times T}) \cong$ $\cong H^n(Y \times T, L^{-n}_{|Y} \boxtimes \mathcal{O}_T)$. If L is sufficiently positive, it follows as above that this is isomorphic to $H^n(Y, L^{-n}_{|Y}) \otimes H^0(T, \mathcal{O}_T)$. Therefore, the map $\mu' \circ h'$ is identified with H^0 of the following map of $\mathcal{O}_T$-modules:

$$H^n(Y, L^{-n}_{|Y}) \otimes \mathcal{O}_T \to \mathcal{H}om_T(\pi_*(Q \otimes \nu^* L^{\otimes n}), R^n\pi_* Q_{|Y\times T}). \tag{33}$$

Hence the injectivity of $\mu' \circ h'$ holds as soon as (33) is injective at a general fiber. For a closed point $t \in T$, let $Q_t = Q_{|X\times\{t\}}$. By base change, the map (33) at a general fiber $X \times \{t\}$ is

$$H^n(Y, L^{\otimes -n}_{|Y}) \to H^0(X, Q_t \otimes L^{\otimes n})^\vee \otimes H^n(Y, Q_{t|Y}), \tag{34}$$

which is the Serre dual of the multiplication map of global sections

$$H^0(X, Q_t \otimes L^{\otimes n}) \otimes H^0(Y, (\omega_X \otimes Q_t^{-1} \otimes L)_{|Y}) \to H^0(Y, (\omega_X \otimes L^{\otimes n+1})_{|Y}). \tag{35}$$

At this point a standard argument with Serre vanishing shows that (35) is surjective as soon as L is sufficiently positive.[13] This proves the injectivity of $\mu' \circ h'$ and concludes also the proof of the lemma. □

3 Proof of Corollary 3

The deduction of Corollary 3 from Theorem 2 is a standard argument with Serre vanishing. However, there are some complications due to the weakness of the assumptions on the singularities of the variety X.

A Gaussian map on the ambient variety $X \times T$. The argument makes use of a (dual) Gaussian map defined on the ambient variety $X \times T$ itself. Namely, for a line bundle A on X we define $\mathcal{M}^X_{A,Q}$ as the kernel of the relative evaluation map

$$\pi^*\pi_*(Q \otimes \nu^* A) \to Q \otimes \nu^* A.$$

As in (4) and Section 2.1, there is the isomorphism

$$\mathcal{M}^X_{A,Q} \cong p_{X*}(\mathcal{I}_{\tilde{\Delta}_X} \otimes q_X^*(Q \otimes \nu^* A)) \tag{36}$$

(where p_X, q_X, and $\tilde{\Delta}_X$ denote the projections and the diagonal of $(X \times Y) \times_T (X \times T)$). There is also the differentiation map $\mathcal{M}^X_{A,Q} \to Q \otimes \nu^*(\Omega^1_X \otimes A)$.

Now, taking as $A = L^{\otimes n}$ and taking $\mathrm{Ext}^{n+1}_{X\times T}(\,\cdot\,, Q \otimes \nu^* L^\vee)$, we get the desired dual Gaussian map on X:

$$g_X : \mathrm{Ext}^{n+1}_X(\Omega^1_X \otimes L^{\otimes n+1}, \mathcal{O}_X) \to \mathrm{Ext}^{n+1}_{X\times T}(\mathcal{M}_{L^{\otimes n},Q}\,, Q \otimes \nu^* L^\vee).$$

Note that there are natural maps $\mathcal{M}_{L^{\otimes n},Q} \to \mathcal{F}^X \to \mathcal{F}^Y$ (see Notation 2).

First step. We consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^n_Y(\Omega^1_Y \otimes N_C^{\otimes n}, \mathcal{O}_Y) & \xrightarrow{\;g\;} & \mathrm{Ext}^n_{Y\times T}(\mathcal{F}^Y_{N_C^{\otimes n}, Q_{|Y\times T}}, Q_{|Y\times T}) \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \eta} \\
\mathrm{Ext}^{n+1}_X(\Omega^1_X \otimes L^{\otimes n+1}, \mathcal{O}_X) & \xrightarrow{\;g_X\;} & \mathrm{Ext}^{n+1}_{X\times T}(\mathcal{M}^X_{L^{\otimes n},Q}, Q \otimes \nu^* L^\vee)
\end{array} \tag{37}$$

where, as in the previous section, g denotes the main character, namely the (dual) Gaussian map $g_{N_C^{\otimes n}, Q_{Y\times T}}$. The maps μ and η are the natural ones, and the

[13] In brief, one shows that the desired surjectivity follows from the surjectivity of $H^0(X, Q_t \otimes L^{\otimes n}) \otimes H^0(X, \omega_X \otimes Q_t^{-1} \otimes L) \to H^0(X, \omega_X \otimes L^{\otimes n+1})$. This in turn is proved by interpreting such a multiplication map as the H^0 of a restriction-to-diagonal map of $\mathcal{O}_{X\times X}$-modules.

definition is left to the reader.[14] However such maps are more easily understood by considering the following commutative diagram, whose maps are the natural ones:

$$\begin{array}{ccc} H^d(\mathcal{M}^X_{L^{\otimes n},Q}\otimes Q^\vee\otimes((\omega_X\otimes L)\boxtimes\omega_T))) & \xrightarrow{g'_X} & H^0(\Omega^1_X\otimes L^{\otimes n+1}\otimes\omega_X)\otimes H^d(\omega_T) \\ \downarrow{\eta'} & & \downarrow{\mu'} \\ H^d(\mathcal{F}^Y_{N_C^{\otimes n},Q}\otimes Q^\vee\otimes(\omega_Y\boxtimes\omega_T)) & \xrightarrow{g'} & H^0(\Omega^1_Y\otimes L^{\otimes n}\otimes\omega_Y)\otimes H^d(\omega_T) \end{array} \tag{38}$$

where $d=\dim T$. (Note that, since X is Cohen–Macaulay, adjunction formulas for dualizing sheaves do hold.) Notice also that, if X and T are Gorenstein, then (38) is the dual of diagram (37).

As is easy to see, after tensoring with $\omega_C\otimes N_C$ the restricted normal sequence (5) remains exact:

$$0\to\omega_C\to\Omega^1_X\otimes L\otimes\omega_C\to\Omega^1_Y\otimes N_C\otimes\omega_C\to 0. \tag{39}$$

Therefore e defines naturally a linear functional on $H^0(\Omega^1_Y\otimes N_C\otimes\omega_C)$ (cf. (44) below), still denoted by e. We have

Claim 11 *If L is sufficiently positive, then the map g'_X is surjective, while cokerμ' is 1-dimensional, with (cokerμ')$^\vee$ spanned by e.*

Proof Serre vanishing ensures the surjectivity of the restriction

$$H^0(\Omega^1_X\otimes L^{\otimes n+1}\otimes\omega_X)\to H^0(\Omega^1_X\otimes L\otimes\omega_C).$$

Since the map μ' is the composition of the above map with H^0 of the right arrow of sequence (39), the claim for μ' follows.

Concerning the surjectivity of the map g'_X, we first note that by Serre vanishing,

$$R^ip_*(\mathcal{I}_{\widetilde{\Delta}_X}\otimes q_X^*(\nu^*L^{\otimes n}\otimes Q))=\begin{cases}0 & \text{for } i>0,\\ \text{locally free} & \text{for } i=0.\end{cases} \tag{40}$$

Now we project on T. A standard computation using (40), base change, Serre vanishing, Leray spectral sequence, and Künneth decomposition shows that the map g'_X is identified as

[14] For example, μ is defined by sending $\mathrm{Ext}^n_Y(\Omega^1_Y\otimes N_C^{\otimes n},O_Y)$ to $\mathrm{Ext}^n_X(\Omega^1_Y\otimes N_C^{\otimes n},O_Y)$ and then composing with the natural map $\Omega^1_X\otimes L^{\otimes n}\to\Omega^1_Y\otimes N_C^{\otimes n}$ on the left, and with the natural extension $0\to L^\vee\to O_X\to O_Y\to 0$ on the right.

$$H^d(T, \pi_*(p_*(\mathcal{I}_{\widetilde{\Delta}_X} \otimes q^*(\nu^* L^{\otimes n} \otimes \mathcal{Q})) \otimes \mathcal{Q}^\vee \otimes ((L \otimes \omega_X) \boxtimes \omega_T))) \\ \to H^d(T, H^0(X, \Omega^1_X \otimes L^{\otimes n+1} \otimes \omega_X) \otimes \omega_T). \quad (41)$$

This is the H^d of a map of coherent sheaves on the q-dimensional variety T. Hence the surjectivity of (41) is implied by the generic surjectivity of the map of sheaves itself. By base change, at a generic fiber $X \times t$ the map of sheaves is the Gaussian map

$$\gamma_t : H^0(X, p_*(\mathcal{I}_{\Delta_X} \otimes q^*(L^{\otimes n} \otimes \mathcal{Q}_t)) \otimes L \otimes \mathcal{Q}_t^\vee \otimes \omega_X) \\ \to H^0(X, \Omega^1_X \otimes L^{\otimes n+1} \otimes \omega_X).$$

The map γ_t is defined by restriction to the diagonal in the usual way. Once again it follows from relative Serre vanishing (on $(X \times T) \times_T (X \times T)$) that, as soon as L is sufficiently positive, γ_t is surjective for all t. This proves the surjectivity of the g'_X and concludes the proof of the claim. □

Last step. If C is Gorenstein, Claim 11 achieves the proof of Corollary 3. Indeed, diagram (37) is dual to diagram (38) and it follows that the kernel of our map $g = g^Y_{N_C^{\otimes n}, \mathcal{Q}}$ is at most 1-dimensional, spanned by e. In the general case, Corollary 3 follows in the same way once we have proved the following:

Claim 12 *As soon as L is sufficiently positive, the maps g'_X and μ' are respectively Serre duals of the maps g_X and μ.*

Proof To prove this assertion for g'_X we note that, concerning its source, the sheaf $\mathcal{M}^X_{L^{\otimes n}, \mathcal{Q}}$ is locally free by (40). Therefore,

$$\mathrm{Ext}^{n+1}_{X \times T}(\mathcal{M}^X_{L^{\otimes n}, \mathcal{Q}}, \mathcal{Q} \otimes \nu^* L^\vee) \cong H^{n+1}((\mathcal{M}^X_{L^{\otimes n}, \mathcal{Q}})^\vee \otimes \mathcal{Q} \otimes \nu^* L^\vee) \\ \cong H^d(\mathcal{M}^X_{L^{\otimes n}, \mathcal{Q}} \otimes \mathcal{Q}^\vee \otimes ((L \otimes \omega_X) \boxtimes \omega_T))^\vee. \quad (42)$$

Next, we show the Serre duality

$$H^0(X, \Omega^1_X \otimes \omega_X \otimes L^{\otimes n+1})^\vee \cong \mathrm{Ext}^{n+1}_X(\Omega^1_X \otimes L^{\otimes n}, L^\vee). \quad (43)$$

By definition of a dualizing complex (see, e.g., [9, Chapter V, Section 2, Proposition 2.1 on p. 258]), in the derived category of X we have that $\mathcal{O}_X = \mathbf{R}\mathcal{H}om(\omega_X, \omega_X)$. Therefore it follows that

$$\mathbf{R}\mathrm{Hom}_X(\Omega^1_X \otimes L^{\otimes n+1}, \mathcal{O}_X) = \mathbf{R}\mathrm{Hom}_X(\Omega^1_X \otimes L^{\otimes n+1}, \mathbf{R}\mathcal{H}om(\omega_X, \omega_X)) \\ = \mathbf{R}\mathrm{Hom}_X(\Omega^1_X \otimes \mathcal{O}_X \underline{\otimes}^{\mathbf{L}} \omega_X \otimes L^{\otimes n+1}, \omega_X).$$

By Serre–Grothendieck duality, this is isomorphic to

$$\mathbf{R}\mathrm{Hom}_k(\mathbf{R}\Gamma(X, \Omega^1_X \underline{\otimes}^{\mathbf{L}} \omega_X \otimes L^{\otimes n+1}[n+1]), k).$$

The spectral sequence computing $\mathbf{R}\Gamma(X, \Omega^1_X \underline{\otimes}^{\mathbf{L}} \omega_X \otimes L^{\otimes n+1})$ degenerates to the isomorphisms

$$H^i(X, \Omega^1_X \underline{\otimes}^{\mathbf{L}} \omega_X \otimes L^{\otimes n+1}) \cong \bigoplus_i H^0(X, tor_i^X(\Omega^1_X, \omega_X) \otimes L^{\otimes n+1})$$

(if L is sufficiently positive, by Serre vanishing there are only H^0s). Therefore, (43) follows. By (42) and (43) we have proved the part of the claim concerning g'_X.

Concerning μ', at this point it is enough to prove the Serre duality

$$\mathrm{Ext}^n_Y(\Omega^1_{X|C} \otimes L^n, \mathcal{O}_Y) \cong \mathrm{Ext}^1_C(\Omega^1_{Y|C} \otimes L, \mathcal{O}_C) \cong H^0(\Omega^1_C \otimes N \otimes \omega_C)^\vee \qquad (44)$$

where the first isomorphism is (6). Arguing as above, it is enough to prove that the $\mathcal{O}_C$-modules

$$tor^i_C((\Omega^1_Y)_{|C}, \omega_C) \otimes N_C$$

have vanishing higher cohomology for all i. For $i > 0$ this follows simply because they are supported on points. For $i = 0$ note that, by the exact sequence (39), it is enough to show that

$$H^1(\Omega^1_X \otimes N_C \otimes \omega_C) = 0. \qquad (45)$$

To prove this, we tensor the Koszul resolution of $\mathcal{O}_C$ as an $\mathcal{O}_X$-module with $\Omega^1_X \otimes \omega_X \otimes L^{\otimes n+1}$, getting a complex (exact at the last step on the right)

$$0 \to \Omega^1_X \otimes \omega_X \otimes L \to \cdots \to (\Omega^1_X \otimes \omega_X \otimes L^{\otimes n})^{\oplus n}$$
$$\to \Omega^1_X \otimes \omega_X \otimes L^{\otimes n+1} \to \Omega^1_X \otimes \omega_C \otimes N_C \to 0.$$

Since C is not contained in the singular locus of X, the cohomology sheaves are supported on points. Therefore the required vanishing (45) follows from Serre vanishing via a diagram chase. This concludes the proof of Claim 12 and of Corollary 3. □

4 Gaussian maps and the Fourier–Mukai transform

In this section we describe the setup of the proof of Theorem 5. We show that when the variety X is a subvariety of an abelian variety A, the parameter variety T is the dual abelian variety $\widehat{A}$, and the line bundle Q is the restriction to $X \times \widehat{A}$ of the Poincaré line bundle then the (dual) Gaussian map $g_{N_C^{\otimes n}, Q_{|Y\times T}}$ of the Introduction can naturally be interpreted as a piece of a (relative version of) the classical Fourier–Mukai transform associated with the Poincaré line bundle, applied to a certain space of morphisms.

Notation/Assumptions 1 We keep all the notation and hypotheses of the Introduction. Explicitly:

- Let X be an $(n+1)$-dimensional normal Cohen–Macaulay subvariety of a d-dimensional abelian variety A. As usual we choose an ample line bundle L on X such that we can find n irreducible divisors $Y = Y_1, Y_2, \ldots, Y_n \in X$ such that their intersection is an irreducible curve C. We assume also that C is not contained in the singular loci of X and Y. The line bundle $L_{|C}$ is denoted N_C.
- Let $\mathcal{P}$ be a Poincaré line bundle on $A \times \widehat{A}$. We denote

$$\mathcal{Q} = \mathcal{P}_{|X\times\widehat{A}} \qquad \text{and} \qquad \mathcal{R} = \mathcal{P}_{|Y\times\widehat{A}}.$$

- ν and π are the projections of $Y \times \widehat{A}$.
- We assume that the line bundle $\nu^*L^{\otimes n} \otimes \mathcal{Q}$ is relatively base-point-free, namely the evaluation map $\pi^*\pi_*(\nu^*L^{\otimes n} \otimes \mathcal{Q}) \to \nu^*L^{\otimes n} \otimes \mathcal{Q}$ is surjective (here ν and π denote also the projection of $X \times \widehat{A} \to \widehat{A}$).
- p, q, and $\widetilde{\Delta}$ are the projections and the diagonal of $(Y \times \widehat{A}) \times_{\widehat{A}} (Y \times \widehat{A})$.
- The Gaussian map of the Introduction (see (7)) is

$$g = g_{N_C^{\otimes n},\mathcal{R}} : \mathrm{Ext}^n_Y(\Omega^1_Y \otimes N_C^{\otimes n}, \mathcal{O}_Y) \to \mathrm{Ext}^n_{Y\times\widehat{A}}(p_*(q^*(\mathcal{I}_{\widetilde{\Delta}} \otimes \mathcal{R} \otimes \nu^* N_C^{\otimes n})), \mathcal{R}) \quad (46)$$

obtained as (the restriction to the relevant Künneth direct summand of) $\mathrm{Ext}^n_{Y\times\widehat{A}}(\,\cdot\,, \mathcal{R})$ of the differentiation (i.e., restriction to the diagonal) map (see Section 2.1). We recall also the identification of the source:

$$\mathrm{Ext}^n_Y(\Omega^1_Y \otimes N_C^{\otimes n}, \mathcal{O}_Y) \cong \mathrm{Ext}^1_C(\Omega^1_Y \otimes N_C, \mathcal{O}_C)$$

(see (6) and Section 2.1).
- The projections of $Y \times A$ will be denoted p_1 and p_2.

Remark 1 Since the variety X is assumed to be smooth in codimension 1, and our arguments concern a sufficiently positive line bundle L, we could have assumed from the beginning that the curve C is smooth and the divisor Y is smooth along C. However, we preferred to assume the smoothness of C only where needed, namely at the end of the proof. See also Remarks 2 and 5 below.

Fourier–Mukai transform. Now we consider the trivial abelian scheme $Y \times A \to Y$ and its dual $Y \times \widehat{A} \to Y$. The Poincaré line bundle $\mathcal{P}$ induces naturally a Poincaré line bundle $\tilde{\mathcal{P}}$ on $(Y \times A) \times_Y (Y \times \widehat{A})$ (namely the pull-back of $\mathcal{P}$ to $Y \times A \times \widehat{A}$) and we consider the functors

$$\mathbf{R}\Phi : \mathbf{D}(Y \times A) \to \mathbf{D}(Y \times \widehat{A}) \qquad \text{and} \qquad \mathbf{R}\Psi : \mathbf{D}(Y \times \widehat{A}) \to \mathbf{D}(Y \times A)$$

defined respectively by $\mathbf{R}\pi_{Y\times\widehat{A}*}(\pi^*_{Y\times A}(\,\cdot\,)\otimes\tilde{P})$ and $\mathbf{R}\pi_{Y\times A*}(\pi^*_{Y\times\widehat{A}}(\,\cdot\,)\otimes\tilde{P})$. By Mukai's theorem [14, Theorem 1.1] they are equivalences of categories, more precisely

$$\mathbf{R}\Psi\circ\mathbf{R}\Phi\cong(-1)^*[-q]\qquad\text{and}\qquad\mathbf{R}\Phi\circ\mathbf{R}\Psi\cong(-1)^*[-q].\tag{47}$$

In particular it follows that, given $\mathcal{O}_{Y\times A}$-modules $\mathcal{F}$ and $\mathcal{G}$, we have the functorial isomorphism

$$FM_i:\operatorname{Ext}^i_{Y\times A}(\mathcal{F},\mathcal{G})\xrightarrow{\cong}\operatorname{Ext}^i_{Y\times\widehat{A}}(\mathbf{R}\Phi(\mathcal{F}),\mathbf{R}\Phi(\mathcal{G}))\tag{48}$$

(note that the Ext-spaces on the right are usually hyperexts).

The Gaussian map. Now we focus on the target of the Gaussian map (46). Let $\Delta_Y\subset Y\times A$ be the graph of the embedding $Y\hookrightarrow A$. In other words, Δ_Y is the diagonal of $Y\times Y$, seen as a subscheme of $Y\times A$. It follows from the definitions that

$$\mathbf{R}\Phi(\mathcal{O}_{\Delta_Y})=\mathcal{P}_{|Y\times\widehat{A}}=\mathcal{R}.\tag{49}$$

Moreover, we have that

$$p_*(q^*(\mathcal{I}_{\widetilde{\Delta}}\otimes\mathcal{R}\otimes\nu^*N_C^{\otimes n}))\cong R^0\Phi(\mathcal{I}_{\Delta_Y}\otimes p_2^*N_C^{\otimes n}).\tag{50}$$

This is because of the natural isomorphisms

$$(Y\times Y)\times_Y(Y\times\widehat{A})\cong Y\times Y\times\widehat{A}\cong(Y\times\widehat{A})\times_{\widehat{A}}(Y\times\widehat{A})$$

yielding the identifications $\tilde{\mathcal{P}}_{|(Y\times Y)\times_Y(Y\times\widehat{A})}\cong q^*(\mathcal{P}_{|Y\times\widehat{A}})=q^*(\mathcal{R})$. Moreover, for any sheaf $\mathcal{F}$ supported on $Y\times C$ (as $\mathcal{I}_{\Delta_Y}\otimes p_2^*N_C^{\otimes n}$), we have that $R^i\Phi(\mathcal{F})=0$ for $i>1$. Therefore the fourth-quadrant spectral sequence

$$\operatorname{Ext}^p_{Y\times\widehat{A}}(R^q\Phi(\mathcal{F}),\mathcal{R})\Rightarrow\operatorname{Ext}^{p-q}(\mathbf{R}\Phi(\mathcal{F}),\mathcal{R})$$

is reduced to a long exact sequence

$$\begin{aligned}\cdots\to\operatorname{Ext}^{i-1}_{Y\times\widehat{A}}(R^0\Phi(\mathcal{F}),\mathcal{R})\to\operatorname{Ext}^{i+1}_{Y\times\widehat{A}}(R^1\Phi(\mathcal{F}),\mathcal{R})&\\ \to\operatorname{Ext}^i_{Y\times\widehat{A}}(\mathbf{R}\Phi(\mathcal{F}),\mathcal{R})\to\operatorname{Ext}^i_{Y\times\widehat{A}}(R^0\Phi(\mathcal{F}),\mathcal{R})\to\cdots&\end{aligned}\tag{51}$$

Putting all that together we get the following diagram, with right column exact in the middle:

$$\begin{array}{ccc}
\mathrm{Ext}^n_Y(\Omega^1_Y \otimes N_C^{\otimes n}, \mathcal{O}_Y) & & \\
\Big\downarrow {\scriptstyle =} & & \\
\mathrm{Ext}^n_{\Delta_Y}((\mathcal{I}_{\Delta_Y} \otimes p_2^* N_C^{\otimes n})_{|\Delta_Y}, \mathcal{O}_{\Delta_Y}) & & \mathrm{Ext}^{n+1}_{Y\times \widehat{A}}(R^1\Phi_{\widetilde{\mathcal{P}}}(\mathcal{I}_{\Delta_Y} \otimes p_2^* N_C^{\otimes n}), \mathcal{R}) \\
\Big\downarrow {\scriptstyle u} & & \Big\downarrow {\scriptstyle \alpha} \\
\mathrm{Ext}^n_{Y\times A}(\mathcal{I}_{\Delta_Y} \otimes p_2^* N_C^{\otimes n}, \mathcal{O}_{\Delta_Y}) & \xrightarrow[\cong]{FM_n} & \mathrm{Ext}^n_{Y\times \widehat{A}}(\mathbf{R}\Phi_{\widetilde{\mathcal{P}}}(\mathcal{I}_{\Delta_Y} \otimes p_2^* N_C^{\otimes n}), \mathcal{R}) \\
 & & \Big\downarrow {\scriptstyle \beta} \\
 & & \mathrm{Ext}^n_{Y\times \widehat{A}}(R^0\Phi_{\widetilde{\mathcal{P}}}(\mathcal{I}_{\Delta_Y} \otimes p_2^* N_C^{\otimes n}), \mathcal{R})
\end{array} \tag{52}$$

where u is the natural map (see also (65) below). In conclusion, the kernel of the Gaussian map can be described as follows:

Lemma 13 *The Gaussian map $g = g_{N_C^{\otimes n}, \mathcal{R}}$ of (46) is the composition $\beta \circ FM_n \circ u$. Therefore*

$$\ker(g) \cong Im(FM_n \circ u) \cap Im(\alpha).$$

Proof The identification of the two maps follows using (50), simply because they are defined in the same way. □

5 Cohomological computations on $Y \times A$

In this section we describe the source of the Fourier–Mukai map FM_n of diagram (52) above, together with other related cohomology groups. We use the Grothendieck duality (or change of rings) spectral sequence

$$\mathrm{Ext}^i_{Y\times C}(\mathcal{F}, \mathcal{E}xt^j_{Y\times A}(\mathcal{O}_{Y\times C}, \mathcal{O}_{\Delta_Y})) \Rightarrow \mathrm{Ext}^{i+j}_{Y\times A}(\mathcal{F}, \mathcal{O}_{\Delta_Y}). \tag{53}$$

With this in mind, we compute the sheaves $\mathcal{E}xt^i_{Y\times A}(\mathcal{O}_{Y\times C}, \mathcal{O}_{\Delta_Y})$ in Proposition 14 below.

5.1 Preliminaries

The following standard identifications will be useful:

$$\bigoplus_i \mathcal{E}xt^i_{Y\times A}(\mathcal{O}_{\Delta_Y}, \mathcal{O}_{\Delta_Y}) \cong \bigoplus_i \delta_*(\Lambda^i T_{A,0} \otimes \mathcal{O}_Y) \tag{54}$$

(as graded algebras), where $T_{A,0}$ is the tangent space of A at 0 and δ denotes the diagonal embedding

$$\delta : Y \hookrightarrow Y \times A.$$

This holds because Δ_Y is the pre-image of 0 via the difference map $Y \times A \to A$, $(y, x) \mapsto y - x$ (which is flat), and $\mathcal{E}xt^\bullet_A(k(0), k(0))$ is $\Lambda^\bullet T_{A,0} \otimes k(0)$.

Moreover, letting $\Delta_C \subset Y \times C \subset Y \times A$ the diagonal of $C \times C$ (seen as a subscheme of $Y \times A$), we have

$$\bigoplus_i \mathcal{E}xt^i_{Y\times A}(\mathcal{O}_{\Delta_C}, \mathcal{O}_{\Delta_Y}) \cong \bigoplus_i \delta_*(\Lambda^{i-n+1} T_{A,0} \otimes N_C^{\otimes n-1}) \tag{55}$$

(as graded modules on the above algebra). This is seen as follows: since C is the complete intersection of $n-1$ divisors of Y, all of them in $|L_{|Y}|$, then $\mathcal{E}xt^j_{\Delta_Y}(\mathcal{O}_{\Delta_C}, \mathcal{O}_{\Delta_Y}) = 0$ if $j \neq n-1$ and $\mathcal{E}xt^{n-1}_{\Delta_Y}(\mathcal{O}_{\Delta_C}, \mathcal{O}_{\Delta_Y}) = \delta_* N_C^{\otimes n-1}$. Therefore (55) follows from (54) and the spectral sequence

$$\mathcal{E}xt^h_{\Delta_Y}(\mathcal{O}_{\Delta_C}, \mathcal{E}xt^j_{Y\times A}(\mathcal{O}_{\Delta_Y}, \mathcal{O}_{\Delta_Y})) \Rightarrow \mathcal{E}xt^{h+j}_{Y\times A}(\mathcal{O}_{\Delta_C}, \mathcal{O}_{\Delta_Y}).$$

5.2 The (equisingular) restricted normal sheaf

We consider the $\mathcal{O}_C$-module $\mathcal{N}'$ defined by the sequence

$$0 \to (\mathcal{T}_Y)_{|C} \to (\mathcal{T}_A)_{|C} \to \mathcal{N}' \to 0. \tag{56}$$

When $Y = C$ the sheaf $\mathcal{N}'$ is usually called the *equisingular normal sheaf* [19, Proposition 1.1.9]. Therefore we refer to $\mathcal{N}'$ as the restricted equisingular normal sheaf.

Remark 2 Note that, since X is non-singular in codimension 1, the curve C can be taken to be smooth and the divisor Y smooth along C so that $\mathcal{N}'$ is locally free and it is the restriction to C of the normal sheaf of Y. Eventually we will make this assumption in the last section. However, the computations of the present section work in the more general setting.

The sheaves $\mathcal{E}xt^j_{Y\times A}(\mathcal{O}_{Y\times C}, \mathcal{O}_{\Delta_Y})$ appearing in (53) are described as follows:

Proposition 14 *(a)* $\bigoplus_i \mathcal{E}xt^i_{Y\times A}(\mathcal{O}_{Y\times C}, \mathcal{O}_{\Delta_Y}) \cong \bigoplus_i \delta_*(\overset{i-n+1}{\Lambda} \mathcal{N}' \otimes N_C^{\otimes n-1})$ *(as graded modules on the algebra (54)). In particular, the LHS is zero for* $i < n-1$.

(b) $\mathcal{E}xt^{d-1}_{Y\times A}(\mathcal{O}_{Y\times C}, \mathcal{O}_{\Delta_Y}) \cong \delta_*\omega_C$.

Proof (a) We apply $\mathcal{H}om_{Y\times A}(\,\cdot\,, \mathcal{O}_{\Delta_Y})$ to the basic exact sequence

$$0 \to \mathcal{I}_{\Delta_C/Y\times C} \to \mathcal{O}_{Y\times C} \to \mathcal{O}_{\Delta_C} \to 0 \tag{57}$$

where $\mathcal{I}_{\Delta_C/Y\times C}$ denotes the ideal of Δ_C in $Y\times C$. Since Δ_C is the intersection (in $Y\times A$) of Δ_Y and $Y\times C$, the resulting long exact sequence is chopped into short exact sequences (where we plug in the isomorphism (55))

$$0\to \mathcal{E}xt^{i-1}_{Y\times A}(\mathcal{I}_{\Delta_C/Y\times C},\mathcal{O}_{\Delta_Y})\to \delta_*(\overset{i-n+1}{\Lambda} T_{A,0}\otimes N_C^{\otimes n-1}) \\ \to \mathcal{E}xt^{i}_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y})\to 0. \quad (58)$$

This proves that

$$\mathcal{E}xt^{i}_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y}) = \begin{cases} 0 & \text{if } i<n-1,\\ \delta_* N_C^{\otimes n-1} & \text{if } i=n-1.\end{cases} \quad (59)$$

For $i=n$ it follows from (59) and the spectral sequence (53) applied to $\mathcal{I}_{\Delta_C/Y\times C}$ that

$$\mathcal{E}xt^{n-1}_{Y\times A}(\mathcal{I}_{\Delta_C/Y\times C},\mathcal{O}_{\Delta_Y}) \\ \cong \mathcal{H}om_{Y\times C}(\mathcal{I}_{\Delta_C/Y\times C},\mathcal{E}xt^{n-1}_{Y\times A}(\mathcal{O}_{Y\times C},\Delta_Y))\cong \delta_*(\mathcal{T}_Y\otimes N_C^{\otimes n-1})$$

and that δ_* identifies (58) with (56), tensored with $N_{|C}^{n-1}$, i.e.,

$$0\to \mathcal{T}_Y\otimes N_C^{\otimes n-1}\to T_{A,0}\otimes N_C^{\otimes n-1}\to \mathcal{N}'\otimes N_C^{\otimes n-1}\to 0. \quad (60)$$

This proves the statement for $i=n$. For $i>n$, Proposition 14 follows by induction. Indeed $\mathcal{E}xt^{\bullet}_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y})$ is naturally a graded module over the exterior algebra $\mathcal{E}xt^{\bullet}_{Y\times A}(\mathcal{O}_{\Delta_Y},\mathcal{O}_{\Delta_Y})\cong \delta_*(\Lambda^{\bullet}T_{A,0}\otimes \mathcal{O}_Y)$ (see (54)). Assume that the statement of the present proposition holds for the positive integer $i-1$. Because of the action of the exterior algebra, sequences (58) and (60) yield that the kernel of the map

$$\delta_*(\overset{i-n+1}{\Lambda} T_{A,0}\otimes N_C^{\otimes n-1})\to \mathcal{E}xt^{i}_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta})\to 0$$

is surjected (up to twisting with $N_C^{\otimes n-1}$) by $\delta_*(\Lambda^{i-n}T_{A,0}\otimes(\mathcal{T}_Y)_{|C})$. This presentation yields that $\mathcal{E}xt^{i}_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta})$ is equal to $\delta_*((\overset{i-n+1}{\Lambda}\mathcal{N}')\otimes N_C^{\otimes n-1})$. This proves (a).

(b) If Y is smooth along C then $\mathcal{N}'$ is locally free (coinciding with the restricted normal bundle) (see Remark 2). In this case (b) follows at once from (a). In the general case the proof is as follows. We claim that for each i the LHS of (a) can alternatively be described as

$$\mathcal{E}xt^{i}_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y})\cong \mathcal{T}or^{Y\times A}_{d-1-i}(p_2^*\omega_C,\mathcal{O}_{\Delta_Y}).$$

This is proved by means of the isomorphism of functors

$$\mathbf{R}\,\mathrm{Hom}_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y})\cong \mathbf{R}\,\mathrm{Hom}_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{Y\times A})\underline{\otimes}^{\mathbf{L}}_{Y\times A}\mathcal{O}_{\Delta_Y}$$

and the corresponding spectral sequences. In fact, since C is Cohen–Macaulay, we have that $\mathcal{E}xt^i_{Y\times A}(\mathcal{O}_{Y\times C}, \mathcal{O}_{Y\times A}) = 0$ for $i \neq d-1$ and equal to $p_2^*\omega_C$ for $i = d-1$. Thus the spectral sequence computing the RHS degenerates, proving the claim. In particular, for $i = d-1$, we have $\mathcal{E}xt^{d-1}_{Y\times A}(\mathcal{O}_{Y\times C}, \mathcal{O}_{\Delta_Y}) \cong (p_2^*\omega_C)\otimes \mathcal{O}_{\Delta_Y} \cong \delta_*\omega_C$. □

5.3 Reduction of the statement of Theorem 5

As a first application of Proposition 14, we reduce the statement of Theorem 5 – in the equivalent formulation provided by Lemma 13 – to a simpler one. This will involve the issue of comparing two spaces of first-order deformations mentioned in the Introduction (Section 1.5), and it will be the content of Proposition 15 and Corollary 16 below.

Notation 4 We consider the first spectral sequence (53) applied to $\mathcal{F} = p_2^*N_C^{\otimes n}$, rather than to $\mathcal{I}_\Delta \otimes p_2^*N_C^{\otimes n}$. Plugging the identification provided by Lemma 14, we get

$$H^j(C, \Lambda^{i-n+1}\mathcal{N}'\otimes N_C^{-1}) \Rightarrow \mathrm{Ext}^{j+i}_{Y\times A}(p_2^*N_C^{\otimes n}, \mathcal{O}_{\Delta_Y}).$$

Since the H^is on the left are zero for $i \neq 0, 1$, the spectral sequence is reduced to short exact sequences

$$0 \to H^1(C, \Lambda^{i-n}\mathcal{N}'\otimes N_C^{-1}) \xrightarrow{v_i} \mathrm{Ext}^i_{Y\times A}(p_2^*N_C^{\otimes n}, \mathcal{O}_{\Delta_Y}) \xrightarrow{w_i} H^0(C, \Lambda^{i-n+1}\mathcal{N}'\otimes N_C^{-1}) \to 0. \quad (61)$$

In particular, for $i = n$ we have the exact sequence

$$0 \to H^1(C, N_C^{-1}) \xrightarrow{v_n} \mathrm{Ext}^n_{Y\times A}(p_2^*N_C^{\otimes n}, \mathcal{O}_{\Delta_Y}) \xrightarrow{w_n} H^0(C, \mathcal{N}'\otimes N_C^{-1}) \to 0.$$

Combining with the exact sequence coming from the spectral sequence (51), applied to $\mathcal{F} = p_2^*N_C^{\otimes n-1}$ we get

$$\begin{array}{ccc} H^1(C, N_C^{-1}) & & \mathrm{Ext}^{n+1}_{Y\times\widehat{A}}(R^1\Phi(p_2^*N_C^{\otimes n}), \mathcal{R}) \\ \downarrow v_n & & \downarrow a_n \\ \mathrm{Ext}^n_{Y\times A}(p_2^*N_C^{\otimes n}, \mathcal{O}_{\Delta_Y}) & \xrightarrow[\cong]{FM_n} & \mathrm{Ext}^n_{Y\times\widehat{A}}(\mathbf{R}\Phi(p_2^*N_C^{\otimes n}), \mathcal{R}) \\ \downarrow w_n & & \downarrow b_n \\ H^0(C, \mathcal{N}'\otimes N_C^{-1}) & & \mathrm{Ext}^n_{Y\times\widehat{A}}(R^0\Phi(p_2^*N_C^{\otimes n}), \mathcal{R}) \end{array} \quad (62)$$

Remark 3 Note that, as shown by the exact sequence (56) defining the restricted equisingular normal sheaf, we get that

$$H^0(C, \mathcal{N}'\otimes N_C^\vee) = \ker(H^1(C, \mathcal{T}_Y\otimes N_C^\vee) \xrightarrow{G} H^1(C, \mathcal{T}_A\otimes N_C^\vee)).$$

This map G is the restriction to $H^1(C, \mathcal{T}_Y \otimes N_C^\vee)$ of the map G_A^Y of (8) in the Introduction (see also Remark 4 below).

Proposition 15 *In diagram (62), if the map a_n is nonzero then the map $w_n \circ FM_n^{-1} \circ a_n$ is nonzero and its image is contained in the kernel of the Gaussian map (46).*

Combining with Theorem 2, and noting that the assumptions in Notation/Assumptions 1 are certainly satisfied by a sufficiently positive line bundle L on the variety X, we get

Corollary 16 *If the map a_n is nonzero then $R^{n-1}\pi_* \mathcal{Q} = 0$.*

Proof of Proposition 15 We apply $\mathrm{Ext}^n_{Y\times A}(\cdot, \mathcal{O}_{\Delta_Y})$ to the usual exact sequence

$$0 \to \mathcal{I}_{\Delta_C/Y\times C} \otimes p_2^* N_C^{\otimes n} \to p_2^* N_C^{\otimes n} \to \delta_* N_C^{\otimes n} \to 0. \tag{63}$$

Using the spectral sequence (53) and the isomorphisms provided by Proposition 14, we get the commutative exact diagram

$$\begin{array}{ccccc}
H^1(N_C^{-1}) \otimes \Lambda^0 T_{A,0} & \xrightarrow{=} & H^1(N_C^{-1}) \otimes \Lambda^0 T_{A,0} & & \\
\downarrow{=} & & \downarrow & & \\
H^1(N_C^{-1}) \otimes \Lambda^0 T_{A,0} & \overset{v_n}{\hookrightarrow} & \mathrm{Ext}^n_{Y\times A}(p_2^* N_C^{\otimes n}, \mathcal{O}_{\Delta_Y}) & \overset{w_n}{\twoheadrightarrow} & H^0(\mathcal{N}' \otimes N_C^{-1}) \\
 & & \downarrow f & & \uparrow \\
 & & \mathrm{Ext}^n_{Y\times A}(\mathcal{I} \otimes p_2^* N_C^{\otimes n}, \mathcal{O}_{\Delta_Y}) & \overset{u}{\hookleftarrow} & H^1(T_Y \otimes N_C^{-1}) \\
 & & \downarrow & & \downarrow G_A^Y \\
 & & H^1(N_C^{-1}) \otimes T_{A,0} & \xrightarrow{=} & H^1(N_C^{-1}) \otimes T_{A,0}
\end{array} \tag{64}$$

where:

- We have used (55) to compute

$$\mathrm{Ext}^i_{Y\times A}(\delta_{C*} N_C^{\otimes n}, \mathcal{O}_{\Delta_Y}) \cong H^1(C, N_C^{-1}) \otimes \Lambda^{i-n+1} T_A.$$

- For typographical brevity we have denoted

$$\mathcal{I} := \mathcal{I}_{\Delta_C/Y\times C}$$

and the map

$$u : H^1(\mathcal{T}_Y \otimes N_C^{-1}) \rightarrowtail \mathrm{Ext}^n_{Y\times A}(\mathcal{I} \otimes p_2^* N_C^{\otimes n}, \mathcal{O}_{\Delta_Y}) \tag{65}$$

is the composition of the natural inclusion

$$\begin{aligned} H^1(\mathcal{T}_Y \otimes N_C^{-1}) = H^1(\mathcal{H}om(\mathcal{I} \otimes p_2^* N_C, \mathcal{O}_{\Delta_C})) &\hookrightarrow \mathrm{Ext}^1_{Y\times C}(\mathcal{I}, \delta_{C*} N_C^{-1}) \\ &\cong \mathrm{Ext}^1_{Y\times C}(\mathcal{I} \otimes p_2^* N_C^{\otimes n}, \mathcal{E}xt^{n-1}_{Y\times A}(\mathcal{O}_{Y\times C}, \mathcal{O}_{\Delta_Y})) \end{aligned}$$

and of the natural injection, arising (by Proposition 14, as the last isomorphism) in the beginning of the spectral sequence (53),

$$\mathrm{Ext}^1_{Y\times C}(\mathcal{I}_{\Delta_C/Y\times C}\otimes p_2^*N_C^{\otimes n}, \mathcal{E}xt^{n-1}_{Y\times A}(\mathcal{O}_{Y\times C},\mathcal{O}_{\Delta_Y})) \\ \to \mathrm{Ext}^n_{Y\times A}(\mathcal{I}_{\Delta_C/Y\times C}\otimes p_2^*N_C^{\otimes n},\mathcal{O}_{\Delta_Y})\ .$$

Next, we look at the Fourier–Mukai image of the central column of (64). In order to do so, we first apply the Fourier–Mukai transform $\mathbf{R}\Phi$ to sequence (63). Then we apply $\mathbf{R}\mathrm{Hom}_{Y\times\widehat{A}}(\cdot\ ,\ \mathcal{R})$ and the spectral sequence on the $Y\times\widehat{A}$ side, namely (51).

We claim that applying the Fourier–Mukai transform $\mathbf{R}\Phi$ to sequence (63), we get the exact sequence

$$0\to R^0\Phi(\mathcal{I}\otimes p_2^*N_C^{\otimes n})\to R^0\Phi(p_2^*N_C^{\otimes n})\to \nu^*(N_C^{\otimes n})\otimes\mathcal{R}\to 0 \qquad (66)$$

and the isomorphism

$$R^1\Phi(\mathcal{I}\otimes p_2^*N_C^{\otimes n})\ \tilde{\to}\ R^1\Phi(p_2^*N_C^{\otimes n}). \qquad (67)$$

Indeed we have that $R^i\Phi(\delta_*(N_C^{\otimes n}))=\nu^*(N_C^{\otimes n})\otimes\mathcal{R}$ for $i=0$ and zero otherwise. The map $R^0\Phi(p_2^*N_C^{\otimes n})\to\nu^*(N_C^{\otimes n})\otimes\mathcal{R}$ is nothing else but the relative evaluation map

$$\pi^*\pi_*(\nu^*(N_C^{\otimes n})\otimes\mathcal{R})\to\nu^*(N_C^{\otimes n})\otimes\mathcal{R}$$

and its surjectivity follows from the assumptions (see Notation/Assumptions 1). This proves what was claimed.

Eventually we get the following exact diagram, whose central column is the Fourier–Mukai transform of the central column of (64) and whose right column is (part of) the long cohomology sequence of $\mathbf{R}\mathrm{Hom}_{Y\times\widehat{A}}(\cdot\ ,\ \mathcal{R})$ applied to the exact sequence (66):

$$\begin{array}{ccccc}
 & & H^1(N_C^\vee)\otimes H^0(\mathcal{O}_{\widehat{A}}) & \xrightarrow{=} & H^1(N_C^\vee)\otimes H^0(\mathcal{O}_{\widehat{A}}) \\
 & & \downarrow & & \downarrow \\
\mathrm{Ext}^{n+1}(R^1\Phi(p_2^*N_C^{\otimes n}),\mathcal{R}) & \xrightarrow{a_n} & \mathrm{Ext}^n(\mathbf{R}\Phi(p_2^*N_C^{\otimes n}),\mathcal{R}) & \xrightarrow{b_n} & \mathrm{Ext}^n(R^0\Phi(p_2^*N_C^{\otimes n}),\mathcal{R}) \\
\downarrow\cong & & \downarrow FM_n(f) & & \downarrow \\
\mathrm{Ext}^{n+1}(R^1\Phi(p_2^*N_C^{\otimes n}),\mathcal{R}) & \xrightarrow{\alpha} & \mathrm{Ext}^n(\mathbf{R}\Phi(\mathcal{I}\otimes p_2^*N_C^{\otimes n}),\mathcal{R}) & \xrightarrow{\beta} & \mathrm{Ext}^n(R^0\Phi(\mathcal{I}\otimes p_2^*N_C^{\otimes n}),\mathcal{R}) \\
 & & \downarrow & & \downarrow \\
 & & H^1(N_C^\vee)\otimes H^1(\mathcal{O}_{\widehat{A}}) & \xrightarrow{=} & H^1(N_C^\vee)\otimes H^1(\mathcal{O}_{\widehat{A}})
\end{array} \qquad (68)$$

For brevity, at the place on the left of the third row we have plugged the isomorphism (67). From (66) and (67) it follows, in particular, that the map $FM_n(f)$ induces the isomorphism of the images of a_n and α:

$$FM_n(f) : \mathrm{im}(a_n) \xrightarrow{\cong} \mathrm{im}(\alpha). \tag{69}$$

An easy diagram chase in (64) and (68) proves the first part of the proposition, namely that if the map a_n is nonzero then the map $w_n \circ FM_n^{-1} \circ a_n$ is nonzero. The second part follows at once from the first one, (69), and Lemma 13. □

6 Proof of Theorem 5

The strategy of proof of Theorem 5 is to see the two vertical exact sequences of diagram (62) as the first homogeneous pieces of two exact sequences of graded modules over the exterior algebra. Namely, for each $i \geq n$ we have

$$\begin{array}{ccc}
\bigoplus_i \mathrm{Ext}^1_C(N_C^{\otimes n}, \Lambda^{i-n}\mathcal{N}' \otimes N_C^{\otimes n-1}) & & \bigoplus_i \mathrm{Ext}^{i+1}_{Y\times\widehat{A}}(R^1\Phi(p_2^* N_C^{\otimes n}), \mathcal{R}) \\
\downarrow v_i & & \downarrow a_i \\
\bigoplus_i \mathrm{Ext}^i_{Y\times A}(p_2^* N_C^{\otimes n}, \mathcal{O}_{\Delta_Y}) & \xrightarrow[\cong]{FM_i} & \bigoplus_i \mathrm{Ext}^i_{Y\times\widehat{A}}(\mathbf{R}\Phi(p_2^* N_C^{\otimes n}), \mathcal{R}) \\
\downarrow w_i & & \downarrow b_i \\
\bigoplus_i \mathrm{Hom}_C(N_C^{\otimes n}, \Lambda^{i-n+1}\mathcal{N}' \otimes N_C^{\otimes n-1}) & & \bigoplus_i \mathrm{Ext}^i_{Y\times\widehat{A}}(R^0\Phi(p_2^* N_C^{\otimes n}), \mathcal{R})
\end{array} \tag{70}$$

The exterior algebra acts on the LHS as $\Lambda^\bullet T_{A,0} \hookrightarrow \mathrm{Ext}^\bullet_{Y\times A}(\mathcal{O}_{\Delta_Y}, \mathcal{O}_{\Delta_Y})$ (see (54) and (55)). After the Fourier–Mukai transform, it acts on the RHS as $\Lambda^\bullet H^1(\mathcal{O}_{\widehat{A}}) \hookrightarrow \mathrm{Ext}^\bullet_{Y\times\widehat{A}}(\mathcal{R}, \mathcal{R})$.

6.1 Computations in degree $d-1$

In this section we will make some explicit calculations in degree $d-1$, where we have the special feature that the Hom space at the bottom of the left column is naturally isomorphic to $\mathrm{Hom}(N_C^{\otimes n}, \omega_C)$ (Proposition 14, part (b)). The following proposition shows that what we want to prove in degree n, namely that the map $w_n \circ FM_n^{-1} \circ a_n$ is nonzero, is true, in strong form, in degree $d-1$.

Proposition 17 *The map w_{d-1} has a canonical (up to scalar) section σ and the injective map $(FM_{d-1})_{|Im(\sigma)}$ factorizes through a_{d-1}. Summarizing, in degree $i = d-1$ diagram (70) specializes to*

$$\begin{array}{ccc}
\mathrm{Ext}^1_C(N_C^{\otimes n}, \Lambda^{d-1-n}\mathcal{N}' \otimes N_C^{\otimes n-1}) & \overset{\tau}{\nearrow} & \mathrm{Ext}^d_{Y\times\widehat{A}}(R^1\Phi(p_2^* N_C^{\otimes n}), \mathcal{R}) \\
\downarrow v_{d-1} & & \downarrow a_{d-1} \\
\mathrm{Ext}^{d-1}_{Y\times A}(p_2^* N_C^{\otimes n}, \mathcal{O}_{\Delta_Y}) & \underset{\cong}{\overset{FM_{d-1}}{\longleftrightarrow}} & \mathrm{Ext}^{d-1}_{Y\times\widehat{A}}(\mathbf{R}\Phi(p_2^* N_C^{\otimes n}), \mathcal{R}) \\
\sigma \curvearrowleft \quad \downarrow w_{d-1} & & \downarrow b_{d-1} \\
\mathrm{Hom}_C(N_C^{\otimes n}, \omega_C) & & \mathrm{Ext}^d_{Y\times\widehat{A}}(R^0\Phi(p_2^* N_C^{\otimes n}), \mathcal{R})
\end{array} \tag{71}$$

Proof The section σ is given (up to scalar) by the product map

$$\mathrm{Ext}^{d-1}_{Y\times A}(p_2^*N_C^{\otimes n},\mathcal{O}_{Y\times A})\otimes \mathrm{Hom}_{Y\times A}(\mathcal{O}_{Y\times A},\mathcal{O}_{\Delta_Y})\xrightarrow{\sigma}\mathrm{Ext}^{d-1}_{Y\times A}(p_2^*N_C^{\otimes n},\mathcal{O}_{\Delta_Y}). \quad (72)$$

In fact, note that $\mathrm{Ext}^{d-1}_{Y\times A}(p_2^*N_C^{\otimes n},\mathcal{O}_{Y\times A})\cong p_1^*H^0(\mathcal{O}_Y)\otimes p_2^*\,\mathrm{Ext}^{d-1}_A(N_C^{\otimes n},\mathcal{O}_A)\cong p_1^*H^0(\mathcal{O}_Y)\otimes p_2^*\mathrm{Hom}_C(N_C^{\otimes n},\omega_C)$. The fact that s is a section of w_{q-1} is clear, as the latter is the natural map

$$\mathrm{Ext}^{d-1}_{Y\times A}(p_2^*N_C^{\otimes n},\mathcal{O}_{\Delta_Y})\to H^0(\mathcal{E}xt^{d-1}_{Y\times A}(p_2^*N_C^{\otimes n}),\mathcal{O}_{\Delta_Y})\cong$$

$$\cong H^0(\mathcal{E}xt^{d-1}_{Y\times A}(p_2^*N_C^{\otimes n},\mathcal{O}_{Y\times A})\otimes\mathcal{O}_{\Delta_Y})\cong\mathrm{Ext}^{d-1}_{Y\times A}(p_2^*N_C^{\otimes n},\mathcal{O}_{Y\times A})\otimes H^0(\mathcal{O}_{\Delta_Y}).$$

Next, we prove the second part of the statement. On the $Y\times\widehat{A}$ side, we consider the following product map:

$$\begin{array}{ccc}\mathrm{Hom}_{Y\times\widehat{A}}(R^1\Phi(p_2^*N_C^{\otimes n}),\mathcal{O}_{Y\times\hat{0}})\otimes\mathrm{Ext}^d_{Y\times\widehat{A}}(\mathcal{O}_{Y\times\hat{0}},\mathcal{R}) & \to & \mathrm{Ext}^d_{Y\times\widehat{A}}(R^1\Phi(p_2^*N_C^{\otimes n}),\mathcal{R})\\ \downarrow\cong & & \downarrow a_{d-1}\\ \mathrm{Ext}^{-1}_{Y\times\widehat{A}}(\mathbf{R}\Phi(p_2^*N_C^{\otimes n}),\mathcal{O}_{Y\times\hat{0}})\otimes\mathrm{Ext}^d_{Y\times\widehat{A}}(\mathcal{O}_{Y\times\hat{0}},\mathcal{R}) & \longrightarrow & \mathrm{Ext}^{d-1}_{Y\times\widehat{A}}(\mathbf{R}\Phi(p_2^*N_C^{\otimes n}),\mathcal{R})\end{array} \quad (73)$$

where the vertical isomorphism comes from the usual spectral sequence (51). By (47) the inverse of the Fourier–Mukai transform is $(-1)_A^*\circ\mathbf{R}\Psi[q]$. By (49) we have that

$$(-1)_A^*\circ\mathbf{R}\Psi(\mathcal{O}_{Y\times\hat{0}})=(-1)_A^*\circ R^0\Psi(\mathcal{O}_{Y\times\hat{0}})=\mathcal{O}_{Y\times A},$$

$$(-1)_A^*\circ\mathbf{R}\Psi(\mathcal{R})=(-1)_A^*\circ R^d\Psi(\mathcal{R})[-d]=\mathcal{O}_{\Delta_Y}.$$

Therefore, thanks to Mukai's inversion theorem (47), the Fourier–Mukai transform identifies – on the $Y\times A$ side – the sources of both rows in diagram (73) to

$$\mathrm{Ext}^{-1}_{Y\times A}(p_2^*N_C^{\otimes n}[-d],\mathcal{O}_{Y\times A})\otimes\mathrm{Ext}^d(\mathcal{O}_{Y\times A},\mathcal{O}_{\Delta_Y}[-d])$$

$$\cong\mathrm{Ext}^{d-1}_{Y\times A}(p_2^*N_C^{\otimes n},\mathcal{O}_{Y\times A})\otimes\mathrm{Hom}_{Y\times A}(\mathcal{O}_{Y\times A},\mathcal{O}_{\Delta_Y}).$$

This concludes the proof of the proposition. □

6.2 Conclusion of the proof of Theorem 5

Notation 5 We introduce the following typographical abbreviations on diagram (70): the isomorphic (via the Fourier–Mukai transform) spaces of the central row of diagram (70) are identified to vector spaces E_i, and we denote by V_i, E_i, W_i the spaces appearing in the left column of diagram (70) (from top to bottom), and by A_i, E_i, B_i the spaces appearing in the right column (from top to bottom). We denote also by $\Lambda^\bullet T_{A,0}$ the acting exterior algebra. The structure of $\Lambda^\bullet T_{A,0}$-graded modules induces a natural map of diagrams (we focus on degrees n and $d-1$ as they are the relevant ones in our argument)

$$
\begin{array}{c}
\begin{array}{ccccc}
 & & A_n & & \\
 & & \downarrow{\scriptstyle a_n} & & \\
V_n & \xrightarrow{v_n} & E_n & \xrightarrow{w_n} & W_n \\
 & & \downarrow{\scriptstyle b_n} & & \\
 & & B_n & &
\end{array} \\
\downarrow^{\phi} \\
\begin{array}{ccccc}
 & & \stackrel{d-1-n}{\Lambda} T^\vee_{A,0} \otimes A_{d-1} & & \\
 & & \downarrow{\scriptstyle \tilde{a}_{d-1}} & & \\
\stackrel{d-1-n}{\Lambda} T^\vee_{A,0} \otimes V_{d-1} & \xrightarrow{\tilde{v}_{d-1}} & \stackrel{d-1-n}{\Lambda} T^\vee_{A,0} \otimes E_{d-1} & \xrightarrow{\tilde{w}_{d-1}} & \stackrel{d-1-n}{\Lambda} T^\vee_{A,0} \otimes W_{d-1} \\
 & & \downarrow{\scriptstyle \tilde{b}_{d-1}} & & \\
 & & \stackrel{d-1-n}{\Lambda} T^\vee_{A,0} \otimes B_{d-1} & &
\end{array}
\end{array}
\tag{74}
$$

where we have denoted $\tilde{v}_{d-1} = \mathrm{id} \otimes v_{d-1}$ and so on. We denote also

$$\phi_{A_n} : A_n \to \stackrel{d-1-n}{\Lambda} T^\vee_{A,0} \otimes A_{d-1}$$

and, similarly, ϕ_{B_n}, ϕ_{V_n}, ϕ_{W_n}, ϕ_{E_n}.

At this point we make the following assumption:
(*) *The extension class e of the restricted cotangent sequence*

$$0 \to N_C^\vee \to (\Omega^1_X)_{|C} \to (\Omega^1_Y)_{|C} \to 0 \tag{75}$$

belongs to the subspace $H^1(\mathcal{T}_Y \otimes N_C^\vee)$ *of* $\mathrm{Ext}^1_C(\Omega^1_Y \otimes N_C, \mathcal{O}_C)$.[15] Note that if C is smooth and Y is smooth along C this is obvious, since the two spaces coincide.

Remark 4 Note that, if (*) holds then e belongs to the subspace $H^0(\mathcal{N}' \otimes N_C^\vee)$ of $H^1(\mathcal{T}_Y \otimes N_C^\vee)$: as mentioned in Remark 3, from the exact sequence defining the restricted equisingular normal sheaf (56) we get that

$$H^0(C, \mathcal{N}' \otimes N_C^\vee) = \ker(\ H^1(C, \mathcal{T}_Y \otimes N_C^\vee) \xrightarrow{G} H^1(C, \mathcal{T}_A \otimes N_C^\vee)).$$

15 These are the locally trivial first-order deformations.

The fact that e belongs to $H^0(\mathcal{N}' \otimes N_C^\vee)$ essentially follows from the deformation-theoretic interpretation of this map G (it is the restriction (to $H^1(C, \mathcal{T}_Y \otimes N_C^\vee)$) of the map G_Z^Y of (8) in the Introduction, with $Z = A$). More formally: the target of G is $\mathrm{Hom}_k(\Omega^1_{A,0}, H^1(C, N_C^\vee))$ and G takes an extension class f to the map $\Omega^1_{A,0} \to H^1(N_C^{-1})$ obtained by composing the coboundary map of f with the map $\Omega^1_{A,0} \to H^0((\Omega^1_Y)_{|C})$. If the extension class is (75) then this map factorizes through $H^0((\Omega^1_X)_{|C})$, hence $e \in \ker G$.

From diagram (74) we have the map

$$\phi_{W_n} : H^0(C, \mathcal{N}' \otimes N_C^\vee) = \ker(G) \to \mathrm{Hom}(\overset{d-1-n}{\Lambda} T_{A,0}, H^0(\omega_C \otimes N_C^{\otimes -n})).$$

Lemma 18 $\phi_{W_n}(e) \neq 0.$

Proof We make the identification $\overset{d-1-n}{\Lambda} T_{A,0} \cong \overset{n+1}{\Lambda} \Omega^1_{A,0}$. Accordingly $\phi_{W_n}(e)$ is identified with a map

$$\phi_{W_n}(e) : \overset{n+1}{\Lambda} \Omega^1_{A,0} \to H^0(\omega_C \otimes N_C^{\otimes -n}).$$

We consider the map

$$\overset{n+1}{\Lambda} \Omega^1_{A,0} \to H^0((\overset{n+1}{\Lambda} \Omega^1_X)_{|C}) \tag{76}$$

obtained as H^0 of Λ^{n+1} of the codifferential $\Omega^1_{A,0} \otimes \mathcal{O}_C \to (\Omega^1_X)_{|C}$. Since the codifferential is surjective, the map (76) is nonzero. If C is smooth and X and Y are smooth along C then the target of (76) is $H^0((\omega_X)_{|C}) = H^0(\omega_C \otimes N_C^{\otimes -n})$. Via the above identifications, the map $\phi_{W_n}(e)$ coincides, up to a scalar, with (76). The lemma follows in this case. Even if X is not smooth along C, $\phi_{W_n}(e)$ is the composition of the map (76) and the H^0 of the canonical map $\Lambda^{n+1}((\Omega^1_X)_{|C}) \to (\omega_X)_{|C} \cong \omega_C \otimes N_C^{\otimes -n}$. Such a composition is clearly nonzero and the lemma follows as above. □

At this point, the line of the argument is clear. The class $\phi_{W_n}(e)$ is nonzero, and, in the splitting of Proposition 17, it belongs to the direct summand (of E_n) $\sigma(W_n) \subset \mathrm{Im}\,(\bar{a}_{d-1})$. Therefore the projection of $\mathrm{Im}\,(\phi_{E_n})$ onto $\sigma(W_n)$ is nonzero. This implies that $\mathrm{Im}\,(a_n)$ is nonzero, since otherwise E_n would be isomorphic to a subspace of B_n and $\mathrm{Im}\,(\phi_{E_n})$ would be contained in the direct summand complementary to $\sigma(W_n)$.

By Corollary 16 this proves that $R^n\pi_*Q = 0$.

To prove the vanishing of $R^i\pi_*Q$ for $0 < i < n$ one takes a sufficiently positive ample line bundle M on X and an $(i+1)$-dimensional complete intersection of divisors in $|M|$, say X'. It follows from relative Serre vanishing that

$R^i\pi_*(Q) = R^i\pi_*(Q_{|X'\times\widehat{A}})$. Therefore the desired vanishing follows by the previous step. The vanishing of $R^0\pi_*Q$ is standard: as it is a torsion-free sheaf, it is enough to show that its support is a proper subvariety of $\widehat{A}$. By a base change, this is contained in the locus of $\alpha \in \widehat{A}$ such that $h^0(X, \alpha_{|X}) > 0$, i.e., the kernel of the homomorphism $\mathrm{Pic}^0 A \to \mathrm{Pic} X$, which is easily seen to be a proper subvariety of $\mathrm{Pic}^0 A$.[16] This concludes the proof of Theorem 5. □

Remark 5 The hypothesis that X is smooth in codimension 1 is used to ensure that assumption *(*)* can be made.

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