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# Rational equivalence of 0-cycles on $K3$ surfaces and conjectures of Huybrechts and O'Grady

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## Abstract

We give a new interpretation of O'Grady's filtration on the $CH_0$ group of a $K3$ surface $X$. In particular, we get a new characterization of the canonical 0-cycles $kc_X$: given $k \geq 0$, $kc_X$ is the only 0-cycle of degree $k$ on $X$ whose orbit under rational equivalence is of dimension $k$. Using this, we extend the results of Huybrechts and O'Grady concerning Chern classes of simple vector bundles on $K3$ surfaces.

## 1 Introduction

Let $X$ be a projective $K3$ surface. In [1], Beauville and the author proved that $X$ carries a canonical 0-cycle $c_X$ of degree 1, which is the class in $CH_0(X)$ of any point of $X$ lying on a (possibly singular) rational curve on $X$. This cycle has the property that for any divisors $D$, $D'$ on $X$, we have

$$D \cdot D' = \deg(D \cdot D')\, c_X \text{ in } CH_0(X).$$

In recent works of Huybrechts [5] and O'Grady [11], this 0-cycle appeared to have other characterizations. Huybrechts proves, for example, the following result (which is proved in [5] to have much more general consequences on spherical objects and autoequivalences of the derived category of $X$):

**Theorem 1** (Huybrechts [5]) *Let $X$ be a projective complex $K3$ surface. Let $F$ be a simple vector bundle on $X$ such that $H^1(\mathrm{End}\, F) = 0$ (such an $F$ is*

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*called spherical in [5]). Then $c_2(F)$ is proportional to $c_X$ in $CH_0(X)$ if one of the following conditions holds:*

*(1) The Picard number of X is at least* 2*.*
*(2) The Picard group of X is $\mathbf{Z}H$ and the determinant of F is equal to kH with $k = \pm 1 \mod r := \operatorname{rank} F$.*

This result is extended in the following way by O'Grady. In [11], he introduces the following increasing filtration of $CH_0(X)$:

$$S_0(X) \subset S_1(X) \subset \cdots \subset S_d(X) \subset \cdots \subset CH_0(X),$$

where $S_d(X)$ is defined as the set of classes of cycles of the form $z + z'$, with $z$ effective of degree $d$ and $z'$ a multiple of $c_X$. It is also convenient to introduce $S_d^k(X)$, which will by definition be the set of degree-$k$ 0-cycles on $X$ which lie in $S_d(X)$. Thus by definition

$$S_d^k(X) = \{z \in CH_0(X),\ z = z' + (k-d)c_X\},$$

where $z'$ is effective of degree $d$.

Consider a torsion-free or more generally a pure sheaf $\mathcal{F}$ on $X$ which is $H$-stable with respect to a polarization $H$. Let $2d(v_{\mathcal{F}})$ be the dimension of the space of deformations of $\mathcal{F}$, where $v_{\mathcal{F}}$ is the Mukai vector of $\mathcal{F}$ (cf. [6]). We recall that $v_{\mathcal{F}} \in H^*(X, \mathbf{Z})$ is the triple

$$(r, l, s) \in H^0(X, \mathbf{Z}) \oplus H^2(X, \mathbf{Z}) \oplus H^4(X, \mathbf{Z}),$$

with $r = \operatorname{rank} \mathcal{F}$, $l = c_1^{top}(\det \mathcal{F})$, and $s \in H^4(X, \mathbf{Z})$ is defined as

$$v_{\mathcal{F}} = ch(\mathcal{F})\sqrt{td(X)}. \tag{1}$$

With this notation we get, by the Riemann–Roch formula, that

$$\sum_i (-1)^i \dim Ext^i(\mathcal{F}, \mathcal{F}) =< v_{\mathcal{F}}, v_{\mathcal{F}}^* >= 2rs - l^2 = 2 - 2d(v_{\mathcal{F}}),$$

where $<,>$ is the intersection pairing on $H^*(X, \mathbf{Z})$, and $v^* = (r, -l, s)$ is the Mukai vector of $\mathcal{F}^*$ (if $\mathcal{F}$ is locally free).

In particular $d(v_{\mathcal{F}}) = 0$ if $\mathcal{F}$ satisfies $\operatorname{End} \mathcal{F} = \mathbb{C}$ and $\operatorname{Ext}^1(\mathcal{F}, \mathcal{F}) = 0$, so that $\mathcal{F}$ is spherical as in Huybrechts' theorem. Noticing that $S_0(X) = \mathbf{Z}c_X$, one can then rephrase Huybrechts' statement by saying that if $\mathcal{F}$ satisfies $\operatorname{End}(\mathcal{F}) = \mathbb{C}$, $d(v_{\mathcal{F}}) = 0$, then $c_2(\mathcal{F}) \in S_0(X)$, assuming the Picard number of $X$ is at least 2.

O'Grady then extends Huybrechts' results as follows:

**Theorem 2** (O'Grady [11]) *Assuming $\mathcal{F}$ is H-stable, one has $c_2(\mathcal{F}) \in S_{d(v_{\mathcal{F}})}(X)$, $v_{\mathcal{F}} = (r, l, s)$, if furthermore one of the following conditions holds:*

*(1) $l = H$, $l$ is primitive and $s \geq 0$.*
*(2) The Picard number of $X$ is at least 2, $r$ is coprime to the divisibility of $l$, and $H$ is $v$-generic.*
*(3) $r \leq 2$ and moreover $H$ is $v$-generic if $r = 2$.*

In fact, O'Grady's result is stronger, as he also shows that $S^k_{d(v)}(X)$, $k = \deg c_2(v)$, is equal to the set of classes $c_2(\mathcal{G})$ with $\mathcal{G}$ a deformation of $\mathcal{F}$. O'Grady indeed proves, by a nice argument involving the rank of the Mukai holomorphic 2-form on the moduli space of deformations of $\mathcal{F}$, the following result:

**Proposition 3** (O'Grady [11, Proposition 1.3]) *If there is a $H$-stable torsion-free sheaf $\mathcal{F}$ with $v = v(\mathcal{F})$, and the conclusion of Theorem 2 holds for the deformations of $\mathcal{F}$, then*

$$\{c_2(\mathcal{G}),\ \mathcal{G} \in \overline{\mathcal{M}^{st}}(X, H, v)\ \} = S^k_{d(v)}(X),\ k = \deg c_2(\mathcal{F}).$$

In this statement, $\overline{\mathcal{M}^{st}}(X, H, v)$ is any smooth completion of the moduli space of $H$-stable sheaves with Mukai vector $v$.

Our results in this paper are of two kinds: First of all we provide another description of $S^k_d(X)$ for any $d \geq 0$, $k \geq d$. In order to state this result, let us introduce the following notation: Given an integer $k \geq 0$, and a cycle $z \in CH_0(X)$ of degree $k$, the subset $O_z$ of $X^{(k)}$ consisting of effective cycles $z' \in X^{(k)}$ which are rationally equivalent to $z$ is a countable union of closed algebraic subsets of $X^{(k)}$ (see [13, Lemma 10.7]). This is the "effective orbit" of $z$ under rational equivalence, and the analogue of $|D|$ for a divisor $D \in CH^1(W)$ on any variety $W$. We define $\dim O_z$ as the supremum of the dimensions of the components of $O_z$. This is the analogue of $r(D) = \dim |D|$ for a divisor $D \in CH^1(W)$ on any variety $W$. We will prove the following:

**Theorem 4** *Let $X$ be a projective $K3$ surface. Let $k \geq d \geq 0$. We have the following characterization of $S^k_d(X)$:*

$$S^k_d(X) = \{z \in CH_0(X),\ O_z \text{ non-empty},\ \dim O_z \geq k - d\}.$$

*Remark 1* The inclusion $S^k_d(X) \subset \{z \in CH_0(X),\ O_z$ non-empty, $\dim O_z \geq k - d\}$ is easy since the cycle $(k - d)c_X$ has its orbit of dimension $\geq k - d$ (e.g., $C^{(k-d)} \subset X^{(k-d)}$, for any rational curve $C \subset X$, is contained in the orbit of $(k - d)c_X$). Hence any cycle of the form $z + (k - d)c_X$ with $z$ effective of degree $d$ has an orbit of dimension $\geq k - d$.

A particular case of the theorem above is the case where $d(v) = 0$. By definition, $S_0(X)$ is the subgroup $\mathbf{Z}c_X \subset CH_0(X)$. We thus have:

**Corollary 5** *For $k > 0$, the cycle $kc_X$ is the unique 0-cycle $z$ of degree $k$ on $X$ such that* $\dim O_z \geq k$.

*Remark 2* We have in fact $\dim O_z = k$, $z = kc_X$ since by Mumford's theorem [10], any component $L$ of $O_z$ is Lagrangian for the holomorphic symplectic form on $S^{(k)}_{reg}$, hence of dimension $\leq k$ if $L$ intersects $S^{(k)}_{reg}$. If $L$ is contained in the singular locus of $S^{(k)}$, we can consider the minimal multiplicity-stratum of $S^{(k)}$ containing $L$, which is determined by the multiplicities $n_i$ of the general cycle $\sum_i n_i x_i$, $x_i$ distinct, parametrized by $L$ and apply the same argument.

*Remark 3* We will give in Section 2 an alternative proof of Corollary 5, using the remark above, and the fact that any Lagrangian subvariety of $X^k$ intersects a product $D_1 \times \ldots \times D_k$ of ample divisors on $X$.

Our main application of Theorem 4 is the following result, which generalizes O'Grady's and Huybrechts' Theorems 2, 1 in the case of simple vector bundles (instead of semistable torsion-free sheaves). We do not need any of the assumptions appearing in Theorems 2, 1, but our results, unlike those of O'Grady, are restricted to the locally free case.

**Theorem 6** *Let $X$ be a projective $K3$ surface. Let $F$ be a simple vector bundle on $X$ with Mukai vector $v = v(F)$. Then*

$$c_2(F) \in S_{d(v)}(X).$$

A particular case of this statement is the case where $d = 0$. The corollary below proves Huybrechts' Theorem 1 without any assumption on the Picard group of the $K3$ surface or on the determinant of $F$. It is conjectured in [5].

**Corollary 7** *Let $F$ be a simple rigid vector bundle on a $K3$ surface. Then the class $c_2(F)$ in $CH_0(X)$ is a multiple of $c_X$.*

We also deduce the following corollary, in the same spirit (and with essentially the same proof) as Proposition 3:

**Corollary 8** *Let $v \in H^*(X, \mathbf{Z})$ be a Mukai vector, with $k = c_2(v)$. Assume there exists a simple vector bundle $F$ on $X$ with Mukai vector $v$. Then*

$$S^k_d(X) = \{c_2(G),\ G \text{ a simple vector bundle on } X,\ v_G = v\},$$

*where* $k = c_2(v) := c_2^{\mathrm{top}}(F) = \deg c_2(F)$.

These results answer, for simple vector bundles on $K3$ surfaces, questions asked by O'Grady (see [11, Section 5]) for simple sheaves.

The paper is organized as follows: in Section 2, we prove Theorem 4. We also show a variant concerning a family of subschemes (rather than 0-cycles) of given length in a constant rational equivalence class. In Section 3, Theorem 6 and Corollary 8 are proved.

## 2 An alternative description of O'Grady's filtration

This section is devoted to the proof of Theorem 4, which we state in the following form:

**Theorem 9** *Let $k \geq d$ and let $Z \subset X^{(k)}$ be a Zariski closed irreducible algebraic subset of dimension $k - d$. Assume that all cycles of $X$ parameterized by $Z$ are rationally equivalent in $X$. Then the class of these cycles belongs to $S_d^k(X)$.*

We will need for the proof the following simple lemma, which already appears in [12]:

**Lemma 10** *Let $X$ be a projective $K3$ surface and let $C \subset S$ be a (possibly singular) curve such that all points of $C$ are rationally equivalent in $X$. Then any point of $C$ is rationally equivalent to $c_X$.*

*Proof* Let $L$ be an ample line bundle on $X$. Then $c_1(L)_{|C}$ is a 0-cycle on $C$ and our assumptions imply that $j_*(c_1(L)_{|C}) = \deg(c_1(L)_{|C})\, c$, for any point $c$ of $C$.

Furthermore, we have

$$j_*(c_1(L)_{|C}) = c_1(L) \cdot C \text{ in } CH_0(X)$$

and thus, by [1], $j_*(c_1(L)_{|C}) = \deg(c_1(L)_{|C})\, c_X$ in $CH_0(X)$. Hence we have

$$\deg(c_1(L)_{|C})\, c = \deg(c_1(L)_{|C})\, c_X \text{ in } CH_0(X).$$

This concludes the proof, since $c$ is arbitrary, $\deg(c_1(L)_{|C}) \neq 0$, and $CH_0(X)$ has no torsion. □

**Lemma 11** *The union of curves $C$ satisfying the property stated in Lemma 10 is Zariski dense in $X$.*

*Proof* The 0-cycle $c_X$ is represented by any point lying on a (singular) rational curve $C \subset X$ (see [1]), so the result is clear if one knows that there are infinitely many distinct rational curves contained in $X$. This result is to our knowledge known only for general $K3$ surfaces but not for all $K3$ surfaces (see however [2] for results in this direction). In any case, we can use the following argument which already appears in [8]. By [9], there is a 1-parameter family of (singular) elliptic curves $E_t$ on $X$. Let $C$ be a rational curve on $X$ which meets the fibers $E_t$. For any integer $N$, and any point $t$, consider the points $y \in \widetilde{E_t}$ (the desingularization of $E_t$), which are rationally equivalent in $\widetilde{E_t}$ to the sum of a point $x_t \in E_t \cap C$ (hence rationally equivalent to $c_X$) and an $N$-torsion 0-cycle on $\widetilde{E_t}$.

As $CH_0(X)$ has no torsion, the images $y_t$ of these points in $X$ are all rationally equivalent to $c_X$ in $X$. Their images are clearly parameterized for $N$ large enough by a (maybe reducible) curve $C_N \subset X$. Finally, the union over all $N$ of the points $y_t$ above is Zariski dense in each $\widetilde{E_t}$, hence the union of the curves $C_N$ is Zariski dense in $X$. □

*Proof of Theorem 9* The proof is by induction on $k$, the case $k = 1$, $d = 0$ being Lemma 10 (the case $k = 1$, $d = 1$ is trivial). Let $Z'$ be an irreducible component of the inverse image of $Z$ in $X^k$. Let $p\colon Z' \to X$ be the first projection. We distinguish two cases and note that they exhaust all possibilities, up to replacing $Z'$ by another component $Z''$ deduced from $Z'$ by letting the symmetric group $\mathfrak{S}_k$ act.

*Case 1. The morphism $p\colon Z' \to X$ is dominant.* For a curve $C \subset X$ parameterizing points rationally equivalent to $c_X$, consider the hypersurface

$$Z'_C := p^{-1}(C) \subset Z'.$$

Let $q\colon Z' \to X^{k-1}$ be the projection on the product of the $k-1$ last factors. Assume first that $\dim q(Z'_C) = \dim Z'_C = k - d - 1$. Note that all cycles of $X$ parameterized by $q(Z'_C)$ are rationally equivalent in $X$. Indeed, an element $z$ of $Z'_C$ is of the form $(c, z')$ with $c \in C$ so that $c = c_X$ in $CH_0(X)$. So, the rational equivalence class of $z'$ is equal to $z - c_X$ and is independent of $z' \in Z'_C$. Thus the induction assumption applies and the cycles of degree $k - 1$ parameterized by $\operatorname{Im} q$ belong to $S_d^{k-1}(X)$. It follows in turn that the classes of the cycles parameterized by $Z'$ (or $Z$) belong to $S_d^k(X)$. Indeed, as just mentioned above, a 0-cycle $z$ parameterized by $Z'$ is rationally equivalent to $z = c_X + z'$ where $z' \in S_d^{k-1}(X)$, so $z'$ is rationally equivalent to $(k - d - 1)c_X + z''$, $z'' \in X^{(d)}$. Hence $z$ is rationally equivalent in $X$ to $(k-d)c_X + z''$, for some $z'' \in X^{(d)}$. Thus $z \in S_d^k(X)$.

Assume to the contrary that $\dim q(Z'_C) < \dim Z'_C = k-d-1$ for any curve $C$ as above. We use now the fact (see Lemma 11) that these curves $C$ are Zariski dense in $X$. We can thus assume that there is a point $x \in Z'_C$ which is generic in $Z'$, so that both $Z'$ and $Z'_C$ are smooth at $x$, of respective dimensions $k - d$ and $k - d - 1$. The fact that $\dim q(Z'_C) < k - d - 1$ implies that $q$ is not of maximal rank $k - d$ at $x$ and as $x$ is generic in $Z'$, we conclude that $q$ is of rank $< k - d$ everywhere on $Z'_{reg}$, so that $\dim \operatorname{Im} q \le k - d - 1$.

Now recall that all 0-cycles parameterized by $Z'$ are rationally equivalent. It follows that for any fiber $F$ of $q$, all points in $p(F)$ are rationally equivalent in $X$. This implies that all these points are rationally equivalent to $c_X$ by Lemma 10. This contradicts the fact that $p$ is surjective.

*Case 2. None of the projections $pr_i$, $i = 1, \dots, k$, from $X^k$ to its factors restricts to a dominant map $p_i : Z' \to X$.* Let $C_i := \operatorname{Im} p_i \subset X$ if $\operatorname{Im} pr_i$ is a curve, and any curve containing $\operatorname{Im} p_i$ if $\operatorname{Im} p_i$ is a point. Thus $Z'$ is contained in $C_1 \times \cdots \times C_k$.

Let $C$ be a non-necessarily irreducible ample curve such that all points in $C$ are rationally equivalent to $c_X$. Observe that the line bundle $pr_1^*\mathcal{O}_X(C) \otimes \cdots \otimes pr_k^*\mathcal{O}_X(C)$ on $X^k$ has its restriction to $C_1 \times \cdots \times C_k$ ample and that its $(k-d)$th self-intersection on $C_1 \times \ldots \times C_k$ is a complete intersection of ample divisors and is equal to

$$W := (k-d)! \sum_{i_1 < \ldots < i_{k-d}} p_{i_1}^* \mathcal{O}_{C_1}(C) \cdot \ldots \cdot p_{i_{k-d}}^* \mathcal{O}_{C_{k-d}}(C) \tag{2}$$

in $CH^{k-d}(C_1 \times \ldots \times C_k)$, where the $p_i$ are the projections from $\prod_i C_i$ to its factors.

The cycle $W$ of (2) is also the restriction to $C_1 \times \ldots \times C_k$ of the effective cycle

$$W' := (k-d)! \sum_{i_1 < \ldots < i_{k-d}} pr_{i_1}^* C \cdot \ldots \cdot pr_{i_{k-d}}^* C. \tag{3}$$

As the $(k-d)$-dimensional subvariety $Z'$ of $C_1 \times \cdots \times C_k$ has a nonzero intersection with $W$, it follows that the intersection number of $Z'$ with $W'$ is nonzero in $X^k$, hence that

$$Z' \cap pr_{i_1}^* C \cdot \ldots \cdot pr_{i_{k-d}}^* C \neq \emptyset$$

for some choice of indices $i_1 < \cdots < i_{k-d}$. This means that there exists a cycle in $Z$ which is of the form

$$z = z' + z''$$

with $z' \in C^{(k-d)}$ and $z'' \in X^{(d)}$. As $z'$ is supported on $C$, it is equal to $(k-d)c_X$ in $CH_0(X)$ and we conclude that $z \in S_d^k(X)$. □

Let us now prove the following variant of Theorem 9. Instead of a family of 0-cycles (that is, elements of $X^{(k)}$), we now consider families of 0-dimensional *subschemes* (that is, elements of $X^{[k]}$):

**Variant 12** *Let $k \geq d$ and let $Z \subset X^{[k]}$ be a Zariski closed irreducible algebraic subset of dimension $k-d$. Assume that all cycles of $X$ parameterized by $Z$ are rationally equivalent in $X$. Then the class of these cycles belongs to $S_d^k(X)$.*

*Proof* Let $z \in Z$ be a general point. The cycle $c(z)$ of $z$, where $c\colon X^{[k]} \to X^{(k)}$ is the Hilbert–Chow morphism, is of the form $\sum_i k_i x_i$, with $\sum_i k_i = k$, where $x_i$ are $k'$ distinct points of $X$. We have of course

$$k' = k - \sum_i (k_i - 1). \tag{4}$$

The fiber of $c$ over a cycle of the form $\sum_i k_i x_i$ as above is of dimension $\sum_i (k_i - 1)$ (see, e.g., [3]). It follows that the image $Z_1$ of $Z$ in $X^{(k)}$ is of dimension $\geq k - d - \sum_i (k_i - 1)$. By definition, $Z_1$ is contained in a multiplicity-stratum of $X^{(k)}$ where the support of the considered cycles has cardinality $\leq k'$. Let $Z_1' \subset X^{k'}$ be the set of $(x_1, \ldots, x_{k'})$ such that $\sum_i k_i x_i \in c(Z)$. Then the morphism

$$Z_1' \to Z_1,\ (x_1, \ldots, x_{k'}) \mapsto \sum_i k_i x_i$$

is finite and surjective, so that

$$\dim Z_1' = \dim Z_1 \geq k - d - \sum_i (k_i - 1), \tag{5}$$

which by (4) can be rewritten as

$$\dim Z_1' = \dim Z_1 \geq k' - d.$$

Note that by construction, $Z_1'$ parameterizes $k'$-uples $(x_1, \ldots, x_{k'})$ with the property that $\sum_i k_i x_i$ is rationally equivalent to a constant cycle.

The proof of Variant 12 then concludes with the following statement:

**Proposition 13** *Let $l$ be a positive integer, $k_1 > 0, \ldots, k_l > 0$ be positive multiplicities. Let $Z$ be a closed algebraic subset of $X^l$. Assume that* $\dim Z \geq l - d$ *and the cycles $\sum_i k_i x_i$, $(x_1, \ldots, x_l) \in Z$, are all rationally equivalent in $X$. Then the class of the cycles $\sum_i k_i x_i$, $(x_1, \ldots, x_l) \in Z$, belongs to $S_d^k(X)$, where $k = \sum_i k_i$.*

□

For the proof of Proposition 13, we have to start with the following lemma:

**Lemma 14** *Let $x_1, \ldots, x_d \in X$ and let $k_i \in \mathbf{Z}$. Then $\sum_i k_i x_i \in S_d^k(X)$, $k = \sum_i k_i$.*

*Proof* We use the following characterization of $S_d(X)$ given by O'Grady:

**Proposition 15** (O'Grady [11]) *A cycle $z \in CH_0(X)$ belongs to $S_d(X)$ if and only if there exists a (possibly singular, possibly reducible) curve $j\colon C \subset X$, such that the genus of the desingularization of $C$ (or the sum of the genera of its components if $C$ is reducible) is not greater than $d$ and $z$ belongs to* $\operatorname{Im}(j_*\colon CH_0(C) \to CH_0(X))$.

Let now $x_1, \dots, x_d$ be as above. There exists by [9] a curve $C \subset X$, whose desingularization has genus $\leq d$ and containing $x_1, \dots, x_d$. Thus for any $k_i$, the cycle $\sum_i k_i x_i$ is supported on $C$, which proves the lemma by Proposition 15. □

*Proof of Proposition 13* Proposition 13 is proved exactly as Theorem 9, by induction on $l$. In case 1 considered in the induction step, we apply the same argument as in that proof. In case 2 considered in the induction step, using the same notation as in that proof, we conclude that there is in $Z$ an $l$-uple $(x_1, \dots, x_l)$ satisfying (up to permutation of the indices)

$$x_{d+1} \ \dots, x_l \in C,$$

and as any point of $C$ is rationally equivalent to $c_X$, we find that

$$\sum_i k_i x_i = (\sum_{i>d} k_i) c_X + \sum_{l \leq i \leq d} k_i x_i.$$

By Lemma 14, $\sum_{1 \leq i \leq d} k_i x_i \in S_d(X)$, so that $\sum_i k_i x_i \in S_d(X)$. □

As mentioned in the Introduction, Theorem 9 in the case $d = 0$ provides the following characterization of the cycle $kc_X$, $k > 0$: it is the only degree-$k$ 0-cycle $z$ of $X$, whose orbit $O_z \subset X^{(k)}$ is $k$-dimensional (cf. Corollary 5). Let us give a slightly more direct proof in this case. We use the following Lemma 16. Let $V$ be a 2-dimensional complex vector space. Let $\eta \in \bigwedge^2 V^*$ be a nonzero generator, and let $\omega \in \bigwedge_{\mathbf{R}}^{1,1}(V^*)$ be a positive real $(1,1)$-form on $V$.

**Lemma 16** *Let $W \subset V^k$ be a $k$-dimensional complex vector subspace which is Lagrangian for the nondegenerate 2-form $\eta_k := \sum_i pr_i^* \eta$ on $V^k$, where the $pr_i$s are the projections from $V^k$ to $V$. Then $\prod_i pr_i^* \omega$ restricts to a volume form on $W$.*

*Proof* The proof is by induction on $k$. Let $\pi \colon W \to V^{k-1}$ be the projector on the product of the last $k-1$ summands. We can clearly assume, up to changing the order of factors, that $\dim \operatorname{Ker} \pi < 2$. As $\dim \operatorname{Ker} \pi \leq 1$, we can choose a linear form $\mu$ on $V$ such that the $(k-1)$-dimensional vector space $W_\mu := \ker pr_1^* \mu_{|W}$ is sent injectively by $\pi$ to a $(k-1)$-dimensional subspace $W'$ of $V^{k-1}$. Furthermore, since $W$ is Lagrangian for $\eta_k$, $W'$ is Lagrangian for $\eta_{k-1}$ because $W_\mu \subset \operatorname{Ker} \mu \times V^{k-1}$, and on $\operatorname{Ker} \mu \times V^{k-1}$, $\eta_k = \pi^* \eta_{k-1}$. By the induction hypothesis, the form $\prod_{i>1} pr_i^* \omega$ restricts to a volume form on $W'$, where the projections here are considered as restricted to $0 \times V^{k-1}$, and it follows that

$$pr_1^*(\sqrt{-1}\mu \wedge \overline{\mu}) \wedge \prod_{i>1} pr_i^* \omega$$

restricts to a volume form on $W$. It follows immediately that $\prod_{i\geq 1} pr_i^*\omega$ restricts to a volume form on $W$ since for a positive number $\alpha$, we have

$$\omega \geq \alpha\sqrt{-1}\mu \wedge \overline{\mu}$$

as real $(1,1)$-forms on $V$. □

*Proof of Corollary 5* Let $z \in CH_0(X)$ be a cycle of degree $k$ such that $\dim O_z \geq k$. Let $\Gamma \subset X^k$ be an irreducible component of the inverse image of a $k$-dimensional component of $O_z \subset X^{(k)}$ via the map $X^k \to X^{(k)}$. By Mumford's theorem [10], using the fact that all the 0-cycles parameterized by $\Gamma$ are rationally equivalent in $X$, $\Gamma$ is Lagrangian for the symplectic form $\sum_i pr_i^*\eta_X$ on $X^k$, where $\eta_X \in H^{2,0}(X)$ is a generator. Let $L$ be an ample line bundle on $X$ such that there is a curve $D \subset X$ in the linear system $|L|$, all of whose components are rational. We claim that

$$\Gamma \cap D^k \neq \emptyset.$$

Indeed, it suffices to prove that the intersection number

$$[\Gamma]\cdot[D^k] \tag{6}$$

is positive. Let $\omega_L \in H^{1,1}(X)$ be a positive representative of $c_1(L)$. Then (6) is equal to

$$\int_{\Gamma_{reg}} \prod_i pr_i^*\omega_L. \tag{7}$$

By Lemma 16, the form $\prod_i pr_i^*\omega_L$ restricts to a volume form on $\Gamma$ at any smooth point of $\Gamma$ and the integral (7) is thus positive. □

## 3 Second Chern class of simple vector bundles

This section is devoted to the proof of Theorem 6. Recall first from [11] that, in order to prove the result for a vector bundle $F$ on $X$, it suffices to prove it for $F \otimes L$, where $L$ is a line bundle on $X$. Choosing $L$ sufficiently ample, we can thus assume that $F$ is generated by global sections, and furthermore that

$$H^1(X, F^*) = 0. \tag{8}$$

Let $r = \operatorname{rank} F$. Choose a general $(r-1)$-dimensional subspace $W$ of $H^0(X, F)$, and consider the evaluation morphism

$$e_W : W \otimes O_X \to F.$$

The following result is well known (cf. [6, 5.1]):

**Lemma 17** *The morphism $e_W$ is generically injective, and the locus $Z \subset X$ where its rank is $< r-1$ consists of k distinct reduced points, where $k = c_2^{top}(F)$.*

*Proof* Let $G = Grass(r-1, H^0(X,F))$ be the Grassmannian of $(r-1)$-dimension subspaces of $H^0(X,F)$. Consider the following universal subvariety of $G \times X$:

$$G_{deg} := \{(W,x) \in G \times X, \operatorname{rank} e_{W,x} < r-1\}.$$

Since $F$ is generated by sections, $G_{deg}$ is a fibration over $X$, with fibers smooth away from the singular locus

$$G_{deg}^{sing} := \{(W,x) \in G \times X, \operatorname{rank} e_{W,x} < r-2\}.$$

Furthermore, we have

$$\dim G_{deg} = \dim(G \times X) - 2 = \dim G$$

and $\dim G_{deg}^{sing} < \dim G$.

Consider the first projection: $p_1 : G_{deg} \to G$. It follows from the observations above and from Sard's theorem that for general $W \in G$, $p_1^{-1}(W)$ avoids $G_{deg}^{sing}$ and consists of finitely many reduced points in $X$. The statement concerning the number $k$ of points follows from [4, 14.3], or from the following argument that we will need later on. Given $W$ such that the morphism $e_W$ is generically injective, and the locus $Z_W$ where its rank is $< r-1$ consists of $k$ distinct reduced points, we have an exact sequence

$$0 \to W \otimes \mathcal{O}_X \to F \to \mathcal{I}_{Z_W} \otimes \mathcal{L} \to 0, \tag{9}$$

where $\mathcal{L} = \det F$. Hence $c_2(F) = c_2(\mathcal{I}_Z \otimes \mathcal{L}) = c_2(\mathcal{I}_Z) = Z$, and in particular $c_2^{top}(F) = \deg Z$. This proves the lemma. □

By Lemma 17, we have a rational map

$$\phi : G \dashrightarrow X^{(k)},\ W \mapsto c(Z_W),$$

where $c : X^{[k]} \to X^{(k)}$ is the Hilbert–Chow morphism.

**Proposition 18** *If $F$ is simple and satisfies assumption (8), the rational map $\phi$ is generically one-to-one on its image.*

*Proof* Let $G^0 \subset G$ be the Zariski open set parameterizing the subspaces $W \subset H^0(X,F)$ of dimension $r-1$ satisfying the conclusions of Lemma 17. Note that $c$ is a local isomorphism at a point $Z_W$ of $X^{[k]}$ consisting of $k$ distinct points, so that the dimension of the image of $\phi$ is equal to the dimension of the image of the rational map $G \dashrightarrow X^{[k]}$, $W \mapsto Z_W$, which we will also denote by $\phi$. This $\phi$ is a morphism on $G^0$ and it suffices to show that the map $\phi^0 := \phi_{|G^0}$

is injective. Let $W \in G^0$, $Z := \phi(W)$. For any $W' \in {\phi^0}^{-1}(Z)$, we have an exact sequence as in (9):

$$0 \to W' \otimes \mathcal{O}_X \to F \to \mathcal{I}_Z \otimes \mathcal{L} \to 0, \tag{10}$$

so that $W'$ determines a morphism

$$t_{W'} : F \to \mathcal{I}_Z \otimes \mathcal{L},$$

and conversely, we recover $W'$ from the data of $t_{W'}$ up to a scalar as the space of sections of $\operatorname{Ker} t_{W'} \subset F$. We thus have an injection of the fiber ${\phi^0}^{-1}(Z)$ into $\mathbf{P}(\operatorname{Hom}(F, \mathcal{I}_Z \otimes \mathcal{L}))$.

In order to compute $\operatorname{Hom}(F, \mathcal{I}_Z \otimes \mathcal{L})$, we tensor by $F^*$ the exact sequence (9). We then get the long exact sequence

$$\cdots \to \operatorname{Hom}(F, F) \to \operatorname{Hom}(F, \mathcal{I}_Z \otimes \mathcal{L}) \to H^1(X, F^* \otimes W). \tag{11}$$

By the vanishing (8), we conclude that the map

$$\operatorname{Hom}(F, F) \to \operatorname{Hom}(F, \mathcal{I}_Z \otimes \mathcal{L})$$

is surjective. As $F$ is simple, the LHS is generated by $Id_F$, so the RHS is generated by $t_W$. The fiber ${\phi^0}^{-1}(Z)$ thus consists of one point. □

*Proof of Theorem 6* Let $F$ be a simple nontrivial globally generated vector bundle of rank $r$, with $h^1(F) = 0$ and with Mukai vector

$$v = v_F = (r, l, s) \in H^*(X, \mathbf{Z}).$$

This means that $r = \operatorname{rank} F$, $l = c_1^{top}(F) \in H^2(X, \mathbf{Z})$, and

$$\chi(X, \operatorname{End} F) = < v, v^* > = 2rs - l^2. \tag{12}$$

The Riemann–Roch formula applied to $\operatorname{End} F$ gives

$$\chi(X, \operatorname{End} F) = 2r^2 + (r-1)l^2 - 2rc_2^{top}(F), \tag{13}$$

hence we get the formula (which can also be derived from the definition (1))

$$s = r + \frac{l^2}{2} - c_2^{top}(F). \tag{14}$$

We have by definition of $d(v)$

$$\chi(X, \operatorname{End} F) = 2 - 2d(v),$$

and thus by (12)

$$d(v) = 1 - rs + \frac{l^2}{2}. \tag{15}$$

The Riemann–Roch formula applied to $F$ gives, furthermore,

$$\chi(X,F) = 2r + \frac{l^2}{2} - c_2^{top}(F) \tag{16}$$

which by (14) gives

$$\chi(X,F) = r + s. \tag{17}$$

As we assume $h^1(F) = 0$ and we have $h^2(F) = 0$, since $F$ is nontrivial, generated by sections and simple, we thus get

$$h^0(X,F) = r + s. \tag{18}$$

With the notation introduced above, we conclude that

$$\dim G = (r-1)(s+1).$$

By Proposition 18, as all cycles parameterized by $\operatorname{Im}\phi$ are rationally equivalent in $X$, the orbit under rational equivalence of $c_2(F)$ in $X^{(k)}$, $k = c_2^{top}(F)$, has dimension greater than or equal to

$$(r-1)(s+1) = rs - s + r - 1.$$

But we have by (14) and (15):

$$k - d(v) = r - s + rs - 1.$$

By Theorem 9, we conclude that $c_2(F) \in S^k_{d(v)}(X)$. □

*Remark 4* Instead of proving that the general $Z_W$ is reduced and applying Theorem 9, we could as well apply Variant 12 directly to the family of subschemes $Z_W$.

For completeness, we conclude this section with the proof of Corollary 8, although a large part of it mimics the proof of Proposition 3 in [11].

We recall for convenience the statement:

**Corollary 19** *Let $v \in H^*(X,\mathbf{Z})$ be a Mukai vector. Assume there exists a simple vector bundle $F$ on $X$ with Mukai vector $v$. Then*

$$S^k_d(X) = \{c_2(G),\ G \text{ a simple vector bundle on } X,\ v_G = v\},$$

*where $d = d(v)$, $k = c_2(v) := c_2^{top}(F)$, $v_F = v$.*

*Proof* The inclusion

$$\{c_2(G),\ G \text{ a simple vector bundle on } X,\ v_G = v\} \subset S^k_d(X) \tag{19}$$

is the content of Theorem 6.

For the reverse inclusion, we first prove that there exists a Zariski open set $U \subset X^{(d)}$ such that

$$cl(U) + (k - d(v))c_X \subset \{c_2(G),\ G \text{ a simple vector bundle on } X,\ v_G = v\} \tag{20}$$

where $cl\colon X^{(d)} \to CH_0(X)$ is the cycle map.

As $F$ is simple, the local deformations of $F$ are unobstructed. Hence there exist a smooth connected quasi-projective variety $Y$, a locally free sheaf $\mathcal{F}$ on $Y \times X$, and a point $y_0 \in Y$ such that $\mathcal{F}_{y_0} \cong F$ and the Kodaira–Spencer map

$$\rho\colon T_{Y,y_0} \to H^1(X, \operatorname{End} F)$$

is an isomorphism.

As $\mathcal{F}_{y_0}$ is simple, so is $\mathcal{F}_y$ for $y$ in a dense Zariski open set of $Y$. Shrinking $Y$ if necessary, $\mathcal{F}_y$ is simple for all $y \in Y$. By Theorem 6, we have $c_2(\mathcal{F}_y) \in S_{d(v)}(X)$ for all $y \in Y$.

Let $\Gamma := c_2(\mathcal{F}) \in CH^2(Y \times X)$. Consider the following set $R \subset Y \times X^{(d(v))}$

$$R = \{(y, z),\ \Gamma_*(y) = c_2(\mathcal{F}_y) = cl(z) + (k - d(v))c_X \text{ in } CH_0(X)\},$$

where $cl\colon X^{(d(v))} \to CH_0(X)$ is the cycle map and $k = c_2(v)$.

$R$ is a countable union of closed algebraic subsets of $Y \times X^{(d)}$ and by the above inclusion (19), the first projection

$$R \to Y$$

is surjective. By a Baire category argument, it follows that for some component $R_0 \subset R$, the first projection is dominant.

We claim that the second projection $R_0 \to X^{(d(v))}$ is also dominant. This follows from the fact that by Mumford's theorem, the pull-backs to $R_0$ of the holomorphic 2-forms on $Y$ and $X^{(d(v))}$ are equal. As the first projection is dominant and the Mukai form on $Y$ has rank $2d(v)$, the same is true for its pull-back to $R_0$ (or rather its smooth locus). Hence the pull-back to $R_0$ of the symplectic form on $X^{(d(v))}$ by the second projection also has rank $2d(v)$. This implies that the second projection is dominant, and hence that its image contains a Zariski open set. Thus (20) is proved. The proof of Corollary 19 is then concluded with Lemma 20 below. □

**Lemma 20** *Let $X$ be a K3 surface and $d > 0$ an integer. Then for any open set (in the analytic or Zariski topology) $U \subset X^{(d)}$, we have*

$$cl(U) = cl(X^{(d)}) \subset CH_0(X).$$

*Proof* It clearly suffices to prove the result for $d = 1$. It is proved in [8] that for any point $x \in X$, the set of points $y \in X$ rationally equivalent to $x$ in $X$ is dense in $X$ for the usual topology. This set thus meets $U$, so $x \in cl(U)$. □

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