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CK-12 Algebra I Concepts

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Contents

1 Equations and Functions 1
1.1 Definition of Variable . . . 2
1.2 Expressions with One or More Variables . . . 6
1.3 Algebra Expressions with Exponents . . . 9
1.4 Order of Operations . . . 12
1.5 Algebra Expressions with Fraction Bars . . . 19
1.6 Patterns and Expressions . . . 23
1.7 Words that Describe Patterns . . . 32
1.8 Checking Solutions to Equations . . . 38
1.9 Checking Solutions to Inequalities . . . 41
1.10 Domain and Range of a Function . . . 44
1.11 Functions that Describe Situations . . . 50
1.12 Functions on a Cartesian Plane . . . 54
1.13 Graphs of Functions based on Rules . . . 60
1.14 Function Rules based on Graphs . . . 65
1.15 Algebraic Functions . . . 71
1.16 Problem-Solving Models . . . 75
1.17 Comparison of Problem-Solving Models . . . 84

2 Real Numbers 93
2.1 Properties of Rational Numbers . . . 94
2.2 Additive Inverses and Absolute Values . . . 101
2.3 Addition of Rational Numbers . . . 105
2.4 Rational Numbers in Applications . . . 111
2.5 Multiplication of Rational Numbers . . . 116
2.6 Division of Rational Numbers . . . 121
2.7 Mixed Numbers in Applications . . . 125
2.8 Distributive Property . . . 129
2.9 Square Roots and Irrational Numbers . . . 134
2.10 Properties of Rational Numbers versus Irrational Numbers . . . 139
2.11 Guess and Check, Work Backward . . . 143

3 Equations of Lines 149
3.1 One-Step Equations and Inverse Operations . . . 150
3.2 One-Step Equations Transformed by Multiplication/Division . . . 154
3.3 Applications of One-Step Equations . . . 157
3.4 Two-Step Equations and Properties of Equality . . . 160
3.5 Multi-Step Equations with Like Terms . . . 164
3.6 Solving Real-World Problems with Two-Step Equations . . . 167
3.7 Multi-Step Equations . . . 171
3.8 Solving Real-World Problems Using Multi-Step Equations . . . 174
3.9 Equations with Variables on Both Sides . . . 178

3.10 Ratios . . . 184
3.11 Proportions . . . 187
3.12 Scale and Indirect Measurement Applications . . . 191
3.13 Conversion of Decimals, Fractions, and Percent . . . 196
3.14 Percent Equations . . . 200
3.15 Percent of Change . . . 203

4 Graphs of Equations and Functions 208
4.1 Points in the Coordinate Plane . . . 209
4.2 Graphs in the Coordinate Plane . . . 216
4.3 Graphs of Linear Equations . . . 222
4.4 Horizontal and Vertical Line Graphs . . . 229
4.5 Intercepts and the Cover-Up Method . . . 233
4.6 Slope . . . 243
4.7 Rates of Change . . . 249
4.8 Graphs Using Slope-Intercept Form . . . 254
4.9 Graphs of Linear Models of Direct Variation . . . 262
4.10 Graphs of Linear Functions . . . 269
4.11 Problem Solving with Linear Graphs . . . 275

5 Writing Linear Equations 283
5.1 Determining the Equation of a Line . . . 284
5.2 Forms of Linear Equations . . . 291
5.3 Applications Using Linear Models . . . 298
5.4 Comparing Equations of Parallel and Perpendicular Lines . . . 302
5.5 Families of Lines . . . 307
5.6 Fitting Lines to Data . . . 313
5.7 Linear Interpolation and Extrapolation . . . 323

6 Linear Inequalities 333
6.1 Inequality Expressions . . . 334
6.2 Linear Inequalities . . . 338
6.3 Multi-Step Inequalities . . . 342
6.4 Applications with Inequalities . . . 346
6.5 Compound Inequalities . . . 350
6.6 Solutions to Compound Inequalities . . . 356
6.7 Absolute Value . . . 361
6.8 Absolute Value Equations . . . 365
6.9 Graphs of Absolute Value Equations . . . 369
6.10 Absolute Value Inequalities . . . 374
6.11 Graphs of Inequalities in One Variable . . . 379
6.12 Linear Inequalities in Two Variables . . . 384

7 Solving Systems of Equations and Inequalities 390
7.1 Graphs of Linear Systems . . . 391
7.2 Systems Using Substitution . . . 403
7.3 Mixture Problems . . . 410
7.4 Linear Systems with Addition or Subtraction . . . 415
7.5 Linear Systems with Multiplication . . . 420
7.6 Comparing Methods for Solving Linear Systems . . . 426
7.7 Consistent and Inconsistent Linear Systems . . . 431
7.8 Determining the Type of Linear System . . . 436

7.9 Applications of Linear Systems 441
7.10 Systems of Linear Inequalities 447
7.11 Linear Programming 457

8 Exponential Functions 467
8.1 Exponential Properties Involving Products 468
8.2 Exponential Terms Raised to an Exponent 472
8.3 Exponential Properties Involving Quotients 476
8.4 Exponent of a Quotient 479
8.5 Negative Exponents 483
8.6 Fractional Exponents 487
8.7 Evaluating Exponential Expressions 491
8.8 Scientific Notation 495
8.9 Scientific Notation with a Calculator 501
8.10 Geometric Sequences and Exponential Functions 505
8.11 Graphs of Exponential Functions 511
8.12 Applications of Exponential Functions 521

9 Polynomials 531
9.1 Polynomials in Standard Form 532
9.2 Addition and Subtraction of Polynomials 538
9.3 Multiplication of Monomials by Polynomials 544
9.4 Multiplication of Polynomials by Binomials 547
9.5 Special Products of Polynomials 554
9.6 Monomial Factors of Polynomials 561
9.7 Zero Product Principle 565
9.8 Factorization of Quadratic Expressions 571
9.9 Factorization of Quadratic Expressions with Negative Coefficients 575
9.10 Factorization using Difference of Squares 580
9.11 Factorization using Perfect Square Trinomials 584
9.12 Factoring Completely 590
9.13 Factoring by Grouping 594
9.14 Solving Problems by Factoring 599

10 Quadratic Equations and Quadratic Functions 604
10.1 Quadratic Functions and Their Graphs 605
10.2 Graphs of Quadratic Functions in Intercept Form 614
10.3 Use Graphs to Solve Quadratic Equations 623
10.4 Use Square Roots to Solve Quadratic Equations 632
10.5 Square Root Applications 637
10.6 Completing the Square 641
10.7 Vertex Form of a Quadratic Equation 647
10.8 Quadratic Formula 653
10.9 Comparing Methods for Solving Quadratics 659
10.10 Solutions Using the Discriminant 668
10.11 Linear, Exponential, and Quadratic Models 673
10.12 Applications of Function Models 683

11 Algebra and Geometry Connections 693
11.1 Graphs of Square Root Functions 694
11.2 Shifts of Square Root Functions 700
11.3 Raising a Product or Quotient to a Power 705

11.4 Simplification of Radical Expressions . . . 711
11.5 Applications Using Radicals . . . 716
11.6 Radical Equations . . . 723
11.7 Equations with Radicals on Both Sides . . . 728
11.8 Pythagorean Theorem and its Converse . . . 733
11.9 Solving Equations Using the Pythagorean Theorem . . . 738
11.10 Applications Using the Pythagorean Theorem . . . 742
11.11 Distance Formula . . . 747
11.12 Midpoint Formula . . . 753

12 Rational Equations and Functions 757
12.1 Inverse Variation Models . . . 758
12.2 Graphs of Rational Functions . . . 764
12.3 Horizontal and Vertical Asymptotes . . . 770
12.4 Determining Asymptotes by Division . . . 775
12.5 Division of Polynomials . . . 782
12.6 Inverse Variation Problems . . . 789
12.7 Excluded Values for Rational Expressions . . . 792
12.8 Multiplication of Rational Expressions . . . 797
12.9 Division of Rational Expressions . . . 800
12.10 Addition and Subtraction of Rational Expressions . . . 804
12.11 Applications of Adding and Subtracting Rational Expressions . . . 810
12.12 Rational Equations Using Proportions . . . 815
12.13 Applications Using Rational Equations . . . 821

13 Probability and Statistics 827
13.1 Measurement of Probability . . . 828
13.2 Empirical Probability . . . 833
13.3 Permutations . . . 838
13.4 Probability and Permutations . . . 843
13.5 Combinations . . . 847
13.6 Probability and Combinations . . . 852
13.7 Mutually Exclusive Events . . . 855
13.8 Independence versus Dependence . . . 860
13.9 Measures of Central Tendency and Dispersion . . . 863
13.10 Measures of Spread/Dispersion . . . 868
13.11 Stem-and-Leaf Plots and Histograms . . . 874
13.12 Box-and-Whisker Plots . . . 886
13.13 Sampling methods . . . 894
13.14 Planning and Conducting Surveys . . . 900

Back Matter 913

CHAPTER 1

Equations and Functions

Chapter Outline

- 1.1 DEFINITION OF VARIABLE
- 1.2 EXPRESSIONS WITH ONE OR MORE VARIABLES
- 1.3 ALGEBRA EXPRESSIONS WITH EXPONENTS
- 1.4 ORDER OF OPERATIONS
- 1.5 ALGEBRA EXPRESSIONS WITH FRACTION BARS
- 1.6 PATTERNS AND EXPRESSIONS
- 1.7 WORDS THAT DESCRIBE PATTERNS
- 1.8 CHECKING SOLUTIONS TO EQUATIONS
- 1.9 CHECKING SOLUTIONS TO INEQUALITIES
- 1.10 DOMAIN AND RANGE OF A FUNCTION
- 1.11 FUNCTIONS THAT DESCRIBE SITUATIONS
- 1.12 FUNCTIONS ON A CARTESIAN PLANE
- 1.13 GRAPHS OF FUNCTIONS BASED ON RULES
- 1.14 FUNCTION RULES BASED ON GRAPHS
- 1.15 ALGEBRAIC FUNCTIONS
- 1.16 PROBLEM-SOLVING MODELS
- 1.17 COMPARISON OF PROBLEM-SOLVING MODELS

Introduction

Equations and functions are the basic building blocks of algebra. You will use the concepts you learn in this chapter not only in the chapters to follow but also in your everyday life. Every time you encounter a situation with an unknown value, like the amount of interest your bank account is earning, you can write and manipulate an equation to find that value. Likewise, when you encounter mathematical rules, graphs, and tables, you can use a function to represent the relationships explained by them. Chemists, teachers, lab technicians, computer programmers, insurance professionals, and engineers are just some of the many professions that make use of equations and functions.

Throughout this chapter, you will learn the language of algebra, you will evaluate expressions, you will write equations and inequalities, you will check your solutions, you will write function rules and graph functions, and you will devise problem-solving strategies for real-world situations.

1.1 Definition of Variable

Here you'll learn how to represent unknown quantities with variables so you can write algebraic expressions and equations for real-world situations.

What if you had a jar filled with dimes and quarters? You know that the total of the coins in the jar is \$8.60. How could you write an equation to represent this situation? After completing this Concept, you'll be able to use variables to write equations like this one with unknown quantities.

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CK-12 Foundation: 0101S Languageof Algebra

Guidance

No one likes doing the same problem over and over again—that's why mathematicians invented algebra. Algebra takes the basic principles of math and makes them more general, so we can solve a problem once and then use that solution to solve a group of similar problems.

In arithmetic, you've dealt with numbers and their arithmetical operations (such as $+$, $-$, $\times$, $\div$). In algebra, we use symbols called **variables** (which are usually letters, such as x, y, a, b, c, $\ldots$) to represent numbers and sometimes processes.

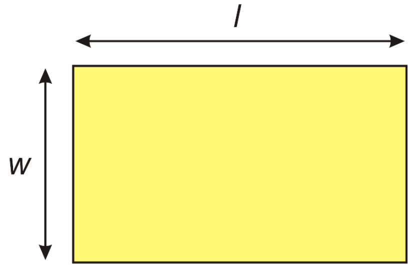
For example, we might use the letter x to represent some number we don't know yet, which we might need to figure out in the course of a problem. Or we might use two letters, like x and y, to show a relationship between two numbers without needing to know what the actual numbers are. The same letters can represent a wide range of possible numbers, and the same letter may represent completely different numbers when used in two different problems.

Using variables offers advantages over solving each problem "from scratch." With variables, we can:

- Formulate arithmetical laws such as $a+b=b+a$ for all real numbers a and b.
- Refer to "unknown" numbers. For instance: find a number x such that $3x+1=10$.
- Write more compactly about functional relationships such as, "If you sell x tickets, then your profit will be $3x-10$ dollars, or "$f(x)=3x-10$," where "f" is the profit function, and x is the input (i.e. how many tickets you sell).

Example A

Write an algebraic equation for the perimeter and area of the rectangle below.



To find the perimeter, we add the lengths of all 4 sides. We can still do this even if we don't know the side lengths in numbers, because we can use variables like l and w to represent the unknown length and width. If we start at the top left and work clockwise, and if we use the letter P to represent the perimeter, then we can say:

$$P = l + w + l + w$$

We are adding 2 l's and 2 w's, so we can say that:

$$P = 2 \cdot l + 2 \cdot w$$

It's customary in algebra to omit multiplication symbols whenever possible. For example, $11x$ means the same thing as $11 \cdot x$ or $11 \times x$. We can therefore also write:

$$P = 2l + 2w$$

Area is *length multiplied by width*. In algebraic terms we get:

$$A = l \times w \rightarrow A = l \cdot w \rightarrow A = lw$$

Note: $2l + 2w$ by itself is an example of a **variable expression**; $P = 2l + 2w$ is an example of an **equation**. The main difference between expressions and **equa**tions is the presence of an **equa**ls sign (=).

In the above example, we found the simplest possible ways to express the perimeter and area of a rectangle when we don't yet know what its length and width actually are. Now, when we encounter a rectangle whose dimensions we do know, we can simply substitute (or **plug in**) those values in the above equations. In this chapter, we will encounter many expressions that we can evaluate by plugging in values for the variables involved.

Example B

Eric has some money in his savings account. How much more money does he need in order to buy a game that costs $98?

Solution:

Let M be the money that Eric still needs and let S be the money that Eric has in his savings account. Then, by subtracting the money he already has from the total money needed, we can figure out how much money he still needs:

$M = 98 - S$.

Example C

Write an equation for the sum of 3 times some number and 5.

Solution:

Let S be the total sum. Let n be *some number.* Then 3 times some number is $3 \cdot n$ and then the sum of that and 5 is:

$S = 3n + 5$.

Watch this video for help with the Examples above.

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Click image to the left for more content.

CK-12 Foundation: The Language of Algebra

Vocabulary

- We use symbols called **variables** (which are usually letters, such as x, y, a, b, c, $\ldots$) to represent numbers and sometimes processes.
- $2l + 2w$ by itself is an example of a **variable expression**; $P = 2l + 2w$ is an example of an **equation**. The main difference between expressions and **equa**tions is the presence of an **equal**s sign (=).

Guided Practice

Alex has a certain amount of nickels and dimes in a jar. Write an algebraic equation for how much money she has, in terms of how many nickles and dimes she has.

Solution:

Let n be the number of nickels and d be the number of dimes that Alex has in the jar. Since each nickel is worth \$0.05, the amount of money she has in nickels will be:

$0.05 \cdot n$

Since each dime is worth \$0.10, the amount of money she has in dimes will be:

$0.10 \cdot d$

This means that the total amount of money M that Alex has will be:

$M = 0.05 \cdot n + 0.10 \cdot d$.

Simplifying the expressions, we get:

$M = 0.05n + 0.10d$.

Practice

For 1-4, write the following in a more condensed form by leaving out a multiplication symbol.

1. $2 \times 11x$

2. $1.35 \cdot y$
3. $3 \times \frac{1}{4}$
4. $\frac{1}{4} \cdot z$

For 5-10, write an equation for the following situations.

5. The amount of money Andrea has in a jar full of quarters and dimes.
6. The amount of money Michelle has in her coin purse if it only contains quarters, dimes and pennies.
7. The sum of 7 and 6 times some number.
8. 4 less than 20 times some number.
9. The amount of money you will earn if you are paid \$10.25 an hour and spend \$4.00 round trip to get too and from work.
10. A father earns a \$2000 dividend from an oil investment and distributes it equally amongst his children.

1.2 Expressions with One or More Variables

Here you'll learn how to evaluate algebraic expressions by plugging in specific values for its variable(s).

What if the paycheck for your summer job were represented by the algebraic expression $10h+25$, where h is the number of hours you work? If you worked 20 hours last week, how could you find the value of your paycheck? After completing this Concept, you'll be able to evaluate algebraic expressions like this one.

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CK-12 Foundation: 0102s evaluatealgebraic expressions

Guidance

When we are given an algebraic expression, one of the most common things we might have to do with it is **evaluate** it for some given value of the variable. The following example illustrates this process.

Example A

Let $x = 12$. *Find the value of* $2x-7$.

Solution:

To find the solution, we substitute 12 for x in the given expression. Every time we see x, we replace it with 12.

$$\begin{aligned}2x-7 &= 2(12)-7\\ &= 24-7\\ &= 17\end{aligned}$$

Note: At this stage of the problem, we place the substituted value in parentheses. We do this to make the written-out problem easier to follow, and to avoid mistakes. (If we didn't use parentheses and also forgot to add a multiplication sign, we would end up turning $2x$ into 212 instead of 2 times 12!)

Example B

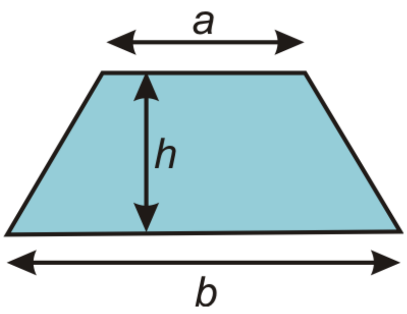
Let $y=-2$. *Find the value of* $\frac{7}{y}-11y+2$.

Solution

$$\begin{aligned}\frac{7}{(-2)}-11(-2)+2&=-3\frac{1}{2}+22+2\\&=24-3\frac{1}{2}\\&=20\frac{1}{2}\end{aligned}$$

Many expressions have more than one variable in them. For example, the formula for the perimeter of a rectangle in the introduction has two variables: length (l) and width (w). In these cases, be careful to substitute the appropriate value in the appropriate place.

Example C



The area of a trapezoid is given by the equation $A=\frac{h}{2}(a+b)$. *Find the area of a trapezoid with bases* $a=10$ *cm and* $b=15$ *cm and height* $h=8$ *cm.*

Solution:

To find the solution to this problem, we simply take the values given for the variables a, b, and h, and plug them in to the expression for A:

$$\begin{aligned}&A=\frac{h}{2}(a+b) && \text{Substitute 10 for } a,\ 15 \text{ for } b, \text{ and 8 for } h.\\&A=\frac{8}{2}(10+15) && \text{Evaluate piece by piece. } 10+15=25;\ \frac{8}{2}=4.\\&A=4(25)=100\end{aligned}$$

The area of the trapezoid is 100 square centimeters.

Watch this video for help with the Examples above.

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CK-12 Foundation: Evaluate AlgebraicExpressions

Vocabulary

- When given an algebraic expression, one of the most common things we might have to do with it is **evaluate** it for some given value of the variable. We substitute the value in for the variable and simplify the expression.

Guided Practice

Let $x = 3$ *and* $y = -2$. *Find the value of* $3xy + \frac{6}{y} - 2x$.

Solution

$$\begin{aligned} 3xy + \frac{6}{y} - 2x &= 3(3)(-2) + \frac{6}{-2} - 2(3) \\ &= -18 - 3 - 6) \\ &= -27 \end{aligned}$$

Practice

Evaluate 1-8 using $a = -3, \ b = 2, \ c = 5$, and $d = -4$.

1. $2a + 3b$
2. $4c + d$
3. $5ac - 2b$
4. $\frac{2a}{c-d}$
5. $\frac{3b}{d}$
6. $\frac{a-4b}{3c+2d}$
7. $\frac{1}{a+b}$
8. $\frac{ab}{cd}$

For 9-11, the weekly cost C of manufacturing x remote controls is given by the formula $C = 2000 + 3x$, where the cost is given in dollars.

9. What is the cost of producing 1000 remote controls?
10. What is the cost of producing 2000 remote controls?
11. What is the cost of producing 2500 remote controls?

1.3 Algebra Expressions with Exponents

Here you'll evaluate algebraic expressions containing exponents for specific variable values.

What if you knew the volume of a cube was represented by the formula $V = s^3$, where s the length of a side. You measure the cube's side to be 4 inches. How could you find its volume? After completing this Concept, you'll be able to evaluate exponential expressions like this one.

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CK-12 Foundation: 0103S EvaluateAlgebraic Expressions with Exponents

For a more detailed review of exponents and their properties, check out the video at http://www.mathvids.com/lesson/mathhelp/863-exponents—basics.

Guidance

Many formulas and equations in mathematics contain exponents. Exponents are used as a short-hand notation for repeated multiplication. For example:

$$\begin{aligned} 2 \cdot 2 &= 2^2 \\ 2 \cdot 2 \cdot 2 &= 2^3 \end{aligned}$$

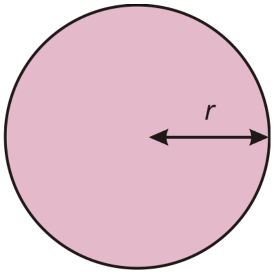
The exponent stands for how many times the number is used as a factor (multiplied). When we deal with integers, it is usually easiest to simplify the expression. We simplify:

$$\begin{aligned} 2^2 &= 4 \\ 2^3 &= 8 \end{aligned}$$

However, we need exponents when we work with variables, because it is much easier to write x^8 than $x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$.

To evaluate expressions with exponents, substitute the values you are given for each variable and simplify. It is especially important in this case to substitute using parentheses in order to make sure that the simplification is done correctly.

Example A



The area of a circle is given by the formula $A = \pi r^2$. *Find the area of a circle with radius* $r = 17$ *inches.*

Substitute values into the equation.

$$\begin{aligned} A &= \pi r^2 && \text{Substitute 17 for } r. \\ A &= \pi(17)^2 && \pi \cdot 17 \cdot 17 \approx 907.9202\ldots \text{ Round to 2 decimal places.} \end{aligned}$$

The area is approximately 907.92 square inches.

Example B

Find the value of $\frac{x^2y^3}{x^3+y^2}$, *for* $x = 2$ *and* $y = -4$.

Substitute the values of x and y in the following.

$$\frac{x^2y^3}{x^3+y^2} = \frac{(2)^2(-4)^3}{(2)^3+(-4)^2} \qquad \text{Substitute 2 for } x \text{ and } -4 \text{ for } y.$$

$$\frac{4(-64)}{8+16} = \frac{-256}{24} = \frac{-32}{3}$$

Evaluate expressions: $(2)^2 = (2)(2) = 4$ and $(2)^3 = (2)(2)(2) = 8$. $(-4)^2 = (-4)(-4) = 16$ and $(-4)^3 = (-4)(-4)(-4) = -64$.

Example C

The height (h) *of a ball in flight is given by the formula* $h = -32t^2 + 60t + 20$, *where the height is given in feet and the time* (t) *is given in seconds. Find the height of the ball at time* $t = 2$ *seconds.*

Solution

$$\begin{aligned} h &= -32t^2 + 60t + 20 \\ &= -32(2)^2 + 60(2) + 20 && \text{Substitute 2 for } t. \\ &= -32(4) + 60(2) + 20 \\ &= 12 \end{aligned}$$

The height of the ball is 12 feet.

Watch this video for help with the Examples above.

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CK-12 Foundation: Evaluate AlgebraicExpressionswith Exponents

Vocabulary

- **Exponents** are used as a short-hand notation for repeated multiplication. For example:

$$2 \cdot 2 = 2^2$$
$$2 \cdot 2 \cdot 2 = 2^3$$

The exponent stands for how many times the number is used as a factor (multiplied).

Guided Practice

Find the value of $\frac{a^2+b^2}{a^2-b^2}$, *for* $a = -1$ *and* 5.

Substitute the values of x and y in the following.

$$\frac{a^2+b^2}{a^2-b^2} = \frac{(-1)^2+(5)^2}{(-1)^2-(5)^2} \qquad \text{Substitute } -1 \text{ for } a \text{ and } 5 \text{ for } b.$$
$$\frac{1+25}{1-25} = \frac{26}{24} = \frac{13}{12} \qquad \text{Evaluate and simplify expressions.}$$

Practice

Evaluate 1-8 using $x = -1,\ y = 2,\ z = -3$, and $w = 4$.

1. $8x^3$
2. $\frac{5x^2}{6z^3}$
3. $3z^2 - 5w^2$
4. $x^2 - y^2$
5. $\frac{z^3+w^3}{z^3-w^3}$
6. $2x^3 - 3x^2 + 5x - 4$
7. $4w^3 + 3w^2 - w + 2$
8. $3 + \frac{1}{z^2}$

For 9-10, use the fact that the volume of a box without a lid is given by the formula $V = 4x(10-x)^2$, where x is a length in inches and V is the volume in cubic inches.

9. What is the volume when $x = 2$?
10. What is the volume when $x = 3$?

1.4 Order of Operations

Here you'll learn how to apply the order of operations to decide which operations take precedence over others when evaluating algebraic expressions.

What if you had a mathematical expression with multiple operations like $17-4 \div 2+3 \times 5$? How could you find its value? After completing this Concept, you'll be able to use the order of operations to evaluate expressions like this one.

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CK-12 Foundation: 0104S EvaluateAlgebraic Expressions with GroupingSymbols

Guidance

Look at and evaluate the following expression:

$$2+4 \times 7-1=?$$

How many different ways can we interpret this problem, and how many different answers could someone possibly find for it?

The *simplest* way to evaluate the expression is simply to start at the left and work your way across:

$$\begin{aligned}&2+4 \times 7-1\\&=6 \times 7-1\\&=42-1\\&=41\end{aligned}$$

This is the answer you would get if you entered the expression into an ordinary calculator. But if you entered the expression into a scientific calculator or a graphing calculator you would probably get 29 as the answer.

In mathematics, the order in which we perform the various **operations** (such as adding, multiplying, etc.) is important. In the expression above, the operation of **multiplication** takes precedence over **addition**, so we evaluate it first. Let's re-write the expression, but put the multiplication in brackets to show that it is to be evaluated first.

$$2+(4 \times 7)-1=?$$

First evaluate the brackets: $4 \times 7 = 28$. Our expression becomes:

$$2 + (28) - 1 = ?$$

When we have only addition and subtraction, we start at the left and work across:

$$\begin{aligned} &2 + 28 - 1 \\ &= 30 - 1 \\ &= 29 \end{aligned}$$

Algebra students often use the word **"PEMDAS"** to help remember the order in which we evaluate the mathematical expressions: **P**arentheses, **E**xponents, **M**ultiplication, **D**ivision, **A**ddition and **S**ubtraction.

Order of Operations

1. Evaluate expressions within **P**arentheses (also all brackets [ ] and braces { }) first.
2. Evaluate all **E**xponents (terms such as 3^2 or x^3) next.
3. **M**ultiplication *and* **D**ivision is next - work from left to right completing **both** multiplication and division in the order that they appear.
4. Finally, evaluate **A**ddition *and* **S**ubtraction - work from left to right completing **both** addition and subtraction in the order that they appear.

The first step in the order of operations is called **parentheses**, but we include all **grouping symbols** in this step—not just parentheses (), but also square brackets [ ] and curly braces { }.

Example A

Evaluate the following:

a) $4 - 7 - 11 + 2$

b) $4 - (7 - 11) + 2$

c) $4 - [7 - (11 + 2)]$

Each of these expressions has the same numbers and the same mathematical operations, in the same order. The placement of the various grouping symbols means, however, that we must evaluate everything in a different order each time. Let's look at how we evaluate each of these examples.

a) This expression doesn't have parentheses, exponents, multiplication, or division. **PEMDAS** states that we treat addition and subtraction as they appear, starting at the left and working right (it's NOT addition *then* subtraction).

$$\begin{aligned} 4 - 7 - 11 + 2 &= -3 - 11 + 2 \\ &= -14 + 2 \\ &= -12 \end{aligned}$$

b) This expression has parentheses, so we first evaluate $7 - 11 = -4$. Remember that when we subtract a negative it is equivalent to adding a positive:

$$\begin{aligned}4-(7-11)+2 &= 4-(-4)+2\\ &= 8+2\\ &= 10\end{aligned}$$

c) An expression can contain any number of sets of parentheses. Sometimes expressions will have sets of parentheses **inside** other sets of parentheses. When faced with **nested parentheses**, start at the innermost parentheses and work outward.

Brackets may also be used to group expressions which already contain parentheses. This expression has both brackets and parentheses. We start with the innermost group: $11+2=13$. Then we complete the operation in the brackets.

$$\begin{aligned}4-[7-(11+2)] &= 4-[7-(13)]\\ &= 4-[-6]\\ &= 10\end{aligned}$$

Example B

Evaluate the following:

a) $3\times5-7\div2$

b) $3\times(5-7)\div2$

a) There are no grouping symbols. **PEMDAS** dictates that we multiply and divide first, working from left to right: $3\times5=15$ and $7\div2=3.5$. (NOTE: It's not multiplication *then* division.) Next we subtract:

$$\begin{aligned}3\times5-7\div2 &= 15-3.5\\ &= 11.5\end{aligned}$$

b) First, we evaluate the expression inside the parentheses: $5-7=-2$. Then work from left to right:

$$\begin{aligned}3\times(5-7)\div2 &= 3\times(-2)\div2\\ &= (-6)\div2\\ &= -3\end{aligned}$$

We can also use the order of operations to simplify an expression that has variables in it, after we substitute specific values for those variables.

Example C

Use the order of operations to evaluate the following:

a) $2-(3x+2)$ when $x=2$

b) $3y^2+2y+1$ when $y=-3$

a) The first step is to substitute the value for x into the expression. We can put it in parentheses to clarify the resulting expression.

$$2-(3(2)+2)$$

(Note: $3(2)$ is the same as 3×2.)

Follow **PEMDAS** - first parentheses. Inside parentheses follow **PEMDAS** again.

$2-(3 \times 2+2)=2-(6+2)$ Inside the parentheses, we multiply first.

$2-8=-6$ Next we add inside the parentheses, and finally we subtract.

b) The first step is to substitute the value for y into the expression.

$$3 \times (-3)^2+2 \times (-3)+1$$

Follow **PEMDAS**: we cannot simplify the expressions in parentheses, so exponents come next.

$3 \times (-3)^2+2 \times (-3)+1$ Evaluate exponents: $(-3)^2=9$

$=3 \times 9+2 \times (-3)+1$ Evaluate multiplication: $3 \times 9=27;\ 2 \times -3=-6$

$=27+(-6)+1$ Add and subtract in order from left to right.

$=27-6+1$

$=22$

In part (b) we left the parentheses around the negative numbers to clarify the problem. They did not affect the order of operations, but they did help avoid confusion when we were multiplying negative numbers.

Graphing Calculators

A graphing calculator is a very useful tool in evaluating algebraic expressions. Like a scientific calculator, a graphing calculator follows **PEMDAS**. In this section we will explain two ways of evaluating expressions with the graphing calculator.

Example D

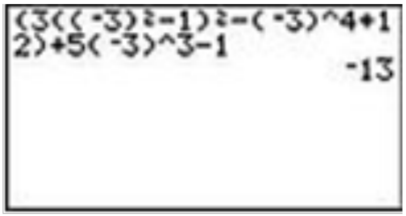
Evaluate $\left[3(x^2-1)^2-x^4+12\right]+5x^3-1$ *when* $x=-3$.

Method 1: Substitute for the variable first. Then evaluate the numerical expression with the calculator.

Substitute the value $x=-3$ *into the expression.*

$$\left[3((-3)^2-1)^2-(-3)^4+12\right]+5(-3)^3-1$$

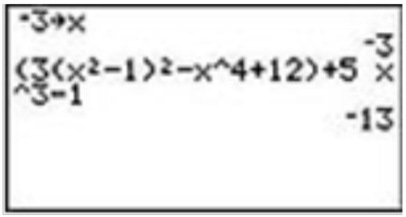
Input this in the calculator just as it is and press **[ENTER]**. *(Note: use* $\wedge$ *to enter exponents)*



```
(3((⁻3)²-1)²-(⁻3)^4+1
2)+5(⁻3)^3-1
                   ⁻13
```

The answer is -13.

Method 2: Input the original expression in the calculator first and then evaluate.

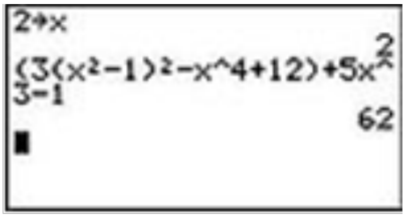


```
⁻3→x
                    ⁻3
(3(x²-1)²-x^4+12)+5 x
^3-1
                   ⁻13
```

First, store the value $x = -3$ *in the calculator. Type -3* **[STO]** *x (The letter x can be entered using the x*−**[VAR]** *button or* **[ALPHA] + [STO]**). *Then type the original expression in the calculator and press* **[ENTER]**.

The answer is -13.

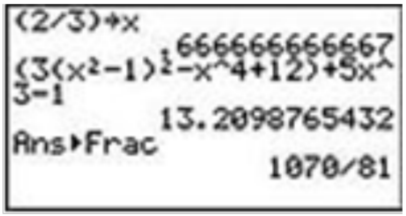
The second method is better because you can easily evaluate the same expression for any value you want. For example, let's evaluate the same expression using the values $x = 2$ and $x = \frac{2}{3}$.



```
2→x
                     2
(3(x²-1)²-x^4+12)+5x^
3-1
                    62
▮
```

For $x = 2$, store the value of x in the calculator: 2**[STO]** x. Press **[2nd] [ENTER]** twice to get the previous expression you typed in on the screen without having to enter it again. Press **[ENTER]** to evaluate the expression.

The answer is 62.



```
(2/3)→x
       .6666666666667
(3(x²-1)²-x^4+12)+5x^
3-1
        13.2098765432
Ans▸Frac
              1070/81
```

For $x = \frac{2}{3}$, store the value of x in the calculator: $\frac{2}{3}$**[STO]** x. Press **[2nd] [ENTER]** twice to get the expression on the screen without having to enter it again. Press **[ENTER]** to evaluate.

The answer is 13.21, or $\frac{1070}{81}$ in fraction form.

Note: On graphing calculators there is a difference between the minus sign and the negative sign. When we stored the value negative three, we needed to use the negative sign which is to the left of the **[ENTER]** button on the calculator. On the other hand, to perform the subtraction operation in the expression we used the minus sign. The minus sign is right above the plus sign on the right.

Watch this video for help with the Examples above.

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CK-12 Foundation: Evaluate with GroupingSymbols

Vocabulary

- Algebra students often use the word **"PEMDAS"** to help remember the order in which we evaluate the mathematical expressions: **P**arentheses, **E**xponents, **M**ultiplication, **D**ivision, **A**ddition and **S**ubtraction.

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4. Finally, evaluate **A**ddition *and* **S**ubtraction - work from left to right completing **both** addition and subtraction in the order that they appear.

- The first step in the order of operations is called **parentheses**, but we include all **grouping symbols** in this step—not just parentheses (), but also square brackets [ ] and curly braces { }.

- Sometimes expressions will have sets of parentheses **inside** other sets of parentheses. These are called **nested parentheses.**

Guided Practice

Use the order of operations to evaluate the following:

a) $(3 \times 5) - (7 \div 2)$

b) $2 - (t-7)^2 \times (u^3 - v)$ when $t = 19,\ u = 4$, and $v = 2$

Solutions:

a) First, we evaluate the expressions inside parentheses: $3 \times 5 = 15$ and $7 \div 2 = 3.5$. Then work from left to right:

$$\begin{aligned}(3 \times 5) - (7 \div 2) &= 15 - 3.5\\ &= 11.5\end{aligned}$$

Note that adding parentheses didn't change the expression in part c, but did make it easier to read. Parentheses can be used to change the order of operations in an expression, but they can also be used simply to make it easier to understand.

b) The first step is to substitute the values for $t,\ u$, and v into the expression.

$$2-(19-7)^2\times(4^3-2)$$

Follow **PEMDAS**:

$$\begin{aligned}
&2-(19-7)^2\times(4^3-2) && \text{Evaluate parentheses: } (19-7)=12;\ (4^3-2)=(64-2)=62\\
&=2-12^2\times 62 && \text{Evaluate exponents: } 12^2=144\\
&=2-144\times 62 && \text{Multiply: } 144\times 62=8928\\
&=2-8928 && \text{Subtract.}\\
&=-8926
\end{aligned}$$

Part (b) in the last example shows another interesting point. When we have an expression inside the parentheses, we use **PEMDAS** to determine the order in which we evaluate the contents.

Practice

1. Evaluate the following expressions involving variables.
   a. $2y^2$ when $x=1$ and $y=5$
   b. $3x^2+2x+1$ when $x=5$
2. Use the order of operations to evaluate the following expressions.
   a. $2+7\times 11-12\div 3$
3. Evaluate the following expressions involving variables.
   a. $(y^2-x)^2$ when $x=2$ and $y=1$

For 4-6, use the order of operations to evaluate the following expressions.

4. $8-(19-(2+5)-7)$
5. $(3+7)\div(7-12)$
6. $(4-1)^2+3^2\cdot 2$

For 7-10, insert parentheses in each expression to make a true equation.

7. $5-2\times 6-5+2=5$
8. $12\div 4+10-3\times 3+7=11$
9. $22-32-5\times 3-6=30$
10. $12-8-4\times 5=-8$

For 11-12, evaluate each expression using a graphing calculator.

11. $x^2+2x-xy$ when $x=250$ and $y=-120$
12. $(xy-y^4)^2$ when $x=0.02$ and $y=-0.025$

1.5 Algebra Expressions with Fraction Bars

Here you'll learn how to apply the order of operations to evaluate algebraic expressions containing fraction bars for specific variable values.

What if you had a mathematical expression containing fraction bars, like $\frac{(7-3)^2}{6-4}+5$? How could you find its value? After completing this Concept, you'll be able to use the order of operations to evaluate expressions like this one.

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CK-12 Foundation: 0105S EvaluateAlgebraic Expressions with FractionBars

Try This

For more practice, you can play an algebra game involving order of operations online at http://www.funbrain.com/algebra/index.html.

Guidance

Fraction bars count as grouping symbols for **PEMDAS**, so we evaluate them in the first step of solving an expression. All numerators and all denominators can be treated as if they have invisible parentheses around them. When **real** parentheses are also present, remember that the innermost grouping symbols come first. If, for example, parentheses appear on a numerator, they would take precedence over the fraction bar. If the parentheses appear outside of the fraction, then the fraction bar takes precedence.

Example A

Use the order of operations to evaluate the following expression: $\frac{z+3}{4}-1$ when $z=2$

Solution:

We substitute the value for z into the expression.

$$\frac{2+3}{4}-1$$

Although this expression has no parentheses, the fraction bar is also a grouping symbol—it has the same effect as a set of parentheses. We can write in the "invisible parentheses" for clarity:

$$\frac{(2+3)}{4}-1$$

Using **PEMDAS**, we first evaluate the numerator:

$$\frac{5}{4}-1$$

We can convert $\frac{5}{4}$ to a mixed number:

$$\frac{5}{4}=1\frac{1}{4}$$

Then evaluate the expression:

$$\frac{5}{4}-1=1\frac{1}{4}-1=\frac{1}{4}$$

Example B

Use the order of operations to evaluate the following expression: $\left(\frac{a+2}{b+4}-1\right)+b$ when $a=3$ and $b=1$

Solution:

We substitute the values for a and b into the expression:

$$\left(\frac{3+2}{1+4}-1\right)+1$$

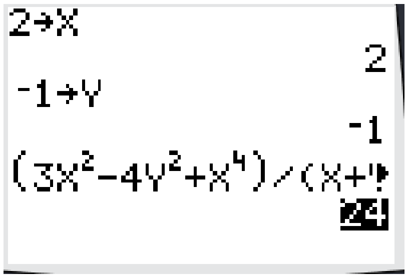
This expression has nested parentheses (remember the effect of the fraction bar). The innermost grouping symbol is provided by the fraction bar. We evaluate the numerator $(3+2)$ and denominator $(1+4)$ first.

$$\begin{aligned}\left(\frac{3+2}{1+4}+1\right)+1 &= \left(\frac{5}{5}-1\right)+1 && \text{Next we evaluate the inside of the parentheses. First we divide.}\\ &=(1-1)+1 && \text{Next we subtract.}\\ &=0+1=1\end{aligned}$$

Example C

Evaluate the expression $\frac{3x^2-4y^2+x^4}{(x+y)^{\frac{1}{2}}}$ *for* $x=2,\ y=-1$.

Solution



```
2→X
                2
-1→Y
               -1
(3X²-4Y²+X⁴)/(X+Y
               24
```

Store the values of x and y: 2 **[STO]** x, -1 **[STO]** y. (The letters x and y can be entered using **[ALPHA] + [KEY]**.) Input the expression in the calculator. When an expression includes a fraction, be sure to use parentheses: $\frac{(\text{numerator})}{(\text{denominator})}$.

Press **[ENTER]** to obtain the answer 24.

Watch this video for help with the Examples above.

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CK-12 Foundation: Evaluate Expressions with FractionBars

Vocabulary

- **Fraction bars** count as grouping symbols for **PEMDAS**, so we evaluate them in the first step of solving an expression. All numerators and all denominators can be treated as if they have invisible parentheses around them.

Guided Practice

Use the order of operations to evaluate the following expression: $2 \times \left(\frac{w+(x-2z)}{(y+2)^2} - 1\right)$ when $w = 11, x = 3, y = 1$, and $z = -2$

Solution:

We substitute the values for w, x, y, and z into the expression:

$2 \times \left(\frac{11+(3-2(-2))}{(1+2)^2} - 1\right)$

This complicated expression has several layers of nested parentheses. One method for ensuring that we start with the innermost parentheses is to use more than one type of parentheses. Working from the outside, we can leave the outermost brackets as parentheses (). Next will be the "invisible brackets" from the fraction bar; we will write these as [ ]. The third level of nested parentheses will be the { }. We will leave negative numbers in round brackets.

$$2 \times \left(\frac{[11+\{3-2(-2)\}]}{\left[\{1+2\}^2\right]} - 1 \right)$$

Start with the innermost grouping sign: $\{\}$.

$$\{1+2\} = 3;\ \{3-2(-2)\} = 3+4 = 7$$

$$= 2\left(\frac{[11+7]}{[3^2]} - 1 \right)$$

Next, evaluate the square brackets.

$$= 2\left(\frac{18}{9} - 1 \right)$$

Next, evaluate the round brackets. Start with division.

$$= 2(2-1)$$

Finally, do the addition and subtraction.

$$= 2(1) = 2$$

Practice

For 1-3, use the order of operations to evaluate the following expressions.

1. $\frac{2\cdot(3+(2-1))}{4-(6+2)} - (3-5)$
2. $\frac{4+7(3)}{9-4} + \frac{12-3\cdot 2}{2}$
3. $\frac{(2^2+5)^2}{5^2-4^2} \div (2+1)$

For 4-5, evaluate the following expressions involving variables.

4. $\frac{jk}{j+k}$ when $j = 6$ and $k = 12$
5. $\frac{x+y^2}{y-x}$ when $x = 2$ and $y = 3$

For 6-9, evaluate the following expressions involving variables.

6. $\frac{4x}{9x^2-3x+1}$ when $x = 2$
7. $\frac{z^2}{x+y} + \frac{x^2}{x-y}$ when $x = 1,\ y = -2$, and $z = 4$
8. $\frac{4xyz}{y^2-x^2}$ when $x = 3,\ y = 2$, and $z = 5$
9. $\frac{x^2-z^2}{xz-2x(z-x)}$ when $x = -1$ and $z = 3$

For 10-14, evaluate each expression using a graphing calculator.

10. $x^2+2x-xy$ when $x = 250$ and $y = -120$
11. $(xy-y^4)^2$ when $x = 0.02$ and $y = -0.025$
12. $\frac{x+y-z}{xy+yz+xz}$ when $x = \frac{1}{2},\ y = \frac{3}{2}$, and $z = -1$
13. $\frac{(x+y)^2}{4x^2-y^2}$ when $x = 3$ and $y = -5$
14. $\frac{(x-y)^3}{x^3-y} + \frac{(x+y)^2}{x+y^4}$ when $x = 4$ and $y = -2$

1.6 Patterns and Expressions

Here you'll learn how to write and evaluate algebraic equations to solve real-world problems.

What if you knew the Booster Club sold 855 spaghetti dinners and collected \$6840? How could you write an equation to find the amount each diner paid? After completing this Concept, you'll be able to write equations like this one.

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CK-12 Foundation: 0106S Write anEquation]

Guidance

In mathematics, and especially in algebra, we look for patterns in the numbers we see. The tools of algebra help us describe these patterns with words and with **equations** (formulas or functions). An equation is a mathematical recipe that gives the value of one variable in terms of another.

For example, if a theme park charges \$12 admission, then the number of people who enter the park every day and the amount of money taken in by the ticket office are related mathematically, and we can write a rule to find the amount of money taken in by the ticket office.

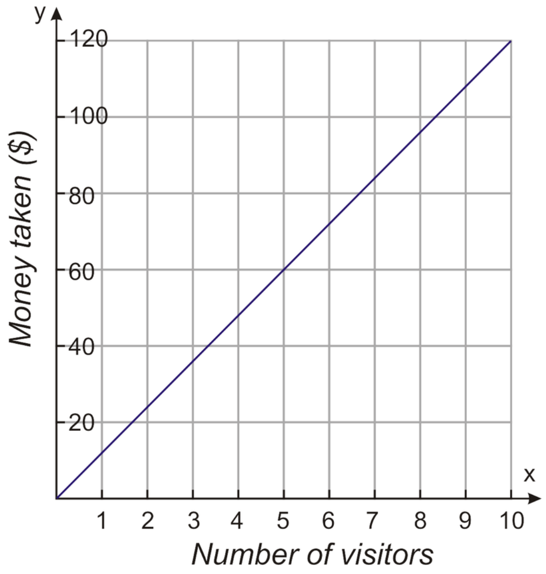
In words, we might say "The amount of money taken in is equal to twelve times the number of people who enter the park."

We could also make a table. The following table relates the number of people who visit the park and the total money taken in by the ticket office.

Number of visitors	1	2	3	4	5	6	7
Money taken in (\$)	12	24	36	48	60	72	84

Clearly, we would need a **big** table to cope with a busy day in the middle of a school vacation!

A third way we might relate the two quantities (visitors and money) is with a graph. If we plot the money taken in on the **vertical axis** and the number of visitors on the **horizontal axis**, then we would have a graph that looks like the one shown below. Note that this graph shows a smooth line that includes non-whole number values of x (e.g. $x = 2.5$). In real life this would not make sense, because fractions of people can't visit a park. This is an issue of domain and range, something we will talk about later.



The method we will examine in detail in this lesson is closer to the first way we chose to describe the relationship. In words we said that *"The amount of money taken in is twelve times the number of people who enter the park."* In mathematical terms we can describe this sort of relationship with **variables**. A variable is a letter used to represent an unknown quantity. We can see the beginning of a mathematical formula in the words:

The amount of money taken in **is** twelve **times** the number of people who enter the park.

This can be translated to:

$$\text{the amount of money taken in} = 12 \times (\text{the number of people who enter the park})$$

We can now see which quantities can be assigned to **letters**. First we must state which letters (or **variables**) relate to which quantities. We call this **defining the variables**:

Let $x =$ the number of people who enter the theme park.

Let $y =$ the total amount of money taken in at the ticket office.

We now have a fourth way to describe the relationship: with an algebraic equation.

$$y = 12x$$

Writing a mathematical equation using variables is very convenient. You can perform all of the operations necessary to solve this problem without having to write out the known and unknown quantities over and over again. At the end of the problem, you just need to remember which quantities x and y represent.

Write an Equation

An equation is a term used to describe a collection of **numbers** and **variables** related through mathematical **operators**. An **algebraic equation** will contain letters that represent real quantities. For example, if we wanted to use the algebraic equation in the example above to find the money taken in for a certain number of visitors, we would substitute that number for x and then solve the resulting equation for y.

Example A

A theme park charges $12 entry to visitors. Find the money taken in if 1296 people visit the park.

Let's break the solution to this problem down into steps. This will be a useful strategy for all the problems in this lesson.

Step 1: Extract the important information.

$$\begin{aligned}(\text{number of dollars taken in}) &= 12 \times (\text{number of visitors})\\ (\text{number of visitors}) &= 1296\end{aligned}$$

Step 2: Translate into a mathematical equation. To do this, we pick variables to stand for the numbers.

$$\begin{aligned}&\text{Let } y = (\text{number of dollars taken in}).\\ &\text{Let } x = (\text{number of visitors}).\end{aligned}$$

$$\begin{aligned}(\text{number of dollars taken in}) &= 12 \times (\text{number of visitors})\\ y &= 12 \times x\end{aligned}$$

Step 3: Substitute in any known values for the variables.

$$\begin{aligned}y &= 12 \times x\\ x &= 1296\\ &\therefore\\ y &= 12 \times 1296\end{aligned}$$

Step 4: Solve the equation.

$$y = 12 \times 1296 = 15552$$

The amount of money taken in is \$15552.

Step 5: Check the result.

If \$15552 is taken at the ticket office and tickets are \$12, then we can divide the total amount of money collected by the price per individual ticket.

$$(\text{number of people}) = \frac{15552}{12} = 1296$$

1296 is indeed the number of people who entered the park. **The answer checks out.**

Example B

The following table shows the relationship between two quantities. First, write an equation that describes the relationship. Then, find out the value of b when a is 750.

a	0	10	20	30	40	50
b	20	40	60	80	100	120

Step 1: Extract the important information.

We can see from the table that every time a increases by 10, b increases by 20. However, b is not simply twice the value of a. We can see that when $a = 0,\ b = 20$, and this gives a clue as to what rule the pattern follows. The rule linking a and b is:

"To find b, double the value of a and add 20."

Step 2: Translate into a mathematical equation:

TABLE 1.1:

Text	Translates to	Mathematical Expression
"To find b"	→	$b =$
"double the value of a"	→	$2a$
"add 20"	→	+ 20

Our equation is $b = 2a + 20$.

Step 3: Solve the equation.

The original problem asks for the value of b when a is 750. When a is 750, $b = 2a + 20$ becomes $b = 2(750) + 20$. Following the order of operations, we get:

$$\begin{aligned} b &= 2(750) + 20 \\ &= 1500 + 20 \\ &= 1520 \end{aligned}$$

Step 4: Check the result.

In some cases you can check the result by plugging it back into the original equation. Other times you must simply double-check your math. In either case, checking your answer is *always* a good idea. In this case, we can plug our answer for b into the equation, along with the value for a, and see what comes out. $1520 = 2(750) + 20$ is TRUE because both sides of the equation are equal. A true statement means that **the answer checks out**.

Solve Problems Using Equations

Let's solve the following real-world problem by using the given information to write a mathematical equation that can be solved for a solution.

Example C

A group of students are in a room. After 25 students leave, it is found that $\frac{2}{3}$ of the original group is left in the room. How many students were in the room at the start?

Step 1: Extract the important information

We know that 25 students leave the room.

We know that $\frac{2}{3}$ of the original number of students are left in the room.

We need to find how many students were in the room at the start.

Step 2: Translate into a mathematical equation. Initially we have an unknown number of students in the room. We can refer to this as the original number.

Let's define the variable $x =$ the original number of students in the room. After 25 students leave the room, the number of students in the room is $x - 25$. We also know that the number of students left is $\frac{2}{3}$ of x. So we have two expressions for the number of students left, and those two expressions are equal because they represent the same number. That means our equation is:

$$\frac{2}{3}x = x - 25$$

Step 3: Solve the equation.

Add 25 to both sides.

$$\begin{aligned} x - 25 &= \frac{2}{3}x \\ x - 25 + 25 &= \frac{2}{3}x + 25 \\ x &= \frac{2}{3}x + 25 \end{aligned}$$

Subtract $\frac{2}{3}x$ from both sides.

$$\begin{aligned} x - \frac{2}{3}x &= \frac{2}{3}x - \frac{2}{3}x + 25 \\ \frac{1}{3}x &= 25 \end{aligned}$$

Multiply both sides by 3.

$$\begin{aligned} 3 \cdot \frac{1}{3}x &= 3 \cdot 25 \\ x &= 75 \end{aligned}$$

Remember that x represents the original number of students in the room. So, there were 75 students in the room to start with.

Step 4: Check the answer:

If we start with 75 students in the room and 25 of them leave, then there are $75 - 25 = 50$ students left in the room.

$\frac{2}{3}$ of the original number is $\frac{2}{3} \cdot 75 = 50$.

This means that the number of students who are left over equals $\frac{2}{3}$ of the original number. **The answer checks out**.

The method of defining variables and writing a mathematical equation is the method you will use the most in an algebra course. This method is often used together with other techniques such as making a table of values, creating a graph, drawing a diagram and looking for a pattern.

Write a Verbal Equation

In the examples above, we had a **rule**, written in words, and from that developed an algebraic **equation**. In the following example, we will develop a verbal equation based on a table, and use that to write an algebraic equation.

Example D

The following table shows the values of two related quantities. Write an equation that describes the relationship mathematically.

TABLE 1.2:

$x-$**value**	$y-$**value**
-2	10
0	0
2	-10
4	-20
6	-30

Step 1: Extract the important information.

We can see from the table that y is five times bigger than x. The value for y is negative when x is positive, and it is positive when x is negative. Here is the rule that links x and y:

"y is the negative of five times the value of x"

Step 2: Translate this statement into a mathematical equation.

TABLE 1.3:

Text	**Translates to**	**Mathematical Expression**
"y is"	→	$y =$
"negative 5 times the value of x"	→	$-5x$

Our equation is $y = -5x$.

Step 3: There is nothing in this problem to **solve** for. We can move to Step 4.

Step 4: Check the result.

In this case, the way we would check our answer is to use the equation to generate our own xy pairs. If they match the values in the table, then we know our equation is correct. We will plug in -2, 0, 2, 4, and 6 for x and solve for y:

TABLE 1.4:

x	y
-2	$-5(-2) = 10$
0	$-5(0) = 0$
2	$-5(2) = -10$
4	$-5(4) = -20$
6	$-5(6) = -30$

The $y-$values in this table match the ones in the earlier table. **The answer checks out**.

Watch this video for help with the Examples above.

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CK-12 Foundation: Write an Equation

Vocabulary

An equation is a term used to describe a collection of **numbers** and **variables** related through mathematical **operators**. An **algebraic equation** will contain letters that represent real quantities.

Guided Practice

Zarina has a $100 gift card, and she has been spending money on the card in small regular amounts. She checks the balance on the card weekly and records it in the following table.

TABLE 1.5:

Week Number	Balance ($)
1	100
2	78
3	56
4	34

Write an equation for the money remaining on the card in any given week.

Solution:

Step 1: Extract the important information.

The balance remaining on the card is not just a constant multiple of the week number; 100 is 100 times 1, but 78 is not 100 times 2. But there is still a pattern: the balance decreases by 22 whenever the week number increases by 1. This suggests that the balance is somehow related to the amount "-22 times the week number."

In fact, the balance equals "-22 times the week number, plus *something*." To determine what that *something* is, we can look at the values in one row on the table—for example, the first row, where we have a balance of $100 for week number 1.

Step 2: Translate into a mathematical equation.

First, we define our variables. Let n stand for the week number and b for the balance.

Then we can translate our verbal expression as follows:

TABLE 1.6:

Text	Translates to	Mathematical Expression
Balance equals -22 times the week number, plus *something*.	$\rightarrow$	$b = -22n + ?$

To find out what that ? represents, we can plug in the values from that first row of the table, where $b = 100$ and $n = 1$. This gives us $100 = -22(1) + ?$.

So what number gives 100 when you add -22 to it? The answer is 122, so that is the number the ? stands for. Now our final equation is:

$$b = -22n + 122$$

Step 3: All we were asked to **find** was the expression. We weren't asked to solve it, so we can move to Step 4.

Step 4: Check the result.

To check that this equation is correct, we see if it really reproduces the data in the table. To do that we plug in values for n:

$$\begin{aligned}
n = 1 \rightarrow b &= -22(1) + 122 = 122 - 22 = 100\\
n = 2 \rightarrow b &= -22(2) + 122 = 122 - 44 = 78\\
n = 3 \rightarrow b &= -22(3) + 122 = 122 - 66 = 56\\
n = 4 \rightarrow b &= -22(4) + 122 = 122 - 88 = 34
\end{aligned}$$

The equation perfectly reproduces the data in the table. **The answer checks out**.

Practice

TABLE 1.7:

Day	Profit
1	20
2	40
3	60
4	80
5	100

For 1-3, use the above table, which depicts the profit in dollars taken in by a store each day.

1. Write a mathematical equation that describes the relationship between the variables in the table.
2. What is the profit on day 10?
3. If the profit on a certain day is $200, what is the profit on the next day?

For 4-6, Write a mathematical equation that describes each situation below, assuming the the cookie jar starts with 24 cookies.

4. How many cookies are left in the jar after you have eaten some?
5. How many cookies are in the jar after you have eaten 9 cookies?
6. How many cookies are in the jar after you have eaten 9 cookies and then eaten 3 more?

For 7-12, write a mathematical equation for the following situations and solve.

7. Seven times a number is 35. What is the number?
8. Three times a number, plus 15, is 24. What is the number?
9. Twice a number is three less than five times another number. Three times the second number is 15. What are the numbers?
10. One number is 25 more than 2 times another number. If each number were multiplied by five, their sum would be 350. What are the numbers?
11. The sum of two consecutive integers is 35. What are the numbers?
12. Peter is three times as old as he was six years ago. How old is Peter?

For 13-16, Jae just took a math test with 20 questions, each worth an equal number of points. The test is worth 100 points total.

13. Write an equation relating the number of questions Jae got right to the total score he will get on the test.
14. If a score of 70 points earns a grade of $C-$, how many questions would Jae need to get right to get a $C-$ on the test?
15. If a score of 83 points earns a grade of B, how many questions would Jae need to get right to get a B on the test?
16. Suppose Jae got a score of 60% and then was allowed to retake the test. On the retake, he got all the questions right that he got right the first time, and also got half the questions right that he got wrong the first time. What is his new score?

For 17-22, solve the problem by writing an equation.

17. How much water should be added to one liter of pure alcohol to make a mixture of 25% alcohol?
18. A mixture of 50% alcohol and 50% water has 4 liters of water added to it. It is now 25% alcohol. What was the total volume of the original mixture?
19. In Crystal's silverware drawer there are twice as many spoons as forks. If Crystal adds nine forks to the drawer, there will be twice as many forks as spoons. How many forks and how many spoons are in the drawer right now?
    1. Mia drove to Javier's house at 40 miles per hour. Javier's house is 20 miles away. Mia arrived at Javier's house at 2:00 pm. What time did she leave?
    2. Mia left Javier's house at 6:00 pm to drive home. This time she drove 25% faster. What time did she arrive home?
    3. The next day, Mia took the expressway to Javier's house. This route was 24 miles long, but she was able to drive at 60 miles per hour. How long did the trip take?
    4. When Mia took the same route back, traffic on the expressway was 20% slower. How long did the return trip take?
20. The price of an mp3 player decreased by 20% from last year to this year. This year the price of the player is \$120. What was the price last year?
21. SmartCo sells deluxe widgets for \$60 each, which includes the cost of manufacture plus a 20% markup. What does it cost SmartCo to manufacture each widget?

1.7 Words that Describe Patterns

Here you'll learn how to define the variables for and translate expressions into equations and inequalities.

What if you were given a word problem like "It took the Eagle Scouts one hour to wash 3 cars. How long did it take them to wash one car?" or "The distance from the East Coast to the West Coast is more than 2500 miles."? How could you write these sentences in algebraic form? After completing this Concept, you'll be able to write equations and inequalities for situations like these.

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CK-12 Foundation: 0107S Write Equations and Inequalities

Guidance

In algebra, an **equation** is a mathematical expression that contains an equals sign. It tells us that two expressions represent the same number. For example, $y = 12x$ is an equation. An **inequality** is a mathematical expression that contains inequality signs. For example, $y \leq 12x$ is an inequality. Inequalities are used to tell us that an expression is either larger or smaller than another expression. Equations and inequalities can contain both **variables** and **constants**.

Variables are usually given a letter and they are used to represent unknown values. These quantities can change because they depend on other numbers in the problem.

Constants are quantities that remain unchanged. Ordinary numbers like 2, -3, $\frac{3}{4}$, and π are constants.

Equations and inequalities are used as a shorthand notation for situations that involve numerical data. They are very useful because most problems require several steps to arrive at a solution, and it becomes tedious to repeatedly write out the situation in words.

Here are some examples of equations:

$$3x-2=5 \qquad x+9=2x+5 \qquad \frac{x}{3}=15 \qquad x^2+1=10$$

To write an inequality, we use the following symbols:

$>$greater than

$\geq$greater than or equal to

$<$less than

$\leq$less than or equal to

$\neq$not equal to

Here are some examples of inequalities:

$$3x < 5 \qquad 4 - x \leq 2x \qquad x^2 + 2x - 1 > 0 \qquad \frac{3x}{4} \geq \frac{x}{2} - 3$$

The most important skill in algebra is the ability to translate a word problem into the correct equation or inequality so you can find the solution easily. The first two steps are **defining the variables** and **translating** the word problem into a mathematical equation.

Defining the variables means that we assign letters to any unknown quantities in the problem.

Translating means that we change the word expression into a mathematical expression containing variables and mathematical operations with an equal sign or an inequality sign.

Example A

Define the variables and translate the following expressions into equations.

a) A number plus 12 is 20.

b) 9 less than twice a number is 33.

c) $20 was one quarter of the money spent on the pizza.

Solution

a) **Define**

Let $n =$ the number we are seeking.

Translate

A number plus 12 is 20.

$$n + 12 = 20$$

b) **Define**

Let $n =$ the number we are seeking.

Translate

9 less than twice a number is 33.

This means that twice the number, minus 9, is 33.

$$2n - 9 = 33$$

c) **Define**

Let $m =$ the money spent on the pizza.

Translate

$20 was one quarter of the money spent on the pizza.

$$20 = \frac{1}{4}m$$

Often word problems need to be reworded before you can write an equation.

Example B

Find the solution to the following problems.

a) Shyam worked for two hours and packed 24 boxes. How much time did he spend on packing one box?

b) After a 20% discount, a book costs \$12. How much was the book before the discount?

Solution

a) **Define**

Let $t =$ time it takes to pack one box.

Translate

Shyam worked for two hours and packed 24 boxes. This means that two hours is 24 times the time it takes to pack one box.

$$2 = 24t$$

Solve

$$t = \frac{2}{24} = \frac{1}{12} \text{ hours}$$
$$\frac{1}{12} \times 60 \text{ minutes} = 5 \text{ minutes}$$

Answer

Shyam takes 5 minutes to pack a box.

b) **Define**

Let $p =$ the price of the book before the discount.

Translate

After a 20% discount, the book costs \$12. This means that the price minus 20% of the price is \$12.

$$p - 0.20p = 12$$

Solve

$$p - 0.20p = 0.8p, \text{ so } 0.8p = 12$$
$$p = \frac{12}{0.8} = 15$$

Answer

The price of the book before the discount was $15.

Check

If the original price was $15, then the book was discounted by 20% of $15, or $3. $\$15 - 3 = \$12$. **The answer checks out**.

Example C

Define the variables and translate the following expressions into inequalities.

a) The sum of 5 and a number is less than or equal to 2.

b) The distance from San Diego to Los Angeles is less than 150 miles.

c) Diego needs to earn more than an 82 on his test to receive a *B* in his algebra class.

d) A child needs to be 42 inches or more to go on the roller coaster.

Solution

a) **Define**

Let $n =$ the unknown number.

Translate

$$5 + n \leq 2$$

b) **Define**

Let $d =$ the distance from San Diego to Los Angeles in miles.

Translate

$$d < 150$$

c) **Define**

Let $x =$ Diego's test grade.

Translate

$$x > 82$$

d) **Define**

Let $h =$ the height of child in inches.

Translate:

$$h \geq 42$$

Watch this video for help with the Examples above.

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CK-12 Foundation: Write Equations and Inequalities

Vocabulary

- To write an inequality, we use the following symbols:

$>$greater than

$\geq$greater than or equal to

$<$less than

$\leq$less than or equal to

$\neq$not equal to

Guided Practice

Define the variables and translate the following expressions into inequalities.

a) Jose took 3 train trips in a day, some of which cost \$2.75 and some of which cost \$3.95. His total cost was \$9.45.

b) The product of 3 and some number is more than the sum of 24 and that number.

Solution:

a) Let t be the number of train rides that cost \$2.75. Then $5-t$ is the number of train rides that cost \$3.95. Then we get:

$2.75t+3.95(5-t)=9.45.$

b) Let n be "some number." Then the product of 3 and n is $3n$. The sume of 24 and n is $24+n$. Together we get:

$3n>24+n$

Practice

For 1-10, define the variables and translate the following expressions into equations.

1. Peter's Lawn Mowing Service charges \$10 per job and \$0.20 per square yard. Peter earns \$25 for a job.
2. Renting the ice-skating rink for a birthday party costs \$200 plus \$4 per person. The rental costs \$324 in total.
3. Renting a car costs \$55 per day plus \$0.45 per mile. The cost of the rental is \$100.
4. Nadia gave Peter 4 more blocks than he already had. He already had 7 blocks.
5. A bus can seat 65 passengers or fewer.
6. The sum of two consecutive integers is less than 54.
7. The product of a number and 3 is greater than 30.
8. An amount of money is invested at 5% annual interest. The interest earned at the end of the year is greater than or equal to \$250.
9. You buy hamburgers at a fast food restaurant. A hamburger costs \$0.49. You have at most \$3 to spend. Write an inequality for the number of hamburgers you can buy.

10. Mariel needs at least 7 extra credit points to improve her grade in English class. Additional book reports are worth 2 extra credit points each. Write an inequality for the number of book reports Mariel needs to do.

1.8 Checking Solutions to Equations

Here you'll learn how to check that a given number is a solution to an equation.

What if you were given an equation like $2x^2 - 8 = 0$ and told that one of its solutions was $x = -2$? How could you determine if that solution were correct? After completing this Concept, you'll be able to check the solutions to equations like this one.

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CK-12 Foundation: 0108S Check Solutions to Equations

Guidance

You will often need to check solutions to equations in order to check your work. In a math class, checking that you arrived at the correct solution is very good practice. We check the solution to an equation by replacing the variable in an equation with the value of the solution. A solution should result in a true statement when plugged into the equation.

Example A

Check that the given number is a solution to the equation: $y = -1$; $3y + 5 = -2y$

Solution

Replace the variable in each equation with the given value.

$$\begin{aligned} 3(-1) + 5 &= -2(-1) \\ -3 + 5 &= 2 \\ 2 &= 2 \end{aligned}$$

This is a true statement. This means that $y = -1$ is a solution to $3y + 5 = -2y$.

Example B

Check that the given number is a solution to the equation: $z = 3$; $z^2 + 2z = 8$

Solution:

$$\begin{aligned}3^2+2(3)&=8\\9+6&=8\\15&=8\end{aligned}$$

This is not a true statement. This means that $z=3$ is **not a solution** to $z^2+2z=8$.

Let's use what we have learned about defining variables, writing equations and writing inequalities to solve some real-world problems.

Example C

Tomatoes cost $0.50 each and avocados cost $2.00 each. Anne buys six more tomatoes than avocados. Her total bill is $8. How many tomatoes and how many avocados did Anne buy? Check your answer!

Solution

Define

Let $a=$ the number of avocados Anne buys.

Translate

Anne buys six more tomatoes than avocados. This means that $a+6=$ the number of tomatoes.

Tomatoes cost $0.50 each and avocados cost $2.00 each. Her total bill is $8. This means that .50 times the number of tomatoes plus 2 times the number of avocados equals 8.

$$\begin{aligned}0.5(a+6)+2a&=8\\0.5a+0.5\cdot 6+2a&=8\\2.5a+3&=8\\2.5a&=5\\a&=2\end{aligned}$$

Remember that $a=$ the number of avocados, so Anne buys two avocados. The number of tomatoes is $a+6=2+6=8$.

Answer

Anne bought 2 avocados and 8 tomatoes.

Check

If Anne bought two avocados and eight tomatoes, the total cost is: $(2\times 2)+(8\times 0.5)=4+4=8$. **The answer checks out.**

Watch this video for help with the Examples above.

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CK-12 Foundation: Check Solutions toEquations

Vocabulary

- A **solution an equation** should result in a true statement when plugged into the equation.

Guided Practice

Check that the given number is a solution to the equation: $x = -\frac{1}{2};\ 3x + 1 = x$

Solution:

$$\begin{aligned}3\left(-\frac{1}{2}\right) + 1 &= -\frac{1}{2}\\ \left(-\frac{3}{2}\right) + 1 &= -\frac{1}{2}\\ -\frac{1}{2} &= -\frac{1}{2}\end{aligned}$$

This is a true statement. This means that $x = -\frac{1}{2}$ is a solution to $3x + 1 = x$.

Practice

For 1-9, check whether the given number is a solution to the corresponding equation.

1. $a = -3;\ 4a + 3 = -9$
2. $x = \frac{4}{3};\ \frac{3}{4}x + \frac{1}{2} = \frac{3}{2}$
3. $y = 2;\ 2.5y - 10.0 = -5.0$
4. $z = -5;\ 2(5 - 2z) = 20 - 2(z - 1)$
5. $a = 10;\ 5a - 7 = 43$
6. $x = \frac{2}{3}; 3x + 5 = 7$
7. $y = -9;\ \frac{x}{3} \cdot 10 = -30$
8. $z = \frac{1}{2};\ 2z = 1.5 - z$
9. $z = 0.5;\ z(1 - 2z) = 6 + 6(4z - 3)$
10. The cost of a Ford Focus is 27% of the price of a Lexus GS 450h. If the price of the Ford is $15000, what is the price of the Lexus?

1.9 Checking Solutions to Inequalities

Here you'll learn to check that a given number is a solution to an inequality.

What if you were given an inequality like $-3x^3 < -81$and told that one of its solutions was $x > 3$? How could you determine if that solution were correct? After completing this Concept, you'll be able to check the solutions to inequations like this one.

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CK-12 Foundation: 0109S Check Solutions to Inequalities

Try This

For more practice solving inequalities, check out http://www.aaastudy.com/equ725x7.htm.

Guidance

To check the solution to an inequality, we replace the variable in the inequality with the value of the solution. A solution to an inequality produces a true statement when substituted into the inequality.

Example A

Check that the given number is a solution to the inequality: $a = 10$; $20a \leq 250$

Solution

Replace the variable in the inequality with the given value.

$$\begin{aligned}20(10) &\leq 250\\ 200 &\leq 250\end{aligned}$$

This statement is true. This means that $a = 10$ is a solution to the inequality $20a \leq 250$.

Note that $a = 10$ is not the only solution to this inequality. If we divide both sides of the inequality by 20, we can write it as $a \leq 12.5$. This means that any number less than or equal to 12.5 is also a solution to the inequality.

Example B

Check that the given number is a solution to the inequality: $b = -0.5; \frac{3-b}{b} > -4$

Solution:

$$\begin{aligned} \frac{3-(-0.5)}{(-0.5)} &> -4 \\ \frac{3+0.5}{-0.5} &> -4 \\ -\frac{3.5}{0.5} &> -4 \\ -7 &> -4 \end{aligned}$$

This statement is false. This means that $b = -0.5$ is not a solution to the inequality $\frac{3-b}{b} > -4$.

Example C

To organize a picnic Peter needs at least two times as many hamburgers as hot dogs. He has 24 hot dogs. What is the possible number of hamburgers Peter has?

Solution:

Define

Let $x =$ number of hamburgers

Translate

Peter needs at least two times as many hamburgers as hot dogs. He has 24 hot dogs.

This means that twice the number of hot dogs is less than or equal to the number of hamburgers.

$$2 \times 24 \leq x, \text{ or } 48 \leq x$$

Answer

Peter needs at least 48 hamburgers.

Check

48 hamburgers is twice the number of hot dogs. So more than 48 hamburgers is more than twice the number of hot dogs. **The answer checks out**.

Watch this video for help with the Examples above.

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Click image to the left for more content.

CK-12 Foundation: Check Solutions toInequalities

Vocabulary

- A **solution to an inequality** produces a true statement when substituted into the inequality.

Guided Practice

Check that the given number is a solution to the inequality: $x = \frac{3}{4}$; $4x+5 \leq 8$

Solution:

$$\begin{aligned} 4\left(\frac{3}{4}\right)+5 &\geq 8 \\ 3+5 &\geq 8 \\ 8 &\geq 8 \end{aligned}$$

This statement is true. It is true because this inequality includes an equals sign; since 8 is equal to itself, it is also "greater than or equal to" itself. This means that $x = \frac{3}{4}$ is a solution to the inequality $4x+5 \leq 8$.

Practice

For 1-4, check whether the given number is a solution to the corresponding inequality.

1. $x = 12$; $2(x+6) \leq 8x$
2. $z = -9$; $1.4z+5.2 > 0.4z$
3. $y = 40$; $-\frac{5}{2}y+\frac{1}{2} < -18$
4. $t = 0.4$; $80 \geq 10(3t+2)$
5. On your new job you can be paid in one of two ways. You can either be paid \$1000 per month plus 6% commission of total sales or be paid \$1200 per month plus 5% commission on sales over \$2000. For what amount of sales is the first option better than the second option? Assume there are always sales over \$2000.

For 6-14, suppose a phone company offers a choice of three text-messaging plans. Plan A gives you unlimited text messages for \$10 a month; Plan B gives you 60 text messages for \$5 a month and then charges you \$0.05 for each additional message; and Plan C has no monthly fee but charges you \$0.10 per message.

6. If m is the number of messages you send per month, write an expression for the monthly cost of each of the three plans.
7. For what values of m is Plan A cheaper than Plan B?
8. For what values of m is Plan A cheaper than Plan C?
9. For what values of m is Plan B cheaper than Plan C?
10. For what values of m is Plan A the cheapest of all? (Hint: for what values is A both cheaper than B and cheaper than C?)
11. For what values of m is Plan B the cheapest of all? (Careful—for what values is B cheaper than A?)
12. For what values of m is Plan C the cheapest of all?
13. If you send 30 messages per month, which plan is cheapest?
14. What is the cost of each of the three plans if you send 30 messages per month?

1.10 Domain and Range of a Function

Here you'll learn how to find the domain and range of a function and you'll make a table of values for a given function.

What if you were given a rule that relates two variables, like $f(x) = 5x^2 + 1$? How could you find the domain and range of the function defined by that rule? After completing this Concept, you'll be able to identify the domain and range of functions like this one.

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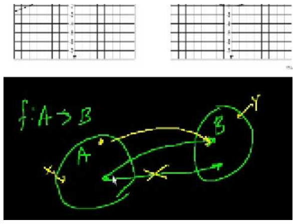
CK-12 Foundation: 0110S Introductionto Functions

For another look at the domain of a function, see the following video, where the narrator solves a sample problem from the California Standards Test about finding the domain of an unusual function.



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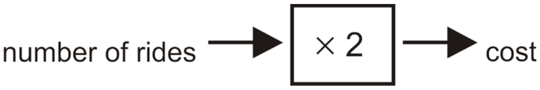
KhanAcademy: CAAlgebraI: Functions

Guidance

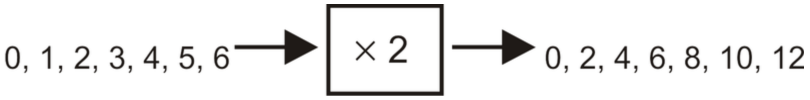
A **function** is a rule for relating two or more variables. For example, the price you pay for phone service may depend on the number of minutes you talk on the phone. We would say that the cost of phone service is a *function* of the number of minutes you talk. Consider the following situation.

Josh goes to an amusement park where he pays $2 per ride.

There is a relationship between the number of rides Josh goes on and the total amount he spends that day: To figure out the amount he spends, we multiply the number of rides by two. This rule is an example of a **function**. Functions usually—*but not always*—are rules based on mathematical operations. You can think of a function as a box or a machine that contains a mathematical operation.



Whatever number we feed into the function box is changed by the given operation, and a new number comes out the other side of the box. When we input different values for the number of rides Josh goes on, we get different values for the amount of money he spends.

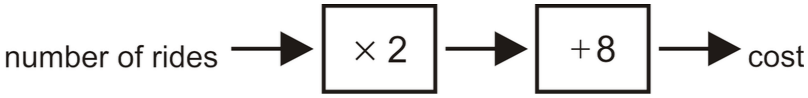


The input is called the **independent variable** because its value can be any number. The output is called the **dependent variable** because its value depends on the input value.

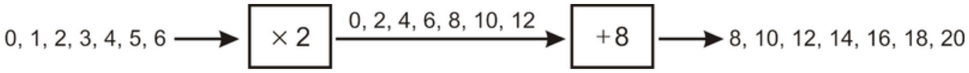
Functions usually contain more than one mathematical operation. Here is a situation that is slightly more complicated than the example above.

Jason goes to an amusement park where he pays $8 admission and $2 per ride.

The following function represents the total amount Jason pays. The rule for this function is "multiply the number of rides by 2 and add 8."



When we input different values for the number of rides, we arrive at different outputs (costs).



These flow diagrams are useful in visualizing what a function is. However, they are cumbersome to use in practice. In algebra, we use the following short-hand notation instead:

$$\overset{\overset{input}{\downarrow}}{\underbrace{f(x)}_{\substack{function\\box}}} = y \leftarrow output$$

First, we define the variables:

$x =$ the number of rides Jason goes on

$y =$ the total amount of money Jason spends at the amusement park.

So, x represents the input and y represents the output. The notation $f()$ represents the function or the mathematical operations we use on the input to get the output. In the last example, the cost is 2 times the number of rides plus 8. This can be written as a function:

$$f(x) = 2x + 8$$

In algebra, the notations y and $f(x)$ are typically used interchangeably. Technically, though, $f(x)$ represents the function itself and y represents the output of the function.

Identify the Domain and Range of a Function

In the last example, we saw that we can input the number of rides into the function to give us the total cost for going to the amusement park. The set of all values that we can use for the input is called the **domain** of the function, and the set of all values that the output could turn out to be is called the **range** of the function. In many situations the **domain** and **range** of a function are both simply the set of all real numbers, but this isn't always the case. Let's look at our amusement park example.

Example A

Find the domain and range of the function that describes the situation:

Jason goes to an amusement park where he pays $8 admission and $2 per ride.

Solution

Here is the function that describes this situation:

$$f(x) = 2x + 8 = y$$

In this function, x is the number of rides and y is the total cost. To find the domain of the function, we need to determine which numbers make sense to use as the input (x).

- The values have to be zero or positive, because Jason can't go on a negative number of rides.
- The values have to be integers because, for example, Jason could not go on 2.25 rides.
- Realistically, there must be a maximum number of rides that Jason can go on because the park closes, he runs out of money, etc. However, since we aren't given any information about what that maximum might be, we must consider that all non-negative integers are possible values regardless of how big they are.

Answer For this function, the domain is the set of all non-negative integers.

To find the range of the function we must determine what the values of y will be when we apply the function to the input values. The domain is the set of all non-negative integers: $\{0, 1, 2, 3, 4, 5, 6, ...\}$. Next we plug these values into the function for x. If we plug in 0, we get 8; if we plug in 1, we get 10; if we plug in 2, we get 12, and so on, counting by 2s each time. Possible values of y are therefore 8, 10, 12, 14, 16, 18, 20... or in other words all even integers greater than or equal to 8.

Answer The range of this function is the set of all even integers greater than or equal to 8.

Example B

Find the domain and range of the following functions.

a) A ball is dropped from a height and it bounces up to 75% of its original height.

b) $y = x^2$

Solution

a) Let's define the variables:

$x =$ original height

$y =$ bounce height

A function that describes the situation is $y = f(x) = 0.75x$. x can represent any real value greater than zero, since you can drop a ball from any height greater than zero. A little thought tells us that y can also represent any real value greater than zero.

Answer

The domain is the set of all real numbers greater than zero. The range is also the set of all real numbers greater than zero.

b) Since there is no word problem attached to this equation, we can assume that we can use any real number as a value of x. When we square a real number, we always get a non-negative answer, so y can be any non-negative real number.

Answer

The domain of this function is all real numbers. The range of this function is all non-negative real numbers.

In the functions we've looked at so far, x is called the **independent variable** because it can be any of the values from the domain, and y is called the **dependent variable** because its value depends on x. However, any letters or symbols can be used to represent the dependent and independent variables. Here are three different examples:

$$\begin{aligned} y &= f(x) = 3x \\ R &= f(w) = 3w \\ v &= f(t) = 3t \end{aligned}$$

These expressions all represent the same function: a function where the dependent variable is three times the independent variable. Only the symbols are different. In practice, we usually pick symbols for the dependent and independent variables based on what they represent in the real world—like t for time, d for distance, v for velocity, and so on. But when the variables don't represent anything in the real world—or even sometimes when they do—we traditionally use y for the dependent variable and x for the independent variable.

Make a Table For a Function

A table is a very useful way of arranging the data represented by a function. We can match the input and output values and arrange them as a table. For example, the values from Example 1 above can be arranged in a table as follows:

x	0	1	2	3	4	5	6
y	8	10	12	14	16	18	20

A table lets us organize our data in a compact manner. It also provides an easy reference for looking up data, and it gives us a set of coordinate points that we can plot to create a graph of the function.

Example C

Make a table of values for the function $f(x) = \frac{1}{x}$. Use the following numbers for input values: -1, -0.5, -0.2, -0.1, -0.01, 0.01, 0.1, 0.2, 0.5, 1.

Solution

Make a table of values by filling the first row with the input values and the next row with the output values calculated using the given function.

x	-1	-0.5	-0.2	-0.1	-0.01	0.01	0.1	0.2	0.5	1
$f(x) = \frac{1}{x}$	$\frac{1}{-1}$	$\frac{1}{-0.5}$	$\frac{1}{-0.2}$	$\frac{1}{-0.1}$	$\frac{1}{-0.01}$	$\frac{1}{0.01}$	$\frac{1}{0.1}$	$\frac{1}{0.2}$	$\frac{1}{0.5}$	$\frac{1}{1}$
y	-1	-2	-5	-10	-100	100	10	5	2	1

When you're given a function, you won't usually be told what input values to use; you'll need to decide for yourself what values to pick based on what kind of function you're dealing with. We will discuss how to pick input values throughout these lessons.

Watch this video for help with the Examples above.

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CK-12 Foundation: Introduction to Functions

Vocabulary

A **function** is a rule for relating two or more variables, one of which is the input variable and the other is the output variable. The input is called the **independent variable** because its value can be any number. The output is called the **dependent variable** because its value depends on the input value. The set of all values that we can use for the input is called the **domain** of the function, and the set of all values that the output could turn out to be is called the **range** of the function.

Guided Practice

Identify the domain and then make a table of values for the function $f(x) = \frac{1}{\sqrt{x}}$. Use the following numbers for input values: 0.01, 0.16, 0.25, 1, 4.

Solution

Since you cannot compute the square root of negative numbers, these cannot be in the domain. Since we cannot have 0 in the denominator, 0 is also not in the domain. This means that the domain is all real numbers greater than zero.

Make a table of values by filling the first row with the input values and the next row with the output values calculated using the given function.

x	0.01	0.16	0.25	1	4
$f(x) = \frac{1}{x}$	$\frac{1}{\sqrt{0.01}}$	$\frac{1}{\sqrt{0.16}}$	$\frac{1}{\sqrt{0.25}}$	$\frac{1}{\sqrt{1}}$	$\frac{1}{\sqrt{4}}$
y	10	2.5	2	1	0.5

Practice

For 1-6, identify the domain and range of the following functions.

1. Dustin charges $10 per hour for mowing lawns.
2. Maria charges $25 per hour for tutoring math, with a minimum charge of $15.
3. $f(x) = 15x - 12$
4. $f(x) = 2x^2 + 5$
5. $f(x) = \frac{1}{x}$
6. $f(x) = \sqrt[3]{x}$

7. What is the range of the function $y = x^2 - 5$ when the domain is -2, -1, 0, 1, 2?
8. What is the range of the function $y = 2x - \frac{3}{4}$ when the domain is -2.5, -1.5, 5?
9. What is the domain of the function $y = 3x$ when the range is 9, 12, 15?
10. What is the range of the function $y = 3x$ when the domain is 9, 12, 15?
11. Angie makes \$6.50 per hour working as a cashier at the grocery store. Make a table that shows how much she earns if she works 5, 10, 15, 20, 25, or 30 hours.
12. The area of a triangle is given by the formula $A = \frac{1}{2}bh$. If the base of the triangle measures 8 centimeters, make a table that shows the area of the triangle for heights 1, 2, 3, 4, 5, and 6 centimeters.
13. Make a table of values for the function $f(x) = \sqrt{2}x + 3$ for input values -1, 0, 1, 2, 3, 4, 5.

1.11 Functions that Describe Situations

Here you'll learn how to write a function rule for a table of values and to represent real-world situations.

What if you were given a table of x and y values? How could you write a rule to describe the relationship between the two variables? After completing this Concept, you'll be able to write a function rule for tables like this.

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CK-12 Foundation: 0111S Write a FunctionRule

Guidance

In many situations, we collect data by conducting a survey or an experiment, and then organize the data in a table of values. Most often, we want to find the function rule or formula that fits the set of values in the table, so we can use the rule to predict what could happen for values that are not in the table.

Example A

Write a function rule for the following table:

Number of CDs	2	4	6	8	10
Cost in $	24	48	72	96	120

Solution

You pay \$24 for 2 CDs, \$48 for 4 CDs, \$120 for 10 CDs. That means that each CD costs \$12.

We can write a function rule:

Cost $=$ \$12 $\times$ (number of CDs) or $f(x) = 12x$

Example B

Write a function rule for the following table:

x	-3	-2	-1	0	1	2	3
y	3	2	1	0	1	2	3

Solution

You can see that a negative number turns into the same number, only positive, while a non-negative number stays the same. This means that the function being used here is the absolute value function: $f(x) = |x|$.

Coming up with a function based on a set of values really is as tricky as it looks. There's no rule that will tell you the function every time, so you just have to think of all the types of functions you know and guess which one might be a good fit, and then check if your guess is right. In this book, though, we'll stick to writing functions for linear relationships, which are the simplest type of function.

Represent a Real-World Situation with a Function

Let's look at a few real-world situations that can be represented by a function.

Example C

Maya has an internet service that currently has a monthly access fee of $11.95 and a connection fee of $0.50 per hour. Represent her monthly cost as a function of connection time.

Solution

Define

Let $x =$ the number of hours Maya spends on the internet in one month

Let $y =$ Maya's monthly cost

Translate

The cost has two parts: the one-time fee of \$11.95 and the per-hour charge of \$0.50. So the total cost is the flat fee + the charge per hour × the number of hours.

Answer

The function is $y = f(x) = 11.95 + 0.50x$.

Example D

Alfredo wants a deck build around his pool. The dimensions of the pool are $12\ feet \times 24\ feet$ *and the decking costs \$3 per square foot. Write the cost of the deck as a function of the width of the deck.*

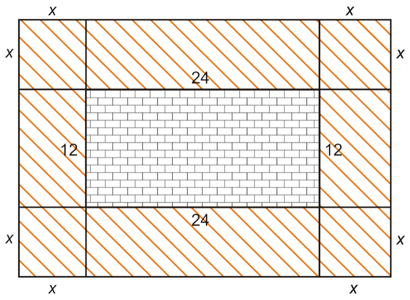
Solution

Define

Let $x =$ width of the deck

Let $y =$ cost of the deck

Make a sketch and label it



Translate

You can look at the decking as being formed by several rectangles and squares. We can find the areas of all the separate pieces and add them together:

$$\begin{aligned}\text{Area} &= 12x+12x+24x+24x+x^2+x^2+x^2+x^2\\ &= 72x+4x^2\end{aligned}$$

To find the total cost, we then multiply the area by the cost per square foot ($3).

Answer

$$f(x)=3(72x+4x^2)=216x+12x^2$$

Watch this video for help with the Examples above.

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CK-12 Foundation: Write a Function Rule

Vocabulary

- Most often, we want to find the **function rule** or formula that fits the set of values in the table, so we can use the rule to **predict** what could happen for values that are not in the table.

Guided Practice

A cell phone company sells two million phones in their first year of business. The number of phones they sell doubles each year. Write a function that gives the number of phones that are sold per year as a function of how old the company is.

Solution

Define

Let $x =$ age of company in years

Let $y =$ number of phones that are sold per year

Make a table

Age (years)	1	2	3	4	5	6	7
Millions of phones	2	4	8	16	32	64	128

Write a function rule

The number of phones sold per year doubles every year, so the first year the company sells 2 million phones, the next year it sells 2×2 million, the next year it sells $2 \times 2 \times 2$ million, and so on. You might remember that when we multiply a number by itself several times we can use exponential notation: $2 = 2^1$, $2 \times 2 = 2^2$, $2 \times 2 \times 2 = 2^3$, and so on. In this problem, the exponent just happens to match the company's age in years, which makes our function easy to describe.

Answer

$$y = f(x) = 2^x$$

Practice

1. Write a function rule for the following table:

x	3	4	5	6
y	9	16	15	36

2. Write a function rule for the following table:

Hours	0	1	2	3
Cost	15	20	25	30

3. Write a function rule for the following table:

x	0	1	2	3
y	24	12	6	3

4. Write a function that represents the number of cuts you need to cut a ribbon into x pieces.
5. Write a function that represents the number of cuts you need to divide a pizza into x slices.

For 6-8, suppose Solomon charges a $40 flat rate plus $25 per hour to repair a leaky pipe.

6. Write a function that represents the total fee charged as a function of hours worked.
7. How much does Solomon earn for a 3-hour job?
8. How much does he earn for three separate 1-hour jobs?

For 9-12, suppose Rochelle has invested $2500 in a jewelry making kit. She makes bracelets that she can sell for $12.50 each.

9. Write a function that shows how much money Rochelle makes from selling b bracelets.
10. Write a function that shows how much money Rochelle has after selling b bracelets, minus her investment in the kit.
11. How many bracelets does Rochelle need to make before she breaks even?
12. If she buys a $50 display case for her bracelets, how many bracelets does she now need to sell to break even?

1.12 Functions on a Cartesian Plane

Here you'll learn how to graph coordinate points on the Cartesian plane and a function from a table of values.

What if you were given a table of x and y values that represented a function? How could you use those values to graph the function? After completing this Concept, you'll be able to graph functions like this in the coordinate plane.

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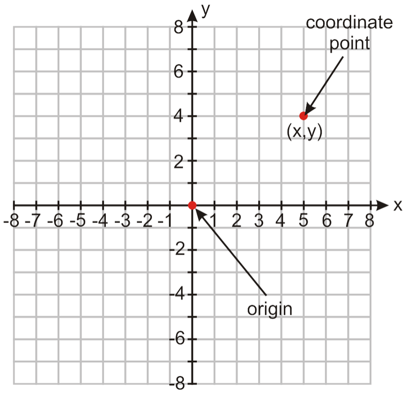
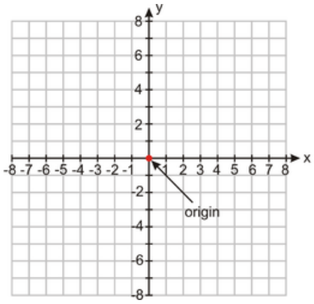
CK-12 Foundation: 0112S Graph a Functionfrom a Table

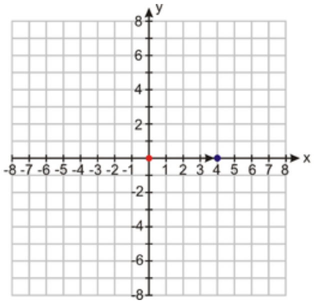
Guidance

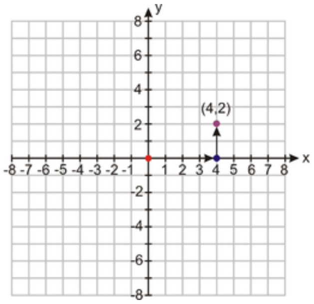


We represent functions graphically by plotting points on a **coordinate plane** (also sometimes called the **Cartesian plane**). The coordinate plane is a grid formed by a horizontal number line and a vertical number line that cross at a point called the **origin**. The origin has this name because it is the "starting" location; every other point on the grid is described in terms of how far it is from the origin.

The horizontal number line is called the $x-$**axis** and the vertical line is called the $y-$**axis**. We can represent each value of a function as a point on the plane by representing the $x-$value as a distance along the $x-$axis and the $y-$value as a distance along the $y-$axis. For example, if the $y-$value of a function is 2 when the $x-$value is 4, we can represent this pair of values with a point that is 4 units to the right of the origin (that is, 4 units along the $x-$axis) and 2 units up (2 units in the $y-$direction).







We write the location of this point as (4, 2).

Example A

Plot the following coordinate points on the Cartesian plane.

a) (5, 3)

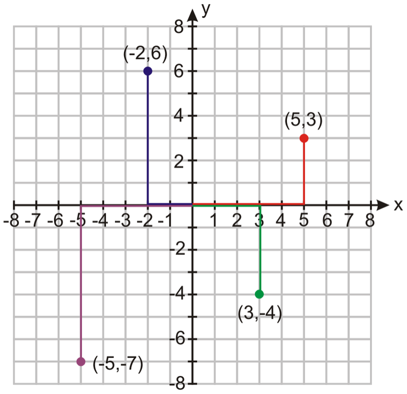
b) (-2, 6)

c) (3, -4)

d) (-5, -7)

Solution

Here are all the coordinate points on the same plot.



Notice that we move to the right for a positive $x-$value and to the left for a negative one, just as we would on a single number line. Similarly, we move up for a positive $y-$value and down for a negative one.

The $x-$ and $y-$axes divide the coordinate plane into four **quadrants**. The quadrants are numbered counter-clockwise starting from the upper right, so the plotted point for (a) is in the **first** quadrant, (b) is in the **second** quadrant, (c) is in the **fourth** quadrant, and (d) is in the **third** quadrant.

Graph a Function From a Table

If we know a rule or have a table of values that describes a function, we can draw a graph of the function. A table of values gives us coordinate points that we can plot on the Cartesian plane.

Example B

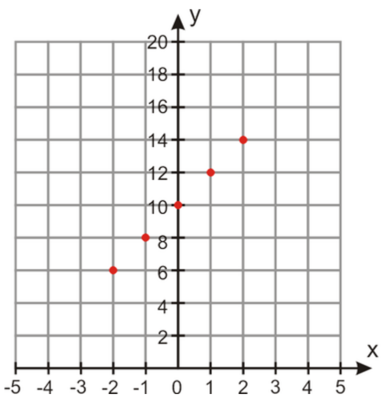
Graph the function that has the following table of values.

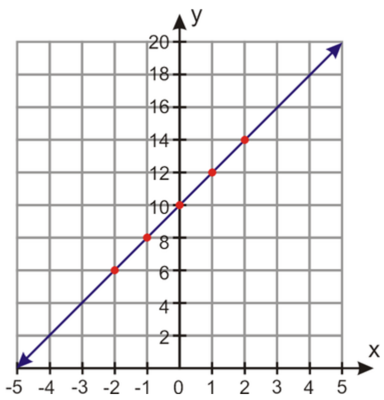
x	-2	-1	0	1	2
y	6	8	10	12	14

Solution

The table gives us five sets of coordinate points: (-2, 6), (-1, 8), (0, 10), (1, 12), and (2, 14).

To graph the function, we plot all the coordinate points. Since we are not told the domain of the function or given a real-world context, we can just assume that the domain is the set of all real numbers. To show that the function holds for all values in the domain, we connect the points with a smooth line (which, we understand, continues infinitely in both directions).





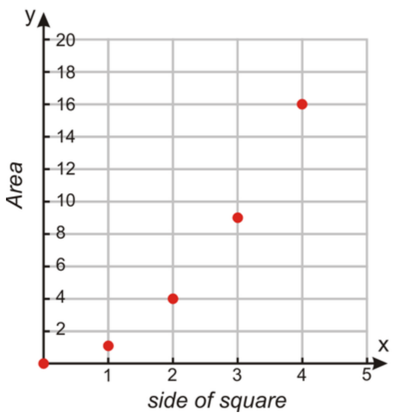
Example C

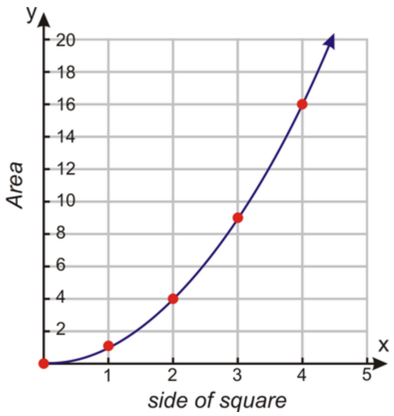
Graph the function that has the following table of values.

Side of square	0	1	2	3	4
Area of square	0	1	4	9	16

The table gives us five sets of coordinate points: (0, 0), (1, 1), (2, 4), (3, 9), and (4, 16).

To graph the function, we plot all the coordinate points. Since we are not told the domain of the function, we can assume that the domain is the set of all non-negative real numbers. To show that the function holds for all values in the domain, we connect the points with a smooth curve. The curve does not make sense for negative values of the independent variable, so it stops at $x = 0$, but it continues infinitely in the positive direction.





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CK-12 Foundation: Graph a Function from a Table

Vocabulary

- We represent functions graphically by plotting points on a **coordinate plane** (also sometimes called the **Cartesian plane**). The coordinate plane is a grid formed by a horizontal number line and a vertical number line that cross at a point called the **origin**. The horizontal number line is called the $x-$**axis** and the vertical line is called the $y-$**axis**.

- The $x-$ and $y-$axes divide the coordinate plane into four **quadrants**. The quadrants are numbered counter-clockwise starting from the upper right, so the plotted point for (a) is in the **first** quadrant, (b) is in the **second** quadrant, (c) is in the **fourth** quadrant, and (d) is in the **third** quadrant.

Guided Practice

Graph the function that has the following table of values.

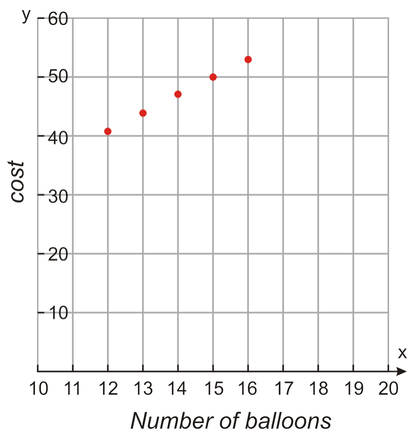
Number of balloons	12	13	14	15	16
Cost	41	44	47	50	53

This function represents the total cost of the balloons delivered to your house. Each balloon is \$3 and the store delivers if you buy a dozen balloons or more. The delivery charge is a \$5 flat fee.

Solution

The table gives us five sets of coordinate points: (12, 41), (13, 44), (14, 47), (15, 50), and (16, 53).

To graph the function, we plot all the coordinate points. Since the $x-$values represent the number of balloons for 12 balloons or more, the domain of this function is all integers greater than or equal to 12. In this problem, the points are not connected by a line or curve because it doesn’tmake sense to have non-integer values of balloons.



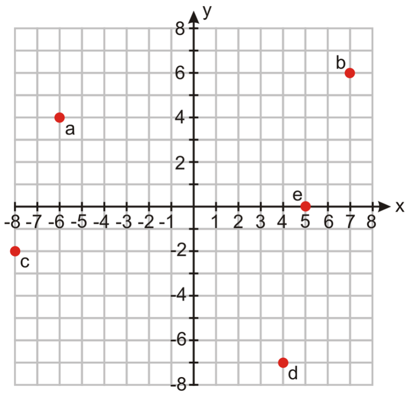
In order to draw a graph of a function given the function rule, we must first make a table of values to give us a set of points to plot. Choosing good values for the table is a skill you'l develop throughout this course. When you pick values, here are some of the things you should keep in mind.

- Pick only values from the domain of the function.
- If the domain is the set of real numbers or a subset of the real numbers, the graph will be a continuous curve.
- If the domain is the set of integers of a subset of the integers, the graph will be a set of points not connected by a curve.
- Picking integer values is best because it makes calculations easier, but sometimes we need to pick other values to capture all the details of the function.
- Often we start with one set of values. Then after drawing the graph, we realize that we need to pick different values and redraw the graph.

Practice

For 1-5, plot the coordinate points on the Cartesian plane.

1. (4, -4)
2. (2, 7)
3. (-3, -5)
4. (6, 3)
5. (-4, 3)
6. Give the coordinates for each point in this Cartesian plane.



For 7-10, graph the function that has the following table of values.

7.

x	-10	-5	0	5	10
y	-3	-0.5	2	4.5	7

8.

Side of cube (in.)	0	1	2	3
Volume (in^3)	0	1	8	27

9.

Time (hours)	-2	-1	0	1	2
Distance from town center (miles)	50	25	0	25	50

10.

x	-2	-1	0	2	1
y	-400	100	200	300	800

1.13 Graphs of Functions based on Rules

Here you'll learn how to graph a function from a given rule.

What if you were given a function rule like $f(x) = \sqrt{2x^2+1}$. How could you graph that function? After completing this Concept, you'll be able to create a table of values to graph functions like this one in the coordinate plane.

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CK-12 Foundation: 0113S Graph a Functionfrom a Rule

Try This

Once you've had some practice graphing functions by hand, you may want to use a graphing calculator to make graphing easier. If you don't have one, you can also use the applet at http://rechneronline.de/function-graphs/. Just type a function in the blank and press Enter. You can use the options under Display Properties to zoom in or pan around to different parts of the graph.

Guidance

Of course, we can always make a graph from a function rule, by substituting values in for the variable and getting the corresponding output value.

Example A

Graph the following function: $f(x) = |x-2|$

Solution

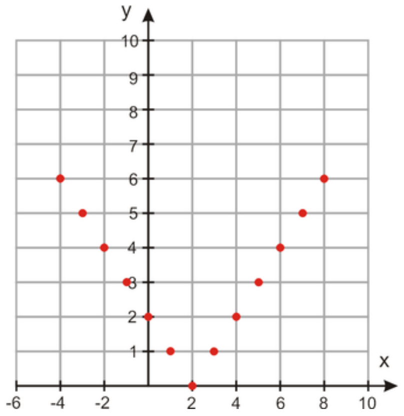
Make a table of values. Pick a variety of negative and positive values for x. Use the function rule to find the value of y for each value of x. Then, graph each of the coordinate points.

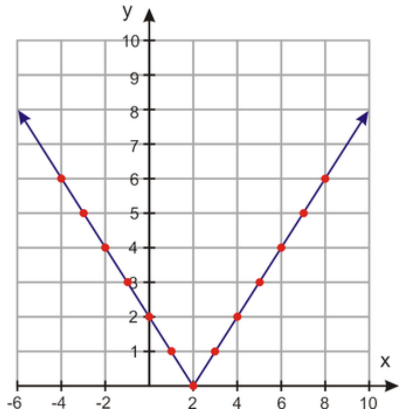
TABLE 1.8:

x	$y = f(x) = \mid x-2 \mid$
-4	$\mid -4-2 \mid = \mid -6 \mid = 6$
-3	$\mid -3-2 \mid = \mid -5 \mid = 5$
-2	$\mid -2-2 \mid = \mid -4 \mid = 4$
-1	$\mid -1-2 \mid = \mid -3 \mid = 3$
0	$\mid 0-2 \mid = \mid -2 \mid = 2$

TABLE 1.8: (continued)

x	$y = f(x) = \mid x-2 \mid$
1	$\mid 1-2 \mid = \mid -1 \mid = 1$
2	$\mid 2-2 \mid = \mid 0 \mid = 0$
3	$\mid 3-2 \mid = \mid 1 \mid = 1$
4	$\mid 4-2 \mid = \mid 2 \mid = 2$
5	$\mid 5-2 \mid = \mid 3 \mid = 3$
6	$\mid 6-2 \mid = \mid 4 \mid = 4$
7	$\mid 7-2 \mid = \mid 5 \mid = 5$
8	$\mid 8-2 \mid = \mid 6 \mid = 6$





It is wise to work with a lot of values when you begin graphing. As you learn about different types of functions, you will start to only need a few points in the table of values to create an accurate graph.

Example B

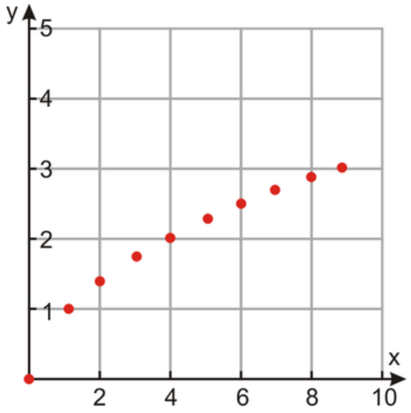
Graph the following function: $f(x) = \sqrt{x}$

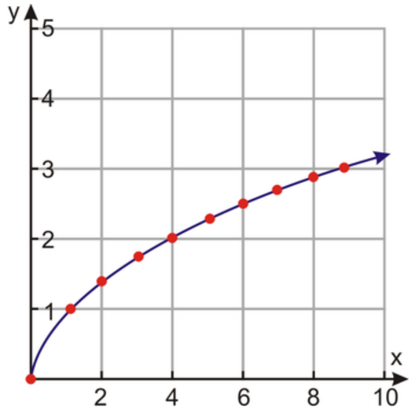
Solution

Make a table of values. We know x can't be negative because we can't take the square root of a negative number. The domain is all positive real numbers, so we pick a variety of positive integer values for x. Use the function rule to find the value of y for each value of x.

TABLE 1.9:

x	$y = f(x) = \sqrt{x}$
0	$\sqrt{0} = 0$
1	$\sqrt{1} = 1$
2	$\sqrt{2} \approx 1.41$
3	$\sqrt{3} \approx 1.73$
4	$\sqrt{4} = 2$
5	$\sqrt{5} \approx 2.24$
6	$\sqrt{6} \approx 2.45$
7	$\sqrt{7} \approx 2.65$
8	$\sqrt{8} \approx 2.83$
9	$\sqrt{9} = 3$





Note that the range is all positive real numbers.

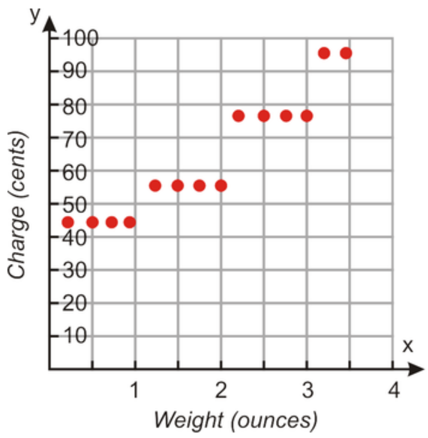
Example C

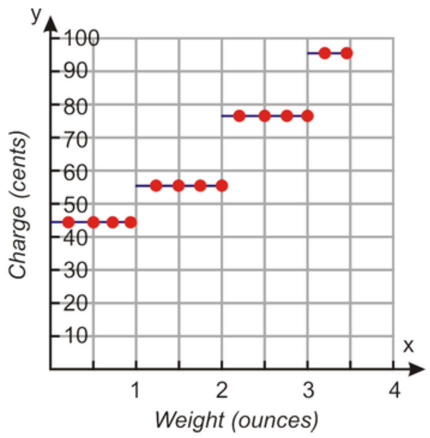
The post office charges 41 cents to send a letter that is one ounce or less and an extra 17 cents for each additional ounce or fraction of an ounce. This rate applies to letters up to 3.5 ounces.

Solution

Make a table of values. We can't use negative numbers for x because it doesn't make sense to have negative weight. We pick a variety of positive values for x, making sure to include some decimal values because prices can be decimals too. Then we use the function rule to find the value of y for each value of x.

x	0	0.2	0.5	0.8	1	1.2	1.5	1.8	2	2.2	2.5	2.8	3	3.2	3.5
y	0	41	41	41	41	58	58	58	58	75	75	75	75	92	92





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CK-12 Foundation: Graph a Function from a Rule

Vocabulary

We represent functions graphically by plotting points on a **coordinate plane** (also sometimes called the **Cartesian plane**). The coordinate plane is a grid formed by a horizontal number line and a vertical number line that cross at a point called the **origin**. The horizontal number line is called the $x-$**axis** and the vertical line is called the $y-$**axis**.

The $x-$ and $y-$axes divide the coordinate plane into four **quadrants**. The quadrants are numbered counter-clockwise starting from the upper right, so the plotted point for (a) is in the **first** quadrant, (b) is in the **second** quadrant, (c) is in the **fourth** quadrant, and (d) is in the **third** quadrant.

Guided Practice

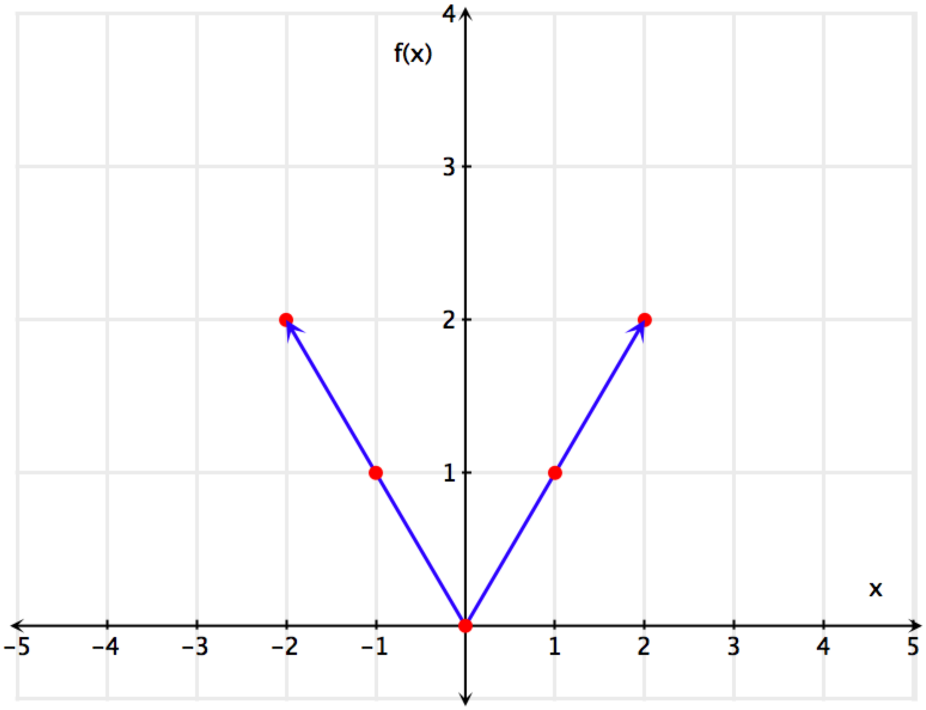
Graph the following function: $f(x) = \sqrt{x^2}$

Solution

Make a table of values. Even though x can't be negative inside the square root, because we are squaring x first, the domain is all real numbers. So we integer values for x which are on either side of zero. Use the function rule to find the value of y for each value of x.

TABLE 1.10:

x	$y = f(x) = \sqrt{x^2}$
-2	$\sqrt{(-2)^2} = 2$
-1	$\sqrt{(-1)^2} = 1$
0	$\sqrt{0^2} = 0$
1	$\sqrt{1^2} = 1$
2	$\sqrt{2^2} = 2$



Note that the range is all positive real numbers, and that this looks like an absolute value function.

Practice

Graph the following functions.

1. Vanson spends \$20 a month on his cat.
2. Brandon is a member of a movie club. He pays a \$50 annual membership and \$8 per movie.
3. $f(x) = (x-2)^2$
4. $f(x) = 3.2^x$
5. $f(t) = 27t - t^2$
6. $f(w) = \frac{w}{4} + 5$
7. $f(x) = t + 2t^2 + 3t^3$
8. $f(x) = (x-1)(x+3)$
9. $f(x) = \frac{x}{3} + \frac{x^2}{5}$
10. $f(x) = \sqrt{2x}$

1.14 Function Rules based on Graphs

Here you'll learn how to write the function rule that describes a function shown in a graph. You'll also interpret the meaning of a graph's coordinate points.

What if you were given the coordinate graph of a function and you wanted to find the function rule shown by the graph? How could you write that rule? After completing this Concept, you'll be able to create a table of values from a graph to help you write function rules like this one.

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CK-12 Foundation: 0114S Write a FunctionRule from aGraph

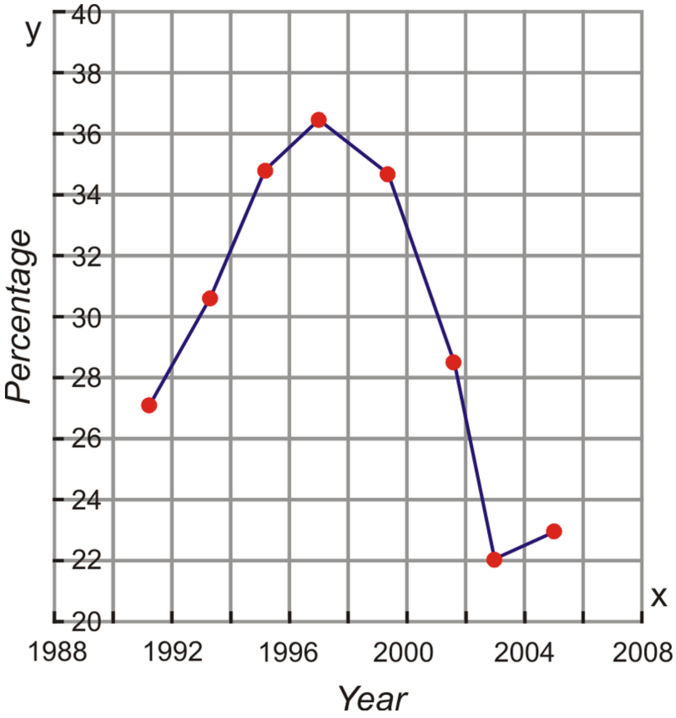
Guidance

In the last two Concepts, you learned how to graph a function from a table and from a function rule. Now, you will learn how to to find coordinate points on a graph and to interpret the meaning. Recall that each point on the graph has an x-value and y-value. When given an x-value, you will be asked to find its y-value.

Example A

The students at a local high school took The Youth Risk Behavior Survey. The graph below shows the percentage of high school students who reported that they were current smokers. (A current smoker is anyone who has smoked one or more cigarettes in the past 30 days.) What percentage of high-school students were current smokers in the following years?

1. 1991
2. 2004



Solution:

1. First, find the value 1991 on the x-axis. This appears to be the first red point on the left side of the graph. By looking at the y-axis, it looks like this point has a y-value of approximately 27. This means that in 1991, approximately 27% of high school students reported that they were current smokers.

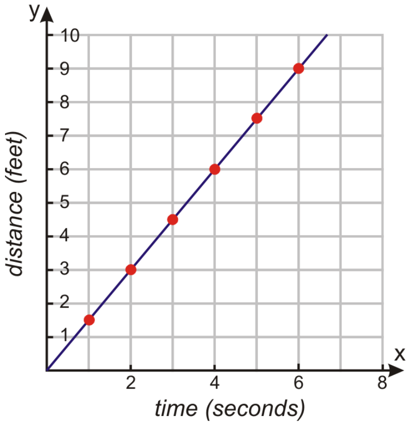
2. Find the value 2004 on the x-axis, which appears to be between the two red points on the right. This has an approximate y-value of 22.5. This means that in 2004, approximately 22.5% of high school students reported that they were current smokers.

Write a Function Rule from a Graph

Sometimes you'll need to find the equation or rule of a function by looking at the graph of the function. Points that are on the graph can give you values of dependent and independent variables that are related to each other by the function rule. However, you must make sure that the rule works for all the points on the curve. In this course you will learn to recognize different kinds of functions and discover the rules for all of them. For now we'll look at some simple examples and find patterns that will help us figure out how the dependent and independent variables are related.

Example B

The graph to the right shows the distance that an ant covers over time. Find the function rule that shows how distance and time are related to each other.



Solution

Let's make a table of values of several coordinate points to see if we can spot how they are related to each other.

Time	0	1	2	3	4	5	6
Distance	0	1.5	3	4.5	6	7.5	9

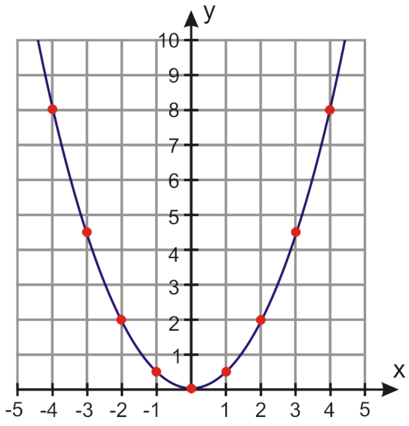
We can see that for every second the distance increases by 1.5 feet. We can write the function rule as

$$\text{Distance} = 1.5 \times \text{time}$$

The equation of the function is $f(x) = 1.5x$.

Example C

Find the function rule that describes the function shown in the graph.



Solution

Again, we can make a table of values of several coordinate points to identify how they are related to each other.

x	-4	-3	-2	-1	0	1	2	3	4
y	8	4.5	2	.5	0	.5	2	4.5	8

Notice that the values of y are half of perfect squares: 8 is half of 16 (which is 4 squared), 4.5 is half of 9 (which is 3 squared), and so on. So the equation of the function is $f(x) = \frac{1}{2}x^2$.

Watch this video for help with the Examples above.

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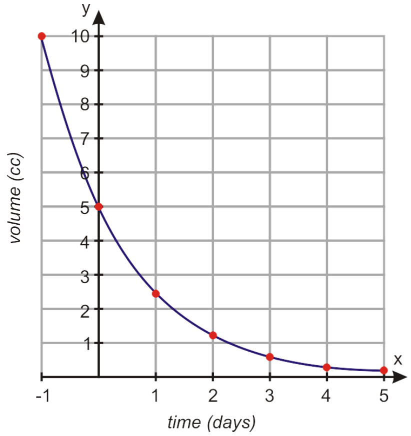
CK-12 Foundation: Write a Function Rule froma Graph

Vocabulary

Most often, we want to find the **function rule** or formula that fits the set of values in the table, so we can use the rule to **predict** what could happen for values that are not in the graph.

Guided Practice

Find the function rule that shows the volume of a balloon at different times, based on the following graph:



(Notice that the graph shows negative time. The negative time can represent what happened on days before you started measuring the volume.)

Solution

Once again, we make a table to spot the pattern:

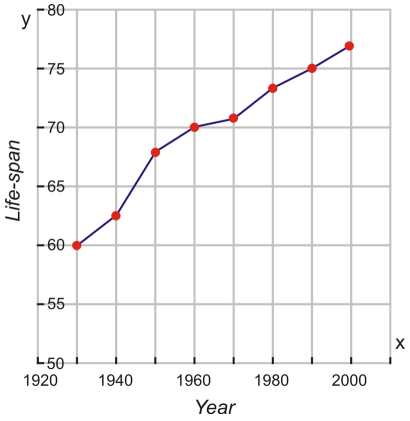
Time	-1	0	1	2	3	4	5
Volume	10	5	2.5	1.2	0.6	0.3	0.15

We can see that every day, the volume of the balloon is half what it was the previous day. On day 0, the volume is 5; on day 1, the volume is $5 \times \frac{1}{2}$; on day 2, it is $5 \times \frac{1}{2} \times \frac{1}{2}$, and in general, on day x it is $5 \times \left(\frac{1}{2}\right)^x$. The equation of the function is $f(x) = 5 \times \left(\frac{1}{2}\right)^x$.

Practice

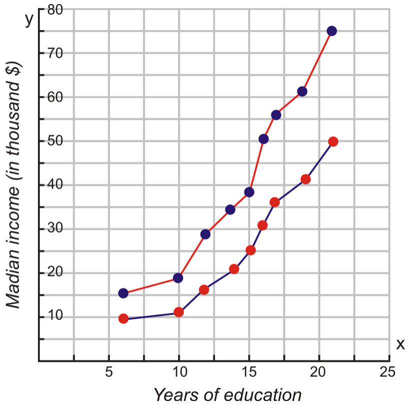
For 1-4, the graph below shows the average life-span of people based on the year in which they were born. This information comes from the National Vital Statistics Report from the Center for Disease Control.

1. What is the average life-span of a person born in 1940?
2. What is the average life-span of a person born in 1955?
3. What is the average life-span of a person born in 1980?
4. What is the average life-span of a person born in 1995?



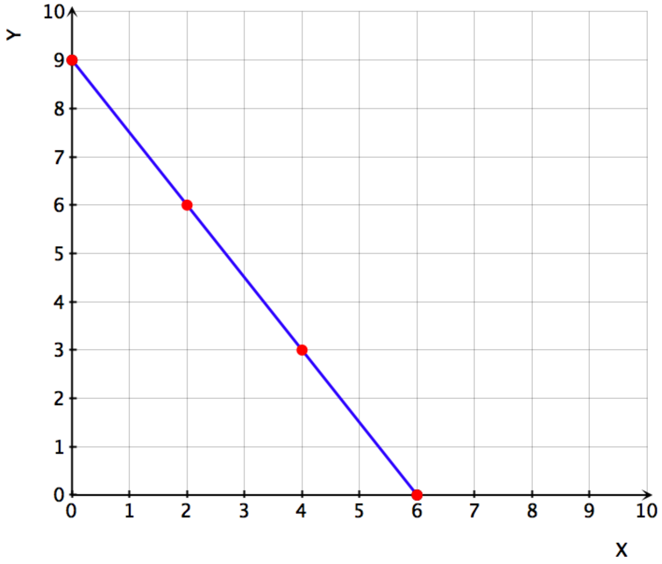
For 5-8, the graph below shows the median income of an individual based on his/her number of years of education. The top curve shows the median income for males and the bottom curve shows the median income for females. (Source: US Census, 2003.)

5. What is the median income of a male that has 10 years of education?
6. What is the median income of a male that has 17 years of education
7. What is the median income of a female that has 10 years of education?
8. What is the median income of a female that has 17 years of education?



For 9-11, write the function rule for each graph.

9.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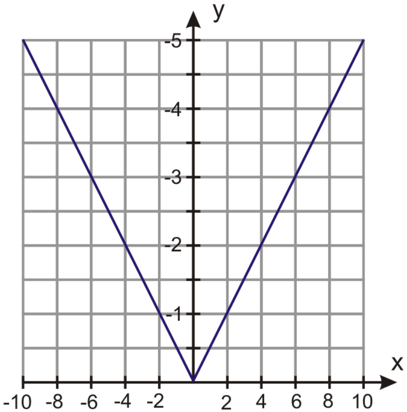
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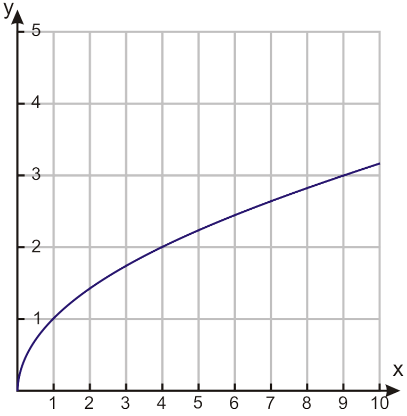
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1.15 Algebraic Functions

Here you'll learn how to determine whether a relation is a function given its domain and range or its graph.

What if you were given a set of x and y values? How could you determine whether the relation between those values represented a function? After completing this Concept you'll be able to analyze the domain and range of a relation to determine if it represents a function.

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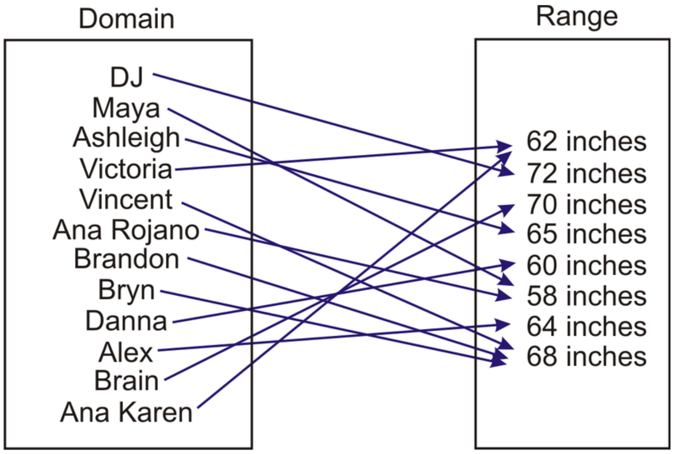
CK-12 Foundation: 0115S Relations and Functions

Guidance

A function is a special kind of **relation**. In a function, for each input there is exactly one output; in a relation, there can be more than one output for a given input.

Example A

Consider the relation that shows the heights of all students in a class. The domain is the set of people in the class and the range is the set of heights. This relation is a function because each person has exactly one height. If any person had more than one height, the relation would not be a function.



Notice that even though the same person can't have more than one height, it's okay for more than one person to have the same height. In a function, more than one input can have the same output, as long as more than one output never comes from the same input.

Example B

Determine if the relation is a function.

a) (1, 3), (-1, -2), (3, 5), (2, 5), (3, 4)

b) (-3, 20), (-5, 25), (-1, 5), (7, 12), (9, 2)

c)

x	2	1	0	1	2
y	12	10	8	6	4

Solution

The easiest way to figure out if a relation is a function is to look at all the $x-$values in the list or the table. If a value of x appears more than once, and it's paired up with different $y-$values, then the relation is not a function.

a) You can see that in this relation there are two different $y-$values paired with the $x-$value of 3. This means that this relation is **not** a function.

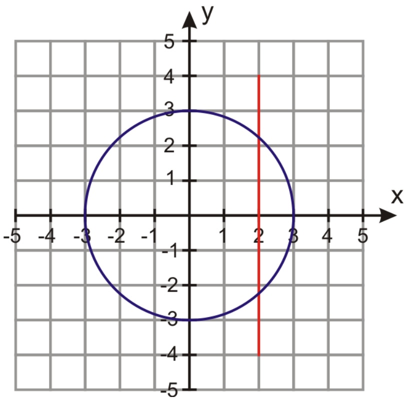
b) Each value of x has exactly one $y-$value. The relation is a function.

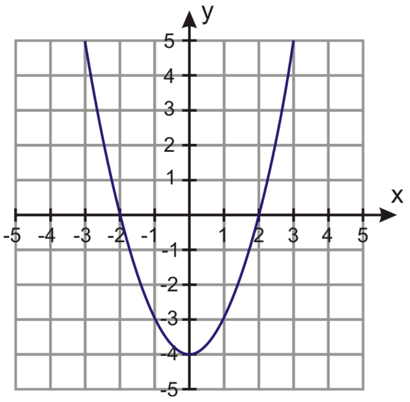
c) In this relation there are two different $y-$values paired with the $x-$value of 2 and two $y-$values paired with the $x-$value of 1. The relation is **not** a function.

When a relation is represented graphically, we can determine if it is a function by using the **vertical line test**. If you can draw a vertical line that crosses the graph in more than one place, then the relation is not a function.

Example C

For the following graphs, determine whether they are functions.





Solution:

1. Not a function. It fails the vertical line test.

2. A function. No vertical line will cross more than one point on the graph.

Watch this video for help with the Examples above.

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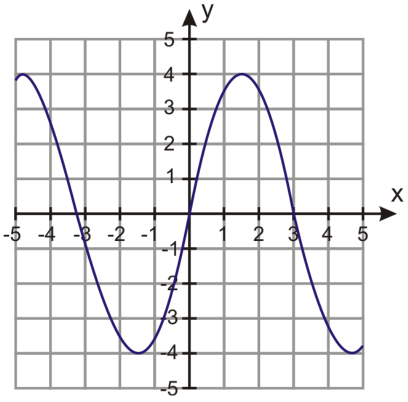
CK-12 Foundation: Relations and Functions

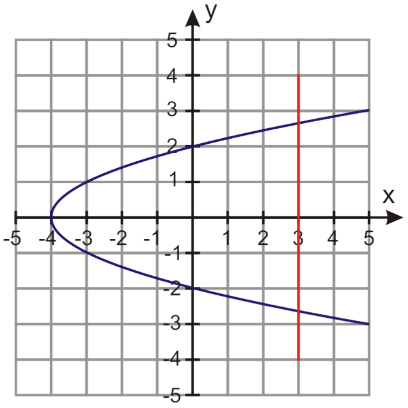
Vocabulary

- A **function** is a special kind of **relation**. In a function, for each input there is exactly one output; in a relation, there can be more than one output for a given input.

Guided Practice

For the following graphs, determine whether they are functions.





Solution:

1. A function. No vertical line will cross more than one point on the graph.

2. Not a function. It fails the vertical line test.

Practice

In 1-8, determine whether each relation is a function:

1. (1, 7), (2, 7), (3, 8), (4, 8), (5, 9)
2. (1, 1), (1, -1), (4, 2), (4, -2), (9, 3), (9, -3)
3.

x	-4	-3	-2	-1	0
y	16	9	4	1	0

4. (2, -6), (1, -3), (0, 0), (1, 3), (2, 6)
5. (-2, -8), (-1, -1), (0, 0), (1, 1), (2, 8)
6. (-5, 10), (-1, 5), (0, 10), (1, 5), (5, 10)
7.

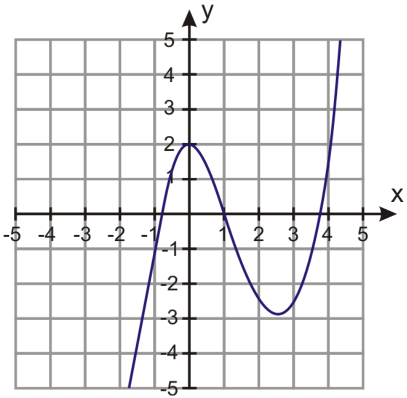
x	0	1	10	100	1000
y	2	-2	2	-2	2

8.

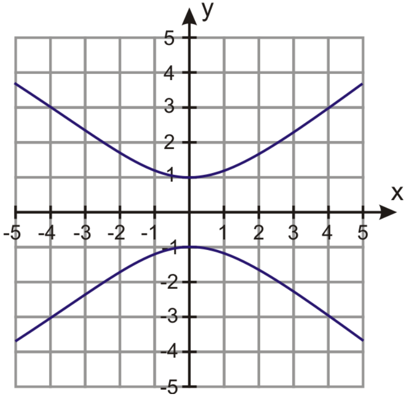
Age	20	25	25	30	35
Number of jobs by that age	3	4	7	4	2

In 9-10, use the vertical line test to determine whether each relation is a function.

9.



10.



1.16 Problem-Solving Models

Here you'll learn a variety of problem-solving strategies you can use to solve a problem situation. Once you've devised a plan, you'll then solve problems and check your results.

What if you were given a word problem like "A taxi cab charges \$3 plus \$0.75 per quarter mile. If you take a 3-mile cab ride, how much do you owe?" How could you devise a plan to solve this problem? After completing this Concept, you'll be able to compare alternative approaches to solving problems like this one.

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CK-12 Foundation: 0116S Problem Solving Plan

Check This Out

The problem-solving plan used here is based on the ideas of George Pólya, who describes his useful problem-solving strategies in more detail in the book *How to Solve It*. Some of the techniques in the book can also be found on Wikipedia, in the entry http://en.wikipedia.org/wiki/How_to_Solve_It.

Guidance

We always think of mathematics as the subject in school where we solve lots of problems. Problem solving is necessary in all aspects of life. Buying a house, renting a car, or figuring out which is the better sale are just a few examples of situations where people use problem-solving techniques. In this book, you will learn different strategies and approaches to solving problems. In this section, we will introduce a problem-solving plan that will be useful throughout this book.

Read and Understand a Given Problem Situation

The first step to solving a word problem is to **read and understand** the problem. Here are a few questions that you should be asking yourself:

- What am I trying to find out?
- What information have I been given?
- Have I ever solved a similar problem?

This is also a good time to define any variables. When you identify your **knowns** and **unknowns**, it is often useful to assign them a letter to make notation and calculations easier.

Make a Plan to Solve the Problem

The next step in the problem-solving plan is to **develop a strategy**. How can the information you know assist you in figuring out the unknowns?

Here are some common strategies that you will learn:

- Drawing a diagram.
- Making a table.
- Looking for a pattern.
- Using guess and check.
- Working backwards.
- Using a formula.
- Reading and making graphs.
- Writing equations.
- Using linear models.
- Using dimensional analysis.
- Using the right type of function for the situation.

In most problems, you will use a combination of strategies. For example, looking for patterns is a good strategy for most problems, and making a table and drawing a graph are often used together. The "writing an equation" strategy is the one you will work with the most in your study of algebra.

Solve the Problem and Check the Results

Once you develop a plan, you can implement it and **solve the problem**, carrying out all operations to arrive at the answer you are seeking.

The last step in solving any problem should always be to **check and interpret** the answer. Ask yourself:

- Does the answer make sense?
- If you plug the answer back into the problem, do all the numbers work out?
- Can you get the same answer through another method?

Compare Alternative Approaches to Solving the Problem

Sometimes one specific method is best for solving a problem. Most problems, however, can be solved by using several different strategies. When you are familiar with all of the problem-solving strategies, it is up to you to choose the methods that you are most comfortable with and that make sense to you. In this book, we will often use more than one method to solve a problem, so we can demonstrate the strengths and weakness of different strategies for solving different types of problems.

Whichever strategy you are using, you should always implement the problem-solving plan when you are solving word problems. Here is a summary of the problem-solving plan.

Step 1:

Understand the problem

Read the problem carefully. Once the problem is read, list all the components and data that are involved. This is where you will be assigning your variables.

Step 2:

Devise a plan - Translate

Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart or construct a table as a start to solve your problem solving plan.

Step 3:

Carry out the plan - Solve

This is where you solve the equation you developed in Step 2.

Step 4:

Look - Check and Interpret

Check to see if you used all your information. Then look to see if the answer makes sense.

The most difficult parts of problem-solving are most often the first two steps in our problem-solving plan. You need to read the problem and make sure you understand what you are being asked. Once you understand the problem, you can devise a strategy to solve it.

Let's apply the first two steps to the following problem.

Example A

Six friends are buying pizza together and they are planning to split the check equally. After the pizza was ordered, one of the friends had to leave suddenly, before the pizza arrived. Everyone left had to pay $1 extra as a result. How much was the total bill?

Solution

Understand

We want to find how much the pizza cost.

We know that five people had to pay an extra $1 each when one of the original six friends had to leave.

Strategy

We can start by making a list of possible amounts for the total bill.

We divide the amount by six and then by five. The total divided by five should equal $1 more than the total divided by six.

Look for any patterns in the numbers that might lead you to the correct answer.

In the rest of this section you will learn how to make a table or look for a pattern to figure out a solution for this type of problem. After you finish reading the rest of the section, you can finish solving this problem for homework.

Develop and Use the Strategy: Make a Table

The method "Make a Table" is helpful when solving problems involving numerical relationships. When data is organized in a table, it is easier to recognize patterns and relationships between numbers. Let's apply this strategy to the following example.

Example B

Josie takes up jogging. On the first week she jogs for 10 minutes per day, on the second week she jogs for 12 minutes per day. Each week, she wants to increase her jogging time by 2 minutes per day. If she jogs six days each week, what will be her total jogging time on the sixth week?

Solution

Understand

We know in the first week Josie jogs 10 minutes per day for six days.

We know in the second week Josie jogs 12 minutes per day for six days.

Each week, she increases her jogging time by 2 minutes per day and she jogs 6 days per week.

We want to find her total jogging time in week six.

Strategy

A good strategy is to list the data we have been given in a table and use the information we have been given to find new information.

We are told that Josie jogs 10 minutes per day for six days in the first week and 12 minutes per day for six days in the second week. We can enter this information in a table:

TABLE 1.11:

Week	Minutes per Day	Minutes per Week
1	10	60
2	12	72

You are told that each week Josie increases her jogging time by 2 minutes per day and jogs 6 times per week. We can use this information to continue filling in the table until we get to week six.

TABLE 1.12:

Week	Minutes per Day	Minutes per Week
1	10	60
2	12	72
3	14	84
4	16	96
5	18	108
6	20	120

Apply strategy/solve

To get the answer we read the entry for week six.

Answer: In week six Josie jogs a total of 120 minutes.

Check

Josie increases her jogging time by two minutes per day. She jogs six days per week. This means that she increases her jogging time by 12 minutes per week.

Josie starts at 60 minutes per week and she increases by 12 minutes per week for five weeks.

That means the total jogging time is $60+12 \times 5 = 120 \ minutes$.

The answer checks out.

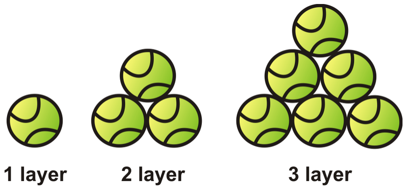
You can see that making a table helped us organize and clarify the information we were given, and helped guide us in the next steps of the problem. We solved this problem solely by making a table; in many situations, we would combine this strategy with others to get a solution.

Develop and Use the Strategy: Look for a Pattern

Looking for a pattern is another strategy that you can use to solve problems. The goal is to look for items or numbers that are repeated or a series of events that repeat. The following problem can be solved by finding a pattern.

Example C

You arrange tennis balls in triangular shapes as shown. How many balls will there be in a triangle that has 8 rows?



Solution

Understand

We know that we arrange tennis balls in triangles as shown.

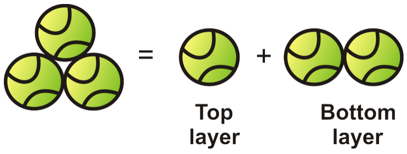
We want to know how many balls there are in a triangle that has 8 rows.

Strategy

A good strategy is to make a table and list how many balls are in triangles of different rows.

One row: It is simple to see that a triangle with one row has only one ball.

Two rows: For a triangle with two rows, we add the balls from the top row to the balls from the bottom row. It is useful to make a sketch of the separate rows in the triangle.



$$3 = 1 + 2$$

Three rows: We add the balls from the top triangle to the balls from the bottom row.

$$6 = 3 + 3$$

Now we can fill in the first three rows of a table.

TABLE 1.13:

Number of Rows	Number of Balls
1	1
2	3
3	6

We can see a pattern.

To create the next triangle, we add a new bottom row to the existing triangle.

The new bottom row has the same number of balls as there are rows. (For example, a triangle with 3 rows has 3 balls in the bottom row.)

To get the total number of balls for the new triangle, we add the number of balls in the old triangle to the number of balls in the new bottom row.

Apply strategy/solve:

We can complete the table by following the pattern we discovered.

Number of balls = number of balls in previous triangle + number of rows in the new triangle

TABLE 1.14:

Number of Rows	Number of Balls
1	1
2	3
3	6
4	$6+4=10$
5	$10+5=15$
6	$15+6=21$
7	$21+7=28$
8	$28+8=36$

Answer There are 36 balls in a triangle arrangement with 8 rows.

Check

Each row of the triangle has one more ball than the previous one. In a triangle with 8 rows,

row 1 has 1 ball, row 2 has 2 balls, row 3 has 3 balls, row 4 has 4 balls, row 5 has 5 balls, row 6 has 6 balls, row 7 has 7 balls, row 8 has 8 balls.

When we add these we get: $1+2+3+4+5+6+7+8=36\ balls$

The answer checks out.

Notice that in this example we made tables and drew diagrams to help us organize our information and find a pattern. Using several methods together is a very common practice and is very useful in solving word problems.

Watch this video for help with the Examples above.

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CK-12 Foundation: Problem Solving Plan

Vocabulary

Whichever strategy you are using, you should always implement the problem-solving plan when you are solving word problems. Here is a summary of the problem-solving plan.

- **Step 1:**

Understand the problem

Read the problem carefully. Once the problem is read, list all the components and data that are involved. This is where you will be assigning your variables.

- **Step 2:**

Devise a plan - Translate

Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart or construct a table as a start to solve your problem solving plan.

- **Step 3:**

Carry out the plan - Solve

This is where you solve the equation you developed in Step 2.

- **Step 4:**

Look - Check and Interpret

Check to see if you used all your information. Then look to see if the answer makes sense.

Guided Practice

Casey is twice as old as Marietta, who is two years younger than Jake. If Casey is 14, how old is Jake?

Solution:

Let C be the age of Casey, M be the age of Marietta and J be the age of Jake. We can write the following equations:

$C = 2M$

and

$M = J - 2.$

We can substitute the second equation into the first, getting:

$C = 2M = 2(J - 2) = 2J - 4.$

This gives us $C = 2J - 4$.

What are possible ages for Jake that would make Casey's age 14? We can make a table based on the equation:

J	2	3	4	5	6	7	8	9
C	0	2	4	6	8	10	12	14

Looking at the table, when Casey's age is 14, Jake's age is 9.

To check the answer, evaluate the equation for $J = 9$:

$C = 2J - 4$ Start with the equation.

$C = 2(9) - 4$ Substitute in J=9 .

$C = 18 - 4$ Simplify.

$C = 14$ It's correct!

Practice

1. A sweatshirt costs $35. Find the total cost if the sales tax is 7.75%.
2. This year you got a 5% raise. If your new salary is $45,000, what was your salary before the raise?
3. Mariana deposits $500 in a savings account that pays 3% simple interest per year. How much will be in her account after three years?
4. It costs $250 to carpet a room that is 14 ft by 18 ft. How much does it cost to carpet a room that is 9 ft by 10 ft?
5. A department store has a 15% discount for employees. Suppose an employee has a coupon worth $10 off any item and she wants to buy a $65 purse. What is the final cost of the purse if the employee discount is applied before the coupon is subtracted?
6. To host a dance at a hotel you must pay $250 plus $20 per guest. How much money would you have to pay for 25 guests?
7. Yusef's phone plan costs $10 a month plus $0.05 per minute. If his phone bill for last month was $25.80, how many minutes did he spend on the phone?
8. It costs $12 to get into the San Diego County Fair and $1.50 per ride.
   a. If Rena spent $24 in total, how many rides did she go on?
   b. How much would she have spent in total if she had gone on five more rides?
9. An ice cream shop sells a small cone for $2.95, a medium cone for $3.50, and a large cone for $4.25. Last Saturday, the shop sold 22 small cones, 26 medium cones and 15 large cones. How much money did the store earn?
10. In Lise's chemistry class, there are two midterm exams, each worth 30% of her total grade, and a final exam worth 40%. If Lise scores 90% on both midterms and 80% on the final exam, what is her overall score in the class?
11. The sum of the angles in a triangle is 180 degrees. If the second angle is twice the size of the first angle and the third angle is three times the size of the first angle, what are the measures of the angles in the triangle?
12. A television that normally costs $120 goes on sale for 20% off. What is the new price?
13. A cake recipe calls for $1\frac{3}{4}$ cup of flour. Jeremy wants to make four cakes. How many cups of flour will he need?
14. Kylie is mowing lawns to earn money for a new bike. After mowing four lawns, she still needs $40 more to pay for the bike. After mowing three more lawns, she has $5 more than she needs to pay for the bike.
   a. How much does she earn per lawn?
   b. What is the cost of the bike?
15. Jared goes trick-or-treating with his brother and sister. At the first house they stop at, they collect three pieces of candy each; at the next three houses, they collect two pieces of candy each. Then they split up and go down different blocks, where Jared collects 12 pieces of candy and his brother and sister collect 14 each.
   a. How many pieces of candy does Jared end up with?
   b. How many pieces of candy do all three of them together end up with?
16. Marco's daughter Elena has four boxes of toy blocks, with 50 blocks in each one. One day she dumps them all out on the floor, and some of them get lost. When Marco tries to put them away again, he ends up with 45 blocks in one box, 53 in another, 46 in a third, and 51 in the fourth. How many blocks are missing?

17. A certain hour-long TV show usually includes 16 minutes of commercials. If the season finale is two and a half hours long, how many minutes of commercials should it include to keep the same ratio of commercial time to show time?
18. Karen and Chase bet on a baseball game: if the home team wins, Karen owes Chase fifty cents for every run scored by both teams, and Chase owes Karen the same amount if the visiting team wins. The game runs nine innings, and the home team scores one run in every odd-numbered inning, while the visiting team scores two runs in the third inning and two in the sixth. Who owes whom how much?
19. Kelly, Chris, and Morgan are playing a card game. In this game, the first player to empty their hand scores points for all the cards left in the other players' hands as follows: aces are worth one point, face cards ten points, and all other cards are face value. When Kelly empties her hand, Morgan is holding two aces, a king, and a three; Chris is holding a five, a seven, and a queen. How many points does Kelly score?
20. A local club rents out a social hall to host an event. The hall rents for $350, and they hope to make back the rental price by charging $15 admission per person. How many people need to attend for the club to break even?
21. You plan to host a barbecue, and you expect 10 friends, 8 neighbors, and 7relatives to show up.
    a. If you expect each person (including yourself) to eat about two ounces of potato salad, how many half-pound containers of potato salad should you buy?
    b. If hot dogs come in ten-packs that cost $4.80 apiece and hot dog buns come in eight-packs that cost $2.80 apiece, how much will you need to spend to have hot dogs and buns for everyone?

1.17 Comparison of Problem-Solving Models

Here you'll learn how to devise and compare various problem-solving strategies. You'll then use the best approach to solve real-world problems.

What if you were given a real-world problem with two unknowns like "You have only dimes and nickels in your pocket that total \$1.25. You have a total of 14 coins in your pocket. How many nickels and dimes do you have?" How could you devise a problem-solving plan to solve it? After completing this Concept, you'll be able to make a table or look for patterns to help you solve problems like this one.

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CK-12 Foundation: 0117S Compare Strategies for Solving

Guidance

In this section, we will use the problem solving methods learned in the last Concept. We will also compare the methods of "Making a Table" and "Looking for a Pattern" by using each method in turn to solve a problem.

Example A

A coffee maker is on sale at 50% off the regular ticket price. On the "Sunday Super Sale" the same coffee maker is on sale at an ***additional*** *40% off. If the final price is \$21, what was the original price of the coffee maker?*

Solution

Step 1: Understand

We know: A coffee maker is discounted 50% and then 40%. The final price is \$21.

We want: The original price of the coffee maker.

Step 2: Strategy

Let's look at the given information and try to find the relationship between the information we know and the information we are trying to find.

50% off the original price means that the sale price is half of the original or $0.5 \times$ original price.

So, the first sale price $= 0.5 \times$ original price

A savings of 40% off the new price means you pay 60% of the new price, or $0.6 \times$ new price.

$0.6 \times (0.5 \times \text{original price}) = 0.3 \times \text{original price}$ is the price after the second discount.

We know that after two discounts, the final price is \$21.

So $0.3 \times$ original price $= \$21$.

Step 3: Solve

Since $0.3 \times$ original price $= \$21$, we can find the original price by dividing \$21 by 0.3.

Original price $= \$21 \div 0.3 = \$70$.

The original price of the coffee maker was \$70.

Step 4: Check

We found that the original price of the coffee maker is \$70.

To check that this is correct, let's apply the discounts.

50% of $\$70 = .5 \times \$70 = \$35$ savings. So the price after the first discount is original price $-$ savings or $\$70 - 35 = \$35$.

Then 40% of that is $.4 \times \$35 = \$14$. So after the second discount, the price is $\$35 - 14 = \$21$.

The answer checks out.

Example B

Andrew cashes a \$180 check and wants the money in \$10 and \$20 bills. The bank teller gives him 12 bills. How many of each kind of bill does he receive?

Solution

Method 1: Making a Table

Understand

Andrew gives the bank teller a \$180 check.

The bank teller gives Andrew 12 bills. These bills are a mix of \$10 bills and \$20 bills.

We want to know how many of each kind of bill Andrew receives.

Strategy

Let's start by making a table of the different ways Andrew can have twelve bills in tens and twenties.

Andrew could have twelve \$10 bills and zero \$20 bills, or eleven \$10 bills and one \$20 bill, and so on.

We can calculate the total amount of money for each case.

Apply strategy/solve

TABLE 1.15:

\$10 bills	**\$ 20 bills**	**Total amount**
12	0	$\$10(12)+\$20(0) = \$120$
11	1	$\$10(11)+\$20(1) = \$130$
10	2	$\$10(10)+\$20(2) = \$140$
9	3	$\$10(9)+\$20(3) = \$150$
8	4	$\$10(8)+\$20(4) = \$160$
7	5	$\$10(7)+\$20(5) = \$170$
6	6	$\$10(6)+\$20(6) = \$180$
5	7	$\$10(5)+\$20(7) = \$190$
4	8	$\$10(4)+\$20(8) = \$200$
3	9	$\$10(3)+\$20(9) = \$210$
2	10	$\$10(2)+\$20(10) = \$220$

TABLE 1.15: (continued)

\$10 bills	\$ 20 bills	Total amount
1	11	$\$10(1)+\$20(11)=\$230$
0	12	$\$10(0)+\$20(12)=\$240$

In the table we listed all the possible ways you can get twelve \$10 bills and \$20 bills and the total amount of money for each possibility. The correct amount is given when Andrew has six \$10 bills and six \$20 bills.

Answer: Andrew gets six \$10 bills and six \$20 bills.

Check

Six \$10 bills and six \$20 bills → $6(\$10)+6(\$20)=\$60+\$120=\$180$

The answer checks out.

Let's solve the same problem using the method "Look for a Pattern."

Method 2: Looking for a Pattern

Understand

Andrew gives the bank teller a \$180 check.

The bank teller gives Andrew 12 bills. These bills are a mix of \$10 bills and \$20 bills.

We want to know how many of each kind of bill Andrew receives.

Strategy

Let's start by making a table just as we did above. However, this time we will look for patterns in the table that can be used to find the solution.

Apply strategy/solve

Let's fill in the rows of the table until we see a pattern.

TABLE 1.16:

\$10 bills	\$20 bills	Total amount
12	0	$\$10(12)+\$20(0)=\$120$
11	1	$\$10(11)+\$20(1)=\$130$
10	2	$\$10(10)+\$20(2)=\$140$

We see that every time we reduce the number of \$10 bills by one and increase the number of \$20 bills by one, the total amount increases by \$10. The last entry in the table gives a total amount of \$140, so we have \$40 to go until we reach our goal. This means that we should reduce the number of \$10 bills by four and increase the number of \$20 bills by four. That would give us six \$10 bills and six \$20 bills.

$$6(\$10)+6(\$20)=\$60+120=\$180$$

Answer: Andrew gets six \$10 bills and six \$20 bills.

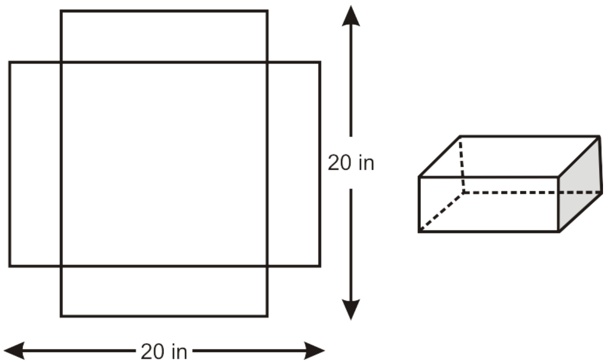
Check

Six \$10 bills and six \$20 bills → $6(\$10)+6(\$20)=\$60+120=\$180$

The answer checks out.

You can see that the second method we used for solving the problem was less tedious. In the first method, we listed all the possible options and found the answer we were seeking. In the second method, we started by listing the options, but we found a pattern that helped us find the solution faster. The methods of "Making a Table" and "Looking for a Pattern" are both more powerful if used alongside other problem-solving methods.

Solve Real-World Problems Using Selected Strategies as Part of a Plan



Example C

Anne is making a box without a lid. She starts with a 20 in. square piece of cardboard and cuts out four equal squares from each corner of the cardboard as shown. She then folds the sides of the box and glues the edges together. How big does she need to cut the corner squares in order to make the box with the biggest volume?

Solution

Step 1:

Understand

Anne makes a box out of a $20\ in \times 20\ in$ piece of cardboard.

She cuts out four equal squares from the corners of the cardboard.

She folds the sides and glues them to make a box.

How big should the cut out squares be to make the box with the biggest volume?

Step 2:

Strategy

We need to remember the formula for the volume of a box.

$\text{Volume} = \text{Area of base} \times \text{height}$

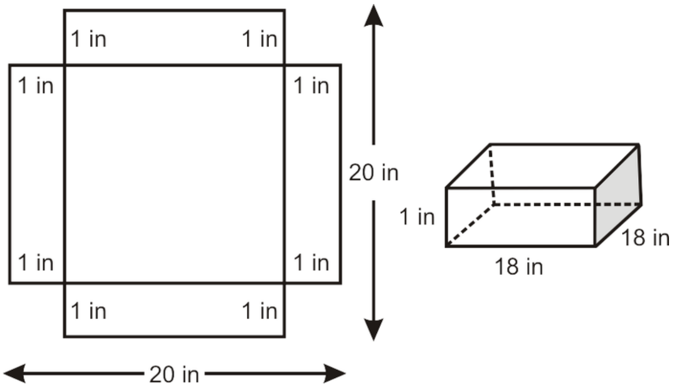
$\text{Volume} = \text{width} \times \text{length} \times \text{height}$

Make a table of values by picking different values for the side of the squares that we are cutting out and calculate the volume.

Step 3:

Apply strategy/solve

Let's "make" a box by cutting out four corner squares with sides equal to 1 inch. The diagram will look like this:



You see that when we fold the sides over to make the box, the height becomes 1 inch, the width becomes 18 inches and the length becomes 18 inches.

Volume $=$ width $\times$ length $\times$ height

Volume $= 18 \times 18 \times 1 = 324$ in^3

Let's make a table that shows the value of the box for different square sizes:

TABLE 1.17:

Side of Square	Box Height	Box Width	Box Length	Volume
1	1	18	18	$18 \times 18 \times 1 = 324$
2	2	16	16	$16 \times 16 \times 2 = 512$
3	3	14	14	$14 \times 14 \times 3 = 588$
4	4	12	12	$12 \times 12 \times 4 = 576$
5	5	10	10	$10 \times 10 \times 5 = 500$
6	6	8	8	$8 \times 8 \times 6 = 384$
7	7	6	6	$6 \times 6 \times 7 = 252$
8	8	4	4	$4 \times 4 \times 8 = 128$
9	9	2	2	$2 \times 2 \times 9 = 36$
10	10	0	0	$0 \times 0 \times 10 = 0$

We stop at a square of 10 inches because at this point we have cut out all of the cardboard and we can't make a box any more. From the table we see that we can make the biggest box if we cut out squares with a side length of three inches. This gives us a volume of 588 in^3.

Answer The box of greatest volume is made if we cut out squares with a side length of three inches.

Step 4:

Check

We see that 588 in^3 is the largest volume appearing in the table. We picked integer values for the sides of the squares that we are cut out. Is it possible to get a larger value for the volume if we pick non-integer values? Since we get the largest volume for the side length equal to three inches, let's make another table with values close to three inches that is split into smaller increments:

TABLE 1.18:

Side of Square	Box Height	Box Width	Box Length	Volume
2.5	2.5	15	15	$15 \times 15 \times 2.5 = 562.5$

TABLE 1.18: (continued)

Side of Square	Box Height	Box Width	Box Length	Volume
2.6	2.6	14.8	14.8	$14.8 \times 14.8 \times 2.6 = 569.5$
2.7	2.7	14.6	14.6	$14.6 \times 14.6 \times 2.7 = 575.5$
2.8	2.8	14.4	14.4	$14.4 \times 14.4 \times 2.8 = 580.6$
2.9	2.9	14.2	14.2	$14.2 \times 14.2 \times 2.9 = 584.8$
3	3	14	14	$14 \times 14 \times 3 = 588$
3.1	3.1	13.8	13.8	$13.8 \times 13.8 \times 3.1 = 590.4$
3.2	3.2	13.6	13.6	$13.6 \times 13.6 \times 3.2 = 591.9$
3.3	3.3	13.4	13.4	$13.4 \times 13.4 \times 3.3 = 592.5$
3.4	3.4	13.2	13.2	$13.2 \times 13.2 \times 3.4 = 592.4$
3.5	3.5	13	13	$13 \times 13 \times 3.5 = 591.5$

Notice that the largest volume is not when the side of the square is three inches, but rather when the side of the square is 3.3 inches.

Our original answer was not incorrect, but it was not as accurate as it could be. We can get an even more accurate answer if we take even smaller increments of the side length of the square. To do that, we would choose smaller measurements that are in the neighborhood of 3.3 inches.

Meanwhile, our first answer checks out if we want it rounded to zero decimal places, but **a more accurate answer is** 3.3 inches.

Watch this video for help with the Examples above.

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Click image to the left for more content.

CK-12 Foundation: Compare StrategiesforSolvingReal-World Problems

Vocabulary

Whichever strategy you are using, you should always implement the problem-solving plan when you are solving word problems. Here is a summary of the problem-solving plan.

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Devise a plan - Translate

Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart or construct a table as a start to solve your problem solving plan.

- **Step 3:**

Carry out the plan - Solve

This is where you solve the equation you developed in Step 2.

- **Step 4:**

Look - Check and Interpret

Check to see if you used all your information. Then look to see if the answer makes sense.

Guided Practice

Tickets to an event go on sale for $20 six weeks before the event, and go up in price by $5 each week. What is the price of tickets one week before the event?

Solution:

We want to know the price one week before the event. We know the price six weeks before the event is $20, and that it goes up $5 each week.

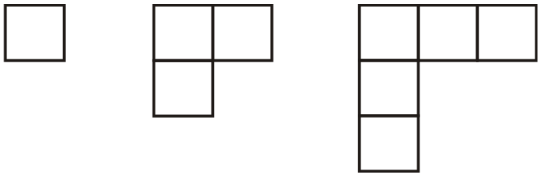
TABLE 1.19:

Weeks before event.	Price of tickets.
6	$\$20$
5	$\$20+\$5=\$25$
4	$\$25+\$5=\$30$
3	$\$30+\$5=\$35$
2	$\$35+\$5=\$40$
1	$\$40+\$5=\$45$

One week before the event, the tickets will cost $45.

Practice

1. Britt has $2.25 in nickels and dimes. If she has 40 coins in total, how many of each coin does she have?
2. Jeremy divides a 160-square-foot garden into plots that are either 10 or 12 square feet each. If there are 14 plots in all, how many plots are there of each size?
3. A pattern of squares is put together as shown. How many squares are in the 12^{th} diagram?



4. In Harrisville, local housing laws specify how many people can live in a house or apartment: the maximum number of people allowed is twice the number of bedrooms, plus one. If Jan, Pat, and their four children want to rent a house, how many bedrooms must it have?
5. A restaurant hosts children's birthday parties for a cost of $120 for the first six children (including the birthday child) and $30 for each additional child. If Jaden's parents have a budget of $200 to spend on his birthday party, how many guests can Jaden invite?
6. A movie theater with 200 seats charges $8 general admission and $5 for students. If the 5:00 showing is sold out and the theater took in $1468 for that showing, how many of the seats are occupied by students?
7. Oswald is trying to cut down on drinking coffee. His goal is to cut down to 6 cups per week. If he starts with 24 cups the first week, then cuts down to 21 cups the second week and 18 cups the third week, how many weeks will it take him to reach his goal?
8. Taylor checked out a book from the library and it is now 5 days late. The late fee is 10 cents per day. How much is the fine?
9. Mikhail is filling a sack with oranges.
   a. If each orange weighs 5 ounces and the sack will hold 2 pounds, how many oranges will the sack hold before it bursts?
   b. Mikhail plans to use these oranges to make breakfast smoothies. If each smoothie requires $\frac{3}{4}$ cup of orange juice, and each orange will yield half a cup, how many smoothies can he make?
10. Jessamyn takes out a $150 loan from an agency that charges 12% of the original loan amount in interest each week. If she takes five weeks to pay off the loan, what is the total amount (loan plus interest) she will need to pay back?
11. How many hours will a car traveling at 75 miles per hour take to catch up to a car traveling at 55 miles per hour if the slower car starts two hours before the faster car?
12. Grace starts biking at 12 miles per hour. One hour later, Dan starts biking at 15 miles per hour, following the same route. How long will it take him to catch up with Grace?
13. A new theme park opens in Milford. On opening day, the park has 120 visitors; on each of the next three days, the park has 10 more visitors than the day before; and on each of the three days after that, the park has 20 more visitors than the day before.
   a. How many visitors does the park have on the seventh day?
   b. How many total visitors does the park have all week?
14. Lemuel wants to enclose a rectangular plot of land with a fence. He has 24 feet of fencing. What is the largest possible area that he could enclose with the fence?
15. Quizzes in Keiko's history class are worth 20 points each. Keiko scored 15 and 18 points on her last two quizzes. What score does she need on her third quiz to get an average score of 17 on all three?
16. Mark is three years older than Janet, and the sum of their ages is 15. How old are Mark and Janet?
17. In a one-on-one basketball game, Jane scored $1\frac{1}{2}$ times as many points as Russell. If the two of them together scored 10 points, how many points did Jane score?
18. Scientists are tracking two pods of whales during their migratory season. On the first day of June, one pod is 120 miles north of a certain group of islands, and every day thereafter it gets 15 miles closer to the islands. The second pod starts out 160 miles east of the islands on June 3, and heads toward the islands at a rate of 20 miles a day.
   a. Which pod will arrive at the islands first, and on what day?
   b. How long after that will it take the other pod to reach the islands?
   c. Suppose the pod that reaches the islands first immediately heads south from the islands at a rate of 15 miles a day, and the pod that gets there second also heads south from there at a rate of 25 miles a day. On what day will the second pod catch up with the first?
   d. How far will both pods be from the islands on that day?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9611.

Summary

This chapter first deals with with writing expressions and equations using variables. It then moves on to evaluating algebraic expressions. Building on this knowledge, you will next write, solve, and check the solution of equations and inequalities. Functions are then covered in detail, with special emphasis given to writing function rules and graphing functions from both rules and tables. From there, you will learn how to determine if a given relation is a function. Finally, the chapter concludes by pulling all this knowledge together into techniques for devising and selecting a problem-solving strategy for real-world scenarios.

CHAPTER 2 Real Numbers

Chapter Outline

2.1 PROPERTIES OF RATIONAL NUMBERS
2.2 ADDITIVE INVERSES AND ABSOLUTE VALUES
2.3 ADDITION OF RATIONAL NUMBERS
2.4 RATIONAL NUMBERS IN APPLICATIONS
2.5 MULTIPLICATION OF RATIONAL NUMBERS
2.6 DIVISION OF RATIONAL NUMBERS
2.7 MIXED NUMBERS IN APPLICATIONS
2.8 DISTRIBUTIVE PROPERTY
2.9 SQUARE ROOTS AND IRRATIONAL NUMBERS
2.10 PROPERTIES OF RATIONAL NUMBERS VERSUS IRRATIONAL NUMBERS
2.11 GUESS AND CHECK, WORK BACKWARD

Introduction

Given a number like 5, 0.75, or $\sqrt{2}$, how would you classify it beyond a real number? In this chapter you'll learn three subsets of real numbers – rational numbers, irrational number, and integers – that you'll find every day in your daily life. The number of students in your class – that's an integer. The interest rate on your car loan – that's a rational number. The ratio of a circle's circumference to its diameter – that's an irrational number. This chapter differentiates these various types of real numbers and explains important properties and rules that apply to them. It then shows you how to perform operations and solve problems involving rational numbers. Finally, it equips you with problem-solving strategies to solve problems involving rational numbers.

2.1 Properties of Rational Numbers

Here you'll learn how to classify and simplify rational numbers and how to plot them on a number line. You'll also learn how to order them from least to greatest.

What if you wanted to order the numbers 2, $-\frac{5}{2}$, and $\frac{5}{2}$ from least to greatest? After completing this Concept, you'll be able to plot numbers like these on a number line to compare them.

Watch This

Multimedia

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CK-12 Foundation: 0201S IntegersandRational Numbers

Try This

To make graphing rational numbers easier, try using the number line generator at http://themathworksheetsite.com/numline.html. You can use it to create a number line divided into whatever units you want, as long as you express the units in decimal form.

Guidance

One day, Jason leaves his house and starts walking to school. After three blocks, he stops to tie his shoe and leaves his lunch bag sitting on the curb. Two blocks farther on, he realizes his lunch is missing and goes back to get it. After picking up his lunch, he walks six more blocks to arrive at school. How far is the school from Jason's house? And how far did Jason actually walk to get there?

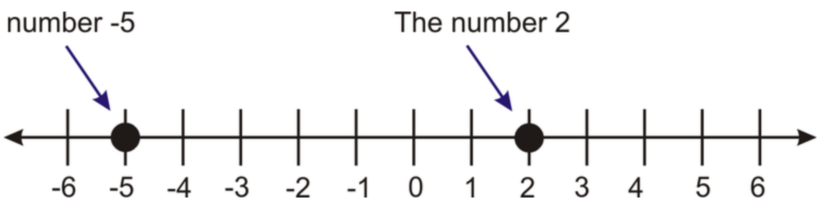
Graph and Compare Integers

Integers are the counting numbers (1, 2, 3...), the negative opposites of the counting numbers (-1, -2, -3...), and zero. There are an infinite number of integers and examples are 0, 3, 76, -2, -11, and 995.

Example A

Compare the numbers 2 *and* -5.

When we plot numbers on a number line, the **greatest** number is farthest to the right, and the **least** is farthest to the left.



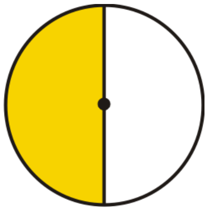
In the diagram above, we can see that 2 is farther to the right on the number line than -5, so we say that 2 is greater than -5. We use the symbol ">" to mean "greater than", so we can write $2 > -5$.

Classifying Rational Numbers

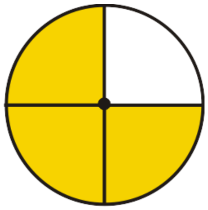
When we divide an integer *a* by another integer *b* (as long as *b* is not zero) we get a **rational number**. It's called this because it is the **ratio** of one number to another, and we can write it in fraction form as $\frac{a}{b}$. (You may recall that the top number in a fraction is called the **numerator** and the bottom number is called the **denominator**.)

You can think of a rational number as a fraction of a cake. If you cut the cake into *b* slices, your share is *a* of those slices.

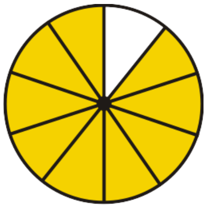
For example, when we see the rational number $\frac{1}{2}$, we can imagine cutting the cake into two parts. Our share is one of those parts. Visually, the rational number $\frac{1}{2}$ looks like this:



With the rational number $\frac{3}{4}$, we cut the cake into four parts and our share is three of those parts. Visually, the rational number $\frac{3}{4}$ looks like this:



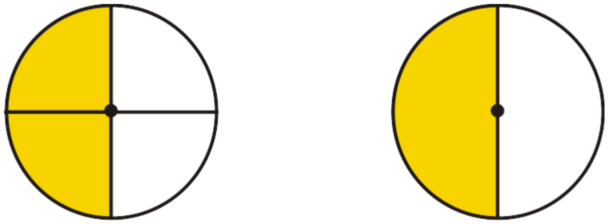
The rational number $\frac{9}{10}$ represents nine slices of a cake that has been cut into ten pieces. Visually, the rational number $\frac{9}{10}$ looks like this:



Proper fractions are rational numbers where the numerator is less than the denominator. A proper fraction represents a number less than one.

Improper fractions are rational numbers where the numerator is greater than or equal to the denominator. An improper fraction can be rewritten as a mixed number – an integer plus a proper fraction. For example, $\frac{9}{4}$ can be written as $2\frac{1}{4}$. An improper fraction represents a number greater than or equal to one.

Equivalent fractions are two fractions that represent the same amount. For example, look at a visual representation of the rational number $\frac{2}{4}$, and one of the number $\frac{1}{2}$.



You can see that the shaded regions are the same size, so the two fractions are equivalent. We can convert one fraction into the other by **reducing** the fraction, or writing it in lowest terms. To do this, we write out the prime factors of both the numerator and the denominator and cancel matching factors that appear in both the numerator **and** denominator.

$$\frac{2}{4} = \frac{2 \cdot 1}{2 \cdot 2 \cdot 1} = \frac{1}{2 \cdot 1} = \frac{1}{2}$$

Reducing a fraction doesn't change the value of the fraction—it just simplifies the way we write it. Once we've canceled all common factors, the fraction is in its **simplest form**.

Example B

Classify and simplify the following rational numbers

a) $\frac{3}{7}$

b) $\frac{9}{3}$

Solution

a) 3 and 7 are both prime, so we can't factor them. That means $\frac{3}{7}$ is already in its simplest form. It is also a proper fraction.

b) $\frac{9}{3}$ is an improper fraction because $9 > 3$. To simplify it, we factor the numerator and denominator and cancel: $\frac{3 \cdot 3}{3 \cdot 1} = \frac{3}{1} = 3$.

Order Rational Numbers

Ordering rational numbers is simply a matter of arranging them by increasing value—least first and greatest last.

Example C

Put the following fractions in order from least to greatest: $\frac{1}{2}, \frac{3}{4}, \frac{2}{3}$

Solution

$\frac{1}{2} < \frac{2}{3} < \frac{3}{4}$

Simple fractions are easy to order—we just know, for example, that one-half is greater than one quarter, and that two thirds is bigger than one-half. But how do we compare more complex fractions?

Example D

Which is greater, $\frac{3}{7}$ *or* $\frac{4}{9}$ *?*

In order to determine this, we need to rewrite the fractions so we can compare them more easily. If we rewrite them as equivalent fractions that have the same denominators, then we can compare them directly. To do this, we need to find the **lowest common denominator** (LCD), or the least common multiple of the two denominators.

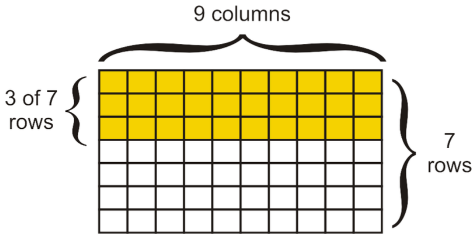
The lowest common multiple of 7 and 9 is 63. Our fraction will be represented by a shape divided into 63 sections. This time we will use a rectangle cut into 9 by $7 = 63$ pieces.

7 divides into 63 nine times, so $\frac{3}{7} = \frac{9 \cdot 3}{9 \cdot 7} = \frac{27}{63}$.

We can multiply the numerator and the denominator both by 9 because that's really just the opposite of reducing the fraction—to get back from $\frac{27}{63}$ to $\frac{3}{7}$, we'd just cancel out the 9's. Or, to put that in more formal terms:

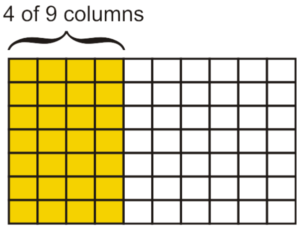
The fractions $\frac{a}{b}$ and $\frac{c \cdot a}{c \cdot b}$ are equivalent as long as $c \neq 0$.

Therefore, $\frac{27}{63}$ is an equivalent fraction to $\frac{3}{7}$. Here it is shown visually:

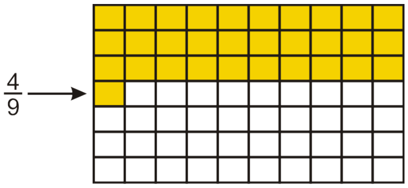


9 divides into 63 seven times, so $\frac{4}{9} = \frac{7 \cdot 4}{7 \cdot 9} = \frac{28}{63}$.

$\frac{28}{63}$ is an equivalent fraction to $\frac{4}{9}$. Here it is shown visually:



By writing the fractions with a **common denominator** of 63, we can easily compare them. If we take the 28 shaded boxes out of 63 (from our image of $\frac{4}{9}$ above) and arrange them in rows instead of columns, we can see that they take up more space than the 27 boxes from our image of $\frac{3}{7}$:



Solution

Since $\frac{28}{63}$ is greater than $\frac{27}{63}$, $\frac{4}{9}$ is greater than $\frac{3}{7}$.

Graph and Order Rational Numbers

To plot non-integer rational numbers (fractions) on the number line, we can convert them to mixed numbers (graphing is one of the few occasions in algebra when it's better to use mixed numbers than improper fractions), or we can convert them to decimal form.

Example E

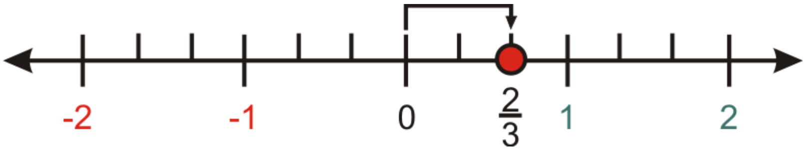
Plot the following rational numbers on the number line.

a) $\frac{2}{3}$

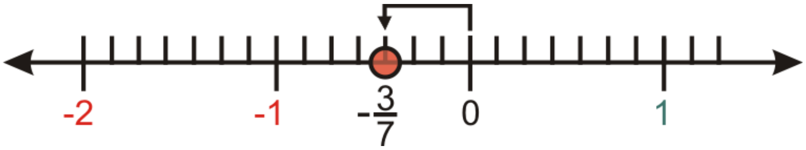
b) $-\frac{3}{7}$

If we divide up the number line into sub-intervals based on the denominator of the fraction, we can look at the fraction's numerator to determine how many of these sub-intervals we need to include.

a) $\frac{2}{3}$ falls between 0 and 1. Because the denominator is 3, we divide the interval between 0 and 1 into three smaller units. Because the numerator is 2, we count two units over from 0.



b) $-\frac{3}{7}$ falls between 0 and -1. We divide the interval into seven units, and move left from zero by three of those units.



Watch this video for help with the Examples above.

MEDIA

Click image to the left for more content.

CK-12 Foundation: Integers and Rational Numbers

Vocabulary

- **Integers** (or **whole numbers**) are the counting numbers (1, 2, 3, ...), the negative counting numbers (-1, -2, -3, ...), and zero.
- A **rational number** is the **ratio** of one integer to another, like $\frac{3}{5}$ or $\frac{a}{b}$. The top number is called the **numerator** and the bottom number (which can't be zero) is called the **denominator**.
- **Proper fractions** are rational numbers where the numerator is less than the denominator.
- **Improper fractions** are rational numbers where the numerator is greater than the denominator.
- **Equivalent fractions** are two fractions that equal the same numerical value. The fractions $\frac{a}{b}$ and $\frac{c \cdot a}{c \cdot b}$ are equivalent as long as $c \neq 0$.
- To **reduce** a fraction (write it in **simplest form**), write out all prime factors of the numerator and denominator, cancel common factors, then recombine.
- To compare two fractions it helps to write them with a **common denominator**.

Guided Practice

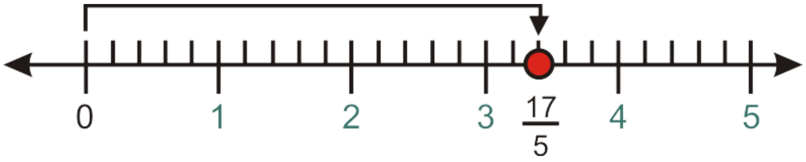
1. *Classify and simplify the rational number* $\frac{50}{60}$.

2. *Plot the rational number* $\frac{17}{5}$ *on the number line.*

Solution

1. $\frac{50}{60}$ is a proper fraction, and we can simplify it as follows: $\frac{50}{60} = \frac{5 \cdot 5 \cdot 2}{5 \cdot 3 \cdot 2 \cdot 2} = \frac{5}{3 \cdot 2} = \frac{5}{6}$.

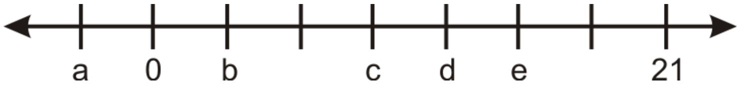
2. $\frac{17}{5}$ as a mixed number is $3\frac{2}{5}$ and falls between 3 and 4. We divide the interval into five units, and move over two units.



Another way to graph this fraction would be as a decimal. $3\frac{2}{5}$ is equal to 3.4, so instead of dividing the interval between 3 and 4 into 5 units, we could divide it into 10 units (each representing a distance of 0.1) and then count over 4 units. We would end up at the same place on the number line either way.

Practice

1. Solve the problem posed in the Introduction.
2. The tick-marks on the number line represent evenly spaced integers. Find the values of a, b, c, d and e.

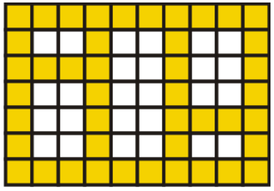


In 3-5, determine what fraction of the whole each shaded region represents.

3.

4.

5.



For 6-10, place the following sets of rational numbers in order, from least to greatest.

6. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

7. $\frac{1}{10}, \frac{1}{2}, \frac{2}{5}, \frac{1}{4}, \frac{7}{20}$
8. $\frac{39}{60}, \frac{49}{80}, \frac{59}{100}$
9. $\frac{7}{11}, \frac{8}{13}, \frac{12}{19}$
10. $\frac{9}{5}, \frac{22}{15}, \frac{4}{3}$

For 11-15, find the simplest form of the following rational numbers.

11. $\frac{22}{44}$
12. $\frac{9}{27}$
13. $\frac{12}{18}$
14. $\frac{315}{420}$
15. $\frac{244}{168}$

2.2 Additive Inverses and Absolute Values

Here you'll learn the property that makes two numbers opposites of each other. You'll also learn how to evaluate absolute value expressions.

What if you had a number like $-\frac{3}{4}$. How could you find its opposite and its absolute value? After completing this Concept, you'll be able to find both values for any number.

Watch This

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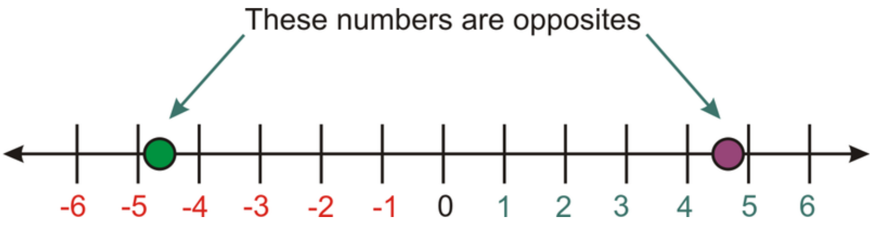
Click image to the left for more content.

CK-12 Foundation: 0202S Opposites and Absolute Values

Guidance

Every number has an opposite. On the number line, a number and its opposite are, predictably, *opposite* each other. In other words, they are the same distance from zero, but on opposite sides of the number line. The opposite of zero is defined to be simply zero.

Example A



The sum of a number and its opposite is always zero, as shown in Example B.

Example B

The numbers 3 and -3 are opposites because: $3+-3=0$

The numbers 4.2 and -4.2 are opposites because: $4.2+-4.2=0$

This is because adding 3 and -3 is like moving 3 steps to the right along the number line, and then 3 steps back to the left. The number and its opposite cancel each other out, leaving zero.

Another way to think of the opposite of a number is that it is simply the original number multiplied by -1.

Example C

The opposite of 4 is 4×-1 or -4, and the opposite of -2.3 is -2.3×-1 or just 2.3.

Another term for the opposite of a number is the **additive inverse**.

Example D

Find the opposite of each of the following:

a) 19.6

b) $-\frac{4}{9}$

c) x

d) xy^2

e) $(x-3)$

Solution

Since we know that opposite numbers are on opposite sides of zero, we can simply multiply each expression by -1. This changes the sign of the number to its opposite—if it's negative, it becomes positive, and vice versa.

a) The opposite of 19.6 is -19.6.

b) The opposite of is $-\frac{4}{9}$ is $\frac{4}{9}$.

c) The opposite of x is $-x$.

d) The opposite of xy^2 is $-xy^2$.

e) The opposite of $(x-3)$ is $-(x-3)$, or $(3-x)$.

Note: With the last example you must multiply the **entire expression** by -1. A common mistake in this example is to assume that the opposite of $(x-3)$ is $(x+3)$. Avoid this mistake!

Find Absolute Values

When we talk about absolute value, we are talking about distances on the number line. For example, the number 7 is 7 units away from zero—and so is the number -7. The absolute value of a number is the distance it is from zero, so the absolute value of 7 and the absolute value of -7 are both 7.

We **write** the absolute value of -7 as $|-7|$. We **read** the expression $|x|$ as "the absolute value of x."

- Treat absolute value expressions like parentheses. If there is an operation inside the absolute value symbols, evaluate that operation first.
- The absolute value of a number or an expression is **always** positive or zero. It cannot be negative. With absolute value, we are only interested in how far a number is from zero, and not in which direction.

Example E

Evaluate the following absolute value expressions.

a) $|5+4|$

b) $-|7-22|$

(Remember to treat any expressions inside the absolute value sign as if they were inside parentheses, and evaluate them first.)

Solution

a) $|5+4| = |9| = 9$

b) $-|7-22| = -|-15| = -(15) = -15$

Watch this video for help with the Examples above.

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Click image to the left for more content.

CK-12 Foundation: Opposites and AbsoluteValues

Vocabulary

- The **absolute value** of a number is the distance it is from zero on the number line. The absolute value of any expression will always be positive or zero.
- Two numbers are **opposites** if they are the same distance from zero on the number line and on opposite sides of zero. The opposite of an expression can be found by multiplying **the entire expression** by -1.

Guided Practice

1. *Find the opposite of each of the following:*

a) x

b) xy^2

2. *Evaluate the following absolute value expressions.*

a) $3-|4-9|$

b) $|-5-11|$

Solution

1. Since we know that opposite numbers are on opposite sides of zero, we can simply multiply each expression by -1. This changes the sign of the number to its opposite—if it's negative, it becomes positive, and vice versa.

a) The opposite of x is $-x$.

b) The opposite of xy^2 is $-xy^2$.

2. a) $3-|4-9| = 3-|-5| = 3-5 = -2$

b) $|-5-11| = |-16| = 16$

Practice

Find the opposite of each of the following.

1. 1.001
2. $(5-11)$
3. $(x+y)$
4. $(x-y)$
5. $(x+y-4)$

6. $(-x+2y)$

Simplify the following absolute value expressions.

7. $11-|-4|$
8. $|4-9|-|-5|$
9. $|-5-11|$
10. $7-|22-15-19|$
11. $-|-7|$
12. $|-2-88|-|88+2|$

2.3 Addition of Rational Numbers

Here you'll learn how to find the sum and difference of rational numbers by applying the properties of addition and subtraction.

What if you had two numbers like $\frac{5}{8}$ and $\frac{1}{4}$? How could you add and subtract them? After completing this Concept, you be able to perform addition and subtraction on rational numbers like these.

Watch This

Watch this video for help with the Examples above.

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Click image to the left for more content.

CK-12 Foundation: 0203S Adding and Subtracting Rational Numbers

Try This

For more practice adding and subtracting fractions, try playing the math games at http://www.mathplayground.com/fractions_add.html and http://www.mathplayground.com/fractions_sub.html, or the one at http://www.aaamath.com/fra66kx2.htm.

Guidance

In the last Concept, we learned how to represent numbers on a number line. To add numbers on a number line, we start at the position of the first number, and then move to the right by a number of units equal to the second number.

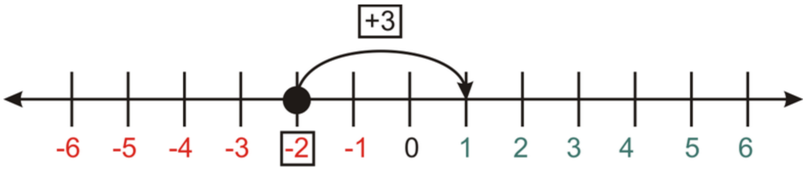
Example A

Represent the sum $-2+3$ on a number line.

We start at the number -2, and then move 3 units to the right. We thus end at +1.

Solution

$-2+3=1$



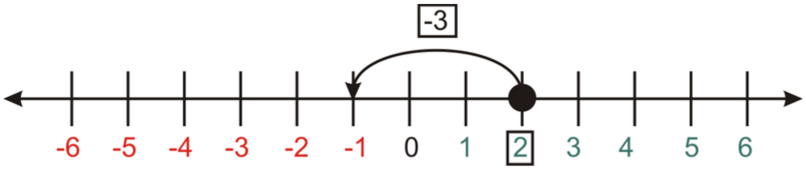
Example B

Represent the sum 2 - 3 on a number line.

Subtracting a number is basically just **adding a negative number**. Instead of moving to the right, we move to the left. Starting at the number 2, and then moving 3 to the left, means we end at -1.

Solution

$2-3=-1$



Adding and Subtracting Rational Numbers

When we add or subtract two fractions, the denominators must match before we can find the sum or difference. We have already seen how to find a common denominator for two rational numbers.

Example C

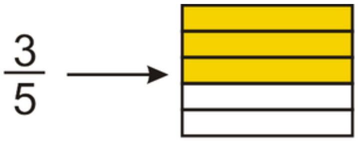
Simplify $\frac{3}{5}+\frac{1}{6}$.

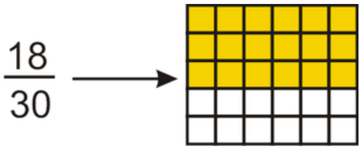
To combine these fractions, we need to rewrite them over a common denominator. We are looking for the **lowest common denominator** (LCD). We need to identify the **lowest common multiple** or **least common multiple** (LCM) of 5 and 6. That is the smallest number that both 5 and 6 divide into evenly (that is, without a remainder).

The lowest number that 5 and 6 both divide into evenly is 30. The LCM of 5 and 6 is 30, so the lowest common denominator for our fractions is also 30.

We need to rewrite our fractions as new **equivalent fractions** so that the denominator in each case is 30.

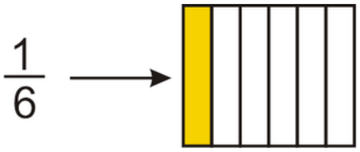
If you think back to our idea of a cake cut into a number of slices, $\frac{3}{5}$ means 3 slices of a cake that has been cut into 5 pieces. You can see that if we cut the same cake into 30 pieces (6 times as many) we would need 6 times as many slices to make up an equivalent fraction of the cake—in other words, 18 slices instead of 3.

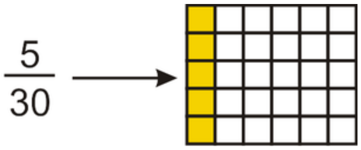




$\frac{3}{5}$ is equivalent to $\frac{18}{30}$.

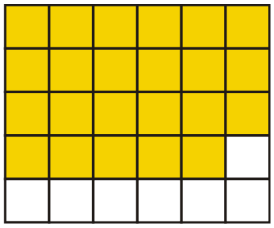
By a similar argument, we can rewrite the fraction $\frac{1}{6}$ as a share of a cake that has been cut into 30 pieces. If we cut it into 5 times as many pieces, we need 5 times as many slices.





$\frac{1}{6}$ is equivalent to $\frac{5}{30}$.

Now that both fractions have the same denominator, we can add them. If we add 18 pieces of cake to 5 pieces, we get a total of 23 pieces. 23 pieces of a cake that has been cut into 30 pieces means that our answer is $\frac{23}{30}$.



$$\frac{3}{5}+\frac{1}{6}=\frac{18}{30}+\frac{5}{30}=\frac{23}{30}$$

Notice that when we have fractions with a common denominator, we **add the numerators** but we **leave the denominators alone**. Here is this information in algebraic terms.

When adding fractions: $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$

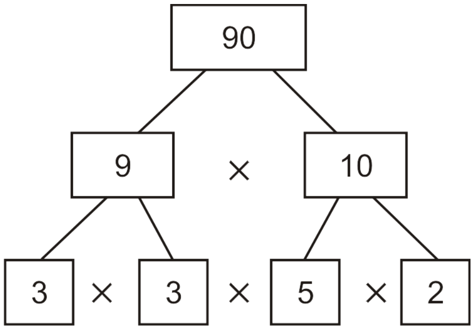
So far, we've only dealt with examples where it's easy to find the least common multiple of the denominators. With larger numbers, it isn't so easy to be sure that we have the LCD. We need a more systematic method. In the next example, we will use the method of **prime factors** to find the least common denominator.

Example D

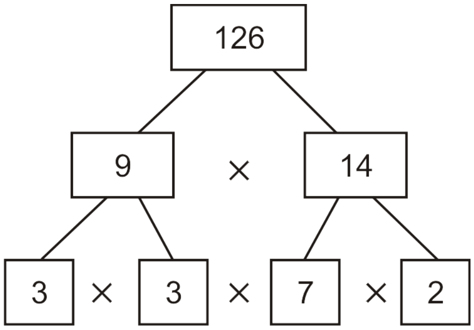
Simplify $\frac{29}{90}-\frac{13}{126}$.

To find the lowest common multiple of 90 and 126, we first find the prime factors of 90 and 126. We do this by continually dividing the number by factors until we can't divide any further. You may have seen a factor tree before. (For practice creating factor trees, try the Factor Tree game at http://www.mathgoodies.com/factors/prime_factors.html.)

The factor tree for 90 looks like this:



The factor tree for 126 looks like this:



The LCM for 90 and 126 is made from the **smallest possible collection of primes** that enables us to construct either of the two numbers. We take only enough instances of each prime to make the number with the greater number of instances of that prime in its factor tree.

TABLE 2.1:

Prime	**Factors in 90**	**Factors in 126**	**We Need**
2	1	1	1
3	2	2	2
5	1	0	1
7	0	1	1

So we need one 2, two 3's, one 5 and one 7. That gives us $2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 = 630$ as the lowest common multiple of 90 and 126. So 630 is the LCD for our calculation.

90 divides into 630 seven times (notice that 7 is the only factor in 630 that is missing from 90), so $\frac{29}{90} = \frac{7 \cdot 29}{7 \cdot 90} = \frac{203}{630}$.

126 divides into 630 five times (notice that 5 is the only factor in 630 that is missing from 126), so $\frac{13}{126} = \frac{5 \cdot 13}{5 \cdot 126} = \frac{65}{630}$.

Now we complete the problem: $\frac{29}{90} - \frac{13}{126} = \frac{203}{630} - \frac{65}{630} = \frac{138}{630}$.

This fraction **simplifies**. To be sure of finding the **simplest form** for $\frac{138}{630}$, we write out the prime factors of the numerator and denominator. We already know the prime factors of 630. The prime factors of 138 are 2, 3 and 23.

$\frac{138}{630} = \frac{2 \cdot 3 \cdot 23}{2 \cdot 3 \cdot 3 \cdot 5 \cdot 7}$; one factor of 2 and one factor of 3 cancels out, leaving $\frac{23}{3 \cdot 5 \cdot 7}$ or $\frac{23}{105}$ as our answer.

Identify and Apply Properties of Addition

Three mathematical properties which involve addition are the **commutative, associative,** and the **additive identity properties**.

Commutative property: When two numbers are added, the sum is the same even if the order of the items being added changes.

Example: $3 + 2 = 2 + 3$

Associative Property: When three or more numbers are added, the sum is the same regardless of how they are grouped.

Example: $(2 + 3) + 4 = 2 + (3 + 4)$

Additive Identity Property: The sum of any number and zero is the original number.

Example: $5 + 0 = 5$

Watch this video for help with the Examples above.

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CK-12 Foundation: AddingandSubtractingRational Numbers

Vocabulary

- **Subtracting** a number is the same as adding the **opposite** (or **additive inverse**) of the number.
- To add fractions, rewrite them over the **lowest common denominator (LCD)**. The lowest common denominator is the **lowest** (or **least**) **common multiple (LCM)** of the two denominators.
- When **adding fractions:** $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$
- When **subtracting fractions:** $\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$
- **Commutative property:** the sum of two numbers is the same even if the order of the items to be added changes.
- **Associative Property:** When three or more numbers are added, the sum is the same regardless of how they are grouped.
- **Additive Identity Property:** The sum of any number and zero is the original number.
- The number one is sometimes called the **invisible denominator**, as every whole number can be thought of as a rational number whose denominator is one.
- The **difference** between two values is the **change** in that quantity.

Guided Practice

Simplify $\frac{1}{3}-\frac{1}{9}$.

Solution:

The lowest common multiple of 9 and 3 is 9, so 9 is our common denominator. That means we don't have to alter the second fraction at all.

3 divides into 9 three times, so $\frac{1}{3}=\frac{3\cdot 1}{3\cdot 3}=\frac{3}{9}$. Our sum becomes $\frac{3}{9}-\frac{1}{9}$. We can subtract fractions with a common denominator by subtracting their numerators, just like adding. In other words:

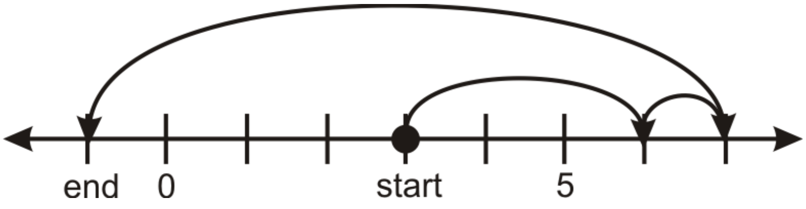
When subtracting fractions: $\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$

$\frac{1}{3}-\frac{1}{9}=\frac{2}{9}$

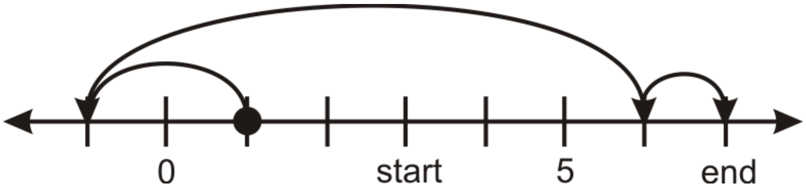
Practice

1. Write the sum that the following moves on a number line represent.

a.



b.



For 2-7, add the following rational numbers. Write each answer in its **simplest form**.

2. $\frac{3}{7}+\frac{2}{7}$
3. $\frac{3}{10}+\frac{1}{5}$
4. $\frac{5}{16}+\frac{5}{12}$
5. $\frac{3}{8}+\frac{9}{16}$
6. $\frac{8}{25}+\frac{7}{10}$
7. $\frac{1}{6}+\frac{1}{4}$

For 8-14, subtract the following rational numbers. Be sure that your answer is in the **simplest form**.

8. $\frac{3}{4}-\frac{1}{3}$
9. $\frac{15}{11}-\frac{9}{7}$
10. $\frac{2}{13}-\frac{1}{11}$
11. $\frac{7}{27}-\frac{9}{39}$
12. $\frac{6}{11}-\frac{3}{22}$
13. $\frac{13}{64}-\frac{7}{40}$
14. $\frac{11}{70}-\frac{11}{30}$
15. Consider the equation $y=3x+2$. Determine the change in y between $x=3$ and $x=7$.
16. Consider the equation $y=\frac{2}{3}x+\frac{1}{2}$. Determine the change in y between $x=1$ and $x=2$.

2.4 Rational Numbers in Applications

Here you'll apply the properties of addition and subtraction to solve real-world problems involving rational numbers.

What if you sent out invitations to a birthday party to 64 of your friends? One week later $\frac{1}{2}$ have responded that they will attend. Two weeks later, another $\frac{1}{4}$ have responded that they will attend. At the last minute, $\frac{1}{8}$ of those who originally said they would attend change their mind. How could you determine how many of your friends will be in attendance? After completing this Concept, you'll be able to solve real-world problems like this one.

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CK-12 Foundation: 0204S Solving Real-World Problems withAddition and Subtraction

Guidance

Let's use the skills we learned in the last concept to solve some real-world problems.

Example A

Peter is hoping to travel on a school trip to Europe. The ticket costs $2400. Peter has several relatives who have pledged to help him with the ticket cost. His parents have told him that they will cover half the cost. His grandma Zenoviea will pay one sixth, and his grandparents in Florida will send him one fourth of the cost. What fraction of the cost can Peter count on his relatives to provide?

The first thing we need to do is extract the relevant information. Peter's parents will provide $\frac{1}{2}$ the cost; his grandma Zenoviea will provide $\frac{1}{6}$; and his grandparents in Florida $\frac{1}{4}$. We need to find the sum of those numbers, or $\frac{1}{2}+\frac{1}{6}+\frac{1}{4}$.

To determine the sum, we first need to find the LCD. The LCM of 2, 6 and 4 is 12, so that's our LCD. Now we can find equivalent fractions:

$$\frac{1}{2}=\frac{6\cdot 1}{6\cdot 2}=\frac{6}{12}$$
$$\frac{1}{6}=\frac{2\cdot 1}{2\cdot 6}=\frac{2}{12}$$
$$\frac{1}{4}=\frac{3\cdot 1}{3\cdot 4}=\frac{3}{12}$$

Putting them all together: $\frac{6}{12}+\frac{2}{12}+\frac{3}{12}=\frac{11}{12}$.

Peter will get $\frac{11}{12}$ the cost of the trip, or $2200 out of $2400, from his family.

Example B

A property management firm is buying parcels of land in order to build a small community of condominiums. It has just bought three adjacent plots of land. The first is four-fifths of an acre, the second is five-twelfths of an acre, and the third is nineteen-twentieths of an acre. The firm knows that it must allow one-sixth of an acre for utilities and a small access road. How much of the remaining land is available for development?

The first thing we need to do is extract the relevant information. The plots of land measure $\frac{4}{5}, \frac{5}{12}$, and $\frac{19}{20}$ acres, and the firm can use all of that land except for $\frac{1}{6}$ of an acre. The total amount of land the firm can use is therefore $\frac{4}{5}+\frac{5}{12}+\frac{19}{20}-\frac{1}{6}$ acres.

We can add and subtract multiple fractions at once just by finding a common denominator for all of them. The factors of 5, 9, 20, and 6 are as follows:

5	5
12	$2 \cdot 2 \cdot 3$
20	$2 \cdot 2 \cdot 5$
6	$2 \cdot 3$

We need a 5, two 2's, and a 3 in our LCD. $2 \cdot 2 \cdot 3 \cdot 5 = 60$, so that's our common denominator. Now to convert the fractions:

$$\begin{aligned}
\frac{4}{5} &= \frac{12 \cdot 4}{12 \cdot 5} = \frac{48}{60}\\
\frac{5}{12} &= \frac{5 \cdot 5}{5 \cdot 12} = \frac{25}{60}\\
\frac{19}{20} &= \frac{3 \cdot 19}{3 \cdot 20} = \frac{57}{60}\\
\frac{1}{6} &= \frac{10 \cdot 1}{10 \cdot 6} = \frac{10}{60}
\end{aligned}$$

We can rewrite our sum as $\frac{48}{60}+\frac{25}{60}+\frac{57}{60}-\frac{10}{60}=\frac{48+25+57-10}{60}=\frac{120}{60}$.

Next, we need to reduce this fraction. We can see immediately that the numerator is twice the denominator, so this fraction reduces to $\frac{2}{1}$ or simply 2. One is sometimes called the **invisible denominator**, because every whole number can be thought of as a rational number whose denominator is one.

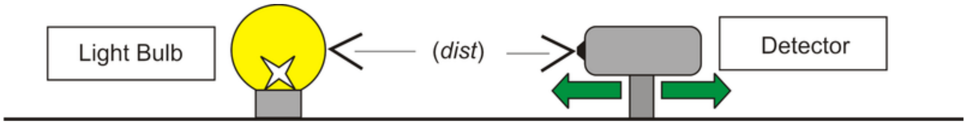
Solution

The property firm has two acres available for development.

Evaluate Change Using a Variable Expression

When we write algebraic expressions to represent a real quantity, the difference between two values is the **change** in that quantity.

Example C



The intensity of light hitting a detector when it is held a certain distance from a bulb is given by this equation:

$$Intensity = \frac{3}{d^2}$$

where d is the distance measured in ***meters****, and intensity is measured in* ***lumens****. Calculate the change in intensity when the detector is moved from two meters to three meters away.*

We first find the values of the intensity at distances of two and three meters.

$$\begin{aligned}\text{Intensity }(2) &= \frac{3}{(2)^2} = \frac{3}{4}\\ \text{Intensity }(3) &= \frac{3}{(3)^2} = \frac{3}{9} = \frac{1}{3}\end{aligned}$$

The **difference** in the two values will give the **change** in the intensity. We move **from** two meters **to** three meters away.

Change $=$ Intensity $(3) -$ Intensity $(2) = \frac{1}{3} - \frac{3}{4}$

To find the answer, we will need to write these fractions over a common denominator.

The LCM of 3 and 4 is 12, so we need to rewrite each fraction with a denominator of 12:

$$\begin{aligned}\frac{1}{3} &= \frac{4\cdot 1}{4\cdot 3} = \frac{4}{12}\\ \frac{3}{4} &= \frac{3\cdot 3}{3\cdot 4} = \frac{9}{12}\end{aligned}$$

So we can rewrite our equation as $\frac{4}{12} - \frac{9}{12} = -\frac{5}{12}$. The negative value means that the intensity decreases as we move from 2 to 3 meters away.

Solution

When moving the detector from two meters to three meters, the intensity falls by $\frac{5}{12}$ lumens.

Watch this video for help with the Examples above.

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CK-12 Foundation: Solving Real-WorldProblems Using AdditionandSubtraction

Vocabulary

- **Subtracting** a number is the same as adding the **opposite** (or **additive inverse**) of the number.
- To add fractions, rewrite them over the **lowest common denominator (LCD)**. The lowest common denominator is the **lowest** (or **least**) **common multiple (LCM)** of the two denominators.
- When **adding fractions:** $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$

- When **subtracting fractions:** $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$
- **Commutative property:** the sum of two numbers is the same even if the order of the items to be added changes.
- **Associative Property:** When three or more numbers are added, the sum is the same regardless of how they are grouped.
- **Additive Identity Property:** The sum of any number and zero is the original number.
- The number one is sometimes called the **invisible denominator**, as every whole number can be thought of as a rational number whose denominator is one.
- The **difference** between two values is the **change** in that quantity.

Guided Practice

Elsa baked a small cake for her family. First her sister ate one quarter and her mom ate one third. How much was left for Elsa?

Solution:

The whole cake is represented by 1. To solve this problem, we subtract the fraction that each person ate.

$1 - \frac{1}{4} - \frac{1}{3}$.

To complete this problem, we must give the terms common denominators. Since the denominators do not share any factors, we simply multiply them together: $4 \cdot 3 = 12$.

$$\begin{aligned}
&1-\frac{1}{4}-\frac{1}{3} && \text{Start with the original expression.}\\
=&1\cdot\frac{12}{12}-\frac{1}{4}\cdot\frac{3}{3}-\frac{1}{3}\cdot\frac{4}{4} && \text{Give each term a common denominator.}\\
=&\frac{12}{12}-\frac{3}{12}-\frac{4}{12} && \text{Simplify.}\\
=&\frac{12-3-4}{12}=\frac{5}{12}
\end{aligned}$$

There is $\frac{5}{12}$ of the original cake left for Elsa.

Practice

Which property of addition does each situation involve?

1. Whichever order your groceries are scanned at the store, the total will be the same.
2. However many shovel-loads it takes to move 1 ton of gravel, the number of rocks moved is the same.
3. If Julia has no money, then Mark and Julia together have just as much money as Mark by himself has.

In 4-7, practice your addition and subtraction skills.

4. $\frac{7}{15} + \frac{2}{9}$
5. $\frac{5}{19} + \frac{2}{27}$
6. $\frac{5}{12} - \frac{9}{18}$
7. $\frac{2}{3} - \frac{1}{4}$
8. Ilana buys two identically sized cakes for a party. She cuts the chocolate cake into 24 pieces and the vanilla cake into 20 pieces, and lets the guests serve themselves. Martin takes three pieces of chocolate cake and one of vanilla, and Sheena takes one piece of chocolate and two of vanilla. Which of them gets more cake?

9. Nadia, Peter and Ian are pooling their money to buy a gallon of ice cream. Nadia is the oldest and gets the greatest allowance. She contributes half of the cost. Ian is next oldest and contributes one third of the cost. Peter, the youngest, gets the smallest allowance and contributes one fourth of the cost. They figure that this will be enough money. When they get to the check-out, they realize that they forgot about sales tax and worry there will not be enough money. Amazingly, they have exactly the right amount of money. What fraction of the cost of the ice cream was added as tax?
10. The time taken to commute from San Diego to Los Angeles is given by the equation $time = \frac{120}{speed}$ where *time* is measured in **hours** and *speed* is measured in **miles per hour** (mph). Calculate the change in time that a rush hour commuter would see when switching from traveling by bus to traveling by train, if the bus averages 40 mph and the train averages 90 mph.

2.5 Multiplication of Rational Numbers

Here you'll learn how to evaluate and simplify rational expressions involving multiplication. You'll also learn how to identify and apply the properties of multiplication.

What if you had two numbers like $\frac{5}{6}$ and $\frac{2}{5}$? How could you multiply them so that your answer was in simplest form? After completing this Concept, you'll be able to solve multiplication problems like this one.

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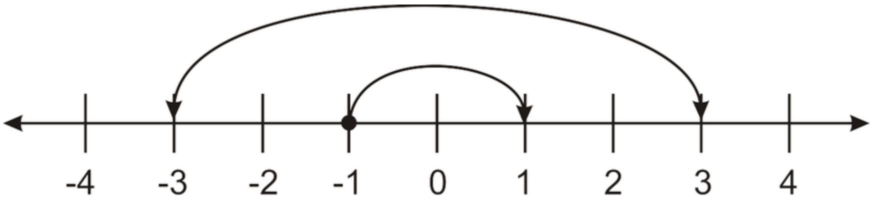
CK-12 Foundation: 0205S Multiplying Rationals

Try This

For more practice multiplying fractions, try playing the fraction game at http://www.aaamath.com/fra66mx2.htm, or the one at http://www.mathplayground.com/fractions_mult.html.

Guidance

Whenever we multiply a number by negative one, the sign of the number changes. In more mathematical terms, multiplying by negative one maps a number onto its opposite. The number line below shows two examples: $3 \cdot -1 = 3$ and $-1 \cdot -1 = 1$.



When we multiply a number by negative one, the absolute value of the new number is the same as the absolute value of the old number, since both numbers are the same distance from zero.

The product of a number "x" and negative one is $-x$. This does not mean that $-x$ is necessarily less than zero! If x itself is negative, then $-x$ will be positive because a negative times a negative (negative one) is a positive.

When you multiply an expression by negative one, remember to multiply the **entire expression** by negative one.

Example A

Multiply the following by negative one.

a) 79.5

b) π

c) $(x+1)$

d) $|x|$

Solution

a) -79.5

b) $-\pi$

c) $-(x+1)$ or $-x-1$

d) $-|x|$

Note that in the last case the negative sign **outside** the absolute value symbol applies **after** the absolute value. Multiplying the **argument** of an absolute value equation (the term inside the absolute value symbol) does not change the absolute value. $|x|$ is always positive. $|-x|$ is always positive. $-|x|$ is always negative.

Whenever you are working with expressions, you can check your answers by substituting in numbers for the variables. For example, you could check part *d* of Example 1 by letting $x = -3$. Then you'd see that $|-3| \neq -|3|$, because $|-3| = 3$ and $-|3| = -3$.

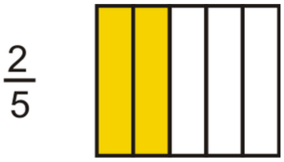
Careful, though—plugging in numbers can tell you if your answer is wrong, but it won't always tell you for sure if your answer is right!

Multiply Rational Numbers

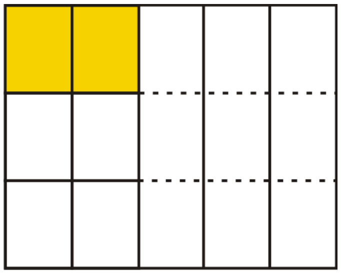
Example B

Simplify $\frac{1}{3} \cdot \frac{2}{5}$.

One way to solve this is to think of money. For example, we know that *one third of sixty dollars* is written as $\frac{1}{3} \cdot \$60$. We can read the above problem as *one-third of two-fifths*. Here is a visual picture of the fractions ***one-third*** and ***two-fifths***.



If we divide our rectangle into thirds one way and fifths the other way, here's what we get:



Here is the intersection of the two shaded regions. The whole has been divided into five pieces width-wise and three pieces height-wise. We get two pieces out of a total of fifteen pieces.

Solution

$\frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}$

Notice that $1 \cdot 2 = 2$ and $3 \cdot 5 = 15$. This turns out to be true in general: when you multiply rational numbers, the numerators multiply together and the denominators multiply together. Or, to put it more formally:

When multiplying fractions: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

This rule doesn't just hold for the product of two fractions, but for any number of fractions.

Example C

Evaluate and simplify $\frac{12}{25} \cdot \frac{35}{42}$.

Solution

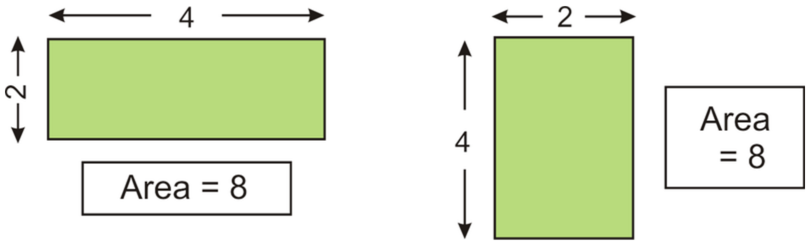
We can see that 12 and 42 are both multiples of six, 25 and 35 are both multiples of five, and 35 and 42 are both multiples of 7. That means we can write the whole product as $\frac{6 \cdot 2}{5 \cdot 5} \cdot \frac{5 \cdot 7}{6 \cdot 7} = \frac{6 \cdot 2 \cdot 5 \cdot 7}{5 \cdot 5 \cdot 6 \cdot 7}$. Then we can cancel out the 5, the 6, and the 7, leaving $\frac{2}{5}$.

Identify and Apply Properties of Multiplication

The four mathematical properties which involve multiplication are the **Commutative, Associative, Multiplicative Identity** and **Distributive Properties**.

Commutative property: When two numbers are multiplied together, the product is the same regardless of the order in which they are written.

Example: $4 \cdot 2 = 2 \cdot 4$



We can see a geometrical interpretation of **The Commutative Property of Multiplication** *to the right. The Area of the shape* $(length \times width)$ *is the same no matter which way we draw it.*

Associative Property: When three or more numbers are multiplied, the product is the same regardless of their grouping.

Example: $2 \cdot (3 \cdot 4) = (2 \cdot 3) \cdot 4$

Multiplicative Identity Property: The product of one and any number is that number.

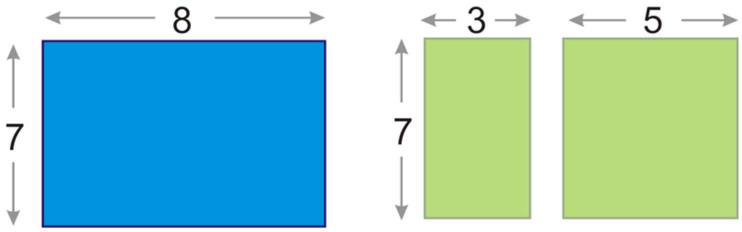
Example: $5 \cdot 1 = 5$

Distributive property: The multiplication of a number and the sum of two numbers is equal to the first number times the second number plus the first number times the third number.

Example: $4(6+3) = 4 \cdot 6 + 4 \cdot 3$

Example D

A gardener is planting vegetables for the coming growing season. He wishes to plant potatoes and has a choice of a single 8×7 *meter plot, or two smaller plots of* 3×7 *and* 5×7 *meters. Which option gives him the largest area for* *his potatoes?*



Solution

In the first option, the gardener has a total area of (8×7) or 56 square meters.

In the second option, the gardener has (3×7) or 21 square meters, plus (5×7) or 35 square meters. $21 + 35 = 56$, so the area is the same as in the first option.

Watch this video for help with the Examples above.

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CK-12 Foundation: Multiplying Rational Numbers

Vocabulary

When multiplying an expression by negative one, remember to multiply the **entire expression** by negative one.

To multiply fractions, multiply the numerators and multiply the denominators: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

The **multiplicative properties** are:

- **Commutative Property:** The product of two numbers is the same whichever order the items to be multiplied are written. **Example:** $2 \cdot 3 = 3 \cdot 2$
- **Associative Property:** When three or more numbers are multiplied, the sum is the same regardless of how they are grouped. **Example:** $2 \cdot (3 \cdot 4) = (2 \cdot 3) \cdot 4$
- **Multiplicative Identity Property:** The product of any number and one is the original number. **Example:** $2 \cdot 1 = 2$
- **Distributive property:** The multiplication of a number and the sum of two numbers is equal to the first number times the second number plus the first number times the third number. **Example:** $4(2+3) = 4(2) + 4(3)$

Guided Practice

Multiply the following rational numbers:

a) $\frac{2}{5} \cdot \frac{5}{9}$

b) $\frac{1}{3} \cdot \frac{2}{7} \cdot \frac{2}{5}$

c) $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5}$

Solution

a) With this problem, we can cancel the fives: $\frac{2}{5} \cdot \frac{5}{9} = \frac{2 \cdot 5}{5 \cdot 9} = \frac{2}{9}$.

b) With this problem, we multiply **all the numerators** and **all the denominators**:

$$\frac{1}{3} \cdot \frac{2}{7} \cdot \frac{2}{5} = \frac{1 \cdot 2 \cdot 2}{3 \cdot 7 \cdot 5} = \frac{4}{105}$$

c) With this problem, we multiply all the numerators and all the denominators, and then we can cancel most of them. The 2's, 3's, and 4's all cancel out, leaving $\frac{1}{5}$.

With multiplication of fractions, we can simplify before or after we multiply. The next example uses factors to help simplify before we multiply.

Practice

In 1-4, multiply the following expressions by negative one.

1. 25
2. -105
3. x^2
4. $(3+x)$

In 5-10, multiply the following rational numbers. Write your answer in the **simplest form**.

5. $\frac{5}{12} \times \frac{9}{10}$
6. $\frac{2}{3} \times \frac{1}{4}$
7. $\frac{3}{4} \times \frac{1}{3}$
8. $\frac{15}{11} \times \frac{9}{7}$
9. $\frac{1}{13} \times \frac{1}{11}$
10. $\frac{12}{15} \times \frac{35}{13} \times \frac{10}{2} \times \frac{26}{36}$

2.6 Division of Rational Numbers

Here you'll learn how to find the multiplicative inverse of a rational number. You'll also learn how to evaluate and simplify rational expressions involving division.

What if you had two numbers like $\frac{6}{5}$ and $\frac{2}{5}$? How could you divide the first one by the second one so that your answer was in simplest form? After completing this Concept, you'll be able to solve division problems like this one.

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CK-12 Foundation: 0206S DividingRationals

Try This

For more practice dividing fractions, try the game at http://www.aaamath.com/div66ox2.htm or the one at http://www.mathplayground.com/fractions_div.html.

Guidance

An **identity element** is a number which, when combined with a mathematical operation on a number, leaves that number unchanged. For example, the **identity element** for addition and subtraction is **zero**, because adding or subtracting zero to a number doesn't change the number. And zero is also what you get when you add together a number and its opposite, like 3 and -3.

Multiplicative Inverses

The **inverse operation** of addition is subtraction—when you add a number and then subtract that same number, you end up back where you started. Also, adding a number's opposite is the same as subtracting it—for example, $4+(-3)$ is the same as $4-3$.

Multiplication and division are also inverse operations to each other—when you multiply by a number and then divide by the same number, you end up back where you started. Multiplication and division also have an identity element: when you multiply or divide a number by **one**, the number doesn't change.

Just as the **opposite** of a number is the number you can add to it to get zero, the **reciprocal** of a number is the number you can multiply it by to get one. And finally, just as adding a number's opposite is the same as subtracting the number, multiplying by a number's reciprocal is the same as dividing by the number.

The reciprocal of a number x is also called the **multiplicative inverse**. Any number times its own multiplicative inverse equals one, and the multiplicative inverse of x is written as $\frac{1}{x}$.

To find the multiplicative inverse of a rational number, we simply ***invert the fraction***—that is, flip it over. In other words:

The multiplicative inverse of $\frac{a}{b}$ is $\frac{b}{a}$, as long as $a \neq 0$.

You'll see why in the following exercise.

Example A

Find the multiplicative inverse of each of the following.

a) $\frac{3}{7}$

b) $\frac{4}{9}$

c) $3\frac{1}{2}$

d) $-\frac{x}{y}$

e) $\frac{1}{11}$

Solution

a) When we invert the fraction $\frac{3}{7}$, we get $\frac{7}{3}$. Notice that if we multiply $\frac{3}{7} \cdot \frac{7}{3}$, the 3's and the 7's both cancel out and we end up with $\frac{1}{1}$, or just 1.

b) Similarly, the inverse of $\frac{4}{9}$ is $\frac{9}{4}$; if we multiply those two fractions together, the 4's and the 9's cancel out and we're left with 1. That's why the rule "invert the fraction to find the multiplicative inverse" works: the numerator and the denominator always end up canceling out, leaving 1.

c) To find the multiplicative inverse of $3\frac{1}{2}$ we first need to convert it to an improper fraction. Three wholes is six halves, so $3\frac{1}{2} = \frac{6}{2} + \frac{1}{2} = \frac{7}{2}$. That means the inverse is $\frac{2}{7}$.

d) Don't let the negative sign confuse you. The multiplicative inverse of a negative number is also negative! Just ignore the negative sign and flip the fraction as usual.

The multiplicative inverse of $-\frac{x}{y}$ is $-\frac{y}{x}$.

e) The multiplicative inverse of $\frac{1}{11}$ is $\frac{11}{1}$, or simply 11.

Look again at the last example. When we took the multiplicative inverse of $\frac{1}{11}$ we got a whole number, 11. That's because we can treat that whole number like a fraction with a denominator of 1. Any number, even a non-rational one, can be treated this way, so we can always find a number's multiplicative inverse using the same method.

Divide Rational Numbers

Earlier, we mentioned that multiplying by a number's reciprocal is the same as dividing by the number. That's how we can divide rational numbers; to divide by a rational number, just multiply by that number's reciprocal. In more formal terms:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

Example B

Divide the following rational numbers, giving your answer in the **simplest form**.

a) $\frac{1}{2} \div \frac{1}{4}$

b) $\frac{7}{3} \div \frac{2}{3}$

c) $\frac{x}{2} \div \frac{1}{4y}$

d) $\frac{11}{2x} \div \left(-\frac{x}{y}\right)$

Solution

a) Replace $\frac{1}{4}$ with $\frac{4}{1}$ and multiply: $\frac{1}{2} \times \frac{4}{1} = \frac{4}{2} = 2$.

b) Replace $\frac{2}{3}$ with $\frac{3}{2}$ and multiply: $\frac{7}{3} \times \frac{3}{2} = \frac{7 \cdot 3}{3 \cdot 2} = \frac{7}{2}$.

c) $\frac{x}{2} \div \frac{1}{4y} = \frac{x}{2} \times \frac{4y}{1} = \frac{4xy}{2} = \frac{2xy}{1} = 2xy$

d) $\frac{11}{2x} \div \left(-\frac{x}{y}\right) = \frac{11}{2x} \times \left(-\frac{y}{x}\right) = -\frac{11y}{2x^2}$

Solve Real-World Problems Using Division

Speed, Distance and Time

An object moving at a certain **speed** will cover a fixed **distance** in a set **time**. The quantities speed, distance and time are related through the equation $\text{Speed} = \frac{\text{Distance}}{\text{Time}}$.

Example C

Anne runs a mile and a half in a quarter hour. What is her speed in miles per hour?

Solution

We already have the distance and time in the correct units (miles and hours), so we just need to write them as fractions and plug them into the equation.

$$\text{Speed} = \frac{1\frac{1}{2}}{\frac{1}{4}} = \frac{3}{2} \div \frac{1}{4} = \frac{3}{2} \times \frac{4}{1} = \frac{3 \cdot 4}{2 \cdot 1} = \frac{12}{2} = 6$$

Anne runs at 6 miles per hour.

Watch this video for help with the Examples above.

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CK-12 Foundation: Dividing Rationals

Vocabulary

The **multiplicative inverse** of a number is the number which produces 1 when multiplied by the original number. The multiplicative inverse of x is the reciprocal $\frac{1}{x}$. To find the multiplicative inverse of a fraction, simply **invert the fraction**: $\frac{a}{b}$ inverts to $\frac{b}{a}$.

To divide fractions, invert the divisor and multiply: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

An object moving at a certain **speed** will cover a fixed **distance** in a set **time**. The quantities speed, distance and time are related through the equation $\text{Speed} = \frac{\text{Distance}}{\text{Time}}$.

Guided Practice

Divide the following rational numbers, giving your answer in the **simplest form**.

a) $\frac{3}{10} \div \frac{7}{5}$

b) $\frac{9x}{5} \div \frac{9}{5}$

Solution

a) Replace $\frac{7}{5}$ with $\frac{5}{7}$ and multiply: $\frac{3}{10} \times \frac{5}{7} = \frac{15}{70} = \frac{3}{10}$.

b) Replace $\frac{9}{5}$ with $\frac{5}{9}$ and multiply: $\frac{9x}{5} \times \frac{5}{9} = \frac{45x}{45} = x$.

Practice

For 1-5, find the multiplicative inverse of each of the following.

1. 100
2. $\frac{2}{8}$
3. $-\frac{19}{21}$
4. 7
5. $-\frac{z^3}{2xy^2}$

For 6-10, divide the following rational numbers. Write your answer in the simplest form.

6. $\frac{5}{2} \div \frac{1}{4}$
7. $\frac{1}{2} \div \frac{7}{9}$
8. $\frac{5}{11} \div \frac{6}{7}$
9. $\frac{1}{2} \div \frac{1}{2}$
10. $-\frac{x}{2} \div \frac{5}{7}$

2.7 Mixed Numbers in Applications

Here you'll learn how to apply the properties of multiplication and division to solve real-world problems involving rational numbers.

What if you had a jar of pennies? You take out a handful that is equal to $\frac{1}{10}$ the total. Your friend Zoey then takes a handful that is equal to $\frac{2}{9}$ the amount that remains. What fraction of the total does Zoey take? After completing this Concept, you'll be able to solve real-world problems like this one.

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CK-12 Foundation: 0207S Real-World Multiplication and Division (H264)

Guidance

Using the skills learned in the last two concepts, you now ready to solve real-world problems.

Example A

In the chemistry lab there is a bottle with two liters of a 15% solution of hydrogen peroxide (H_2O_2). *John removes one-fifth of what is in the bottle, and puts it in a beaker. He measures the amount of* H_2O_2 *and adds twice that amount of water to the beaker. Calculate the following measurements.*

a) The amount of H_2O_2 left in the bottle.

b) The amount of diluted H_2O_2 in the beaker.

c) The concentration of the H_2O_2 in the beaker.

Solution

a) To determine the amount of H_2O_2 left in the bottle, we first determine the amount that was removed. That amount was $\frac{1}{5}$ of the amount in the bottle (2 liters). $\frac{1}{5}$ of 2 is $\frac{2}{5}$.

The amount remaining is $2-\frac{2}{5}$, or $\frac{10}{5}-\frac{2}{5}=\frac{8}{5}$ liter (or 1.6 liters).

There are 1.6 liters left in the bottle.

b) We determined that the amount of the 15% H_2O_2 solution removed was $\frac{2}{5}$ liter. The amount of water added was twice this amount, or $\frac{4}{5}$ liter. So the total amount of solution in the beaker is now $\frac{2}{5}+\frac{4}{5}=\frac{6}{5}$ liter, or 1.2 liters.

There are 1.2 liters of diluted H_2O_2 in the beaker.

c) The new concentration of H_2O_2 can be calculated.

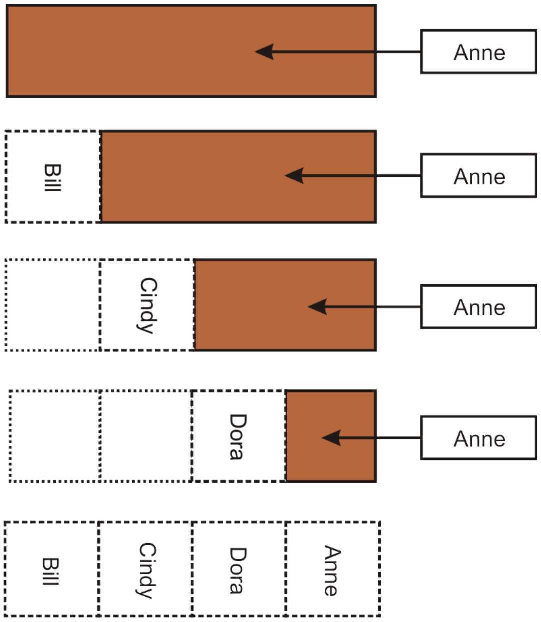
John started with $\frac{2}{5}$ liter of 15% H_2O_2 solution, so the amount of **pure** H_2O_2 is 15% of $\frac{2}{5}$ liters, or $0.15 \times 0.40 = 0.06$ liters.

After he adds the water, there is 1.2 liters of solution in the beaker, so the concentration of H_2O_2 is $\frac{0.06}{1.2} = \frac{1}{20}$ or 0.05. To convert to a percent we multiply this number by 100, so the beaker's contents are 5% H_2O_2.

Example B

Anne has a bar of chocolate and she offers Bill a piece. Bill quickly breaks off $\frac{1}{4}$ of the bar and eats it. Another friend, Cindy, takes $\frac{1}{3}$ of what was left. Anne splits the remaining candy bar into two equal pieces which she shares with a third friend, Dora. How much of the candy bar does each person get?

First, let's look at this problem visually.



Anne starts with a full candy bar.

Bill breaks off $\frac{1}{4}$ of the bar.

Cindy takes $\frac{1}{3}$ of what was left.

Dora gets half of the remaining candy bar.

We can see that the candy bar ends up being split four ways, with each person getting an equal amount.

Solution

Each person gets exactly $\frac{1}{4}$ of the candy bar.

We can also examine this problem using rational numbers. We keep a running total of what fraction of the bar remains. Remember, when we read a fraction followed by *of* in the problem, it means we multiply by that fraction.

We start with 1 bar. Then Bill takes $\frac{1}{4}$ of it, so there is $1 - \frac{1}{4} = \frac{3}{4}$ of a bar left.

Cindy takes $\frac{1}{3}$ of what's left, or $\frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$ of a whole bar. That leaves $\frac{3}{4} - \frac{1}{4} = \frac{2}{4}$, or $\frac{1}{2}$ of a bar.

That half bar gets split between Anne and Dora, so they each get half of a half bar: $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

So each person gets exactly $\frac{1}{4}$ of the candy bar.

Extension: If each person's share is 3 oz, how much did the original candy bar weigh?

Example C

Newton's second law $(F = ma)$ *relates the force applied to a body in Newtons* (F), *the mass of the body in kilograms* (m) *and the acceleration in meters per second squared* (a). *Calculate the resulting acceleration if a Force of* $7\frac{1}{3}$ *Newtons is applied to a mass of* $\frac{1}{5}kg$.

Solution

First, we rearrange our equation to isolate the acceleration, a. If $F = ma$, dividing both sides by m gives us $a = \frac{F}{m}$. Then we substitute in the known values for F and m:

$$a = \frac{7\frac{1}{3}}{\frac{1}{5}} = \frac{22}{3} \div \frac{1}{5} = \frac{22}{3} \times \frac{5}{1} = \frac{110}{3}$$

The resultant acceleration is $36\frac{2}{3}\ m/s^2$.

Watch this video for help with the Examples above.

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CK-12 Foundation: Solving Real-WorldProblems Using Multiplication and Division

Vocabulary

When multiplying an expression by negative one, remember to multiply the **entire expression** by negative one.

To multiply fractions, multiply the numerators and multiply the denominators: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

The **multiplicative properties** are:

- **Commutative Property:** The product of two numbers is the same whichever order the items to be multiplied are written. **Example:** $2 \cdot 3 = 3 \cdot 2$
- **Associative Property:** When three or more numbers are multiplied, the sum is the same regardless of how they are grouped. **Example:** $2 \cdot (3 \cdot 4) = (2 \cdot 3) \cdot 4$
- **Multiplicative Identity Property:** The product of any number and one is the original number. **Example:** $2 \cdot 1 = 2$
- **Distributive property:** The multiplication of a number and the sum of two numbers is equal to the first number times the second number plus the first number times the third number. **Example:** $4(2+3) = 4(2) + 4(3)$

The **multiplicative inverse** of a number is the number which produces 1 when multiplied by the original number. The multiplicative inverse of x is the reciprocal $\frac{1}{x}$. To find the multiplicative inverse of a fraction, simply **invert the fraction**: $\frac{a}{b}$ inverts to $\frac{b}{a}$.

To divide fractions, invert the divisor and multiply: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

An object moving at a certain **speed** will cover a fixed **distance** in a set **time**. The quantities speed, distance and time are related through the equation $\text{Speed} = \frac{\text{Distance}}{\text{Time}}$.

Guided Practice

Andrew is driving down the freeway. He passes mile marker 27 at exactly mid-day. At 12:35 he passes mile marker 69. At what speed, in miles per hour, is Andrew traveling?

Solution

To find the speed, we need the distance traveled and the time taken. If we want our speed to come out in miles per hour, we'll need distance in **miles** and time in **hours**.

The distance is $69 - 27$ or 42 miles. The time is 35 minutes, or $\frac{35}{60}$ hours, which reduces to $\frac{7}{12}$. Now we can *plug in* the values for distance and time into our equation for speed.

$$\text{Speed} = \frac{42}{\frac{7}{12}} = 42 \div \frac{7}{12} = \frac{42}{1} \times \frac{12}{7} = \frac{6 \cdot 7 \cdot 12}{1 \cdot 7} = \frac{6 \cdot 12}{1} = 72$$

Andrew is driving at 72 miles per hour.

Practice

For 1-8, perform the operations of multiplication and division.

1. $(3 - x)$
2. $\frac{1}{13} \times \frac{1}{11}$
3. $\frac{7}{27} \times \frac{9}{14}$
4. $\left(\frac{3}{5}\right)^2$
5. $\frac{1}{2} \div \frac{x}{4y}$
6. $\left(-\frac{1}{3}\right) \div \left(-\frac{3}{5}\right)$
7. $\frac{7}{2} \div \frac{7}{4}$
8. $11 \div \frac{-x}{4}$

For 9-11, solve the real-world problems using multiplication and division.

9. The label on a can of paint says that it will cover 50 square feet per pint. If I buy a $\frac{1}{8}$ pint sample, it will cover a square two feet long by three feet high. Is the coverage I get more, less or the same as that stated on the label?
10. The world's largest trench digger, "Bagger 288", moves at $\frac{3}{8}$ mph. How long will it take to dig a trench $\frac{2}{3}$ mile long?
11. A $\frac{2}{7}$ Newton force applied to a body of unknown mass produces an acceleration of $\frac{3}{10}$ m/s^2. Calculate the mass of the body.

2.8 Distributive Property

Here you'll learn how to apply the distributive property to simplify rational expressions and solve real-world problems.

What if you had an expression that involved addition or subraction like $(2x^2 - 4)$ and you wanted to multiply it by a number like $\frac{1}{2}$? How would you go about it? After completing this Concept, you'll be able to use the Distributive Property to evaluate problems like this one.

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CK-12 Foundation: 0208S The DistributiveProperty (H264)

Try This

For more practice using the Distributive Property, try playing the Battleship game at http://www.quia.com/ba/15357.html.

Guidance

At the end of the school year, an elementary school teacher makes a little gift bag for each of his students. Each bag contains one class photograph, two party favors and five pieces of candy. The teacher will distribute the bags among his 28 students. How many of each item does the teacher need?

Apply the Distributive Property

When we have a problem like the one posed in the introduction, **The Distributive Property** can help us solve it. First, we can write an expression for the contents of each bag: Items = (photo + 2 favors + 5 candies), or simply $I = (p+2f+5c)$.

For all 28 students, the teacher will need 28 times that number of items, so $I = 28(p+2f+5c)$.

Next, **the Distributive Property of Multiplication** tells us that when we have a single term multiplied by a sum of several terms, we can rewrite it by multiplying the single term by each of the other terms separately. In other words, $28(p+2f+5c) = 28(p)+28(2f)+28(5c)$, which simplifies to $28p+56f+140c$. So the teacher needs 28 class photos, 56 party favors and 140 pieces of candy.

You can see why the Distributive Property works by looking at a simple problem where we just have numbers inside the parentheses, and considering the **Order of Operations**.

Example A

Determine the value of 11(2 - 6) using both the Order of Operations and the Distributive Property.

Solution

Order of Operations tells us to evaluate the amount inside the parentheses first:

$$11(2-6) = 11(-4) = -44$$

Now let's try it with the Distributive Property:

$$11(2-6) = 11(2) - 11(6) = 22 - 66 = -44$$

Note: When applying the Distributive Property you **MUST** take note of any **negative signs!**

Example B

Use the Distributive Property to determine the following.

a) $11(2x+6)$

b) $\frac{2x}{7}\left(3y^2 - \frac{11}{xy}\right)$

Solution

a) $11(2x+6) = 11(2x) + 11(6) = 22x + 66$

b) $\frac{2x}{7}\left(3y^2 - \frac{11}{xy}\right) = \frac{2x}{7}(3y^2) + \frac{2x}{7}\left(-\frac{11}{xy}\right) = \frac{6xy^2}{7} - \frac{22x}{7xy}$

We can simplify this answer by canceling the x's in the second fraction, so we end up with $\frac{6xy^2}{7} - \frac{22}{7y}$.

Identify Expressions That Involve the Distributive Property

The Distributive Property can also appear in expressions that don't include parentheses. In Lesson 1.2, we saw how the fraction bar also acts as a grouping symbol. Now we'll see how to use the Distributive Property with fractions.

Example C

Simplify the following expressions.

a) $\frac{2x+8}{4}$

b) $\frac{9y-2}{3}$

Solution

Even though these expressions aren't written in a form we usually associate with the Distributive Property, remember that we treat the numerator of a fraction as if it were in parentheses, and that means we can use the Distributive Property here too.

a) $\frac{2x+8}{4}$ can be re-written as $\frac{1}{4}(2x+8)$. Then we can distribute the $\frac{1}{4}$:

$$\frac{1}{4}(2x+8) = \frac{2x}{4} + \frac{8}{4} = \frac{x}{2} + 2$$

b) $\frac{9y-2}{3}$ can be re-written as $\frac{1}{3}(9y-2)$, and then we can distribute the $\frac{1}{3}$:

$$\frac{1}{3}(9y-2)=\frac{9y}{3}-\frac{2}{3}=3y-\frac{2}{3}$$

Solve Real-World Problems Using the Distributive Property

The Distributive Property is one of the most common mathematical properties used in everyday life. Any time we have two or more groups of objects, the Distributive Property can help us solve for an unknown.

Example D

Each student on a field trip into a forest is to be given an emergency survival kit. The kit is to contain a flashlight, a first aid kit, and emergency food rations. Flashlights cost $12 each, first aid kits are $7 each and emergency food rations cost $2 per day. There is $500 available for the kits and 17 students to provide for. How many days worth of rations can be provided with each kit?

The unknown quantity in this problem is the number of days' rations. This will be x in our expression.

Each kit will contain **one** $12 flashlight, **one** $7 first aid kit, and x times $2 worth of rations, for a total cost of $(12+7+2x)$ dollars. With 17 kits, therefore, the total cost will be $17(12+7+2x)$ dollars.

We can use the Distributive Property on this expression:

$$17(12+7+2x)=204+119+34x$$

Since the total cost can be at most $500, we set the expression equal to 500 and solve for x. (You'll learn in more detail how to solve equations like this in the next chapter.)

$$\begin{aligned}204+119+34x&=500\\323+34x&=500\\323+34x-323&=500-323\\34x&=177\\\frac{34x}{34}&=\frac{177}{34}\\x&\approx 5.206\end{aligned}$$

Since this represents the number of days' worth of rations that can be bought, we must **round to the next lowest whole number**. We wouldn't have enough money to buy a sixth day of supplies.

Solution

Five days worth of emergency rations can be purchased for each survival kit.

Watch this video for help with the Examples above.

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Click image to the left for more content.

CK-12 Foundation: The Distributive Property

Vocabulary

- **Distributive Property** The product of a number and the sum of two numbers is equal to the first number times the second number plus the first number times the third number.
- When applying the Distributive Property you **MUST** take note of any **negative signs!**

Guided Practice

Simplify the following expressions:

a) $7(3x-5)$

b) $\frac{2}{7}(3y^2-11)$

c) $\frac{z+6}{2}$

Solution:

a) Note the negative sign on the second term.

$$7(3x-5) = 21x-35$$

b) $\frac{2}{7}(3y^2-11) = \frac{2}{7}(3y^2)+\frac{2}{7}(-11) = \frac{6y^2}{7}-\frac{22}{7}$, or $\frac{6y^2-22}{7}$

c) Rewrite $\frac{z+6}{2}$ as $\frac{1}{2}(z+6)$, and distribute the $\frac{1}{2}$:

$$\frac{1}{2}(z+6) = \frac{z}{2}+\frac{6}{2} = \frac{z}{2}+3$$

Practice

For 1-7, use the Distributive Property to simplify the following expressions.

1. $(x+4)-2(x+5)$
2. $\frac{1}{2}(4z+6)$
3. $(4+5)-(5+2)$
4. $x(x+7)$
5. $y(x+7)$
6. $x\left(\frac{3}{x}+5\right)$
7. $xy\left(\frac{1}{x}+\frac{2}{y}\right)$

For 8-15, use the Distributive Property to remove the parentheses from the following expressions.

8. $\frac{1}{2}(x-y)-4$
9. $0.6(0.2x+0.7)$
10. $6+(x-5)+7$
11. $6-(x-5)+7$
12. $4(m+7)-6(4-m)$
13. $-5(y-11)+2y$
14. $-(x-3y)+\frac{1}{2}(z+4)$
15. $\frac{a}{b}\left(\frac{2}{a}+\frac{3}{b}+\frac{b}{5}\right)$

For 16-23, use the Distributive Property to simplify the following fractions.

16. $\frac{8x+12}{4}$
17. $\frac{9x+12}{3}$
18. $\frac{11x+12}{2}$
19. $\frac{3y+2}{6}$
20. $-\frac{6z-2}{3}$
21. $\frac{7-6p}{3}$
22. $\frac{3d-4}{6d}$
23. $\frac{12g+8h}{4gh}$
24. A bookcase has five shelves, and each shelf contains seven poetry books and eleven novels. How many of each type of book does the bookcase contain?
25. Amar is making giant holiday cookies for his friends at school. He makes each cookie with 6 oz of cookie dough and decorates them with macadamia nuts. If Amar has 5 lbs of cookie dough $(1\ lb = 16\ oz)$ and 60 macadamia nuts, calculate the following.
    a. How many (full) cookies he can make?
    b. How many macadamia nuts he can put on each cookie, if each is to be identical?
    c. If 4 cups of flour and 1 cup of sugar went into each pound of cookie dough, how much of each did Amar use to make the 5 pounds of dough?

2.9 Square Roots and Irrational Numbers

Here you'll learn how to find and approximate square roots. You'll also learn how to simplify expressions involving square roots.

What if you had a number like 1000 and you wanted to find its square root? After completing this concept, you'll be able to find square roots like this one by hand and with a calculator.

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CK-12 Foundation: 0209S Square Roots(H264)

Try This

You can also work out square roots by hand using a method similar to long division. (See the web page at http://www.homeschoolmath.net/teaching/square-root-algorithm.php for an explanation of this method.)

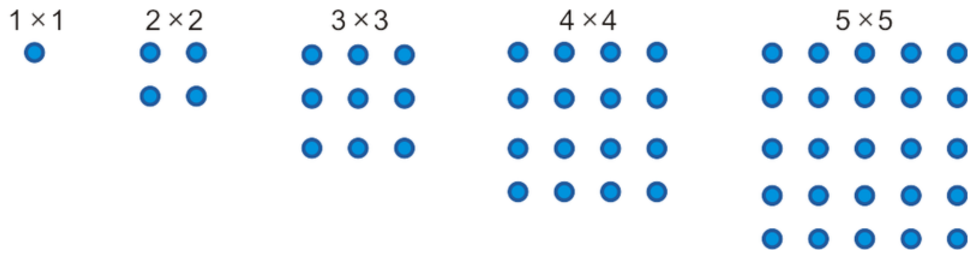
Guidance

The **square root** of a number is a number which, when multiplied by itself, gives the original number. In other words, if $a = b^2$, we say that b is the square root of a.

Note: Negative numbers and positive numbers both yield positive numbers when squared, so each positive number has both a positive and a negative square root. (For example, 3 and -3 can both be squared to yield 9.) The positive square root of a number is called the **principal square root**.

The square root of a number x is written as $\sqrt{x}$ or sometimes as $\sqrt[2]{x}$. The symbol $\sqrt{\ }$ is sometimes called a **radical sign**.

Numbers with whole-number square roots are called **perfect squares**. The first five perfect squares (1, 4, 9, 16, and 25) are shown below.



You can determine whether a number is a perfect square by looking at its prime factors. If every number in the factor tree appears an even number of times, the number is a perfect square. To find the square root of that number, simply take one of each pair of matching factors and multiply them together.

Example A

Find the principal square root of each of these perfect squares.

a) 121

b) 225

c) 324

Solution

a) $121 = 11 \times 11$, so $\sqrt{121} = 11$.

b) $225 = (5 \times 5) \times (3 \times 3)$, so $\sqrt{225} = 5 \times 3 = 15$.

c) $324 = (2 \times 2) \times (3 \times 3) \times (3 \times 3)$, so $\sqrt{324} = 2 \times 3 \times 3 = 18$.

For more practice matching numbers with their square roots, try the Flash games at http://www.quia.com/jg/65631.html.

When the prime factors don't pair up neatly, we "factor out" the ones that do pair up and leave the rest under a radical sign. We write the answer as $a\sqrt{b}$, where a is the product of half the paired factors we pulled out and b is the product of the leftover factors.

Example B

Find the principal square root of the following numbers.

a) 8

b) 48

c) 75

Solution

a) $8 = 2 \times 2 \times 2$. This gives us one pair of 2's and one leftover 2, so $\sqrt{8} = 2\sqrt{2}$.

b) $48 = (2 \times 2) \times (2 \times 2) \times 3$, so $\sqrt{48} = 2 \times 2 \times \sqrt{3}$, or $4\sqrt{3}$.

c) $75 = (5 \times 5) \times 3$, so $\sqrt{75} = 5\sqrt{3}$.

Note that in the last example we collected the paired factors first, **then** we collected the unpaired ones under a single radical symbol. Here are the four rules that govern how we treat square roots.

- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
- $A\sqrt{a} \times B\sqrt{b} = AB\sqrt{ab}$
- $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
- $\frac{A\sqrt{a}}{B\sqrt{b}} = \frac{A}{B}\sqrt{\frac{a}{b}}$

Example C

Simplify the following square root problems

a) $\sqrt{8} \times \sqrt{2}$

b) $3\sqrt{4} \times 4\sqrt{3}$

c) $\sqrt{12} \div \sqrt{3}$

d) $12\sqrt{10} \div 6\sqrt{5}$

Solution

a) $\sqrt{8} \times \sqrt{2} = \sqrt{16} = 4$

b) $3\sqrt{4} \times 4\sqrt{3} = 12\sqrt{12} = 12\sqrt{(2 \times 2) \times 3} = 12 \times 2\sqrt{3} = 24\sqrt{3}$

c) $\sqrt{12} \div \sqrt{3} = \sqrt{\frac{12}{3}} = \sqrt{4} = 2$

d) $12\sqrt{10} \div 6\sqrt{5} = \frac{12}{6}\sqrt{\frac{10}{5}} = 2\sqrt{2}$

Approximate Square Roots

Terms like $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{7}$ (square roots of prime numbers) cannot be written as **rational numbers**. That is to say, they cannot be expressed as the ratio of two integers. We call them **irrational numbers**. In decimal form, they have an unending, seemingly random, string of numbers after the decimal point.

To find approximate values for square roots, we use the $\sqrt{\ }$ or $\sqrt{x}$ button on a calculator. When the number we plug in is a perfect square, or the square of a rational number, we will get an exact answer. When the number is a non-perfect square, the answer will be irrational and will look like a random string of digits. Since the calculator can only show some of the infinitely many digits that are actually in the answer, it is really showing us an **approximate answer**—not exactly the right answer, but as close as it can get.

Example D

Use a calculator to find the following square roots. Round your answer to three decimal places.

a) $\sqrt{99}$

b) $\sqrt{5}$

c) $\sqrt{0.5}$

d) $\sqrt{1.75}$

Solution

a) ≈ 9.950

b) ≈ 2.236

c) ≈ 0.707

d) ≈ 1.323

Watch this video for help with the Examples above.

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Click image to the left for more content.

CK-12 Foundation: SquareRoots

Vocabulary

- The **square root** of a number is a number which gives the original number when multiplied by itself. In algebraic terms, for two numbers a and b, if $a = b^2$, then $b = \sqrt{a}$.
- A square root can have two possible values: a positive value called the **principal square root**, and a negative value (the opposite of the positive value).
- A **perfect square** is a number whose square root is an integer.
- Some mathematical properties of square roots are:
  - $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
  - $A\sqrt{a} \times B\sqrt{b} = AB\sqrt{ab}$
  - $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
  - $\frac{A\sqrt{a}}{B\sqrt{b}} = \frac{A}{B}\sqrt{\frac{a}{b}}$
- Square roots of numbers that are not perfect squares (or ratios of perfect squares) are **irrational numbers**. They cannot be written as rational numbers (the ratio of two integers). In decimal form, they have an unending, seemingly random, string of numbers after the decimal point.
- Computing a square root on a calculator will produce an **approximate solution** since the calculator only shows a finite number of digits after the decimal point.

Guided Practice

Find the square root of each number.

a) 576

b) 216

Solution

a) $576 = (2 \times 2) \times (2 \times 2) \times (2 \times 2) \times (3 \times 3)$, so $\sqrt{576} = 2 \times 2 \times 2 \times 3 = 24$.

b) $216 = (2 \times 2) \times 2 \times (3 \times 3) \times 3$, so $\sqrt{216} = 2 \times 3 \times \sqrt{2 \times 3}$, or $6\sqrt{6}$.

Practice

For 1-10, find the following square roots **exactly without using a calculator**, giving your answer in the simplest form.

1. $\sqrt{25}$
2. $\sqrt{24}$
3. $\sqrt{20}$
4. $\sqrt{200}$
5. $\sqrt{2000}$
6. $\sqrt{\frac{1}{4}}$ (Hint: The division rules you learned can be applied backwards!)
7. $\sqrt{\frac{9}{4}}$
8. $\sqrt{0.16}$
9. $\sqrt{0.1}$
10. $\sqrt{0.01}$

For 11-20, use a calculator to find the following square roots. Round to two decimal places.

11. $\sqrt{13}$
12. $\sqrt{99}$
13. $\sqrt{123}$
14. $\sqrt{2}$
15. $\sqrt{2000}$
16. $\sqrt{.25}$
17. $\sqrt{1.35}$
18. $\sqrt{0.37}$
19. $\sqrt{0.7}$
20. $\sqrt{0.01}$

2.10 Properties of Rational Numbers versus Irrational Numbers

Here you'll learn how to differentiate between rational and irrational numbers. You'll also learn how to classify and order real numbers.

What if you wanted to identify a number like $\sqrt{2}$? Would you classify it as rational or irrational? After completing this Concept, you'll be able to decide which category numbers like this one fall into.

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CK-12 Foundation: 0210S Irrational Numbers (H264)

Guidance

Not all square roots are irrational, but any square root that can't be reduced to a form with no radical signs in it is irrational. For example, $\sqrt{49}$ is rational because it equals 7, but $\sqrt{50}$ can't be reduced farther than $5\sqrt{2}$. That factor of $\sqrt{2}$ is irrational, making the whole expression irrational.

Example A

Identify which of the following are rational numbers and which are irrational numbers.

a) 23.7

b) 2.8956

c) π

d) $\sqrt{6}$

Solution

a) 23.7 can be written as $23\frac{7}{10}$, so it is rational.

b) 2.8956 can be written as $2\frac{8956}{10000}$, so it is rational.

c) $\pi = 3.141592654\ldots$ We know from the definition of π that the decimals do not terminate or repeat, so π is irrational.

d) $\sqrt{6} = \sqrt{2} \times \sqrt{3}$. We can't reduce it to a form without radicals in it, so it is irrational.

Repeating Decimals

Any number whose decimal representation has a finite number of digits is rational, since each decimal place can be expressed as a fraction. For example, $3.\overline{27} = 3.272727272727\ldots$ This decimal goes on forever, but it's not random; it repeats in a predictable pattern. Repeating decimals are always rational; this one can actually be expressed as $\frac{36}{11}$.

Example B

Express the following decimals as fractions.

a.) 0.439

b.) $0.25\overline{38}$

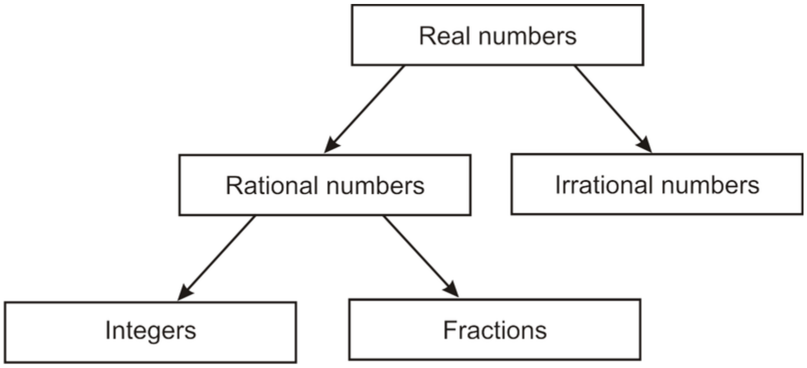
Solution:

a.) 0.439 can be expressed as $\frac{4}{10} + \frac{3}{100} + \frac{9}{1000}$, or just $\frac{439}{1000}$. Also, any decimal that repeats is rational, and can be expressed as a fraction.

b.) $0.25\overline{38}$ can be expressed as $\frac{25}{100} + \frac{38}{9900}$, which is equivalent to $\frac{2513}{9900}$.

Classify Real Numbers

We can now see how real numbers fall into one of several categories.



If a real number can be expressed as a rational number, it falls into one of two categories. If the denominator of its **simplest form** is one, then it is an **integer**. If not, it is a fraction (this term also includes decimals, since they can be written as fractions.)

If the number cannot be expressed as the ratio of two integers (i.e. as a fraction), it is **irrational**.

Example C

Classify the following real numbers.

a) 0

b) -1

c) $\frac{\pi}{3}$

d) $\frac{\sqrt{2}}{3}$

e) $\frac{\sqrt{36}}{9}$

Solution

a) Integer

b) Integer

c) Irrational (Although it's written as a fraction, π is irrational, so any fraction with π in it is also irrational.)

d) Irrational

e) Rational (It simplifies to $\frac{6}{9}$, or $\frac{2}{3}$.)

Watch this video for help with the Examples above.

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Click image to the left for more content.

CK-12 Foundation: IrrationalNumbers

Vocabulary

- The **square root** of a number is a number which gives the original number when multiplied by itself. In algebraic terms, for two numbers a and b, if $a = b^2$, then $b = \sqrt{a}$.
- A square root can have two possible values: a positive value called the **principal square root**, and a negative value (the opposite of the positive value).
- A **perfect square** is a number whose square root is an integer.
- Some mathematical properties of square roots are:
  - $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
  - $A\sqrt{a} \times B\sqrt{b} = AB\sqrt{ab}$
  - $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
  - $\frac{A\sqrt{a}}{B\sqrt{b}} = \frac{A}{B}\sqrt{\frac{a}{b}}$
- Square roots of numbers that are not perfect squares (or ratios of perfect squares) are **irrational numbers**. They cannot be written as rational numbers (the ratio of two integers). In decimal form, they have an unending, seemingly random, string of numbers after the decimal point.
- Computing a square root on a calculator will produce an **approximate solution** since the calculator only shows a finite number of digits after the decimal point.

Guided Practice

Place the following numbers in numerical order, from lowest to highest.

$\frac{100}{99} \quad \frac{\sqrt{3}}{3} \quad -\sqrt{.075} \quad \frac{2\pi}{3}$

Solution:

Since $-\sqrt{.075}$ is the only negative number, it is the smallest.

Since $100 > 99$, $\frac{100}{99} > 1$.

Since the $\sqrt{3} < s$, then $\frac{\sqrt{3}}{3} < 1$.

Since $\pi > 3$, then $\frac{\pi}{3} > 1 \Rightarrow \frac{2\pi}{3} > 2$

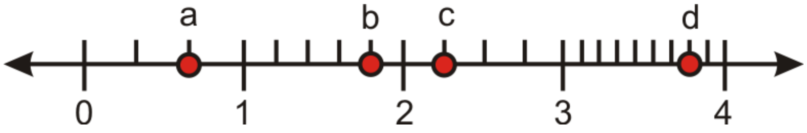
This means that the ordering is:

$-\sqrt{.075}, \frac{\sqrt{3}}{3}, \frac{100}{99}, \frac{2\pi}{3}$

Practice

For questions 1-7, classify the following numbers as an integer, a rational number or an irrational number.

1. $\sqrt{0.25}$
2. $\sqrt{1.35}$
3. $\sqrt{20}$
4. $\sqrt{25}$
5. $\sqrt{100}$
6. $\sqrt{\pi^2}$
7. $\sqrt{2 \cdot 18}$
8. Write 0.6278 as a fraction.
9. Place the following numbers in numerical order, from lowest to highest. $\frac{\sqrt{6}}{2}$ $\frac{61}{50}$ $\sqrt{1.5}$ $\frac{16}{13}$
10. Use the marked points on the number line and identify each proper fraction.



2.11 Guess and Check, Work Backward

Here you'll learn how to apply the problem-solving strategies of Guess and Check and Work Backword to solve real-world problems.

What if you wanted to find two numbers? You are told that one number is twice the other and that their sum is 48. How could you find what the two numbers are? After completing this Concept, you'll be able to use the strategies of Guess and Check and Work Backward to solve problems like this one.

Guidance

In this section, you will learn about the methods of **Guess and Check** and **Working Backwards**. These are very powerful strategies in problem solving and probably the most commonly used in everyday life. Let's review our problem-solving plan.

Step 1

Understand the problem.

Read the problem carefully. Then list all the components and data involved, and assign your variables.

Step 2

Devise a plan – Translate

Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart or construct a table.

Step 3

Carry out the plan – Solve

This is where you solve the equation you came up with in Step 2.

Step 4

Look – Check and Interpret

Check that the answer makes sense.

Let's now look at some strategies we can use as part of this plan.

Develop and Use the Strategy "Guess and Check"

The strategy for the method "Guess and Check" is to guess a solution and then plug the guess back into the problem to see if you get the correct answer. If the answer is too big or too small, make another guess that will get you closer to the goal, and continue guessing until you arrive at the correct solution. The process might sound long, but often you will find patterns that you can use to make better guesses along the way.

Here is an example of how this strategy is used in practice.

Example A

Nadia takes a ribbon that is 48 inches long and cuts it in two pieces. One piece is three times as long as the other. How long is each piece?

Solution

Step 1: Understand

We need to find two numbers that add up to 48. One number is three times the other number.

Step 2: Strategy

We guess two random numbers, one three times bigger than the other, and find the sum.

If the sum is too small we guess larger numbers, and if the sum is too large we guess smaller numbers.

Then, we see if any patterns develop from our guesses.

Step 3: Apply Strategy/Solve

$$\begin{array}{llll}
\text{Guess} & 5 \text{ and } 15 & 5+15=20 & \text{sum is too small} \\
\text{Guess} & 6 \text{ and } 18 & 6+18=24 & \text{sum is too small}
\end{array}$$

Our second guess gives us a sum that is exactly half of 48. What if we double that guess?

$$12+36=48$$

There's our answer. The pieces are 12 and 36 inches long.

Step 4: Check

$$\begin{aligned}
12+36 &= 48 && \text{The pieces add up to 48 inches.} \\
36 &= 3(12) && \text{One piece is three times as long as the other.}
\end{aligned}$$

The answer checks out.

Develop and Use the Strategy "Work Backward"

The "Work Backward" method works well for problems where a series of operations is done on an unknown number and you're only given the result. To use this method, start with the result and apply the operations in reverse order until you find the starting number.

Example B

Anne has a certain amount of money in her bank account on Friday morning. During the day she writes a check for $24.50, makes an ATM withdrawal of $80 and deposits a check for $235. At the end of the day she sees that her balance is $451.25. How much money did she have in the bank at the beginning of the day?

Step 1: Understand

We need to find the money in Anne's bank account at the beginning of the day on Friday.

She took out $24.50 and $80 and put in $235.

She ended up with $451.25 at the end of the day.

Step 2: Strategy

We start with an unknown amount, do some operations, and end up with a known amount.

We need to start with the result and apply the operations in reverse.

Step 3: Apply Strategy/Solve

Start with $451.25. Subtract $235, add $80, and then add $24.50.

$451.25 - 235 + 80 + 24.50 = 320.75$

Anne had \$320.75 in her account at the beginning of the day on Friday.

Step 4: Check

Anne starts with	\$320.75
She writes a check for \$24.50.	$\$320.75 - 24.50 = \$296.25$
She withdraws \$80.	$\$296.25 - 80 = \$216.25$
She deposits \$235.	$\$216.25 + 235 = \$451.25$

The answer checks out.

Plan and Compare Alternative Approaches to Solving Problems

Most word problems can be solved in more than one way. Often one method is more straightforward than others, but which method is best can depend on what kind of problem you are facing.

Example C

Nadia's father is 36. He is 16 years older than four times Nadia's age. How old is Nadia?

Solution

This problem can be solved with either of the strategies you learned in this section. Let's solve it using both strategies.

Guess and Check Method

Step 1: Understand

We need to find Nadia's age.

We know that her father is 16 years older than four times her age, or $4 \times (\text{Nadia's age}) + 16$.

We know her father is 36 years old.

Step 2: Strategy

We guess a random number for Nadia's age.

We multiply the number by 4 and add 16 and check to see if the result equals 36.

If the answer is too small, we guess a larger number, and if the answer is too big, we guess a smaller number.

We keep guessing until we get the answer to be 36.

Step 3: Apply strategy/Solve

Guess Nadia's age	10	$4(10) + 16 = 56$	too big for her father's age
Guess a smaller number	9	$4(9) + 16 = 52$	still too big

Guessing 9 for Nadia's age gave us a number that is 16 years too great to be her father's age. But notice that when we decreased Nadia's age by one, her father's age decreased by four. That suggests that we can decrease our final answer by 16 years if we decrease our guess by 4 years.

4 years less than 9 is 5. $4(5) + 16 = 36$, which is the right age.

Answer: Nadia is 5 years old.

Step 4: Check

Nadia is 5 years old. Her father's age is $4(5)+16=36$. This is correct. **The answer checks out.**

You will be asked to try the Work Backward method in the Guided Practice section.

Watch this video for help with the Examples above.

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CK-12 Foundation: Problem Solving Strategies- Guess and Check and Work Backwards

Vocabulary

The four steps of the **problem solving plan** are:

- Understand the problem
- Devise a plan – Translate
- Carry out the plan – Solve
- Look – Check and Interpret

Two common problem solving strategies are:

Guess and Check

Guess a solution and use the guess in the problem to see if you get the correct answer. If the answer is too big or too small, then make another guess that will get you closer to the goal.

Work Backward

This method works well for problems in which a series of operations is applied to an unknown quantity and you are given the resulting number. Start with the result and apply the operations in reverse order until you find the unknown.

Guided Practice

Nadia's father is 36. He is 16 years older than four times Nadia's age. Determine Nadia's age by using the Work Backward method.

Solution:

Work Backward Method

Step 1: Understand

We need to find Nadia's age.

We know her father is 16 years older than four times her age, or $4\times(\text{Nadia's age})+16$.

We know her father is 36 years old.

Step 2: Strategy

To get from Nadia's age to her father's age, we multiply Nadia's age by four and add 16.

Working backwards means we start with the father's age, subtract 16 and divide by 4.

Step 3: Apply Strategy/Solve

Start with the father's age	36
Subtract 16	$36-16=20$
Divide by 4	$20 \div 4 = 5$

Answer Nadia is 5 years old.

Step 4: Check

Nadia is 5 years old. Her father's age is $4(5)+16=36$. This is correct. **The answer checks out.**

You see that in this problem, the "Work Backward" strategy is more straightforward than the Guess and Check method. The Work Backward method always works best when we know the result of a series of operations, but not the starting number. In the next chapter, you will learn algebra methods based on the Work Backward method.

Practice

1. Nadia is at home and Peter is at school which is 6 miles away from home. They start traveling towards each other at the same time. Nadia is walking at 3.5 miles per hour and Peter is skateboarding at 6 miles per hour. When will they meet and how far from home is their meeting place?
2. Peter bought several notebooks at Staples for \$2.25 each; then he bought a few more notebooks at Rite-Aid for \$2 each. He spent the same amount of money in both places and he bought 17 notebooks in all. How many notebooks did Peter buy in each store?
3. Andrew took a handful of change out of his pocket and noticed that he was only holding dimes and quarters in his hand. He counted and found that he had 22 coins that amounted to \$4. How many quarters and how many dimes does Andrew have?
4. Anne wants to put a fence around her rose bed that is one and a half times as long as it is wide. She uses 50 feet of fencing. What are the dimensions of the garden?
5. Peter is outside looking at the pigs and chickens in the yard. Nadia is indoors and cannot see the animals. Peter gives her a puzzle. He tells her that he can see 13 heads and 36 feet and asks her how many pigs and how many chickens are in the yard. Help Nadia find the answer.
6. Andrew invests \$8000 in two types of accounts: a savings account that pays 5.25% interest per year and a more risky account that pays 9% interest per year. At the end of the year he has \$450 in interest from the two accounts. Find the amount of money invested in each account.
7. 450 tickets are sold for a concert: balcony seats for \$35 each and orchestra seats for \$25 each. If the total box office take is \$13,000, how many of each kind of ticket were sold?
8. There is a bowl of candy sitting on our kitchen table. One morning Nadia takes one-sixth of the candy. Later that morning Peter takes one-fourth of the candy that's left. That afternoon, Andrew takes one-fifth of what's left in the bowl and finally Anne takes one-third of what is left in the bowl. If there are 16 candies left in the bowl at the end of the day, how much candy was there at the beginning of the day?
9. Nadia can completely mow the lawn by herself in 30 minutes. Peter can completely mow the lawn by himself in 45 minutes. How long does it take both of them to mow the lawn together?
10. Three monkeys spend a day gathering coconuts together. When they have finished, they are very tired and fall asleep. The following morning, the first monkey wakes up. Not wishing to disturb his friends, he decides to divide the coconuts into three equal piles. There is one left over, so he throws this odd one away, helps himself to his share, and goes home. A few minutes later, the second monkey awakes. Not realizing that the first has already gone, he too divides the coconuts into three equal heaps. He finds one left over, throws the odd one away, helps himself to his fair share, and goes home. In the morning, the third monkey wakes to find that he is alone. He spots the two discarded coconuts, and puts them with the pile, giving him a total of twelve coconuts.
    a. How many coconuts did the first two monkeys take?

   b. How many coconuts did the monkeys gather in all?

11. Two prime numbers have a product of 51. What are the numbers?
12. Two prime numbers have a product of 65. What are the numbers?
13. The square of a certain positive number is eight more than twice the number. What is the number?
14. Is 91 prime? (Hint: if it's *not* prime, what are its prime factors?)
15. Is 73 prime?
16. Alison's school day starts at 8:30, but today Alison wants to arrive ten minutes early to discuss an assignment with her English teacher. If she is also giving her friend Sherice a ride to school, and it takes her 12 minutes to get to Sherice's house and another 15 minutes to get to school from there, at what time does Alison need to leave her house?
17. At her retail job, Kelly gets a raise of 10% every six months. After her third raise, she now makes $13.31 per hour. How much did she make when she first started out?
18. Three years ago, Kevin's little sister Becky had her fifth birthday. If Kevin was eight when Becky was born, how old is he now?
19. A warehouse is full of shipping crates; half of them are headed for Boston and the other half for Philadelphia. A truck arrives to pick up 20 of the Boston-bound crates, and then another truck carries away one third of the Philadelphia-bound crates. An hour later, half of the remaining crates are moved onto the loading dock outside. If there are 40 crates left in the warehouse, how many were there originally?
20. Gerald is a bus driver who takes over from another bus driver one day in the middle of his route. He doesn't pay attention to how many passengers are on the bus when he starts driving, but he does notice that three passengers get off at the next stop, a total of eight more get on at the next three stops, two get on and four get off at the next stop, and at the stop after that, a third of the passengers get off.
    a. If there are now 14 passengers on the bus, how many were there when Gerald first took over the route?
    b. If half the passengers who got on while Gerald was driving paid the full adult fare of $1.50, and the other half were students or seniors who paid a discounted fare of $1.00, how much cash was in the bus's fare box at the beginning of Gerald's shift if there is now $73.50 in it?
    c. When Gerald took over the route, all the passengers currently on the bus had paid full fare. However, some of the passengers who had previously gotten on and off the bus were students or seniors who had paid the discounted fare. Based on the amount of money that was in the cash box, if 28 passengers had gotten on the bus and gotten off before Gerald arrived (in addition to the passengers who had gotten on and were still there when he arrived), how many of those passengers paid the discounted fare?
    d. How much money would currently be in the cash box if all the passengers throughout the day had paid the full fare?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9612.

Summary

This chapter explains the difference between integers, rational numbers, and irrational numbers. It then focuses on adding, subtracting, multiplying, and dividing rational numbers. It also covers number opposites, absolute values, and the distributive property. Square roots are discussed as well. Finally, it introduces you to many real-world problems and provides the strategies of guess and check and working backward to solve them.

CHAPTER 3 Equations of Lines

Chapter Outline

3.1 ONE-STEP EQUATIONS AND INVERSE OPERATIONS
3.2 ONE-STEP EQUATIONS TRANSFORMED BY MULTIPLICATION/DIVISION
3.3 APPLICATIONS OF ONE-STEP EQUATIONS
3.4 TWO-STEP EQUATIONS AND PROPERTIES OF EQUALITY
3.5 MULTI-STEP EQUATIONS WITH LIKE TERMS
3.6 SOLVING REAL-WORLD PROBLEMS WITH TWO-STEP EQUATIONS
3.7 MULTI-STEP EQUATIONS
3.8 SOLVING REAL-WORLD PROBLEMS USING MULTI-STEP EQUATIONS
3.9 EQUATIONS WITH VARIABLES ON BOTH SIDES
3.10 RATIOS
3.11 PROPORTIONS
3.12 SCALE AND INDIRECT MEASUREMENT APPLICATIONS
3.13 CONVERSION OF DECIMALS, FRACTIONS, AND PERCENT
3.14 PERCENT EQUATIONS
3.15 PERCENT OF CHANGE

Introduction

Solving equations is one of the most important mathematical skills you will learn. And in order to solve a equation, you'll need to manipulate it to isolate its variable. That isolation may require the application of one mathematical operation or rule, two operations/rules, or more than two operations/rules.

Some day-to-day situations in which you might have to use equations are figuring out the amount a cab ride will cost you, determining the final cost of a restaurant bill with the tip included, and converting temperature from Celsius to Fahrenheit and vice versa.

In this chapter, you will learn how to apply math operations and rules to linear equations so that you can solve for a particular variable. You will also be introduced to ratio, proportion, percentage, and percent change problems.

3.1 One-Step Equations and Inverse Operations

Here you'll learn how to use addition and subtraction to solve algebraic equations for their unknown variable.

What if you had an algebraic equation involving additon or subtraction like $x-\frac{2}{5}=\frac{5}{8}$? How could you solve it for the unknown variable x? After completing this Concept, you'll be able to solve equations like this one.

Watch This

MEDIA

Click image to the left for more content.

CK-12 Foundation: 0301S Solving Equations with Addition and Subtraction (H264)

Guidance

Nadia is buying a new mp3 player. Peter watches her pay for the player with a \$100 bill. She receives \$22.00 in change, and from only this information, Peter works out how much the player cost. How much was the player?

In algebra, we can solve problems like this using an **equation**. An **equation** is an algebraic expression that involves an **equals** sign. If we use the letter x to represent the cost of the mp3 player, we can write the equation $x+22=100$. This tells us that the value of the player **plus** the value of the change received is **equal** to the \$100 that Nadia paid.

Another way we could write the equation would be $x=100-22$. This tells us that the value of the player is **equal** to the total amount of money Nadia paid $(100-22)$. This equation is mathematically equivalent to the first one, but it is easier to solve.

In this chapter, we will learn how to solve for the variable in a one-variable linear equation. **Linear equations** are equations in which each term is either a constant, or a constant times a single variable (raised to the first power). The term linear comes from the word line, because the graph of a linear equation is always a line.

We'll start with simple problems like the one in the last example.

Solving Equations Using Addition and Subtraction

When we work with an algebraic equation, it's important to remember that the two sides have to stay equal for the equation to stay true. We can change the equation around however we want, but whatever we do to one side of the equation, we have to do to the other side. In the introduction above, for example, we could get from the first equation to the second equation by subtracting 22 from both sides:

$$\begin{aligned} x+22&=100\\ x+22-22&=100-22\\ x&=100-22 \end{aligned}$$

Similarly, we can add numbers to each side of an equation to help solve for our unknown.

Example A

Solve $x-3=9$.

Solution

To solve an equation for x, we need to **isolate** $x-$that is, we need to get it by itself on one side of the equals sign. Right now our x has a 3 subtracted from it. To reverse this, we'll add 3—but we must add 3 to **both sides.**

$$\begin{aligned} x-3&=9\\ x-3+3&=9+3\\ x+0&=9+3\\ x&=12 \end{aligned}$$

Example B

Solve $z-9.7=-1.026$

Solution

It doesn't matter what the variable is—the solving process is the same.

$$\begin{aligned} z-9.7&=-1.026\\ z-9.7+9.7&=-1.026+9.7\\ z&=8.674 \end{aligned}$$

Make sure you understand the addition of decimals in this example!

Example C

Solve $x+\frac{4}{7}=\frac{9}{5}$.

Solution

To isolate x, we need to subtract $\frac{4}{7}$ from both sides.

$$\begin{aligned} x+\frac{4}{7}&=\frac{9}{5}\\ x+\frac{4}{7}-\frac{4}{7}&=\frac{9}{5}-\frac{4}{7}\\ x&=\frac{9}{5}-\frac{4}{7} \end{aligned}$$

Now we have to subtract fractions, which means we need to find the LCD. Since 5 and 7 are both prime, their lowest common multiple is just their product, 35.

$$\begin{aligned}
x &= \frac{9}{5} - \frac{4}{7} \\
x &= \frac{7 \cdot 9}{7 \cdot 5} - \frac{4 \cdot 5}{7 \cdot 5} \\
x &= \frac{63}{35} - \frac{20}{35} \\
x &= \frac{63 - 20}{35} \\
x &= \frac{43}{35}
\end{aligned}$$

Make sure you're comfortable with decimals and fractions! To master algebra, you'll need to work with them frequently.

Watch this video for help with the Examples above.

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CK-12 Foundation: Solving Equations withAddition and Subtraction

Vocabulary

- An equation in which each term is either a constant or the product of a constant and a single variable is a **linear equation**.
- We can add, subtract, multiply, or divide both sides of an equation by the same value and still have an **equivalent equation**.
- To solve an equation, **isolate** the unknown variable on one side of the equation by applying one or more arithmetic operations to both sides.

Guided Practice

Solve $x + 10 = 17$.

Solution

To solve an equation for x, we need to **isolate** x—that is, we need to get it by itself on one side of the equals sign. Right now our x has 10 added to it. To reverse this, we'll subtract 10—but we must subtract 10 to **both sides.**

$$\begin{aligned}
x + 10 &= 17 \\
x + 10 - 10 &= 17 - 10 \\
x + 0 &= 17 - 10 \\
x &= 7
\end{aligned}$$

Practice

For 1-5, solve the following equations for x.

1. $x-11=7$
2. $x-1.1=3.2$
3. $x+0.257=1$
4. $x+\frac{5}{2}=\frac{2}{3}$
5. $x-\frac{5}{6}=\frac{3}{8}$

For 6-10, solve the following equations for the unknown variable.

6. $q-13=-13$
7. $z+1.1=3.0001$
8. $r+1=\frac{2}{5}$
9. $t+\frac{1}{2}=\frac{1}{3}$
10. $\frac{3}{4}=-\frac{1}{2}-y$

3.2 One-Step Equations Transformed by Multiplication/Division

Here you'll learn how to use multiplication and division to solve algebraic equations for their unknown variable.

What if you had an algebraic equation involving multiplication or division like $-5x = 3$? How could you solve it for the unknown variable x? After completing this Concept, you'll be able to solve equations like this one.

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CK-12 Foundation: 0302S Solving Equations with Multiplication and Division (H264)

Guidance

Suppose you are selling pizza for \$1.50 a slice and you can get eight slices out of a single pizza. How much money do you get for a single pizza? It shouldn't take you long to figure out that you get $8 \times \$1.50 = \$12.00$. You solved this problem by multiplying. Here's how to do the same thing algebraically, using x to stand for the cost in dollars of the whole pizza.

Example A

Solve $\frac{1}{8} \cdot x = 1.5$.

Our x is being multiplied by one-eighth. To cancel that out and get x by itself, we have to multiply by the reciprocal, 8. Don't forget to multiply **both sides** of the equation.

$$8\left(\frac{1}{8} \cdot x\right) = 8(1.5)$$
$$x = 12$$

Example B

Solve $0.25x = 5.25$.

0.25 is the decimal equivalent of one fourth, so to cancel out the 0.25 factor we would multiply by 4.

$$\begin{aligned} 4(0.25x) &= 4(5.25) \\ x &= 21 \end{aligned}$$

Solving by division is another way to isolate x. Suppose you buy five identical candy bars, and you are charged \$3.25. How much did each candy bar cost? You might just divide \$3.25 by 5, but let's see how this problem looks in algebra.

Example C

Solve $5x = 3.25$.

To cancel the 5, we divide both sides by 5.

$$\begin{aligned} \frac{5x}{5} &= \frac{3.25}{5} \\ x &= 0.65 \end{aligned}$$

Example D

Solve $1.375x = 1.2$.

Divide by 1.375

$$\begin{aligned} x &= \frac{1.2}{1.375} \\ x &= 0.8\overline{72} \end{aligned}$$

Notice the bar above the final two decimals; it means that those digits recur, or repeat. The full answer is 0.872727272727272....

To see more examples of one - and two-step equation solving, watch the Khan Academy video series starting at http://www.youtube.com/watch?v=bAerID24QJ0.

Watch this video for help with the Examples above.

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CK-12 Foundation: Solving Equations withMultiplication and Division

Vocabulary

- An equation in which each term is either a constant or the product of a constant and a single variable is a **linear equation**.
- We can add, subtract, multiply, or divide both sides of an equation by the same value and still have an **equivalent equation**.
- To solve an equation, **isolate** the unknown variable on one side of the equation by applying one or more arithmetic operations to both sides.

Guided Practice

Solve:

a) $\frac{9x}{5} = 5$.

b) $7x = \frac{5}{11}$.

Solutions:

a) $\frac{9x}{5}$ is equivalent to $\frac{9}{5} \cdot x$, so to cancel out that $\frac{9}{5}$, we multiply by the reciprocal, $\frac{5}{9}$.

$$\frac{5}{9}\left(\frac{9x}{5}\right) = \frac{5}{9}(5)$$
$$x = \frac{25}{9}$$

b) Divide both sides by 7.

$$x = \frac{5}{11.7}$$
$$x = \frac{5}{77}$$

Practice

For 1-5, solve the following equations for x.

1. $7x = 21$
2. $4x = 1$
3. $\frac{5x}{12} = \frac{2}{3}$
4. $0.01x = 11$
5. $\frac{-2x}{9} = \frac{10}{3}$

For 6-10, solve the following equations for the unknown variable.

6. $21s = 3$
7. $-7a = -5$
8. $\frac{7f}{11} = \frac{7}{11}$
9. $6r = \frac{3}{8}$
10. $\frac{9b}{16} = \frac{3}{8}$

3.3 Applications of One-Step Equations

Here you'll learn how to apply arithmetic operations to write and solve real-world one-step equations.

What if The Perfect Pizza charges \$1.50 for a slice of pizza? However, the restaurant offers a \$2.00 discount off the per-slice cost if you buy a whole pizza. A whole pizza costs \$10. How could you find how many slices are in a whole pizza? After completing this Concept, you'll be able to solve real-world problems like this one.

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CK-12 Foundation: 0303S Solving Real-World Problems Using Equations (H264)

Guidance

Let's use the skills we learned in the last two concepts, to solve applications of one-step problems.

Example A

In the year 2017, Anne will be 45 years old. In what year was Anne born?

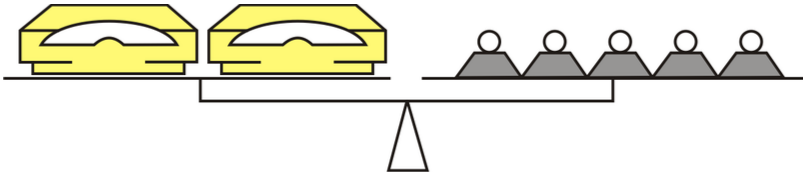
The unknown here is the year Anne was born, so that's our variable x. Here's our equation:

$$\begin{aligned}x+45&=2017\\x+45-45&=2017-45\\x&=1972\end{aligned}$$

Anne was born in 1972.

Example B

A mail order electronics company stocks a new mini DVD player and is using a balance to determine the shipping weight. Using only one-pound weights, the shipping department found that the following arrangement balances:



How much does each DVD player weigh?

Solution

Since the system balances, the total weight on each side must be equal. To write our equation, we'll use x for the weight of one DVD player, which is unknown. There are two DVD players, weighing a total of $2x$ pounds, on the left side of the balance, and on the right side are 5 1-pound weights, weighing a total of 5 pounds. So our equation is $2x = 5$. Dividing both sides by 2 gives us $x = 2.5$.

Each DVD player weighs 2.5 pounds.

Example C

In 2004, Takeru Kobayashi of Nagano, Japan, ate 53.5 hot dogs in 12 minutes. This was 3 more hot dogs than his own previous world record, set in 2002. Calculate how many minutes it took him to eat one hot dog.

Solution

We know that the total time for 53.5 hot dogs is 12 minutes. We want to know the time for one hot dog, so that's x. Our equation is $53.5x = 12$. Then we divide both sides by 53.5 to get $x = \frac{12}{53.5}$, or $x = 0.224 \, minutes$.

We can also multiply by 60 to get the time in seconds; 0.224 minutes is about 13.5 seconds. So that's how long it took Takeru to eat one hot dog.

Watch this video for help with the Examples above.

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CK-12 Foundation: Solving Real WorldProblems Using Equations

Vocabulary

- An equation in which each term is either a constant or the product of a constant and a single variable is a **linear equation**.
- We can add, subtract, multiply, or divide both sides of an equation by the same value and still have an **equivalent equation.**
- To solve an equation, **isolate** the unknown variable on one side of the equation by applying one or more arithmetic operations to both sides.

Guided Practice

Calculate the following, using the problem from Example C:

a) *How many hot dogs he ate per minute.*

b) *What his old record was.*

Solution:

a) For this questions we're looking for hot dogs per minute instead of minutes per hot dog, as in Example C. We'll use the variable y instead of x this time so we don't get the two confused. 12 minutes, times the number of hot dogs per minute, equals the total number of hot dogs, so $12y = 53.5$. Dividing both sides by 12 gives us $y = \frac{53.5}{12}$, or $y = 4.458$ hot dogs per minute.

b) We know that his new record is 53.5, and we know that's three more than his old record. If we call his old record z, we can write the following equation: $z + 3 = 53.5$. Subtracting 3 from both sides gives us $z = 50.5$. So Takeru's old record was 50.5 hot dogs in 12 minutes.

Practice

Peter is collecting tokens on breakfast cereal packets in order to get a model boat. In eight weeks he has collected 10 tokens. He needs 25 tokens for the boat. Write an equation and determine the following information.

1. How many more tokens he needs to collect, n.
2. How many tokens he collects per week, w.
3. How many more weeks remain until he can send off for his boat, r.

Juan has baked a cake and wants to sell it in his bakery. He is going to cut it into 12 slices and sell them individually. He wants to sell it for three times the cost of making it. The ingredients cost him $8.50, and he allowed $1.25 to cover the cost of electricity to bake it. Write equations that describe the following statements.

4. The amount of money that he sells the cake for (u).
5. The amount of money he charges for each slice (c).
6. The total profit he makes on the cake (w).

Jane is baking cookies for a large party. She has a recipe that will make one batch of two dozen cookies, and she decides to make five batches. To make five batches, she finds that she will need 12.5 cups of flour and 15 eggs.

7. How many cookies will she make in all?
8. How many cups of flour go into one batch?
9. How many eggs go into one batch?
10. If Jane only has a dozen eggs on hand, how many more does she need to make five batches?
11. If she doesn't go out to get more eggs, how many batches can she make? How many cookies will that be?

3.4 Two-Step Equations and Properties of Equality

Here you'll learn how to apply arithmetic operations to write and solve real-world two-step equations.

What if you had an algebraic equation like $7x+5=40$ that required two operations (in this case subtraction and then division) to solve it? After completing this Concept, you'll be able to solve two-step equations like this one.

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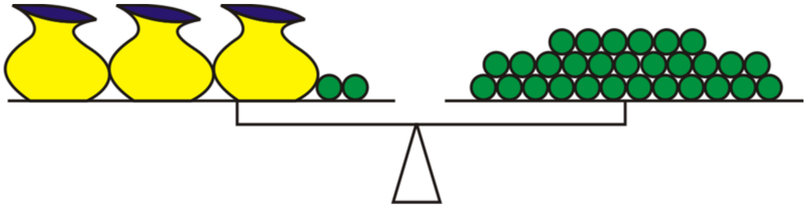
CK-12 Foundation: 0304S Solve Two-Step Equations(H264)

Guidance

We've seen how to solve for an unknown by isolating it on one side of an equation and then evaluating the other side. Now we'll see how to solve equations where the variable takes more than one step to isolate.

Example A

Rebecca has three bags containing the same number of marbles, plus two marbles left over. She places them on one side of a balance. Chris, who has more marbles than Rebecca, adds marbles to the other side of the balance. He finds that with 29 marbles, the scales balance. How many marbles are in each bag? Assume the bags weigh nothing.



Solution

We know that the system balances, so the weights on each side must be equal. If we use x to represent the number of marbles in each bag, then we can see that on the left side of the scale we have three bags (each containing x marbles) plus two extra marbles, and on the right side of the scale we have 29 marbles. The balancing of the scales is similar to the balancing of the following equation.

$$3x+2=29$$

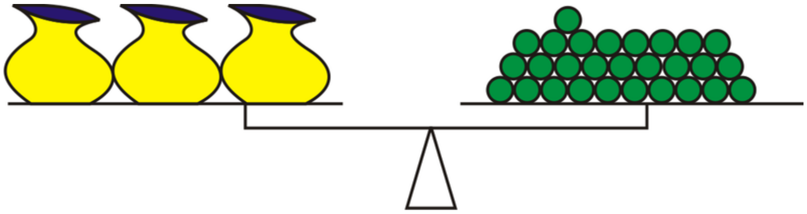
"Three bags plus two marbles **equals** 29 marbles"

To solve for x, we need to first get all the variables (terms containing an x) alone on one side of the equation. We've already got all the x's on one side; now we just need to isolate them.

$$\begin{aligned}
3x+2 &= 29 \\
3x+2-2 &= 29-2 && \text{Get rid of the 2 on the left by subtracting it from both sides.} \\
3x &= 27 \\
\frac{3x}{3} &= \frac{27}{3} && \text{Divide both sides by 3.} \\
x &= 9
\end{aligned}$$

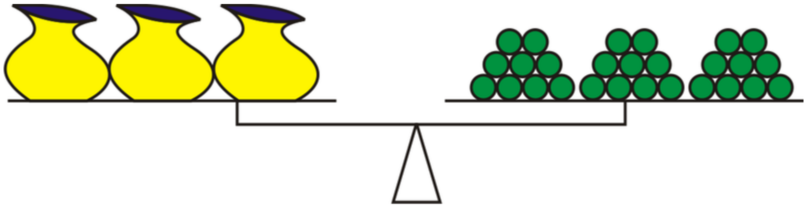
There are nine marbles in each bag.

We can do the same with the real objects as we did with the equation. Just as we subtracted 2 from both sides of the equals sign, we could remove two marbles from each side of the scale. Because we removed the same number of marbles from each side, we know the scales will still balance.



Then, because there are three bags of marbles on the left-hand side of the scale, we can divide the marbles on the right-hand side into three equal piles. You can see that there are nine marbles in each.

Three bags of marbles ***balances*** *three piles of nine marbles.*



So each bag of marbles balances nine marbles, meaning that each bag contains nine marbles.

Check out http://www.mste.uiuc.edu/pavel/java/balance/ for more interactive balance beam activities!

Example B

Solve $6(x+4) = 12$.

This equation has the x buried in parentheses. To dig it out, we can proceed in one of two ways: we can either distribute the six on the left, or divide both sides by six to remove it from the left. Since the right-hand side of the equation is a multiple of six, it makes sense to divide. That gives us $x+4=2$. Then we can subtract 4 from both sides to get $x=-2$.

Example C

Solve $\frac{x-3}{5} = 7$.

It's always a good idea to get rid of fractions first. Multiplying both sides by 5 gives us $x-3=35$, and then we can add 3 to both sides to get $x=38$.

Watch this video for help with the Examples above.

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CK-12 Foundation: Solve Two-StepEquations

Vocabulary

- Terms with the same variable in them (or no variable in them) are **like terms. Combine like terms** (adding or subtracting them from each other) to simplify the expression and solve for the unknown.

Guided Practice

Solve $\frac{5}{9}(x+1)=\frac{2}{7}$.

Solution:

First, we'll cancel the fraction on the left by multiplying by the reciprocal (the multiplicative inverse).

$$\begin{aligned}\frac{9}{5}\cdot\frac{5}{9}(x+1)&=\frac{9}{5}\cdot\frac{2}{7}\\(x+1)&=\frac{18}{35}\end{aligned}$$

Then we subtract 1 from both sides. ($\frac{35}{35}$ is equivalent to 1.)

$$\begin{aligned}x+1&=\frac{18}{35}\\x+1-1&=\frac{18}{35}-\frac{35}{35}\\x&=\frac{18-35}{35}\\x&=\frac{-17}{35}\end{aligned}$$

These examples are called **two-step equations**, because we need to perform two separate operations on the equation to isolate the variable.

Practice

Solve the following equations for the unknown variable.

1. $6x-1.3=3.2$

2. $4(x+3) = 1$
3. $5q-7 = \frac{2}{3}$
4. $\frac{3}{5}x+\frac{5}{2} = \frac{2}{3}$
5. $0.1y+11 = 0$
6. $\frac{5q-7}{12} = \frac{2}{3}$
7. $\frac{5(q-7)}{12} = \frac{2}{3}$
8. $33t-99 = 0$
9. $5p-2 = 32$
10. $10y+5 = 10$
11. $10(y+5) = 10$

3.5 Multi-Step Equations with Like Terms

Here you'll learn how to add and subtract like terms to simplify two-step equations and solve for their unknown.

What if you had an equation in which the same variable appeared twice, like $2(x-4)+4x=-23$? How could you simplify the equation so that the variable appears only once in order to solve for it? After completing this Concept, you'll be able to combine like terms to solve two-step equations like this one.

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CK-12 Foundation: 0305S Combining Like Termsin Two-StepEquations

Guidance

When we look at a linear equation we see two kinds of terms: those that contain the unknown variable, and those that don't. When we look at an equation that has an x on both sides, we know that in order to solve it, we need to get all the $x-$terms on one side of the equation. This is called **combining like terms**. The terms with an x in them are **like terms** because they contain the same variable (or, as you will see in later chapters, the same combination of variables).

TABLE 3.1:

Like Terms	Unlike Terms
$4x, 10x, -3.5x$, and $\frac{x}{12}$	$3x$ and $3y$
$3y, 0.000001y$, and y	$4xy$ and $4x$
$xy, 6xy$, and $2.39xy$	$0.5x$ and 0.5

Example A

To add or subtract like terms, we can use the Distributive Property of Multiplication.

$$\begin{aligned}3x+4x&=(3+4)x=7x\\0.03xy-0.01xy&=(0.03-0.01)xy=0.02xy\\-y+16y+5y&=(-1+16+5)y=10y\\5z+2z-7z&=(5+2-7)z=0z=0\end{aligned}$$

To solve an equation with two or more like terms, we need to combine the terms first.

Example B

Solve $(x+5)-(2x-3)=6$.

There are two like terms: the x and the $-2x$ (don't forget that the negative sign applies to everything in the parentheses). So we need to get those terms together. The associative and distributive properties let us rewrite the equation as $x+5-2x+3=6$, and then the commutative property lets us switch around the terms to get $x-2x+5+3=6$, or $(x-2x)+(5+3)=6$.

$(x-2x)$ is the same as $(1-2)x$, or $-x$, so our equation becomes $-x+8=6$

Subtracting 8 from both sides gives us $-x=-2$.

And finally, multiplying both sides by -1 gives us $x=2$.

Example C

Solve $\frac{x}{2}-\frac{x}{3}=6$.

This problem requires us to deal with fractions. We need to write all the terms on the left over a common denominator of six.

$$\frac{3x}{6}-\frac{2x}{6}=6$$

Then we subtract the fractions to get $\frac{x}{6}=6$.

Finally we multiply both sides by 6 to get $x=36$.

Watch this video for help with the Examples above.

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CK-12 Foundation: Combing Like Termsin 2 Step Equations

Vocabulary

- Terms with the same variable in them (or no variable in them) are **like terms. Combine like terms** (adding or subtracting them from each other) to simplify the expression and solve for the unknown.

Guided Practice

Solve $\frac{2x}{5}-\frac{3x}{2}=11$.

This problem requires us to deal with fractions. We need to write all the terms on the left over a common denominator of ten.

$$\frac{4x}{10}-\frac{15x}{10}=11$$

Then we subtract the fractions to get $-\frac{11x}{10} = 11$.

Finally we multiply both sides by $-\frac{10}{11}$:

$-\frac{11x}{10} \cdot -\frac{10}{11} = 11 \cdot -\frac{10}{11}$

to get $x = -10$.

Practice

Solve the following equations for the unknown variable.

1. $1.3x - 0.7x = 12$
2. $-10a - 2(a+5) = 14$
3. $5(2y - 3y) = -20$
4. $\frac{2}{3}x - \frac{1}{5}x = \frac{14}{15}$
5. $5x - (3x+2) = 1$
6. $s - \frac{3s}{8} = \frac{5}{6}$
7. $10(y+5y) = 10$
8. $2.3x + 2(0.75x - 3.5) = 7.5$
9. $3(x+2) + 5(2-x) = -32$
10. $6x + 2(5x-2) = 12$

3.6 Solving Real-World Problems with Two-Step Equations

Here you'll learn how to translate words into to two-step equations. You'll then solve such equations for their unknown variable.

What if Marisha's target heart rate is 0.70 times the quantity 220 minus her age. Her target heart rate is 140. What is her age? After completing this Concept, you'll be able to solve real-world applications like this one that involve two-step equations.

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CK-12 Foundation: 0306S Solving RealWorld Problems withTwoStep Equations (H264)

Guidance

The hardest part of solving word problems is translating from words to an equation. First, you need to look to see what the equation is asking. What is the **unknown** for which you have to solve? That will be what your **variable** stands for. Then, follow what is going on with your variable all the way through the problem.

Example A

An emergency plumber charges $65 as a call-out fee plus an additional $75 per hour. He arrives at a house at 9:30 and works to repair a water tank. If the total repair bill is $196.25, at what time was the repair completed?

In order to solve this problem, we collect the information from the text and convert it to an equation.

Unknown: time taken in hours – this will be our x

The bill is made up of two parts: a call out fee and a per-hour fee. The call out is a flat fee, and independent of x—it's the same no matter how many hours the plumber works. The per-hour part depends on the number of hours (x). So the total fee is $65 (no matter what) plus 75x$ (where x is the number of hours), or $65+75x$.

Looking at the problem again, we also can see that the total bill is $196.25. So our final equation is $196.25 = 65+75x$.

Solving for x:

$$\begin{aligned} 196.25 &= 65+75x && \text{Subtract 65 from both sides.} \\ 131.25 &= 75x && \text{Divide both sides by 75.} \\ 1.75 &= x && \text{The job took 1.75 hours.} \end{aligned}$$

Solution

The repair job was completed 1.75 hours after 9:30, so it was completed at 11:15AM.

Example B

When Asia was young her Daddy marked her height on the door frame every month. Asia's Daddy noticed that between the ages of one and three, he could predict her height (in inches) by taking her age in months, adding 75 inches and multiplying the result by one-third. Use this information to answer the following:

a) *Write an equation linking her predicted height, h, with her age in months, m.*

b) *Determine her predicted height on her second birthday.*

c) *Determine at what age she is predicted to reach three feet tall.*

Solution

a) To convert the text to an equation, first determine the type of equation we have. We are going to have an equation that links **two variables**. Our unknown will change, depending on the information we are given. For example, we could solve for height given age, or solve for age given height. However, the text gives us a way to determine **height**. Our equation will start with "$h =$".

The text tells us that we can predict her height by taking her age in months, adding 75, and multiplying by $\frac{1}{3}$. So our equation is $h = (m+75)\cdot\frac{1}{3}$, or $h = \frac{1}{3}(m+75)$.

b) To predict Asia's height on her second birthday, we substitute $m = 24$ into our equation (because 2 years is 24 months) and solve for h.

$$\begin{aligned} h &= \frac{1}{3}(24+75) \\ h &= \frac{1}{3}(99) \\ h &= 33 \end{aligned}$$

Asia's height on her second birthday was predicted to be 33 inches.

c) To determine the predicted age when she reached three feet, substitute $h = 36$ into the equation and solve for m.

$$\begin{aligned} 36 &= \frac{1}{3}(m+75) \\ 108 &= m+75 \\ 33 &= m \end{aligned}$$

Asia was predicted to be 33 months old when her height was three feet.

Example C

To convert temperatures in Fahrenheit to temperatures in Celsius, follow the following steps: Take the temperature in degrees Fahrenheit and subtract 32. Then divide the result by 1.8 and this gives the temperature in degrees Celsius.

a) *Write an equation that shows the conversion process.*

b) *Convert 50 degrees Fahrenheit to degrees Celsius.*

a) The text gives the process to convert Fahrenheit to Celsius. We can write an equation using two variables. We will use f for temperature in Fahrenheit, and c for temperature in Celsius.

First we take the temperature in Fahrenheit and subtract 32.	$f-32$
Then divide by 1.8.	$\frac{f-32}{1.8}$
This equals the temperature in Celsius.	$c=\frac{f-32}{1.8}$

In order to convert from one temperature scale to another, simply substitute in for whichever temperature you know, and solve for the one you don't know.

b) To convert 50 degrees Fahrenheit to degrees Celsius, substitute $f=50$ into the equation.

$$\begin{aligned}
c &= \frac{50-32}{1.8}\\
c &= \frac{18}{1.8}\\
c &= 10
\end{aligned}$$

50 degrees Fahrenheit is equal to 10 degrees Celsius.

Watch this video for help with the Examples above.

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CK-12 Foundation: Solving Real-WorldProblems with Two-Step Equations

Vocabulary

- Some equations require more than one operation to solve. Generally it, is good to go from the outside in. If there are parentheses around an expression with a variable in it, cancel what is outside the parentheses first.
- Terms with the same variable in them (or no variable in them) are **like terms. Combine like terms** (adding or subtracting them from each other) to simplify the expression and solve for the unknown.

Guided Practice

To convert temperatures in Fahrenheit to temperatures in Celsius, follow the following steps: Take the temperature in degrees Fahrenheit and subtract 32. Then divide the result by 1.8 and this gives the temperature in degrees Celsius.

a) *Convert 25 degrees Celsius to degrees Fahrenheit.*

b) *Convert -40 degrees Celsius to degrees Fahrenheit.*

Solution:

a) To convert 25 degrees Celsius to degrees Fahrenheit, substitute $c = 25$ into the equation:

$$\begin{aligned}25 &= \frac{f-32}{1.8}\\45 &= f-32\\77 &= f\end{aligned}$$

25 degrees Celsius is equal to 77 degrees Fahrenheit.

b) To convert -40 degrees Celsius to degrees Fahrenheit, substitute $c = -40$ into the equation.

$$\begin{aligned}-40 &= \frac{f-32}{1.8}\\-72 &= f-32\\-40 &= f\end{aligned}$$

-40 degrees Celsius is equal to -40 degrees Fahrenheit. (No, that's not a mistake! This is the one temperature where they are equal.)

Practice

1. Jade is stranded downtown with only \$10 to get home. Taxis cost \$0.75 per mile, but there is an additional \$2.35 hire charge. Write a formula and use it to calculate how many miles she can travel with her money.
2. Jasmin's Dad is planning a surprise birthday party for her. He will hire a bouncy castle, and will provide party food for all the guests. The bouncy castle costs \$150 for the afternoon, and the food will cost \$3 per person. Andrew, Jasmin's Dad, has a budget of \$300. Write an equation and use it to determine the maximum number of guests he can invite.

The local amusement park sells summer memberships for \$50 each. Normal admission to the park costs \$25; admission for members costs \$15.

3. If Darren wants to spend no more than \$100 on trips to the amusement park this summer, how many visits can he make if he buys a membership with part of that money?
4. How many visits can he make if he does not?
5. If he increases his budget to \$160, how many visits can he make as a member?
6. And how many as a non-member?

For an upcoming school field trip, there must be one adult supervisor for every five children.

7. Write an expression for the number of children and adults who will go on the field trip.
8. If the bus seats 40 people, how many children can go on the trip?
9. How many children can go if a second 40-person bus is added?
10. Four of the adult chaperones decide to arrive separately by car. Now how many children can go in the two buses?

3.7 Multi-Step Equations

Here you'll learn how to solve equations that take several steps to isolate the unknown variable.

What if you had an equation like $3(x-2)-x=6$ that required more than two steps to solve for the variable? After completing this Concept, you'll be able to solve multi-step equations like this one that require using the distributive property and combining like terms.

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CK-12 Foundation: 0307S Multi-Step Equations

Guidance

We've seen that when we solve for an unknown variable, it can take just one or two steps to get the terms in the right places. Now we'll look at solving equations that take several steps to isolate the unknown variable. Such equations are referred to as **multi-step equations**.

In this section, we'll simply be combining the steps we already know how to do. Our goal is to end up with all the constants on one side of the equation and all the variables on the other side. We'll do this by collecting like terms. Don't forget, like terms have the same combination of variables in them.

Example A

Solve $\frac{3x+4}{3}-5x=6$.

Before we can combine the variable terms, we need to get rid of that fraction.

First let's put all the terms on the left over a common denominator of three: $\frac{3x+4}{3}-\frac{15x}{3}=6$.

Combining the fractions then gives us $\frac{3x+4-15x}{3}=6$.

Combining like terms in the numerator gives us $\frac{4-12x}{3}=6$.

Multiplying both sides by 3 gives us $4-12x=18$.

Subtracting 4 from both sides gives us $-12x=14$.

And finally, dividing both sides by -12 gives us $x=-\frac{14}{12}$, which reduces to $x=-\frac{7}{6}$.

Solving Multi-Step Equations Using the Distributive Property

You may have noticed that when one side of the equation is multiplied by a constant term, we can either distribute it or just divide it out. If we can divide it out without getting awkward fractions as a result, then that's usually the better choice, because it gives us smaller numbers to work with. But if dividing would result in messy fractions, then it's usually better to distribute the constant and go from there.

Example B

Solve $7(2x-5)=21$.

The first thing we want to do here is get rid of the parentheses. We could use the Distributive Property, but it just so happens that 7 divides evenly into 21. That suggests that dividing both sides by 7 is the easiest way to solve this problem.

If we do that, we get $2x-5=\frac{21}{7}$ or just $2x-5=3$. Then all we need to do is add 5 to both sides to get $2x=8$, and then divide by 2 to get $x=4$.

Example C

Solve $17(3x+4)=7$.

Once again, we want to get rid of those parentheses. We could divide both sides by 17, but that would give us an inconvenient fraction on the right-hand side. In this case, distributing is the easier way to go.

Distributing the 17 gives us $51x+68=7$. Then we subtract 68 from both sides to get $51x=-61$, and then we divide by 51 to get $x=\frac{-61}{51}$. (Yes, that's a messy fraction too, but since it's our final answer and we don't have to do anything else with it, we don't really care how messy it is.)

Example D

Solve $4(3x-4)-7(2x+3)=3$.

Before we can collect like terms, we need to get rid of the parentheses using the Distributive Property. That gives us $12x-16-14x-21=3$, which we can rewrite as $(12x-14x)+(-16-21)=3$. This in turn simplifies to $-2x-37=3$.

Next we add 37 to both sides to get $-2x=40$.

And finally, we divide both sides by -2 to get $x=-20$.

Watch this video for help with the Examples above.

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CK-12 Foundation: Multi-StepEquations

Vocabulary

- If dividing a number outside of parentheses will produce fractions, it is often better to use the **Distributive Property** to expand the terms and then combine like terms to solve the equation.

Guided Practice

Solve the following equation for x: $0.1(3.2+2x)+\frac{1}{2}\left(3-\frac{x}{5}\right)=0$

Solution:

This function contains both fractions and decimals. We should convert all terms to one or the other. It's often easier to convert decimals to fractions, but in this equation the fractions are easy to convert to decimals—and with decimals we don't need to find a common denominator!

In decimal form, our equation becomes $0.1(3.2+2x)+0.5(3-0.2x)=0$.

Distributing to get rid of the parentheses, we get $0.32+0.2x+1.5-0.1x=0$.

Collecting and combining like terms gives us $0.1x+1.82=0$.

Then we can subtract 1.82 from both sides to get $0.1x=-1.82$, and finally divide by 0.1 (or multiply by 10) to get $x=-18.2$.

Practice

Solve the following equations for the unknown variable.

1. $3(x-1)-2(x+3)=0$
2. $3(x+3)-2(x-1)=0$
3. $7(w+20)-w=5$
4. $5(w+20)-10w=5$
5. $9(x-2)-3x=3$
6. $12(t-5)+5=0$
7. $2(2d+1)=\frac{2}{3}$
8. $2\left(5a-\frac{1}{3}\right)=\frac{2}{7}$
9. $\frac{2}{9}\left(i+\frac{2}{3}\right)=\frac{2}{5}$
10. $4\left(v+\frac{1}{4}\right)=\frac{35}{2}$
11. $\frac{g}{10}=\frac{6}{3}$
12. $\frac{m+3}{2}-\frac{m}{4}=\frac{1}{3}$
13. $5\left(\frac{k}{3}+2\right)=\frac{32}{3}$
14. $\frac{3}{z}=\frac{2}{5}$
15. $\frac{2}{r}+2=\frac{10}{3}$
16. $\frac{12}{5}=\frac{3+z}{z}$

3.8 Solving Real-World Problems Using Multi-Step Equations

Here you'll learn how to translate words into to multi-step equations. You'll then solve such equations for their unknown variable.

What if you were told that 10 less than $\frac{3}{2}$ a number plus the number was equal to 5? How could you find the number? After completing this Concept, you'll be able to solve real-world problems like this one that involve multi-step equations.

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CK-12 Foundation: 0308S Solving Real-World Problems Using Multi-StepEquations

Guidance

We can now use strategies for solving multi-step equations to solve real world equations.

Example A

A growers' cooperative has a farmer's market in the town center every Saturday. They sell what they have grown and split the money into several categories. 8.5% of all the money taken in is set aside for sales tax. $150 goes to pay the rent on the space they occupy. What remains is split evenly between the seven growers. How much total money is taken in if each grower receives a $175 share?

Let's translate the text above into an equation. The unknown is going to be the total money taken in dollars. We'll call this x.

"8.5% of all the money taken in is set aside for sales tax." This means that 91.5% of the money remains. This is $0.915x$.

"$150 goes to pay the rent on the space they occupy." This means that what's left is $0.915x - 150$.

"What remains is split evenly between the 7 growers." That means each grower gets $\frac{0.915x-150}{7}$.

If each grower's share is $175, then our equation to find x is $\frac{0.915x-150}{7} = 175$.

First we multiply both sides by 7 to get $0.915x - 150 = 1225$.

Then add 150 to both sides to get $0.915x = 1375$.

Finally divide by 0.915 to get $x \approx 1502.7322$. Since we want our answer in dollars and cents, we round to two decimal places, or $1502.73.

The workers take in a total of $1502.73.

Ohm's Law

The electrical current, I (amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship $V = I \cdot R$ where R is the resistance measured in Ohms (Ω).

Example B

A scientist is trying to deduce the resistance of an unknown component. He labels the resistance of the unknown component x Ω. *The resistance of a circuit containing a number of these components is* $(5x+20)\Omega$. *If a 120 volt potential difference across the circuit produces a current of 2.5 amps, calculate the resistance of the unknown component.*

Solution

To solve this, we need to start with the equation $V = I \cdot R$ and substitute in $V = 120, I = 2.5$, and $R = 5x+20$. That gives us $120 = 2.5(5x+20)$.

Distribute the 2.5 to get $120 = 12.5x+50$.

Subtract 50 from both sides to get $70 = 12.5x$.

Finally, divide by 12.5 to get $5.6 = x$.

The unknown components have a resistance of $5.6\ \Omega$**.**

Distance, Speed and Time

The speed of a body is the distance it travels per unit of time. That means that we can also find out how far an object moves in a certain amount of time if we know its speed: we use the equation "distance = speed × time."

Example C

Shanice's car is traveling 10 miles per hour slower than twice the speed of Brandon's car. She covers 93 miles in 1 hour 30 minutes. How fast is Brandon driving?

Solution

Here, we don't know either Brandon's speed or Shanice's, but since the question asks for Brandon's speed, that's what we'll use as our variable x.

The distance Shanice covers in miles is 93, and the time in hours is 1.5. Her speed is 10 less than twice Brandon's speed, or $2x-10$ miles per hour. Putting those numbers into the equation gives us $93 = 1.5(2x-10)$.

First we distribute, to get $93 = 3x-15$.

Then we add 15 to both sides to get $108 = 3x$.

Finally we divide by 3 to get $36 = x$.

Brandon is driving at 36 miles per hour.

We can check this answer by considering the situation another way: we can solve for Shanice's speed instead of Brandon's and then check that against Brandon's speed. We'll use y for Shanice's speed since we already used x for Brandon's.

The equation for Shanice's speed is simply $93 = 1.5y$. We can divide both sides by 1.5 to get $62 = y$, so Shanice is traveling at 62 miles per hour.

The problem tells us that Shanice is traveling 10 mph slower than twice Brandon's speed; that would mean that 62 is equal to 2 times 36 minus 10. Is that true? Well, 2 times 36 is 72, minus 10 is 62. The answer checks out.

In algebra, there's almost always more than one method of solving a problem. If time allows, it's always a good idea to try to solve the problem using two different methods just to confirm that you've got the answer right.

Speed of Sound

The speed of sound in dry air, v, is given by the equation $v = 331 + 0.6T$, where T is the temperature in Celsius and v is the speed of sound in meters per second.

Example D

Tashi hits a drainpipe with a hammer and 250 meters away Minh hears the sound and hits his own drainpipe. Unfortunately, there is a one second delay between him hearing the sound and hitting his own pipe. Tashi accurately measures the time between her hitting the pipe and hearing Mihn's pipe at 2.46 seconds. What is the temperature of the air?

This is a complex problem and we need to be careful in writing our equations. First of all, the distance the sound travels is equal to the speed of sound multiplied by the time, and the speed is given by the equation above. So the distance equals $(331 + 0.6T) \times$ time, and the time is $2.46 - 1$ (because for 1 second out of the 2.46 seconds measured, there was no sound actually traveling). We also know that the distance is 250×2 (because the sound traveled from Tashi to Minh and back again), so our equation is $250 \times 2 = (331 + 0.6T)(2.46 - 1)$, which simplifies to $500 = 1.46(331 + 0.6T)$.

Distributing gives us $500 = 483.26 + 0.876T$, and subtracting 483.26 from both sides gives us $16.74 = 0.876T$. Then we divide by 0.876 to get $T \approx 19.1$.

The temperature is about 19.1 degrees Celsius.

Watch this video for help with the Examples above.

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CK-12 Foundation: Solving Real-WorldProblems Using Multi-Step Equations

Vocabulary

- If dividing a number outside of parentheses will produce fractions, it is often better to use the **Distributive Property** to expand the terms and then combine like terms to solve the equation.

Guided Practice

A factory manager is packing engine components into wooden crates to be shipped on a small truck. The truck is designed to hold sixteen crates, and will safely carry a 1200 lb cargo. Each crate weighs 12 lbs empty. How much weight should the manager instruct the workers to put in each crate in order to get the shipment weight as close as possible to 1200 lbs?

Solution:

The unknown quantity is the weight to put in each box, so we'll call that x.

Each crate when full will weigh $x + 12$ *lbs*, so all 16 crates together will weigh $16(x + 12)$ *lbs*.

We also know that all 16 crates together should weigh 1200 lbs, so we can say that $16(x + 12) = 1200$.

To solve this equation, we can start by dividing both sides by 16: $x+12 = \frac{1200}{16} = 75$.

Then subtract 12 from both sides: $x = 63$.

The manager should tell the workers to put 63 lbs of components in each crate.

Practice

For 1-6, solve for the variable in the equation.

1. $\frac{s-4}{11} = \frac{2}{5}$
2. $\frac{2k}{7} = \frac{3}{8}$
3. $\frac{7x+4}{3} = \frac{9}{2}$
4. $\frac{9y-3}{6} = \frac{5}{2}$
5. $\frac{r}{3} + \frac{r}{2} = 7$
6. $\frac{p}{16} - \frac{2p}{3} = \frac{1}{9}$
7. An engineer is building a suspended platform to raise bags of cement. The platform has a mass of 200 kg, and each bag of cement is 40 kg. He is using two steel cables, each capable of holding 250 kg. Write an equation for the number of bags he can put on the platform at once, and solve it.
8. A scientist is testing a number of identical components of unknown resistance which he labels $x\Omega$. He connects a circuit with resistance $(3x+4)\Omega$ to a steady 12 volt supply and finds that this produces a current of 1.2 amps. What is the value of the unknown resistance?
9. Lydia inherited a sum of money. She split it into five equal parts. She invested three parts of the money in a high-interest bank account which added 10% to the value. She placed the rest of her inheritance plus $500 in the stock market but lost 20% on that money. If the two accounts end up with exactly the same amount of money in them, how much did she inherit?
10. Pang drove to his mother's house to drop off her new TV. He drove at 50 miles per hour there and back, and spent 10 minutes dropping off the TV. The entire journey took him 94 minutes. How far away does his mother live?

3.9 Equations with Variables on Both Sides

Here you'll learn how to manipulate equations with variables on both sides of the equal sign so that all variable terms appear on one side.

What if you had an equation like $9(x-2) = 6-3x$ in which the variable was on both sides of the equal sign? How could you solve this equation for x? After completing this Concept, you'll be able to solve equations like this one where the variable is on both sides.

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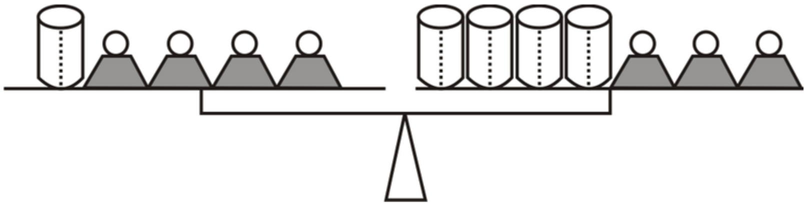
CK-12 Foundation: 0309S Equations with Variableson BothSides

Guidance

When a variable appears on both sides of the equation, we need to manipulate the equation so that all variable terms appear on one side, and only constants are left on the other.

Example A

Dwayne was told by his chemistry teacher to measure the weight of an empty beaker using a balance. Dwayne found only one lb weights, and so devised the following way of balancing the scales.



Knowing that each weight is one lb, calculate the weight of one beaker.

Solution

We know that the system balances, so the weights on each side must be equal. We can write an algebraic expression based on this fact. The unknown quantity, the weight of the beaker, will be our x. We can see that on the left hand scale we have one beaker and four weights. On the right scale, we have four beakers and three weights. The balancing of the scales is analogous to the balancing of the following equation:

$$x+4 = 4x+3$$

"One beaker plus 4 lbs **equals** 4 beakers plus 3 lbs"

To solve for the weight of the beaker, we want all the constants (numbers) on one side and all the variables (terms with x in them) on the other side. Since there are more beakers on the right and more weights on the left, we'll try to move all the x terms (beakers) to the right, and the constants (weights) to the left.

First we subtract 3 from both sides to get $x+1=4x$.

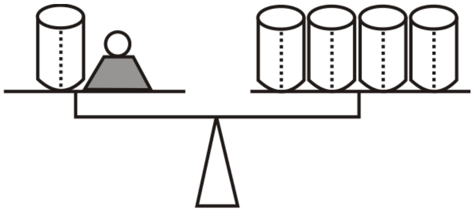
Then we subtract x from both sides to get $1=3x$.

Finally we divide by 3 to get $\frac{1}{3}=x$.

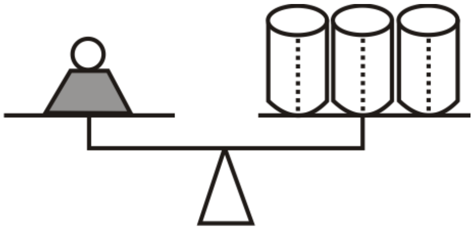
The weight of the beaker is one-third of a pound.

We can do the same with the real objects as we did with the equation. Just as we subtracted amounts from each side of the equation, we could remove a certain number of weights or beakers from each scale. Because we remove the same number of objects from each side, we know the scales will still balance.

First, we could remove three weights from each scale. This would leave one beaker and one weight on the left and four beakers on the right (in other words $x+1=4x$):



Then we could remove one beaker from each scale, leaving only one weight on the left and three beakers on the right, to get $1=3x$:



Looking at the balance, it is clear that the weight of one beaker is one-third of a pound.

To see more examples of solving equations with variables on both sides of the equation, see the Khan Academy video at http://www.youtube.com/watch?v=Zn-GbH2S0Dk.

Solve an Equation with Grouping Symbols

As you've seen, we can solve equations with variables on both sides even when some of the variables are in parentheses; we just have to get rid of the parentheses, and then we can start combining like terms. We use the same technique when dealing with fractions: first we multiply to get rid of the fractions, and then we can shuffle the terms around by adding and subtracting.

Example B

Solve $3x+2=\frac{5x}{3}$.

Solution

The first thing we'll do is get rid of the fraction. We can do this by multiplying both sides by 3, leaving $3(3x+2)=5x$.

Then we distribute to get rid of the parentheses, leaving $9x+6=5x$.

We've already got all the constants on the left side, so we'll move the variables to the right side by subtracting $9x$ from both sides. That leaves us with $6=-4x$.

And finally, we divide by -4 to get $-\frac{3}{2}=x$, or $x=-1.5$.

Example C

Solve the following equation for x: $\frac{14x}{(x+3)}=7$

Solution

The form of the left hand side of this equation is known as a **rational function** because it is the ratio of two other functions: $14x$ and $(x+3)$. But we can solve it just like any other equation involving fractions.

First we multiply both sides by $(x+3)$ to get rid of the fraction. Now our equation is $14x=7(x+3)$.

Then we distribute: $14x=7x+21$.

Then subtract $7x$ from both sides: $7x=21$.

And divide by 7: $x=3$.

Solve Real-World Problems Using Equations with Variables on Both Sides

Here's another chance to practice translating problems from words to equations. What is the equation asking? What is the **unknown** variable? What quantity will we use for our variable?

The text explains what's happening. Break it down into small, manageable chunks, and follow what's going on with our variable all the way through the problem.

More on Ohm's Law

Recall that the electrical current, I (amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship $V=I\cdot R$ where R is the resistance measured in Ohms (Ω).

The resistance R of a number of components wired in a **series** (one after the other) is simply the sum of all the resistances of the individual components.

Example D

In an attempt to find the resistance of a new component, a scientist tests it in series with standard resistors. A fixed voltage causes a 4.8 amp current in a circuit made up from the new component plus a 15Ω *resistor in series. When the component is placed in a series circuit with a* 50Ω *resistor, the same voltage causes a 2.0 amp current to flow. Calculate the resistance of the new component.*

This is a complex problem to translate, but once we convert the information into equations it's relatively straightforward to solve. First, we are trying to find the resistance of the new component (in Ohms, Ω). This is our x. We don't know the voltage that is being used, but we can leave that as a variable, V. Our first situation has a total resistance that equals the unknown resistance plus 15Ω. The current is 4.8 amps. Substituting into the formula $V=I\cdot R$, we get $V=4.8(x+15)$.

Our second situation has a total resistance that equals the unknown resistance plus 50Ω. The current is 2.0 amps. Substituting into the same equation, this time we get $V=2(x+50)$.

We know the voltage is fixed, so the V in the first equation must equal the V in the second. That means we can set the right-hand sides of the two equations equal to each other: $4.8(x+15)=2(x+50)$. Then we can solve for x.

Distribute the constants first: $4.8x+72=2x+100$.

Subtract $2x$ from both sides: $2.8x + 72 = 100$.

Subtract 72 from both sides: $2.8x = 28$.

Divide by 2.8: $x = 10$.

The resistance of the component is 10Ω.

Watch this video for help with the Examples above.

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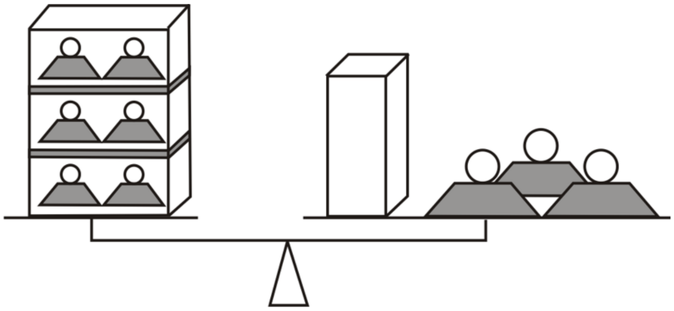
CK-12 Foundation: Variables on Both Sides

Vocabulary

An equation has the **variable on both sides** if the variable appears somewhere on each side of the equation. Distribute as necessary and then simplify the equation to have the unknown on only one side.

Guided Practice

1. *Sven was told to find the weight of an empty box with a balance. Sven found some one lb weights and five lb weights. He placed two one lb weights in three of the boxes and with a fourth empty box found the following way of balancing the scales:*



Knowing that small weights are one lb and big weights are five lbs, calculate the weight of one box.

2. *Solve* $7x + 2 = \frac{5x-3}{6}$.

Solutions:

1. We know that the system balances, so the weights on each side must be equal. We can write an algebraic expression based on this equality. The unknown quantity—the weight of each empty box, in pounds—will be our x. A box with two 1 lb weights in it weighs $(x+2)$ pounds. Our equation, based on the picture, is $3(x+2) = x + 3(5)$.

Distributing the 3 and simplifying, we get $3x + 6 = x + 15$.

Subtracting x from both sides, we get $2x + 6 = 15$.

Subtracting 6 from both sides, we get $2x = 9$.

And finally we can divide by 2 to get $x = \frac{9}{2}$, or $x = 4.5$.

Each box weighs 4.5 lbs.

2. Again we start by eliminating the fraction. Multiplying both sides by 6 gives us $6(7x+2) = 5x-3$, and distributing gives us $42x+12 = 5x-3$.

Subtracting $5x$ from both sides gives us $37x+12 = -3$.

Subtracting 12 from both sides gives us $37x = -15$.

Finally, dividing by 37 gives us $x = -\frac{15}{37}$.

Practice

For 1-11, solve the following equations for the unknown variable.

1. $3(x-1) = 2(x+3)$
2. $7(x+20) = x+5$
3. $9(x-2) = 3x+3$
4. $2\left(a-\frac{1}{3}\right) = \frac{2}{5}\left(a+\frac{2}{3}\right)$
5. $\frac{2}{7}\left(t+\frac{2}{3}\right) = \frac{1}{5}\left(t-\frac{2}{3}\right)$
6. $\frac{1}{7}\left(v+\frac{1}{4}\right) = 2\left(\frac{3v}{2}-\frac{5}{2}\right)$
7. $\frac{y-4}{11} = \frac{2}{5}\cdot\frac{2y+1}{3}$
8. $\frac{z}{16} = \frac{2(3z+1)}{9}$
9. $\frac{q}{16}+\frac{q}{6} = \frac{(3q+1)}{9}+\frac{3}{2}$
10. $\frac{3}{x} = \frac{2}{x+1}$
11. $\frac{5}{2+p} = \frac{3}{p-8}$
12. Manoj and Tamar are arguing about a number trick they heard. Tamar tells Andrew to think of a number, multiply it by five and subtract three from the result. Then Manoj tells Andrew to think of a number, add five and multiply the result by three. Andrew says that whichever way he does the trick he gets the same answer.
    a. What was the number Andrew started with?
    b. What was the result Andrew got both times?
    c. Name another set of steps that would have resulted in the same answer if Andrew started with the same number.
13. Manoj and Tamar try to come up with a harder trick. Manoj tells Andrew to think of a number, double it, add six, and then divide the result by two. Tamar tells Andrew to think of a number, add five, triple the result, subtract six, and then divide the result by three.
    a. Andrew tries the trick both ways and gets an answer of 10 each time. What number did he start out with?
    b. He tries again and gets 2 both times. What number did he start out with?
    c. Is there a number Andrew can start with that will *not* give him the same answer both ways?
    d. **Bonus:** Name another set of steps that would give Andrew the same answer every time as he would get from Manoj's and Tamar's steps.
14. I have enough money to buy five regular priced CDs and have \$6 left over. However, all CDs are on sale today, for \$4 less than usual. If I borrow \$2, I can afford nine of them.
    a. How much are CDs on sale for today?
    b. How much would I have to borrow to afford nine of them if they weren't on sale?
15. Five identical electronics components were connected in series. A fixed but unknown voltage placed across them caused a 2.3 amp current to flow. When two of the components were replaced with standard 10Ω resistors, the current dropped to 1.9 amps. What is the resistance of each component?
16. Solve the following resistance problems. Assume the same voltage is applied to all circuits.
    a. Three unknown resistors plus 20Ω give the same current as one unknown resistor plus 70Ω.

b. One unknown resistor gives a current of 1.5 amps and a 15Ω resistor gives a current of 3.0 amps.
c. Seven unknown resistors plus 18Ω gives twice the current of two unknown resistors plus 150Ω.
d. Three unknown resistors plus 1.5Ω gives a current of 3.6 amps and seven unknown resistors plus seven 12Ω resistors gives a current of 0.2 amps.

3.10 Ratios

Here you'll learn how to write and simplify comparisons of two numbers, measurements, or quantities.

What if you were told that every two inches on a map represented 500 miles? How could you write this comparison mathematically? After completing this Concept, you'll be able to write and understand ratios like this one.

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CK-12 Foundation: 0309S Ratios (H264)

Guidance

A **ratio** is a way to compare two numbers, measurements or quantities. When we write a ratio, we divide one number by another and express the answer as a fraction. There are two distinct ratios in the example below.

Example A

Nadia is counting out money with her little brother. She gives her brother all the nickels and pennies. She keeps the quarters and dimes for herself. Nadia has four quarters and six dimes. Her brother has fifteen nickels and five pennies and is happy because he has more coins than his big sister. How would you explain to him that he is actually getting a bad deal?

Solution:

The ratio of the **number** of Nadia's coins to her brother's is $\frac{4+6}{15+5}$, or $\frac{10}{20} = \frac{1}{2}$. (Ratios should always be simplified.) In other words, Nadia has half as many coins as her brother.

Another ratio we could look at is the **value** of the coins. The value of Nadia's coins is $(4 \times 25) + (6 \times 10) = 160\ cents$. The value of her brother's coins is $(15 \times 5) + (5 \times 1) = 80\ cents$. The ratio of the **value** of Nadia's coins to her brother's is $\frac{160}{80} = \frac{2}{1}$. So the value of Nadia's money is twice the value of her brother's.

Notice that even though the denominator is one, we still write it out and leave the ratio as a fraction instead of a whole number. A ratio with a denominator of one is called a **unit rate**.

Example B

The price of a Harry Potter Book on Amazon.com is $10.00. The same book is also available used for $6.50. Find two ways to compare these prices.

Solution

We could compare the numbers by expressing the difference between them: $\$10.00 - \$6.50 = \$3.50$.

We can also use a ratio to compare them: $\frac{10.00}{6.50} = \frac{100}{65} = \frac{20}{13}$ (after multiplying by 10 to remove the decimals, and then simplifying).

So we can say that **the new book is $3.50 more than the used book**, or we can say that **the new book costs** $\frac{20}{13}$**times as much as the used book**.

Example C

A tournament size shuffleboard table measures 30 inches wide by 14 feet long. Compare the length of the table to its width and express the answer as a ratio.

Solution

We could just write the ratio as $\frac{14\ feet}{30\ inches}$. But since we're comparing two lengths, it makes more sense to convert all the measurements to the same units. 14 feet is $14 \times 12 = 168$ *inches*, so our new ratio is $\frac{168}{30} = \frac{28}{5}$.

Watch this video for help with the Examples above.

MEDIA

Click image to the left for more content.

CK-12 Foundation: Ratios

Vocabulary

- A **ratio** is a way to compare two numbers, measurements or quantities by dividing one number by the other and expressing the answer as a fraction.
- **A proportion** is formed when two ratios are set equal to each other.
- **Cross multiplication** is useful for solving equations in the form of proportions. To cross multiply, multiply the bottom of each ratio by the top of the other ratio and set them equal. For instance, cross multiplying

  results in $11 \times 3 = 5x$.
- **Scale** is a proportion that relates map distance to real life distance.

Guided Practice

A family car is being tested for fuel efficiency. It drives non-stop for 100 miles and uses 3.2 gallons of gasoline. Write the ratio of distance traveled to fuel used as a **unit rate**.

Solution

The ratio of distance to fuel is $\frac{100\ miles}{3.2\ gallons}$. But a unit rate has to have a denominator of one, so to make this ratio a unit rate we need to divide both numerator and denominator by 3.2. $\frac{\frac{100}{3.2}\ miles}{\frac{3.2}{3.2}\ gallons} = \frac{31.25\ miles}{1\ gallon}$ or **31.25 miles per gallon.**

Practice

Write the following comparisons as ratios. Simplify fractions where possible.

1. \$150 to \$3.
2. 150 boys to 175 girls.
3. 200 minutes to 1 hour.
4. 10 days to 2 weeks.
5. Write the following ratios as a unit rate.
6. 54 hotdogs to 12 minutes.
7. 5000 lbs to 250 square inches.
8. 20 computers to 80 students.
9. 180 students to 6 teachers .
10. 12 meters to 4 floors .
11. 18 minutes to 15 appointments.

3.11 Proportions

Here you'll learn how to use cross products to solve for unknown values in proportions.

What if you had two ratios that you knew were equal to one another like $\frac{5}{x} = \frac{10}{3}$ in which one of the numbers was unknown. How could you solve for x? After completing this Concept, you'll be able to write and solve proportions like this one.

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CK-12 Foundation: 0311S Proportions (H264)

Guidance

When two ratios are equal to each other, we call it a proportion. For example, the equation $\frac{10}{5} = \frac{6}{9}$ is a proportion. We know it's true because we can reduce both fractions to $\frac{2}{3}$.

(Check this yourself to make sure!)

We often use proportions in science and business—for example, when scaling up the size of something. We generally use them to solve for an unknown, so we use algebra and label the unknown variable x.

Example A

A small fast food chain operates 60 stores and makes \$1.2 million profit every year. How much profit would the chain make if it operated 250 stores?

Solution

First, we need to write a **ratio:** the ratio of profit to number of stores. That would be $\frac{\$1{,}200{,}000}{60}$.

Now we want to know how much profit 250 stores would make. If we label that profit x, then the ratio of profit to stores in that case is $\frac{x}{250}$.

Since we're assuming the profit is proportional to the number of stores, the ratios are equal and our proportion is $\frac{1{,}200{,}000}{60} = \frac{x}{250}$.

(Note that we can drop the units – not because they are the same in the numerator and denominator, but because they are the same on both sides of the equation.)

To solve this equation, first we simplify the left-hand fraction to get $20,000 = \frac{x}{250}$. Then we multiply both sides by 250 to get $5,000,000 = x$.

If the chain operated 250 stores, the annual profit would be 5 million dollars.

Solve Proportions Using Cross Products

One neat way to simplify proportions is to cross multiply. Consider the following proportion:

$$\frac{16}{4} = \frac{20}{5}$$

If we want to eliminate the fractions, we could multiply both sides by 4 and then multiply both sides by 5. But suppose we just do both at once?

$$\begin{gathered} 4 \times 5 \times \frac{16}{4} = 4 \times 5 \times \frac{20}{5} \\ 5 \times 16 = 4 \times 20 \end{gathered}$$

Now comparing this to the proportion we started with, we see that the denominator from the left hand side ends up being multiplied by the numerator on the right hand side. You can also see that the denominator from the *right* hand side ends up multiplying the numerator on the *left* hand side.

In effect the two denominators have multiplied across the equal sign:

becomes $5 \times 16 = 4 \times 20$.

This movement of denominators is known as **cross multiplying**. It is extremely useful in solving proportions, especially when the unknown variable is in the denominator.

Example B

Solve this proportion for x: $\frac{4}{3} = \frac{9}{x}$

Solution

Cross multiply to get $4x = 9 \times 3$, or $4x = 27$. Then divide both sides by 4 to get $x = \frac{27}{4}$, or $x = 6.75$.

Example C

Solve the following proportion for x: $\frac{0.5}{3} = \frac{56}{x}$

Solution

Cross multiply to get $0.5x = 56 \times 3$, or $0.5x = 168$. Then divide both sides by 0.5 to get $x = 336$.

Solve Real-World Problems Using Proportions

Example D

A cross-country train travels at a steady speed. It covers 15 miles in 20 minutes. How far will it travel in 7 hours assuming it continues at the same speed?

Solution

We've done speed problems before; remember that speed is just the ratio $\frac{\text{distance}}{\text{time}}$, so that ratio is the one we'll use for our proportion. We can see that the speed is $\frac{15\ miles}{20\ minutes}$, and that speed is also equal to $\frac{x\ miles}{7\ hours}$.

To set up a proportion, we first have to get the units the same. 20 minutes is $\frac{1}{3}$ of an hour, so our proportion will be $\frac{15}{\frac{1}{3}} = \frac{x}{7}$. This is a very awkward looking ratio, but since we'll be cross multiplying, we can leave it as it is.

Cross multiplying gives us $7 \times 15 = \frac{1}{3}x$. Multiplying both sides by 3 then gives us $3 \times 7 \times 15 = x$, or $x = 315$.

The train will travel 315 miles in 7 hours.

Example E

In the United Kingdom, Alzheimer's disease is said to affect one in fifty people over 65 years of age. If approximately 250000 people over 65 are affected in the UK, how many people over 65 are there in total?

Solution

The fixed ratio in this case is the 1 person in 50. The unknown quantity x is the total number of people over 65. Note that in this case we don't need to include the units, as they will cancel between the numerator and denominator.

Our proportion is $\frac{1}{50} = \frac{250000}{x}$. Each ratio represents

$$\frac{\text{people with Alzheimer's}}{\text{total people}}$$

.

Cross multiplying, we get $1 \cdot x = 250000 \cdot 50$, or $x = 12,500,000$.

There are approximately 12.5 million people over the age of 65 in the UK.

Watch this video for help with the Examples above.

MEDIA

Click image to the left for more content.

CK-12 Foundation: Proportions

Vocabulary

- A **ratio** is a way to compare two numbers, measurements or quantities by dividing one number by the other and expressing the answer as a fraction.
- **A proportion** is formed when two ratios are set equal to each other.
- **Cross multiplication** is useful for solving equations in the form of proportions. To cross multiply, multiply the bottom of each ratio by the top of the other ratio and set them equal. For instance, cross multiplying

  results in $11 \times 3 = 5x$.
- **Scale** is a proportion that relates map distance to real life distance.

Guided Practice

A chemical company makes up batches of copper sulfate solution by adding 250 kg of copper sulfate powder to 1000 liters of water. A laboratory chemist wants to make a solution of identical concentration, but only needs 350 mL (0.35 liters) of solution. How much copper sulfate powder should the chemist add to the water?

Solution

The ratio of powder to water in the first case, in kilograms per liter, is $\frac{250}{1000}$, which reduces to $\frac{1}{4}$. In the second case, the unknown amount is how much powder to add. If we label that amount x, the ratio is $\frac{x}{0.35}$. So our proportion is $\frac{1}{4} = \frac{x}{0.35}$.

To solve for x, first we multiply both sides by 0.35 to get $\frac{0.35}{4} = x$, or $x = 0.0875$.

The mass of copper sulfate that the chemist should add is 0.0875 kg, or 87.5 grams.

Practice

Solve the following proportions.

1. $\frac{13}{6} = \frac{5}{x}$
2. $\frac{1.25}{7} = \frac{3.6}{x}$
3. $\frac{6}{19} = \frac{x}{11}$
4. $\frac{1}{x} = \frac{0.01}{5}$
5. $\frac{300}{4} = \frac{x}{99}$
6. $\frac{2.75}{9} = \frac{x}{\left(\frac{2}{9}\right)}$
7. $\frac{1.3}{4} = \frac{x}{1.3}$
8. $\frac{0.1}{1.01} = \frac{1.9}{x}$
9. $\frac{5}{36} = \frac{x}{30}$
10. $\frac{10}{3} = \frac{6.9}{x}$

3.12 Scale and Indirect Measurement Applications

Here you'll learn how to use scales to indirectly measure distance and length.

What if you had a 1:80 scale model of the Eiffel Tower. The model stands 4 meters tall. How could you find the height of the actual Eiffel Tower? After completing this Concept, you'll be able to use indirect measurements to solve scale problems like this one.

Watch This

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Click image to the left for more content.

CK-12 Foundation: 0312S Scale and Indirect Measurement (H264)

For some more advanced ratio problems and applications, watch the Khan Academy video at http://www.youtube.com/watch?v=PASSD2OcU0c.

Guidance

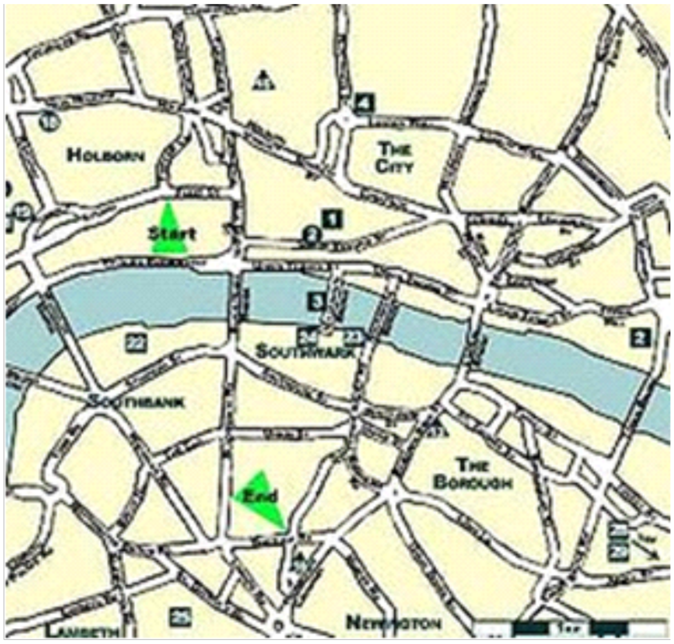
One place where ratios are often used is in making maps. The **scale** of a map describes the relationship between distances on a map and the corresponding distances on the earth's surface. These measurements are expressed as a fraction or a ratio.

So far we have only written ratios as fractions, but outside of mathematics books, ratios are often written as two numbers separated by a colon (:). For example, instead of $\frac{2}{3}$, we would write 2:3.

Ratios written this way are used to express the relationship between a map and the area it represents. For example, a map with a scale of 1:1000 would be a map where one unit of measurement (such as a centimeter) on the map would represent 1000 of the same unit (1000 centimeters, or 10 meters) in real life.

Example A

Anne is visiting a friend in London, and is using the map below to navigate from Fleet Street to Borough Road. She is using a 1:100,000 scale map, where 1 cm on the map represents 1 km in real life. Using a ruler, she measures the distance on the map as 8.8 cm. How far is the real distance from the start of her journey to the end?



Solution

The scale is the ratio of distance on the map to the corresponding distance in real life. Written as a fraction, it is $\frac{1}{100000}$. We can also write an equivalent ratio for the distance Anne measures on the map and the distance in real life that she is trying to find: $\frac{8.8}{x}$. Setting these two ratios equal gives us our proportion: $\frac{1}{100000} = \frac{8.8}{x}$. Then we can cross multiply to get $x = 880000$.

That's how many *centimeters* it is from Fleet Street to Borough Road; now we need to convert to kilometers. There are 100000 cm in a km, so we have to divide our answer by 100000.

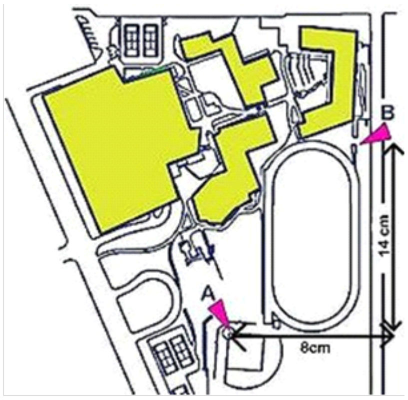
$$\frac{880000}{100000} = 8.8.$$

The distance from Fleet Street to Borough Road is 8.8 km.

In this problem, we could have just used our intuition: the $1\ cm = 1\ km$ scale tells us that any number of cm on the map is equal to the same number of km in real life. But not all maps have a scale this simple. You'll usually need to refer to the map scale to convert between measurements on the map and distances in real life!

Example B

Antonio is drawing a map of his school for a project in math. He has drawn out the following map of the school buildings and the surrounding area



He is trying to determine the scale of his figure. He knows that the distance from the point marked A on the baseball diamond to the point marked B on the athletics track is 183 meters. Use the dimensions marked on the drawing to determine the scale of his map.

Solution

We know that the real-life distance is 183 m, and the scale is the ratio $\frac{\text{distance on map}}{\text{distance in real life}}$.

To find the distance on the map, we use Pythagoras' Theorem: $a^2+b^2=c^2$, where a and b are the horizontal and vertical lengths and c is the diagonal between points A and B.

$$\begin{aligned}
8^2+14^2&=c^2\\
64+196&=c^2\\
260&=c^2\\
\sqrt{260}&=c\\
16.12&\approx c
\end{aligned}$$

So the distance on the map is about 16.12 cm. The distance in real life is 183 m, which is 18300 cm. Now we can divide:

$$\text{Scale}=\frac{16.12}{18300}\approx\frac{1}{1135.23}$$

The scale of Antonio's map is approximately 1:1100.

Another visual use of ratio and proportion is in **scale drawings**. Scale drawings (often called **plans**) are used extensively by architects. The equations governing scale are the same as for maps; the scale of a drawing is the ratio $\frac{\text{distance on diagram}}{\text{distance in real life}}$.

Example C

Oscar is trying to make a scale drawing of the Titanic, which he knows was 883 ft long. He would like his drawing to be at a 1:500 scale. How many inches long does his sheet of paper need to be?

Solution

We can reason intuitively that since the scale is 1:500, the paper must be $\frac{883}{500}=1.766\ feet$ long. Converting to inches means the length is $12(1.766)=21.192\ inches$.

Oscar's paper should be at least 22 inches long.

Watch this video for help with the Examples above.

MEDIA

Click image to the left for more content.

CK-12 Foundation: Scale and IndirectMeasurement

Vocabulary

- A **ratio** is a way to compare two numbers, measurements or quantities by dividing one number by the other and expressing the answer as a fraction.
- **A proportion** is formed when two ratios are set equal to each other.
- **Cross multiplication** is useful for solving equations in the form of proportions. To cross multiply, multiply the bottom of each ratio by the top of the other ratio and set them equal. For instance, cross multiplying

$$\frac{11}{5}\substack{\nwarrow\!\nearrow\\ \swarrow\!\searrow}\frac{x}{3}$$

  results in $11 \times 3 = 5x$.
- **Scale** is a proportion that relates map distance to real life distance.

Guided Practice

The Rose Bowl stadium in Pasadena, California measures 880 feet from north to south and 695 feet from east to west. A scale diagram of the stadium is to be made. If 1 inch represents 100 feet, what would be the dimensions of the stadium drawn on a sheet of paper? Will it fit on a standard 8.5×11 *inch sheet of paper?*

Solution

Instead of using a proportion, we can simply use the following equation: (distance on diagram) = (distance in real life) × (scale). (We can derive this from the fact that scale $= \frac{\text{distance on diagram}}{\text{distance in real life}}$.)

Plugging in, we get

height on paper $= 880\ feet \times \frac{1\ inch}{100\ feet} = 8.8\ inches$

width on paper $= 695\ feet \times \frac{1\ inch}{100\ feet} = 6.95\ inches$

The scale diagram will be $8.8\ in \times 6.95\ in$**. It will fit on a standard sheet of paper.**

Practice

1. A restaurant serves 100 people per day and takes in $908. If the restaurant were to serve 250 people per day, how much money would it take in?
2. The highest mountain in Canada is Mount Yukon. It is $\frac{298}{67}$ the size of Ben Nevis, the highest peak in Scotland. Mount Elbert in Colorado is the highest peak in the Rocky Mountains. Mount Elbert is $\frac{220}{67}$ the height of Ben Nevis and $\frac{11}{12}$ the size of Mont Blanc in France. Mont Blanc is 4800 meters high. How high is Mount Yukon?
3. At a large high school it is estimated that two out of every three students have a cell phone, and one in five of all students have a cell phone that is one year old or less. Out of the students who own a cell phone, what proportion owns a phone that is more than one year old?

For 4-6, suppose a map of Ratio City has a scale of 1:1,000,000, where 1 centimeter on the map represents 10 kilometers in real life. Use that scale to determine the real-life distances in kilometers.

4. The distance on the map between city hall and high school is 1.2cm.
5. The distance on the map between city hall and the main library is 0.6cm.
6. The distance on the map between the main library and the high school is 0.4cm.

For 7-10, use the map in Example A. Using the scale printed on the map, determine the distances (rounded to the nearest half km) between:

7. Points 1 and 4
8. Points 22 and 25
9. Points 18 and 13
10. Tower Bridge and London Bridge

3.13 Conversion of Decimals, Fractions, and Percent

Here you'll learn how to convert percents to fractions, fractions to percents, decimals to percents, and percents to decimals.

What if you knew that 75% of your classmates owned a pet. How could you convert this number to a fraction or a decimal? After completing this Concept, you'll be able to switch back and forth between percents, fractions, and decimals.

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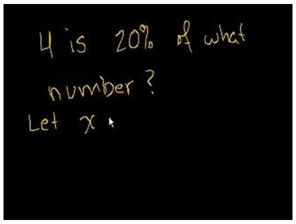
Click image to the left for more content.

CK-12 Foundation: 0313S Percentages (H264)

Watch This



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KhanAcademy: TakingPercentages

Guidance

A **percent** is simply a ratio with a base unit of 100. When we write a ratio as a fraction, the percentage we want to represent is the numerator, and the denominator is 100. For example, 43% is another way of writing $\frac{43}{100}$. $\frac{43}{1000}$, on the other hand, is equal to $\frac{4.3}{100}$, so it would be equivalent to 4.3%. $\frac{2}{5}$ is equal to $\frac{40}{100}$, or 40%. To convert any fraction to a percent, just convert it to an equivalent fraction with a denominator of 100, and then take the numerator as your percent value.

To convert a percent to a decimal, just move the decimal point two spaces to the right:

$$67\% = 0.67$$
$$0.2\% = 0.002$$
$$150\% = 1.5$$

And to convert a decimal to a percent, just move the decimal point two spaces to the left:

$$\begin{aligned}2.3 &= 230\%\\0.97 &= 97\%\\0.00002 &= 0.002\%\end{aligned}$$

Finding and Converting Percentages

Before we work with percentages, we need to know how to convert between percentages, decimals and fractions.

Converting percentages to fractions is the easiest. The word "percent" simply means "per 100"—so, for example, 55% means 55 per 100, or $\frac{55}{100}$. This fraction can then be simplified to $\frac{11}{20}$.

Example A

Convert 32.5% to a fraction.

Solution

32.5% is equal to 32.5 per 100, or $\frac{32.5}{100}$. To reduce this fraction, we first need to multiply it by $\frac{10}{10}$ to get rid of the decimal point. $\frac{325}{1000}$ then reduces to $\frac{13}{40}$.

Converting fractions to percentages can be a little harder. To convert a fraction directly to a percentage, you need to express it as an equivalent fraction with a denominator of 100.

Example B

Convert $\frac{7}{8}$ to a percent.

Solution

To get the denominator of this fraction equal to 100, we have to multiply it by 12.5. Multiplying the numerator by 12.5 also, we get $\frac{87.5}{100}$, which is equivalent to 87.5%.

But what about a fraction like $\frac{1}{6}$, where there's no convenient number to multiply the denominator by to get 100? In a case like this, it's easier to do the division problem suggested by the fraction in order to convert the fraction to a decimal, and *then* convert the decimal to a percent. 1 divided by 6 works out to 0.166666.... Moving the decimal two spaces to the right tells us that this is equivalent to about 16.7%.

Why can we convert from decimals to percents just by moving the decimal point? Because of what decimal places represent. 0.1 is another way of representing one tenth, and 0.01 is equal to one hundredth—and one hundredth is one percent. By the same logic, 0.02 is 2 percent, 0.35 is 35 percent, and so on.

Example C

Convert 2.64 to a percent.

Solution

To convert to a percent, simply move the decimal two places to the right. $2.64 = 264\%$.

Does a percentage greater than 100 even make sense? Sure it does—percentages greater than 100 come up in real life all the time. For example, a business that made 10 million dollars last year and 13 million dollars this year would have made 130% as much money this year as it did last year.

The only situation where a percentage greater than 100 doesn't make sense is when you're talking about dividing up something that you only have a fixed amount of—for example, if you took a survey and found that 56% of the respondents gave one answer and 72% gave another answer (for a total of 128%), you'd know something went wrong with your math somewhere, because there's no way you could have gotten answers from more than 100% of the people you surveyed.

Converting percentages to decimals is just as easy as converting decimals to percentages—simply move the decimal to the left instead of to the right.

Example D

Convert 58% to a decimal.

Solution

The decimal point here is invisible—it's right after the 8. So moving it to the left two places gives us 0.58.

It can be hard to remember which way to move the decimal point when converting from decimals to percents or vice versa. One way to check if you're moving it the right way is to check whether your answer is a bigger or smaller number than you started out with. If you're converting from percents to decimals, you should end up with a smaller number—just think of how a number like 50 percent, where 50 is **greater** than 1, represents a fraction like $\frac{1}{2}$ (or 0.50 in decimal form), where $\frac{1}{2}$ is **less** than 1. Conversely, if you're converting from decimals to percents, you should end up with a bigger number.

One way you might remember this is by remembering that a percent sign is bigger than a decimal point—so percents should be bigger numbers than decimals.

Watch this video for help with the Examples above.

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Click image to the left for more content.

CK-12 Foundation: Percentages

Vocabulary

- A **percent** is simply a ratio with a base unit of 100—for example, $13\% = \frac{13}{100}$.
- The **percent equation** is Rate $\times$ Total $=$ Part, or R% of Total is Part.
- The percent change equation is Percent change $= \frac{\text{final amount - original amount}}{\text{original amount}} \times 100\%$. A **positive** percent change means the value **increased**, while a **negative** percent change means the value **decreased**.

Guided Practice

a) *Convert 3.4 to a percent.*

b) *Convert $\frac{2}{7}$ to a percent.*

Solution

a) If you move the decimal point to the left, you get 0.034%. That's a smaller number than you started out with, but you're moving from decimals to percents, so you want the number to get bigger, not smaller. Move it to the right instead to get 340%.

b) $\frac{2}{7}$ doesn't convert easily unless you change it to a decimal first. 2 divided by 7 is approximately 0.285714..., and moving the decimal and rounding gives us 28.6%.

Practice

Express the following decimals as a percent.

1. 0.011
2. 0.001
3. 0.91
4. 1.75
5. 20

Express the following percentages in decimal form

6. 15%
7. 0.08%
8. 222%
9. 3.5%
10. 341.9%

Express the following fractions as a percent (round to two decimal places when necessary)

11. $\frac{1}{6}$
12. $\frac{5}{24}$
13. $\frac{6}{7}$
14. $\frac{11}{7}$
15. $\frac{13}{97}$

Express the following percentages as a reduced fraction.

16. 11%
17. 65%
18. 16%
19. 12.5%
20. 87.5%

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9613.

3.14 Percent Equations

Here you'll learn how to use the percent equation to find the rate, total, or part.

What if you knew that 25% of a number was equal to 24? How could you find that number? After completing this Concept, you'll be able to use the percent equation to solve problems like this one.

Watch This

MEDIA

Click image to the left for more content.

CK-12 Foundation: 0314S The Percent Equation(H264)

Guidance

The percent equation is often used to solve problems. It goes like this:

$$\text{Rate} \times \text{Total} = \text{Part}$$
$$\text{or}$$
$$R\% \text{ of Total is Part}$$

Rate is the ratio that the percent represents ($R\%$ in the second version).

Total is often called the ***base unit***.

Part is the amount we are comparing with the base unit.

Example A

Find 25% of $80.

Solution

We are looking for the ***part***. The ***total*** is $80. 'of' means multiply. $R\%$ is 25%, so we can use the second form of the equation: 25% of $80 is Part, or $0.25 \times 80 = \text{Part}$.

$0.25 \times 80 = 20$, so the Part we are looking for is **$20**.

Example B

Express $90 as a percentage of $160.

Solution

This time we are looking for the ***rate***. We are given the ***part*** (\$90) and the ***total*** (\$160). Using the rate equation, we get $\text{Rate} \times 160 = 90$. Dividing both sides by 160 tells us that the rate is 0.5625, or 56.25%.

Example C

\$50 is 15% of what total sum?

Solution

This time we are looking for the ***total***. We are given the ***part*** (\$50) and the ***rate*** (15%, or 0.15). Using the rate equation, we get $0.15 \times \text{Total} = \$50$. Dividing both sides by 0.15, we get $\text{Total} = \frac{50}{0.15} \approx 333.33$. So **\$50 is 15% of \$333.33.**

Watch this video for help with the Examples above.

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CK-12 Foundation: The Percent Equation

Vocabulary

- A **percent** is simply a ratio with a base unit of 100—for example, $13\% = \frac{13}{100}$.
- The **percent equation** is Rate × Total = Part, or R% of Total is Part.
- The percent change equation is $\text{Percent change} = \frac{\text{final amount - original amount}}{\text{original amount}} \times 100\%$. A **positive** percent change means the value **increased**, while a **negative** percent change means the value **decreased**.

Guided Practice

\$96 is 12% of what total sum?

Solution:

This time we are looking for the ***total***. We are given the ***part*** (\$96) and the ***rate*** (12%, or 0.12). Using the rate equation, we get $0.12 \times \text{Total} = \$96$. Dividing both sides by 0.15, we get $\text{Total} = \frac{96}{0.12} = 800$. So **\$96 is 12% of \$800.**

Practice

Find the following.

1. 30% of 90
2. 27% of 19
3. 16.7% of 199
4. 11.5% of 10.01
5. 0.003% of 1,217.46
6. 250% of 67
7. 34.5% of y

8. 17.02% of y
9. x% of 280
10. a% of 0.332
11. y% of $3x$

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9613.

3.15 Percent of Change

Here you'll learn how to use the percent change equation to find how much a value increases or decreases.

What if a printer that normally cost \$125 were marked down to \$100. How could you calculate the percent it was marked down by? After completing this Concept, you'll be able to determine the percent of change in problems like this one.

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CK-12 Foundation: 0315S Percent of Change (H264)

Guidance

A useful way to express changes in quantities is through percents. You've probably seen signs such as "20% extra free," or "save 35% today." When we use percents to represent a change, we generally use the formula

$$\text{Percent change} = \frac{\text{final amount - original amount}}{\text{original amount}} \times 100\%$$

or

$$\frac{\text{percent change}}{100} = \frac{\text{actual change}}{\text{original amount}}$$

This means that a **positive** percent change is an **increase**, while a **negative** change is a **decrease**.

Example A

A school of 500 students is expecting a 20% increase in students next year. How many students will the school have?

Solution

First let's solve this using the first formula. Since the 20% change is an increase, we represent it in the formula as 20 (if it were a decrease, it would be -20.) Plugging in all the numbers, we get

$$20\% = \frac{\text{final amount} - 500}{500} \times 100\%$$

Dividing both sides by 100%, we get $0.2 = \frac{\text{final amount}-500}{500}$.

Multiplying both sides by 500 gives us $100 = \text{final amount} - 500$.

Then adding 500 to both sides gives us 600 as the final number of students.

How about if we use the second formula? Then we get $\frac{20}{100} = \frac{\text{actual change}}{500}$. (Reducing the first fraction to $\frac{1}{5}$ will make the problem easier, so let's rewrite the equation as $\frac{1}{5} = \frac{\text{actual change}}{500}$.

Cross multiplying is our next step; that gives us $500 = 5 \times (\text{actual change})$. Dividing by 5 tells us the change is equal to 100. We were told this was an increase, so if we start out with 500 students, after an increase of 100 we know there will be a total of 600.

Markup

A **markup** is an increase from the price a store pays for an item from its supplier to the retail price it charges to the public. For example, a 100% mark-up (commonly known in business as *keystone*) means that the price is doubled. Half of the retail price covers the cost of the item from the supplier, half is profit.

Example B

A furniture store places a 30% markup on everything it sells. It offers its employees a 20% discount from the sales price. The employees are demanding a 25% discount, saying that the store would still make a profit. The manager says that at a 25% discount from the sales price would cause the store to lose money. Who is right?

Solution

We'll consider this problem two ways. First, let's consider an item that the store buys from its supplier for a certain price, say \$1000. The markup would be 30% of 1000, or \$300, so the item would sell for \$1300 and the store would make a \$300 profit.

And what if an employee buys the product? With a discount of 20%, the employee would pay 80% of the \$1300 retail price, or $0.8 \times \$1300 = \$1040$.

But with a 25% discount, the employee would pay 75% of the retail price, or $0.75 \times \$1300 = \$975$.

So with a 20% employee discount, the store still makes a \$40 profit on the item they bought for \$1000—but with a 25% employee discount, the store loses \$25 on the item.

Now let's use algebra to see how this works for an item of any price. If x is the price of an item, then the store's markup is 30% of x, or $0.3x$, and the retail price of the item is $x + 0.3x$, or $1.3x$. An employee buying the item at a 20% discount would pay $0.8 \times 1.3x = 1.04x$, while an employee buying it at a 25% discount would pay $0.75 \times 1.3x = 0.975x$.

So the manager is right: a 20% employee discount still allows the store to make a profit, while a 25% employee discount would cause the store to lose money.

It may not seem to make sense that the store would lose money after applying a 30% markup and only a 25% discount. The reason it does work out that way is that the discount is bigger in absolute dollars after the markup is factored in. That is, an employee getting 25% off an item is getting 25% off the original price *plus* 25% off the 30% markup, and those two numbers together add up to more than 30% of the original price.

Solve Real-World Problems Using Percents

Example C

In 2004 the US Department of Agriculture had 112071 employees, of which 87846 were Caucasian. Of the remaining minorities, African-American and Hispanic employees had the two largest demographic groups, with 11754 and

*6899 employees respectively.**

a) *Calculate the total percentage of minority (non-Caucasian) employees at the USDA.*

b) *Calculate the percentage of African-American employees at the USDA.*

c) *Calculate the percentage of minority employees who were neither African-American nor Hispanic.*

Solution

a) Use the percent equation $\text{Rate} \times \text{Total} = \text{Part}$.

The *total* number of employees is 112071. We know that the number of Caucasian employees is 87846, which means that there must be $112071 - 87646 = 24225$ non-Caucasian employees. This is the *part*. Plugging in the total and the part, we get $\text{Rate} \times 112071 = 24225$.

Divide both sides by 112071 to get $\text{Rate} = \frac{24225}{112071} \approx 0.216$. Multiply by 100 to get this as a percent: 21.6%.

21.6% of USDA employees in 2004 were from minority groups.

b) Here, the total is still 112071 and the part is 11754, so we have $\text{Rate} \times 112071 = 11754$. Dividing, we get $\text{Rate} = \frac{11754}{112071} \approx 0.105$, or 10.5%.

10.5% of USDA employees in 2004 were African-American.

c) Here, our total is just the number of non-Caucasian employees, which we found out is 24225. Subtracting the African-American and Hispanic employees leaves $24225 - 11754 - 6899 = 5572$ employees in the group we're looking at.

So with 24225 for the whole and 5572 for the part, our equation is $\text{Rate} \times 24225 = 5572$, or $\text{Rate} = \frac{5572}{24225} \approx 0.230$, or 23%.

23% of USDA minority employees in 2004 were neither African-American nor Hispanic.

Watch this video for help with the Examples above.

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CK-12 Foundation: Percent ofChange

Vocabulary

- A **percent** is simply a ratio with a base unit of 100—for example, $13\% = \frac{13}{100}$.
- The **percent equation** is $\text{Rate} \times \text{Total} = \text{Part}$, or R% of Total is Part.
- The percent change equation is $\text{Percent change} = \frac{\text{final amount - original amount}}{\text{original amount}} \times 100\%$. A **positive** percent change means the value **increased**, while a **negative** percent change means the value **decreased**.

Guided Practice

In 1995 New York had 18136000 residents. There were 827025 reported crimes, of which 152683 were violent. By 2005 the population was 19254630 and there were 85839 violent crimes out of a total of 491829 reported crimes. (Source: New York Law Enforcement Agency Uniform Crime Reports.) Calculate the percentage change from 1995 to 2005 in:

a) *Population of New York*

b) *Total reported crimes*

c) *violent crimes*

Solution

This is a percentage change problem. Remember the formula for percentage change:

$$\text{Percent change} = \frac{\text{final amount - original amount}}{\text{original amount}} \times 100\%$$

In these problems, the final amount is the 2005 statistic, and the initial amount is the 1995 statistic.

a) Population:

$$\begin{aligned}\text{Percent change} &= \frac{19254630 - 18136000}{18136000} \times 100\% \\ &= \frac{1118630}{18136000} \times 100\% \\ &\approx 0.0617 \times 100\% \\ &= 6.17\%\end{aligned}$$

The population grew by 6.17%.

b) Total reported crimes:

$$\begin{aligned}\text{Percent change} &= \frac{491829 - 827025}{827025} \times 100\% \\ &= \frac{-335196}{827025} \times 100\% \\ &\approx -0.4053 \times 100\% \\ &= -40.53\%\end{aligned}$$

The total number of reported crimes fell by 40.53%.

c) Violent crimes:

$$\begin{aligned}\text{Percent change} &= \frac{85839 - 152683}{152683} \times 100\% \\ &= \frac{-66844}{152683} \times 100\% \\ &\approx -0.4377 \times 100\% \\ &= -43.77\%\end{aligned}$$

The total number of violent crimes fell by 43.77%.

Practice

For questions 1-3, a hair stylist charges $70 for a haircut. Depending on how much you tip, what will be the total cost of the haircut?

1. You tip 15%.
2. You tip 20%.
3. You tip 25%.
4. 250 is what percentage of 195?
5. 0.0032 is what percentage of 0.045?
6. An employee at a store is currently paid $9.50 per hour. If she works a full year she gets a 12% pay raise. What will her new hourly rate be after the raise?
7. A TV is advertised on sale. It is 35% off and now costs $195. What was the pre-sale price?
8. A TV was advertised on sale. If you saved $40, and bought it for $160, what percentage off was it?
9. Another TV is advertised on sale. If this TV is also $40 cheaper than the pre-sale price, was it also the same percentage off as the TV in the question above? Explain!
10. Store A and Store B both sell bikes, and both buy bikes from the same supplier at the same prices. Store A has a 40% mark-up for their prices, while store B has a 250% mark-up. Store B has a permanent sale and will always sell at 60% off the marked-up prices. Which store offers the better deal?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9613.

Summary

This chapter begins with one-step equations involving addition, subtraction, multiplication, and division. It then moves on to more sophisticated equations that require two or multiple steps to solve. Here, you'll learn how to combine like terms. Next, it introduces equations that have variables on both sides of the equal sign. Ratios and proportions are also discussed as are problems that entail scales and indirect measurement. The chapter concludes with a thorough discussion of percentages, an introduction to the percent equation, and a collection of real-life problems that involve percent of change.

CHAPTER 4 Graphs of Equations and Functions

Chapter Outline

4.1 POINTS IN THE COORDINATE PLANE
4.2 GRAPHS IN THE COORDINATE PLANE
4.3 GRAPHS OF LINEAR EQUATIONS
4.4 HORIZONTAL AND VERTICAL LINE GRAPHS
4.5 INTERCEPTS AND THE COVER-UP METHOD
4.6 SLOPE
4.7 RATES OF CHANGE
4.8 GRAPHS USING SLOPE-INTERCEPT FORM
4.9 GRAPHS OF LINEAR MODELS OF DIRECT VARIATION
4.10 GRAPHS OF LINEAR FUNCTIONS
4.11 PROBLEM SOLVING WITH LINEAR GRAPHS

Introduction

Given a table or a function rule, how can you represent the data visually in two dimensions? Learning how to graph equations is not just a skill you will use in algebra. It is one you will carry with you throughout your mathematics studies.

This chapter will focus on graphing linear equations. After learning how to graph lines from tables and functions, you'll be introduced to a line's intercepts. You'll also use graphs to find the slope of a line and the rate of change of a linear function.

Conversions, like dollars to Euros, kilograms to pounds, and Fahrenheit to Celsius, can all be modeled using linear equations. This chapter focuses on these applications and many more.

4.1 Points in the Coordinate Plane

Here you'll learn how to identify the coordinates of points, and given their coordinates, you'll plot points in the coordinate plane.

What if you were given the *x*- and *y*-coordinates of a point like (-2, 3). How could you determine in which quadrant of the coordinate plane this point would lie? After completing this Concept, you'll be able to plot points like this one given their coordinates.

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CK-12 Foundation: 0401S Points in the CoordinatePlane (H264)

Try This

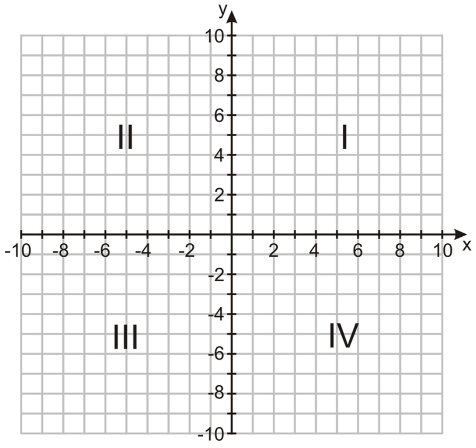
For more practice locating and naming points on the coordinate plane, try playing the Coordinate Plane Game at http://www.math-play.com/Coordinate%20Plane%20Game/Coordinate%20Plane%20Game.html.

Guidance

Lydia lives 2 blocks north and one block east of school; Travis lives three blocks south and two blocks west of school. What's the shortest line connecting their houses?

The Coordinate Plane

We've seen how to represent numbers using number lines; now we'll see how to represent sets of numbers using a **coordinate plane.** The coordinate plane can be thought of as two number lines that meet at right angles. The horizontal line is called the $x-$**axis** and the vertical line is the $y-$**axis**. Together the lines are called the **axes**, and the point at which they cross is called the **origin**. The axes split the coordinate plane into four **quadrants**, which are numbered sequentially (I, II, III, IV) moving counter-clockwise from the upper right.



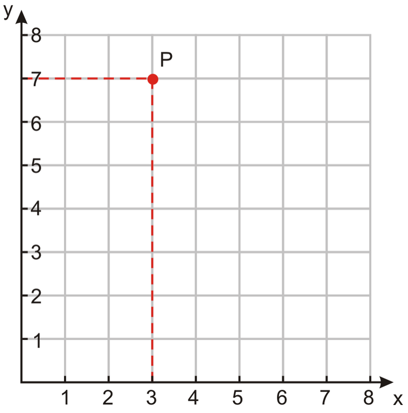
Identify Coordinates of Points

When given a point on a coordinate plane, it's easy to determine its **coordinates**. The coordinates of a point are two numbers - written together they are called an **ordered pair**. The numbers describe how far along the $x-$axis and $y-$axis the point is. The ordered pair is written in parentheses, with the $x-$**coordinate** (also called the **abscissa**) first and the $y-$**coordinate** (or the **ordinate**) second.

$(1,7)$	An ordered pair with an $x-$ value of one and a $y-$ value of seven
$(0,5)$	An ordered pair with an $x-$ value of zero and a $y-$ value of five
$(-2.5,4)$	An ordered pair with an $x-$ value of -2.5 and a $y-$ value of four
$(-107.2,-.005)$	An ordered pair with an $x-$ value of -107.2 and a $y-$ value of $-.005$

Identifying coordinates is just like reading points on a number line, except that now the points do not actually lie **on** the number line! Look at the following example.

Example A



Find the coordinates of the point labeled P in the diagram above

Solution

Imagine you are standing at the origin (the point where the $x-$axis meets the $y-$axis). In order to move to a position where P was directly above you, you would move 3 units to the **right** (we say this is in the **positive** $x-$direction).

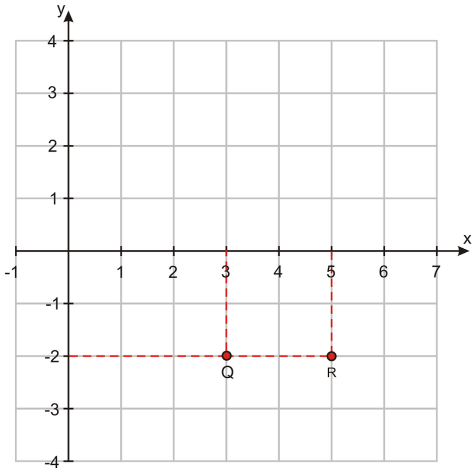
The $x-$coordinate of P is +3.

Now if you were standing at the 3 marker on the $x-$axis, point P would be 7 units **above** you (above the axis means it is in the **positive** y direction).

The $y-$coordinate of P is +7.

The coordinates of point P are (3, 7).

Example B



Find the coordinates of the points labeled Q and R in the diagram to the right.

Solution

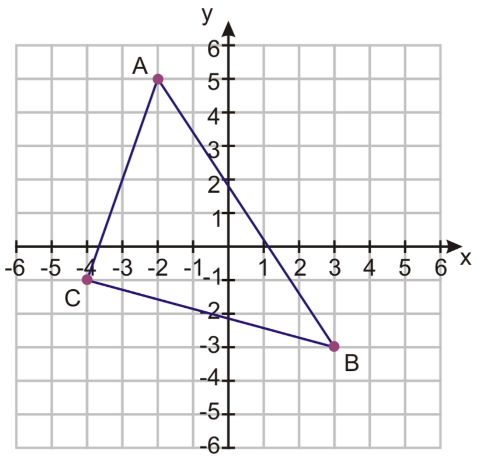
In order to get to Q we move three units to the right, in the positive $x-$direction, then two units **down**. This time we are moving in the **negative** $y-$direction. The $x-$coordinate of Q is +3, the $y-$coordinate of Q is -2.

The coordinates of R are found in a similar way. The $x-$coordinate is +5 (five units in the positive $x-$direction) and the $y-$coordinate is again -2.

The coordinates of Q are (3, -2). The coordinates of R are (5, -2).

Example C

Triangle ABC is shown in the diagram to the right. Find the coordinates of the vertices A, B and C.



Point A:

$x-\text{coordinate} = -2$

$y-\text{coordinate} = +5$

Point B:

$x-\text{coordinate} = +3$

$y-\text{coordinate} = -3$

Point C:

$x-\text{coordinate} = -4$

$y-\text{coordinate} = -1$

Solution

$A(-2,5)$

$B(3,-3)$

$C(-4,-1)$

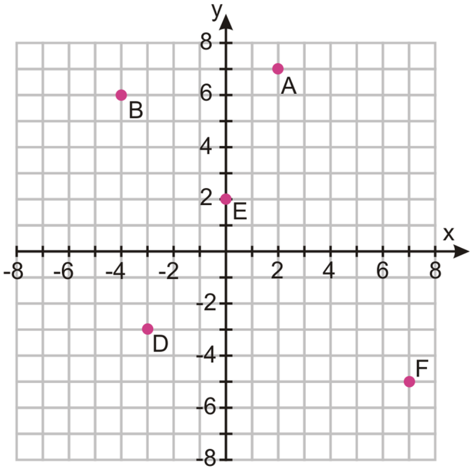
Plot Points in a Coordinate Plane

Plotting points is simple, once you understand how to read coordinates and read the scale on a graph. As a note on scale, in the next two examples pay close attention to the labels on the axes.

Example D

Plot the following points on the coordinate plane.

$A(2,7) \quad B(-4,6) \quad D(-3,-3) \quad E(0,2) \quad F(7,-5)$



Point $A(2,7)$ is 2 units right, 7 units up. It is in Quadrant I.

Point $B(-4,6)$ is 4 units left, 6 units up. It is in Quadrant II.

Point $D(-3,-3)$ is 3 units left, 3 units down. It is in Quadrant III.

Point $E(0,2)$ is 2 units up from the origin. It is right on the $y-$axis, between Quadrants I and II.

Point $F(7,-5)$ is 7 units right, 5 units down. It is in Quadrant IV.

Watch this video for help with the Examples above.

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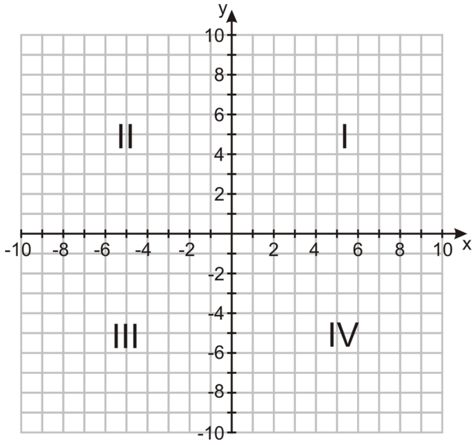
CK-12 Foundation: Pointsin the Coordinate Plane

Vocabulary



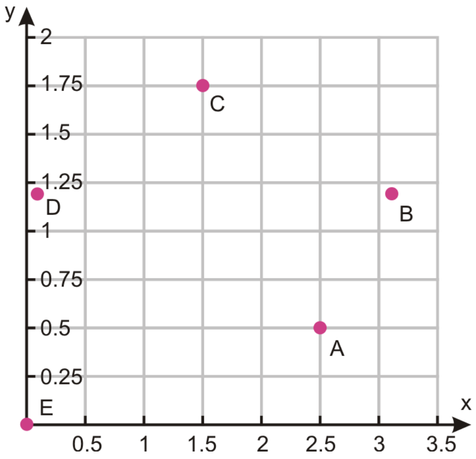
- The **coordinate plane** is a two-dimensional space defined by a horizontal number line (the $x-$**axis**) and a vertical number line (the $y-$**axis**). The **origin** is the point where these two lines meet. Four areas, or **quadrants,** are formed as shown in the diagram above.
- Each point on the coordinate plane has a set of **coordinates,** two numbers written as an **ordered pair** which describe how far along the $x-$axis and $y-$axis the point is. The $x-$**coordinate** is always written first, then the $y-$**coordinate**, in the form (x,y).
- **Functions** are a way that we can relate one quantity to another. Functions can be plotted on the coordinate plane.

Guided Practice

Plot the following points on the coordinate plane.

$A(2.5,0.5)$ $B(\pi,1.2)$ $C(2,1.75)$ $D(0.1,1.2)$ $E(0,0)$

Solution:



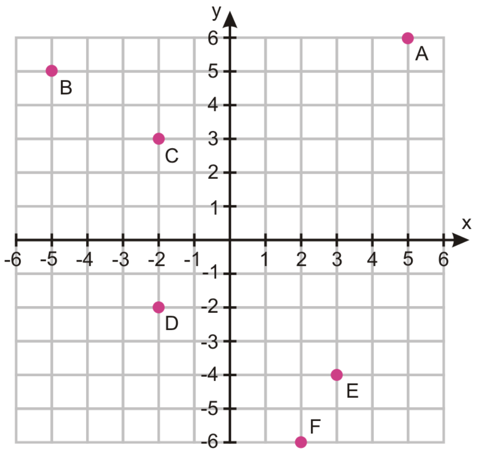
Here we see the importance of choosing the right scale and range for the graph. In Example 4, our points were scattered throughout the four quadrants. In this case, all the coordinates are positive, so we don't need to show the negative values of x or y. Also, there are no $x-$values bigger than about 3.14, and 1.75 is the largest value of y. We can therefore show just the part of the coordinate plane where $0 \leq x \leq 3.5$ and $0 \leq y \leq 2$.

Here are some other important things to notice about this graph:

- The tick marks on the axes don't correspond to unit increments (i.e. the numbers do not go up by one each time). This is so that we can plot the points more precisely.
- The scale on the $x-$axis is different than the scale on the $y-$axis, so distances that look the same on both axes are actually greater in the $x-$direction. Stretching or shrinking the scale in one direction can be useful when the points we want to plot are farther apart in one direction than the other.

Practice

1. Identify the coordinates of each point, $A-F$, on the graph below.



2. Draw a line on the above graph connecting point B with the origin. Where does that line intersect the line connecting points C and D?

Plot the following points on a graph and identify which quadrant each point lies in:

3. (4, 2)
4. (-3, 5.5)
5. (4, -4)

6. (-2, -3)

Without graphing the following points, identify which quadrant each lies in:

7. (5, 3)
8. (-3, -5)
9. (-4, 2)
10. (2, -4)

4.2 Graphs in the Coordinate Plane

Here you'll learn how to make a table of values and graph a function given the function's rule.

What if you were given a function rule like $y = 2x - 3$? How could you graph that function in the coordinate plane? After completing this Concept, you'll be able to graph functions like this one by either creating a table of values or using its slope and intercept.

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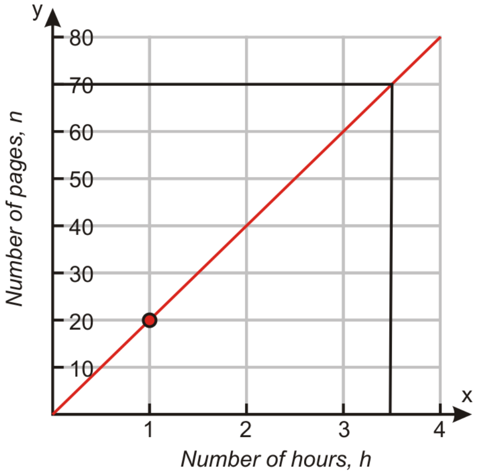
CK-12 Foundation: 0402S Graphs in the CoordinatePlane (H264)-1

Guidance

Once we know how to plot points on a coordinate plane, we can think about how we'd go about plotting a relationship between $x-$and $y-$values. So far we've just been plotting sets of ordered pairs. A set like that is a **relation**, and there isn't necessarily a relationship between the $x-$values and $y-$values. If there is a relationship between the $x-$and $y-$values, and each $x-$value corresponds to exactly one $y-$value, then the relation is called a **function**. Remember that a function is a particular way to relate one quantity to another.

Example A

If you're reading a book and can read twenty pages an hour, there is a relationship between how many hours you read and how many pages you read. You may even know that you could write the formula as either $n = 20h$ or $h = \frac{n}{20}$, where h is the number of hours you spend reading and n is the number of pages you read. To find out, for example, how many pages you could read in $3\frac{1}{2}$ hours, or how many hours it would take you to read 46 pages, you could use one of those formulas. Or, you could make a graph of the function:



Once you know how to graph a function like this, you can simply read the relationship between the $x-$and $y-$values off the graph. You can see in this case that you could read 70 pages in $3\frac{1}{2}$ hours, and it would take you about $2\frac{1}{3}$ hours to read 46 pages.

Generally, the graph of a **function** appears as a line or curve that goes through all points that have the relationship that the function describes. If the domain of the function (the set of $x-$values we can plug into the function) is all real numbers, then we call it a **continuous function**. If the domain of the function is a particular set of values (such as whole numbers only), then it is called a **discrete function**. The graph will be a series of dots, but they will still often fall along a line or curve.

In graphing equations, we assume the domain is all real numbers, unless otherwise stated. Often, though, when we look at data in a table, the domain will be whole numbers (number of presents, number of days, etc.) and the function will be discrete. But sometimes we'll still draw the graph as a continuous line to make it easier to interpret. Be aware of the difference between discrete and continuous functions as you work through the examples.

Example B

Sarah is thinking of the number of presents she receives as a function of the number of friends who come to her birthday party. She knows she will get a present from her parents, one from her grandparents and one each from her uncle and aunt. She wants to invite up to ten of her friends, who will each bring one present. She makes a table of how many presents she will get if one, two, three, four or five friends come to the party. Plot the points on a coordinate plane and graph the function that links the number of presents with the number of friends. Use your graph to determine how many presents she would get if eight friends show up.

TABLE 4.1:

Number of Friends	Number of Presents
0	4
1	5
2	6
3	7
4	8
5	9

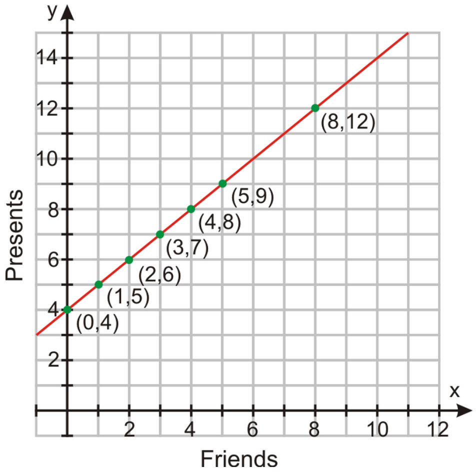
The first thing we need to do is decide how our graph should appear. We need to decide what the independent variable is, and what the dependant variable is. Clearly in this case, the number of friends can vary independently, but the number of presents must depend on the number of friends who show up.

So we'll plot friends on the $x-$axis and presents on the $y-$axis. Let's add another column to our table containing the coordinates that each (friends, presents) ordered pair gives us.

TABLE 4.2:

Friends (x)	**Presents** (y)	**Coordinates** (x,y)
0	4	(0, 4)
1	5	(1, 5)
2	6	(2, 6)
3	7	(3, 7)
4	8	(4, 8)
5	9	(5, 9)

Next we need to set up our axes. It is clear that the number of friends and number of presents both must be positive, so we only need to show points in Quadrant I. Now we need to choose a suitable scale for the $x-$ and $y-$axes. We only need to consider eight friends (look again at the question to confirm this), but it always pays to allow a little extra room on your graph. We also need the $y-$scale to accommodate the presents for eight people. We can see that this is still going to be under 20!



The scale of this graph has room for up to 12 friends and 15 presents. This will be fine, but there are many other scales that would be equally good!

Now we proceed to plot the points. The first five points are the coordinates from our table. You can see they all lie on a straight line, so the function that describes the relationship between x and y will be linear. To graph the function, we simply draw a line that goes through all five points. This line represents the function.

This is a discrete problem since Sarah can only invite a positive whole number of friends. For instance, it would be impossible for 2.4 or -3 friends to show up. So although the line helps us see where the other values of the function are, the only points on the line that actually *are* values of the function are the ones with positive whole-number coordinates.

The graph easily lets us find other values for the function. For example, the question asks how many presents Sarah would get if eight friends come to her party. Don't forget that x represents the number of friends and y represents the number of presents. If we look at the graph where $x = 8$, we can see that the function has a $y-$value of 12.

Solution

If 8 friends show up, Sarah will receive a total of 12 presents.

Graph a Function Given a Rule

If we are given a rule instead of a table, we can proceed to graph the function in either of two ways. We will use the following example to show each way.

Example C

Ali is trying to work out a trick that his friend showed him. His friend started by asking him to think of a number, then double it, then add five to the result. Ali has written down a rule to describe the first part of the trick. He is using the letter x to stand for the number he thought of and the letter y to represent the final result of applying the rule. He wrote his rule in the form of an equation: $y = 2x + 5$.

Help him visualize what is going on by graphing the function that this rule describes.

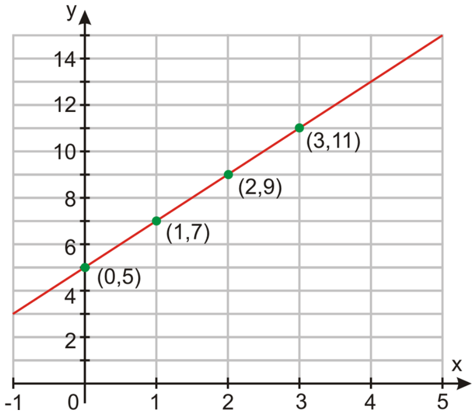
Method One - Construct a Table of Values

If we wish to plot a few points to see what is going on with this function, then the best way is to construct a table and populate it with a few (x, y) pairs. We'll use 0, 1, 2 and 3 for $x-$values.

TABLE 4.3:

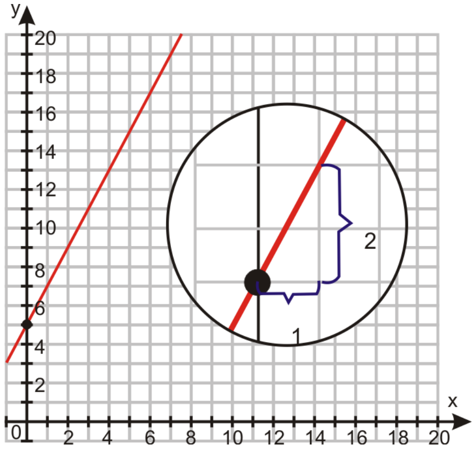
x	y
0	5
1	7
2	9
3	11

Next, we plot the points and join them with a line.



This method is nice and simple—especially with linear relationships, where we don't need to plot more than two or three points to see the shape of the graph. In this case, the function is continuous because the domain is all real numbers—that is, Ali could think of any real number, even though he may only be thinking of positive whole numbers.

Method Two - Intercept and Slope



Another way to graph this function (one that we'll learn in more detail in a later lesson) is the slope-intercept method. To use this method, follow these steps:

1. Find the y value when $y = 0$.

$y(0) = 2 \cdot 0 + 5 = 5$, *so our* $y-$*intercept is* $(0, 5)$.

2. Look at the coefficient multiplying the x.

Every time we increase x by one, y increases by two, so our slope is +2.

3. Plot the line with the given slope that goes through the intercept. We start at the point (0, 5) and move over one in the $x-$direction, then up two in the $y-$direction. This gives the slope for our line, which we extend in both directions.

We will properly examine this last method later in this chapter!

Watch this video for help with the Examples above.

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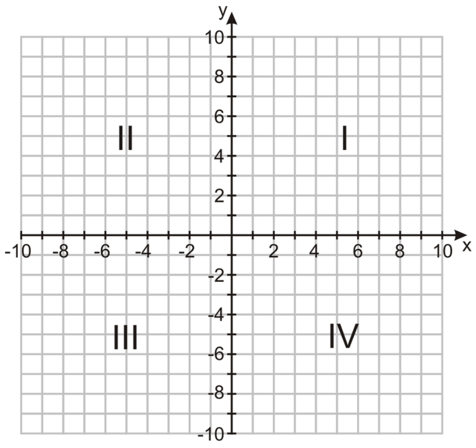
CK-12 Foundation: Graphsin the Coordinate Plane

Vocabulary



- The **coordinate plane** is a two-dimensional space defined by a horizontal number line (the $x-$**axis**) and a vertical number line (the $y-$**axis**). The **origin** is the point where these two lines meet. Four areas, or **quadrants,** are formed as shown in the diagram above.
- Each point on the coordinate plane has a set of **coordinates,** two numbers written as an **ordered pair** which describe how far along the $x-$axis and $y-$axis the point is. The $x-$**coordinate** is always written first, then the $y-$**coordinate**, in the form (x,y).
- **Functions** are a way that we can relate one quantity to another. Functions can be plotted on the coordinate plane.

Guided Practice

1. *The point (0, -2) is the boundary of which two quadrants?*

2. *If you move the point (-3, 4) down 5, what quadrant would it be in?*

Solutions:

1. Since the x-value is 0, the point is on the y-axis. Since the y-value is negative, the point is on the lower half of the y-axis. This is the boundary between the 3rd and 4th quadrants.

2. Moving the point down 5 is equivalent to subtracting 5 from the y-value. $(-3,4-5)=(-3,-1)$. Since both coordinates are now negative, this is in the 3rd quadrant.

Practice

1. Consider the graph of the equation $y=3$. Which quadrants does it pass through?
2. Consider the graph of the equation $y=x$. Which quadrants does it pass through?
3. Consider the graph of the equation $y=x+3$. Which quadrants does it pass through?
4. The point (4, 0) is on the boundary between which two quadrants?
5. The point (0, -5) is on the boundary between which two quadrants?
6. If you moved the point (3, 2) five units to the left, what quadrant would it be in?
7. The following three points are three vertices of square $ABCD$. Plot them on a graph, then determine what the coordinates of the fourth point, D, would be. Plot that point and label it.

$$A(-4,-4)\ B(3,-4)\ C(3,3)$$

8. In what quadrant is the center of the square from problem 10? (You can find the center by drawing the square's diagonals.)
9. What point is halfway between (1, 3) and (1, 5)?
10. What point is halfway between (2, 8) and (6, 8)?
11. What point is halfway between the origin and (10, 4)?
12. What point is halfway between (3, -2) and (-3, 2)?
13. Becky has a large bag of MMs that she knows she should share with Jaeyun. Jaeyun has a packet of Starburst. Becky tells Jaeyun that for every Starburst he gives her, she will give him three MMs in return. If x is the number of Starburst that Jaeyun gives Becky, and y is the number of MMs he gets in return, then complete each of the following.
    a. Write an algebraic rule for y in terms of x.

    a. Make a table of values for y with x-values of 0, 1, 2, 3, 4, 5.

    a. Plot the function linking x and y on the following scale: $0\leq x\leq 10,\ 0\leq y\leq 10$.

4.3 Graphs of Linear Equations

Here you'll learn how to graph linear equations and make predictions by analyzing such graphs.

What if a bowling alley charged \$3 for the first game and \$2 for each additional game? How could you graph this functional relationship and use it to find the cost of playing 5 games? After completing this Concept, you'll be able to graph and analyze linear functions like this one.

Watch This

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Click image to the left for more content.

CK-12 Foundation: 0403S Graphs of LinearEquations (H264)

Guidance

You're stranded downtown late at night with only \$8 in your pocket, and your home is 6 miles away. Two cab companies serve this area; one charges \$1.20 per mile with an additional \$1 fee, and the other charges \$0.90 per mile with an additional \$2.50 fee. Which cab will be able to get you home?

Graph a Linear Equation

At the end of Lesson 4.1 we looked at ways to graph a function from a rule. A rule is a way of writing the relationship between the two quantities we are graphing. In mathematics, we tend to use the words **formula** and **equation** to describe the rules we get when we express relationships algebraically. Interpreting and graphing these equations is an important skill that you'll use frequently in math.

Example A

A taxi costs more the further you travel. Taxis usually charge a fee on top of the per-mile charge to cover hire of the vehicle. In this case, the taxi charges \$3 as a set fee and \$0.80 per mile traveled. Here is the equation linking the cost in dollars (y) *to hire a taxi and the distance traveled in miles* (x).

$$y = 0.8x + 3$$

Graph the equation and use your graph to estimate the cost of a seven-mile taxi ride.

Solution

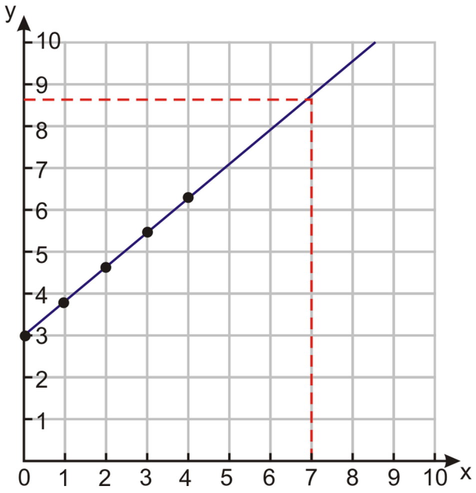
We'll start by making a table of values. We will take a few values for x (0, 1, 2, 3, and 4), find the corresponding $y-$values, and then plot them. Since the question asks us to find the cost for a seven-mile journey, we need to choose a scale that can accommodate this.

First, here's our table of values:

TABLE 4.4:

x	y
0	3
1	3.8
2	4.6
3	5.4
4	6.2

And here's our graph:



To find the cost of a seven-mile journey, first we find $x = 7$ on the horizontal axis and draw a line up to our graph. Next, we draw a horizontal line across to the $y-$axis and read where it hits. It appears to hit around half way between $y = 8$ and $y = 9$. Let's call it 8.5.

A seven mile taxi ride would cost approximately $8.50 ($8.60 exactly).

Here are some things you should notice about this graph and the formula that generated it:

- The graph is a straight line (this means that the equation is **linear**), although the function is **discrete** and really just consists of a series of points.
- The graph crosses the $y-$axis at $y = 3$ (notice that there's $a + 3$ in the equation—that's not a coincidence!). This is the base cost of the taxi.
- Every time we move **over** by one square we move **up** by 0.8 squares (notice that that's also the coefficient of x in the equation). This is the rate of charge of the taxi (cost per mile).
- If we move over by three squares, we move up by 3×0.8 squares.

Example B

A small business has a debt of $500,000 incurred from start-up costs. It predicts that it can pay off the debt at a rate of $85,000 per year according to the following equation governing years in business (x) *and debt measured in thousands of dollars* (y).

$$y = -85x + 500$$

Graph the above equation and use your graph to predict when the debt will be fully paid.

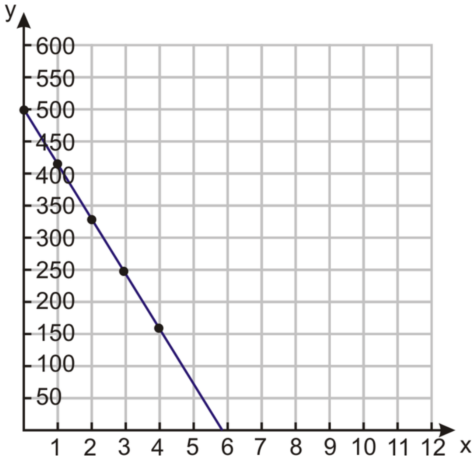
Solution

First, we start with our table of values:

TABLE 4.5:

x	y
0	500
1	415
2	330
3	245
4	160

Then we plot our points and draw the line that goes through them:



Notice the scale we've chosen here. There's no need to include any points above $y = 500$, but it's still wise to allow a little extra.

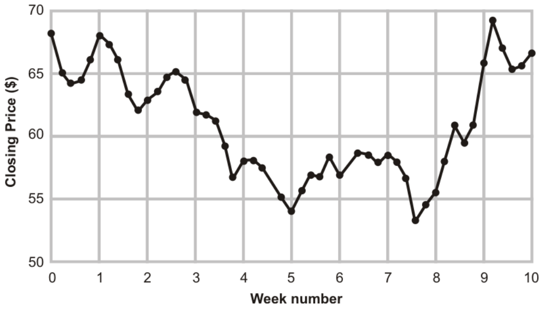
Next we need to determine how many years it takes the debt to reach zero, or in other words, what $x-$value will make the $y-$value equal 0. We know it's greater than four (since at $x = 4$ the $y-$value is still positive), so we need an $x-$scale that goes well past $x = 4$. Here we've chosen to show the $x-$values from 0 to 12, though there are many other places we could have chosen to stop.

To read the time that the debt is paid off, we simply read the point where the line hits $y = 0$ (the $x-$axis). It looks as if the line hits pretty close to $x = 6$. So **the debt will definitely be paid off in six years.**

To see more simple examples of graphing linear equations by hand, see the Khan Academy video on graphing lines at http://www.youtube.com/watch?v=2UrcUfBizyw. The narrator shows how to graph several linear equations, using a table of values to plot points and then connecting the points with a line.

Analyze Graphs of Linear Functions

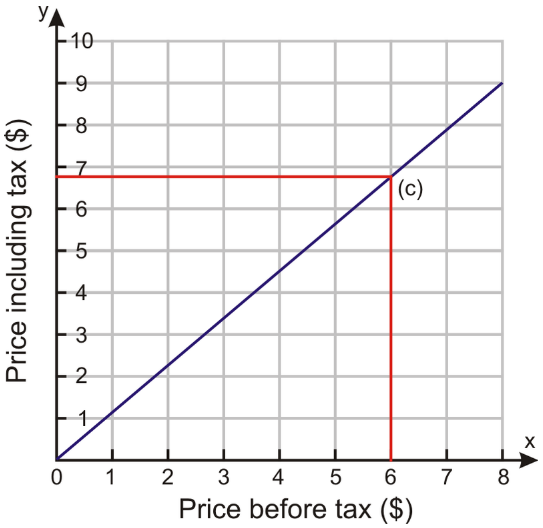
We often use graphs to represent relationships between two linked quantities. It's useful to be able to interpret the information that graphs convey. For example, the chart below shows a fluctuating stock price over ten weeks. You can read that the index closed the first week at about \$68, and at the end of the third week it was at about \$62. You may also see that in the first five weeks it lost about 20% of its value, and that it made about 20% gain between weeks seven and ten. Notice that this relationship is discrete, although the dots are connected to make the graph easier to interpret.



Analyzing graphs is a part of life - whether you are trying to decide to buy stock, figure out if your blog readership is increasing, or predict the temperature from a weather report. Many graphs are very complicated, so for now we'll start off with some simple linear conversion graphs. Algebra starts with basic relationships and builds to more complicated tasks, like reading the graph above.

Example C

Below is a graph for converting marked prices in a downtown store into prices that include sales tax. Use the graph to determine the cost including sales tax for a \$6.00 pen in the store.



To find the relevant price with tax, first find the correct pre-tax price on the $x-$axis. This is the point $x = 6$.

Draw the line $x = 6$ up until it meets the function, then draw a horizontal line to the $y-$axis. This line hits at $y \approx 6.75$ (about three fourths of the way from $y = 6$ to $y = 7$).

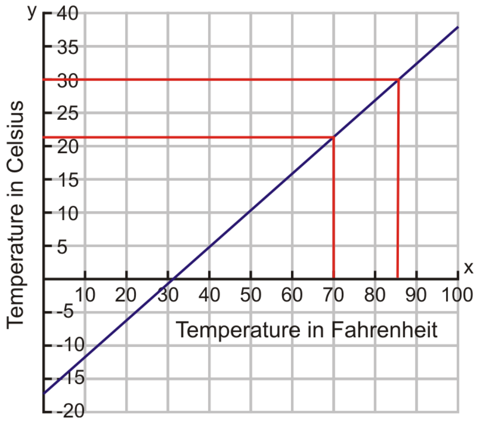
The approximate cost including tax is \$6.75.

Example D

The graph for converting temperature from Fahrenheit to Celsius is shown below. Use the graph to convert the following:

a) 70° Fahrenheit to Celsius

b) 0° Celsius to Fahrenheit



Solution:

a) To find 70° Fahrenheit, we look along the Fahrenheit-axis (in other words the $x-$axis) and draw the line $x = 70$ up to the function. Then we draw a horizontal line to the Celsius-axis ($y-$axis). The horizontal line hits the axis at a little over 20 (21 or 22).

70° **Fahrenheit is approximately equivalent to** 21° **Celsius.**

b) To find 0° Celsius, we look at the Fahrenheit-axis (the line $y = 0$). The function hits the $x-$axis just right of 30.

0° **Celsius is equivalent to** 32° **Fahrenheit.**

Watch this video for help with the Examples above.

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CK-12 Foundation: Graphsof Linear Equations

Vocabulary

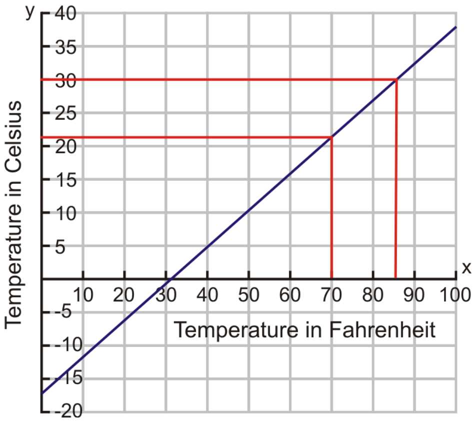
- Be aware that although we graph the function as a line to make it easier to interpret, the function may actually be **discrete**.

Guided Practice

The graph for converting temperature from Fahrenheit to Celsius is shown below. Use the graph to convert the following:

a) 0° Fahrenheit to Celsius

b) 30° Celsius to Fahrenheit



Solution:

a) To find 0° Fahrenheit, we just look at the $y-$axis. (Don't forget that this axis is simply the line $x = 0$.) The line hits the $y-$axis just below the half way point between -15 and -20.

0° **Fahrenheit is approximately equivalent to** $-18°$ **Celsius.**

b) To find 30° Celsius, we look up the Celsius-axis and draw the line $y = 30$ along to the function. When this horizontal line hits the function, we draw a line straight down to the Fahrenheit-axis. The line hits the axis at approximately 85.

30° **Celsius is approximately equivalent to** 85° **Fahrenheit.**

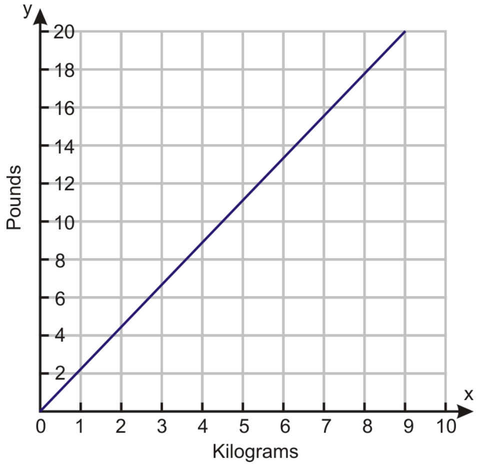
Practice

For 1-3, make a table of values for the following equations and then graph them.

1. $y = 2x + 7$
2. $y = 0.7x - 4$
3. $y = 6 - 1.25x$

4. *"Think of a number. Multiply it by 20, divide the answer by 9, and then subtract seven from the result."*
   a. Make a table of values and plot the function that represents this sentence.
   b. If you picked 0 as your starting number, what number would you end up with?
   c. To end up with 12, what number would you have to start out with?
5. At the airport, you can change your money from dollars into euros. The service costs \$5, and for every additional dollar you get 0.7 euros.
   a. Make a table for this and plot the function on a graph.
   b. Use your graph to determine how many euros you would get if you give the office \$50.
   c. To get 35 euros, how many dollars would you have to pay?
   d. The exchange rate drops so that you can only get 0.5 euros per additional dollar. Now how many dollars do you have to pay for 35 euros?

For 6-9, the graph below shows a conversion chart for converting between weight in kilograms and weight in pounds. Use it to convert the following measurements.



6. 4 kilograms into weight in pounds
7. 9 kilograms into weight in pounds
8. 12 pounds into weight in kilograms
9. 17 pounds into weight in kilograms

For 10-12, use the graph from problems 6-9 to answer the following questions.

10. An employee at a sporting goods store is packing 3-pound weights into a box that can hold 8 kilograms. How many weights can she place in the box?
11. After packing those weights, there is some extra space in the box that she wants to fill with one-pound weights. How many of those can she add?
12. After packing those, she realizes she misread the label and the box can actually hold 9 kilograms. How many more one-pound weights can she add?

4.4 Horizontal and Vertical Line Graphs

Here you'll learn how to write the equations of horizontal and vertical lines and graph such lines.

What if you were given the graph of a vertical or horizontal line? How could you write the equation of this line? After completing this Concept, you'll be able to write horizontal and vertical linear equations and graph them in the coordinate plane.

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Click image to the left for more content.

CK-12 Foundation: 0404S Graphs of HorizontalandVertical Lines (H264)

Guidance

How do you graph equations of horizontal and vertical lines? See how in the example below.

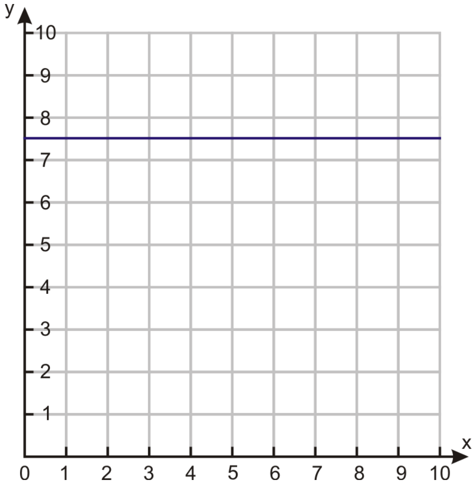
Example A

"Mad-cabs" have an unusual offer going on. They are charging \$7.50 for a taxi ride of any length within the city limits. Graph the function that relates the cost of hiring the taxi (y) *to the length of the journey in miles* (x).

To proceed, the first thing we need is an **equation**. You can see from the problem that the cost of a journey doesn't depend on the length of the journey. It should come as no surprise that the equation then, does not have x in it. Since any value of x results in the same value of $y(7.5)$, the value you choose for x doesn't matter, so it isn't included in the equation. Here is the equation:

$y = 7.5$

The graph of this function is shown below. You can see that it's simply a horizontal line.



Any time you see an equation of the form "$y =$ constant," the graph is a horizontal line that intercepts the $y-$axis at the value of the constant.

Similarly, when you see an equation of the form $x =$ constant, then the graph is a vertical line that intercepts the $x-$axis at the value of the constant. (Notice that that kind of equation is a relation, and not a function, because each $x-$value (there's only one in this case) corresponds to many (actually an infinite number) $y-$values.)

Example B

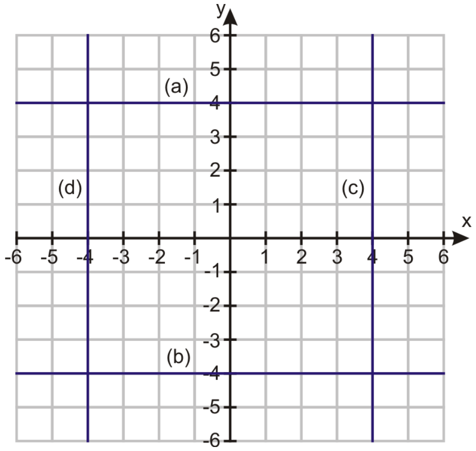
Plot the following graphs.

(a) $y = 4$

(b) $y = -4$

(c) $x = 4$

(d) $x = -4$



(a) $y = 4$ is a horizontal line that crosses the $y-$axis at 4.

(b) $y = -4$ is a horizontal line that crosses the $y-$axis at -4.

(c) $x = 4$ is a vertical line that crosses the $x-$axis at 4.

(d) $x = -4$ is a vertical line that crosses the $x-$axis at -4.

Example C

Find an equation for the $x-$axis and the $y-$axis.

Look at the axes on any of the graphs from previous examples. We have already said that they intersect at the origin (the point where $x = 0$ and $y = 0$). The following definition could easily work for each axis.

*$x-$**axis:** A horizontal line crossing the $y-$axis at zero.*

*$y-$**axis:** A vertical line crossing the $x-$axis at zero.*

So using example 3 as our guide, we could define the $x-$axis as the line $y = 0$ and the $y-$axis as the line $x = 0$.

Watch this video for help with the Examples above.

MEDIA

Click image to the left for more content.

CK-12 Foundation: Graphsof Horizontal and Vertical Lines

Vocabulary

- **Horizontal lines** are defined by the equation $y =$ constant and **vertical lines** are defined by the equation $x =$ constant.
- Be aware that although we graph the function as a line to make it easier to interpret, the function may actually be **discrete**.

Guided Practice

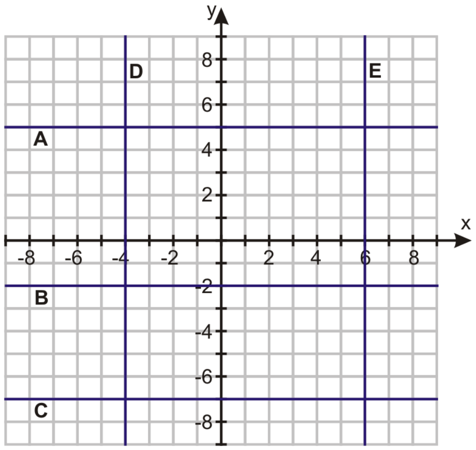
Write the equation of the horizontal line that is 3 units below the x-axis.

Solution:

The horizontal line that is 3 units below the x-axis will intercept the y-axis at $y = -3$. No matter what the value of x, the y value of the line will always be -3. This means that the equations for the line is $y = -3$.

Practice

1. Write the equations for the five lines (A through E) plotted in the graph below.



For 2-10, use the graph above to determine at what points the following lines intersect.

2. A and E
3. A and D
4. C and D
5. B and the $y-$axis
6. E and the $x-$axis
7. C and the line $y = x$
8. E and the line $y = \frac{1}{2}x$
9. A and the line $y = x + 3$
10. B and the line $y = -2x$

4.5 Intercepts and the Cover-Up Method

Here you'll learn how to find the intercepts of an equation. You'll then use those intercepts to graph the equation.

What if you were given the equation of a line like $4y - 3x = 8$? How could you find the x- and y-intercepts to help you graph the line? After completing this Concept, you'll be able to find the intercepts of linear equations like this one.

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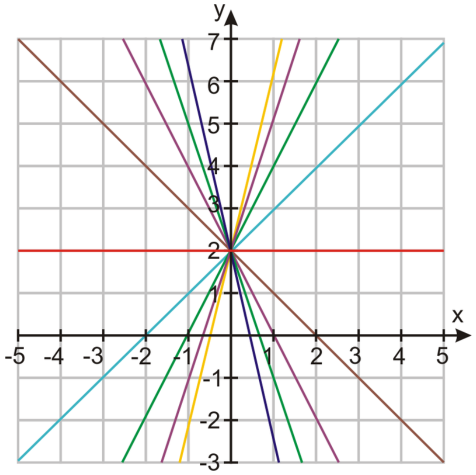
CK-12 Foundation: 0405S GraphingUsing Intercepts (H264)

Try This

To learn more about equations in standard form, try the Java applet at http://www.analyzemath.com/line/line.htm (scroll down and click the "click here to start" button.) You can use the sliders to change the values of a, b, and c and see how that affects the graph.

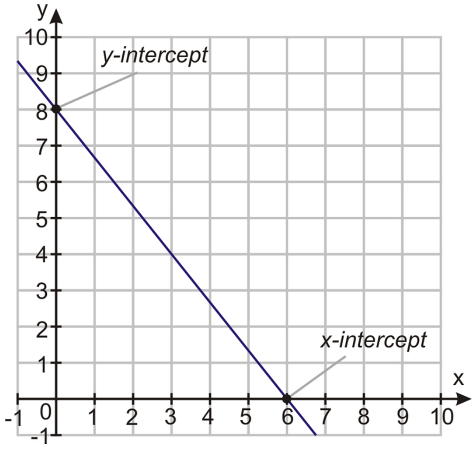
Guidance

Sanjit's office is 25 miles from home, and in traffic he expects the trip home to take him an hour if he starts at 5 PM. Today he hopes to stop at the post office along the way. If the post office is 6 miles from his office, when will Sanjit get there?



If you know just one of the points on a line, you'll find that isn't enough information to plot the line on a graph. As you can see in the graph above, there are many lines—in fact, infinitely many lines—that pass through a single point. But what if you know two points that are both on the line? Then there's only one way to graph that line; all you need to do is plot the two points and use a ruler to draw the line that passes through both of them.

There are a lot of options for choosing which two points on the line you use to plot it. In this lesson, we'll focus on two points that are rather convenient for graphing: the points where our line crosses the $x-$and $y-$axes, or **intercepts.** We'll see how to find intercepts algebraically and use them to quickly plot graphs.



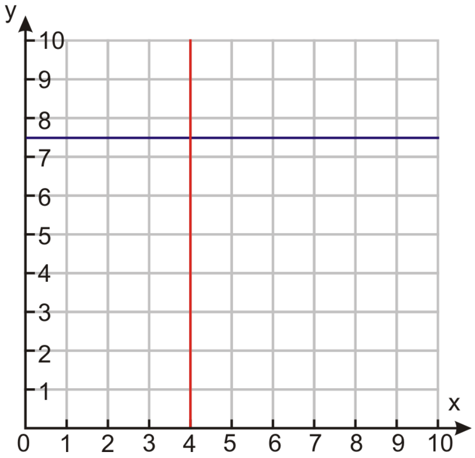
Look at the graph above. The $y-$**intercept** occurs at the point where the graph crosses the $y-$axis. The $y-$value at this point is 8, and the $x-$value is 0.

Similarly, the $x-$**intercept** occurs at the point where the graph crosses the $x-$axis. The $x-$value at this point is 6, and the $y-$value is 0.

So we know the coordinates of two points on the graph: (0, 8) and (6, 0). If we'd just been given those two coordinates out of the blue, we could quickly plot those points and join them with a line to recreate the above graph.

Note: Not all lines will have both an $x-$and a $y-$intercept, but most do. However, horizontal lines never cross the $x-$axis and vertical lines never cross the $y-$axis.

For examples of these special cases, see the graph below.



Finding Intercepts by Substitution

Example A

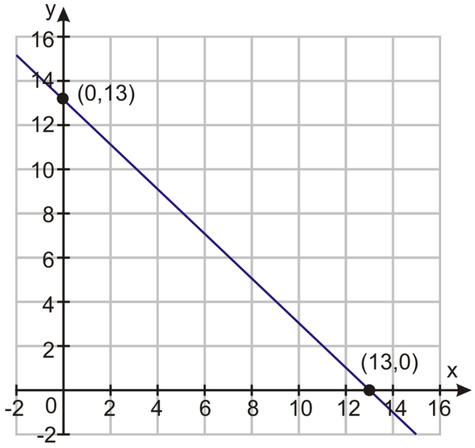
Find the intercepts of the line $y = 13 - x$ *and use them to graph the function.*

Solution

The first intercept is easy to find. The $y-$intercept occurs when $x = 0$. Substituting gives us $y = 13 - 0 = 13$, so the $y-$intercept is (0, 13).

Similarly, the $x-$intercept occurs when $y = 0$. Plugging in 0 for y gives us $0 = 13 - x$, and adding x to both sides gives us $x = 13$. So (13, 0) is the $x-$intercept.

To draw the graph, simply plot these points and join them with a line.



Example B

Graph the following functions by finding intercepts.

a) $y = 2x + 3$

b) $y = 7 - 2x$

c) $4x - 2y = 8$

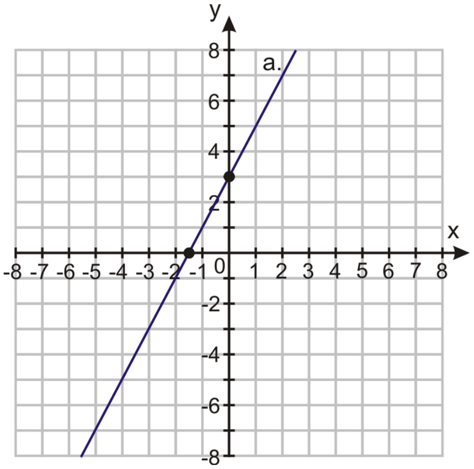
Solution

a) Find the $y-$intercept by plugging in $x = 0:$

$$y = 2 \cdot 0 + 3 = 3 \qquad - \text{the } y - \text{intercept is } (0,3)$$

Find the $x-$intercept by plugging in $y = 0:$

$$\begin{aligned} 0 &= 2x + 3 && - subtract\ 3\ from\ both\ sides: \\ -3 &= 2x && - divide\ by\ 2: \\ -\frac{3}{2} &= x && - \text{the } x - \text{intercept is } (-1.5, 0) \end{aligned}$$

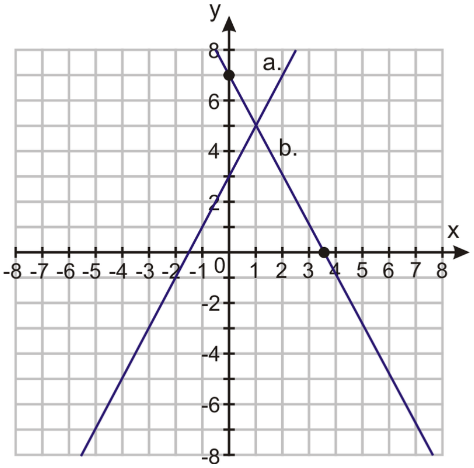


b) Find the $y-$intercept by plugging in $x = 0$:

$$y = 7 - 2 \cdot 0 = 7 \qquad - \text{the } y - \text{intercept is } (0,7)$$

Find the $x-$intercept by plugging in $y = 0$:

$$\begin{aligned} 0 &= 7 - 2x && - subtract\ 7\ from\ both\ sides: \\ -7 &= -2x && - divide\ by\ -2: \\ \frac{7}{2} &= x && - \text{the } x - \text{intercept is } (3.5,0) \end{aligned}$$

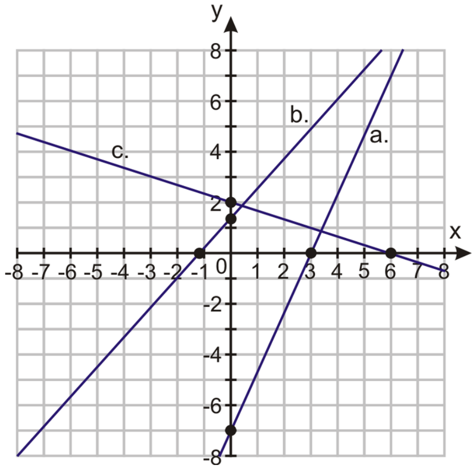


c) Find the $y-$intercept by plugging in $x = 0$:

$$\begin{aligned} 4 \cdot 0 - 2y &= 8 \\ -2y &= 8 && - divide\ by\ -2 \\ y &= -4 && - \text{the } y - \text{intercept is } (0,-4) \end{aligned}$$

Find the $x-$intercept by plugging in $y = 0$:

$$\begin{aligned}4x-2\cdot 0&=8\\ 4x&=8 && -\mathit{divide\ by}\ 4:\\ x&=2 && -\text{the } x-\text{intercept is } (2,0)\end{aligned}$$



Finding Intercepts for Standard Form Equations Using the Cover-Up Method

Look at the last two equations in example 2. These equations are written in ***standard form.*** Standard form equations are always written "**coefficient** times x plus (or minus) **coefficient** times y equals **value**". In other words, they look like this:

$$ax+by=c$$

where a has to be positive, but b and c do not.

There is a neat method for finding intercepts in standard form, often referred to as the cover-up method.

Example C

Find the intercepts of the following equations:

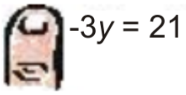
a) $7x-3y=21$

b) $12x-10y=-15$

Solution

To solve for each intercept, we realize that at the intercepts the value of **either** x**or** y is zero, and so any terms that contain that variable effectively drop out of the equation. To make a term disappear, simply cover it (a finger is an excellent way to cover up terms) and solve the resulting equation.

a) To solve for the $y-$intercept we set $x=0$ and cover up the $x-$term:



$$-3y = 21$$

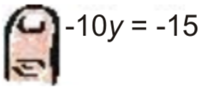
$$y = -7 \qquad (0,-7) \text{ is the } y-\text{intercept.}$$

Now we solve for the $x-$intercept:

$$7x = 21$$

$$x = 3 \qquad (3,0) \text{ is the } x-\text{intercept.}$$

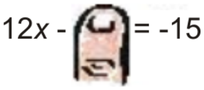
b) To solve for the $y-$intercept $(x = 0)$, cover up the $x-$term:



$$-10y = -15$$

$$y = 1.5 \qquad (0,1.5) \text{ is the } y-\text{intercept.}$$

Now solve for the $x-$intercept $(y = 0)$:



$$12x = -15$$

$$x = -\frac{5}{4} \qquad (-1.25,0) \text{ is the } x-\text{intercept.}$$

Solving Real-World Problems Using Intercepts of a Graph

Example D

Jesus has $30 to spend on food for a class barbecue. Hot dogs cost $0.75 each (including the bun) and burgers cost $1.25 (including the bun). Plot a graph that shows all the combinations of hot dogs and burgers he could buy for the barbecue, without spending more than $30.

This time we will find an equation first, and then we can think logically about finding the intercepts.

If the number of burgers that Jesus buys is x, then the money he spends on burgers is $1.25x$

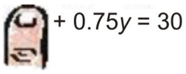
If the number of hot dogs he buys is y, then the money he spends on hot dogs is $0.75y$

So the total cost of the food is $1.25x + 0.75y$.

The total amount of money he has to spend is \$30, so if he is to spend it ALL, we can use the following equation:

$$1.25x + 0.75y = 30$$

We can solve for the intercepts using the cover-up method. First the $y-$intercept $(x = 0)$:

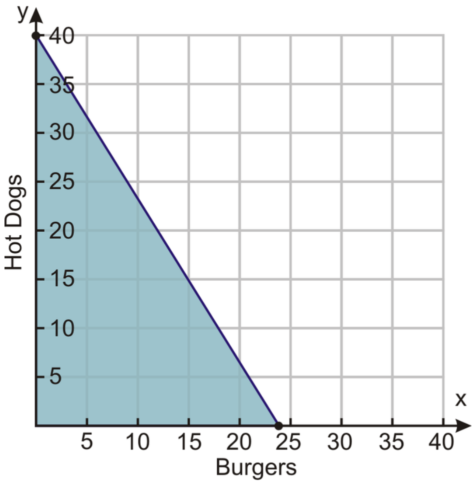


$$0.75y = 30$$
$$y = 40 \qquad y - \text{intercept: } (0,40)$$

Then the $x-$intercept $(y = 0)$:

$$1.25x = 30$$
$$x = 24 \qquad x - \text{intercept: } (24,0)$$

Now we plot those two points and join them to create our graph, shown here:



We could also have created this graph without needing to come up with an equation. We know that if John were to spend ALL the money on hot dogs, he could buy $\frac{30}{.75} = 40$ hot dogs. And if he were to buy only burgers he could buy $\frac{30}{1.25} = 24$ burgers. From those numbers, we can get 2 intercepts: (0 burgers, 40 hot dogs) and (24 burgers, 0 hot dogs). We could plot these just as we did above and obtain our graph that way.

As a final note, we should realize that Jesus' problem is really an example of an inequality. He can, in fact, spend any amount up to \$30. The only thing he cannot do is spend more than \$30. The graph above reflects this: the line is the set of solutions that involve spending exactly \$30, and the shaded region shows solutions that involve spending less than \$30. We'll work with inequalities some more in Chapter 6.

Watch this video for help with the Examples above.

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CK-12 Foundation: Graphing UsingIntercepts

Vocabulary

- A $y-$**intercept** occurs at the point where a graph crosses the $y-$axis (where $x = 0$) and an $x-$**intercept** occurs at the point where a graph crosses the $x-$axis (where $y = 0$).
- The $y-$intercept can be found by **substituting** $x = 0$ into the equation and solving for y. Likewise, the $x-$intercept can be found by **substituting** $y = 0$ into the equation and solving for x.
- A linear equation is in **standard form** if it is written as "positive coefficient times x plus coefficient times y equals value". Equations in standard form can be solved for the intercepts by covering up the x (or y) term and solving the equation that remains.

Guided Practice

1. *Graph* $2x+3y = -6$ *by finding intercepts.*

2. *Find the intercepts of* $x+3y = 6$ *using the cover-up method.*

Solution

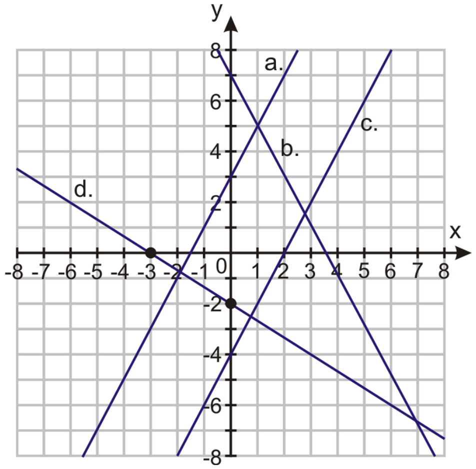
1. Find the $y-$intercept by plugging in $x = 0:$

$$\begin{aligned} 2\cdot 0+3y &= -6 \\ 3y &= -6 && -\text{divide by } 3: \\ y &= -2 && -\text{the } y-\text{intercept is } (0,-2) \end{aligned}$$

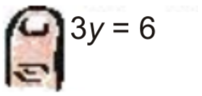
Find the $x-$intercept by plugging in $y = 0:$

$$\begin{aligned} 2x+3\cdot 0 &= -6 \\ 2x &= -6 && -\text{divide by } 2: \\ x &= -3 && -\text{the } x-\text{intercept is } (-3,0) \end{aligned}$$

The graph of this line is the line labeled d, the two intercepts are marked by dots.



2. To solve for the $y-$intercept $(x=0)$, cover up the $x-$term:



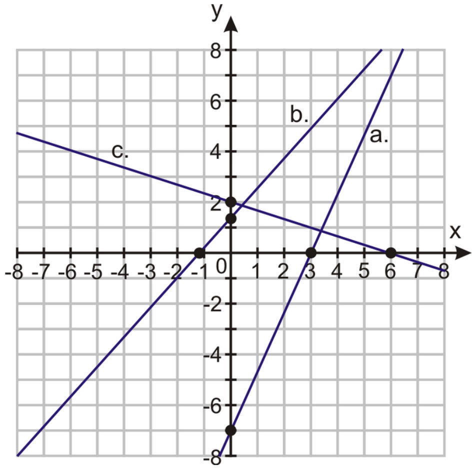
$$3y = 6$$

$$y = 2 \qquad (0,2) \text{ is the } y-\text{intercept.}$$

Solve for the $y-$intercept:

$$x = 6 \qquad (6,0) \text{ is the } x-\text{intercept.}$$

The graph of this function and the intercepts is line c:



Practice

For 1-8, find the intercepts for the following equations using substitution.

1. $y = 3x - 6$
2. $y = -2x + 4$
3. $y = 14x - 21$
4. $y = 7 - 3x$
5. $y = 2.5x - 4$
6. $y = 1.1x + 2.2$
7. $y = \frac{3}{8}x + 7$
8. $y = \frac{5}{9} - \frac{2}{7}x$

For 9-16, find the intercepts of the following equations using the cover-up method.

9. $5x - 6y = 15$
10. $3x - 4y = -5$
11. $2x + 7y = -11$
12. $5x + 10y = 25$
13. $5x - 1.3y = 12$
14. $1.4x - 3.5y = 7$
15. $\frac{3}{5}x + 2y = \frac{2}{5}$
16. $\frac{3}{4}x - \frac{2}{3}y = \frac{1}{5}$

For 17-20, use any method to find the intercepts and then graph the following equations.

17. $y = 2x + 3$
18. $6(x - 1) = 2(y + 3)$
19. $x - y = 5$
20. $x + y = 8$
21. At the local grocery store strawberries cost $3.00 per pound and bananas cost $1.00 per pound.
    a. If I have $10 to spend on strawberries and bananas, draw a graph to show what combinations of each I can buy and spend exactly $10.
    b. Plot the point representing 3 pounds of strawberries and 2 pounds of bananas. Will that cost more or less than $10?
    c. Do the same for the point representing 1 pound of strawberries and 5 pounds of bananas.
22. A movie theater charges $7.50 for adult tickets and $4.50 for children. If the theater takes in $900 in ticket sales for a particular screening, draw a graph which depicts the possibilities for the number of adult tickets and the number of child tickets sold.
23. Why can't we use the intercept method to graph the following equation? $3(x + 2) = 2(y + 3)$
24. Name two more equations that we can't use the intercept method to graph.

4.6 Slope

Here you'll learn how to find the slope of a line given the line's graph or two of its points.

What if you were given two points that a line passes through like (-1, 0) and (2, 2)? How could you find the slope of that line? After completing this Concept, you'll be able to find the slope of any line.

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CK-12 Foundation: 0406S Slope ofa Line (H264)

Guidance

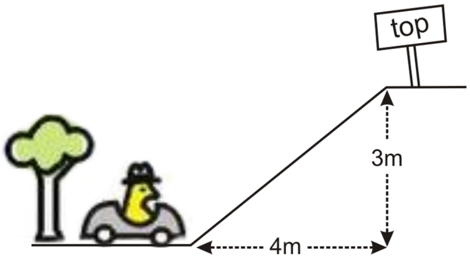
Wheelchair ramps at building entrances must have a slope between $\frac{1}{16}$ and $\frac{1}{20}$. If the entrance to a new office building is 28 inches off the ground, how long does the wheelchair ramp need to be?

We come across many examples of slope in everyday life. For example, a slope is in the pitch of a roof, the grade or incline of a road, or the slant of a ladder leaning on a wall. In math, we use the word **slope** to define steepness in a particular way.

$$\text{Slope} = \frac{\text{distance moved vertically}}{\text{distance moved horizontally}}$$

To make it easier to remember, we often word it like this:

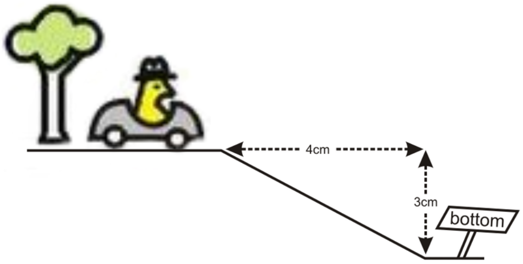
$$\text{Slope} = \frac{\text{rise}}{\text{run}}$$



In the picture above, the slope would be the ratio of the **height** of the hill to the horizontal **length** of the hill. In other words, it would be $\frac{3}{4}$, or 0.75.

If the car were driving to the **right** it would **climb** the hill - we say this is a positive slope. Any time you see the graph of a line that goes up as you move to the right, the slope is **positive**.

If the car kept driving after it reached the top of the hill, it might go down the other side. If the car is driving to the **right** and **descending**, then we would say that the slope is **negative**.



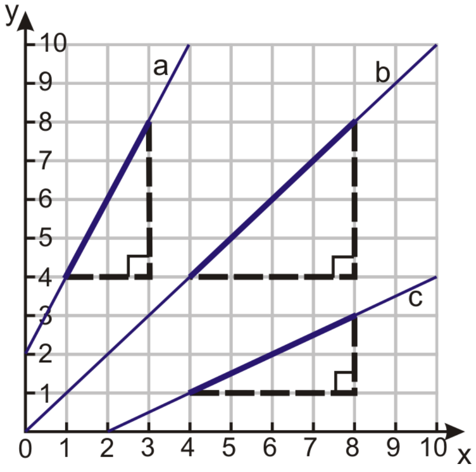
Here's where it gets tricky: If the car turned around instead and drove back down the left side of the hill, the slope of that side would still be positive. This is because the rise would be -3, but the run would be -4 (think of the $x-$axis - if you move from right to left you are moving in the **negative** $x-$direction). That means our slope ratio would be $\frac{-3}{-4}$, and the negatives cancel out to leave 0.75, the same slope as before. In other words, the slope of a line is the same no matter which direction you travel along it.

Find the Slope of a Line

A simple way to find a value for the slope of a line is to draw a right triangle whose hypotenuse runs along the line. Then we just need to measure the distances on the triangle that correspond to the rise (the vertical dimension) and the run (the horizontal dimension).

Example A

Find the slopes for the three graphs shown.



Solution

There are already right triangles drawn for each of the lines - in future problems you'll do this part yourself. Note that it is easiest to make triangles whose vertices are **lattice points** (i.e. points whose coordinates are all integers).

a) The rise shown in this triangle is 4 units; the run is 2 units. The slope is $\frac{4}{2} = 2$.

b) The rise shown in this triangle is 4 units, and the run is also 4 units. The slope is $\frac{4}{4} = 1$.

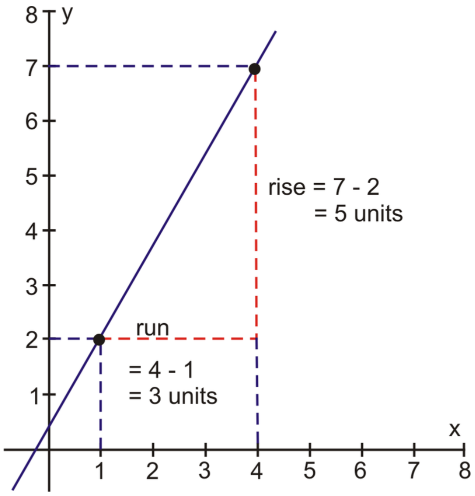
c) The rise shown in this triangle is 2 units, and the run is 4 units. The slope is $\frac{2}{4} = \frac{1}{2}$.

Example B

Find the slope of the line that passes through the points (1, 2) and (4, 7).

Solution

We already know how to graph a line if we're given two points: we simply plot the points and connect them with a line. Here's the graph:



Since we already have coordinates for the vertices of our right triangle, we can quickly work out that the rise is $7-2=5$ and the run is $4-1=3$ (see diagram). So the slope is $\frac{7-2}{4-1}=\frac{5}{3}$.

If you look again at the calculations for the slope, you'll notice that the 7 and 2 are the $y-$coordinates of the two points and the 4 and 1 are the $x-$coordinates. This suggests a pattern we can follow to get a general formula for the slope between two points (x_1,y_1) and (x_2,y_2):

Slope between (x_1,y_1) and $(x_2,y_2)=\frac{y_2-y_1}{x_2-x_1}$

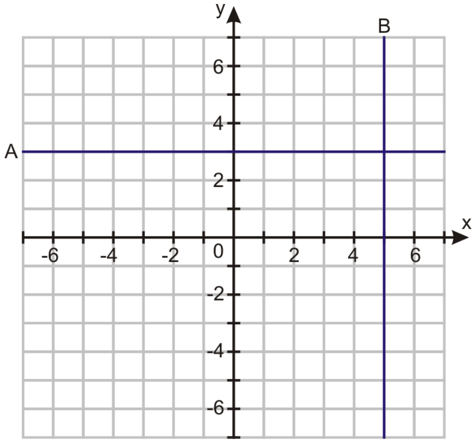
or $m=\frac{\Delta y}{\Delta x}$.

In the second equation the letter m denotes the slope (this is a mathematical convention you'll see often) and the Greek letter delta (Δ) means ***change***. So another way to express slope is *change in y* divided by *change in x*. In the next section, you'll see that it doesn't matter which point you choose as point 1 and which you choose as point 2.

Find the Slopes of Horizontal and Vertical lines

Example C

Determine the slopes of the two lines on the graph below.



Solution

There are 2 lines on the graph: $A(y=3)$ and $B(x=5)$.

Let's pick 2 points on line A—say, $(x_1,y_1)=(-4,3)$ and $(x_2,y_2)=(5,3)$—and use our equation for slope:

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{(3)-(3)}{(5)-(-4)}=\frac{0}{9}=0.$$

If you think about it, this makes sense - if y doesn't change as x increases then there is no slope, or rather, the slope is zero. You can see that this must be true for all horizontal lines.

Horizontal lines (y = *constant*) all have a slope of 0.

Now let's consider line B. If we pick the points $(x_1,y_1)=(5,-3)$ and $(x_2,y_2)=(5,4)$, our slope equation is $m=\frac{y_2-y_1}{x_2-x_1}=\frac{(4)-(-3)}{(5)-(5)}=\frac{7}{0}$. But dividing by zero isn't allowed!

In math we often say that a term which involves division by zero is **undefined.** (Technically, the answer can also be said to be infinitely large—or infinitely small, depending on the problem.)

Vertical lines ($x=$ *constant*) all have an infinite (or undefined) slope.

Watch this video for help with the Examples above.

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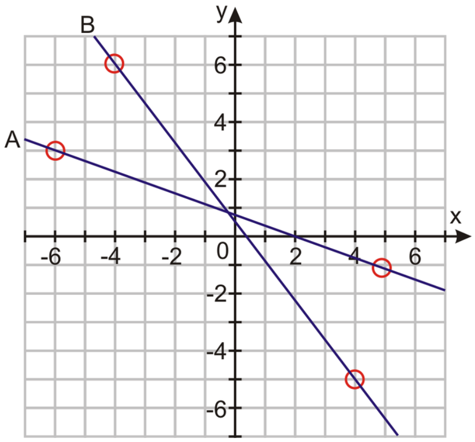
CK-12 Foundation: The Slope of aLine

Vocabulary

- **Slope** is a measure of change in the vertical direction for each step in the horizontal direction. Slope is often represented as "m".
- Slope can be expressed as $\frac{\text{rise}}{\text{run}}$, or $\frac{\Delta y}{\Delta x}$.
- The slope between two points (x_1,y_1) and (x_2,y_2) is equal to $\frac{y_2-y_1}{x_2-x_1}$.
- **Horizontal lines** (where $y=a$ constant) all have a slope of 0.
- **Vertical lines** (where $x=a$ constant) all have an infinite (or undefined) slope.
- The slope (or **rate of change**) of a distance-time graph is a **velocity.**

Guided Practice

Find the slopes of the lines on the graph below.



Solution

Look at the lines - they both slant down (or decrease) as we move from left to right. Both these lines have **negative slope.**

The lines don't pass through very many convenient lattice points, but by looking carefully you can see a few points that look to have integer coordinates. These points have been circled on the graph, and we'll use them to determine the slope. We'll also do our calculations twice, to show that we get the same slope whichever way we choose point 1 and point 2.

For Line A:

$$(x_1,y_1)=(-6,3) \qquad (x_2,y_2)=(5,-1) \qquad\qquad (x_1,y_1)=(5,-1) \qquad (x_2,y_2)=(-6,3)$$

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{(-1)-(3)}{(5)-(-6)}=\frac{-4}{11}\approx -0.364 \qquad\qquad m=\frac{y_2-y_1}{x_2-x_1}=\frac{(3)-(-1)}{(-6)-(5)}=\frac{4}{-11}\approx -0.364$$

For Line B

$$(x_1,y_1)=(-4,6) \qquad (x_2,y_2)=(4,-5) \qquad\qquad (x_1,y_1)=(4,-5) \qquad (x_2,y_2)=(-4,6)$$

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{(-5)-(6)}{(4)-(-4)}=\frac{-11}{8}=-1.375 \qquad\qquad m=\frac{y_2-y_1}{x_2-x_1}=\frac{(6)-(-5)}{(-4)-(4)}=\frac{11}{-8}=-1.375$$

You can see that whichever way round you pick the points, the answers are the same. Either way, **Line A has slope -0.364, and Line B has slope -1.375.**

Practice

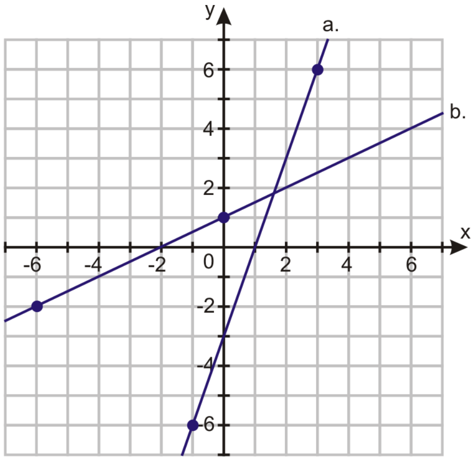
Use the slope formula to find the slope of the line that passes through each pair of points.

1. (-5, 7) and (0, 0)
2. (-3, -5) and (3, 11)
3. (3, -5) and (-2, 9)

4. (-5, 7) and (-5, 11)
5. (9, 9) and (-9, -9)
6. (3, 5) and (-2, 7)
7. (2.5, 3) and (8, 3.5)

For each line in the graphs below, use the points indicated to determine the slope.

8.



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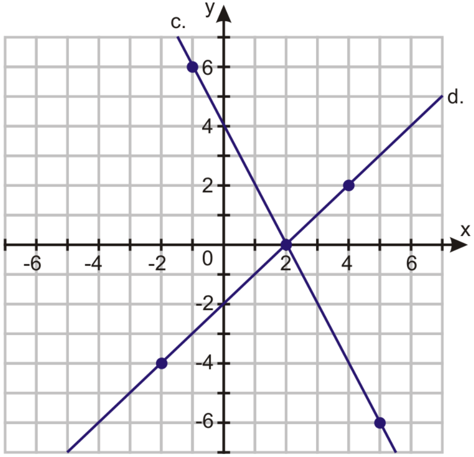

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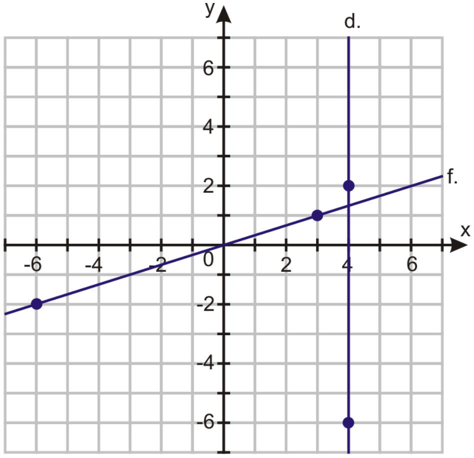

11. For each line in the graphs above, imagine another line with the same slope that passes through the point (1, 1), and name one more point on that line.

4.7 Rates of Change

Here you'll learn how to find the rate of change of a function. You'll also compare rates of change on a graph to interpret what is happening.

What if when you bought a koi fish it measured 4 inches long? Three months later you measure it again and it is 6 inches long? At what rate is the fish growing? After completing this Concept, you'll be able to find the rate of change taking place in problems like this one.

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CK-12 Foundation: 0407S Rateof Change (H264)

Guidance

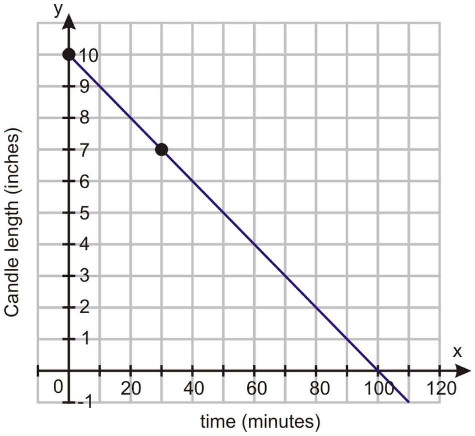
The slope of a function that describes real, measurable quantities is often called a **rate of change**. In that case the slope refers to a change in one quantity (y)**per** unit change in another quantity (x). (This is where the equation $m = \frac{\Delta y}{\Delta x}$ comes in—remember that Δy and Δx represent the change in y and x respectively.)

Example A

A candle has a starting length of 10 inches. 30 minutes after lighting it, the length is 7 inches. Determine the rate of change in length of the candle as it burns. Determine how long the candle takes to completely burn to nothing.

Solution

First we'll graph the function to visualize what is happening. We have 2 points to start with: we know that at the moment the candle is lit $(time = 0)$ the length of the candle is 10 inches, and after 30 minutes $(time = 30)$ the length is 7 inches. Since the candle length depends on the time, we'll plot time on the horizontal axis, and candle length on the vertical axis.



The rate of change of the candle's length is simply the slope of the line. Since we have our 2 points $(x_1, y_1) = (0, 10)$ and $(x_2, y_2) = (30, 7)$, we can use the familiar version of the slope formula:

$$\begin{aligned}\text{Rate of change} &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{(7 \text{ inches}) - (10 \text{ inches})}{(30 \text{ minutes}) - (0 \text{ minutes})}\\ &= \frac{-3 \text{ inches}}{30 \text{ minutes}}\\ &= -0.1 \text{ inches per minute}\end{aligned}$$

Note that the slope is negative. A negative rate of change means that the quantity is decreasing with time—just as we would expect the length of a burning candle to do.

To find the point when the candle reaches zero length, we can simply read the $x-$intercept off the graph (100 minutes). We can use the rate equation to verify this algebraically:

$$\begin{aligned}\text{Length burned} &= \text{rate} \times \text{time}\\ 10 &= 0.1 \times 100\end{aligned}$$

Since the candle length was originally 10 inches, our equation confirms that 100 minutes is the time taken.

Example B

The population of fish in a certain lake increased from 370 to 420 over the months of March and April. At what rate is the population increasing?

Solution

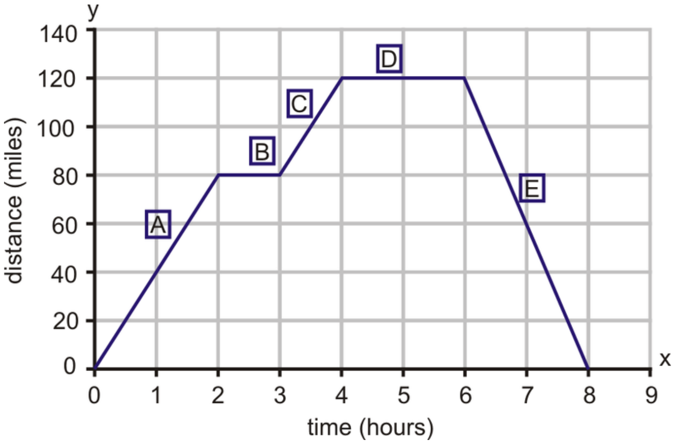
Here we don't have two points from which we can get $x-$ and $y-$coordinates for the slope formula. Instead, we'll need to use the alternate formula, $m = \frac{\Delta y}{\Delta x}$.

The change in $y-$values, or Δy, is the change in the number of fish, which is $420 - 370 = 50$. The change in $x-$values, Δx, is the amount of time over which this change took place: two months. So $\frac{\Delta y}{\Delta x} = \frac{50 \text{ fish}}{2 \text{ months}}$, or **25 fish per month.**

Interpret a Graph to Compare Rates of Change

Example C

The graph below represents a trip made by a large delivery truck on a particular day. During the day the truck made two deliveries, one taking an hour and the other taking two hours. Identify what is happening in the first 3 stages of the trip (stages A through C).



Solution:

The first 3 stages of the trip are:

A. The truck sets off and travels 80 miles in 2 hours.

B. The truck covers no distance for 2 hours.

C. The truck covers $(120-80)=40$ miles in 1 hour.

A. Rate of change $=\frac{\Delta y}{\Delta x}=\frac{80 \text{ miles}}{2 \text{ hours}}=40$ miles per hour

Notice that the rate of change is a **speed—**or rather, a **velocity.** (The difference between the two is that velocity has a direction, and speed does not. In other words, velocity can be either positive or negative, with negative velocity representing travel in the opposite direction. You'll see the difference more clearly in part E.)

Since velocity equals distance divided by time, the slope (or rate of change) of a distance-time graph is always a velocity.

So during the first part of the trip, the truck travels at a constant speed of 40 mph for 2 hours, covering a distance of 80 miles.

B. The slope here is 0, so the rate of change is 0 mph. The truck is stationary for one hour. This is the first delivery stop.

C. Rate of change $=\frac{\Delta y}{\Delta x}=\frac{(120-80) \text{ miles}}{(4-3) \text{ hours}}=40$ miles per hour. The truck is traveling at 40 mph.

Watch this video for help with the Examples above.

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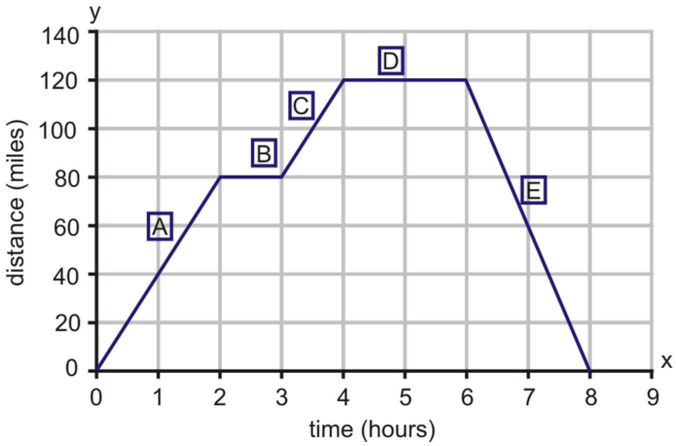
CK-12 Foundation: Rate of Change

Vocabulary

- **Slope** is a measure of change in the vertical direction for each step in the horizontal direction. Slope is often represented as "m".
- Slope can be expressed as $\frac{\text{rise}}{\text{run}}$, or $\frac{\Delta y}{\Delta x}$.
- The slope between two points (x_1, y_1) and (x_2, y_2) is equal to $\frac{y_2-y_1}{x_2-x_1}$.
- **Horizontal lines** (where $y = a$ constant) all have a slope of 0.
- **Vertical lines** (where $x = a$ constant) all have an infinite (or undefined) slope.
- The slope (or **rate of change**) of a distance-time graph is a **velocity.**

Guided Practice

Continue where we left of in Example C, by identifying what is happening in the last two stages of the trip (stages D and E).



Solution:

The last two stages of the trip are:

D. The truck covers no distance for 1 hour.

E. The truck covers -120 miles in 2 hours.

Let's find the slopes:

D. Here the slope is 0, so the rate of change is 0 mph. The truck is stationary for two hours. This is the second delivery stop. At this point the truck is 120 miles from the start position.

E.

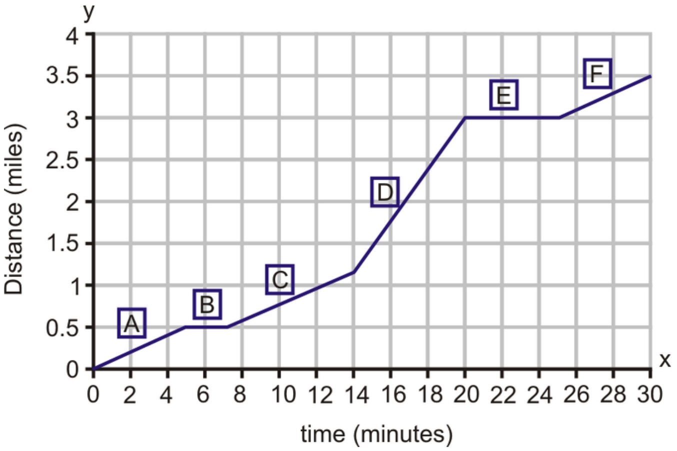
$$\begin{aligned}\text{Rate of change} &= \frac{\Delta y}{\Delta x}\\ &= \frac{(0-120)\text{ miles}}{(8-6)\text{ hours}}\\ &= \frac{-120\text{ miles}}{2\text{ hours}}\\ &= -60\text{ miles per hour.}\end{aligned}$$

The truck is traveling at *negative* 60 mph.

Wait – a negative speed? Does that mean that the truck is reversing? Well, probably not. It's actually the *velocity* and not the speed that is negative, and a negative velocity simply means that the distance *from the starting position* is decreasing with time. The truck is driving in the opposite direction – back to where it started from. Since it no longer has 2 heavy loads, it travels faster (60 mph instead of 40 mph), covering the 120 mile return trip in 2 hours. Its *speed* is 60 mph, and its *velocity* is -60 mph, because it is traveling in the opposite direction from when it started out.

Practice

For 1-6, the graph below is a distance-time graph for Mark's three and a half mile cycle ride to school. During this ride, he rode on cycle paths but the terrain was hilly. He rode slower up hills and faster down them. He stopped once at a traffic light and at one point he stopped to mend a punctured tire. The graph shows his distance from home at any given time. Identify each section of the graph accordingly.



1. Section A.
2. Section B.
3. Section C.
4. Section D.
5. Section E.
6. Section F.

For 7-12, approximate the slope of each part of Mark's ride.

7. Section A.
8. Section B.
9. Section C.
10. Section D.
11. Section E.
12. Section F.

4.8 Graphs Using Slope-Intercept Form

Here you'll learn how to identify the slope and $y-$intercept of equations. You'll also learn how to graph equations that are in slope-intercept form.

What if you were given the equation of a line like $y = -2x - 7$? How could you determine its slope and y-intercept? After completing this Concept, you'll be able to identify the slope and $y-$intercept of linear equations like this one.

Watch This

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CK-12 Foundation: 0408S Graphs UsingSlope-InterceptForm (H264)

Try This

To get a better understanding of what happens when you change the slope or the $y-$intercept of a linear equation, try playing with the Java applet at http://standards.nctm.org/document/eexamples/chap7/7.5/index.htm.

Guidance

The total profit of a business is described by the equation $y = 15000x - 80000$, where x is the number of months the business has been running. How much profit is the business making per month, and what were its start-up costs? How much profit will it have made in a year?

Identify Slope and $y-$intercept

So far, we've been writing a lot of our equations in **slope-intercept form—**that is, we've been writing them in the form $y = mx + b$, where m and b are both constants. It just so happens that m is the slope and the point $(0, b)$ is the $y-$intercept of the graph of the equation, which gives us enough information to draw the graph quickly.

Example A

Identify the slope and $y-$intercept of the following equations.

a) $y = 3x + 2$

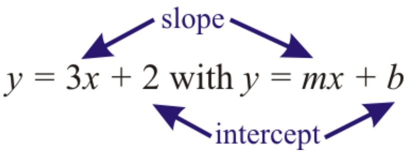
b) $y = 0.5x - 3$

c) $y = -7x$

d) $y = -4$

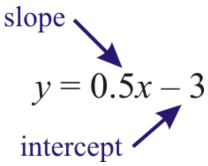
Solution

a) Comparing



, we can see that $m = 3$ and $b = 2$. So $y = 3x+2$ has a **slope of 3** and a $y-$**intercept of (0, 2).**

b)



has a **slope of 0.5** and a $y-$**intercept of (0, -3).**

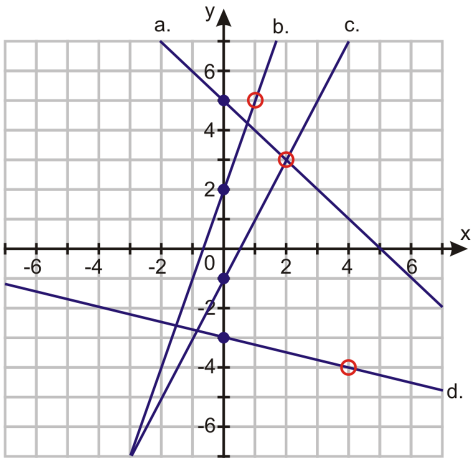
Notice that the intercept is **negative**. The $b-$term includes the sign of the operator (plus or minus) in front of the number—for example, $y = 0.5x - 3$ is identical to $y = 0.5x + (-3)$, and that means that b is -3, not just 3.

c) At first glance, this equation doesn't look like it's in slope-intercept form. But we can rewrite it as $y = -7x + 0$, and that means it has a **slope of -7** and a $y-$**intercept of (0, 0).** Notice that the slope is negative and the line passes through the origin.

d) We can rewrite this one as $y = 0x - 4$, giving us a **slope of 0** and a $y-$**intercept of (0, -4).** This is a horizontal line.

Example B

Identify the slope and $y-$*intercept of the lines on the graph shown below.*



The intercepts have been marked, as well as some convenient lattice points that the lines pass through.

Solution

a) **The** $y-$**intercept is (0, 5).** The line also passes through (2, 3), so the slope is $\frac{\Delta y}{\Delta x} = \frac{-2}{2} = -1$.

b) **The** $y-$**intercept is (0, 2).** The line also passes through (1, 5), so the slope is $\frac{\Delta y}{\Delta x} = \frac{3}{1} = 3$.

c) **The** $y-$**intercept is (0, -1).** The line also passes through (2, 3), so the slope is $\frac{\Delta y}{\Delta x} = \frac{4}{2} = 2$.

d) **The** $y-$**intercept is (0, -3).** The line also passes through (4, -4), so the slope is $\frac{\Delta y}{\Delta x} = \frac{-1}{4} = -\frac{1}{4}$ or -0.25.

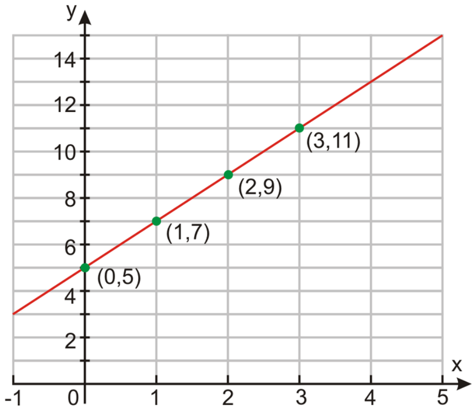
Graph an Equation in Slope-Intercept Form

Once we know the slope and intercept of a line, it's easy to graph it. Just remember what slope means. Let's look back at this example from Lesson 4.1.

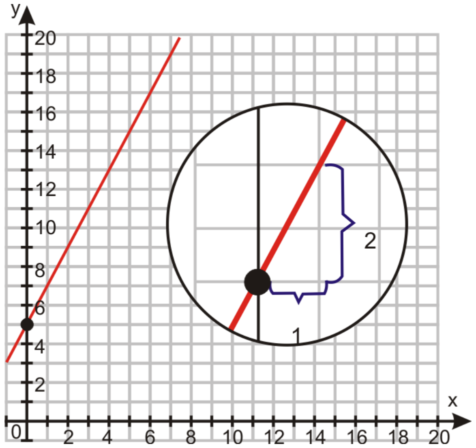
Ali is trying to work out a trick that his friend showed him. His friend started by asking him to think of a number, then double it, then add five to the result. Ali has written down a rule to describe the first part of the trick. He is using the letter x to stand for the number he thought of and the letter y to represent the final result of applying the rule. He wrote his rule in the form of an equation: $y = 2x + 5$.

Help him visualize what is going on by graphing the function that this rule describes.

In that example, we constructed a table of values, and used that table to plot some points to create our graph.



We also saw another way to graph this equation. Just by looking at the equation, we could see that the $y-$intercept was (0, 5), so we could start by plotting that point. Then we could also see that the slope was 2, so we could find another point on the graph by going over 1 unit and up 2 units. The graph would then be the line between those two points.



Here's another problem where we can use the same method.

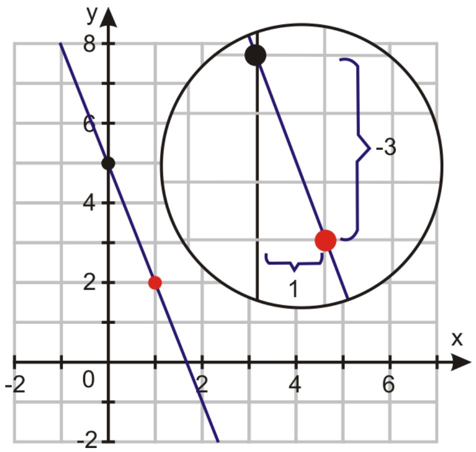
Example C

Graph the following function: $y = -3x + 5$

Solution

To graph the function without making a table, follow these steps:

1. Identify the $y-$intercept: $b = 5$
2. Plot the intercept: (0, 5)
3. Identify the slope: $m = -3$. (This is equal to $\frac{-3}{1}$, so the **rise** is -3 and the **run** is 1.)
4. Move **over** 1 unit and **down** 3 units to find another point on the line: (1, 2)
5. Draw the line through the points (0, 5) and (1, 2).



Notice that to graph this equation based on its slope, we had to find the rise and run—and it was easiest to do that when the slope was expressed as a fraction. That's true in general: to graph a line with a particular slope, it's easiest to first express the slope as a fraction in simplest form, and then read off the numerator and the denominator of the fraction to get the rise and run of the graph.

Example D

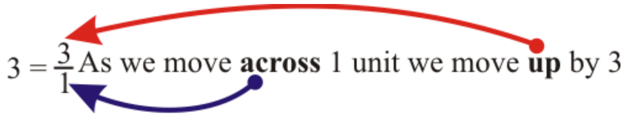
Find integer values for the ***rise*** *and* ***run*** *of the following slopes, then graph lines with corresponding slopes.*

a) $m = 3$

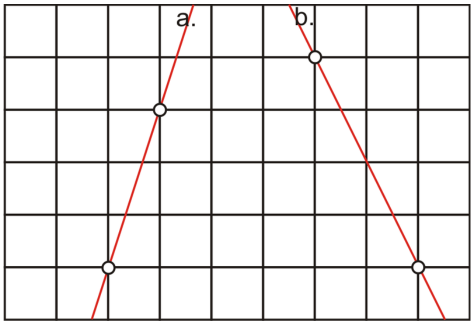
b) $m = -2$

Solution

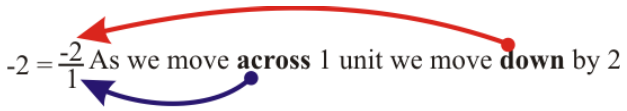
a)

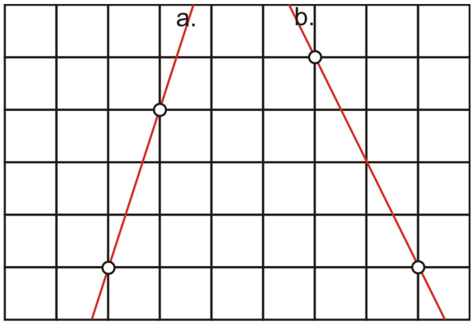


$3 = \frac{3}{1}$ As we move **across** 1 unit we move **up** by 3



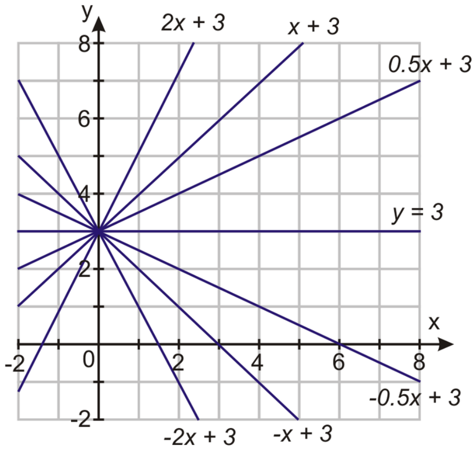
b)





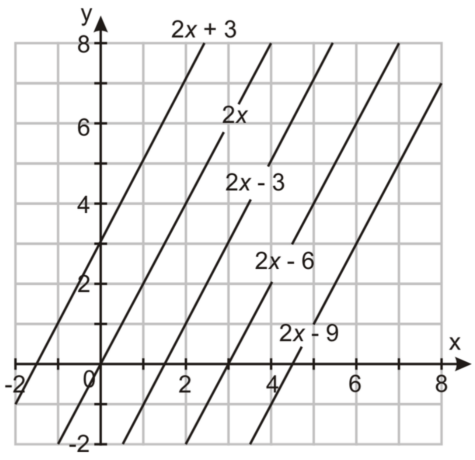
Changing the Slope or Intercept of a Line

The following graph shows a number of lines with different slopes, but all with the same $y-$intercept: (0, 3).



You can see that all the functions with positive slopes increase as we move from left to right, while all functions with negative slopes decrease as we move from left to right. Another thing to notice is that the greater the slope, the steeper the graph.

This graph shows a number of lines with the same slope, but different $y-$intercepts.



Notice that changing the intercept simply translates (shifts) the graph up or down. Take a point on the graph of $y = 2x$, such as (1, 2). The corresponding point on $y = 2x+3$ would be (1, 5). Adding 3 to the $y-$intercept means we also add 3 to every other $y-$value on the graph. Similarly, the corresponding point on the $y = 2x - 3$ line would be (1, -1); we would subtract 3 from the $y-$value and from every other $y-$value.

Notice also that these lines all appear to be parallel. Are they truly parallel?

To answer that question, we'll use a technique that you'll learn more about in a later chapter. We'll take 2 of the equations—say, $y = 2x$ and $y = 2x + 3$—and solve for values of x and y that satisfy both equations. That will tell us at what point those two lines intersect, if any. (Remember that **parallel lines**, by definition, are lines that don't intersect.)

So what values would satisfy both $y = 2x$ and $y = 2x + 3$? Well, if both of those equations were true, then y would be equal to both $2x$ and $2x + 3$, which means those two expressions would also be equal to each other. So we can get our answer by solving the equation $2x = 2x + 3$.

But what happens when we try to solve that equation? If we subtract $2x$ from both sides, we end up with $0 = 3$. That can't be true no matter what x equals. And that means that there just isn't any value for x that will make both of the equations we started out with true. In other words, there isn't any point where those two lines intersect. They are parallel, just as we thought.

And we'd find out the same thing no matter which two lines we'd chosen. In general, since changing the intercept of a line just results in shifting the graph up or down, the new line will always be parallel to the old line as long as the slope stays the same.

Any two lines with identical slopes are parallel.

Watch this video for help with the Examples above.

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CK-12 Foundation: GraphsUsing Slope-Intercept Form

Vocabulary

- A common form of a line (linear equation) is **slope-intercept form:** $y = mx + b$, where m is the slope and the point $(0, b)$ is the $y-$intercept
- Graphing a line in slope-intercept form is a matter of first plotting the $y-$intercept $(0, b)$, then finding a second point based on the slope, and using those two points to graph the line.
- Any two lines with identical slopes are **parallel.**

Guided Practice

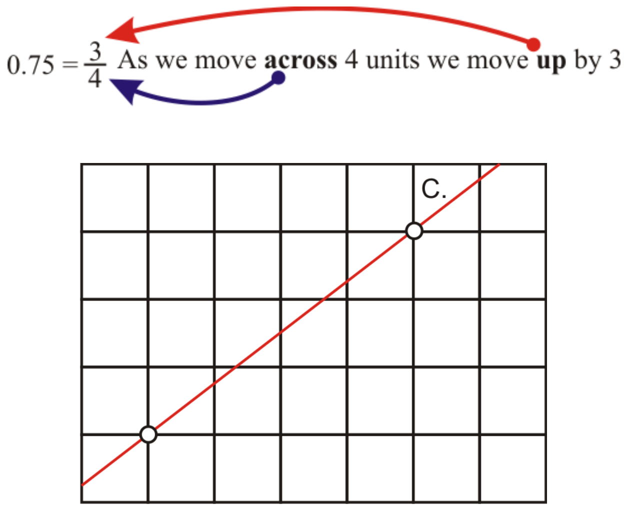
Find integer values for the ***rise*** *and* ***run*** *of the following slopes, then graph lines with corresponding slopes.*

a) $m = 0.75$

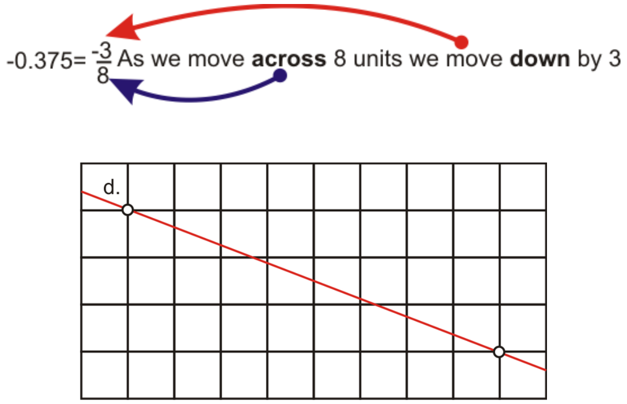
b) $m = -0.375$

Solution:

a)



b)

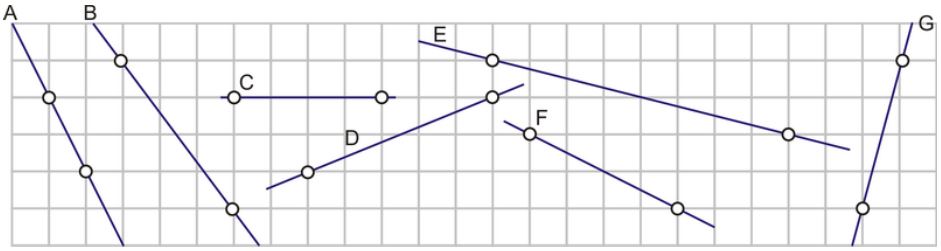


Practice

Identify the slope and $y-$intercept for the following equations.

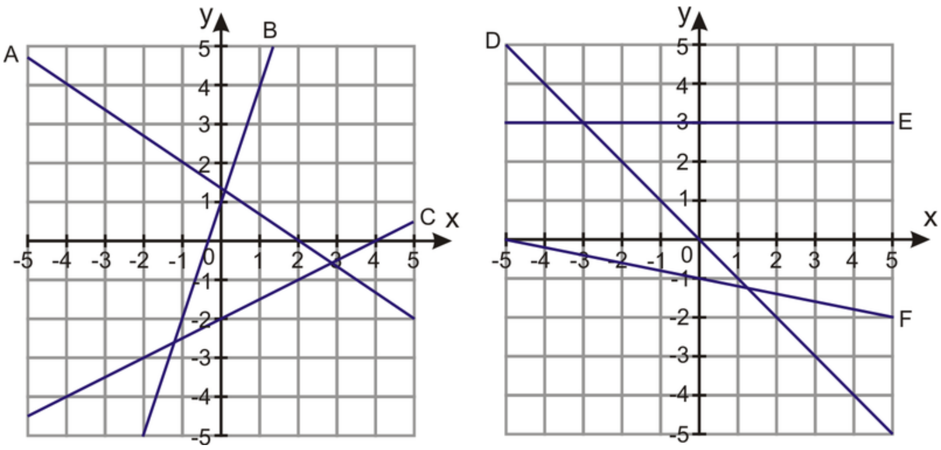
1. $y = 2x + 5$
2. $y = -0.2x + 7$
3. $y = x$
4. $y = 3.75$

Identify the slope of the following lines.

5. 

Identify the slope and $y-$intercept for the following functions.

6.



For 7-10, plot the following functions on a graph.

7. $y = 2x + 5$
8. $y = -0.2x + 7$
9. $y = x$
10. $y = 3.75$

11. Which two of the following lines are parallel?
    a. $y = 2x + 5$
    b. $y = -0.2x + 7$
    c. $y = x$
    d. $y = 3.75$
    e. $y = -\frac{1}{5}x - 11$
    f. $y = -5x + 5$
    g. $y = -3x + 11$
    h. $y = 3x + 3.5$
12. What is the $y-$intercept of the line passing through (1, -4) and (3, 2)?
13. What is the $y-$intercept of the line with slope -2 that passes through (3, 1)?
14. Line *A* passes through the points (2, 6) and (-4, 3). Line *B* passes through the point (3, 2.5), and is parallel to line *A*
    a. Write an equation for line *A* in slope-intercept form.
    b. Write an equation for line *B* in slope-intercept form.
15. Line *C* passes through the points (2, 5) and (1, 3.5). Line *D* is parallel to line *C*, and passes through the point (2, 6). Name another point on line *D*. (Hint: you can do this without graphing or finding an equation for either line.)

4.9 Graphs of Linear Models of Direct Variation

Here you'll learn how to identify and plot direct variations. You'll also learn how to determine the constant of proportionality and solve real-world problems using direct variation models.

What if the amount of your paycheck varied directly with the number of hours you worked? If your paycheck were \$150 and you worked 10 hours, how could you find your hourly rate, or the constant of proportionality? After completing this Concpet, you'll be able to solve direct variation applications like this one.

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CK-12 Foundation: 0409S Direct VariationModels (H264)

Guidance

Suppose you see someone buy five pounds of strawberries at the grocery store. The clerk weighs the strawberries and charges \$12.50 for them. Now suppose you wanted two pounds of strawberries for yourself. How much would you expect to pay for them?

Identify Direct Variation

The preceding problem is an example of a **direct variation**. We would expect that the strawberries are priced on a "per pound" basis, and that if you buy two-fifths the amount of strawberries, you would pay two-fifths of \$12.50 for your strawberries, or \$5.00.

Similarly, if you bought 10 pounds of strawberries (twice the amount) you would pay twice \$12.50, and if you did not buy any strawberries you would pay nothing.

If variable y varies directly with variable x, then we write the relationship as $y = k \cdot x$. k is called the **constant of proportionality.**

If we were to graph this function, you can see that it would pass through the origin, because $y = 0$ when $x = 0$,whatever the value of k. So we know that a direct variation, when graphed, has a single intercept at (0, 0).

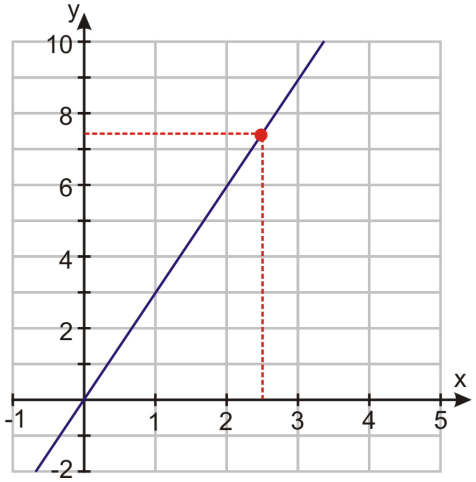
Example A

If y varies directly with x according to the relationship $y = k \cdot x$, and $y = 7.5$ when $x = 2.5$, determine the constant of proportionality, k.

Solution

We can solve for the constant of proportionality using substitution. Substitute $x = 2.5$ and $y = 7.5$ into the equation $y = k \cdot x$ to get $7.5 = k(2.5)$. Then divide both sides by 2.5 to get $k = \frac{7.5}{2.5} = 3$. **The constant of proportionality, k, is 3.**

We can graph the relationship quickly, using the intercept (0, 0) and the point (2.5, 7.5). The graph is shown below. It is a straight line with slope 3.



The graph of a direct variation always passes through the origin, and always has a slope that is equal to the constant of proportionality, k.

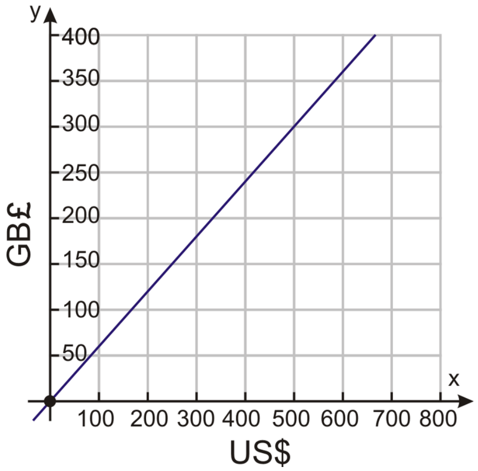
Example B

The graph shown below is a conversion chart used to convert U.S. dollars (US$) to British pounds (GB£) in a bank on a particular day. Use the chart to determine:

a) the number of pounds you could buy for $600

b) the number of dollars it would cost to buy £200

c) the exchange rate in pounds per dollar



Solution

We can read the answers to a) and b) right off the graph. It looks as if at $x = 600$ the graph is about one fifth of the way between £350 and £400. So $600 would buy £360.

Similarly, the line $y = 200$ appears to intersect the graph about a third of the way between $300 and $400. We can round this to $330, so it would cost approximately $330 to buy £200.

To solve for the exchange rate, we should note that as this is a direct variation - the graph is a straight line passing through the origin. The slope of the line gives the constant of proportionality (in this case the **exchange rate**) and it is equal to the ratio of the $y-$value to $x-$value at any point. Looking closely at the graph, we can see that the line passes through one convenient lattice point: (500, 300). This will give us the most accurate value for the slope and so the exchange rate.

$$y = k \cdot x \Rightarrow = \frac{y}{x},$$

$$\text{so rate} = \frac{300 \text{ pounds}}{500 \text{ dollars}} = 0.60 \text{ pounds per dollar.}$$

Graph Direct Variation Equations

We know that all direct variation graphs pass through the origin, and also that the slope of the line is equal to the constant of proportionality, k. Graphing is a simple matter of using the point-slope or point-point methods discussed earlier in this chapter.

Example C

Plot the following direct variations on the same graph.

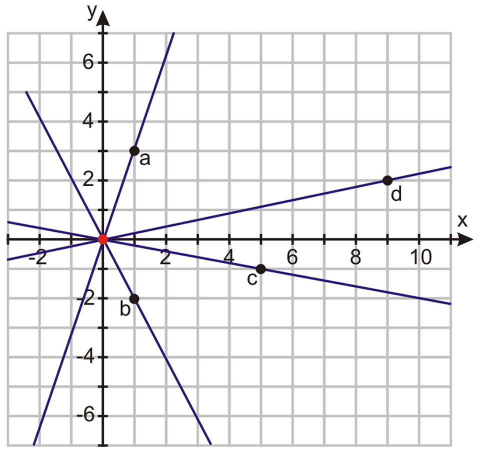
a) $y = 3x$

b) $y = -2x$

c) $y = -0.2x$

d) $y = \frac{2}{9}x$

Solution



a) The line passes through (0, 0), as will all these functions. This function has a slope of 3. When we move across by one unit, the function increases by three units.

b) The line has a slope of -2. When we move across the graph by one unit, the function **falls** by two units.

c) The line has a slope of -0.2. As a fraction this is equal to $-\frac{1}{5}$. When we move across by five units, the function **falls** by one unit.

d) The line passes through (0, 0) and has a slope of $\frac{2}{9}$. When we move across the graph by 9 units, the function increases by two units.

For more examples of how to plot and identify direct variation functions, see the video at http://neaportal.k12.ar.us/index.php/2010/06/slope-and-direct-variation/.

Solve Real-World Problems Using Direct Variation Models

Direct variations are seen everywhere in everyday life. Any time one quantity increases at the same rate another quantity increases (for example, doubling when it doubles and tripling when it triples), we say that they follow a direct variation.

Newton's Second Law

In 1687 Sir Isaac Newton published the famous *Principia Mathematica.* It contained, among other things, his second law of motion. This law is often written as $F = m \cdot a$, where a force of F Newtons applied to a mass of m kilograms results in acceleration of a meters per second^2. Notice that if the mass stays constant, then this formula is basically the same as the direct variation equation, just with different variables—and m is the constant of proportionality.

Example D

If a 175 Newton force causes a shopping cart to accelerate down the aisle with an acceleration of $2.5\ m/s^2$, *calculate:*

a) *The mass of the shopping cart.*

b) *The force needed to accelerate the same cart at* $6\ m/s^2$.

Solution

a) We can solve for m (the mass) by plugging in our given values for force and acceleration. $F = m \cdot a$ becomes $175 = m(2.5)$, and then we divide both sides by 2.5 to get $70 = m$.

So **the mass of the shopping cart is 70 kg.**

b) Once we have solved for the mass, we simply substitute that value, plus our required acceleration, back into the formula $F = m \cdot a$ and solve for F. We get $F = 70 \times 6 = 420$.

So **the force needed to accelerate the cart at** $6\ m/s^2$ **is 420 Newtons.**

Ohm's Law

The electrical current, I (amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship $V = I \cdot R$, where R is the resistance (measured in Ohms). The resistance is considered to be a constant for all values of V and I, so once again, this formula is a version of the direct variation formula, with R as the constant of proportionality.

Example E

A certain electronics component was found to pass a current of 1.3 amps at a voltage of 2.6 volts. When the voltage was increased to 12.0 volts the current was found to be 6.0 amps.

a) *Does the component obey Ohm's law?*

b) *What would the current be at 6 volts?*

Solution

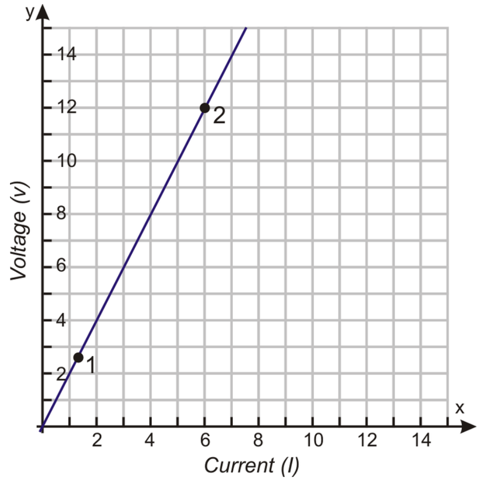
Ohm's law is a simple direct proportionality law, with the resistance R as our constant of proportionality. To know if this component obeys Ohm's law, we need to know if it follows a direct proportionality rule. In other words, **is** V **directly proportional to** I**?**

We can determine this in two different ways.

Graph It: If we plot our two points on a graph and join them with a line, does the line pass through (0, 0)?

Voltage is the independent variable and current is the dependent variable, so normally we would graph V on the horizontal axis and I on the vertical axis. However, if we swap the variables around just this once, we'll get a graph whose slope conveniently happens to be equal to the resistance, R. So we'll treat I as the independent variable, and our two points will be (1.3, 2.6) and (6, 12).

Plotting those points and joining them gives the following graph:



The graph does appear to pass through the origin, so **yes, the component obeys Ohm's law.**

Solve for R: If this component does obey Ohm's law, the constant of proportionality (R) should be the same when we plug in the second set of values as when we plug in the first set. Let's see if it is. (We can quickly find the value of R in each case; since $V = I \cdot R$, that means $R = \frac{V}{I}$.)

$$\text{Case 1: } R = \frac{V}{I} = \frac{2.6}{1.3} = 2 \text{ Ohms}$$
$$\text{Case 2: } R = \frac{V}{I} = \frac{12}{6} = 2 \text{ Ohms}$$

The values for R agree! This means that we are indeed looking at a direct variation. **The component obeys Ohm's law.**

b) Now to find the current at 6 volts, simply substitute the values for V and R into $V = I \cdot R$. We found that $R = 2$, so we plug in 2 for R and 6 for V to get $6 = I(2)$, and divide both sides by 2 to get $3 = I$.

So **the current through the component at a voltage of 6 volts is 3 amps.**

Watch this video for help with the Examples above.

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CK-12 Foundation: DirectVariation Models

Vocabulary

- If a variable y varies *directly* with variable x, then we write the relationship as $y = k \cdot x$, where k is a constant called the **constant of proportionality.**

Guided Practice

The volume of water in a fish-tank, V, varies directly with depth, d. If there are 15 gallons in the tank when the depth is 8 inches, calculate how much water is in the tank when the depth is 20 inches.

Solution

This is a good example of a direct variation, but for this problem we'll have to determine the equation of the variation ourselves. Since the volume, V, depends on depth, d, we'll use an equation of the form $y = k \cdot x$, but in place of y we'll use V and in place of x we'll use d:

$$V = k \cdot d$$

We know that when the depth is 8 inches the volume is 15 gallons, so to solve for k, we plug in 15 for V and 8 for d to get $15 = k(8)$. Dividing by 8 gives us $k = \frac{15}{8} = 1.875$.

Now to find the volume of water at the final depth, we use $V = k \cdot d$ again, but this time we can plug in our new d and the value we found for k:

$$\begin{aligned} V &= 1.875 \times 20 \\ V &= 37.5 \end{aligned}$$

At a depth of 20 inches, the volume of water in the tank is 37.5 gallons.

Practice

For 1-4, plot the following direct variations on the same graph.

1. $y = \frac{4}{3}x$
2. $y = -\frac{2}{3}x$
3. $y = -\frac{1}{6}x$
4. $y = 1.75x$
5. Dasan's mom takes him to the video arcade for his birthday.
   a. In the first 10 minutes, he spends \$3.50 playing games. If his allowance for the day is \$20, how long can he keep playing games before his money is gone?
   b. He spends the next 15 minutes playing Alien Invaders. In the first two minutes, he shoots 130 aliens. If he keeps going at this rate, how many aliens will he shoot in fifteen minutes?
   c. The high score on this machine is 120000 points. If each alien is worth 100 points, will Dasan beat the high score? What if he keeps playing for five more minutes?
6. The current standard for low-flow showerheads is 2.5 gallons per minute.
   a. How long would it take to fill a 30-gallon bathtub using such a showerhead to supply the water?
   b. If the bathtub drain were not plugged all the way, so that every minute 0.5 gallons ran out as 2.5 gallons ran in, how long would it take to fill the tub?
   c. After the tub was full and the showerhead was turned off, how long would it take the tub to empty through the partly unplugged drain?
   d. If the drain were immediately unplugged all the way when the showerhead was turned off, so that it drained at a rate of 1.5 gallons per minute, how long would it take to empty?
7. Amin is using a hose to fill his new swimming pool for the first time. He starts the hose at 10 PM and leaves it running all night.

   a. At 6 AM he measures the depth and calculates that the pool is four sevenths full. At what time will his new pool be full?
   b. At 10 AM he measures again and realizes his earlier calculations were wrong. The pool is still only three quarters full. When will it actually be full?
   c. After filling the pool, he needs to chlorinate it to a level of 2.0 ppm (parts per million). He adds two gallons of chlorine solution and finds that the chlorine level is now 0.7 ppm. How many more gallons does he need to add?
   d. If the chlorine level in the pool decreases by 0.05 ppm per day, how much solution will he need to add each week?

8. Land in Wisconsin is for sale to property investors. A 232-acre lot is listed for sale for $200,500.
   a. Assuming the same price per acre, how much would a 60-acre lot sell for?
   b. Again assuming the same price, what size lot could you purchase for $100,000?
9. The force (F) needed to stretch a spring by a distance x is given by the equation $F = k \cdot x$, where k is the spring constant (measured in Newtons per centimeter, or N/cm). If a 12 Newton force stretches a certain spring by 10 cm, calculate:
   a. The spring constant, k
   b. The force needed to stretch the spring by 7 cm.
   c. The distance the spring would stretch with a 23 Newton force.
10. Angela's cell phone is completely out of power when she puts it on the charger at 3 PM. An hour later, it is 30% charged. When will it be completely charged?
11. It costs $100 to rent a recreation hall for three hours and $150 to rent it for five hours.
   a. Is this a direct variation?
   b. Based on the cost to rent the hall for three hours, what would it cost to rent it for six hours, assuming it is a direct variation?
   c. Based on the cost to rent the hall for five hours, what would it cost to rent it for six hours, assuming it is a direct variation?
   d. Plot the costs given for three and five hours and graph the line through those points. Based on that graph, what would you expect the cost to be for a six-hour rental?

4.10 Graphs of Linear Functions

Here you'll learn how to write equations in function form. You'll also learn how to evaluate and graph linear functions. Finally, you'll learn how to find the common difference of arithmetic progressions.

What if you were given an equation like $y = \frac{1}{2}x + 3$? How could you write it in function notation, evaluate it for a specific function value, and graph it? After completing this Concept, you'll be able to perform tasks like these.

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CK-12 Foundation: 0410S Linear Function Graphs (H264)

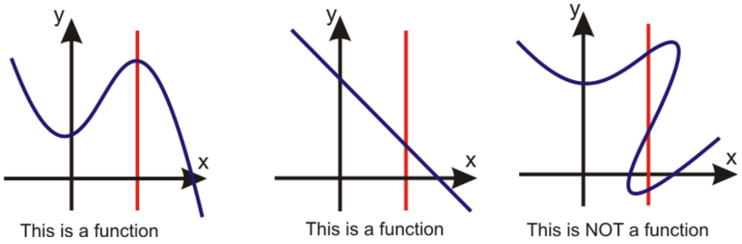
Guidance

The highly exclusive Fellowship of the Green Mantle allows in only a limited number of new members a year. In its third year of membership it has 28 members, in its fourth year it has 33, and in its fifth year it has 38. How many members are admitted a year, and how many founding members were there?

Functions

So far we've used the term **function** to describe many of the equations we've been graphing, but in mathematics it's important to remember that not all equations are functions. In order to be a function, a relationship between two variables, x and y, must map each $x-$value to **exactly one** $y-$value.

Visually this means the graph of y versus x must pass the ***vertical line test,*** meaning that a vertical line drawn through the graph of the function must never intersect the graph in more than one place:



Use Function Notation

When we write functions we often use the notation "$f(x) =$" in place of "$y =$". $f(x)$ is pronounced "f of x".

Example A

Rewrite the following equations so that y is a function of x and is written $f(x)$:

a) $y = 2x + 5$

b) $y = -0.2x + 7$

c) $x = 4y - 5$

d) $9x + 3y = 6$

Solution

a) Simply replace y with $f(x) : f(x) = 2x + 5$

b) Again, replace y with $f(x) : f(x) = -0.2x + 7$

c) First we need to solve for y. Starting with $x = 4y - 5$, we add 5 to both sides to get $x + 5 = 4y$, divide by 4 to get $\frac{x+5}{4} = y$, and then replace y with $f(x) : f(x) = \frac{x+5}{4}$.

d) Solve for y : take $9x + 3y = 6$, subtract $9x$ from both sides to get $3y = 6 - 9x$, divide by 3 to get $y = \frac{6-9x}{3} = 2 - 3x$, and express as a function: $f(x) = 2 - 3x$.

Using the functional notation in an equation gives us more information. For instance, the expression $f(x) = mx + b$ shows clearly that x is the independent variable because you ***plug in*** values of x into the function and perform a series of operations on the value of x in order to calculate the values of the dependent variable, y.

We can also plug in expressions rather than just numbers. For example, if our function is $f(x) = x + 2$, we can plug in the expression $(x + 5)$. We would express this as $f(x + 5) = (x + 5) + 2 = x + 7$.

Example B

A function is defined as $f(x) = 6x - 36$. *Evaluate the following:*

a) $f(2)$

b) $f(0)$

c) $f(z)$

d) $f(x + 3)$

e) $f(2r - 1)$

Solution

a) Substitute $x = 2$ into the function $f(x)$: $f(2) = 6 \cdot 2 - 36 = 12 - 36 = -24$

b) Substitute $x = 0$ into the function $f(x)$: $f(0) = 6 \cdot 0 - 36 = 0 - 36 = -36$

c) Substitute $x = z$ into the function $f(x)$: $f(z) = 6z + 36$

d) Substitute $x = (x + 3)$ into the function $f(x)$: $f(x + 3) = 6(x + 3) + 36 = 6x + 18 + 36 = 6x + 54$

e) Substitute $x = (2r + 1)$ into the function $f(x)$: $f(2r + 1) = 6(2r + 1) + 36 = 12r + 6 + 36 = 12r + 42$

Graph a Linear Function

Since the notations "$f(x) =$" and "$y =$" are interchangeable, we can use all the concepts we have learned so far to graph functions.

Example C

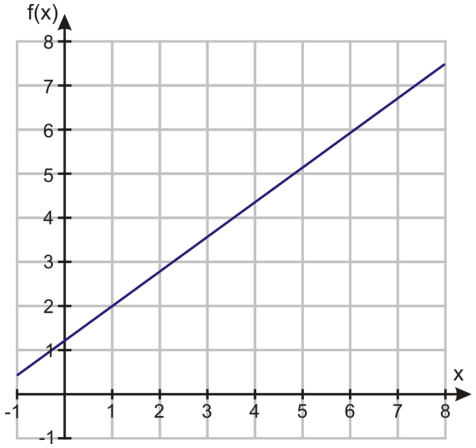
Graph the function $f(x) = \frac{3x+5}{4}$.

Solution

We can write this function in **slope-intercept** form:

$$f(x) = \frac{3}{4}x + \frac{5}{4} = 0.75x + 1.25$$

So our graph will have a $y-$intercept of (0, 1.25) and a slope of 0.75.



Arithmetic Progressions

You may have noticed that with linear functions, when you increase the $x-$value by 1 unit, the $y-$value increases by a fixed amount, equal to the slope. For example, if we were to make a table of values for the function $f(x) = 2x + 3$, we might start at $x = 0$ and then add 1 to x for each row:

TABLE 4.6:

x	$f(x)$
0	3
1	5
2	7
3	9
4	11

Notice that the values for $f(x)$ go up by 2 (the slope) each time. When we repeatedly add a fixed value to a starting number, we get a sequence like {3, 5, 7, 9, 11....}. We call this an ***arithmetic progression***, and it is characterized by the fact that each number is bigger (or smaller) than the preceding number by a fixed amount. This amount is called the **common difference**. We can find the common difference for a given sequence by taking 2 consecutive terms in the sequence and subtracting the first from the second.

Example D

Find the common difference for the following arithmetic progressions:

a) {7, 11, 15, 19, ...}

b) {12, 1, -10, -21, ...}

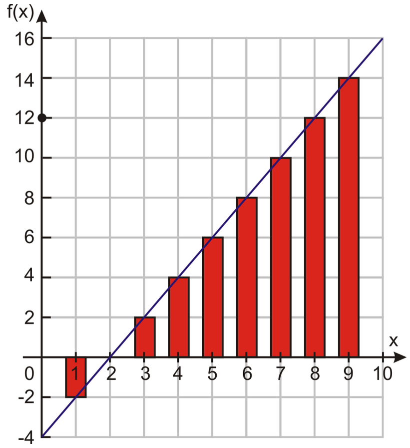
c) {7, __, 12, __, 17, ...}

Solution

a) $11 - 7 = 4;\ 15 - 11 = 4;\ 19 - 15 = 4$. The common difference is **4**.

b) $1-12=-11$. The common difference is **-11**.

c) There are not 2 consecutive terms here, but we know that to get the term after 7 we would add the common difference, and then to get to 12 we would add the common difference again. So *twice* the common difference is $12-7=5$, and so the common difference is **2.5.**



Arithmetic sequences and linear functions are very closely related. To get to the next term in a arithmetic sequence, you add the *common difference* to the last term; similarly, when the $x-$value of a linear function increases by one, the $y-$value increases by the amount of the *slope*. So arithmetic sequences are very much like linear functions, with the common difference playing the same role as the slope.

The graph below shows the arithmetic progression {-2, 0, 2, 4, 6...} along with the function $y=2x-4$. The only major difference between the two graphs is that an arithmetic sequence is **discrete** while a linear function is **continuous.**

We can write a formula for an arithmetic progression: if we define the first term as a_1 and d as the common difference, then the other terms are as follows:

$$\begin{array}{ccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & & a_n \\ a_1 & a_1+d & a_1+2d & a_1+3d & a_1+4d & \ldots & a_1+(n-1)\cdot d \end{array}$$

The online calculator at http://planetcalc.com/177/ will tell you the nth term in an arithmetic progression if you tell it the first term, the common difference, and what value to use for n (in other words, which term in the sequence you want to know). It will also tell you the sum of all the terms up to that point. Finding sums of sequences is something you will learn to do in future math classes.

Watch this video for help with the Examples above.

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CK-12 Foundation: LinearFunction Graphs

Vocabulary

- In order for an equation to be a **function**, the relationship between the two variables, x and y, must map each $x-$value to *exactly* one $y-$value.
- The graph of a function of y versus x must pass the **vertical line test:** any vertical line will only cross the graph of the function in one place.
- Functions can be expressed in function notation using $f(x) =$ in place of $y =$.
- The sequence of $f(x)$ values for a linear function form an arithmetic progression. Each number is greater than (or less than) the preceding number by a fixed amount, or **common difference.**

Guided Practice

Graph the function $f(x) = \frac{7(5-x)}{5}$.

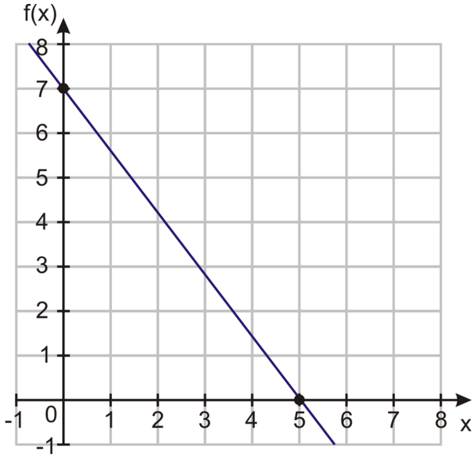
Solution

This time we'll solve for the $x-$ and $y-$intercepts.

To solve for the $y-$intercept, plug in $x = 0 : f(0) = \frac{7(5-0)}{5} = \frac{35}{5} = 7$, so the $x-$intercept is **(0, 7)**.

To solve for the $x-$intercept, set $f(x) = 0 : 0 = \frac{7(5-x)}{5}$, so $0 = 35-7x$, therefore $7x = 35$ and $x = 5$. The $y-$intercept is **(5, 0)**.

We can graph the function from those two points:



Review Questions

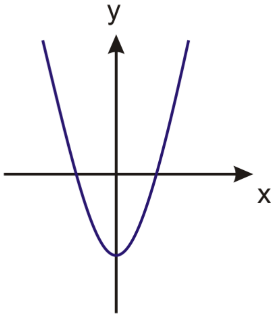
1. When an object falls under gravity, it gains speed at a constant rate of 9.8 m/s every second. An item dropped from the top of the Eiffel Tower, which is 300 meters tall, takes 7.8 seconds to hit the ground. How fast is it moving on impact?
2. A prepaid phone card comes with $20 worth of calls on it. Calls cost a flat rate of $0.16 per minute.
   a. Write the value left on the card as a function of minutes used so far.
   b. Use the function to determine how many minutes of calls you can make with the card.

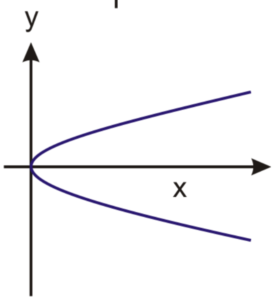
For questions 3-5, evaluate the function for $f(-3)$, $f(0)$, $f(z)$, $f(x+3)$, $f(2n)$, $f(3y+8)$, and $f\left(\frac{q}{2}\right)$.

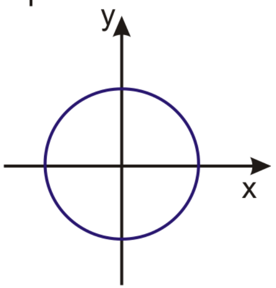
3. $f(x) = -2x+3$
4. $f(x) = 0.7x+3.2$

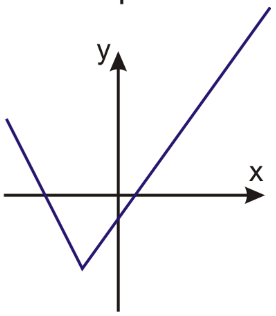
5. $f(x) = \frac{5(2-x)}{11}$

For questions 6-9, determine whether the graph could be a **functions.**

6. 

7. 

8. 

9. 

10. The roasting guide for a turkey suggests cooking for 100 minutes plus an additional 8 minutes per pound.
    a. Write a function for the roasting time the given the turkey weight in pounds (x).
    b. Determine the time needed to roast a 10 lb turkey.
    c. Determine the time needed to roast a 27 lb turkey.
    d. Determine the maximum size turkey you could roast in 4.5 hours.

For questions 11-13, determine the missing terms in the following arithmetic progressions.

11. {-11, 17, __, 73}
12. {2, __, -4}
13. {13, __, __, __, 0}

4.11 Problem Solving with Linear Graphs

Here you'll learn how to follow a problem-solving plan that includes graphing to solve real-world problems.

What if you borrowed \$2000 from your parents for a used car? At the end of the summer you pay them back \$750 that you earned from your summer job. Throughout the school year, you will pay them another \$125 per month from your part-time job. How many months will it take you to pay them back? After completing this Concept, you'll be able to solve problems like this one using graphs.

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CK-12 Foundation: 0411S Problem-Solving Strategies -Graphs (H264)

Guidance

In this chapter, we've been solving problems where quantities are linearly related to each other. In this section, we'll look at a few examples of linear relationships that occur in real-world problems, and see how we can solve them using graphs. Remember back to our Problem Solving Plan:

1. **Understand the Problem**
2. **Devise a Plan—Translate**
3. **Carry Out the Plan—Solve**
4. **Look—Check and Interpret**

Example A

A cell phone company is offering its costumers the following deal: You can buy a new cell phone for \$60 and pay a monthly flat rate of \$40 per month for unlimited calls. How much money will this deal cost you after 9 months?

Solution

Let's follow the problem solving plan.

Step 1: The phone costs \$60; the calling plan costs \$40 per month.

Let $x =$ number of months.

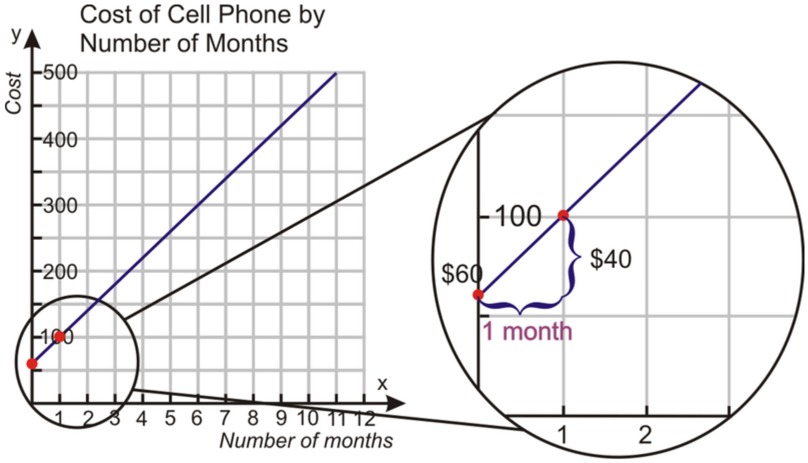
Let $y =$ total cost.

Step 2: We can solve this problem by making a graph that shows the number of months on the horizontal axis and the cost on the vertical axis.

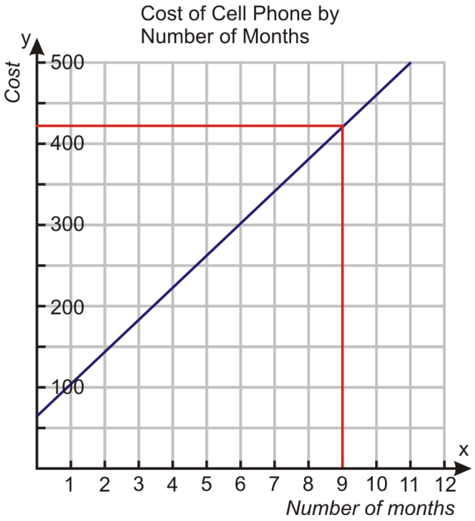
Since you pay \$60 when you get the phone, the $y-$intercept is (0, 60).

You pay \$40 for each month, so the cost rises by \$40 for 1 month, so the slope is 40.

We can graph this line using the slope-intercept method.



Step 3: The question was "How much will this deal cost after 9 months?" We can now read the answer from the graph. We draw a vertical line from 9 months until it meets the graph, and then draw a horizontal line until it meets the vertical axis.



We see that after 9 months **you pay approximately \$420.**

Step 4: To check if this is correct, let's think of the deal again.

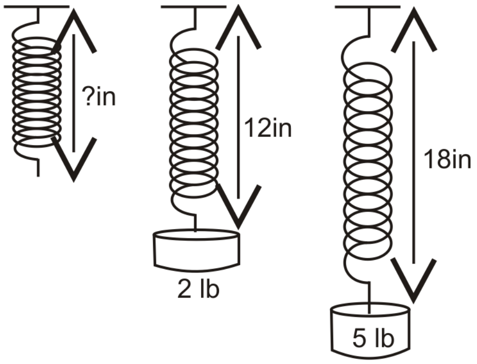
Originally, you pay \$60 and then \$40 a month for 9 months.

$$\begin{aligned}\text{Phone} &= \$60\\ \text{Calling plan} &= \$40 \times 9 = \$360\\ \text{Total cost} &= \$420.\end{aligned}$$

The answer checks out.

Example B

A stretched spring has a length of 12 inches when a weight of 2 lbs is attached to the spring. The same spring has a length of 18 inches when a weight of 5 lbs is attached to the spring. What is the length of the spring when no weights are attached?



Solution

Step 1: We know: the length of the spring = 12 inches when weight = 2 lbs

the length of the spring = 18 inches when weight = 5 lbs

We want: the length of the spring when weight = 0 lbs

Let $x =$ the weight attached to the spring.

Let $y =$ the length of the spring.

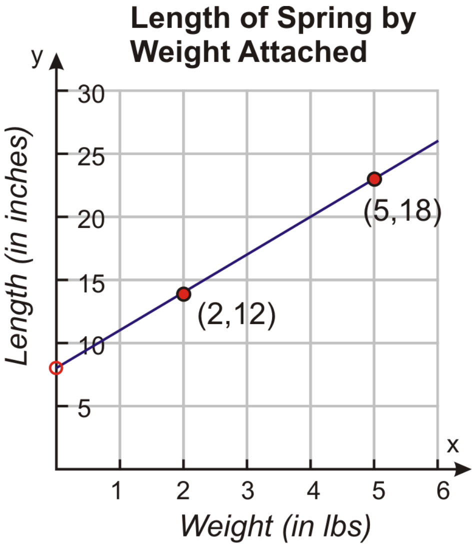
Step 2: We can solve this problem by making a graph that shows the weight on the horizontal axis and the length of the spring on the vertical axis.

We have two points we can graph:

When the weight is 2 lbs, the length of the spring is 12 inches. This gives point (2, 12).

When the weight is 5 lbs, the length of the spring is 18 inches. This gives point (5, 18).

Graphing those two points and connecting them gives us our line.



Step 3: The question was: *"What is the length of the spring when no weights are attached?*

We can answer this question by reading the graph we just made. When there is no weight on the spring, the $x-$value equals zero, so we are just looking for the $y-$intercept of the graph. On the graph, the $y-$intercept appears to be approximately 8 inches.

Step 4: To check if this correct, let's think of the problem again.

You can see that the length of the spring goes up by 6 inches when the weight is increased by 3 lbs, so the slope of the line is $\frac{6\ inches}{3\ lbs} = 2\ inches/lb$.

To find the length of the spring when there is no weight attached, we can look at the spring when there are 2 lbs attached. For each pound we take off, the spring will shorten by 2 inches. If we take off 2 lbs, the spring will be shorter by 4 inches. So, the length of the spring with no weights is 12 inches - 4 inches = 8 inches.

The answer checks out.

Example C

Christine took 1 hour to read 22 pages of Harry Potter. *She has 100 pages left to read in order to finish the book. How much time should she expect to spend reading in order to finish the book?*

Solution

Step 1: We know - Christine takes 1 hour to read 22 pages.

We want - How much time it takes to read 100 pages.

Let $x =$ the time expressed in hours.

Let $y =$ the number of pages.

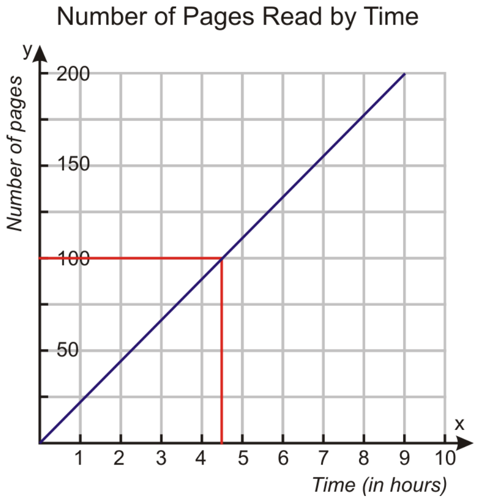
Step 2: We can solve this problem by making a graph that shows the number of hours spent reading on the horizontal axis and the number of pages on the vertical axis.

We have two points we can graph:

Christine takes 1 hour to read 22 pages. This gives point (1, 22).

A second point is not given, but we know that Christine would take 0 hours to read 0 pages. This gives point (0, 0).

Graphing those two points and connecting them gives us our line.



Step 3: The question was: "How much time should Christine expect to spend reading 100 pages?" We can find the answer from reading the graph - we draw a horizontal line from 100 pages until it meets the graph and then we draw

the vertical until it meets the horizontal axis. We see that it takes **approximately 4.5 hours** to read the remaining 100 pages.

Step 4: To check if this correct, let's think of the problem again.

We know that Christine reads 22 pages per hour - this is the slope of the line or the rate at which she is reading. To find how many hours it takes her to read 100 pages, we divide the number of pages by the rate. In this case, $\frac{100\ pages}{22\ pages/hour} = 4.54\ hours$. This is very close to the answer we got from reading the graph.

The answer checks out.

Watch this video for help with the Examples above.

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CK-12 Foundation: Problem Solving Strategies- Graphs

Vocabulary

The four steps of the **problem solving plan** when using graphs are:

1. **Understand the Problem**
2. **Devise a Plan—Translate:** Make a graph.
3. **Carry Out the Plan—Solve:** Use the graph to answer the question asked.
4. **Look—Check and Interpret**

Guided Practice

Aatif wants to buy a surfboard that costs $249. He was given a birthday present of $50 and he has a summer job that pays him $6.50 per hour. To be able to buy the surfboard, how many hours does he need to work?

Solution

Step 1: We know - The surfboard costs $249.

Aatif has $50.

His job pays $6.50 per hour.

We want - How many hours Aatif needs to work to buy the surfboard.

Let $x =$ the time expressed in hours

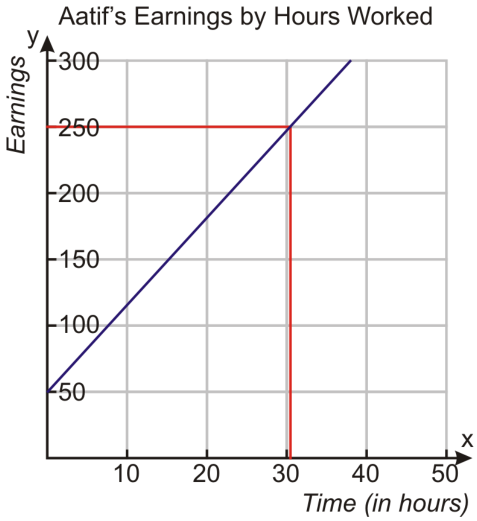
Let $y =$ Aatif's earnings

Step 2: We can solve this problem by making a graph that shows the number of hours spent working on the horizontal axis and Aatif's earnings on the vertical axis.

Aatif has $50 at the beginning. This is the $y-$intercept: (0, 50).

He earns $6.50 per hour. This is the slope of the line.

We can graph this line using the slope-intercept method. We graph the $y-$intercept of (0, 50), and we know that for each unit in the horizontal direction, the line rises by 6.5 units in the vertical direction. Here is the line that describes this situation.



Step 3: The question was: *"How many hours does Aatif need to work to buy the surfboard?"*

We find the answer from reading the graph - since the surfboard costs \$249, we draw a horizontal line from \$249 on the vertical axis until it meets the graph and then we draw a vertical line downwards until it meets the horizontal axis. We see that it takes **approximately 31 hours** to earn the money.

Step 4: To check if this correct, let's think of the problem again.

We know that Aatif has \$50 and needs \$249 to buy the surfboard. So, he needs to earn $\$249 - \$50 = \$199$ from his job.

His job pays \$6.50 per hour. To find how many hours he need to work we divide: $\frac{\$199}{\$6.50/hour} = 30.6\ hours$. This is very close to the answer we got from reading the graph.

The answer checks out.

Practice

Solve the following problems by making a graph and reading it.

1. A gym is offering a deal to new members. Customers can sign up by paying a registration fee of \$200 and a monthly fee of \$39.
    a. How much will this membership cost a member by the end of the year?
    b. The old membership rate was \$49 a month with a registration fee of \$100. How much more would a year's membership cost at that rate?
    c. **Bonus:** For what number of months would the two membership rates be the same?
2. A candle is burning at a linear rate. The candle measures five inches two minutes after it was lit. It measures three inches eight minutes after it was lit.
    a. What was the original length of the candle?
    b. How long will it take to burn down to a half-inch stub?
    c. Six half-inch stubs of candle can be melted together to make a new candle measuring $2\frac{5}{6}$ inches (a little wax gets lost in the process). How many stubs would it take to make three candles the size of the original candle?
3. A dipped candle is made by taking a wick and dipping it repeatedly in melted wax. The candle gets a little bit thicker with each added layer of wax. After it has been dipped three times, the candle is 6.5 mm thick. After it has been dipped six times, it is 11 mm thick.
    a. How thick is the wick before the wax is added?

   b. How many times does the wick need to be dipped to create a candle 2 cm thick?

4. Tali is trying to find the thickness of a page of his telephone book. In order to do this, he takes a measurement and finds out that 55 pages measures $\frac{1}{8}$ *inch*. What is the thickness of one page of the phone book?
5. Bobby and Petra are running a lemonade stand and they charge 45 cents for each glass of lemonade. In order to break even they must make $25.
   a. How many glasses of lemonade must they sell to break even?
   b. When they've sold $18 worth of lemonade, they realize that they only have enough lemons left to make 10 more glasses. To break even now, they'll need to sell those last 10 glasses at a higher price. What does the new price need to be?
6. Dale is making cookies using a recipe that calls for 2.5 cups of flour for two dozen cookies. How many cups of flour does he need to make five dozen cookies?
7. To buy a car, Jason makes a down payment of $1500 and pays $350 per month in installments.
   a. How much money has Jason paid at the end of one year?
   b. If the total cost of the car is $8500, how long will it take Jason to finish paying it off?
   c. The resale value of the car decreases by $100 each month from the original purchase price. If Jason sells the car as soon as he finishes paying it off, how much will he get for it?
8. Anne transplants a rose seedling in her garden. She wants to track the growth of the rose so she measures its height every week. On the third week, she finds that the rose is 10 inches tall and on the eleventh week she finds that the rose is 14 inches tall. Assuming the rose grows linearly with time, what was the height of the rose when Anne planted it?
9. Ravi hangs from a giant spring whose length is 5 m. When his child Nimi hangs from the spring its length is 2 m. Ravi weighs 160 lbs and Nimi weighs 40 lbs. Write the equation for this problem in slope-intercept form. What should we expect the length of the spring to be when his wife Amardeep, who weighs 140 lbs, hangs from it?
10. Nadia is placing different weights on a spring and measuring the length of the stretched spring. She finds that for a 100 gram weight the length of the stretched spring is 20 cm and for a 300 gram weight the length of the stretched spring is 25 cm.
    a. What is the unstretched length of the spring?
    b. If the spring can only stretch to twice its unstretched length before it breaks, how much weight can it hold?
11. Andrew is a submarine commander. He decides to surface his submarine to periscope depth. It takes him 20 minutes to get from a depth of 400 feet to a depth of 50 feet.
    a. What was the submarine's depth five minutes after it started surfacing?
    b. How much longer would it take at that rate to get all the way to the surface?
12. Kiersta's phone has completely run out of battery power when she puts it on the charger. Ten minutes later, when the phone is 40% recharged, Kiersta's friend Danielle calls and Kiersta takes the phone off the charger to talk to her. When she hangs up 45 minutes later, her phone has 10% of its charge left. Then she gets another call from her friend Kwan.
    a. How long can she spend talking to Kwan before the battery runs out again?
    b. If she puts the phone back on the charger afterward, how long will it take to recharge completely?
13. Marji is painting a 75-foot fence. She starts applying the first coat of paint at 2 PM, and by 2:10 she has painted 30 feet of the fence. At 2:15, her husband, who paints about $\frac{2}{3}$ as fast as she does, comes to join her.
    a. How much of the fence has Marji painted when her husband joins in?
    b. When will they have painted the whole fence?
    c. How long will it take them to apply the second coat of paint if they work together the whole time?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9614.

Summary

This chapter begins with points and graphs in the coordinate plane. It then builds on these concepts with graphs of linear equations, vertical lines, and horizontal lines. It then discusses the intercepts of a line—how to find them and how to use them to graph and equation. Next it defines slope and rate of change and discusses how to find the slope of a line/the rate of change of a function. As part of this topic, direct variation models will be introduced, with an emphasis on real-world applications. Function notation, the vertical line test, and arithmetic progression will also be covered. Finally, the chapter concludes with the problem-solving strategies Read a Graph and Make a Graph.

CHAPTER 5 Writing Linear Equations

Chapter Outline

5.1 DETERMINING THE EQUATION OF A LINE
5.2 FORMS OF LINEAR EQUATIONS
5.3 APPLICATIONS USING LINEAR MODELS
5.4 COMPARING EQUATIONS OF PARALLEL AND PERPENDICULAR LINES
5.5 FAMILIES OF LINES
5.6 FITTING LINES TO DATA
5.7 LINEAR INTERPOLATION AND EXTRAPOLATION

Introduction

What if you were given select pieces of information about a line like its y-intercept and its slope, or two of its points? How could you determine the equation of that line? The goal of this chapter is to find linear equations that model real-life situations. Why is this important? Because given the equation of a line, you can find the value of its variable(s). Based on the information available to you, there are various ways to determine the equation that best represents a real-world scenario. This chapter focuses on several methods for writing linear equations, including slope-intercept form, point-slope form, and standard form. It also teaches you how to determine if two lines are parallel or perpendicular, how to write the equation of a line that is parallel or perpendicular to another, and how to find the line of best fit for a data set.

5.1 Determining the Equation of a Line

Here you'll learn how to write the equations of lines given their slope and y-intercept or two of their points.

What if you were given the slope of a line and either its y-intercept or one of its points? Or what if you were given two of its points? How could you write the equation of that line? After completing this Concept, you'll be able to write and graph equations from such information.

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CK-12 Foundation: 0501S Linear Equations(H264)

Try This

Another applet at http://www.cut-the-knot.org/Curriculum/Calculus/StraightLine.shtml lets you create multiple lines and see how they intersect. Each line is defined by two points; you can change the slope of a line by moving either of the points, or just drag the whole line around without changing its slope. To create another line, just click Duplicate and then drag one of the lines that are already there.

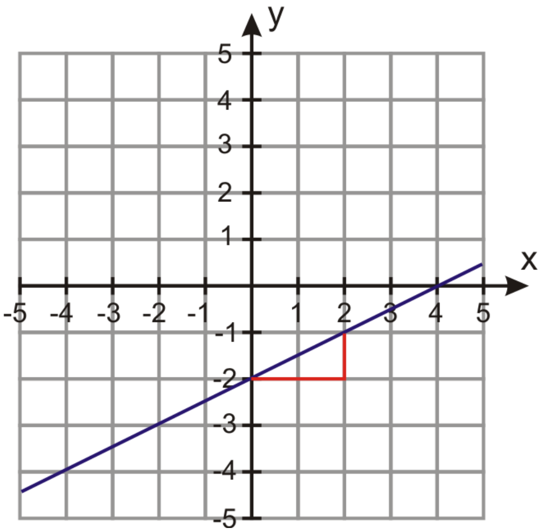
Guidance

We saw in the last chapter that many real-world situations can be described with linear graphs and equations. In this chapter, we'll see how to find those equations in a variety of situations.

Write an Equation Given Slope and $y-$Intercept

You've already learned how to write an equation in slope–intercept form: simply start with the general equation for the slope-intercept form of a line, $y = mx + b$, and then plug the given values of m and b into the equation. For example, a line with a slope of 4 and a $y-$intercept of -3 would have the equation $y = 4x - 3$.

If you are given just the graph of a line, you can read off the slope and $y-$intercept from the graph and write the equation from there. For example, on the graph below you can see that the line rises by 1 unit as it moves 2 units to the right, so its slope is $\frac{1}{2}$. Also, you can see that the $y-$intercept is -2, so the equation of the line is $y = \frac{1}{2}x - 2$.



Write an Equation Given the Slope and a Point

Often, we don't know the value of the $y-$intercept, but we know the value of y for a non-zero value of x. In this case, it's often easier to write an equation of the line in **point-slope form.** An equation in point-slope form is written as $y-y_0 = m(x-x_0)$, where m is the slope and (x_0, y_0) is a point on the line.

Example A

A line has a slope of $\frac{3}{5}$*, and the point (2, 6) is on the line. Write the equation of the line in point-slope form.*

Solution

Start with the formula $y-y_0 = m(x-x_0)$.

Plug in $\frac{3}{5}$ for m, 2 for x_0 and 6 for y_0.

The equation in point-slope form is $y-6 = \frac{3}{5}(x-2)$.

Notice that the equation in point-slope form is not solved for y. If we did solve it for y, we'd have it in $y-$intercept form. To do that, we would just need to distribute the $\frac{3}{5}$ and add 6 to both sides. That means that the equation of this line in slope-intercept form is $y = \frac{3}{5}x - \frac{6}{5} + 6$, or simply $y = \frac{3}{5}x + \frac{24}{5}$.

Write an Equation Given Two Points

Point-slope form also comes in useful when we need to find an equation given just two points on a line.

For example, suppose we are told that the line passes through the points (-2, 3) and (5, 2). To find the equation of the line, we can start by finding the slope.

Starting with the slope formula, $m = \frac{y_2-y_1}{x_2-x_1}$, we plug in the $x-$ and $y-$values of the two points to get $m = \frac{2-3}{5-(-2)} = \frac{-1}{7}$. We can plug that value of m into the point-slope formula to get $y-y_0 = -\frac{1}{7}(x-x_0)$.

Now we just need to pick one of the two points to plug into the formula. Let's use (5, 2); that gives us $y-2 = -\frac{1}{7}(x-5)$.

What if we'd picked the other point instead? Then we'd have ended up with the equation $y-3 = -\frac{1}{7}(x+2)$, which doesn't look the same. That's because there's more than one way to write an equation for a given line in point-slope form. But let's see what happens if we solve each of those equations for y.

Starting with $y-2 = -\frac{1}{7}(x-5)$, we distribute the $-\frac{1}{7}$ and add 2 to both sides. That gives us $y = -\frac{1}{7}x + \frac{5}{7} + 2$, or $y = -\frac{1}{7}x + \frac{19}{7}$.

On the other hand, if we start with $y-3 = -\frac{1}{7}(x+2)$, we need to distribute the $-\frac{1}{7}$ and add 3 to both sides. That gives us $y = -\frac{1}{7}x - \frac{2}{7} + 3$, which also simplifies to $y = -\frac{1}{7}x + \frac{19}{7}$.

So whichever point we choose to get an equation in point-slope form, the equation is still mathematically the same, and we can see this when we convert it to $y-$intercept form.

Example B

A line contains the points (3, 2) and (-2, 4). Write an equation for the line in point-slope form; then write an equation in y−intercept form.

Solution

Find the slope of the line: $m = \frac{y_2-y_1}{x_2-x_1} = \frac{4-2}{-2-3} = -\frac{2}{5}$

Plug in the value of the slope: $y - y_0 = -\frac{2}{5}(x - x_0)$.

Plug point (3, 2) into the equation: $y - 2 = -\frac{2}{5}(x - 3)$.

The equation in point-slope form is $y - 2 = -\frac{2}{5}(x - 3)$.

To convert to $y-$intercept form, simply solve for y:

$$\begin{aligned}
y - 2 = -\frac{2}{5}(x - 3) &\rightarrow y - 2 = -\frac{2}{5}x + \frac{6}{5}\\
&\rightarrow y = -\frac{2}{5}x + \frac{6}{5} + 2\\
&\rightarrow y = -\frac{2}{5}x + 3\frac{1}{5}.
\end{aligned}$$

The equation in $y-$**intercept form is** $y = -\frac{2}{5}x + 3\frac{1}{5}$**.**

Graph an Equation in Point-Slope Form

Another useful thing about point-slope form is that you can use it to graph an equation without having to convert it to slope-intercept form. From the equation $y - y_0 = m(x - x_0)$, you can just read off the slope m and the point (x_0, y_0). To draw the graph, all you have to do is plot the point, and then use the slope to figure out how many units up and over you should move to find another point on the line.

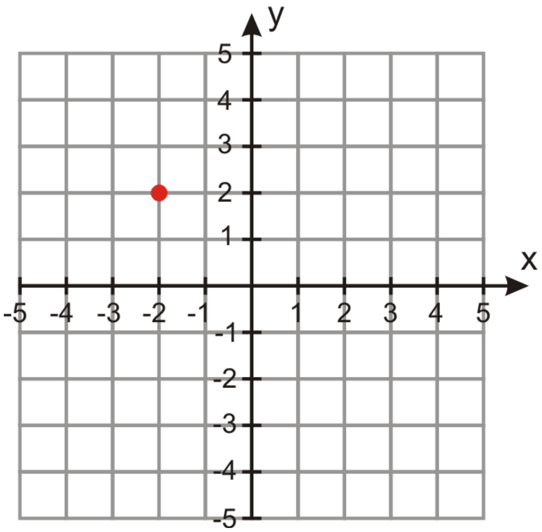
Example C

Make a graph of the line given by the equation $y + 2 = \frac{2}{3}(x - 2)$.

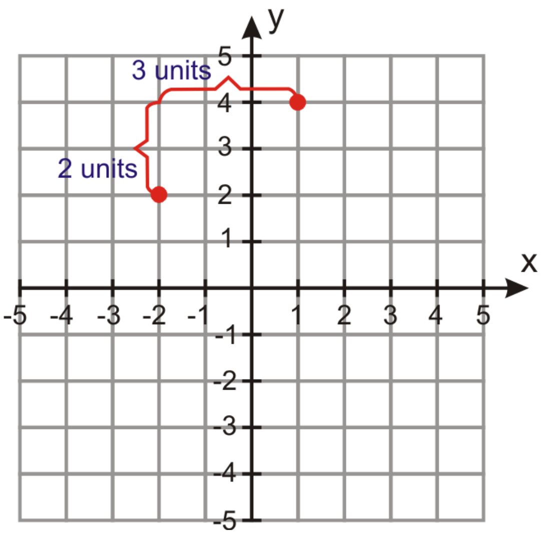
Solution

To read off the right values, we need to rewrite the equation slightly: $y - (-2) = \frac{2}{3}(x - 2)$. Now we see that point (2, -2) is on the line and that the slope is $\frac{2}{3}$.

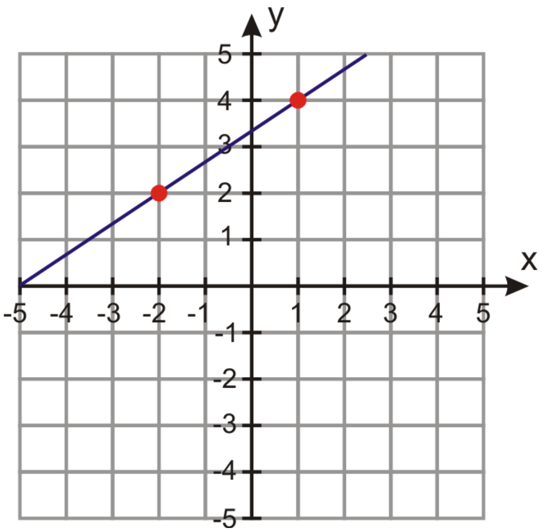
First plot point (2, -2) on the graph:



A slope of $\frac{2}{3}$ tells you that from that point you should move 2 units up and 3 units to the right and draw another point:



Now draw a line through the two points and extend it in both directions:



Watch this video for help with the Examples above.

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CK-12 Foundation: LinearEquations

Vocabulary

- Often, we don't know the value of the $y-$intercept, but we know the value of y for a non-zero value of x. In this case, it's often easier to write an equation of the line in **point-slope form.** An equation in point-slope form is written as $y-y_0 = m(x-x_0)$, where m is the slope and (x_0, y_0) is a point on the line.

Guided Practice

A line contains the points (1, -2) and (0, 0). Write an equation for the line in point-slope form; then write an equation in y−intercept form.

Solution

Find the slope of the line: $m = \frac{y_2-y_1}{x_2-x_1} = \frac{-2-0}{1-0} = \frac{-2}{1} = -2$

Plug in the value of the slope: $y-y_0 = -2(x-x_0)$.

Plug point (1, -2) into the equation: $y-(-2) = -2(x-1)$.

The equation in point-slope form is $y+2 = -2(x-1)$.

To convert to $y-$intercept form, simply solve for y:

$$\begin{aligned} y+2 = -2(x-1) &\rightarrow y+2 = -2x+2 \\ &\rightarrow y = -2x+2-2 \\ &\rightarrow y = -2x. \end{aligned}$$

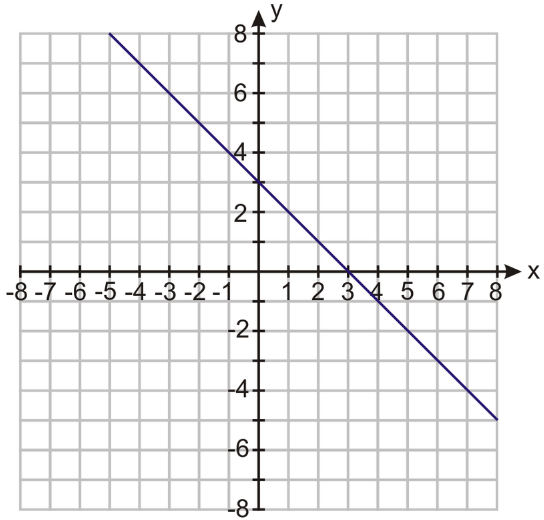
The equation in $y-$intercept form is $y = -2x$.

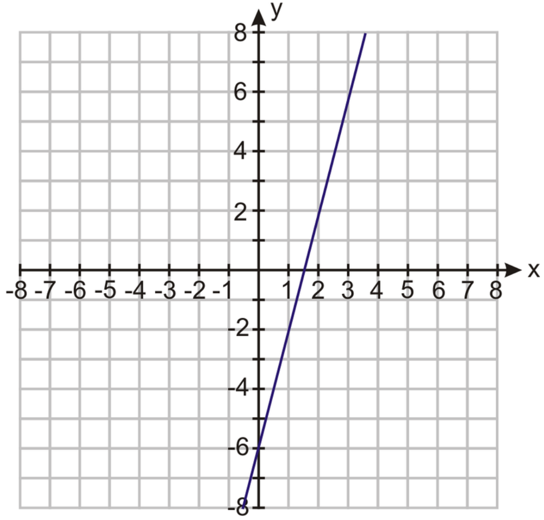
Practice

Find the equation of each line in slope–intercept form.

1. The line has a slope of 7 and a $y-$intercept of -2.
2. The line has a slope of -5 and a $y-$intercept of 6.
3. The line has a slope of $-\frac{1}{4}$ and contains the point (4, -1).
4. The line contains points (3, 5) and (-3, 0).
5. The line contains points (10, 15) and (12, 20).

Write the equation of each line in slope-intercept form.

6. 

7. 

Find the equation of each linear function in slope–intercept form.

8. $m = 5, f(0) = -3$
9. $m = -7, f(2) = -1$
10. $m = \frac{1}{3}, f(-1) = \frac{2}{3}$
11. $m = 4.2, f(-3) = 7.1$
12. $f\left(\frac{1}{4}\right) = \frac{3}{4}, f(0) = \frac{5}{4}$
13. $f(1.5) = -3, f(-1) = 2$

Write the equation of each line in point-slope form.

14. The line has slope $-\frac{1}{10}$ and goes through the point (10, 2).
15. The line has slope -75 and goes through the point (0, 125).
16. The line has slope 10 and goes through the point (8, -2).
17. The line goes through the points (-2, 3) and (-1, -2).
18. The line contains the points (10, 12) and (5, 25).
19. The line goes through the points (2, 3) and (0, 3).
20. The line has a slope of $\frac{3}{5}$ and a $y-$intercept of -3.
21. The line has a slope of -6 and a $y-$intercept of 0.5.

Write the equation of each linear function in point-slope form.

22. $m = -\frac{1}{5}$ and $f(0) = 7$
23. $m = -12$ and $f(-2) = 5$
24. $f(-7) = 5$ and $f(3) = -4$
25. $f(6) = 0$ and $f(0) = 6$
26. $m = 3$ and $f(2) = -9$
27. $m = -\frac{9}{5}$ and $f(0) = 32$

5.2 Forms of Linear Equations

Here you'll learn how to write equations of lines in the standard form of $ax+by=c$. You'll also learn how to find the slope and y-intercept of lines written in standard form.

In this Concept, you will learn how to write equations in standard form.

Watch This

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Click image to the left for more content.

CK-12 Foundation: 0502S StandardForm ofLinear Equations (H264)

Try This

Now that you've worked with equations in all three basic forms, check out the Java applet at http://www.ronblond.com/M10/lineAP/index.html. You can use it to manipulate graphs of equations in all three forms, and see how the graphs change when you vary the terms of the equations.

Guidance

You've already encountered another useful form for writing linear equations: **standard form.** An equation in standard form is written $ax+by=c$, where a, b, and c are all integers and a is positive. (Note that the b in the standard form is different than the b in the slope-intercept form.)

One useful thing about standard form is that it allows us to write equations for vertical lines, which we can't do in slope-intercept form.

For example, let's look at the line that passes through points (2, 6) and (2, 9). How would we find an equation for that line in slope-intercept form?

First we'd need to find the slope: $m = \frac{9-6}{0-0} = \frac{3}{0}$. But that slope is undefined because we can't divide by zero. And if we can't find the slope, we can't use point-slope form either.

If we just graph the line, we can see that x equals 2 no matter what y is. There's no way to express that in slope-intercept or point-slope form, but in standard form we can just say that $x+0y=2$, or simply $x=2$.

Converting to Standard Form

To convert an equation from another form to standard form, all you need to do is rewrite the equation so that all the variables are on one side of the equation and the coefficient of x is not negative.

Example A

Rewrite the following equations in standard form:

a) $y = 5x - 7$

b) $y - 2 = -3(x + 3)$

c) $y = \frac{2}{3}x + \frac{1}{2}$

Solution

We need to rewrite each equation so that all the variables are on one side and the coefficient of x is not negative.

a) $y = 5x - 7$

Subtract y from both sides to get $0 = 5x - y - 7$.

Add 7 to both sides to get $7 = 5x - y$.

Flip the equation around to put it in standard form: $5x - y = 7$.

b) $y - 2 = -3(x + 3)$

Distribute the –3 on the right-hand-side to get $y - 2 = -3x - 9$.

Add $3x$ to both sides to get $y + 3x - 2 = -9$.

Add 2 to both sides to get $y + 3x = -7$. Flip that around to get $3x + y = -7$.

c) $y = \frac{2}{3}x + \frac{1}{2}$

Find the common denominator for all terms in the equation – in this case that would be 6.

Multiply all terms in the equation by 6: $6\left(y = \frac{2}{3}x + \frac{1}{2}\right) \Rightarrow 6y = 4x + 3$

Subtract $6y$ from both sides: $0 = 4x - 6y + 3$

Subtract 3 from both sides: $-3 = 4x - 6y$

The equation in standard form is $4x - 6y = -3$.

Graphing Equations in Standard Form

When an equation is in slope-intercept form or point-slope form, you can tell right away what the slope is. How do you find the slope when an equation is in standard form?

Well, you could rewrite the equation in slope-intercept form and read off the slope. But there's an even easier way. Let's look at what happens when we rewrite an equation in standard form.

Starting with the equation $ax + by = c$, we would subtract ax from both sides to get $by = -ax + c$. Then we would divide all terms by b and end up with $y = -\frac{a}{b}x + \frac{c}{b}$.

That means that the slope is $-\frac{a}{b}$ and the $y-$intercept is $\frac{c}{b}$. So next time we look at an equation in standard form, we don't have to rewrite it to find the slope; we know the slope is just $-\frac{a}{b}$, where a and b are the coefficients of x and y in the equation.

Example B

Find the slope and the y−intercept of the following equations written in standard form.

a) $3x + 5y = 6$

b) $2x - 3y = -8$

c) $x - 5y = 10$

Solution

a) $a = 3, b = 5$, and $c = 6$, so the slope is $-\frac{a}{b} = -\frac{3}{5}$, and the $y-$intercept is $\frac{c}{b} = \frac{6}{5}$.

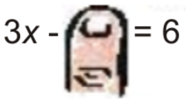
b) $a = 2, b = -3$, and $c = -8$, so the slope is $-\frac{a}{b} = \frac{2}{3}$, and the $y-$intercept is $\frac{c}{b} = \frac{8}{3}$.

c) $a = 1, b = -5$, and $c = 10$, so the slope is $-\frac{a}{b} = \frac{1}{5}$, and the $y-$intercept is $\frac{c}{b} = \frac{10}{-5} = -2$.

Once we've found the slope and $y-$intercept of an equation in standard form, we can graph it easily. But if we start with a graph, how do we find an equation of that line in standard form?

First, remember that we can also use the cover-up method to graph an equation in standard form, by finding the intercepts of the line. For example, let's graph the line given by the equation $3x - 2y = 6$.

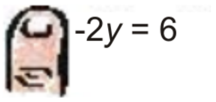
To find the $x-$intercept, cover up the y term (remember, the $x-$intercept is where $y = 0$):



$3x = 6 \Rightarrow x = 2$

The $x-$ intercept is (2, 0).

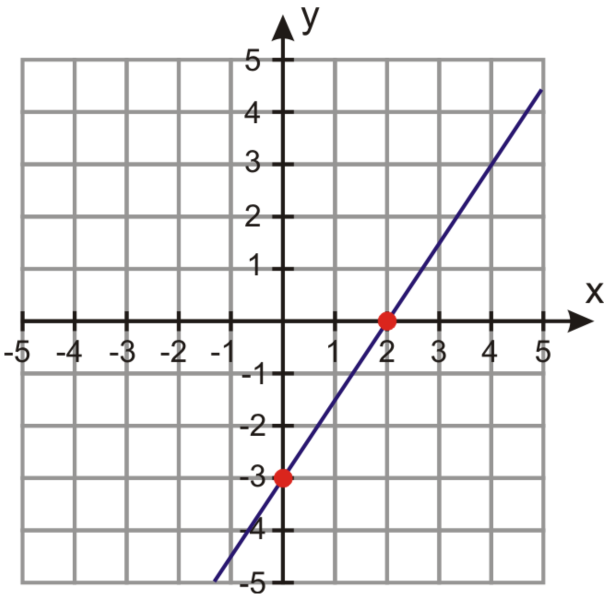
To find the $y-$intercept, cover up the x term (remember, the $y-$intercept is where $x = 0$):



$-2y = 6 \Rightarrow y = -3$

The $y-$intercept is (0, -3).

We plot the intercepts and draw a line through them that extends in both directions:

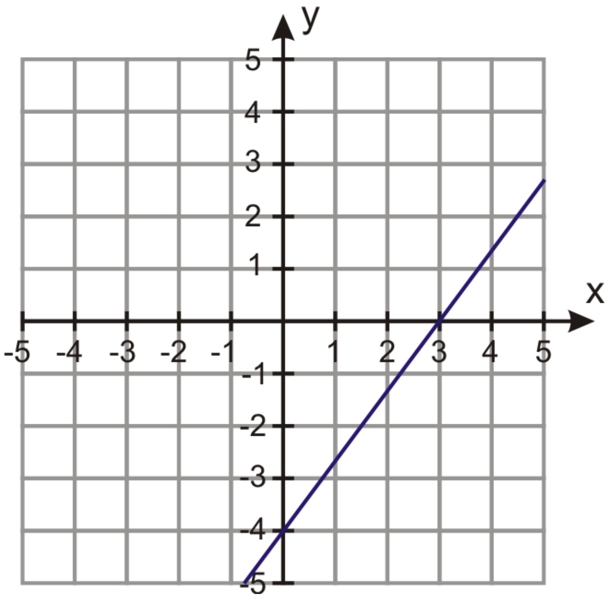


Now we want to apply this process in reverse—to start with the graph of the line and write the equation of the line in standard form.

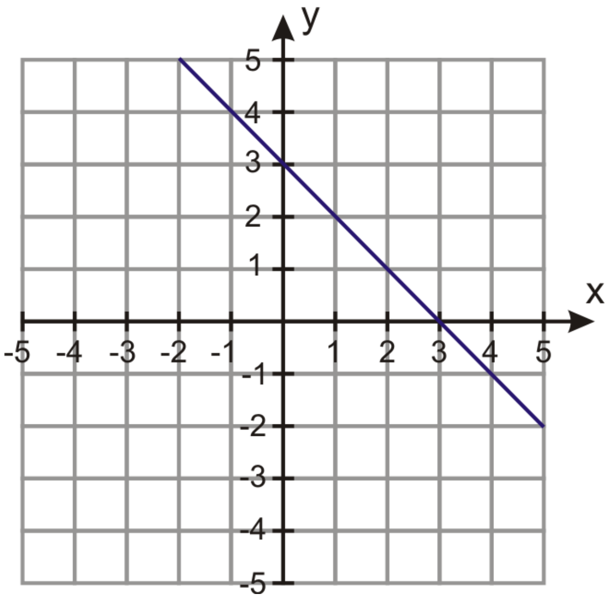
Example C

Find the equation of each line and write it in standard form.

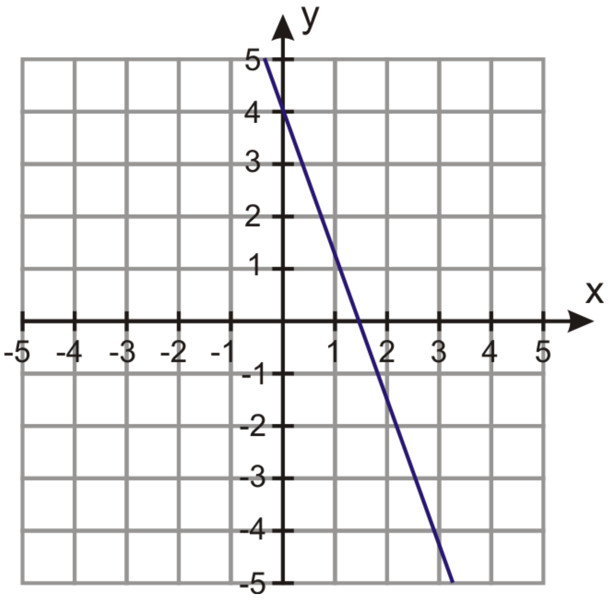
a)



b)



c)



Solution

a) We see that the $x-$intercept is $(3,0) \Rightarrow x = 3$ and the $y-$intercept is $(0,-4) \Rightarrow y = -4$

We saw that in standard form $ax+by = c$: if we "cover up" the y term, we get $ax = c$, and if we "cover up" the x term, we get $by = c$.

So we need to find values for a and b so that we can plug in 3 for x and -4 for y and get the same value for c in both cases. This is like finding the least common multiple of the $x-$ and $y-$intercepts.

In this case, we see that multiplying $x = 3$ by 4 and multiplying $y = -4$ by –3 gives the same result:

$$(x = 3) \times 4 \Rightarrow 4x = 12 \quad \text{and} \quad (y = -4) \times (-3) \Rightarrow -3y = 12$$

Therefore, $a = 4, b = -3$ and $c = 12$ and **the equation in standard form is** $4x - 3y = 12$.

b) We see that the $x-$intercept is $(3,0) \Rightarrow x = 3$ and the $y-$intercept is $(0,3) \Rightarrow y = 3$

The values of the intercept equations are already the same, so $a = 1, b = 1$ and $c = 3$. **The equation in standard form is** $x+y = 3$.

c) We see that the $x-$intercept is $\left(\frac{3}{2},0\right) \Rightarrow x = \frac{3}{2}$ and the $y-$intercept is $(0,4) \Rightarrow y = 4$

Let's multiply the $x-$intercept equation by $2 \Rightarrow 2x = 3$

Then we see we can multiply the $x-$intercept again by 4 and the $y-$intercept by 3, so we end up with $8x = 12$ and $3y = 12$.

The equation in standard form is $8x+3y = 12$.

Watch this video for help with the Examples above.

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CK-12 Foundation: Standard Form of Linear Equations

Vocabulary

- An equation in **standard form** is written $ax+by=c$, where a,b, and c are all integers and a is positive. (Note that the b in the standard form is different than the b in the slope-intercept form.)

Guided Practice

Find the slope and the $y-$intercept of the following equations written in standard form.

a) $10x+2y=5$

b) $21x-3y=-9$

Solution:

a) $a=10,b=2$, and $c=5$, so the slope is $-\frac{a}{b}=-\frac{10}{2}=-5$, and the $y-$intercept is $\frac{5}{2}=2.5$.

b) $a=21,b=-3$, and $c=-9$, so the slope is $-\frac{a}{b}=-\frac{21}{-3}=7$, and the $y-$intercept is $\frac{c}{b}=\frac{-9}{-3}=3$.

Practice

For 1-6, rewrite the following equations in standard form.

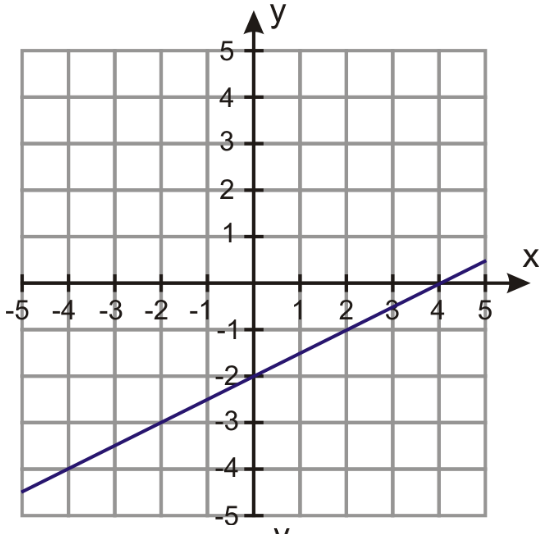
1. $y=3x-8$
2. $y-7=-5(x-12)$
3. $2y=6x+9$
4. $y=\frac{9}{4}x+\frac{1}{4}$
5. $y+\frac{3}{5}=\frac{2}{3}(x-2)$
6. $3y+5=4(x-9)$

For 7-12, find the slope and $y-$intercept of the following lines.

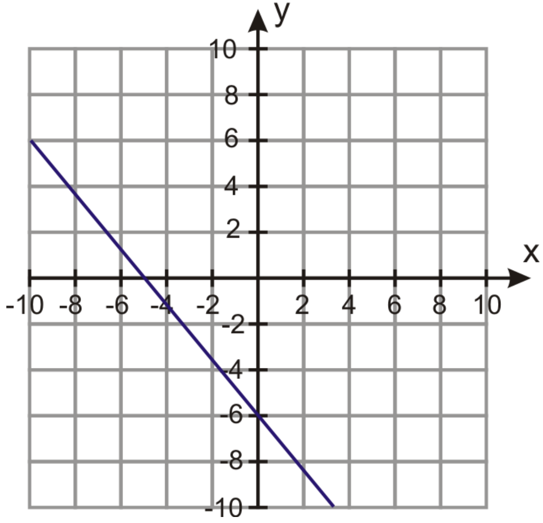
7. $5x-2y=15$
8. $3x+6y=25$
9. $x-8y=12$
10. $3x-7y=20$
11. $9x-9y=4$
12. $6x+y=3$

For 13-14, find the equation of each line and write it in standard form.

13.



14.



5.3 Applications Using Linear Models

Here you'll learn how to solve real-world problems whose equations are straight lines in either point-slope, slope-intercept, or standard form.

What if your car rental company charges $25 per day plus $0.25 per mile? When the car is returned to you the trip odometer reads 324 miles and the customer's bill totals $156. How could you determine the number of days the customer rented the car? In this Concept, you'll be able to solve real-world problems like this one.

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CK-12 Foundation: 0503S Solving Real-World Problems withLinear Equations (H264)

Guidance

Let's solve some word problems where we need to write the equation of a straight line in point-slope form.

Example A

Marciel rented a moving truck for the day. Marciel only remembers that the rental truck company charges $40 per day and some number of cents per mile. Marciel drives 46 miles and the final amount of the bill (before tax) is $63. What is the amount per mile the truck rental company charges? Write an equation in point-slope form that describes this situation. How much would it cost to rent this truck if Marciel drove 220 miles?

Solution

Let's define our variables:

$$x = \text{distance in miles}$$
$$y = \text{cost of the rental truck}$$

Peter pays a flat fee of $40 for the day; this is the $y-$intercept.

He pays $63 for 46 miles; this is the coordinate point (46,63).

Start with the point-slope form of the line: $y - y_0 = m(x - x_0)$

Plug in the coordinate point: $63 - y_0 = m(46 - x_0)$

Plug in the point (0, 40): $63 - 40 = m(46 - 0)$

Solve for the slope: $23 = 46m \rightarrow m = \frac{23}{46} = 0.5$

The slope is 0.5 dollars per mile, so the truck company charges 50 cents per mile ($0.5 = 50 cents). Plugging in the slope and the $y-$intercept, the equation of the line is $y = 0.5x + 40$.

To find out the cost of driving the truck 220 miles, we plug in $x = 220$ to get $y - 40 = 0.5(220) \Rightarrow y = \$150$.

Driving 220 miles would cost $150.

Example B

Anne got a job selling window shades. She receives a monthly base salary and a $6 commission for each window shade she sells. At the end of the month she adds up sales and she figures out that she sold 200 window shades and made $2500. Write an equation in point-slope form that describes this situation. How much is Anne's monthly base salary?

Solution

Let's define our variables:

$$x = \text{number of window shades sold}$$
$$y = \text{Anne's earnings}$$

We see that we are given the slope and a point on the line:

Nadia gets $6 for each shade, so the slope is 6.

She made $2500 when she sold 200 shades, so the point is (200, 2500).

Start with the point-slope form of the line: $y - y_0 = m(x - x_0)$

Plug in the slope: $y - y_0 = 6(x - x_0)$

Plug in the point (200, 2500): $y - 2500 = 6(x - 200)$

To find Anne's base salary, we plug in $x = 0$ and get $y - 2500 = -1200 \Rightarrow y = \$1300$.

Anne's monthly base salary is $1300.

Solving Real-World Problems Using Linear Models in Standard Form

Here are two examples of real-world problems where the standard form of an equation is useful.

Example C

Nadia buys fruit at her local farmer's market. This Saturday, oranges cost $2 per pound and cherries cost $3 per pound. She has $12 to spend on fruit. Write an equation in standard form that describes this situation. If she buys 4 pounds of oranges, how many pounds of cherries can she buy?

Solution

Let's define our variables:

$$x = \text{pounds of oranges}$$
$$y = \text{pounds of cherries}$$

The equation that describes this situation is $2x + 3y = 12$.

If she buys 4 pounds of oranges, we can plug $x = 4$ into the equation and solve for y:

$$2(4)+3y=12 \Rightarrow 3y=12-8 \Rightarrow 3y=4 \Rightarrow y=\frac{4}{3}$$

Nadia can buy $1\frac{1}{3}$ pounds of cherries.

Watch this video for help with the Examples above.

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CK-12 Foundation: Solving Real-WorldProblems with Linear Equations

Vocabulary

- A common form of a line (linear equation) is **slope-intercept form:** $y=mx+b$, where m is the slope and the point $(0,b)$ is the $y-$intercept.
- Often, we don't know the value of the $y-$intercept, but we know the value of y for a non-zero value of x. In this case, it's often easier to write an equation of the line in **point-slope form.** An equation in point-slope form is written as $y-y_0=m(x-x_0)$, where m is the slope and (x_0,y_0) is a point on the line.
- An equation in **standard form** is written $ax+by=c$, where a,b, and c are all integers and a is positive. (Note that the b in the standard form is different than the b in the slope-intercept form.)

Guided Practice

Peter skateboards part of the way to school and walks the rest of the way. He can skateboard at 7 miles per hour and he can walk at 3 miles per hour. The distance to school is 6 miles. Write an equation in standard form that describes this situation. If he skateboards for $\frac{1}{2}$ an hour, how long does he need to walk to get to school?

Solution

Let's define our variables:

$$\begin{aligned}x &= \text{time Peter skateboards}\\ y &= \text{time Peter walks}\end{aligned}$$

The equation that describes this situation is: $7x+3y=6$

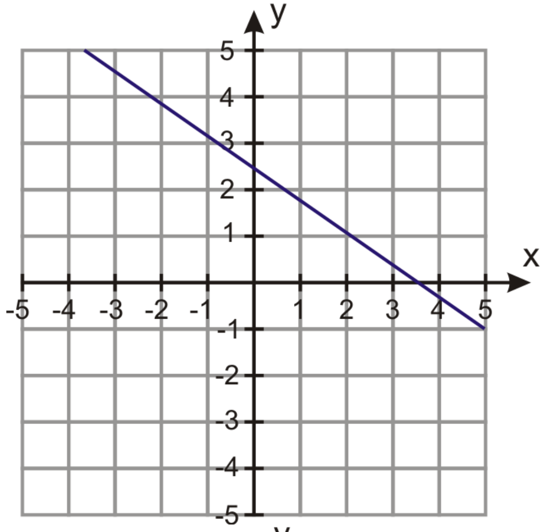
If Peter skateboards $\frac{1}{2}$ an hour, we can plug $x=0.5$ into the equation and solve for y:

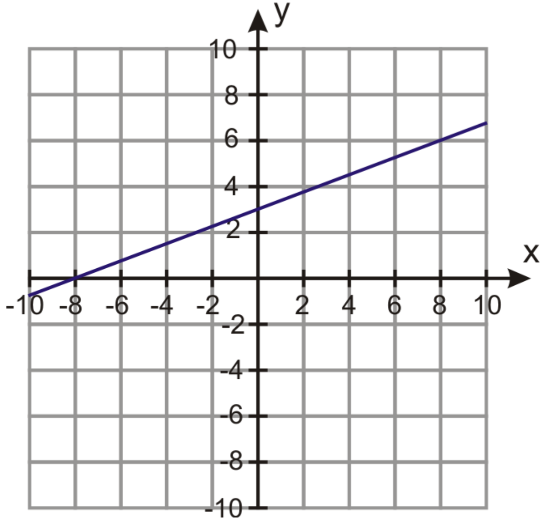
$$7(0.5)+3y=6 \Rightarrow 3y=6-3.5 \Rightarrow 3y=2.5 \Rightarrow y=\frac{5}{6}$$

Peter must walk $\frac{5}{6}$ of an hour.

Practice

For 1-8, write the equation in slope-intercept, point-slope and standard forms.

1. The line has a slope of $\frac{2}{3}$ and contains the point $\left(\frac{1}{2},1\right)$.
2. The line has a slope of -1 and contains the point $\left(\frac{4}{5},0\right)$.
3. The line has a slope of 2 and contains the point $\left(\frac{1}{3},10\right)$.
4. The line contains points (2, 6) and (5, 0).
5. The line contains points (5, -2) and (8, 4).
6. The line contains points (-2, -3) and (-5, 1).
7. 

8. 

For 9-10, solve the problem.

9. Andrew has two part time jobs. One pays \$6 per hour and the other pays \$10 per hour. He wants to make \$366 per week. Write an equation in standard form that describes this situation. If he is only allowed to work 15 hours per week at the \$10 per hour job, how many hours does he need to work per week in his \$6 per hour job in order to achieve his goal?
10. Anne invests money in two accounts. One account returns 5% annual interest and the other returns 7% annual interest. In order not to incur a tax penalty, she can make no more than \$400 in interest per year. Write an equation in standard form that describes this problem. If she invests \$5000 in the 5% interest account, how much money does she need to invest in the other account?

5.4 Comparing Equations of Parallel and Perpendicular Lines

Here you'll learn how to use slopes to determine whether two lines are parallel or perpendicular.

What if you had two lines? One line passes through the points (1, -2) and (3, 5). The other passes through the points (0, 2) and (7, 0). How could you determine if the two lines are parallel or perpendicular? After completing this Concept, you'll be able to make such a determination.

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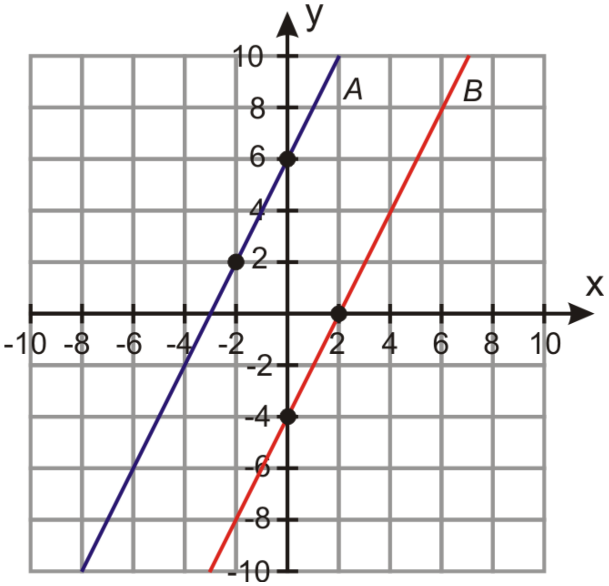
CK-12 Foundation: 0504S Determine Parallel and PerpendicularLines (H264)

Guidance

In this section you will learn how **parallel lines** and **perpendicular lines** are related to each other on the coordinate plane. Let's start by looking at a graph of two parallel lines.



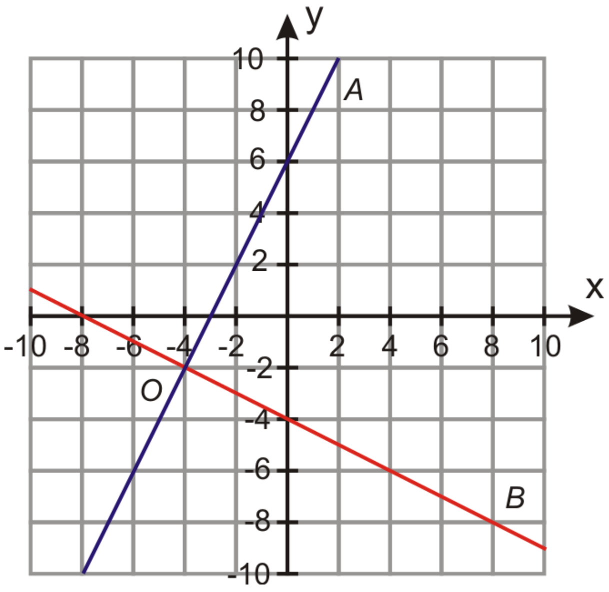
We can clearly see that the two lines have different $y-$intercepts: 6 and –4.

How about the slopes of the lines? The slope of line A is $\frac{6-2}{0-(-2)} = \frac{4}{2} = 2$, and the slope of line B is $\frac{0-(-4)}{2-0} = \frac{4}{2} = 2$. The slopes are the same.

Is that significant? Yes. By definition, parallel lines never meet. That means that when one of them slopes up by a certain amount, the other one has to slope up by the same amount so the lines will stay the same distance apart. If you look at the graph above, you can see that for any $x-$value you pick, the $y-$values of lines A and B are the same vertical distance apart—which means that both lines go up by the same vertical distance every time they go across by the same horizontal distance. In order to stay parallel, their slopes must stay the same.

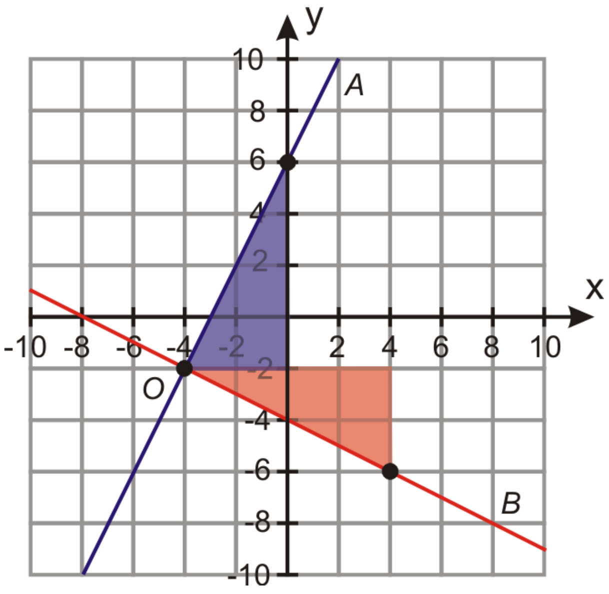
All parallel lines have the same slopes and different $y-$intercepts.

Now let's look at a graph of two perpendicular lines.



We can't really say anything about the $y-$intercepts. In this example, the $y-$intercepts are different, but if we moved the lines four units to the right, they would both intercept the $y-$axis at (0, -2). So perpendicular lines can have the same or different $y-$intercepts.

What about the relationship between the slopes of the two lines?



To find the slope of line A, we pick two points on the line and draw the blue (upper) right triangle. The legs of the triangle represent the rise and the run. We can see that the slope is $\frac{8}{4}$, or 2.

To find the slope of line B, we pick two points on the line and draw the red (lower) right triangle. Notice that the two triangles are identical, only rotated by 90°. Where line A goes 8 units up and 4 units right, line B goes 8 units right and 4 units down. Its slope is $-\frac{4}{8}$, or $-\frac{1}{2}$.

This is always true for perpendicular lines; where one line goes a units up and b units right, the other line will go a units right and b units down, so the slope of one line will be $\frac{a}{b}$ and the slope of the other line will be $-\frac{b}{a}$.

The slopes of **perpendicular lines** are always negative reciprocals of each other.

The Java applet at http://members.shaw.ca/ron.blond/perp.APPLET/index.html lets you drag around a pair of perpendicular lines to see how their slopes change. Click "Show Grid" to see the $x-$ and $y-$axes, and click "Show Constructors" to see the triangles that are being used to calculate the slopes of the lines (you can then drag the circle to make it bigger or smaller, and click on a triangle to see the slope calculations in detail.)

Determine When Lines are Parallel or Perpendicular

You can find whether lines are parallel or perpendicular by comparing the slopes of the lines. If you are given points on the lines, you can find their slopes using the formula. If you are given the equations of the lines, re-write each equation in a form that makes it easy to read the slope, such as the slope-intercept form.

Example A

Determine whether the lines are parallel or perpendicular or neither. One line passes through the points (2, 11) and (-1, 2); another line passes through the points (0, -4) and (-2, -10).

Solution

Find the slope of each line and compare them.

$m_1 = \frac{2-11}{-1-2} = \frac{-9}{-3} = 3 \quad \text{and} \quad m_2 = \frac{-10-(-4)}{-2-0} = \frac{-6}{-2} = 3$

The slopes are equal, so **the lines are parallel.**

Example B

Determine whether the lines are parallel or perpendicular or neither. One line passes through the points (-2, -7) and (1, 5); another line passes through the points (4, 1) and (-8, 4).

Solution:

$m_1 = \frac{5-(-7)}{1-(-2)} = \frac{12}{3} = 4 \quad \text{and} \quad m_2 = \frac{4-1}{-8-4} = \frac{3}{-12} = -\frac{1}{4}$

The slopes are negative reciprocals of each other, so **the lines are perpendicular.**

Example C

Determine whether the lines are parallel or perpendicular or neither. One line passes through the points (3, 1) and (-2, -2); another line passes through the points (5, 5) and (4, -6).

Solution:

$m_1 = \frac{-2-1}{-2-3} = \frac{-3}{-5} = \frac{3}{5} \quad \text{and} \quad m_2 = \frac{-6-5}{4-5} = \frac{-13}{-1} = 13$

The slopes are not the same or negative reciprocals of each other, so **the lines are neither parallel nor perpendicular.**

Watch this video for help with the Examples above.

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CK-12 Foundation: Determine ParallelandPerpendicular Lines

Vocabulary

- **All parallel lines** have the same slopes and different $y-$intercepts.
- The slopes of **perpendicular lines** are always negative reciprocals of each other.

Guided Practice

Determine whether the lines are parallel or perpendicular or neither:

a) $3x+4y=2$ and $8x-6y=5$

b) $2x=y-10$ and $y=-2x+5$

c) $7y+1=7x$ and $x+5=y$

Solution

Write each equation in slope-intercept form:

a) line 1: $3x+4y=2 \Rightarrow 4y=-3x+2 \Rightarrow y=-\frac{3}{4}x+\frac{1}{2} \Rightarrow \text{ slope}=-\frac{3}{4}$

line 2: $8x-6y=5 \Rightarrow 8x-5=6y \Rightarrow y=\frac{8}{6}x-\frac{5}{6} \Rightarrow y=\frac{4}{3}x-\frac{5}{6} \Rightarrow \text{ slope}=\frac{4}{3}$

The slopes are negative reciprocals of each other, so **the lines are perpendicular.**

b) line 1: $2x=y-10 \Rightarrow y=2x+10 \Rightarrow \text{ slope}=2$

line 2: $y=-2x+5 \Rightarrow \text{ slope}=-2$

The slopes are not the same or negative reciprocals of each other, so **the lines are neither parallel nor perpendicular.**

c) line 1: $7y+1=7x \Rightarrow 7y=7x-1 \Rightarrow y=x-\frac{1}{7} \Rightarrow \text{ slope}=1$

line 2: $x+5=y \Rightarrow y=x+5 \Rightarrow \text{ slope}=1$

The slopes are the same, so **the lines are parallel.**

Practice

For 1-10, determine whether the lines are parallel, perpendicular or neither.

1. One line passes through the points (-1, 4) and (2, 6); another line passes through the points (2, -3) and (8, 1).
2. One line passes through the points (4, -3) and (-8, 0); another line passes through the points (-1, -1) and (-2, 6).
3. One line passes through the points (-3, 14) and (1, -2); another line passes through the points (0, -3) and (-2, 5).
4. One line passes through the points (3, 3) and (-6, -3); another line passes through the points (2, -8) and (-6, 4).
5. One line passes through the points (2, 8) and (6, 0); another line has the equation $x-2y=5$.

6. One line passes through the points (-5, 3) and (2, -1); another line has the equation $2x+3y=6$.
7. Both lines pass through the point (2, 8); one line also passes through (3, 5), and the other line has slope 3.
8. Line 1: $4y+x=8$ Line 2: $12y+3x=1$
9. Line 1: $5y+3x=1$ Line 2: $6y+10x=-3$
10. Line 1: $2y-3x+5=0$ Line 2: $y+6x=-3$
11. Lines A, B, C, D, and E all pass through the point (3, 6). Line A also passes through (7, 12); line B passes through (8, 4); line C passes through (-1, -3); line D passes through (1, 1); and line E passes through (6, 12).
    a. Are any of these lines perpendicular? If so, which ones? If not, why not?
    b. Are any of these lines parallel? If so, which ones? If not, why not?

5.5 Families of Lines

Here you'll learn how to write the equation of a line that is parallel or perpendicular to a second line given that second line's equation and one of the points it passes through. You'll also investigate families of lines.

What if you were given the equation of a line like $y = -4x - 3$ and you wanted to find the equation of a line that is parallel or perpendicular to it that passes through the point (2, 1). How could you find the equation of this line? After completing this Concept, you'll be able to write equations of perpendicular and parallel lines.

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CK-12 Foundation: 0505S Equations ofParallel and Perpendicular Lines (H264)

Guidance

We can use the properties of parallel and perpendicular lines to write an equation of a line parallel or perpendicular to a given line. You might be given a line and a point, and asked to find the line that goes through the given point and is parallel or perpendicular to the given line. Here's how to do this:

1. Find the slope of the given line from its equation. (You might need to re-write the equation in a form such as the slope-intercept form.)
2. Find the slope of the parallel or perpendicular line—which is either the same as the slope you found in step 1 (if it's parallel), or the negative reciprocal of the slope you found in step 1 (if it's perpendicular).
3. Use the slope you found in step 2, along with the point you were given, to write an equation of the new line in slope-intercept form or point-slope form.

Example A

Find an equation of the line perpendicular to the line $y = -3x + 5$ that passes through the point (2, 6).

Solution

The slope of the given line is -3, so the perpendicular line will have a slope of $\frac{1}{3}$.

Now to find the equation of a line with slope $\frac{1}{3}$ that passes through (2, 6):

Start with the slope-intercept form: $y = mx + b$.

Plug in the slope: $y = \frac{1}{3}x + b$.

Plug in the point (2, 6) to find b: $6 = \frac{1}{3}(2) + b \Rightarrow b = 6 - \frac{2}{3} \Rightarrow b = \frac{16}{3} \rightarrow 5\frac{1}{3}$.

The equation of the line is $y = \frac{1}{3}x + 5\frac{1}{3}$.

Example B

Find the equation of the line parallel to $6x-5y=12$ that passes through the point (-5, -3).

Solution

Rewrite the equation in slope-intercept form: $6x-5y=12 \Rightarrow 5y=6x-12 \Rightarrow y=\frac{6}{5}x-\frac{12}{5}$.

The slope of the given line is $\frac{6}{5}$, so we are looking for a line with slope $\frac{6}{5}$ that passes through the point (-5, -3).

Start with the slope-intercept form: $y=mx+b$.

Plug in the slope: $y=\frac{6}{5}x+b$.

Plug in the point (-5, -3): $n-3=\frac{6}{5}(-5)+b \Rightarrow -3=-6+b \Rightarrow b=3$

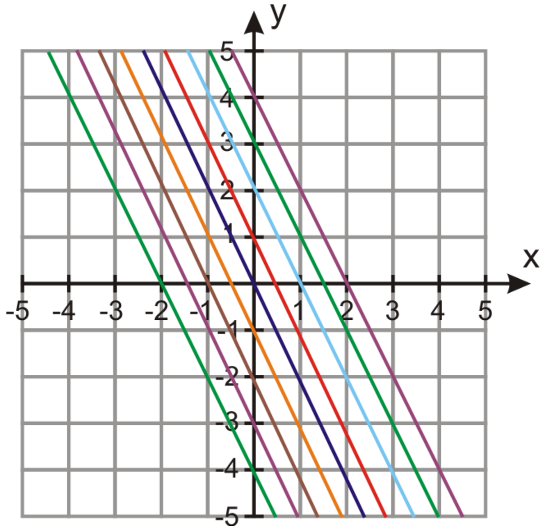
The equation of the line is $y=\frac{6}{5}x+3$.

Investigate Families of Lines

A **family of lines** is a set of lines that have something in common with each other. Straight lines can belong to two types of families: one where the slope is the same and one where the $y-$intercept is the same.

Family 1: Keep the slope unchanged and vary the $y-$intercept.

The figure below shows the family of lines with equations of the form $y=-2x+b$:

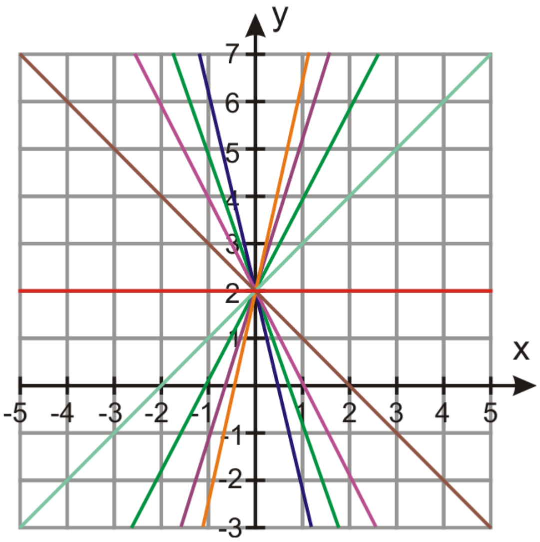


All the lines have a slope of –2, but the value of b is different for each line.

Notice that in such a family all the lines are parallel. All the lines look the same, except that they are shifted up and down the $y-$axis. As b gets larger the line rises on the $y-$axis, and as b gets smaller the line goes lower on the $y-$axis. This behavior is often called a **vertical shift.**

Family 2: Keep the $y-$intercept unchanged and vary the slope.

The figure below shows the family of lines with equations of the form $y=mx+2$:



All the lines have a $y-$intercept of two, but the slope is different for each line. The steeper lines have higher values of m.

Example C

Write the equation of the family of lines satisfying the given condition.

a) parallel to the $x-$axis

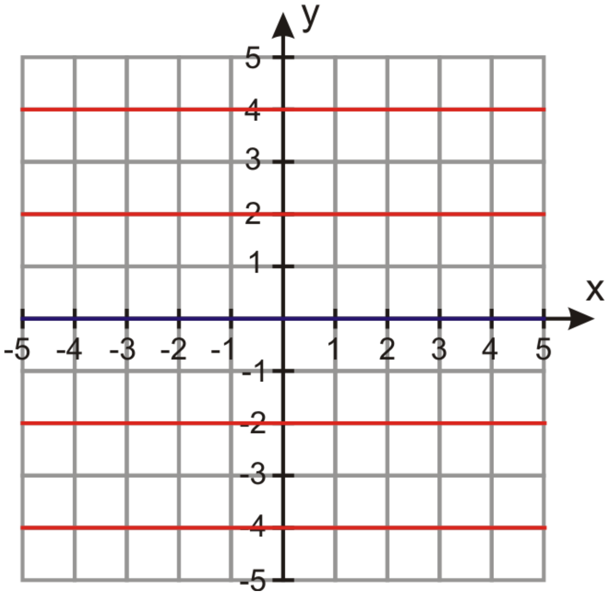
b) through the point (0, -1)

c) perpendicular to $2x+7y-9=0$

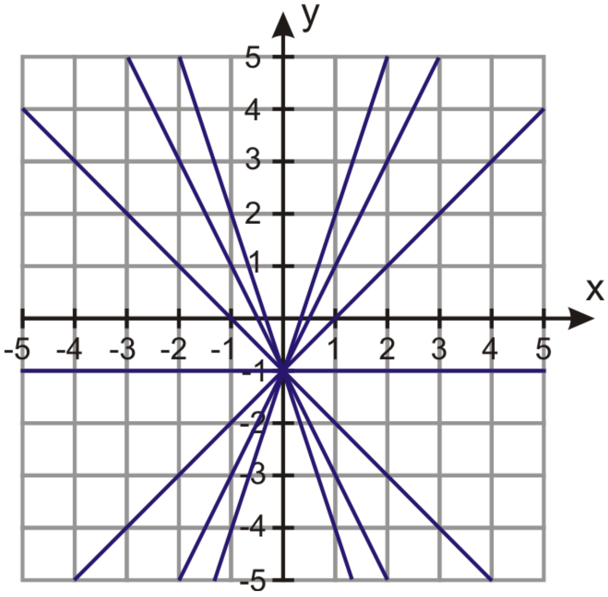
d) parallel to $x+4y-12=0$

Solution

a) All lines parallel to the $x-$axis have a slope of zero; the $y-$intercept can be anything. So the family of lines is $y=0x+b$ or just $y=b$.

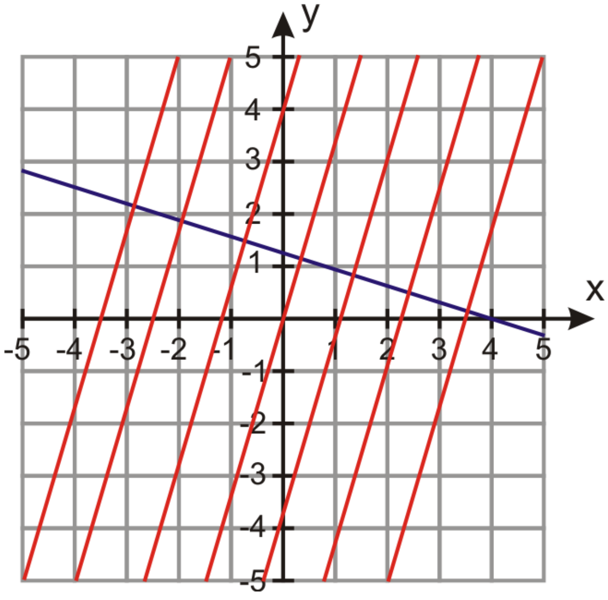


b) All lines passing through the point (0, -1) have the same $y-$intercept, $b=-1$. The family of lines is: $y=mx-1$.



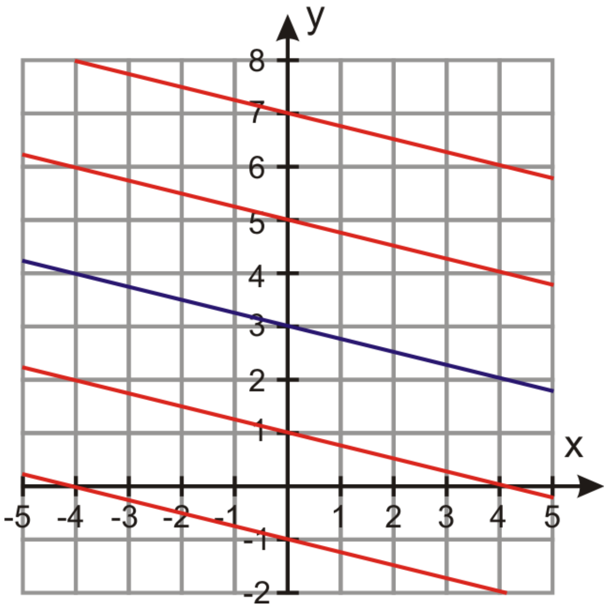
c) First we need to find the slope of the given line. Rewriting $2x+7y-9=0$ in slope-intercept form, we get $y=-\frac{2}{7}x+\frac{9}{7}$. The slope of the line is $-\frac{2}{7}$, so we're looking for the family of lines with slope $\frac{7}{2}$.

The family of lines is $y=\frac{7}{2}x+b$.



d) Rewrite $x+4y-12=0$ in slope-intercept form: $y=-\frac{1}{4}x+3$. The slope is $-\frac{1}{4}$, so that's also the slope of the family of lines we are looking for.

The family of lines is $y=-\frac{1}{4}x+b$.



Watch this video for help with the Examples above.

MEDIA

Click image to the left for more content.

CK-12 Foundation: Equations of Parallel and Perpendicular Lines

Vocabulary

- A **family of lines** is a set of lines that have something in common with each other. Straight lines can belong to two types of families: one where the slope is the same and one where the $y-$intercept is the same.

- Notice that in such a family all the lines are parallel. All the lines look the same, except that they are shifted up and down the $y-$axis. As b gets larger the line rises on the $y-$axis, and as b gets smaller the line goes lower on the $y-$axis. This behavior is often called a **vertical shift.**

Guided Practice

Find the equation of the line perpendicular to $x-5y=15$ *that passes through the point (-2, 5).*

Solution

Re-write the equation in slope-intercept form: $x-5y=15 \Rightarrow -5y=-x+15 \Rightarrow y=\frac{1}{5}x-3$.

The slope of the given line is $\frac{1}{5}$, so we're looking for a line with slope -5.

Start with the slope-intercept form: $y=mx+b$.

Plug in the slope: $y=-5x+b$.

Plug in the point (-2, 5): $5=-5(-2)+b \Rightarrow b=5-10 \Rightarrow b=-5$

The equation of the line is $y=-5x-5$.

Practice

1. Find the equation of the line parallel to $5x-2y=2$ that passes through point (3, -2).
2. Find the equation of the line perpendicular to $y=-\frac{2}{5}x-3$ that passes through point (2, 8).
3. Find the equation of the line parallel to $7y+2x-10=0$ that passes through the point (2, 2).
4. Find the equation of the line perpendicular to $y+5=3(x-2)$ that passes through the point (6, 2).
5. Line S passes through the points (2, 3) and (4, 7). Line T passes through the point (2, 5). If Lines S and T are parallel, name one more point on line T. (**Hint:** you don't need to find the slope of either line.)
6. Lines P and Q both pass through (-1, 5). Line P also passes through (-3, -1). If P and Q are perpendicular, name one more point on line Q. (This time you will have to find the slopes of both lines.)
7. Write the equation of the family of lines satisfying the given condition.
    a. All lines that pass through point (0, 4).
    b. All lines that are perpendicular to $4x+3y-1=0$.
    c. All lines that are parallel to $y-3=4x+2$.
    d. All lines that pass through the point (0, -1).
8. Name two lines that pass through the point (3, -1) and are perpendicular to each other.
9. Name two lines that are each perpendicular to $y=-4x-2$. What is the relationship of those two lines to each other?
10. Name two perpendicular lines that both pass through the point (3, -2). Then name a line parallel to one of them that passes through the point (-2, 5).

5.6 Fitting Lines to Data

Here you'll learn how to make a scatter plot of a set of data. You'll also learn how to find the line that best fits that data.

What if you had a graph with many random ordered pairs plotted on it? How could you find the line that best describes those plotted points? After completing this Concept, you'll be able to find the line of best fit for scattered data.

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CK-12 Foundation: 0506S Fitting a Line to Data (H264)

Guidance

Katja has noticed that sales are falling off at her store lately. She plots her sales figures for each week on a graph and sees that the points are trending downward, but they don't quite make a straight line. How can she predict what her sales figures will be over the next few weeks?

In real-world problems, the relationship between our dependent and independent variables is linear, but not perfectly so. We may have a number of data points that don't quite fit on a straight line, but we may still want to find an equation representing those points. In this lesson, we'll learn how to find linear equations to fit real-world data.

Make a Scatter Plot

A **scatter plot** is a plot of all the ordered pairs in a table. Even when we expect the relationship we're analyzing to be linear, we usually can't expect that all the points will fit perfectly on a straight line. Instead, the points will be "scattered" about a straight line.

There are many reasons why the data might not fall perfectly on a line. Small errors in measurement are one reason; another reason is that the real world isn't always as simple as a mathematical abstraction, and sometimes math can only describe it approximately.

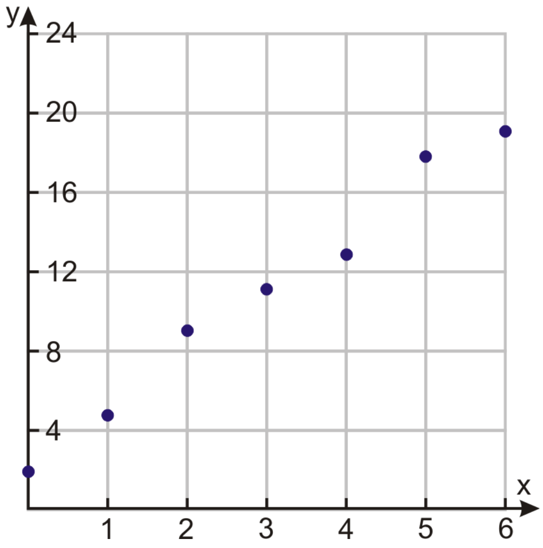
Example A

Make a scatter plot of the following ordered pairs:

(0, 2); (1, 4.5); (2, 9); (3, 11); (4, 13); (5, 18); (6, 19.5)

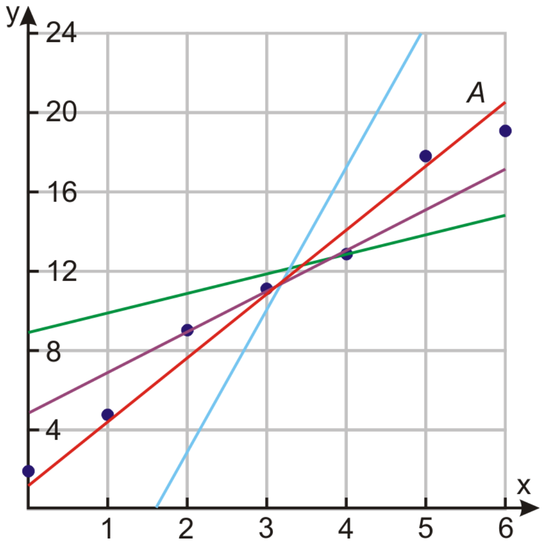
Solution

We make a scatter plot by graphing all the ordered pairs on the coordinate axis:

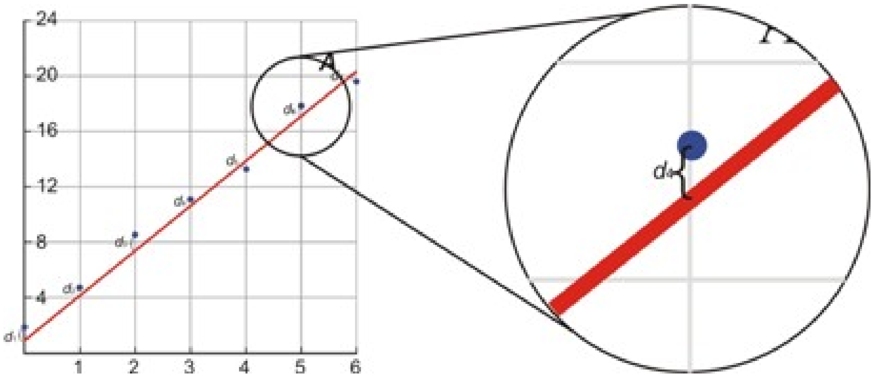


Fit a Line to Data

Notice that the points look like they might be part of a straight line, although they wouldn't fit perfectly on a straight line. If the points were perfectly lined up, we could just draw a line through any two of them, and that line would go right through all the other points as well. When the points aren't lined up perfectly, we just have to find a line that is as close to all the points as possible.



Here you can see that we could draw many lines through the points in our data set. However, the red line *A* is the line that best fits the points. To prove this mathematically, we would measure all the distances from each data point to line *A*: and then we would show that the sum of all those distances—or rather, the square root of the sum of the squares of the distances—is less than it would be for any other line.



Actually proving this is a lesson for a much more advanced course, so we won't do it here. And finding the best fit line in the first place is even more complex; instead of doing it by hand, we'll use a graphing calculator or just "eyeball" the line, as we did above—using our visual sense to guess what line fits best.

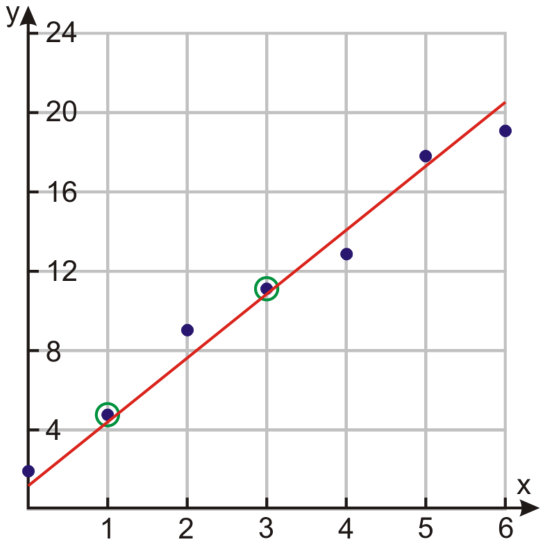
For more practice eyeballing lines of best fit, try the Java applet at http://mste.illinois.edu/activity/regression/. Click on the green field to place up to 50 points on it, then use the slider to adjust the slope of the red line to try and make it fit the points. (The thermometer shows how far away the line is from the points, so you want to try to make the thermometer reading as *low* as possible.) Then click "Show Best Fit" to show the actual best fit line in blue. Refresh the page or click "Reset" if you want to try again. For more of a challenge, try scattering the points in a less obvious pattern.

Write an Equation For a Line of Best Fit

Once you draw the line of best fit, you can find its equation by using two points on the line. Finding the equation of the line of best fit is also called **linear regression.**

Caution: Make sure you don't get caught making a common mistake. Sometimes the line of best fit won't pass straight through any of the points in the original data set. This means that you can't just use two points from the data set – **you need to use two points that are on the line**, which might not be in the data set at all.

In Example 1, it happens that two of the data points *are* very close to the line of best fit, so we can just use these points to find the equation of the line: (1, 4.5) and (3, 11).



Start with the slope-intercept form of a line: $y = mx + b$

Find the slope: $m = \frac{11-4.5}{3-1} = \frac{6.5}{2} = 3.25.$

So $y = 3.25x + b.$

Plug (3, 11) into the equation: $11 = 3.25(3) + b \Rightarrow b = 1.25$

So **the equation for the line that fits the data best is** $y = 3.25x + 1.25$.

Perform Linear Regression With a Graphing Calculator

The problem with eyeballing a line of best fit, of course, is that you can't be sure how accurate your guess is. To get the most accurate equation for the line, we can use a graphing calculator instead. The calculator uses a mathematical algorithm to find the line that minimizes the sum of the squares.

Example B

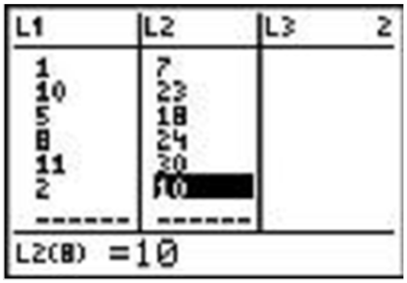
Use a graphing calculator to find the equation of the line of best fit for the following data:

(3, 12), (8, 20), (1, 7), (10, 23), (5, 18), (8, 24), (11, 30), (2, 10)

Solution

Step 1: *Input the data in your calculator.*

Press **[STAT]** and choose the **[EDIT]** option. Input the data into the table by entering the $x-$values in the first column and the $y-$values in the second column.



```
L1     |L2     |L3      2
       |       |
 1     |7      |
 10    |23     |
 5     |18     |
 8     |24     |
 11    |30     |
 2     |10     |
------ |------ |
L2(8) =10
```

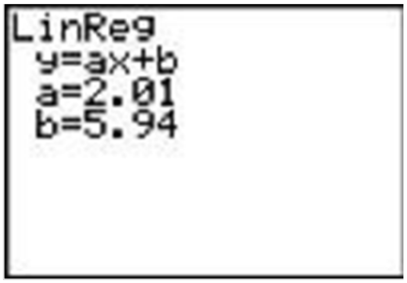
Step 2: *Find the equation of the line of best fit.*

Press **[STAT]** again use right arrow to select **[CALC]** at the top of the screen.

Chose option number 4, $LinReg(ax+b)$, and press **[ENTER]**

The calculator will display $LinReg(ax+b)$.

Press **[ENTER]** and you will be given the $a-$ and $b-$values.



```
LinReg
 y=ax+b
 a=2.01
 b=5.94
```

Here a represents the slope and b represents the $y-$intercept of the equation. The linear regression line is $y = 2.01x + 5.94$.

Step 3. *Draw the scatter plot.*

To draw the scatter plot press **[STATPLOT] [2nd] [Y=].**

Plot1 Plot2 Plot3
On Off
Type:
Xlist:L1
Ylist:L2
Mark: ▫ + ·

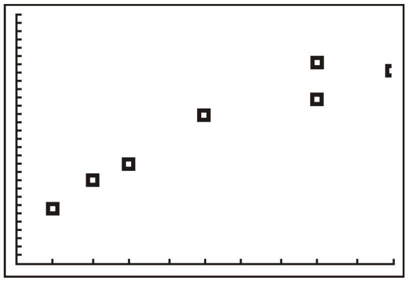
Choose Plot 1 and press **[ENTER]**.

Press the On option and set the Type as scatter plot (the one highlighted in black).

Make sure that the X list and Y list names match the names of the columns of the table in Step 1.

Choose the box or plus as the mark, since the simple dot may make it difficult to see the points.

Press **[GRAPH]** and adjust the window size so you can see all the points in the scatter plot.

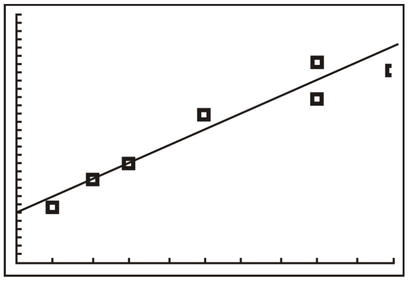


Step 4. *Draw the line of best fit through the scatter plot.*

Press **[Y=]**

Enter the equation of the line of best fit that you just found: $y = 2.01x + 5.94$.

Press **[GRAPH]**.



Solve Real-World Problems Using Linear Models of Scattered Data

Once we've found the line of best fit for a data set, we can use the equation of that line to predict other data points.

Example C

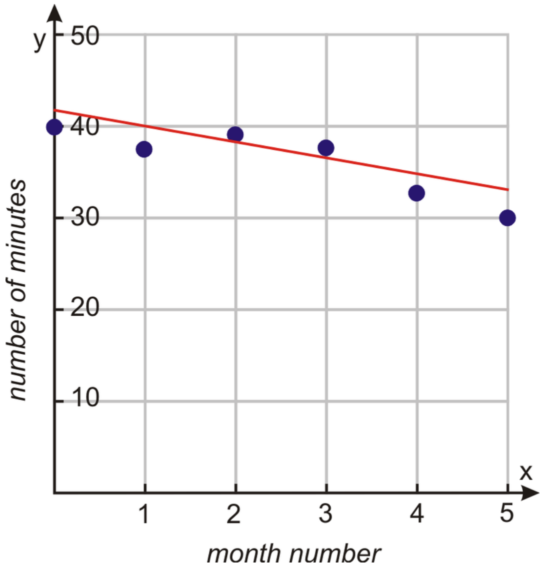
Nadia is training for a 5K race. The following table shows her times for each month of her training program. Find an equation of a line of fit. Predict her running time if her race is in August.

TABLE 5.1:

Month	Month number	Average time (minutes)
January	0	40
February	1	38
March	2	39
April	3	38
May	4	33
June	5	30

Solution

Let's make a scatter plot of Nadia's running times. The independent variable, x, is the month number and the dependent variable, y, is the running time. We plot all the points in the table on the coordinate plane, and then sketch a line of fit.



Two points on the line are (0, 42) and (4, 34). We'll use them to find the equation of the line:

$$\begin{aligned} m &= \frac{34-42}{4-0} = -\frac{8}{4} = -2 \\ y &= -2x+b \\ 42 &= -2(0)+b \Rightarrow b = 42 \\ y &= -2x+42 \end{aligned}$$

In a real-world problem, the slope and $y-$intercept have a physical significance. In this case, the slope tells us how Nadia's running time changes each month she trains. Specifically, it decreases by 2 minutes per month. Meanwhile, the $y-$intercept tells us that when Nadia started training, she ran a distance of 5K in 42 minutes.

The problem asks us to predict Nadia's running time in August. Since June is defined as month number 5, August will be month number 7. We plug $x = 7$ into the equation of the line of best fit:

$$y = -2(7)+42 = -14+42 = 28$$

The equation predicts that **Nadia will run the 5K race in 28 minutes.**

In this solution, we eyeballed a line of fit. Using a graphing calculator, we can find this equation for a line of fit instead: $y = -2.2x + 43.7$

If we plug $x = 7$ into this equation, we get $y = -2.2(7) + 43.7 = 28.3$. This means that **Nadia will run her race in 28.3 minutes.** You see that the graphing calculator gives a different equation and a different answer to the question. The graphing calculator result is more accurate, but the line we drew by hand still gives a good approximation to the result. And of course, there's no guarantee that Nadia will actually finish the race in that exact time; both answers are estimates, it's just that the calculator's estimate is slightly more likely to be right.

Watch this video for help with the Examples above.

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Click image to the left for more content.

CK-12 Foundation: Fitting a Lineto Data

Vocabulary

- A **scatter plot** is a plot of all the ordered pairs in a table. Even when we expect the relationship we're analyzing to be linear, we usually can't expect that all the points will fit perfectly on a straight line. Instead, the points will be "scattered" about a straight line.

- Once you draw the line of best fit, you can find its equation by using two points on the line. Finding the equation of the line of best fit is also called **linear regression.**

Guided Practice

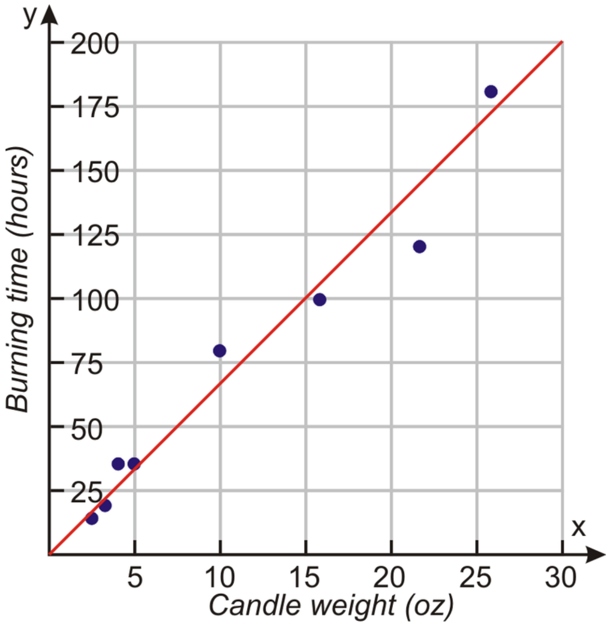
Peter is testing the burning time of "BriteGlo" candles. The following table shows how long it takes to burn candles of different weights. Assume it's a linear relation, so we can use a line to fit the data. If a candle burns for 95 hours, what must be its weight in ounces?

TABLE 5.2:

Candle weight (oz)	Time (hours)
2	15
3	20
4	35
5	36
10	80
16	100
22	120
26	180

Solution

Let's make a scatter plot of the data. The independent variable, x, is the candle weight and the dependent variable, y, is the time it takes the candle to burn. We plot all the points in the table on the coordinate plane, and draw a line of fit.



Two convenient points on the line are (0,0) and (30, 200). Find the equation of the line:

$$\begin{aligned}m &= \frac{200}{30} = \frac{20}{3}\\ y &= \frac{20}{3}x+b\\ 0 &= \frac{20}{3}(0)+b \Rightarrow b=0\\ y &= \frac{20}{3}x\end{aligned}$$

A slope of $\frac{20}{3} = 6\frac{2}{3}$ tells us that for each extra ounce of candle weight, the burning time increases by $6\frac{2}{3}$ hours. A $y-$intercept of zero tells us that a candle of weight 0 oz will burn for 0 hours.

The problem asks for the weight of a candle that burns 95 hours; in other words, what's the $x-$value that gives a $y-$value of 95? Plugging in $y = 95$:

$$y = \frac{20}{3}x \Rightarrow 95 = \frac{20}{3}x \Rightarrow x = \frac{285}{20} = \frac{57}{4} = 14\frac{1}{4}$$

A candle that burns 95 hours weighs 14.25 oz.

A graphing calculator gives the linear regression equation as $y = 6.1x + 5.9$ and a result of **14.6 oz.**

Practice

For problems 1-4, draw the scatter plot and find an equation that fits the data set by hand.

1. (57, 45); (65, 61); (34, 30); (87, 78); (42, 41); (35, 36); (59, 35); (61, 57); (25, 23); (35, 34)
2. (32, 43); (54, 61); (89, 94); (25, 34); (43, 56); (58, 67); (38, 46); (47, 56); (39, 48)

3. (12, 18); (5, 24); (15, 16); (11, 19); (9, 12); (7, 13); (6, 17); (12, 14)
4. (3, 12); (8, 20); (1, 7); (10, 23); (5, 18); (8, 24); (2, 10)
5. Use the graph from problem 1 to predict the $y-$values for two $x-$values of your choice that are not in the data set.
6. Use the graph from problem 2 to predict the $x-$values for two $y-$values of your choice that are not in the data set.
7. Use the equation from problem 3 to predict the $y-$values for two $x-$values of your choice that are not in the data set.
8. Use the equation from problem 4 to predict the $x-$values for two $y-$values of your choice that are not in the data set.

For problems 9-11, use a graphing calculator to find the equation of the line of best fit for the data set.

9. (57, 45); (65, 61); (34, 30); (87, 78); (42, 41); (35, 36); (59, 35); (61, 57); (25, 23); (35, 34)
10. (32, 43); (54, 61); (89, 94); (25, 34); (43, 56); (58, 67); (38, 46); (47, 56); (95, 105); (39, 48)
11. (12, 18); (3, 26); (5, 24); (15, 16); (11, 19); (0, 27); (9, 12); (7, 13); (6, 17); (12, 14)
12. Graph the best fit line on top of the scatter plot for problem 10. Then pick a data point that's close to the line, and change its $y-$value to move it much farther from the line.
    a. Calculate the new best fit line with that one point changed; write the equation of that line along with the coordinates of the new point.
    b. How much did the slope of the best fit line change when you changed that point?
13. Graph the scatter plot from problem 11 and change one point as you did in the previous problem.
    a. Calculate the new best fit line with that one point changed; write the equation of that line along with the coordinates of the new point.
    b. Did changing that one point seem to affect the slope of the best fit line more or less than it did in the previous problem? What might account for this difference?
14. Shiva is trying to beat the samosa-eating record. The current record is 53.5 samosas in 12 minutes. Each day he practices and the following table shows how many samosas he eats each day for the first week of his training.

TABLE 5.3:

Day	No. of samosas
1	30
2	34
3	36
4	36
5	40
6	43
7	45

(a) Draw a scatter plot and find an equation to fit the data.

(b) Will he be ready for the contest if it occurs two weeks from the day he started training?

(c) What are the meanings of the slope and the $y-$intercept in this problem?

15. Anne is trying to find the elasticity coefficient of a Superball. She drops the ball from different heights and measures the maximum height of the ball after the bounce. The table below shows the data she collected.

TABLE 5.4:

Initial height (cm)	Bounce height (cm)
30	22
35	26
40	29
45	34
50	38
55	40
60	45
65	50
70	52

(a) Draw a scatter plot and find the equation.

(b) What height would she have to drop the ball from for it to bounce 65 cm?

(c) What are the meanings of the slope and the $y-$intercept in this problem?

(d) Does the $y-$intercept make sense? Why isn't it (0, 0)?

16. The following table shows the median California family income from 1995 to 2002 as reported by the US Census Bureau.

TABLE 5.5:

Year	Income
1995	53,807
1996	55,217
1997	55,209
1998	55,415
1999	63,100
2000	63,206
2001	63,761
2002	65,766

(a) Draw a scatter plot and find the equation.

(b) What would you expect the median annual income of a Californian family to be in year 2010?

(c) What are the meanings of the slope and the $y-$intercept in this problem?

(d) Inflation in the U.S. is measured by the Consumer Price Index, which increased by 20% between 1995 and 2002. Did the median income of California families keep up with inflation over that time period? (In other words, did it increase by at least 20%?)

5.7 Linear Interpolation and Extrapolation

Here you'll learn how to use linear interpolation to fill in gaps in data and linear extrapolation to estimate values outside a data set's range.

What if you were given a table of values that showed the average lifespan for Americans for each decade from 1950 to 2010? How could you use that data to estimate the average lifespan in 2020 or 2030? After completing this Concept, you'll be able to make predictions from linear models like this one.

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CK-12 Foundation: 0507S Predicting with Linear Models (H264)

Guidance

Katja's sales figures were trending downward quickly at first, and she used a line of best fit to describe the numbers. But now they seem to be decreasing more slowly, and fitting the line less and less accurately. How can she make a more accurate prediction of what next week's sales will be?

In the last lesson we saw how to find the equation of a line of best fit and how to use this equation to make predictions. The line of "best fit" is a good method if the relationship between the dependent and the independent variables is linear. In this section you will learn other methods that are useful even when the relationship isn't linear.

Linear Interpolation

We use linear interpolation to fill in gaps in our data—that is, to estimate values that fall in between the values we already know. To do this, we use a straight line to connect the known data points on either side of the unknown point, and use the equation of that line to estimate the value we are looking for.

Example A

The following table shows the median ages of first marriage for men and women, as gathered by the U.S. Census Bureau.

TABLE 5.6:

Year	Median age of males	Median age of females
1890	26.1	22.0
1900	25.9	21.9
1910	25.1	21.6
1920	24.6	21.2
1930	24.3	21.3

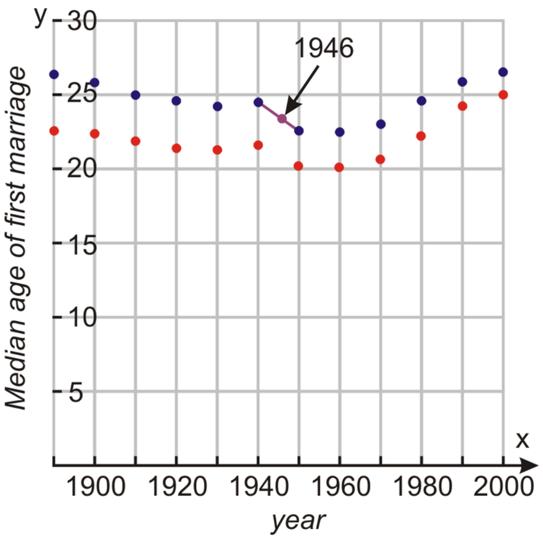
TABLE 5.6: (continued)

Year	Median age of males	Median age of females
1940	24.3	21.5
1950	22.8	20.3
1960	22.8	20.3
1970	23.2	20.8
1980	24.7	22.0
1990	26.1	23.9
2000	26.8	25.1

Estimate the median age for the first marriage of a male in the year 1946.

Solution

We connect the two points on either side of 1946 with a straight line and find its equation. Here's how that looks on a scatter plot:



We find the equation by plugging in the two data points:

$$\begin{aligned} m &= \frac{22.8-24.3}{1950-1940} = \frac{-1.5}{10} = -0.15 \\ y &= -0.15x+b \\ 24.3 &= -0.15(1940)+b \\ b &= 315.3 \end{aligned}$$

Our equation is $y = -0.15x + 315.3$.

To estimate the median age of marriage of males in the year 1946, we plug $x = 1946$ into the equation we just found:

$y = -0.15(1946) + 315.3 = 23.4$**years old**

For non-linear data, linear interpolation is often not accurate enough for our purposes. If the points in the data set change by a large amount in the interval you're interested in, then linear interpolation may not give a good estimate. In that case, it can be replaced by **polynomial interpolation,** which uses a curve instead of a straight line to estimate values between points. But that's beyond the scope of this lesson.

Linear Extrapolation

Linear extrapolation can help us estimate values that are outside the range of our data set. The strategy is similar to linear interpolation: we pick the two data points that are closest to the one we're looking for, find the equation of the line between them, and use that equation to estimate the coordinates of the missing point.

Example B

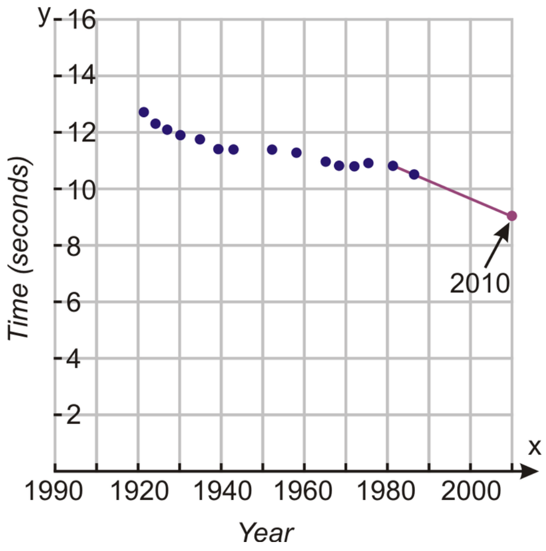
The winning times for the women's 100 meter race are given in the following table. Estimate the winning time in the year 2010. Is this a good estimate?

TABLE 5.7:

Winner	Country	Year	Time (seconds)
Mary Lines	UK	1922	12.8
Leni Schmidt	Germany	1925	12.4
Gerturd Glasitsch	Germany	1927	12.1
Tollien Schuurman	Netherlands	1930	12.0
Helen Stephens	USA	1935	11.8
Lulu Mae Hymes	USA	1939	11.5
Fanny Blankers-Koen	Netherlands	1943	11.5
Marjorie Jackson	Australia	1952	11.4
Vera Krepkina	Soviet Union	1958	11.3
Wyomia Tyus	USA	1964	11.2
Barbara Ferrell	USA	1968	11.1
Ellen Strophal	East Germany	1972	11.0
Inge Helten	West Germany	1976	11.0
Marlies Gohr	East Germany	1982	10.9
Florence Griffith Joyner	USA	1988	10.5

Solution

We start by making a scatter plot of the data; then we connect the last two points on the graph and find the equation of the line.



$$\begin{aligned}m &= \frac{10.5-10.9}{1988-1982} = \frac{-0.4}{6} = -0.067\\ y &= -0.067x+b\\ 10.5 &= -0.067(1988)+b\\ b &= 143.7\end{aligned}$$

Our equation is $y = -0.067x + 143.7$.

The winning time in year 2010 is estimated to be:

$y = -0.067(2010) + 143.7 = 9.03$**seconds.**

Unfortunately, this estimate actually isn't very accurate. This example demonstrates the weakness of linear extrapolation; it uses only a couple of points, instead of using *all* the points like the best fit line method, so it doesn't give as accurate results when the data points follow a linear pattern. In this particular example, the last data point clearly doesn't fit in with the general trend of the data, so the slope of the extrapolation line is much steeper than it would be if we'd used a line of best fit. (As a historical note, the last data point corresponds to the winning time for Florence Griffith Joyner in 1988. After her race she was accused of using performance-enhancing drugs, but this fact was never proven. In addition, there was a question about the accuracy of the timing: some officials said that tail-wind was not accounted for in this race, even though all the other races of the day were affected by a strong wind.)

Here's an example of a problem where linear extrapolation does work better than the line of best fit method.

Example C

A cylinder is filled with water to a height of 73 centimeters. The water is drained through a hole in the bottom of the cylinder and measurements are taken at 2 second intervals. The following table shows the height of the water level in the cylinder at different times.

TABLE 5.8:

Time (seconds)	Water level (cm)
0.0	73
2.0	63.9
4.0	55.5
6.0	47.2
8.0	40.0
10.0	33.4
12.0	27.4
14.0	21.9
16.0	17.1
18.0	12.9
20.0	9.4
22.0	6.3
24.0	3.9
26.0	2.0
28.0	0.7
30.0	0.1

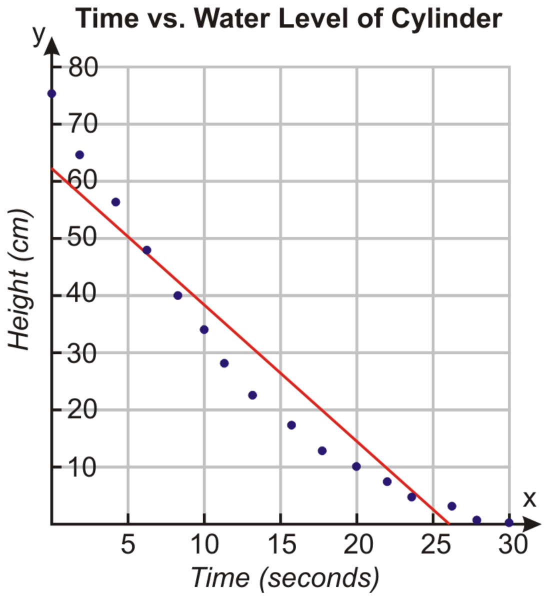
a) *Find the water level at time 15 seconds.*

b) *Find the water level at time 27 seconds*

c) *What would be the original height of the water in the cylinder if the water takes 5 extra seconds to drain? (Find the height at time of –5 seconds.)*

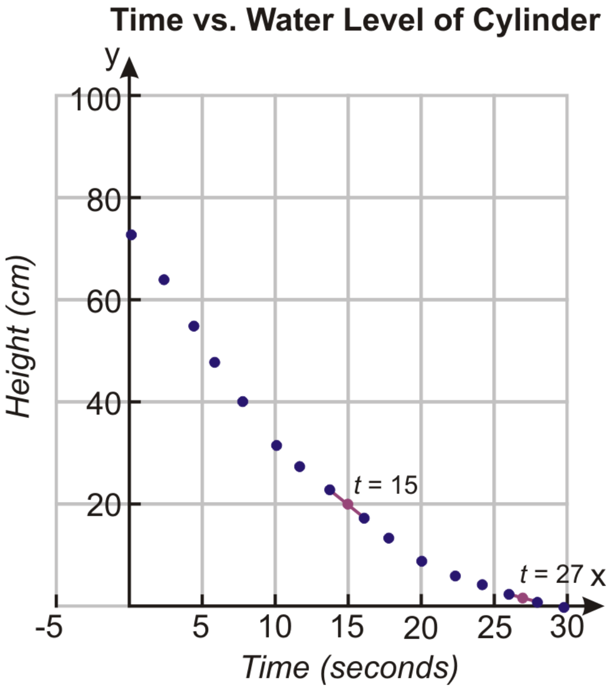
Solution

Here's what the line of best fit would look like for this data set:



Notice that the data points don't really make a line, and so the line of best fit still isn't a terribly good fit. Just a glance tells us that we'd estimate the water level at 15 seconds to be about 27 cm, which is *more* than the water level at 14 seconds. That's clearly not possible! Similarly, at 27 seconds we'd estimate the water to have all drained out, which it clearly hasn't yet.

So let's see what happens if we use linear extrapolation and interpolation instead. First, here are the lines we'd use to interpolate between 14 and 16 seconds, and between 26 and 28 seconds.



a) The slope of the line between points (14, 21.9) and (16, 17.1) is $m = \frac{17.1-21.9}{16-14} = \frac{-4.8}{2} = -2.4$. So $y = -2.4x + b \Rightarrow 21.9 = -2.4(14) + b \Rightarrow b = 55.5$, and the equation is $y = -2.4x + 55.5$.

Plugging in $x = 15$ gives us $y = -2.4(15) + 55.5 = 19.5\ cm$.

b) The slope of the line between points (26, 2) and (28, 0.7) is $m = \frac{0.7-2}{28-26} = \frac{-1.3}{2} = -.65$, so $y = -.65x + b \Rightarrow 2 = -.65(26) + b \Rightarrow b = 18.9$, and the equation is $y = -.65x + 18.9$.

Plugging in $x = 27$, we get $y = -.65(27) + 18.9 = 1.35\ cm$.

c) Finally, we can use extrapolation to estimate the height of the water at -5 seconds. The slope of the line between points (0, 73) and (2, 63.9) is $m = \frac{63.9-73}{2-0} = \frac{-9.1}{2} = -4.55$, so the equation of the line is $y = -4.55x + 73$.

Plugging in $x = -5$ gives us $y = -4.55(-5) + 73 = 95.75\ cm$.

To make linear interpolation easier in the future, you might want to use the calculator at http://www.ajdesigner.com/phpinterpolation/linear_interpolation_equation.php. Plug in the coordinates of the first known data point in the blanks labeled x_1 and y_1, and the coordinates of the second point in the blanks labeled x_3 and y_3; then enter the $x-$coordinate of the point in between in the blank labeled x_2, and the $y-$coordinate will be displayed below when you click "Calculate."

Watch this video for help with the Examples above.

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CK-12 Foundation: Predictingwith LinearModels

Vocabulary

- The **line of best fit** is a good method if the relationship between the dependent and the independent variables is linear. In this section you will learn other methods that are useful even when the relationship isn't linear.
- We use **linear interpolation** to fill in gaps in our data—that is, to estimate values that fall in between the values we already know. To do this, we use a straight line to connect the known data points on either side of the unknown point, and use the equation of that line to estimate the value we are looking for.
- **For non-linear data,** linear interpolation is often not accurate enough for our purposes. If the points in the data set change by a large amount in the interval you're interested in, then linear interpolation may not give a good estimate. In that case, it can be replaced by **polynomial interpolation,** which uses a curve instead of a straight line to estimate values between points. But that's beyond the scope of this lesson.
- **Linear extrapolation** can help us estimate values that are outside the range of our data set. The strategy is similar to linear interpolation: we pick the two data points that are closest to the one we're looking for, find the equation of the line between them, and use that equation to estimate the coordinates of the missing point.

Guided Practice

The Center for Disease Control collects information about the health of the American people and behaviors that might lead to bad health. The following table shows the percent of women who smoke during pregnancy.

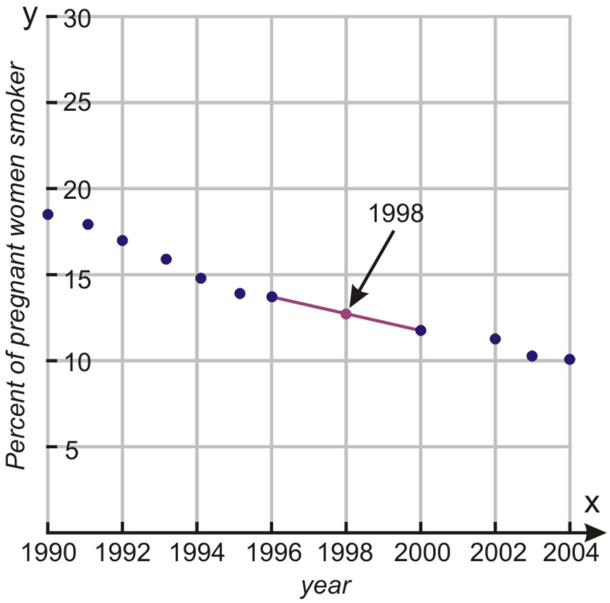
TABLE 5.9:

Year	Percent of pregnant women smokers
1990	18.4
1991	17.7
1992	16.9
1993	15.8
1994	14.6
1995	13.9
1996	13.6
2000	12.2
2002	11.4
2003	10.4
2004	10.2

Estimate the percentage of pregnant women that were smoking in the year 1998.

Solution

We connect the two points on either side of 1998 with a straight line and find its equation. Here's how that looks on a scatter plot:



We find the equation by plugging in the two data points:

$$\begin{aligned} m &= \frac{12.2-13.6}{2000-1996} = \frac{-1.4}{4} = -0.35 \\ y &= -0.35x+b \\ 12.2 &= -0.35(2000)+b \\ b &= 712.2 \end{aligned}$$

Our equation is $y = -0.35x + 712.2$.

To estimate the percentage of pregnant women who smoked in the year 1998, we plug $x = 1998$ into the equation we just found:

$$y = -0.35(1998) + 712.2 = 12.9\%$$

Practice

1. Use the data from Example 1 (*Median age at first marriage*) to estimate the age at marriage for females in 1946. Fit a line, by hand, to the data before 1970.
2. Use the data from Example 1 (*Median age at first marriage*) to estimate the age at marriage for females in 1984. Fit a line, by hand, to the data from 1970 on in order to estimate this accurately.
3. Use the data from Example 1 (*Median age at first marriage*) to estimate the age at marriage for males in 1995. Use linear interpolation between the 1990 and 2000 data points.
4. Use the data from Example 2 (*Pregnant women and smoking*) to estimate the percentage of pregnant smokers in 1997. Use linear interpolation between the 1996 and 2000 data points.
5. Use the data from Example 2 (*Pregnant women and smoking*) to estimate the percentage of pregnant smokers in 2006. Use linear extrapolation with the final two data points.
6. Use the data from Example 3 (*Winning times*) to estimate the winning time for the female 100-meter race in 1920. Use linear extrapolation because the first two or three data points have a different slope than the rest of the data.
7. The table below shows the highest temperature vs. the hours of daylight for the 15^{th} day of each month in the year 2006 in San Diego, California.

TABLE 5.10:

Hours of daylight	**High temperature (F)**
10.25	60
11.0	62
12	62
13	66
13.8	68
14.3	73
14	86
13.4	75
12.4	71
11.4	66
10.5	73
10	61

(a) What would be a better way to organize this table if you want to make the relationship between daylight hours and temperature easier to see?

(b) Estimate the high temperature for a day with 13.2 hours of daylight using linear interpolation.

(c) Estimate the high temperature for a day with 9 hours of daylight using linear extrapolation. Is the prediction accurate?

(d) Estimate the high temperature for a day with 9 hours of daylight using a line of best fit.

The table below lists expected life expectancies based on year of birth (US Census Bureau). Use it to answer questions 8-15.

TABLE 5.11:

Birth year	Life expectancy in years
1930	59.7
1940	62.9
1950	68.2
1960	69.7
1970	70.8
1980	73.7
1990	75.4
2000	77

8. Make a scatter plot of the data.
9. Use a line of best fit to estimate the life expectancy of a person born in 1955.
10. Use linear interpolation to estimate the life expectancy of a person born in 1955.
11. Use a line of best fit to estimate the life expectancy of a person born in 1976.
12. Use linear interpolation to estimate the life expectancy of a person born in 1976.
13. Use a line of best fit to estimate the life expectancy of a person born in 2012.
14. Use linear extrapolation to estimate the life expectancy of a person born in 2012.
15. Which method gives better estimates for this data set? Why?

The table below lists the high temperature for the fist day of the month for the year 2006 in San Diego, California (Weather Underground). Use it to answer questions 16-21.

TABLE 5.12:

Month number	Temperature (F)
1	63
2	66
3	61
4	64
5	71
6	78
7	88
8	78
9	81
10	75
11	68
12	69

16. Draw a scatter plot of the data.
17. Use a line of best fit to estimate the temperature in the middle of the 4^{th} month (month 4.5).
18. Use linear interpolation to estimate the temperature in the middle of the 4^{th} month (month 4.5).
19. Use a line of best fit to estimate the temperature for month 13 (January 2007).
20. Use linear extrapolation to estimate the temperature for month 13 (January 2007).
21. Which method gives better estimates for this data set? Why?
22. Name a real-world situation where you might want to make predictions based on available data. Would linear extrapolation/interpolation or the best fit method be better to use in that situation? Why?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9615.

Summary

This chapter begins by thoroughly covering the various ways to write a linear equation given bits and pieces about the line. The methods covered include slope-intercept form, point-slope form, and standard form. Real-world situations that can be modeled and solved by linear equations in these various forms are highlighted as well. The chapter then moves on to parallel and perpendicular lines, with a focus on using slopes to write the equations of lines that are parallel or perpendicular to another line. It wraps up with scatter plots of data, finding the line of best fit for a data set, and making predictions by using equations.

CHAPTER 6

Linear Inequalities

Chapter Outline

6.1 INEQUALITY EXPRESSIONS
6.2 LINEAR INEQUALITIES
6.3 MULTI-STEP INEQUALITIES
6.4 APPLICATIONS WITH INEQUALITIES
6.5 COMPOUND INEQUALITIES
6.6 SOLUTIONS TO COMPOUND INEQUALITIES
6.7 ABSOLUTE VALUE
6.8 ABSOLUTE VALUE EQUATIONS
6.9 GRAPHS OF ABSOLUTE VALUE EQUATIONS
6.10 ABSOLUTE VALUE INEQUALITIES
6.11 GRAPHS OF INEQUALITIES IN ONE VARIABLE
6.12 LINEAR INEQUALITIES IN TWO VARIABLES

Introduction

You've learned how to solve equations for a specific value, but what if the solution you were looking for involved a set of many answers. For example, you can pass a math test with any grade that is at least 65. Also, anyone under the age of 55 does not qualify for a senior discount. Both of these situations have many values that make them true statements, and inequalities are what you will use and solve to represent scenarios like these.

6.1 Inequality Expressions

Here you'll learn how to write and graph inequalities in one variable on a number line.

What if the maximum occupancy of an elevator were listed at 20 people? How could you graph the number of allowable occupants on the elevator? After completing this Concept, you'll be able to write and graph inequalities like this one.

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CK-12 Foundation: 0601S GraphingInequalities (H264)

Guidance

Dita has a budget of $350 to spend on a rental car for an upcoming trip, but she wants to spend as little of that money as possible. If the trip will last five days, what range of daily rental rates should she be willing to consider?

Like equations, inequalities show a relationship between two expressions. We solve and graph inequalities in a similar way to equations—but when we solve an inequality, the answer is usually a set of values instead of just one value.

When writing inequalities we use the following symbols:

$>$is greater than

$\geq$ is greater than or equal to

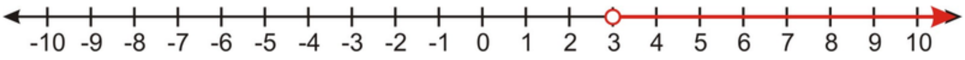
$<$is less than

$\leq$ is less than or equal to

Write and Graph Inequalities in One Variable on a Number Line

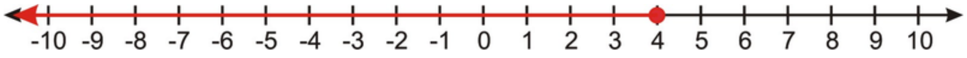
Let's start with the simple inequality $x > 3$.

We read this inequality as "x is greater than 3." The solution is the set of all real numbers that are greater than three. We often represent the solution set of an inequality with a number line graph.



Consider another simple inequality: $x \leq 4$.

We read this inequality as "x is less than or equal to 4." The solution is the set of all real numbers that are equal to four or less than four. We can graph this solution set on the number line.



Notice that we use an empty circle for the endpoint of a strict inequality (like $x > 3$), and a filled circle for one where the equals sign is included (like $x \leq 4$).

Example A

Graph the following inequalities on the number line.

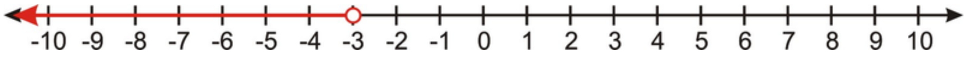
a) $x < -3$

b) $x \geq 6$

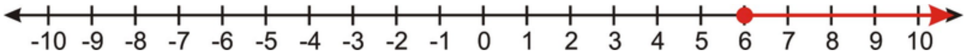
c) $x > 0$

Solution

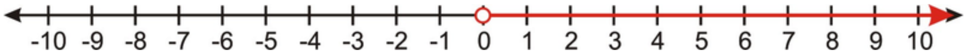
a) The inequality $x < -3$ represents all numbers that are less than -3. The number -3 is not included in the solution, so it is represented by an open circle on the graph.



b) The inequality $x \geq 6$ represents all numbers that are greater than or equal to 6. The number 6 is included in the solution, so it is represented by a closed circle on the graph.



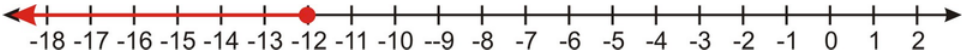
c) The inequality $x > 0$ represents all numbers that are greater than 0. The number 0 is not included in the solution, so it is represented by an open circle on the graph.



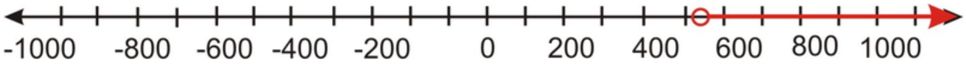
Example B

Write the inequality that is represented by each graph.

a)



b)



c)

Solution

a) $x \leq -12$

b) $x > 540$

c) $x < 6.5$

Inequalities appear everywhere in real life. Here are some simple examples of real-world applications.

Example C

Write each statement as an inequality and graph it on the number line.

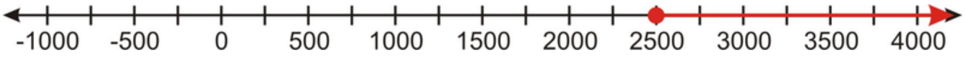
a) You must maintain a balance of at least \$2500 in your checking account to get free checking.

b) You must be at least 48 inches tall to ride the "Thunderbolt" Rollercoaster.

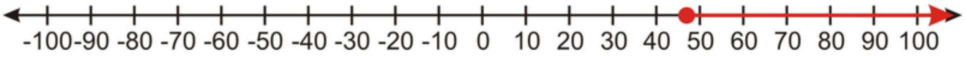
c) You must be younger than 3 years old to get free admission at the San Diego Zoo.

Solution

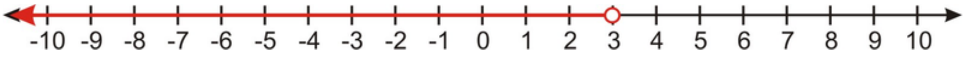
a) The words "at least" imply that the value of \$2500 is included in the solution set, so the inequality is written as $x \geq 2500$.



b) The words "at least" imply that the value of 48 inches is included in the solution set, so the inequality is written as $x \geq 48$.



c) The inequality is written as $x < 3$.



Watch this video for help with the Examples above.

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Click image to the left for more content.

CK-12 Foundation: Graphing Inequalities

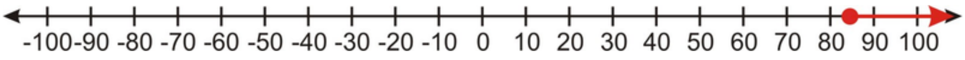
Vocabulary

- The answer to an **inequality** is usually an **interval of values**.

Guided Practice

1. *Graph the inequality* $x \leq 8$ *on the number line.*

2. *Write the inequality that is represented by the graph below.*



3. *Write the statement, "the speed limit on the interstate is 65 miles per hour or less" as an inequality .*

Solution

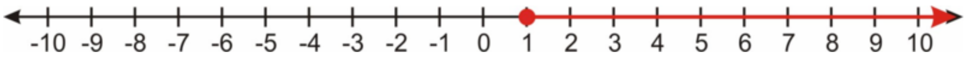
1. The inequality $x \leq 8$ represents all numbers that are less than or equal to 8. The number 8 is included in the solution, so it is represented by a closed circle on the graph.

2. $x \geq 85$

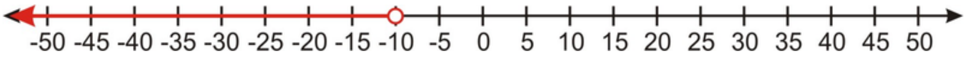
3. Speed limit means the highest allowable speed, so the inequality is written as $x \leq 65$.

Practice

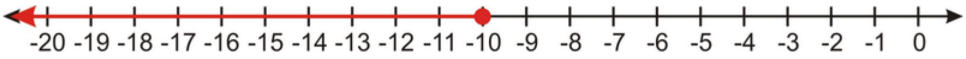
1. Write the inequality represented by the graph.



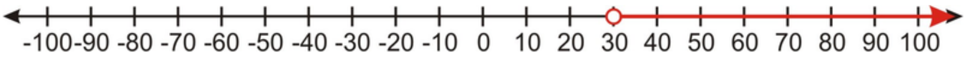
2. Write the inequality represented by the graph.



3. Write the inequality represented by the graph.



4. Write the inequality represented by the graph.



Graph each inequality on the number line.

5. $x < -35$
6. $x > -17$
7. $x \geq 20$
8. $x \leq 3$
9. $x \geq -5$
10. $x > 20$

6.2 Linear Inequalities

Here you'll learn how to solve inequalities by isolating the variable on one side of the inequality sign. You'll also learn how to graph their solution set.

What if you had an inequality with an unknown variable like $x - 12 > -5$? How could you isolate the variable to find its value? After completing this Concept, you'll be able to solve one-step inequalities like this one.

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CK-12 Foundation: 0602S Solving One-StepInequalities (H264)

Guidance

To solve an inequality we must isolate the variable on one side of the inequality sign. To isolate the variable, we use the same basic techniques used in solving equations.

We can solve some inequalities by adding or subtracting a constant from one side of the inequality.

Example A

Solve the inequality and graph the solution set.

$x - 3 < 10$

Solution

Starting inequality: $x - 3 < 10$

Add **3** to both sides of the inequality: $x - 3 + 3 < 10 + 3$

Simplify: $x < 13$

Example B

Solve the inequality and graph the solution set.

$x - 20 \leq 14$

Solution:

Starting inequality: $x - 20 \leq 14$

Add **20** to both sides of the inequality: $x-20+20 \leq 14+20$

Simplify: $x \leq 34$

Solving Inequalities Using Multiplication and Division

We can also solve inequalities by multiplying or dividing both sides by a constant. For example, to solve the inequality $5x < 3$, we would divide both sides by 5 to get $x < \frac{3}{5}$.

However, something different happens when we multiply or divide by a negative number. We know, for example, that 5 is greater than 3. But if we multiply both sides of the inequality $5 > 3$ by -2, we get $-10 > -6$. And we know that's not true; -10 is less than -6.

This happens whenever we multiply or divide an inequality by a negative number, and so we have to flip the sign around to make the inequality true. For example, to multiply $2 < 4$ by -3, first we multiply the 2 and the 4 each by -3, and then we change the <sign to a >sign, so we end up with $-6 > -12$.

The same principle applies when the inequality contains variables.

Example C

Solve the inequality.

$4x < 24$

Solution:

Original problem: $4x < 24$

Divide both sides by 4: $\frac{4x}{4} < \frac{24}{4}$

Simplify: $x < 6$

Example D

Solve the inequality.

$-5x \leq 21$

Solution:

Original problem: $-5x \leq 21$

Divide both sides by -5 : $\frac{-5x}{-5} \geq \frac{21}{-5}$***Flip the inequality sign.***

Simplify: $x \geq -\frac{21}{5}$

Watch this video for help with the Examples above.

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CK-12 Foundation: Solving One-Step Inequalities

Vocabulary

- The answer to an **inequality** is usually an **interval of values**.
- Solving inequalities works just like solving an equation. To solve, we isolate the variable on one side of the equation.
- When multiplying or dividing both sides of an inequality by a negative number, you need to ***reverse the inequality***.

Guided Practice

Solve each inequality.

a) $x+8 \leq -7$

b) $x+4 > 13$

c) $\frac{x}{25} < \frac{3}{2}$

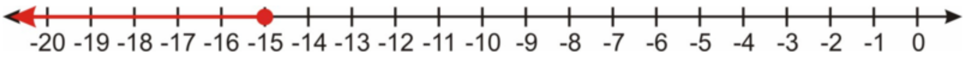
d) $\frac{x}{-7} \geq 9$

Solutions:

a) Starting inequality: $x+8 \leq -7$

Subtract **8** from both sides of the inequality: $x+8-8 \leq -7-8$

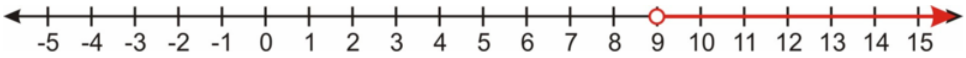
Simplify: $x \leq -15$



b) Starting inequality: $x+4 > 13$

Subtract **4** from both sides of the inequality: $x+4-4 > 13-4$

Simplify: $x > 9$



c) Original problem: $\frac{x}{25} < \frac{3}{2}$

Multiply both sides by 25: $25 \cdot \frac{x}{25} < \frac{3}{2} \cdot 25$

Simplify: $x < \frac{75}{2}$ or $x < 37.5$

d) Original problem: $\frac{x}{-7} \geq 9$

Multiply both sides by -7: $-7 \cdot \frac{x}{-7} \leq 9 \cdot (-7)$***Flip the inequality sign.***

Simplify: $x \leq -63$

Practice

For 1-8, solve each inequality and graph the solution on the number line.

1. $x-5 < 35$
2. $x+15 \geq -60$
3. $x-2 \leq 1$

4. $x-8>-20$
5. $x+11>13$
6. $x+65<100$
7. $x-32\leq 0$
8. $x+68\geq 75$

For 9-11, solve each inequality. Write the solution as an inequality and graph it.

9. $3x\leq 6$
10. $\frac{x}{5}>-\frac{3}{10}$
11. $-10x>250$
12. $\frac{x}{-7}\geq -5$

6.3 Multi-Step Inequalities

Here you'll learn how to solve inequalities that require several steps to arrive at the solution. You'll also graph their solution set.

What if you had an inequality with an unknown variable on both sides like $2(x-2) > 3x-5$? How could you isolate the variable to find its value? After completing this Concept, you'll be able to solve multi-step inequalities like this one.

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CK-12 Foundation: 0603S Solving Multi-Step Inequalities (H264)

Try This

For additional practice solving inequalities, try the online game at http://www.aaamath.com/equ725x7.htm#section2 . If you're having a hard time with multi-step inequalities, the video at http://www.schooltube.com/video/aa66df49e0af4f85a5e9/MultiStep-Inequalities will walk you through a few.

Guidance

In the last two sections, we considered very simple inequalities which required one step to obtain the solution. However, most inequalities require several steps to arrive at the solution. As with solving equations, we must use the order of operations to find the correct solution. In addition, remember that **when we multiply or divide the inequality by a negative number, the direction of the inequality changes.**

The general procedure for solving multi-step inequalities is almost exactly like the procedure for solving multi-step equations:

1. Clear parentheses on both sides of the inequality and collect like terms.
2. Add or subtract terms so the variable is on one side and the constant is on the other side of the inequality sign.
3. Multiply and divide by whatever constants are attached to the variable. Remember to change the direction of the inequality if you multiply or divide by a negative number.

Example A

Solve the inequality $\frac{9x}{5}-7 \ge -3x+12$ *and graph the solution set.*

Solution

Original problem: $\frac{9x}{5}-7 \ge -3x+12$

Add $3x$ to both sides: $\frac{9x}{5}+3x-7 \geq -3x+3x+12$

Simplify: $\frac{24x}{5}-7 \geq 12$

Add 7 to both sides: $\frac{24x}{5}-7+7 \geq 12+7$

Simplify: $\frac{24x}{5} \geq 19$

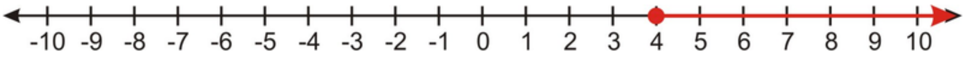
Multiply 5 to both sides: $5 \cdot \frac{24x}{5} \geq 5 \cdot 19$

Simplify: $24x \geq 95$

Divide both sides by 24: $\frac{24x}{24} \geq \frac{95}{24}$

Simplify: $x \geq \frac{95}{24}$**Answer**

Graph:



Example B

Solve the inequality $-25x+12 \leq -10x-12$ *and graph the solution set.*

Solution:

Original problem: $-25x+12 \leq -10x-12$

Add $10x$ to both sides: $-25x+10x+12 \leq -10x+10x-12$

Simplify: $-15x+12 \leq -12$

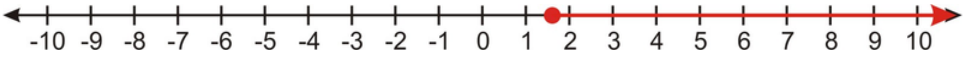
Subtract 12: $-15x+12-12 \leq -12-12$

Simplify: $-15x \leq -24$

Divide both sides by -15: $\frac{-15x}{-15} \geq \frac{-24}{-15}$***flip the inequality sign***

Simplify: $x \geq \frac{8}{5}$**Answer**

Graph:



Example C

Solve the inequality $4x-2(3x-9) \leq -4(2x-9)$.

Solution:

Original problem: $4x-2(3x-9) \leq -4(2x-9)$

Simplify parentheses: $4x-6x+18 \leq -8x+36$

Collect like terms: $-2x+18 \leq -8x+36$

Add $8x$ to both sides: $-2x+8x+18 \leq -8x+8x+36$

Simplify: $6x+18 \leq 36$

Subtract 18: $6x+18-18 \leq 36-18$

Simplify: $6x \leq 18$

Divide both sides by 6: $\frac{6x}{6} \leq \frac{18}{6}$

Simplify: $x \leq 3$**Answer**

Watch this video for help with the Examples above.

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CK-12 Foundation: Solving Multi-StepInequalities

Vocabulary

- The answer to an **inequality** is usually an **interval of values**.
- Solving inequalities works just like solving an equation. To solve, we isolate the variable on one side of the equation.
- When multiplying or dividing both sides of an inequality by a negative number, you need to ***reverse the inequality***.

Guided Practice

Solve the inequality $\frac{5x-1}{4} > -2(x+5)$.

Solution:

Original problem: $\frac{5x-1}{4} > -2(x+5)$

Simplify parenthesis: $\frac{5x-1}{4} > -2x-10$

Multiply both sides by 4: $4 \cdot \frac{5x-1}{4} > 4(-2x-10)$

Simplify: $5x-1 > -8x-40$

Add $8x$ to both sides: $5x+8x-1 > -8x+8x-40$

Simplify: $13x-1 > -40$

Add 1 to both sides: $13x-1+1 > -40+1$

Simplify: $13x > -39$

Divide both sides by 13: $\frac{13x}{13} > -\frac{39}{13}$

Simplify: $x > -3$**Answer**

Practice

Solve each multi-step inequality.

1. $3x-5 < x+3$
2. $x-5 > 2x+3$
3. $2(x-3) \leq 3x-2$
4. $3(x+1) \geq 2x+5$
5. $2(x-9) \geq -1(4x+7)$
6. $\frac{x}{3} < x+7$

7. $\frac{x}{4} < 2x - 21$
8. $\frac{3(x-4)}{12} \le \frac{2x}{3}$
9. $2\left(\frac{x}{4} + 3\right) > 6(x - 1)$
10. $9x + 4 \le -2\left(x + \frac{1}{2}\right)$

6.4 Applications with Inequalities

Here you'll learn how to express the solutions of an inequality in four different ways. You'll also learn how to dentify the number of solutions an inequality has and you'll solve real-world problems using inequalities.

What if you solved an inequality and came up with the solution $x > -3$? How else could you express this solution? After completing this Concept, you'll be able to express the solution of an inequality in inequality notation, set notation, interval notation, and as a solution graph.

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CK-12 Foundation: 0604S Using Inequalities (H264)

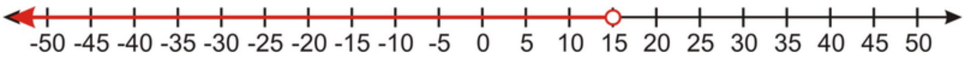
Guidance

Ms. Jerome wants to buy identical boxes of art supplies for her 25 students. If she can spend no more than $375 on art supplies, what inequality describes the price can she afford for each individual box of supplies?

Expressing Solutions of an Inequality

The solution of an inequality can be expressed in four different ways:

1. **Inequality notation** The answer is simply expressed as $x < 15$.
2. **Set notation** The answer is expressed as a set: $\{x|x < 15\}$. The brackets indicate a set and the vertical line means "such that," so we read this expression as "the set of all values of x such that x is a real number less than 15".
3. **Interval notation** uses brackets to indicate the range of values in the solution. For example, the answer to our problem would be expressed as $(-\infty, 15)$, meaning "the interval containing all the numbers from $-\infty$ to 15 but not actually including $-\infty$ or 15".
    a. Square or **closed brackets** "[" and "]" indicate that the number next to the bracket is included in the solution set.
    b. Round or **open brackets** "(" and ")" indicate that the number next to the bracket is not included in the solution set. When using **infinity** and **negative infinity** (∞ and $-\infty$), we always use open brackets, because infinity isn't an actual number and so it can't ever really be included in an interval.
4. **Solution graph** shows the solution on the real number line. A closed circle on a number indicates that the number is included in the solution set, while an open circle indicates that the number is not included in the set. For our example, the solution graph is:



Example A

a) [-4, 6] means that the solution is all numbers between -4 and 6 **including** -4 and 6.

b) (8, 24) means that the solution is all numbers between 8 and 24 **not including** the numbers 8 and 24.

c) [3, 12) means that the solution is all numbers between 3 and 12, **including** 3 but **not including** 12.

d) $(-10, \infty)$ means that the solution is all numbers greater than -10, **not including** -10.

e) $(-\infty, \infty)$ means that the solution is all real numbers.

Identify the Number of Solutions of an Inequality

Inequalities can have:

- A set that has an infinite number of solutions.
- A set that has a discrete number of solutions.
- No solutions.

The inequalities we have solved so far all have an infinite number of solutions, at least in theory. For example, the inequality $\frac{5x-1}{4} > -2(x+5)$ has the solution $x > -3$. This solution says that all real numbers greater than -3 make this inequality true, and there are infinitely many such numbers.

However, in real life, sometimes we are trying to solve a problem that can only have positive integer answers, because the answers describe numbers of discrete objects.

For example, suppose you are trying to figure out how many $8 CDs you can buy if you want to spend less than $50. An inequality to describe this situation would be $8x < 50$, and if you solved that inequality you would get $x < \frac{50}{8}$, or $x < 6.25$.

But could you really buy *any* number of CDs as long as it's less than 6.25? No; you couldn't really buy 6.1 CDs, or -5 CDs, or any other fractional or negative number of CDs. So if we wanted to express our solution in set notation, we couldn't express it as the set of all numbers less than 6.25, or $\{x|x < 6.25\}$. Instead, the solution is just the set containing all the *nonnegative whole numbers* less than 6.25, or {0, 1, 2, 3, 4, 5, 6}. When we're solving a real-world problem dealing with discrete objects like CDs, our solution set will often be a finite set of numbers instead of an infinite interval.

An inequality can also have no solutions at all. For example, consider the inequality $x-5 > x+6$. When we subtract x from both sides, we end up with $-5 > 6$, which is not true for any value of x. We say that this inequality has no solution.

The opposite can also be true. If we flip the inequality sign in the above inequality, we get $x-5 < x+6$, which simplifies to $-5 < 6$. That's always true no matter what x is, so the solution to that inequality would be all real numbers, or $(-\infty, \infty)$.

Solve Real-World Problems Using Inequalities

Solving real-world problems that involve inequalities is very much like solving problems that involve equations.

Example B

In order to get a bonus this month, Leon must sell at least 120 newspaper subscriptions. He sold 85 subscriptions in the first three weeks of the month. How many subscriptions must Leon sell in the last week of the month?

Solution

Let $x =$ the number of subscriptions Leon sells in the last week of the month. The total number of subscriptions for the month must be greater than 120, so we write $85 + x \geq 120$. We solve the inequality by subtracting 85 from both sides: $x \geq 35$.

Leon must sell 35 or more subscriptions in the last week to get his bonus.

To check the answer, we see that $85+35=120$. If he sells 35 or more subscriptions, the total number of subscriptions he sells that month will be 120 or more. **The answer checks out**.

Example C

Virena's Scout troop is trying to raise at least \$650 this spring. How many boxes of cookies must they sell at \$4.50 per box in order to reach their goal?

Solution

Let $x =$ number of boxes sold. Then the inequality describing this problem is $4.50x \geq 650$.

We solve the inequality by dividing both sides by 4.50: $x \geq 144.44$.

We round up the answer to 145 since only whole boxes can be sold.

Virena's troop must sell at least 145 boxes.

If we multiply 145 by \$4.50 we obtain \$652.50, so if Virena's troop sells more than 145 boxes they will raise more than \$650. But if they sell 144 boxes, they will only raise \$648, which is not enough. So they must indeed sell at least 145 boxes. **The answer checks out**.

Watch this video for help with the Examples above.

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CK-12 Foundation: Using Inequalities

Vocabulary

- Inequalities can have infinite solutions, no solutions, or discrete solutions.
- There are four ways to represent an inequality: *Equation notation, set notation, interval notation,* and *solution graph.*

Guided Practice

The width of a rectangle is 20 inches. What must the length be if the perimeter is at least 180 inches?

Solution

Let $x =$ length of the rectangle. The formula for perimeter is

$$\text{Perimeter} = 2 \times \text{length} + 2 \times \text{width}$$

Since the perimeter must be at least 180 inches, we have $2x+2(20) \geq 180$.

Simplify: $2x+40 \geq 180$

Subtract 40 from both sides: $2x \geq 140$

Divide both sides by 2: $x \geq 70$

The length must be at least 70 inches.

If the length is at least 70 inches and the width is 20 inches, then the perimeter is at least $2(70)+2(20) = 180$ *inches*. **The answer checks out**.

Practice

Solve each inequality. Give the solution in inequality notation and interval notation.

1. $x+15 < 12$
2. $x-4 \geq 13$
3. $9x > -\frac{3}{4}$
4. $-\frac{x}{15} \leq 5$
5. $620x > 2400$
6. $\frac{x}{20} \geq -\frac{7}{40}$
7. $\frac{3x}{5} > \frac{3}{5}$
8. $x+3 > x-2$

Solve each inequality. Give the solution in inequality notation and set notation.

9. $x+17 < 3$
10. $x-12 \geq 80$
11. $-0.5x \leq 7.5$
12. $75x \geq 125$
13. $\frac{x}{-3} > -\frac{10}{9}$
14. $\frac{x}{-15} < 8$
15. $\frac{x}{4} > \frac{5}{4}$
16. $3x-7 \geq 3(x-7)$

Solve the following inequalities, give the solution in set notation, and show the solution graph.

17. $4x+3 < -1$
18. $2x < 7x-36$
19. $5x > 8x+27$
20. $5-x < 9+x$
21. $4-6x \leq 2(2x+3)$
22. $5(4x+3) \geq 9(x-2)-x$
23. $2(2x-1)+3 < 5(x+3)-2x$
24. $8x-5(4x+1) \geq -1+2(4x-3)$
25. $9.\ 2(7x-2)-3(x+2) < 4x-(3x+4)$
26. $\frac{2}{3}x-\frac{1}{2}(4x-1) \geq x+2(x-3)$
27. At the San Diego Zoo you can either pay \$22.75 for the entrance fee or \$71 for the yearly pass which entitles you to unlimited admission.
    a. At most how many times can you enter the zoo for the \$22.75 entrance fee before spending more than the cost of a yearly membership?
    b. Are there infinitely many or finitely many solutions to this inequality?
28. Proteek's scores for four tests were 82, 95, 86, and 88. What will he have to score on his fifth and last test to average at least 90 for the term?

6.5 Compound Inequalities

Here you'll learn how to write inequalities with more than one constraint on the possible values the solution can have. You'll also learn how to graph such inequalities on a number line.

What if you had an inequality with more than one inequality symbol, like $-4 < x < 5$ or $x > -2$ *or* $x < -7$? How could you graph such inequalities? After completing this Concept, you'll be able to graph compound inequalities like these on a number line.

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CK-12 Foundation: 0605S Graphs of Compound Inequalities (H264)

Guidance

In this section, we'll solve compound inequalities—inequalities with more than one constraint on the possible values the solution can have.

There are two types of compound inequalities:

1. Inequalities joined by the word *"and,"* where the solution is a set of values greater than a number *and* less than another number. We can write these inequalities in the form "$x > a$ and $x < b$," but usually we just write "$a < x < b$." Possible values for x are ones that will make *both* inequalities true.
2. Inequalities joined by the word *"or,"* where the solution is a set of values greater than a number *or* less than another number. We write these inequalities in the form "$x > a$ or $x < b$." Possible values for x are ones that will make *at least one* of the inequalities true.

You might wonder why the variable x has to be *greater than* one number and/or *less than* the other number; why can't it be greater than both numbers, or less than both numbers? To see why, let's take an example.

Consider the compound inequality "$x > 5$ and $x > 3$." Are there any numbers greater than 5 that are *not* greater than 3? No! Since 5 is greater than 3, everything greater than 5 is also greater than 3. If we say x is greater than both 5 and 3, that doesn't tell us any more than if we just said x is greater than 5. So this compound inequality isn't really compound; it's equivalent to the simple inequality $x > 5$. And that's what would happen no matter which two numbers we used; saying that x is greater than both numbers is just the same as saying that x is greater than the bigger number, and saying that x is less than both numbers is just the same as saying that x is less than the smaller number.

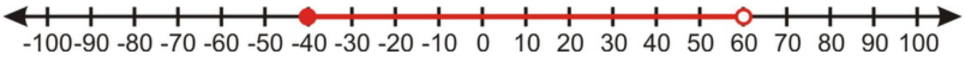
Compound inequalities with "or" work much the same way. Every number that's greater than 3 *or* greater than 5 is also just plain greater than 3, and every number that's greater than 3 is certainly greater than 3 *or* greater than 5—so if we say "$x > 5$ or $x > 3$," that's the same as saying just "$x > 3$." Saying that x is greater than at least one of two numbers is just the same as saying that x is greater than the smaller number, and saying that x is less than at least one of two numbers is just the same as saying that x is less than the greater number.

Write and Graph Compound Inequalities on a Number Line

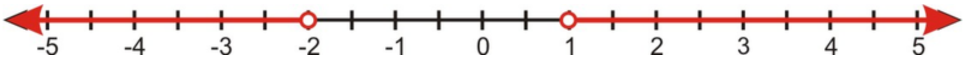
Example A

Write the inequalities represented by the following number line graphs.

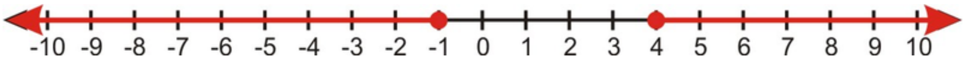
a)



b)



c)



Solution

a) The solution graph shows that the solution is any value between -40 and 60, including -40 but not 60.

Any value in the solution set satisfies both $x \geq -40$ *and* $x < 60$.

This is usually written as $-40 \leq x < 60$.

b) The solution graph shows that the solution is any value greater than 1 (not including 1) or any value less than -2 (not including -2). You can see that there can be no values that can satisfy both these conditions at the same time. We write: $x > 1$ *or* $x < -2$.

c) The solution graph shows that the solution is any value greater than 4 (including 4) or any value less than -1 (including - 1). We write: $x \geq 4$ *or* $x \leq -1$.

Example B

Graph the following compound inequalities on a number line.

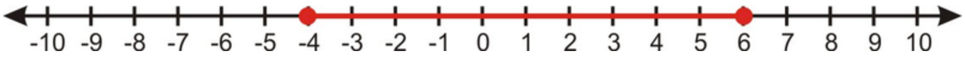
a) $-4 \leq x \leq 6$

b) $x < 0$ or $x > 2$

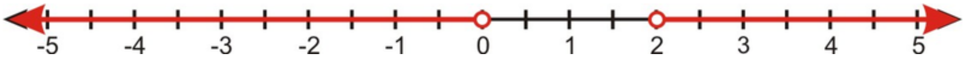
c) $x \geq -8$ or $x \leq -20$

Solution

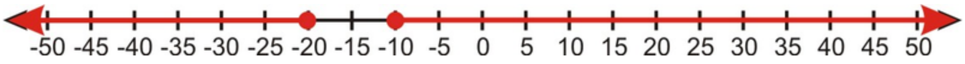
a) The solution is all numbers between -4 and 6, including both -4 and 6.



b) The solution is all numbers less than 0 or greater than 2, not including 0 or 2.



c) The solution is all numbers greater than or equal to -8 or less than or equal to -20.



Solve Compound Inequalities Using a Graphing Calculator (TI-83/84 family)

Graphing calculators can show you the solution to an inequality in the form of a graph. This can be especially useful when dealing with compound inequalities.

Example C

Solve the following inequalities using a graphing calculator.

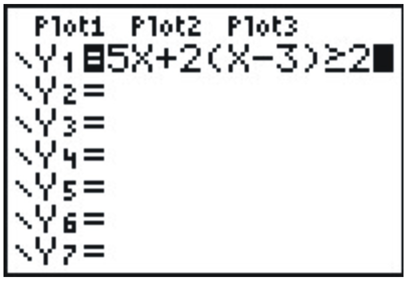
a) $5x+2(x-3) \geq 2$

b) $7x-2 < 10x+1 < 9x+5$

c) $3x+2 \leq 10$ or $3x+2 \geq 15$

Solution

a) Press the **[Y=]** button and enter the inequality on the first line of the screen.

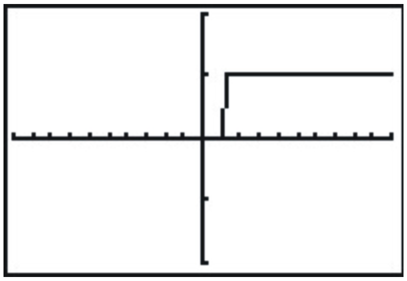


```
 Plot1 Plot2 Plot3
\Y1=5X+2(X-3)≥2
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

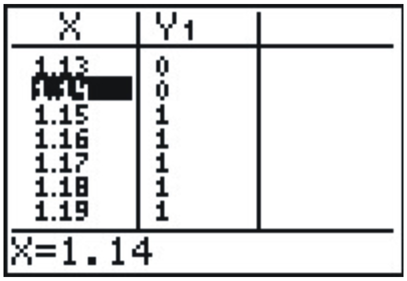
(To get the $\geq$ symbol, press **[TEST] [2nd] [MATH]** and choose option 4.)

Then press the **[GRAPH]** button.

Because the calculator uses the number 1 to mean "true" and 0 to mean "false," you will see a step function with the $y-$value jumping from 0 to 1.



The solution set is the values of x for which the graph shows $y = 1$—in other words, the set of $x-$values that make the inequality true.



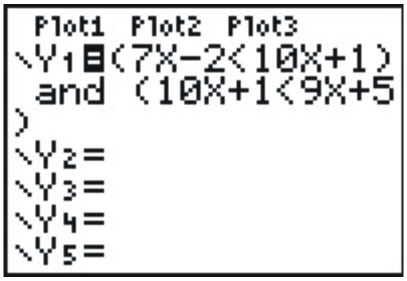
X	Y1
1.13	0
1.14	0
1.15	1
1.16	1
1.17	1
1.18	1
1.19	1

X=1.14

Note: You may need to press the **[WINDOW]** key or the **[ZOOM]** key to adjust the window to see the full graph.

The solution is $x > \frac{8}{7}$, which is why you can see the $y-$value changing from 0 to 1 at about 1.14.

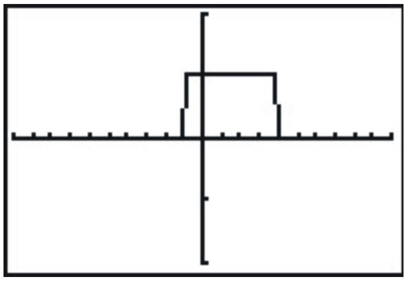
b) This is a compound inequality: $7x - 2 < 10x + 1$ and $10x + 1 < 9x + 5$. You enter it like this:



```
 Plot1  Plot2  Plot3
\Y1=(7X-2<10X+1)
 and (10X+1<9X+5
)
\Y2=
\Y3=
\Y4=
\Y5=
```

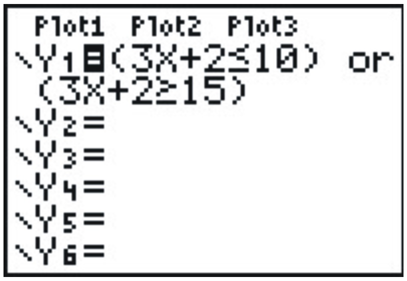
(To find the **[AND]** symbol, press **[TEST]**, choose **[LOGIC]** on the top row and choose option 1.)

The resulting graph should look like this:



The solution are the values of x for which $y = 1$; in this case that would be $-1 < x < 4$.

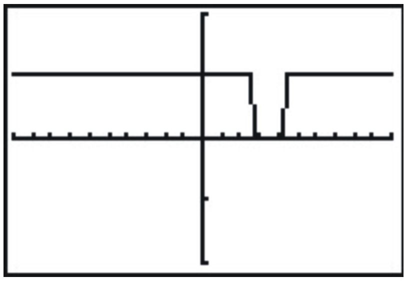
c) This is another compound inequality.



```
 Plot1  Plot2  Plot3
\Y1=(3X+2≤10) or
 (3X+2≥15)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

(To enter the **[OR]** symbol, press **[TEST]**, choose **[LOGIC]** on the top row and choose option 2.)

The resulting graph should look like this:



The solution are the values of x for which $y = 1$–in this case, $x \leq 2.7$ or $x \geq 4.3$.

Watch this video for help with the Examples above.

MEDIA

Click image to the left for more content.

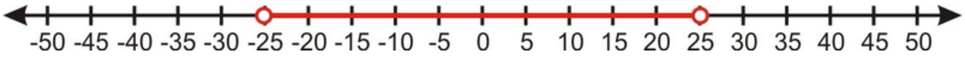
CK-12 Foundation: Graphsof CompoundInequalities

Vocabulary

- **Compound inequalities** combine two or more inequalities with **"and"** or **"or."**
- **"And"** combinations mean that only solutions for *both* inequalities will be solutions to the compound inequality.
- **"Or"** combinations mean solutions to *either* inequality will also be solutions to the compound inequality.

Guided Practice

1. *Write the inequality represented by the following number line graph.*



2. *Graph the following compound inequality on a number line.*

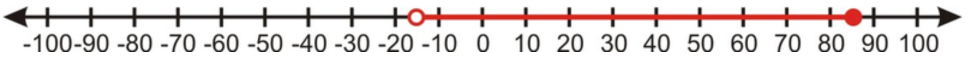
$-15 < x \leq 85$

Solution

1. The solution graph shows that the solution is any value that is both less than 25 (not including 25) and greater than -25 (not including -25). Any value in the solution set satisfies both $x > -25$ *and* $x < 25$.

This is usually written as $-25 < x < 25$.

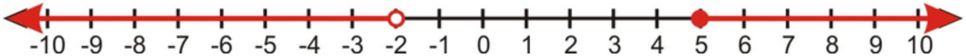
2. The solution is all numbers between -15 and 85, not including -15 but including 85.



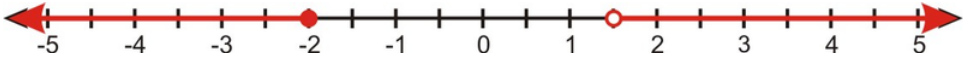
Practice

Write the compound inequalities represented by the following graphs.

1.


2.


3.


4.


Graph the following compound inequalities on a number line.

5. $-2 \leq x \leq 20$
6. $x < 7$ or $x > 25$
7. $x \geq -100$ or $x \leq -50$
8. $-1 < x < 200$
9. $2000 < x \leq 2001$
10. $x \leq 1.56$ or $x > 1.78$
11. $x > 0.0005$ or $x \leq -0.03$

6.6 Solutions to Compound Inequalities

Here you'll learn how to separate compound inequalities with *"and"* or *"or"* and solve them separately. You'll then learn how to combine your answers into a single solution and graph the solution set.

What if you had a compound inequality like $0 \leq 2x+6 \leq 6$? How could you solve it and graph its solution set? After completing this Concept, you'll be able to graph the solution set of compound inequalities like this one on a number line.

Watch This

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CK-12 Foundation: 0606S Solving CompoundInequalities (H264)

Guidance

When we solve compound inequalities, we separate the inequalities and solve each of them separately. Then, we combine the solutions at the end.

Example A

Solve the following compound inequalities and graph the solution set.

a) $-2 < 4x-5 \leq 11$

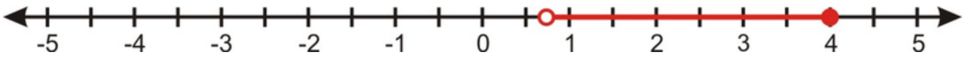
b) $3x-5 < x+9 \leq 5x+13$

Solution

a) First we re-write the compound inequality as two separate inequalities with *and*. Then solve each inequality separately.

$$\begin{aligned} -2 &< 4x-5 & & & 4x-5 &\leq 11 \\ 3 &< 4x & & \textit{and} & 4x &\leq 16 \\ \frac{3}{4} &< x & & & x &\leq 4 \end{aligned}$$

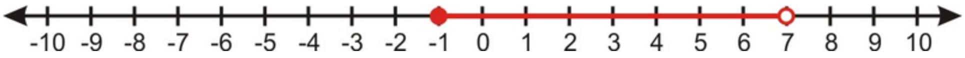
Answer: $\frac{3}{4} < x$ and $x \leq 4$. This can be written as $\frac{3}{4} < x \leq 4$.



b) Re-write the compound inequality as two separate inequalities with *and*. Then solve each inequality separately.

$$\begin{array}{rcr} 3x-5 < x+9 & & x+9 \le 5x+13 \\ 2x < 14 & \textit{and} & -4 \le 4x \\ x < 7 & & -1 \le x \end{array}$$

Answer: $x < 7$ and $x \ge -1$. This can be written as: $-1 \le x < 7$.



Example B

Solve the following compound inequalities and graph the solution set.

a) $9-2x \le 3$ or $3x+10 \le 6-x$

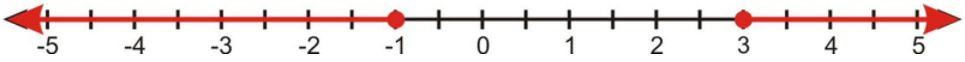
b) $\frac{x-2}{6} \le 2x-4$ or $\frac{x-2}{6} > x+5$

Solution

a) Solve each inequality separately:

$$\begin{array}{rcr} 9-2x \le 3 & & 3x+10 \le 6-x \\ -2x \le -6 & \textit{or} & 4x \le -4 \\ x \ge 3 & & x \le -1 \end{array}$$

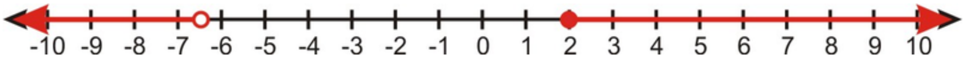
Answer: $x \ge 3$ or $x \le -1$



b) Solve each inequality separately:

$$\begin{array}{rcr} \frac{x-2}{6} \le 2x-4 & & \frac{x-2}{6} > x+5 \\ x-2 \le 6(2x-4) & & x-2 > 6(x+5) \\ x-2 \le 12x-24 & \textit{or} & x-2 > 6x+30 \\ 22 \le 11x & & -32 > 5x \\ 2 \le x & & -6.4 > x \end{array}$$

Answer: $x \ge 2$ or $x < -6.4$



One thing you may notice in the video for this Concept is that in the second problem, the two solutions joined with "or" overlap, and so the solution ends up being the set of all real numbers, or $(-\infty, \infty)$. This happens sometimes with compound inequalities that involve "or"; for example, if the solution to an inequality ended up being "$x < 5$ or $x > 1$," the solution set would be all real numbers. This makes sense if you think about it: all real numbers are either a) less than 5, or b) greater than or equal to 5, and the ones that are greater than or equal to 5 are also greater than 1—so all real numbers are either a) less than 5 or b) greater than 1.

Compound inequalities with "and," meanwhile, can turn out to have *no* solutions. For example, the inequality "$x < 3$ and $x > 4$" has no solutions: no number is both greater than 4 and less than 3. If we write it as $4 < x < 3$ it's even more obvious that it has no solutions; $4 < x < 3$ implies that $4 < 3$, which is false.

Solve Real-World Problems Using Compound Inequalities

Many application problems require the use of compound inequalities to find the solution.

Example C

The speed of a golf ball in the air is given by the formula $v = -32t + 80$. When is the ball traveling between 20 ft/sec and 30 ft/sec?

Solution

First we set up the inequality $20 \leq v \leq 30$, and then replace v with the formula $v = -32t + 80$ to get $20 \leq -32t + 80 \leq 30$.

Then we separate the compound inequality and solve each separate inequality:

$$\begin{array}{rcr} 20 \leq -32t + 80 & & -32t + 80 \leq 30 \\ 32t \leq 60 & \text{and} & 50 \leq 32t \\ t \leq 1.875 & & 1.56 \leq t \end{array}$$

Answer: $1.56 \leq t \leq 1.875$

To check the answer, we plug in the minimum and maximum values of t into the formula for the speed.

For $t = 1.56, \ v = -32t + 80 = -32(1.56) + 80 = 30 \ ft/sec$

For $t = 1.875, \ v = -32t + 80 = -32(1.875) + 80 = 20 \ ft/sec$

So the speed is between 20 and 30 ft/sec. **The answer checks out.**

Watch this video for help with the Examples above.

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CK-12 Foundation: Solving Compound Inequalities

Vocabulary

- **Compound inequalities** combine two or more inequalities with **"and"** or **"or."**
- **"And"** combinations mean that only solutions for *both* inequalities will be solutions to the compound inequality.
- **"Or"** combinations mean solutions to *either* inequality will also be solutions to the compound inequality.

Guided Practice

William's pick-up truck gets between 18 to 22 miles per gallon of gasoline. His gas tank can hold 15 gallons of gasoline. If he drives at an average speed of 40 miles per hour, how much driving time does he get on a full tank of gas?

Solution

Let $t =$ driving time. We can use dimensional analysis to get from time per tank to miles per gallon:

$$\frac{t\ hours}{1\ tank} \times \frac{1\ tank}{15\ gallons} \times \frac{40\ miles}{1\ hour} \times \frac{40t\ \ miles}{15\ gallon}$$

Since the truck gets between 18 and 22 miles/gallon, we set up the compound inequality $18 \le \frac{40t}{15} \le 22$. Then we separate the compound inequality and solve each inequality separately:

$$\begin{array}{rcr} 18 \le \dfrac{40t}{15} & & \dfrac{40t}{15} \le 22 \\ 270 \le 40t & \text{and} & 40t \le 330 \\ 6.75 \le t & & t \le 8.25 \end{array}$$

Answer: $6.75 \le t \le 8.25$.

Andrew can drive between 6.75 and 8.25 hours on a full tank of gas.

If we plug in $t = 6.75$ we get $\frac{40t}{15} = \frac{40(6.75)}{15} = 18\ miles\ per\ gallon.$

If we plug in $t = 8.25$ we get $\frac{40t}{15} = \frac{40(8.25)}{15} = 22\ miles\ per\ gallon.$

The answer checks out.

Practice

Solve the following compound inequalities and graph the solution on a number line.

1. $-5 \le x - 4 \le 13$
2. $1 \le 3x + 5 \le 4$
3. $-12 \le 2 - 5x \le 7$
4. $\frac{3}{4} \le 2x + 9 \le \frac{3}{2}$
5. $-2 \le \frac{2x-1}{3} < -1$
6. $4x - 1 \ge 7$ or $\frac{9x}{2} < 3$
7. $3 - x < -4$ or $3 - x > 10$
8. $\frac{2x+3}{4} < 2$ or $-\frac{x}{5} + 3 < \frac{2}{5}$
9. $2x - 7 \le -3$ or $2x - 3 > 11$
10. $4x + 3 < 9$ or $-5x + 4 \le -12$
11. How would you express the answer to problem 1 as a set?
12. How would you express the answer to problem 1 as an interval?
13. How would you express the answer to problem 6 as a set?
    1. Could you express the answer to problem 6 as a single interval? Why or why not?
    2. How would you express the first part of the solution in interval form?
    3. How would you express the second part of the solution in interval form?

14. Express the answers to problems 2 through 5 in interval notation.
15. Solve the inequality "$x \geq -3$ or $x < 1$" and express the answer in interval notation.
16. How many solutions does the inequality "$x \geq 2$ and $x \leq 2$" have?
17. To get a grade of B in her Algebra class, Stacey must have an average grade greater than or equal to 80 and less than 90. She received the grades of 92, 78, 85 on her first three tests.
    a. Between which scores must her grade on the final test fall if she is to receive a grade of B for the class? (Assume all four tests are weighted the same.)
    b. What range of scores on the final test would give her an overall grade of C, if a C grade requires an average score greater than or equal to 70 and less than 80?
    c. If an A grade requires a score of at least 90, and the maximum score on a single test is 100, is it possible for her to get an A in this class? (Hint: look again at your answer to part a.)

6.7 Absolute Value

Here you'll learn how to find the distance between two values on a number line and solve equations involving absolute values.

What if you were given two points like -8 and 12? How could you find the distance between them on a number line? After completing this Concept, you'll be able to use absolute value properties to solve problems like this one.

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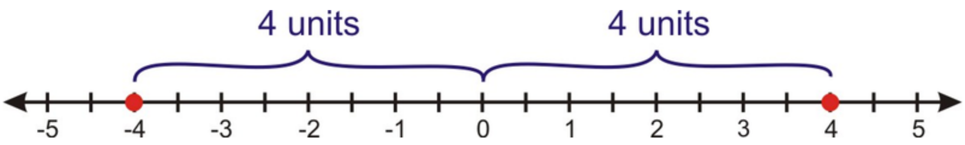
CK-12 Foundation: 0607S AbsoluteValue Equations(H264)

Guidance

Timmy is trying out his new roller skates. He's not allowed to cross the street yet, so he skates back and forth in front of his house. If he skates 20 yards east and then 10 yards west, how far is he from where he started? What if he skates 20 yards west and then 10 yards east?

The **absolute value** of a number is its distance from zero on a number line. There are always two numbers on the number line that are the same distance from zero. For instance, the numbers 4 and -4 are each a distance of 4 units away from zero.



$|4|$ represents the distance from 4 to zero, which equals 4.

$|-4|$ represents the distance from -4 to zero, which also equals 4.

In fact, for any real number x:

$|x| = x$ if x is not negative, and $|x| = -x$ if x is negative.

Absolute value has no effect on a positive number, but changes a negative number into its positive inverse.

Example A

Evaluate the following absolute values.

a) $|25|$

b) $|-120|$

c) $|-3|$

d) $|55|$

e) $\left|-\frac{5}{4}\right|$

Solution

a) $|25| = 25$ Since 25 is a positive number, the absolute value does not change it.

b) $|-120| = 120$ Since -120 is a negative number, the absolute value makes it positive.

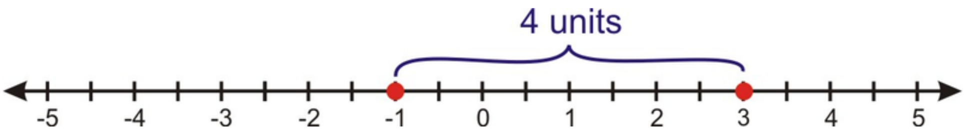
c) $|-3| = 3$ Since -3 is a negative number, the absolute value makes it positive.

d) $|55| = 55$ Since 55 is a positive number, the absolute value does not change it.

e) $\left|-\frac{5}{4}\right| = \frac{5}{4}$ Since $-\frac{5}{4}$ is a negative number, the absolute value makes it positive.

Absolute value is very useful in finding the distance between two points on the number line. The **distance** between any two points a and b on the number line is $|a-b|$ or $|b-a|$.

For example, the distance from 3 to -1 on the number line is $|3-(-1)| = |4| = 4$.



We could have also found the distance by subtracting in the opposite order: $|-1-3| = |-4| = 4$. This makes sense because the distance is the same whether you are going from 3 to -1 or from -1 to 3.

Example B

Find the distance between the following points on the number line.

a) 6 and 15

b) -5 and 8

c) -3 and -12

Solution

Distance is the absolute value of the difference between the two points.

a) distance $= |6-15| = |-9| = 9$

b) distance $= |-5-8| = |-13| = 13$

c) distance $= |-3-(-12)| = |9| = 9$

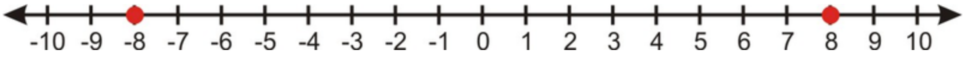
Remember: When we computed the change in x and the change in y as part of the slope computation, these values were positive or negative, depending on the direction of movement. In this discussion, "distance" means a positive distance only.

Solve an Absolute Value Equation

We now want to solve equations involving absolute values. Consider the following equation:

$$|x| = 8$$

This means that the distance from the number x to zero is 8. There are two numbers that satisfy this condition: 8 and -8.



When we solve absolute value equations we always consider two possibilities:

1. The expression inside the absolute value sign is not negative.
2. The expression inside the absolute value sign is negative.

Then we solve each equation separately.

Example C

Solve the following absolute value equations.

a) $|x| = 3$

b) $|x| = 10$

Solution

a) There are two possibilities: $x = 3$ **and** $x = -3$.

b) There are two possibilities: $x = 10$ **and** $x = -10$.

Watch this video for help with the Examples above.

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CK-12 Foundation: Absolute ValueEquations

Vocabulary

- The absolute value of a number is its distance from zero on a number line.
- $|x| = x$ if x is not negative, and $|x| = -x$ if x is negative.
- An equation or inequality with an absolute value in it **splits into two equations,** one where the expression inside the absolute value sign is positive and one where it is negative. When the expression within the absolute value is **positive**, then the absolute value signs do nothing and can be omitted. When the expression within the absolute value is **negative,** then the expression within the absolute value signs must be negated before removing the signs.
- Inequalities of the type $|x| < a$ can be rewritten as "$-a < x < a$."
- Inequalities of the type $|x| > b$ can be rewritten as "$x < -b$ or $x > b$."

Guided Practice

Find the distance between the values $-\frac{1}{3}$ *and* $\frac{1}{5}$ *on the number line.*

Solution:

The distance is the absolute value of the difference:

$$\begin{aligned}
\left|-\frac{1}{3}-\frac{1}{5}\right| &= \text{Set up the absolute value.}\\
\left|-\frac{5}{15}-\frac{3}{15}\right| &= \text{Give the two terms common denominators.}\\
\left|\frac{-5-3}{15}\right| &= \text{Combine the terms.}\\
\left|\frac{-8}{15}\right| &= \text{Simplify.}\\
\frac{8}{15} &= \text{Evaluate.}
\end{aligned}$$

The distance between the two points is $\frac{8}{15}$.

Practice

Evaluate the absolute values.

1. $|250|$
2. $|-12|$
3. $|-0.003|$
4. $\left|-\frac{2}{5}\right|$
5. $\left|\frac{1}{10}\right|$

Find the distance between the points.

6. 12 and -11
7. 5 and 22
8. -9 and -18
9. -2 and 3
10. -0.012 and 1.067
11. $-\frac{2}{3}$ and $\frac{7}{8}$

6.8 Absolute Value Equations

Here you'll learn how to how to solve more complicated absolute value equations and interpret your answers.

What if you were asked to solve an absolute value equation like $|3x-4|=5$? How could you interpret the solution? After completing this Concept, you'll be able to interpret the solutions to absolute value equations like this one by graphing them on a number line.

Watch This

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CK-12 Foundation: 0608S Analyze Solutions toAbsolute Value Equations (H264)

Guidance

In the previous concept, we saw how to solve simple absolute value equations. In this concept, you will see how to solve more complicated absolute value equations.

Example A

Solve the equation $|x-4|=5$ *and interpret the answers.*

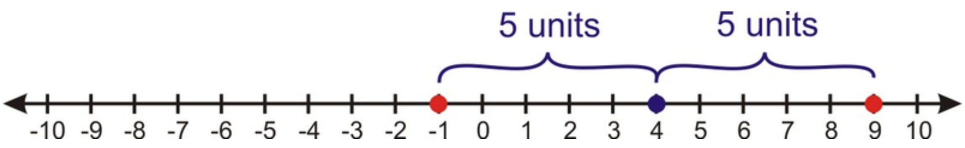
Solution

We consider two possibilities: the expression inside the absolute value sign is non-negative or is negative. Then we solve each equation separately.

$$\begin{aligned} x-4 &= 5 \quad \text{and} \quad x-4=-5 \\ x &= 9 \qquad\qquad\qquad x=-1 \end{aligned}$$

$x=9$ and $x=-1$ are the solutions.

The equation $|x-4|=5$ can be interpreted as "what numbers on the number line are 5 units away from the number 4?" If we draw the number line we see that there are two possibilities: 9 and -1.



Example B

Solve the equation $|x+3| = 2$ *and interpret the answers.*

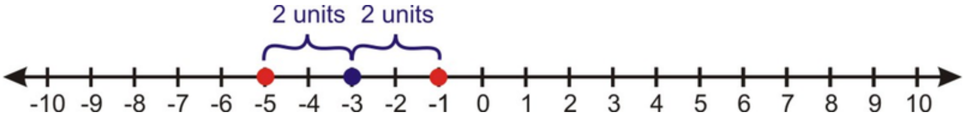
Solution

Solve the two equations:

$$\begin{aligned} x+3 &= 2 \quad \text{and} \quad & x+3 &= -2 \\ x &= -1 & x &= -5 \end{aligned}$$

$x = -5$ and $x = -1$ are the answers.

The equation $|x+3| = 2$ can be re-written as: $|x-(-3)| = 2$. We can interpret this as "what numbers on the number line are 2 units away from -3?" There are two possibilities: -5 and -1.



Solve Real-World Problems Using Absolute Value Equations

Example C

A company packs coffee beans in airtight bags. Each bag should weigh 16 ounces, but it is hard to fill each bag to the exact weight. After being filled, each bag is weighed; if it is more than 0.25 ounces overweight or underweight, it is emptied and repacked. What are the lightest and heaviest acceptable bags?

Solution

The weight of each bag is allowed to be 0.25 ounces away from 16 ounces; in other words, the *difference* between the bag's weight and 16 ounces is allowed to be 0.25 ounces. So if x is the weight of a bag in ounces, then the equation that describes this problem is $|x-16| = 0.25$.

Now we must consider the positive and negative options and solve each equation separately:

$$\begin{aligned} x-16 &= 0.25 \quad \text{and} \quad & x-16 &= -0.25 \\ x &= 16.25 & x &= 15.75 \end{aligned}$$

The lightest acceptable bag weighs 15.75 ounces and the heaviest weighs 16.25 ounces.

We see that $16.25-16 = 0.25 \ ounces$ and $16-15.75 = 0.25 \ ounces$. The answers are 0.25 ounces bigger and smaller than 16 ounces respectively.

The answer checks out.

The answer you just found describes the lightest and heaviest acceptable bags of coffee beans. But how do we describe the total possible range of acceptable weights? That's where inequalities become useful once again.

Watch this video for help with the Examples above.

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CK-12 Foundation: Analyze Solutions to Absolute Value Equations

Vocabulary

- The absolute value of a number is its distance from zero on a number line.
- $|x| = x$ if x is not negative, and $|x| = -x$ if x is negative.
- An equation or inequality with an absolute value in it **splits into two equations,** one where the expression inside the absolute value sign is positive and one where it is negative. When the expression within the absolute value is **positive**, then the absolute value signs do nothing and can be omitted. When the expression within the absolute value is **negative,** then the expression within the absolute value signs must be negated before removing the signs.
- Inequalities of the type $|x| < a$ can be rewritten as "$-a < x < a$."
- Inequalities of the type $|x| > b$ can be rewritten as "$x < -b$ or $x > b$."

Guided Practice

Solve the equation $|2x-7| = 6$ *and interpret the answers.*

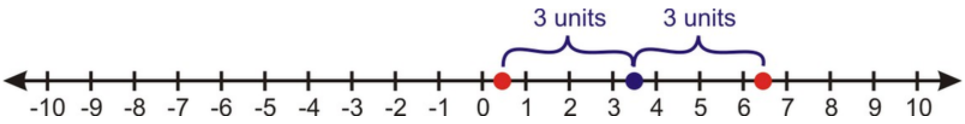
Solution

Solve the two equations:

$$\begin{aligned} 2x-7 &= 6 & \qquad & & 2x-7 &= -6 \\ 2x &= 13 & & \text{and} & 2x &= 1 \\ x &= \frac{13}{2} & & & x &= \frac{1}{2} \end{aligned}$$

Answer: $x = \frac{13}{2}$ and $x = \frac{1}{2}$.

The interpretation of this problem is clearer if the equation $|2x-7| = 6$ is divided by 2 on both sides to get $\frac{1}{2}|2x-7| = 3$. Because $\frac{1}{2}$ is nonnegative, we can distribute it over the absolute value sign to get $\left|x - \frac{7}{2}\right| = 3$. The question then becomes "What numbers on the number line are 3 units away from $\frac{7}{2}$?" There are two answers: $\frac{13}{2}$ and $\frac{1}{2}$.



Practice

Solve the absolute value equations and interpret the results by graphing the solutions on the number line.

1. $|x-5| = 10$
2. $|x+2| = 6$
3. $|5x-2| = 3$

4. $|x-4| = -3$
5. $\left|2x-\frac{1}{2}\right| = 10$
6. $|-x+5| = \frac{1}{5}$
7. $\left|\frac{1}{2}x-5\right| = 100$
8. $|10x-5| = 15$
9. $|0.1x+3| = 0.015$
10. $|27-2x| = 3x+2$

6.9 Graphs of Absolute Value Equations

Here you'll learn how to make a table of values for absolute value functions so you can graph them.

What if you were given an absolute value function like $y = |x - 8|$? How could you graph this function? After completing this Concept, you'll be able to make a table of values to graph absolute value functions like this one.

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CK-12 Foundation: 0609S Graphs of Absolute ValueEquations (H264)

Guidance

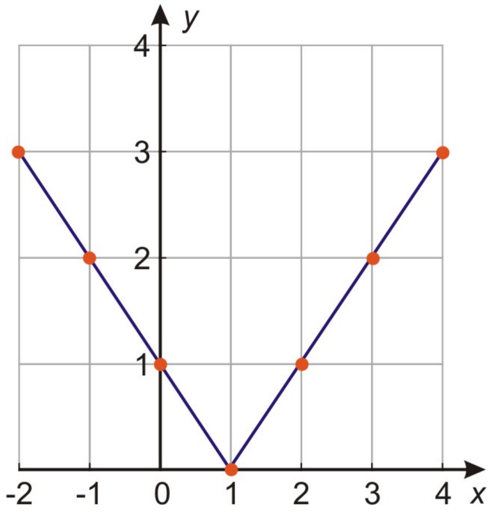
Now let's look at how to graph absolute value functions.

Example A

Consider the function $y = |x - 1|$. We can graph this function by making a table of values:

TABLE 6.1:

x	$y = \|x - 1\|$
-2	$y = \|-2 - 1\| = \|-3\| = 3$
-1	$y = \|-1 - 1\| = \|-2\| = 2$
0	$y = \|0 - 1\| = \|-1\| = 1$
1	$y = \|1 - 1\| = \|0\| = 0$
2	$y = \|2 - 1\| = \|1\| = 1$
3	$y = \|3 - 1\| = \|2\| = 2$
4	$y = \|4 - 1\| = \|3\| = 3$



You can see that the graph of an absolute value function makes a big "V". It consists of two line rays (or line segments), one with positive slope and one with negative slope, joined at the **vertex** or **cusp**.

We've already seen that to solve an absolute value equation we need to consider two options:

1. The expression inside the absolute value is not negative.
2. The expression inside the absolute value is negative.

Combining these two options gives us the two parts of the graph.

For instance, in the above example, the expression inside the absolute value sign is $x-1$. By definition, this expression is nonnegative when $x-1 \geq 0$, which is to say when $x \geq 1$. When the expression inside the absolute value sign is nonnegative, we can just drop the absolute value sign. So for all values of x greater than or equal to 1, the equation is just $y = x-1$.

On the other hand, when $x-1 < 0$ — in other words, when $x < 1$ — the expression inside the absolute value sign is negative. That means we have to drop the absolute value sign but also multiply the expression by -1. So for all values of x less than 1, the equation is $y = -(x-1)$, or $y = -x+1$.

These are both graphs of straight lines, as shown above. They meet at the point where $x-1 = 0$ — that is, at $x = 1$.

We can graph absolute value functions by breaking them down algebraically as we just did, or we can graph them using a table of values. However, when the absolute value equation is linear, the easiest way to graph it is to combine those two techniques, as follows:

1. Find the vertex of the graph by setting the expression inside the absolute value equal to zero and solving for x.
2. Make a table of values that includes the vertex, a value smaller than the vertex, and a value larger than the vertex. Calculate the corresponding values of y using the equation of the function.
3. Plot the points and connect them with two straight lines that meet at the vertex.

Example B

Graph the absolute value function $y = |x+5|$.

Solution

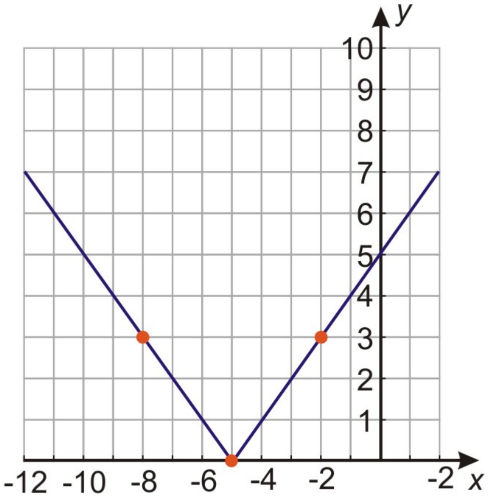
Step 1: Find the vertex by solving $x+5 = 0$. The vertex is at $x = -5$.

Step 2: Make a table of values:

TABLE 6.2:

x	$y = \lvert x+5 \rvert$
-8	$y = \lvert -8+5 \rvert = \lvert -3 \rvert = 3$
-5	$y = \lvert -5+5 \rvert = \lvert 0 \rvert = 0$
-2	$y = \lvert -2+5 \rvert = \lvert 3 \rvert = 3$

Step 3: Plot the points and draw two straight lines that meet at the vertex:



Example C

Graph the absolute value function: $y = \lvert 3x-12 \rvert$

Solution

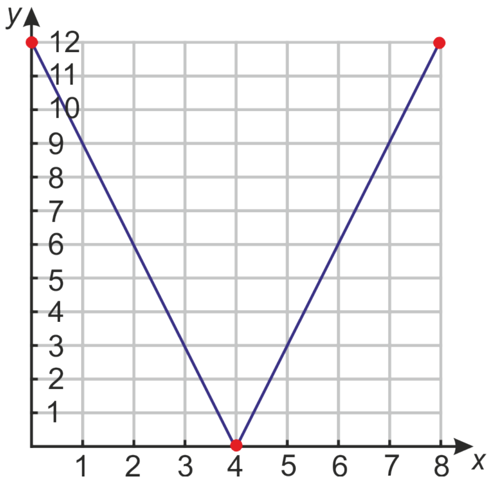
Step 1: Find the vertex by solving $3x-12 = 0$. The vertex is at $x = 4$.

Step 2: Make a table of values:

TABLE 6.3:

x	$y = \lvert 3x-12 \rvert$
0	$y = \lvert 3(0)-12 \rvert = \lvert -12 \rvert = 12$
4	$y = \lvert 3(4)-12 \rvert = \lvert 0 \rvert = 0$
8	$y = \lvert 3(8)-12 \rvert = \lvert 12 \rvert = 12$

Step 3: Plot the points and draw two straight lines that meet at the vertex.



Watch this video for help with the Examples above.

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CK-12 Foundation: Graphsof AbsoluteValue Equations

Vocabulary

- The absolute value of a number is its distance from zero on a number line.
- $|x| = x$ if x is not negative, and $|x| = -x$ if x is negative.
- An equation or inequality with an absolute value in it **splits into two equations,** one where the expression inside the absolute value sign is positive and one where it is negative. When the expression within the absolute value is **positive**, then the absolute value signs do nothing and can be omitted. When the expression within the absolute value is **negative,** then the expression within the absolute value signs must be negated before removing the signs.
- Inequalities of the type $|x| < a$ can be rewritten as "$-a < x < a$."
- Inequalities of the type $|x| > b$ can be rewritten as "$x < -b$ or $x > b$."

Guided Practice

Graph the absolute value function: $y = 3|x-4|$

Solution

Step 1: Find the vertex by solving $x-4=0$. The vertex is at $x=4$.

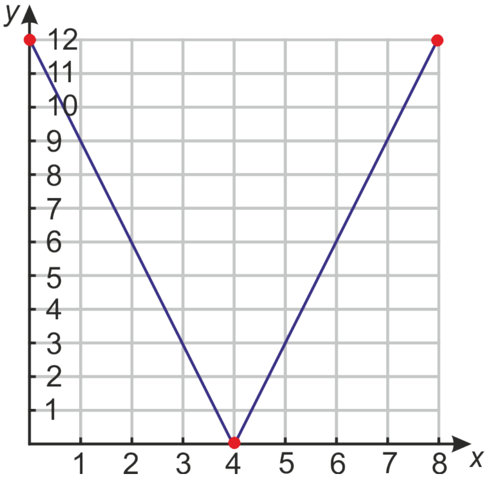
Step 2: Make a table of values:

TABLE 6.4:

x	$y = 3\|x-4\|$
0	$y = 3\|0-4\| = 3\|-4\| = 3 \cdot 4 = 12$
4	$y = 3\|4-4\| = 3\|0\| = 3 \cdot 0 = 0$
8	$y = 3\|8-4\| = 3\|4\| = 3 \cdot 4 = 12$

Notice this is the same table as Example C. The function $y = 3|x-4|$ is equivalent to the function $y = |3x - 12|$.</math>. This is because positive numbers can be factored out, or distributed into the absolute value function.

Step 3: Plot the points and draw two straight lines that meet at the vertex.



Practice

Graph the absolute value functions.

1. $y = |x+3|$
2. $y = |x-6|$
3. $y = |4x+2|$
4. $y = |5-6x|$
5. $y = |2x-1|$
6. $y = 3|2x-7|$
7. $y = 0.05|x-1.25|$
8. $y = \frac{1}{2}|x+10|$
9. $y = \left|\frac{x}{3} - 4\right|$
10. $y = -2\left|\frac{x}{2} - 5\right|$

6.10 Absolute Value Inequalities

Here you'll learn how to solve absolute value inequalities and show their solution graph. You'll also solve real-world problems involving absolute value inequalities.

What if you were given an absolute value inequality like $|2x| \leq 16$? How could you solve it? After completing this Concept, you'll be able to find the solution set and show the solution graph of inequalities like this one.

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CK-12 Foundation: 0610S AbsoluteValue Inequalities (H264)

Guidance

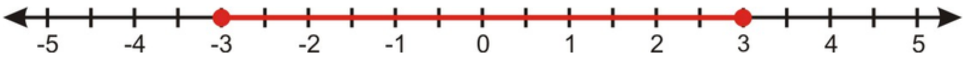
Absolute value inequalities are solved in a similar way to absolute value equations. In both cases, you must consider the same two options:

1. The expression inside the absolute value is not negative.
2. The expression inside the absolute value is negative.

Then you must solve each inequality separately.

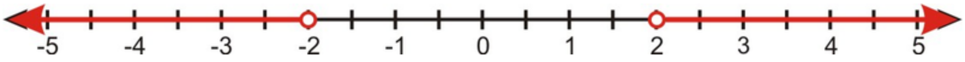
Solve Absolute Value Inequalities

Consider the inequality $|x| \leq 3$. Since the absolute value of x represents the distance from zero, the solutions to this inequality are those numbers whose distance from zero is less than or equal to 3. The following graph shows this solution:



Notice that this is also the graph for the compound inequality $-3 \leq x \leq 3$.

Now consider the inequality $|x| > 2$. Since the absolute value of x represents the distance from zero, the solutions to this inequality are those numbers whose distance from zero are more than 2. The following graph shows this solution.



Notice that this is also the graph for the compound inequality $x < -2$ or $x > 2$.

Example A

Solve the following inequalities and show the solution graph.

a) $|x| < 5$

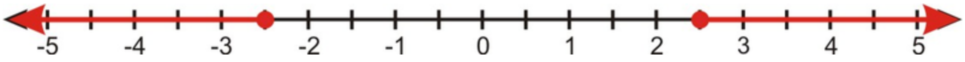
b) $|x| \geq 2.5$

Solution

a) $|x| < 5$ represents all numbers whose distance from zero is less than 5.

This answer can be written as "$-5 < x < 5$".

b) $|x| \geq 2.5$ represents all numbers whose distance from zero is more than or equal to 2.5



This answer can be written as "$x \leq -2.5$ or $x \geq 2.5$".

Rewrite and Solve Absolute Value Inequalities as Compound Inequalities

In the last section you saw that absolute value inequalities are compound inequalities.

Inequalities of the type $|x| < a$ can be rewritten as "$-a < x < a$".

Inequalities of the type $|x| > b$ can be rewritten as "$x < -b$ or $x > b$."

To solve an absolute value inequality, we separate the expression into two inequalities and solve each of them individually.

Example B

Solve the inequality $|x-3| < 7$ and show the solution graph.

Solution

Re-write as a compound inequality: $-7 < x-3 < 7$

Write as two separate inequalities: $x-3 < 7$ and $x-3 > -7$

Solve each inequality: $x < 10$ and $x > -4$

Re-write solution: $-4 < x < 10$

The solution graph is

We can think of the question being asked here as "What numbers are within 7 units of 3?"; the answer can then be expressed as "All the numbers between -4 and 10."

Example C

Solve the inequality $|4x+5| \leq 13$ and show the solution graph.

Solution

Re-write as a compound inequality: $-13 \leq 4x+5 \leq 13$

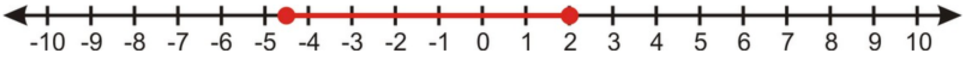
Write as two separate inequalities: $4x+5 \leq 13$ and $4x+5 \geq -13$

Solve each inequality: $4x \leq 8$ and $4x \geq -18$

$x \leq 2$ and $x \geq -\frac{9}{2}$

Re-write solution: $-\frac{9}{2} \leq x \leq 2$

The solution graph is



Example D

Solve the inequality $|x+12|>2$ *and show the solution graph.*

Solution

Re-write as a compound inequality: $x+12<-2$ or $x+12>2$

Solve each inequality: $x<-14$ or $x>-10$

The solution graph is

Solve Real-World Problems Using Absolute Value Inequalities

Absolute value inequalities are useful in problems where we are dealing with a range of values.

Example E

The velocity of an object is given by the formula $v = 25t - 80$*, where the time is expressed in seconds and the velocity is expressed in feet per second. Find the times for which the magnitude of the velocity is greater than or equal to 60 feet per second.*

Solution

The *magnitude* of the velocity is the absolute value of the velocity. If the velocity is $25t-80$ feet per second, then its magnitude is $|25t-80|$ feet per second. We want to find out when that magnitude is greater than or equal to 60, so we need to solve $|25t-80| \geq 60$ for t.

First we have to split it up: $25t-80 \geq 60$ or $25t-80 \leq -60$

Then solve: $25t \geq 140$ or $25t \leq 20$

$t \geq 5.6$ or $t \leq 0.8$

The magnitude of the velocity is greater than 60 ft/sec for times **less than 0.8 seconds** and for times **greater than 5.6 seconds.**

When $t = 0.8 \; seconds, \; v = 25(0.8) - 80 = -60 \; ft/sec$. The magnitude of the velocity is 60 ft/sec. (The negative sign in the answer means that the object is moving backwards.)

When $t = 5.6 \; seconds, \; v = 25(5.6) - 80 = 60 \; ft/sec$.

To find where the magnitude of the velocity is **greater** than 60 ft/sec, check some arbitrary values in each of the following time intervals: $t \leq 0.8, \; 0.8 \leq t \leq 5.6$ and $t \geq 5.6$.

Check $t = 0.5: \; v = 25(0.5) - 80 = -67.5 \; ft/sec$

Check $t = 2: \; v = 25(2) - 80 = -30 \; ft/sec$

Check $t = 6: \; v = 25(6) - 80 = -70 \; ft/sec$

You can see that the magnitude of the velocity is greater than 60 ft/sec only when $t \geq 5.6$ or when $t \leq 0.8$.

The answer checks out.

Watch this video for help with the Examples above.

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CK-12 Foundation: Absolute ValueInequalities

Vocabulary

- The absolute value of a number is its distance from zero on a number line.
- $|x| = x$ if x is not negative, and $|x| = -x$ if x is negative.
- An equation or inequality with an absolute value in it **splits into two equations,** one where the expression inside the absolute value sign is positive and one where it is negative. When the expression within the absolute value is **positive**, then the absolute value signs do nothing and can be omitted. When the expression within the absolute value is **negative,** then the expression within the absolute value signs must be negated before removing the signs.
- Inequalities of the type $|x| < a$ can be rewritten as "$-a < x < a$."
- Inequalities of the type $|x| > b$ can be rewritten as "$x < -b$ or $x > b$."

Guided Practice

Solve the inequality $|8x - 15| \geq 9$ *and show the solution graph.*

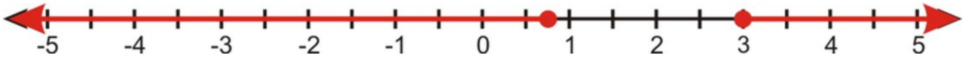
Solution

Re-write as a compound inequality: $8x - 15 \leq -9$ or $8x - 15 \geq 9$

Solve each inequality: $8x \leq 6$ or $8x \geq 24$

$x \leq \frac{3}{4}$ or $x \geq 3$

The solution graph is



Practice

Solve the following inequalities and show the solution graph.

1. $|x| \leq 6$
2. $|x| > 3.5$

3. $|x| < 12$
4. $|x| > 10$
5. $|7x| \geq 21$
6. $|x-5| > 8$
7. $|x+7| < 3$
8. $\left|x-\frac{3}{4}\right| \leq \frac{1}{2}$
9. $|2x-5| \geq 13$
10. $|5x+3| < 7$
11. $\left|\frac{x}{3}-4\right| \leq 2$
12. $\left|\frac{2x}{7}+9\right| > \frac{5}{7}$
    1. How many solutions does the inequality $|x| \leq 0$ have?
    2. How about the inequality $|x| \geq 0$?
13. A company manufactures rulers. Their 12-inch rulers pass quality control if they are within $\frac{1}{32}$ *inches* of the ideal length. What is the longest and shortest ruler that can leave the factory?
14. A three month old baby boy weighs an average of 13 pounds. He is considered healthy if he is at most 2.5 lbs. more or less than the average weight. Find the weight range that is considered healthy for three month old boys.

6.11 Graphs of Inequalities in One Variable

Here you'll learn how to graph linear inequalities in one variable on the coordinate plane. You'll also learn how to find their solution plane.

What if you were given a linear inequality like $|y| \geq -5$? How could you graph that inequality in the coordinate plane? After completing this Concept, you'll be able to graph linear inequalities in one variable like this one.

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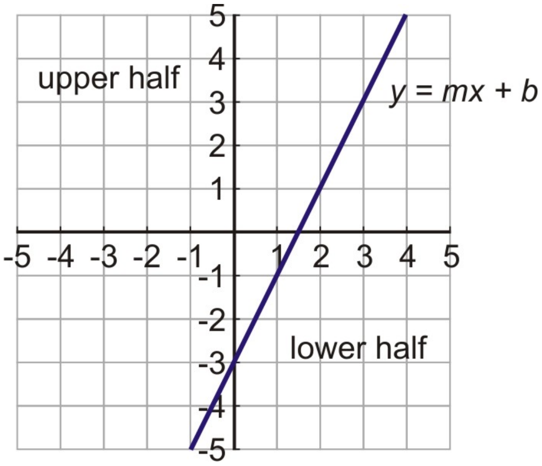
CK-12 Foundation: 0611S GraphingLinear Inequalitiesin the Coordinate Plane(H264)

Guidance

A **linear inequality** in two variables takes the form $y > mx + b$ or $y < mx + b$. Linear inequalities are closely related to graphs of straight lines; recall that a straight line has the equation $y = mx + b$.

When we graph a line in the coordinate plane, we can see that it divides the plane in half:



The solution to a linear inequality includes all the points in one half of the plane. We can tell which half by looking at the inequality sign:

$>$The solution set is the half plane above the line.

$\geq$ The solution set is the half plane above the line and also all the points on the line.

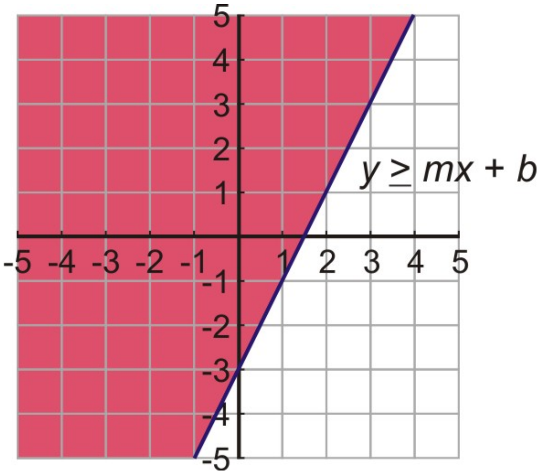
$<$The solution set is the half plane below the line.

$\leq$ The solution set is the half plane below the line and also all the points on the line.

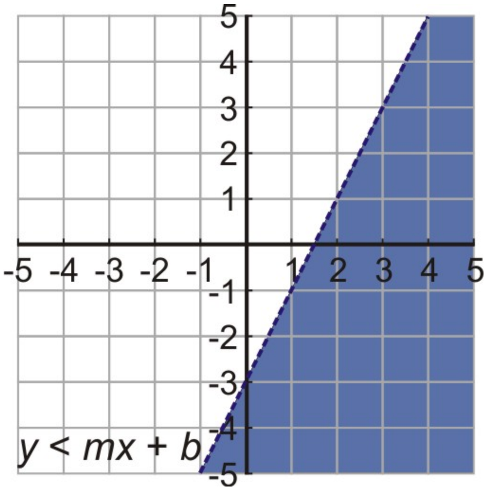
For a strict inequality, we draw a **dashed line** to show that the points in the line *are not* part of the solution. For an inequality that includes the equals sign, we draw a **solid line** to show that the points on the line *are* part of the solution.

Example A

This is a graph of $y \geq mx+b$; the solution set is the line and the half plane above the line.



This is a graph of $y < mx+b$; the solution set is the half plane above the line, not including the line itself.



Graph Linear Inequalities in One Variable in the Coordinate Plane

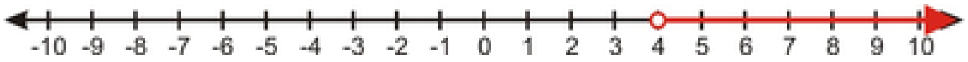
In the last few sections we graphed inequalities in one variable on the number line. We can also graph inequalities in one variable on the coordinate plane. We just need to remember that when we graph an equation of the type $x = a$ we get a vertical line, and when we graph an equation of the type $y = b$ we get a horizontal line.

Example B

Graph the inequality $x > 4$ *on the coordinate plane.*

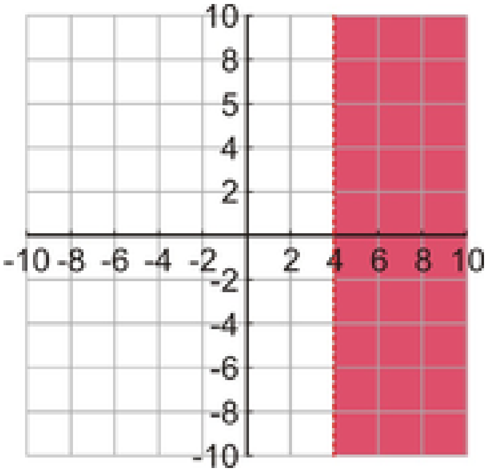
Solution

First let's remember what the solution to $x > 4$ looks like on the number line.



The solution to this inequality is the set of all real numbers x that are bigger than 4, not including 4. The solution is represented by a line.

In two dimensions, the solution still consists of all the points to the right of $x = 4$, but for all possible $y-$values as well. This solution is represented by the half plane to the right of $x = 4$. (You can think of it as being like the solution graphed on the number line, only stretched out vertically.)



The line $x = 4$ is dashed because the equals sign is not included in the inequality, meaning that points on the line are not included in the solution.

Example C

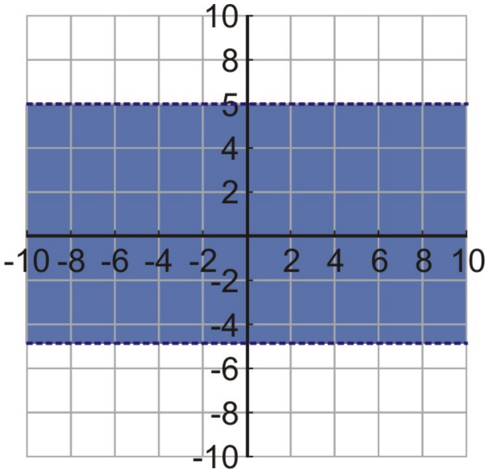
Graph the inequality $|y| < 5$

Solution

The absolute value inequality $|y| < 5$ can be re-written as $-5 < y < 5$. This is a compound inequality which can be expressed as

$$y > -5 \quad \text{and} \quad y < 5$$

In other words, the solution is all the coordinate points for which the value of y is larger than -5 **and** smaller than 5. The solution is represented by the plane between the horizontal lines $y = -5$ and $y = 5$.



Both horizontal lines are dashed because points on the lines are not included in the solution.

Watch this video for help with the Examples above.

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CK-12 Foundation: Graphing Inequalities in the Coordinate Plane

Vocabulary

- Inequalities of the type $|x| < a$ can be rewritten as "$-a < x < a$."
- Inequalities of the type $|x| > b$ can be rewritten as "$x < -b$ or $x > b$."
- **Horizontal lines** are defined by the equation $y =$ constant and **vertical lines** are defined by the equation $x =$ constant.
- For a strict inequality, we draw a **dashed line** to show that the points in the line *are not* part of the solution. For an inequality that includes the equals sign, we draw a **solid line** to show that the points on the line *are* part of the solution.

- The solution to a linear inequality includes all the points in one half of the plane. We can tell which half by looking at the inequality sign:

>The solution set is the half plane above the line.

$\geq$ The solution set is the half plane above the line and also all the points on the line.

<The solution set is the half plane below the line.

$\leq$ The solution set is the half plane below the line and also all the points on the line.

Guided Practice

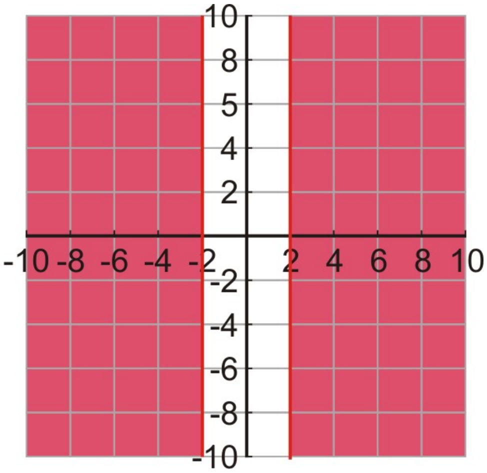
Graph the inequality $|x| \geq 2$.

Solution:

The absolute value inequality $|x| \geq 2$ can be re-written as a compound inequality:

$$x \leq -2 \quad \text{or} \quad x \geq 2$$

In other words, the solution is all the coordinate points for which the value of x is smaller than or equal to -2 **or** greater than or equal to 2. The solution is represented by the plane to the left of the vertical line $x = -2$ and the plane to the right of line $x = 2$.



Both vertical lines are solid because points on the lines are included in the solution.

Practice

Graph the following inequalities on the coordinate plane.

1. $x < 20$
2. $y \geq -5$
3. $x > 0.5$
4. $x \leq \frac{1}{2}$
5. $y > -\frac{2}{3}$
6. $y < -0.2$
7. $|x| > 10$
8. $|y| \leq 7$
9. $|y| < \frac{1}{3}$
10. $|x| \geq -10$

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9616.

6.12 Linear Inequalities in Two Variables

Here you'll learn how to graph linear inequalities in two variables of the form $y > mx+b$ or $y < mx+b$. You'll also solve real-world problems involving such inequalities.

What if you were given a linear inequality like $2x-3y \leq 5$? How could you graph that inequality in the coordinate plane? After completing this Concept, you'll be able to graph linear inequalities in two variables like this one.

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CK-12 Foundation: 0612S Linear Inequalities in Two Variables(H264)

Guidance

The general procedure for graphing inequalities in two variables is as follows:

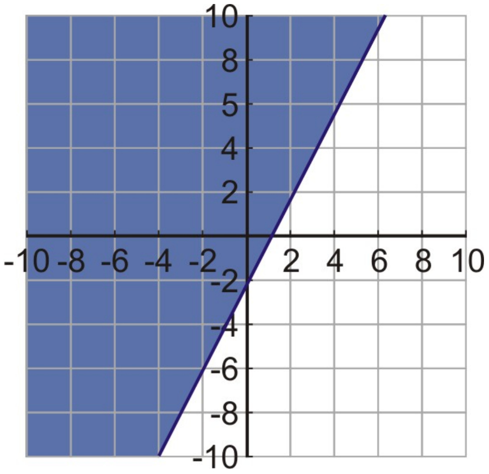
1. Re-write the inequality in slope-intercept form: $y = mx+b$. Writing the inequality in this form lets you know the direction of the inequality.
2. Graph the line of the equation $y = mx+b$ using your favorite method (plotting two points, using slope and $y-$intercept, using $y-$intercept and another point, or whatever is easiest). Draw the line as a dashed line if the equals sign is not included and a solid line if the equals sign is included.
3. Shade the half plane above the line if the inequality is "greater than." Shade the half plane under the line if the inequality is "less than."

Example A

Graph the inequality $y \geq 2x-3$.

Solution

The inequality is already written in slope-intercept form, so it's easy to graph. First we graph the line $y = 2x-3$; then we shade the half-plane above the line. The line is solid because the inequality includes the equals sign.



Example B

Graph the inequality $5x - 2y > 4$.

Solution

First we need to rewrite the inequality in slope-intercept form:

$$-2y > -5x + 4$$
$$y < \frac{5}{2}x - 2$$

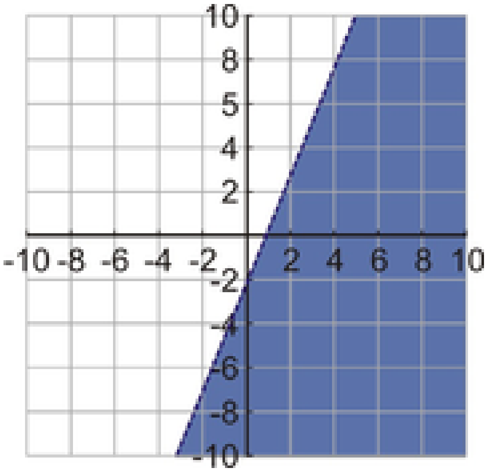
Notice that the inequality sign changed direction because we divided by a negative number.

To graph the equation, we can make a table of values:

TABLE 6.5:

x	y
-2	$\frac{5}{2}(-2) - 2 = -7$
0	$\frac{5}{2}(0) - 2 = -2$
2	$\frac{5}{2}(2) - 2 = 3$

After graphing the line, we shade the plane **below** the line because the inequality in slope-intercept form is **less than**. The line is dashed because the inequality does not include an equals sign.



Solve Real-World Problems Using Linear Inequalities

In this section, we see how linear inequalities can be used to solve real-world applications.

Example C

A retailer sells two types of coffee beans. One type costs \$9 per pound and the other type costs \$7 per pound. Find all the possible amounts of the two different coffee beans that can be mixed together to get a quantity of coffee beans costing \$8.50 or less.

Solution

Let $x =$ weight of \$9 per pound coffee beans in pounds.

Let $y =$ weight of \$7 per pound coffee beans in pounds.

The cost of a pound of coffee blend is given by $9x+7y$.

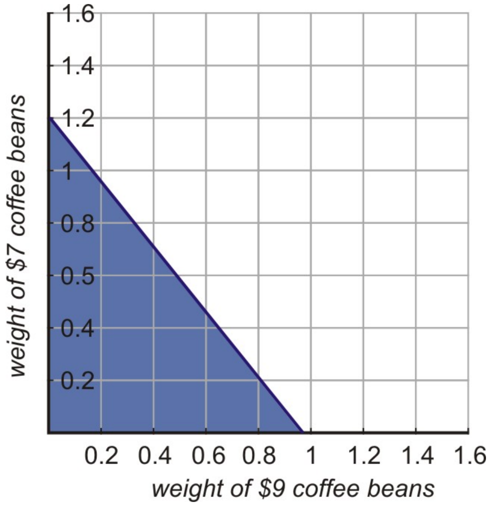
We are looking for the mixtures that cost \$8.50 or less. We write the inequality $9x+7y \leq 8.50$.

Since this inequality is in standard form, it's easiest to graph it by finding the $x-$ and $y-$intercepts. When $x = 0$, we have $7y = 8.50$ or $y = \frac{8.50}{7} \approx 1.21$. When $y = 0$, we have $9x = 8.50$ or $x = \frac{8.50}{9} \approx 0.94$. We can then graph the line that includes those two points.

Now we have to figure out which side of the line to shade. In $y-$intercept form, we shade the area **below** the line when the inequality is "less than." But in standard form that's not always true. We could convert the inequality to $y-$intercept form to find out which side to shade, but there is another way that can be easier.

The other method, which works for any linear inequality in any form, is to plug a random point into the inequality and see if it makes the inequality true. Any point that's not on the line will do; the point (0, 0) is usually the most convenient.

In this case, plugging in 0 for x and y would give us $9(0)+7(0) \leq 8.50$, which is true. That means we should shade the half of the plane that includes (0, 0). If plugging in (0, 0) gave us a false inequality, that would mean that the solution set is the part of the plane that does *not* contain (0, 0).



Notice also that in this graph we show only the first quadrant of the coordinate plane. That's because weight values in the real world are always nonnegative, so points outside the first quadrant don't represent real-world solutions to this problem.

Watch this video for help with the Examples above.

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CK-12 Foundation: LinearInequalities inTwoVariables

Vocabulary

- For a strict inequality, we draw a **dashed line** to show that the points in the line *are not* part of the solution. For an inequality that includes the equals sign, we draw a **solid line** to show that the points on the line *are* part of the solution.
- The solution to a linear inequality includes all the points in one half of the plane. We can tell which half by looking at the inequality sign:

$>$The solution set is the half plane above the line.

$\geq$ The solution set is the half plane above the line and also all the points on the line.

$<$The solution set is the half plane below the line.

$\leq$ The solution set is the half plane below the line and also all the points on the line.

Guided Practice

Julius has a job as an appliance salesman. He earns a commission of $60 for each washing machine he sells and $130 for each refrigerator he sells. How many washing machines and refrigerators must Julius sell in order to make $1000 or more in commissions?

Solution

Let $x =$ number of washing machines Julius sells.

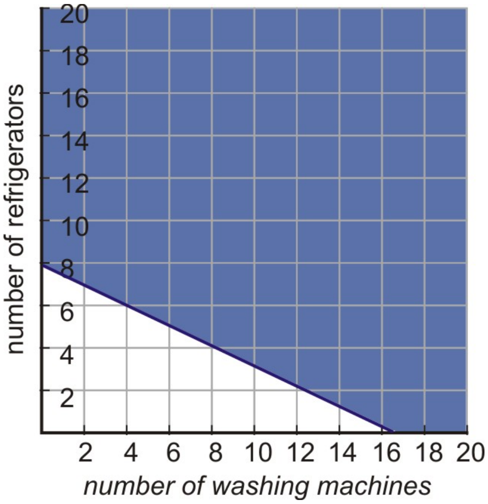
Let $y =$ number of refrigerators Julius sells.

The total commission is $60x + 130y$.

We're looking for a total commission of $1000 or more, so we write the inequality $60x + 130y \geq 1000$.

Once again, we can do this most easily by finding the $x-$ and $y-$intercepts. When $x = 0$, we have $130y = 1000$, or $y = \frac{1000}{30} \approx 7.69$. When $y = 0$, we have $60x = 1000$, or $x = \frac{1000}{60} \approx 16.67$.

We draw a solid line connecting those points, and shade above the line because the inequality is "greater than." We can check this by plugging in the point (0, 0): selling 0 washing machines and 0 refrigerators would give Julius a commission of $0, which is *not* greater than or equal to $1000, so the point (0, 0) is *not* part of the solution; instead, we want to shade the side of the line that does *not* include it.



Notice also that we show only the first quadrant of the coordinate plane, because Julius's commission should be non-negative.

Practice

Graph the following inequalities on the coordinate plane.

1. $y \leq 4x + 3$
2. $y > -\frac{x}{2} - 6$
3. $3x - 4y \geq 12$
4. $x + 7y < 5$
5. $6x + 5y > 1$
6. $y + 5 \leq -4x + 10$
7. $x - \frac{1}{2}y \geq 5$
8. $6x + y < 20$
9. $30x + 5y < 100$
10. Remember what you learned in the last chapter about families of lines.
    a. What do the graphs of $y > x + 2$ and $y < x + 5$ have in common?
    b. What do you think the graph of $x + 2 < y < x + 5$ would look like?
11. How would the answer to problem 6 change if you subtracted 2 from the right-hand side of the inequality?

12. How would the answer to problem 7 change if you added 12 to the right-hand side?
13. How would the answer to problem 8 change if you flipped the inequality sign?
14. A phone company charges 50 cents per minute during the daytime and 10 cents per minute at night. How many daytime minutes and nighttime minutes could you use in one week if you wanted to pay less than $20?
15. Suppose you are graphing the inequality $y > 5x$.
    a. Why can't you plug in the point (0, 0) to tell you which side of the line to shade?
    b. What happens if you do plug it in?
    c. Try plugging in the point (0, 1) instead. Now which side of the line should you shade?
16. A theater wants to take in at least $2000 for a certain matinee. Children's tickets cost $5 each and adult tickets cost $10 each.
    a. If x represents the number of adult tickets sold and y represents the number of children's tickets, write an inequality describing the number of tickets that will allow the theater to meet their minimum take.
    b. If 100 children's tickets and 100 adult tickets have already been sold, what inequality describes how many *more* tickets of both types the theater needs to sell?
    c. If the theater has only 300 seats (so only 100 are still available), what inequality describes the *maximum* number of additional tickets of both types the theater can sell?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9616.

Summary

To help you visualize inequalities, this chapter begins with graphing them on a number line. It then moves into solving inequalities that entail one and multiple operation(s) to isolate the variable. Next, writing, graphing, and solving compound inequalities are introduced. From there, the chapter transitions into absolute value equations and inequalities, with special attention paid to analysis and graphing of them. It concludes with graphing of linear inequalities in two variables and solving real-world problems using linear inequalities.

CHAPTER 7

Solving Systems of Equations and Inequalities

Chapter Outline

- 7.1 GRAPHS OF LINEAR SYSTEMS
- 7.2 SYSTEMS USING SUBSTITUTION
- 7.3 MIXTURE PROBLEMS
- 7.4 LINEAR SYSTEMS WITH ADDITION OR SUBTRACTION
- 7.5 LINEAR SYSTEMS WITH MULTIPLICATION
- 7.6 COMPARING METHODS FOR SOLVING LINEAR SYSTEMS
- 7.7 CONSISTENT AND INCONSISTENT LINEAR SYSTEMS
- 7.8 DETERMINING THE TYPE OF LINEAR SYSTEM
- 7.9 APPLICATIONS OF LINEAR SYSTEMS
- 7.10 SYSTEMS OF LINEAR INEQUALITIES
- 7.11 LINEAR PROGRAMMING

Introduction

Up until this point, you've solved single equations and inequalities for one unknown, but what if you had more than one equation or inequality with the same but multiple unknowns? For example, what if you wanted to find the number of paid adults AND the number of paid children at a movie premiere? How could you find the value of both variables? This chapter will teach you how to do so by introducing you to systems of equations and inequalities.

A system is nothing more than a set of equations or inequalities with the same variables. To solve such systems, you can use various methods including graphing, substitution, and elimination.

7.1 Graphs of Linear Systems

Here you'll learn how to determine whether an ordered pair is a solution to a system of equations. You'll also learn how to solve a system of equations by graphing. Finally, you'll solve word problems involving systems of equations.

What if you were given a set of linear equations like $y = -3x + 4$ and $y = 6x - 1$? How could you determine the solution(s) both equations have in common? After completing this Concept, you'll be able to determine if an ordered pair is a solution to a system of equations and you'll find such solutions by graphing.

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CK-12 Foundation: 0701S Linear Systems by Graphing (H264)

Guidance

In this Concept, first we'll discover methods to determine if an ordered pair is a solution to a system of two equations. Then we'll learn to solve the two equations graphically and confirm that the solution is the point where the two lines intersect. Finally, we'll look at real-world problems that can be solved using the methods described in this chapter.

Determine Whether an Ordered Pair is a Solution to a System of Equations

A linear system of equations is a set of equations that must be solved together to find the one solution that fits them both.

Consider this system of equations:

$$
\begin{aligned}
y &= x + 2\\
y &= -2x + 1
\end{aligned}
$$

Since the two lines are in a system, we deal with them together by graphing them on the same coordinate axes. We can use any method to graph them; let's do it by making a table of values for each line.

Line 1: $y = x + 2$

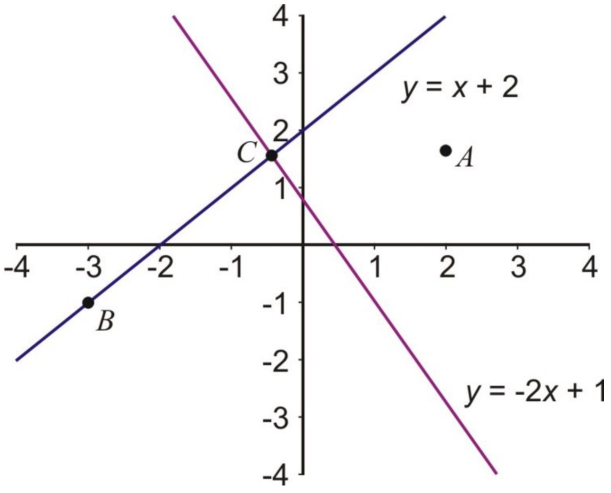
TABLE 7.1:

x	y
0	2
1	3

Line 2: $y = -2x + 1$

TABLE 7.2:

x	y
0	1
1	-1



We already know that any point that lies on a line is a solution to the equation for that line. That means that any point that lies on *both* lines in a system is a solution to both equations.

So in this system:

- Point A is not a solution to the system because it does not lie on either of the lines.
- Point B is not a solution to the system because it lies only on the blue line but not on the red line.
- Point C is a solution to the system because it lies on both lines at the same time.

In fact, point C is the only solution to the system, because it is the only point that lies on both lines. For a system of equations, the geometrical solution is the intersection of the two lines in the system. The algebraic solution is the ordered pair that solves both equations—in other words, the coordinates of that intersection point.

You can confirm the solution by plugging it into the system of equations, and checking that the solution works in each equation.

Example A

Determine which of the points (1, 3), (0, 2), or (2, 7) is a solution to the following system of equations:

$$\begin{aligned} y &= 4x - 1 \\ y &= 2x + 3 \end{aligned}$$

Solution

To check if a coordinate point is a solution to the system of equations, we plug each of the x and y values into the equations to see if they work.

Point (1, 3):

$$\begin{aligned} y &= 4x-1 \\ 3 &\overset{?}{=} 4(1)-1 \\ 3 &= 3 \text{ solution checks} \\ \\ y &= 2x+3 \\ 3 &\overset{?}{=} 2(1)+3 \\ 3 &\neq 5 \text{ solution does not check} \end{aligned}$$

Point (1, 3) is on the line $y = 4x - 1$, but it is not on the line $y = 2x + 3$, so it is not a solution to the system.

Point (0, 2):

$$\begin{aligned} y &= 4x-1 \\ 2 &\overset{?}{=} 4(0)-1 \\ 2 &\neq -1 \text{ solution does not check} \end{aligned}$$

Point (0, 2) is not on the line $y = 4x - 1$, so it is not a solution to the system. Note that it is not necessary to check the second equation because the point needs to be on both lines for it to be a solution to the system.

Point (2, 7):

$$\begin{aligned} y &= 4x-1 \\ 7 &\overset{?}{=} 4(2)-1 \\ 7 &= 7 \text{ solution checks} \\ \\ y &= 2x+3 \\ 7 &\overset{?}{=} 2(2)+3 \\ 7 &= 7 \text{ solution checks} \end{aligned}$$

Point (2, 7) is a solution to the system since it lies on both lines.

The solution to the system is the point (2, 7).

Determine the Solution to a Linear System by Graphing

The solution to a linear system of equations is the point, (if there is one) that lies on both lines. In other words, the solution is the point where the two lines intersect.

We can solve a system of equations by graphing the lines on the same coordinate plane and reading the intersection point from the graph.

This method most often offers only approximate solutions, so it's not sufficient when you need an exact answer. However, graphing the system of equations can be a good way to get a sense of what's really going on in the problem you're trying to solve, especially when it's a real-world problem.

Example B

Solve the following system of equations by graphing:

$$y = 3x - 5$$
$$y = -2x + 5$$

Solution

Graph both lines on the same coordinate axis using any method you like.

In this case, let's make a table of values for each line.

Line 1: $y = 3x - 5$

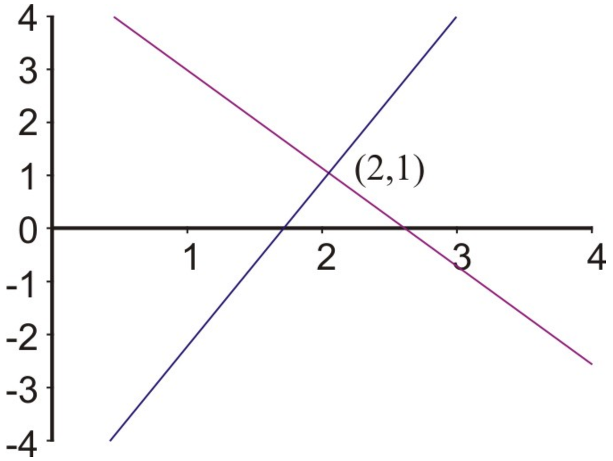
TABLE 7.3:

x	y
1	-2
2	1

Line 2: $y = -2x + 5$

TABLE 7.4:

x	y
1	3
2	1



The solution to the system is given by the intersection point of the two lines. The graph shows that the lines intersect at point (2, 1). So **the solution is** $x = 2, y = 1$**or (2, 1).**

Solving a System of Equations Using a Graphing Calculator

As an alternative to graphing by hand, you can use a graphing calculator to find or check solutions to a system of equations.

Example C

Solve the following system of equations using a graphing calculator.

$$\begin{aligned} x-3y&=4\\ 2x+5y&=8 \end{aligned}$$

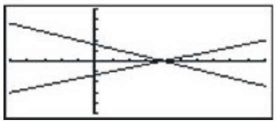
To input the equations into the calculator, you need to rewrite them in slope-intercept form (that is, $y = mx+b$ form).

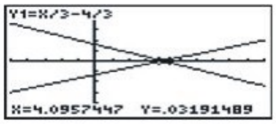
$$\begin{aligned} x-3y&=4 & y&=\frac{1}{3}x-\frac{4}{3}\\ & & &\Rightarrow\\ 2x+5y&=8 & y&=-\frac{2}{5}x+\frac{8}{5} \end{aligned}$$

Press the **[y=]** button on the graphing calculator and enter the two functions as:

$$\begin{aligned} Y_1&=\frac{x}{3}-\frac{4}{3}\\ Y_2&=\frac{-2x}{5}+\frac{8}{5} \end{aligned}$$

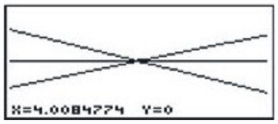
Now press **[GRAPH]**. Here's what the graph should look like on a TI-83 family graphing calculator with the window set to $-5 \le x \le 10$ and $-5 \le y \le 5$.





Y1=X/3-4/3

X=4.0957447 Y=.03191489



X=4.0084774 Y=0

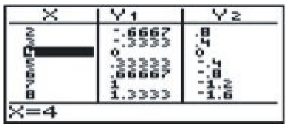
There are a few different ways to find the intersection point.

Option 1: Use **[TRACE]** and move the cursor with the arrows until it is on top of the intersection point. The values of the coordinate point will be shown on the bottom of the screen. The second screen above shows the values to be $X = 4.0957447$ and $Y = 0.03191489$.

Use the **[ZOOM]** function to zoom into the intersection point and find a more accurate result. The third screen above shows the system of equations after zooming in several times. A more accurate solution appears to be $X = 4$ and $Y = 0$.

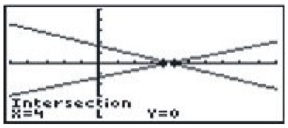
Option 2 Look at the table of values by pressing **[2nd] [GRAPH]**. The first screen below shows a table of values for this system of equations. Scroll down until the $Y-$values for the two functions are the same. In this case this occurs at $X = 4$ and $Y = 0$.

(Use the **[TBLSET]** function to change the starting value for your table of values so that it is close to the intersection point and you don't have to scroll too long. You can also improve the accuracy of the solution by setting the value of Δ Table smaller.)



X	Y1	Y2
2	-.6667	.8
3	-.3333	.4
4	0	0
5	.33333	-.4
6	.66667	-.8
7	1	-1.2
8	1.3333	-1.6

X=4



Intersection

X=4 Y=0

Option 3 Using the **[2nd] [TRACE]** function gives the second screen shown above.

Scroll down and select "intersect."

The calculator will display the graph with the question **[FIRSTCURVE]**? Move the cursor along the first curve until it is close to the intersection and press **[ENTER]**.

The calculator now shows **[SECONDCURVE]**?

Move the cursor to the second line (if necessary) and press **[ENTER]**.

The calculator displays **[GUESS]**?

Press **[ENTER]** and the calculator displays the solution at the bottom of the screen (see the third screen above).

The point of intersection is $X = 4$ and $Y = 0$. Note that with this method, the calculator works out the intersection point for you, which is generally more accurate than your own visual estimate.

Solve Real-World Problems Using Graphs of Linear Systems

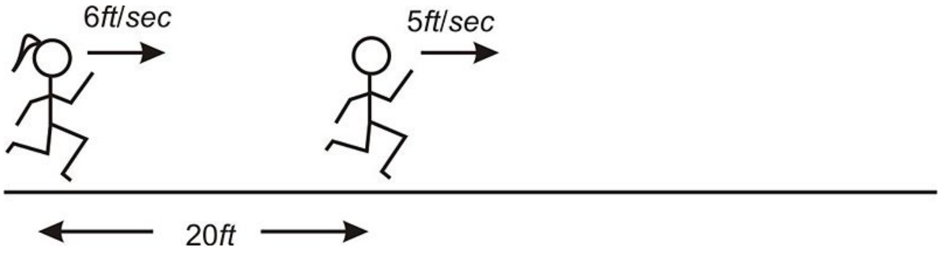
Consider the following problem:

Example D

Peter and Nadia like to race each other. Peter can run at a speed of 5 feet per second and Nadia can run at a speed of 6 feet per second. To be a good sport, Nadia likes to give Peter a head start of 20 feet. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?

Solution:

Let's start by drawing a sketch. Here's what the race looks like when Nadia starts running; we'll call this time $t = 0$.



Now let's define two variables in this problem:

$t =$ the time from when Nadia starts running

$d =$ the distance of the runners from the starting point.

Since there are two runners, we need to write equations for each of them. That will be the *system of equations* for this problem.

For each equation, we use the formula: distance = speed × time

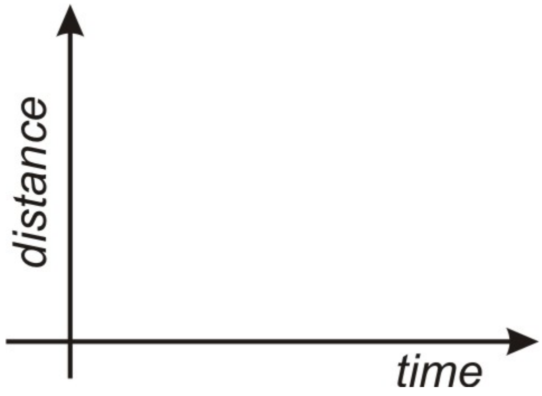
Nadia's equation: $d = 6t$

Peter's equation: $d = 5t + 20$

(Remember that Peter was already 20 feet from the starting point when Nadia started running.)

Let's graph these two equations on the same coordinate axes.

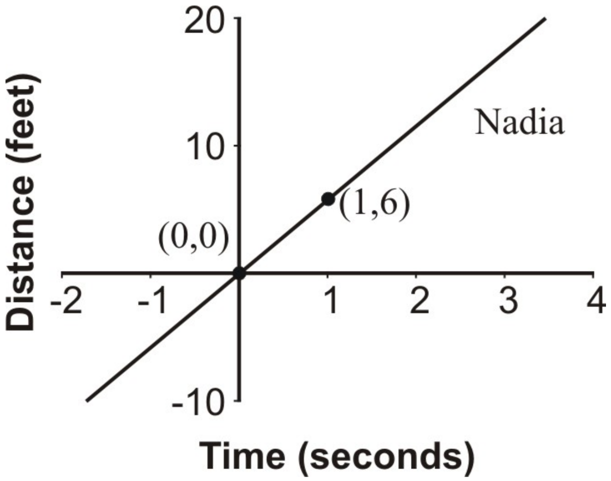
Time should be on the horizontal axis since it is the independent variable. Distance should be on the vertical axis since it is the dependent variable.



We can use any method for graphing the lines, but in this case we'll use the *slope–intercept* method since it makes more sense physically.

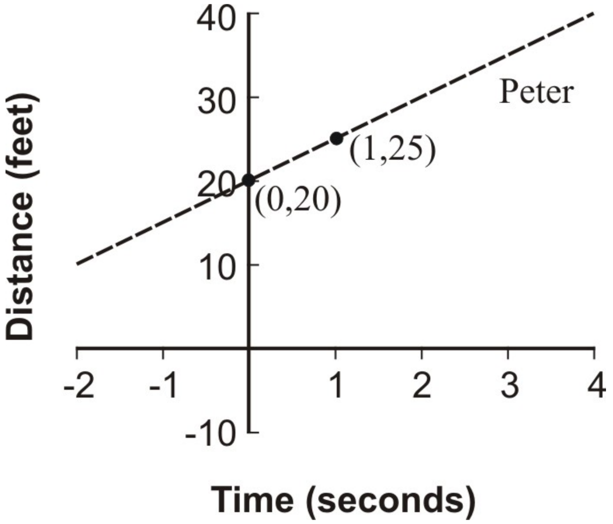
To graph the line that describes Nadia's run, start by graphing the $y-$intercept: (0, 0). (If you don't see that this is the $y-$intercept, try plugging in the test-value of $x = 0$.)

The slope tells us that Nadia runs 6 feet every one second, so another point on the line is (1, 6). Connecting these points gives us Nadia's line:

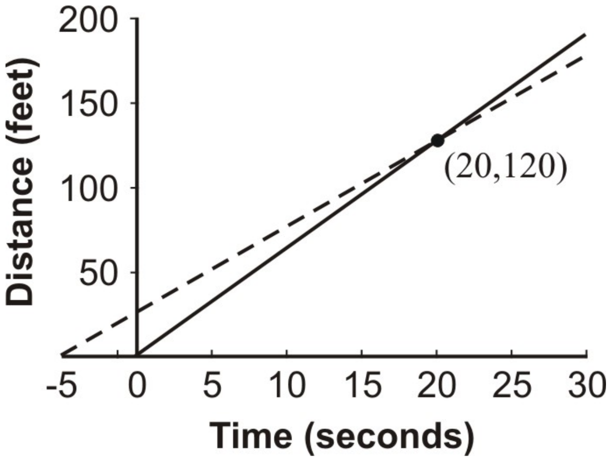


To graph the line that describes Peter's run, again start with the $y-$intercept. In this case this is the point (0, 20).

The slope tells us that Peter runs 5 feet every one second, so another point on the line is (1, 25). Connecting these points gives us Peter's line:



In order to find when and where Nadia and Peter meet, we'll graph both lines on the same graph and extend the lines until they cross. The crossing point is the solution to this problem.



The graph shows that Nadia and Peter meet **20 seconds after Nadia starts running, and 120 feet from the starting point.**

These examples are great at demonstrating that the solution to a system of linear equations means the point at which the lines intersect. This is, in fact, the greatest strength of the graphing method because it offers a very visual representation of system of equations and its solution. You can also see, though, that finding the solution from a graph requires very careful graphing of the lines, and is really only practical when you're sure that the solution gives integer values for x and y. Usually, this method can only offer approximate solutions to systems of equations, so we need to use other methods to get an exact solution.

Watch this video for help with the Examples above.

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CK-12 Foundation: LinearSystemsby Graphing

Vocabulary

- A **linear system of equations** is a set of equations that must be solved together to find the one solution that fits them both.

Guided Practice

Solve the following system of equations by graphing:

$$\begin{aligned} 2x+3y&=6 \\ 4x-y&=-2 \end{aligned}$$

Solution

Since the equations are in standard form, this time we'll graph them by finding the $x-$ and $y-$intercepts of each of the lines.

Line 1: $2x+3y=6$

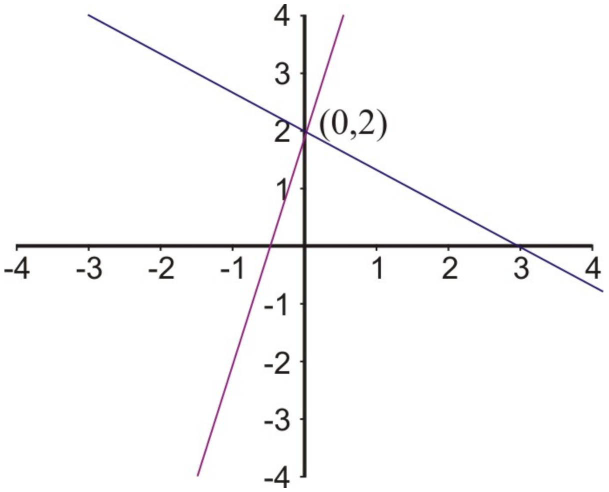
$x-$intercept: set $y=0 \Rightarrow 2x=6 \Rightarrow x=3$ so the intercept is (3, 0)

$y-$intercept: set $x=0 \Rightarrow 3y=6 \Rightarrow y=2$ so the intercept is (0, 2)

Line 2: $-4x+y=2$

$x-$intercept: set $y=0 \Rightarrow -4x=2 \Rightarrow x=-\frac{1}{2}$ so the intercept is $\left(-\frac{1}{2},0\right)$

$y-$intercept: set $x=0 \Rightarrow y=2$ so the intercept is (0, 2)



The graph shows that the lines intersect at (0, 2). Therefore, **the solution to the system of equations is** $x=0, y=2$.

Practice

Determine which ordered pair satisfies the system of linear equations.

1.

$$\begin{aligned} y&=3x-2 \\ y&=-x \end{aligned}$$

   a. (1, 4)
   b. (2, 9)
   c.

$$\left(\frac{1}{2},\frac{-1}{2}\right)$$

2.

$$y = 2x - 3$$
$$y = x + 5$$

   a. (8, 13)
   b. (-7, 6)
   c. (0, 4)

3.

$$2x + y = 8$$
$$5x + 2y = 10$$

   a. (-9, 1)
   b. (-6, 20)
   c. (14, 2)

4.

$$3x + 2y = 6$$
$$y = \frac{1}{2}x - 3$$

   a. $\left(3, \frac{-3}{2}\right)$
   b. (-4, 3)
   c.

$$\left(\frac{1}{2},4\right)$$

5.

$$2x - y = 10$$
$$3x + y = -5$$

   a. (4, -2)
   b. (1, -8)
   c. (-2, 5)

Solve the following systems using the graphing method.

6.

$$y = x + 3$$
$$y = -x + 3$$

7.

$$y = 3x - 6$$
$$y = -x + 6$$

8.

$$\begin{aligned} 2x &= 4 \\ y &= -3 \end{aligned}$$

9.

$$\begin{aligned} y &= -x+5 \\ -x+y &= 1 \end{aligned}$$

10.

$$\begin{aligned} x+2y &= 8 \\ 5x+2y &= 0 \end{aligned}$$

11.

$$\begin{aligned} 3x+2y &= 12 \\ 4x-y &= 5 \end{aligned}$$

12.

$$\begin{aligned} 5x+2y &= -4 \\ x-y &= 2 \end{aligned}$$

13.

$$\begin{aligned} 2x+4 &= 3y \\ x-2y+4 &= 0 \end{aligned}$$

14.

$$\begin{aligned} y &= \frac{1}{2}x-3 \\ 2x-5y &= 5 \end{aligned}$$

15.

$$\begin{aligned} y &= 4 \\ x &= 8-3y \end{aligned}$$

16. Try to solve the following system using the graphing method:

$$\begin{aligned} y &= \frac{3}{5}x+5 \\ y &= -2x-\frac{1}{2} \end{aligned}$$

.

a. What does it look like the $x-$coordinate of the solution should be?
b. Does that coordinate really give the same $y-$value when you plug it into both equations?
c. Why is it difficult to find the real solution to this system?

17. Try to solve the following system using the graphing method:

$$y = 4x + 8$$
$$y = 5x + 1$$

. Use a grid with $x-$values and $y-$values ranging from -10 to 10.

    a. Do these lines appear to intersect?
    b. Based on their equations, are they parallel?
    c. What would we have to do to find their intersection point?

18. Try to solve the following system using the graphing method:

$$y = \frac{1}{2}x + 4$$
$$y = \frac{4}{9}x + \frac{9}{2}$$

. Use the same grid as before.

    a. Can you tell exactly where the lines cross?
    b. What would we have to do to make it clearer?

Solve the following problems by using the graphing method.

19. Mary's car has broken down and it will cost her \$1200 to get it fixed—or, for \$4500, she can buy a new, more efficient car instead. Her present car uses about \$2000 worth of gas per year, while gas for the new car would cost about \$1500 per year. After how many years would the total cost of fixing the car equal the total cost of replacing it?
20. A tortoise and hare decide to race 30 feet. The hare, being much faster, decides to give the tortoise a 20 foot head start. The tortoise runs at 0.5 feet/sec and the hare runs at 5.5 feet per second. How long until the hare catches the tortoise?

7.2 Systems Using Substitution

Here you'll learn how to use subtitution to solve systems of linear equations in two variables. You'll then solve real-world problems involving such systems.

What if you were given a system of linear equations like $x - y = 7$ and $3x - 4y = -3$? How could you substitute one equation into the other to solve for the variables? After completing this Concept, you'll be able to solve a system of linear equations by substitution.

Try This

For lots more practice solving linear systems, check out this web page: http://www.algebra.com/algebra/homework/coordinate/practice-linear-system.epl

After clicking to see the solution to a problem, you can click the back button and then click Try Another Practice Linear System to see another problem.

Guidance

In this lesson, we'll learn to solve a system of two equations using the method of substitution.

Solving Linear Systems Using Substitution of Variable Expressions

Let's look again at the problem about Peter and Nadia racing.

Peter and Nadia like to race each other. Peter can run at a speed of 5 feet per second and Nadia can run at a speed of 6 feet per second. To be a good sport, Nadia likes to give Peter a head start of 20 feet. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?

In that example we came up with two equations:

Nadia's equation: $d = 6t$

Peter's equation: $d = 5t + 20$

Each equation produced its own line on a graph, and to solve the system we found the point at which the lines intersected—the point where the values for d and t satisfied **both** relationships. When the values for d and t are equal, that means that Peter and Nadia are at the same place at the same time.

But there's a faster way than graphing to solve this system of equations. Since we want the value of d to be the same in both equations, we could just set the two right-hand sides of the equations equal to each other to solve for t. That is, if $d = 6t$ and $d = 5t + 20$, and the two d's are equal to each other, then by the transitive property we have $6t = 5t + 20$. We can solve this for t:

$$\begin{aligned}
6t &= 5t + 20 && subtract\ 5t\ from\ both\ sides:\\
t &= 20 && substitute\ this\ value\ for\ t\ into\ Nadia's\ equation:\\
d &= 6 \cdot 20 = 120
\end{aligned}$$

Even if the equations weren't so obvious, we could use simple algebraic manipulation to find an expression for one variable in terms of the other. If we rearrange Peter's equation to isolate t:

$$\begin{aligned} d &= 5t+20 && subtract\ 20\ from\ both\ sides: \\ d-20 &= 5t && divide\ by\ 5: \\ \frac{d-20}{5} &= t \end{aligned}$$

We can now ***substitute*** this expression for t into Nadia's equation $(d = 6t)$ to solve:

$$\begin{aligned} d &= 6\left(\frac{d-20}{5}\right) && multiply\ both\ sides\ by\ 5: \\ 5d &= 6(d-20) && distribute\ the\ 6: \\ 5d &= 6d-120 && subtract\ 6d\ from\ both\ sides: \\ -d &= -120 && divide\ by\ -1: \\ d &= 120 && substitute\ value\ for\ d\ into\ our\ expression\ for\ t: \\ t &= \frac{120-20}{5} = \frac{100}{5} = 20 \end{aligned}$$

So we find that Nadia and Peter meet 20 seconds after they start racing, at a distance of 120 feet away.

The method we just used is called the **Substitution Method.** In this lesson you'll learn several techniques for isolating variables in a system of equations, and for using those expressions to solve systems of equations that describe situations like this one.

Example A

Let's look at an example where the equations are written in **standard form.**

Solve the system

$$\begin{aligned} 2x+3y &= 6 \\ -4x+y &= 2 \end{aligned}$$

Again, we start by looking to isolate one variable in either equation. If you look at the second equation, you should see that the coefficient of y is 1. So the easiest way to start is to use this equation to solve for y.

Solve the second equation for y:

$$\begin{aligned} -4x+y &= 2 && add\ 4x\ to\ both\ sides: \\ y &= 2+4x \end{aligned}$$

Substitute this expression into the first equation:

$$\begin{aligned} 2x+3(2+4x) &= 6 && distribute\ the\ 3: \\ 2x+6+12x &= 6 && collect\ like\ terms: \\ 14x+6 &= 6 && subtract\ 6\ from\ both\ sides: \\ 14x &= 0 && and\ hence: \\ x &= 0 \end{aligned}$$

Substitute back into our expression for y:

$$y = 2 + 4 \cdot 0 = 2$$

As you can see, we end up with the same solution $(x = 0, y = 2)$ that we found when we graphed these functions back in Lesson 7.1. So long as you are careful with the algebra, the substitution method can be a very efficient way to solve systems.

Next, let's look at a more complicated example. Here, the values of x and y we end up with aren't whole numbers, so they would be difficult to read off a graph!

Example B

Solve the system

$$\begin{aligned} 2x + 3y &= 3 \\ 2x - 3y &= -1 \end{aligned}$$

Again, we start by looking to isolate one variable in either equation. In this case it doesn't matter which equation we use—all the variables look about equally easy to solve for.

So let's solve the first equation for x:

$$\begin{aligned} 2x + 3y &= 3 && \textit{subtract } 3y \textit{ from both sides}: \\ 2x &= 3 - 3y && \textit{divide both sides by } 2: \\ x &= \frac{1}{2}(3 - 3y) \end{aligned}$$

Substitute this expression into the second equation:

$$\begin{aligned} \cancel{2} \cdot \frac{1}{2}(3 - 3y) - 3y &= -1 && \textit{cancel the fraction and re-write terms}: \\ 3 - 3y - 3y &= -1 && \textit{collect like terms}: \\ 3 - 6y &= -1 && \textit{subtract } 3 \textit{ from both sides}: \\ -6y &= -4 && \textit{divide by } -6: \\ y &= \frac{2}{3} \end{aligned}$$

Substitute into the expression we got for x:

$$\begin{aligned} x &= \frac{1}{2}\left(3 - \cancel{3}\left(\frac{2}{\cancel{3}}\right)\right) \\ x &= \frac{1}{2} \end{aligned}$$

So our solution is $x = \frac{1}{2}, y = \frac{2}{3}$. You can see how the graphical solution $\left(\frac{1}{2}, \frac{2}{3}\right)$ might have been difficult to read accurately off a graph!

Solving Real-World Problems Using Linear Systems

Simultaneous equations can help us solve many real-world problems. We may be considering a purchase—for example, trying to decide whether it's cheaper to buy an item online where you pay shipping or at the store where you do not. Or you may wish to join a CD music club, but aren't sure if you would really save any money by buying a new CD every month in that way. Or you might be considering two different phone contracts. Let's look at an example of that now.

Example C

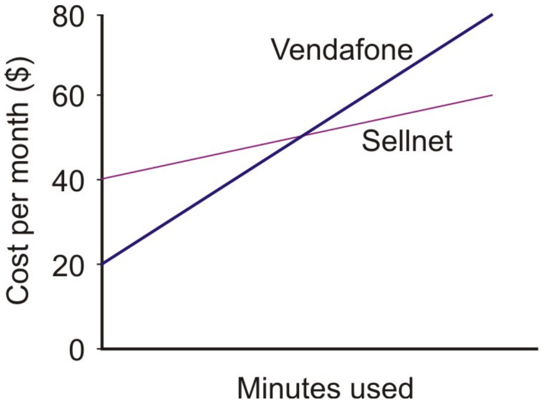
Anne is trying to choose between two phone plans. The first plan, with Vendafone, costs $20 per month, with calls costing an additional 25 cents per minute. The second company, Sellnet, charges $40 per month, but calls cost only 8 cents per minute. Which should she choose?

You should see that Anne's choice will depend upon how many minutes of calls she expects to use each month. We start by writing two equations for the cost in dollars in terms of the minutes used. Since the *number of minutes* is the independent variable, it will be our x. Cost is *dependent* on minutes – the *cost per month* is the dependent variable and will be assigned y.

For Vendafone: $y = 0.25x + 20$

For Sellnet: $y = 0.08x + 40$

By writing the equations in slope-intercept form $(y = mx + b)$, you can sketch a graph to visualize the situation:



The line for Vendafone has an intercept of 20 and a slope of 0.25. The Sellnet line has an intercept of 40 and a slope of 0.08 (which is roughly a third of the Vendafone line's slope). In order to help Anne decide which to choose, we'll find where the two lines cross, by solving the two equations as a system.

Since equation 1 gives us an expression for $y(0.25x + 20)$, we can substitute this expression directly into equation 2:

$$\begin{aligned}
0.25x + 20 &= 0.08x + 40 && subtract\ 20\ from\ both\ sides:\\
0.25x &= 0.08x + 20 && subtract\ 0.08x\ from\ both\ sides:\\
0.17x &= 20 && divide\ both\ sides\ by\ 0.17:\\
x &= 117.65\ minutes && rounded\ to\ 2\ decimal\ places.
\end{aligned}$$

So if Anne uses 117.65 minutes a month (although she can't really do *exactly* that, because phone plans only count whole numbers of minutes), the phone plans will cost the same. Now we need to look at the graph to see which plan is better if she uses more minutes than that, and which plan is better if she uses fewer. You can see that the Vendafone plan costs more when she uses more minutes, and the Sellnet plan costs more with fewer minutes.

So, **if Anne will use 117 minutes or less every month she should choose *Vendafone*. If she plans on using 118 or more minutes she should choose *Sellnet*.**

Watch this video for help with the Examples above.

Multimedia

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Click image to the left for more content.

CK-12 Foundation: LinearSystemsby Substitution

Vocabulary

- Solving linear systems **by substitution** means to solve for one variable in one equation, and then to substitute it into the other equation, solving for the other variable.

Guided Practice

Solve the system

$$\begin{aligned} 8x+10y &= 2 \\ 4x-15y &= -19 \end{aligned}$$

Solution:

Again, we start by looking to isolate one variable in either equation. In this case it doesn't matter which equation we use—all the variables look about equally easy to solve for.

So let's solve the first equation for x:

$$\begin{aligned} 8x+10y &= 2 && subtract\ 10y\ from\ both\ sides: \\ 8x &= 2-10y && divide\ both\ sides\ by\ 8: \\ x &= \frac{1}{8}(2-10y) \end{aligned}$$

Substitute this expression into the second equation:

$$\begin{aligned} 4\cdot\frac{1}{8}(2-10y)-15y &= -19 && simplify\ the\ fraction: \\ \frac{1}{2}(2-10y)-15y &= -19 && distribute\ the\ fraction\ and\ re-write\ terms: \\ 1-5y-15y &= -19 && collect\ like\ terms: \\ 1-20y &= -19 && subtract\ 1\ from\ both\ sides: \\ -20y &= -20 && divide\ by\ -20: \\ y &= 1 \end{aligned}$$

Substitute into the expression we got for x:

$$
\begin{aligned}
x &= \frac{1}{8}(2-10y) && \textit{Substitute the } y-\textit{value into the } x \textit{ equation}: \\
x &= \frac{1}{8}(2-10(1)) && \textit{Simplify}: \\
x &= \frac{1}{8}(2-10) \\
x &= \frac{1}{8}(-8) \\
x &= -1
\end{aligned}
$$

So our solution is $x=-1, y=1$.

Practice

1. Solve the system:

$$\begin{aligned} x+2y &= 9 \\ 3x+5y &= 20 \end{aligned}$$

2. Solve the system:

$$\begin{aligned} x-3y &= 10 \\ 2x+y &= 13 \end{aligned}$$

3. Solve the system:

$$\begin{aligned} 2x+0.5y &= -10 \\ x-y &= -10 \end{aligned}$$

4. Solve the system:

$$\begin{aligned} 2x+0.5y &= 3 \\ x+2y &= 8.5 \end{aligned}$$

5. Solve the system:

$$\begin{aligned} 3x+5y &= -1 \\ x+2y &= -1 \end{aligned}$$

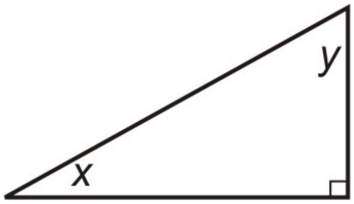
6. Solve the system:

$$\begin{aligned} 3x+5y &= -3 \\ x+2y &= -\frac{4}{3} \end{aligned}$$

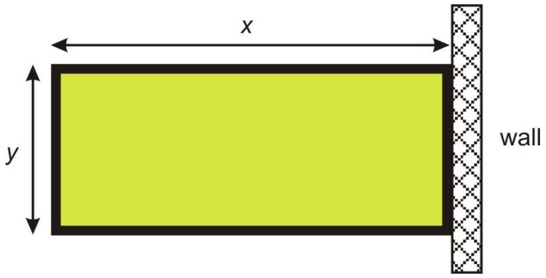
7. Solve the system:

$$\begin{aligned} x-y &= -\frac{12}{5} \\ 2x+5y &= -2 \end{aligned}$$

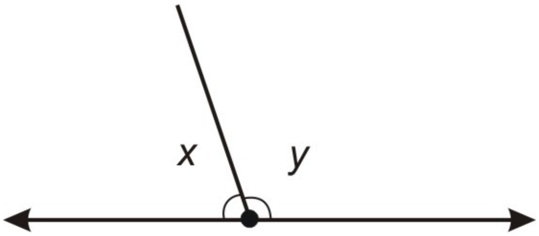
8. Of the two non-right angles in a right angled triangle, one measures twice as many degrees as the other. What are the angles?



9. The sum of two numbers is 70. They differ by 11. What are the numbers?
10. A number plus half of another number equals 6; twice the first number minus three times the second number equals 4. What are the numbers?
11. A rectangular field is enclosed by a fence on three sides and a wall on the fourth side. The total length of the fence is 320 yards. If the field has a total perimeter of 400 yards, what are the dimensions of the field?



12. A ray cuts a line forming two angles. The difference between the two angles is 18°. What does each angle measure?



13. Jason is five years older than Becky, and the sum of their ages is 23. What are their ages?

7.3 Mixture Problems

Here you'll learn how to use subtitution to solve real-world systems of linear equations in two variables that involve mixtures.

What if you had $90 to buy 8 pizzas for a party. Plain pizzas cost $10 and pepperoni pizzas cost $12. How could you figure out how many of each you can buy? After completing this Concept, you'll be able to solve mixture problems like this one using systems of equations.

Watch This

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CK-12 Foundation: 0703S Solving Mixture Problemsby Substitution(H264)

Guidance

Systems of equations crop up frequently in problems that deal with mixtures of two things—chemicals in a solution, nuts and raisins, or even the change in your pocket! Let's look at some examples of these.

Example A

Janine empties her purse and finds that it contains only nickels (worth 5 cents each) and dimes (worth 10 cents each). If she has a total of 7 coins and they have a combined value of 55 cents, how many of each coin does she have?

Since we have 2 types of coins, let's call the number of nickels x and the number of dimes y. We are given two key pieces of information to make our equations: the number of coins and their value.

$$\begin{array}{lll}\text{of coins equation:} & x+y=7 & (number\ of\ nickels)+(number\ of\ dimes)\\ \text{value equation:} & 5x+10y=55 & (since\ nickels\ are\ worth\ 5c\ and\ dimes\ 10c)\end{array}$$

We can quickly rearrange the first equation to isolate x:

$$\begin{aligned}
x &= 7-y && \textit{now substitute into equation } 2: \\
5(7-y)+10y &= 55 && \textit{distribute the } 5: \\
35-5y+10y &= 55 && \textit{collect like terms}: \\
35+5y &= 55 && \textit{subtract } 35 \textit{ from both sides}: \\
5y &= 20 && \textit{divide by } 5: \\
\underline{y=4} & && \textit{substitute back into equation } 1: \\
x+4 &= 7 && \textit{subtract } 4 \textit{ from both sides}: \\
\underline{x=3} & &&
\end{aligned}$$

Janine has 3 nickels and 4 dimes.

Sometimes a question asks you to determine (from concentrations) how much of a particular substance to use. The substance in question could be something like coins as above, or it could be a chemical in solution, or even heat. In such a case, you need to know the amount of whatever substance is in each part. There are several common situations where to get one equation you simply add two given quantities, but to get the second equation you need to use a **product.** Three examples are below.

TABLE 7.5:

Type of mixture	First equation	Second equation
Coins (items with $ value)	total number of items (n_1+n_2)	total ***value*** (item value × no. of items)
Chemical solutions	total solution volume (V_1+V_2)	amount of ***solute*** (vol × concentration)
Density of two substances	total amount or volume of mix	total ***mass*** (volume × density)

For example, when considering mixing chemical solutions, we will most likely need to consider the total amount of *solute* in the individual parts and in the final mixture. (A solute is the chemical that is dissolved in a solution. An example of a solute is salt when added to water to make a brine.) To find the total amount, simply multiply the amount of the mixture by the **fractional concentration**. To illustrate, let's look at an example where you are given amounts relative to the whole.

Example B

A chemist needs to prepare 500 ml of copper-sulfate solution with a 15% concentration. She wishes to use a high concentration solution (60%) and dilute it with a low concentration solution (5%) in order to do this. How much of each solution should she use?

Solution

To set this problem up, we first need to define our variables. Our unknowns are the amount of concentrated solution (x) and the amount of dilute solution (y). We will also convert the percentages (60%, 15% and 5%) into decimals (0.6, 0.15 and 0.05). The two pieces of critical information are the final volume (500 ml) and the final amount of solute (15% of 500 $ml = 75$ ml). Our equations will look like this:

Volume equation: $x+y=500$

Solute equation: $0.6x+0.05y=75$

To isolate a variable for substitution, we can see it's easier to start with equation 1:

$$\begin{aligned}
&x+y=500 && \textit{subtract y from both sides}:\\
&x=500-y && \textit{now substitute into equation } 2:\\
&0.6(500-y)+0.05y=75 && \textit{distribute the } 0.6:\\
&300-0.6y+0.05y=75 && \textit{collect like terms}:\\
&300-0.55y=75 && \textit{subtract } 300 \textit{ from both sides}:\\
&-0.55y=-225 && \textit{divide both sides by } -0.55:\\
&\underline{y=409\ ml} && \textit{substitute back into equation for } x:\\
&x=500-409=\underline{91\ ml}
\end{aligned}$$

So the chemist should mix 91 ml of the 60% solution with 409 ml of the 5% solution.

Example C

A coffee company makes a product which is a mixture of two coffees, using a coffee that costs $10.20 per pound and another coffee that costs $6.80 per pound. In order to make 20 pounds of a mixture that costs $8.50 per pound, how much of each type of coffee should it use?

Solution:

Let m be the amount of the $10.20 coffee, and let n be the amount needed of the $6.80 coffee. Since we want 20 pounds of coffee that costs $8.50 per pound, the total cost for all 20 pounds is $20 \cdot \$8.50 = \$170$. The cost for the 20 pounds of mixture is equal to the cost of each type of coffee added together: $10.20 \cdot m + 6.8 \cdot n = 170$.

Also, the amount of each type of coffee added together equals 20 pounds: $m+n=20$.

The system is: $\begin{cases} m+n=20 \\ 10.20 \cdot m + 6.8 \cdot n = 170 \end{cases}$.

We can isolate one variable and use substitution to solve the system:

$m = 20-n$

Now solve for n.

$$\begin{aligned}
10.20(20-n)+6.8n &= 170 \\
204-10.20n+6.8n &= 0170 && \text{Distributive Property} \\
204-3.4n &= 170 && \text{Add like terms.} \\
-3.4n &= -34 && \text{Subtract 204.} \\
n &= 10 && \text{Divide by } -3.4.
\end{aligned}$$

Since $n=10$, we can plug that into $m+n=20$.

$m+10=20 \Rightarrow m=10$.

The coffee company needs to use 10 pounds of each type of coffee in order to have a 20 pound mixture that costs $8.50 per pound.

Watch this video for help with the Examples above.

MEDIA

Click image to the left for more content.

CK-12 Foundation: Mixture Problems by Substitution

Vocabulary

- Solving linear systems **by substitution** means to solve for one variable in one equation, and then to substitute it into the other equation, solving for the other variable.

Guided Practice

A light green latex paint that is 20% yellow paint is combined with a darker green latex paint that is 45% yellow paint. How many gallons of each paint must be used to create 15 gallons of a green paint that is 25% yellow paint?

Solution:

Let x be the number of gallons of the 20% yellow paint and let y be the number of gallons of the 40% yellow paint. This means that we want those two numbers to add up to 15: $x+y=15$

Now if we want 15 gallons of 25% yellow paint, that means we want $0.25 \cdot 15 = 3.75$ gallons of pure yellow pigment. The expression $0.20 \cdot x$ represents the amount of pure yellow pigment in the x gallons of 20% yellow paint. The expression $0.45 \cdot y$ represents the amount of pure yellow pigment in the y gallons of 45% yellow paint. Combing the last two adds up to the 3.75 gallons of pure pigment in the final mixture:

$0.20x+0.40y=3.75$

The system is: $\begin{cases} x+y=15 \\ 0.20x+0.45y=3.75 \end{cases}$.

We can isolate one variable and use substitution to solve the system:

$x=15-y$

Now solve for y.

$$\begin{aligned} 0.20(15-y)+0.45y &= 3.75 \\ 3-0.20y+0.45y &= 3.75 && \text{Distributive Property} \\ 3+0.2y &= 3.75 && \text{Add like terms.} \\ 0.25y &= 0.75 && \text{Subtract 3.} \\ y &= 3 && \text{Divide by 0.25.} \end{aligned}$$

Now we can plug in $y=3$ into $x+y=15$:

$x+y=15 \Rightarrow x+3=15 \Rightarrow x=12.$

This means 12 gallons of 20% yellow paint should be mixed with 3 gallons of 45% yellow paint in order to get 15 gallons of 25% yellow paint.

Practice

1. I have \$15 and wish to buy five pounds of mixed nuts for a party. Peanuts cost \$2.20 per pound. Cashews cost \$4.70 per pound.
   a. How many pounds of each should I buy?
   b. If I suddenly realize I need to set aside \$5 to buy chips, can I still buy 5 pounds of nuts with the remaining \$10?
   c. What's the greatest amount of nuts I can buy?
2. A chemistry experiment calls for one liter of sulfuric acid at a 15% concentration, but the supply room only stocks sulfuric acid in concentrations of 10% and 35%.
   a. How many liters of each should be mixed to give the acid needed for the experiment?
   b. How many liters should be mixed to give *two* liters at a 15% concentration?
3. Bachelle wants to know the density of her bracelet, which is a mix of gold and silver. Density is total mass divided by total volume. The density of gold is 19.3 g/cc and the density of silver is 10.5 g/cc. The jeweler told her that the volume of silver in the bracelet was 10 cc and the volume of gold was 20 cc. Find the combined density of her bracelet.
4. Tickets to a show cost \$10 in advance and \$15 at the door. If 120 tickets are sold for a total of \$1390, how many of the tickets were bought in advance?
5. A light purple latex paint that is 40% blue paint is combined with a blue latex paint that is 100% blue paint. How many gallons of each paint must be used to create 15 gallons of a dark purple paint that is 60% blue paint?

In 6-10, the multiple-choice questions on a test are worth 2 points each, and the short-answer questions are worth 5 points each.

6. If the whole test is worth 100 points and has 35 questions, how many of the questions are multiple-choice and how many are short-answer?
7. If Kwan gets 31 questions right and ends up with a score of 86 on the test, how many questions of each type did she get right? (Assume there is no partial credit.)
8. If Ashok gets 5 questions wrong and ends up with a score of 87 on the test, how many questions of each type did he get wrong? (Careful!)
9. What are two ways you could have set up the equations for part c?
10. How could you have set up part b differently?

7.4 Linear Systems with Addition or Subtraction

Here you'll learn how to solve systems of linear equations in two variables by eliminating one of the variables. You'll then solve real-world problems involving such systems.

What if you were given a system of linear equations like $x+4y=7$ and $3x-4y=-3$? How could you solve for one of the variables by eliminating the other? After completing this Concept, you'll be able to solve a system of linear equations by elimination.

Watch This

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CK-12 Foundation: 0704S Solving Linear Systems by Elimination (H264)

Guidance

In this lesson, we'll see how to use simple addition and subtraction to simplify our system of equations to a single equation involving a single variable. Because we go from two unknowns (x and y) to a single unknown (either x or y), this method is often referred to by ***solving by elimination***. We eliminate one variable in order to make our equations solvable! To illustrate this idea, let's look at the simple example of buying apples and bananas.

Example A

If one apple plus one banana costs \$1.25 and one apple plus 2 bananas costs \$2.00, how much does one banana cost? One apple?

It shouldn't take too long to discover that each banana costs \$0.75. After all, the second purchase just contains 1 more banana than the first, and costs \$0.75 more, so that one banana must cost \$0.75.

Here's what we get when we describe this situation with algebra:

$$\begin{aligned} a+b &= 1.25 \\ a+2b &= 2.00 \end{aligned}$$

Now we can subtract the number of apples and bananas in the first equation from the number in the second equation, and also subtract the cost in the first equation from the cost in the second equation, to get the *difference* in cost that corresponds to the *difference* in items purchased.

$$(a+2b)-(a+b)=2.00-1.25 \rightarrow b=0.75$$

That gives us the cost of one banana. To find out how much one apple costs, we subtract \$0.75 from the total cost of one apple and one banana.

$$a+0.75=1.25 \rightarrow a=1.25-0.75 \rightarrow a=0.50$$

So an apple costs 50 cents.

To solve systems using addition and subtraction, we'll be using exactly this idea – by looking at the ***sum*** or ***difference*** of the two equations we can determine a value for one of the unknowns.

Solving Linear Systems Using Addition of Equations

Often considered the easiest and most powerful method of solving systems of equations, the addition (or elimination) method lets us combine two equations in such a way that the resulting equation has only one variable. We can then use simple algebra to solve for that variable. Then, if we need to, we can substitute the value we get for that variable back into either one of the original equations to solve for the other variable.

Example B

Solve this system by addition:

$$\begin{aligned}3x+2y&=11\\5x-2y&=13\end{aligned}$$

Solution

We will add **everything** on the left of the equals sign from both equations, and this will be equal to the sum of everything on the right:

$$(3x+2y)+(5x-2y)=11+13 \rightarrow 8x=24 \rightarrow x=3$$

A simpler way to visualize this is to keep the equations as they appear above, and to add them together vertically, going down the columns. However, just like when you add units, tens and hundreds, you MUST be sure to keep the x's and y's in their own columns. You may also wish to use terms like "$0y''$ as a placeholder!

$$\begin{array}{rr} & 3x+2y=11 \\ + & (5x-2y)=13 \\ \hline & 8x+0y=24 \end{array}$$

Again we get $8x=24$, or $x=3$. To find a value for y, we simply substitute our value for x back in.

Substitute $x=3$ into the second equation:

$$\begin{aligned}5\cdot 3-2y&=13 && since\ 5\times 3=15,\ we\ subtract\ 15\ from\ both\ sides\\-2y&=-2 && divide\ by\ -2\\y&=1\end{aligned}$$

The reason this method worked is that the $y-$coefficients of the two equations were opposites of each other: 2 and -2. Because they were opposites, they canceled each other out when we added the two equations together, so our final equation had no $y-$term in it and we could just solve it for x.

Solving Linear Systems Using Subtraction of Equations

Another, very similar method for solving systems is subtraction. When the $x-$ or $y-$coefficients in both equations are the same (including the sign) instead of being opposites, you can **subtract** one equation from the other.

If you look again at Example 3, you can see that the coefficient for x in both equations is +1. Instead of adding the two equations together to get rid of the y's, you could have subtracted to get rid of the x's:

$$\begin{aligned}
&(x+y)-(x-y)=7-1.5 \Rightarrow 2y=5.5 \Rightarrow y=2.75\\
&\qquad \text{or...}\\
&\qquad x+y=7\\
&\underline{-\ \ (x-y)=-1.5}\\
&\quad 0x+2y=5.5
\end{aligned}$$

So again we get $y=2.75$, and we can plug that back in to determine x.

The method of subtraction is just as straightforward as addition, so long as you remember the following:

- Always put the equation you are subtracting in parentheses, and distribute the negative.
- Don't forget to **subtract** the numbers on the right-hand side.
- Always remember that subtracting a negative is the same as adding a positive.

Example C

Peter examines the coins in the fountain at the mall. He counts 107 coins, all of which are either pennies or nickels. The total value of the coins is \$3.47. How many of each coin did he see?

Solution

We have 2 types of coins, so let's call the number of pennies x and the number of nickels y. The total value of all the pennies is just x, since they are worth 1¢ each. The total value of the nickels is $5y$. We are given two key pieces of information to make our equations: the number of coins and their value in cents.

$$\begin{aligned}
\text{of coins equation} &: x+y=107\\
\text{value equation} &: x+5y=347
\end{aligned}$$

We'll jump straight to subtracting the two equations:

$$\begin{aligned}
x+y&=107\\
\underline{-\ \ (x+5y)}&\underline{=-347}\\
-4y&=-240\\
y&=60
\end{aligned}$$

Substituting this value back into the first equation:

$$\begin{aligned} x+60 &= 107 && \mathit{subtract}\ 60\ \mathit{from\ both\ sides}: \\ x &= 47 \end{aligned}$$

So Peter saw 47 pennies (worth 47 cents) and 60 nickels (worth \$3.00) making a total of \$3.47.

Watch this video for help with the Examples above.

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CK-12 Foundation: LinearSystemsby Elimination

Vocabulary

- The purpose of the **elimination method** to solve a system is to cancel, or eliminate, a variable by either adding or subtracting the two equations. Sometimes the equations must be multiplied by scalars first, in order to cancel out a variable.

Guided Practice

Andrew is paddling his canoe down a fast-moving river. Paddling downstream he travels at 7 miles per hour, relative to the river bank. Paddling upstream, he moves slower, traveling at 1.5 miles per hour. If he paddles equally hard in both directions, how fast is the current? How fast would Andrew travel in calm water?

Solution

First we convert our problem into equations. We have two unknowns to solve for, so we'll call the speed that Andrew paddles at x, and the speed of the river y. When traveling downstream, Andrew speed is boosted by the river current, so his total speed is his paddling speed *plus* the speed of the river $(x+y)$. Traveling upstream, the river is working against him, so his total speed is his paddling speed *minus* the speed of the river $(x-y)$.

Downstream Equation: $x+y=7$

Upstream Equation: $x-y=1.5$

Next we'll eliminate one of the variables. If you look at the two equations, you can see that the coefficient of y is +1 in the first equation and -1 in the second. Clearly $(+1)+(-1)=0$, so this is the variable we will eliminate. To do this we simply add equation 1 to equation 2. We must be careful to collect like terms, and make sure that everything on the left of the equals sign stays on the left, and everything on the right stays on the right:

$$(x+y)+(x-y)=7+1.5 \Rightarrow 2x=8.5 \Rightarrow x=4.25$$

Or, using the column method we used in example 2:

$$\begin{array}{rl} & x+y=7 \\ + & x-y=1.5 \\ \hline & 2x+0y=8.5 \end{array}$$

Again we get $2x = 8.5$, or $x = 4.25$. To find a corresponding value for y, we plug our value for x into either equation and isolate our unknown. In this example, we'll plug it into the first equation:

$$\begin{aligned} 4.25 + y &= 7 && subtract\ 4.25\ from\ both\ sides: \\ y &= 2.75 \end{aligned}$$

Andrew paddles at 4.25 miles per hour. The river moves at 2.75 miles per hour.

Practice

1. Solve the system:

$$\begin{aligned} 3x + 4y &= 2.5 \\ 5x - 4y &= 25.5 \end{aligned}$$

2. Solve the system:

$$\begin{aligned} 2x + -y &= 10 \\ 3x + y &= -5 \end{aligned}$$

3. Solve the system:

$$\begin{aligned} 5x + 7y &= -31 \\ 5x - 9y &= 17 \end{aligned}$$

4. Solve the system:

$$\begin{aligned} 3y - 4x &= -33 \\ 5x - 3y &= 40.5 \end{aligned}$$

5. Nadia and Peter visit the candy store. Nadia buys three candy bars and four fruit roll-ups for $2.84. Peter also buys three candy bars, but can only afford one additional fruit roll-up. His purchase costs $1.79. What is the cost of a candy bar and a fruit roll-up individually?
6. A small plane flies from Los Angeles to Denver with a tail wind (the wind blows in the same direction as the plane) and an air-traffic controller reads its ground-speed (speed measured relative to the ground) at 275 miles per hour. Another, identical plane, moving in the opposite direction has a ground-speed of 227 miles per hour. Assuming both planes are flying with identical air-speeds, calculate the speed of the wind.
7. An airport taxi firm charges a pick-up fee, plus an additional per-mile fee for any rides taken. If a 12-mile journey costs $14.29 and a 17-mile journey costs $19.91, calculate:
   a. the pick-up fee
   b. the per-mile rate
   c. the cost of a seven mile trip
8. Calls from a call-box are charged per minute at one rate for the first five minutes, then a different rate for each additional minute. If a 7-minute call costs $4.25 and a 12-minute call costs $5.50, find each rate.
9. A plumber and a builder were employed to fit a new bath, each working a different number of hours. The plumber earns $35 per hour, and the builder earns $28 per hour. Together they were paid $330.75, but the plumber earned $106.75 more than the builder. How many hours did each work?
10. Paul has a part time job selling computers at a local electronics store. He earns a fixed hourly wage, but can earn a bonus by selling warranties for the computers he sells. He works 20 hours per week. In his first week, he sold eight warranties and earned $220. In his second week, he managed to sell 13 warranties and earned $280. What is Paul's hourly rate, and how much extra does he get for selling each warranty?

7.5 Linear Systems with Multiplication

Here you'll learn how to solve systems of linear equations in two variables by first multiplying and then eliminating one of the variables. You'll then solve real-world problems involving such systems.

What if you were given a system of linear equations like $x-2y=7$ and $3x-4y=-3$? How could you solve for one of the variables by eliminating the other? After completing this Concept, you'll be able to solve a system of linear equations by multiplication and then elimination.

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CK-12 Foundation: 0705S Solving Linear Systems by Elimination and Multiplication(H264)

Guidance

So far, we've seen that the elimination method works well when the coefficient of one variable happens to be the same (or opposite) in the two equations. But what if the two equations don't have any coefficients the same?

It turns out that we can still use the elimination method; we just have to *make* one of the coefficients match. We can accomplish this by multiplying one or both of the equations by a constant.

Here's a quick review of how to do that. Consider the following questions:

1. If 10 apples cost \$5, how much would 30 apples cost?
2. If 3 bananas plus 2 carrots cost \$4, how mush would 6 bananas plus 4 carrots cost?

If you look at the first equation, it should be obvious that each apple costs \$0.50. So 30 apples should cost \$15.00.

The second equation is trickier; it isn't obvious what the individual price for either bananas or carrots is. Yet we know that the answer to question 2 is \$8.00. How?

If we look again at question 1, we see that we can write an equation: $10a=5$ (a being the cost of 1 apple). So to find the cost of 30 apples, we *could* solve for a and then multiply by 30—but we could also just multiply both sides of the equation by 3. We would get $30a=15$, and that tells us that 30 apples cost \$15.

And we can do the same thing with the second question. The equation for this situation is $3b+2c=4$, and we can see that we need to solve for $(6b+4c)$, which is simply 2 times $(3b+2c)$! So algebraically, we are simply multiplying the entire equation by 2:

$$\begin{aligned} 2(3b+2c) &= 2\cdot 4 && \textit{distribute and multiply}: \\ 6b+4c &= 8 \end{aligned}$$

So when we multiply an equation, all we are doing is multiplying every term in the equation by a fixed amount.

Solving a Linear System by Multiplying One Equation

If we can multiply every term in an equation by a fixed number (a **scalar**), that means we can use the addition method on a whole new set of linear systems. We can manipulate the equations in a system to ensure that the coefficients of one of the variables match.

This is easiest to do when the coefficient as a variable in one equation is a multiple of the coefficient in the other equation.

Example A

Solve the system:

$$\begin{aligned}7x+4y&=17\\5x-2y&=11\end{aligned}$$

Solution

You can easily see that if we multiply the second equation by 2, the coefficients of y will be +4 and -4, allowing us to solve the system by addition:

2 times equation 2:

$$\begin{aligned}10x-4y&=22 && \textit{now add to equation one}:\\ \underline{+\ (7x+4y)}&\underline{=17} &&\\ 17x&=34 &&\end{aligned}$$

$$\textit{divide by } 17 \textit{ to get}: \qquad x=2$$

Now simply substitute this value for x back into equation 1:

$$\begin{aligned}7\cdot2+4y&=17 && since\ 7\times2=14,\ subtract\ 14\ from\ both\ sides:\\ 4y&=3 && divide\ by\ 4:\\ y&=0.75 &&\end{aligned}$$

Example B

Anne is rowing her boat along a river. Rowing downstream, it takes her 2 minutes to cover 400 yards. Rowing upstream, it takes her 8 minutes to travel the same 400 yards. If she was rowing equally hard in both directions, calculate, in yards per minute, the speed of the river and the speed Anne would travel in calm water.

Solution

Step one: first we convert our problem into equations. We know that *distance traveled* is equal to *speed* × *time*. We have two unknowns, so we'll call the speed of the river x, and the speed that Anne rows at y. When traveling downstream, her total speed is her rowing speed plus the speed of the river, or $(x+y)$. Going upstream, her speed is hindered by the speed of the river, so her speed upstream is $(x-y)$.

Downstream Equation: $2(x+y) = 400$

Upstream Equation: $8(x-y) = 400$

Distributing gives us the following system:

$$\begin{aligned} 2x+2y &= 400 \\ 8x-8y &= 400 \end{aligned}$$

Right now, we can't use the method of elimination because none of the coefficients match. But if we multiplied the top equation by 4, the coefficients of y would be +8 and -8. Let's do that:

$$\begin{array}{r} 8x+8y = 1,600 \\ +\ \ (8x-8y) = 400 \\ \hline 16x = 2,000 \end{array}$$

Now we divide by 16 to obtain $x = 125$.

Substitute this value back into the first equation:

$$\begin{aligned} 2(125+y) &= 400 && \textit{divide both sides by } 2: \\ 125+y &= 200 && \textit{subtract } 125 \textit{ from both sides}: \\ y &= 75 \end{aligned}$$

Anne rows at 125 yards per minute, and the river flows at 75 yards per minute.

Solving a Linear System by Multiplying Both Equations

So what do we do if none of the coefficients match and none of them are simple multiples of each other? We do the same thing we do when we're adding fractions whose denominators aren't simple multiples of each other. Remember that when we add fractions, we have to find a **lowest common denominator**—that is, the lowest common multiple of the two denominators—and sometimes we have to rewrite not just one, but both fractions to get them to have a common denominator. Similarly, sometimes we have to multiply both equations by different constants in order to get one of the coefficients to match.

Example C

Andrew and Anne both use the I-Haul truck rental company to move their belongings from home to the dorm rooms on the University of Chicago campus. I-Haul has a charge per day and an additional charge per mile. Andrew travels from San Diego, California, a distance of 2060 miles in five days. Anne travels 880 miles from Norfolk, Virginia, and it takes her three days. If Anne pays $840 and Andrew pays $1845, what does I-Haul charge

a) *per day?*

b) *per mile traveled?*

Solution

First, we'll set up our equations. Again we have 2 unknowns: the **daily rate** (we'll call this x), and the **per-mile rate** (we'll call this y).

Anne's equation: $3x+880y = 840$

Andrew's Equation: $5x+2060y=1845$

We can't just multiply a single equation by an integer number in order to arrive at matching coefficients. But if we look at the coefficients of x (as they are easier to deal with than the coefficients of y), we see that they both have a common multiple of 15 (in fact 15 is the ***lowest common multiple***). So we can multiply both equations.

Multiply the top equation by 5:

$$15x+4400y=4200$$

Multiply the lower equation by 3:

$$15x+6180y=5535$$

Subtract:

$$\begin{array}{rr} & 15x+4400y=4200 \\ - & (15x+6180y)=5535 \\ \hline & -1780y=-1335 \end{array}$$

$$\text{Divide by } -1780: \ y=0.75$$

Substitute this back into the top equation:

$$\begin{aligned} 3x+880(0.75) &= 840 \\ 3x &= 180 \\ x &= 60 \end{aligned}$$

I-Haul charges \$60 per day plus \$0.75 per mile.

Watch this video for help with the Examples above.

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CK-12 Foundation: LinearSystemsby Elimination and Multiplication

Vocabulary

- **Elimination method:** The purpose of the **elimination method** to solve a system is to cancel, or eliminate, a variable by either adding or subtracting the two equations. Sometimes the equations must be multiplied by scalars first, in order to cancel out a variable.
- **Least common multiple:** The ***least common multiple*** is the smallest value that is divisible by two or more quantities **without a remainder**.

Guided Practice

Solve the system $\begin{cases} 4x+7y=6 \\ 6x+5y=20 \end{cases}$.

Solution:

Neither x nor y have additive inverse coefficients, but the x-variables do share a common factor of 2. Thus we can most easily eliminate x.

In order to make x have the same coefficient in each equation, we must multiply one equation by the factor not shared in the coefficient of x in the other equation. We need to multiply the first equation by 3 and the second equation by 2, making one of them negative as well:

$$\begin{cases} 3(4x+7y=6) \\ -2(6x+5y=20) \end{cases} \rightarrow \begin{cases} 12x+21y=18 \\ -12x-10y=-40 \end{cases}$$

$$\begin{aligned} \text{Add the two equations.} \quad & 11y=-22 \\ \text{Divide by } 11. \quad & y=-2 \end{aligned}$$

To find the $x-$value, use the Substitution Property in either equation.

$$\begin{aligned} 4x+7(2)&=6 \\ 4x+14&=6 \\ 4x&=-8 \\ x&=-2 \end{aligned}$$

The solution to this system is $(-2,-2)$.

Practice

Solve the following systems using multiplication.

1.

$$\begin{aligned} 5x-10y&=15 \\ 3x-2y&=3 \end{aligned}$$

2.

$$\begin{aligned} 5x-y&=10 \\ 3x-2y&=-1 \end{aligned}$$

3.

$$\begin{aligned} 5x+7y&=15 \\ 7x-3y&=5 \end{aligned}$$

4.

$$\begin{aligned} 9x + 5y &= 9 \\ 12x + 8y &= 12.8 \end{aligned}$$

5.

$$\begin{aligned} 4x - 3y &= 1 \\ 3x - 4y &= 4 \end{aligned}$$

6.

$$\begin{aligned} 7x - 3y &= -3 \\ 6x + 4y &= 3 \end{aligned}$$

7.

$$\begin{aligned} &x = 3y \\ &x - 2y = -3 \end{aligned}$$

8.

$$\begin{aligned} &y = 3x + 2 \\ &y = -2x + 7 \end{aligned}$$

9.

$$\begin{aligned} &5x - 5y = 5 \\ &5x + 5y = 35 \end{aligned}$$

10.

$$\begin{aligned} &y = -3x - 3 \\ &3x - 2y + 12 = 0 \end{aligned}$$

11.

$$\begin{aligned} &3x - 4y = 3 \\ &4y + 5x = 10 \end{aligned}$$

12.

$$\begin{aligned} &9x - 2y = -4 \\ &2x - 6y = 1 \end{aligned}$$

7.6 Comparing Methods for Solving Linear Systems

Here you'll learn how to decide which method of solving a system of linear equations is the best one for the situation.

What if you had a system of linear equations like $x - y = 10$ and $2y = 3x + 5$? How could you decide upon the best method to solve it? After completing this Concept, you'll be able to solve a system of linear equations using the method of your choosing.

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CK-12 Foundation: 0706S Comparing Methods for Solving LinearSystemsby Elimination (H264)

Guidance

Now that we've covered the major methods for solving linear equations, let's review them. For simplicity, we'll look at them in table form. This should help you decide which method would be best for a given situation.

TABLE 7.6:

Method:	**Best used when you...**	**Advantages:**	**Comment:**
Graphing	...don't need an accurate answer.	Often easier to see number and quality of intersections on a graph. With a graphing calculator, it can be the fastest method since you don't have to do any computation.	Can lead to imprecise answers with non-integer solutions.
Substitution	...have an *explicit* equation for one variable (e.g. $y = 14x + 2$)	Works on all systems. Reduces the system to one variable, making it easier to solve.	You are not often given explicit functions in systems problems, so you may have to do extra work to get one of the equations into that form.
Elimination by Addition or Subtraction	...have matching coefficients for one variable in both equations.	Easy to combine equations to eliminate one variable. Quick to solve.	It is not very likely that a given system will have matching coefficients.
Elimination by Multiplication and then Addition and Subtraction	...do not have any variables defined explicitly or any matching coefficients.	Works on all systems. Makes it possible to combine equations to eliminate one variable.	Often more algebraic manipulation is needed to prepare the equations.

The table above is only a guide. You might prefer to use the graphical method for every system in order to better understand what is happening, or you might prefer to use the multiplication method even when a substitution would work just as well.

Example A

Two angles are ***complementary*** *when the sum of their angles is* 90°. *Angles A and B are complementary angles, and twice the measure of angle A is* 9° *more than three times the measure of angle B. Find the measure of each angle.*

Solution

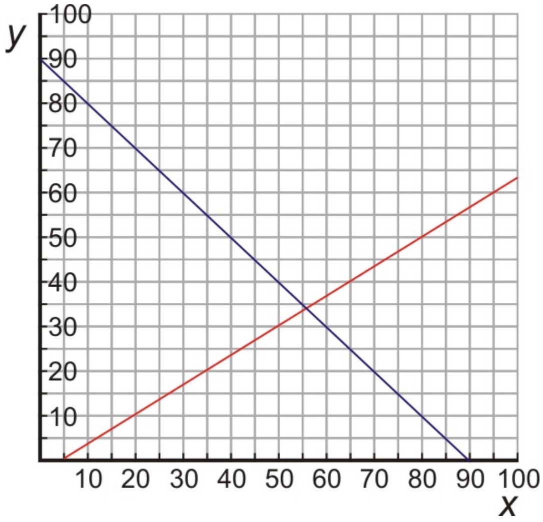
First we write out our 2 equations. We will use x to be the measure of angle A and y to be the measure of angle B. We get the following system:

$$\begin{aligned} x+y&=90 \\ 2x&=3y+9 \end{aligned}$$

First, we'll solve this system with the graphical method. For this, we need to convert the two equations to $y=mx+b$ form:

$$\begin{aligned} x+y=90 \qquad &\Rightarrow y=-x+90 \\ 2x=3y+9 \qquad &\Rightarrow y=\frac{2}{3}x-3 \end{aligned}$$

The first line has a slope of -1 and a $y-$intercept of 90, and the second line has a slope of $\frac{2}{3}$ and a $y-$intercept of -3. The graph looks like this:



In the graph, it appears that the lines cross at around $x=55, y=35$, but it is difficult to tell exactly! Graphing by hand is not the best method in this case!

Example B

In this example, we'll try solving by substitution. Let's look again at the system:

$$\begin{aligned} x+y&=90\\ 2x&=3y+9 \end{aligned}$$

We've already seen that we can start by solving either equation for y, so let's start with the first one:

$$y=90-x$$

Substitute into the second equation:

$$\begin{aligned} &2x=3(90-x)+9 && \textit{distribute the } 3:\\ &2x=270-3x+9 && \textit{add } 3x \textit{ to both sides}:\\ &5x=270+9=279 && \textit{divide by } 5:\\ &x=55.8^\circ \end{aligned}$$

Substitute back into our expression for y:

$$y=90-55.8=34.2^\circ$$

Angle A**measures** 55.8°; **angle** B**measures** 34.2°.

Example C

Finally, in this example, we'll try solving by elimination (with multiplication):

Rearrange equation one to standard form:

$$x+y=90 \qquad \Rightarrow 2x+2y=180$$

Multiply equation two by 2:

$$2x=3y+9 \qquad \Rightarrow 2x-3y=9$$

Subtract:

$$\begin{aligned} 2x+2y&=180\\ -\quad(2x-3y)&=-9\\ \hline 5y&=171 \end{aligned}$$

Divide by 5 to obtain $y=34.2^\circ$

Substitute this value into the very first equation:

$$\begin{aligned} x+34.2 &= 90 && \textit{subtract } 34.2 \textit{ from both sides}: \\ x &= 55.8^\circ \end{aligned}$$

Angle A measures 55.8°; **angle B measures** 34.2°.

Even though this system looked ideal for substitution, the method of multiplication worked well too. Once the equations were rearranged properly, the solution was quick to find. You'll need to decide yourself which method to use in each case you see from now on. Try to master all the techniques, and recognize which one will be most efficient for each system you are asked to solve.

Watch this video for help with the Examples above.

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CK-12 Foundation: Comparing Methods for Solving Linear Systems

Vocabulary

- A **linear system of equations** is a set of equations that must be solved together to find the one solution that fits them both.
- Solving linear systems **by substitution** means to solve for one variable in one equation, and then to substitute it into the other equation, solving for the other variable.
- The purpose of the **elimination method** to solve a system is to cancel, or eliminate, a variable by either adding or subtracting the two equations. Sometimes the equations must be multiplied by scalars first, in order to cancel out a variable.

Guided Practice

$$\textit{Solve the system } \begin{cases} 5s+2t=6 \\ 9s+2t=22 \end{cases}.$$

Solution:

Since these equations are both written in standard form, and both have the term $2t$ in them, we will will use elimination by subtracting. This will cause the t terms to cancel out and we will be left with one variable, s, which we can then isolate.

$$\begin{aligned} 5s+2t &= 6 \\ \underline{-(9s+2t} &\underline{= 22)} \\ -4s+0t &= -16 \\ -4s &= -16 \\ s &= 4 \end{aligned}$$

$$\begin{aligned}5(4)+2t &= 6\\ 20+2t &= 6\\ 2t &= -14\\ t &= -7\end{aligned}$$

The solution is $(4,-7)$.

Practice

Solve the following systems using any method.

1.

$$\begin{aligned}x &= 3y\\ x-2y &= -3\end{aligned}$$

2.

$$\begin{aligned}y &= 3x+2\\ y &= -2x+7\end{aligned}$$

3.

$$\begin{aligned}5x-5y &= 5\\ 5x+5y &= 35\end{aligned}$$

4.

$$\begin{aligned}y &= -3x-3\\ 3x-2y+12 &= 0\end{aligned}$$

5.

$$\begin{aligned}3x-4y &= 3\\ 4y+5x &= 10\end{aligned}$$

6.

$$\begin{aligned}9x-2y &= -4\\ 2x-6y &= 1\end{aligned}$$

7. Supplementary angles are two angles whose sum is 180°. Angles A and B are supplementary angles. The measure of Angle A is 18° less than twice the measure of Angle B. Find the measure of each angle.
8. A farmer has fertilizer in 5% and 15% solutions. How much of each type should he mix to obtain 100 liters of fertilizer in a 12% solution?
9. A 150-yard pipe is cut to provide drainage for two fields. If the length of one piece is three yards less that twice the length of the second piece, what are the lengths of the two pieces?
10. Mr. Stein invested a total of $100,000 in two companies for a year. Company A's stock showed a 13% annual gain, while Company B showed a 3% loss for the year. Mr. Stein made an 8% return on his investment over the year. How much money did he invest in each company?

7.7 Consistent and Inconsistent Linear Systems

Here you'll learn the difference between three special types of linear systems: **inconsistent linear systems**, **consistent linear systems**, and **dependent linear systems**. You'll then use that information to determine the number of solutions a system has.

What if you were given a system of equations like $2x - y = 5$ and $10x - 5y = 25$? How could you rewrite these equations to determine the number of solutions the system has? After completing this Concept, you'll be able to identify whether a system of equations like this one is an inconsistent one, a consistent one, or a dependent one.

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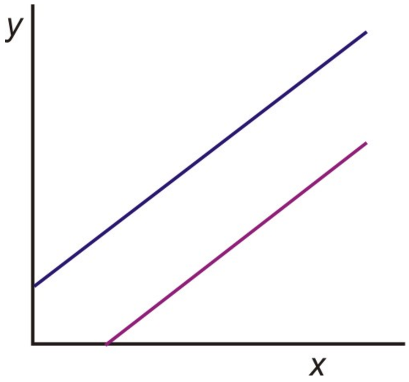
CK-12 Foundation: 0707S Special Types ofLinear Systems (H264)

Guidance

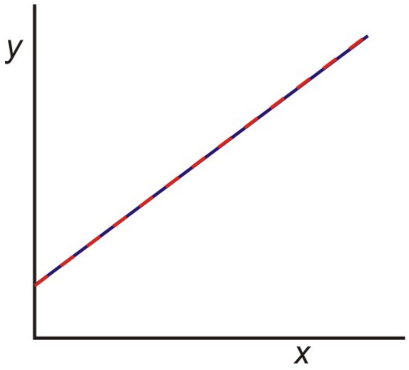
As we saw in Section 7.1, a system of linear equations is a set of linear equations which must be solved together. The lines in the system can be graphed together on the same coordinate graph and the solution to the system is the point at which the two lines intersect.

Or at least that's what usually happens. But what if the lines turn out to be parallel when we graph them?



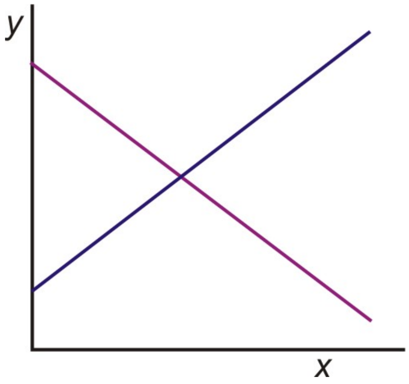
If the lines are parallel, they won't ever intersect. That means that the system of equations they represent has no solution. A system with no solutions is called an **inconsistent system.**

And what if the lines turn out to be identical?



If the two lines are the same, then *every* point on one line is also on the other line, so every point on the line is a solution to the system. The system has an **infinite number** of solutions, and the two equations are really just different forms of the same equation. Such a system is called a **dependent system**.

But usually, two lines cross at exactly one point and the system has exactly one solution:



A system with exactly one solution is called a **consistent system**.

To identify a system as **consistent, inconsistent**, or **dependent**, we can graph the two lines on the same graph and see if they intersect, are parallel, or are the same line. But sometimes it is hard to tell whether two lines are parallel just by looking at a roughly sketched graph.

Another option is to write each line in slope-intercept form and compare the slopes and $y-$ intercepts of the two lines. To do this we must remember that:

- Lines with different slopes always intersect.
- Lines with the same slope but different $y-$intercepts are parallel.
- Lines with the same slope and the same $y-$intercepts are identical.

Example A

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned}2x-5y&=2\\4x+y&=5\end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form

$2x-5y=2 \Rightarrow -5y=-2x+2 \Rightarrow y=\frac{2}{5}x-\frac{2}{5}$

$4x+y=5 \Rightarrow y=-4x+5$

The slopes of the two equations are different; therefore the lines must cross at a single point and the system has exactly one solution. This is a **consistent system.**

Example B

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned} 3x&=5-4y \\ 6x+8y&=7 \end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form

$3x=5-4y \Rightarrow 4y=-3x+5 \Rightarrow y=-\frac{3}{4}x+\frac{5}{4}$

$6x+8y=7 \Rightarrow 8y=-6x+7 \Rightarrow y=-\frac{3}{4}x+\frac{7}{8}$

The slopes of the two equations are the same but the $y-$intercepts are different; therefore the lines are parallel and the system has no solutions. This is an **inconsistent system.**

Example C

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned} x+y&=3 \\ 3x+3y&=9 \end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form

$x+y=3 \Rightarrow y=-x+3$

$3x+3y=9 \Rightarrow 3y=-3x+9 \Rightarrow y=-x+3$

The lines are identical; therefore the system has an infinite number of solutions. It is a **dependent system.**

Watch this video for help with the Examples above.

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CK-12 Foundation: Special Types of Linear Systems

Vocabulary

- A system with no solutions is called an **inconsistent system.** For linear equations, this occurs with parallel lines.
- A system where the two equations overlap at one, multiple, or infinitely many points is called a **consistent system.**
- **Coincident lines** are lines with the same slope and $y-$intercept. The lines completely overlap.
- When solving a system of coincident lines, the resulting equation will be without variables and the statement will be true. You can conclude the system has an infinite number of solutions. This is called a **consistent-dependent system**.

Guided Practice

Determine whether the following system of linear equations has zero, one, or infinitely many solutions:

$$\begin{cases} 2y+6x=20 \\ y=-3x+7 \end{cases}$$

What kind of system is this?

Solution:

It is easier to compare equations when they are in the same form. We will rewrite the first equation in slope-intercept form.

$2y+6x=20 \Rightarrow y+3x=10 \Rightarrow y=-3x+10$

Since the two equations have the same slope, but different y-intercepts, they are different but parallel lines. Parallel lines never intersect, so they have no solutions.

Since the lines are parallel, it is an inconsistent system.

Practice

Express each equation in slope-intercept form. Without graphing, state whether the system of equations is consistent, inconsistent or dependent.

1.

$$\begin{aligned} 3x-4y&=13 \\ y&=-3x-7 \end{aligned}$$

2.

$$\begin{aligned} \frac{3}{5}x+y&=3 \\ 1.2x+2y&=6 \end{aligned}$$

3.

$$\begin{aligned} 3x-4y&=13 \\ y&=-3x-7 \end{aligned}$$

4.

$$\begin{aligned} 3x-3y&=3 \\ x-y&=1 \end{aligned}$$

5.

$$\begin{aligned} 0.5x-y&=30 \\ 0.5x-y&=-30 \end{aligned}$$

6.

$$\begin{aligned} 4x-2y&=-2 \\ 3x+2y&=-12 \end{aligned}$$

7.

$$\begin{aligned} 3x+y&=4 \\ y&=5-3x \end{aligned}$$

8.

$$\begin{aligned} x-2y&=7 \\ 4y-2x&=14 \end{aligned}$$

9.

$$\begin{aligned} -2y+4x&=8 \\ y-2x&=-4 \end{aligned}$$

10.

$$\begin{aligned} x-\frac{y}{2}&=\frac{3}{2} \\ 3x+y&=6 \end{aligned}$$

11.

$$\begin{aligned} 0.05x+0.25y&=6 \\ x+y&=24 \end{aligned}$$

12.

$$\begin{aligned} x+\frac{2y}{3}&=6 \\ 3x+2y&=2 \end{aligned}$$

7.8 Determining the Type of Linear System

Here you'll learn how to solve a system of equations and use the result as a guide in determining the type of system it is.

What if you were given a system of equations like $2x+y=-1$ and $3x-2y=-5$? How could you use the solution to this system to determine if the system is consistent, inconsistent, or dependent? After completing this Concept, you'll be able to determine the type of a system algebraically.

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CK-12 Foundation: 0708S Determining the Typeof Linear Systems (H264)

Guidance

A third option for identifying systems as consistent, inconsistent or dependent is to just solve the system and use the result as a guide.

Example A

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned} 10x-3y&=3 \\ 2x+y&=9 \end{aligned}$$

Solution

Let's solve this system using the substitution method.

Solve the second equation for y:

$$2x+y=9 \Rightarrow y=-2x+9$$

Substitute that expression for y in the first equation:

$$\begin{aligned}10x-3y&=3\\10x-3(-2x+9)&=3\\10x+6x-27&=3\\16x&=30\\x&=\frac{15}{8}\end{aligned}$$

Substitute the value of x back into the second equation and solve for y:

$$2x+y=9\Rightarrow y=-2x+9\Rightarrow y=-2\cdot\frac{15}{8}+9\Rightarrow y=\frac{21}{4}$$

The solution to the system is $\left(\frac{15}{8},\frac{21}{4}\right)$. The system is **consistent** since it has only one solution.

Example B

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned}3x-2y&=4\\9x-6y&=1\end{aligned}$$

Solution

Let's solve this system by the method of multiplication.

Multiply the first equation by 3:

$$\begin{array}{ccc}3(3x-2y=4) & & 9x-6y=12\\ & & \Rightarrow\\ 9x-6y=1 & & 9x-6y=1\end{array}$$

Add the two equations:

$$\begin{aligned}9x-6y&=4\\ \underline{9x-6y}&\underline{=1}\\ 0&=13 \quad \text{This statement is not true.}\end{aligned}$$

If our solution to a system turns out to be a statement that is not true, then the system doesn't really have a solution; it is **inconsistent.**

Example C

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned} 4x+y&=3 \\ 12x+3y&=9 \end{aligned}$$

Solution

Let's solve this system by substitution.

Solve the first equation for y:

$$4x+y=3 \Rightarrow y=-4x+3$$

Substitute this expression for y in the second equation:

$$\begin{aligned} 12x+3y&=9 \\ 12x+3(-4x+3)&=9 \\ 12x-12x+9&=9 \\ 9&=9 \end{aligned}$$

This statement is always true.

If our solution to a system turns out to be a statement that is always true, then the system is **dependent.**

A second glance at the system in this example reveals that the second equation is three times the first equation, so the two lines are identical. The system has an infinite number of solutions because they are really the same equation and trace out the same line.

Let's clarify this statement. An infinite number of solutions does not mean that *any* ordered pair (x,y) satisfies the system of equations. Only ordered pairs that solve the equation in the system (either one of the equations) are also solutions to the system. There are infinitely many of these solutions to the system because there are infinitely many points on any one line.

For example, (1, -1) is a solution to the system in this example, and so is (-1, 7). Each of them fits both the equations because both equations are really the same equation. But (3, 5) doesn't fit either equation and is not a solution to the system.

In fact, for every $x-$value there is just one $y-$value that fits both equations, and for every $y-$value there is exactly one $x-$value—just as there is for a single line.

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CK-12 Foundation: Determining the Type of LinearSystem

Vocabulary

- A **consistent system** will always give exactly one solution.

- An **inconsistent system** will yield a statement that is *always false* (like $0 = 13$).
- A **dependent system** will yield a statement that is *always true* (like $9 = 9$).

Guided Practice

Identify the system as consistent, inconsistent, or consistent-dependent.

$$\begin{aligned} 3x-2y&=4\\ 9x-6y&=1 \end{aligned}$$

Solution: Because both equations are in standard form, elimination is the best method to solve this system.

Multiply the first equation by 3.

$$\begin{aligned} 3(3x-2y&=4) & & 9x-6y=12\\ & &\Rightarrow & \\ 9x-6y&=1 & & 9x-6y=1 \end{aligned}$$

Subtract the two equations.

$$\begin{aligned} 9x-6y&=12\\ \underline{9x-6y}&\underline{=1}\\ 0&=11 \quad \text{This Statement is not true.} \end{aligned}$$

This is an untrue statement; therefore, you can conclude:

1. These lines are parallel.
2. The system has no solution.
3. The system is inconsistent.

Practice

Find the solution of each system of equations using the method of your choice. State if the system is inconsistent or dependent.

1. $$\begin{aligned} 3x+2y&=4\\ -2x+2y&=24 \end{aligned}$$

2. $$\begin{aligned} 5x-2y&=3\\ 2x-3y&=10 \end{aligned}$$

3.

$$\begin{aligned} 3x-4y &= 13 \\ y &= -3x-7 \end{aligned}$$

4.

$$\begin{aligned} 5x-4y &= 1 \\ -10x+8y &= -30 \end{aligned}$$

5.

$$\begin{aligned} 4x+5y &= 0 \\ 3x &= 6y+4.5 \end{aligned}$$

6.

$$\begin{aligned} -2y+4x &= 8 \\ y-2x &= -4 \end{aligned}$$

7.

$$\begin{aligned} x-\frac{1}{2}y &= \frac{3}{2} \\ 3x+y &= 6 \end{aligned}$$

8.

$$\begin{aligned} 0.05x+0.25y &= 6 \\ x+y &= 24 \end{aligned}$$

9.

$$\begin{aligned} x+\frac{2}{3}y &= 6 \\ 3x+2y &= 2 \end{aligned}$$

10.

$$\begin{aligned} 3x-4y &= 13 \\ y &= -3x-7 \end{aligned}$$

11.

$$\begin{aligned} 4x+y &= 3 \\ 12x+3y &= 9 \end{aligned}$$

12.

$$\begin{aligned} 10x-3y &= 3 \\ 2x+y &= 9 \end{aligned}$$

7.9 Applications of Linear Systems

Here you'll learn how consistent, inconsistent, and dependent systems arise in real-world applications and you'll solve such problems.

What if you were playing a game in which you collected houses and hotels. Three houses and one hotel are worth \$1750. One house and three hotels are worth \$3250. How could you find the value of each house and each hotel? After completing this Concept, you'll be able to solve real-world applications like this one that involve linear systems.

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CK-12 Foundation: 0709S Applicationsof Linear Systems

Guidance

In this section, we'll see how consistent, inconsistent and dependent systems might arise in applications.

Example A

The movie rental store CineStar offers customers two choices. Customers can pay a yearly membership of \$45 and then rent each movie for \$2 or they can choose not to pay the membership fee and rent each movie for \$3.50. How many movies would you have to rent before the membership becomes the cheaper option?

Solution

Let's translate this problem into algebra. Since there are two different options to consider, we can write two different equations and form a system.

The choices are "membership" and "no membership." We'll call the number of movies you rent x and the total cost of renting movies for a year y.

TABLE 7.7:

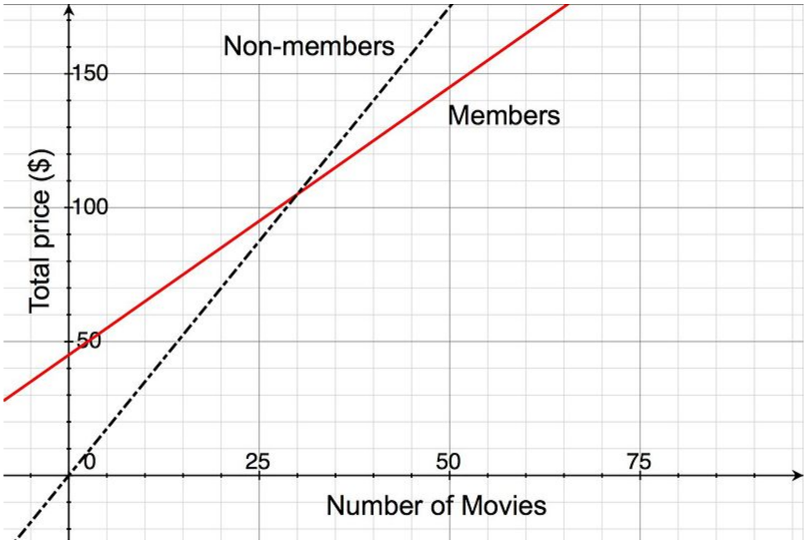
	flat fee	rental fee	total
membership	\$45	$2x$	$y = 45 + 2x$
no membership	\$0	$3.50x$	$y = 3.5x$

The flat fee is the dollar amount you pay per year and the rental fee is the dollar amount you pay when you rent a movie. For the membership option the rental fee is $2x$, since you would pay \$2 for each movie you rented; for the no membership option the rental fee is $3.50x$, since you would pay \$3.50 for each movie you rented.

Our system of equations is:

$$\begin{aligned} y &= 45+2x \\ y &= 3.50x \end{aligned}$$

Here's a graph of the system:



Now we need to find the exact intersection point. Since each equation is already solved for y, we can easily solve the system with substitution. Substitute the second equation into the first one:

$$\begin{aligned} y &= 45+2x \\ \Rightarrow 3.50x = 45+2x \Rightarrow 1.50x = 45 \Rightarrow x &= 30 \text{ movies} \\ y &= 3.50x \end{aligned}$$

You would have to rent **30 movies per year** before the membership becomes the better option.

This example shows a real situation where a consistent system of equations is useful in finding a solution. Remember that for a consistent system, the lines that make up the system intersect at single point. In other words, the lines are not parallel or the slopes are different.

In this case, the slopes of the lines represent the price of a rental per movie. The lines cross because the price of rental per movie is different for the two options in the problem

Now let's look at a situation where the system is inconsistent. From the previous explanation, we can conclude that the lines will not intersect if the slopes are the same (and the $y-$intercept is different). Let's change the previous problem so that this is the case.

Example B

Two movie rental stores are in competition. Movie House charges an annual membership of $30 and charges $3 per movie rental. Flicks for Cheap charges an annual membership of $15 and charges $3 per movie rental. After how many movie rentals would Movie House become the better option?

Solution

It should already be clear to see that Movie House will never become the better option, since its membership is more expensive and it charges the same amount per movie as Flicks for Cheap.

The lines on a graph that describe each option have different $y-$intercepts—namely 30 for Movie House and 15 for Flicks for Cheap—but the same slope: 3 dollars per movie. This means that the lines are parallel and so the system is inconsistent.

Now let's see how this works algebraically. Once again, we'll call the number of movies you rent x and the total cost of renting movies for a year y.

TABLE 7.8:

	flat fee	**rental fee**	**total**
Movie House	\$30	$3x$	$y = 30 + 3x$
Flicks for Cheap	\$15	$3x$	$y = 15 + 3x$

The system of equations that describes this problem is:

$$\begin{aligned} y &= 30 + 3x \\ y &= 15 + 3x \end{aligned}$$

Let's solve this system by substituting the second equation into the first equation:

$$\begin{aligned} &y = 30 + 3x \\ \Rightarrow 15 + 3x = 30 + 3x \Rightarrow 15 = 30 \qquad &\text{This statement is always false.} \\ &y = 15 + 3x \end{aligned}$$

This means that the system is **inconsistent.**

Example C

Peter buys two apples and three bananas for \$4. Nadia buys four apples and six bananas for \$8 from the same store. How much does one banana and one apple costs?

Solution

We must write two equations: one for Peter's purchase and one for Nadia's purchase.

Let's say a is the cost of one apple and b is the cost of one banana.

TABLE 7.9:

	cost of apples	**cost of bananas**	**total cost**
Peter	$2a$	$3b$	$2a + 3b = 4$
Nadia	$4a$	$6b$	$4a + 6b = 8$

The system of equations that describes this problem is:

$$\begin{aligned} 2a+3b&=4\\ 4a+6b&=8 \end{aligned}$$

Let’s solve this system by multiplying the first equation by -2 and adding the two equations:

$$\begin{array}{lr} -2(2a+3b=4) & -4a-6b=-8 \\ & \Rightarrow \\ 4a+6b=8 & \underline{4a+6b=8} \\ & 0+0=0 \end{array}$$

This statement is always true. This means that the system is **dependent.**

Looking at the problem again, we can see that we were given exactly the same information in both statements. If Peter buys two apples and three bananas for $4, it makes sense that if Nadia buys twice as many apples (four apples) and twice as many bananas (six bananas) she will pay twice the price ($8). Since the second equation doesn’t give us any new information, it doesn’t make it possible to find out the price of each fruit.

Watch this video for help with the Examples above.

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CK-12 Foundation: Applications of LinearSystems

Vocabulary

- A **linear system of equations** is a set of equations that must be solved together to find the one solution that fits them both.
- Solving linear systems **by substitution** means to solve for one variable in one equation, and then to substitute it into the other equation, solving for the other variable.
- The purpose of the **elimination method** to solve a system is to cancel, or eliminate, a variable by either adding or subtracting the two equations. Sometimes the equations must be multiplied by scalars first, in order to cancel out a variable.
- A **consistent system** will always give exactly one solution.
- An **inconsistent system** will yield a statement that is *always false* (like $0 = 13$).
- A **dependent system** will yield a statement that is *always true* (like $9 = 9$).

Guided Practice

A baker sells plain cakes for \$7 and decorated cakes for \$11. On a busy Saturday the baker started with 120 cakes, and sold all but three. His takings for the day were \$991. How many plain cakes did he sell that day, and how many were decorated before they were sold?

Solution:

TABLE 7.10:

	plain cakes	**decorated cakes**	**total**
Cakes sold	p	d	$120-3=117$
Cost of cakes	7p	11d	\$991

The system of equations that describes this problem is:

$$\begin{aligned} p+d &= 117 \\ 7p+11d &= 991 \end{aligned}$$

Let's solve this system by substituting the second equation into the first equation:

$p+d=117 \Rightarrow p=117-d$

$$\begin{aligned} 7p+11d=991 &\Rightarrow 7(117-d)+11d=991 \\ &\Rightarrow 819-7d+11d=991 \\ &\Rightarrow 819+4d=991 \\ &\Rightarrow 4d=172 \\ &\Rightarrow d=43 \end{aligned}$$

We can substitute d into the the first equation to solve for p.

$p=117-d=117-(43)=74$

The baker sold 74 plain cakes and 43 decorated cakes.

Practice

1. Twice John's age plus five times Claire's age is 204. Nine times John's age minus three times Claire's age is also 204. How old are John and Claire?
2. Juan is considering two cell phone plans. The first company charges \$120 for the phone and \$30 per month for the calling plan that Juan wants. The second company charges \$40 for the same phone but charges \$45 per month for the calling plan that Juan wants. After how many months would the total cost of the two plans be the same?
3. Jamal placed two orders with an internet clothing store. The first order was for 13 ties and 4 pairs of suspenders, and totaled \$487. The second order was for 6 ties and 2 pairs of suspenders, and totaled \$232. The bill does not list the per-item price, but all ties have the same price and all suspenders have the same price. What is the cost of one tie and of one pair of suspenders?
4. An airplane took four hours to fly 2400 miles in the direction of the jet-stream. The return trip against the jet-stream took five hours. What were the airplane's speed in still air and the jet-stream's speed?

For questions 5-7, a movie theater charges \$4.50 for children and \$8.00 for adults.

5. On a certain day, 1200 people enter the theater and \$8375 is collected. How many children and how many adults attended?
6. The next day, the manager announces that she wants to see them take in \$10000 in tickets. If there are 240 seats in the house and only five movie showings planned that day, is it possible to meet that goal?
7. At the same theater, a 16-ounce soda costs \$3 and a 32-ounce soda costs \$5. If the theater sells 12,480 ounces of soda for \$2100, how many people bought soda? (**Note:** Be careful in setting up this problem!)

For questions 8-10, consider the situation: Nadia told Peter that she went to the farmer's market and bought two apples and one banana, and that it cost her \$2.50. She thought that Peter might like some fruit, so she went back to the seller and bought four more apples and two more bananas. Peter thanked Nadia, but told her that he did not like bananas, so he would only pay her for four apples. Nadia told him that the second time she paid \$6.00 for the fruit.

8. What did Peter find when he tried to figure out the price of four apples?
9. Nadia then told Peter she had made a mistake, and she actually paid \$5.00 on her second trip. Now what answer did Peter get when he tried to figure out how much to pay her?
10. Alicia then showed up and told them she had just bought 3 apples and 2 bananas from the same seller for \$4.25. Now how much should Peter pay Nadia for four apples?

7.10 Systems of Linear Inequalities

Here you'll learn how to graph and solve a system of two or more linear inequalities. You'll also determine if such systems are consistent or inconsistent.

What if you were given a system of linear inequalities like $6x-2y \geq 3$ and $2y-3x \leq 7$? How could you determine its solution? After completing this Concept, you'll be able to find the solution region of systems of linear inequalities like this one.

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CK-12 Foundation: 0710S Systems of Linear Inequalities

Guidance

In the last chapter you learned how to graph a linear inequality in two variables. To do that, you graphed the equation of the straight line on the coordinate plane. The line was solid for $\leq$ or $\geq$ signs (where the equals sign is included), and the line was dashed for <or >signs (where the equals sign is not included). Then you shaded above the line (if the inequality began with $y >$ or $y \geq$) or below the line (if it began with $y <$ or $y \leq$).

In this section, we'll see how to graph two or more linear inequalities on the same coordinate plane. The inequalities are graphed separately on the same graph, and the solution for the system is the common shaded region between all the inequalities in the system. One linear inequality in two variables divides the plane into two **half-planes**. A **system** of two or more linear inequalities can divide the plane into more complex shapes.

Let's start by solving a system of two inequalities.

Graph a System of Two Linear Inequalities

Example A

Solve the following system:

$$\begin{aligned}2x+3y &\leq 18\\ x-4y &\leq 12\end{aligned}$$

Solution

Solving systems of linear inequalities means graphing and finding the intersections. So we graph each inequality, and then find the intersection *regions* of the solution.

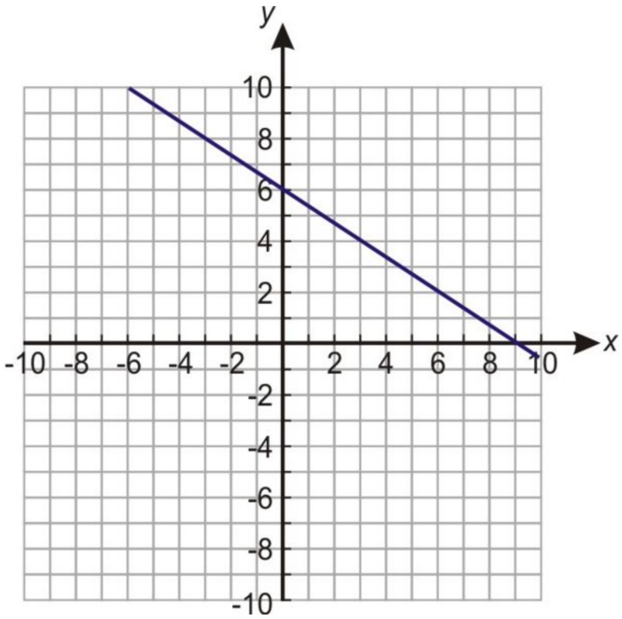
First, let's rewrite each equation in slope-intercept form. (Remember that this form makes it easier to tell which region of the coordinate plane to shade.) Our system becomes

$$\begin{aligned} 3y &\le -2x+18 & \qquad y &\le -\frac{2}{3}x+6 \\ & & &\Rightarrow \\ -4y &\le -x+12 & \qquad y &\ge \frac{x}{4}-3 \end{aligned}$$

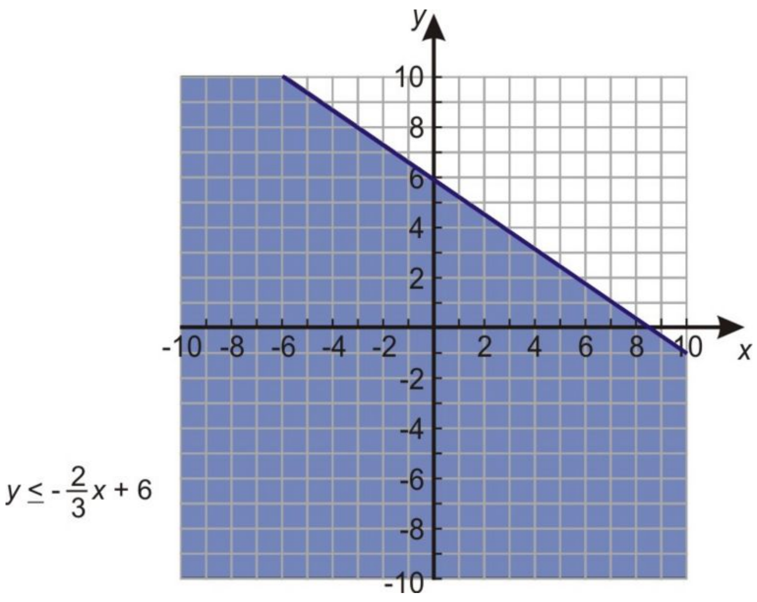
Notice that the inequality sign in the second equation changed because we divided by a negative number!

For this first example, we'll graph each inequality separately and then combine the results.

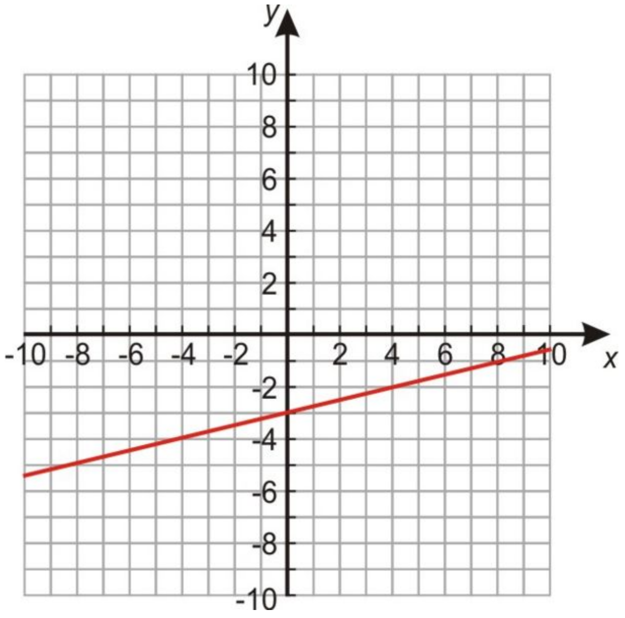
Here's the graph of the first inequality:



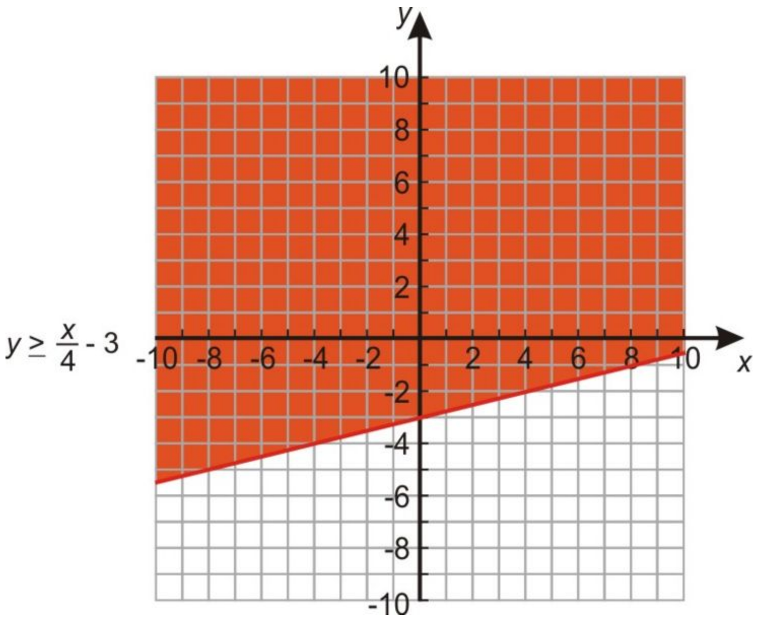
The line is solid because the equals sign is included in the inequality. Since the inequality is **less** than or equal to, we shade **below** the line.



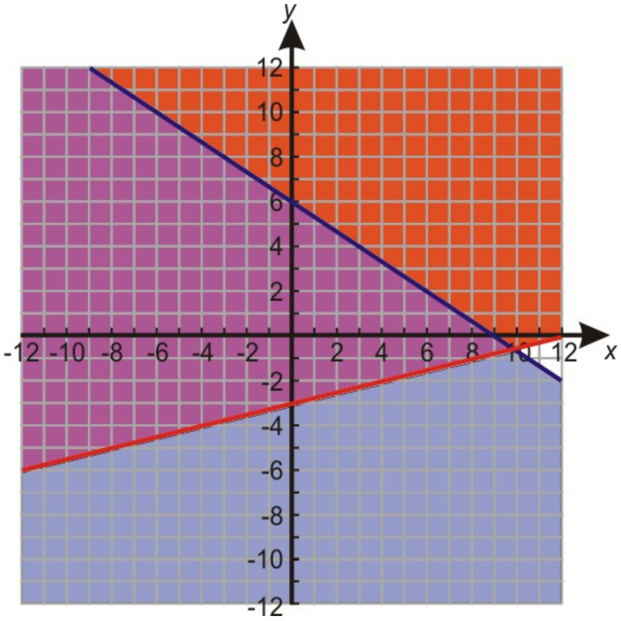
And here's the graph of the second inequality:



The line is solid again because the equals sign is included in the inequality. We now shade **above** the line because *y* is **greater** than or equal to.



When we combine the graphs, we see that the blue and red shaded regions overlap. The area where they overlap is the area where both inequalities are true. Thus that area (shown below in purple) is the solution of the system.



The kind of solution displayed in this example is called **unbounded**, because it continues forever in at least one direction (in this case, forever upward and to the left).

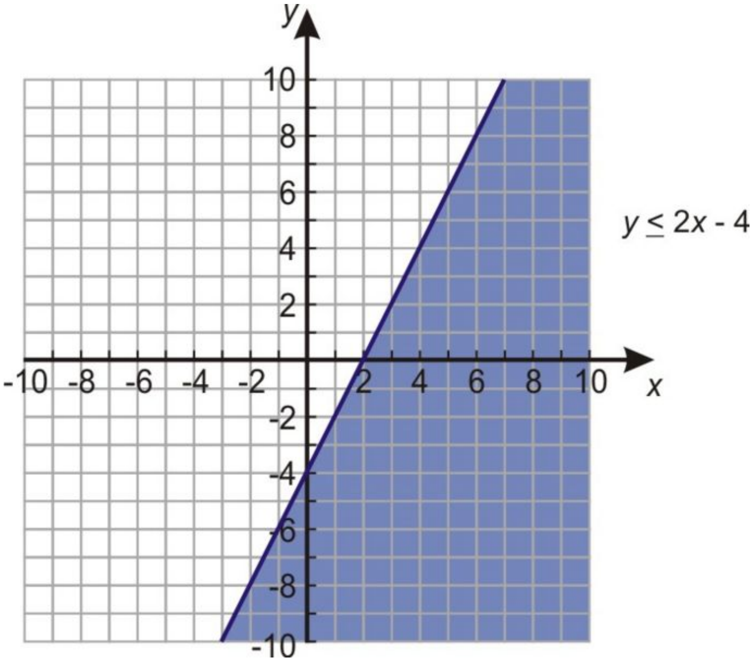
Example B

There are also situations where a system of inequalities has no solution. For example, let's solve this system.

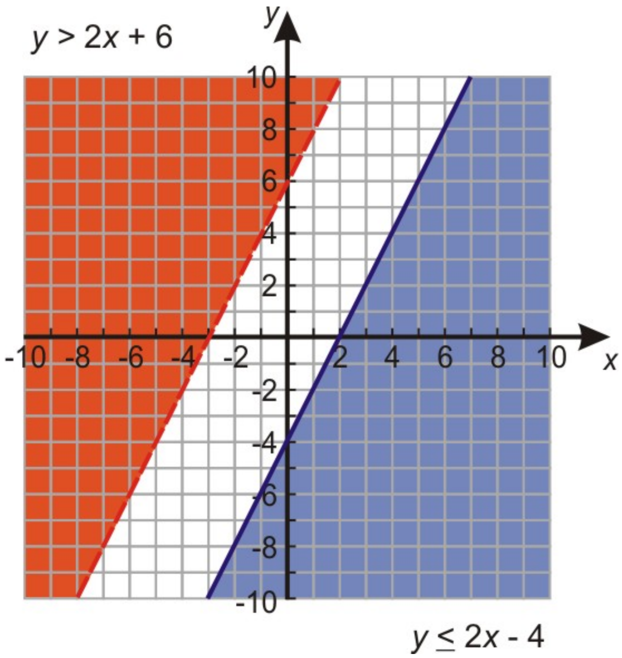
$$y \leq 2x - 4$$
$$y > 2x + 6$$

Solution

We start by graphing the first line. The line will be solid because the equals sign is included in the inequality. We must shade downwards because y is less than.



Next we graph the second line on the same coordinate axis. This line will be dashed because the equals sign is not included in the inequality. We must shade upward because *y* is greater than.



It doesn't look like the two shaded regions overlap at all. The two lines have the same slope, so we know they are parallel; that means that the regions indeed won't ever overlap since the lines won't ever cross. So this system of inequalities has no solution.

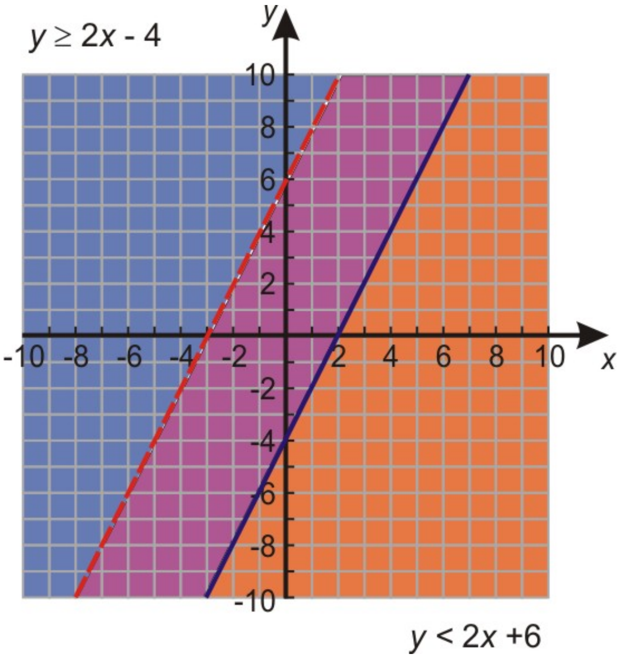
But a system of inequalities can sometimes have a solution even if the lines are parallel. For example, what happens if we swap the directions of the inequality signs in the system we just graphed?

To graph the system

$$
\begin{aligned}
y &\geq 2x-4 \\
y &< 2x+6
\end{aligned}
$$

,

we draw the same lines we drew for the previous system, but we shade *upward* for the first inequality and *downward* for the second inequality. Here is the result:



You can see that this time the shaded regions overlap. The area between the two lines is the solution to the system.

Graph a System of More Than Two Linear Inequalities

When we solve a system of just two linear inequalities, the solution is always an **unbounded** region—one that continues infinitely in at least one direction. But if we put together a system of more than two inequalities, sometimes we can get a solution that is **bounded**—a finite region with three or more sides.

Let's look at a simple example.

Example C

Find the solution to the following system of inequalities.

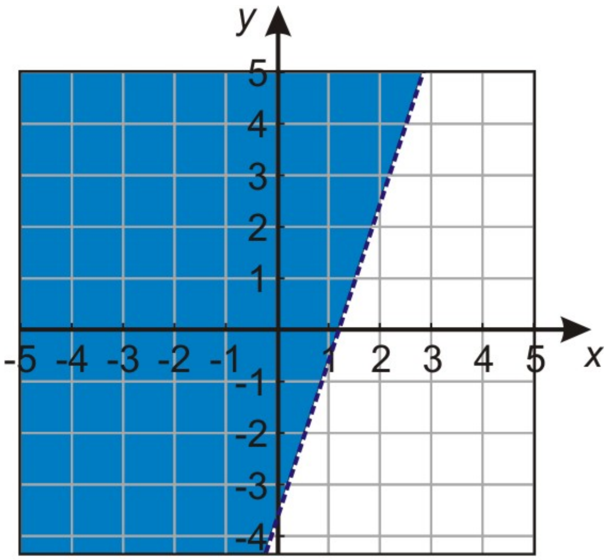
$$
\begin{aligned}
3x-y &< 4 \\
4y+9x &< 8 \\
x &\geq 0 \\
y &\geq 0
\end{aligned}
$$

Solution

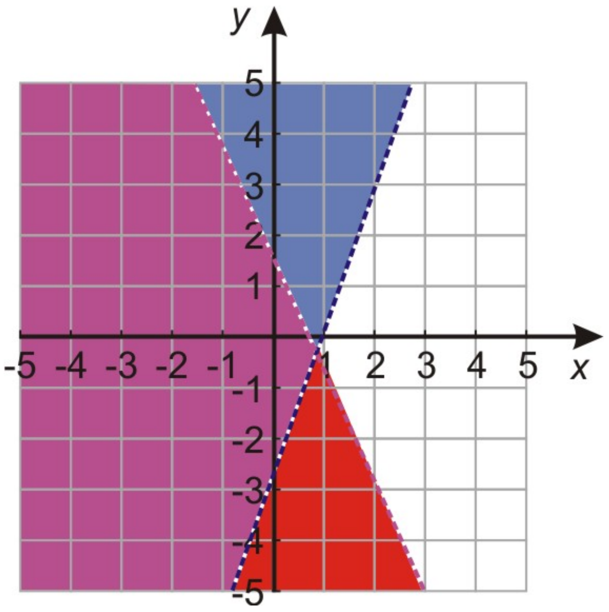
Let's start by writing our inequalities in slope-intercept form.

$$y > 3x - 4$$
$$y < -\frac{9}{4}x + 2$$
$$x \geq 0$$
$$y \geq 0$$

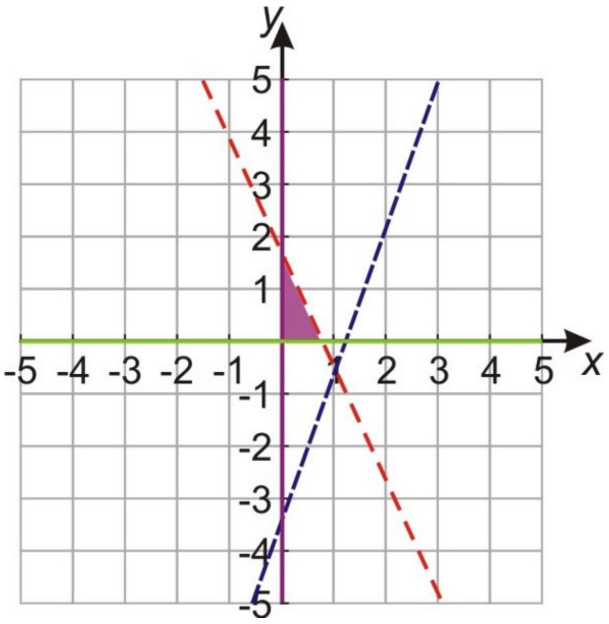
Now we can graph each line and shade appropriately. First we graph $y > 3x - 4$:



Next we graph $y < -\frac{9}{4}x + 2$:



Finally we graph $x \geq 0$ and $y \geq 0$, and we're left with the region below; this is where all four inequalities overlap.



The solution is **bounded** because there are lines on all sides of the solution region. In other words, the solution region is a bounded geometric figure, in this case a triangle.

Notice, too, that only three of the lines we graphed actually form the boundaries of the region. Sometimes when we graph multiple inequalities, it turns out that some of them don't affect the overall solution; in this case, the solution would be the same even if we'd left out the inequality $y > 3x-4$. That's because the solution region of the system formed by the other three inequalities is completely contained within the solution region of that fourth inequality; in other words, any solution to the other three inequalities is *automatically* a solution to that one too, so adding that inequality doesn't narrow down the solution set at all.

But that wasn't obvious until we actually drew the graph!

Watch this video for help with the Examples above.

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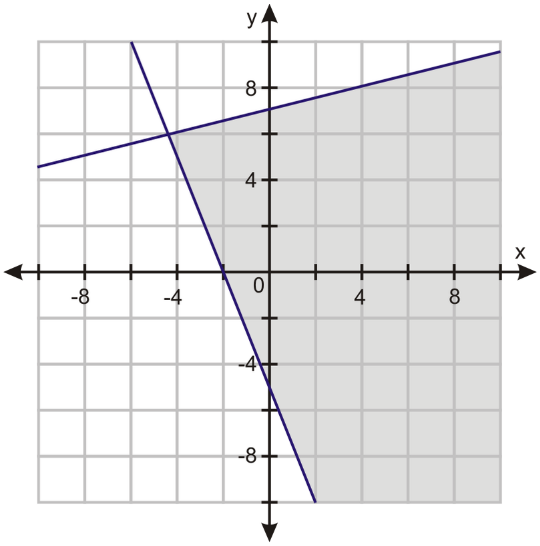
CK-12 Foundation: Systems ofLinear Inequalities

Vocabulary

- **Solution for the system of inequalities:** The ***solution for the system of inequalities*** is the common shaded region between all the inequalities in the system.
- **Feasible region:** The common shaded region of the system of inequalities is called the ***feasible region***.
- **Optimization:** The goal is to locate the feasible region of the system and use it to answer a profitability, or ***optimization***, question.

Guided Practice

Write the system of inequalities shown below.



Solution:

There are two boundary lines, so there are two inequalities. Write each one in slope-intercept form.

$$y \leq \frac{1}{4}x + 7$$
$$y \geq -\frac{5}{2}x - 5$$

Practice

1. Consider the system

$$y < 3x - 5$$
$$y > 3x - 5$$

   . Is it consistent or inconsistent? Why?
2. Consider the system

$$y \leq 2x + 3$$
$$y \geq 2x + 3$$

   . Is it consistent or inconsistent? Why?
3. Consider the system

$$y \leq -x + 1$$
$$y > -x + 1$$

   . Is it consistent or inconsistent? Why?
4. In example 3 in this lesson, we solved a system of four inequalities and saw that one of the inequalities, $y > 3x - 4$, didn't affect the solution set of the system.

   a. What would happen if we changed that inequality to $y < 3x - 4$?
   b. What's another inequality that we could add to the original system without changing it? Show how by sketching a graph of that inequality along with the rest of the system.
   c. What's another inequality that we could add to the original system to make it inconsistent? Show how by sketching a graph of that inequality along with the rest of the system.

5. Recall the compound inequalities in one variable that we worked with back in chapter 6. Compound inequalities with "and" are simply systems like the ones we are working with here, except with one variable instead of two.
   a. Graph the inequality $x > 3$ in two dimensions. What's another inequality that could be combined with it to make an inconsistent system?
   b. Graph the inequality $x \leq 4$ on a number line. What two-dimensional system would have a graph that looks just like this one?

Find the solution region of the following systems of inequalities.

6.
$$\begin{aligned} x - y &< -6 \\ 2y &\geq 3x + 17 \end{aligned}$$

7.
$$\begin{aligned} 4y - 5x &< 8 \\ -5x &\geq 16 - 8y \end{aligned}$$

8.
$$\begin{aligned} 5x - y &\geq 5 \\ 2y - x &\geq -10 \end{aligned}$$

9.
$$\begin{aligned} 5x + 2y &\geq -25 \\ 3x - 2y &\leq 17 \\ x - 6y &\geq 27 \end{aligned}$$

10.
$$\begin{aligned} 2x - 3y &\leq 21 \\ x + 4y &\leq 6 \\ 3x + y &\geq -4 \end{aligned}$$

11.
$$\begin{aligned} 12x - 7y &< 120 \\ 7x - 8y &\geq 36 \\ 5x + y &\geq 12 \end{aligned}$$

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9617.

7.11 Linear Programming

Here you'll learn how to analyze and find the feasible solution(s) to a system of inequalities under a given set of constraints.

What if you had an equation like $z = x + y$ in which a set of contraints like $x - y \leq 4$, $x + y \leq 2$, and $2x + 3y \geq -3$ were placed on it. How could you find the minimum and maximum values of z? After completing this Concept, you'll be able to analyze a system of inequalities to make the best decisions given the constraints of the situation.

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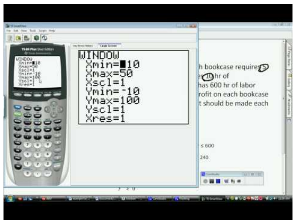
CK-12 Foundation: 0711S Linear Programming

Graphing calculators can be very useful for problems that involve this many inequalities. The following video shows a real-world linear programming problem worked through in detail on a graphing calculator, although the methods used there can also be used for pencil-and paper solving.



```
WINDOW
 Xmin=■10
 Xmax=50
 Xscl=1
 Ymin=-10
 Ymax=100
 Yscl=1
 Xres=1
```

h bookcase requires 5
es 10 hr of
has 600 hr of labor
rofit on each bookcase
t should be made each

≤ 600

240

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Stacy Reagan: LinearProgramming

Guidance

A lot of interesting real-world problems can be solved with systems of linear inequalities.

For example, you go to your favorite restaurant and you want to be served by your best friend who happens to work there. However, your friend only waits tables in a certain region of the restaurant. The restaurant is also known for its great views, so you want to sit in a certain area of the restaurant that offers a good view. Solving a system of linear inequalities will allow you to find the area in the restaurant where you can sit to get the best view and be served by your friend.

Often, systems of linear inequalities deal with problems where you are trying to find the best possible situation given a set of constraints. Most of these application problems fall in a category called **linear programming** problems.

Linear programming is the process of taking various linear inequalities relating to some situation, and finding the *best* possible value under those conditions. A typical example would be taking the limitations of materials and labor at a factory, then determining the best production levels for maximal profits under those conditions. These kinds of problems are used every day in the organization and allocation of resources. These real-life systems can have dozens or hundreds of variables, or more. In this section, we'll only work with the simple two-variable linear case.

The general process is to:

- Graph the inequalities (called **constraints**) to form a bounded area on the coordinate plane (called **the feasibility region**).
- Figure out the coordinates of the corners (or vertices) of this feasibility region by solving the system of equations that applies to each of the intersection points.
- Test these corner points in the formula (called the **optimization equation**) for which you're trying to find the **maximum** or **minimum** value.

Example A

If $z = 2x+5y$, *find the maximum and minimum values of* z *given these constraints:*

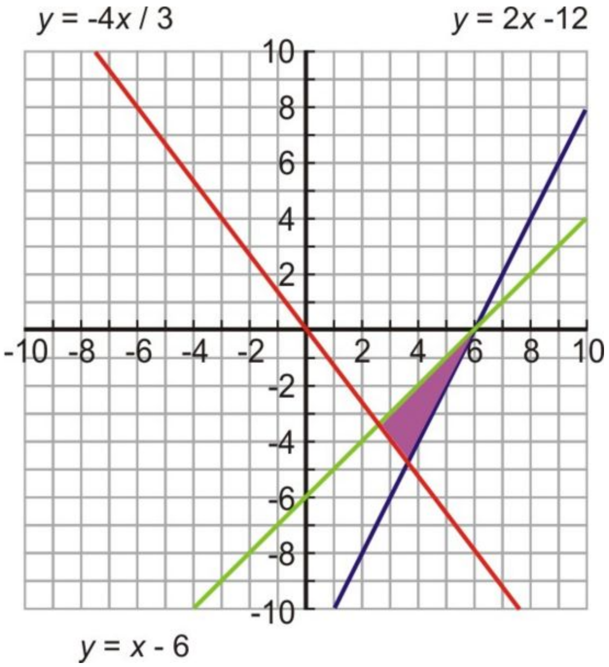
$$\begin{aligned} 2x-y &\leq 12 \\ 4x+3y &\geq 0 \\ x-y &\geq 6 \end{aligned}$$

Solution

First, we need to find the solution to this system of linear inequalities by graphing and shading appropriately. To graph the inequalities, we rewrite them in slope-intercept form:

$$\begin{aligned} y &\geq 2x-12 \\ y &\geq -\frac{4}{3}x \\ y &\leq x-6 \end{aligned}$$

These three linear inequalities are called the **constraints**, and here is their graph:



The shaded region in the graph is called the **feasibility region**. All possible solutions to the system occur in that region; now we must try to find the maximum and minimum values of the variable z within that region. In other words, which values of x and y within the feasibility region will give us the greatest and smallest overall values for the expression $2x+5y$?

Fortunately, we don't have to test every point in the region to find that out. It just so happens that the minimum or maximum value of the optimization equation in a linear system like this will always be found at one of the vertices (the corners) of the feasibility region; we just have to figure out *which* vertices. So for each vertex—each point where two of the lines on the graph cross—we need to solve the system of just those two equations, and then find the value of z at that point.

The first system consists of the equations $y=2x-12$ and $y=-\frac{4}{3}x$. We can solve this system by substitution:

$$-\frac{4}{3}x = 2x-12 \Rightarrow -4x = 6x-36 \Rightarrow -10x = -36 \Rightarrow x = 3.6$$
$$y = 2x-12 \Rightarrow y = 2(3.6)-12 \Rightarrow y = -4.8$$

The lines intersect at the point (3.6, -4.8).

The second system consists of the equations $y=2x-12$ and $y=x-6$. Solving this system by substitution:

$$x-6 = 2x-12 \Rightarrow 6 = x \Rightarrow x = 6$$
$$y = x-6 \Rightarrow y = 6-6 \Rightarrow y = 0$$

The lines intersect at the point (6, 0).

The third system consists of the equations $y=-\frac{4}{3}x$ and $y=x-6$. Solving this system by substitution:

$$x-6 = -\frac{4}{3}x \Rightarrow 3x-18 = -4x \Rightarrow 7x = 18 \Rightarrow x = 2.57$$
$$y = x-6 \Rightarrow y = 2.57-6 \Rightarrow y = -3.43$$

The lines intersect at the point (2.57, -3.43).

So now we have three different points that might give us the maximum and minimum values for z. To find out which ones actually do give the maximum and minimum values, we can plug the points into the optimization equation $z = 2x + 5y$.

When we plug in (3.6, -4.8), we get $z = 2(3.6) + 5(-4.8) = -16.8$.

When we plug in (6, 0), we get $z = 2(6) + 5(0) = 12$.

When we plug in (2.57, -3.43), we get $z = 2(2.57) + 5(-3.43) = -12.01$.

So we can see that **the point (6, 0) gives us the maximum possible value for z and the point (3.6, –4.8) gives us the minimum value.**

In the previous example, we learned how to apply the method of linear programming in the abstract. In the next example, we'll look at a real-life application.

Example B

You have $10,000 to invest, and three different funds to choose from. The municipal bond fund has a 5% return, the local bank's CDs have a 7% return, and a high-risk account has an expected 10% return. To minimize risk, you decide not to invest any more than $1,000 in the high-risk account. For tax reasons, you need to invest at least three times as much in the municipal bonds as in the bank CDs. What's the best way to distribute your money given these constraints?

Solution:

Let's define our variables:

x is the amount of money invested in the municipal bond at 5% return

y is the amount of money invested in the bank's CD at 7% return

$10000 - x - y$ is the amount of money invested in the high-risk account at 10% return

z is the total interest returned from all the investments, so $z = .05x + .07y + .1(10000 - x - y)$ or $z = 1000 - 0.05x - 0.03y$. This is the amount that we are trying to maximize. Our goal is to find the values of x and y that maximizes the value of z.

Now, let's write inequalities for the *constraints*:

You decide not to invest more than $1000 in the high-risk account—that means:

$$10000 - x - y \leq 1000$$

You need to invest at least three times as much in the municipal bonds as in the bank CDs—that means:

$$3y \leq x$$

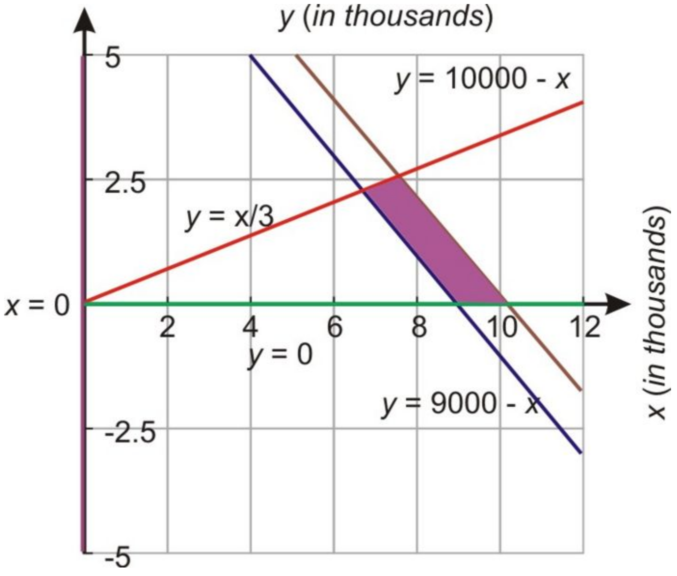
Also, you can't invest less than zero dollars in each account, so:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 10000 - x - y &\geq 0 \end{aligned}$$

To summarize, we must maximize the expression $z = 1000 - .05x - .03y$ using the constraints:

$$\begin{array}{lcl} 10000 - x - y \leq 1000 & & y \geq 9000 - x \\ 3y \leq x & & y \leq \frac{x}{3} \\ x \geq 0 & \text{Or in slope-intercept form:} & x \geq 0 \\ y \geq 0 & & y \geq 0 \\ 10000 - x - y \geq 0 & & y \leq 10000 - x \end{array}$$

Step 1: Find the solution region to the set of inequalities by graphing each line and shading appropriately. The following figure shows the overlapping region:



The purple region is the feasibility region where all the possible solutions can occur.

Step 2: Next we need to find the corner points of the feasibility region. Notice that there are four corners. To find their coordinates, we must pair up the relevant equations and solve each resulting system.

System 1:

$$\begin{aligned} y &= \frac{x}{3} \\ y &= 10000 - x \end{aligned}$$

Substitute the first equation into the second equation:

$$\begin{aligned} &\frac{x}{3} = 10000 - x \Rightarrow x = 30000 - 3x \Rightarrow 4x = 30000 \Rightarrow x = 7500 \\ &y = \frac{x}{3} \Rightarrow y = \frac{7500}{3} \Rightarrow y = 2500 \end{aligned}$$

The intersection point is (7500, 2500).

System 2:

$$y = \frac{x}{3}$$
$$y = 9000 - x$$

Substitute the first equation into the second equation:

$$\frac{x}{3} = 9000 - x \Rightarrow x = 27000 - 3x \Rightarrow 4x = 27000 \Rightarrow x = 6750$$
$$y = \frac{x}{3} \Rightarrow y = \frac{6750}{3} \Rightarrow y = 2250$$

The intersection point is (6750, 2250).

System 3:

$$y = 0$$
$$y = 10000 - x$$

The intersection point is (10000, 0).

System 4:

$$y = 0$$
$$y = 9000 - x$$

The intersection point is (9000, 0).

Step 3: In order to find the maximum value for z, we need to plug all the intersection points into the equation for z and find which one yields the largest number.

(7500, 2500): $z = 1000 - 0.05(7500) - 0.03(2500) = 550$

(6750, 2250): $z = 1000 - 0.05(6750) - 0.03(2250) = 595$

(10000, 0): $z = 1000 - 0.05(10000) - 0.03(0) = 500$

(9000, 0): $z = 1000 - 0.05(9000) - 0.03(0) = 550$

The maximum return on the investment of $595 occurs at the point (6750, 2250). This means that:

$6,750 is invested in the municipal bonds.

$2,250 is invested in the bank CDs.

$1,000 is invested in the high-risk account.

Example C

James is trying to expand his pastry business to include cupcakes and personal cakes. He has 40 hours available to decorate the new items and can use no more than 22 pounds of cake mix. Each personal cake requires 2 pounds of cake mix and 2 hours to decorate. Each cupcake order requires one pound of cake mix and 4 hours to decorate. If he can sell each personal cake for $14.99 and each cupcake order for $16.99, how many personal cakes and cupcake orders should James make to make the most revenue?

There are four inequalities in this situation. First, state the variables. Let $p =$ *the number of personal cakes* and $c =$ *the number of cupcake orders*.

Translate this into a system of inequalities.

$2p+1c \leq 22$ – This is the amount of available cake mix.

$2p+4c \leq 40$ – This is the available time to decorate.

$p \geq 0$ – You cannot make negative personal cakes.

$c \geq 0$ – You cannot make negative cupcake orders.

Now graph each inequality and determine the feasible region.

The feasible region has four vertices: {(0, 0),(0, 10),(11, 0),(8, 6)}. According to our theorem, the optimization answer will **only occur** at one of these vertices.

Write the optimization equation. How much of each type of order should James make to bring in the most revenue?

$$14.99p+16.99c = \textit{maximum revenue}$$

Substitute each ordered pair to determine which makes the most money.

$$\begin{aligned}(0,0) &\rightarrow \$0.00\\(0,10) &\rightarrow 14.99(0)+16.99(10) = \$169.90\\(11,0) &\rightarrow 14.99(11)+16.99(0) = \$164.89\\(8,6) &\rightarrow 14.99(8)+16.99(6) = \$221.86\end{aligned}$$

To make the most revenue, James should make 8 personal cakes and 6 cupcake orders.

Watch this video for help with the Examples above.

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CK-12 Foundation: LinearProgramming

Vocabulary

- **Linear programming** is the mathematical process of analyzing a system of inequalities to make the best decisions given the constraints of the situation.
- **Constraints** are the particular restrictions of a situation due to time, money, or materials.
- In an **optimization** problem, the goal is to locate the feasible region of the system and use it to answer a profitability, or ***optimization***, question.
- The **solution for the system of inequalities** is the common shaded region between all the inequalities in the system.
- The common shaded region of the system of inequalities is called the **feasible region**.

Guided Practice

Graph the solution to the following system:

$$\begin{aligned} x-y &< -6 \\ 2y &\geq 3x+17 \end{aligned}$$

Solution:

First we will rewrite the equations in slope-intercept form in order to graph them:

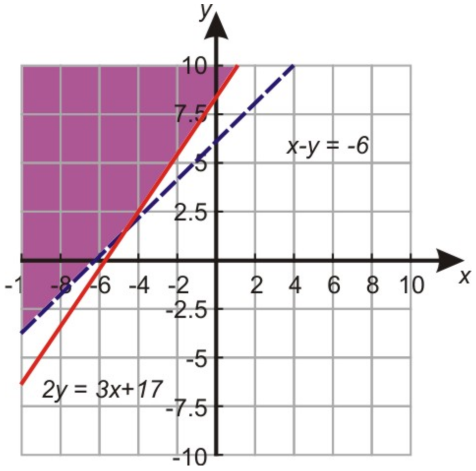
Inequality 1

$$\begin{aligned} x-y &< -6 && \text{Solve for y.} \\ -y &< -x-6 && \text{Subtract x from each side.} \\ y &> x+6 && \text{Multiply each side by -1, flipping the inequality.} \end{aligned}$$

Inequality 2

$$\begin{aligned} 2y \geq 3x+17 &< && \text{Solve for y.} \\ y \geq \frac{3}{2}x+8.5 &< && \text{Divide each side by 2.} \end{aligned}$$

Graph each equation and shade accordingly:



Practice

Solve the following linear programming problems.

1. Given the following constraints, find the maximum and minimum values of $z = -x + 5y$:

$$\begin{aligned} x + 3y &\le 0 \\ x - y &\ge 0 \\ 3x - 7y &\le 16 \end{aligned}$$

Santa Claus is assigning elves to work an eight-hour shift making toy trucks. Apprentice elves draw a wage of five candy canes per hour worked, but can only make four trucks an hour. Senior elves can make six trucks an hour and are paid eight candy canes per hour. There's only room for nine elves in the truck shop, and due to a candy-makers' strike, Santa Claus can only pay out 480 candy canes for the whole 8-hour shift.

2. How many senior elves and how many apprentice elves should work this shift to maximize the number of trucks that get made?
3. How many trucks will be made?
4. Just before the shift begins, the apprentice elves demand a wage increase; they insist on being paid seven candy canes an hour. Now how many apprentice elves and how many senior elves should Santa assign to this shift?
5. How many trucks will now get made, and how many candy canes will Santa have left over?

In Adrian's Furniture Shop, Adrian assembles both bookcases and TV cabinets. Each type of furniture takes her about the same time to assemble. She figures she has time to make at most 18 pieces of furniture by this Saturday. The materials for each bookcase cost her \$20 and the materials for each TV stand costs her \$45. She has \$600 to spend on materials. Adrian makes a profit of \$60 on each bookcase and a profit of \$100 on each TV stand.

6. Set up a system of inequalities. What $x-$ and $y-$values do you get for the point where Adrian's profit is maximized? Does this solution make sense in the real world?
7. What two possible real-world $x-$values and what two possible real-world $y-$values would be closest to the values in that solution?
8. With two choices each for x and y, there are four possible combinations of $x-$ and $y-$values. Of those four combinations, which ones actually fall within the feasibility region of the problem?
9. Which one of those feasible combinations seems like it would generate the most profit? Test out each one to confirm your guess. How much profit will Adrian make with that combination?

10. Based on Adrian's previous sales figures, she doesn't think she can sell more than 8 TV stands. Now how many of each piece of furniture should she make, and what will her profit be?
11. Suppose Adrian is confident she can sell all the furniture she can make, but she doesn't have room to display more than 7 bookcases in her shop. Now how many of each piece of furniture should she make, and what will her profit be?
12. Here's a "linear programming" problem on a line instead of a plane: Given the constraints $x \leq 5$ and $x \geq -2$, maximize the value of y where $y = x + 3$.

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9617.

Summary

This chapter focuses on solving systems of linear equations. First it provides strategies for determining if an ordered pair is a solution to a system. It then moves into methods to solve such systems, including graphing, substitution, and elimination through addition, subtraction, and multiplication. The chapter then distinguishes between dependent, consistent, and inconsistent linear systems. Systems of linear inequalities are addressed as well. Finally, it ends with an overview of linear programming, or the mathematical process of analyzing a system of inequalities to make the best decisions given the constraints of the situation

CHAPTER 8 Exponential Functions

Chapter Outline

8.1 EXPONENTIAL PROPERTIES INVOLVING PRODUCTS
8.2 EXPONENTIAL TERMS RAISED TO AN EXPONENT
8.3 EXPONENTIAL PROPERTIES INVOLVING QUOTIENTS
8.4 EXPONENT OF A QUOTIENT
8.5 NEGATIVE EXPONENTS
8.6 FRACTIONAL EXPONENTS
8.7 EVALUATING EXPONENTIAL EXPRESSIONS
8.8 SCIENTIFIC NOTATION
8.9 SCIENTIFIC NOTATION WITH A CALCULATOR
8.10 GEOMETRIC SEQUENCES AND EXPONENTIAL FUNCTIONS
8.11 GRAPHS OF EXPONENTIAL FUNCTIONS
8.12 APPLICATIONS OF EXPONENTIAL FUNCTIONS

Introduction

In the world of real estate, the size of homes and buildings is measured in “square feet.” What does this terminology mean? One square foot equals 1 foot x 1 foot or 1^2 foot. The two in this expression is an exponent and it represents repeated multiplication. The various exponential properties that allow you to combine multiple exponents into a single one are the focus of this chapter. Once you’ve mastered these techniques, you’ll be able to evaluate what at first appear to be complicated exponential expressions.

8.1 Exponential Properties Involving Products

Here you'll learn how to write repeated multiplication in exponential form. You'll also learn how to multiply and simplify exponential expressions.

What if you wanted to simplify a mathematical expression containing exponents, like $4^3 \cdot 4^2$? How would you do so? After completing this Concept, you'll be able to use the product of powers property to simplify exponential expressions like this one.

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CK-12 Foundation: 0801S Product of Powers

Guidance

Back in chapter 1, we briefly covered expressions involving exponents, like 3^5 or x^3. In these expressions, the number on the bottom is called the **base** and the number on top is the **power** or **exponent**. The whole expression is equal to the base multiplied by itself a number of times equal to the exponent; in other words, the exponent tells us how many copies of the base number to multiply together.

Example A

Write in exponential form.

a) $2 \cdot 2$

b) $(-3)(-3)(-3)$

c) $y \cdot y \cdot y \cdot y \cdot y$

d) $(3a)(3a)(3a)(3a)$

Solution

a) $2 \cdot 2 = 2^2$ because we have 2 factors of 2

b) $(-3)(-3)(-3) = (-3)^3$ because we have 3 factors of (-3)

c) $y \cdot y \cdot y \cdot y \cdot y = y^5$ because we have 5 factors of y

d) $(3a)(3a)(3a)(3a) = (3a)^4$ because we have 4 factors of $3a$

When the base is a variable, it's convenient to leave the expression in exponential form; if we didn't write x^7, we'd have to write $x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$ instead. But when the base is a number, we can simplify the expression further than that; for example, 2^7 equals $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, but we can multiply all those 2's to get 128.

Let's simplify the expressions from Example A.

Example B

Simplify.

a) 2^2

b) $(-3)^3$

c) y^5

d) $(3a)^4$

Solution

a) $2^2 = 2 \cdot 2 = 4$

b) $(-3)^3 = (-3)(-3)(-3) = -27$

c) y^5 is already simplified

d) $(3a)^4 = (3a)(3a)(3a)(3a) = 3 \cdot 3 \cdot 3 \cdot 3 \cdot a \cdot a \cdot a \cdot a = 81a^4$

Be careful when taking powers of negative numbers. Remember these rules:

$$(negative\ number) \cdot (positive\ number) = negative\ number$$
$$(negative\ number) \cdot (negative\ number) = positive\ number$$

So **even powers of negative numbers** are always positive. Since there are an even number of factors, we pair up the negative numbers and all the negatives cancel out.

$$\begin{aligned}(-2)^6 &= (-2)(-2)(-2)(-2)(-2)(-2)\\ &= \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4}\\ &= +64\end{aligned}$$

And **odd powers of negative numbers** are always negative. Since there are an odd number of factors, we can still pair up negative numbers to get positive numbers, but there will always be one negative factor left over, so the answer is negative:

$$\begin{aligned}(-2)^5 &= (-2)(-2)(-2)(-2)(-2)\\ &= \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)(-2)}_{+4} \cdot \underbrace{(-2)}_{-2}\\ &= -32\end{aligned}$$

Use the Product of Powers Property

So what happens when we multiply one power of x by another? Let's see what happens when we multiply x ***to the power of 5*** by x ***cubed***. To illustrate better, we'll use the full factored form for each:

$$\underbrace{(x \cdot x \cdot x \cdot x \cdot x)}_{x^5} \cdot \underbrace{(x \cdot x \cdot x)}_{x^3} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^8}$$

So $x^5 \times x^3 = x^8$. You may already see the pattern to multiplying powers, but let's confirm it with another example. We'll multiply x ***squared*** by x ***to the power of 4***:

$$\underbrace{(x \cdot x)}_{x^2} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^6}$$

So $x^2 \times x^4 = x^6$. Look carefully at the powers and how many factors there are in each calculation. 5 x's times 3 x's equals $(5+3) = 8$ x's. 2 x's times 4 x's equals $(2+4) = 6$ x's.

You should see that when we take the product of two powers of x, the number of x's in the answer is the total number of x's in all the terms you are multiplying. In other words, the exponent in the answer is the sum of the exponents in the product.

Product Rule for Exponents: $x^n \cdot x^m = x^{(n+m)}$

There are some easy mistakes you can make with this rule, however. Let's see how to avoid them.

Example C

Multiply $2^2 \cdot 2^3$.

Solution

$2^2 \cdot 2^3 = 2^5 = 32$

Note that when you use the product rule you **don't multiply the bases**. In other words, you must avoid the common error of writing $2^2 \cdot 2^3 = 4^5$. You can see this is true if you multiply out each expression: 4 times 8 is definitely 32, not 1024.

Example D

Multiply $2^2 \cdot 3^3$.

Solution

$2^2 \cdot 3^3 = 4 \cdot 27 = 108$

In this case, we can't actually use the product rule at all, because it only applies to terms that have the *same base*. In a case like this, where the bases are different, we just have to multiply out the numbers by hand—the answer is *not* 2^5 or 3^5 or 6^5 or anything simple like that.

Watch this video for help with the Examples above.

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CK-12 Foundation: Products of Powers

Vocabulary

- An ***exponent*** is a power of a number that shows how many times that number is multiplied by itself. An example would be 2^3. You would multiply 2 by itself 3 times: $2 \times 2 \times 2$. The number 2 is the ***base*** and the number 3 is the ***exponent.*** The value 2^3 is called the ***power.***

- **Product Rule for Exponents:** $x^n \cdot x^m = x^{(n+m)}$

Guided Practice

Simplify the following exponents:

a. $(-2)^5$

b. $(10x)^2$

Solutions:

a. $(-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32$

b. $(10x)^2 = 10^2 \cdot x^2 = 100x^2$

Practice

Write in exponential notation:

1. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
2. $3x \cdot 3x \cdot 3x$
3. $(-2a)(-2a)(-2a)(-2a)$
4. $6 \cdot 6 \cdot 6 \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y$
5. $2 \cdot x \cdot y \cdot 2 \cdot 2 \cdot y \cdot x$

Find each number.

6. 5^4
7. $(-2)^6$
8. $(0.1)^5$
9. $(-0.6)^3$
10. $(1.2)^2 + 5^3$
11. $3^2 \cdot (0.2)^3$

Multiply and simplify:

12. $6^3 \cdot 6^6$
13. $2^2 \cdot 2^4 \cdot 2^6$
14. $3^2 \cdot 4^3$
15. $x^2 \cdot x^4$
16. $(-2y^4)(-3y)$
17. $(4a^2)(-3a)(-5a^4)$

8.2 Exponential Terms Raised to an Exponent

Here you'll learn how to simplify exponential expressions that are raised to another secondary power.

What if you had an exponential expression that was raised to a secondary power, like $(2^3)^2$? How could you simplify it? After completing this Concept, you'll be able to use the power of a product property to simplify exponential expressions like this one.

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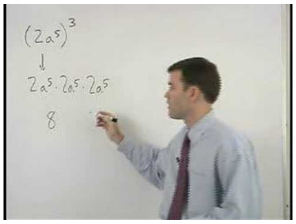
Foundation: 0802S Power of aProduct

The following video from YourTeacher.com may make it clearer how the power rule works for a variety of exponential expressions:



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YourTeacher:Power Rule-MultiplyingExponents

Guidance

What happens when we raise a whole expression to a power? Let's take x ***to the power of 4*** and ***cube it***. Again we'll use the full factored form for each expression:

$$(x^4)^3 = x^4 \times x^4 \times x^4 \qquad 3\ factors\ of\ \{x\ to\ the\ power\ 4\}$$
$$(x \cdot x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x = x^{12}$$

So $(x^4)^3 = x^{12}$. You can see that when we raise a power of x to a new power, the powers multiply.

Power Rule for Exponents: $(x^n)^m = x^{(n \cdot m)}$

If we have a product of more than one term inside the parentheses, then we have to distribute the exponent over all the factors, like distributing multiplication over addition. For example:

$$(x^2y)^4 = (x^2)^4 \cdot (y)^4 = x^8y^4.$$

Or, writing it out the long way:

$$\begin{aligned}(x^2y)^4 = (x^2y)(x^2y)(x^2y)(x^2y) &= (x \cdot x \cdot y)(x \cdot x \cdot y)(x \cdot x \cdot y)(x \cdot x \cdot y)\\ &= x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y = x^8y^4\end{aligned}$$

Note that this does NOT work if you have a sum or difference inside the parentheses! For example, $(x+y)^2 \neq x^2+y^2$. This is an easy mistake to make, but you can avoid it if you remember what an exponent means: if you multiply out $(x+y)^2$ it becomes $(x+y)(x+y)$, and that's not the same as x^2+y^2. We'll learn how we can simplify this expression in a later chapter.

Example A

Simplify the following expressions.

a) $3^5 \cdot 3^7$

b) $2^6 \cdot 2$

c) $(4^2)^3$

Solution

When we're just working with numbers instead of variables, we can use the product rule and the power rule, or we can just do the multiplication and then simplify.

a) We can use the product rule first and then evaluate the result: $3^5 \cdot 3^7 = 3^{12} = 531441$.

OR we can evaluate each part separately and then multiply them: $3^5 \cdot 3^7 = 243 \cdot 2187 = 531441$.

b) We can use the product rule first and then evaluate the result: $2^6 \cdot 2 = 2^7 = 128$.

OR we can evaluate each part separately and then multiply them: $2^6 \cdot 2 = 64 \cdot 2 = 128$.

c) We can use the power rule first and then evaluate the result: $(4^2)^3 = 4^6 = 4096$.

OR we can evaluate the expression inside the parentheses first, and then apply the exponent outside the parentheses: $(4^2)^3 = (16)^3 = 4096$.

Example B

Simplify the following expressions.

a) $x^2 \cdot x^7$

b) $(y^3)^5$

Solution

When we're just working with variables, all we can do is simplify as much as possible using the product and power rules.

a) $x^2 \cdot x^7 = x^{2+7} = x^9$

b) $(y^3)^5 = y^{3 \times 5} = y^{15}$

Example C

Simplify the following expressions.

a) $(3x^2y^3)\cdot(4xy^2)$

b) $(4xyz)\cdot(x^2y^3)\cdot(2yz^4)$

c) $(2a^3b^3)^2$

Solution

When we have a mix of numbers and variables, we apply the rules to each number and variable separately.

a) First we group like terms together: $(3x^2y^3)\cdot(4xy^2) = (3\cdot 4)\cdot(x^2\cdot x)\cdot(y^3\cdot y^2)$

Then we multiply the numbers or apply the product rule on each grouping: $= 12x^3y^5$

b) Group like terms together: $(4xyz)\cdot(x^2y^3)\cdot(2yz^4) = (4\cdot 2)\cdot(x\cdot x^2)\cdot(y\cdot y^3\cdot y)\cdot(z\cdot z^4)$

Multiply the numbers or apply the product rule on each grouping: $= 8x^3y^5z^5$

c) Apply the power rule for each separate term in the parentheses: $(2a^3b^3)^2 = 2^2\cdot(a^3)^2\cdot(b^3)^2$

Multiply the numbers or apply the power rule for each term $= 4a^6b^6$

Watch this video for help with the Examples above.

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CK-12 Foundation: Power of aProduct

Vocabulary

- **Exponent:** An ***exponent*** is a power of a number that shows how many times that number is multiplied by itself. An example would be 2^3. You would multiply 2 by itself 3 times: $2\times 2\times 2$. The number 2 is the ***base*** and the number 3 is the ***exponent.*** The value 2^3 is called the ***power.***

- **Product of Powers Property:** For all real numbers χ,

$\chi^n\cdot\chi^m = \chi^{n+m}$.

- **Power of a Product Property:** For all real numbers χ,

$$(\chi^n)^m = \chi^{n\cdot m}$$

.

Guided Practice

Simplify the following expressions.

a) $(x^2)^2 \cdot x^3$

b) $(2x^2y) \cdot (3xy^2)^3$

c) $(4a^2b^3)^2 \cdot (2ab^4)^3$

Solution

In problems where we need to apply the product and power rules together, we must keep in mind the order of operations. Exponent operations take precedence over multiplication.

a) We apply the power rule first: $(x^2)^2 \cdot x^3 = x^4 \cdot x^3$

Then apply the product rule to combine the two terms: $x^4 \cdot x^3 = x^7$

b) Apply the power rule first: $(2x^2y) \cdot (3xy^2)^3 = (2x^2y) \cdot (27x^3y^6)$

Then apply the product rule to combine the two terms: $(2x^2y) \cdot (27x^3y^6) = 54x^5y^7$

c) Apply the power rule on each of the terms separately: $(4a^2b^3)^2 \cdot (2ab^4)^3 = (16a^4b^6) \cdot (8a^3b^{12})$

Then apply the product rule to combine the two terms: $(16a^4b^6) \cdot (8a^3b^{12}) = 128a^7b^{18}$

Practice

Simplify:

1. $(a^3)^4$
2. $(xy)^2$
3. $(-5y)^3$
4. $(3a^2b^3)^4$
5. $(-2xy^4z^2)^5$
6. $(-8x)^3(5x)^2$
7. $(-x)^2(xy)^3$
8. $(4a^2)(-2a^3)^4$
9. $(12xy)(12xy)^2$
10. $(2xy^2)(-x^2y)^2(3x^2y^2)$

8.3 Exponential Properties Involving Quotients

Here you'll learn how to simplify one exponential expression that is divided by another.

What if you had a fractional expression like $\frac{x^5}{x^2}$ in which both the numerator and denominator contained exponents? How could you simplify it? After completing this Concept, you'll be able to use the quotient of powers property to simplify exponential expressions like this one.

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CK-12 Foundation: 0803S Quotientof Powers

Guidance

The rules for simplifying quotients of exponents are a lot like the rules for simplifying products.

Example A

Let's look at what happens when we divide x^7 by x^4:

$$\frac{x^7}{x^4} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = \frac{x \cdot x \cdot x}{1} = x^3$$

You can see that when we divide two powers of x, the number of x's in the solution is the number of x's in the top of the fraction minus the number of x's in the bottom. In other words, when dividing expressions with the same base, we keep the same base and simply subtract the exponent in the denominator from the exponent in the numerator.

Quotient Rule for Exponents: $\frac{x^n}{x^m} = x^{(n-m)}$

When we have expressions with more than one base, we apply the quotient rule separately for each base:

Now let's see what happens if the exponent in the denominator is bigger than the exponent in the numerator. For example, what happens when we apply the quotient rule to $\frac{x^4}{x^7}$?

The quotient rule tells us to subtract the exponents. 4 minus 7 is -3, so our answer is x^{-3}. A negative exponent! What does that mean?

Example B

$$\frac{x^5y^3}{x^3y^2} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}} \cdot \frac{\cancel{y} \cdot \cancel{y} \cdot y}{\cancel{y} \cdot \cancel{y}} = \frac{x \cdot x}{1} \cdot \frac{y}{1} = x^2y$$

OR

$$\frac{x^5y^3}{x^3y^2} = x^{5-3} \cdot y^{3-2} = x^2y$$

Well, let's look at what we get when we do the division longhand by writing each term in factored form:

$$\frac{x^4}{x^7} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x} = \frac{1}{x \cdot x \cdot x} = \frac{1}{x^3}$$

Even when the exponent in the denominator is bigger than the exponent in the numerator, we can still subtract the powers. The x's that are left over after the others have been canceled out just end up in the denominator instead of the numerator. Just as $\frac{x^7}{x^4}$ would be equal to $\frac{x^3}{1}$ (or simply x^3), $\frac{x^4}{x^7}$ is equal to $\frac{1}{x^3}$. And you can also see that $\frac{1}{x^3}$ is equal to x^{-3}. We'll learn more about negative exponents shortly.

Example C

Simplify the following expressions, leaving all exponents positive.

a) $\frac{x^2}{x^6}$

b) $\frac{a^2b^6}{a^5b}$

Solution

a) Subtract the exponent in the numerator from the exponent in the denominator and leave the x's in the denominator: $\frac{x^2}{x^6} = \frac{1}{x^{6-2}} = \frac{1}{x^4}$

b) Apply the rule to each variable separately: $\frac{a^2b^6}{a^5b} = \frac{1}{a^{5-2}} \cdot \frac{b^{6-1}}{1} = \frac{b^5}{a^3}$

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CK-12 Foundation: Quotient of Powers

Vocabulary

Quotient of Powers Property: For all real numbers x,

$\frac{x^n}{x^m} = x^{n-m}$.

Guided Practice

Simplify each of the following expressions using the quotient rule.

a) $\frac{x^{10}}{x^5}$

b) $\frac{a^6}{a}$

c) $\frac{a^5b^4}{a^3b^2}$

Solution

a) $\frac{x^{10}}{x^5} = x^{10-5} = x^5$

b) $\frac{a^6}{a} = a^{6-1} = a^5$

c) $\frac{a^5b^4}{a^3b^2} = a^{5-3} \cdot b^{4-2} = a^2b^2$

Review Questions

Evaluate the following expressions.

1. $\frac{5^6}{5^2}$
2. $\frac{6^7}{6^3}$
3. $\frac{3^4}{3^{10}}$
4. $\frac{2^2 \cdot 3^2}{5^2}$
5. $\frac{3^3 \cdot 5^2}{3^7}$

Simplify the following expressions.

6. $\frac{a^3}{a^2}$
7. $\frac{x^5}{x^9}$
8. $\frac{x^6y^2}{x^2y^5}$
9. $\frac{6a^3}{2a^2}$
10. $\frac{15x^5}{5x}$
11. $\frac{25yx^6}{20y^5x^2}$

8.4 Exponent of a Quotient

Here you'll learn how to simplify a fraction with exponential expressions in both its numerator and denominator that is raised to another secondary power.

What if you had a fractional expression containing exponents that was raised to a secondary power, like $\left(\frac{x^8}{x^4}\right)^5$? How could you simplify it? After completing this Concept, you'll be able to use the power of a quotient property to simplify exponential expressions like this one.

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CK-12 Foundation: 0804S Power ofa Quotient

Guidance

When we raise a whole quotient to a power, another special rule applies. Here is an example:

$$\begin{aligned}\left(\frac{x^3}{y^2}\right)^4 &= \left(\frac{x^3}{y^2}\right)\cdot\left(\frac{x^3}{y^2}\right)\cdot\left(\frac{x^3}{y^2}\right)\cdot\left(\frac{x^3}{y^2}\right)\\ &= \frac{(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)}{(y\cdot y)\cdot(y\cdot y)\cdot(y\cdot y)\cdot(y\cdot y)}\\ &= \frac{x^{12}}{y^8}\end{aligned}$$

Notice that the exponent outside the parentheses is multiplied by the exponent in the numerator and the exponent in the denominator, separately. This is called the power of a quotient rule:

Power Rule for Quotients: $\left(\frac{x^n}{y^m}\right)^p = \frac{x^{n\cdot p}}{y^{m\cdot p}}$

Let's apply these new rules to a few examples.

Example A

Simplify the following expressions.

a) $\frac{4^5}{4^2}$

b) $\frac{5^3}{5^7}$

c) $\left(\frac{3^4}{5^2}\right)^2$

Solution

Since there are just numbers and no variables, we can evaluate the expressions and get rid of the exponents completely.

a) We can use the quotient rule first and then evaluate the result: $\frac{4^5}{4^2} = 4^{5-2} = 4^3 = 64$

OR we can evaluate each part separately and then divide: $\frac{4^5}{4^2} = \frac{1024}{16} = 64$

b) Use the quotient rule first and hen evaluate the result: $\frac{5^3}{5^7} = \frac{1}{5^4} = \frac{1}{625}$

OR evaluate each part separately and then reduce: $\frac{5^3}{5^7} = \frac{125}{78125} = \frac{1}{625}$

Notice that it makes more sense to apply the quotient rule first for examples (a) and (b). Applying the exponent rules to simplify the expression *before* plugging in actual numbers means that we end up with smaller, easier numbers to work with.

c) Use the power rule for quotients first and then evaluate the result: $\left(\frac{3^4}{5^2}\right)^2 = \frac{3^8}{5^4} = \frac{6561}{625}$

OR evaluate inside the parentheses first and then apply the exponent: $\left(\frac{3^4}{5^2}\right)^2 = \left(\frac{81}{25}\right)^2 = \frac{6561}{625}$

Example B

Simplify the following expressions:

a) $\frac{x^{12}}{x^5}$

b) $\left(\frac{x^4}{x}\right)^5$

Solution

a) Use the quotient rule: $\frac{x^{12}}{x^5} = x^{12-5} = x^7$

b) Use the power rule for quotients and then the quotient rule: $\left(\frac{x^4}{x}\right)^5 = \frac{x^{20}}{x^5} = x^{15}$

OR use the quotient rule inside the parentheses first, then apply the power rule: $\left(\frac{x^4}{x}\right)^5 = (x^3)^5 = x^{15}$

Example C

Simplify the following expressions.

a) $\frac{6x^2y^3}{2xy^2}$

b) $\left(\frac{2a^3b^3}{8a^7b}\right)^2$

Solution

When we have a mix of numbers and variables, we apply the rules to each number or each variable separately.

a) Group like terms together: $\frac{6x^2y^3}{2xy^2} = \frac{6}{2} \cdot \frac{x^2}{x} \cdot \frac{y^3}{y^2}$

Then reduce the numbers and apply the quotient rule on each fraction to get $3xy$.

b) Apply the quotient rule inside the parentheses first: $\left(\frac{2a^3b^3}{8a^7b}\right)^2 = \left(\frac{b^2}{4a^4}\right)^2$

Then apply the power rule for quotients: $\left(\frac{b^2}{4a^4}\right)^2 = \frac{b^4}{16a^8}$

Watch this video for help with the Examples above.

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CK-12 Foundation: Power of aQuotient

Vocabulary

- **Quotient of Powers Property:** For all real numbers x,

$\frac{x^n}{x^m} = x^{n-m}$.

- **Power of a Quotient Property:**

$\left(\frac{x^n}{y^m}\right)^p = \frac{x^{n \cdot p}}{y^{m \cdot p}}$

Guided Practice

Simplify the following expressions.

a) $(x^2)^2 \cdot \frac{x^6}{x^4}$

b) $\left(\frac{16a^2}{4b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$

Solution

In problems where we need to apply several rules together, we must keep the order of operations in mind.

a) We apply the power rule first on the first term:

$$(x^2)^2 \cdot \frac{x^6}{x^4} = x^4 \cdot \frac{x^6}{x^4}$$

Then apply the quotient rule to simplify the fraction:

$$x^4 \cdot \frac{x^6}{x^4} = x^4 \cdot x^2$$

And finally simplify with the product rule:

$$x^4 \cdot x^2 = x^6$$

b) $\left(\frac{16a^2}{4b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$

Simplify inside the parentheses by reducing the numbers:

$$\left(\frac{4a^2}{b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$$

Then apply the power rule to the first fraction:

$$\left(\frac{4a^2}{b^5}\right)^3 \cdot \frac{b^2}{a^{16}} = \frac{64a^6}{b^{15}} \cdot \frac{b^2}{a^{16}}$$

Group like terms together:

$$\frac{64a^6}{b^{15}} \cdot \frac{b^2}{a^{16}} = 64 \cdot \frac{a^6}{a^{16}} \cdot \frac{b^2}{b^{15}}$$

And apply the quotient rule to each fraction:

$$64 \cdot \frac{a^6}{a^{16}} \cdot \frac{b^2}{b^{15}} = \frac{64}{a^{10}b^{13}}$$

Practice

Evaluate the following expressions.

1. $\left(\frac{3}{8}\right)^2$
2. $\left(\frac{2^2}{3^3}\right)^3$
3. $\left(\frac{2^3 \cdot 4^2}{2^4}\right)^2$

Simplify the following expressions.

4. $\left(\frac{a^3b^4}{a^2b}\right)^3$
5. $\left(\frac{18a^4}{15a^{10}}\right)^4$
6. $\left(\frac{x^6y^2}{x^4y^4}\right)^3$
7. $\left(\frac{6a^2}{4b^4}\right)^2 \cdot \frac{5b}{3a}$
8. $\frac{(2a^2bc^2)(6abc^3)}{4ab^2c}$
9. $\frac{(2a^2bc^2)(6abc^3)}{4ab^2c}$ for $a = 2, b = 1$, and $c = 3$
10. $\left(\frac{3x^2y}{2z}\right)^3 \cdot \frac{z^2}{x}$ for $x = 1, y = 2$, and $z = -1$
11. $\frac{2x^3}{xy^2} \cdot \left(\frac{x}{2y}\right)^2$ for $x = 2, y = -3$
12. $\frac{2x^3}{xy^2} \cdot \left(\frac{x}{2y}\right)^2$ for $x = 0, y = 6$
13. If $a = 2$ and $b = 3$, simplify $\frac{(a^2b)(bc)^3}{a^3c^2}$ as much as possible.

8.5 Negative Exponents

Here you'll learn how to simplify expressions that contain negative exponents.

What if you had a mathematical expression like $\frac{x^{-2}}{x^{-6}}$ that contained negative exponents? How could you simplify it so that none of the exponents were negative? After completing this Concept, you'll be able to simplify expressions with negative exponents like this one.

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Foundation: 0805S Negative Exponents

Guidance

The product and quotient rules for exponents lead to many interesting concepts. For example, so far we've mostly just considered positive, whole numbers as exponents, but you might be wondering what happens when the exponent isn't a positive whole number. What does it mean to raise something to the power of zero, or -1, or $\frac{1}{2}$? In this lesson, we'll find out.

Simplify Expressions With Negative Exponents

When we learned the quotient rule for exponents $\left(\frac{x^n}{x^m} = x^{(n-m)}\right)$, we saw that it applies even when the exponent in the denominator is bigger than the one in the numerator. Canceling out the factors in the numerator and denominator leaves the leftover factors in the denominator, and subtracting the exponents leaves a negative number. So negative exponents simply represent fractions with exponents in the denominator. This can be summarized in a rule:

Negative Power Rule for Exponents: $x^{-n} = \frac{1}{x^n}$, where $x \neq 0$

Negative exponents can be applied to products and quotients also. Here's an example of a negative exponent being applied to a product:

$$
\begin{aligned}
(x^3y)^{-2} &= x^{-6}y^{-2} && \text{using the power rule}\\
x^{-6}y^{-2} &= \frac{1}{x^6} \cdot \frac{1}{y^2} = \frac{1}{x^6y^2} && \text{using the negative power rule separately on each variable}
\end{aligned}
$$

And here's one applied to a quotient:

$\left(\frac{a}{b}\right)^{-3}=\frac{a^{-3}}{b^{-3}}$	using the power rule for quotients
$\frac{a^{-3}}{b^{-3}}=\frac{a^{-3}}{1}\cdot\frac{1}{b^{-3}}=\frac{1}{a^3}\cdot\frac{b^3}{1}$	using the negative power rule on each variable separately
$\frac{1}{a^3}\cdot\frac{b^3}{1}=\frac{b^3}{a^3}$	simplifying the division of fractions
$\frac{b^3}{a^3}=\left(\frac{b}{a}\right)^3$	using the power rule for quotients in reverse.

That last step wasn't really necessary, but putting the answer in that form shows us something useful: $\left(\frac{a}{b}\right)^{-3}$ is equal to $\left(\frac{b}{a}\right)^3$. This is an example of a rule we can apply more generally:

Negative Power Rule for Fractions: $\left(\frac{x}{y}\right)^{-n}=\left(\frac{y}{x}\right)^n$, where $x \neq 0, y \neq 0$

This rule can be useful when you want to write out an expression without using fractions.

Example A

Write the following expressions without fractions.

a) $\frac{1}{x}$

b) $\frac{2}{x^2}$

c) $\frac{x^2}{y^3}$

d) $\frac{3}{xy}$

Solution

a) $\frac{1}{x}=x^{-1}$

b) $\frac{2}{x^2}=2x^{-2}$

c) $\frac{x^2}{y^3}=x^2y^{-3}$

d) $\frac{3}{xy}=3x^{-1}y^{-1}$

Example B

Simplify the following expressions and write them without fractions.

a) $\frac{4a^2b^3}{2a^5b}$

b) $\left(\frac{x}{3y^2}\right)^3\cdot\frac{x^2y}{4}$

Solution

a) Reduce the numbers and apply the quotient rule to each variable separately:

$$\frac{4a^2b^3}{2a^5b}=2\cdot a^{2-5}\cdot b^{3-1}=2a^{-3}b^2$$

b) Apply the power rule for quotients first:

$$\left(\frac{2x}{y^2}\right)^3 \cdot \frac{x^2y}{4} = \frac{8x^3}{y^6} \cdot \frac{x^2y}{4}$$

Then simplify the numbers, and use the product rule on the x's and the quotient rule on the y's:

$$\frac{8x^3}{y^6} \cdot \frac{x^2y}{4} = 2 \cdot x^{3+2} \cdot y^{1-6} = 2x^5y^{-5}$$

You can also use the negative power rule the other way around if you want to write an expression without negative exponents.

Example C

Write the following expressions without negative exponents.

a) $3x^{-3}$

b) $a^2b^{-3}c^{-1}$

c) $4x^{-1}y^3$

d) $\frac{2x^{-2}}{y^{-3}}$

Solution

a) $3x^{-3} = \frac{3}{x^3}$

b) $a^2b^{-3}c^{-1} = \frac{a^2}{b^3c}$

c) $4x^{-1}y^3 = \frac{4y^3}{x}$

d) $\frac{2x^{-2}}{y^{-3}} = \frac{2y^3}{x^2}$

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CK-12 Foundation: Negative Exponents

Vocabulary

- **Negative Power Rule for Exponents:** $x^{-n} = \frac{1}{x^n}$, where $x \neq 0$.

Guided Practice

Simplify the following expressions and write the answers without negative powers.

a) $\left(\frac{ab^{-2}}{b^3}\right)^2$

b) $\frac{x^{-3}y^2}{x^2y^{-2}}$

Solution

a) Apply the quotient rule inside the parentheses: $\left(\frac{ab^{-2}}{b^3}\right)^2 = (ab^{-5})^2$

Then apply the power rule: $(ab^{-5})^2 = a^2b^{-10} = \frac{a^2}{b^{10}}$

b) Apply the quotient rule to each variable separately: $\frac{x^{-3}y^2}{x^2y^{-2}} = x^{-3-2}y^{2-(-2)} = x^{-5}y^4 = \frac{y^4}{x^5}$

Practice

Simplify the following expressions in such a way that there aren't any negative exponents in the answer.

1. $x^{-1}y^2$
2. x^{-4}
3. $\frac{x^{-3}}{x^{-7}}$
4. $\frac{x^{-3}y^{-5}}{z^{-7}}$
5. $\left(\frac{a}{b}\right)^{-2}$
6. $(3a^{-2}b^2c^3)^3$

Simplify the following expressions in such a way that there aren't any fractions in the answer.

7. $\frac{a^{-3}(a^5)}{a^{-6}}$
8. $\frac{5x^6y^2}{x^8y}$
9. $\frac{(4ab^6)^3}{(ab)^5}$
10. $\frac{(3x^3)(4x^4)}{(2y)^2}$
11. $\frac{a^{-2}b^{-3}}{c^{-1}}$

8.6 Fractional Exponents

Here you'll learn how to simplify expressions that contain zero and fractional exponents.

What if you had a mathematical expression like $\frac{x^{\frac{5}{8}}}{x^{\frac{1}{3}}}$ that contained fractional exponents? How could you simplify it? After completing this Concept, you'll be able to to simplify expressions like this one with fractional or zero exponents.

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CK-12 Foundation: 0806S Fractional Exponents

For more on zero and negative exponents, watch the following video at squidoo.com: http://www.google.com/url?sa=t&source=video&cd=4&ved=0CFMQtwIwAw&url=http%3A%2F%2Fwww.youtube.com%2Fwatch%3Fv%3D9svqGWwyN8Q&rct=j&q=negative%20exponents%20applet&ei=1fH6TP2IGoX4sAOnlbT3DQ&usg=AFQjCNHzLF4_-2aeo0dMWsa2wJ_CwzckXNA&cad=rja.

Guidance

In a previous concept we looked at the quotient rule for exponents: $\left(\frac{x^n}{x^m} = x^{(n-m)}\right)$. Consider what happens when $n = m$.

Example A

For example, what happens when we divide x^4 by x^4? Applying the quotient rule tells us that $\frac{x^4}{x^4} = x^{(4-4)} = x^0$—so what does that zero mean?

Well, we first discovered the quotient rule by considering how the factors of x cancel in such a fraction. Let's do that again with our example of x^4 divided by x^4:

$$\frac{x^4}{x^4} = \frac{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}}{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}} = 1$$

So $x^0 = 1$! You can see that this works for any value of the exponent, not just 4:

$$\frac{x^n}{x^n} = x^{(n-n)} = x^0$$

Since there is the same number of x's in the numerator as in the denominator, they cancel each other out and we get $x^0 = 1$. This rule applies for all expressions:

Zero Rule for Exponents: $x^0 = 1$, where $x \neq 0$

Simplify Expressions With Fractional Exponents

So far we've only looked at expressions where the exponents are positive and negative integers. The rules we've learned work exactly the same if the powers are fractions or irrational numbers—but what does a fractional exponent even mean? Let's see if we can figure that out by using the rules we already know.

Suppose we have an expression like $9^{\frac{1}{2}}$—how can we relate this expression to one that we already know how to work with? For example, how could we turn it into an expression that doesn't have any fractional exponents?

Well, the power rule tells us that if we raise an exponential expression to a power, we can multiply the exponents.

Example B

For example, if we raise $9^{\frac{1}{2}}$ to the power of 2, we get $\left(9^{\frac{1}{2}}\right)^2 = 9^{2\cdot\frac{1}{2}} = 9^1 = 9$.

So if $9^{\frac{1}{2}}$ *squared* equals 9, what does $9^{\frac{1}{2}}$ itself equal? Well, 3 is the number whose square is 9 (that is, it's the square root of 9), so $9^{\frac{1}{2}}$ must equal 3. And that's true for all numbers and variables: a number raised to the power of $\frac{1}{2}$ is just the square root of the number. We can write that as $\sqrt{x} = x^{\frac{1}{2}}$, and then we can see that's true because $\left(\sqrt{x}\right)^2 = x$ just as $\left(x^{\frac{1}{2}}\right)^2 = x$.

Similarly, a number to the power of $\frac{1}{3}$ is just the cube root of the number, and so on. In general, $x^{\frac{1}{n}} = \sqrt[n]{x}$. And when we raise a number to a power and then take the root of it, we still get a fractional exponent; for example, $\sqrt[3]{x^4} = \left(x^4\right)^{\frac{1}{3}} = x^{\frac{4}{3}}$. In general, the rule is as follows:

Rule for Fractional Exponents: $\sqrt[m]{a^n} = a^{\frac{n}{m}}$ and $\left(\sqrt[m]{a}\right)^n = a^{\frac{n}{m}}$

We'll examine roots and radicals in detail in a later chapter. In this section, we'll focus on how exponent rules apply to fractional exponents.

Example C

Simplify the following expressions.

a) $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}}$

b) $\left(a^{\frac{1}{3}}\right)^2$

c) $\frac{a^{\frac{5}{2}}}{a^{\frac{1}{2}}}$

d) $\left(\frac{x^2}{y^3}\right)^{\frac{1}{3}}$

Solution

a) Apply the product rule: $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{2}+\frac{1}{3}} = a^{\frac{5}{6}}$

b) Apply the power rule: $\left(a^{\frac{1}{3}}\right)^2 = a^{\frac{2}{3}}$

c) Apply the quotient rule: $\frac{a^{\frac{5}{2}}}{a^{\frac{1}{2}}} = a^{\frac{5}{2}-\frac{1}{2}} = a^{\frac{4}{2}} = a^2$

d) Apply the power rule for quotients: $\left(\frac{x^2}{y^3}\right)^{\frac{1}{3}} = \frac{x^{\frac{2}{3}}}{y}$

Watch this video for help with the Examples above.

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CK-12 Foundation: Zero and Fractional Exponents

Vocabulary

- **Zero Rule for Exponents:** $x^0 = 1$, where $x \neq 0$
- **Rule for Fractional Exponents:** $\sqrt[m]{a^n} = a^{\frac{n}{m}}$ and $\left(\sqrt[m]{a}\right)^n = a^{\frac{n}{m}}$

Guided Practice

Simplify the following expressions.

a) $\left(x^{\frac{2}{5}}\right)^5$

b) $\frac{y^{\frac{3}{4}}}{y^{\frac{1}{8}}}$

c) $\left(\frac{x^{2a}}{y^{4b}}\right)^{\frac{1}{2}}$

Solution

a) Apply the power rule: $\left(x^{\frac{2}{5}}\right)^5 = x^{\frac{2}{5}\cdot 5} = x^2$

b) Apply the quotient rule: $\frac{y^{\frac{3}{4}}}{y^{\frac{1}{8}}} = y^{\frac{3}{4}-\frac{1}{8}} = y^{\frac{6}{8}-\frac{1}{8}} = y^{\frac{5}{8}}$

c) Apply the power rule for quotients: $\left(\frac{x^{2a}}{y^{4b}}\right)^{\frac{1}{2}} = \frac{x^{2a\cdot\frac{1}{2}}}{y^{4b\cdot\frac{1}{2}}} = \frac{x^a}{y^{2b}}$

Practice

Simplify the following expressions in such a way that there aren't any negative exponents in the answer.

1. $(x^{\frac{1}{2}}y^{\frac{-2}{3}})(x^2y^{\frac{1}{3}})$
2. $x^{-3} \cdot x^3$
3. $y^5 \cdot y^{-5}$
4. $\frac{x^2y^{-3}}{x^{-4}y^{-2}} \cdot x^2$

Simplify the following expressions in such a way that there aren't any fractions in the answer.

5. $x^{\frac{1}{2}}y^{\frac{5}{2}}$
6. $\left(\frac{a}{b}\right)^{3/4}$
7. $\frac{3x^2y^{\frac{3}{2}}}{xy^{\frac{1}{2}}}$

8. $\frac{x^{\frac{1}{2}}y^{\frac{5}{2}}}{x^{\frac{3}{2}}y^{\frac{3}{2}}}$
9. $\left(\frac{a^2b^{\frac{1}{3}}}{a^3b}\right)^{\frac{1}{2}}$
10. $\left(\frac{a^{\frac{1}{2}}b}{ab^{\frac{1}{4}}}\right)^2$

8.7 Evaluating Exponential Expressions

Here you'll learn how to apply the order of operations to exponential expressions. You'll also learn how to evaluate exponential expressions for given values.

What if you had an exponential expression requiring multiple operations, like $2\left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^3$? How could you simplify it? After completing this Concept, you'll be able to use the order of operations to evaluate exponential expressions like this one.

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Foundation: 02807S Evaluating Exponential Expressions

Guidance

When evaluating expressions we must keep in mind the order of operations. You must remember **PEMDAS:**

1. Evaluate inside the **Parentheses**.
2. Evaluate **E**xponents.
3. Perform **M**ultiplication and **D**ivision operations from left to right.
4. Perform **A**ddition and **S**ubtraction operations from left to right.

Example A

Evaluate the following expressions.

a) 5^0

b) $\left(\frac{2}{3}\right)^3$

c) $16^{\frac{1}{2}}$

d) $8^{-\frac{1}{3}}$

Solution

a) $5^0 = 1$ A number raised to the power 0 is always 1.

b) $\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3} = \frac{8}{27}$

c) $16^{\frac{1}{2}} = \sqrt{16} = 4$ Remember that an exponent of $\frac{1}{2}$ means taking the square root.

d) $8^{-\frac{1}{3}} = \frac{1}{8^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$ Remember that an exponent of $\frac{1}{3}$ means taking the cube root.

Example B

Evaluate the following expressions.

a) $3 \cdot 5^2 - 10 \cdot 5 + 1$

b) $\frac{2 \cdot 4^2 - 3 \cdot 5^2}{3^2 - 2^2}$

c) $\left(\frac{3^3}{2^2}\right)^{-2} \cdot \frac{3}{4}$

Solution

a) Evaluate the exponent: $3 \cdot 5^2 - 10 \cdot 5 + 1 = 3 \cdot 25 - 10 \cdot 5 + 1$

Perform multiplications from left to right: $3 \cdot 25 - 10 \cdot 5 + 1 = 75 - 50 + 1$

Perform additions and subtractions from left to right: $75 - 50 + 1 = 26$

b) Treat the expressions in the numerator and denominator of the fraction like they are in parentheses: $\frac{(2 \cdot 4^2 - 3 \cdot 5^2)}{(3^2 - 2^2)} = \frac{(2 \cdot 16 - 3 \cdot 25)}{(9-4)} = \frac{(32-75)}{5} = \frac{-43}{5}$

c) $\left(\frac{3^3}{2^2}\right)^{-2} \cdot \frac{3}{4} = \left(\frac{2^2}{3^3}\right)^2 \cdot \frac{3}{4} = \frac{2^4}{3^6} \cdot \frac{3}{4} = \frac{2^4}{3^6} \cdot \frac{3}{2^2} = \frac{2^2}{3^5} = \frac{4}{243}$

Example C

Evaluate the following expressions for $x = 2, y = -1, z = 3$.

a) $2x^2 - 3y^3 + 4z$

b) $(x^2 - y^2)^2$

c) $\left(\frac{3x^2y^5}{4z}\right)^{-2}$

Solution

a)

$$\begin{aligned}2x^2 - 3y^3 + 4z &= 2 \cdot 2^2 - 3 \cdot (-1)^3 + 4 \cdot 3\\ &= 2 \cdot 4 - 3 \cdot (-1) + 4 \cdot 3 = 8 + 3 + 12\\ &= 23\end{aligned}$$

b) $(x^2 - y^2)^2 = (2^2 - (-1)^2)^2 = (4-1)^2 = 3^2 = 9$

c)

$$\begin{aligned}\left(\frac{3x^2y^5}{4z}\right)^{-2} &= \left(\frac{3\cdot 2^2\cdot(-1)^5}{4\cdot 3}\right)^{-2}\\ &= \left(\frac{3\cdot 4\cdot(-1)}{12}\right)^{-2}\\ &= \left(\frac{-12}{12}\right)^{-2}\\ &= \left(\frac{-1}{1}\right)^{-2}\\ &= \left(\frac{1}{-1}\right)^{2}\\ &= (-1)^2\\ &= 1\end{aligned}$$

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CK-12 Foundation: EvaluatingExponentialExpressions

Vocabulary

- When evaluating expressions we must keep in mind the order of operations. You must remember **PEMDAS:**

1. Evaluate inside the **Parentheses**.
2. Evaluate **E**xponents.
3. Perform **M**ultiplication and **D**ivision operations from left to right.
4. Perform **A**ddition and **S**ubtraction operations from left to right.

Guided Practice

Evaluate the following expression for $x = 3, y = -2, z = -1$.

$2z((x+1)^{\frac{1}{2}} - y^3)^2$

Solution:

$$\begin{aligned}2z((x+1)^{\frac{1}{2}} - y^3)^2 &= 2(-1)(((3)+1)^{\frac{1}{2}} - (-2)^3)^2\\ &= -2(4^{\frac{1}{2}} + 8)^2\\ &= -2(2+8)^2\\ &= -2(10)^2\\ &= -200\end{aligned}$$

Practice

Evaluate the following expressions to a single number.

1. 3^{-2}
2. -4^{-3}
3. $(6.2)^0$
4. $8^{-4} \cdot 8^6$
5. $\left(16^{\frac{1}{2}}\right)^3$
6. $x^2 \cdot 4x^3 \cdot y^4 \cdot 4y^2$, if $x = 2$ and $y = -1$
7. $a^4(b^2)^3 + 2ab$, if $a = -2$ and $b = 1$
8. $5x^2 - 2y^3 + 3z$, if $x = 3, y = 2$, and $z = 4$
9. $\left(\frac{a^2}{b^3}\right)^{-2}$, if $a = 5$ and $b = 3$
10. $\left(\frac{x^{-2}}{y^4}\right)^{\frac{1}{2}}$, if $x = -3$ and $y = 2$

8.8 Scientific Notation

Here you'll learn how to write very large and very small numbers so that they are easier to work with and evaluate.

What if you knew that the population of the United States was 308,000,000? How could you simplify this number so that it is easier to work with? After completing this Concept, you'll be able to write very large and very small numbers like this one in scientific notation.

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Foundation: 0808S ScientificNotation

Guidance

Consider the number six hundred and forty three thousand, two hundred and ninety seven. We write it as 643,297 and each digit's position has a "value" assigned to it. You may have seen a table like this before:

hundred-thousands	ten-thousands	thousands	hundreds	tens	units
6	4	3	2	9	7

We've seen that when we write an exponent above a number, it means that we have to multiply a certain number of copies of that number together. We've also seen that a zero exponent always gives us 1, and negative exponents give us fractional answers.

Look carefully at the table above. Do you notice that all the column headings are powers of ten? Here they are listed:

$$\begin{aligned}100,000 &= 10^5\\10,000 &= 10^4\\1,000 &= 10^3\\100 &= 10^2\\10 &= 10^1\end{aligned}$$

Even the "units" column is really just a power of ten. ***Unit*** means 1, and 1 is 10^0.

If we divide 643,297 by 100,000 we get 6.43297; if we multiply 6.43297 by 100,000 we get 643, 297. But we have just seen that 100,000 is the same as 10^5, so if we multiply 6.43297 by 10^5 we should also get 643,297. In other words,

$$643,297 = 6.43297 \times 10^5$$

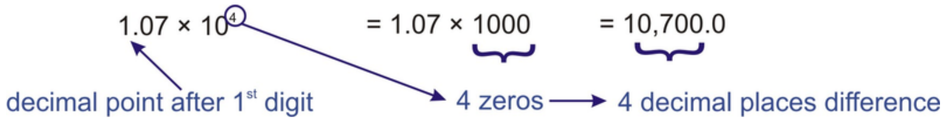
Writing Numbers in Scientific Notation

In scientific notation, numbers are always written in the form $a \times 10^b$, where b is an integer and a is between 1 and 10 (that is, it has exactly 1 nonzero digit before the decimal). This notation is especially useful for numbers that are either very small or very large.

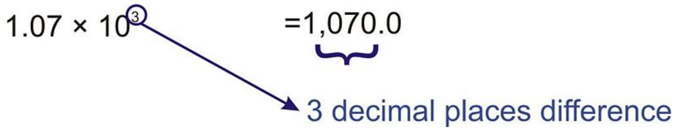
Here's a set of examples:

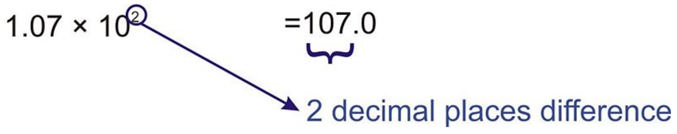
$$\begin{aligned}
1.07 \times 10^4 &= 10,700 \\
1.07 \times 10^3 &= 1,070 \\
1.07 \times 10^2 &= 107 \\
1.07 \times 10^1 &= 10.7 \\
1.07 \times 10^0 &= 1.07 \\
1.07 \times 10^{-1} &= 0.107 \\
1.07 \times 10^{-2} &= 0.0107 \\
1.07 \times 10^{-3} &= 0.00107 \\
1.07 \times 10^{-4} &= 0.000107
\end{aligned}$$

Look at the first example and notice where the decimal point is in both expressions.



So the exponent on the ten acts to move the decimal point over to the right. An exponent of 4 moves it 4 places and an exponent of 3 would move it 3 places.





This makes sense because each time you multiply by 10, you move the decimal point one place to the right. 1.07 times 10 is 10.7, times 10 again is 107.0, and so on.

Similarly, if you look at the later examples in the table, you can see that a negative exponent on the 10 means the decimal point moves that many places to the left. This is because multiplying by 10^{-1} is the same as multiplying by $\frac{1}{10}$, which is like dividing by 10. So instead of moving the decimal point one place to the right for every multiple of 10, we move it one place to the left for every multiple of $\frac{1}{10}$.

That's how to convert numbers from scientific notation to standard form. When we're converting numbers *to* scientific notation, however, we have to apply the whole process backwards. First we move the decimal point until it's immediately after the first nonzero digit; then we count how many places we moved it. If we moved it to the *left,* the exponent on the 10 is positive; if we moved it to the *right,* it's negative.

So, for example, to write 0.000032 in scientific notation, we'd first move the decimal five places to the right to get 3.2; then, since we moved it right, the exponent on the 10 should be *negative* five, so the number in scientific notation is 3.2×10^{-5}.

You can double-check whether you've got the right direction by comparing the number in scientific notation with the number in standard form, and thinking "Does this represent a *big* number or a *small* number?" A positive exponent on the 10 represents a number bigger than 10 and a negative exponent represents a number smaller than 10, and you can easily tell if the number in standard form is bigger or smaller than 10 just by looking at it.

For more practice, try the online tool at http://hotmath.com/util/hm_flash_movie.html?movie=/learning_activities/interactivities/sciNotation.swf. Click the arrow buttons to move the decimal point until the number in the middle is written in proper scientific notation, and see how the exponent changes as you move the decimal point.

Example A

Write the following numbers in scientific notation.

a) 63

b) 9,654

c) 653,937,000

d) 0.003

e) 0.000056

f) 0.00005007

Solution

a) $63 = 6.3 \times 10 = 6.3 \times 10^1$

b) $9,654 = 9.654 \times 1,000 = 9.654 \times 10^3$

c) $653,937,000 = 6.53937000 \times 100,000,000 = 6.53937 \times 10^8$

d) $0.003 = 3 \times \frac{1}{1000} = 3 \times 10^{-3}$

e) $0.000056 = 5.6 \times \frac{1}{100,000} = 5.6 \times 10^{-5}$

f) $0.00005007 = 5.007 \times \frac{1}{100,000} = 5.007 \times 10^{-5}$

Example B

Evaluate the following expressions and write your answer in scientific notation.

a) $(3.2 \times 10^6) \cdot (8.7 \times 10^{11})$

b) $(5.2 \times 10^{-4}) \cdot (3.8 \times 10^{-19})$

c) $(1.7 \times 10^6) \cdot (2.7 \times 10^{-11})$

Solution

The key to evaluating expressions involving scientific notation is to group the powers of 10 together and deal with them separately.

a) $(3.2\times10^6)(8.7\times10^{11}) = \underbrace{3.2\times8.7}_{27.84}\times\underbrace{10^6\times10^{11}}_{10^{17}} = 27.84\times10^{17}$. But 27.84×10^{17} isn't in proper scientific notation, because it has more than one digit before the decimal point. We need to move the decimal point one more place to the left and add 1 to the exponent, which gives us 2.784×10^{18}.

b)

$$\begin{aligned}(5.2\times10^{-4})(3.8\times10^{-19}) &= \underbrace{5.2\times3.8}_{19.76}\times\underbrace{10^{-4}\times10^{-19}}_{10^{-23}}\\ &= 19.76\times10^{-23}\\ &= 1.976\times10^{-22}\end{aligned}$$

c) $(1.7\times10^6)(2.7\times10^{-11}) = \underbrace{1.7\times2.7}_{4.59}\times\underbrace{10^6\times10^{-11}}_{10^{-5}} = 4.59\times10^{-5}$

When we use scientific notation in the real world, we often round off our calculations. Since we're often dealing with very big or very small numbers, it can be easier to round off so that we don't have to keep track of as many digits—and scientific notation helps us with that by saving us from writing out all the extra zeros. For example, if we round off 4,227, 457,903 to 4,200,000,000, we can then write it in scientific notation as simply 4.2×10^9.

When rounding, we often talk of **significant figures** or **significant digits**. Significant figures include

- all nonzero digits
- all zeros that come *before* a nonzero digit and *after* either a decimal point or another nonzero digit

For example, the number 4000 has one significant digit; the zeros don't count because there's no nonzero digit after them. But the number 4000.5 has five significant digits: the 4, the 5, and all the zeros in between. And the number 0.003 has three significant digits: the 3 and the two zeros that come between the 3 and the decimal point.

Example C

Evaluate the following expressions. Round to 3 significant figures and write your answer in scientific notation.

a) $(3.2\times10^6)\div(8.7\times10^{11})$

b) $(5.2\times10^{-4})\div(3.8\times10^{-19})$

Solution

It's easier if we convert to fractions and THEN separate out the powers of 10.

a)

$$\begin{aligned}(3.2\times10^6)\div(8.7\times10^{11}) &= \frac{3.2\times10^6}{8.7\times10^{11}} && -separate\ out\ the\ powers\ of\ 10:\\ &= \frac{3.2}{8.7}\times\frac{10^6}{10^{11}} && -evaluate\ each\ fraction\ (round\ to\ 3\ s.f.):\\ &= 0.368\times10^{(6-11)}\\ &= 0.368\times10^{-5} && -remember\ how\ to\ write\ scientific\ notation!\\ &= 3.68\times10^{-6}\end{aligned}$$

b)

$$\begin{aligned}
(5.2\times10^{-4})\div(3.8\times10^{-19}) &= \frac{5.2\times10^{-4}}{3.8\times10^{-19}} && -\mathit{separate\ the\ powers\ of}\ 10:\\
&= \frac{5.2}{3.8}\times\frac{10^{-4}}{10^{-19}} && -\mathit{evaluate\ each\ fraction}\ (\mathit{round\ to}\ 3\ s.f.)\\
&= 1.37\times10^{((-4)-(-19))}\\
&= 1.37\times10^{15}
\end{aligned}$$

Watch this video for help with the Examples above.

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CK-12 Foundation: ScientificNotation

Vocabulary

- In **scientific notation**, numbers are always written in the form $a\times10^{b}$, where b is an integer and a is between 1 and 10 (that is, it has exactly 1 nonzero digit before the decimal).

Guided Practice

Evaluate the following expression. Round to 3 significant figures and write your answer in scientific notation.

$(1.7\times10^{6})\div(2.7\times10^{-11})$

Solution:

$$\begin{aligned}
(1.7\times10^{6})\div(2.7\times10^{-11}) &= \frac{1.7\times10^{6}}{2.7\times10^{-11}} && -\mathit{next\ we\ separate\ the\ powers\ of}\ 10:\\
&= \frac{1.7}{2.7}\times\frac{10^{6}}{10^{-11}} && -\mathit{evaluate\ each\ fraction}\ (\mathit{round\ to}\ 3\ s.f.)\\
&= 0.630\times10^{(6-(-11))}\\
&= 0.630\times10^{17}\\
&= 6.30\times10^{16}
\end{aligned}$$

Note that we have to leave in the final zero to indicate that the result has been rounded.

Practice

Write the numerical value of the following.

1. 3.102×10^{2}
2. 7.4×10^{4}

3. 1.75×10^{-3}
4. 2.9×10^{-5}
5. 9.99×10^{-9}

Write the following numbers in scientific notation.

6. 120,000
7. 1,765,244
8. 12
9. 0.00281
10. 0.000000027

How many significant digits are in each of the following?

11. 38553000
12. 2754000.23
13. 0.0000222
14. 0.0002000079

Round each of the following to two significant digits.

15. 3.0132
16. 82.9913

8.9 Scientific Notation with a Calculator

Here you'll learn how to use a calculator to evaluate scientific notation expressions, and how to work with real-world applications involving scientific notation.

What if you knew that a milligram was one-millionth of a kilogram? How could you express this relationship exponentially? After completing this Concept, you'll be able to solve real-world problems like this one that involve scientific notation.

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Foundation: 02809S Applications Using ScientificNotation

Guidance

Let's look at some real-world applications involving scientific notation.

Example A

The mass of a single lithium atom is approximately one percent of one millionth of one billionth of one billionth of one kilogram. Express this mass in scientific notation.

Solution

We know that a *percent* is $\frac{1}{100}$, and so our calculation for the mass (in kg) is:

$$\frac{1}{100} \times \frac{1}{1,000,000} \times \frac{1}{1,000,000,000} \times \frac{1}{1,000,000,000} = 10^{-2} \times 10^{-6} \times 10^{-9} \times 10^{-9}$$

Next we use the product of powers rule we learned earlier:

$$10^{-2} \times 10^{-6} \times 10^{-9} \times 10^{-9} = 10^{((-2)+(-6)+(-9)+(-9))} = 10^{-26}\ kg.$$

The mass of one lithium atom is approximately $1 \times 10^{-26}\ kg$.

Example B

You could fit about 3 million E. coli bacteria on the head of a pin. If the size of the pin head in question is $1.2 \times 10^{-5}\ m^2$, *calculate the area taken up by one E. coli bacterium. Express your answer in scientific notation*

Solution

Since we need our answer in scientific notation, it makes sense to convert 3 million to that format first:

$$3,000,000 = 3 \times 10^6$$

Next we need an expression involving our unknown, the area taken up by one bacterium. Call this A.

$$3 \times 10^6 \cdot A = 1.2 \times 10^{-5} \qquad - since\ 3\ million\ of\ them\ make\ up\ the\ area\ of\ the\ pin-head$$

Isolate A:

$$\begin{aligned}
A &= \frac{1}{3 \times 10^6} \cdot 1.2 \times 10^{-5} && - rearranging\ the\ terms\ gives: \\
A &= \frac{1.2}{3} \cdot \frac{1}{10^6} \times 10^{-5} && - then\ using\ the\ definition\ of\ a\ negative\ exponent: \\
A &= \frac{1.2}{3} \cdot 10^{-6} \times 10^{-5} && - evaluate\ \&\ \ combine\ exponents\ using\ the\ product\ rule: \\
A &= 0.4 \times 10^{-11} && - but\ we\ can't\ leave\ our\ answer\ like\ this,\ so\ldots
\end{aligned}$$

The area of one bacterium is $4.0 \times 10^{-12}\ m^2$.

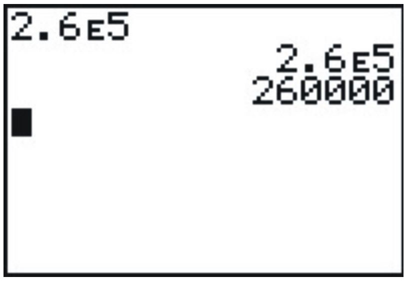
(Notice that we had to move the decimal point over one place to the right, subtracting 1 from the exponent on the 10.)

Evaluate Expressions in Scientific Notation Using a Graphing Calculator

All scientific and graphing calculators can use scientific notation, and it's very useful to know how.

To insert a number in scientific notation, use the **[EE]** button. This is **[2nd] [,]** on some TI models.

For example, to enter 2.6×10^5, enter 2.6 **[EE]** 5. When you hit **[ENTER]** the calculator displays 2.6E5 if it's set in **Scientific** mode, or 260000 if it's set in **Normal** mode.



```
2.6E5
                2.6E5
               260000
█
```

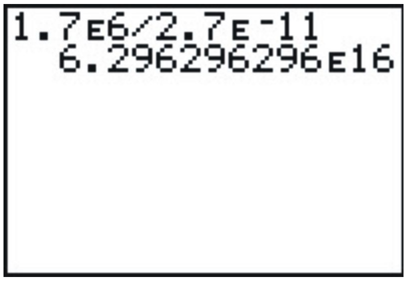
(To change the mode, press the 'Mode' key.)

Example C

Evaluate $(2.3 \times 10^6) \times (4.9 \times 10^{-10})$ *using a graphing calculator.*

Solution

Enter 2.3 **[EE]** 6 × 4.9**[EE]** - 10 and press **[ENTER]**.



```
1.7E6/2.7E-11
   6.296296296E16
```

The calculator displays 6.296296296E16 whether it's in Normal mode or Scientific mode. That's because the number is so big that even in Normal mode it won't fit on the screen. The answer displayed instead isn't the precisely correct answer; it's rounded off to 10 significant figures.

Since it's a repeating decimal, though, we can write it more efficiently *and* more precisely as $6.\overline{296} \times 10^{16}$.

Watch this video for help with the Examples above.

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CK-12 Foundation: Applications usingScientific Notation

Vocabulary

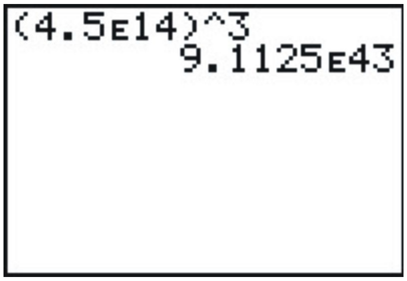
- In **scientific notation**, numbers are always written in the form $a \times 10^b$, where b is an integer and a is between 1 and 10 (that is, it has exactly 1 nonzero digit before the decimal).

Guided Practice

Evaluate $(4.5 \times 10^{14})^3$ *using a graphing calculator.*

Solution

Enter (4.5 **[EE]** 14)^3 and press **[ENTER]**.



```
(4.5E14)^3
        9.1125E43
```

The calculator displays 9.1125E43. The answer is 9.1125×10^{43}.

Practice

For questions 1-9, use a calculator to evaluate the expression.

1. $(3.5 \times 10^4) \cdot (2.2 \times 10^7)$
2. $\frac{2.1 \times 10^9}{3 \times 10^2}$
3. $(3.1 \times 10^{-3}) \cdot (1.2 \times 10^{-5})$
4. $\frac{7.4 \times 10^{-5}}{3.7 \times 10^{-2}}$
5. $12,000,000 \times 400,000$
6. $3,000,000 \times 0.00000000022$
7. $\frac{17,000}{680,000,000}$
8. $\frac{25,000,000}{0.000000005}$
9. $\frac{0.0000000000042}{0.00014}$

10. The moon is approximately a sphere with radius $r = 1.08 \times 10^3 \ miles$. Use the formula Surface Area $= 4\pi r^2$ to determine the surface area of the moon, in square miles. Express your answer in scientific notation, rounded to two significant figures.
11. The charge on one electron is approximately 1.60×10^{19} coulombs. One **Faraday** is equal to the total charge on 6.02×10^{23} electrons. What, in coulombs, is the charge on one Faraday?
12. Proxima Centauri, the next closest star to our Sun, is approximately 2.5×10^{13} miles away. If light from Proxima Centauri takes 3.7×10^4 hours to reach us from there, calculate the speed of light in miles per hour. Express your answer in scientific notation, rounded to 2 significant figures.

8.10 Geometric Sequences and Exponential Functions

Here you'll learn how to find missing terms in geometric sequences. You'll also learn how to solve real-world applications involving geometric sequences.

What if you had a series of numbers like 1, 3, 9, 27 ... in which each number was found by multiplying the previous number by a fixed amount? How could you find the fifth and sixth numbers in this series? After completing this Concept, you'll be able to solve problems like this one that involve geometric sequences.

Try This

For more practice finding the terms in geometric sequences, try the browser game at http://www.ies-math.com/math/java/misc/sum/sum.html.

Watch This

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Click image to the left for more content.

Foundation: 0810S LessonGeometric Sequences

Guidance

Consider the following question:

Which would you prefer, being given one million dollars, or one penny the first day, double that penny the next day, and then double the previous day's pennies and so on for a month?

At first glance it's easy to say "Give me the million!" But why don't we do a few calculations to see how the other choice stacks up?

You start with a penny the first day and keep doubling each day. Doubling means that we keep multiplying by 2 each day for one month (30 days).

On the first day, you get 1 penny, or 2^0 pennies.

On the second day, you get 2 pennies, or 2^1 pennies.

On the third day, you get 4 pennies, or 2^2 pennies. Do you see the pattern yet?

On the fourth day, you get 8 pennies, or 2^3 pennies. Each day, the exponent is one less than the number of that day.

So on the thirtieth day, you get 2^{29} pennies, which is 536,870,912 pennies, or $5,368,709.12. That's a lot more than a million dollars, even just counting the amount you get on that one day!

This problem is an example of a geometric sequence. In this section, we'll find out what a geometric sequence is and how to solve problems involving geometric sequences.

Identify a Geometric Sequence

A **geometric sequence** is a sequence of numbers in which each number in the sequence is found by multiplying the previous number by a fixed amount called the **common ratio.** In other words, the ratio between any term and the term before it is always the same. In the previous example the common ratio was 2, as the number of pennies doubled each day.

The common ratio, r, in any geometric sequence can be found by dividing any term by the preceding term.

Here are some examples of geometric sequences and their common ratios.

$$\begin{array}{lll} 4, 16, 64, 256, \ldots & r = 4 & (\text{divide 16 by 4 to get 4}) \\ 15, 30, 60, 120, \ldots & r = 2 & (\text{divide 30 by 15 to get 2}) \\ 11, \frac{11}{2}, \frac{11}{4}, \frac{11}{8}, \frac{11}{16}, \ldots & r = \frac{1}{2} & \left(\text{divide } \frac{11}{2} \text{ by 11 to get } \frac{1}{2}\right) \\ 25, -5, 1, -\frac{1}{5}, \frac{1}{25}, \ldots & r = -\frac{1}{5} & \left(\text{divide 1 by -5 to get} -\frac{1}{5}\right) \end{array}$$

If we know the common ratio r, we can find the next term in the sequence just by multiplying the last term by r. Also, if there are any terms missing in the sequence, we can find them by multiplying the term before each missing term by the common ratio.

Example A

Fill in the missing terms in each geometric sequence.

a) 1, ___, 25, 125, ___

b) 20, ___, 5, ___, 1.25

Solution

a) First we can find the common ratio by dividing 125 by 25 to obtain $r = 5$.

To find the first missing term, we multiply 1 by the common ratio: $1 \cdot 5 = 5$

To find the second missing term, we multiply 125 by the common ratio: $125 \cdot 5 = 625$

Sequence (a) becomes: **1, 5, 25, 125, 625,...**

b) We need to find the common ratio first, but how do we do that when we have no terms next to each other that we can divide?

Well, we know that to get from 20 to 5 in the sequence we must multiply 20 by the common ratio *twice*: once to get to the second term in the sequence, and again to get to five. So we can say $20 \cdot r \cdot r = 5$, or $20 \cdot r^2 = 5$.

Dividing both sides by 20, we get $r^2 = \frac{5}{20} = \frac{1}{4}$, or $r = \frac{1}{2}$ (because $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$).

To get the first missing term, we multiply 20 by $\frac{1}{2}$ and get 10.

To get the second missing term, we multiply 5 by $\frac{1}{2}$ and get 2.5.

Sequence (b) becomes: **20, 10, 5, 2.5, 1.25,...**

You can see that if we keep multiplying by the common ratio, we can find any term in the sequence that we want—the tenth term, the fiftieth term, the thousandth term.... However, it would be awfully tedious to keep multiplying over and over again in order to find a term that is a long way from the start. What could we do instead of just multiplying repeatedly?

Let's look at a geometric sequence that starts with the number 7 and has common ratio of 2.

The 1^{st} term is:	$7 = 7 \cdot 2^0$
We obtain the 2^{nd} term by multiplying by 2 :	$7 \cdot 2 = 7 \cdot 2^1$
We obtain the 3^{rd} term by multiplying by 2 again:	$7 \cdot 2 \cdot 2 = 7 \cdot 2^2$
We obtain the 4^{th} term by multiplying by 2 again:	$7 \cdot 2 \cdot 2 \cdot 2 = 7 \cdot 2^3$
We obtain the 5^{th} term by multiplying by 2 again:	$7 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 7 \cdot 2^4$
The nth term would be:	$7 \cdot 2^{n-1}$

The nth term is $7 \cdot 2^{n-1}$ because the 7 is multiplied by 2 once for the 2^{nd} term, twice for the third term, and so on—for each term, one less time than the term's place in the sequence. In general, we write a geometric sequence with n terms like this:

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}$$

The formula for finding a specific term in a geometric sequence is:

n^{th} term in a geometric sequence: $a_n = a_1 r^{n-1}$

(a_1 = first term, r = common ratio)

Example B

For each of these geometric sequences, find the eighth term in the sequence.

a) 1, 2, 4,...

b) 16, -8, 4, -2, 1,...

Solution

a) First we need to find the common ratio: $r = \frac{2}{1} = 2$.

The eighth term is given by the formula $a_8 = a_1 r^7 = 1 \cdot 2^7 = 128$

In other words, to get the eighth term we start with the first term, which is 1, and then multiply by 2 seven times.

b) The common ratio is $r = \frac{-8}{16} = \frac{-1}{2}$

The eighth term in the sequence is $a_8 = a_1 r^7 = 16 \cdot \left(\frac{-1}{2}\right)^7 = 16 \cdot \frac{(-1)^7}{2^7} = 16 \cdot \frac{-1}{2^7} = \frac{-16}{128} = -\frac{1}{8}$

Let's take another look at the terms in that second sequence. Notice that they alternate **positive, negative, positive, negative** all the way down the list. When you see this pattern, you know the common ratio is negative; multiplying by a negative number each time means that the sign of each term is opposite the sign of the previous term.

Solve Real-World Problems Involving Geometric Sequences

Let's solve two application problems involving geometric sequences.

Example C

A courtier presented the Indian king with a beautiful, hand-made chessboard. The king asked what he would like in return for his gift and the courtier surprised the king by asking for one grain of rice on the first square, two grains on the second, four grains on the third, etc. The king readily agreed and asked for the rice to be brought. (From Meadows et al. 1972, via Porritt 2005) How many grains of rice does the king have to put on the last square?

Solution

A chessboard is an 8×8 square grid, so it contains a total of 64 squares.

The courtier asked for one grain of rice on the first square, 2 grains of rice on the second square, 4 grains of rice on the third square and so on. We can write this as a geometric sequence:

1, 2, 4,...

The numbers double each time, so the common ratio is $r = 2$.

The problem asks how many grains of rice the king needs to put on the last square, so we need to find the 64^{th} term in the sequence. Let's use our formula:

$a_n = a_1 r^{n-1}$, where a_n is the nth term, a_1 is the first term and r is the common ratio.

$a_{64} = 1 \cdot 2^{63} = 9,223,372,036,854,775,808$**grains of rice.**

The problem we just solved has real applications in business and technology. In technology strategy, the **Second Half of the Chessboard** is a phrase, coined by a man named Ray Kurzweil, in reference to the point where an exponentially growing factor begins to have a significant economic impact on an organization's overall business strategy.

The total number of grains of rice on the **first half** of the chessboard is $1+2+4+8+16+32+64+128+256+512+1024\ldots+2,147,483,648$, for a total of exactly 4,294,967,295 grains of rice, or about 100,000 kg of rice (the mass of one grain of rice being roughly 25 mg). This total amount is about $\frac{1}{1,000,000^{th}}$ of total rice production in India in the year 2005 and is an amount the king could surely have afforded.

The total number of grains of rice on the **second half** of the chessboard is $2^{32}+2^{33}+2^{34}\ldots+2^{63}$, for a total of 18, 446, 744, 069, 414, 584, 320 grains of rice. This is about 460 billion tons, or 6 times the entire weight of all living matter on Earth. The king didn't realize what he was agreeing to—perhaps he should have studied algebra! [Wikipedia; GNU-FDL]

Watch this video for help with the Examples above.

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CK-12 Foundation: Geometric Sequences

Vocabulary

- A **geometric sequence** is a sequence of numbers in which each number in the sequence is found by multiplying the previous number by a fixed amount called the **common ratio.** In other words, the ratio between any term and the term before it is always the same. In the previous example the common ratio was 2, as the number of pennies doubled each day.
- The **common ratio**, r, in any geometric sequence can be found by dividing any term by the preceding term.

Guided Practice

A super-ball has a 75% rebound ratio—that is, when it bounces repeatedly, each bounce is 75% as high as the previous bounce. When you drop it from a height of 20 feet:

a) *how high does the ball bounce after it strikes the ground for the third time?*

b) *how high does the ball bounce after it strikes the ground for the seventeenth time?*

Solution

We can write a geometric sequence that gives the height of each bounce with the common ratio of $r = \frac{3}{4}$:

$$20, 20 \cdot \frac{3}{4}, 20 \cdot \left(\frac{3}{4}\right)^2, 20 \cdot \left(\frac{3}{4}\right)^3 \ldots$$

a) The ball starts at a height of 20 feet; after the first bounce it reaches a height of $20 \cdot \frac{3}{4} = 15\ feet$.

After the second bounce it reaches a height of $20 \cdot \left(\frac{3}{4}\right)^2 = 11.25\ feet$.

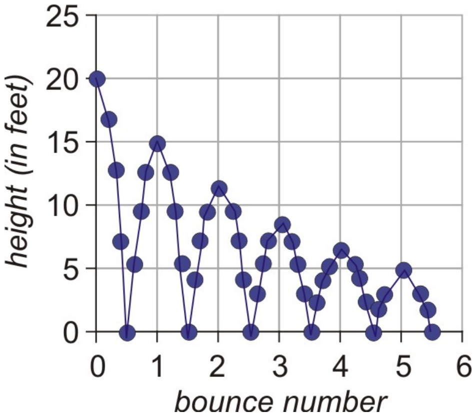
After the third bounce it reaches a height of $20 \cdot \left(\frac{3}{4}\right)^3 = 8.44\ feet$.

b) Notice that the height after the first bounce corresponds to the second term in the sequence, the height after the second bounce corresponds to the third term in the sequence and so on.

This means that the height after the seventeenth bounce corresponds to the 18^{th} term in the sequence. You can find the height by using the formula for the 18^{th} term:

$$a_{18} = 20 \cdot \left(\frac{3}{4}\right)^{17} = 0.15\ feet.$$

Here is a graph that represents this information. (The heights at points other than the top of each bounce are just approximations.)



Practice

Determine the first five terms of each geometric sequence.

1. $a_1 = 2, r = 3$
2. $a_1 = 90, r = -\frac{1}{3}$
3. $a_1 = 6, r = -2$
4. $a_1 = 1, r = 5$
5. $a_1 = 5, r = 5$
6. $a_1 = 25, r = 5$

7. What do you notice about the last three sequences?

Find the missing terms in each geometric sequence.

8. 3, __ , 48, 192, __
9. 81, __ , __ , __ , 1
10. $\frac{9}{4}$, __ , __ , $\frac{2}{3}$, __
11. 2, __ , __ , -54, 162

Find the indicated term of each geometric sequence.

12. $a_1 = 4, r = 2$; find a_6
13. $a_1 = -7, r = -\frac{3}{4}$; find a_4
14. $a_1 = -10, r = -3$; find a_{10}
15. In a geometric sequence, $a_3 = 28$ and $a_5 = 112$; find r and a_1.
16. In a geometric sequence, $a_2 = 28$ and $a_5 = 112$; find r and a_1.
17. As you can see from the previous two questions, the same terms can show up in sequences with different ratios.
    a. Write a geometric sequence that has 1 and 9 as two of the terms (not necessarily the first two).
    b. Write a different geometric sequence that also has 1 and 9 as two of the terms.
    c. Write a geometric sequence that has 6 and 24 as two of the terms.
    d. Write a different geometric sequence that also has 6 and 24 as two of the terms.
    e. What is the common ratio of the sequence whose first three terms are 2, 6, 18?
    f. What is the common ratio of the sequence whose first three terms are 18, 6, 2?
    g. What is the relationship between those ratios?
18. Anne goes bungee jumping off a bridge above water. On the initial jump, the bungee cord stretches by 120 feet. On the next bounce the stretch is 60% of the original jump and each additional bounce the rope stretches by 60% of the previous stretch.
    a. What will the rope stretch be on the third bounce?
    b. What will be the rope stretch be on the 12^{th} bounce?

8.11 Graphs of Exponential Functions

Here you'll learn how to graph exponential functions and how to compare the graphs of exponential functions on the same coordinate axes.

What if you had an exponential function like $y = 2 \cdot (3^x)$.? How could you graph this function? After completing this Concept, you'll be able to graph and compare the graphs of exponential functions like this one.

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Foundation: 08011S Exponential Functions

Guidance

A colony of bacteria has a population of three thousand at noon on Monday. During the next week, the colony's population doubles every day. What is the population of the bacteria colony just before midnight on Saturday?

At first glance, this seems like a problem you could solve using a geometric sequence. And you could, if the bacteria population doubled all at once every day; since it doubled every day for five days, the final population would be 3000 times 2^5.

But bacteria don't reproduce all at once; their population grows slowly over the course of an entire day. So how do we figure out the population after five *and a half* days?

Exponential Functions

Exponential functions are a lot like geometrical sequences. The main difference between them is that a geometric sequence is **discrete** while an exponential function is **continuous**.

Discrete means that the sequence has values only at distinct points (the 1st term, 2nd term, etc.)

Continuous means that the function has values for all possible values of x. The integers are included, but also all the numbers in between.

The problem with the bacteria is an example of a continuous function. Here's an example of a discrete function:

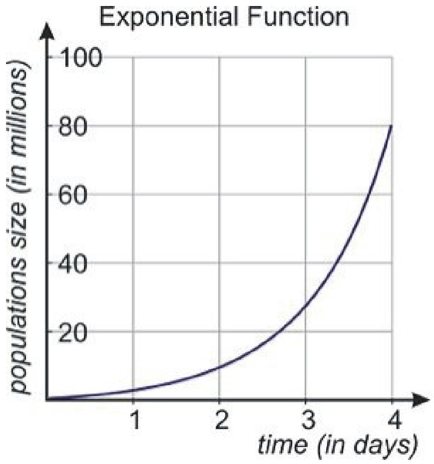
An ant walks past several stacks of Lego blocks. There is one block in the first stack, 3 blocks in the 2^{nd} *stack and 9 blocks in the* 3^{rd} *stack. In fact, in each successive stack there are triple the number of blocks than in the previous stack.*

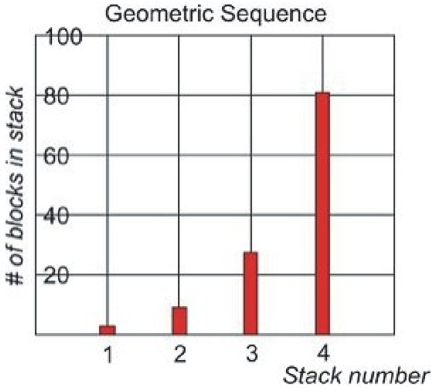
In this example, each stack has a distinct number of blocks and the next stack is made by adding a certain number of whole pieces all at once. More importantly, however, there are no values of the sequence **between** the stacks. You can't ask how high the stack is between the 2^{nd} and 3^{rd} stack, as no stack exists at that position!

As a result of this difference, we use a geometric series to describe quantities that have values at discrete points, and we use exponential functions to describe quantities that have values that change continuously.

When we graph an exponential function, we draw the graph with a solid curve to show that the function has values at any time during the day. On the other hand, when we graph a geometric sequence, we draw discrete points to signify that the sequence only has value at those points but not in between.

Here are graphs for the two examples above:





The formula for an exponential function is similar to the formula for finding the terms in a geometric sequence. An exponential function takes the form

$$y = A \cdot b^x$$

where A is the starting amount and b is the amount by which the total is multiplied every time. For example, the bacteria problem above would have the equation $y = 3000 \cdot 2^x$.

Compare Graphs of Exponential Functions

Let's graph a few exponential functions and see what happens as we change the constants in the formula. The basic shape of the exponential function should stay the same—but it may become steeper or shallower depending on the constants we are using.

First, let's see what happens when we change the value of A.

Example A

Compare the graphs of $y = 2^x$ *and* $y = 3 \cdot 2^x$.

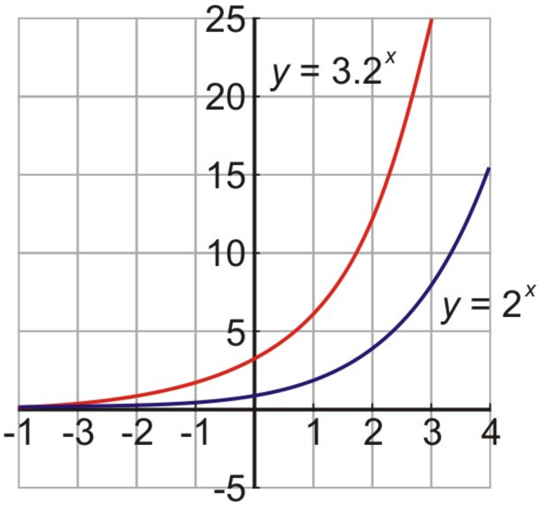
Solution

Let's make a table of values for both functions.

TABLE 8.1:

x	$y = 2^x$	$y = 3 \cdot 2^x$
-3	$\frac{1}{8}$	$y = 3 \cdot 2^{-3} = 3 \cdot \frac{1}{2^3} = \frac{3}{8}$
-2	$\frac{1}{4}$	$y = 3 \cdot 2^{-2} = 3 \cdot \frac{1}{2^2} = \frac{3}{4}$
-1	$\frac{1}{2}$	$y = 3 \cdot 2^{-1} = 3 \cdot \frac{1}{2^1} = \frac{3}{2}$
0	1	$y = 3 \cdot 2^0 = 3$
1	2	$y = 3 \cdot 2^1 = 6$
2	4	$y = 3 \cdot 2^2 = 3 \cdot 4 = 12$
3	8	$y = 3 \cdot 2^3 = 3 \cdot 8 = 24$

Now let's use this table to graph the functions.



We can see that the function $y = 3 \cdot 2^x$ is bigger than the function $y = 2^x$. In both functions, the value of y doubles every time x increases by one. However, $y = 3 \cdot 2^x$ "starts" with a value of 3, while $y = 2^x$ "starts" with a value of 1, so it makes sense that $y = 3 \cdot 2^x$ would be bigger as its values of y keep getting doubled.

Similarly, if the starting value of A is smaller, the values of the entire function will be smaller.

Example B

Compare the graphs of $y = 2^x$ *and* $y = \frac{1}{3} \cdot 2^x$.

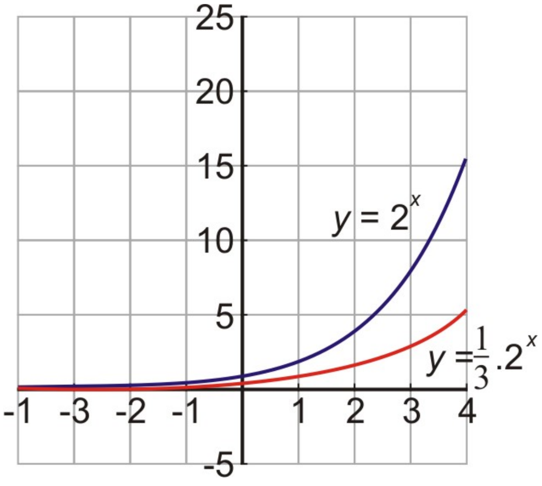
Solution

Let's make a table of values for both functions.

TABLE 8.2:

x	$y = 2^x$	$y = \frac{1}{3} \cdot 2^x$
-3	$\frac{1}{8}$	$y = \frac{1}{3} \cdot 2^{-3} = \frac{1}{3} \cdot \frac{1}{2^3} = \frac{1}{24}$
-2	$\frac{1}{4}$	$y = \frac{1}{3} \cdot 2^{-2} = \frac{1}{3} \cdot \frac{1}{2^2} = \frac{1}{12}$
-1	$\frac{1}{2}$	$y = \frac{1}{3} \cdot 2^{-1} = \frac{1}{3} \cdot \frac{1}{2^1} = \frac{1}{6}$
0	1	$y = \frac{1}{3} \cdot 2^0 = \frac{1}{3}$
1	2	$y = \frac{1}{3} \cdot 2^1 = \frac{2}{3}$
2	4	$y = \frac{1}{3} \cdot 2^2 = \frac{1}{3} \cdot 4 = \frac{4}{3}$
3	8	$y = \frac{1}{3} \cdot 2^3 = \frac{1}{3} \cdot 8 = \frac{8}{3}$

Now let's use this table to graph the functions.



As we expected, the exponential function $y = \frac{1}{3} \cdot 2^x$ is smaller than the exponential function $y = 2^x$.

So what happens if the starting value of A is negative? Let's find out.

Example C

Graph the exponential function $y = -5 \cdot 2^x$.

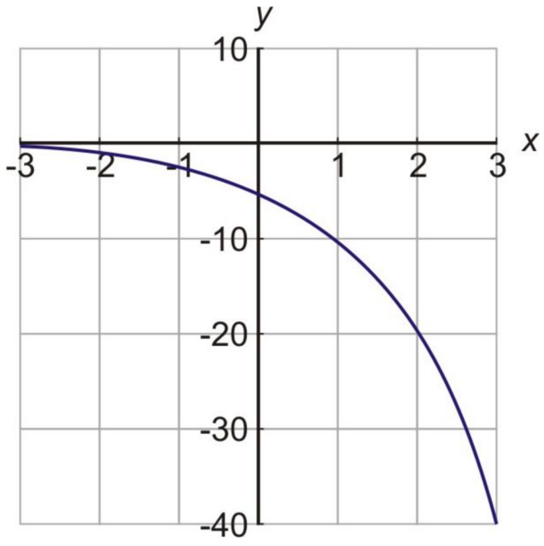
Solution

Let's make a table of values:

TABLE 8.3:

x	$y = -5 \cdot 2^x$
-2	$-\frac{5}{4}$
-1	$-\frac{5}{2}$
0	-5
1	-10
2	-20
3	-40

Now let's graph the function:



This result shouldn't be unexpected. Since the starting value is negative and keeps doubling over time, it makes sense that the value of y gets farther from zero, but in a negative direction. The graph is basically just like the graph of $y = 5 \cdot 2^x$, only mirror-reversed about the $x-$axis.

Now, let's compare exponential functions whose bases (b) are different.

Example D

Graph the following exponential functions on the same graph: $y = 2^x, y = 3^x, y = 5^x, y = 10^x$.

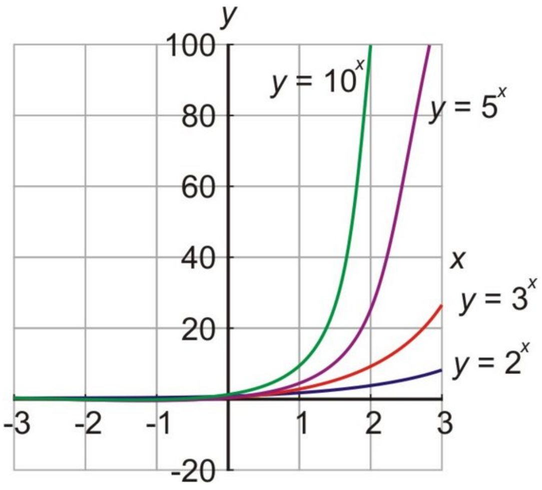
Solution

First we'll make a table of values for all four functions.

TABLE 8.4:

x	$y = 2^x$	$y = 3^x$	$y = 5^x$	$y = 10^x$
-2	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{25}$	$\frac{1}{100}$
-1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{10}$
0	1	1	1	1
1	2	3	5	10
2	4	9	25	100
3	8	27	125	1000

Now let's graph the functions.



Notice that for $x = 0$, all four functions equal 1. They all "start out" at the same point, but the ones with higher values for b grow faster when x is positive—and also shrink faster when x is negative.

Finally, let's explore what happens for values of b that are less than 1.

Example E

Graph the exponential function $y = 5 \cdot \left(\frac{1}{2}\right)^x$.

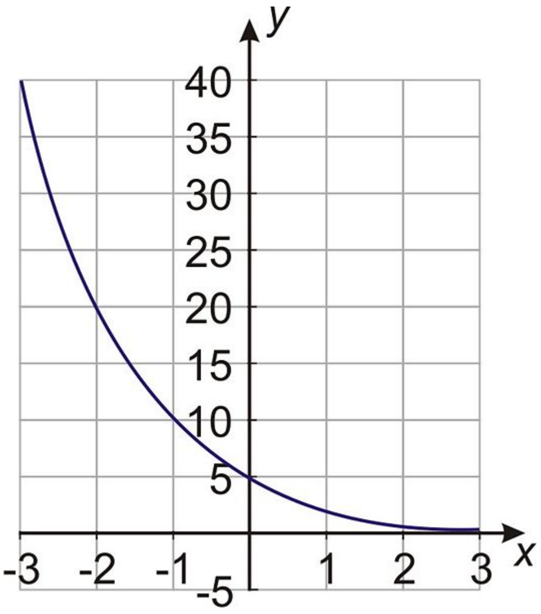
Solution

Let's start by making a table of values. (Remember that a fraction to a negative power is equivalent to its reciprocal to the same positive power.)

TABLE 8.5:

x	$y = 5 \cdot \left(\frac{1}{2}\right)^x$
-3	$y = 5 \cdot \left(\frac{1}{2}\right)^{-3} = 5 \cdot 2^3 = 40$
-2	$y = 5 \cdot \left(\frac{1}{2}\right)^{-2} = 5 \cdot 2^2 = 20$
-1	$y = 5 \cdot \left(\frac{1}{2}\right)^{-1} = 5 \cdot 2^1 = 10$
0	$y = 5 \cdot \left(\frac{1}{2}\right)^0 = 5 \cdot 1 = 5$
1	$y = 5 \cdot \left(\frac{1}{2}\right)^1 = \frac{5}{2}$
2	$y = 5 \cdot \left(\frac{1}{2}\right)^2 = \frac{5}{4}$

Now let's graph the function.



This graph looks very different than the graphs from the previous example! What's going on here?

When we raise a number greater than 1 to the power of x, it gets bigger as x gets bigger. But when we raise a number smaller than 1 to the power of x, it gets *smaller* as x gets bigger—as you can see from the table of values above. This makes sense because multiplying any number by a quantity less than 1 always makes it smaller.

So, when the base b of an exponential function is between 0 and 1, the graph is like an ordinary exponential graph, only decreasing instead of increasing. Graphs like this represent **exponential decay** instead of **exponential growth**. Exponential decay functions are used to describe quantities that decrease over a period of time.

When b can be written as a fraction, we can use the Property of Negative Exponents to write the function in a different form. For instance, $y = 5 \cdot \left(\frac{1}{2}\right)^x$ is equivalent to $5 \cdot 2^{-x}$. These two forms are both commonly used, so it's important to know that they are equivalent.

Watch this video for help with the Examples above.

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CK-12 Foundation: Exponential Functions

Vocabulary

- **General Form of an Exponential Function:** $y = A(b)^x$, where $A =$ *initial value* and $b =$ multiplication factor.

Guided Practice

a.) *Graph the exponential function* $y = 8 \cdot 3^{-x}$.

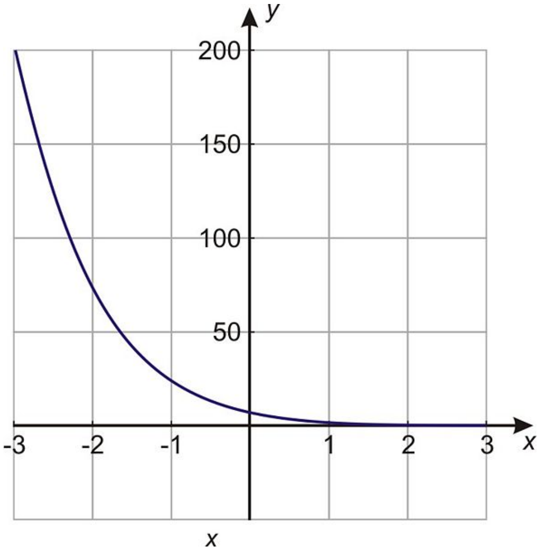
b.) *Graph the functions* $y = 4^x$ *and* $y = 4^{-x}$ *on the same coordinate axes.*

Solution:

a.) Here is our table of values and the graph of the function.

TABLE 8.6:

x	$y = 8 \cdot 3^{-x}$
-3	$y = 8 \cdot 3^{-(-3)} = 8 \cdot 3^3 = 216$
-2	$y = 8 \cdot 3^{-(-2)} = 8 \cdot 3^2 = 72$
-1	$y = 8 \cdot 3^{-(-1)} = 8 \cdot 3^1 = 24$
0	$y = 8 \cdot 3^0 = 8$
1	$y = 8 \cdot 3^{-1} = \frac{8}{3}$
2	$y = 8 \cdot 3^{-2} = \frac{8}{9}$

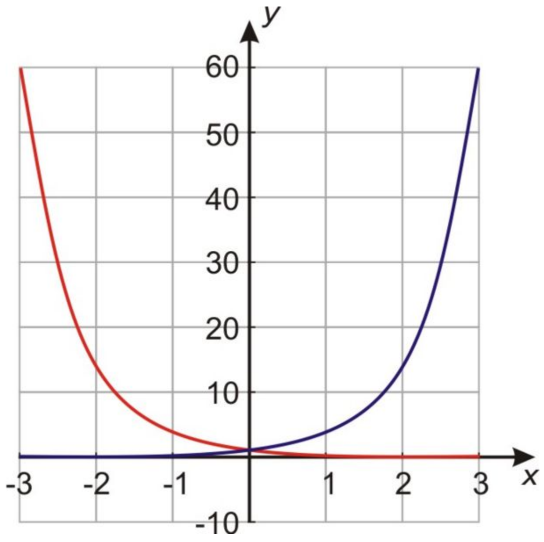


b.) Here is the table of values for the two functions. Looking at the values in the table, we can see that the two functions are "backwards" of each other, in the sense that the values for the two functions are reciprocals.

TABLE 8.7:

x	$y = 4^x$	$y = 4^{-x}$
-3	$y = 4^{-3} = \frac{1}{64}$	$y = 4^{-(-3)} = 64$
-2	$y = 4^{-2} = \frac{1}{16}$	$y = 4^{-(-2)} = 16$
-1	$y = 4^{-1} = \frac{1}{4}$	$y = 4^{-(-1)} = 4$
0	$y = 4^0 = 1$	$y = 4^0 = 1$
1	$y = 4^1 = 4$	$y = 4^{-1} = \frac{1}{4}$
2	$y = 4^2 = 16$	$y = 4^{-2} = \frac{1}{16}$
3	$y = 4^3 = \frac{1}{64}$	$y = 4^{-3} = \frac{1}{64}$

Here is the graph of the two functions. Notice that the two functions are mirror images of each other if the mirror is placed vertically on the $y-$axis.



In the next lesson, you'll see how exponential growth and decay functions can be used to represent situations in the real world.

Practice

Graph the following exponential functions by making a table of values.

1. $y = 3^x$
2. $y = 5 \cdot 3^x$
3. $y = 40 \cdot 4^x$
4. $y = 3 \cdot 10^x$

Graph the following exponential functions.

5. $y = \left(\frac{1}{5}\right)^x$
6. $y = 4 \cdot \left(\frac{2}{3}\right)^x$
7. $y = 3^{-x}$
8. $y = \frac{3}{4} \cdot 6^{-x}$
9. Which two of the eight graphs above are mirror images of each other?
10. What function would produce a graph that is the mirror image of the one in problem 4?
11. How else might you write the exponential function in problem 5?
12. How else might you write the function in problem 6?

Solve the following problems.

13. A chain letter is sent out to 10 people telling everyone to make 10 copies of the letter and send each one to a new person.
    a. Assume that everyone who receives the letter sends it to ten new people and that each cycle takes a week. How many people receive the letter on the sixth week?
    b. What if everyone only sends the letter to 9 new people? How many people will then get letters on the sixth week?
14. Nadia received \$200 for her 10^{th} birthday. If she saves it in a bank account with 7.5% interest compounded yearly, how much money will she have in the bank by her 21^{st} birthday?

8.12 Applications of Exponential Functions

Here you'll learn how to apply a problem-solving plan to problems involving exponential functions. You'll also learn how to solve real-world applications involving exponential growth and decay.

What if you won \$500 in a spelling bee competition and invested it into a mutual fund that pays 8% interest compounded annually? How much money would you have after 5 years? After completing this Concept, you'll be able to solve real-world problems like this one that involve exponential functions.

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CK-12 Foundation: 0812S Applicationsof Exponential Functions

Guidance

For her eighth birthday, Shelley's grandmother gave her a full bag of candy. Shelley counted her candy and found out that there were 160 pieces in the bag. As you might suspect, Shelley loves candy, so she ate half the candy on the first day. Then her mother told her that if she eats it at that rate, the candy will only last one more day—so Shelley devised a clever plan. She will always eat half of the candy that is left in the bag each day. She thinks that this way she can eat candy every day and never run out.

How much candy does Shelley have at the end of the week? Will the candy really last forever?

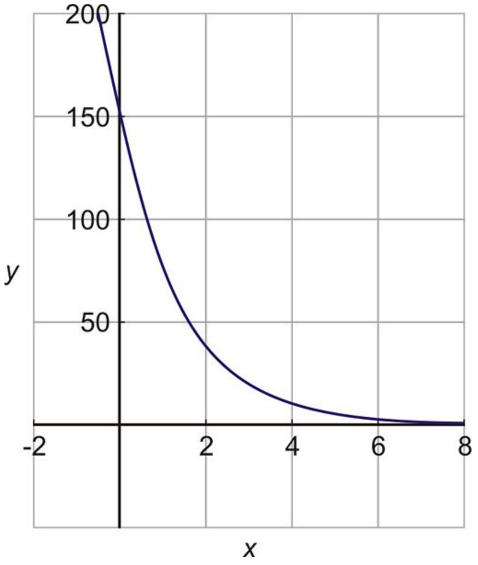
Let's make a table of values for this problem.

Day	0	1	2	3	4	5	6	7
of candies	160	80	40	20	10	5	2.5	1.25

You can see that if Shelley eats half the candies each day, then by the end of the week she only has 1.25 candies left in her bag.

Let's write an equation for this exponential function. Using the formula $y = A \cdot b^x$, we can see that A is 160 (the number of candies she starts out with and b is $\frac{1}{2}$, so our equation is $y = 160 \cdot \left(\frac{1}{2}\right)^x$.)

Now let's graph this function. The resulting graph is shown below.



So, will Shelley's candy last forever? We saw that by the end of the week she has 1.25 candies left, so there doesn't seem to be much hope for that. But if you look at the graph, you'll see that the graph never really gets to zero. Theoretically there will always be *some* candy left, but Shelley will be eating very tiny fractions of a candy every day after the first week!

This is a fundamental feature of an exponential decay function. Its values get smaller and smaller but never quite reach zero. In mathematics, we say that the function has an **asymptote** at $y = 0$; in other words, it gets closer and closer to the line $y = 0$ but never quite meets it.

Problem-Solving Strategies

Remember our problem-solving plan from earlier?

1. Understand the problem.
2. Devise a plan – Translate.
3. Carry out the plan – Solve.
4. Look – Check and Interpret.

We can use this plan to solve application problems involving exponential functions. Compound interest, loudness of sound, population increase, population decrease or radioactive decay are all applications of exponential functions. In these problems, we'll use the methods of constructing a table and identifying a pattern to help us devise a plan for solving the problems.

Example A

Suppose \$4000 is invested at 6% interest compounded annually. How much money will there be in the bank at the end of 5 years? At the end of 20 years?

Solution

Step 1: Read the problem and summarize the information.

\$4000 is invested at 6% interest compounded annually; we want to know how much money we have in five years.

Assign variables:

Let $x =$ time in years

Let $y =$ amount of money in investment account

Step 2: Look for a pattern.

We start with \$4000 and each year we add 6% interest to the amount in the bank.

Start:	\$4000
1^{st} year:	Interest $= 4000 \times (0.06) = \$240$
	This is added to the previous amount: $\$4000 + \$4000 \times (0.06)$
	$= \$4000(1+0.06)$
	$= \$4000(1.06)$
	$= \$4240$
2^{nd} year	Previous amount + interest on the previous amount
	$= \$4240(1+0.06)$
	$= \$4240(1.06)$
	$= \$4494.40$

The pattern is that each year we multiply the previous amount by the factor of 1.06.

Let's fill in a table of values:

Time (years)	0	1	2	3	4	5
Investments amount(\$)	4000	4240	4494.4	4764.06	5049.90	5352.9

We see that **at the end of five years we have \$5352.90 in the investment account**.

Step 3: Find a formula.

We were able to find the amount after 5 years just by following the pattern, but rather than follow that pattern for another 15 years, it's easier to use it to find a general formula. Since the original investment is multiplied by 1.06 each year, we can use exponential notation. Our formula is $y = 4000 \cdot (1.06)^x$, where x is the number of years since the investment.

To find the amount after 5 years we plug $x = 5$ into the equation:

$$y = 4000 \cdot (1.06)^5 = \$5352.90$$

To find the amount after 20 years we plug $x = 20$ into the equation:

$$y = 4000 \cdot (1.06)^{20} = \$12828.54$$

Step 4: Check.

Looking back over the solution, we see that we obtained the answers to the questions we were asked and the answers make sense.

To check our answers, we can plug some low values of x into the formula to see if they match the values in the table:

$x = 0: \quad y = 4000 \cdot (1.06)^0 = 4000$

$x = 1: \quad y = 4000 \cdot (1.06)^1 = 4240$

$x = 2: \quad y = 4000 \cdot (1.06)^2 = 4494.4$

The answers match the values we found earlier. The amount of increase gets larger each year, and that makes sense because the interest is 6% of an amount that is larger every year.

Example B

In 2002 the population of schoolchildren in a city was 90,000. This population decreases at a rate of 5% each year. What will be the population of school children in year 2010?

Solution

Step 1: Read the problem and summarize the information.

The population is 90,000; the rate of decrease is 5% each year; we want the population after 8 years.

Assign variables:

Let $x =$ time since 2002 (in years)

Let $y =$ population of school children

Step 2: Look for a pattern.

Let's start in 2002, when the population is 90,000.

The rate of decrease is 5% each year, so the amount in 2003 is 90,000 minus 5% of 90,000, or 95% of 90,000.

$$\begin{aligned} &\text{In } 2003: && \text{Population} = 90,000 \times 0.95 \\ &\text{In } 2004: && \text{Population} = 90,000 \times 0.95 \times 0.95 \end{aligned}$$

The pattern is that for each year we multiply by a factor of 0.95

Let's fill in a table of values:

Year	2002	2003	2004	2005	2006	2007
Population	90,000	85,500	81,225	77,164	73,306	69,640

Step 3: Find a formula.

Since we multiply by 0.95 every year, our exponential formula is $y = 90000 \cdot (0.95)^x$, where x is the number of years since 2002. To find the population in 2010 (8 years after 2002), we plug in $x = 8$:

$y = 90000 \cdot (0.95)^8 = 59,708$ schoolchildren.

Step 4: Check.

Looking back over the solution, we see that we answered the question we were asked and that it makes sense. The answer makes sense because the numbers decrease each year as we expected. We can check that the formula is correct by plugging in the values of x from the table to see if the values match those given by the formula.

$$\begin{aligned} &\text{Year } 2002, x = 0: && \text{Population} = y = 90000 \cdot (0.95)^0 = 90,000 \\ &\text{Year } 2003, x = 1: && \text{Population} = y = 90000 \cdot (0.95)^1 = 85,500 \\ &\text{Year } 2004, x = 2: && \text{Population} = y = 90000 \cdot (0.95)^2 = 81,225 \end{aligned}$$

Solve Real-World Problems Involving Exponential Growth

Now we'll look at some more real-world problems involving exponential functions. We'll start with situations involving exponential growth.

Example C

The population of a town is estimated to increase by 15% per year. The population today is 20 thousand. Make a graph of the population function and find out what the population will be ten years from now.

Solution

First, we need to write a function that describes the population of the town.

The general form of an exponential function is $y = A \cdot b^x$.

Define y as the population of the town.

Define x as the number of years from now.

A is the initial population, so $A = 20$ (thousand).

Finally we must find what b is. We are told that the population increases by 15% each year. To calculate percents we have to change them into decimals: 15% is equivalent to 0.15. So each year, the population increases by 15% of A, or $0.15A$.

To find the total population for the following year, we must add the *current* population to the *increase* in population. In other words, $A + 0.15A = 1.15A$. So the population must be multiplied by a factor of 1.15 each year. This means that the base of the exponential is $b = 1.15$.

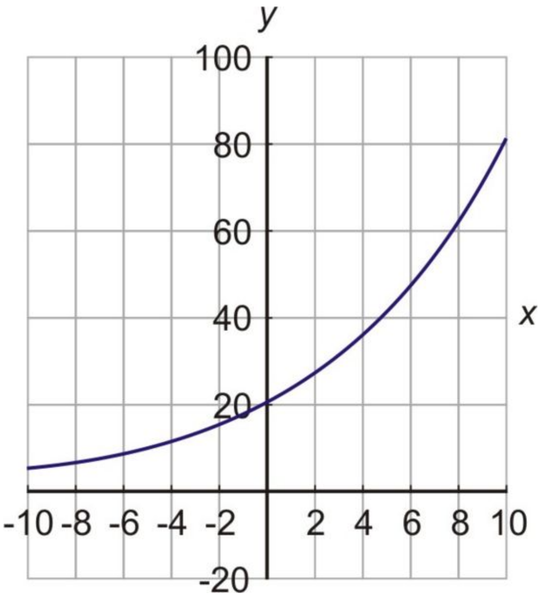
The formula that describes this problem is $y = 20 \cdot (1.15)^x$.

Now let's make a table of values.

TABLE 8.8:

x	$y = 20 \cdot (1.15)^x$
-10	4.9
-5	9.9
0	20
5	40.2
10	80.9

Now we can graph the function.



Notice that we used negative values of x in our table of values. Does it make sense to think of negative time? Yes; negative time can represent time in the past. For example, $x = -5$ in this problem represents the population from five years ago.

The question asked in the problem was: *what will be the population of the town ten years from now?* To find that number, we plug $x = 10$ into the equation we found: $y = 20 \cdot (1.15)^{10} = 80,911$.

The town will have 80,911 people ten years from now.

Example D

Peter earned \$1500 last summer. If he deposited the money in a bank account that earns 5% interest compounded yearly, how much money will he have after five years?

Solution

This problem deals with interest which is compounded yearly. This means that each year the interest is calculated on the amount of money you have in the bank. That interest is added to the original amount and next year the interest is calculated on this new amount, so you get paid interest on the interest.

Let's write a function that describes the amount of money in the bank.

The general form of an exponential function is $y = A \cdot b^x$.

Define y as the amount of money in the bank.

Define x as the number of years from now.

A is the initial amount, so $A = 1500$.

Now we have to find what b is.

We're told that the interest is 5% each year, which is 0.05 in decimal form. When we add $0.05A$ to A, we get $1.05A$, so that is the factor we multiply by each year. The base of the exponential is $b = 1.05$.

The formula that describes this problem is $y = 1500 \cdot 1.05^x$. To find the total amount of money in the bank at the end of five years, we simply plug in $x = 5$.

$$y = 1500 \cdot (1.05)^5 = \$1914.42$$

Solve Real-World Problems Involving Exponential Decay

Exponential decay problems appear in several application problems. Some examples of these are **half-life problems** and **depreciation problems**. Let's solve an example of each of these problems.

Example E

A radioactive substance has a half-life of one week. In other words, at the end of every week the level of radioactivity is half of its value at the beginning of the week. The initial level of radioactivity is 20 counts per second.

Draw the graph of the amount of radioactivity against time in weeks.

Find the formula that gives the radioactivity in terms of time.

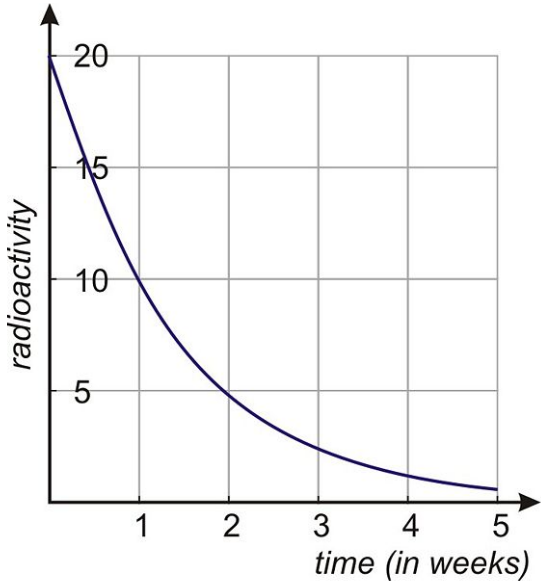
Find the radioactivity left after three weeks.

Solution

Let's start by making a table of values and then draw the graph.

TABLE 8.9:

Time	Radioactivity
0	20
1	10
2	5
3	2.5
4	1.25
5	0.625



Exponential decay fits the general formula $y = A \cdot b^x$. In this case:

y is the amount of radioactivity

x is the time in weeks

$A = 20$ is the starting amount

$b = \frac{1}{2}$ since the substance losses half its value each week

The formula for this problem is $y = 20 \cdot \left(\frac{1}{2}\right)^x$ or $y = 20 \cdot 2^{-x}$. To find out how much radioactivity is left after three weeks, we plug $x = 3$ into this formula.

$$y = 20 \cdot \left(\frac{1}{2}\right)^3 = 20 \cdot \left(\frac{1}{8}\right) = 2.5$$

Watch this video for help with the Examples above.

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CK-12 Foundation: Applications of Exponential Functions

Vocabulary

- **General Form of an Exponential Function:** $y = A(b)^x$, where $A =$ *initial value* and

$$b =$$

multiplication factor.

Guided Practice

The cost of a new car is \$32,000. It depreciates at a rate of 15% per year. This means that it loses 15% of each value each year.

Draw the graph of the car's value against time in year.

Find the formula that gives the value of the car in terms of time.

Find the value of the car when it is four years old.

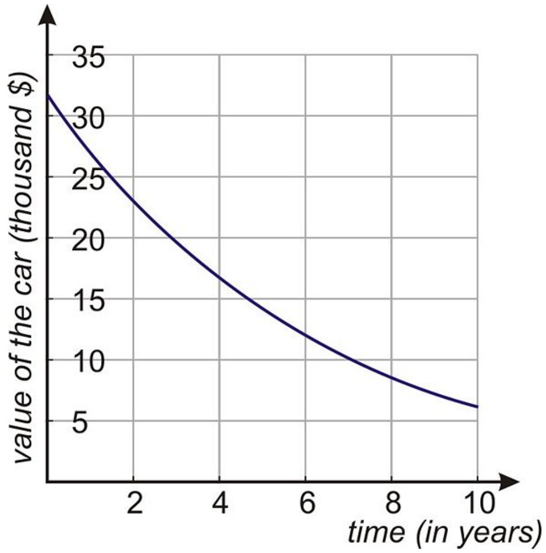
Solution

Let's start by making a table of values. To fill in the values we start with 32,000 at time $t = 0$. Then we multiply the value of the car by 85% for each passing year. (Since the car loses 15% of its value, that means it keeps 85% of its value). Remember that 85% means that we multiply by the decimal .85.

TABLE 8.10:

Time	Value (thousands)
0	32
1	27.2
2	23.1
3	19.7
4	16.7
5	14.2

Now draw the graph:



Let's start with the general formula $y = A \cdot b^x$

In this case:

y is the value of the car,

x is the time in years,

$A = 32$ is the starting amount in thousands,

$b = 0.85$ since we multiply the amount by this factor to get the value of the car next year

The formula for this problem is $y = 32 \cdot (0.85)^x$.

Finally, to find the value of the car when it is four years old, we plug $x = 4$ into that formula: $y = 32 \cdot (0.85)^4 = 16.7$ thousand dollars, or **$16,704** if we don't round.

Practice

Solve the following problems involving exponential growth.

1. Nadia received $200 for her 10^{th} birthday. If she saves it in a bank with a 7.5% interest rate compounded yearly, how much money will she have in the bank by her 21^{st} birthday?
2. Suppose again that Nadia received $200 for her 10^{th} birthday. But what if she saves it in a bank, also with a 7.5% interest rate, but this bank compounds compounds quarterly - how much money will she have in the bank by her 21^{st} birthday?
3. The population of a city grows 15% each year. If the town started with 105 people, how many people will there be in 10 years?

4. **Half-life:** Suppose a radioactive substance decays at a rate of 3.5% per hour.
   a. What percent of the substance is left after 6 hours?
   b. What percent is left after 12 hours?
   c. The substance is safe to handle when at least 50% of it has decayed. Make a guess as to how many hours this will take.
   d. Test your guess. How close were you?

5. **Population decrease:** In 1990 a rural area has 1200 bird species.
    a. If species of birds are becoming extinct at the rate of 1.5% per decade (ten years), how many bird species will be left in the year 2020?
    b. At that same rate, how many were there in 1980?
6. **Growth:** Janine owns a chain of fast food restaurants that operated 200 stores in 1999. If the rate of increase is 8% annually, how many stores does the restaurant operate in 2007?
7. **Investment:** Paul invests \$360 in an account that pays 7.25% compounded annually.
    a. What is the total amount in the account after 12 years?
    b. If Paul invests an equal amount in an account that pays 5% compounded quarterly (four times a year), what will be the amount in that account after 12 years?
    c. Which is the better investment?
8. The cost of a new ATV (all-terrain vehicle) is \$7200. It depreciates at 18% per year.
    a. Draw the graph of the vehicle's value against time in years.
    b. Find the formula that gives the value of the ATV in terms of time.
    c. Find the value of the ATV when it is ten years old.
9. The percentage of light visible at d meters is given by the function $V(d) = 0.70^d$.
    a. What is the multiplication factor?
    b. What is the initial value?
    c. Find the percentage of light visible at 25 meters.
10. A person is infected by a certain bacterial infection. When he goes to the doctor the population of bacteria is 2 million. The doctor prescribes an antibiotic that reduces the bacteria population to $\frac{1}{4}$ of its size each day.
    a. Draw the graph of the size of the bacteria population against time in days.
    b. Find the formula that gives the size of the bacteria population in terms of time.
    c. Find the size of the bacteria population ten days after the drug was first taken.
    d. Find the size of the bacteria population after 2 weeks (14 days).

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9618.

Summary

This chapter begins with four important properties for combining exponents: Product of Powers, Power of a Product, Quotient of Powers, and Power of a Quotient. Negative, zero, and fractional exponents are discussed next. All these properties come together in the evaluation of exponential expressions. A way of representing very large and very small numbers more simply, known as scientific notation, is next covered. The chapter wraps up with discussion of geometric sequences and exponential functions, with an emphasis on real-world applications.

CHAPTER 9 Polynomials

Chapter Outline

9.1 POLYNOMIALS IN STANDARD FORM
9.2 ADDITION AND SUBTRACTION OF POLYNOMIALS
9.3 MULTIPLICATION OF MONOMIALS BY POLYNOMIALS
9.4 MULTIPLICATION OF POLYNOMIALS BY BINOMIALS
9.5 SPECIAL PRODUCTS OF POLYNOMIALS
9.6 MONOMIAL FACTORS OF POLYNOMIALS
9.7 ZERO PRODUCT PRINCIPLE
9.8 FACTORIZATION OF QUADRATIC EXPRESSIONS
9.9 FACTORIZATION OF QUADRATIC EXPRESSIONS WITH NEGATIVE COEFFICIENTS
9.10 FACTORIZATION USING DIFFERENCE OF SQUARES
9.11 FACTORIZATION USING PERFECT SQUARE TRINOMIALS
9.12 FACTORING COMPLETELY
9.13 FACTORING BY GROUPING
9.14 SOLVING PROBLEMS BY FACTORING

Introduction

An object with an initial velocity of 100 feet per second has a height of $-16t^2 + 100t$ after t seconds. This height is an example of a polynomial expression. How could you describe this polynomial? How could you factor it and solve it? In this chapter, you'll learn the vocabulary of polynomials, how to add, subtract, and multiply polynomials like this one, how to factor them, and how to solve them.

9.1 Polynomials in Standard Form

Here you'll learn how to identify polynomials and find their degree. You'll also learn how to write polynomial expressions in standard form and simplify them by combining like terms.

What if you were given an algebraic expression like $3x - 2x^2 + 5 - x + 6x^2$? How could you simplify it and find its degree? After completing this Concept, you'll be able to combine like terms to simplify polynomial expressions like this one and classify them by degree.

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Foundation: 0901S LessonPolynomial Expressions

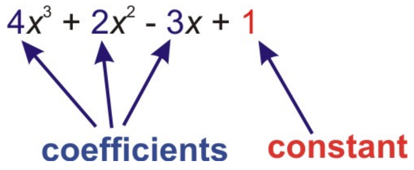
Guidance

So far we've seen functions described by straight lines (linear functions) and functions where the variable appeared in the exponent (exponential functions). In this section we'll introduce polynomial functions. A **polynomial** is made up of different terms that contain **positive integer** powers of the variables. Here is an example of a polynomial:

$$4x^3 + 2x^2 - 3x + 1$$

Each part of the polynomial that is added or subtracted is called a **term** of the polynomial. The example above is a polynomial with *four terms*.

The numbers appearing in each term in front of the variable are called the **coefficients**. The number appearing all by itself without a variable is called a **constant**.



In this case the coefficient of x^3 is **4**, the coefficient of x^2 is **2**, the coefficient of x is **-3** and the constant is **1**.

Degrees of Polynomials and Standard Form

Each term in the polynomial has a different **degree**. The degree of the term is the power of the variable in that term.

$4x^3$	has degree 3 and is called a cubic term or 3^{rd} order term.
$2x^2$	has degree 2 and is called a quadratic term or 2^{nd} order term.
$-3x$	has degree 1 and is called a linear term or 1^{st} order term.
1	has degree 0 and is called the constant.

By definition, **the degree of the polynomial** is the same as the degree of the term with the highest degree. This example is a polynomial of degree 3, which is also called a "cubic" polynomial. (Why do you think it is called a cubic?).

Polynomials can have more than one variable. Here is another example of a polynomial:

$$t^4-6s^3t^2-12st+4s^4-5$$

This is a polynomial because all the exponents on the variables are positive integers. This polynomial has five terms. Let's look at each term more closely.

Note: *The degree of a term is the sum of the powers on each variable in the term.* In other words, the degree of each term is the number of variables that are multiplied together in that term, whether those variables are the same or different.

t^4	has a degree of 4, so it's a 4^{th} order term
$-6s^3t^2$	has a degree of 5, so it's a 5^{th} order term.
$-12st$	has a degree of 2, so it's a 2^{nd} order term.
$4s^4$	has a degree of 4, so it's a 4^{th} order term.
-5	is a constant, so its degree is 0.

Since the highest degree of a term in this polynomial is 5, then this is polynomial of degree 5^{th} or a 5^{th} order polynomial.

A polynomial that has only one term has a special name. It is called a **monomial** (*mono* means one). A monomial can be a constant, a variable, or a product of a constant and one or more variables. You can see that each term in a polynomial is a monomial, so a polynomial is just the sum of several monomials. Here are some examples of monomials:

$$b^2 \qquad -2ab^2 \qquad 8 \qquad \frac{1}{4}x^4 \qquad -29xy$$

Example A

For the following polynomials, identify the coefficient of each term, the constant, the degree of each term and the degree of the polynomial.

a) $x^5-3x^3+4x^2-5x+7$

b) $x^4-3x^3y^2+8x-12$

Solution

a) $x^5 - 3x^3 + 4x^2 - 5x + 7$

The coefficients of each term in order are 1, -3, 4, and -5 and the constant is 7.

The degrees of each term are 5, 3, 2, 1, and 0. Therefore the degree of the polynomial is 5.

b) $x^4 - 3x^3y^2 + 8x - 12$

The coefficients of each term in order are 1, -3, and 8 and the constant is -12.

The degrees of each term are 4, 5, 1, and 0. Therefore the degree of the polynomial is 5.

Example B

Identify the following expressions as polynomials or non-polynomials.

a) $5x^5 - 2x$

b) $3x^2 - 2x^{-2}$

c) $x\sqrt{x} - 1$

d) $\frac{5}{x^3+1}$

e) $4x^{\frac{1}{3}}$

f) $4xy^2 - 2x^2y - 3 + y^3 - 3x^3$

Solution

a) This ***is*** a polynomial.

b) This is ***not*** a polynomial because it has a negative exponent.

c) This is ***not*** a polynomial because it has a radical.

d) This is ***not*** a polynomial because the power of x appears in the denominator of a fraction (and there is no way to rewrite it so that it does not).

e) This is ***not*** a polynomial because it has a fractional exponent.

f) This ***is*** a polynomial.

Often, we arrange the terms in a polynomial in order of decreasing power. This is called **standard form**.

The following polynomials are in standard form:

$$4x^4 - 3x^3 + 2x^2 - x + 1$$

$$a^4b^3 - 2a^3b^3 + 3a^4b - 5ab^2 + 2$$

The first term of a polynomial in standard form is called the **leading term**, and the coefficient of the leading term is called the **leading coefficient**.

The first polynomial above has the leading term $4x^4$, and the leading coefficient is 4.

The second polynomial above has the leading term a^4b^3, and the leading coefficient is 1.

Example C

Rearrange the terms in the following polynomials so that they are in standard form. Indicate the leading term and leading coefficient of each polynomial.

a) $7-3x^3+4x$

b) $ab-a^3+2b$

c) $-4b+4+b^2$

Solution

a) $7-3x^3+4x$ becomes $-3x^3+4x+7$. Leading term is $-3x^3$; leading coefficient is -3.

b) $ab-a^3+2b$ becomes $-a^3+ab+2b$. Leading term is $-a^3$; leading coefficient is -1.

c) $-4b+4+b^2$ becomes b^2-4b+4. Leading term is b^2; leading coefficient is 1.

Simplifying Polynomials

A polynomial is simplified if it has no terms that are alike. **Like terms** are terms in the polynomial that have the same variable(s) with the same exponents, whether they have the same or different coefficients.

For example, $2x^2y$ and $5x^2y$ are like terms, but $6x^2y$ and $6xy^2$ are not like terms.

When a polynomial has like terms, we can simplify it by combining those terms.

$$x^2+\underline{6xy}-\underline{4xy}+y^2$$

$$\nearrow \quad \nwarrow$$

Like terms

We can simplify this polynomial by combining the like terms $6xy$ and $-4xy$ into $(6-4)xy$, or $2xy$. The new polynomial is $x^2+2xy+y^2$.

Example D

Simplify the following polynomials by collecting like terms and combining them.

a) $2x-4x^2+6+x^2-4+4x$

b) $a^3b^3-5ab^4+2a^3b-a^3b^3+3ab^4-a^2b$

Solution

a) Rearrange the terms so that like terms are grouped together: $(-4x^2+x^2)+(2x+4x)+(6-4)$

Combine each set of like terms: $-3x^2+6x+2$

b) Rearrange the terms so that like terms are grouped together: $(a^3b^3-a^3b^3)+(-5ab^4+3ab^4)+2a^3b-a^2b$

Combine each set of like terms: $0-2ab^4+2a^3b-a^2b=-2ab^4+2a^3b-a^2b$

Watch this video for help with the Examples above.

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CK-12 Foundation: PolynomialExpressions

Vocabulary

- A ***polynomial*** is an expression made with constants, variables, and *positive integer* exponents of the variables.
- In a polynomial, the number appearing in each term in front of the variables is called the **coefficient.**
- In a polynomial, the number appearing all by itself without a variable is called the **constant.**
- A **monomial** is a one-termed polynomial. It can be a constant, a variable, or a variable with a coefficient.
- The **degree of a polynomial** is the largest degree of the terms. The **degree of a term** is the power of the variable, or if the term has more than one variable, it is the sum of the powers on each variable.
- We arrange the terms in a polynomial in **standard form** in which the term with the highest degree is first and is followed by the other terms in order of decreasing powers.
- **Like terms** are terms in the polynomial that have the same variable(s) with the same exponents, but they can have different coefficients.

Guided Practice

Simplify and rewrite the following polynomial in standard form. State the degree of the polynomial.

$16x^2y^3-3xy^5-2x^3y^2+2xy-7x^2y^3+2x^3y^2$

Solution:

Start by simplifying by combining like terms:

$16x^2y^3-3xy^5-2x^3y^2+2xy-7x^2y^3+2x^3y^2$

is equal to

$(16x^2y^3-7x^2y^3)-3xy^5+(-2x^3y^2+2x^3y^2)+2xy$

which simplifies to

$9x^2y^3-3xy^5+2xy.$

In order to rewrite in standard form, we need to determine the degree of each term. The first term has a degree of $2+3=5$, the second term has a degree of $1+5=6$, and the last term has a degree of $1+1=2$. We will rewrite the terms in order from largest degree to smallest degree:

$-3xy^5+9x^2y^3+2xy$

The degree of a polynomial is the largest degree of all the terms. In this case that is 6.

Practice

Indicate whether each expression is a polynomial.

1. $x^2+3x^{\frac{1}{2}}$
2. $\frac{1}{3}x^2y-9y^2$

3. $3x^{-3}$
4. $\frac{2}{3}t^2 - \frac{1}{t^2}$
5. $\sqrt{x} - 2x$
6. $\left(x^{\frac{3}{2}}\right)^2$

Express each polynomial in standard form. Give the degree of each polynomial.

7. $3 - 2x$
8. $8 - 4x + 3x^3$
9. $-5 + 2x - 5x^2 + 8x^3$
10. $x^2 - 9x^4 + 12$
11. $5x + 2x^2 - 3x$

9.2 Addition and Subtraction of Polynomials

Here you'll learn how to add and subtract polynomials and simplify your answers. You'll also solve real-world problems using addition and subtraction of polynomials.

What if you had two polynomials like $4x^2 - 5$ and $13x + 2$? How could you add and and subtract them? After completing this Concept, you'll be able to perform addition and subraction on polynomials like these.

Try This

For more practice adding and subtracting polynomials, try playing the Battleship game at http://www.quia.com/ba/28820.html. (The problems get harder as you play; watch out for trick questions!)

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CK-12 Foundation: 0902S Lesson Addition and Subtraction of Polynomials

Guidance

To add two or more polynomials, write their sum and then simplify by combining like terms.

Example A

Add and simplify the resulting polynomials.

a) Add $3x^2 - 4x + 7$ and $2x^3 - 4x^2 - 6x + 5$

b) Add $x^2 - 2xy + y^2$ and $2y^2 - 3x^2$ and $10xy + y^3$

Solution

a)

$$\begin{aligned} &(3x^2 - 4x + 7) + (2x^3 - 4x^2 - 6x + 5) \\ \text{Group like terms: } &= 2x^3 + (3x^2 - 4x^2) + (-4x - 6x) + (7 + 5) \\ \text{Simplify: } &= 2x^3 - x^2 - 10x + 12 \end{aligned}$$

b)

$$\begin{aligned} &(x^2 - 2xy + y^2) + (2y^2 - 3x^2) + (10xy + y^3) \\ \text{Group like terms: } &= (x^2 - 3x^2) + (y^2 + 2y^2) + (-2xy + 10xy) + y^3 \\ \text{Simplify: } &= -2x^2 + 3y^2 + 8xy + y^3 \end{aligned}$$

To subtract one polynomial from another, add the opposite of each term of the polynomial you are subtracting.

Example B

a) Subtract $x^3-3x^2+8x+12$ from $4x^2+5x-9$

b) Subtract $5b^2-2a^2$ from $4a^2-8ab-9b^2$

Solution

a)

$$\begin{aligned}(4x^2+5x-9)-(x^3-3x^2+8x+12) &= (4x^2+5x-9)+(-x^3+3x^2-8x-12)\\ \text{Group like terms:} &= -x^3+(4x^2+3x^2)+(5x-8x)+(-9-12)\\ \text{Simplify:} &= -x^3+7x^2-3x-21\end{aligned}$$

b)

$$\begin{aligned}(4a^2-8ab-9b^2)-(5b^2-2a^2) &= (4a^2-8ab-9b^2)+(-5b^2+2a^2)\\ \text{Group like terms:} &= (4a^2+2a^2)+(-9b^2-5b^2)-8ab\\ \text{Simplify:} &= 6a^2-14b^2-8ab\end{aligned}$$

Note: An easy way to check your work after adding or subtracting polynomials is to substitute a convenient value in for the variable, and check that your answer and the problem both give the same value. For example, in part (b) above, if we let $a=2$ and $b=3$, then we can check as follows:

Given	Solution
$(4a^2-8ab-9b^2)-(5b^2-2a^2)$	$6a^2-14b^2-8ab$
$(4(2)^2-8(2)(3)-9(3)^2)-(5(3)^2-2(2)^2)$	$6(2)^2-14(3)^2-8(2)(3)$
$(4(4)-8(2)(3)-9(9))-(5(9)-2(4))$	$6(4)-14(9)-8(2)(3)$
$(-113)-37$	$24-126-48$
-150	-150

Since both expressions evaluate to the same number when we substitute in arbitrary values for the variables, we can be reasonably sure that our answer is correct.

Note: When you use this method, do not choose 0 or 1 for checking since these can lead to common problems.

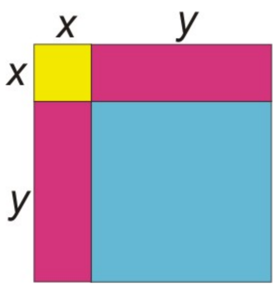
Problem Solving Using Addition or Subtraction of Polynomials

One way we can use polynomials is to find the area of a geometric figure.

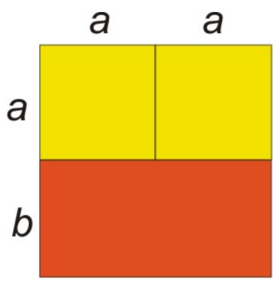
Example C

Write a polynomial that represents the area of each figure shown.

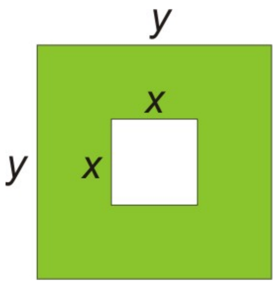
a)



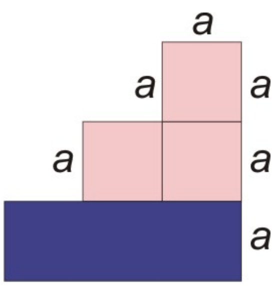
b)



c)



d)



Solution

a) This shape is formed by two squares and two rectangles.

The blue square has area $y \times y = y^2$.

The yellow square has area $x \times x = x^2$.

The pink rectangles each have area $x \times y = xy$.

To find the total area of the figure we add all the separate areas:

$$\begin{aligned} Total\ area &= y^2+x^2+xy+xy \\ &= y^2+x^2+2xy \end{aligned}$$

b) This shape is formed by two squares and one rectangle.

$$\begin{aligned} \text{The yellow squares each have area } a\times a &= a^2. \\ \text{The orange rectangle has area } 2a\times b &= 2ab. \end{aligned}$$

To find the total area of the figure we add all the separate areas:

$$\begin{aligned} Total\ area &= a^2+a^2+2ab \\ &= 2a^2+2ab \end{aligned}$$

c) To find the area of the green region we find the area of the big square and subtract the area of the little square.

$$\begin{aligned} \text{The big square has area} : y\times y &= y^2. \\ \text{The little square has area} : x\times x &= x^2. \\ Area\ of\ the\ green\ region &= y^2-x^2 \end{aligned}$$

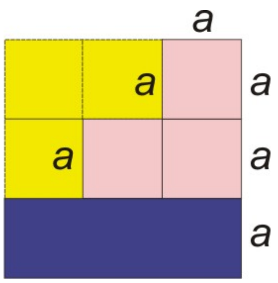
d) To find the area of the figure we can find the area of the big rectangle and add the areas of the pink squares.

$$\begin{aligned} \text{The pink squares each have area } a\times a &= a^2. \\ \text{The blue rectangle has area } 3a\times a &= 3a^2. \end{aligned}$$

To find the total area of the figure we add all the separate areas:

$$Total\ area = a^2+a^2+a^2+3a^2 = 6a^2$$

Another way to find this area is to find the area of the big square and subtract the areas of the three yellow squares:



$$\begin{aligned} \text{The big square has area } 3a\times 3a &= 9a^2. \\ \text{The yellow squares each have area } a\times a &= a^2. \end{aligned}$$

To find the total area of the figure we subtract:

$$\begin{aligned} Area &= 9a^2 - (a^2 + a^2 + a^2) \\ &= 9a^2 - 3a^2 \\ &= 6a^2 \end{aligned}$$

Watch this video for help with the Examples above.

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CK-12 Foundation: Addition and Subtraction of Polynomials

Vocabulary

- A **polynomial** is an expression made with constants, variables, and *positive integer* exponents of the variables.
- In a polynomial, the number appearing in each term in front of the variables is called the **coefficient.**
- In a polynomial, the number appearing all by itself without a variable is called the **constant.**
- **Like terms** are terms in the polynomial that have the same variable(s) with the same exponents, but they can have different coefficients.

Guided Practice

Subtract $4t^2 + 7t^3 - 3t - 5$ from $6t + 3 - 5t^3 + 9t^2$.

Solution:

When subtracting polynomials, we have to remember to subtract each term. If the term is already negative, subtracting a negative term is the same thing as adding:

$$\begin{aligned} 6t + 3 - 5t^3 + 9t^2 - (4t^2 + 7t^3 - 3t - 5) &= \\ 6t + 3 - 5t^3 + 9t^2 - (4t^2) - (7t^3) - (-3t) - (-5) &= \\ 6t + 3 - 5t^3 + 9t^2 - 4t^2 - 7t^3 + 3t + 5 &= \\ (6t + 3t) + (3 + 5) + (-5t^3 - 7t^3) + (9t^2 - 4t^2) &= \\ 9t + 8 - 12t^3 + 5t^2 &= \\ -12t^3 + 5t^2 + 9t + 8 & \end{aligned}$$

The final answer is in standard form.

Practice

Add and simplify.

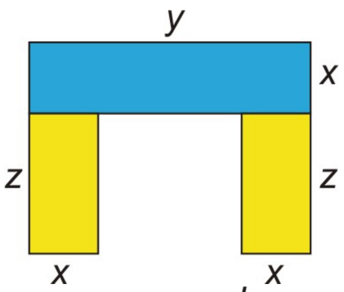
1. $(x+8)+(-3x-5)$
2. $(-2x^2+4x-12)+(7x+x^2)$
3. $(2a^2b-2a+9)+(5a^2b-4b+5)$
4. $(6.9a^2-2.3b^2+2ab)+(3.1a-2.5b^2+b)$
5. $\left(\frac{3}{5}x^2-\frac{1}{4}x+4\right)+\left(\frac{1}{10}x^2+\frac{1}{2}x-2\frac{1}{5}\right)$

Subtract and simplify.

6. $(-t+5t^2)-(5t^2+2t-9)$
7. $(-y^2+4y-5)-(5y^2+2y+7)$
8. $(-5m^2-m)-(3m^2+4m-5)$
9. $(2a^2b-3ab^2+5a^2b^2)-(2a^2b^2+4a^2b-5b^2)$
10. $(3.5x^2y-6xy+4x)-(1.2x^2y-xy+2y-3)$

Find the area of the following figures.

11.



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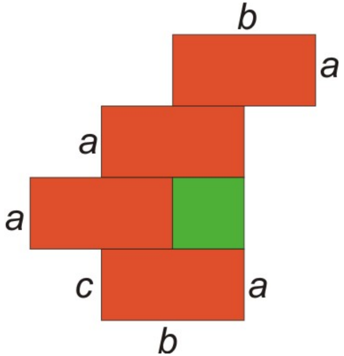

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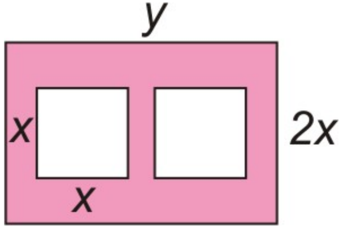

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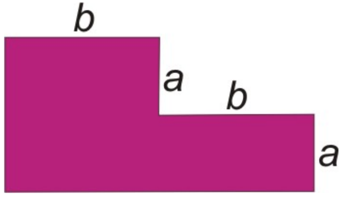

9.3 Multiplication of Monomials by Polynomials

Here you'll learn how to use the Distributive Property to multiply a polynomial by a monomial.

What if you had a monomial and polynomial like $3x^3$ and x^2+4? How could you multiply them? After completing this Concept, you'll be able multiply a polynomial by a monomial.

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CK-12 Foundation: Multiplying a Polynomial by a Monomial

Guidance

Just as we can add and subtract polynomials, we can also multiply them. The Distributive Property and the techniques you've learned for dealing with exponents will be useful here.

Multiplying a Polynomial by a Monomial

When multiplying polynomials, we must remember the exponent rules that we learned in the last chapter. Especially important is the product rule: $x^n \cdot x^m = x^{n+m}$.

If the expressions we are multiplying have coefficients and more than one variable, we multiply the coefficients just as we would any number and we apply the product rule on each variable separately.

Example A

Multiply the following monomials.

a) $(2x^2)(5x^3)$

b) $(-3y^4)(2y^2)$

c) $(3xy^5)(-6x^4y^2)$

d) $(-12a^2b^3c^4)(-3a^2b^2)$

Solution

a) $(2x^2)(5x^3) = (2 \cdot 5) \cdot (x^2 \cdot x^3) = 10x^{2+3} = 10x^5$

b) $(-3y^4)(2y^2) = (-3 \cdot 2) \cdot (y^4 \cdot y^2) = -6y^{4+2} = -6y^6$

c) $(3xy^5)(-6x^4y^2) = -18x^{1+4}y^{5+2} = -18x^5y^7$

d) $(-12a^2b^3c^4)(-3a^2b^2) = 36a^{2+2}b^{3+2}c^4 = 36a^4b^5c^4$

To multiply a polynomial by a monomial, we have to use the **Distributive Property**. Remember, that property says that $a(b+c) = ab+ac$.

Example B

Multiply:

a) $3(x^2+3x-5)$

b) $4x(3x^2-7)$

c) $-7y(4y^2-2y+1)$

Solution

a) $3(x^2+3x-5)=3(x^2)+3(3x)-3(5)=3x^2+9x-15$

b) $4x(3x^2-7)=(4x)(3x^2)+(4x)(-7)=12x^3-28x$

c)

$$\begin{aligned}-7y(4y^2-2y+1)&=(-7y)(4y^2)+(-7y)(-2y)+(-7y)(1)\\&=-28y^3+14y^2-7y\end{aligned}$$

Notice that when we use the Distributive Property, the problem becomes a matter of just multiplying monomials by monomials and adding all the separate parts together.

Example C

Multiply:

a) $2x^3(-3x^4+2x^3-10x^2+7x+9)$

b) $-7a^2bc^3(5a^2-3b^2-9c^2)$

Solution

a)

$$\begin{aligned}2x^3(-3x^4+2x^3-10x^2+7x+9)&=(2x^3)(-3x^4)+(2x^3)(2x^3)+(2x^3)(-10x^2)+(2x^3)(7x)+(2x^3)(9)\\&=-6x^7+4x^6-20x^5+14x^4+18x^3\end{aligned}$$

b)

$$\begin{aligned}-7a^2bc^3(5a^2-3b^2-9c^2)&=(-7a^2bc^3)(5a^2)+(-7a^2bc^3)(-3b^2)+(-7a^2bc^3)(-9c^2)\\&=-35a^4bc^3+21a^2b^3c^3+63a^2bc^5\end{aligned}$$

Watch this video for help with the Examples above.

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Click image to the left for more content.

CK-12 Foundation: Multiplying a Polynomial by a Monomial

Vocabulary

- **Distributive Property:** For any expressions $a,\ b$, and c, $a(b+c) = ab+ac$.

Guided Practice

Multiply $-2a^2b^4(3ab^2+7a^3b-9a+3)$.

Solution:

Multiply the monomial by each term inside the parenthesis:

$$\begin{aligned}
&-2a^2b^4(3ab^2+7a^3b-9a+3)\\
&=(-2a^2b^4)(3ab^2)+(-2a^2b^4)(7a^3b)+(-2a^2b^4)(-9a)+(-2a^2b^4)(3)\\
&=-6a^3b^6-14a^5b^5+18a^5b^4-6a^2b^4
\end{aligned}$$

Practice

Multiply the following monomials.

1. $(2x)(-7x)$
2. $(10x)(3xy)$
3. $(4mn)(0.5nm^2)$
4. $(-5a^2b)(-12a^3b^3)$
5. $(3xy^2z^2)(15x^2yz^3)$

Multiply and simplify.

6. $17(8x-10)$
7. $2x(4x-5)$
8. $9x^3(3x^2-2x+7)$
9. $3x(2y^2+y-5)$
10. $10q(3q^2r+5r)$
11. $-3a^2b(9a^2-4b^2)$

9.4 Multiplication of Polynomials by Binomials

Here you'll learn how to multiply one polynomial by another and simplify your answer. You'l also learn how to solve real-world problems using multiplication of polynomials.

What if you had two polynomials like $3x^3 + 2x^2$ and $x^2 - 1$? How could you multiply them? After completing this Concept, you'll be able to use the Distributive Property to multiply one polynomial by another.

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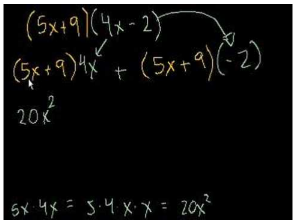
CK-12 Foundation: 0904S Multiplying a Polynomialby a Polynomial

This Khan Academy video shows how multiplying two binomials together is related to the distributive property.



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KhanAcademy: Level 1 multiplying expressions

Guidance

Let's start by multiplying two binomials together. A binomial is a polynomial with two terms, so a product of two binomials will take the form $(a+b)(c+d)$.

We can still use the Distributive Property here if we do it cleverly. First, let's think of the first set of parentheses as one term. The Distributive Property says that we can multiply that term by c, multiply it by d, and then add those two products together: $(a+b)(c+d) = (a+b)\cdot c + (a+b)\cdot d$.

We can rewrite this expression as $c(a+b)+d(a+b)$. Now let's look at each half separately. We can apply the distributive property again to each set of parentheses in turn, and that gives us $c(a+b)+d(a+b) = ca+cb+da+db$.

What you should notice is that when multiplying any two polynomials, *every term in one polynomial is multiplied by every term in the other polynomial.*

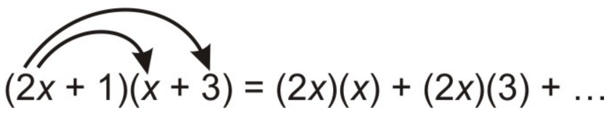
Example A

Multiply and simplify: $(2x+1)(x+3)$

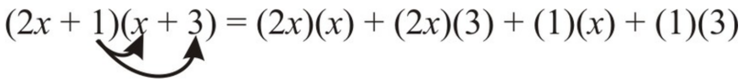
Solution

We must multiply each term in the first polynomial by each term in the second polynomial. Let's try to be systematic to make sure that we get all the products.

First, multiply the first term in the first set of parentheses by all the terms in the second set of parentheses.



Now we're done with the first term. Next we multiply the second term in the first parenthesis by all terms in the second parenthesis and add them to the previous terms.



Now we're done with the multiplication and we can simplify:

$(2x)(x)+(2x)(3)+(1)(x)+(1)(3) = 2x^2+6x+x+3 = 2x^2+7x+3$

This way of multiplying polynomials is called **in-line** multiplication or **horizontal** multiplication. Another method for multiplying polynomials is to use **vertical** multiplication, similar to the vertical multiplication you learned with regular numbers.

Example B

Multiply and simplify:

a) $(4x-5)(x-20)$

b) $(3x-2)(3x+2)$

c) $(3x^2+2x-5)(2x-3)$

d) $(x^2-9)(4x^4+5x^2-2)$

Solution

a) With horizontal multiplication this would be

$$\begin{aligned}(4x-5)(x-20) &= (4x)(x)+(4x)(-20)+(-5)(x)+(-5)(-20)\\ &= 4x^2-80x-5x+100\\ &= 4x^2-85x+100\end{aligned}$$

To do vertical multiplication instead, we arrange the polynomials on top of each other with like terms in the same columns:

$$\begin{array}{r} 4x-5 \\ \underline{\qquad x-20} \\ -80x+100 \\ \underline{4x^2-5x\qquad\;\;} \\ 4x^2-85x+100 \end{array}$$

Both techniques result in the same answer: $4x^2-85x+100$. We'll use vertical multiplication for the other problems.

b)

$$\begin{array}{r}
3x-2\\
3x+2\\
\hline
6x-4\\
9x^2-6x\\
\hline
9x^2+0x-4
\end{array}$$

The answer is $9x^2-4$.

c) It's better to place the smaller polynomial on the bottom:

$$\begin{array}{r}
3x^2+2x-5\\
2x-3\\
\hline
-9x^2-6x+15\\
6x^3+4x^2-10x\\
\hline
6x^3-5x^2-16x+15
\end{array}$$

The answer is $6x^3-5x^2-16x+15$.

d) Set up the multiplication vertically and leave gaps for missing powers of x:

$$\begin{array}{r}
4x^4+5x^2-2\\
x^2-9\\
\hline
-36x^4-45x^2+18\\
4x^6+\ 5x^4\ -\ 2x^2\\
\hline
4x^6-31x^4-47x^2+18
\end{array}$$

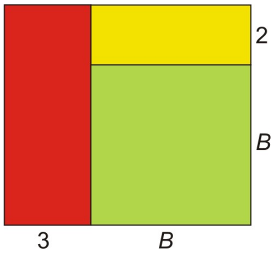
The answer is $4x^6-31x^4-47x^2+18$.

Solve Real-World Problems Using Multiplication of Polynomials

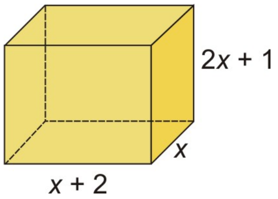
In this section, we'll see how multiplication of polynomials is applied to finding the areas and volumes of geometric shapes.

Example C

a) *Find the areas of the figure:*



b) *Find the volumes of the figure:*



Solutions:

a) We use the formula for the area of a rectangle: Area = length × width.

For the big rectangle:

$$\begin{aligned}\text{Length} &= b+3,\ \text{Width} = b+2\\ \text{Area} &= (b+3)(b+2)\\ &= b^2+2b+3b+6\\ &= b^2+5b+6\end{aligned}$$

b) The volume of this shape = (area of the base)(height).

$$\begin{aligned}\text{Area of the base} &= x(x+2)\\ &= x^2+2x\\ \text{Height} &= 2x+1\\ \text{Volume} &= (x^2+2x)(2x+1)\\ &= 2x^3+x^2+4x^2+2x\\ &= 2x^3+5x^2+2x\end{aligned}$$

Watch this video for help with the Examples above.

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CK-12 Foundation: Multiplying a Polynomial by a Polynomial

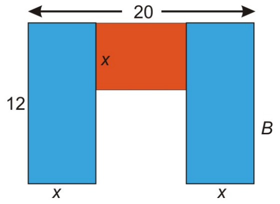
Vocabulary

- A ***binomial*** is a polynomial with two terms.
- **The Distributive Property for Binomials:** The Distributive Property says that the term in front of the parentheses multiplies with each term inside the parentheses separately. Then, we add the results of the products.

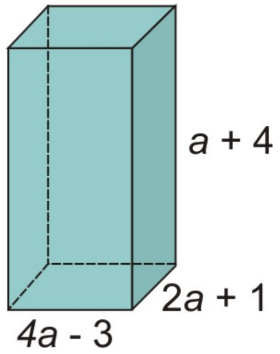
$$\begin{aligned}(a+b)(c+d) &= c\cdot(a+b)+d\cdot(a+b)\\ &= c\cdot a+c\cdot b+d\cdot a+d\cdot b\\ &= \ ca+cb+da+db\end{aligned}$$

Guided Practice

1. *Find the areas of the figure:*

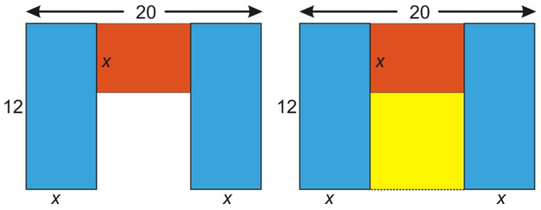


2. *Find the volumes of the figure:*



Solutions:

1. We could add up the areas of the blue and orange rectangles, but it's easier to just find the area of the whole *big* rectangle and subtract the area of the yellow rectangle.



$$\begin{aligned}\text{Area of big rectangle} &= 20(12)=240\\ \text{Area of yellow rectangle} &= (12-x)(20-2x)\\ &= 240-24x-20x+2x^2\\ &= 240-44x+2x^2\\ &= 2x^2-44x+240\end{aligned}$$

The desired area is the difference between the two:

$$\begin{aligned}
\text{Area} &= 240-(2x^2-44x+240)\\
&= 240+(-2x^2+44x-240)\\
&= 240-2x^2+44x-240\\
&= -2x^2+44x
\end{aligned}$$

2. The volume of this shape = (area of the base)(height).

$$\begin{aligned}
\text{Area of the base} &= (4a-3)(2a+1)\\
&= 8a^2+4a-6a-3\\
&= 8a^2-2a-3\\
\text{Height} &= a+4\\
\text{Volume} &= (8a^2-2a-3)(a+4)
\end{aligned}$$

Let's multiply using the vertical method:

$$\begin{array}{rrrr}
 & 8a^2 & -2a & -3\\
 & & a & +4\\ \hline
 & 32a^2 & -8a & -12\\
8a^3 & -\ 2a^2 & -3a & \\ \hline
8a^3+ & 30a^2 & -11a & -12
\end{array}$$

The volume is $8a^3+30a^2-11a-12$.

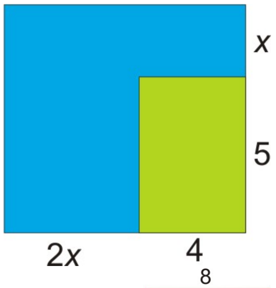
Practice

Multiply and simplify.

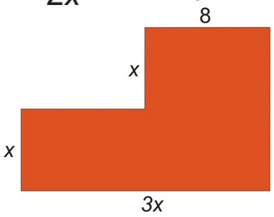
1. $(x-3)(x+2)$
2. $(a+b)(a-5)$
3. $(x+2)(x^2-3)$
4. $(a^2+2)(3a^2-4)$
5. $(7x-2)(9x-5)$
6. $(2x-1)(2x^2-x+3)$
7. $(3x+2)(9x^2-6x+4)$
8. $(a^2+2a-3)(a^2-3a+4)$
9. $3(x-5)(2x+7)$
10. $5x(x+4)(2x-3)$

Find the areas of the following figures.

11.

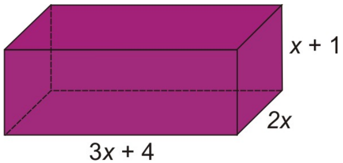


12.



Find the volumes of the following figures.

13.



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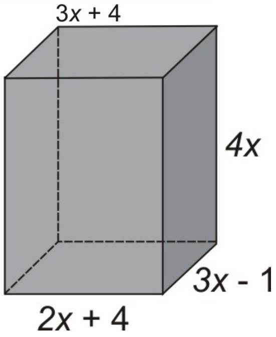

9.5 Special Products of Polynomials

Here you'll learn how to find two special polynomial products: 1) the square of a binomial and 2) two binomials where the sum and difference formula can be applied. You'll also learn how to apply special products of polynomials to solve real-world problems.

What if you wanted to multiply two binomials that were exactly the same, like $(x^2-2)(x^2-2)$? Similarly what if you wanted to multiply two binomials in which the sign between the two terms was the opposite in one from the other, like $(x^2-2)(x^2+2)$? What shortcuts could you use? After completing this Concept, you'll be able to find the square of a binomial as well as the product of binomials using the sum and difference formula.

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CK-12 Foundation: 0905S Special Productsof Polynomials

Guidance

We saw that when we multiply two binomials we need to make sure to multiply each term in the first binomial with each term in the second binomial. Let's look at another example.

Multiply two linear binomials (binomials whose degree is 1):

$$(2x+3)(x+4)$$

When we multiply, we obtain a quadratic polynomial (one with degree 2) with four terms:

$$2x^2+8x+3x+12$$

The middle terms are like terms and we can combine them. We simplify and get $2x^2+11x+12$. This is a quadratic, or second-degree, **trinomial** (polynomial with three terms).

You can see that every time we multiply two linear binomials with one variable, we will obtain a quadratic polynomial. In this section we'll talk about some special products of binomials.

Find the Square of a Binomial

One special binomial product is the **square of a binomial**. Consider the product $(x+4)(x+4)$.

Since we are multiplying the same expression by itself, that means we are squaring the expression. $(x+4)(x+4)$ is the same as $(x+4)^2$.

When we multiply it out, we get $x^2+4x+4x+16$, which simplifies to $x^2+8x+16$.

Notice that the two middle terms—the ones we added together to get $8x$—were the same. Is this a coincidence? In order to find that out, let's square a general linear binomial.

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) = a^2+ab+ab+b^2\\ &= a^2+2ab+b^2\end{aligned}$$

Sure enough, the middle terms are the same. How about if the expression we square is a difference instead of a sum?

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b) = a^2-ab-ab+b^2\\ &= a^2-2ab+b^2\end{aligned}$$

It looks like the middle two terms are the same in general whenever we square a binomial. The general pattern is: to square a binomial, take the square of the first term, add or subtract twice the product of the terms, and add the square of the second term. You should remember these formulas:

$$\begin{gathered}(a+b)^2 = a^2+2ab+b^2\\ \text{and}\\ (a-b)^2 = a^2-2ab+b^2\end{gathered}$$

Remember! Raising a polynomial to a power means that we multiply the polynomial by itself however many times the exponent indicates. For instance, $(a+b)^2=(a+b)(a+b)$. **Don't make the common mistake of thinking that** $(a+b)^2=a^2+b^2$**!** To see why that's not true, try substituting numbers for a and b into the equation (for example, $a=4$ and $b=3$), and you will see that it is *not* a true statement. The middle term, $2ab$, is needed to make the equation work.

We can apply the formulas for squaring binomials to any number of problems.

Example A

Square each binomial and simplify.

a) $(x+10)^2$

b) $(2x-3)^2$

c) $(x^2+4)^2$

Solution

Let's use the square of a binomial formula to multiply each expression.

a) $(x+10)^2$

If we let $a=x$ and $b=10$, then our formula $(a+b)^2=a^2+2ab+b^2$ becomes $(x+10)^2=x^2+2(x)(10)+10^2$, which simplifies to $x^2+20x+100$.

b) $(2x-3)^2$

If we let $a=2x$ and $b=3$, then our formula $(a-b)^2=a^2-2ab+b^2$ becomes $(2x-3)^2=(2x^2)-2(2x)(3)+(3)^2$, which simplifies to $4x^2-12x+9$.

c) $(x^2+4)^2$

If we let $a = x^2$ and $b = 4$, then

$$\begin{aligned}(x^2+4)^2 &= (x^2)^2+2(x^2)(4)+(4)^2\\ &= x^4+8x^2+16\end{aligned}$$

Find the Product of Binomials Using Sum and Difference Patterns

Another special binomial product is the product of a sum and a difference of terms. For example, let's multiply the following binomials.

$$\begin{aligned}(x+4)(x-4) &= x^2-4x+4x-16\\ &= x^2-16\end{aligned}$$

Notice that the middle terms are opposites of each other, so they *cancel out* when we collect like terms. This is not a coincidence. This always happens when we multiply a sum and difference of the same terms. In general,

$$\begin{aligned}(a+b)(a-b) &= a^2-ab+ab-b^2\\ &= a^2-b^2\end{aligned}$$

When multiplying a sum and difference of the same two terms, the middle terms cancel out. We get the square of the first term minus the square of the second term. You should remember this formula.

Sum and Difference Formula: $(a+b)(a-b) = a^2-b^2$

Let's apply this formula to a few examples.

Example B

Multiply the following binomials and simplify.

a) $(x+3)(x-3)$

b) $(5x+9)(5x-9)$

c) $(2x^3+7)(2x^3-7)$

Solution

a) Let $a = x$ and $b = 3$, then:

$$\begin{aligned}(a+b)(a-b) &= a^2-b^2\\ (x+3)(x-3) &= x^2-3^2\\ &= x^2-9\end{aligned}$$

b) Let $a = 5x$ and $b = 9$, then:

$$\begin{aligned}(a+b)(a-b) &= a^2-b^2\\(5x+9)(5x-9) &= (5x)^2-9^2\\&= 25x^2-81\end{aligned}$$

c) Let $a=2x^3$ and $b=7$, then:

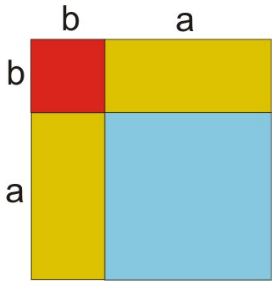
$$\begin{aligned}(2x^3+7)(2x^3-7) &= (2x^3)^2-(7)^2\\&= 4x^6-49\end{aligned}$$

Solve Real-World Problems Using Special Products of Polynomials

Now let's see how special products of polynomials apply to geometry problems and to mental arithmetic.

Example C

Find the area of the following square:



Solution

The length of each side is $(a+b)$, so the area is $(a+b)(a+b)$.

Notice that this gives a visual explanation of the square of a binomial. The blue square has area a^2, the red square has area b^2, and each rectangle has area ab, so added all together, the area $(a+b)(a+b)$ is equal to $a^2+2ab+b^2$.

The next example shows how you can use the special products to do fast mental calculations.

Example D

Use the difference of squares and the binomial square formulas to find the products of the following numbers without using a calculator.

a) 43×57

b) 45^2

c) 481×319

Solution

The key to these mental "tricks" is to rewrite each number as a sum or difference of numbers you know how to square easily.

a) Rewrite 43 as $(50-7)$ and 57 as $(50+7)$.

Then $43 \times 57 = (50-7)(50+7) = (50)^2 - (7)^2 = 2500 - 49 = 2451$

b) $45^2 = (40+5)^2 = (40)^2 + 2(40)(5) + (5)^2 = 1600 + 400 + 25 = 2025$

c) Rewrite 481 as $(400+81)$ and 319 as $(400-81)$.

Then $481 \times 319 = (400+81)(400-81) = (400)^2 - (81)^2$

$(400)^2$ is easy - it equals 160000.

$(81)^2$ is not easy to do mentally, so let's rewrite 81 as $80+1$.

$(81)^2 = (80+1)^2 = (80)^2 + 2(80)(1) + (1)^2 = 6400 + 160 + 1 = 6561$

Then $481 \times 319 = (400)^2 - (81)^2 = 160000 - 6561 = 153439$

Watch this video for help with the Examples above.

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CK-12 Foundation: Special Products of Polynomials

Vocabulary

- **Square of a binomial:** $(a+b)^2 = a^2 + 2ab + b^2$, *and* $(a-b)^2 = a^2 - 2ab + b^2$
- **Sum and difference formula:** $(a+b)(a-b) = a^2 - b^2$

Guided Practice

1. *Square the binomial and simplify:* $(5x-2y)^2$.

2. *Multiply* $(4x+5y)(4x-5y)$ *and simplify.*

3. *Use the difference of squares and the binomial square formulas to find the product of* 112×88 *without using a calculator.*

Solutions:

1.) $(5x-2y)^2$

If we let $a = 5x$ and $b = 2y$, then

$$\begin{aligned}(5x-2y)^2 &= (5x)^2 - 2(5x)(2y) + (2y)^2\\ &= 25x^2 - 20xy + 4y^2\end{aligned}$$

2.) Let $a = 4x$ and $b = 5y$, then:

$$\begin{aligned}(4x+5y)(4x-5y) &= (4x)^2 - (5y)^2\\ &= 16x^2 - 25y^2\end{aligned}$$

3.) The key to these mental “tricks” is to rewrite each number as a sum or difference of numbers you know how to square easily.

Rewrite 112 as $(100+12)$ and 88 as $(100-12)$.

Then

$$\begin{aligned}112\times 88 &= (100+12)(100-12)\\ &= (100)^2-(12)^2\\ &= 10000-144\\ &= 9856\end{aligned}$$

Practice

Use the special product rule for squaring binomials to multiply these expressions.

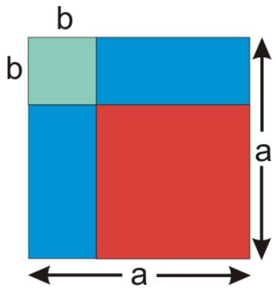
1. $(x+9)^2$
2. $(3x-7)^2$
3. $(5x-y)^2$
4. $(2x^3-3)^2$
5. $(4x^2+y^2)^2$
6. $(8x-3)^2$
7. $(2x+5)(5+2x)$
8. $(xy-y)^2$

Use the special product of a sum and difference to multiply these expressions.

9. $(2x-1)(2x+1)$
10. $(x-12)(x+12)$
11. $(5a-2b)(5a+2b)$
12. $(ab-1)(ab+1)$
13. $(z^2+y)(z^2-y)$
14. $(2q^3+r^2)(2q^3-r^2)$
15. $(7s-t)(t+7s)$
16. $(x^2y+xy^2)(x^2y-xy^2)$

Find the area of the lower right square in the following figure.

17.



Multiply the following numbers using special products.

18. 45×55

19. 56^2
20. 1002×998
21. 36×44
22. 10.5×9.5
23. 100.2×9.8
24. -95×-105
25. 2×-2

9.6 Monomial Factors of Polynomials

Here you'll learn how to factor out the greatest common monomial from a polynomial.

What if you had a polynomial like $3x^3 - 9x^2 + 6x$? How could you factor it completely? After completing this Concept, you'll be able to find a polynomial's greatest common monomial factor.

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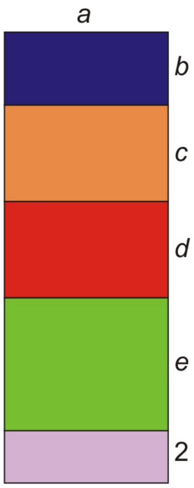
CK-12 Foundation: 0906S GreatestCommon MonomialFactors

Guidance

In the last few sections, we learned how to multiply polynomials by using the Distributive Property. All the terms in one polynomial had to be multiplied by all the terms in the other polynomial. In this section, you'll start learning how to do this process in reverse. The reverse of distribution is called **factoring**.



The total area of the figure above can be found in two ways.

We could find the areas of all the small rectangles and add them: $ab + ac + ad + ae + 2a$.

Or, we could find the area of the big rectangle all at once. Its width is a and its length is $b + c + d + e + 2$, so its area is $a(b + c + d + e + 2)$.

Since the area of the rectangle is the same no matter what method we use, those two expressions must be equal.

$$ab + ac + ad + ae + 2a = a(b + c + d + e + 2)$$

To turn the right-hand side of this equation into the left-hand side, we would use the distributive property. To turn the left-hand side into the right-hand side, we would need to **factor** it. Since polynomials can be multiplied just like numbers, they can also be factored just like numbers—and we'll see later how this can help us solve problems.

Find the Greatest Common Monomial Factor

You will be learning several factoring methods in the next few sections. In most cases, factoring takes several steps to complete because we want to **factor completely**. That means that we factor until we can't factor any more.

Let's start with the simplest step: finding the greatest monomial factor. When we want to factor, we always look for common monomials first. Consider the following polynomial, written in expanded form:

$$ax+bx+cx+dx$$

A common factor is any factor that appears in all terms of the polynomial; it can be a number, a variable or a combination of numbers and variables. Notice that in our example, the factor x appears in all terms, so it is a common factor.

To factor out the x, we write it outside a set of parentheses. Inside the parentheses, we write what's left when we divide each term by x:

$$x(a+b+c+d)$$

Let's look at more examples.

Example A

Factor:

a) $2x+8$

b) $15x-25$

c) $3a+9b+6$

Solution

a) We see that the factor 2 divides evenly into both terms: $2x+8=2(x)+2(4)$

We factor out the 2 by writing it in front of a parenthesis: 2( )

Inside the parenthesis we write what is left of each term when we divide by 2: $2(x+4)$

b) We see that the factor of 5 divides evenly into all terms: $15x-25=5(3x)-5(5)$

Factor out the 5 to get: $5(3x-5)$

c) We see that the factor of 3 divides evenly into all terms: $3a+9b+6=3(a)+3(3b)+3(2)$

Factor 3 to get: $3(a+3b+2)$

Example B

Find the greatest common factor:

a) a^3-3a^2+4a

b) $12a^4-5a^3+7a^2$

Solution

a) Notice that the factor a appears in all terms of $a^3 - 3a^2 + 4a$, but each term has a raised to a different power. The greatest common factor of all the terms is simply a.

So first we rewrite $a^3 - 3a^2 + 4a$ as $a(a^2) + a(-3a) + a(4)$.

Then we factor out the a to get $a(a^2 - 3a + 4)$.

b) The factor a appears in all the terms, and it's always raised to at least the second power. So the greatest common factor of all the terms is a^2.

We rewrite the expression $12a^4 - 5a^3 + 7a^2$ as $(12a^2 \cdot a^2) - (5a \cdot a^2) + (7 \cdot a^2)$

Factor out the a^2 to get $a^2(12a^2 - 5a + 7)$.

Example C

Factor completely:

a) $3ax + 9a$

b) $x^3y + xy$

c) $5x^3y - 15x^2y^2 + 25xy^3$

Solution

a) Both terms have a common factor of 3, but they also have a common factor of a. It's simplest to factor these both out at once, which gives us $3a(x+3)$.

b) Both x and y are common factors. When we factor them both out at once, we get $xy(x^2 + 1)$.

c) The common factors are 5, x, and y. Factoring out $5xy$ gives us $5xy(x^2 - 3xy + 5xy^2)$.

Watch this video for help with the Examples above.

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CK-12 Foundation: Greatest Common Monomial Factors

Vocabulary

- A **common factor** can be a number, a variable, or a combination of numbers and variables that appear in **every term** of the polynomial.

Guided Practice

Find the greatest common factor.

$16x^2y^3z^2 + 4x^3yz + 8x^2y^4z^5$

Solution:

First, look at the coefficients to see if they share any common factors. They do: 4.

Next, look for the lowest power of each variable, because that is the most you can factor out. The lowest power of x is x^2. The lowest powers of y and z are to the first power.

This means we can factor out $4x^2yz$. Now, we have to determine what is left in each term after we factor out $4x^2yz$:

$16x^2y^3z^2+4x^3yz+8x^2y^4z^5=4x^2yz(4y^2z+x+2y^3z^4)$

Practice

Factor out the greatest common factor in the following polynomials.

1. $2x^2-5x$
2. $3x^3-21x$
3. $5x^6+15x^4$
4. $4x^3+10x^2-2x$
5. $-10x^6+12x^5-4x^4$
6. $12xy+24xy^2+36xy^3$
7. $5a^3-7a$
8. $3y+6z$
9. $10a^3-4ab$
10. $45y^{12}+30y^{10}$
11. $16xy^2z+4x^3y$
12. $2a-4a^2+6$
13. $5xy^2-10xy+5y^2$

9.7 Zero Product Principle

Here you'll learn how to apply the zero-product property and how to factor polynomials to solve for their unknown variables.

What if you had a polynomial equation like $3x^2+4x-4=0$? How could you factor the polynomial to solve the equation? After completing this Concept, you'll be able to solve polynomial equations by factoring and by using the zero-product property.

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CK-12 Foundation: 0907S Factoring toSolve Polynomials

Guidance

The most useful thing about factoring is that we can use it to help solve polynomial equations.

Example A

Consider an equation like $2x^2+5x-42=0$. How do you solve for x?

Solution:

There's no good way to isolate x in this equation, so we can't solve it using any of the techniques we've already learned. But the left-hand side of the equation can be factored, making the equation $(x+6)(2x-7)=0$.

How is this helpful? The answer lies in a useful property of multiplication: if two numbers multiply to zero, then at least one of those numbers must be zero. This is called the **Zero-Product Property.**

What does this mean for our polynomial equation? Since the product equals 0, then at least one of the factors on the left-hand side must equal zero. So we can find the two solutions by setting each factor equal to zero and solving each equation separately.

Setting the factors equal to zero gives us:

$$(x+6)=0 \qquad \text{OR} \qquad (2x-7)=0$$

Solving both of those equations gives us:

$$\begin{array}{lcl} x+6=0 & & 2x-7=0 \\ \underline{\underline{x=-6}} & \text{OR} & 2x=7 \\ & & \underline{\underline{x=\frac{7}{2}}} \end{array}$$

Notice that the solution is $x=-6$**OR** $x=\frac{7}{2}$. The **OR** means that either of these values of x would make the product of the two factors equal to zero. Let's plug the solutions back into the equation and check that this is correct.

$$\begin{array}{ll} Check: x=-6; & Check: x=\frac{7}{2} \\ (x+6)(2x-7)= & (x+6)(2x-7)= \\ (-6+6)(2(-6)-7)= & \left(\frac{7}{2}+6\right)\left(2\cdot\frac{7}{2}-7\right)= \\ (0)(-19)=0 & \left(\frac{19}{2}\right)(7-7)= \\ & \left(\frac{19}{2}\right)(0)=0 \end{array}$$

Both solutions check out.

Factoring a polynomial is very useful because the Zero-Product Property allows us to break up the problem into simpler separate steps. When we can't factor a polynomial, the problem becomes harder and we must use other methods that you will learn later.

As a last note in this section, keep in mind that the Zero-Product Property only works when a product equals zero. For example, if you multiplied two numbers and the answer was nine, that wouldn't mean that one or both of the numbers must be nine. In order to use the property, the factored polynomial must be equal to zero.

Example B

Solve each equation:

a) $(x-9)(3x+4)=0$

b) $x(5x-4)=0$

c) $4x(x+6)(4x-9)=0$

Solution

Since all the polynomials are in factored form, we can just set each factor equal to zero and solve the simpler equations separately

a) $(x-9)(3x+4)=0$ can be split up into two linear equations:

$$\begin{array}{lcl} x-9=0 & & 3x+4=0 \\ \underline{\underline{x=9}} & \text{or} & 3x=-4 \\ & & \underline{\underline{x=-\frac{4}{3}}} \end{array}$$

b) $x(5x-4)=0$ can be split up into two linear equations:

$$\underline{\underline{x=0}} \qquad \text{or} \qquad \begin{aligned} 5x-4&=0\\ 5x&=4\\ \underline{\underline{x=\frac{4}{5}}} \end{aligned}$$

c) $4x(x+6)(4x-9)=0$ can be split up into three linear equations:

$$\begin{aligned} 4x&=0\\ x&=\frac{0}{4}\\ \underline{\underline{x=0}} \end{aligned} \qquad \text{or} \qquad \begin{aligned} x+6&=0\\ x&=-6 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 4x-9&=0\\ 4x&=9\\ \underline{\underline{x=\frac{9}{4}}} \end{aligned}$$

Solve Simple Polynomial Equations by Factoring

Now that we know the basics of factoring, we can solve some simple polynomial equations. We already saw how we can use the Zero-Product Property to solve polynomials in factored form—now we can use that knowledge to solve polynomials by factoring them first. Here are the steps:

a) If necessary, **rewrite** the equation in standard form so that the right-hand side equals zero.

b) **Factor** the polynomial completely.

c) Use the zero-product rule to **set each factor equal to zero.**

d) **Solve** each equation from step 3.

e) **Check** your answers by substituting your solutions into the original equation

Example C

Solve the following polynomial equations.

a) $x^2-2x=0$

b) $2x^2=5x$

c) $9x^2y-6xy=0$

Solution

a) $x^2-2x=0$

Rewrite: this is not necessary since the equation is in the correct form.

Factor: The common factor is x, so this factors as $x(x-2)=0$.

Set each factor equal to zero:

$$x=0 \qquad \text{or} \qquad x-2=0$$

Solve:

$$\underline{x=0} \qquad \text{or} \qquad \underline{x=2}$$

Check: Substitute each solution back into the original equation.

$$\begin{aligned} &x=0 \Rightarrow (0)^2-2(0)=0 && \text{works out}\\ &x=2 \Rightarrow (2)^2-2(2)=4-4=0 && \text{works out} \end{aligned}$$

Answer: $x=0, x=2$

b) $2x^2=5x$

Rewrite: $2x^2=5x \Rightarrow 2x^2-5x=0$

Factor: The common factor is x, so this factors as $x(2x-5)=0$.

Set each factor equal to zero:

$$x=0 \qquad \text{or} \qquad 2x-5=0$$

Solve:

$$\underline{x=0} \qquad \text{or} \qquad \begin{aligned} 2x&=5\\ \underline{x}&\underline{=\frac{5}{2}} \end{aligned}$$

Check: Substitute each solution back into the original equation.

$$\begin{aligned} &x=0 \Rightarrow 2(0)^2=5(0) \Rightarrow 0=0 && \text{works out}\\ &x=\frac{5}{2} \Rightarrow 2\left(\frac{5}{2}\right)^2=5\cdot\frac{5}{2} \Rightarrow 2\cdot\frac{25}{4}=\frac{25}{2} \Rightarrow \frac{25}{2}=\frac{25}{2} && \text{works out} \end{aligned}$$

Answer: $x=0, x=\frac{5}{2}$

c) $9x^2y-6xy=0$

Rewrite: not necessary

Factor: The common factor is $3xy$, so this factors as $3xy(3x-2)=0$.

Set each factor equal to zero:

$3=0$ is never true, so this part does not give a solution. The factors we have left give us:

$$x=0 \qquad \text{or} \qquad y=0 \qquad \text{or} \qquad 3x-2=0$$

Solve:

$$\underline{x=0} \qquad \text{or} \qquad \underline{y=0} \qquad \text{or} \qquad 3x=2$$

$$\underline{x=\frac{2}{3}}$$

Check: Substitute each solution back into the original equation.

$$\begin{aligned} &x=0 \Rightarrow 9(0)y-6(0)y=0-0=0 && \text{works out}\\ &y=0 \Rightarrow 9x^2(0)-6x(0)=0-0=0 && \text{works out}\\ &x=\frac{2}{3} \Rightarrow 9\cdot\left(\frac{2}{3}\right)^2 y-6\cdot\frac{2}{3}y=9\cdot\frac{4}{9}y-4y=4y-4y=0 && \text{works out}\end{aligned}$$

Answer: $x=0, y=0, x=\frac{2}{3}$

Watch this video for help with the Examples above.

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CK-12 Foundation: Factoring to SolvePolynomials

Vocabulary

- Polynomials can be written in **expanded form** or in **factored form**. Expanded form means that you have sums and differences of different terms:
- The **factored form** of a polynomial means it is written as a product of its factors.
- **Zero Product Property:** The only way a product is zero is if one or more of the terms are equal to zero:

$a\cdot b=0 \Rightarrow a=0$ or $b=0$.

Guided Practice

Solve the following polynomial equation.

$$9x^2-3x=0$$

Solution: $9x^2-3x=0$

Rewrite: This is not necessary since the equation is in the correct form.

Factor: The common factor is $3x$, so this factors as: $3x(3x-1)=0$.

Set each factor equal to zero.

$$3x = 0 \qquad \text{or} \qquad x - 2 = 0$$

Solve:

$$x = 0 \qquad \text{or} \qquad x = 2$$

Check: Substitute each solution back into the original equation.

$$\begin{aligned} x &= 0 & (0)^2 - 2(0) &= 0 \\ x &= 2 & (2)^2 - 2(2) &= 0 \end{aligned}$$

Answer $x = 0, \ x = 2$

Practice

Solve the following polynomial equations.

1. $x(x+12) = 0$
2. $(2x+1)(2x-1) = 0$
3. $(x-5)(2x+7)(3x-4) = 0$
4. $2x(x+9)(7x-20) = 0$
5. $x(3+y) = 0$
6. $x(x-2y) = 0$
7. $18y - 3y^2 = 0$
8. $9x^2 = 27x$
9. $4a^2 + a = 0$
10. $b^2 - \frac{5}{3}b = 0$
11. $4x^2 = 36$
12. $x^3 - 5x^2 = 0$

9.8 Factorization of Quadratic Expressions

Here you'll learn how to factor second-degree polynomials, also known as quadratic polynomials, in which all of the terms have positive coefficients.

What if you had a quadratic expression like $x^2+9x+14$ in which all the coefficients were positive? How could you factor that expression? After completing this Concept, you'll be able to factor quadratic expressions like this one with positive coefficient values.

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CK-12 Foundation: 0908S Factoring Quadratic Expressions

Guidance

Quadratic polynomials are polynomials of the 2^{nd} degree. The standard form of a quadratic polynomial is written as

$$ax^2+bx+c$$

where a, b, and c stand for constant numbers. Factoring these polynomials depends on the values of these constants. In this section we'll learn how to factor quadratic polynomials for different values of a, b, and c. (When none of the coefficients are zero, these expressions are also called quadratic **trinomials**, since they are polynomials with three terms.)

You've already learned how to factor quadratic polynomials where $c = 0$. For example, for the quadratic ax^2+bx, the common factor is x and this expression is factored as $x(ax+b)$. Now we'll see how to factor quadratics where c is nonzero.

Factor when a = 1, b is Positive, and c is Positive

First, let's consider the case where $a = 1, b$ is positive and c is positive. The quadratic trinomials will take the form

$$x^2+bx+c$$

You know from multiplying binomials that when you multiply two factors $(x+m)(x+n)$, you get a quadratic polynomial. Let's look at this process in more detail. First we use distribution:

$$(x+m)(x+n) = x^2+nx+mx+mn$$

Then we simplify by combining the like terms in the middle. We get:

$$(x+m)(x+n) = x^2+(n+m)x+mn$$

So to factor a quadratic, we just need to do this process in reverse.

$$\begin{array}{ll} \text{We see that} & x^2+(n+m)x+mn \\ \text{is the same form as} & x^2+bx+c \end{array}$$

This means that we need to find two numbers m and n where

$$n+m=b \qquad \text{and} \qquad mn=c$$

The factors of x^2+bx+c are always two binomials

$$(x+m)(x+n)$$

such that $n+m=b$ and $mn=c$.

Example A

Factor x^2+5x+6.

Solution

We are looking for an answer that is a product of two binomials in parentheses:

$$(x \quad)(x \quad)$$

We want two numbers m and n that multiply to 6 and add up to 5. A good strategy is to list the possible ways we can multiply two numbers to get 6 and then see which of these numbers add up to 5:

$$\begin{array}{llll} 6=1\cdot 6 & \text{and} & 1+6=7 & \\ 6=2\cdot 3 & \text{and} & 2+3=5 & \textit{This is the correct choice.} \end{array}$$

So the answer is $(x+2)(x+3)$.

We can check to see if this is correct by multiplying $(x+2)(x+3)$:

$$\begin{array}{r} x+2 \\ \underline{\quad x+3} \\ 3x+6 \\ \underline{x^2+2x \quad\quad} \\ x^2+5x+6 \end{array}$$

The answer checks out.

Example B

Factor $x^2+7x+12$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

The number 12 can be written as the product of the following numbers:

$$\begin{array}{llll}
12 = 1 \cdot 12 & \text{and} & 1+12=13 & \\
12 = 2 \cdot 6 & \text{and} & 2+6=8 & \\
12 = 3 \cdot 4 & \text{and} & 3+4=7 & This\ is\ the\ correct\ choice.
\end{array}$$

The answer is $(x+3)(x+4)$.

Example C

Factor $x^2+8x+12$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

The number 12 can be written as the product of the following numbers:

$$\begin{array}{llll}
12 = 1 \cdot 12 & \text{and} & 1+12=13 & \\
12 = 2 \cdot 6 & \text{and} & 2+6=8 & This\ is\ the\ correct\ choice. \\
12 = 3 \cdot 4 & \text{and} & 3+4=7 &
\end{array}$$

The answer is $(x+2)(x+6)$.

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CK-12 Foundation: Factoring Quadratic Expressions

Vocabulary

- A quadratic of the form x^2+bx+c factors as a product of two binomials in parentheses: $(x+m)(x+n)$
- If b and c are positive, then both m and n are positive.

Guided Practice

Factor $x^2 + 12x + 36$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

The number 36 can be written as the product of the following numbers:

$$\begin{array}{lll l}
36 = 1 \cdot 36 & \text{and} & 1 + 36 = 37 & \\
36 = 2 \cdot 18 & \text{and} & 2 + 18 = 20 & \\
36 = 3 \cdot 12 & \text{and} & 3 + 12 = 15 & \\
36 = 4 \cdot 9 & \text{and} & 4 + 9 = 13 & \\
36 = 6 \cdot 6 & \text{and} & 6 + 6 = 12 & \textit{This is the correct choice.}
\end{array}$$

The answer is $(x + 6)(x + 6)$.

Practice

Factor the following quadratic polynomials.

1. $x^2 + 10x + 9$
2. $x^2 + 15x + 50$
3. $x^2 + 10x + 21$
4. $x^2 + 16x + 48$
5. $x^2 + 14x + 45$
6. $x^2 + 15x + 50$
7. $x^2 + 22x + 40$
8. $x^2 + 15x + 56$
9. $x^2 + 2x + 1$
10. $x^2 + 10x + 24$
11. $x^2 + 17x + 72$
12. $x^2 + 25x + 150$

9.9 Factorization of Quadratic Expressions with Negative Coefficients

Here you'll learn how to factor quadratic polynomials in which some of the terms have negative coefficients.

What if you had a quadratic expression like $x^2 - 3x - 10$ or $-x^2 - 4x - 4$ in which some or all the coefficients were negative? How could you factor that expression? After completing this Concept, you'll be able to factor quadratic expressions like these for various negative coefficient values.

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CK-12 Foundation: 0909S Factoring Quadratic Expressions withNegative Coefficients

Guidance

In the previous concept, we saw how to factor quadratic expressions whose coefficients were all positive. In this concept we will now see what happens when we factor quadratic expressions where some of the coefficients are negative.

Factor when a = 1, b is Negative and c is Positive

Now let's see how this method works if the middle coefficient is negative.

Example A

Factor $x^2 - 6x + 8$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

When negative coefficients are involved, we have to remember that negative factors may be involved also. The number 8 can be written as the product of the following numbers:

$$8 = 1 \cdot 8 \qquad \text{and} \qquad 1 + 8 = 9$$

but also

$$8 = (-1) \cdot (-8) \qquad \text{and} \qquad -1 + (-8) = -9$$

and

$$8 = 2 \cdot 4 \qquad \text{and} \qquad 2 + 4 = 6$$

but also

$$8 = (-2) \cdot (-4) \qquad \text{and} \qquad -2 + (-4) = -6.$$

The last option is the correct choice. The answer is $(x-2)(x-4)$. We can check to see if this is correct by multiplying $(x-2)(x-4)$:

$$\begin{array}{rrr}
 & x & -2 \\
 & x & -4 \\
\hline
 & -4x & +8 \\
x^2 & -2x & \\
\hline
x^2 & -6x & +8
\end{array}$$

The answer checks out.

Example B

Factor $x^2 - 17x + 16$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

The number 16 can be written as the product of the following numbers:

$$\begin{array}{llll}
16 = 1 \cdot 16 & \text{and} & 1 + 16 = 17 & \\
16 = (-1) \cdot (-16) & \text{and} & -1 + (-16) = -17 & (\textit{Correct choice}) \\
16 = 2 \cdot 8 & \text{and} & 2 + 8 = 10 & \\
16 = (-2) \cdot (-8) & \text{and} & -2 + (-8) = -10 & \\
16 = 4 \cdot 4 & \text{and} & 4 + 4 = 8 & \\
16 = (-4) \cdot (-4) & \text{and} & -4 + (-4) = -8 &
\end{array}$$

The answer is $(x-1)(x-16)$.

In general, whenever b is negative and a and c are positive, the two binomial factors will have minus signs instead of plus signs.

Factor when a = 1 and c is Negative

Now let's see how this method works if the constant term is negative.

Example C

Factor $x^2 + 2x - 15$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

Once again, we must take the negative sign into account. The number -15 can be written as the product of the following numbers:

$$\begin{array}{llll}
-15=-1\cdot 15 & \text{and} & -1+15=14 & \\
-15=1\cdot(-15) & \text{and} & 1+(-15)=-14 & \\
-15=-3\cdot 5 & \text{and} & -3+5=2 & (Correct\ choice) \\
-15=3\cdot(-5) & \text{and} & 3+(-5)=-2 &
\end{array}$$

The answer is $(x-3)(x+5)$.

We can check to see if this is correct by multiplying:

$$\begin{array}{r}
x-\ 3 \\
\underline{x+\ 5} \\
5x-15 \\
\underline{x^2-3x\quad\ \ } \\
x^2+2x-15
\end{array}$$

The answer checks out.

Example D

Factor $x^2-10x-24$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

The number -24 can be written as the product of the following numbers:

$$\begin{array}{llll}
-24=-1\cdot 24 & \text{and} & -1+24=23 & \\
-24=1\cdot(-24) & \text{and} & 1+(-24)=-23 & \\
-24=-2\cdot 12 & \text{and} & -2+12=10 & \\
-24=2\cdot(-12) & \text{and} & 2+(-12)=-10 & (Correct\ choice) \\
-24=-3\cdot 8 & \text{and} & -3+8=5 & \\
-24=3\cdot(-8) & \text{and} & 3+(-8)=-5 & \\
-24=-4\cdot 6 & \text{and} & -4+6=2 & \\
-24=4\cdot(-6) & \text{and} & 4+(-6)=-2 &
\end{array}$$

The answer is $(x-12)(x+2)$.

Factor when a = - 1

When $a=-1$, the best strategy is to factor the common factor of -1 from all the terms in the quadratic polynomial and then apply the methods you learned so far in this section

Example E

Factor $-x^2+x+6$.

Solution

First factor the common factor of -1 from each term in the trinomial. Factoring -1 just changes the signs of each term in the expression:

$$-x^2+x+6=-(x^2-x-6)$$

We're looking for a product of two binomials in parentheses: $-(x\quad)(x\quad)$

Now our job is to factor x^2-x-6.

The number -6 can be written as the product of the following numbers:

$$\begin{aligned}
-6&=-1\cdot 6 && \text{and} && -1+6=5 \\
-6&=1\cdot(-6) && \text{and} && 1+(-6)=-5 \\
-6&=-2\cdot 3 && \text{and} && -2+3=1 \\
-6&=2\cdot(-3) && \text{and} && 2+(-3)=-1 && (\mathit{Correct\ choice})
\end{aligned}$$

The answer is $-(x-3)(x+2)$.

Watch this video for help with the Examples above.

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CK-12 Foundation: Factoring Quadratic Expressions with Negative Coefficients

Vocabulary

- A quadratic of the form x^2+bx+c factors as a product of two binomials in parentheses: $(x+m)(x+n)$
- If b and c are positive, then both m and n are positive.
- If b is negative and c is positive, then both m and n are negative.
- If c is negative, then either m is positive and n is negative or vice-versa.
- If there is a negative in front of x^2, factor out -1 from each term in the trinomial and then factor as usual. The answer will have the form: $-(x+m)(x+n)$

Guided Practice

Factor $x^2+34x-35$.

Solution

We are looking for an answer that is a product of two binomials in parentheses: $(x \quad)(x \quad)$

The number -35 can be written as the product of the following numbers:

$$\begin{array}{llll} -35=-1\cdot 35 & \text{and} & -1+35=34 & (Correct\ choice) \\ -35=1\cdot(-35) & \text{and} & 1+(-35)=-34 & \\ -35=-5\cdot 7 & \text{and} & -5+7=2 & \\ -35=5\cdot(-7) & \text{and} & 5+(-7)=-2 & \end{array}$$

The answer is $(x-1)(x+35)$.

Practice

Factor the following quadratic polynomials.

1. $x^2-11x+24$
2. $x^2-13x+42$
3. $x^2-14x+33$
4. $x^2-9x+20$
5. $x^2+5x-14$
6. $x^2+6x-27$
7. $x^2+7x-78$
8. $x^2+4x-32$
9. $x^2-12x-45$
10. $x^2-5x-50$
11. $x^2-3x-40$
12. x^2-x-56
13. $-x^2-2x-1$
14. $-x^2-5x+24$
15. $-x^2+18x-72$
16. $-x^2+25x-150$
17. $x^2+21x+108$
18. $-x^2+11x-30$
19. $x^2+12x-64$
20. $x^2-17x-60$
21. $x^2+5x-36$

9.10 Factorization using Difference of Squares

Here you'll learn how to factor polynomials that are the difference of two squares.

What if you had a quadratic expression like $9x^2 - 4y^2$ in which one square term were subtracted from another? How could you factor that expression? After completing this Concept, you'll be able to factor the difference of two squares like this one.

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CK-12 Foundation: 0910S Factoring the Differenceof Squares

Guidance

When you learned how to multiply binomials we talked about two special products.

$$\begin{aligned} \text{The sum and difference formula:} \quad & (a+b)(a-b) = a^2 - b^2 \\ \text{The square of a binomial formulas:} \quad & (a+b)^2 = a^2 + 2ab + b^2 \\ & (a-b)^2 = a^2 - 2ab + b^2 \end{aligned}$$

In this section we'll learn how to recognize and factor these special products.

Factor the Difference of Two Squares

We use the sum and difference formula to factor a difference of two squares. A difference of two squares is any quadratic polynomial in the form $a^2 - b^2$, where a and b can be variables, constants, or just about anything else. The factors of $a^2 - b^2$ are always $(a+b)(a-b)$; the key is figuring out what the a and b terms are.

Example A

Factor the difference of squares:

a) $x^2 - 9$

b) $x^2 - 100$

c) $x^2 - 1$

Solution

a) Rewrite $x^2 - 9$ as $x^2 - 3^2$. Now it is obvious that it is a difference of squares.

The difference of squares formula is:

$a^2 - b^2 = (a+b)(a-b)$

Let's see how our problem matches with the formula:

$x^2 - 3^2 = (x+3)(x-3)$

The answer is:

$x^2 - 9 = (x+3)(x-3)$

We can check to see if this is correct by multiplying $(x+3)(x-3)$:

$$\begin{array}{r} x+3 \\ \underline{x-3} \\ -3x-9 \\ \underline{x^2+3x} \\ x^2+0x-9 \end{array}$$

The answer checks out.

Note: We could factor this polynomial without recognizing it as a difference of squares. With the methods we learned in the last section we know that a quadratic polynomial factors into the product of two binomials:

$$(x \quad)(x \quad)$$

We need to find two numbers that multiply to -9 and add to 0 (since there is no $x-$term, that's the same as if the $x-$term had a coefficient of 0). We can write -9 as the following products:

$$\begin{array}{llll} -9 = -1 \cdot 9 & \text{and} & -1+9 = 8 & \\ -9 = 1 \cdot (-9) & \text{and} & 1+(-9) = -8 & \\ -9 = 3 \cdot (-3) & \text{and} & 3+(-3) = 0 & \textit{These are the correct numbers.} \end{array}$$

We can factor $x^2 - 9$ as $(x+3)(x-3)$, which is the same answer as before. You can always factor using the methods you learned in the previous section, but recognizing special products helps you factor them faster.

b) Rewrite $x^2 - 100$ as $x^2 - 10^2$. This factors as $(x+10)(x-10)$.

c) Rewrite $x^2 - 1$ as $x^2 - 1^2$. This factors as $(x+1)(x-1)$.

Example B

Factor the difference of squares:

a) $16x^2 - 25$

b) $4x^2 - 81$

c) $49x^2 - 64$

Solution

a) Rewrite $16x^2 - 25$ as $(4x)^2 - 5^2$. This factors as $(4x+5)(4x-5)$.

b) Rewrite $4x^2-81$ as $(2x)^2-9^2$. This factors as $(2x+9)(2x-9)$.

c) Rewrite $49x^2-64$ as $(7x)^2-8^2$. This factors as $(7x+8)(7x-8)$.

Example C

Factor the difference of squares:

a) x^2-y^2

b) $9x^2-4y^2$

c) x^2y^2-1

Solution

a) x^2-y^2 factors as $(x+y)(x-y)$.

b) Rewrite $9x^2-4y^2$ as $(3x)^2-(2y)^2$. This factors as $(3x+2y)(3x-2y)$.

c) Rewrite x^2y^2-1 as $(xy)^2-1^2$. This factors as $(xy+1)(xy-1)$.

Watch this video for help with the Examples above.

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CK-12 Foundation: Factoring the Difference of Squares

Vocabulary

- The **difference of two squares** has the form

$$a^2-b^2=(a+b)(a-b)$$

.

Guided Practice

Factor the difference of squares:

a) x^4-25

b) $16x^4-y^2$

c) $x^2y^8-64z^2$

Solution

a) Rewrite x^4-25 as $(x^2)^2-5^2$. This factors as $(x^2+5)(x^2-5)$.

b) Rewrite $16x^4-y^2$ as $(4x^2)^2-y^2$. This factors as $(4x^2+y)(4x^2-y)$.

c) Rewrite $x^2y^4-64z^2$ as $(xy^2)^2-(8z)^2$. This factors as $(xy^2+8z)(xy^2-8z)$.

Practice

Factor the following differences of squares.

1. $x^2 - 4$
2. $x^2 - 36$
3. $-x^2 + 100$
4. $x^2 - 400$
5. $9x^2 - 4$
6. $25x^2 - 49$
7. $9a^2 - 25b^2$
8. $-36x^2 + 25$
9. $4x^2 - y^2$
10. $16x^2 - 81y^2$

9.11 Factorization using Perfect Square Trinomials

Here you'll learn how to factor polynomials that are perfect square trinomials. You'll also learn how to solve quadratic polynomial equations by factoring.

What if you had a trinomial expression like $x^2+10x+25$ in which the first and third terms were perfect squares and the second term was twice the product of the square roots of the first and third terms? How could you factor that expression? After completing this Concept, you'll be able to factor perfect square trinomials like this one.

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CK-12 Foundation: 0911S Factoring Perfect SquareTrinomials

For more examples of factoring perfect square trinomials, watch the videos at http://www.onlinemathlearning.com/perfect-square-trinomial.html.

Guidance

We use the square of a binomial formula to factor perfect square trinomials. A perfect square trinomial has the form $a^2+2ab+b^2$ or $a^2-2ab+b^2$.

In these special kinds of trinomials, the first and last terms are perfect squares and the middle term is twice the product of the square roots of the first and last terms. In a case like this, the polynomial factors into perfect squares:

$$a^2+2ab+b^2=(a+b)^2$$
$$a^2-2ab+b^2=(a-b)^2$$

Once again, the key is figuring out what the a and b terms are.

Example A

Factor the following perfect square trinomials:

a) $x^2+8x+16$

b) x^2-4x+4

c) $x^2+14x+49$

Solution

a) The first step is to recognize that this expression is a perfect square trinomial.

First, we can see that the first term and the last term are perfect squares. We can rewrite $x^2+8x+16$ as x^2+8x+4^2.

Next, we check that the middle term is twice the product of the square roots of the first and the last terms. This is true also since we can rewrite $x^2+8x+16$ as $x^2+2\cdot 4\cdot x+4^2$.

This means we can factor $x^2+8x+16$ as $(x+4)^2$. We can check to see if this is correct by multiplying $(x+4)^2 = (x+4)(x+4)$:

$$\begin{array}{r}
x+4 \\
\underline{x+4} \\
4x+16 \\
\underline{x^2+4x} \\
x^2+8x+16
\end{array}$$

The answer checks out.

Note: We could factor this trinomial without recognizing it as a perfect square. We know that a trinomial factors as a product of two binomials:

$$(x \quad)(x \quad)$$

We need to find two numbers that multiply to 16 and add to 8. We can write 16 as the following products:

$$\begin{array}{llll}
16 = 1\cdot 16 & \text{and} & 1+16 = 17 & \\
16 = 2\cdot 8 & \text{and} & 2+8 = 10 & \\
16 = 4\cdot 4 & \text{and} & 4+4 = 8 & \textit{These are the correct numbers}
\end{array}$$

So we can factor $x^2+8x+16$ as $(x+4)(x+4)$, which is the same as $(x+4)^2$.

Once again, you can factor perfect square trinomials the normal way, but recognizing them as perfect squares gives you a useful shortcut.

b) Rewrite x^2+4x+4 as $x^2+2\cdot(-2)\cdot x+(-2)^2$.

We notice that this is a perfect square trinomial, so we can factor it as $(x-2)^2$.

c) Rewrite $x^2+14x+49$ as $x^2+2\cdot 7\cdot x+7^2$.

We notice that this is a perfect square trinomial, so we can factor it as $(x+7)^2$.

Example B

Factor the following perfect square trinomials:

a) $4x^2+20x+25$

b) $9x^2-24x+16$

c) $x^2+2xy+y^2$

Solution

a) Rewrite $4x^2+20x+25$ as $(2x)^2+2\cdot 5\cdot(2x)+5^2$.

We notice that this is a perfect square trinomial and we can factor it as $(2x+5)^2$.

b) Rewrite $9x^2-24x+16$ as $(3x)^2+2\cdot(-4)\cdot(3x)+(-4)^2$.

We notice that this is a perfect square trinomial and we can factor it as $(3x-4)^2$.

We can check to see if this is correct by multiplying $(3x-4)^2=(3x-4)(3x-4)$:

$$\begin{array}{r}
3x-4 \\
3x-4 \\ \hline
-12x+16 \\
9x^2-12x \quad\; \\ \hline
9x^2-24x+16
\end{array}$$

The answer checks out.

c) $x^2+2xy+y^2$

We notice that this is a perfect square trinomial and we can factor it as $(x+y)^2$.

Solve Quadratic Polynomial Equations by Factoring

With the methods we've learned in the last two sections, we can factor many kinds of quadratic polynomials. This is very helpful when we want to solve them. Remember the process we learned earlier:

1. If necessary, **rewrite** the equation in standard form so that the right-hand side equals zero.
2. Factor the polynomial completely.
3. Use the zero-product rule to set each factor equal to zero.
4. Solve each equation from step 3.
5. Check your answers by substituting your solutions into the original equation

We can use this process to solve quadratic polynomials using the factoring methods we just learned.

Example C

Solve the following polynomial equations.

a) $x^2+12x+36=0$

b) $x^2-24x=-144$

Solution

a) **Rewrite:**

The equation is already in the correct form.

Factor:

Rewrite $x^2+12x+36=0$ as $x^2+2(6x)+6^2=0$. We notice that this is a perfect square trinomial and we can factor it as $(x+6)^2$.

Set the factor equal to zero:

$$x+6=0$$

Solve:

$$\underline{\underline{x=-6}}$$

Check: Substitute each solution back into the original equation.

$$\begin{aligned}
&(-6)^2+12(-6)+36= && \text{Substitute in -6.}\\
&36+-72+36= && \text{Simplify.}\\
&72+-72=0 && \text{Checks out.}
\end{aligned}$$

b) **Rewrite:** $x^2-24x=-144$ is rewritten as $x^2-24x+144=0$

Factor:

$x^2-24x+144=x^2+2(-12)x+(-12)^2=(x-12)^2$

Set the factor equal to zero:

$$x-12=0$$

Solve:

$$\underline{\underline{x=12}}$$

Check: Substitute the solution back into the original equation.

$$\begin{aligned}
&(12)^2-24(12)+144= && \text{Substitute in 12.}\\
&144-288+144= && \text{Simplify.}\\
&288-288=0 && \text{Checks out.}
\end{aligned}$$

Watch this video for help with the Examples above.

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CK-12 Foundation: Factoring Perfect Square Trinomials

Vocabulary

- A **perfect square trinomial** has the form

$$a^2+2ab+b^2=(a+b)^2 \qquad \text{or} \qquad a^2-2ab+b^2=(a-b)^2.$$

Guided Practice

Solve the following polynomial equations:

a) $x^2+x+0.25=0$

b) $x^2-81=0$

Solution

a) $x^2+x+0.25=0$

Rewrite: The equation is in the correct form already.

Factor: Rewrite $x^2+x+0.25=0$ as $x^2+2\cdot(0.5)x+(0.5)^2$.

We recognize this as a perfect square. This factors as $(x+0.5)^2=0$ or $(x+0.5)(x+0.5)=0$

Set the factor equal to zero:

$$x+0.25=0$$

Solve:

$$\underline{\underline{x=-0.5}}$$

Check: Substitute the solution back into the original equation.

$$\begin{aligned}(-0.5)^2+-0.5+0.25 &= && \text{Substitute in -0.5}\\ 0.25+-0.5+0.25 &= && \text{Simplify.}\\ 0.5-0.5 &= 0 && \text{Checks out.}\end{aligned}$$

b) $x^2-81=0$

Rewrite: this is not necessary since the equation is in the correct form already

Factor: Rewrite x^2-81 as x^2-9^2.

We recognize this as a difference of squares. This factors as $(x-9)(x+9)=0$.

Set each factor equal to zero:

$$x-9=0 \qquad \text{or} \qquad x+9=0$$

Solve:

$$\underline{\underline{x=9}} \qquad \text{or} \qquad \underline{\underline{x=-9}}$$

Check: Substitute each solution back into the original equation.

$$\begin{aligned}&x=9 && 9^2-81=81-81=0 && \text{checks out}\\ &x=-9 && (-9)^2-81=81-81=0 && \text{checks out}\end{aligned}$$

c) $x^2+20x+100=0$

Rewrite: this is not necessary since the equation is in the correct form already

Factor: Rewrite $x^2+20x+100$ as $x^2+2\cdot10\cdot x+10^2$.

We recognize this as a perfect square. This factors as $(x+10)^2=0$ or $(x+10)(x+10)=0$

Set each factor equal to zero:

$$x+10=0 \qquad \text{or} \qquad x+10=0$$

Solve:

$$\underline{\underline{x=-10}} \qquad \text{or} \qquad \underline{\underline{x=-10}} \qquad \text{This is a double root.}$$

Check: Substitute each solution back into the original equation.

$$x=10 \qquad (-10)^2+20(-10)+100=100-200+100=0 \qquad \text{checks out}$$

Practice

Factor the following perfect square trinomials.

1. $x^2+8x+16$
2. $x^2-18x+81$
3. $-x^2+24x-144$
4. $x^2+14x+49$
5. $4x^2-4x+1$
6. $25x^2+60x+36$
7. $4x^2-12xy+9y^2$
8. x^4+22x^2+121

Solve the following quadratic equations using factoring.

9. $x^2-11x+30=0$
10. $x^2+4x=21$
11. $x^2+49=14x$
12. $x^2-64=0$
13. $x^2-24x+144=0$
14. $4x^2-25=0$
15. $x^2+26x=-169$
16. $-x^2-16x-60=0$

9.12 Factoring Completely

Here you'll learn how to factor polynomials completely so that they can't be factored any further.

What if you had a polynomial like $3x^2 - 27$ with multiple factors? How could you factor it completely? After completing this Concept, you'll be able to factor out common monomials and binomials from polynomials like this one.

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CK-12 Foundation: 0912S Factoring Polynomials Completely

The WTAMU Virtual Math Lab has a detailed page on factoring polynomials here: http://www.wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/col_alg_tut7_factor.htm. This page contains many videos showing example problems being solved.

Guidance

We say that a polynomial is **factored completely** when we can't factor it any more. Here are some suggestions that you should follow to make sure that you factor completely:

- Factor all common monomials first.
- Identify special products such as difference of squares or the square of a binomial. Factor according to their formulas.
- If there are no special products, factor using the methods we learned in the previous sections.
- Look at each factor and see if any of these can be factored further.

Example A

Factor the following polynomials completely.

a) $6x^2 - 30x + 24$

b) $2x^2 - 8$

c) $x^3 + 6x^2 + 9x$

Solution

a) Factor out the common monomial. In this case 6 can be divided from each term:

$$6(x^2 - 5x - 6)$$

There are no special products. We factor x^2-5x+6 as a product of two binomials: $(x\ )(x\ )$

The two numbers that multiply to 6 and add to -5 are -2 and -3, so:

$$6(x^2-5x+6)=6(x-2)(x-3)$$

If we look at each factor we see that we can factor no more.

The answer is $6(x-2)(x-3)$.

b) Factor out common monomials: $2x^2-8=2(x^2-4)$

We recognize x^2-4 as a difference of squares. We factor it as $(x+2)(x-2)$.

If we look at each factor we see that we can factor no more.

The answer is $2(x+2)(x-2)$.

c) Factor out common monomials: $x^3+6x^2+9x=x(x^2+6x+9)$

We recognize x^2+6x+9 as a perfect square and factor it as $(x+3)^2$.

If we look at each factor we see that we can factor no more.

The answer is $x(x+3)^2$.

Example B

Factor the following polynomials completely:

a) $-2x^4+162$

b) x^5-8x^3+16x

Solution

a) Factor out the common monomial. In this case, factor out -2 rather than 2. (It's always easier to factor out the negative number so that the highest degree term is positive.)

$$-2x^4+162=-2(x^4-81)$$

We recognize expression in parenthesis as a difference of squares. We factor and get:

$$-2(x^2-9)(x^2+9)$$

If we look at each factor we see that the first parenthesis is a difference of squares. We factor and get:

$$-2(x+3)(x-3)(x^2+9)$$

If we look at each factor now we see that we can factor no more.

The answer is $-2(x+3)(x-3)(x^2+9)$.

b) Factor out the common monomial: $x^5-8x^3+14x=x(x^4-8x^2+16)$

We recognize x^4-8x^2+16 as a perfect square and we factor it as $x(x^2-4)^2$.

We look at each term and recognize that the term in parentheses is a difference of squares.

We factor it and get $((x+2)(x-2))^2$, which we can rewrite as $(x+2)^2(x-2)^2$.

If we look at each factor now we see that we can factor no more.

The final answer is $x(x+2)^2(x-2)^2$.

Factor out a Common Binomial

The first step in the factoring process is often factoring out the common monomials from a polynomial. But sometimes polynomials have common terms that are binomials. For example, consider the following expression:

$$x(3x+2)-5(3x+2)$$

Since the term $(3x+2)$ appears in both terms of the polynomial, we can factor it out. We write that term in front of a set of parentheses containing the terms that are left over:

$$(3x+2)(x-5)$$

This expression is now completely factored.

Let's look at some more examples.

Example C

Factor out the common binomials.

a) $3x(x-1)+4(x-1)$

b) $x(4x+5)+(4x+5)$

Solution

a) $3x(x-1)+4(x-1)$ has a common binomial of $(x-1)$.

When we factor out the common binomial we get $(x-1)(3x+4)$.

b) $x(4x+5)+(4x+5)$ has a common binomial of $(4x+5)$.

When we factor out the common binomial we get $(4x+5)(x+1)$.

Watch this video for help with the Examples above.

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CK-12 Foundation: Factoring Polynomials Completely

Vocabulary

- We say that a polynomial is **factored completely** when we factor as much as we can and we are unable to factor any more.

Guided Practice

Factor completely: $24x^3 - 28x^2 + 8x$.

Solution:

First, notice that each term has $4x$ as a factor. Start by factoring out $4x$:

$24x^3 - 28x^2 + 8x = 4x(6x^2 - 7x + 2)$

Next, factor the trinomial in the parenthesis. Since $a \neq 1$ find $a \cdot c$: $6 \cdot 2 = 12$. Find the factors of 12 that add up to -7. Since 12 is positive and -7 is negative, the two factors should be negative:

$$\begin{aligned} 12 &= -1 \cdot -12 && \textit{and} && -1 + -12 = -13 \\ 12 &= -2 \cdot -6 && \textit{and} && -2 + -6 = -8 \\ 12 &= -3 \cdot -4 && \textit{and} && -3 + -4 = -7 \end{aligned}$$

Rewrite the trinomial using $-7x = -3x - 4x$, and then factor by grouping:

$$\begin{aligned} 6x^2 - 7x + 2 &= 6x^2 - 3x - 4x + 2 \\ &= 3x(2x - 1) - 2(2x - 1) \\ &= (3x - 2)(2x - 1) \end{aligned}$$

The final factored answer is:

$4x(3x - 2)(2x - 1)$

Practice

Factor completely.

1. $2x^2 + 16x + 30$
2. $5x^2 - 70x + 245$
3. $-x^3 + 17x^2 - 70x$
4. $2x^4 - 512$
5. $25x^4 - 20x^3 + 4x^2$
6. $12x^3 + 12x^2 + 3x$
7. $12c^2 - 75$
8. $6x^2 - 600$
9. $-5t^2 - 20t - 20$
10. $6x^2 + 18x - 24$
11. $-n^2 + 10n - 21$
12. $2a^2 - 14a - 16$

9.13 Factoring by Grouping

Here you'll learn how to group a polynomial's terms to help you factor it.

What if you had a polynomial expression like $3x^2-6x+2x-4$ in which some of the terms shared a common factor but not all of them? How could you factor this expression? After completing this Concept, you'll be able to factor polynomials like this one by grouping.

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CK-12 Foundation: 0913S Factoring ByGrouping

Guidance

Sometimes, we can factor a polynomial containing four or more terms by factoring common monomials from groups of terms. This method is called **factor by grouping.**

The next example illustrates how this process works.

Example A

Factor $2x+2y+ax+ay$.

Solution

There is no factor common to all the terms. However, the first two terms have a common factor of 2 and the last two terms have a common factor of a. Factor 2 from the first two terms and factor a from the last two terms:

$$2x+2y+ax+ay=2(x+y)+a(x+y)$$

Now we notice that the binomial $(x+y)$ is common to both terms. We factor the common binomial and get:

$$(x+y)(2+a)$$

Example B

Factor $3x^2+6x+4x+8$.

Solution

We factor 3x from the first two terms and factor 4 from the last two terms:

$$3x(x+2)+4(x+2)$$

Now factor $(x+2)$ from both terms: $(x+2)(3x+4)$.

Now the polynomial is factored completely.

Factor Quadratic Trinomials Where a $\neq$ 1

Factoring by grouping is a very useful method for factoring quadratic trinomials of the form ax^2+bx+c, where $a \neq 1$.

A quadratic like this doesn't factor as $(x \pm m)(x \pm n)$, so it's not as simple as looking for two numbers that multiply to c and add up to b. Instead, we also have to take into account the coefficient in the first term.

To factor a quadratic polynomial where $a \neq 1$, we follow these steps:

1. We find the product ac.
2. We look for two numbers that multiply to ac and add up to b.
3. We rewrite the middle term using the two numbers we just found.
4. We factor the expression by grouping.

Let's apply this method to the following examples.

Example C

Factor the following quadratic trinomials by grouping.

a) $3x^2+8x+4$

b) $6x^2-11x+4$

Solution:

Let's follow the steps outlined above:

a) $3x^2+8x+4$

Step 1: $ac = 3 \cdot 4 = 12$

Step 2: The number 12 can be written as a product of two numbers in any of these ways:

$$\begin{array}{llll} 12 = 1 \cdot 12 & \text{and} & 1+12=13 & \\ 12 = 2 \cdot 6 & \text{and} & 2+6=8 & This\ is\ the\ correct\ choice. \\ 12 = 3 \cdot 4 & \text{and} & 3+4=7 & \end{array}$$

Step 3: Re-write the middle term: $8x = 2x+6x$, so the problem becomes:

$$3x^2+8x+4 = 3x^2+2x+6x+4$$

Step 4: Factor an x from the first two terms and a 2 from the last two terms:

$$x(3x+2)+2(3x+2)$$

Now factor the common binomial $(3x+2)$:

$$(3x+2)(x+2) \qquad \textit{This is the answer.}$$

To check if this is correct we multiply $(3x+2)(x+2)$:

$$\begin{array}{r} 3x+2 \\ \underline{\quad\quad x+2} \\ 6x+4 \\ \underline{3x^2+2x\quad\;} \\ 3x^2+8x+4 \end{array}$$

The solution checks out.

b) $6x^2-11x+4$

Step 1: $ac = 6 \cdot 4 = 24$

Step 2: The number 24 can be written as a product of two numbers in any of these ways:

$24 = 1 \cdot 24$	and	$1+24 = 25$	
$24 = -1 \cdot (-24)$	and	$-1+(-24) = -25$	
$24 = 2 \cdot 12$	and	$2+12 = 14$	
$24 = -2 \cdot (-12)$	and	$-2+(-12) = -14$	
$24 = 3 \cdot 8$	and	$3+8 = 11$	
$24 = -3 \cdot (-8)$	and	$-3+(-8) = -11$	(*Correct choice*)
$24 = 4 \cdot 6$	and	$4+6 = 10$	
$24 = -4 \cdot (-6)$	and	$-4+(-6) = -10$	

Step 3: Re-write the middle term: $-11x = -3x-8x$, so the problem becomes:

$$6x^2-11x+4 = 6x^2-3x-8x+4$$

Step 4: Factor by grouping: factor a $3x$ from the first two terms and a -4 from the last two terms:

$$3x(2x-1)-4(2x-1)$$

Now factor the common binomial $(2x-1)$:

$$(2x-1)(3x-4) \qquad \textit{This is the answer.}$$

Watch this video for help with the Examples above.

Multimedia

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CK-12 Foundation: Factoring By Grouping

Vocabulary

- It is possible to factor a polynomial containing four or more terms by factoring common monomials from groups of terms. This method is called **factoring by grouping**.

Guided Practice

Factor $5x^2-6x+1$ *by grouping.*

Solution:

Let's follow the steps outlined above:

$5x^2-6x+1$

Step 1: $ac=5\cdot 1=5$

Step 2: The number 5 can be written as a product of two numbers in any of these ways:

$$\begin{array}{lll l}5=1\cdot 5 & \text{and} & 1+5=6 & \\ 5=-1\cdot(-5) & \text{and} & -1+(-5)=-6 & (Correct\ choice)\end{array}$$

Step 3: Re-write the middle term: $-6x=-x-5x$, so the problem becomes:

$$5x^2-6x+1=5x^2-x-5x+1$$

Step 4: Factor by grouping: factor an x from the first two terms and $a-1$ from the last two terms:

$$x(5x-1)-1(5x-1)$$

Now factor the common binomial $(5x-1)$:

$$(5x-1)(x-1) \qquad This\ is\ the\ answer.$$

Practice

Factor by grouping.

1. $6x^2-9x+10x-15$
2. $5x^2-35x+x-7$

3. $9x^2 - 9x - x + 1$
4. $4x^2 + 32x - 5x - 40$
5. $2a^2 - 6ab + 3ab - 9b^2$
6. $5x^2 + 15x - 2xy - 6y$

Factor the following quadratic trinomials by grouping.

7. $4x^2 + 25x - 21$
8. $6x^2 + 7x + 1$
9. $4x^2 + 8x - 5$
10. $3x^2 + 16x + 21$
11. $6x^2 - 2x - 4$
12. $8x^2 - 14x - 15$

9.14 Solving Problems by Factoring

Here you'll learn how to apply polynomial factoring techniques to solve real-world applications involving polynomial equations.

What if you had a triangle in which one leg was one unit shorter than the other and the hypotenuse was 5 units? How could you find the length of the two legs? After completing this Concept, you'll be able to solve real-world applications like this one that involve factoring polynomial equations.

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CK-12 Foundation: 0914S Solving Real-World Problems By Factoring

Guidance

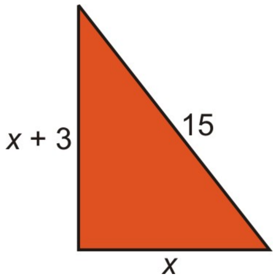
Now that we know most of the factoring strategies for quadratic polynomials, we can apply these methods to solving real world problems.

Example A

One leg of a right triangle is 3 feet longer than the other leg. The hypotenuse is 15 feet. Find the dimensions of the triangle.

Solution

Let $x =$ the length of the short leg of the triangle; then the other leg will measure $x+3$.



Use the Pythagorean Theorem: $a^2+b^2=c^2$, where a and b are the lengths of the legs and c is the length of the hypotenuse. When we substitute the values from the diagram, we get $x^2+(x+3)^2=15^2$.

In order to solve this equation, we need to get the polynomial in standard form. We must first distribute, collect like terms and **rewrite** in the form "polynomial = 0."

$$\begin{aligned} x^2+x^2+6x+9 &= 225 \\ 2x^2+6x+9 &= 225 \\ 2x^2+6x-216 &= 0 \end{aligned}$$

Factor out the common monomial: $2(x^2+3x-108)=0$

To factor the trinomial inside the parentheses, we need two numbers that multiply to -108 and add to 3. It would take a long time to go through all the options, so let's start by trying some of the bigger factors:

$$\begin{array}{llll} -108=-12\cdot 9 & \text{and} & -12+9=-3 & \\ -108=12\cdot(-9) & \text{and} & 12+(-9)=3 & (Correct\ choice) \end{array}$$

We factor the expression as $2(x-9)(x+12)=0$.

Set each term equal to zero and **solve:**

$$\begin{array}{ccc} x-9=0 & & x+12=0 \\ & \text{or} & \\ \underline{\underline{x=9}} & & \underline{\underline{x=-12}} \end{array}$$

It makes no sense to have a negative answer for the length of a side of the triangle, so the answer must be $x=9$. That means **the short leg is 9 feet and the long leg is 12 feet.**

Check: $9^2+12^2=81+144=225=15^2$, so the answer checks.

Example B

The product of two positive numbers is 60. Find the two numbers if one number is 4 more than the other.

Solution

Let $x=$ one of the numbers; then $x+4$ is the other number.

The product of these two numbers is 60, so we can write the equation $x(x+4)=60$.

In order to solve we must write the polynomial in standard form. Distribute, collect like terms and **rewrite:**

$$\begin{aligned} x^2+4x &= 60 \\ x^2+4x-60 &= 0 \end{aligned}$$

Factor by finding two numbers that multiply to -60 and add to 4. List some numbers that multiply to -60:

$$\begin{array}{llll} -60=-4\cdot 15 & \text{and} & -4+15=11 & \\ -60=4\cdot(-15) & \text{and} & 4+(-15)=-11 & \\ -60=-5\cdot 12 & \text{and} & -5+12=7 & \\ -60=5\cdot(-12) & \text{and} & 5+(-12)=-7 & \\ -60=-6\cdot 10 & \text{and} & -6+10=4 & (Correct\ choice) \\ -60=6\cdot(-10) & \text{and} & 6+(-10)=-4 & \end{array}$$

The expression factors as $(x+10)(x-6)=0$.

Set each term equal to zero and **solve:**

$$\begin{array}{ccc} x+10=0 & & x-6=0 \\ & \text{or} & \\ \underline{\underline{x=-10}} & & \underline{\underline{x=6}} \end{array}$$

Since we are looking for positive numbers, the answer must be $x=6$. **One number is 6, and the other number is 10.**

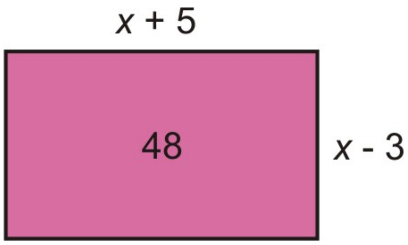
Check: $6 \cdot 10 = 60$, so the answer checks.

Example C

A rectangle has sides of length $x+5$ and $x-3$. What is x if the area of the rectangle is 48?

Solution

Make a sketch of this situation:



Using the formula Area = length × width, we have $(x+5)(x-3)=48$.

In order to solve, we must write the polynomial in standard form. Distribute, collect like terms and **rewrite:**

$$x^2+2x-15=48$$
$$x^2+2x-63=0$$

Factor by finding two numbers that multiply to -63 and add to 2. List some numbers that multiply to -63:

$$\begin{array}{cccc} -63=-7\cdot 9 & \text{and} & -7+9=2 & (\textit{Correct choice}) \\ -63=7\cdot(-9) & \text{and} & 7+(-9)=-2 & \end{array}$$

The expression factors as $(x+9)(x-7)=0$.

Set each term equal to zero and **solve:**

$$\begin{array}{ccc} x+9=0 & & x-7=0 \\ & \text{or} & \\ \underline{\underline{x=-9}} & & \underline{\underline{x=7}} \end{array}$$

Since we are looking for positive numbers the answer must be $x = 7$. So **the width is** $x - 3 = 4$**and the length is** $x + 5 = 12$.

Check: $4 \cdot 12 = 48$, so the answer checks.

Watch this video for help with the Examples above.

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CK-12 Foundation: Solving Real-WorldProblems byFactoring

Vocabulary

- Polynomials can be written in **expanded form** or in **factored form**. Expanded form means that you have sums and differences of different terms:
- The **factored form** of a polynomial means it is written as a product of its factors.
- **Zero Product Property:** The only way a product is zero is if one or more of the terms are equal to zero: $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$.
- We say that a polynomial is ***factored completely*** when we factor as much as we can and we are unable to factor any more.

Guided Practice

Consider the rectangle in Example C with sides of length $x+5$ and $x-3$. What is x if the area of the rectangle is now 20?

Solution

Make a sketch of this situation:

Using the formula Area = length × width, we have $(x+5)(x-3) = 20$.

In order to solve, we must write the polynomial in standard form. Distribute, collect like terms and **rewrite:**

$$\begin{aligned} x^2 + 2x - 15 &= 20 \\ x^2 + 2x - 35 &= 0 \end{aligned}$$

Factor by finding two numbers that multiply to -35 and add to 2. List some numbers that multiply to -35:

$$\begin{array}{lcr} -35 = -7 \cdot 5 & \text{and} & -7 + 5 = -2 \\ -35 = 7 \cdot (-5) & \text{and} & 7 + (-5) = 2 \end{array}$$

The expression factors as $(x+7)(x-5) = 0$.

Set each term equal to zero and **solve:**

$$x+7=0 \qquad \text{or} \qquad x-5=0$$

$$\underline{\underline{x=-7}} \qquad\qquad \underline{\underline{x=5}}$$

Since we are looking for positive numbers the answer must be $x=5$. So **the width is** $x-3=2$**and the length is** $x+5=10$.

Check: $2\cdot 10=20$, so the answer checks.

Practice

Solve the following application problems:

1. One leg of a right triangle is 1 feet longer than the other leg. The hypotenuse is 5. Find the dimensions of the right triangle.
2. One leg of a right triangle is 7 feet longer than the other leg. The hypotenuse is 13. Find the dimensions of the right triangle.
3. A rectangle has sides of $x+2$ and $x-1$. What value of x gives an area of 108?
4. A rectangle has sides of $x-1$ and $x+1$. What value of x gives an area of 120?
5. The product of two positive numbers is 120. Find the two numbers if one numbers is 7 more than the other.
6. A rectangle has a 50-foot diagonal. What are the dimensions of the rectangle if it is 34 feet longer than it is wide?
7. Two positive numbers have a sum of 8, and their product is equal to the larger number plus 10. What are the numbers?
8. Two positive numbers have a sum of 8, and their product is equal to the smaller number plus 10. What are the numbers?
9. Framing Warehouse offers a picture framing service. The cost for framing a picture is made up of two parts: glass costs \$1 per square foot and the frame costs \$2 per foot. If the frame has to be a square, what size picture can you get framed for \$20?

Summary

This chapter begins with an introduction to the special vocabulary associated with polynomials. It then focuses on adding, subtracting, and multiplying polynomials. Special products like the square of a binomial and the product of binomials using the sum and difference formula are covered as well. The remainder of the chapter is devoted to various methods for factoring polynomials so that they can be solved. Real-world geometry problems involving polynomials are also given consideration.

CHAPTER 10 Quadratic Equations and Quadratic Functions

Chapter Outline

10.1 QUADRATIC FUNCTIONS AND THEIR GRAPHS
10.2 GRAPHS OF QUADRATIC FUNCTIONS IN INTERCEPT FORM
10.3 USE GRAPHS TO SOLVE QUADRATIC EQUATIONS
10.4 USE SQUARE ROOTS TO SOLVE QUADRATIC EQUATIONS
10.5 SQUARE ROOT APPLICATIONS
10.6 COMPLETING THE SQUARE
10.7 VERTEX FORM OF A QUADRATIC EQUATION
10.8 QUADRATIC FORMULA
10.9 COMPARING METHODS FOR SOLVING QUADRATICS
10.10 SOLUTIONS USING THE DISCRIMINANT
10.11 LINEAR, EXPONENTIAL, AND QUADRATIC MODELS
10.12 APPLICATIONS OF FUNCTION MODELS

Introduction

Did you ever look at a rainbow and wonder about its shape? The arc a rainbow makes in the sky is a special type of curved function known as a parabola. A parabola is the graph of a quadratic function, which is the focus of this chapter. If you look closely, you'll see parabolas in many aspects of everyday life. The hill on a roller coaster—that's a parabola. The trajectory of water as it comes out of a drinking fountain—that's a parabola. To model these real-life situations, you will use quadratic functions. This chapter will teach you how to graph functions and solve quadratic equations.

10.1 Quadratic Functions and Their Graphs

Here you'll learn how to make a table of values to graph the curved lines of quadratic functions called parabolas. You'll also learn how to describe what parabolas will look like.

What if you had a quadratic function like $5+2x-3x^2$? What would its graph look like? Would the graph of $5+2x-x^2$ be wider or narrower than it? After completing this Concept, you'll be able to graph and compare graphs of quadratic functions like these.

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CK-12 Foundation: 1001S Graphs of Quadratic Functions

Try This

Meanwhile, if you want to explore further what happens when you change the coefficients of a quadratic equation, the page at http://www.analyzemath.com/quadraticg/quadraticg.htm has an applet you can use. Click on the "Click here to start" button in section A, and then use the sliders to change the values of a, b, and c.

Guidance

The graphs of quadratic functions are curved lines called **parabolas**. You don't have to look hard to find parabolic shapes around you. Here are a few examples:

- The path that a ball or a rocket takes through the air.
- Water flowing out of a drinking fountain.
- The shape of a satellite dish.
- The shape of the mirror in car headlights or a flashlight.
- The cables in a suspension bridge.

Example A

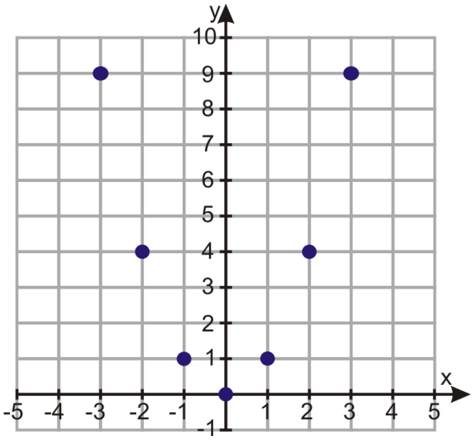
Let's see what a parabola looks like by graphing the simplest quadratic function, $y=x^2$.

We'll graph this function by making a table of values. Since the graph will be curved, we need to plot a fair number of points to make it accurate.

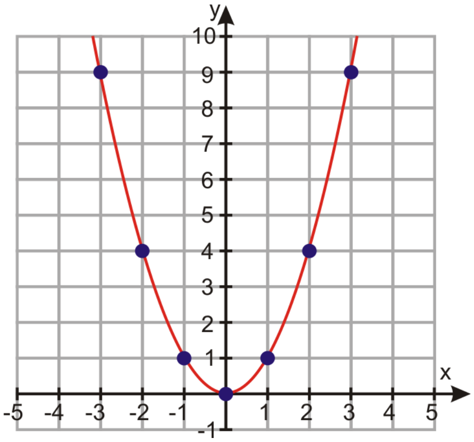
TABLE 10.1:

x	$y = x^2$
-3	$(-3)^2 = 9$
-2	$(-2)^2 = 4$
-1	$(-1)^2 = 1$
0	$(0)^2 = 0$
1	$(1)^2 = 1$
2	$(2)^2 = 4$
3	$(3)^2 = 9$

Here are the points plotted on a coordinate graph:



To draw the parabola, draw a smooth curve through all the points. (Do not connect the points with straight lines).



Let's graph a few more examples.

Example B

Graph the following parabolas.

a) $y = 2x^2 + 4x + 1$

b) $y = -x^2 + 3$

c) $y = x^2 - 8x + 3$

Solution

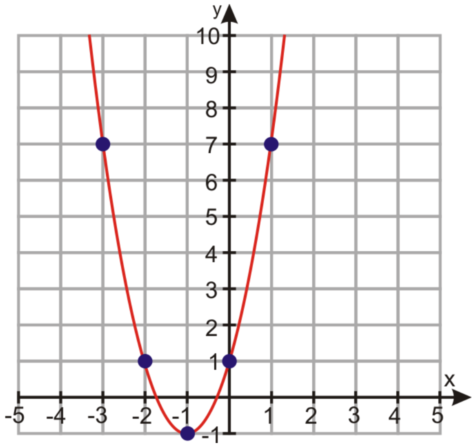
a) $y = 2x^2 + 4x + 1$

Make a table of values:

TABLE 10.2:

x	$y = 2x^2 + 4x + 1$
-3	$2(-3)^2 + 4(-3) + 1 = 7$
–2	$2(-2)^2 + 4(-2) + 1 = 1$
–1	$2(-1)^2 + 4(-1) + 1 = -1$
0	$2(0)^2 + 4(0) + 1 = 1$
1	$2(1)^2 + 4(1) + 1 = 7$
2	$2(2)^2 + 4(2) + 1 = 17$
3	$2(3)^2 + 4(3) + 1 = 31$

Notice that the last two points have very large $y-$values. Since we don't want to make our $y-$scale too big, we'll just skip graphing those two points. But we'll plot the remaining points and join them with a smooth curve.



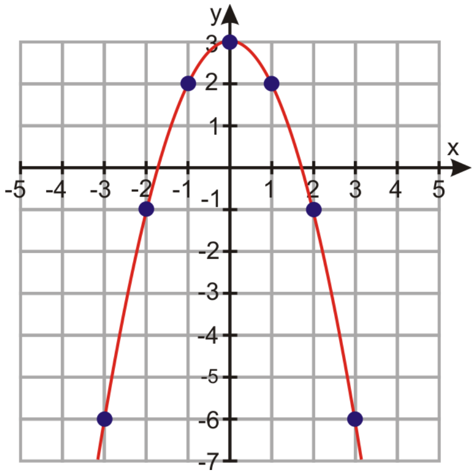
b) $y = -x^2 + 3$

Make a table of values:

TABLE 10.3:

x	$y = -x^2 + 3$
-3	$-(-3)^2 + 3 = -6$
–2	$-(-2)^2 + 3 = -1$
–1	$-(-1)^2 + 3 = 2$
0	$-(0)^2 + 3 = 3$
1	$-(1)^2 + 3 = 2$
2	$-(2)^2 + 3 = -1$
3	$-(3)^2 + 3 = -6$

Plot the points and join them with a smooth curve.



Notice that this time we get an "upside down" parabola. That's because our equation has a negative sign in front of the x^2 term. The sign of the coefficient of the x^2 term determines whether the parabola turns up or down: the parabola turns up if it's positive and down if it's negative.

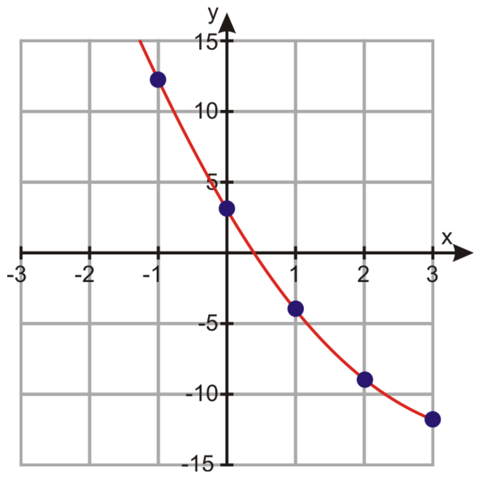
c) $y = x^2 - 8x + 3$

Make a table of values:

TABLE 10.4:

x	$y = x^2 - 8x + 3$
-3	$(-3)^2 - 8(-3) + 3 = 36$
–2	$(-2)^2 - 8(-2) + 3 = 23$
–1	$(-1)^2 - 8(-1) + 3 = 12$
0	$(0)^2 - 8(0) + 3 = 3$
1	$(1)^2 - 8(1) + 3 = -4$
2	$(2)^2 - 8(2) + 3 = -9$
3	$(3)^2 - 8(3) + 3 = -12$

Let's not graph the first two points in the table since the values are so big. Plot the remaining points and join them with a smooth curve.



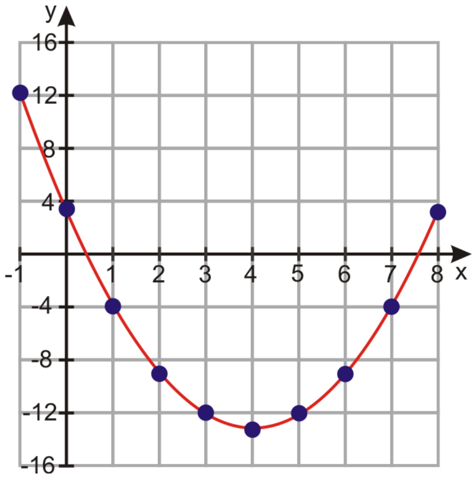
Wait—this doesn't look like a parabola. What's going on here?

Maybe if we graph more points, the curve will look more familiar. For negative values of x it looks like the values of y are just getting bigger and bigger, so let's pick more positive values of x beyond $x = 3$.

TABLE 10.5:

x	$y = x^2 - 8x + 3$
-1	$(-1)^2 - 8(-1) + 3 = 12$
0	$(0)^2 - 8(0) + 3 = 3$
1	$(1)^2 - 8(1) + 3 = -4$
2	$(2)^2 - 8(2) + 3 = -9$
3	$(3)^2 - 8(3) + 3 = -12$
4	$(4)^2 - 8(4) + 3 = -13$
5	$(5)^2 - 8(5) + 3 = -12$
6	$(6)^2 - 8(6) + 3 = -9$
7	$(7)^2 - 8(7) + 3 = -4$
8	$(8)^2 - 8(8) + 3 = 3$

Plot the points again and join them with a smooth curve.



Now we can see the familiar parabolic shape. And now we can see the drawback to graphing quadratics by making a table of values—if we don't pick the right values, we won't get to see the important parts of the graph.

In the next couple of lessons, we'll find out how to graph quadratic equations more efficiently—but first we need to learn more about the properties of parabolas.

Compare Graphs of Quadratic Functions

The **general form** (or **standard form**) of a quadratic function is:

$$y = ax^2 + bx + c$$

Here a, b and c are the **coefficients.** Remember, a coefficient is just a number (a constant term) that can go before a variable or appear alone.

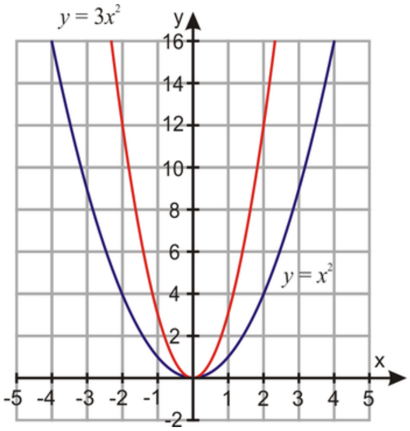
Although the graph of a quadratic equation in standard form is always a parabola, the shape of the parabola depends on the values of the coefficients a, b and c. Let's explore some of the ways the coefficients can affect the graph.

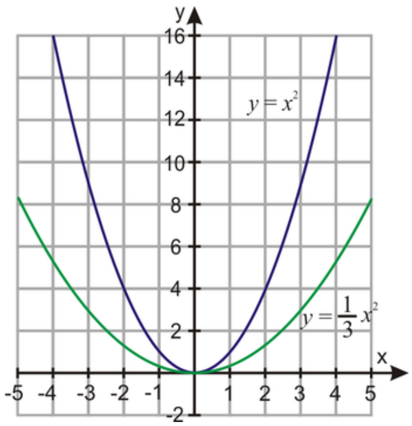
Dilation

Changing the value of a makes the graph "fatter" or "skinnier". Let's look at how graphs compare for different positive values of a.

Example C

The plot on the left shows the graphs of $y = x^2$ and $y = 3x^2$. The plot on the right shows the graphs of $y = x^2$ and $y = \frac{1}{3}x^2$.





Notice that the larger the value of a is, the skinnier the graph is – for example, in the first plot, the graph of $y = 3x^2$ is skinnier than the graph of $y = x^2$. Also, the smaller a is, the fatter the graph is – for example, in the second plot, the graph of $y = \frac{1}{3}x^2$ is fatter than the graph of $y = x^2$. This might seem counterintuitive, but if you think about it, it should make sense. Let's look at a table of values of these graphs and see if we can explain why this happens.

TABLE 10.6:

x	$y = x^2$	$y = 3x^2$	$y = \frac{1}{3}x^2$
-3	$(-3)^2 = 9$	$3(-3)^2 = 27$	$\frac{(-3)^2}{3} = 3$
-2	$(-2)^2 = 4$	$3(-2)^2 = 12$	$\frac{(-2)^2}{3} = \frac{4}{3}$
-1	$(-1)^2 = 1$	$3(-1)^2 = 3$	$\frac{(-1)^2}{3} = \frac{1}{3}$
0	$(0)^2 = 0$	$3(0)^2 = 0$	$\frac{(0)^2}{3} = 0$
1	$(1)^2 = 1$	$3(1)^2 = 3$	$\frac{(1)^2}{3} = \frac{1}{3}$
2	$(2)^2 = 4$	$3(2)^2 = 12$	$\frac{(2)^2}{3} = \frac{4}{3}$
3	$(3)^2 = 9$	$3(3)^2 = 27$	$\frac{(3)^2}{3} = 3$

From the table, you can see that the values of $y = 3x^2$ are bigger than the values of $y = x^2$. This is because each value of y gets multiplied by 3. As a result the parabola will be skinnier because it grows three times faster than $y = x^2$. On the other hand, you can see that the values of $y = \frac{1}{3}x^2$ are smaller than the values of $y = x^2$, because each value of y gets divided by 3. As a result the parabola will be fatter because it grows at one third the rate of $y = x^2$.

Orientation

As the value of a gets smaller and smaller, then, the parabola gets wider and flatter. What happens when a gets all the way down to zero? What happens when it's negative?

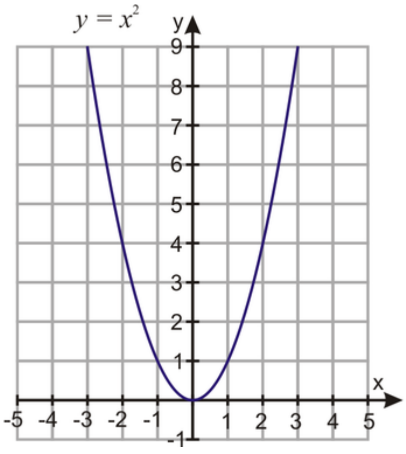
Well, when $a = 0$, the x^2 term drops out of the equation entirely, so the equation becomes linear and the graph is just a straight line. For example, we just saw what happens to $y = ax^2$ when we change the value of a; if we tried to graph $y = 0x^2$, we would just be graphing $y = 0$, which would be a horizontal line.

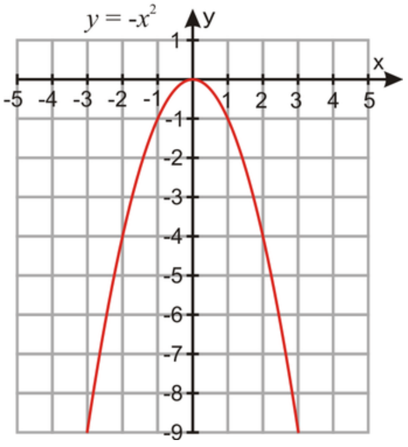
So as a gets smaller and smaller, the graph of $y = ax^2$ gets flattened all the way out into a horizontal line. Then, when a becomes negative, the graph of $y = ax^2$ starts to curve again, only it curves downward instead of upward. This fits with what you've already learned: the graph opens upward if a is positive and downward if a is negative.

Example D

What do the graphs of $y = x^2$ and $y = -x^2$ look like?

Solution:





You can see that the parabola has the same shape in both graphs, but the graph of $y = x^2$ is right-side-up and the graph of $y = -x^2$ is upside-down.

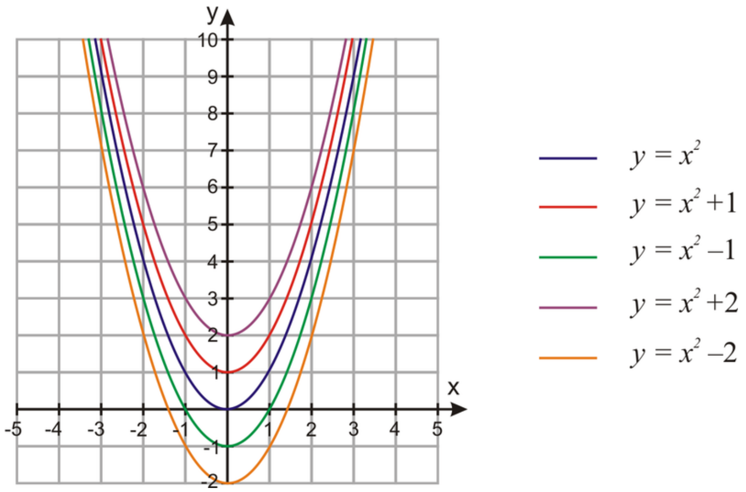
Vertical Shifts

Changing the constant c just shifts the parabola up or down.

Example E

What do the graphs of $y = x^2, y = x^2 + 1, y = x^2 - 1, y = x^2 + 2$, and $y = x^2 - 2$ look like?

Solution:



You can see that when c is positive, the graph shifts up, and when c is negative the graph shifts down; in either case, it shifts by $|c|$ units. In one of the later Concepts, we'll learn about **horizontal shift** (i.e. moving to the right or to the left). Before we can do that, though, we need to learn how to rewrite quadratic equations in different forms - our objective for the next Concept.

Watch this video for help with the Examples above.

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CK-12 Foundation: 1001 Graphs ofQuadratic Functions

Vocabulary

- The **general form** (or **standard form**) of a quadratic function is:

$$y = ax^2 + bx + c$$

Here a, b and c are the **coefficients.**

- Changing the value of a makes the graph "fatter" or "skinnier". This is called **dilation.**
- The vertical movement along a parabola's line of symmetry is called a ***vertical shift.***

Guided Practice

Graph the quadratic function, $y = -x^2 + 2$.

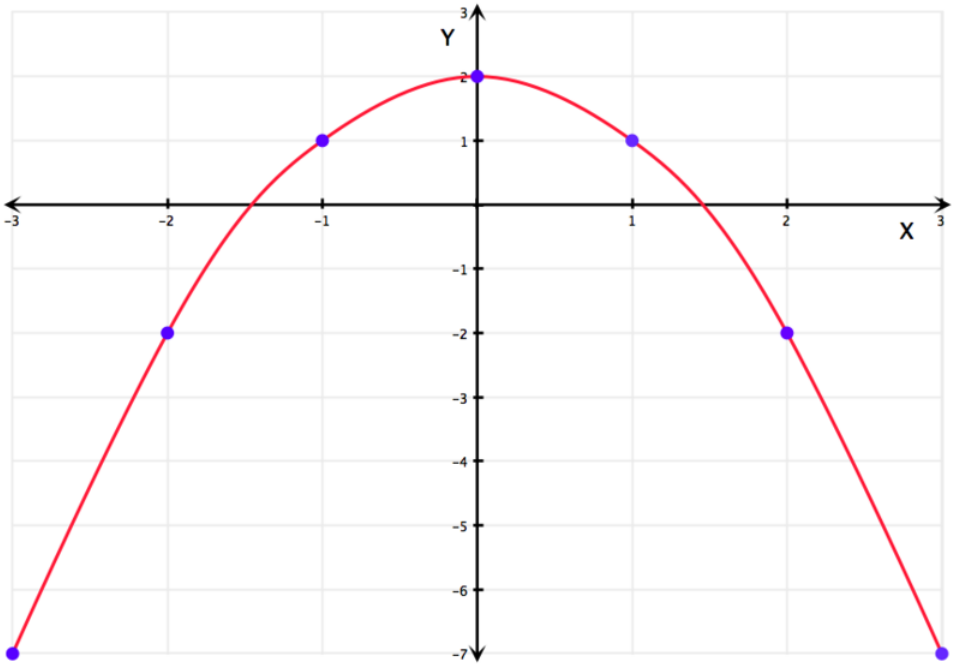
Solution:

We'll graph this function by making a table of values. Since the graph will be curved, we need to plot a fair number of points to make it accurate.

TABLE 10.7:

x	$y = x^2$
-3	$-(-3)^2 + 2 = -7$
-2	$-(-2)^2 + 2 = -2$
-1	$-(-1)^2 + 2 = 1$
0	$-(0)^2 + 2 = 2$
1	$-(1)^2 + 2 = 1$
2	$-(2)^2 + 2 = -2$
3	$-(3)^2 + 2 = -7$

Plot the points and connect them with a smooth curve:



Practice

For 1-5, does the graph of the parabola turn up or down?

1. $y = -2x^2 - 2x - 3$
2. $y = 3x^2$
3. $y = 16 - 4x^2$
4. $y = -100 + 0.25x^2$
5. $y = 3x^2 - 2x - 4x^2 + 3$

For 6-10, which parabola is wider?

6. $y = x^2$ or $y = 4x^2$
7. $y = 2x^2 + 4$ or $y = \frac{1}{2}x^2 + 4$
8. $y = -2x^2 - 2$ or $y = -x^2 - 2$
9. $y = x^2 + 3x^2$ or $y = x^2 + 3$
10. $y = -x^2$ or $y = \frac{1}{10}x^2$

10.2 Graphs of Quadratic Functions in Intercept Form

Here you'll learn how to write and graph quadratic functions in intercept form. You'll also learn how to find the x-intercepts and vertex of quadratic functions.

What if you had a quadratic function like $y = x^2 + 3x + 2$? How could you find its x-intercepts and vertex to help you graph it? After completing this Concept, you'll be able to use the intercept form of quadratic functions to solve problems like this one.

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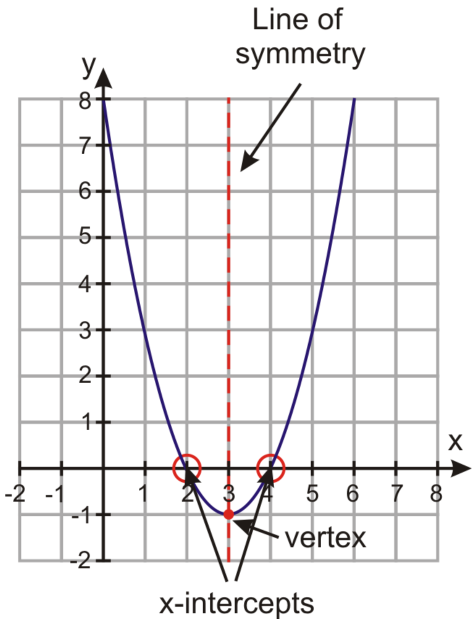
CK-12 Foundation: 1002S Graph Quadratic Functions inIntercept Form

Guidance

Now it's time to learn how to graph a parabola without having to use a table with a large number of points.

Let's look at the graph of $y = x^2 - 6x + 8$.



There are several things we can notice:

- The parabola crosses the $x-$axis at two points: $x = 2$ and $x = 4$. These points are called the $x-$**intercepts** of the parabola.

- The lowest point of the parabola occurs at (3, -1).
    - This point is called the **vertex** of the parabola.
    - The vertex is the lowest point in any parabola that turns upward, or the highest point in any parabola that turns downward.
    - The vertex is ***exactly halfway between the two $x-$intercepts.*** This will always be the case, and you can find the vertex using that property.
- The parabola is **symmetric.** If you draw a vertical line through the vertex, you see that the two halves of the parabola are mirror images of each other. This vertical line is called the **line of symmetry.**

We said that the general form of a quadratic function is $y = ax^2 + bx + c$. When we can factor a quadratic expression, we can rewrite the function in **intercept form:**

$$y = a(x-m)(x-n)$$

This form is very useful because it makes it easy for us to find the $x-$intercepts and the vertex of the parabola. The $x-$intercepts are the values of x where the graph crosses the $x-$axis; in other words, they are the values of x when $y = 0$. To find the $x-$intercepts from the quadratic function, we set $y = 0$ and solve:

$$0 = a(x-m)(x-n)$$

Since the equation is already factored, we use the zero-product property to set each factor equal to zero and solve the individual linear equations:

$$\begin{array}{ccc} x-m=0 & & x-n=0 \\ & \text{or} & \\ x=m & & x=n \end{array}$$

So the $x-$intercepts are at points $(m,0)$ and $(n,0)$.

Once we find the $x-$intercepts, it's simple to find the vertex. The $x-$value of the vertex is halfway between the two $x-$intercepts, so we can find it by taking the average of the two values: $\frac{m+n}{2}$. Then we can find the $y-$value by plugging the value of x back into the equation of the function.

Example A

Find the $x-$intercepts and the vertex of the following quadratic functions:

a) $y = x^2 - 8x + 15$

b) $y = 3x^2 + 6x - 24$

Solution

a) $y = x^2 - 8x + 15$

Write the quadratic function in intercept form by factoring the right hand side of the equation. Remember, to factor we need two numbers whose product is 15 and whose sum is –8. These numbers are –5 and –3.

The function in intercept form is $y = (x-5)(x-3)$

We find the $x-$intercepts by setting $y = 0$.

We have:

$$0 = (x-5)(x-3)$$

$$x-5=0 \quad \text{or} \quad x-3=0$$

$$x=5 \qquad x=3$$

So **the $x-$intercepts are (5, 0) and (3, 0).**

The vertex is halfway between the two $x-$intercepts. We find the $x-$value by taking the average of the two $x-$intercepts: $x = \frac{5+3}{2} = 4$

We find the $y-$value by plugging the $x-$value we just found into the original equation:

$$y = x^2 - 8x + 15 \Rightarrow y = 4^2 - 8(4) + 15 = 16 - 32 + 15 = -1$$

So **the vertex is (4, -1).**

b) $y = 3x^2 + 6x - 24$

Re-write the function in intercept form.

Factor the common term of 3 first: $y = 3(x^2 + 2x - 8)$

Then factor completely: $y = 3(x+4)(x-2)$

Set $y = 0$ and solve:

$$0 = 3(x+4)(x-2) \Rightarrow \quad x+4=0 \quad \text{or} \quad x-2=0$$

$$x=-4 \qquad x=2$$

The $x-$intercepts are (-4, 0) and (2, 0).

For the vertex,

$x = \frac{-4+2}{2} = -1$ and $y = 3(-1)^2 + 6(-1) - 24 = 3 - 6 - 24 = -27$

The vertex is: (-1, -27)

Knowing the vertex and $x-$intercepts is a useful first step toward being able to graph quadratic functions more easily. Knowing the vertex tells us where the middle of the parabola is. When making a table of values, we can make sure to pick the vertex as a point in the table. Then we choose just a few smaller and larger values of x. In this way, we get an accurate graph of the quadratic function without having to have too many points in our table.

Example B

Find the $x-$ intercepts and vertex. Use these points to create a table of values and graph each function.

a) $y = x^2 - 4$

b) $y = -x^2 + 14x - 48$

Solution

a) $y = x^2 - 4$

Let's find the $x-$intercepts and the vertex:

Factor the right-hand side of the function to put the equation in intercept form:

$$y = (x-2)(x+2)$$

Set $y = 0$ and solve:

$$\begin{aligned} &0 = (x-2)(x+2) && \\ &x-2=0 && && x+2=0 \\ & && \text{or} && \\ &x=2 && && x=-2 \end{aligned}$$

The $x-$intercepts are (2, 0) and (-2, 0).

Find the vertex:

$$x = \frac{2-2}{2} = 0 \quad y = (0)^2 - 4 = -4$$

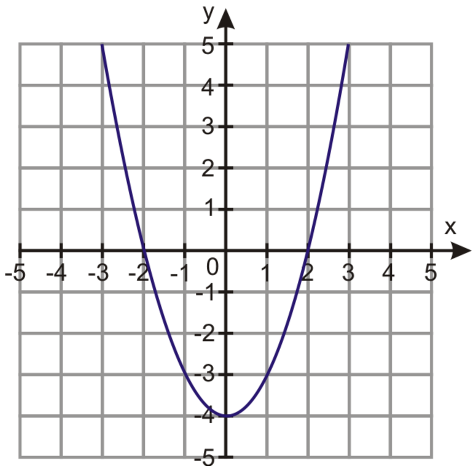
The vertex is (0, -4).

Make a table of values using the vertex as the middle point. Pick a few values of x smaller and larger than $x = 0$. Include the $x-$intercepts in the table.

TABLE 10.8:

x	$y = x^2 - 4$	
-3	$y = (-3)^2 - 4 = 5$	
–2	$y = (-2)^2 - 4 = 0$	$x-$intercept
–1	$y = (-1)^2 - 4 = -3$	
0	$y = (0)^2 - 4 = -4$	vertex
1	$y = (1)^2 - 4 = -3$	
2	$y = (2)^2 - 4 = 0$	$x-$intercept
3	$y = (3)^2 - 4 = 5$	

Then plot the graph:



b) $y = -x^2 + 14x - 48$

Let's find the $x-$intercepts and the vertex:

Factor the right-hand-side of the function to put the equation in intercept form:

$$y = -(x^2 - 14x + 48) = -(x-6)(x-8)$$

Set $y = 0$ and solve:

$$0 = -(x-6)(x-8)$$

$$x - 6 = 0 \quad \text{or} \quad x - 8 = 0$$

$$x = 6 \qquad x = 8$$

The $x-$intercepts are (6, 0) and (8, 0).

Find the vertex:

$$x = \frac{6+8}{2} = 7 \quad y = -(7)^2 + 14(7) - 48 = 1$$

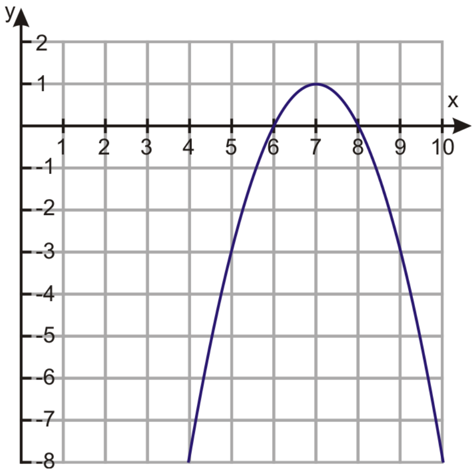
The vertex is (7, 1).

Make a table of values using the vertex as the middle point. Pick a few values of x smaller and larger than $x = 7$. Include the $x-$intercepts in the table.

TABLE 10.9:

x	$y = -x^2 + 14x - 48$
4	$y = -(4)^2 + 14(4) - 48 = -8$
5	$y = -(5)^2 + 14(5) - 48 = -3$
6	$y = -(6)^2 + 14(6) - 48 = 0$
7	$y = -(7)^2 + 14(7) - 48 = 1$
8	$y = -(8)^2 + 14(8) - 48 = 0$
9	$y = -(9)^2 + 14(9) - 48 = -3$
10	$y = -(10)^2 + 14(10) - 48 = -8$

Then plot the graph:



Applications of Quadratic Functions to Real-World Problems

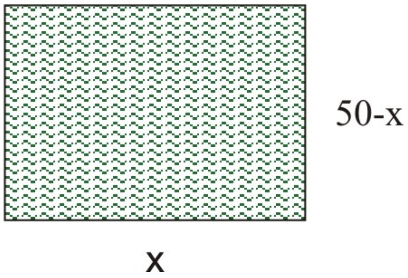
As we mentioned in a previous concept, parabolic curves are common in real-world applications. Here we will look at a few graphs that represent some examples of real-life application of quadratic functions.

Example C

Andrew has 100 feet of fence to enclose a rectangular tomato patch. What should the dimensions of the rectangle be in order for the rectangle to have the greatest possible area?

Solution

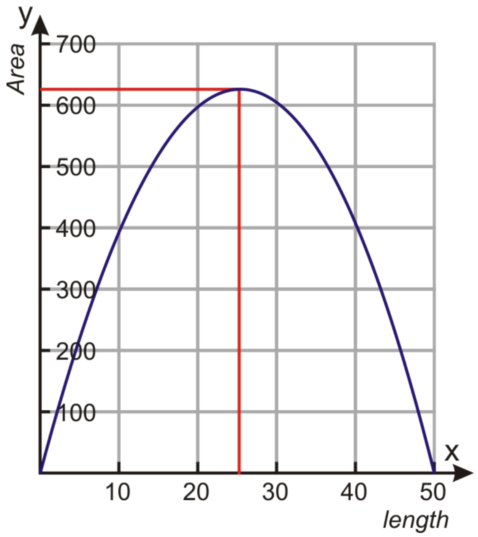
Drawing a picture will help us find an equation to describe this situation:



If the length of the rectangle is x, then the width is $50-x$. (The length and the width add up to 50, not 100, because two lengths and two widths together add up to 100.)

If we let y be the area of the triangle, then we know that the area is length × width, so $y = x(50-x) = 50x - x^2$.

Here's the graph of that function, so we can see how the area of the rectangle depends on the length of the rectangle:



We can see from the graph that the highest value of the area occurs when the length of the rectangle is 25. The area of the rectangle for this side length equals 625. (Notice that the width is also 25, which makes the shape a square with side length 25.)

This is an example of an *optimization problem.* These problems show up often in the real world, and if you ever study calculus, you'll learn how to solve them without graphs.

Watch this video for help with the Examples above.

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CK-12 Foundation: 1002 GraphQuadratic Functionsin Intercept Form

Vocabulary

- A parabola can be divided in half by a vertical line. Because of this, parabolas have ***symmetry***. The vertical line dividing the parabola into two equal portions is called the line of ***symmetry***.

- The point where the parabola crosses the $x-$axis is called the $x-$**intercepts** of the parabola.

- All parabolas have a ***vertex***, the ordered pair that represents the bottom (or the top) of the curve. The line of symmetry always goes through the vertex.
  - The vertex is the lowest point in any parabola that turns upward, or the highest point in any parabola that turns downward.
  - The vertex is ***exactly halfway between the two*** $x-$***intercepts.*** This will always be the case, and you can find the vertex using that property.

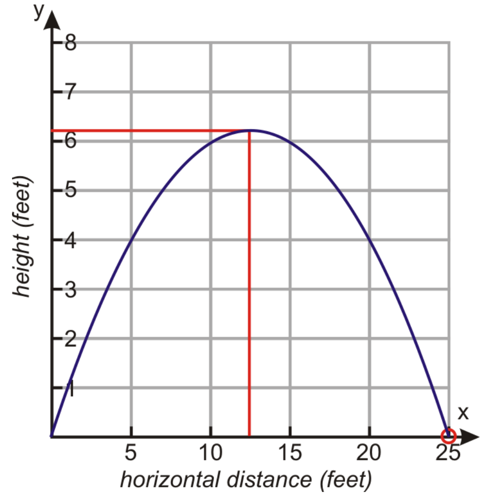
- If the $x-$intercepts are at points $(m,0)$ and $(n,0)$. The $x-$value of the **vertex** is halfway between the two $x-$intercepts, so we can find it by taking the average of the two values: $\frac{m+n}{2}$. Then we can find the $y-$value by plugging the value of x back into the equation of the function.

Guided Practice

Anne is playing golf. On the 4^{th} *tee, she hits a slow shot down the level fairway. The ball follows a parabolic path described by the equation* $y = x - 0.04x^2$*, where y is the ball's height in the air and x is the horizontal distance it has traveled from the tee. The distances are measured in feet. How far from the tee does the ball hit the ground? At what distance from the tee does the ball attain its maximum height? What is the maximum height?*

Solution

Let's graph the equation of the path of the ball:



$x(1 - 0.04x) = 0$ has solutions $x = 0$ and $x = 25$.

From the graph, we see that the ball hits the ground **25 feet from the tee.** (The other $x-$intercept, $x = 0$, tells us that the ball was also on the ground when it was on the tee!)

We can also see that the ball reaches its maximum height of **about 6.25 feet** when it is **12.5 feet from the tee.**

Practice

For 1-4, rewrite the following functions in intercept form. Find the $x-$intercepts and the vertex.

1. $y = x^2 - 2x - 8$
2. $y = -x^2 + 10x - 21$
3. $y = 2x^2 + 6x + 4$
4. $y = 3(x + 5)(x - 2)$

For 5-8, the vertex of which parabola is higher?

5. $y = x^2 + 4$ or $y = x^2 + 1$
6. $y = -2x^2$ or $y = -2x^2 - 2$
7. $y = 3x^2 - 3$ or $y = 3x^2 - 6$
8. $y = 5 - 2x^2$ or $y = 8 - 2x^2$

For 9-14, graph the following functions by making a table of values. Use the vertex and $x-$intercepts to help you pick values for the table.

9. $y = 4x^2 - 4$

10. $y = -x^2 + x + 12$
11. $y = 2x^2 + 10x + 8$
12. $y = \frac{1}{2}x^2 - 2x$
13. $y = x - 2x^2$
14. $y = 4x^2 - 8x + 4$

15. Nadia is throwing a ball to Peter. Peter does not catch the ball and it hits the ground. The graph shows the path of the ball as it flies through the air. The equation that describes the path of the ball is $y = 4 + 2x - 0.16x^2$. Here y is the height of the ball and x is the horizontal distance from Nadia. Both distances are measured in feet.
    a. How far from Nadia does the ball hit the ground?
    b. At what distance x from Nadia, does the ball attain its maximum height?
    c. What is the maximum height?
16. Jasreel wants to enclose a vegetable patch with 120 feet of fencing. He wants to put the vegetable against an existing wall, so he only needs fence for three of the sides. The equation for the area is given by $A = 120x - x^2$. From the graph, find what dimensions of the rectangle would give him the greatest area.

10.3 Use Graphs to Solve Quadratic Equations

Here you'll learn how to identify the number of solutions to a quadratic equation and how to find those solutions. You'll also learn how to use a graphing calculator to find the roots and the vertex of polynomials. Finally, you'll solve real-world problems by graphing quadratic functions.

What if you had a quadratic function like $y = 2x^2 + 5x + 3$? How could you graph it to find its roots? After completing this Concept, you'll be able to determine the number of solutions for a quadratic equation like this one and you'll find the solutions by graphing.

Try This

Now that you've learned how to solve quadratic equations by graphing them, you can sharpen your skills even more by learning how to find an equation from the graph alone. Go to the page linked in the previous section, http://www.analyzemath.com/quadraticg/quadraticg.htm, and scroll down to section E. Read the example there to learn how to find the equation of a quadratic function by reading off a few key values from the graph; then click the "Click here to start" button to try a problem yourself. The "New graph" button will give you a new problem when you finish the first one.

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CK-12 Foundation: 1002S Lesson Solving QuadraticEquations by Graphing

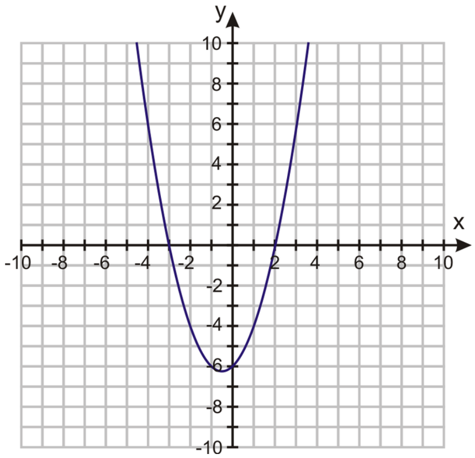
Guidance

Solving a quadratic equation means finding the $x-$values that will make the quadratic function equal zero; in other words, it means finding the points where the graph of the function crosses the $x-$axis. The solutions to a quadratic equation are also called the **roots** or **zeros** of the function, and in this section we'll learn how to find them by graphing the function.

Identify the Number of Solutions of a Quadratic Equation

Three different situations can occur when graphing a quadratic function:

Case 1: The parabola crosses the $x-$axis at two points. An example of this is $y = x^2 + x - 6$:



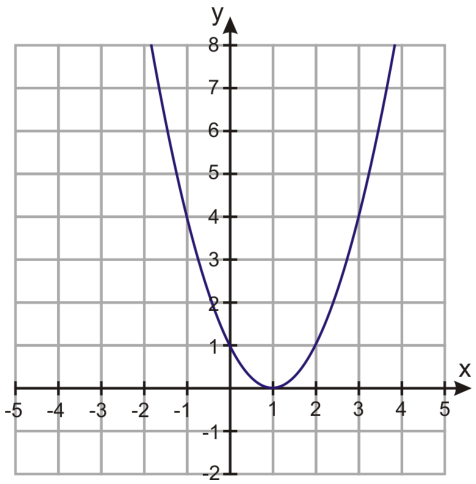
Looking at the graph, we see that the parabola crosses the $x-$axis at $x = -3$ and $x = 2$.

We can also find the solutions to the equation $x^2 + x - 6 = 0$ by setting $y = 0$. We solve the equation by factoring:

$(x+3)(x-2) = 0$, so $x = -3$ or $x = 2$.

When the graph of a quadratic function crosses the $x-$axis at two points, we get **two distinct solutions** to the quadratic equation.

Case 2: The parabola touches the $x-$axis at one point. An example of this is $y = x^2 - 2x + 1$:

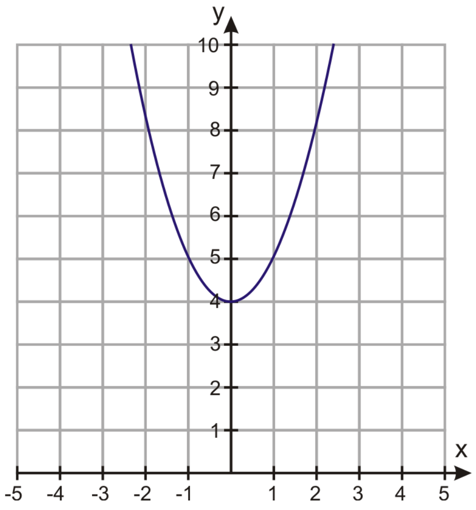


We can see that the graph touches the $x-$axis at $x = 1$.

We can also solve this equation by factoring. If we set $y = 0$ and factor, we obtain $(x-1)^2 = 0$, so $x = 1$. Since the quadratic function is a perfect square, we get only one solution for the equation—it's just the same solution repeated twice over.

When the graph of a quadratic function touches the $x-$axis at one point, the quadratic equation has one solution and the solution is called a **double root.**

Case 3: The parabola does not cross or touch the $x-$axis. An example of this is $y = x^2 + 4$:



If we set $y = 0$ we get $x^2 + 4 = 0$. This quadratic polynomial does not factor.

When the graph of a quadratic function does not cross or touch the $x-$axis, the quadratic equation has **no real solutions.**

Solve Quadratic Equations by Graphing

So far we've found the solutions to quadratic equations using factoring. However, in real life very few functions factor easily. As you just saw, graphing a function gives a lot of information about the solutions. We can find exact or approximate solutions to a quadratic equation by graphing the function associated with it.

Example A

Find the solutions to the following quadratic equations by graphing.

a) $-x^2 + 3 = y$

b) $-x^2 + x - 3 = y$

c) $y = -x^2 + 4x - 4$

Solution

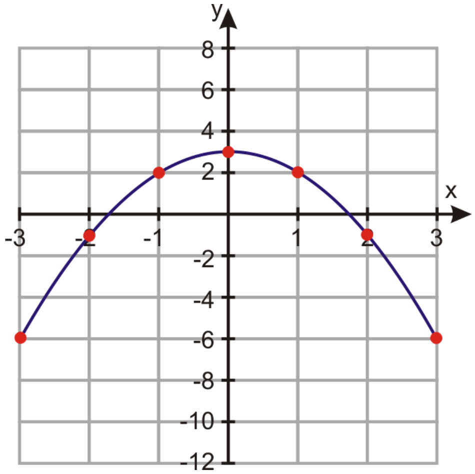
Since we can't factor any of these equations, we won't be able to graph them using intercept form (if we could, we wouldn't need to use the graphs to find the intercepts!) We'll just have to make a table of arbitrary values to graph each one.

a)

TABLE 10.10:

x	$y = -x^2 + 3$
-3	$y = -(-3)^2 + 3 = -6$
-2	$y = -(-2)^2 + 3 = -1$
-1	$y = -(-1)^2 + 3 = 2$
0	$y = -(0)^2 + 3 = 3$
1	$y = -(1)^2 + 3 = 2$
2	$y = -(2)^2 + 3 = -1$
3	$y = -(3)^2 + 3 = -6$

We plot the points and get the following graph:



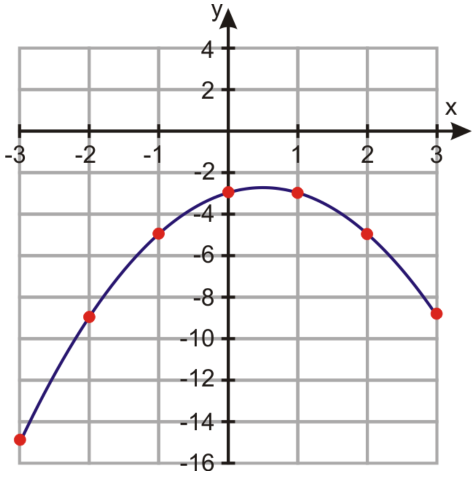
From the graph we can read that the $x-$intercepts are approximately $x = 1.7$**and** $x = -1.7$. These are the solutions to the equation.

b)

TABLE 10.11:

x	$y = -x^2 + x - 3$
-3	$y = -(-3)^2 + (-3) - 3 = -15$
–2	$y = -(-2)^2 + (-2) - 3 = -9$
–1	$y = -(-1)^2 + (-1) - 3 = -5$
0	$y = -(0)^2 + (0) - 3 = -3$
1	$y = -(1)^2 + (1) - 3 = -3$
2	$y = -(2)^2 + (2) - 3 = -5$
3	$y = -(3)^2 + (3) - 3 = -9$

We plot the points and get the following graph:



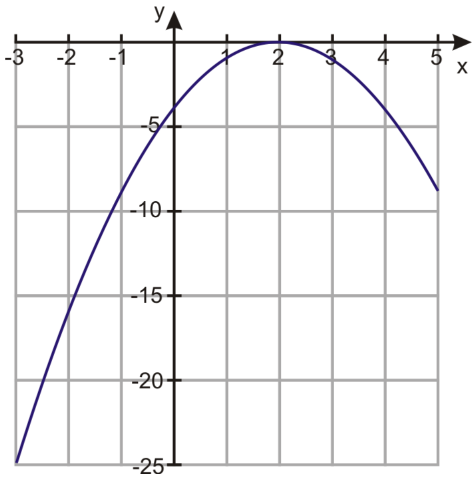
The graph curves up toward the $x-$axis and then back down without ever reaching it. This means that the graph never intercepts the $x-$axis, and so the corresponding equation has **no real solutions.**

c)

TABLE 10.12:

x	$y=-x^2+4x-4$
-3	$y=-(-3)^2+4(-3)-4=-25$
–2	$y=-(-2)^2+4(-2)-4=-16$
–1	$y=-(-1)^2+4(-1)-4=-9$
0	$y=-(0)^2+4(0)-4=-4$
1	$y=-(1)^2+4(1)-4=-1$
2	$y=-(2)^2+4(2)-4=0$
3	$y=-(3)^2+4(3)-4=-1$
4	$y=-(4)^2+4(4)-4=-4$
5	$y=-(5)^2+4(5)-4=-9$

Here is the graph of this function:



The graph just touches the $x-$axis at $x=2$, so the function has a **double root** there. $x=2$ is the only solution to the equation.

’Analyze Quadratic Functions Using a Graphing Calculator

A graphing calculator is very useful for graphing quadratic functions. Once the function is graphed, we can use the calculator to find important information such as the roots or the vertex of the function.

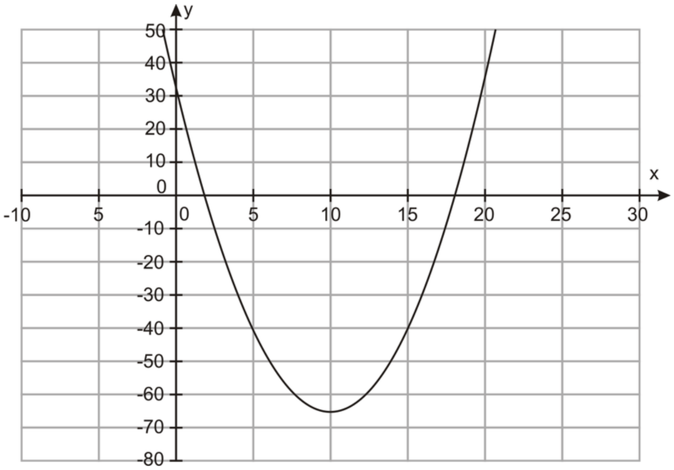
Example B

Use a graphing calculator to analyze the graph of $y=x^2-20x+35$.

Solution

1. **Graph** the function.

Press the **[Y=]** button and enter “$x^2-20x+35$” next to $[Y_1=]$. Press the **[GRAPH]** button. This is the plot you should see:



If this is not what you see, press the **[WINDOW]** button to change the window size. For the graph shown here, the $x-$values should range from -10 to 30 and the $y-$values from -80 to 50.

2. Find the **roots.**

There are at least three ways to find the roots:

Use **[TRACE]** to scroll over the $x-$intercepts. The approximate value of the roots will be shown on the screen. You can improve your estimate by zooming in.

OR

Use **[TABLE]** and scroll through the values until you find values of y equal to zero. You can change the accuracy of the solution by setting the step size with the **[TBLSET]** function.

OR

Use **[2nd] [TRACE]** (i.e. 'calc' button) and use option 'zero'.

Move the cursor to the left of one of the roots and press **[ENTER]**.

Move the cursor to the right of the same root and press **[ENTER]**.

Move the cursor close to the root and press **[ENTER]**.

The screen will show the value of the root. Repeat the procedure for the other root.

Whichever technique you use, you should get about $x = 1.9$**and** $x = 18$ for the two roots.

3. Find the **vertex.**

There are three ways to find the vertex:

Use **[TRACE]** to scroll over the highest or lowest point on the graph. The approximate value of the roots will be shown on the screen.

OR

Use **[TABLE]** and scroll through the values until you find values the lowest or highest value of y. You can change the accuracy of the solution by setting the step size with the **[TBLSET]** function.

OR

Use **[2nd] [TRACE]** and use the option 'maximum' if the vertex is a maximum or 'minimum' if the vertex is a minimum.

Move the cursor to the left of the vertex and press **[ENTER]**.

Move the cursor to the right of the vertex and press **[ENTER]**.

Move the cursor close to the vertex and press **[ENTER]**.

The screen will show the $x-$ and $y-$values of the vertex.

Whichever method you use, you should find that the vertex is at **(10, -65)**.

Solve Real-World Problems by Graphing Quadratic Functions

Here's a real-world problem we can solve using the graphing methods we've learned.

Example C

Andrew is an avid archer. He launches an arrow that takes a parabolic path. The equation of the height of the ball with respect to time is $y = -4.9t^2 + 48t$*, where* y *is the height of the arrow in meters and* t *is the time in seconds since Andrew shot the arrow. Find how long it takes the arrow to come back to the ground.*

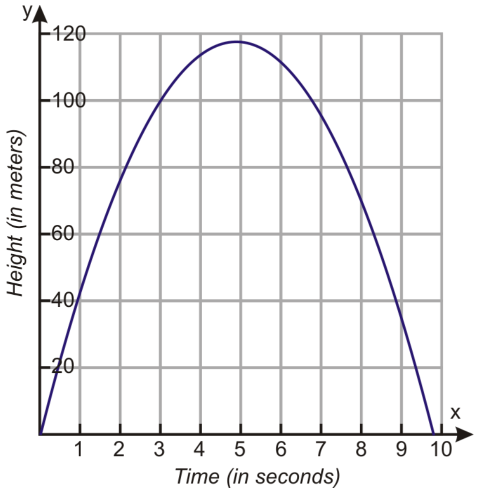
Solution

Let's graph the equation by making a table of values.

TABLE 10.13:

t	$y = -4.9t^2 + 48t$
0	$y = -4.9(0)^2 + 48(0) = 0$
1	$y = -4.9(1)^2 + 48(1) = 43.1$
2	$y = -4.9(2)^2 + 48(2) = 76.4$
3	$y = -4.9(3)^2 + 48(3) = 99.9$
4	$y = -4.9(4)^2 + 48(4) = 113.6$
5	$y = -4.9(5)^2 + 48(5) = 117.6$
6	$y = -4.9(6)^2 + 48(6) = 111.6$
7	$y = -4.9(7)^2 + 48(7) = 95.9$
8	$y = -4.9(8)^2 + 48(8) = 70.4$
9	$y = -4.9(9)^2 + 48(9) = 35.1$
10	$y = -4.9(10)^2 + 48(10) = -10$

Here's the graph of the function:



The roots of the function are approximately $x = 0$ sec and $x = 9.8$ sec. The first root tells us that the height of the arrow was 0 meters when Andrew first shot it. The second root says that it takes approximately **9.8 seconds** for the arrow to return to the ground.

Watch this video for help with the Examples above.

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CK-12 Foundation: 1003 Solving QuadraticEquations by Graphing

Vocabulary

- The **solutions of a quadratic equation** are often called the ***roots*** or ***zeros*.**

Guided Practice

Find the solutions to $2x^2+5x-7=0$ *by graphing.*

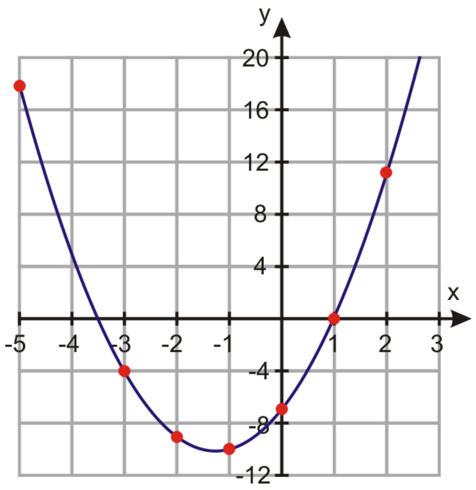
Solution

Since we can't factor this equation, we won't be able to graph it using intercept form (if we could, we wouldn't need to use the graphs to find the intercepts!) We'll just have to make a table of arbitrary values to graph the equation.

TABLE 10.14:

x	$y=2x^2+5x-7$
-5	$y=2(-5)^2+5(-5)-7=18$
-4	$y=2(-4)^2+5(-4)-7=5$
-3	$y=2(-3)^2+5(-3)-7=-4$
-2	$y=2(-2)^2+5(-2)-7=-9$
-1	$y=2(-1)^2+5(-1)-7=-10$
0	$y=2(0)^2+5(0)-7=-7$
1	$y=2(1)^2+5(1)-7=0$
2	$y=2(2)^2+5(2)-7=11$
3	$y=2(3)^2+5(3)-7=26$

We plot the points and get the following graph:



From the graph we can read that the $x-$intercepts are $x = 1$**and** $x = -3.5$. These are the solutions to the equation.

Practice

For 1-6, find the solutions of the following equations by graphing.

1. $x^2 + 3x + 6 = 0$
2. $-2x^2 + x + 4 = 0$
3. $x^2 - 9 = 0$
4. $x^2 + 6x + 9 = 0$
5. $10x - 3x^2 = 0$
6. $\frac{1}{2}x^2 - 2x + 3 = 0$

For 7-12, find the roots of the following quadratic functions by graphing.

7. $y = -3x^2 + 4x - 1$
8. $y = 9 - 4x^2$
9. $y = x^2 + 7x + 2$
10. $y = -x^2 - 10x - 25$
11. $y = 2x^2 - 3x$
12. $y = x^2 - 2x + 5$

For 13-18, use your graphing calculator to find the roots and the vertex of each polynomial.

13. $y = x^2 + 12x + 5$
14. $y = x^2 + 3x + 6$
15. $y = -x^2 - 3x + 9$
16. $y = -x^2 + 4x - 12$
17. $y = 2x^2 - 4x + 8$
18. $y = -5x^2 - 3x + 2$
19. Graph the equations $y = 2x^2 - 4x + 8$ and $y = x^2 - 2x + 4$ on the same screen. Find their roots and vertices.
    a. What is the same about the graphs? What is different?
    b. How are the two equations related to each other? (Hint: factor them.)
    c. What might be another equation with the same roots? Graph it and see.
20. Graph the equations $y = x^2 - 2x + 2$ and $y = x^2 - 2x + 4$ on the same screen. Find their roots and vertices.
    a. What is the same about the graphs? What is different?
    b. How are the two equations related to each other?
21. Phillip throws a ball and it takes a parabolic path. The equation of the height of the ball with respect to time is $y = -16t^2 + 60t$, where y is the height in feet and t is the time in seconds. Find how long it takes the ball to come back to the ground.
22. Use your graphing calculator to solve Ex. C. You should get the same answers as we did graphing by hand, but a lot quicker!

10.4 Use Square Roots to Solve Quadratic Equations

Here you'll learn how to solve quadratic equations in which finding the solutions involves square roots.

What if you had a quadratic equation like $4x^2 - 9 = 0$ in which both terms were perfect squares? How could you solve such an equation? After completing this Concept, you'll be able to solve quadratic equations like this one that involve perfect squares.

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CK-12 Foundation: 1004S Solving Quadratic Equations Using SquareRoots

Guidance

So far you know how to solve quadratic equations by factoring. However, this method works only if a quadratic polynomial can be factored. In the real world, most quadratics can't be factored, so now we'll start to learn other methods we can use to solve them. In this Concept, we'll examine equations in which we can take the square root of both sides of the equation in order to arrive at the result.

Solve Quadratic Equations Involving Perfect Squares

Let's first examine quadratic equations of the type

$$x^2 - c = 0$$

We can solve this equation by isolating the x^2 term: $x^2 = c$

Once the x^2 term is isolated we can take the square root of both sides of the equation. Remember that when we take the square root we get two answers: the positive square root and the negative square root:

$$x = \sqrt{c} \qquad \text{and} \qquad x = -\sqrt{c}$$

Often this is written as $x = \pm\sqrt{c}$.

Example A

Solve the following quadratic equations:

a) $x^2 - 4 = 0$

b) $x^2 - 25 = 0$

Solution

a) $x^2 - 4 = 0$

Isolate the x^2: $x^2 = 4$

Take the square root of both sides: $x = \sqrt{4}$ and $x = -\sqrt{4}$

The solutions are $x = 2$**and** $x = -2$.

b) $x^2 - 25 = 0$

Isolate the x^2: $x^2 = 25$

Take the square root of both sides: $x = \sqrt{25}$ and $x = -\sqrt{25}$

The solutions are $x = 5$**and** $x = -5$**.**

We can also find the solution using the square root when the x^2 term is multiplied by a constant—in other words, when the equation takes the form

$$ax^2 - c = 0$$

We just have to isolate the x^2:

$$\begin{aligned} ax^2 &= b \\ x^2 &= \frac{b}{a} \end{aligned}$$

Then we can take the square root of both sides of the equation:

$$x = \sqrt{\frac{b}{a}} \qquad \text{and} \qquad x = -\sqrt{\frac{b}{a}}$$

Often this is written as: $x = \pm\sqrt{\frac{b}{a}}$.

Example B

Solve the following quadratic equations.

a) $9x^2 - 16 = 0$

b) $81x^2 - 1 = 0$

Solution

a) $9x^2 - 16 = 0$

Isolate the x^2:

$$\begin{aligned} 9x^2 &= 16 \\ x^2 &= \frac{16}{9} \end{aligned}$$

Take the square root of both sides: $x = \sqrt{\frac{16}{9}}$ and $x = -\sqrt{\frac{16}{9}}$

Answer: $x = \frac{4}{3}$ and $x = -\frac{4}{3}$

b) $81x^2 - 1 = 0$

Isolate the x^2:

$$\begin{aligned} 81x^2 &= 1 \\ x^2 &= \frac{1}{81} \end{aligned}$$

Take the square root of both sides: $x = \sqrt{\frac{1}{81}}$ and $x = -\sqrt{\frac{1}{81}}$

Answer: $x = \frac{1}{9}$ and $x = -\frac{1}{9}$

As you've seen previously, some quadratic equations have no real solutions.

Example C

Solve the following quadratic equations.

a) $x^2 + 1 = 0$

b) $4x^2 + 9 = 0$

Solution

a) $x^2 + 1 = 0$

Isolate the x^2: $x^2 = -1$

Take the square root of both sides: $x = \sqrt{-1}$ and $x = -\sqrt{-1}$

Square roots of negative numbers do not give real number results, so there are **no real solutions** to this equation.

b) $4x^2 + 9 = 0$

Isolate the x^2:

$$\begin{aligned} 4x^2 &= -9 \\ x^2 &= -\frac{9}{4} \end{aligned}$$

Take the square root of both sides: $x = \sqrt{-\frac{9}{4}}$ and $x = -\sqrt{-\frac{9}{4}}$

There are **no real solutions.**

We can also use the square root function in some quadratic equations where both sides of an equation are perfect squares. This is true if an equation is of this form:

$$(x - 2)^2 = 9$$

Both sides of the equation are perfect squares. We take the square root of both sides and end up with two equations: $x - 2 = 3$ and $x - 2 = -3$.

Solving both equations gives us $x = 5$ and $x = -1$.

Example D

Solve the following quadratic equations.

a) $(x-1)^2 = 4$

b) $(x+3)^2 = 1$

Solution

a) $(x-1)^2 = 4$

$$\begin{aligned} \text{Take the square root of both sides :} &\qquad x-1=2 \text{ and } x-1=-2\\ \text{Solve each equation :} &\qquad x=3 \text{ and } x=-1 \end{aligned}$$

Answer: $x = 3$ and $x = -1$

b) $(x+3)^2 = 1$

$$\begin{aligned} \text{Take the square root of both sides :} &\qquad x+3=1 \text{ and } x+3=-1\\ \text{Solve each equation :} &\qquad x=-2 \text{ and } x=-4 \end{aligned}$$

Answer: $x = -2$ and $x = -4$

It might be necessary to factor the right-hand side of the equation as a perfect square before applying the method outlined above.

Watch this video for help with the Examples above.

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CK-12 Foundation: 1004 Solving QuadraticEquations UsingSquare Roots

Vocabulary

- The **solutions of a quadratic equation** are often called the ***roots*** or ***zeros.***

Guided Practice

Solve the following quadratic equations.

a) $x^2 + 8x + 16 = 25$

b) $4x^2 - 40x + 25 = 9$

Solution

a) $x^2+8x+16=25$

$$\begin{array}{ll} \text{Factor the right-hand-side :} & x^2+8x+16=(x+4)^2 \quad \text{so} \quad (x+4)^2=25 \\ \text{Take the square root of both sides :} & x+4=5 \text{ and } x+4=-5 \\ \text{Solve each equation :} & x=1 \text{ and } x=-9 \end{array}$$

Answer: $x=1$ and $x=-9$

b) $4x^2-20x+25=9$

$$\begin{array}{ll} \text{Factor the right-hand-side :} & 4x^2-20x+25=(2x-5)^2 \quad \text{so} \quad (2x-5)^2=9 \\ \text{Take the square root of both sides :} & 2x-5=3 \text{ and } 2x-5=-3 \\ \text{Solve each equation :} & 2x=8 \text{ and } 2x=2 \end{array}$$

Answer: $x=4$ and $x=1$

Practice

Solve the following quadratic equations.

1. $x^2-1=0$
2. $x^2-100=0$
3. $x^2+16=0$
4. $9x^2-1=0$
5. $4x^2-49=0$
6. $64x^2-9=0$
7. $x^2-81=0$
8. $25x^2-36=0$
9. $x^2+9=0$
10. $x^2-16=0$
11. $x^2-36=0$
12. $16x^2-49=0$
13. $(x-2)^2=1$
14. $(x+5)^2=16$
15. $(2x-1)^2-4=0$
16. $(3x+4)^2=9$
17. $(x-3)^2+25=0$
18. $x^2-10x+25=9$
19. $x^2+18x+81=1$
20. $4x^2-12x+9=16$
21. $2(x+3)^2=8$

10.5 Square Root Applications

Here you'll learn how to approximate the solutions of quadratic equations involving square roots. You'll also learn how to solve applications using quadratic functions and square roots.

What if a penny were dropped from the top of the Washington Monument at a height of 555 feet? How long would it take for the penny to reach the ground? After completing this Concept, you'll be able to use quadratic functions and square roots to solve real-world applications like this one.

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CK-12 Foundation: 1005S ApplicationsUsing Square Roots

Guidance

We can use the methods we've learned so far in this section to find approximate solutions to quadratic equations, when taking the square root doesn't give an exact answer.

Example A

Solve the following quadratic equations.

a) $x^2-3=0$

b) $2x^2-9=0$

Solution

a)

$$\begin{array}{rr} \text{Isolate the } x^2: & x^2=3 \\ \text{Take the square root of both sides}: & x=\sqrt{3} \text{ and } x=-\sqrt{3} \end{array}$$

Answer: $x \approx 1.73$ and $x \approx -1.73$

b)

$$\begin{array}{rr} \text{Isolate the } x^2: & 2x^2=9 \text{ so } x^2=\frac{9}{2} \\ \text{Take the square root of both sides}: & x=\sqrt{\frac{9}{2}} \text{ and } x=-\sqrt{\frac{9}{2}} \end{array}$$

Answer: $x \approx 2.12$ and $x \approx -2.12$

Example B

Solve the following quadratic equations.

a) $(2x+5)^2 = 10$

b) $x^2 - 2x + 1 = 5$

Solution

a)

$$\begin{aligned} \text{Take the square root of both sides :} \quad & 2x+5 = \sqrt{10} \text{ and } 2x+5 = -\sqrt{10} \\ \text{Solve both equations :} \quad & x = \frac{-5+\sqrt{10}}{2} \text{ and } x = \frac{-5-\sqrt{10}}{2} \end{aligned}$$

Answer: $x \approx -0.92$ and $x \approx -4.08$

b)

$$\begin{aligned} \text{Factor the right-hand-side :} \quad & (x-1)^2 = 5 \\ \text{Take the square root of both sides :} \quad & x-1 = \sqrt{5} \text{ and } x-1 = -\sqrt{5} \\ \text{Solve each equation :} \quad & x = 1+\sqrt{5} \text{ and } x = 1-\sqrt{5} \end{aligned}$$

Answer: $x \approx 3.24$ and $x \approx -1.24$

Solve Applications Using Quadratic Functions and Square Roots

Quadratic equations are needed to solve many real-world problems. In this section, we'll examine problems about objects falling under the influence of gravity. When objects are **dropped** from a height, they have no initial velocity; the force that makes them move towards the ground is due to gravity. The acceleration of gravity on earth is given by the equation

$$g = -9.8\ m/s^2 \quad \text{or} \quad g = -32\ ft/s^2$$

The negative sign indicates a downward direction. We can assume that gravity is constant for the problems we'll be examining, because we will be staying close to the surface of the earth. The acceleration of gravity decreases as an object moves very far from the earth. It is also different on other celestial bodies such as the moon.

The equation that shows the height of an object in free fall is

$$y = \frac{1}{2}gt^2 + y_0$$

The term y_0 represents the initial height of the object, t is time, and g is the constant representing the force of gravity. You then plug in one of the two values for g above, depending on whether you want the answer in feet or meters. Thus the equation works out to $y = -4.9t^2 + y_0$ if you want the height in meters, and $y = -16t^2 + y_0$ if you want it in feet.

Example C

How long does it take a ball to fall from a roof to the ground 25 feet below?

Solution

$$\begin{aligned} \text{Since we are given the height in feet, use equation :} &\qquad y = -16t^2 + y_0 \\ \text{The initial height is } y_0 = 25\ feet, \text{ so :} &\qquad y = -16t^2 + 25 \\ \text{The height when the ball hits the ground is } y = 0, \text{ so :} &\qquad 0 = -16t^2 + 25 \\ \text{Solve for } t : &\qquad 16t^2 = 25 \\ &\qquad t^2 = \frac{25}{16} \\ &\qquad t = \frac{5}{4} \text{ or } t = -\frac{5}{4} \end{aligned}$$

Since only positive time makes sense in this case, **it takes the ball 1.25 seconds to fall to the ground.**

Example D

A rock is dropped from the top of a cliff and strikes the ground 7.2 seconds later. How high is the cliff in meters?

Solution

$$\begin{aligned} \text{Since we want the height in meters, use equation :} &\qquad y = -4.9t^2 + y_0 \\ \text{The time of flight is } t = 7.2\ seconds : &\qquad y = -4.9(7.2)^2 + y_0 \\ \text{The height when the ball hits the ground is } y = 0, \text{ so :} &\qquad 0 = -4.9(7.2)^2 + y_0 \\ \text{Simplify :} &\qquad 0 = -254 + y_0 \text{ so } y_0 = 254 \end{aligned}$$

The cliff is 254 meters high.

Watch this video for help with the Examples above.

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CK-12 Foundation: 1005 Applications Using SquareRoots

Vocabulary

- The **solutions of a quadratic equation** are often called the ***roots*** or ***zeros.***

Guided Practice

Victor throws an apple out of a window on the 10^{th} *floor which is 120 feet above ground. One second later Juan throws an orange out of a* 6^{th} *floor window which is 72 feet above the ground. Which fruit reaches the ground first, and how much faster does it get there?*

Solution

Let's find the time of flight for each piece of fruit.

Apple:

$$\begin{aligned}
&\text{Since we have the height in feet, use this equation :} && y = -16t^2 + y_0\\
&\text{The initial height is } y_0 = 120\ feet: && y = -16t^2 + 120\\
&\text{The height when the ball hits the ground is } y = 0, \text{ so :} && 0 = -16t^2 + 120\\
&\text{Solve for } t: && 16t^2 = 120\\
& && t^2 = \frac{120}{16} = 7.5\\
& && \underline{t = 2.74} \text{ or } t = -2.74\ seconds
\end{aligned}$$

Orange:

$$\begin{aligned}
&\text{The initial height is } y_0 = 72\ feet: && 0 = -16t^2 + 72\\
&\text{Solve for } t: && 16t^2 = 72\\
& && t^2 = \frac{72}{16} = 4.5\\
& && \underline{t = 2.12} \text{ or } t = -2.12\ seconds
\end{aligned}$$

The orange was thrown one second later, so add 1 second to the time of the orange: $t = 3.12$ *seconds*

The apple hits the ground first. It gets there 0.38 seconds faster than the orange.

Practice

Solve the following quadratic equations.

1. $x^2 = 11$
2. $5x^2 = 0.01$
3. $x^2 - 6 = 0$
4. $x^2 - 20 = 0$
5. $3x^2 + 14 = 0$
6. $(x-6)^2 = 5$
7. $(x+10)^2 = 2$
8. Susan drops her camera in the river from a bridge that is 400 feet high. How long is it before she hears the splash?
9. It takes a rock 5.3 seconds to splash in the water when it is dropped from the top of a cliff. How high is the cliff in meters?
10. Nisha drops a rock from the roof of a building 50 feet high. Ashaan drops a quarter from the top story window, 40 feet high, exactly half a second after Nisha drops the rock. Which hits the ground first?

10.6 Completing the Square

Here you'll learn how to complete the square to help you solve quadratic equations. You'll also solve quadratic equations in standard form.

What if you had a quadratic equation like $x^2 + 12x = 13$? How could you solve it by taking the square root of both sides? After completing this Concept, you'll be able to complete the square to solve quadratic equations like this one.

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CK-12 Foundation: 1006S Solving Quadratic Equations by Completing the Square

Guidance

You saw in the last section that if you have a quadratic equation of the form $(x-2)^2 = 5$, you can easily solve it by taking the square root of each side:

$$x-2 = \sqrt{5} \qquad \text{and} \qquad x-2 = -\sqrt{5}$$

Simplify to get:

$$x = 2+\sqrt{5} \approx 4.24 \qquad \text{and} \qquad x = 2-\sqrt{5} \approx -0.24$$

So what do you do with an equation that isn't written in this nice form? In this section, you'll learn how to rewrite any quadratic equation in this form by **completing the square.**

Complete the Square of a Quadratic Expression

Completing the square lets you rewrite a quadratic expression so that it contains a perfect square trinomial that you can factor as the square of a binomial.

Remember that the square of a binomial takes one of the following forms:

$$(x+a)^2 = x^2+2ax+a^2$$
$$(x-a)^2 = x^2-2ax+a^2$$

So in order to have a perfect square trinomial, we need two terms that are perfect squares and one term that is twice the product of the square roots of the other terms.

Example A

Complete the square for the quadratic expression x^2+4x.

Solution

To complete the square we need a constant term that turns the expression into a perfect square trinomial. Since the middle term in a perfect square trinomial is always 2 times the product of the square roots of the other two terms, we re-write our expression as:

$$x^2+2(2)(x)$$

We see that the constant we are seeking must be 2^2 :

$$x^2+2(2)(x)+2^2$$

Answer: By adding 4 to both sides, this can be factored as: $(x+2)^2$

Notice, though, that we just changed the value of the whole expression by adding 4 to it. If it had been an equation, we would have needed to add 4 to the other side as well to make up for this.

Also, this was a relatively easy example because a, the coefficient of the x^2 term, was 1. When that coefficient doesn't equal 1, we have to factor it out from the whole expression before completing the square.

Example B

Complete the square for the quadratic expression $4x^2+32x$.

Solution

Factor the coefficient of the x^2 term:

$4(x^2+8x)$

Re-write the expression:

$4(x^2+2(4)(x))$

We complete the square by adding the constant 4^2:

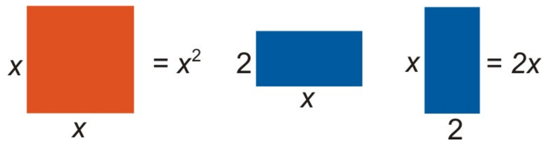
$4(x^2+2(4)(x)+4^2)$

Factor the perfect square trinomial inside the parenthesis:

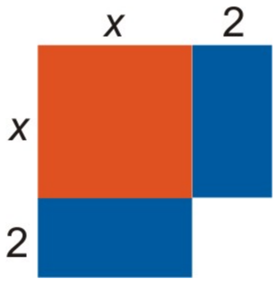
$4(x+4)^2$

The expression **"completing the square"** comes from a geometric interpretation of this situation. Let's revisit the quadratic expression in Example 1: x^2+4x.

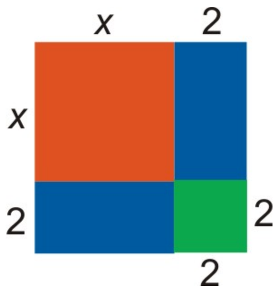
We can think of this expression as the sum of three areas. The first term represents the area of a square of side x. The second expression represents the areas of two rectangles with a length of 2 and a width of x:



We can combine these shapes as follows:



We obtain a square that is not quite complete. To complete the square, we need to add a smaller square of side length 2.



We end up with a square of side length $(x+2)$; its area is therefore $(x+2)^2$. Let's demonstrate the method of **completing the square** with an example.

Example C

Solve the following quadratic equation: $3x^2-10x=-1$

Solution

Divide all terms by the coefficient of the x^2 term:

$$x^2-\frac{10}{3}x=-\frac{1}{3}$$

Rewrite: $x^2-2\left(\frac{5}{3}\right)(x)=-\frac{1}{3}$

In order to have a perfect square trinomial on the right-hand-side we need to add the constant $\left(\frac{5}{3}\right)^2$. Add this constant to **both** sides of the equation:

$$x^2-2\left(\frac{5}{3}\right)(x)+\left(\frac{5}{3}\right)^2=-\frac{1}{3}+\left(\frac{5}{3}\right)^2$$

Factor the perfect square trinomial and simplify:

$$\left(x-\frac{5}{3}\right)^2=-\frac{1}{3}+\frac{25}{9}$$
$$\left(x-\frac{5}{3}\right)^2=\frac{22}{9}$$

Take the square root of both sides:

$$x-\frac{5}{3}=\sqrt{\frac{22}{9}} \quad \text{and} \quad x-\frac{5}{3}=-\sqrt{\frac{22}{9}}$$
$$x=\frac{5}{3}+\sqrt{\frac{22}{9}}\approx 3.23 \quad \text{and} \quad x=\frac{5}{3}-\sqrt{\frac{22}{9}}\approx 0.1$$

Answer: $x=3.23$ and $x=0.1$

Solving Quadratic Equations in Standard Form

If an equation is in standard form $(ax^2+bx+c=0)$, we can still solve it by the method of completing the square. All we have to do is start by moving the constant term to the right-hand-side of the equation.

Example D

Solve the following quadratic equation: $x^2+15x+12=0$

Solution

Move the constant to the other side of the equation:

$$x^2+15x=-12$$

Rewrite: $x^2+2\left(\frac{15}{2}\right)(x)=-12$

Add the constant $\left(\frac{15}{2}\right)^2$ to both sides of the equation:

$$x^2+2\left(\frac{15}{2}\right)(x)+\left(\frac{15}{2}\right)^2=-12+\left(\frac{15}{2}\right)^2$$

Factor the perfect square trinomial and simplify:

$$\left(x+\frac{15}{2}\right)^2=-12+\frac{225}{4}$$
$$\left(x+\frac{15}{2}\right)^2=\frac{177}{4}$$

Take the square root of both sides:

$$x+\frac{15}{2}=\sqrt{\frac{177}{4}} \quad \text{and} \quad x+\frac{15}{2}=-\sqrt{\frac{177}{4}}$$
$$x=-\frac{15}{2}+\sqrt{\frac{177}{4}}\approx -0.85 \quad \text{and} \quad x=-\frac{15}{2}-\sqrt{\frac{177}{4}}\approx -14.15$$

Answer: $x=-0.85$ and $x=-14.15$

Watch this video for help with the Examples above.

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CK-12 Foundation: 1006 Solving QuadraticEquations by CompletingtheSquare

Vocabulary

- A ***perfect square trinomial*** has the form $a^2+2(ab)+b^2$, which factors into $(a+b)^2$.

Guided Practice

Solve the following quadratic equation: $-x^2+22x=5$

Solution

Divide all terms by the coefficient of the x^2 term:

$$x^2-22x=-6$$

Rewrite: $x^2-2(11)(x)=-6$.

In order to have a perfect square trinomial on the right-hand-side we need to add the constant $(11)^2$. Add this constant to **both** sides of the equation:

$$x^2-2(11)(x)+(11)^2=-6+(11)^2$$

Factor the perfect square trinomial and simplify:

$$\begin{aligned}(x-11)^2&=-6+(11)^2\\ \left(x-\frac{5}{3}\right)^2&=16\end{aligned}$$

Take the square root of both sides:

$$\begin{aligned}x-11&=\sqrt{16} && \text{and} && x-11=-\sqrt{16}\\ x&=11+\sqrt{16}=15 && \text{and} && x=11-\sqrt{4}=7\end{aligned}$$

Answer: $x=15$ and $x=7$

Practice

Complete the square for each expression.

1. $x^2 + 5x$
2. $x^2 - 2x$
3. $x^2 + 3x$
4. $x^2 - 4x$
5. $3x^2 + 18x$
6. $2x^2 - 22x$
7. $8x^2 - 10x$
8. $5x^2 + 12x$

Solve each quadratic equation by completing the square.

9. $x^2 - 4x = 5$
10. $x^2 - 5x = 10$
11. $x^2 + 10x + 15 = 0$
12. $x^2 + 15x + 20 = 0$
13. $2x^2 - 18x = 3$
14. $4x^2 + 5x = -1$
15. $10x^2 - 30x - 8 = 0$
16. $5x^2 + 15x - 40 = 0$

10.7 Vertex Form of a Quadratic Equation

Here you'll learn how to find the vertex, the x-intercepts, and the y-intercept of parabolas that are written in vertex form. You'll also learn how to graph such parabolas. Finally, you'll rewrite quadratic functions in vertex form.

What if you had a quadratic function like $y-2=x^2+4x$? How could you rewrite it in vertex form to find its vertex and intercepts? After completing this Concept, you'll be able to rewrite and graph quadratic equations like this one in vertex form.

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CK-12 Foundation: 1007S Graph Quadratic Functions inVertex Form

Guidance

Probably one of the best applications of the method of completing the square is using it to rewrite a quadratic function in vertex form. The vertex form of a quadratic function is

$$y-k=a(x-h)^2$$

This form is very useful for graphing because it gives the vertex of the parabola explicitly. The vertex is at the point (h,k).

It is also simple to find the $x-$intercepts from the vertex form: just set $y=0$ and take the square root of both sides of the resulting equation.

To find the $y-$intercept, set $x=0$ and simplify.

Example A

Find the vertex, the $x-$ intercepts and the $y-$ intercept of the following parabolas:

a) $y-2=(x-1)^2$

b) $y+8=2(x-3)^2$

Solution

a) $y-2=(x-1)^2$

Vertex: $\underline{(1,2)}$

To find the $x-$intercepts,

$$\begin{aligned} \text{Set } y=0: &\quad -2=(x-1)^2 \\ \text{Take the square root of both sides}: &\quad \sqrt{-2}=x-1 \quad \text{and} \quad -\sqrt{-2}=x-1 \end{aligned}$$

The solutions are not real so there are **no $x-$intercepts.**

To find the $y-$intercept,

$$\begin{aligned} \text{Set } x=0: &\quad y-2=(-1)^2 \\ \text{Simplify}: &\quad y-2=1 \Rightarrow \underline{\underline{y=3}} \end{aligned}$$

b) $y+8=2(x-3)^2$

$$\begin{aligned} \text{Rewrite}: &\quad y-(-8)=2(x-3)^2 \\ \text{Vertex}: &\quad \underline{\underline{(3,-8)}} \end{aligned}$$

To find the $x-$intercepts,

$$\begin{aligned} \text{Set } y=0: &\quad 8=2(x-3)^2 \\ \text{Divide both sides by 2}: &\quad 4=(x-3)^2 \\ \text{Take the square root of both sides}: &\quad 4=x-3 \quad \text{and} \quad -4=x-3 \\ \text{Simplify}: &\quad \underline{\underline{x=7}} \quad \text{and} \quad \underline{\underline{x=-1}} \end{aligned}$$

To find the $y-$intercept,

$$\begin{aligned} \text{Set } x=0: &\quad y+8=2(-3)^2 \\ \text{Simplify}: &\quad y+8=18 \Rightarrow \underline{\underline{y=10}} \end{aligned}$$

To graph a parabola, we only need to know the following information:

- the vertex
- the $x-$intercepts
- the $y-$intercept
- whether the parabola turns up or down (remember that it turns up if $a>0$ and down if $a<0$)

Example B

Graph the parabola given by the function $y+1=(x+3)^2$.

Solution

$$\begin{aligned} \text{Rewrite}: &\quad y-(-1)=(x-(-3))^2 \\ \text{Vertex}: &\quad \underline{\underline{(-3,-1)}} \qquad \text{vertex}: (-3,-1) \end{aligned}$$

To find the $x-$intercepts,

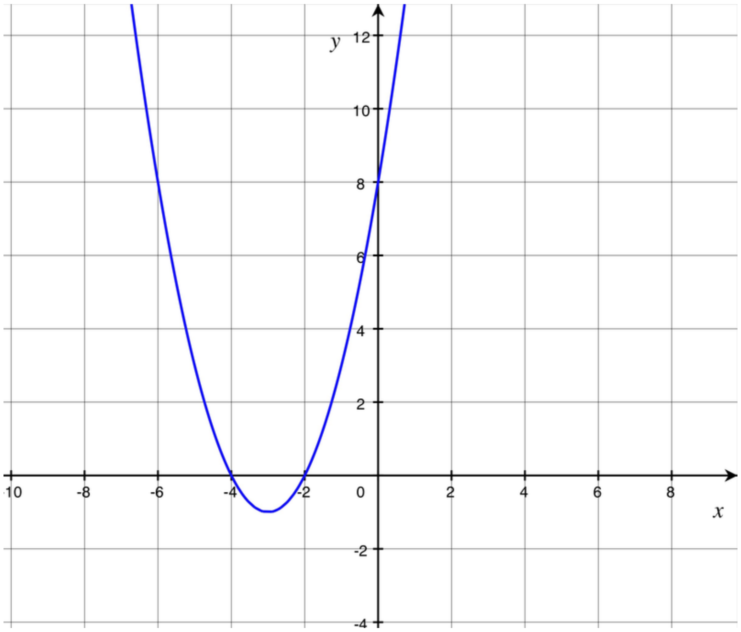
$$\begin{array}{lllll}\text{Set } y=0: & 1=(x+3)^2 & & & \\ \text{Take the square root of both sides :} & 1=x+3 & \text{and} & -1=x+3 & \\ \text{Simplify :} & \underline{\underline{x=-2}} & \text{and} & \underline{\underline{x=-4}} & \\ & x-\text{intercepts : } (-2,0) \text{ and } (-4,0) & & & \end{array}$$

To find the $y-$intercept,

$$\begin{array}{lll}\text{Set } x=0: & y+1=(3)^2 & \\ \text{Simplify:} & \underline{\underline{y=8}} & y-\text{intercept : } (0,8)\end{array}$$

And since $a>0$, the parabola **turns up.**

Graph all the points and connect them with a smooth curve:



Example C

Graph the parabola given by the function $y=-\frac{1}{2}(x-2)^2$.

Solution:

$$\begin{array}{lll}\text{Rewrite} & y-(0)=-\dfrac{1}{2}(x-2)^2 & \\ \text{Vertex:} & \underline{(2,0)} & \text{vertex:}(2,0)\end{array}$$

To find the $x-$intercepts,

$$\begin{aligned} \text{Set } y=0: & \quad 0=-\frac{1}{2}(x-2)^2 \\ \text{Multiply both sides by } -2: & \quad 0=(x-2)^2 \\ \text{Take the square root of both sides}: & \quad 0=x-2 \\ \text{Simplify}: & \quad \underline{\underline{x=2}} \qquad x-\text{intercept:}(2,0) \end{aligned}$$

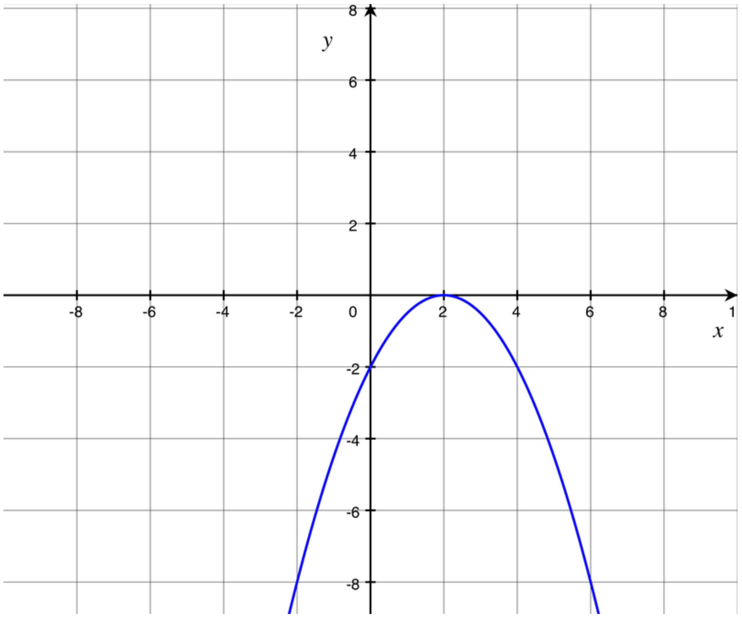
Note: there is only one $x-$intercept, indicating that the vertex is located at this point, (2, 0).

To find the $y-$intercept,

$$\begin{aligned} \text{Set } x=0: & \quad y=-\frac{1}{2}(-2)^2 \\ \text{Simplify:} & \quad y=-\frac{1}{2}(4) \Rightarrow \underline{\underline{y=-2}} \qquad y-\text{intercept:}(0,-2) \end{aligned}$$

Since $a<0$, the parabola **turns down.**

Graph all the points and connect them with a smooth curve:



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CK-12 Foundation: 1007 GraphQuadratic Functionsin Vertex Form

Vocabulary

- The vertex form of a quadratic function is

$$y-k=a(x-h)^2$$

This form is very useful for graphing because it gives the vertex of the parabola explicitly. The vertex is at the point (h,k).

- To find the $x-$intercepts from the vertex form: just set $y=0$ and take the square root of both sides of the resulting equation.

- To find the $y-$intercept, set $x=0$ and simplify.

Guided Practice

Graph the parabola given by the function $y=4(x+2)^2-1$.

Solution:

$$\begin{array}{lll} \text{Rewrite} & y-(-1)=4(x+2)^2 & \\ \text{Simplify} & y+1=4(x+2)^2 & \\ \text{Vertex:} & \underline{(-2,-1)} & \text{vertex:}(-2,-1) \end{array}$$

To find the $x-$intercepts,

$$\begin{array}{rll} \text{Set. } y=0: & 0=4(x+2)^2-1 & \\ \text{Subtract 1 from each side}: & 1=4(x+2)^2 & \\ \text{Divide both sides by } 4: & \frac{1}{4}=(x+2)^2 & \\ \text{Take the square root of both sides}: & \frac{1}{2}=\pm(x+2) & \\ \text{Separate}: & \frac{1}{2}=-(x+2) & \frac{1}{2}=x+2) \\ \text{Simplify}: & \underline{\underline{x=-2.5}} & \underline{\underline{x=-1.5}} \end{array}$$

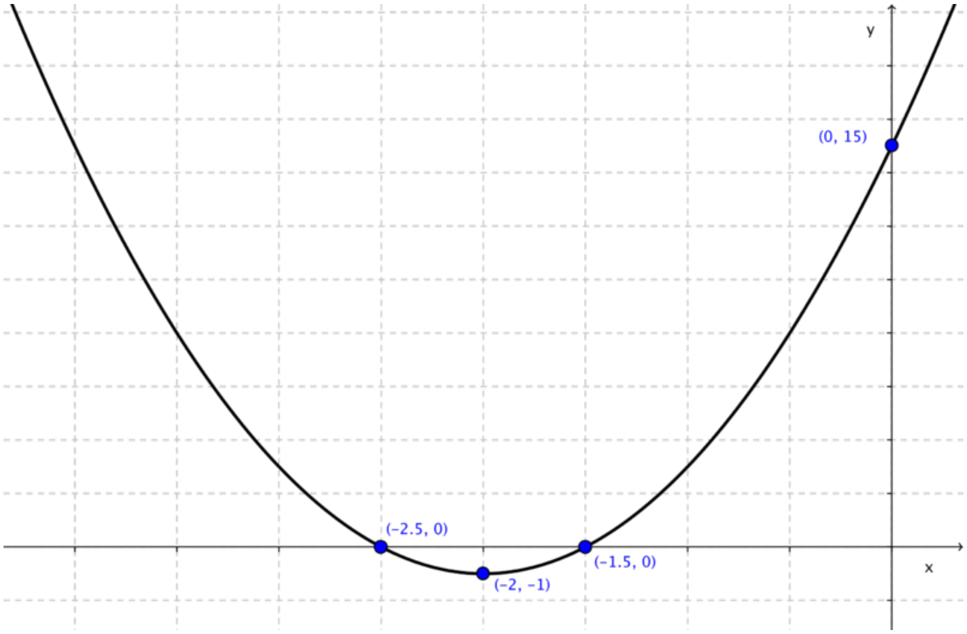
The $x-$intercepts are $(-2.5,0)$ and $(-1.5,0)$.

To find the $y-$intercept,

$$\begin{array}{rll} \text{Set } x=0: & y=4(0+2)^2-1 & \\ \text{Simplify:} & y=15\Rightarrow \underline{\underline{y=15}} & y-\text{intercept:}(0,15) \end{array}$$

Since $a<0$, the parabola **turns up.**

Graph all the points and connect them with a smooth curve:



Practice

Rewrite each quadratic function in vertex form.

1. $y = x^2 - 6x$
2. $y + 1 = -2x^2 - x$
3. $y = 9x^2 + 3x - 10$
4. $y = -32x^2 + 60x + 10$

For each parabola, find the vertex; the $x-$ and $y-$intercepts; and if it turns up or down. Then graph the parabola.

5. $y - 4 = x^2 + 8x$
6. $y = -4x^2 + 20x - 24$
7. $y = 3x^2 + 15x$
8. $y + 6 = -x^2 + x$
9. $x^2 - 10x + 25 = 9$
10. $x^2 + 18x + 81 = 1$
11. $4x^2 - 12x + 9 = 16$
12. $x^2 + 14x + 49 = 3$
13. $4x^2 - 20x + 25 = 9$
14. $x^2 + 8x + 16 = 25$

10.8 Quadratic Formula

Here you'll learn how to use the quadratic formula to find the vertex and solution of quadratic equations.

What if you had a quadratic equation like x^2+5x+2 that you could not easily factor? How could you use its coeffient values to solve it? After completing this Concept, you'll be able to use the quadratic formula to solve equations like this one.

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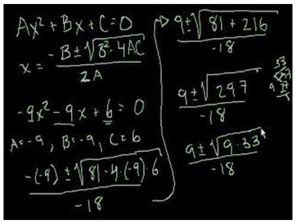
CK-12 Foundation: 1008S The Quadratic Formula

For more examples of solving quadratic equations using the quadratic formula, see the Khan Academy video at



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Guidance

The **Quadratic Formula** is probably the most used method for solving quadratic equations. For a quadratic equation in standard form, $ax^2+bx+c=0$, the quadratic formula looks like this:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

This formula is derived by solving a general quadratic equation using the method of completing the square that you learned in the previous section.

We start with a general quadratic equation: $ax^2+bx+c=0$

Subtract the constant term from both sides: $ax^2+bx=-c$

Divide by the coefficient of the x^2 term:

$x^2+\frac{b}{a}x=-\frac{c}{a}$

Rewrite:

$x^2+2\left(\frac{b}{2a}\right)x=-\frac{c}{a}$

Add the constant $\left(\frac{b}{2a}\right)^2$ to both sides:

$x^2+2\left(\frac{b}{2a}\right)x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$

Factor the perfect square trinomial:

$\left(x+\frac{b}{2a}\right)^2=-\frac{4ac}{4a^2}+\frac{b^2}{4a^2}$

Simplify:

$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$

Take the square root of both sides:

$x+\frac{b}{2a}=\sqrt{\frac{b^2-4ac}{4a^2}}$ and $x+\frac{b}{2a}=-\sqrt{\frac{b^2-4ac}{4a^2}}$

Simplify:

$x+\frac{b}{2a}=\frac{\sqrt{b^2-4ac}}{2a}$ and $x+\frac{b}{2a}=-\frac{\sqrt{b^2-4ac}}{2a}$

$x=-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}$ and $x=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}$

$x=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x=\frac{-b-\sqrt{b^2-4ac}}{2a}$

This can be written more compactly as $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

You can see that the familiar formula comes directly from applying the method of completing the square. Applying the method of completing the square to solve quadratic equations can be tedious, so the quadratic formula is a more straightforward way of finding the solutions.

Solve Quadratic Equations Using the Quadratic Formula

To use the quadratic formula, just plug in the values of a, b, and c.

Example A

Solve the following quadratic equations using the quadratic formula.

a) $2x^2+3x+1=0$

b) $x^2-6x+5=0$

c) $-4x^2+x+1=0$

Solution

Start with the quadratic formula and plug in the values of a, b and c.

a)

$$\begin{aligned}
&\text{Quadratic formula:} && x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
&\text{Plug in the values } a=2,\ b=3,\ c=1 && x=\frac{-3\pm\sqrt{(3)^2-4(2)(1)}}{2(2)}\\
&\text{Simplify:} && x=\frac{-3\pm\sqrt{9-8}}{4}=\frac{-3\pm\sqrt{1}}{4}\\
&\text{Separate the two options:} && x=\frac{-3+1}{4}\ \text{ and }\ x=\frac{-3-1}{4}\\
&\text{Solve:} && x=\frac{-2}{4}=-\frac{1}{2}\text{ and } x=\frac{-4}{4}=-1
\end{aligned}$$

Answer: $x=-\frac{1}{2}$ and $x=-1$

b)

$$\begin{aligned}
&\text{Quadratic formula:} && x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
&\text{Plug in the values } a=1,\ b=-6,\ c=5 && x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(5)}}{2(1)}\\
&\text{Simplify:} && x=\frac{6\pm\sqrt{36-20}}{2}=\frac{6\pm\sqrt{16}}{2}\\
&\text{Separate the two options:} && x=\frac{6+4}{2}\text{ and } x=\frac{6-4}{2}\\
&\text{Solve:} && x=\frac{10}{2}=5\text{ and } x=\frac{2}{2}=1
\end{aligned}$$

Answer: $x=5$ and $x=1$

c)

$$\begin{aligned}
&\text{Quadratic formula:} && x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
&\text{Plug in the values } a=-4,\ b=1,\ c=1 && x=\frac{-1\pm\sqrt{(1)^2-4(-4)(1)}}{2(-4)}\\
&\text{Simplify:} && x=\frac{-1\pm\sqrt{1+16}}{-8}=\frac{-1\pm\sqrt{17}}{-8}\\
&\text{Separate the two options:} && x=\frac{-1+\sqrt{17}}{-8}\text{ and } x=\frac{-1-\sqrt{17}}{-8}\\
&\text{Solve:} && x=-.39\text{ and } x=.64
\end{aligned}$$

Answer: $x=-.39$ and $x=.64$

Often when we plug the values of the coefficients into the quadratic formula, we end up with a negative number inside the square root. Since the square root of a negative number does not give real answers, we say that the equation has no real solutions. In more advanced math classes, you'll learn how to work with "complex" (or "imaginary") solutions to quadratic equations.

Example B

Use the quadratic formula to solve the equation $x^2+2x+7=0$.

Solution

$$\begin{aligned}
\text{Quadratic formula:} \quad & x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
\text{Plug in the values } a=1,\ b=2,\ c=7 \quad & x=\frac{-2\pm\sqrt{(2)^2-4(1)(7)}}{2(1)}\\
\text{Simplify:} \quad & x=\frac{-2\pm\sqrt{4-28}}{2}=\frac{-2\pm\sqrt{-24}}{2}
\end{aligned}$$

Answer: There are no real solutions.

To apply the quadratic formula, we must make sure that the equation is written in standard form. For some problems, that means we have to start by rewriting the equation.

Finding the Vertex of a Parabola with the Quadratic Formula

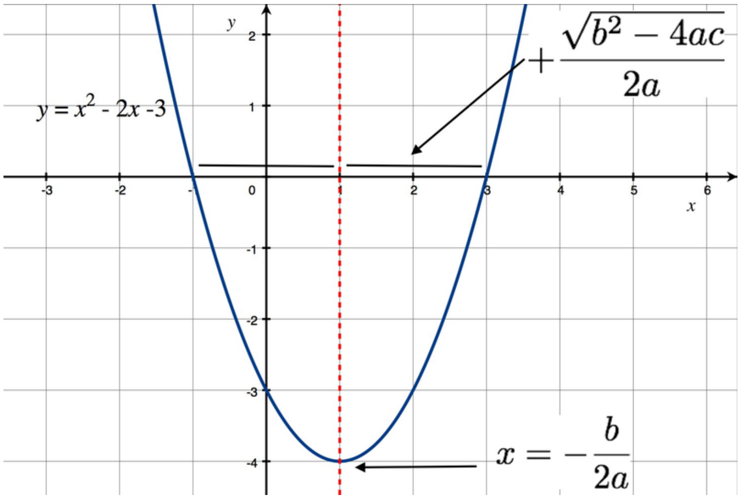
Sometimes a formula gives you even more information than you were looking for. For example, the quadratic formula also gives us an easy way to locate the vertex of a parabola.

Remember that the quadratic formula tells us the **roots** or **solutions** of the equation $ax^2+bx+c=0$. Those roots are $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$,, and we can rewrite that as $x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$.

Recall that the roots are **symmetric** about the vertex. In the form above, we can see that the roots of a quadratic equation are symmetric around the $x-$coordinate $\frac{-b}{2a}$, because they are $\frac{\sqrt{b^2-4ac}}{2a}$ units to the left and right (recall the ± sign) from the vertical line $x=\frac{-b}{2a}$.

Example C

In the equation $x^2-2x-3=0$, the roots -1 and 3 are both 2 units from the vertical line $x=1$, as you can see in the graph below:



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CK-12 Foundation: 1008 The QuadraticFormula

Vocabulary

- For a **quadratic equation** in standard form, $ax^2+bx+c=0$, the **quadratic formula** looks like this:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

- The quadratic formula tells us the **roots** or **solutions** of the equation $ax^2+bx+c=0$. Those roots are $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, and we can rewrite that as $x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$.
- The roots are symmetric about the **vertex**. In the form above, we can see that the roots of a quadratic equation are symmetric around the $x-$coordinate $\frac{-b}{2a}$, because they are $\frac{\sqrt{b^2-4ac}}{2a}$ units to the left and right (recall the $\pm$ sign) from the vertical line $x=\frac{-b}{2a}$.

Guided Practice

Solve the following equations using the quadratic formula.

a) $x^2-6x=10$

b) $-8x^2=5x+6$

Solution

a)

Re-write the equation in standard form: $x^2-6x-10=0$

Quadratic formula: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Plug in the values $a=1,\ b=-6,\ c=-10$ $\quad x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-10)}}{2(1)}$

Simplify: $x=\frac{6\pm\sqrt{36+40}}{2}=\frac{6\pm\sqrt{76}}{2}$

Separate the two options: $x=\frac{6+\sqrt{76}}{2}$ and $x=\frac{6-\sqrt{76}}{2}$

Solve: $x=7.36$ and $x=-1.36$

Answer: $x=7.36$ and $x=-1.36$

b)

Re-write the equation in standard form: $8x^2+5x+6=0$

Quadratic formula: $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Plug in the values $a=8,\ b=5,\ c=6$ $$x=\frac{-5\pm\sqrt{(5)^2-4(8)(6)}}{2(8)}$$

Simplify: $$x=\frac{-5\pm\sqrt{25-192}}{16}=\frac{-5\pm\sqrt{-167}}{16}$$

Answer: no real solutions

Practice

Solve the following quadratic equations using the quadratic formula.

1. $x^2+4x-21=0$
2. $x^2-6x=12$
3. $3x^2-\frac{1}{2}x=\frac{3}{8}$
4. $2x^2+x-3=0$
5. $-x^2-7x+12=0$
6. $-3x^2+5x=2$
7. $4x^2=x$
8. $x^2+2x+6=0$
9. $5x^2-2x+100=0$
10. $100x^2+10x+70=0$

10.9 Comparing Methods for Solving Quadratics

Here you'll learn how to choose the best method (factoring, taking the square root, using the quadratic formula, or completing the square) to solve real-world applications involving quadratic equations.

What if you and a friend started running from the same spot. You ran west and your friend ran south. After two hours, you had run 10 miles. The distance at that point between you and your friend was three times the distance your friend ran plus 2 miles? How could you determine how many miles your friend ran? After completing this Concept, you'll be able to solve real-world quadratic equation problems like this one.

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CK-12 Foundation: 1009S Solving Real-World Problems Using Quadratic Equations

Guidance

In mathematics, you'll need to solve quadratic equations that describe application problems or that are part of more complicated problems. You've learned four ways of solving a quadratic equation:

- Factoring
- Taking the square root
- Quadratic formula
- Completing the square

Usually you'll have to decide for yourself which method to use. However, here are some guidelines as to which methods are better in different situations.

Factoring is always best if the quadratic expression is easily factorable. It is always worthwhile to check if you can factor because this is the fastest method. Many expressions are not factorable so this method is not used very often in practice.

Taking the square root is best used when there is no $x-$term in the equation.

Quadratic formula is the method that is used most often for solving a quadratic equation. When solving directly by taking square root and factoring does not work, this is the method that most people prefer to use.

Completing the square can be used to solve any quadratic equation. This is usually not any better than using the quadratic formula (in terms of difficult computations), but it is very useful if you need to rewrite a quadratic function in vertex form. It's also used to rewrite the equations of circles, ellipses and hyperbolas in standard form (something you'll do in algebra II, trigonometry, physics, calculus, and beyond).

If you are using factoring or the quadratic formula, make sure that the equation is in standard form.

Example A

Solve each quadratic equation.

a) $x^2-4x-5=0$

b) $x^2=8$

c) $-4x^2+x=2$

d) $25x^2-9=0$

e) $3x^2=8x$

Solution

a) This expression if easily factorable so we can factor and apply the zero-product property:

$$\begin{array}{ll} \text{Factor:} & (x-5)(x+1)=0 \\ \text{Apply zero-product property:} & x-5=0 \quad \text{and} \quad x+1=0 \\ \text{Solve:} & x=5 \qquad \text{and} \quad x=-1 \end{array}$$

Answer: $x=5$ and $x=-1$

b) Since the expression is missing the x term we can take the square root:

Take the square root of both sides: $x=\sqrt{8}$ and $x=-\sqrt{8}$

Answer: $x=2.83$ and $x=-2.83$

c) Re-write the equation in standard form: $-4x^2+x-2=0$

It is not apparent right away if the expression is factorable so we will use the quadratic formula:

$$\begin{array}{rl} \text{Quadratic formula:} & x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a} \\ \text{Plug in the values } a=-4,\ b=1,\ c=-2: & x=\dfrac{-1\pm\sqrt{1^2-4(-4)(-2)}}{2(-4)} \\ \text{Simplify:} & x=\dfrac{-1\pm\sqrt{1-32}}{-8}=\dfrac{-1\pm\sqrt{-31}}{-8} \end{array}$$

Answer: no real solution

d) This problem can be solved easily either with factoring or taking the square root. Let's take the square root in this case:

$$\begin{array}{rl} \text{Add 9 to both sides of the equation:} & 25x^2=9 \\ \text{Divide both sides by 25}: & x^2=\dfrac{9}{25} \\ \text{Take the square root of both sides:} & x=\sqrt{\dfrac{9}{25}} \text{ and } x=-\sqrt{\dfrac{9}{25}} \\ \text{Simplify:} & x=\dfrac{3}{5} \text{ and } x=-\dfrac{3}{5} \end{array}$$

Answer: $x=\frac{3}{5}$ and $x=-\frac{3}{5}$

e)

$$\begin{aligned}
\text{Re-write the equation in standard form:} &\quad 3x^2-8x=0\\
\text{Factor out common } x \text{ term:} &\quad x(3x-8)=0\\
\text{Set both terms to zero:} &\quad x=0 \text{ and } 3x=8\\
\text{Solve:} &\quad x=0 \text{ and } x=\frac{8}{3}=2.67
\end{aligned}$$

Answer: $x=0$ and $x=2.67$

Solving Real-World Problems by Completing the Square

In the last section you learned that an object that is dropped falls under the influence of gravity. The equation for its height with respect to time is given by $y=\frac{1}{2}gt^2+y_0$, where y_0 represents the initial height of the object and g is the coefficient of gravity on earth, which equals $-9.8\ m/s^2$ or $-32\ ft/s^2$.

On the other hand, if an object is thrown straight up or straight down in the air, it has an initial vertical velocity. This term is usually represented by the notation v_{0y}. Its value is positive if the object is thrown up in the air and is negative if the object is thrown down. The equation for the height of the object in this case is

$$y=\frac{1}{2}gt^2+v_{0y}t+y_0$$

Plugging in the appropriate value for g turns this equation into

$y=-4.9t^2+v_{0y}t+y_0$ if you wish to have the height in meters

$y=-16t^2+v_{0y}t+y_0$ if you wish to have the height in feet

Example B

An arrow is shot straight up from a height of 2 meters with a velocity of 50 m/s.

a) *How high will the arrow be 4 seconds after being shot? After 8 seconds?*

b) *At what time will the arrow hit the ground again?*

c) *What is the maximum height that the arrow will reach and at what time will that happen?*

Solution

Since we are given the velocity in m/s, use: $y=-4.9t^2+v_{0y}t+y_0$

We know $v_{0y}=50\ m/s$ and $y_0=2$ meters so: $y=-4.9t^2+50t+2$

a) To find how high the arrow will be 4 seconds after being shot we plug in $t=4$:

$$\begin{aligned}
y&=-4.9(4)^2+50(4)+2\\
&=-4.9(16)+200+2=\underline{\underline{123.6\ meters}}
\end{aligned}$$

we plug in $t=8$:

$$\begin{aligned}
y&=-4.9(8)^2+50(8)+2\\
&=-4.9(64)+400+2=\underline{\underline{88.4\ meters}}
\end{aligned}$$

b) The height of the arrow on the ground is $y = 0$, so: $0 = -4.9t^2 + 50t + 2$

Solve for t by completing the square:

$$\begin{aligned}
-4.9t^2 + 50t &= -2 \\
-4.9(t^2 - 10.2t) &= -2 \\
t^2 - 10.2t &= 0.41 \\
t^2 - 2(5.1)t + (5.1)^2 &= 0.41 + (5.1)^2 \\
(t - 5.1)^2 &= 26.43 \\
t - 5.1 &= 5.14 \text{ and } t - 5.1 = -5.14 \\
t &= \underline{\underline{10.2}}\ sec \text{ and } t = -0.04\ sec
\end{aligned}$$

The arrow will hit the ground about 10.2 seconds after it is shot.

c) If we graph the height of the arrow with respect to time we would get an upside down parabola $(a < 0)$. The maximum height and the time when this occurs is really the vertex of this parabola: (t, h).

We re-write the equation in vertex form:
$$\begin{aligned}
y &= -4.9t^2 + 50t + 2 \\
y - 2 &= -4.9t^2 + 50t \\
y - 2 &= -4.9(t^2 - 10.2t)
\end{aligned}$$

Complete the square:
$$\begin{aligned}
y - 2 - 4.9(5.1)^2 &= -4.9\left(t^2 - 10.2t + (5.1)^2\right) \\
y - 129.45 &= -4.9(t - 5.1)^2
\end{aligned}$$

The vertex is at (5.1, 129.45). In other words, **when** $t = 5.1$ *seconds*, **the height is** $y = 129$ *meters*.

Another type of application problem that can be solved using quadratic equations is one where two objects are moving away from each other in perpendicular directions. Here is an example of this type of problem.

Example C

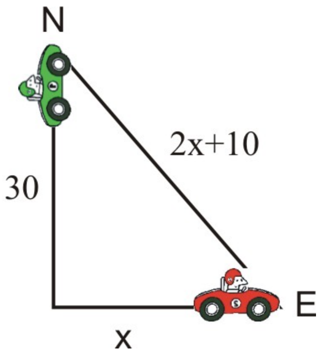
Two cars leave an intersection. One car travels north; the other travels east. When the car traveling north had gone 30 miles, the distance between the cars was 10 miles more than twice the distance traveled by the car heading east. Find the distance between the cars at that time.

Solution

Let $x =$ the distance traveled by the car heading east

Then $2x + 10 =$ the distance between the two cars

Let's make a sketch:



We can use the Pythagorean Theorem to find an equation for x:

$$x^2+30^2=(2x+10)^2$$

Expand parentheses and simplify:

$$\begin{aligned}x^2+900&=4x^2+40x+100\\800&=3x^2+40x\end{aligned}$$

Solve by completing the square:

$$\begin{aligned}\frac{800}{3}&=x^2+\frac{40}{3}x\\\frac{800}{3}+\left(\frac{20}{3}\right)^2&=x^2+2\left(\frac{20}{3}\right)x+\left(\frac{20}{3}\right)^2\\\frac{2800}{9}&=\left(x+\frac{20}{3}\right)^2\\x+\frac{20}{3}&=17.6 \text{ and } x+\frac{20}{3}=-17.6\\x&=11 \text{ and } x=-24.3\end{aligned}$$

Since only positive distances make sense here, the distance between the two cars is: $2(11)+10=32$ *miles*

Solve Applications Using Quadratic Functions by any Method

Here is an application problem that arises from number relationships and geometry applications.

Example D

The product of two positive consecutive integers is 156. Find the integers.

Solution

Define: Let $x=$ the smaller integer

Then $x+1=$ the next integer

Translate: The product of the two numbers is 156. We can write the equation:

$$x(x+1) = 156$$

Solve:

$$\begin{aligned} x^2 + x &= 156 \\ x^2 + x - 156 &= 0 \end{aligned}$$

Apply the quadratic formula with: $a = 1,\ b = 1,\ c = -156$

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1^2 - 4(1)(-156)}}{2(1)} \\ x &= \frac{-1 \pm \sqrt{625}}{2} = \frac{-1 \pm 25}{2} \\ x &= \frac{-1+25}{2} \quad \text{and} \quad x = \frac{-1-25}{2} \\ x &= \frac{24}{2} = 12 \quad \text{and} \quad x = \frac{-26}{2} = -13 \end{aligned}$$

Since we are looking for positive integers, we want $x = 12$. So the numbers are **12 and 13.**

Check: $12 \times 13 = 156$. The answer checks out.

Example E

Suzie wants to build a garden that has three separate rectangular sections. She wants to fence around the whole garden and between each section as shown. The plot is twice as long as it is wide and the total area is $200 ft^2$. *How much fencing does Suzie need?*

Solution

Define: Let $x =$ the width of the plot

Then $2x =$ the length of the plot

Translate: area of a rectangle is $A =$ length × width, so

$$x(2x) = 200$$

Solve: $2x^2 = 200$

Solve by taking the square root:

$$\begin{aligned} x^2 &= 100 \\ x &= \sqrt{100} \text{ and } x = -\sqrt{100} \\ x &= 10 \text{ and } x = -10 \end{aligned}$$

We take $x = 10$ since only positive dimensions make sense.

The plot of land is $10 \ feet \times 20 \ feet$.

To fence the garden the way Suzie wants, we need 2 lengths and 4 widths $= 2(20)+4(10) = 80$**feet of fence.**

Check: $10 \times 20 = 200 \ ft^2$ and $2(20)+4(10) = 80 \ feet$. The answer checks out.

Watch this video for help with the Examples above.

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CK-12 Foundation: 1009 Solving Real-World Problems UsingQuadratic Equations

Vocabulary

- For a **quadratic equation** in standard form, $ax^2+bx+c=0$, the **quadratic formula** looks like this:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

- A ***perfect square trinomial*** has the form $a^2+2(ab)+b^2$, which factors into $(a+b)^2$.

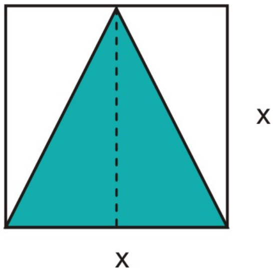
Guided Practice

1. *An isosceles triangle is enclosed in a square so that its base coincides with one of the sides of the square and the tip of the triangle touches the opposite side of the square. If the area of the triangle is* 20 in^2 *what is the length of one side of the square?*

2. *The length of a rectangular pool is 10 meters more than its width. The area of the pool is 875 square meters. Find the dimensions of the pool.*

Solutions:

1. **Draw a sketch:**



Define: Let $x =$ base of the triangle

Then $x =$ height of the triangle

Translate: Area of a triangle is $\frac{1}{2} \times base \times height$, so $\frac{1}{2} \cdot x \cdot x = 20$

Solve: $\frac{1}{2}x^2 = 20$

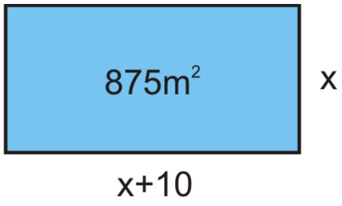
Solve by taking the square root:

$$\begin{aligned}x^2 &= 40\\ x &= \sqrt{40} \text{ and } x = -\sqrt{40}\\ x &= 6.32 \text{ and } x = -6.32\end{aligned}$$

The side of the square is **6.32 inches.** That means the area of the square is $(6.32)^2 = 40\ in^2$, twice as big as the area of the triangle.

Check: It makes sense that the area of the square will be twice that of the triangle. If you look at the figure you can see that you could fit two triangles inside the square.

2. **Draw a sketch:**



Define: Let $x =$ the width of the pool

Then $x+10 =$ the length of the pool

Translate: The area of a rectangle is $A =$ length × width, so we have $x(x+10) = 875$.

Solve:

$$\begin{aligned}x^2+10x &= 875\\ x^2+10x-875 &= 0\end{aligned}$$

Apply the quadratic formula with $a = 1,\ b = 10$ and $c = -875$

$$\begin{aligned}x &= \frac{-10 \pm \sqrt{(10)^2-4(1)(-875)}}{2(1)}\\ x &= \frac{-10 \pm \sqrt{100+3500}}{2}\\ x &= \frac{-10 \pm \sqrt{3600}}{2} = \frac{-10 \pm 60}{2}\\ x &= \frac{-10+60}{2} \text{ and } x = \frac{-10-60}{2}\\ x &= \frac{50}{2} = 25 \text{ and } x = \frac{-70}{2} = -35\end{aligned}$$

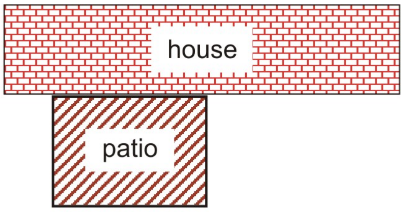
Since the dimensions of the pool should be positive, we want $x = 25$ *meters*. So the pool is 25 *meters* × 35 *meters*.

Check: $25 \times 35 = 875\ m^2$. The answer checks out.

Practice

Solve the following quadratic equations using the method of your choice.

1. $x^2 - x = 6$
2. $x^2 - 12 = 0$
3. $-2x^2 + 5x - 3 = 0$
4. $x^2 + 7x - 18 = 0$
5. $3x^2 + 6x = -10$
6. $-4x^2 + 4000x = 0$
7. $-3x^2 + 12x + 1 = 0$
8. $x^2 + 6x + 9 = 0$
9. $81x^2 + 1 = 0$
10. $-4x^2 + 4x = 9$
11. $36x^2 - 21 = 0$
12. $x^2 - 2x - 3 = 0$
13. The product of two consecutive integers is 72. Find the two numbers.
14. The product of two consecutive odd integers is 1 less than 3 times their sum. Find the integers.
15. The length of a rectangle exceeds its width by 3 inches. The area of the rectangle is 70 square inches, find its dimensions.
16. Angel wants to cut off a square piece from the corner of a rectangular piece of plywood. The larger piece of wood is 4 $feet \times 8$ $feet$ and the cut off part is $\frac{1}{3}$ of the total area of the plywood sheet. What is the length of the side of the square?
17. Mike wants to fence three sides of a rectangular patio that is adjacent the back of his house. The area of the patio is 192 ft^2 and the length is 4 feet longer than the width.



Find how much fencing Mike will need.

18. Sam throws an egg straight down from a height of 25 feet. The initial velocity of the egg is 16 ft/sec. How long does it take the egg to reach the ground?
19. Amanda and Dolvin leave their house at the same time. Amanda walks south and Dolvin bikes east. Half an hour later they are 5.5 miles away from each other and Dolvin has covered three miles more than the distance that Amanda covered. How far did Amanda walk and how far did Dolvin bike?

10.10 Solutions Using the Discriminant

Here you'll learn how to find the discriminant of quadratic equations and you'll use it to tell the nature of the equation's solutions. You'll then solve real-world applications by using quadratic functions and interpreting their discriminant.

What if you were given a quadratic equation like $x^2-3x+1=0$? How could you determine how many real solutions it had without actually solving it? After completing this Concept, you'll be able to find and interpret the discriminant of a quadratic equation like this one.

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CK-12 Foundation: 1010S The Discriminant

Guidance

In the quadratic formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, the expression inside the square root is called the **discriminant.** The discriminant can be used to analyze the types of solutions to a quadratic equation without actually solving the equation. Here's how:

- If $b^2-4ac>0$, the equation has two separate real solutions.
- If $b^2-4ac<0$, the equation has only non-real solutions.
- If $b^2-4ac=0$, the equation has one real solution, a **double root.**

Find the Discriminant of a Quadratic Equation

To find the discriminant of a quadratic equation we calculate $D=b^2-4ac$.

Example A

Find the discriminant of each quadratic equation. Then tell how many solutions there will be to the quadratic equation without solving.

a) $x^2-5x+3=0$

b) $4x^2-4x+1=0$

c) $-2x^2+x=4$

Solution

a) Plug $a=1,\ b=-5$ and $c=3$ into the discriminant formula: $D=(-5)^2-4(1)(3)=13D>0$, so there are **two real solutions.**

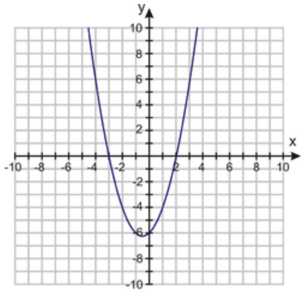
b) Plug $a=4,\ b=-4$ and $c=1$ into the discriminant formula: $D=(-4)^2-4(4)(1)=0D=0$, so there is **one real solution.**

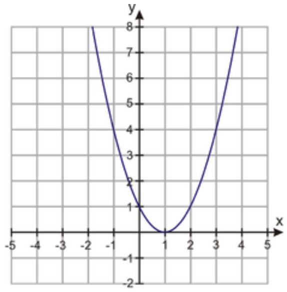
c) Rewrite the equation in standard form: $-2x^2+x-4=0$

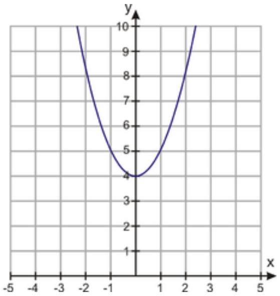
Plug $a=-2,\ b=1$ and $c=-4$ into the discriminant formula: $D=(1)^2-4(-2)(-4)=-31D<0$, so there are **no real solutions.**

Interpret the Discriminant of a Quadratic Equation

The sign of the discriminant tells us the nature of the solutions (or roots) of a quadratic equation. We can obtain two distinct real solutions if $D>0$, two non-real solutions if $D<0$ or one solution (called a double root) if $D=0$. Recall that the number of solutions of a quadratic equation tells us how many times its graph crosses the $x-$axis. If $D>0$, the graph crosses the $x-$axis in two places; if $D=0$ it crosses in one place; if $D<0$ it doesn't cross at all:







Example B

Determine the nature of the solutions of each quadratic equation.

a) $4x^2-1=0$

b) $10x^2-3x=-4$

c) $x^2-10x+25=0$

Solution

Use the value of the discriminant to determine the nature of the solutions to the quadratic equation.

a) Plug $a=4,\ b=0$ and $c=-1$ into the discriminant formula: $D=(0)^2-4(4)(-1)=16$

The discriminant is positive, so the equation has **two distinct real solutions.**

The solutions to the equation are: $\frac{0\pm\sqrt{16}}{8}=\pm\frac{4}{8}=\pm\frac{1}{2}$

b) Re-write the equation in standard form: $10x^2-3x+4=0$

Plug $a=10,\ b=-3$ and $c=4$ into the discriminant formula: $D=(-3)^2-4(10)(4)=-151$

The discriminant is negative, so the equation has **two non-real solutions.**

c) Plug $a=1,\ b=-10$ and $c=25$ into the discriminant formula: $D=(-10)^2-4(1)(25)=0$

The discriminant is 0, so the equation has a **double root.**

The solution to the equation is: $\frac{10\pm\sqrt{0}}{2}=\frac{10}{2}=5$

If the discriminant is a perfect square, then the solutions to the equation are not only real, but also rational. If the discriminant is positive but not a perfect square, then the solutions to the equation are real but irrational.

Example C

Determine the nature of the solutions to each quadratic equation.

a) $2x^2+x-3=0$

b) $5x^2-x-1=0$

Solution

Use the discriminant to determine the nature of the solutions.

a) Plug $a=2,\ b=1$ and $c=-3$ into the discriminant formula: $D=(1)^2-4(2)(-3)=25$

The discriminant is a positive perfect square, so the solutions are **two real rational numbers.**

The solutions to the equation are: $\frac{-1\pm\sqrt{25}}{4}=\frac{-1\pm5}{4}$, so $x=1$ and $x=-\frac{3}{2}$.

b) Plug $a=5,\ b=-1$ and $c=-1$ into the discriminant formula: $D=(-1)^2-4(5)(-1)=21$

The discriminant is positive but not a perfect square, so the solutions are **two real irrational numbers.**

The solutions to the equation are: $\frac{1\pm\sqrt{21}}{10}$, so $x\approx0.56$ and $x\approx-0.36$.

Solve Real-World Problems Using Quadratic Functions and Interpreting the Discriminant

You've seen that calculating the discriminant shows what types of solutions a quadratic equation possesses. Knowing the types of solutions is very useful in applied problems. Consider the following situation.

Example D

Marcus kicks a football in order to score a field goal. The height of the ball is given by the equation $y=-\frac{32}{6400}x^2+x$. *If the goalpost is 10 feet high, can Marcus kick the ball high enough to go over the goalpost? What is the farthest distance that Marcus can kick the ball from and still make it over the goalpost?*

Solution

Define: Let $y=$ height of the ball in feet.

Let $x=$ distance from the ball to the goalpost.

Translate: We want to know if it is possible for the height of the ball to equal 10 feet at some real distance from the goalpost.

Solve:

$$\begin{aligned}&\text{Write the equation in standard form:} && -\frac{32}{6400}x^2+x-10=0\\&\text{Simplify:} && -0.005x^2+x-10=0\\&\text{Find the discriminant:} && D=(1)^2-4(-0.005)(-10)=0.8\end{aligned}$$

Since the discriminant is positive, we know that it is possible for the ball to go over the goal post, if Marcus kicks it from an acceptable distance x from the goalpost.

To find the value of x that will work, we need to use the quadratic formula:

$$x=\frac{-1\pm\sqrt{0.8}}{-0.01}=189.4\ feet \text{ or } 10.56\ feet$$

What does this answer mean? It means that if Marcus is exactly 189.4 feet or exactly 10.56 feet from the goalposts, the ball will just barely go over them. Are these the only distances that will work? No; those are just the distances at which the ball will be exactly 10 feet high, but *between* those two distances, the ball will go even higher than that. (It travels in a downward-opening parabola from the place where it is kicked to the spot where it hits the ground.) This means that Marcus will make the goal if he is anywhere **between 10.56 and 189.4 feet from the goalposts.**

Watch this video for help with the Examples above.

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CK-12 Foundation: 1010 The Discriminant

Vocabulary

In the quadratic formula, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, the expression inside the square root is called the **discriminant.** The discriminant can be used to analyze the types of solutions to a quadratic equation without actually solving the equation. Here's how:

- If $b^2 - 4ac > 0$, the equation has two separate real solutions.
- If $b^2 - 4ac < 0$, the equation has only non-real solutions.
- If $b^2 - 4ac = 0$, the equation has one real solution, a **double root.**

Guided Practice

Emma and Bradon own a factory that produces bike helmets. Their accountant says that their profit per year is given by the function $P = -0.003x^2 + 12x + 27760$, *where x is the number of helmets produced. Their goal is to make a profit of $40,000 this year. Is this possible?*

Solution

We want to know if it is possible for the profit to equal $40,000.

$$40000 = -0.003x^2 + 12x + 27760$$

Write the equation in standard form: $-0.003x^2 + 12x - 12240 = 0$

Find the discriminant: $D = (12)^2 - 4(-0.003)(-12240) = -2.88$

Since the discriminant is negative, we know that **it is not possible** for Emma and Bradon to make a profit of $40,000 this year no matter how many helmets they make.

Practice

Find the discriminant of each quadratic equation.

1. $2x^2 - 4x + 5 = 0$

2. $x^2 - 5x = 8$
3. $4x^2 - 12x + 9 = 0$
4. $x^2 + 3x + 2 = 0$
5. $x^2 - 16x = 32$
6. $-5x^2 + 5x - 6 = 0$
7. $x^2 + 4x = 2$
8. $-3x^2 + 2x + 5 = 0$

Determine the nature of the solutions of each quadratic equation.

9. $-x^2 + 3x - 6 = 0$
10. $5x^2 = 6x$
11. $41x^2 - 31x - 52 = 0$
12. $x^2 - 8x + 16 = 0$
13. $-x^2 + 3x - 10 = 0$
14. $x^2 - 64 = 0$
15. $3x^2 = 7$
16. $x^2 + 30 + 225 = 0$

Without solving the equation, determine whether the solutions will be rational or irrational.

17. $x^2 = -4x + 20$
18. $x^2 + 2x - 3 = 0$
19. $3x^2 - 11x = 10$
20. $\frac{1}{2}x^2 + 2x + \frac{2}{3} = 0$
21. $x^2 - 10x + 25 = 0$
22. $x^2 = 5x$
23. $2x^2 - 5x = 12$
24. Marty is outside his apartment building. He needs to give his roommate Yolanda her cell phone but he does not have time to run upstairs to the third floor to give it to her. He throws it straight up with a vertical velocity of 55 feet/second. Will the phone reach her if she is 36 feet up? (Hint: the equation for the height is $y = -32t^2 + 55t + 4$.)
25. Bryson owns a business that manufactures and sells tires. The revenue from selling the tires in the month of July is given by the function $R = x(200 - 0.4x)$ where x is the number of tires sold. Can Bryson's business generate revenue of $20,000 in the month of July?

10.11 Linear, Exponential, and Quadratic Models

Here you'll learn how to identify a function's type by examining the difference or the ratio of different values of the dependent variable.

What if you were given a table of x and y values? How could you determine if those values represented a linear function, an exponetial function, or a quadratic function? After completing this Concept, you'll be able to identify functions using differences and ratios between their values.

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CK-12 Foundation: 1011S Linear, Exponential and Quadratic Models

Guidance

In this course we've learned about three types of functions, linear, quadratic and exponential.

- Linear functions take the form $y = mx + b$.
- Quadratic functions take the form $y = ax^2 + bx + c$.
- Exponential functions take the form $y = a \cdot b^x$.

In real-world applications, the function that describes some physical situation is not given; it has to be found before the problem can be solved. For example, scientific data such as observations of planetary motion are often collected as a set of measurements given in a table. Part of the scientist's job is to figure out which function best fits the data. In this section, you'll learn some methods that are used to identify which function describes the relationship between the variables in a problem.

Identify Functions Using Differences or Ratios

One method for identifying functions is to look at the difference or the ratio of different values of the dependent variable. For example, **if the difference between values of the dependent variable is the same each time we change the independent variable by the same amount, then the function is *linear*.**

Example A

Determine if the function represented by the following table of values is linear.

TABLE 10.15:

x	y
-2	-4

TABLE 10.15: (continued)

x	y
–1	–1
0	2
1	5
2	8

If we take the difference between consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value always increases by 3.

Since the difference is always the same, **the function is linear.**

When we look at the difference of the $y-$values, we have to make sure that we examine entries for which the $x-$values increase by the same amount.

For example, examine the values in this table:

TABLE 10.16:

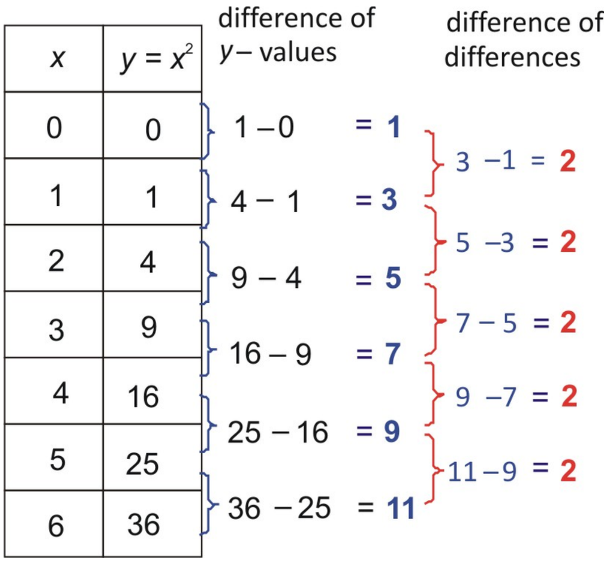
x	y
0	5
1	10
3	20
4	25
6	35

At first glance this function might not look linear, because the difference in the $y-$values is not always the same. But if we look closer, we can see that when the $y-$value increases by 10 instead of 5, it's because the $x-$value increased by 2 instead of 1. Whenever the $x-$value increases by the *same* amount, the $y-$value does too, so the function is linear.

Another way to think of this is in mathematical notation. We can say that a function is linear if $\frac{y_2-y_1}{x_2-x_1}$ is always the same for any two pairs of $x-$ and $y-$values. Notice that the expression we used here is simply the definition of the slope of a line.

Differences can also be used to identify quadratic functions. **For a quadratic function, when we increase the $x-$values by the same amount, the difference between $y-$values will not be the same. However, the difference of the *differences* of the $y-$values will be the same.**

Here are some examples of quadratic relationships represented by tables of values:



x	$y = x^2$
0	0
1	1
2	4
3	9
4	16
5	25
6	36

difference of $y-$ values

$1 - 0 = 1$

$4 - 1 = 3$

$9 - 4 = 5$

$16 - 9 = 7$

$25 - 16 = 9$

$36 - 25 = 11$

difference of differences

$3 - 1 = 2$

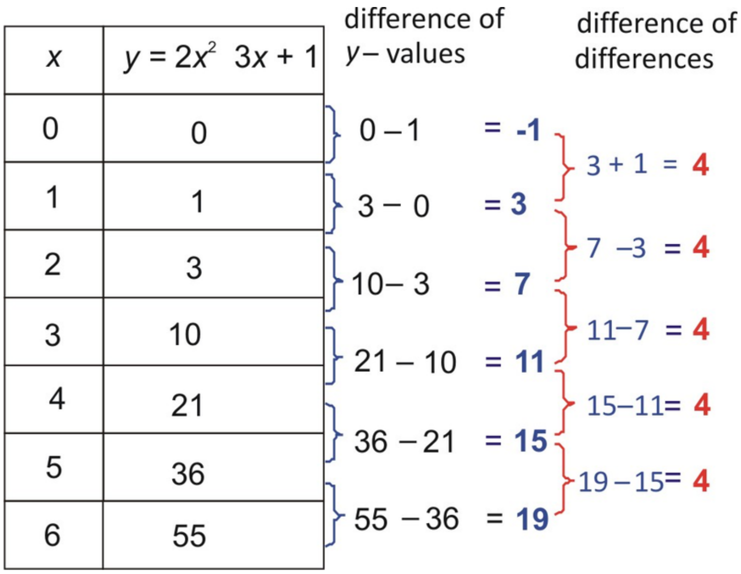
$5 - 3 = 2$

$7 - 5 = 2$

$9 - 7 = 2$

$11 - 9 = 2$

In this quadratic function, $y = x^2$, when we increase the $x-$value by one, the value of y increases by different values. However, it increases at a constant rate, so the difference of the difference is always 2.



x	$y = 2x^2\ 3x + 1$
0	0
1	1
2	3
3	10
4	21
5	36
6	55

difference of $y-$ values

$0 - 1 = -1$

$3 - 0 = 3$

$10 - 3 = 7$

$21 - 10 = 11$

$36 - 21 = 15$

$55 - 36 = 19$

difference of differences

$3 + 1 = 4$

$7 - 3 = 4$

$11 - 7 = 4$

$15 - 11 = 4$

$19 - 15 = 4$

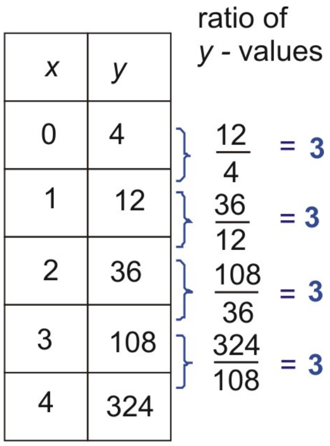
In this quadratic function, $y = 2x^2 - 3x + 1$, when we increase the $x-$value by one, the value of y increases by different values. However, the increase is constant: the difference of the difference is always 4.

To identify exponential functions, we use ratios instead of differences. **If the ratio between values of the dependent variable is the same each time we change the independent variable by the same amount, then the function is *exponential.***

Example B

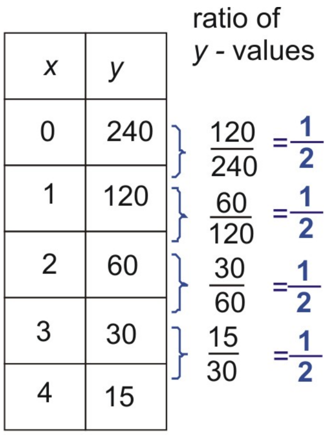
Determine if the function represented by each table of values is exponential.

a)



x	y
0	4
1	12
2	36
3	108
4	324

b)



x	y
0	240
1	120
2	60
3	30
4	15

a) If we take the ratio of consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value is multiplied by 3. Since the ratio is always the same, **the function is exponential.**

b) If we take the ratio of consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value is multiplied by $\frac{1}{2}$. Since the ratio is always the same, **the function is exponential.**

Write Equations for Functions

Once we identify which type of function fits the given values, we can write an equation for the function by starting with the general form for that type of function.

Example C

Determine what type of function represents the values in the following table.

TABLE 10.17:

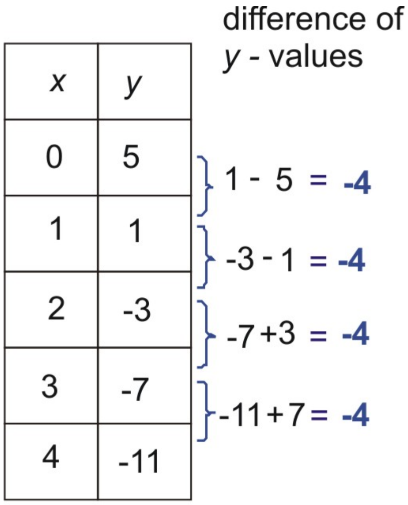
x	y
0	5
1	1
2	-3

TABLE 10.17: (continued)

x	y
3	-7
4	-11

Solution

Let's first check the difference of consecutive values of y.



If we take the difference between consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value always decreases by 4. Since the difference is always the same, **the function is linear.**

To find the equation for the function, we start with the general form of a linear function: $y = mx + b$. Since m is the slope of the line, it's also the quantity by which y increases every time the value of x increases by one. The constant b is the value of the function when $x = 0$. Therefore, the function is $y = -4x + 5$.

Example D

Determine what type of function represents the values in the following table.

TABLE 10.18:

x	y
0	0
1	5
2	20
3	45
4	80
5	125
6	180

Solution

Here, the difference between consecutive $y-$values isn't constant, so the function is not linear. Let's look at those differences more closely.

TABLE 10.19:

x	y	
0	0	
1	5	$5-0=5$
2	20	$20-5=15$
3	45	$45-20=25$
4	80	$80-45=35$
5	125	$125-80=45$
6	180	$180-125=55$

When the $x-$value increases by one, the difference between the $y-$values increases by 10 every time. Since the difference of the differences is constant, the function describing this set of values is **quadratic.**

To find the equation for the function that represents these values, we start with the general form of a quadratic function: $y = ax^2 + bx + c$.

We need to use the values in the table to find the values of the constants: a, b and c.

The value of c represents the value of the function when $x = 0$, so $c = 0$.

$$\begin{aligned}
\text{Plug in the point } (1,5): &\quad 5 = a+b\\
\text{Plug in the point } (2,20): &\quad 20 = 4a+2b \Rightarrow 10 = 2a+b\\
\text{To find } a \text{ and } b, \text{we solve the system of equations:} &\quad 5 = a+b\\
&\quad 10 = 2a+b\\
\text{Solve the first equation for } b: &\quad 5 = a+b \Rightarrow b = 5-a\\
\text{Plug the first equation into the second:} &\quad 10 = 2a+5-a\\
\text{Solve for } a \text{ and } b &\quad a = 5 \text{ and } b = 0
\end{aligned}$$

Therefore the equation of the quadratic function is $y = 5x^2$.

Watch this video for help with the Examples above.

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CK-12 Foundation: 1011 Linear, Exponential and QuadraticModels

Vocabulary

- If the differences of the $y-$values is always the same, **the function is linear.**
- If the difference of the *differences* of the $y-$values is always the same, **the function is quadratic.**
- If the ratio of the $y-$values is always the same, **the function is exponential.**

Guided Practice

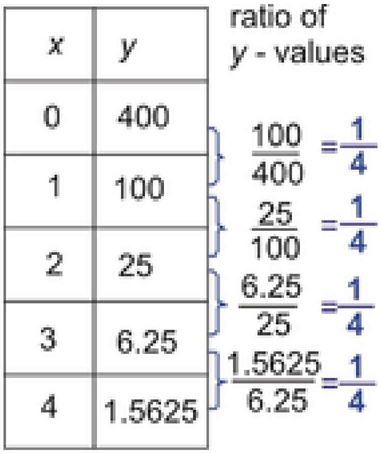
Determine what type of function represents the values in the following table.

TABLE 10.20:

x	y
0	400
1	500
2	25
3	6.25
4	1.5625

Solution

The differences between consecutive $y-$values aren't the same, and the differences between those differences aren't the same either. So let's check the ratios instead.



x	y
0	400
1	100
2	25
3	6.25
4	1.5625

ratio of y - values

$\frac{100}{400} = \frac{1}{4}$

$\frac{25}{100} = \frac{1}{4}$

$\frac{6.25}{25} = \frac{1}{4}$

$\frac{1.5625}{6.25} = \frac{1}{4}$

Each time the $x-$value increases by one, the $y-$value is multiplied by $\frac{1}{4}$. Since the ratio is always the same, **the function is exponential.**

To find the equation for the function that represents these values, we start with the general form of an exponential function, $y = a \cdot b^x$.

Here b is the ratio between the values of y each time x is increased by one. The constant a is the value of the function when $x = 0$. Therefore, the function is $y = 400\left(\frac{1}{4}\right)^x$.

Practice

Determine whether the data in the following tables can be represented by a linear function.

TABLE 10.21:

x	y
-4	10
-3	7
-2	4
-1	1

TABLE 10.21: (continued)

x	y
0	-2
1	-5

TABLE 10.22:

x	y
-2	4
-1	3
0	2
1	3
2	6
3	11

TABLE 10.23:

x	y
0	50
1	75
2	100
3	125
4	150
5	175

Determine whether the data in the following tables can be represented by a quadratic function.

TABLE 10.24:

x	y
-10	10
-5	2.5
0	0
5	2.5
10	10
15	22.5

TABLE 10.25:

x	y
1	4
2	6
3	6
4	4
5	0
6	-6

TABLE 10.26:

x	y
-3	-27
-2	-8
-1	-1
0	0
1	1
2	8
3	27

Determine whether the data in the following tables can be represented by an exponential function.

TABLE 10.27:

x	y
0	200
1	300
2	1800
3	8300
4	25800
5	62700

TABLE 10.28:

x	y
0	120
1	180
2	270
3	405
4	607.5
5	911.25

TABLE 10.29:

x	y
0	4000
1	2400
2	1440
3	864
4	518.4
5	311.04

Determine what type of function represents the values in the following tables and find the equation of each function.

TABLE 10.30:

x	y
0	400
1	500
2	625

TABLE 10.30: (continued)

x	y
3	781.25
4	976.5625

TABLE 10.31:

x	y
-9	-3
-7	-2
-5	-1
-3	0
-1	1
1	2

TABLE 10.32:

x	y
-3	14
-2	4
-1	-2
0	-4
1	-2
2	4
3	14

10.12 Applications of Function Models

Here you'll learn how to perform exponential and quadratic regression to find equations for curves that fit non-linear data sets. You'll also solve real-world problems by comparing function models.

What if you were given a table of values that showed the U.S. deficit for each year from 2000 to 2010? How could you use the information in that table to estimate the deficit for 2015? After completing this Concept, you'll be able to perform regression to model real-world situations like this one.

Try This

If you don't have a graphing calculator, there are resources available on the Internet for finding lines and curves of best fit. For example, the applet at http://science.kennesaw.edu/ plaval/applets/LRegression.html does linear regression on a set of data points; the one at http://science.kennesaw.edu/ plaval/applets/QRegression.html does quadratic regression; and the one at http://science.kennesaw.edu/ plaval/applets/ERegression.html does exponential regression. Also, programs like Microsoft Office or OpenOffice have the ability to create graphs and charts that include lines and curves of best fit.

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CK-12 Foundation: 1012S Comparing Function Models Using Regression

Guidance

Earlier, you learned how to perform linear regression with a graphing calculator to find the equation of a straight line that fits a linear data set. In this section, you'll learn how to perform exponential and quadratic regression to find equations for curves that fit non-linear data sets.

Example A

The following table shows how many miles per gallon a car gets at different speeds.

TABLE 10.33:

Speed (mph)	**Miles per gallon**
30	18
35	20
40	23
45	25

TABLE 10.33: (continued)

Speed (mph)	Miles per gallon
50	28
55	30
60	29
65	25
70	25

Using a graphing calculator:

a) *Draw the scatterplot of the data.*

b) *Find the quadratic function of best fit.*

c) *Draw the quadratic function of best fit on the scatterplot.*

d) *Find the speed that maximizes the miles per gallon.*

e) *Predict the miles per gallon of the car if you drive at a speed of 48 mph.*

Solution

Step 1: Input the data.

Press **[STAT]** and choose the **[EDIT]** option.

Input the values of x in the first column (L_1) and the values of y in the second column (L_2). (**Note:** in order to clear a list, move the cursor to the top so that L_1 or L_2 is highlighted. Then press **[CLEAR]** and then **[ENTER]**.)

Step 2: Draw the scatterplot.

First press **[Y=]** and clear any function on the screen by pressing **[CLEAR]** when the old function is highlighted.

Press **[STATPLOT] [STAT]** and **[Y=]** and choose option 1.

Choose the ON option; after TYPE, choose the first graph type (scatterplot) and make sure that the Xlist and Ylist names match the names on top of the columns in the input table.

Press **[GRAPH]** and make sure that the window is set so you see all the points in the scatterplot. In this case, the settings should be $30 \le x \le 80$ and $0 \le y \le 40$. You can set the window size by pressing the **[WINDOW]** key at the top.

Step 3: Perform quadratic regression.

Press **[STAT]** and use the right arrow to choose **[CALC]**.

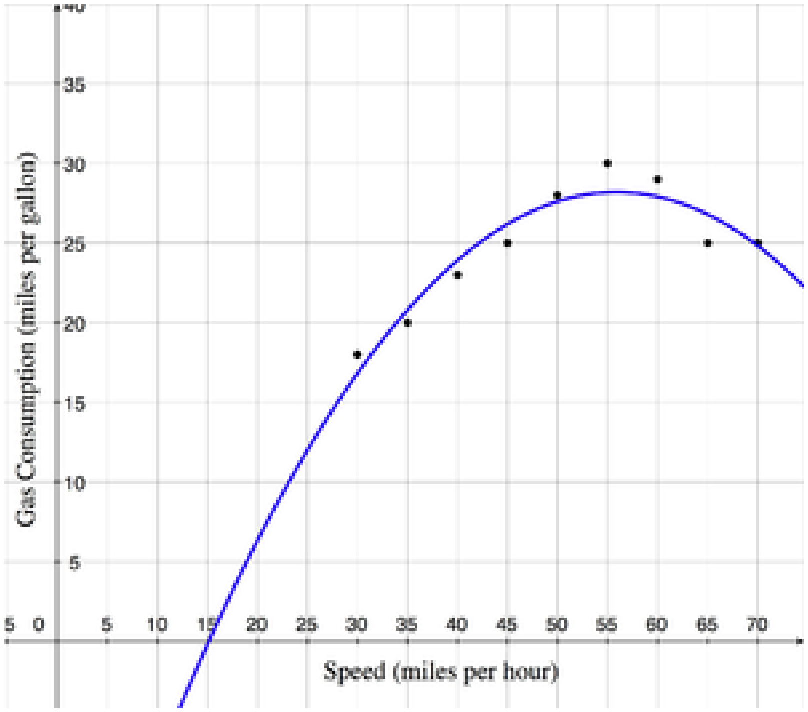
Choose Option 5 (QuadReg) and press **[ENTER]**. You will see “QuadReg” on the screen.

Type in L_1, L_2 after ’QuadReg’ and press **[ENTER]**. The calculator shows the quadratic function: $y = -0.017x^2 + 1.9x - 25$

Step 4: Graph the function.

Press [Y=] and input the function you just found.

Press **[GRAPH]** and you will see the curve fit drawn over the data points.



To find the speed that maximizes the miles per gallon, use **[TRACE]** and move the cursor to the top of the parabola. You can also use **[CALC] [2nd] [TRACE]** and option 4:Maximum, for a more accurate answer. The speed that maximizes miles per gallon is **56 mph.**

Finally, plug $x = 48$ into the equation you found: $y = -0.017(48)^2 + 1.9(48) - 25 = 27.032$ *miles per gallon*.

Note: The image above shows our function plotted on the same graph as the data points from the table. One thing that is clear from this graph is that predictions made with this function won't make sense for all values of x. For example, if $x < 15$, this graph predicts that we will get negative mileage, which is impossible. Part of the skill of using regression on your calculator is being aware of the strengths and limitations of this method of fitting functions to data.

Example B

The following table shows the amount of money an investor has in an account each year for 10 years.

TABLE 10.34:

Year	Value of account
1996	$5000
1997	$5400
1998	$5800
1999	$6300
2000	$6800
2001	$7300
2002	$7900
2003	$8600
2004	$9300
2005	$10000
2006	$11000

Using a graphing calculator:

a) *Draw a scatterplot of the value of the account as the dependent variable, and the number of years since 1996 as the independent variable.*

b) *Find the exponential function that fits the data.*

c) *Draw the exponential function on the scatterplot.*

d) *What will be the value of the account in 2020?*

Solution

Step 1: Input the data.

Press **[STAT]** and choose the **[EDIT]** option.

Input the values of x in the first column (L_1) and the values of y in the second column (L_2).

Step 2: Draw the scatterplot.

First press **[Y=]** and clear any function on the screen.

Press **[GRAPH]** and choose Option 1.

Choose the ON option and make sure that the Xlist and Ylist names match the names on top of the columns in the input table.

Press **[GRAPH]** make sure that the window is set so you see all the points in the scatterplot. In this case the settings should be $0 \le x \le 10$ and $0 \le y \le 11000$.

Step 3: Perform exponential regression.

Press **[STAT]** and use the right arrow to choose **[CALC]**.

Choose Option 0 and press **[ENTER]**. You will see "ExpReg" on the screen.

Press **[ENTER]**. The calculator shows the exponential function: $y = 4975.7(1.08)^x$

Step 4: Graph the function.

Press **[Y=]** and input the function you just found. Press **[GRAPH].**

Finally, plug $x = 2020 - 1996 = 24$ into the function: $y = 4975.7(1.08)^{24} = \underline{\$31551.81}$

In 2020, **the account will have a value of \$31551.81.**

Note: The function above is the curve that comes *closest* to all the data points. It won't return $y-$values that are exactly the same as in the data table, but they will be close. It is actually more accurate to use the curve fit values than the data points.

Solve Applications by Comparing Function Models

Example C

The following table shows the number of students enrolled in public elementary schools in the US (source: US Census Bureau). Make a scatterplot with the number of students as the dependent variable, and the number of years since 1990 as the independent variable. Find which curve fits this data the best and predict the school enrollment in the year 2007.

TABLE 10.35:

Year	Number of students (millions)
1990	26.6
1991	26.6

TABLE 10.35: (continued)

Year	Number of students (millions)
1992	27.1
1993	27.7
1994	28.1
1995	28.4
1996	28.1
1997	29.1
1998	29.3
2003	32.5

Solution

We need to perform linear, quadratic and exponential regression on this data set to see which function represents the values in the table the best.

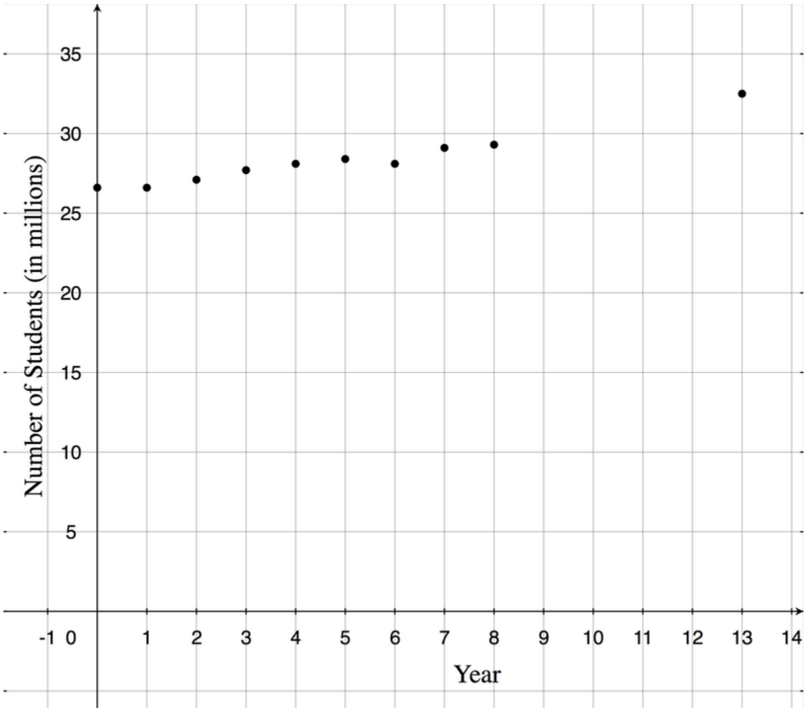
Step 1: Input the data.

Input the values of x in the first column (L_1) and the values of y in the second column (L_2).

Step 2: Draw the scatterplot.

Set the window size: $0 \leq x \leq 10$ and $20 \leq y \leq 40$.

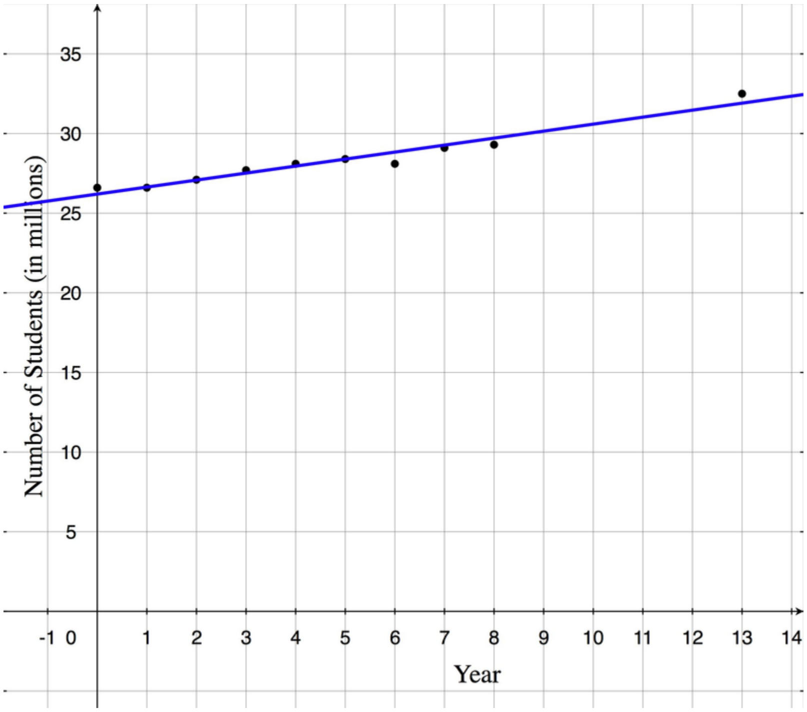
Here is the scatterplot:



Step 3: Perform Regression.

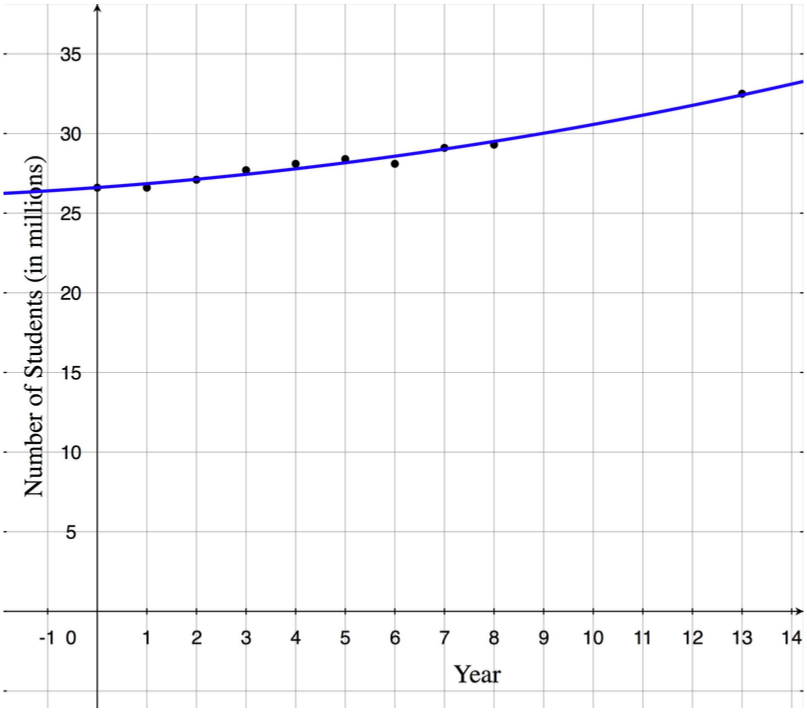
Linear Regression

The function of the line of best fit is $y = 0.51x + 26.1$. Here is the graph of the function on the scatterplot:



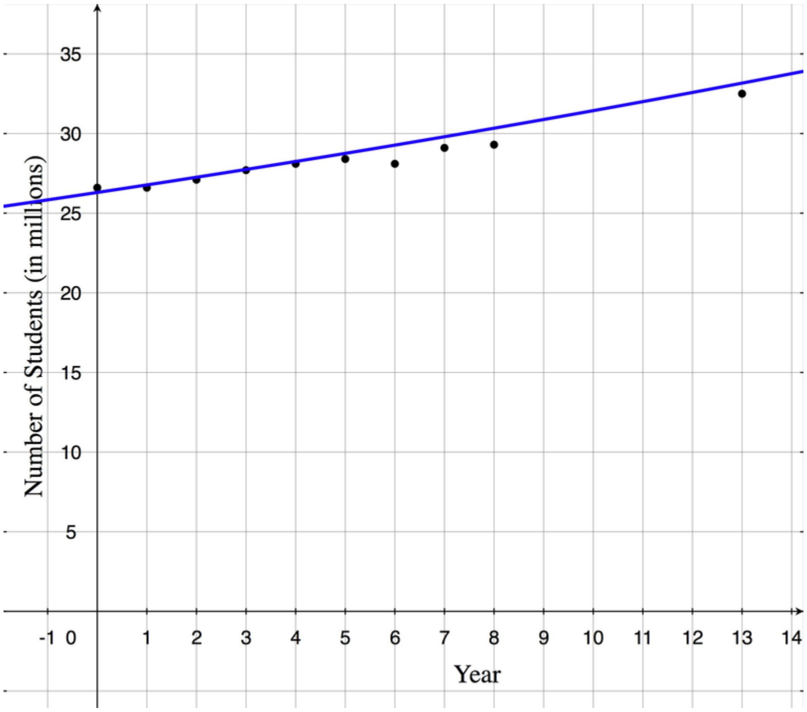
Quadratic Regression

The quadratic function of best fit is $y = 0.064x^2 - .067x + 26.84$. Here is the graph of the function on the scatterplot:



Exponential Regression

The exponential function of best fit is $y = 26.2(1.018)^x$. Here is the graph of the function on the scatterplot:



From the graphs, it looks like the quadratic function is the best fit for this data set. We'll use this function to predict school enrollment in 2007.

$x = 2007 - 1990 = 17$ so $y = 0.064(17)^2 - .067(17) + 26.84$ which is 44.2 million students.

Watch this video for help with the Examples above.

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CK-12 Foundation: 1012 ComparingFunction ModelsUsing Regression

Vocabulary

- **Mathematical modeling** is a process by which we start with a real-life situation and arrive at a quantitative solution.
- If the differences of the $y-$values is always the same, **the function is linear.**
- If the difference of the *differences* of the $y-$values is always the same, **the function is quadratic.**
- If the ratio of the $y-$values is always the same, **the function is exponential.**

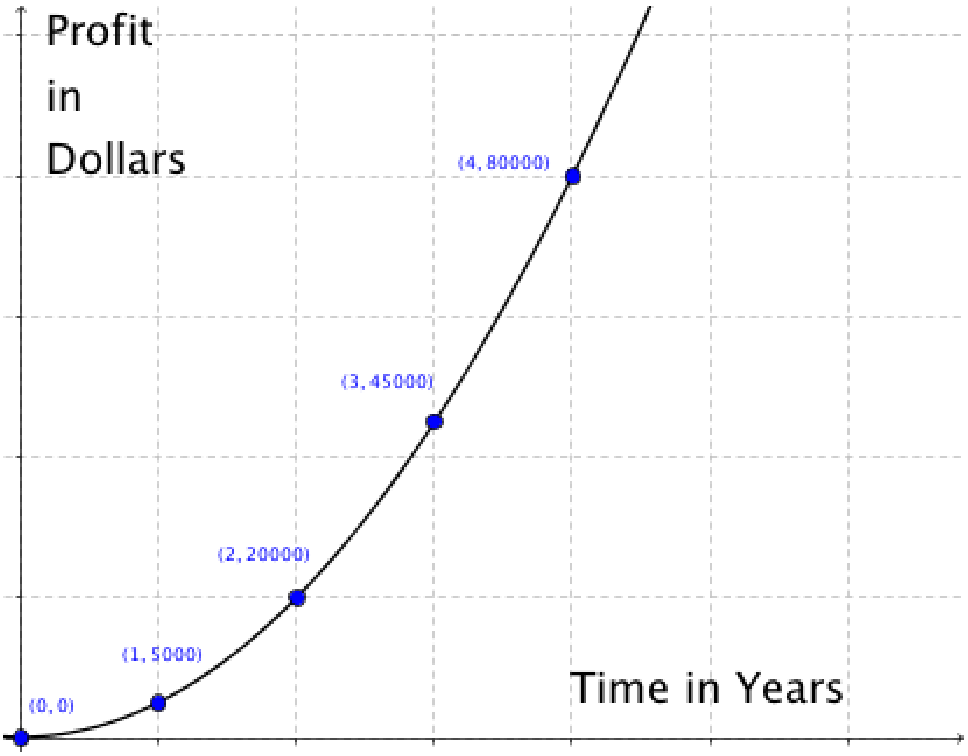
Guided Practice

The profits in dollars of a company are given in the table below. Find the model that describes the relationship to profit as a function of time in years:

Time in Years	0	1	2	3	4
Profit in Dollars	0	5000	20000	45000	80000

Solution:

Start by graphing the points, to get a sense of the shape.



This curve looks like it could be quadratic or exponential. If you check the ratios, you would see they are not the same. Checking the differences of differences yields:

Time in Years	0	1	2	3	4
Profit in Dollars	0	5000	20000	45000	80000
Differences		$5000-0=5000$	$20000-5000=15000$	$45000-20000=25000$	$80000-45000=$
Differences of Differences		$15000-5000=10000$	$25000-15000=10000$	$35000-25000=10000$	

Since the differences of differences are the same, this is a quadratic model.

Practice

For 1-5, suppose as a ball bounces up and down, the maximum height that the ball reaches continually decreases from one bounce to the next. For a given bounce, this table shows the height of the ball with respect to time:

TABLE 10.36:

Time (seconds)	Height (inches)
2	2

TABLE 10.36: (continued)

Time (seconds)	Height (inches)
2.2	16
2.4	24
2.6	33
2.8	38
3.0	42
3.2	36
3.4	30
3.6	28
3.8	14
4.0	6

Using a graphing calculator, answer the following questions:

1. Draw the scatterplot of the data
2. Find the quadratic function of best fit
3. Draw the quadratic function of best fit on the scatterplot
4. Find the maximum height the ball reaches on the bounce
5. Predict how high the ball is at time $t = 2.5\ seconds$

For 6-9, a chemist has a 250 gram sample of a radioactive material. She records the amount of radioactive material remaining in the sample every day for a week and obtains the data in the following table:

TABLE 10.37:

Day	Weight (grams)
0	250
1	208
2	158
3	130
4	102
5	80
6	65
7	50

Using a graphing calculator to answer the following questions:

6. Draw a scatterplot of the data
7. Find the exponential function of best fit
8. Draw the exponential function of best fit on the scatterplot
9. Predict the amount of material after 10 days

For 10-12, use the following table, which shows the rate of pregnancies (per 1000) for US women aged 15 to 19. (source: US Census Bureau).

10. Make a scatterplot with the rate of pregnancies as the dependent variable and the number of years since 1990 as the independent variable.
11. Find which type of curve fits this data best.
12. Predict the rate of teen pregnancies in the year 2010.

TABLE 10.38:

Year	Rate of pregnancy (per 1000)
1990	116.9
1991	115.3
1992	111.0
1993	108.0
1994	104.6
1995	99.6
1996	95.6
1997	91.4
1998	88.7
1999	85.7
2000	83.6
2001	79.5
2002	75.4

Summary

The focus of this chapter is quadratic functions and equations. The chapter first deals with graphs of quadratic functions and then moves of to solving quadratic equations using such graphs. Quadratic functions in intercept form and vertex form are both covered. Other methods of solving quadratic equations, such as using square roots and completing the square, are the next topics of discussion. The chapter then moves on to using the Quadratic Formula to solve equations. A portion of this formula, called the discriminant, is dealt with in detail as a means of determining the nature of a quadratic equation's solutions. The chapter closes with a discussion of linear, exponential, and quadratic models as well as introducing a method of comparing function models: regression.

CHAPTER 11 Algebra and Geometry Connections

Chapter Outline

11.1 GRAPHS OF SQUARE ROOT FUNCTIONS
11.2 SHIFTS OF SQUARE ROOT FUNCTIONS
11.3 RAISING A PRODUCT OR QUOTIENT TO A POWER
11.4 SIMPLIFICATION OF RADICAL EXPRESSIONS
11.5 APPLICATIONS USING RADICALS
11.6 RADICAL EQUATIONS
11.7 EQUATIONS WITH RADICALS ON BOTH SIDES
11.8 PYTHAGOREAN THEOREM AND ITS CONVERSE
11.9 SOLVING EQUATIONS USING THE PYTHAGOREAN THEOREM
11.10 APPLICATIONS USING THE PYTHAGOREAN THEOREM
11.11 DISTANCE FORMULA
11.12 MIDPOINT FORMULA

Introduction

As you've seen, to solve quadratic equations you often have to take the square root of a number. As you might expect then, when graphed, square root functions form half of a parabola. In this chapter you'll graph such functions. Square roots are a type of radical, which is the second focus of this chapter. You'll also perform operations such as addition, subtraction, multiplication, and rationalizing the denominator on a variety of radical expressions and equations to simplify and solve them. You'll then apply the knowledge of square roots and radicals you've gained to solve right triangle problems, to find the distance between two points, and to find the midpoint of a line segment.

11.1 Graphs of Square Root Functions

Here you'll learn how to graph and compare functions that involve square roots.

What if you had a square root function like $y = \sqrt{2x} + 3$. How would you graph that function? After completing this Concept, you'll be able to graph square root functions like this one and compare them to other square root functions.

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CK-12 Foundation: Graphsof Square Root Functions

Guidance

In this chapter you'll learn about a different kind of function called the square root function. You've seen that taking the square root is very useful in solving quadratic equations. For example, to solve the equation $x^2 = 25$ we take the square root of both sides: $\sqrt{x^2} = \pm\sqrt{25}$, so $x = \pm 5$.

A square root function is any function with the form: $y = a\sqrt{f(x)} + c$ —in other words, any function where an expression in terms of x is found inside a square root sign (also called a "radical" sign), although other terms may be included as well.

Graph and Compare Square Root Functions

When working with square root functions, you'll have to consider the domain of the function before graphing. The domain is very important because the function is undefined when the expression inside the square root sign is negative, and as a result there will be no graph in whatever region of $x-$values makes that true.

To discover how the graphs of square root functions behave, let's make a table of values and plot the points.

Example A

Graph the function $y = \sqrt{x}$.

Solution

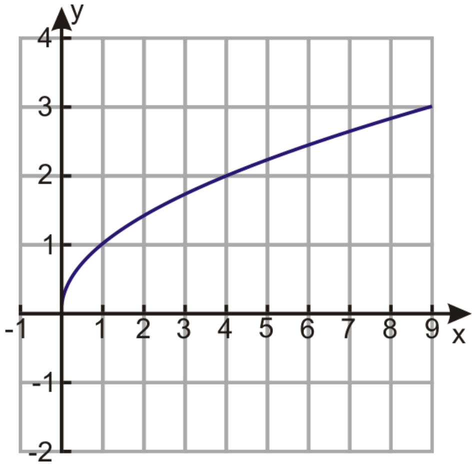
Before we make a table of values, we need to find the domain of this square root function. The domain is found by realizing that the function is only defined when the expression inside the square root is greater than or equal to zero. Since the expression inside the square root is just x, that means the domain is all values of x such that $x \geq 0$.

This means that when we make our table of values, we should pick values of x that are greater than or equal to zero. It is very useful to include zero itself as the first value in the table and also include many values greater than zero. This will help us in determining what the shape of the curve will be.

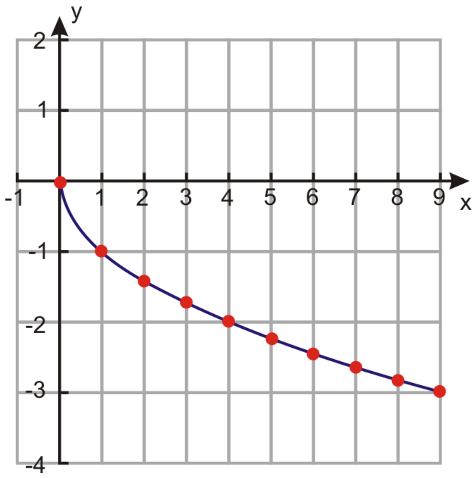
TABLE 11.1:

x	$y = \sqrt{x}$
0	$y = \sqrt{0} = 0$
1	$y = \sqrt{1} = 1$
2	$y = \sqrt{2} = 1.4$
3	$y = \sqrt{3} = 1.7$
4	$y = \sqrt{4} = 2$
5	$y = \sqrt{5} = 2.2$
6	$y = \sqrt{6} = 2.4$
7	$y = \sqrt{7} = 2.6$
8	$y = \sqrt{8} = 2.8$
9	$y = \sqrt{9} = 3$

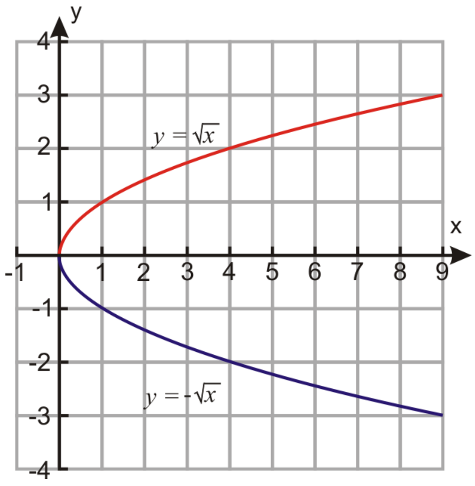
Here is what the graph of this table looks like:



The graphs of square root functions are always curved. The curve above looks like half of a parabola lying on its side, and in fact it is. It's half of the parabola that you would get if you graphed the expression $y^2 = x$. And the graph of $y = -\sqrt{x}$ is the other half of that parabola:



Notice that if we graph the two separate functions on the same coordinate axes, the combined graph is a parabola lying on its side.



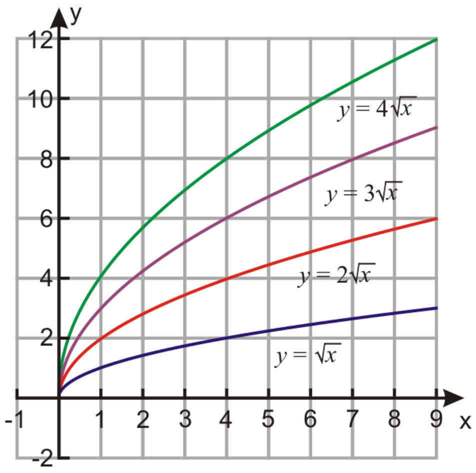
Now let's compare square root functions that are multiples of each other.

Example B

Graph the functions $y = \sqrt{x}, y = 2\sqrt{x}, y = 3\sqrt{x}$, *and* $y = 4\sqrt{x}$ *on the same graph.*

Solution

Here is just the graph without the table of values:

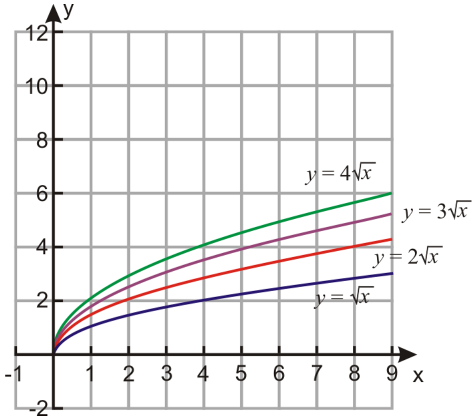


If we multiply the function by a constant bigger than one, the function increases faster the greater the constant is.

Example C

Graph the functions $y = \sqrt{x}, y = \sqrt{2x}, y = \sqrt{3x}$, *and* $y = \sqrt{4x}$ *on the same graph.*

Solution



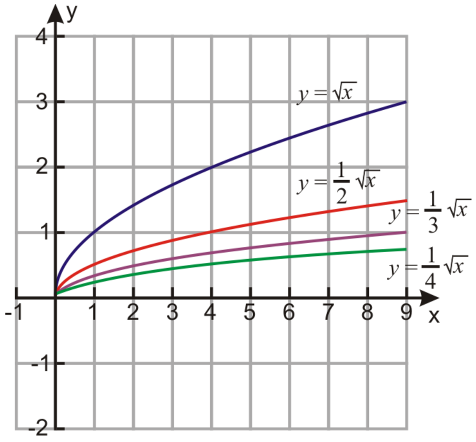
Notice that multiplying the expression *inside* the square root by a constant has the same effect as multiplying by a constant *outside* the square root; the function just increases at a slower rate because the entire function is effectively multiplied by the square root of the constant.

Also note that the graph of $\sqrt{4x}$ is the same as the graph of $2\sqrt{x}$. This makes sense algebraically since $\sqrt{4} = 2$.

Example D

Graph the functions $y = \sqrt{x}, y = \frac{1}{2}\sqrt{x}, y = \frac{1}{3}\sqrt{x}$, *and* $y = \frac{1}{4}\sqrt{x}$ *on the same graph.*

Solution



If we multiply the function by a constant between 0 and 1, the function increases more slowly the smaller the constant is.

Watch this video for help with the Examples above.

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CK-12 Foundation: Graphsof Square Root Functions

Vocabulary

- A **square root function** is any function with the form: $y = a\sqrt{f(x)} + c$ —in other words, any function where an expression in terms of x is found inside a square root sign (also called a "radical" sign).

Guided Practice

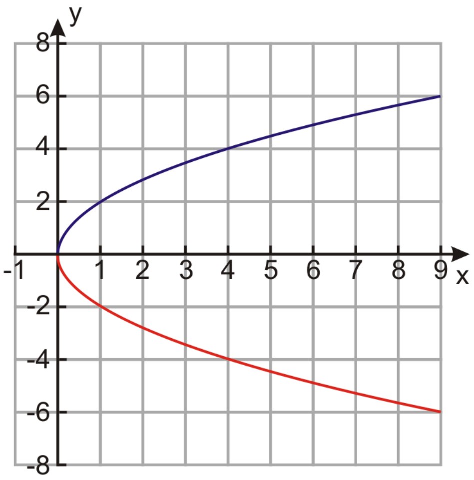
Graph the functions

a) $y = 2\sqrt{x}$ *and* $y = -2\sqrt{x}$ *on the same graph.*

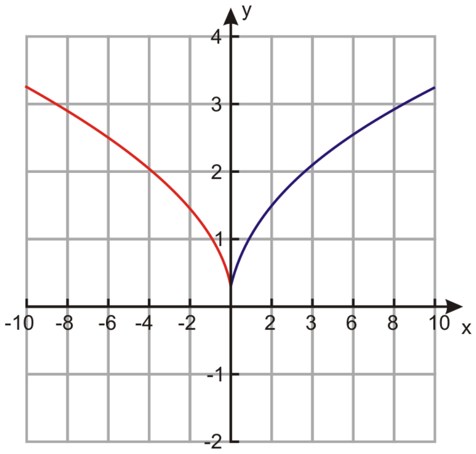
b) $y = \sqrt{x}$ *and* $y = \sqrt{-x}$ *on the same graph.*

Solutions:

a) If we multiply the whole function by -1, the graph is reflected about the $x-$axis.



b)



On the other hand, when just the x is multiplied by -1, the graph is reflected about the $y-$axis. Notice that the function $y = \sqrt{-x}$ has only negative $x-$values in its domain, because when x is negative, the expression under the radical sign is positive.

Practice

Graph the following functions.

1. $y = 3\sqrt{x}$
2. $y = -\frac{1}{2}\sqrt{x}$
3. $y = \sqrt{4x}$
4. $y = \sqrt{x+7}$
5. $y = 2\sqrt{x-5}$
6. $y = -\sqrt{3x+1-2}$

Graph the following functions on the same coordinate axes.

7. $y = \sqrt{x}, y = 2.5\sqrt{x}$ and $y = -2.5\sqrt{x}$
8. $y = \sqrt{x}, y = 0.3\sqrt{x}$ and $y = 0.6\sqrt{x}$
9. $y = \sqrt{x}, y = \sqrt{x-5}$ and $y = \sqrt{x+5}$
10. $y = \sqrt{x}, y = \sqrt{x}+8$ and $y = \sqrt{x}-8$

11.2 Shifts of Square Root Functions

Here you'll learn what shifts result from performing operations both inside and outside the square root sign of square root functions. You'll also learn how to graph such functions.

What if you had the square root function $y = \sqrt{x}$? How would the graph of the function change if you added 5 to the righthand side of the equation or if you multiplied x by 3? After completing this Concept, you'll be able to identify various shifts in square root functions.

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CK-12 Foundation: Shiftsof Square Root Functions

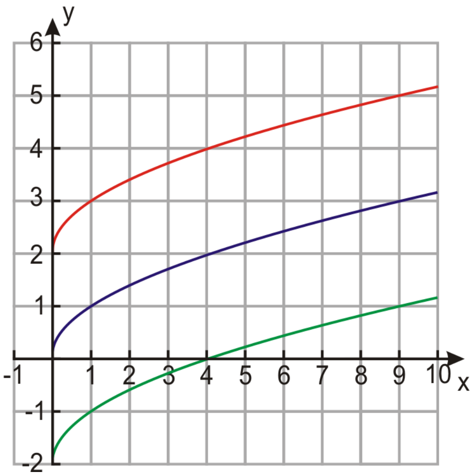
Guidance

We will now look at how graphs are shifted up and down in the Cartesian plane.

Example A

Graph the functions $y = \sqrt{x}, y = \sqrt{x}+2$ *and* $y = \sqrt{x}-2$.

Solution

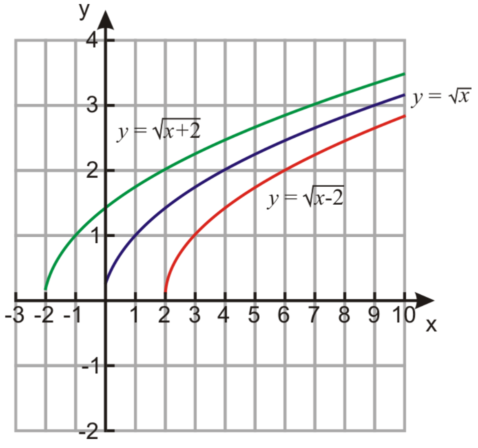


When we add a constant to the right-hand side of the equation, the graph keeps the same shape, but shifts up for a positive constant or down for a negative one.

Example B

Graph the functions $y = \sqrt{x}, y = \sqrt{x-2}$, *and* $y = \sqrt{x+2}$.

Solution



When we add a constant to the **argument** of the function (the part under the radical sign), the function shifts to the left for a positive constant and to the right for a negative constant.

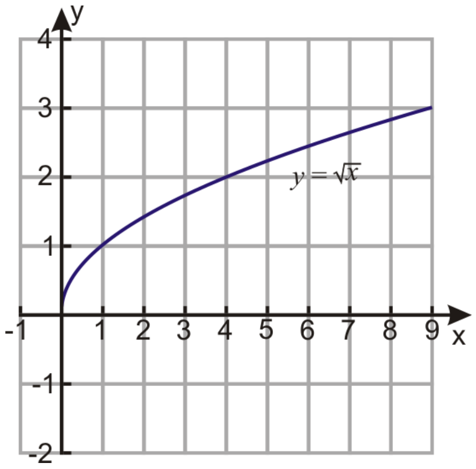
Now let's see how to combine all of the above types of transformations.

Example C

Graph the function $y = 2\sqrt{3x-1}+2$.

Solution

We can think of this function as a combination of shifts and stretches of the basic square root function $y = \sqrt{x}$. We know that the graph of that function looks like this:



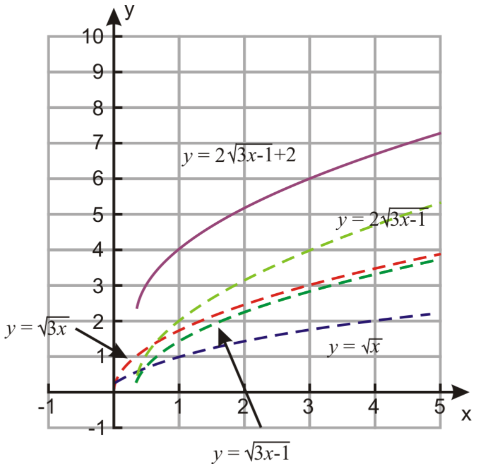
If we multiply the argument by 3 to obtain $y = \sqrt{3x}$, this stretches the curve vertically because the value of y increases faster by a factor of $\sqrt{3}$.

Next, when we subtract 1 from the argument to obtain $y = \sqrt{3x-1}$ this shifts the entire graph to the left by one unit.

Multiplying the function by a factor of 2 to obtain $y = 2\sqrt{3x-1}$ stretches the curve vertically again, because y increases faster by a factor of 2.

Finally we add 2 to the function to obtain $y = 2\sqrt{3x-1}+2$. This shifts the entire function vertically by 2 units.

Each step of this process is shown in the graph below. The purple line shows the final result.



Now we know how to graph square root functions without making a table of values. If we know what the basic function looks like, we can use shifts and stretches to **transform** the function and get to the desired result.

Watch this video for help with the Examples above.

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CK-12 Foundation: Shiftsof Square Root Functions

Vocabulary

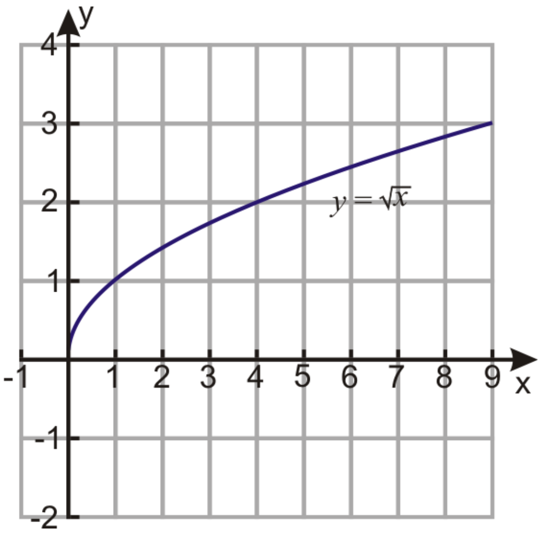
- For the **square root function** with the form: $y = a\sqrt{f(x)}+c$, c is the vertical shift.

Guided Practice

Graph the function $y = -\sqrt{x+3}-5$.

Solution

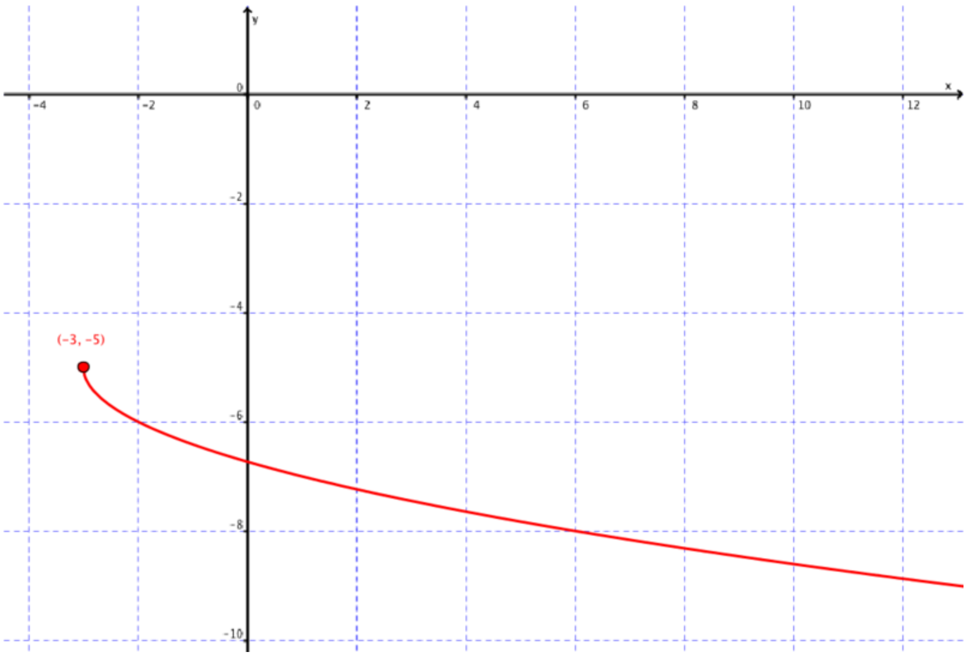
We can think of this function as a combination of shifts and stretches of the basic square root function $y = \sqrt{x}$. We know that the graph of that function looks like this:



Next, when we add 3 to the argument to obtain $y = \sqrt{x+3}$ this shifts the entire graph to the right by 3 units.

Multiplying the function by -1 to obtain $y = -\sqrt{x+3}$ which reflects the function across the *x*-axis.

Finally we subtract 5 from the function to obtain $y = -\sqrt{x+3} - 5$. This shifts the entire function down vertically by 5 units.



Practice

Graph the following functions.

1. $y = \sqrt{2x-1}$
2. $y = \sqrt{x-100}$
3. $y = \sqrt{4x+4}$
4. $y = \sqrt{5-x}$
5. $y = 2\sqrt{x} + 5$
6. $y = 3 - \sqrt{x}$

7. $y = 4 + 2\sqrt{x}$
8. $y = 2\sqrt{2x+3} + 1$
9. $y = 4 + \sqrt{2-x}$
10. $y = \sqrt{x+1} - \sqrt{4x-5}$

11.3 Raising a Product or Quotient to a Power

Here you'll learn how to use the product and quotient properties to evaluate radical expressions. You'll also use those properties to write radicals in simplest radical form.

What if you had a radical expression like $\sqrt{50x^3y^5}$? How could you simplify this expression? After completing this Concept, you'll be able to use the product and quotient properties of radicals to write them in simplest form.

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CK-12 Foundation: Propertiesof Radicals

Guidance

A radical reverses the operation of raising a number to a power. For example, the square of 4 is $4^2 = 4 \cdot 4 = 16$, and so the square root of 16 is 4. The symbol for a square root is $\sqrt{\ }$. This symbol is also called the **radical sign.**

In addition to square roots, we can also take cube roots, fourth roots, and so on. For example, since 64 is the cube of 4, 4 is the cube root of 64.

$$\sqrt[3]{64} = 4 \qquad \text{since} \qquad 4^3 = 4 \cdot 4 \cdot 4 = 64$$

We put an index number in the top left corner of the radical sign to show which root of the number we are seeking. Square roots have an index of 2, but we usually don't bother to write that out.

$$\sqrt[2]{36} = \sqrt{36} = 6$$

The cube root of a number gives a number which when raised to the power three gives the number under the radical sign. The fourth root of number gives a number which when raised to the power four gives the number under the radical sign:

$$\sqrt[4]{81} = 3 \qquad \text{since} \qquad 3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$

And so on for any power we can name.

Even and Odd Roots

Radical expressions that have even indices are called **even roots** and radical expressions that have odd indices are called **odd roots**. There is a very important difference between even and odd roots, because they give drastically different results when the number inside the radical sign is negative.

Any real number raised to an even power results in a positive answer. Therefore, when the index of a radical is even, the number inside the radical sign must be non-negative in order to get a real answer.

On the other hand, a positive number raised to an odd power is positive and a negative number raised to an odd power is negative. Thus, a negative number inside the radical sign is not a problem. It just results in a negative answer.

Example A

Evaluate each radical expression.

a) $\sqrt{121}$

b) $\sqrt[3]{125}$

c) $\sqrt[4]{-625}$

d) $\sqrt[5]{-32}$

Solution

a) $\sqrt{121} = 11$

b) $\sqrt[3]{125} = 5$

c) $\sqrt[3]{-625}$ is not a real number

d) $\sqrt[5]{-32} = -2$

Use the Product and Quotient Properties of Radicals

Radicals can be re-written as rational powers. The radical: $\sqrt[m]{a^n}$ is defined as $a^{\frac{n}{m}}$.

Example B

Write each expression as an exponent with a rational value for the exponent.

a) $\sqrt{5}$

b) $\sqrt[4]{a}$

c) $\sqrt[3]{4xy}$

d) $\sqrt[6]{x^5}$

Solution

a) $\sqrt{5} = 5^{\frac{1}{2}}$

b) $\sqrt[4]{a} = a^{\frac{1}{4}}$

c) $\sqrt[3]{4xy} = (4xy)^{\frac{1}{3}}$

d) $\sqrt[6]{x^5} = x^{\frac{5}{6}}$

As a result of this property, for any non-negative number a we know that $\sqrt[n]{a^n} = a^{\frac{n}{n}} = a$.

Since roots of numbers can be treated as powers, we can use exponent rules to simplify and evaluate radical expressions. Let's review the product and quotient rule of exponents.

Raising a product to a power: $(x \cdot y)^n = x^n \cdot y^n$

Raising a quotient to a power: $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

In radical notation, these properties are written as

Raising a product to a power: $\sqrt[m]{x \cdot y} = \sqrt[m]{x} \cdot \sqrt[m]{y}$

Raising a quotient to a power: $\sqrt[m]{\frac{x}{y}} = \frac{\sqrt[m]{x}}{\sqrt[m]{y}}$

A very important application of these rules is reducing a radical expression to its simplest form. This means that we apply the root on all the factors of the number that are perfect roots and leave all factors that are not perfect roots inside the radical sign.

For example, in the expression $\sqrt{16}$, the number 16 is a perfect square because $16 = 4^2$. This means that we can simplify it as follows:

$$\sqrt{16} = \sqrt{4^2} = 4$$

Thus, the square root disappears completely.

On the other hand, in the expression $\sqrt{32}$, the number 32 is not a perfect square, so we can't just remove the square root. However, we notice that $32 = 16 \cdot 2$, so we can write 32 as the product of a perfect square and another number. Thus,

$$\sqrt{32} = \sqrt{16 \cdot 2}$$

If we apply the "raising a product to a power" rule we get:

$$\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16} \cdot \sqrt{2}$$

Since $\sqrt{16} = 4$, we get: $\sqrt{32} = 4 \cdot \sqrt{2} = \underline{\underline{4\sqrt{2}}}$

Example C

Write the following expressions in the simplest radical form.

a) $\sqrt{8}$

b) $\sqrt{50}$

c) $\sqrt{\frac{125}{72}}$

Solution

The strategy is to write the number under the square root as the product of a perfect square and another number. The goal is to find the highest perfect square possible; if we don't find it right away, we just repeat the procedure until we can't simplify any longer.

a)

$$\begin{aligned}
\text{We can write } 8 = 4 \cdot 2, \text{ so} \quad & \sqrt{8} = \sqrt{4 \cdot 2}. \\
\text{With the “Raising a product to a power” rule, that becomes} \quad & \sqrt{4} \cdot \sqrt{2}. \\
\text{Evaluate } \sqrt{4} \text{ and we’re left with} \quad & \underline{\underline{2\sqrt{2}}}.
\end{aligned}$$

b)

$$\begin{aligned}
\text{We can write } 50 = 25 \cdot 2, \text{ so:} \quad & \sqrt{50} = \sqrt{25 \cdot 2} \\
\text{Use “Raising a product to a power” rule:} \quad & = \sqrt{25} \cdot \sqrt{2} = \underline{\underline{5\sqrt{2}}}
\end{aligned}$$

c)

$$\begin{aligned}
\text{Use “Raising a quotient to a power” rule to separate the fraction:} \quad & \sqrt{\frac{125}{72}} = \frac{\sqrt{125}}{\sqrt{72}} \\
\text{Re-write each radical as a product of a perfect square and another number:} \quad & = \frac{\sqrt{25 \cdot 5}}{\sqrt{36 \cdot 2}} = \frac{5\sqrt{5}}{6\sqrt{2}}
\end{aligned}$$

The same method can be applied to reduce radicals of different indices to their simplest form.

Example D

Write the following expression in the simplest radical form.

a) $\sqrt[3]{40}$

b) $\sqrt[4]{\frac{162}{80}}$

c) $\sqrt[3]{135}$

Solution

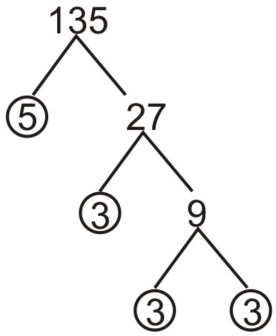
In these cases we look for the highest possible perfect cube, fourth power, etc. as indicated by the index of the radical.

a) Here we are looking for the product of the highest perfect cube and another number. We write: $\sqrt[3]{40} = \sqrt[3]{8 \cdot 5} = \sqrt[3]{8} \cdot \sqrt[3]{5} = 2\sqrt[3]{5}$

b) Here we are looking for the product of the highest perfect fourth power and another number.

$$\begin{aligned}
\text{Re-write as the quotient of two radicals:} \quad & \sqrt[4]{\frac{162}{80}} = \frac{\sqrt[4]{162}}{\sqrt[4]{80}} \\
\text{Simplify each radical separately:} \quad & = \frac{\sqrt[4]{81 \cdot 2}}{\sqrt[4]{16 \cdot 5}} = \frac{\sqrt[4]{81} \cdot \sqrt[4]{2}}{\sqrt[4]{16} \cdot \sqrt[4]{5}} = \frac{3\sqrt[4]{2}}{2\sqrt[4]{5}} \\
\text{Recombine the fraction under one radical sign:} \quad & = \frac{3}{2}\sqrt[4]{\frac{2}{5}}
\end{aligned}$$

c) Here we are looking for the product of the highest perfect cube root and another number. Often it’s not very easy to identify the perfect root in the expression under the radical sign. In this case, we can factor the number under the radical sign completely by using a factor tree:



We see that $135 = 3 \cdot 3 \cdot 3 \cdot 5 = 3^3 \cdot 5$. Therefore $\sqrt[3]{135} = \sqrt[3]{3^3 \cdot 5} = \sqrt[3]{3^3} \cdot \sqrt[3]{5} = 3\sqrt[3]{5}$.

(You can find a useful tool for creating factor trees at http://www.softschools.com/math/factors/factor_tree/. Click on “User Number” to type in your own number to factor, or just click “New Number” for a random number if you want more practice factoring.)

Now let’s see some examples involving variables.

Watch this video for help with the Examples above.

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CK-12 Foundation: Propertiesof Radicals

Vocabulary

- The symbol for a square root is $\sqrt{\ }$. This symbol is also called the **radical sign.**

Guided Practice

Write the following expression in the simplest radical form.

a) $\sqrt{12x^3y^5}$

b) $\sqrt[4]{\frac{1250x^7}{405y^9}}$

Solution

Treat constants and each variable separately and write each expression as the products of a perfect power as indicated by the index of the radical and another number.

a)

Re-write as a product of radicals: $\sqrt{12x^3y^5} = \sqrt{12} \cdot \sqrt{x^3} \cdot \sqrt{y^5}$

Simplify each radical separately: $\left(\sqrt{4 \cdot 3}\right) \cdot \left(\sqrt{x^2 \cdot x}\right) \cdot \left(\sqrt{y^4 \cdot y}\right) = \left(2\sqrt{3}\right) \cdot (x\sqrt{x}) \cdot (y^2\sqrt{y})$

Combine all terms outside and inside the radical sign: $= 2xy^2\sqrt{3xy}$

b)

Re-write as a quotient of radicals: $\sqrt[4]{\frac{1250x^7}{405y^9}} = \frac{\sqrt[4]{1250x^7}}{\sqrt[4]{405y^9}}$

Simplify each radical separately: $= \frac{\sqrt[4]{625 \cdot 2} \cdot \sqrt[4]{x^4 \cdot x^3}}{\sqrt[4]{81 \cdot 5} \cdot \sqrt[4]{y^4 \cdot y^4 \cdot y}} = \frac{5\sqrt[4]{2} \cdot x \cdot \sqrt[4]{x^3}}{3\sqrt[4]{5} \cdot y \cdot y \cdot \sqrt[4]{y}} = \frac{5x\sqrt[4]{2x^3}}{3y^2\sqrt[4]{5y}}$

Recombine fraction under one radical sign: $= \frac{5x}{3y^2}\sqrt[4]{\frac{2x^3}{5y}}$

Practice

Evaluate each radical expression.

1. $\sqrt{169}$
2. $\sqrt[4]{-81}$
3. $\sqrt[3]{-125}$
4. $\sqrt[5]{1024}$

Write each expression as a rational exponent.

5. $\sqrt[3]{14}$
6. $\sqrt[4]{zw}$
7. $\sqrt{a}$
8. $\sqrt[9]{y^3}$

Write the following expressions in simplest radical form.

9. $\sqrt{24}$
10. $\sqrt{300}$
11. $\sqrt[5]{96}$
12. $\sqrt{\frac{240}{567}}$
13. $\sqrt[3]{500}$
14. $\sqrt[6]{64x^8}$
15. $\sqrt[3]{48a^3b^7}$
16. $\sqrt[3]{\frac{16x^5}{135y^4}}$

11.4 Simplification of Radical Expressions

Here you'll learn how to simplify radical expressions by adding, subtracting, multiplying, distributing, and rationalizing the denominator.

What if you wanted to perform an operation on two radical expressions, like $\sqrt{32x}-\sqrt{8x}$, in which the numbers under the radical signs were different? How could you find the difference? After completing this Concept, you'll be able to add, subtract, multiply, and divide radical expressions.

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CK-12 Foundation: Radical Expressions

Guidance

When we add and subtract radical expressions, we can combine radical terms only when they have the same expression under the radical sign. This is a lot like combining like terms in variable expressions.

Example A

Simplify the following expressions as much as possible.

a.) $4\sqrt{2}+5\sqrt{2}$

b.) $2\sqrt{3}-\sqrt{2}+5\sqrt{3}+10\sqrt{2}$

$$4\sqrt{2}+5\sqrt{2}=9\sqrt{2}$$

or

$$2\sqrt{3}-\sqrt{2}+5\sqrt{3}+10\sqrt{2}=7\sqrt{3}+9\sqrt{2}$$

It's important to reduce all radicals to their simplest form in order to make sure that we're combining all possible like terms in the expression. For example, the expression $\sqrt{8}-2\sqrt{50}$ looks like it can't be simplified any more because it has no like terms. However, when we write each radical in its simplest form we get $2\sqrt{2}-10\sqrt{2}$, and we can combine those terms to get $-8\sqrt{2}$.

Example B

Simplify the following expressions as much as possible.

a) $4\sqrt[3]{128}-\sqrt[3]{250}$

b) $3\sqrt{x^3}-4x\sqrt{9x}$

Solution

a)

$$\begin{aligned}\text{Re-write radicals in simplest terms:} \quad & = 4\sqrt[3]{2\cdot 64}-\sqrt[3]{2\cdot 125}=16\sqrt[3]{2}-5\sqrt[3]{2}\\ \text{Combine like terms:} \quad & = 11\sqrt[3]{2}\end{aligned}$$

b)

$$\begin{aligned}\text{Re-write radicals in simplest terms:} \quad & 3\sqrt{x^2\cdot x}-12x\sqrt{x}=3x\sqrt{x}-12x\sqrt{x}\\ \text{Combine like terms:} \quad & =-9x\sqrt{x}\end{aligned}$$

Multiply Radical Expressions

When we multiply radical expressions, we use the "raising a product to a power" rule: $\sqrt[m]{x\cdot y}=\sqrt[m]{x}\cdot\sqrt[m]{y}$. In this case we apply this rule in reverse.

Example C

Simplify the expression

$$\sqrt{6}\cdot\sqrt{8}$$

.

Solution:

$$\sqrt{6}\cdot\sqrt{8}=\sqrt{6\cdot 8}=\sqrt{48}$$

Or, in simplest radical form: $\sqrt{48}=\sqrt{16\cdot 3}=4\sqrt{3}$.

We'll also make use of the fact that: $\sqrt{a}\cdot\sqrt{a}=\sqrt{a^2}=a$.

When we multiply expressions that have numbers on both the outside and inside the radical sign, we treat the numbers outside the radical sign and the numbers inside the radical sign separately. For example, $a\sqrt{b}\cdot c\sqrt{d}=ac\sqrt{bd}$.

Example D

Multiply the following expressions.

a) $\sqrt{2}\left(\sqrt{3}+\sqrt{5}\right)$

b) $2\sqrt{x}(3\sqrt{y}-\sqrt{x})$

c) $\left(2+\sqrt{5}\right)\left(2-\sqrt{6}\right)$

d) $(2\sqrt{x}+1)(5-\sqrt{x})$

Solution

In each case we use distribution to eliminate the parentheses.

a)

$$\begin{aligned} \text{Distribute } \sqrt{2} \text{ inside the parentheses:} && \sqrt{2}\left(\sqrt{3}+\sqrt{5}\right) &= \sqrt{2}\cdot\sqrt{3}+\sqrt{2}\cdot\sqrt{5} \\ \text{Use the ``raising a product to a power'' rule:} && &= \sqrt{2\cdot 3}+\sqrt{2\cdot 5} \\ \text{Simplify:} && &= \sqrt{6}+\sqrt{10} \end{aligned}$$

b)

$$\begin{aligned} \text{Distribute } 2\sqrt{x} \text{ inside the parentheses:} && &= (2\cdot 3)\left(\sqrt{x}\cdot\sqrt{y}\right)-2\cdot\left(\sqrt{x}\cdot\sqrt{x}\right) \\ \text{Multiply:} && &= 6\sqrt{xy}-2\sqrt{x^2} \\ \text{Simplify:} && &= 6\sqrt{xy}-2x \end{aligned}$$

c)

$$\begin{aligned} \text{Distribute:} && (2+\sqrt{5})(2-\sqrt{6}) &= (2\cdot 2)-\left(2\cdot\sqrt{6}\right)+\left(2\cdot\sqrt{5}\right)-\left(\sqrt{5}\cdot\sqrt{6}\right) \\ \text{Simplify:} && &= 4-2\sqrt{6}+2\sqrt{5}-\sqrt{30} \end{aligned}$$

d)

$$\begin{aligned} \text{Distribute:} && \left(2\sqrt{x}-1\right)\left(5-\sqrt{x}\right) &= 10\sqrt{x}-2x-5+\sqrt{x} \\ \text{Simplify:} && &= 11\sqrt{x}-2x-5 \end{aligned}$$

Rationalize the Denominator

Often when we work with radicals, we end up with a radical expression in the denominator of a fraction. It's traditional to write our fractions in a form that doesn't have radicals in the denominator, so we use a process called **rationalizing the denominator** to eliminate them.

Rationalizing is easiest when there's just a radical and nothing else in the denominator, as in the fraction $\frac{2}{\sqrt{3}}$. All we have to do then is multiply the numerator and denominator by a radical expression that makes the expression inside the radical into a perfect square, cube, or whatever power is appropriate. In the example above, we multiply by $\sqrt{3}$:

$$\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$$

Cube roots and higher are a little trickier than square roots.

Example E

How would we rationalize $\frac{7}{\sqrt[3]{5}}$*?*

Solution:

We can't just multiply by $\sqrt[3]{5}$, because then the denominator would be $\sqrt[3]{5^2}$. To make the denominator a whole number, we need to multiply the numerator and the denominator by $\sqrt[3]{5^2}$:

$$\frac{7}{\sqrt[3]{5}}\cdot\frac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}}=\frac{7\sqrt[3]{25}}{\sqrt[3]{5^3}}=\frac{7\sqrt[3]{25}}{5}$$

Trickier still is when the expression in the denominator contains more than one term.

Example F

Consider the expression $\frac{2}{2+\sqrt{3}}$. We can't just multiply by $\sqrt{3}$, because we'd have to distribute that term and then the denominator would be $2\sqrt{3}+3$.

Instead, we multiply by $2-\sqrt{3}$. This is a good choice because the product $\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)$ is a product of a sum and a difference, which means it's a difference of squares. The radicals cancel each other out when we multiply out, and the denominator works out to $\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)=2^2-\left(\sqrt{3}\right)^2=4-3=1$.

When we multiply both the numerator and denominator by $2-\sqrt{3}$, we get:

$$\frac{2}{2+\sqrt{3}}\cdot\frac{2-\sqrt{3}}{2-\sqrt{3}}=\frac{2\left(2-\sqrt{3}\right)}{4-3}=\frac{4-2\sqrt{3}}{1}=4-2\sqrt{3}$$

Now consider the expression $\frac{\sqrt{x}-1}{\sqrt{x}-2\sqrt{y}}$.

In order to eliminate the radical expressions in the denominator we must multiply by $\sqrt{x}+2\sqrt{y}$.

We get: $\frac{\sqrt{x}-1}{\sqrt{x}-2\sqrt{y}}\cdot\frac{\sqrt{x}+2\sqrt{y}}{\sqrt{x}+2\sqrt{y}}=\frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+2\sqrt{y}\right)}{\left(\sqrt{x}-2\sqrt{y}\right)\left(\sqrt{x}+2\sqrt{y}\right)}=\frac{x-2\sqrt{y}-\sqrt{x}+2\sqrt{xy}}{x-4y}$

Watch this video for help with the Examples above.

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CK-12 Foundation: Radical Expressions

Vocabulary

- When we **multiply radical expressions**, we use the "raising a product to a power" rule:

$\sqrt[m]{x\cdot y}=\sqrt[m]{x}\cdot\sqrt[m]{y}$.

- When we multiply expressions that have numbers on both the outside and inside the radical sign, we treat the numbers outside the radical sign and the numbers inside the radical sign separately:

$a\sqrt{b}\cdot c\sqrt{d}=ac\sqrt{bd}$

Guided Practice

Simplify the following expressions as much as possible.

a) $4\sqrt{3}+2\sqrt{12}$

b) $10\sqrt{24}-\sqrt{28}$

Solutions:

a)

Simplify $\sqrt{12}$ to its simplest form: $= 4\sqrt{3} + 2\sqrt{4 \cdot 3} = 4\sqrt{3} + 6\sqrt{3}$

Combine like terms: $= 10\sqrt{3}$

b)

Simplify $\sqrt{24}$ and $\sqrt{28}$ to their simplest form: $= 10\sqrt{6 \cdot 4} - \sqrt{7 \cdot 4} = 20\sqrt{6} - 2\sqrt{7}$

There are no like terms.

Practice

Simplify the following expressions as much as possible.

1. $3\sqrt{8} - 6\sqrt{32}$
2. $\sqrt{180} + \sqrt{405}$
3. $\sqrt{6} - \sqrt{27} + 2\sqrt{54} + 3\sqrt{48}$
4. $\sqrt{8x^3} - 4x\sqrt{98x}$
5. $\sqrt{48a} + \sqrt{27a}$
6. $\sqrt[3]{4x^3} + x \cdot \sqrt[3]{256}$

Multiply the following expressions.

7. $\sqrt{6}\left(\sqrt{10} + \sqrt{8}\right)$
8. $\left(\sqrt{a} - \sqrt{b}\right)\left(\sqrt{a} + \sqrt{b}\right)$
9. $(2\sqrt{x} + 5)(2\sqrt{x} + 5)$

Rationalize the denominator.

10. $\frac{7}{\sqrt{5}}$
11. $\frac{9}{\sqrt{10}}$
12. $\frac{2x}{\sqrt{5x}}$
13. $\frac{\sqrt{5}}{\sqrt{3y}}$
14. $\frac{12}{2-\sqrt{5}}$
15. $\frac{6+\sqrt{3}}{4-\sqrt{3}}$
16. $\frac{\sqrt{x}}{\sqrt{2}+\sqrt{x}}$
17. $\frac{5y}{2\sqrt{y}-5}$

11.5 Applications Using Radicals

Here you'll learn how to solve harmonic motion and other real-world applications involving radical expressions.

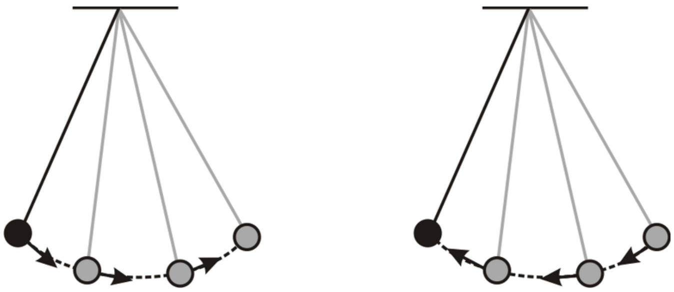
What if your computer screen measured 16 inches long and 12 inches wide? How could you find the screen's diagonal length? After completing this Concept, you'll be able to solve real-world applications involving radicals like this one.

Try This

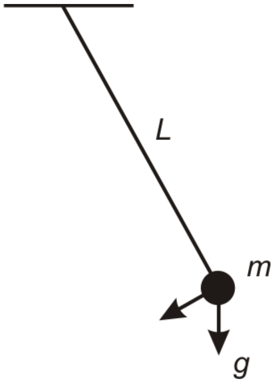
For more equations that describe pendulum motion, check out http://hyperphysics.phy-astr.gsu.edu/hbase/pend.html , where you can also find a tool for calculating the period of a pendulum in different gravities than Earth's.

Guidance

Mathematicians and physicists have studied the motion of pendulums in great detail because this motion explains many other behaviors that occur in nature. This type of motion is called **simple harmonic motion** and it is important because it describes anything that repeats periodically. Galileo was the first person to study the motion of a pendulum, around the year 1600. He found that the time it takes a pendulum to complete a swing doesn't depend on its mass or on its angle of swing (as long as the angle of the swing is small). Rather, it depends only on the length of the pendulum.



The time it takes a pendulum to complete one whole back-and-forth swing is called the **period** of the pendulum. Galileo found that the period of a pendulum is proportional to the square root of its length: $T = a\sqrt{L}$. The proportionality constant, a, depends on the acceleration of gravity: $a = \frac{2\pi}{\sqrt{g}}$. At sea level on Earth, acceleration of gravity is $g = 9.81\ m/s^2$ (meters per second squared). Using this value of gravity, we find $a = 2.0$ with units of $\frac{s}{\sqrt{m}}$ (seconds divided by the square root of meters).



Up until the mid 20^{th} century, all clocks used pendulums as their central time keeping component.

Example A

Graph the period of a pendulum of a clock swinging in a house on Earth at sea level as we change the length of the pendulum. What does the length of the pendulum need to be for its period to be one second?

Solution

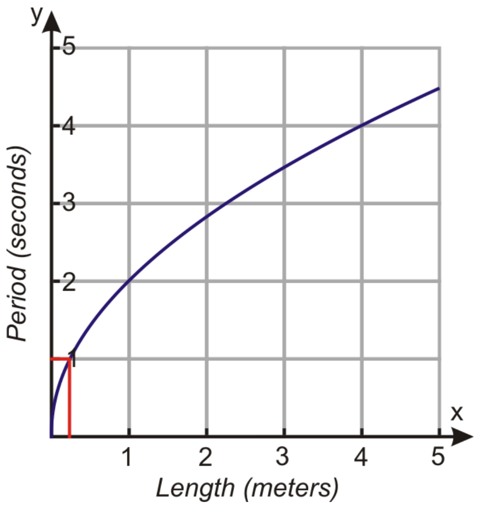
The function for the period of a pendulum at sea level is $T = 2\sqrt{L}$.

We start by making a table of values for this function:

TABLE 11.2:

L	$T = 2\sqrt{L}$
0	$T = 2\sqrt{0} = 0$
1	$T = 2\sqrt{1} = 2$
2	$y = 2\sqrt{2} = 2.8$
3	$y = 2\sqrt{3} = 3.5$
4	$y = 2\sqrt{4} = 4$
5	$y = 2\sqrt{5} = 4.5$

Now let's graph the function. It makes sense to let the horizontal axis represent the length of the pendulum and the vertical axis represent the period of the pendulum.



We can see from the graph that a length of approximately $\frac{1}{4}$ meters gives a period of 1 second. We can confirm this answer by using our function for the period and plugging in $T = 1$ *second*:

$$T = 2\sqrt{L} \Rightarrow 1 = 2\sqrt{L}$$

$$\begin{aligned} &\text{Square both sides of the equation:} && 1 = 4L \\ &\text{Solve for } L: && L = \frac{1}{4} \; meters \end{aligned}$$

Example B

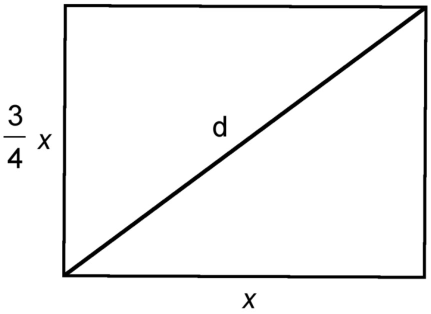
"Square" TV screens have an aspect ratio of 4:3; in other words, the width of the screen is $\frac{4}{3}$ the height. TV "sizes" are traditionally represented as the length of the diagonal of the television screen. Graph the length of the diagonal of a screen as a function of the area of the screen. What is the diagonal of a screen with an area of $180 \; in^2$?

Solution

Let $d =$ length of the diagonal, $x =$ width

Then 4 × height = 3 × width

Or, height = $\frac{3}{4}x$.



The area of the screen is: $A =$ length × width or $A = \frac{3}{4}x^2$

Find how the diagonal length relates to the width by using the Pythagorean theorem:

$$\begin{aligned} x^2 + \left(\frac{3}{4}x\right)^2 &= d^2 \\ x^2 + \frac{9}{16}x^2 &= d^2 \\ \frac{25}{16}x^2 &= d^2 \Rightarrow x^2 = \frac{16}{25}d^2 \Rightarrow x = \frac{4}{5}d \end{aligned}$$

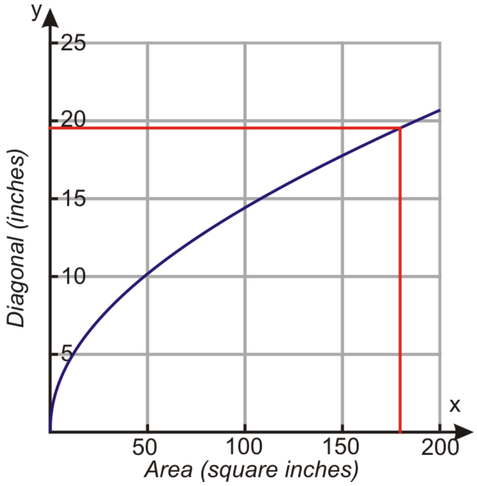
Therefore, the diagonal length relates to the area as follows: $A = \frac{3}{4}\left(\frac{4}{5}d\right)^2 = \frac{3}{4} \cdot \frac{16}{25}d^2 = \frac{12}{25}d^2$.

We can also flip that around to find the diagonal length as a function of the area: $d^2 = \frac{25}{12}A$ or $d = \frac{5}{2\sqrt{3}}\sqrt{A}$.

Now we can make a graph where the horizontal axis represents the area of the television screen and the vertical axis is the length of the diagonal. First let's make a table of values:

TABLE 11.3:

A	$d = \frac{5}{2\sqrt{3}}\sqrt{A}$
0	0
25	7.2
50	10.2
75	12.5
100	14.4
125	16.1
150	17.6
175	19
200	20.4



From the graph we can estimate that when the area of a TV screen is 180 in^2 the length of the diagonal is approximately 19.5 inches. We can confirm this by plugging in $A = 180$ into the formula that relates the diagonal to the area: $d = \frac{5}{2\sqrt{3}}\sqrt{A} = \frac{5}{2\sqrt{3}}\sqrt{180} = 19.4\ inches$.

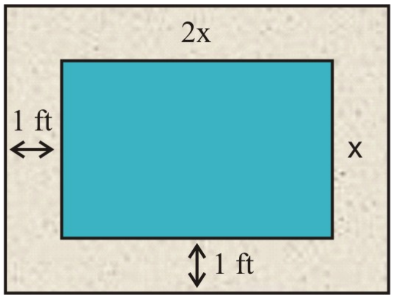
Radicals often arise in problems involving areas and volumes of geometrical figures.

Example C

A pool is twice as long as it is wide and is surrounded by a walkway of uniform width of 1 foot. The combined area of the pool and the walkway is 400 square feet. Find the dimensions of the pool and the area of the pool.

Solution

Make a sketch:



Let $x =$ the width of the pool. Then:

Area $=$ length $\times$ width

Combined length of pool and walkway $= 2x+2$

Combined width of pool and walkway $= x+2$

Area $= (2x+2)(x+2)$

Since the combined area of pool and walkway is 400 ft^2 we can write the equation

$(2x+2)(x+2) = 400$

$$\begin{aligned}
\text{Multiply in order to eliminate the parentheses:} &\quad 2x^2+4x+2x+4 = 400\\
\text{Collect like terms:} &\quad 2x^2+6x+4 = 400\\
\text{Move all terms to one side of the equation:} &\quad 2x^2+6x-396 = 0\\
\text{Divide all terms by 2:} &\quad x^2+3x-198 = 0\\
\text{Use the quadratic formula:} &\quad x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
&\quad x = \frac{-3\pm\sqrt{3^2-4(1)(-198)}}{2(1)}\\
&\quad x = \frac{-3\pm\sqrt{801}}{2} = \frac{-3\pm 28.3}{2}\\
&\quad x = 12.65\ feet
\end{aligned}$$

(The other answer is negative, so we can throw it out because only a positive number makes sense for the width of a swimming pool.)

So the dimensions of the pool are: $length = 12.65$ and $width = 25.3$ (since the width is 2 times the length)

That means that the area of just the pool is $A = 12.65 \cdot 25.3 \rightarrow 320 ft^2$

Check by plugging the result in the area formula:

Area $= (2(12.65)+2)(12.65+2) = 27.3 \cdot 14.65 = 400\ ft^2$.

The answer checks out.

Watch this video for help with the Examples above.

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CK-12 Foundation: Applications UsingRadicals

Vocabulary

- When we **multiply radical expressions**, we use the "raising a product to a power" rule:

$\sqrt[m]{x \cdot y} = \sqrt[m]{x} \cdot \sqrt[m]{y}$.

- When we multiply expressions that have numbers on both the outside and inside the radical sign, we treat the numbers outside the radical sign and the numbers inside the radical sign separately:

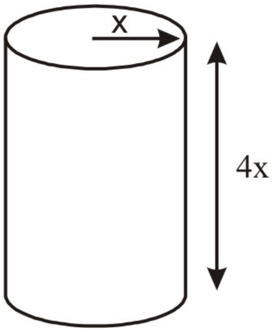
$a\sqrt{b} \cdot c\sqrt{d} = ac\sqrt{bd}$

Guided Practice

The volume of a soda can is 355 cm^3. *The height of the can is four times the radius of the base. Find the radius of the base of the cylinder.*

Solution

Make a sketch:



Let $x =$ the radius of the cylinder base. Then the height of the cylinder is $4x$.

The volume of a cylinder is given by $V = \pi R^2 \cdot h$; in this case, R is x and h is $4x$, and we know the volume is 355.

Solve the equation:

$$\begin{aligned}355 &= \pi x^2 \cdot (4x)\\ 355 &= 4\pi x^3\\ x^3 &= \frac{355}{4\pi}\\ x &= \sqrt[3]{\frac{355}{4\pi}} = 3.046\ cm\end{aligned}$$

Check by substituting the result back into the formula:

$$V = \pi R^2 \cdot h = \pi(3.046)^2 \cdot (4 \cdot 3.046) = 355\ cm^3$$

So the volume is 355 cm^3. **The answer checks out.**

Practice

1. If a certain model of a laptop has a diagonal of 15.4 inches and a length of 14.35 inches, find the width.
2. If a certain model of a laptop has a width of 12.78 inches and an area of 114.25 inches squared, find the diagonal.
3. The acceleration of gravity can also given in feet per second squared. It is $g = 32\ ft/s^2$ at sea level.

   a. Graph the period of a pendulum with respect to its length in feet.
   b. For what length in feet will the period of a pendulum be 2 seconds?

4. The acceleration of gravity on the Moon is 1.6 m/s^2.
   a. Graph the period of a pendulum on the Moon with respect to its length in meters.
   b. For what length, in meters, will the period of a pendulum be 10 seconds?
5. The acceleration of gravity on Mars is 3.69 m/s^2.
   a. Graph the period of a pendulum on the Mars with respect to its length in meters.
   b. For what length, in meters, will the period of a pendulum be 3 seconds?
6. The acceleration of gravity on the Earth depends on the latitude and altitude of a place. The value of g is slightly smaller for places closer to the Equator than places closer to the poles and the value of g is slightly smaller for places at higher altitudes that it is for places at lower altitudes. In Helsinki the value of $g = 9.819\ m/s^2$, in Los Angeles the value of $g = 9.796\ m/s^2$ and in Mexico City the value of $g = 9.779\ m/s^2$.
   a. Graph the period of a pendulum with respect to its length for all three cities on the same graph.
   b. Use the formula to find for what length, in meters, will the period of a pendulum be 8 seconds in each of these cities?
7. The aspect ratio of a wide-screen TV is 2.39:1.
   a. Graph the length of the diagonal of a screen as a function of the area of the screen.
   b. What is the diagonal of a screen with area 150 in^2?

For 8-10, rationalize the denominator.

8. The volume of a soup can is 452 cm^3. The height of the can is three times the radius of the base. Find the radius of the base of the cylinder.
9. The volume of a spherical balloon is 950 cm^3. Find the radius of the balloon. (Volume of a sphere $= \frac{4}{3}\pi R^3$).
10. A rectangular picture is 9 inches wide and 12 inches long. The picture has a frame of uniform width. If the combined area of picture and frame is 180 in^2, what is the width of the frame?

11.6 Radical Equations

Here you'll learn how to solve equations involving radicals. You'll also solve real-world applications that involve such equations.

What if you had a radical equation like $\sqrt{2x+5}-3=0$? How could you find its real solutions? After completing this Concept, you'll be able to solve radical equations like this one.

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CK-12 Foundation: Radical Equations

Guidance

When the variable in an equation appears inside a radical sign, the equation is called a **radical equation.** To solve a radical equation, we need to eliminate the radical and change the equation into a polynomial equation.

A common method for solving radical equations is to isolate the most complicated radical on one side of the equation and raise both sides of the equation to the power that will eliminate the radical sign. If there are any radicals left in the equation after simplifying, we can repeat this procedure until all radical signs are gone. Once the equation is changed into a polynomial equation, we can solve it with the methods we already know.

We must be careful when we use this method, because whenever we raise an equation to a power, we could introduce false solutions that are not in fact solutions to the original problem. These are called **extraneous solutions.** In order to make sure we get the correct solutions, we must always check all solutions in the original radical equation.

Solve a Radical Equation

Let's consider a few simple examples of radical equations where only one radical appears in the equation.

Example A

Find the real solutions of the equation $\sqrt{2x-1}=5$.

Solution

Since the radical expression is already isolated, we can just square both sides of the equation in order to eliminate the radical sign:

$$\left(\sqrt{2x-1}\right)^2=5^2$$

Remember that $\sqrt{a^2} = a$ so the equation simplifies to: $2x - 1 = 25$

Add one to both sides: $2x = 26$

Divide both sides by 2: $\underline{\underline{x = 13}}$

Finally we need to plug the solution in the original equation to see if it is a valid solution.

$\sqrt{2x-1} = \sqrt{2(13)-1} = \sqrt{26-1} = \sqrt{25} = 5$**The solution checks out.**

Example B

Find the real solutions of $\sqrt[3]{3-7x} - 3 = 0$.

Solution

We isolate the radical on one side of the equation: $\sqrt[3]{3-7x} = 3$

Raise each side of the equation to the third power: $\left(\sqrt[3]{3-7x}\right)^3 = 3^3$

Simplify: $3 - 7x = 27$

Subtract 3 from each side: $-7x = 24$

Divide both sides by -7: $\underline{\underline{x = -\frac{24}{7}}}$

Check: $\sqrt[3]{3-7x} - 3 = \sqrt[3]{3 - 7\left(-\frac{24}{7}\right)} - 3 = \sqrt[3]{3+24} - 3 = \sqrt[3]{27} - 3 = 3 - 3 = 0$. The solution checks out.

Example C

Find the real solutions of $\sqrt{10-x^2} - x = 2$.

Solution

We isolate the radical on one side of the equation: $\sqrt{10-x^2} = 2 + x$

Square each side of the equation: $\left(\sqrt{10-x^2}\right)^2 = (2+x)^2$

Simplify: $10 - x^2 = 4 + 4x + x^2$

Move all terms to one side of the equation: $0 = 2x^2 + 4x - 6$

Solve using the quadratic formula: $x = \frac{-4 \pm \sqrt{4^2 - 4(2)(-6)}}{4}$

Simplify: $x = \frac{-4 \pm \sqrt{64}}{4}$

Re-write $\sqrt{24}$ in simplest form: $x = \frac{-4 \pm 8}{4}$

Reduce all terms by a factor of 2: $x = 1$ or $x = -3$

Check: $\sqrt{10 - 1^2} - 1 = \sqrt{9} - 1 = 3 - 1 = 2$ This solution checks out.

$\sqrt{10-(-3)^2}-(-3)=\sqrt{1}+3=1+3=4$ This solution does not check out.

The equation has only one solution, $\underline{\underline{x=1}}$; the solution $x=-3$ is extraneous.

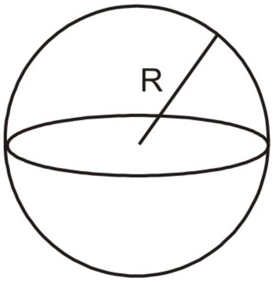
Applications using Radical Equations

Example D

A sphere has a volume of $456\ cm^3$. *If the radius of the sphere is increased by 2 cm, what is the new volume of the sphere?*

Solution

Make a sketch:



Define variables: Let $R=$ the radius of the sphere.

Find an equation: The volume of a sphere is given by the formula $V=\frac{4}{3}\pi R^3$.

Solve the equation:

Plug in the value of the volume: $456=\frac{4}{3}\pi R^3$

Multiply by 3: $1368=4\pi R^3$

Divide by 4π : $108.92=R^3$

Take the cube root of each side: $R=\sqrt[3]{108.92}\Rightarrow R=4.776\ cm$

The new radius is 2 centimeters more: $R=6.776\ cm$

The new volume is: $V=\frac{4}{3}\pi(6.776)^3=\underline{\underline{1302.5}}\ cm^3$

Check: Let's plug in the values of the radius into the volume formula:

$V=\frac{4}{3}\pi R^3=\frac{4}{3}\pi(4.776)^3=456\ cm^3$. **The solution checks out.**

Example E

The kinetic energy of an object of mass m and velocity v is given by the formula: $KE=\frac{1}{2}mv^2$. *A baseball has a mass of 145 kg and its kinetic energy is measured to be 654 Joules* $(kg\cdot m^2/s^2)$ *when it hits the catcher's glove. What is the velocity of the ball when it hits the catcher's glove?*

Solution

Start with the formula: $KE = \frac{1}{2}mv^2$

Plug in the values for the mass and the kinetic energy: $654\frac{kg \cdot m^2}{s^2} = \frac{1}{2}(145\ kg)v^2$

Multiply both sides by 2: $1308\frac{kg \cdot m^2}{s^2} = 145\ kg \cdot v^2$

Divide both sides by 145 kg : $9.02\frac{m^2}{s^2} = v^2$

Take the square root of both sides: $v = \sqrt{9.02}\sqrt{\frac{m^2}{s^2}} = 3.003\ m/s$

Check: Plug the values for the mass and the velocity into the energy formula:

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}(145\ kg)(3.003\ m/s)^2 = 654\ kg \cdot m^2/s^2$$

(To learn more about kinetic energy, watch the video at http://www.youtube.com/watch?v=zhX01toLjZs)

Watch this video for help with the Examples above.

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CK-12 Foundation: Radical Equations

Vocabulary

- For a **quadratic equation** in standard form, $ax^2+bx+c=0$, the **quadratic formula** looks like this:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Guided Practice

Find the real solutions of $\sqrt{3x-9}-1=5$.

Solution

We isolate the radical on one side of the equation: $\sqrt{3x-9} = 6$

Square each side of the equation: $\sqrt{3x-9}^2 = 6^2$

Simplify: $3x-9 = 36$

Add 9 from each side: $3x = 45$

Divide both sides by 3: $\underline{\underline{x = \frac{45}{3} = 15}}$

Check: $\sqrt{3x-9}-1 = \sqrt{3(15)-9}-1 = \sqrt{45-9}-1 = \sqrt{36}-1 = 6-1 = 5$. The solution checks out.

Practice

Find the solution to each of the following radical equations.

1. $\sqrt{x+2}-2=0$
2. $\sqrt{3x-1}=5$
3. $2\sqrt{4-3x}+3=0$
4. $\sqrt[3]{x-3}=1$
5. $\sqrt[4]{x^2-9}=2$
6. $\sqrt[3]{-2-5x}+3=0$
7. $\sqrt{x^2-3}=x-1$
8. $\sqrt{x}=x-6$
9. $\sqrt{x^2-5x}-6=0$
10. $\sqrt{(x+1)(x-3)}=x$
11. $\sqrt{x+6}=x+4$
12. $\sqrt{3x+4}=-6$
13. The area of a triangle is 24 in^2 and the height of the triangle is twice as long as the base. What are the base and the height of the triangle?
14. The length of a rectangle is 7 meters less than twice its width, and its area is 660 m^2. What are the length and width of the rectangle?
15. The area of a circular disk is 124 in^2. What is the circumference of the disk? $(\text{Area} = \pi R^2, \text{Circumference} = 2\pi R)$.
16. The volume of a cylinder is 245 cm^3 and the height of the cylinder is one third of the diameter of the base of the cylinder. The diameter of the cylinder is kept the same but the height of the cylinder is increased by 2 centimeters. What is the volume of the new cylinder? $(\text{Volume} = \pi R^2 \cdot h)$
17. The height of a golf ball as it travels through the air is given by the equation $h=-16t^2+256$. Find the time when the ball is at a height of 120 feet.

11.7 Equations with Radicals on Both Sides

Here you'll learn how to find the real roots and identify the extraneous solutions of radical equations that have radicals on both sides of the equal sign. You'll also solve real-world applications that involve such equations.

What if you had a radical equation like $\sqrt{x+1}+2=\sqrt{2x-5}$ with a radical sign on both sides? How could you find the solutions to this equation? After completing this Concept, you'll be able to solve radical equations like this one.

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CK-12 Foundation: Special Cases withRadicalEquations

Guidance

Often equations have more than one radical expression. The strategy in this case is to start by isolating the most complicated radical expression and raise the equation to the appropriate power. We then repeat the process until all radical signs are eliminated.

Example A

Find the real roots of the equation $\sqrt{2x+1}-\sqrt{x-3}=2$.

Solution

Isolate one of the radical expressions: $\sqrt{2x+1}=2+\sqrt{x-3}$

Square both sides: $\left(\sqrt{2x+1}\right)^2=\left(2+\sqrt{x-3}\right)^2$

Eliminate parentheses: $2x+1=4+4\sqrt{x-3}+x-3$

Simplify: $x=4\sqrt{x-3}$

Square both sides of the equation: $x^2=\left(4\sqrt{x-3}\right)^2$

Eliminate parentheses: $x^2=16(x-3)$

Simplify: $x^2=16x-48$

Move all terms to one side of the equation: $x^2-16x+48=0$

Factor: $(x-12)(x-4)=0$

Solve: $x=12$ or $x=4$

Check: $\sqrt{2(12)+1}-\sqrt{12-3}=\sqrt{25}-\sqrt{9}=5-3=2$. The solution checks out.

$\sqrt{2(4)+1}-\sqrt{4-3}=\sqrt{9}-\sqrt{1}=3-1=2$ The solution checks out.

The equation has two solutions: $x=12$ and $x=4$.

Identify Extraneous Solutions to Radical Equations

We saw in Example 3 that some of the solutions that we find by solving radical equations do not check out when we substitute (or "plug in") those solutions back into the original radical equation. These are called **extraneous solutions.** It is very important to check the answers we obtain by plugging them back into the original equation, so we can tell which of them are real solutions.

Example B

Find the real solutions of the equation $\sqrt{x-3}-\sqrt{x}=1$.

Solution

$$\begin{aligned} \text{Isolate one of the radical expressions:} &\quad \sqrt{x-3}=\sqrt{x}+1\\ \text{Square both sides:} &\quad \left(\sqrt{x-3}\right)^2=\left(\sqrt{x}+1\right)^2\\ \text{Remove parenthesis:} &\quad x-3=\left(\sqrt{x}\right)^2+2\sqrt{x}+1\\ \text{Simplify:} &\quad x-3=x+2\sqrt{x}+1\\ \text{Now isolate the remaining radical:} &\quad -4=2\sqrt{x}\\ \text{Divide all terms by 2:} &\quad -2=\sqrt{x}\\ \text{Square both sides:} &\quad x=4 \end{aligned}$$

Check: $\sqrt{4-3}-\sqrt{4}=\sqrt{1}-2=1-2=-1$ The solution does not check out.

The equation has no real solutions. $x=4$ is an extraneous solution.

Applications using Special Case of Radical Equations

Radical equations often appear in problems involving areas and volumes of objects.

Example C

Anita's square vegetable garden is 21 square feet larger than Fred's square vegetable garden. Anita and Fred decide to pool their money together and buy the same kind of fencing for their gardens. If they need 84 feet of fencing, what is the size of each garden?

Solution

Make a sketch:

Define variables: Let Fred's area be x; then Anita's area is $x+21$.

Find an equation:

Side length of Fred's garden is $\sqrt{x}$

Side length of Anita's garden is $\sqrt{x+21}$

The amount of fencing is equal to the combined perimeters of the two squares:

$$4\sqrt{x}+4\sqrt{x+21}=84$$

Solve the equation:

$$\begin{aligned}
\text{Divide all terms by 4:} \quad & \sqrt{x}+\sqrt{x+21}=21\\
\text{Isolate one of the radical expressions:} \quad & \sqrt{x+21}=21-\sqrt{x}\\
\text{Square both sides:} \quad & \left(\sqrt{x+21}\right)^2=\left(21-\sqrt{x}\right)^2\\
\text{Eliminate parentheses:} \quad & x+21=441-42\sqrt{x}+x\\
\text{Isolate the radical expression:} \quad & 42\sqrt{x}=420\\
\text{Divide both sides by 42:} \quad & \sqrt{x}=10\\
\text{Square both sides:} \quad & x=100\ ft^2
\end{aligned}$$

Check: $4\sqrt{100}+4\sqrt{100+21}=40+44=84$. **The solution checks out.**

Fred's garden is $10\ ft\times 10\ ft=100\ ft^2$ and Anita's garden is $11\ ft\times 11\ ft=121\ ft^2$.

Watch this video for help with the Examples above.

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CK-12 Foundation: Special Cases withRadicalEquations

Vocabulary

- The symbol for a square root is $\sqrt{\ }$. This symbol is also called the **radical sign.**

Guided Practice

Find the real solutions of the equation $\sqrt{9-x}=3+\sqrt{2x}$.

Solution

Isolate one of the radical expressions: $\sqrt{9-x} = 3 + \sqrt{2x}$

Square both sides: $(\sqrt{9-x})^2 = (3 + \sqrt{2x})^2$

Remove parenthesis: $9 - x = 9 + 6\sqrt{2x} + 2x$

Simplify: $x = -2\sqrt{2x}$

Square both sides: $x^2 = 4(2x)$

Simplify: $x^2 = 8x$

Set one side equal to zero: $x^2 - 8x = 0$

Factor: $x(x-8) = 0$

Use the zero product principle: $x = 0$ or $x = 8$

Check: First check $x = 0$:

Start with the original equation: $\sqrt{9-x} = 3 + \sqrt{2x}$

Substitute in the x-value: $\sqrt{9-8} = 3 + \sqrt{2(8)}$

Simplify: $\sqrt{1} = 3 + \sqrt{16)}$

Take the square root: $1 = 3 + 4 = 7$

The solution does not check out.

Then check $x = 8$:

Start with the original equation: $\sqrt{9-x} = 3 + \sqrt{2x}$

Substitute in the x-value: $\sqrt{9-0} = 3 + \sqrt{2(0)}$

Simplify: $\sqrt{9} = 3 + \sqrt{0)}$

Take the square root: $3 = 3 + 0 = 3$

The solution checks out.

The equation has one real solution. $x = 8$ is an extraneous solution.

Practice

Find the solution to each of the following radical equations. Identify extraneous solutions.

1. $\sqrt{x} = \sqrt{x-9} + 1$
2. $\sqrt{x+2} = \sqrt{3x-2}$
3. $5\sqrt{x} = \sqrt{x+12} + 6$
4. $\sqrt{10-5x} + \sqrt{1-x} = 7$
5. $\sqrt{2x-2} - 2\sqrt{x+2} = 0$
6. $\sqrt{2x+5} - 3\sqrt{2x-3} = \sqrt{2-x}$

7. $3\sqrt{x}-9=\sqrt{2x-14}$
8. $\sqrt{x+7}=\sqrt{x+4}+1$
9. $\sqrt{4x}=\sqrt{3-2x}-1$
10. $\sqrt{2x+5}+\sqrt{3-x}=10$

11.8 Pythagorean Theorem and its Converse

Here you'll learn how to use the Pythagorean Theorem to determine if three side lengths make a right triangle.

What if you were told that the side lengths of a triangle were 4, 5, and 7? How could you determine if the triangle were a right triangle? After completing this Concept, you'll be able to use the Pythagorean Theorem and its converse to solve problems like this one.

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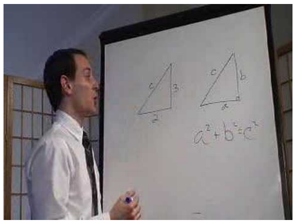
CK-12 Foundation: The Pythagorean Theorem and its Converse

Watch This

The Pythagorean Theorem can also be used to find the missing hypotenuse of a right triangle if we know the legs of the triangle.



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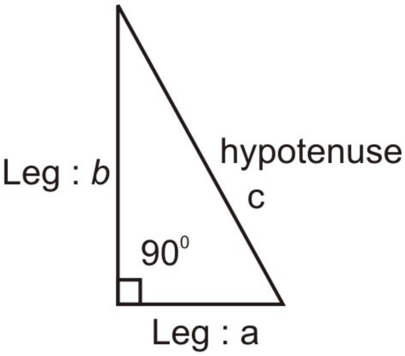
MathCrazy Tutoring:PythagoreanTheoremin 60 Seconds

Guidance

Teresa wants to string a clothesline across her backyard, from one corner to the opposite corner. If the yard measures 22 feet by 34 feet, how many feet of clothesline does she need?

The **Pythagorean Theorem** is a statement of how the lengths of the sides of a right triangle are related to each other. A right triangle is one that contains a 90 degree angle. The side of the triangle opposite the 90 degree angle is called the **hypotenuse** and the sides of the triangle adjacent to the 90 degree angle are called the **legs**.

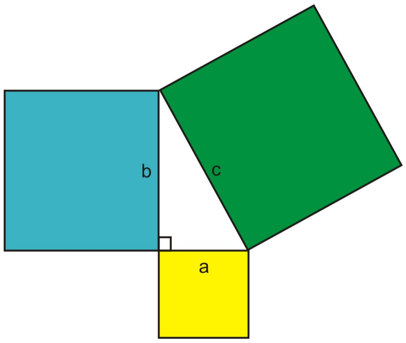


If we let *a* and *b* represent the legs of the right triangle and *c* represent the hypotenuse then the Pythagorean Theorem can be stated as:

In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs. That is: $a^2 + b^2 = c^2$.

This theorem is very useful because if we know the lengths of the legs of a right triangle, we can find the length of the hypotenuse. Also, if we know the length of the hypotenuse and the length of a leg, we can calculate the length of the missing leg of the triangle. When you use the Pythagorean Theorem, it does not matter which leg you call *a* and which leg you call *b*, but the hypotenuse is always called *c*.

Although nowadays we use the Pythagorean Theorem as a statement about the relationship between distances and lengths, originally the theorem made a statement about areas. If we build squares on each side of a right triangle, the Pythagorean Theorem says that the area of the square whose side is the hypotenuse is equal to the sum of the areas of the squares formed by the legs of the triangle.



Use the Pythagorean Theorem and Its Converse

The Pythagorean Theorem can be used to verify that a triangle is a right triangle. If you can show that the three sides of a triangle make the equation $a^2 + b^2 = c^2$ true, then you know that the triangle is a right triangle. This is called the **Converse of the Pythagorean Theorem.**

Note: When you use the Converse of the Pythagorean Theorem, you must make sure that you substitute the correct values for the legs and the hypotenuse. The hypotenuse must be the longest side. The other two sides are the legs, and the order in which you use them is not important.

Example A

Determine if a triangle with sides 5, 12 and 13 is a right triangle.

Solution

The triangle is right if its sides satisfy the Pythagorean Theorem.

If it is a right triangle, the longest side has to be the hypotenuse, so we let $c = 13$.

We then designate the shorter sides as $a = 5$ and $b = 12$.

We plug these values into the Pythagorean Theorem:

$$a^2 + b^2 = c^2 \Rightarrow 5^2 + 12^2 = c^2$$
$$25 + 144 = 169 = c^2 \Rightarrow c = 13$$

The sides of the triangle satisfy the Pythagorean Theorem, thus **the triangle is a right triangle.**

Example B

Determine if a triangle with sides, $\sqrt{10}$, $\sqrt{15}$ *and 5 is a right triangle.*

Solution

The longest side has to be the hypotenuse, so $c = 5$.

We designate the shorter sides as $a = \sqrt{10}$ and $b = \sqrt{15}$.

We plug these values into the Pythagorean Theorem:

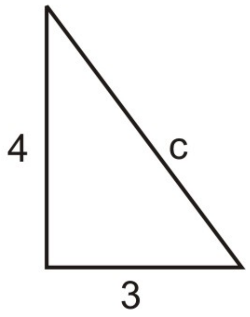
$$a^2 + b^2 = c^2 \Rightarrow \left(\sqrt{10}\right)^2 + \left(\sqrt{15}\right)^2 = c^2$$
$$10 + 15 = 25 = c^2 \Rightarrow c = 5$$

The sides of the triangle satisfy the Pythagorean Theorem, thus **the triangle is a right triangle.**

Example C

In a right triangle one leg has length 4 and the other has length 3. Find the length of the hypotenuse.

Solution



Start with the Pythagorean Theorem:	$a^2 + b^2 = c^2$
Plug in the known values of the legs:	$3^2 + 4^2 = c^2$
Simplify:	$9 + 16 = c^2$
	$25 = c^2$
Take the square root of both sides:	$c = 5$

Watch this video for help with the Examples above.

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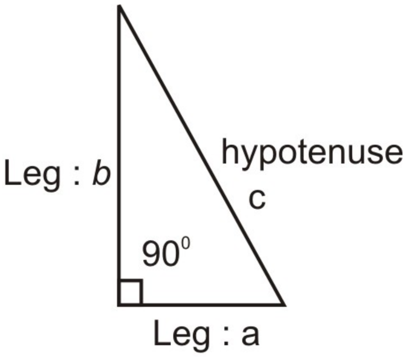
CK-12 Foundation: The Pythagorean Theorem and Its Converse

Vocabulary

- The **Pythagorean Theorem** is a statement of how the lengths of the sides of a right triangle are related to each other. A right triangle is one that contains a 90 degree angle. The side of the triangle opposite the 90 degree angle is called the **hypotenuse** and the sides of the triangle adjacent to the 90 degree angle are called the **legs**.



- If we let a and b represent the legs of the right triangle and c represent the hypotenuse then the **Pythagorean Theorem** can be stated as:

In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs. That is: $a^2+b^2=c^2$.

Guided Practice

Determine if a triangle with sides, 2, $\sqrt{21}$ *and 5 is a right triangle.*

Solution

The longest side has to be the hypotenuse, so $c=5$.

We designate the shorter sides as $a=2$ and $b=\sqrt{21}$.

We plug these values into the Pythagorean Theorem:

$$a^2+b^2=c^2 \Rightarrow (2)^2+\left(\sqrt{21}\right)^2=c^2$$
$$4+21=25=c^2 \Rightarrow c=5$$

The sides of the triangle satisfy the Pythagorean Theorem, thus **the triangle is a right triangle.**

Practice

Determine whether each set of three numbers could be the side lengths of a right triangle.

1. $a = 12, b = 9, c = 15$
2. $a = 6, b = 6, c = 6\sqrt{2}$
3. $a = 8, b = 8\sqrt{3}, c = 16$
4. $a = 2\sqrt{14}, b = 5, c = 9$
5. $a = 13, b = 16, c = 19$
6. $a = 20, b = 99, c = 101$
7. $a = 21, b = 220, c = 221$
8. $a = 7, b = 2, c = \sqrt{50}$
9. $a = 8, b = 6, c = 10$
10. $a = 7, b = \sqrt{404}, c = 25$

11.9 Solving Equations Using the Pythagorean Theorem

Here you'll learn how to apply the Pythagorean Theorem to find the unknown side lengths of right triangles.

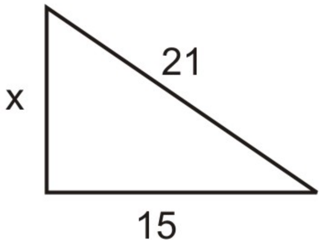
What if you knew the lengths of two sides of a right triangle but not the third? How could you find the length of this missing side? After completing this Concept, you'll be able to use the Pythagorean Theorem to solve problems like this one where variables are involved.

Guidance

In the previous concept, we learned about the Pythagorean theorem and how to use it to find the hypotenuse. In this concept, we will learn how to use the Pythagorean theorem to find any side of a right triangle.

Example A

Determine the value of the missing side. You may assume that each triangle is a right triangle.



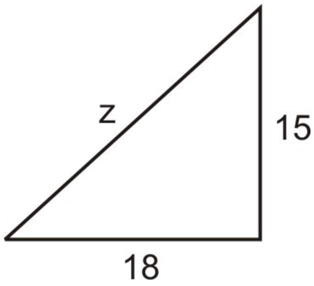
Solution

Apply the Pythagorean Theorem.

$$\begin{aligned} a^2+b^2 &= c^2 \\ x^2+15^2 &= 21^2 \\ x^2+225 &= 441 \\ x^2 &= 216 \Rightarrow \\ x &= \sqrt{216} = 6\sqrt{6} \end{aligned}$$

Example B

Determine the values of the missing sides. You may assume that each triangle is a right triangle.



Solution

Apply the Pythagorean Theorem.

$$\begin{aligned}a^2+b^2&=c^2\\18^2+15^2&=z^2\\324+225&=z\\z^2&=549\Rightarrow\\z&=\sqrt{549}=3\sqrt{61}\end{aligned}$$

Example C

One leg of a right triangle is 5 units longer than the other leg. The hypotenuse is one unit longer than twice the size of the short leg. Find the dimensions of the triangle.

Solution

Let $x =$ length of the short leg.

Then $x+5 =$ length of the long leg

And $2x+1 =$ length of the hypotenuse.

The sides of the triangle must satisfy the Pythagorean Theorem.

$$\begin{aligned}\text{Therefore:}&\quad x^2+(x+5)^2=(2x+1)^2\\\text{Eliminate the parentheses:}&\quad x^2+x^2+10x+25=4x^2+4x+1\\\text{Move all terms to the right hand side of the equation:}&\quad 0=2x^2-6x-24\\\text{Divide all terms by 2 :}&\quad 0=x^2-3x-12\\\text{Solve using the quadratic formula:}&\quad x=\frac{3\pm\sqrt{9+48}}{2}=\frac{3\pm\sqrt{57}}{2}\\&\quad x=\underline{\underline{5.27}}\text{ or }x=-2.27\end{aligned}$$

The negative solution doesn't make sense when we are looking for a physical distance, so we can discard it. Using the positive solution, we get: **short leg** $= 5.27$, **long leg** $= 10.27$ **and hypotenuse** $= 11.54$**.**

Watch this video for help with the Examples above.

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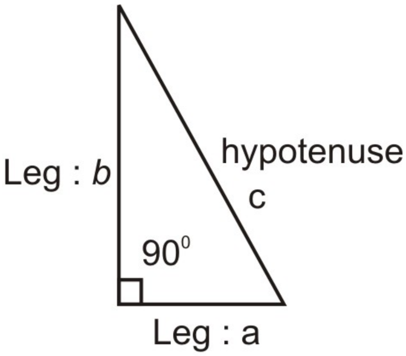
CK-12 Foundation: The Pythagorean Theorem with Variables

Vocabulary

- The **Pythagorean Theorem** is a statement of how the lengths of the sides of a right triangle are related to each other. A right triangle is one that contains a 90 degree angle. The side of the triangle opposite the 90 degree angle is called the **hypotenuse** and the sides of the triangle adjacent to the 90 degree angle are called the **legs**.

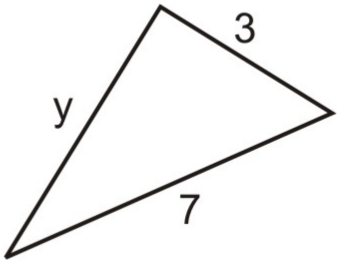


- If we let a and b represent the legs of the right triangle and c represent the hypotenuse then the **Pythagorean Theorem** can be stated as:

In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs. That is: $a^2+b^2=c^2$.

Guided Practice

Determine the values of the missing sides. You may assume that each triangle is a right triangle.



Solution

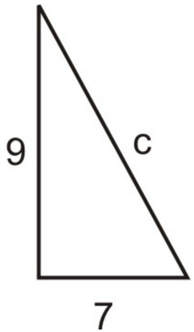
Apply the Pythagorean Theorem.

$$\begin{aligned} a^2+b^2 &= c^2 \\ y^2+3^2 &= 7^2 \\ y^2+9 &= 49 \\ y^2 &= 40 \Rightarrow \\ y &= \sqrt{40} = 2\sqrt{10} \end{aligned}$$

Practice

Find the missing length of each right triangle.

1. $a = 12, b = 16, c = ?$
2. $a = ?, b = 20, c = 30$
3. $a = 4, b = ?, c = 11$
4. $a = 12, b = ?, c = 37$



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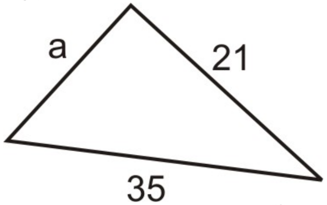

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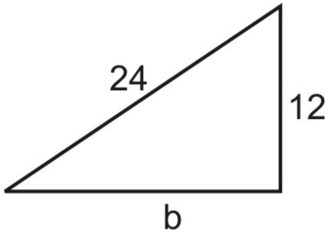


7.

8. One leg of a right triangle is 4 feet less than the hypotenuse. The other leg is 12 feet. Find the lengths of the three sides of the triangle.
9. One leg of a right triangle is 3 more than twice the length of the other. The hypotenuse is 3 times the length of the short leg. Find the lengths of the three legs of the triangle.
10. Two sides of a right triangle are 5 units and 8 units respectively. Those sides could be the legs, or they could be one leg and the hypotenuse. What are the possible lengths of the third side?

11.10 Applications Using the Pythagorean Theorem

Here you'll learn how to apply the Pythagorean Theorem and its converse to solve real-world problems.

What if a fireman needed to rescue a cat from a tree? The cat is 40 feet up from the ground. The fireman places a ladder 30 feet away from the base of the tree? How tall does the ladder need to be for him to reach the cat? After completing this Concept, you'll be able to solve real-world applications like this one using the Pythagorean Theorem and its converse.

Guidance

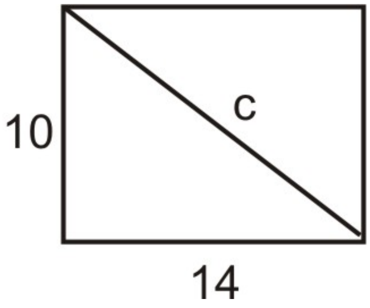
The Pythagorean Theorem and its converse have many applications for finding lengths and distances.

Example A

Maria has a rectangular cookie sheet that measures 10 *inches* × 14 *inches. Find the length of the diagonal of the cookie sheet.*

Solution

Draw a sketch:



Define variables: Let $c =$ length of the diagonal.

Write a formula: Use the Pythagorean Theorem: $a^2+b^2=c^2$

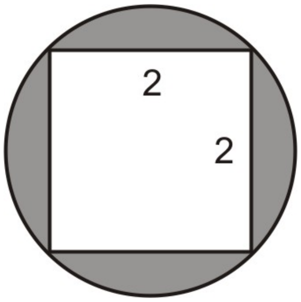
Solve the equation:

$$\begin{aligned}10^2+14^2&=c^2\\100+196&=c^2\\c^2=296\Rightarrow c=\sqrt{296}&\Rightarrow c=2\sqrt{74} \text{ or } c=17.2 \textit{ inches}\end{aligned}$$

Check: $10^2+14^2=100+196=296$ and $c^2=17.2^2=296$. The solution checks out.

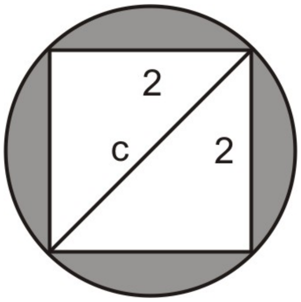
Example B

Find the area of the shaded region in the following diagram:



Solution

Draw the diagonal of the square in the figure:



Notice that the diagonal of the square is also the diameter of the circle.

Define variables: Let $c =$ diameter of the circle.

Write the formula: Use the Pythagorean Theorem: $a^2+b^2=c^2$.

Solve the equation:

$$2^2+2^2=c^2$$
$$4+4=c^2$$
$$c^2=8 \Rightarrow c=\sqrt{8} \Rightarrow c=2\sqrt{2}$$

The diameter of the circle is $2\sqrt{2}$, therefore the radius $R=\sqrt{2}$.

Area of a circle formula: $A=\pi \cdot R^2=\pi\left(\sqrt{2}\right)^2=2\pi$.

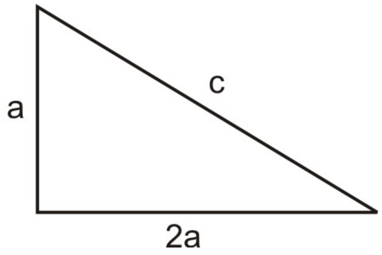
The area of the shaded region is therefore $2\pi-4=2.28$.

Example C

In a right triangle, one leg is twice as long as the other and the perimeter is 28. What are the measures of the sides of the triangle?

Solution

Make a sketch and define variables:



Let: $a =$ length of the short leg

$2a =$ length of the long leg

$c =$ length of the hypotenuse

Write formulas:

The sides of the triangle are related in two different ways.

The perimeter is 28, so $a + 2a + c = 28 \Rightarrow 3a + c = 28$

The triangle is a right triangle, so the measures of the sides must satisfy the Pythagorean Theorem:

$$a^2 + (2a)^2 = c^2 \Rightarrow a^2 + 4a^2 = c^2 \Rightarrow 5a^2 = c^2$$

$$\text{or} \qquad c = a\sqrt{5} = 2.236a$$

Solve the equation:

Plug the value of c we just obtained into the perimeter equation: $3a + c = 28$

$$3a + 2.236a = 28 \Rightarrow 5.236a = 28 \Rightarrow a = 5.35$$

The short leg is: $a = 5.35$

The long leg is: $2a = 10.70$

The hypotenuse is: $c = 11.95$

Check: The legs of the triangle should satisfy the Pythagorean Theorem:

$a^2 + b^2 = 5.35^2 + 10.70^2 = 143.1, c^2 = 11.95^2 = 142.80$. The results are approximately the same.

The perimeter of the triangle should be 28:

$a + b + c = 5.35 + 10.70 + 11.95 = 28$.**The answer checks out.**

Watch this video for help with the Examples above.

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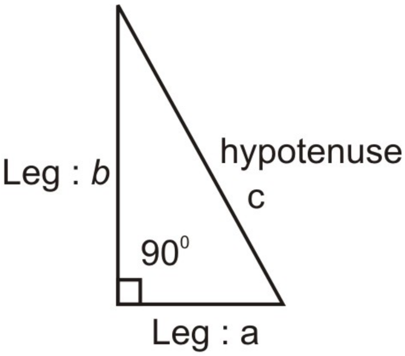
CK-12 Foundation: Applications UsingthePythagoreanTheorem

Vocabulary

- The **Pythagorean Theorem** is a statement of how the lengths of the sides of a right triangle are related to each other. A right triangle is one that contains a 90 degree angle. The side of the triangle opposite the 90 degree angle is called the **hypotenuse** and the sides of the triangle adjacent to the 90 degree angle are called the **legs**.



- If we let a and b represent the legs of the right triangle and c represent the hypotenuse then the **Pythagorean Theorem** can be stated as:

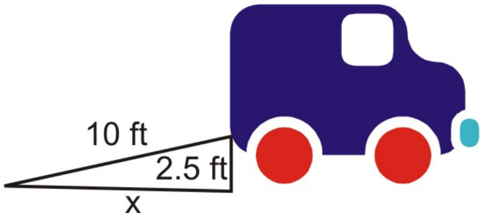
In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs. That is: $a^2+b^2=c^2$.

Guided Practice

Mike is loading a moving van by walking up a ramp. The ramp is 10 feet long and the bed of the van is 2.5 feet above the ground. How far does the ramp extend past the back of the van?

Solution

Make a sketch:



Define variables: Let $x=$ how far the ramp extends past the back of the van.

Write a formula: Use the Pythagorean Theorem: $x^2+2.5^2=10^2$

Solve the equation:

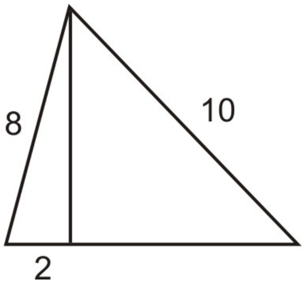
$$x^2+6.25=100$$
$$x^2=93.5$$
$$x=\sqrt{93.5}=9.7\ ft$$

Check by plugging the result in the Pythagorean Theorem:

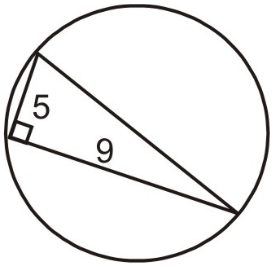
$9.7^2+2.5^2=94.09+6.25=100.34\approx 100$. So the ramp is 10 feet long. **The answer checks out.**

Practice

1. In order to make a ramp that is $3ft$ high and covers $4ft$ of ground, how long must the ramp be?
2. A regulation baseball diamond is a square with 90 feet between bases. How far is second base from home plate?
3. Emanuel has a cardboard box that measures 20 cm long $\times$ 10 cm wide $\times$ 8 cm deep.
   a. What is the length of the diagonal across the bottom of the box?
   b. What is the length of the diagonal from a bottom corner to the opposite top corner?
4. Samuel places a ladder against his house. The base of the ladder is 6 feet from the house and the ladder is 10 feet long.
   a. How high above the ground does the ladder touch the wall of the house?
   b. If the edge of the roof is 10 feet off the ground and sticks out 1.5 feet beyond the wall, how far is it from the edge of the roof to the top of the ladder?
5. Find the area of the triangle below if the area of a triangle is defined as $A = \frac{1}{2}\, base \times height$:



6. Instead of walking along the two sides of a rectangular field, Mario decided to cut across the diagonal. He thus saves a distance that is half of the long side of the field.
   a. Find the length of the long side of the field given that the short side is 123 feet.
   b. Find the length of the diagonal.
7. Marcus sails due north and Sandra sails due east from the same starting point. In two hours Marcus' boat is 35 miles from the starting point and Sandra's boat is 28 miles from the starting point.
   a. How far are the boats from each other?
   b. Sandra then sails 21 miles due north while Marcus stays put. How far is Sandra from the original starting point?
   c. How far is Sandra from Marcus now?
8. Determine the area of the circle below. (Hint: the hypotenuse of the triangle is the diameter of the circle.)



9. A rectangle's length is $1in$ longer than its width and if the diagonal has a length of $29in$, what are the lengths of the sides of the rectangle?
10. For an isosceles triangle with sides of the length given, find the length of hypotenuse:
    a. 1
    b. 2
    c. 3
    d. n

11.11 Distance Formula

Here you'l learn how to use the distance formula to find the distance between two points in the coordinate plane. You'll also learn how to find the missing coordinate of a point given its distance from another known point. Finally, you'll solve real-world applications using the distance formula.

What if you were given the coordinates of two points like (6, 2) and (-3, 0). How could you determine how far apart these two points are? After completing this Concept, you'll be able to find the distance between any two points in the coordinate plane using the Distance Formula.

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CK-12 Foundation: The Distance Formula

Guidance

In the last section, we saw how to use the Pythagorean Theorem to find lengths. In this section, you'll learn how to use the Pythagorean Theorem to find the distance between two coordinate points.

Example A

Find the distance between points $A = (1,4)$ *and* $B = (5,2)$.

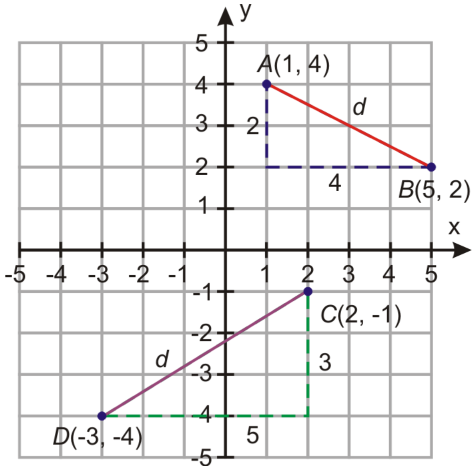
Solution

Plot the two points on the coordinate plane.

In order to get from point $A = (1,4)$ to point $B = (5,2)$, we need to move 4 units to the right and 2 units down. These lines make the legs of a right triangle.

To find the distance between A and B we find the value of the hypotenuse, d, using the Pythagorean Theorem.

$$d^2 = 2^2 + 4^2 = 20$$
$$d = \sqrt{20} = 2\sqrt{5} = 4.47$$



Example B

Find the distance between points $C = (2, -1)$ *and* $D = (-3, -4)$.

Solution

We plot the two points on the graph above.

In order to get from point C to point D, we need to move 3 units down and 5 units to the left.

We find the distance from C to D by finding the length of d with the Pythagorean Theorem.

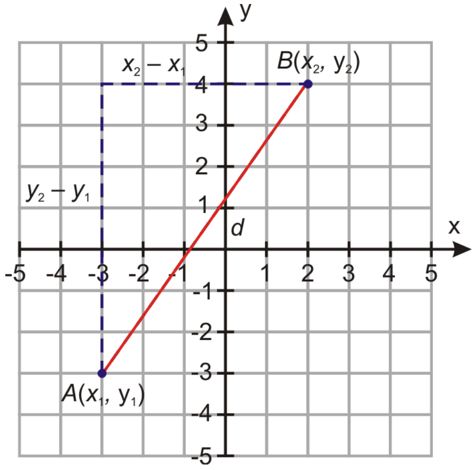
$$d^2 = 3^2 + 5^2 = 34$$
$$d = \sqrt{34} = 5.83$$

The Distance Formula

The procedure we just used can be generalized by using the Pythagorean Theorem to derive a formula for the distance between any two points on the coordinate plane.

Let's find the distance between two general points $A = (x_1, y_1)$ and $B = (x_2, y_2)$.

Start by plotting the points on the coordinate plane:



In order to move from point A to point B in the coordinate plane, we move $x_2 - x_1$ units to the right and $y_2 - y_1$ units up.

We can find the length d by using the Pythagorean Theorem:

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

Therefore, $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. This is called the **Distance Formula.** More formally:

Given any two points (x_1, y_1) and (x_2, y_2), the distance between them is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

We can use this formula to find the distance between any two points on the coordinate plane. Notice that the distance is the same whether you are going from point A to point B or from point B to point A, so it does not matter which order you plug the points into the distance formula.

Let's now apply the distance formula to the following examples.

Example C

Find the distance between the following points.

a) (-3, 5) and (4, -2)

b) (12, 16) and (19, 21)

c) (11.5, 2.3) and (-4.2, -3.9)

Solution

Plug the values of the two points into the distance formula. Be sure to simplify if possible.

a) $d = \sqrt{(-3-4)^2 + (5-(-2))^2} = \sqrt{49+49} = \sqrt{98} = 7\sqrt{2}$

b) $d = \sqrt{(12-19)^2 + (16-21)^2} = \sqrt{49+25} = \sqrt{74}$

c) $d = \sqrt{(11.5+4.2)^2 + (2.3+3.9)^2} = \sqrt{284.93} = 16.88$

Applications Using Distance and Midpoint Formulas

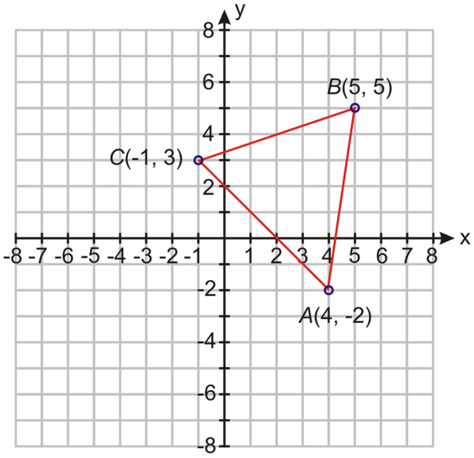
The distance and midpoint formula are useful in geometry situations where we want to find the distance between two points or the point halfway between two points.

Example D

Plot the points $A = (4, -2), B = (5, 5)$*, and* $C = (-1, 3)$ *and connect them to make a triangle. Show that the triangle is isosceles.*

Solution

Let's start by plotting the three points on the coordinate plane and making a triangle:



We use the distance formula three times to find the lengths of the three sides of the triangle.

$$AB = \sqrt{(4-5)^2+(-2-5)^2} = \sqrt{(-1)^2+(-7)^2} = \sqrt{50} = 5\sqrt{2}$$
$$BC = \sqrt{(5+1)^2+(5-3)^2} = \sqrt{(6)^2+(2)^2} = \sqrt{40} = 2\sqrt{10}$$
$$AC = \sqrt{(4+1)^2+(-2-3)^2} = \sqrt{(5)^2+(-5)^2} = \sqrt{50} = 5\sqrt{2}$$

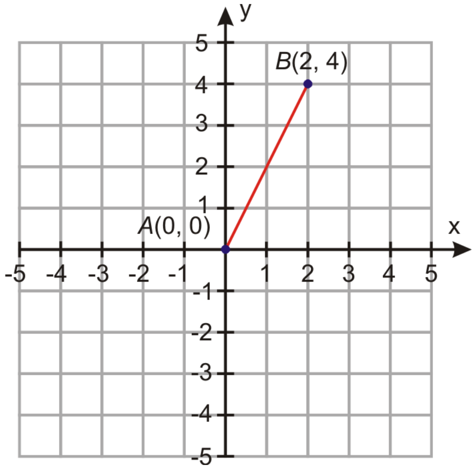
Notice that $AB = AC$, therefore triangle ABC is isosceles.

Example E

At 8 AM one day, Amir decides to walk in a straight line on the beach. After two hours of making no turns and traveling at a steady rate, Amir is two miles east and four miles north of his starting point. How far did Amir walk and what was his walking speed?

Solution

Let's start by plotting Amir's route on a coordinate graph. We can place his starting point at the origin: $A = (0,0)$. Then his ending point will be at $B = (2,4)$.



The distance can be found with the distance formula:

$$d = \sqrt{(2-0)^2+(4-0)^2} = \sqrt{(2)^2+(4)^2} = \sqrt{4+16} = \sqrt{20}$$
$$d = \underline{\underline{4.47\ miles}}$$

Since Amir walked 4.47 miles in 2 hours, his speed is $s = \frac{4.47\ miles}{2\ hours} = \underline{\underline{2.24\ mi/h}}$.

Watch this video for help with the Examples above.

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CK-12 Foundation: The Distance Formula

Vocabulary

- The **Distance Formula** states that given any two points (x_1,y_1) and (x_2,y_2), the distance between them is

$d = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$.

Guided Practice

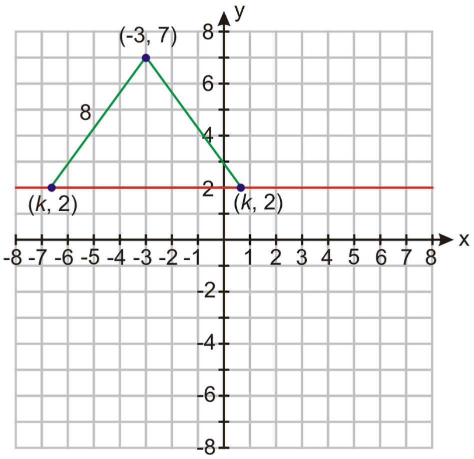
Find all points on the line $y = 2$ *that are exactly 8 units away from the point (-3, 7).*

Solution

Let's make a sketch of the given situation.

Draw line segments from the point (-3, 7) to the line $y = 2$.

Let k be the missing value of x we are seeking.



Let's use the distance formula: $8 = \sqrt{(-3-k)^2+(7-2)^2}$

Square both sides of the equation: $64 = (-3-k)^2+25$

Therefore: $0 = 9+6k+k^2-39$ or $0 = k^2+6k-30$

Use the quadratic formula: $k = \frac{-6 \pm \sqrt{36+120}}{2} = \frac{-6 \pm \sqrt{156}}{2}$

Therefore: $k = 3.24$ or $k = -9.24$

The points are **(-9.24, 2) and (3.24, 2).**

Practice

Find the distance between the two points.

1. (3, -4) and (6, 0)
2. (-1, 0) and (4, 2)
3. (-3, 2) and (6, 2)
4. (0.5, -2.5) and (4, -4)
5. (12, -10) and (0, -6)
6. (-5, -3) and (-2, 11)
7. (2.3, 4.5) and (-3.4, -5.2)
8. Find all points having an $x-$coordinate of -4 whose distance from the point (4, 2) is 10.
9. Find all points having a $y-$coordinate of 3 whose distance from the point (-2, 5) is 8.
10. Find three points that are each 13 units away from the point (3, 2) but do *not* have an $x-$coordinate of 3 or a $y-$coordinate of 2.

Find the midpoint of the line segment joining the two points.

11. Plot the points $A = (1,0), B = (6,4), C = (9,-2)$ and $D = (-6,-4), E = (-1,0), F = (2,-6)$. Prove that triangles ABC and DEF are congruent.
12. Plot the points $A = (4,-3), B = (3,4), C = (-2,-1), D = (-1,-8)$. Show that $ABCD$ is a rhombus (all sides are equal)
13. Plot points $A = (-5,3), B = (6,0), C = (5,5)$. Find the length of each side. Show that ABC is a right triangle. Find its area.
14. Find the area of the circle with center (-5, 4) and the point on the circle (3, 2).
15. Michelle decides to ride her bike one day. First she rides her bike due south for 12 miles and then the direction of the bike trail changes and she rides in the new direction for a while longer. When she stops Michelle is 2 miles south and 10 miles west from her starting point. Find the total distance that Michelle covered from her starting point.

11.12 Midpoint Formula

Here you'll learn how to use the midpoint formula to find the coordinates of the point that is in the middle of the line segment connecting two given points. You'll also use that formula to find one endpoint of a line segment given its other endpoint and its midpoint.

What if you were given the coordinates of two points like (4, 1) and (0, -3)? How could you find the midpoint of the line segment joining the two points? After completing this Concept, you'll be able to find the midpoint of any line segment using the Midpoint Formula.

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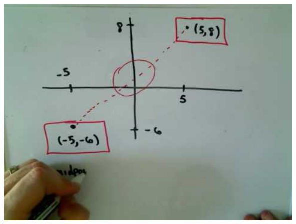
CK-12 Foundation: The Midpoint Formula

For a graphic demonstration of the midpoint formula, watch this video:



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PatrickJMT: The MidpointFormula

Guidance

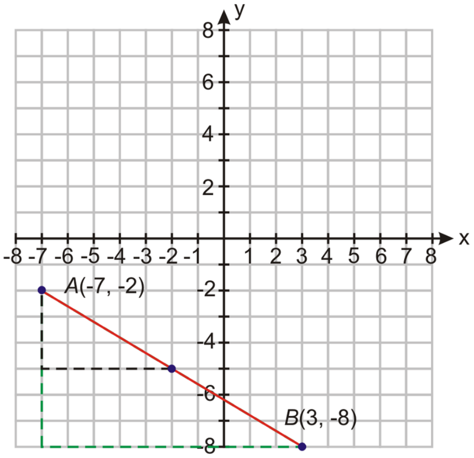
In the last concept, you saw how to find the distance between two points. In this concept, you will learn how to find the point exactly half way between two points.

Example A

Find the coordinates of the point that is in the middle of the line segment connecting the points $A = (-7, -2)$ *and* $B = (3, -8)$.

Solution

Let's start by graphing the two points:



We see that to get from point A to point B we move 6 units down and 10 units to the right.

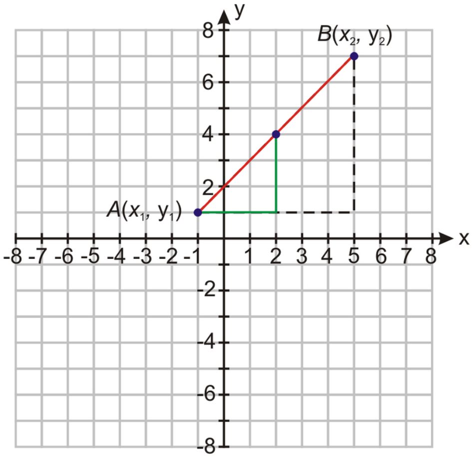
In order to get to the point that is halfway between the two points, it makes sense that we should move half the vertical distance and half the horizontal distance—that is, 3 units down and 5 units to the right from point A.

The midpoint is $M = (-7+5, -2-3) = (-2, -5)$.

The Midpoint Formula

We now want to generalize this method in order to find a formula for the midpoint of a line segment.

Let's take two general points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ and mark them on the coordinate plane:



We see that to get from A to B, we move $x_2 - x_1$ units to the right and $y_2 - y_1$ units up.

In order to get to the half-way point, we need to move $\frac{x_2-x_1}{2}$ units to the right and $\frac{y_2-y_1}{2}$ up from point A. Thus the midpoint M is at $\left(x_1 + \frac{x_2-x_1}{2}, y_1 + \frac{y_2-y_1}{2}\right)$.

This simplifies to $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. This is the **Midpoint Formula:**

The midpoint of the line segment connecting the points (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

It should hopefully make sense that the midpoint of a line is found by taking the average values of the x and $y-$values of the endpoints.

Example B

Find the midpoint between the following points.

a) (-10, 2) and (3, 5)

b) (3, 6) and (7, 6)

Solution

Let's apply the Midpoint Formula: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

a) the midpoint of (-10, 2) and (3, 5) is $\left(\frac{-10+3}{2}, \frac{2+5}{2}\right) = \left(\frac{-7}{2}, \frac{7}{2}\right) = \underline{\underline{(-3.5, 3.5)}}$

b) the midpoint of (3, 6) and (7, 6) is $\left(\frac{3+7}{2}, \frac{6+6}{2}\right) = \left(\frac{10}{2}, \frac{12}{2}\right) = \underline{\underline{(5, 6)}}$

Example C

A line segment whose midpoint is (2, -6) has an endpoint of (9, -2). What is the other endpoint?

Solution

In this problem we know the midpoint and we are looking for the missing endpoint.

The midpoint is (2, -6).

One endpoint is $(x_1, x_2) = (9, -2)$.

Let's call the missing point (x, y).

We know that the $x-$coordinate of the midpoint is 2, so: $2 = \frac{9+x_2}{2} \Rightarrow 4 = 9 + x_2 \Rightarrow x_2 = -5$

We know that the $y-$coordinate of the midpoint is -6, so:

$$-6 = \frac{-2+y_2}{2} \Rightarrow -12 = -2 + y_2 \Rightarrow y_2 = -10$$

The missing endpoint is **(-5, -10).**

Here's another way to look at this problem: To get from the endpoint (9, -2) to the midpoint (2, `[U+2011]`6), we had to go 7 units left and 4 units down. To get from the midpoint to the other endpoint, then, we would need to go 7 more units left and 4 more units down, which takes us to (-5, -10).

Watch this video for help with the Examples above.

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CK-12 Foundation: The Midpoint Formula

Vocabulary

- The **Midpoint Formula** states that the midpoint of the line segment connecting the points (x_1, y_1) and (x_2, y_2) is

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Guided Practice

Find the midpoint between the points (4, -5) and (-4, 5).

Solution

Let's apply the Midpoint Formula: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

The midpoint of (4, -5) and (-4, 5) is $\left(\frac{4-4}{2}, \frac{-5+5}{2}\right) = \left(\frac{0}{2}, \frac{0}{2}\right) = \underline{\underline{(0,0)}}$

Practice

Find the midpoint of the line segment joining the two points.

1. (3, -4) and (6, 1)
2. (2, -3) and (2, 4)
3. (4, -5) and (8, 2)
4. (1.8, -3.4) and (-0.4, 1.4)
5. (5, -1) and (-4, 0)
6. (10, 2) and (2, -4)
7. (3, -3) and (2, 5)
8. An endpoint of a line segment is (4, 5) and the midpoint of the line segment is (3, -2). Find the other endpoint.
9. An endpoint of a line segment is (-10, -2) and the midpoint of the line segment is (0, 4). Find the other endpoint.
10. Find a point that is the same distance from (4, 5) as it is from (-2, -1), but is *not* the midpoint of the line segment connecting them.

Summary

This chapter begins by focusing on the graphs of one type of radical called square root functions. It teaches how to graph simple square root functions and addresses what types of mathematical operations performed on those functions produce vertical and horizontal shifts in their graphs. It then moves on to a more general discussion of radicals, introducing properties and special cases that can be used to solve radical equations. An important geometric application of radicals, the Pythagorean Theorem, is covered next as a way to find the unknown values of a right triangle. The chapter closes with two more geometric applications: the Distance Formula and the Midpoint Formula.

CHAPTER 12

Rational Equations and Functions

Chapter Outline

12.1 INVERSE VARIATION MODELS
12.2 GRAPHS OF RATIONAL FUNCTIONS
12.3 HORIZONTAL AND VERTICAL ASYMPTOTES
12.4 DETERMINING ASYMPTOTES BY DIVISION
12.5 DIVISION OF POLYNOMIALS
12.6 INVERSE VARIATION PROBLEMS
12.7 EXCLUDED VALUES FOR RATIONAL EXPRESSIONS
12.8 MULTIPLICATION OF RATIONAL EXPRESSIONS
12.9 DIVISION OF RATIONAL EXPRESSIONS
12.10 ADDITION AND SUBTRACTION OF RATIONAL EXPRESSIONS
12.11 APPLICATIONS OF ADDING AND SUBTRACTING RATIONAL EXPRESSIONS
12.12 RATIONAL EQUATIONS USING PROPORTIONS
12.13 APPLICATIONS USING RATIONAL EQUATIONS

Introduction

Up until this point, you've dealt with non-fractional polynomials, but what if you were asked to solve a fractional function with a polynomial in the numerator and the denominator? Such functions are known as rational functions. The graphs of these functions grow closer and closer to certain values without ever reaching those values. This is called asymptotic behavior. In this chapter, you'll find the asymptotes of rational functions. You'll also perform operations on rational expressions and solve rational equations. In the real world, rational equations are often used to model electrical circuit and distance problems.

12.1 Inverse Variation Models

Here you'l learn how to graph inverse variation equations. You'll also learn how to write and solve such equations to find unknown values.

What if you were paid $500 per week regardless of the number of hours you worked? The more hours you worked in a week (increasing quantity), the less your hourly rate (decreasing quantity) would be. How could you write and solve a function to model this situation? After completing Concept, you'll be able to write inverse variation equations and solve inverse variation applications like this one.

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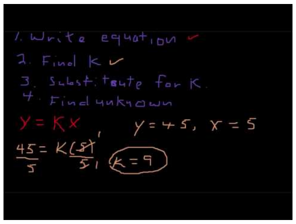
Foundation: 1201S nverseVariation Models

Watch this video to see some more variation problems worked out, including problems involving joint variation.



1. Write equation ✓
2. Find K ✓
3. Substitute for K
4. Find unknown

$y = Kx$, $y = 45$, $x = 5$

$\frac{45}{5} = \frac{K(5)}{5}$, $\boxed{K = 9}$

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Mathphonetutor: Algebra:Direct,Inverse, Joint Variation Problem

Guidance

Many variables in real-world problems are related to each other by variations. A **variation** is an equation that relates a variable to one or more other variables by the operations of multiplication and division. There are three different kinds of variation: **direct variation, inverse variation** and **joint variation.**

Distinguish Direct and Inverse Variation

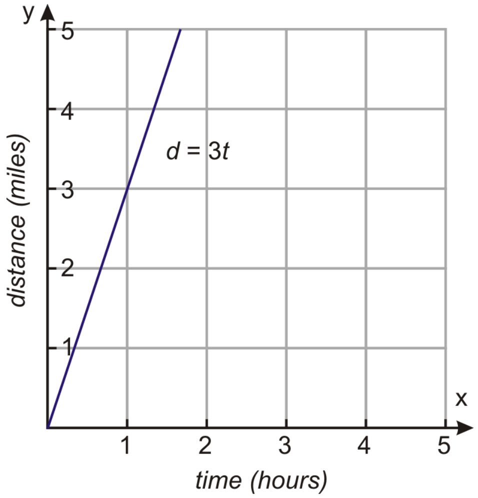
In **direct variation** relationships, the related variables will either increase together or decrease together at a steady rate. For instance, consider a person walking at three miles per hour. As time increases, the distance covered by the person walking also increases, at the rate of three miles each hour. The distance and time are related to each other by a direct variation:

$$distance = speed \times time$$

Since the speed is a constant 3 miles per hour, we can write: $d = 3t$.

The general equation for a direct variation is $y = kx$, where k is called the **constant of proportionality.**

You can see from the equation that a direct variation is a linear equation with a $y-$intercept of zero. The graph of a direct variation relationship is a straight line passing through the origin whose slope is k, the constant of proportionality.



A second type of variation is **inverse variation**. When two quantities are related to each other inversely, one quantity increases as the other one decreases, and vice versa.

For instance, if we look at the formula $distance = speed \times time$ again and solve for time, we obtain:

$$time = \frac{distance}{speed}$$

If we keep the distance constant, we see that as the speed of an object increases, then the time it takes to cover that distance decreases. Consider a car traveling a distance of 90 miles, then the formula relating time and speed is: $t = \frac{90}{s}$.

The general equation for inverse variation is $y = \frac{k}{x}$, where k is the **constant of proportionality**.

In this chapter, we'll investigate how the graphs of these relationships behave.

Another type of variation is a **joint variation**. In this type of relationship, one variable may vary as a product of two or more variables.

For example, the volume of a cylinder is given by:

$$V = \pi R^2 \cdot h$$

In this example the volume varies directly as the product of the square of the radius of the base and the height of the cylinder. The constant of proportionality here is the number π.

In many application problems, the relationship between the variables is a combination of variations. For instance Newton's Law of Gravitation states that the force of attraction between two spherical bodies varies jointly as the masses of the objects and inversely as the square of the distance between them:

$$F = G\frac{m_1 m_2}{d^2}$$

In this example the constant of proportionality is called the gravitational constant, and its value is given by $G = 6.673 \times 10^{-11}\ N \cdot m^2/kg^2$.

Graph Inverse Variation Equations

We saw that the general equation for inverse variation is given by the formula $y = \frac{k}{x}$, where k is a constant of proportionality. We will now show how the graphs of such relationships behave. We start by making a table of values. In most applications, x and y are positive, so in our table we'll choose only positive values of x.

Example A

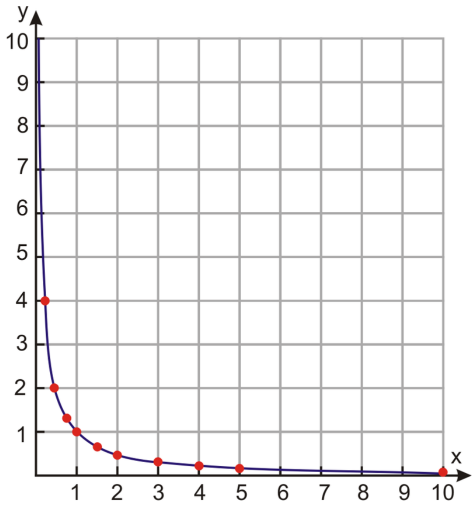
Graph an inverse variation relationship with the proportionality constant $k = 1$.

Solution

TABLE 12.1:

x	$y = \frac{1}{x}$
0	$y = \frac{1}{0} =$ undefined
$\frac{1}{4}$	$y = \frac{1}{\frac{1}{4}} = 4$
$\frac{1}{2}$	$y = \frac{1}{\frac{1}{2}} = 2$
$\frac{3}{4}$	$y = \frac{1}{\frac{3}{4}} = 1.33$
1	$y = \frac{1}{1} = 1$
$\frac{3}{2}$	$y = \frac{1}{\frac{3}{2}} = 0.67$
2	$y = \frac{1}{2} = 0.5$
3	$y = \frac{1}{3} = 0.33$
4	$y = \frac{1}{4} = 0.25$
5	$y = \frac{1}{5} = 0.2$
10	$y = \frac{1}{10} = 0.1$

Here is a graph showing these points connected with a smooth curve.



Both the table and the graph demonstrate the relationship between variables in an inverse variation. As one variable increases, the other variable decreases and vice versa.

Notice that when $x = 0$, the value of y is undefined. The graph shows that when the value of x is very small, the value of y is very big—so it approaches infinity as x gets closer and closer to zero.

Similarly, as the value of x gets very large, the value of y gets smaller and smaller but never reaches zero. We will investigate this behavior in detail throughout this chapter.

Write Inverse Variation Equations

As we saw, an inverse variation fulfills the equation $y = \frac{k}{x}$. In general, we need to know the value of y at a particular value of x in order to find the proportionality constant. Once we know the proportionality constant, we can then find the value of y for any given value of x.

Example B

If y is inversely proportional to x, and if $y = 10$ *when* $x = 5$*, find y when* $x = 2$.

Solution

$$\text{Since } y \text{ is inversely proportional to } x, \text{then:} \qquad y = \frac{k}{x}$$

$$\text{Plug in the values } y = 10 \text{ and } x = 5: \qquad 10 = \frac{k}{5}$$

$$\text{Solve for } k \text{ by multiplying both sides of the equation by } 5: \quad k = 50$$

$$\text{The inverse relationship is given by:} \qquad y = \frac{50}{x}$$

$$\text{When } x = 2: \qquad y = \frac{50}{2} \text{ or } y = 25$$

Compare Graphs of Inverse Variation Equations

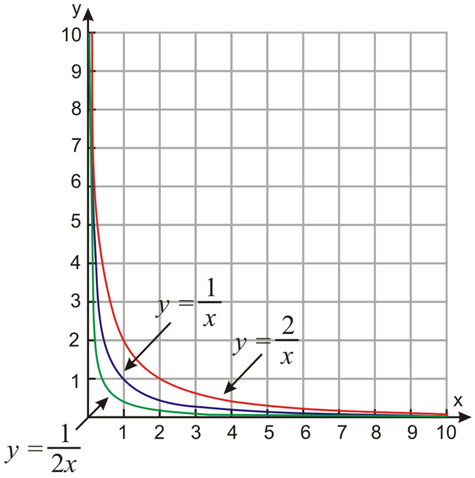
Inverse variation problems are the simplest example of rational functions. We saw that an inverse variation has the general equation: $y = \frac{k}{x}$. In most real-world problems, x and y take only positive values. Below, we will show graphs of three inverse variation functions.

Example C

On the same coordinate grid, graph inverse variation relationships with the proportionality constants $k = 1, k = 2$, *and* $k = \frac{1}{2}$.

Solution

We'll skip the table of values for this problem, and just show the graphs of the three functions on the same coordinate axes. Notice that for larger constants of proportionality, the curve decreases at a slower rate than for smaller constants of proportionality. This makes sense because the value of y is related directly to the proportionality constants, so we should expect larger values of y for larger values of k.



Watch this video for help with the Examples above.

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CK-12 Foundation: Inverse Variation Models

Vocabulary

- The general equation for a direct variation is $y = kx$, where k is called the **constant of proportionality.**
- The general equation for inverse variation is $y = \frac{k}{x}$, where k is the **constant of proportionality**.

Guided Practice

If p is inversely proportional to the square of q, and $p = 64$ *when* $q = 3$*, find p when* $q = 5$.

Solution

Since p is inversely proportional to q^2, then: $$p = \frac{k}{q^2}$$

Plug in the values $p = 64$ and $q = 3$: $$64 = \frac{k}{3^2} \text{ or } 64 = \frac{k}{9}$$

Solve for k by multiplying both sides of the equation by 9: $$k = 576$$

The inverse relationship is given by: $$p = \frac{576}{q^2}$$

When $q = 5$: $$p = \frac{576}{25} \text{ or } y = 23.04$$

Practice

For 1-4, graph the following inverse variation relationships.

1. $y = \frac{3}{x}$
2. $y = \frac{10}{x}$
3. $y = \frac{1}{4x}$
4. $y = \frac{5}{6x}$
5. If z is inversely proportional to w and $z = 81$ when $w = 9$, find w when $z = 24$.
6. If y is inversely proportional to x and $y = 2$ when $x = 8$, find y when $x = 12$.
7. If a is inversely proportional to the square root of b, and $a = 32$ when $b = 9$, find b when $a = 6$.
8. If w is inversely proportional to the square of u and $w = 4$ when $u = 2$, find w when $u = 8$.
9. If a is proportional to both b and c and $a = 7$ when $b = 2$ and $c = 6$, find a when $b = 4$ and $c = 3$.
10. If x is proportional to y and inversely proportional to z, and $x = 2$ when $y = 10$ and $z = 25$, find x when $y = 8$ and $z = 35$.
11. If a varies directly with b and inversely with the square of c, and $a = 10$ when $b = 5$ and $c = 2$, find the value of a when $b = 3$ and $c = 6$.
12. If x varies directly with y and z varies inversely with x, and $z = 3$ when $y = 5$, find z when $y = 10$.

12.2 Graphs of Rational Functions

Here you'll learn how to graph rational functions and find their asymptotes.

What if you had a function like $y = \frac{x+1}{x^2-4}$? How could you graph it and find its asymptotes? After completing this Concept, you'll be able to graph rational functions like this one.

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CK-12 Foundation: 1202S RGraphs of Rational Functions

Guidance

Graphs of rational functions are very distinctive, because they get closer and closer to certain values but never reach those values. This behavior is called asymptotic behavior, and we will see that rational functions can have **horizontal asymptotes, vertical asymptotes** or **oblique (or slant) asymptotes.**

Now we'll extend the domain and range of rational equations to include negative values of x and y. First we'll plot a few rational functions by using a table of values, and then we'll talk about the distinguishing characteristics of rational functions that can help us make better graphs.

As we graph rational functions, we need to always pay attention to values of x that will cause us to divide by 0. Remember that dividing by 0 doesn't give us an actual number as a result.

Example A

Graph the function $y = \frac{1}{x}$.

Solution

Before we make a table of values, we should notice that the function is not defined for $x = 0$. This means that the graph of the function won't have a value at that point. Since the value of $x = 0$ is special, we should make sure to pick enough values close to $x = 0$ in order to get a good idea how the graph behaves.

Let's make two tables: one for $x-$values smaller than zero and one for $x-$values larger than zero.

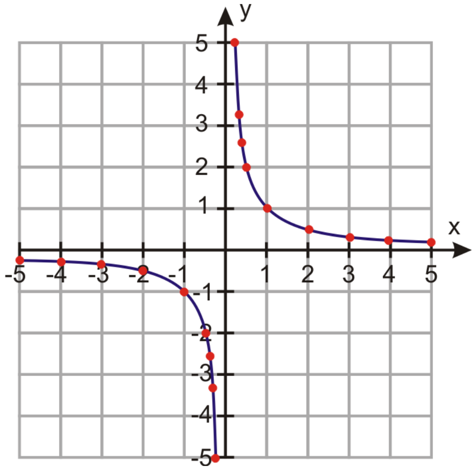
TABLE 12.2:

x	$y = \frac{1}{x}$	x	$y = \frac{1}{x}$
-5	$y = \frac{1}{-5} = -0.2$	0.1	$y = \frac{1}{0.1} = 10$
-4	$y = \frac{1}{-4} = -0.25$	0.2	$y = \frac{1}{0.2} = 5$
-3	$y = \frac{1}{-3} = -0.33$	0.3	$y = \frac{1}{0.3} = 3.3$
-2	$y = \frac{1}{-2} = -0.5$	0.4	$y = \frac{1}{0.4} = 2.5$

TABLE 12.2: (continued)

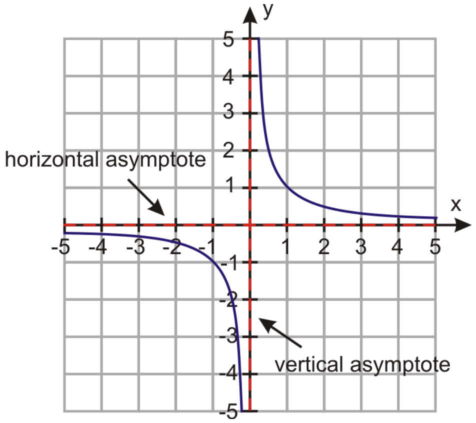
x	$y=\frac{1}{x}$	x	$y=\frac{1}{x}$
-1	$y=\frac{1}{-1}=-1$	0.5	$y=\frac{1}{0.5}=2$
-0.5	$y=\frac{1}{-0.5}=-2$	1	$y=\frac{1}{1}=1$
-0.4	$y=\frac{1}{-0.4}=-2.5$	2	$y=\frac{1}{2}=0.5$
-0.3	$y=\frac{1}{-0.3}=-3.3$	3	$y=\frac{1}{3}=0.33$
-0.2	$y=\frac{1}{-0.2}=-5$	4	$y=\frac{1}{4}=0.25$
-0.1	$y=\frac{1}{-0.1}=-10$	5	$y=\frac{1}{5}=0.2$

We can see that as we pick positive values of x closer and closer to zero, y gets larger, and as we pick negative values of x closer and closer to zero, y gets smaller (or more and more negative).



Notice on the graph that for values of x near 0, the points on the graph get closer and closer to the vertical line $x=0$. The line $x=0$ is called a **vertical asymptote** of the function $y=\frac{1}{x}$.

We also notice that as the absolute values of x get larger in the positive direction or in the negative direction, the value of y gets closer and closer to $y=0$ but will never gain that value. Since $y=\frac{1}{x}$, we can see that there are no values of x that will give us the value $y=0$. The horizontal line $y=0$ is called a **horizontal asymptote** of the function $y=\frac{1}{x}$



Asymptotes are usually denoted as dashed lines on a graph. They are not part of the function; instead, they show values that the function approaches, but never gets to. A horizontal asymptote shows the value of y that the function approaches (but never reaches) as the absolute value of x gets larger and larger. A vertical asymptote shows that the absolute value of y gets larger and larger as x gets closer to a certain value which it can never actually reach.

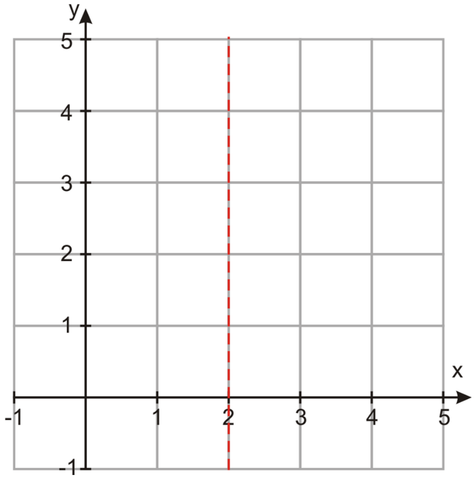
Now we'll show the graph of a rational function that has a vertical asymptote at a non-zero value of x.

Example B

Graph the function $y = \frac{1}{(x-2)^2}$.

Solution

We can see that the function is not defined for $x = 2$, because that would make the denominator of the fraction equal zero. This tells us that there should be a vertical asymptote at $x = 2$, so we can start graphing the function by drawing the vertical asymptote.

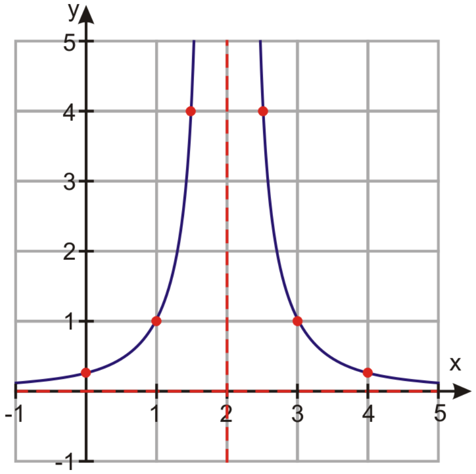


Now let's make a table of values.

TABLE 12.3:

x	$y = \frac{1}{(x-2)^2}$
0	$y = \frac{1}{(0-2)^2} = \frac{1}{4}$
1	$y = \frac{1}{(1-2)^2} = 1$
1.5	$y = \frac{1}{(1.5-2)^2} = 4$
2	undefined
2.5	$y = \frac{1}{(2.5-2)^2} = 4$
3	$y = \frac{1}{(3-2)^2} = 1$
4	$y = \frac{1}{(4-2)^2} = \frac{1}{4}$

Here's the resulting graph:



Notice that we didn't pick as many values for our table this time, because by now we have a pretty good idea what happens near the vertical asymptote.

We also know that for large values of $|x|$, the value of y could approach a constant value. In this case that value is $y = 0$: this is the horizontal asymptote.

A rational function doesn't have to have a vertical or horizontal asymptote. The next example shows a rational function with no vertical asymptotes.

Example C

Graph the function $y = \frac{x^2}{x^2+1}$.

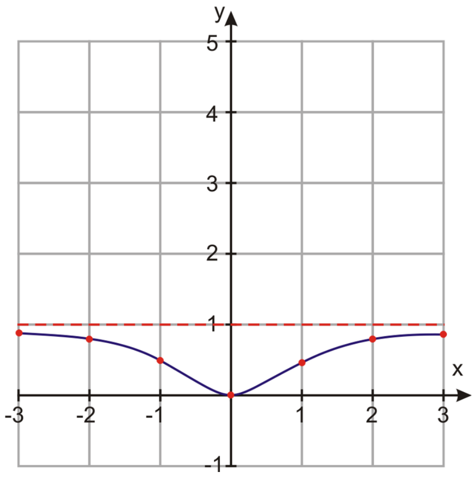
Solution

We can see that this function will have no vertical asymptotes because the denominator of the fraction will never be zero. Let's make a table of values to see if the value of y approaches a particular value for large values of x, both positive and negative.

TABLE 12.4:

x	$y = \frac{x^2}{x^2+1}$
-3	$y = \frac{(-3)^2}{(-3)^2+1} = \frac{9}{10} = 0.9$
-2	$y = \frac{(-2)^2}{(-2)^2+1} = \frac{4}{5} = 0.8$
-1	$y = \frac{(-1)^2}{(-1)^2+1} = \frac{1}{2} = 0.5$
0	$y = \frac{(0)^2}{(0)^2+1} = \frac{0}{1} = 0$
1	$y = \frac{(1)^2}{(1)^2+1} = \frac{1}{2} = 0.5$
2	$y = \frac{(2)^2}{(2)^2+1} = \frac{4}{5} = 0.8$
3	$y = \frac{(3)^2}{(3)^2+1} = \frac{9}{10} = 0.9$

Below is the graph of this function.



The function has no vertical asymptote. However, we can see that as the values of $|x|$ get larger, the value of y gets closer and closer to 1, so the function has a horizontal asymptote at $y = 1$.

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CK-12 Foundation: Graphsof RationalFunctions

Vocabulary

- Graphs of rational functions are very distinctive, because they get closer and closer to certain values but never reach those values. This behavior is called asymptotic behavior, and we will see that rational functions can have **horizontal asymptotes, vertical asymptotes** or **oblique (or slant) asymptotes.**

Guided Practice

Graph the function $y = \frac{3}{x-1}$.

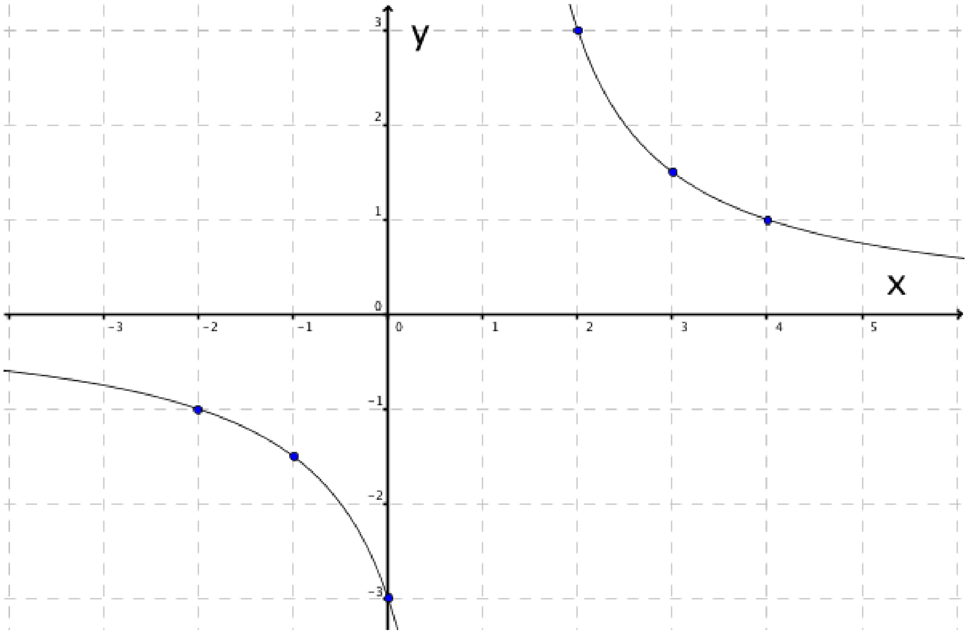
Solution

Start by making a table of values:

TABLE 12.5:

x	$y = \frac{3}{x-1}$
-2	$y = \frac{3}{-2-1} = \frac{3}{-3} = -1$
-1	$y = \frac{3}{-1-1} = \frac{3}{-2} = -1.5$
0	$y = \frac{3}{0-1} = \frac{3}{-1} = -3$
1	undefined
2	$y = \frac{3}{2-1} = \frac{3}{1} = 3$
3	$y = \frac{3}{3-1} = \frac{3}{2} = 1.5$
4	$y = \frac{3}{4-1} = \frac{3}{3} = 1$

Next, graph the points. Recall that the function $y = \frac{1}{x}$ has two curves, that are on either side of the vertical asymptote, which is where the function is undefined. The same is true for this function.



Practice

Graph the following rational functions. Draw dashed vertical and horizontal lines on the graph to denote asymptotes.

1. $y = \frac{2}{x-3}$
2. $y = \frac{3}{x^2}$
3. $y = \frac{x}{x-1}$
4. $y = \frac{2x}{x+1}$
5. $y = \frac{-1}{x^2+2}$
6. $y = \frac{x}{x^2+9}$
7. $y = \frac{x^2}{x^2+1}$
8. $y = \frac{1}{x^2-1}$
9. $y = \frac{2x}{x^2-9}$
10. $y = \frac{x^2}{x^2-16}$
11. $y = \frac{3}{x^2-4x+4}$
12. $y = \frac{x}{x^2-x-6}$

12.3 Horizontal and Vertical Asymptotes

Here you'l learn how to find the horizontal and vertical asymptotes of radical equations.

What if you had a function like $y = \frac{x^2}{2x^2-1}$? How could you find its vertical and horizontal asymptotes? After completing this Concept, you'll be able to find asymptotes of rational functions like this one.

Try This

To explore more graphs of rational functions, try the applets available at http://www.analyzemath.com/rational/rational1.html.

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CK-12 Foundation: 1203S Horizontal and Vertical Asymptotes

Guidance

We said that a horizontal asymptote is the value of y that the function approaches for large values of $|x|$. When we plug in large values of x in our function, higher powers of x get larger much quickly than lower powers of x. For example, consider:

$$y = \frac{2x^2+x-1}{3x^2-4x+3}$$

If we plug in a large value of x, say $x = 100$, we get:

$$y = \frac{2(100)^2+(100)-1}{3(100)^2-4(100)+3} = \frac{20000+100-1}{30000-400+2}$$

We can see that the beginning terms in the numerator and denominator are much bigger than the other terms in each expression. One way to find the horizontal asymptote of a rational function is to ignore all terms in the numerator and denominator except for the highest powers.

In this example the horizontal asymptote is $y = \frac{2x^2}{3x^2}$, which simplifies to $y = \frac{2}{3}$.

In the function above, the highest power of x was the same in the numerator as in the denominator. Now consider a function where the power in the numerator is less than the power in the denominator:

$$y = \frac{x}{x^2+3}$$

As before, we ignore all the terms except the highest power of x in the numerator and the denominator. That gives us $y = \frac{x}{x^2}$, which simplifies to $y = \frac{1}{x}$.

For large values of x, the value of y gets closer and closer to zero. Therefore the horizontal asymptote is $y = 0$.

To summarize:

- Find vertical asymptotes by setting the denominator equal to zero and solving for x.
- For horizontal asymptotes, we must consider several cases:
  - If the highest power of x in the numerator is less than the highest power of x in the denominator, then the horizontal asymptote is at $y = 0$.
  - If the highest power of x in the numerator is the same as the highest power of x in the denominator, then the horizontal asymptote is at $y = \frac{coefficient\ of\ highest\ power\ of\ x}{coefficient\ of\ highest\ power\ of\ x}$.
  - If the highest power of x in the numerator is greater than the highest power of x in the denominator, then we don't have a horizontal asymptote; we could have what is called an oblique (slant) asymptote, or no asymptote at all.

Example A

Find the vertical and horizontal asymptotes for $y = \frac{1}{x-1}$.

Solution

Vertical asymptotes:

Set the denominator equal to zero. $x - 1 = 0 \Rightarrow x = 1$ is the vertical asymptote.

Horizontal asymptote:

Keep only the highest powers of x. $y = \frac{1}{x} \Rightarrow y = 0$ is the horizontal asymptote.

Example B

Find the vertical and horizontal asymptotes for $y = \frac{3x}{4x+2}$.

Solution

Vertical asymptotes:

Set the denominator equal to zero. $4x + 2 = 0 \Rightarrow x = -\frac{1}{2}$ is the vertical asymptote.

Horizontal asymptote:

Keep only the highest powers of x. $y = \frac{3x}{4x} \Rightarrow y = \frac{3}{4}$ is the horizontal asymptote.

Example C

Find the vertical and horizontal asymptotes for $y = \frac{x^3}{x^2-3x+2}$.

Solution

Vertical asymptotes:

Set the denominator equal to zero: $x^2 - 3x + 2 = 0$

Factor: $(x-2)(x-1)=0$

Solve: $x=2$ and $x=1$ are the vertical asymptotes.

Horizontal asymptote. There is no horizontal asymptote because the power of the numerator is larger than the power of the denominator.

Notice the function in part *d* had more than one vertical asymptote. Here's another function with two vertical asymptotes.

Example D

Graph the function $y=\frac{-x^2}{x^2-4}$.

Solution

$$\text{Let's set the denominator equal to zero:} \qquad x^2-4=0$$

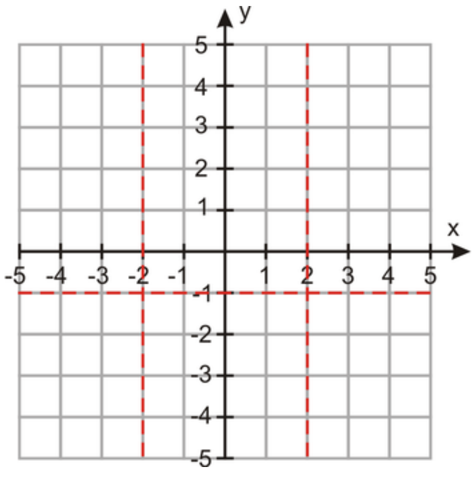
$$\text{Factor:} \qquad (x-2)(x+2)=0$$

$$\text{Solve:} \qquad x=2, x=-2$$

We find that the function is undefined for $x=2$**and** $x=-2$, so we know that there are vertical asymptotes at these values of x.

We can also find the horizontal asymptote by the method we outlined above. It's at $y=\frac{-x^2}{x^2}$, or $y=-1$.

So, we start plotting the function by drawing the vertical and horizontal asymptotes on the graph.



Now, let's make a table of values. Because our function has a lot of detail we must make sure that we pick enough values for our table to determine the behavior of the function accurately. We must make sure especially that we pick values close to the vertical asymptotes.

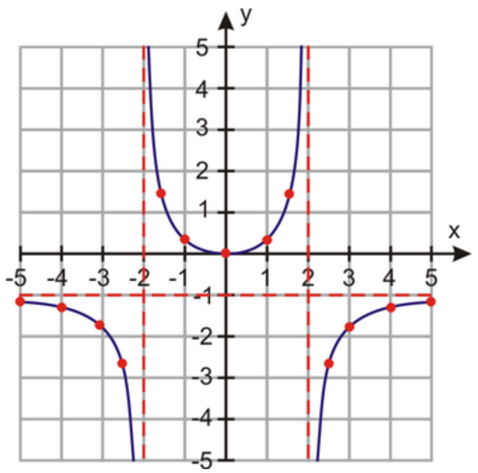
TABLE 12.6:

x	$y=\frac{-x^2}{x^2-4}$
-5	$y=\frac{-(-5)^2}{(-5)^2-4}=\frac{-25}{21}=-1.19$
-4	$y=\frac{-(-4)^2}{(-4)^2-4}=\frac{-16}{12}=-1.33$

TABLE 12.6: (continued)

x	$y=\frac{-x^2}{x^2-4}$
-3	$y=\frac{-(-3)^2}{(-3)^2-4}=\frac{-9}{5}=-1.8$
-2.5	$y=\frac{-(-2.5)^2}{(-2.5)^2-4}=\frac{-6.25}{2.25}=-2.8$
-1.5	$y=\frac{-(-1.5)^2}{(-1.5)^2-4}=\frac{-2.25}{-1.75}=1.3$
-1	$y=\frac{-(-1)^2}{(-1)^2-4}=\frac{-1}{-3}=0.33$
0	$y=\frac{-0^2}{(0)^2-4}=\frac{0}{-4}=0$
1	$y=\frac{-1^2}{(1)^2-4}=\frac{-1}{-3}=0.33$
1.5	$y=\frac{-1.5^2}{(1.5)^2-4}=\frac{-2.25}{-1.75}=1.3$
2.5	$y=\frac{-2.5^2}{(2.5)^2-4}=\frac{-6.25}{2.25}=-2.8$
3	$y=\frac{-3^2}{(3)^2-4}=\frac{-9}{5}=-1.8$
4	$y=\frac{-4^2}{(4)^2-4}=\frac{-16}{12}=-1.33$
5	$y=\frac{-5^2}{(5)^2-4}=\frac{-25}{21}=-1.19$

Here is the resulting graph.



Watch this video for help with the Examples above.

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CK-12 Foundation: HorizontalandVertical Asymptotes

Vocabulary

- Graphs of rational functions are very distinctive, because they get closer and closer to certain values but never reach those values. This behavior is called asymptotic behavior, and we will see that rational functions can have **horizontal asymptotes, vertical asymptotes** or **oblique (or slant) asymptotes.**

Guided Practice

Find the vertical and horizontal asymptotes for $y = \frac{x^2-2}{2x^2+3}$.

Solution

Vertical asymptotes:

Set the denominator equal to zero: $2x^2+3=0 \Rightarrow 2x^2 = -3 \Rightarrow x^2 = -\frac{3}{2}$. Since there are no solutions to this equation, there is no vertical asymptote.

Horizontal asymptote:

Keep only the highest powers of x. $y = \frac{x^2}{2x^2} \Rightarrow y = \frac{1}{2}$ is the horizontal asymptote.

Practice

Find all the vertical and horizontal asymptotes of the following rational functions.

1. $y = \frac{4}{x+2}$
2. $y = \frac{5x-1}{2x-6}$
3. $y = \frac{10}{x}$
4. $y = \frac{2}{x} - 5$
5. $y = \frac{x+1}{x^2}$
6. $y = \frac{4x^2}{4x^2+1}$
7. $y = \frac{2x}{x^2-9}$
8. $y = \frac{3x^2}{x^2-4}$
9. $y = \frac{1}{x^2+4x+3}$
10. $y = \frac{2x+5}{x^2-2x-8}$

12.4 Determining Asymptotes by Division

Here you'l learn how to solve circuitry and other real-wold applications that involve inverse variations and rational functions.

What if you had a circuit with a current of 2 amps and two resistances in parallel of 10 and 20 ohms each? How could you find the circuit's voltage? After completing this Concept, you'll be able to solve real-world applications like this one using rational functions.

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CK-12 Foundation: 1204S Solve Applications UsingRational Functions

Guidance

We are going to investigate some problems that are described by rational functions. Our first example is one which is modeled by inverse variation, which are a special type of rational function. Many formulas in physics are described by inverse variation.

Example A

The frequency, f, of sound varies inversely with wavelength, λ. A sound signal that has a wavelength of 34 meters has a frequency of 10 hertz. What frequency does a sound signal of 120 meters have?

Solution

$$\text{The inverse variation relationship is:} \qquad f = \frac{k}{\lambda}$$

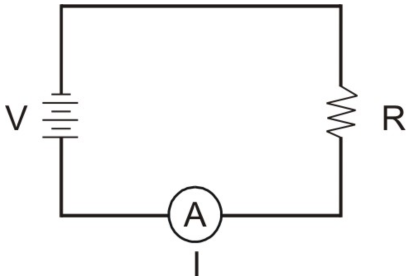
$$\text{Plug in the values: } \lambda = 34 \text{ and } f = 10: \qquad 10 = \frac{k}{34}$$

$$\text{Multiply both sides by } 34: \qquad k = 340$$

$$\text{Thus, the relationship is given by:} \qquad f = \frac{340}{\lambda}$$

$$\text{Plug in } \lambda = 120 \text{ meters:} \qquad f = \frac{340}{120} \Rightarrow f = 2.83 \text{ Hertz}$$

Electrical circuits are commonplace is everyday life—for example, they're in all the electrical appliances in your home. The figure below shows an example of a simple electrical circuit. It consists of a battery which provides a voltage (V, measured in Volts, V), a resistor (R, measured in ohms, Ω) which resists the flow of electricity and an ammeter that measures the current (I, measured in amperes, A) in the circuit.

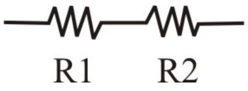


Ohm's Law gives a relationship between current, voltage and resistance. It states that

$$I = \frac{V}{R}$$

Your light bulbs, toaster and hairdryer are all basically simple resistors. In addition, resistors are used in an electrical circuit to control the amount of current flowing through a circuit and to regulate voltage levels. One important reason to do this is to prevent sensitive electrical components from burning out due to too much current or too high a voltage level. Resistors can be arranged in series or in parallel.

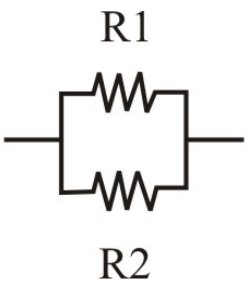
For resistors placed in a series:



the total resistance is just the sum of the resistances of the individual resistors:

$$R_{tot} = R_1 + R_2$$

For resistors placed in parallel:

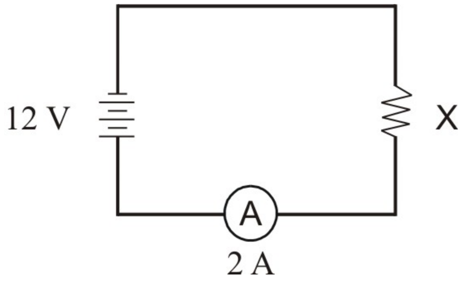


the reciprocal of the total resistance is the sum of the reciprocals of the resistances of the individual resistors:

$$\frac{1}{R_c} = \frac{1}{R_1} + \frac{1}{R_2}$$

Example B

Find the quantity labeled x in the following circuit.



Solution

We use the formula $I = \frac{V}{R}$.

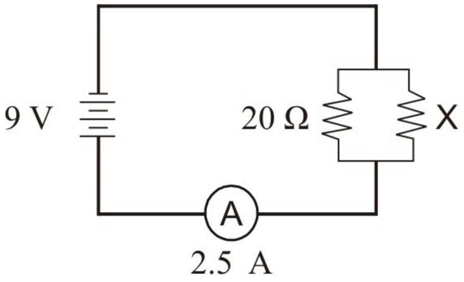
Plug in the known values: $I = 2\ A, V = 12\ V$: $2 = \frac{12}{R}$

Multiply both sides by R : $2R = 12$

Divide both sides by 2 : $R = 6\Omega$ **Answer**

Example C

Find the quantity labeled x in the following circuit.



Solution

Ohm's Law also tells us that $I_{total} = \frac{V_{total}}{R_{total}}$

Plug in the values we know, $I = 2.5\ A$ and $E = 9\ V$: $2.5 = \frac{9}{R_{tot}}$

Multiply both sides by R : $2.5R_{tot} = 9$

Divide both sides by 2.5 : $R_{tot} = 3.6\Omega$

Since the resistors are placed in parallel, the total resistance is given by: $\frac{1}{R_{tot}} = \frac{1}{X} + \frac{1}{20}$

$$\Rightarrow \frac{1}{3.6} = \frac{1}{X} + \frac{1}{20}$$

Multiply all terms by $72X$: $\frac{1}{3.6}(72X) = \frac{1}{X}(72X) + \frac{1}{20}(72X)$

Cancel common factors: $20X = 72 + 3.6X$

Solve: $16.4X = 72$

Divide both sides by 16.4 : $X = 4.39\Omega$ **Answer**

Watch this video for help with the Examples above.

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CK-12 Foundation: Solve ApplicationsUsing Rational Functions

Vocabulary

- **Ohm's Law** gives a relationship between current, voltage and resistance. It states that

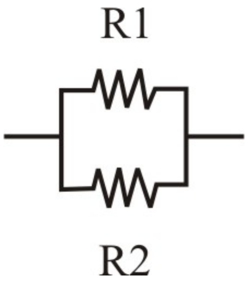
$$I = \frac{V}{R}$$

- For resistors placed in a series:

the total resistance is just the sum of the resistances of the individual resistors:

$$R_{tot} = R_1 + R_2$$

- For resistors placed in parallel:



the reciprocal of the total resistance is the sum of the reciprocals of the resistances of the individual resistors:

$$\frac{1}{R_c} = \frac{1}{R_1} + \frac{1}{R_2}$$

Guided Practice

Electrostatic force is the force of attraction or repulsion between two charges. The electrostatic force is given by the formula $F = \frac{Kq_1q_2}{d^2}$, *where* q_1 *and* q_2 *are the charges of the charged particles, d is the distance between the charges and k is a proportionality constant. The charges do not change, so they too are constants; that means we can combine them with the other constant k to form a new constant K,* so we can rewrite the equation as $F = \frac{K}{d^2}$.

If the electrostatic force is $F = 740$ *Newtons when the distance between charges is* 5.3×10^{-11} *meters, what is F when* $d = 2.0 \times 10^{-10}$ *meters?*

Solution

The inverse variation relationship is: $F = \frac{K}{d^2}$

Plug in the values $F = 740$ and $d = 5.3 \times 10^{-11}$: $740 = \frac{K}{(5.3 \times 10^{-11})^2}$

Multiply both sides by $(5.3 \times 10^{-11})^2$: $K = 740\left(5.3 \times 10^{-11}\right)^2$

$K = 2.08 \times 10^{-18}$

The electrostatic force is given by: $F = \frac{2.08 \times 10^{-18}}{d^2}$

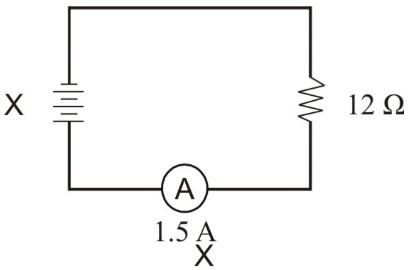
When $d = 2.0 \times 10^{-10}$: $F = \frac{2.08 \times 10^{-18}}{(2.0 \times 10^{-10})^2}$

Use scientific notation to simplify: $F = 52$ Newtons

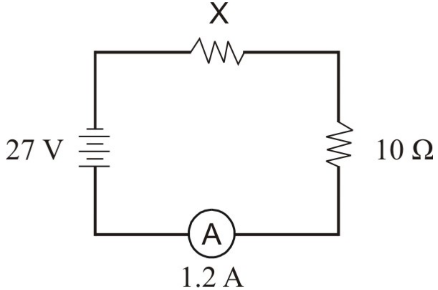
Practice

For 1-4, find the quantity labeled x in each of the following circuits.

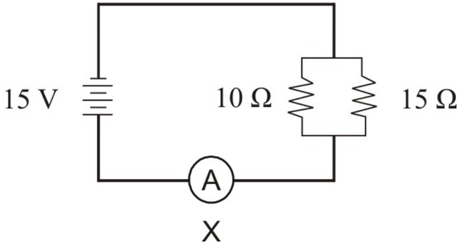
1.

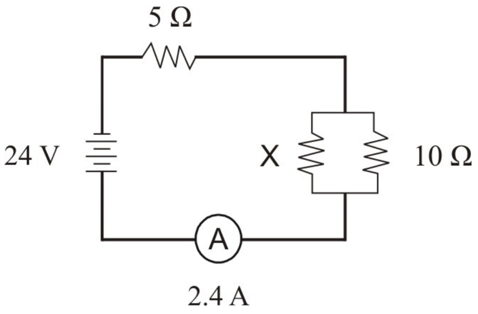


2.



3.



4. 

For 5-7, the intensity of light is inversely proportional to the square of the distance between the light source and the object being illuminated.

5. A light meter that is 10 meters from a light source registers 35 lux. What intensity would it register 25 meters from the light source?
6. A light meter that is registering 40 lux is moved twice as far away from the light source illuminating it. What intensity does it now register? (Hint: let x be the original distance from the light source.)
7. The same light meter is moved twice as far away again (so it is now four times as far from the light source as it started out). What intensity does it register now?

8. Ohm's Law states that current flowing in a wire is inversely proportional to the resistance of the wire. If the current is 2.5 Amperes when the resistance is 20 ohms, find the resistance when the current is 5 Amperes.

For 9-10, the volume of a gas varies directly with its temperature and inversely with its pressure. At 273 degrees Kelvin and pressure of 2 atmospheres, the volume of a certain gas is 24 liters.

9. Find the volume of the gas when the temperature is 220 Kelvin and the pressure is 1.2 atmospheres.
10. Find the temperature when the volume is 24 liters and the pressure is 3 atmospheres.

For 11-13, the volume of a square pyramid varies jointly with the height and the square of the side length of the base. A pyramid whose height is 4 inches and whose base has a side length of 3 inches has a volume of 12 in^3.

11. Find the volume of a square pyramid that has a height of 9 inches and whose base has a side length of 5 inches.
12. Find the height of a square pyramid that has a volume of 49 in^3 and whose base has a side length of 7 inches.
13. A square pyramid has a volume of 72 in^3 and its base has a side length equal to its height. Find the height of the pyramid.

12.5 Division of Polynomials

Here you'll learn how to divide polynomials by monomials and binomials.

What if you had a polynomial like $2x^2+5x-3$ and you wanted to divide it by a monomial like x or a binomial like $x+1$? How would you do so? After completing this Concept, you'll be able to divide polynomials like this one by monomials and binomials.

Try This

To check your answers to long division problems involving polynomials, try the solver at http://calc101.com/webMathematica/long-divide.jsp. It shows the long division steps so you can tell where you may have made a mistake.

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CK-12 Foundation: 1205S Divisionof Polynomials

Guidance

A **rational expression** is formed by taking the quotient of two polynomials.

Some examples of rational expressions are

$$\frac{2x}{x^2-1} \qquad \frac{4x^2-3x+4}{2x} \qquad \frac{9x^2+4x-5}{x^2+5x-1} \qquad \frac{2x^3}{2x+3}$$

Just as with rational numbers, the expression on the top is called the **numerator** and the expression on the bottom is called the **denominator**. In special cases we can simplify a rational expression by dividing the numerator by the denominator.

Divide a Polynomial by a Monomial

We'll start by dividing a polynomial by a monomial. To do this, we divide each term of the polynomial by the monomial. When the numerator has more than one term, the monomial on the bottom of the fraction serves as the **common** denominator to all the terms in the numerator.

Example A

Divide.

a) $\frac{8x^2-4x+16}{2}$

b) $\frac{3x^2+6x-1}{x}$

c) $\frac{-3x^2-18x+6}{9x}$

Solution

a) $\frac{8x^2-4x+16}{2} = \frac{8x^2}{2} - \frac{4x}{2} + \frac{16}{2} = 4x^2 - 2x + 8$

b) $\frac{3x^3+6x-1}{x} = \frac{3x^3}{x} + \frac{6x}{x} - \frac{1}{x} = 3x^2 + 6 - \frac{1}{x}$

c) $\frac{-3x^2-18x+6}{9x} = -\frac{3x^2}{9x} - \frac{18x}{9x} + \frac{6}{9x} = -\frac{x}{3} - 2 + \frac{2}{3x}$

A common error is to cancel the denominator with just one term in the numerator.

Consider the quotient $\frac{3x+4}{4}$.

Remember that the denominator of 4 is common to both the terms in the numerator. In other words we are dividing both of the terms in the numerator by the number 4.

The correct way to simplify is:

$$\frac{3x+4}{4} = \frac{3x}{4} + \frac{4}{4} = \frac{3x}{4} + 1$$

A common mistake is to cross out the number 4 from the numerator and the denominator, leaving just $3x$. This is incorrect, because the entire numerator needs to be divided by 4.

Example B

Divide $\frac{5x^3-10x^2+x-25}{-5x^2}$.

Solution

$$\frac{5x^3-10x^2+x-25}{-5x^2} = \frac{5x^3}{-5x^2} - \frac{10x^2}{-5x^2} + \frac{x}{-5x^2} - \frac{25}{-5x^2}$$

The negative sign in the denominator changes all the signs of the fractions:

$$-\frac{5x^3}{5x^2} + \frac{10x^2}{5x^2} - \frac{x}{5x^2} + \frac{25}{5x^2} = -x + 2 - \frac{1}{5x} + \frac{5}{x^2}$$

Divide a Polynomial by a Binomial

We divide polynomials using a method that's a lot like long division with numbers. We'll explain the method by doing an example.

Example C

Divide $\frac{x^2+4x+5}{x+3}$.

Solution

When we perform division, the expression in the numerator is called the **dividend** and the expression in the denominator is called the **divisor**.

To start the division we rewrite the problem in the following form:

$$x+3\overline{)x^2+4x+5}$$

We start by dividing the first term in the dividend by the first term in the divisor: $\frac{x^2}{x}=x$.

We place the answer on the line above the x term:

$$\begin{array}{r} x \\ x+3\overline{)x^2+4x+5} \end{array}$$

Next, we multiply the x term in the answer by the divisor, $x+3$, and place the result under the dividend, matching like terms. x times $(x+3)$ is x^2+3x, so we put that under the divisor:

$$\begin{array}{r} x \\ x+3\overline{)x^2+4x+5} \\ x^2+3x \end{array}$$

Now we subtract x^2+3x from x^2+4x+5. It is useful to change the signs of the terms of x^2+3x to $-x^2-3x$ and add like terms vertically:

$$\begin{array}{r} x \\ x+3\overline{)x^2+4x+5} \\ \underline{-x^2-3x} \\ x \end{array}$$

Now, we bring down the 5, the next term in the dividend.

$$\begin{array}{r} x \\ x+3\overline{)x^2+4x+5} \\ \underline{-x^2-3x} \\ x+5 \end{array}$$

And now we go through that procedure once more. First we divide the first term of $x+5$ by the first term of the divisor. x divided by x is 1, so we place this answer on the line above the constant term of the dividend:

$$\begin{array}{r} x+1 \\ x+3\overline{)x^2+4x+5} \\ \underline{-x^2-3x} \\ x+5 \end{array}$$

Multiply 1 by the divisor, $x+3$, and write the answer below $x+5$, matching like terms.

$$\begin{array}{r} x+1 \\ x+3\overline{)x^2+4x+5} \\ \underline{-x^2-3x} \\ x+5 \\ x+3 \end{array}$$

Subtract $x+3$ from $x+5$ by changing the signs of $x+3$ to $-x-3$ and adding like terms:

$$\begin{array}{r}
x+1 \\
x+3\overline{)x^2+4x+5} \\
\underline{-x^2-3x} \\
x+5 \\
\underline{-x-3} \\
2
\end{array}$$

Since there are no more terms from the dividend to bring down, we are done. The quotient is $x+1$ and the remainder is 2.

Remember that for a division with a remainder the answer is quotient $+\frac{\text{remainder}}{\text{divisor}}$. So the answer to this division problem is $\frac{x^2+4x+5}{x+3}=x+1+\frac{2}{x+3}$.

Check

To check the answer to a long division problem we use the fact that

$$(\text{divisor}\times\text{quotient})+\text{remainder}=\text{dividend}$$

For the problem above, here's how we apply that fact to check our solution:

$$\begin{aligned}(x+3)(x+1)+2&=x^2+4x+3+2\\&=x^2+4x+5\end{aligned}$$

The answer checks out.

Watch this video for help with the Examples above.

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Click image to the left for more content.

CK-12 Foundation: Division of Polynomials

Vocabulary

- A **rational expression** is formed by taking the quotient of two polynomials.

Guided Practice

Divide $\frac{x^2+8x+17}{x+4}$.

Solution

When we perform division, the expression in the numerator is called the **dividend** and the expression in the denominator is called the **divisor**.

To start the division we rewrite the problem in the following form:

$$x+4\overline{)x^2+8x+17}$$

We start by dividing the first term in the dividend by the first term in the divisor: $\frac{x^2}{x}=x$.

We place the answer on the line above the x term:

$$\begin{array}{r} x \\ x+4\overline{)x^2+8x+17} \end{array}$$

Next, we multiply the x term in the answer by the divisor, $x+4$, and place the result under the dividend, matching like terms. x times $(x+4)$ is x^2+4x, so we put that under the divisor:

$$\begin{array}{r} x \\ x+4\overline{)x^2+8x+17} \\ x^2+4x \end{array}$$

Now we subtract x^2+4x from $x^2+8x+17$. It is useful to change the signs of the terms of x^2+4x to $-x^2-4x$ and add like terms vertically:

$$\begin{array}{r} x \\ x+4\overline{)x^2+8x+17} \\ \underline{-x^2-4x} \\ 4x \end{array}$$

Now, we bring down the 17, the next term in the dividend.

$$\begin{array}{r} x \\ x+4\overline{)x^2+8x+17} \\ \underline{-x^2-4x} \\ 4x+17 \end{array}$$

And now we go through that procedure once more. First we divide the first term of $4x+17$ by the first term of the divisor. $4x$ divided by x is 4, so we place this answer on the line above the constant term of the dividend:

$$\begin{array}{r} x\ +\ 4 \\ x+4\overline{)x^2+8x+17} \\ \underline{-x^2-4x} \\ x+17 \end{array}$$

Multiply 4 by the divisor, $x+4$, and write the answer below $4x+16$, matching like terms.

$$\begin{array}{r} x+4 \\ x+4\overline{)x^2+8x+17} \\ \underline{-x^2-4x\quad} \\ 4x+17 \\ 4x+16 \end{array}$$

Subtract $4x+16$ from $4x+17$ by changing the signs of $4x+16$ to $-4x-16$ and adding like terms:

$$\begin{array}{r} x+4 \\ x+4\overline{)x^2+8x+17} \\ \underline{-x^2-4x\quad} \\ x+17 \\ \underline{-4x-16} \\ 1 \end{array}$$

Since there are no more terms from the dividend to bring down, we are done. The quotient is $x+4$ and the remainder is 1.

Remember that for a division with a remainder the answer is quotient $+\frac{\text{remainder}}{\text{divisor}}$. So the answer to this division problem is $\frac{x^2+8x+17}{x+4}=x+4+\frac{1}{x+4}$.

Check

To check the answer to a long division problem we use the fact that

$$(\text{divisor} \times \text{quotient})+\text{remainder}=\text{dividend}$$

For the problem above, here's how we apply that fact to check our solution:

$$\begin{aligned}(x+4)(x+4)+1 &= x^2+8x+16+1 \\ &= x^2+8x+17\end{aligned}$$

The answer checks out.

Practice

Divide the following polynomials:

1. $\frac{2x+4}{2}$
2. $\frac{x-4}{x}$
3. $\frac{5x-35}{5x}$
4. $\frac{x^2+2x-5}{x}$
5. $\frac{4x^2+12x-36}{-4x}$
6. $\frac{2x^2+10x+7}{2x^2}$

7. $\frac{x^3-x}{-2x^2}$
8. $\frac{5x^4-9}{3x}$
9. $\frac{x^3-12x^2+3x-4}{12x^2}$
10. $\frac{3-6x+x^3}{-9x^3}$
11. $\frac{x^2+3x+6}{x+1}$
12. $\frac{x^2-9x+6}{x-1}$
13. $\frac{x^2+5x+4}{x+4}$
14. $\frac{x^2-10x+25}{x-5}$
15. $\frac{x^2-20x+12}{x-3}$
16. $\frac{3x^2-x+5}{x-2}$
17. $\frac{9x^2+2x-8}{x+4}$
18. $\frac{3x^2-4}{3x+1}$
19. $\frac{5x^2+2x-9}{2x-1}$
20. $\frac{x^2-6x-12}{5x^4}$

12.6 Inverse Variation Problems

Here you'll learn how to use division to determine the asymptotes of rational functions.

What if you had a function like $y = \frac{3x^2-2x+1}{x+2}$? How could you rewrite it to find its asymptotes? After completing this Concept, you'll be able to rewrite rational functions like this one using division.

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CK-12 Foundation: 1206S Rewriting Rational FunctionsUsing Division

Guidance

In the last section we saw how to find vertical and horizontal asymptotes. Remember, the horizontal asymptote shows the value of y that the function approaches for large values of x. Let's review the method for finding horizontal asymptotes and see how it's related to polynomial division.

When it comes to finding asymptotes, there are basically four different types of rational functions.

Case 1: The polynomial in the numerator has a lower degree than the polynomial in the denominator.

Example A

Find the horizontal asymptote of $y = \frac{2}{x-1}$.

Solution:

We can't reduce this fraction, and as x gets larger the denominator of the fraction gets much bigger than the numerator, so the whole fraction approaches zero.

The horizontal asymptote is $y = 0$.

Case 2: The polynomial in the numerator has the same degree as the polynomial in the denominator.

Example B

Find the horizontal asymptote of $y = \frac{3x+2}{x-1}$.

Solution:

In this case we can divide the two polynomials:

$$\begin{array}{r} 3 \\ x-1\overline{)3x+2} \\ \underline{-3x+3} \\ 5 \end{array}$$

So the expression can be written as $y = 3 + \frac{5}{x-1}$.

Because the denominator of the remainder is bigger than the numerator of the remainder, the remainder will approach zero for large values of x. Adding the 3 to that 0 means the whole expression will approach 3.

The horizontal asymptote is $y = 3$.

Case 3: The polynomial in the numerator has a degree that is one more than the polynomial in the denominator.

Example C

Find any asymptotes of $y = \frac{4x^2+3x+2}{x-1}$.

Solution:

We can do long division once again and rewrite the expression as $y = 4x + 7 + \frac{9}{x-1}$. The fraction here approaches zero for large values of x, so the whole expression approaches $4x + 7$.

When the rational function approaches a straight line for large values of x, we say that the rational function has an **oblique asymptote.** In this case, then, **the oblique asymptote is $y = 4x + 7$.**

Case 4: The polynomial in the numerator has a degree that is two or more than the degree in the denominator.

Example D

Find any asymptotes of $y = \frac{x^3}{x-1}$.

This is actually the simplest case of all: **the polynomial has no horizontal or oblique asymptotes.**

Notice that a rational function will either have a horizontal asymptote, an oblique asymptote or neither kind. In other words, a function can't have both; in fact, it can't have more than one of either kind. On the other hand, a rational function can have any number of *vertical* asymptotes at the same time that it has horizontal or oblique asymptotes.

Watch this video for help with the Examples above.

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CK-12 Foundation: Rewriting RationalFunctions UsingDivision

Vocabulary

- When the rational function approaches a straight line for large values of x, we say that the rational function has an **oblique asymptote.**

Guided Practice

Find the horizontal or oblique asymptotes of the following rational functions.

a) $y = \frac{3x^2}{x^2+4}$

b) $y = \frac{x-1}{3x^2-6}$

c) $y = \frac{x^4+1}{x-5}$

d) $y = \frac{x^3-3x^2+4x-1}{x^2-2}$

Solution

a) When we simplify the function, we get $y = 3 - \frac{12}{x^2+4}$. **There is a horizontal asymptote at** $y = 3$**.**

b) We cannot divide the two polynomials. **There is a horizontal asymptote at** $y = 0$**.**

c) The power of the numerator is 3 more than the power of the denominator. **There are no horizontal or oblique asymptotes.**

d) When we simplify the function, we get $y = x - 3 + \frac{6x-7}{x^2-2}$. **There is an oblique asymptote at** $y = x - 3$**.**

Practice

Find all asymptotes of the following rational functions:

1. $\frac{x^2}{x-2}$
2. $\frac{1}{x+4}$
3. $\frac{x^2-1}{x^2+1}$
4. $\frac{x-4}{x^2-9}$
5. $\frac{x^2+2x+1}{4x-1}$
6. $\frac{x^3+1}{4x-1}$
7. $\frac{x-x^3}{x^2-6x-7}$
8. $\frac{x^4-2x}{8x+24}$

Graph the following rational functions. Indicate all asymptotes on the graph:

9. $\frac{x^2}{x+2}$
10. $\frac{x^3-1}{x^2-4}$
11. $\frac{x^2+1}{2x-4}$
12. $\frac{x-x^2}{3x+2}$

12.7 Excluded Values for Rational Expressions

Here you'll learn how to reduce rational expressions to their simplest terms. You'll also learn how to find the excluded values of rational expressions.

What if you had a rational expression like $\frac{x+2}{x^2+3x+2}$? How could you simplify it? After completing this Concept, you'll be able to reduce rational expressions like this one to their simplest terms and find their excluded values.

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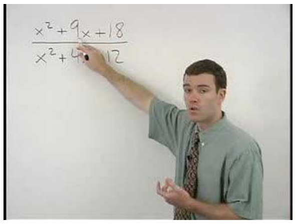
CK-12 Foundation: 1207S RationalExpressions

Watch this video for more examples of how to simplify rational expressions.



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YourTeacher:SimplifyingRational Expressions

Guidance

A simplified rational expression is one where the numerator and denominator have no common factors. In order to simplify an expression to **lowest terms**, we factor the numerator and denominator as much as we can and cancel common factors from the numerator and the denominator.

Simplify Rational Expressions

Example A

Reduce each rational expression to simplest terms.

a) $\frac{4x-2}{2x^2+x-1}$

b) $\frac{x^2-2x+1}{8x-8}$

c) $\frac{x^2-4}{x^2-5x+6}$

Solution

a)

Factor the numerator and denominator completely: $\frac{2(2x-1)}{(2x-1)(x+1)}$

Cancel the common factor $(2x-1)$: $\frac{2}{x+1}$

b)

Factor the numerator and denominator completely: $\frac{(x-1)(x-1)}{8(x-1)}$

Cancel the common factor $(x-1)$: $\frac{x-1}{8}$

c)

Factor the numerator and denominator completely: $\frac{(x-2)(x+2)}{(x-2)(x-3)}$

Cancel the common factor$(x-2)$: $\frac{x+2}{x-3}$

When reducing fractions, you are only allowed to cancel common **factors** from the denominator but NOT common terms. For example, in the expression $\frac{(x+1)\cdot(x-3)}{(x+2)\cdot(x-3)}$, we can cross out the $(x-3)$ factor because $\frac{(x-3)}{(x-3)} = 1$. But in the expression $\frac{x^2+1}{x^2-5}$ we can't just cross out the x^2 terms.

Why can't we do that? When we cross out terms that are part of a sum or a difference, we're violating the order of operations (PEMDAS). Remember, the fraction bar means division. When we perform the operation $\frac{x^2+1}{x^2-5}$, we're really performing the division $(x^2+1) \div (x^2-5)$ — and the order of operations says that we must perform the operations inside the parentheses before we can perform the division.

Using numbers instead of variables makes it more obvious that canceling individual terms doesn't work. You can see that $\frac{9+1}{9-5} = \frac{10}{4} = 2.5$ — but if we canceled out the 9's first, we'd get $\frac{1}{-5}$, or -0.2, instead.

Find Excluded Values of Rational Expressions

Whenever there's a variable expression in the denominator of a fraction, we must remember that the denominator could be zero when the independent variable takes on certain values. Those values, corresponding to the vertical asymptotes of the function, are called **excluded** values. To find the excluded values, we simply set the denominator equal to zero and solve the resulting equation.

Example B

Find the excluded values of the following expressions.

a) $\frac{x}{x+4}$

b) $\frac{2x+1}{x^2-x-6}$

Solution

a)

When we set the denominator equal to zero we obtain: $x+4=0 \Rightarrow x=-4$

So $-\mathbf{4}$ is the excluded value.

b)

When we set the denominator equal to zero we obtain: $x^2-x-6=0$

Solve by factoring: $(x-3)(x+2)=0$

$\Rightarrow x=3 \text{ and } x=-2$

So **3 and** $-\mathbf{2}$ are the excluded values.

Removable Zeros

Removable zeros are those zeros from the original expression, but is not a zero for the simplified version of the expression. However, we have to keep track of them, because they were zeros in the original expression. This is illustrated in the following examples.

Example C

Determine the removable values of $\frac{4x-2}{2x^2+x-1}$.

Solution:

Notice that in the expressions in Example A, we removed a division by zero when we simplified the problem. For instance, we rewrote $\frac{4x-2}{2x^2+x-1}$ as $\frac{2(2x-1)}{(2x-1)(x+1)}$. The denominator of this expression is zero when $x=\frac{1}{2}$ or when $x=-1$.

However, when we cancel common factors, we simplify the expression to $\frac{2}{x+1}$. This reduced form allows the value $x=\frac{1}{2}$, so $x=-1$ is its only excluded value.

Technically the original expression and the simplified expression are not the same. When we reduce a radical expression to its simplest form, we should specify the removed excluded value. In other words, we should write our final answer as $\frac{4x-2}{2x^2+x-1}=\frac{2}{x+1}, x \neq \frac{1}{2}$.

Example D

Determine the removable values of the expressions from Example A parts b and c.

Solution:

We should write the answer from Example A, part b as $\frac{x^2-2x+1}{8x-8}=\frac{x-1}{8}, x \neq 1$.

The answer from Example A, part c as $\frac{x^2-4}{x^2-5x+6}=\frac{x+2}{x-3}, x \neq 2$.

Watch this video for help with the Examples above.

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CK-12 Foundation: Rational Expressions

Vocabulary

- Whenever there's a variable expression in the denominator of a fraction, we must remember that the denominator could be zero when the independent variable takes on certain values. Those values, corresponding to the vertical asymptotes of the function, are called **excluded** values.
- Removable zeros are those zeros from the original expression, but is not a zero for the simplified version of the expression.

Guided Practice

Find the excluded values of $\frac{4}{x^2-5x}$.

Solution

When we set the denominator equal to zero we obtain: $x^2-5x=0$

Solve by factoring: $x(x-5)=0$

$\Rightarrow x=0$ and $x=5$

So **0 and 5** are the excluded values.

Practice

Reduce each fraction to lowest terms.

1. $\frac{4}{2x-8}$
2. $\frac{x^2+2x}{x}$
3. $\frac{9x+3}{12x+4}$
4. $\frac{6x^2+2x}{4x}$
5. $\frac{x-2}{x^2-4x+4}$
6. $\frac{x^2-9}{5x+15}$
7. $\frac{x^2+6x+8}{x^2+4x}$
8. $\frac{2x^2+10x}{x^2+10x+25}$
9. $\frac{x^2+6x+5}{x^2-x-2}$
10. $\frac{x^2-16}{x^2+2x-8}$
11. $\frac{3x^2+3x-18}{2x^2+5x-3}$
12. $\frac{x^3+x^2-20x}{6x^2+6x-120}$

Find the excluded values for each rational expression.

13. $\frac{2}{x}$
14. $\frac{4}{x+2}$
15. $\frac{2x-1}{(x-1)^2}$
16. $\frac{3x+1}{x^2-4}$
17. $\frac{x^2}{x^2+9}$
18. $\frac{2x^2+3x-1}{x^2-3x-28}$
19. $\frac{5x^3-4}{x^2+3x}$
20. $\frac{9}{x^3+11x^2+30x}$
21. $\frac{4x-1}{x^2+3x-5}$
22. $\frac{5x+11}{3x^2-2x-4}$
23. $\frac{x^2-1}{2x^2+x+3}$
24. $\frac{12}{x^2+6x+1}$
25. In an electrical circuit with resistors placed in parallel, the reciprocal of the total resistance is equal to the sum of the reciprocals of each resistance. $\frac{1}{R_c} = \frac{1}{R_1} + \frac{1}{R_2}$. If $R_1 = 25\ \Omega$ and the total resistance is $R_c = 10\ \Omega$, what is the resistance R_2?
26. Suppose that two objects attract each other with a gravitational force of 20 Newtons. If the distance between the two objects is doubled, what is the new force of attraction between the two objects?
27. Suppose that two objects attract each other with a gravitational force of 36 Newtons. If the mass of both objects was doubled, and if the distance between the objects was doubled, then what would be the new force of attraction between the two objects?
28. A sphere with radius R has a volume of $\frac{4}{3}\pi R^3$ and a surface area of $4\pi R^2$. Find the ratio the surface area to the volume of a sphere.
29. The side of a cube is increased by a factor of 2. Find the ratio of the old volume to the new volume.
30. The radius of a sphere is decreased by 4 units. Find the ratio of the old volume to the new volume.

12.8 Multiplication of Rational Expressions

Here you'll learn how to multiply two monomial rational expressions, two polynomial rational expressions, and a rational expression by a polynomial.

What if you had two rational expressions like $\frac{2x^2-3}{x-4}$ and $\frac{x^2-3x+2}{x^2}$ and you wanted to multiply them? How could you do so such that the answer were in simplest terms? After completig this Concept, you'll be able to multiply rational expressions like this one.

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CK-12 Foundation: 1208S Multiplying RationalExpressions

Guidance

The rules for multiplying and dividing rational expressions are the same as the rules for multiplying and dividing rational numbers, so let's start by reviewing multiplication and division of fractions. When we multiply two fractions we multiply the numerators and denominators separately:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Multiply Rational Expressions Involving Monomials

Example A

Multiply the following: $\frac{a}{16b^8} \cdot \frac{4b^3}{5a^2}$.

Solution

Cancel common factors from the numerator and denominator. The common factors are 4, a, and b^3. Canceling them out leaves $\frac{1}{4b^5} \cdot \frac{1}{5a} = \frac{1}{20ab^5}$.

Example B

Multiply $9x^2 \cdot \frac{4y^2}{21x^4}$.

Solution

Rewrite the problem as a product of two fractions: $\frac{9x^2}{1} \cdot \frac{4y^2}{21x^4}$ Then cancel common factors from the numerator and denominator.

The common factors are 3 and x^2. Canceling them out leaves $\frac{3}{1} \cdot \frac{4y^2}{7x^2} = \frac{12y^2}{7x^2}$.

Multiply Rational Expressions Involving Polynomials

When multiplying rational expressions involving polynomials, first we need to factor all polynomial expressions as much as we can. Then we follow the same procedure as before.

Example C

Multiply $\frac{4x+12}{3x^2} \cdot \frac{x}{x^2-9}$.

Solution

Factor all polynomial expressions as much as possible: $\frac{4(x+3)}{3x^2} \cdot \frac{x}{(x+3)(x-3)}$

The common factors are x and $(x+3)$. Canceling them leaves $\frac{4}{3x} \cdot \frac{1}{(x-3)} = \frac{4}{3x(x-3)} = \frac{4}{3x^2-9x}$.

Multiply a Rational Expression by a Polynomial

When we multiply a rational expression by a whole number or a polynomial, we can write the whole number (or polynomial) as a fraction with denominator equal to one. We then proceed the same way as in the previous examples.

Example D

Multiply $\frac{3x+18}{4x^2+19x-5} \cdot (x^2+3x-10)$.

Solution

Rewrite the expression as a product of fractions: $\frac{3x+18}{4x^2+19x-5} \cdot \frac{x^2+3x-10}{1}$

Factor polynomials: $\frac{3(x+6)}{(x+5)(4x-1)} \cdot \frac{(x-2)(x+5)}{1}$

The common factor is $(x+5)$. Canceling it leaves $\frac{3(x+6)}{(4x-1)} \cdot \frac{(x-2)}{1} = \frac{3(x+6)(x-2)}{(4x-1)} = \frac{3x^2+12x-36}{4x-1}$

Watch this video for help with the Examples above.

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CK-12 Foundation: Multiplying Rational Expressions

Vocabulary

- When we multiply two fractions we multiply the numerators and denominators separately:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Guided Practice

Multiply $\frac{12x^2-x-6}{x^2-1} \cdot \frac{x^2+7x+6}{4x^2-27x+18}$.

Solution

Factor polynomials: $\frac{(3x+2)(4x-3)}{(x+1)(x-1)} \cdot \frac{(x+1)(x+6)}{(4x-3)(x-6)}$.

The common factors are $(x+1)$ and $(4x-3)$. Canceling them leaves $\frac{(3x+2)}{(x-1)} \cdot \frac{(x+6)}{(x-6)} = \frac{(3x+2)(x+6)}{(x-1)(x-6)} = \frac{3x^2+20x+12}{x^2-7x+6}$

Practice

Multiply the following rational expressions and reduce the answer to lowest terms.

1. $\frac{x^3}{2y^3} \cdot \frac{2y^2}{x}$
2. $\frac{2x}{y^2} \cdot \frac{4y}{5x}$
3. $2xy \cdot \frac{2y^2}{x^3}$
4. $\frac{4y^2-1}{y^2-9} \cdot \frac{y-3}{2y-1}$
5. $\frac{6ab}{a^2} \cdot \frac{a^3b}{3b^2}$
6. $\frac{33a^2}{-5} \cdot \frac{20}{11a^3}$
7. $\frac{2x^2+2x-24}{x^2+3x} \cdot \frac{x^2+x-6}{x+4}$
8. $\frac{x}{x-5} \cdot \frac{x^2-8x+15}{x^2-3x}$
9. $\frac{5x^2+16x+3}{36x^2-25} \cdot (6x^2+5x)$
10. $\frac{x^2+7x+10}{x^2-9} \cdot \frac{x^2-3x}{3x^2+4x-4}$
11. $\frac{x^2+8x+16}{7x^2+9x+2} \cdot \frac{7x+2}{x^2+4x}$

12.9 Division of Rational Expressions

Here you'll learn how to divide a rational expression by a polynomial. You'll also learn how to solve real-world applications that involve multiplication and division of rational expressions.

What if you had two rational expressions like $\frac{x+5}{x}$ and $\frac{x^2+6x+5}{x-2}$ and you wanted to divide them? How could you do so such that the answer were in simplest terms? After completing this Concept, you'll be able to divide rational expressions like this one.

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CK-12 Foundation: 1209S DividingRational Expressions

Watch this video for more examples of how to multiply and divide rational expressions.

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RobiChaudd: Multiplyor divide rational expressions

Guidance

Just as with ordinary fractions, we first rewrite the division problem as a multiplication problem and then proceed with the multiplication as outlined in the previous example.

Note: Remember that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$. The first fraction remains the same and you take the reciprocal of the *second* fraction. Do not fall into the common trap of flipping the first fraction.

Example A

Divide $\frac{4x^2}{15} \div \frac{6x}{5}$.

Solution

First convert into a multiplication problem by flipping the second fraction and then simplify as usual:

$$\frac{4x^2}{15} \div \frac{6x}{5} = \frac{4x^2}{15} \cdot \frac{5}{6x} = \frac{2x}{3} \cdot \frac{1}{3} = \frac{2x}{9}$$

Divide a Rational Expression by a Polynomial

When we divide a rational expression by a whole number or a polynomial, we can write the whole number (or polynomial) as a fraction with denominator equal to one, and then proceed the same way as in the previous examples.

Example B

Divide $\frac{9x^2-4}{2x-2} \div (21x^2-2x-8)$.

Solution

Rewrite the expression as a division of fractions, and then convert into a multiplication problem by taking the reciprocal of the divisor:

$$\frac{9x^2-4}{2x-2} \div \frac{21x^2-2x-8}{1} = \frac{9x^2-4}{2x-2} \cdot \frac{1}{21x^2-2x-8}$$

Then factor and solve:

$$\frac{9x^2-4}{2x-2} \cdot \frac{1}{21x^2-2x-8} = \frac{(3x-2)(3x+2)}{2(x-1)} \cdot \frac{1}{(3x-2)(7x+4)} = \frac{(3x+2)}{2(x-1)} \cdot \frac{1}{(7x+4)} = \frac{3x+2}{14x^2-6x-8}$$

Solve Applications Involving Multiplication and Division of Rational Expressions

Example C

Suppose Marciel is training for a running race. Marciel's speed (in miles per hour) of his training run each morning is given by the function x^3-9x*, where x is the number of bowls of cereal he had for breakfast. Marciel's training distance (in miles), if he eats x bowls of cereal, is* $3x^2-9x$*. What is the function for Marciel's time, and how long does it take Marciel to do his training run if he eats five bowls of cereal on Tuesday morning?*

Solution

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$\text{time} = \frac{3x^2-9x}{x^3-9x} = \frac{3x(x-3)}{x(x^2-9)} = \frac{3x(x-3)}{x(x+3)(x-3)}$$

$$\text{time} = \frac{3}{x+3}$$

$$\text{If } x = 5, \text{ then}$$

$$\text{time} = \frac{3}{5+3} = \frac{3}{8}$$

Marciel will run for $\frac{3}{8}$ of an hour.

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CK-12 Foundation: 1209 Dividing RationalExpressions

Vocabulary

- When we multiply two fractions we multiply the numerators and denominators separately:

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$$

- When we divide two fractions, we replace the second fraction with its reciprocal and multiply, since that's mathematically the same operation:

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{a\cdot d}{b\cdot c}$$

Guided Practice

Divide $\frac{3x^2-15x}{2x^2+3x-14}\div\frac{x^2-25}{2x^2+13x+21}$.

Solution

$$\frac{3x^2-15x}{2x^2+3x-14}\cdot\frac{2x^2+13x+21}{x^2-25}=\frac{3x(x-5)}{(2x+7)(x-2)}\cdot\frac{(2x+7)(x+3)}{(x-5)(x+5)}=\frac{3x}{(x-2)}\cdot\frac{(x+3)}{(x+5)}=\frac{3x^2+9x}{x^2+3x-10}$$

Practice

Divide the rational functions and reduce the answer to lowest terms.

1. $2xy\div\frac{2x^2}{y}$
2. $\frac{2x^3}{y}\div 3x^2$
3. $\frac{3x+6}{y-4}\div\frac{3y+9}{x-1}$
4. $\frac{x^2}{x-1}\div\frac{x}{x^2+x-2}$
5. $\frac{a^2+2ab+b^2}{ab^2-a^2b}\div(a+b)$
6. $\frac{3-x}{3x-5}\div\frac{x^2-9}{2x^2-8x-10}$
7. $\frac{x^2-25}{x+3}\div(x-5)$
8. $\frac{2x+1}{2x-1}\div\frac{4x^2-1}{1-2x}$

9. $\frac{3x^2+5x-12}{x^2-9} \div \frac{3x-4}{3x+4}$
10. $\frac{x^2+x-12}{x^2+4x+4} \div \frac{x-3}{x+2}$
11. $\frac{x^4-16}{x^2-9} \div \frac{x^2+4}{x^2+6x+9}$
12. Maria's recipe asks for $2\frac{1}{2}$ times more flour than sugar. How many cups of flour should she mix in if she uses $3\frac{1}{3}$ cups of sugar?
13. George drives from San Diego to Los Angeles. On the return trip he increases his driving speed by 15 miles per hour. In terms of his initial speed, by what factor is the driving time decreased on the return trip?
14. Ohm's Law states that in an electrical circuit $I = \frac{V}{R_c}$. The total resistance for resistors placed in parallel is given by: $\frac{1}{R_{tot}} = \frac{1}{R_1} + \frac{1}{R_2}$. Write the formula for the electric current in term of the component resistances: R_1 and R_2.

12.10 Addition and Subtraction of Rational Expressions

Here you'll learn how to add and subtract rational expressions with the same and different denominators.

What if you had two rational expressions like $\frac{x}{x+5}$ and $\frac{3}{x-4}$ with different denominators? How could you add and subtract them. After completing this Concept, you'll be to perform addition and subtraction with rational expressions like these.

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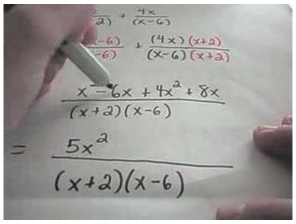
CK-12 Foundation: 1210S Adding and Subtracting Rational Expressions

Watch this video for more examples of how to add and subtract rational expressions.



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PatrickJMT: Adding and Subtracting Rational Expressions

Guidance

Like fractions, rational expressions represent a portion of a quantity. Remember that when we add or subtract fractions we must first make sure that they have the same denominator. Once the fractions have the same denominator, we combine the different portions by adding or subtracting the numerators and writing that answer over the common denominator.

Add and Subtract Rational Expressions with the Same Denominator

Fractions with common denominators combine in the following manner:

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \qquad \text{and} \qquad \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$$

Example A

Simplify.

a) $\frac{8}{7}-\frac{2}{7}+\frac{4}{7}$

b) $\frac{4x^2-3}{x+5}+\frac{2x^2-1}{x+5}$

c) $\frac{x^2-2x+1}{2x+3}-\frac{3x^2-3x+5}{2x+3}$

Solution

a) Since the denominators are the same we combine the numerators:

$$\frac{8}{7}-\frac{2}{7}+\frac{4}{7}=\frac{8-2+4}{7}=\frac{10}{7}$$

b)

Since the denominators are the same we combine the numerators: $\frac{4x^2-3+2x^2-1}{x+5}$

Simplify by collecting like terms: $\frac{6x^2-4}{x+5}$

c) Since the denominators are the same we combine the numerators. Make sure the subtraction sign is distributed to all terms in the second expression:

$$\frac{x^2-2x+1-(3x^2-3x+5)}{2x+3}=\frac{x^2-2x+1-3x^2+3x-5}{2x+3}=\frac{-2x^2+x-4}{2x+3}$$

Find the Least Common Denominator of Rational Expressions

To add and subtract fractions with different denominators, we must first rewrite all fractions so that they have the same denominator. In general, we want to find the **least common denominator**. To find the least common denominator, we find the **least common multiple** (LCM) of the expressions in the denominators of the different fractions. Remember that the least common multiple of two or more integers is the least positive integer that has all of those integers as factors.

The procedure for finding the lowest common multiple of polynomials is similar. We rewrite each polynomial in factored form and we form the LCM by taking each factor to the highest power it appears in any of the separate expressions.

Example B

Find the LCM of $48x^2y$ *and* $60xy^3z$.

Solution

First rewrite the integers in their prime factorization.

$$48=2^4\cdot 3$$
$$60=2^2\cdot 3\cdot 5$$

The two expressions can be written as:

$$48x^2y=2^4\cdot 3\cdot x^2\cdot y$$
$$60xy^3z=2^2\cdot 3\cdot 5\cdot x\cdot y^3\cdot z$$

To find the LCM, take the highest power of each factor that appears in either expression.

$$\text{LCM} = 2^4 \cdot 3 \cdot 5 \cdot x^2 \cdot y^3 \cdot z = 240x^2y^3z$$

Example C

Find the LCM of $2x^2+8x+8$ *and* x^3-4x^2-12x

Solution

Factor the polynomials completely:

$$\begin{aligned} 2x^2+8x+8 &= 2(x^2+4x+4) \\ &= 2(x+2)^2 \end{aligned}$$

$$\begin{aligned} x^3-4x^2-12x &= x(x^2-4x-12) \\ &= x(x+2)(x-6) \end{aligned}$$

To find the LCM, take the highest power of each factor that appears in either expression.

$$\text{LCM} = 2x(x+2)^2(x-6)$$

It's customary to leave the LCM in factored form, because this form is useful in simplifying rational expressions and finding any excluded values.

Add and Subtract Rational Expressions with Different Denominators

Now we're ready to add and subtract rational expressions. We use the following procedure.

1. Find the **least common denominator** (LCD) of the fractions.
2. Express each fraction as an equivalent fraction with the LCD as the denominator.
3. Add or subtract and simplify the result.

Example D

Perform the following operation and simplify: $\frac{2}{x+2} - \frac{3}{2x-5}$

Solution

The denominators can't be factored any further, so the LCD is just the product of the separate denominators: $(x+2)(2x-5)$. That means the first fraction needs to be multiplied by the factor $(2x-5)$ and the second fraction needs to be multiplied by the factor $(x+2)$:

$$\frac{2}{x+2} \cdot \frac{(2x-5)}{(2x-5)} - \frac{3}{2x-5} \cdot \frac{(x+2)}{(x+2)}$$

Combine the numerators and simplify: $\frac{2(2x-5)-3(x+2)}{(x+2)(2x-5)} = \frac{4x-10-3x-6}{(x+2)(2x-5)}$

Combine like terms in the numerator: $\frac{x-16}{(x+2)(2x-5)}$ **Answer**

Example E

Perform the following operation and simplify: $\frac{4x}{x-5} - \frac{3x}{5-x}$.

Solution

Notice that the denominators are almost the same; they just differ by a factor of -1.

Factor out -1 from the second denominator: $\frac{4x}{x-5} - \frac{3x}{-(x-5)}$

The two negative signs in the second fraction cancel: $\frac{4x}{x-5} + \frac{3x}{(x-5)}$

Since the denominators are the same we combine the numerators: $\frac{7x}{x-5}$ **Answer**

Watch this video for help with the Examples above.

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CK-12 Foundation: AddingandSubtractingRational Expressions

Vocabulary

- **Add and Subtract Rational Expressions with the Same Denominator**

Fractions with common denominators combine in the following manner:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \quad \text{and} \quad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

Guided Practice

a.) Find the LCM of $x^2 - 25$ *and* $x^2 + 3x + 2$.

b.) Perform the following operation and simplify: $\frac{2x-1}{x^2-9} - \frac{3x+4}{x^2-9}$.

Solution:

a.) First factor each polynomial to see if they have any common factors:

$x^2-25=(x+5)(x-5)$ and $x^2+3x+2=(x+2)(x+1)$

Since the two polynomials do not have any common factors, this means that the LCM of the two polynomials is:

$(x^2-25)(x^2+3x+2)=x^4+3x^3-23x^2-75x-50$

b.) To subtract the second fraction from the first, subtraction the numerator of the second from the numerator of the first. Make sure to put parenthesis around the numerator of the second fraction, so you remember to subtract each term.

$$\frac{2x-1}{x^2-9}-\frac{3x+4}{x^2-9}=\frac{2x-1-(3x+4)}{x^2-9}=\frac{2x-1-3x-4}{x^2-9}=\frac{-x-5}{x^2-9}$$

Practice

Perform the indicated operation and simplify. Leave the denominator in factored form.

1. $\frac{5}{24}-\frac{7}{24}$
2. $\frac{2x}{13}-\frac{x}{3}$
3. $\frac{5}{2x+3}+\frac{3}{2x+3}$
4. $\frac{1}{5x-7}+\frac{10}{5x-7}$
5. $\frac{3x-1}{x+9}-\frac{4x+3}{x+9}$
6. $\frac{1-7x}{3x+10}-\frac{x+20}{3x+10}$
7. $\frac{4x+7}{2x^2}-\frac{3x-4}{2x^2}$
8. $\frac{10x-5}{9x^2}-\frac{5}{9x^2}$
9. $\frac{x^2}{x+5}-\frac{25}{x+5}$
10. $\frac{.25x^2}{x+100}-\frac{0.1}{x+100}$
11. $\frac{1}{x}+\frac{2}{3x}$
12. $\frac{4}{5x^2}-\frac{2}{7x^3}$
13. $\frac{10}{3x-1}-\frac{7}{1-3x}$
14. $\frac{10}{x+5}+\frac{2}{x+2}$
15. $\frac{2x}{x-3}-\frac{3x}{x+4}$
16. $\frac{4x-3}{2x+1}+\frac{x+2}{x-9}$
17. $\frac{x^2}{x+4}-\frac{3x^2}{4x-1}$
18. $\frac{2}{5x+2}-\frac{x+1}{x^2}$
19. $\frac{x+4}{2x}+\frac{2}{9x}$
20. $\frac{5x+3}{x^2+x}+\frac{2x+1}{x}$
21. $\frac{4}{(x+1)(x-1)}-\frac{5}{(x+1)(x+2)}$
22. $\frac{2x}{(x+2)(3x-4)}+\frac{7x}{(3x-4)^2}$
23. $\frac{3x+5}{x(x-1)}-\frac{9x-1}{(x-1)^2}$
24. $\frac{1}{(x-2)(x-3)}+\frac{4}{(2x+5)(x-6)}$
25. $\frac{3x-2}{x-2}+\frac{1}{x^2-4x+4}$
26. $\frac{-x^3}{x^2-7x+6}+x-4$
27. $\frac{2x}{x^2+10x+25}-\frac{3x}{2x^2+7x-15}$
28. $\frac{1}{x^2-9}+\frac{2}{x^2+5x+6}$

29. $\frac{-x+4}{2x^2-x-15}+\frac{x}{4x^2+8x-5}$
30. $\frac{4}{9x^2-49}-\frac{1}{3x^2+5x-28}$

12.11 Applications of Adding and Subtracting Rational Expressions

Here you'll learn how to solve circuitry and other real-world applications that involve adding and subtracting rational expressions.

What if it took you 3 hours to clean your house by yourself and it took your brother 2 hours to clean it by himself? How long would it take both of you to clean the house if you were working together? After completing this Concept, you'll be able to solve applications like this one involving addition and subtraction of rational expressions.

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CK-12 Foundation: 1211S Solve Applications by AddingandSubtractingRational Expressions

Guidance

In the previous concept, you learned how to add and subtract rational expressions. In this concept, you will use those tools to solve real-world problems.

Example A

In an electrical circuit with two resistors placed in parallel, the reciprocal of the total resistance is equal to the sum of the reciprocals of each resistance: $\frac{1}{R_{tot}} = \frac{1}{R_1} + \frac{1}{R_2}$. *Find an expression for the total resistance,* R_{tot}.

Solution

Let's simplify the expression $\frac{1}{R_1} + \frac{1}{R_2}$.

The lowest common denominator is R_1R_2, so we multiply the first fraction by $\frac{R_2}{R_2}$ and the second fraction by $\frac{R_1}{R_1}$:

$$\frac{R_2}{R_2} \cdot \frac{1}{R_1} + \frac{R_1}{R_1} \cdot \frac{1}{R_2}$$

Simplify:

$$\frac{R_2 + R_1}{R_1R_2}$$

The total resistance is the reciprocal of this expression: $R_{tot} = \frac{R_1R_2}{R_1 + R_2}$ **Answer**

Example B

The sum of a number and its reciprocal is $\frac{53}{14}$*. Find the numbers.*

Solution

Define variables:

Let x be the number; then its reciprocal is $\frac{1}{x}$.

Set up an equation:

The equation that describes the relationship between the numbers is $x+\frac{1}{x}=\frac{53}{14}$

Solve the equation:

$$\text{Find the lowest common denominator:} \qquad \text{LCM} = 14x$$

$$\text{Multiply all terms by } 14x: \qquad 14x \cdot x + 14x \cdot \frac{1}{x} = 14x \cdot \frac{53}{14}$$

(Notice that we're multiplying the terms by $14x$ instead of by $\frac{14x}{14x}$. We can do this because we're multiplying both sides of the equation by the same thing, so we don't have to keep the actual values of the terms the same. We could also multiply by $\frac{14x}{14x}$, but then the denominators would just cancel out a couple of steps later.)

$$\text{Cancel common factors in each term:} \qquad 14x \cdot x + 14x \cdot \frac{1}{x} = 14x \cdot \frac{53}{14}$$

$$\text{Simplify:} \qquad 14x^2+14=53x$$

$$\text{Write all terms on one side of the equation:} \qquad 14x^2-53x+14=0$$

$$\text{Factor:} \qquad (7x-2)(2x-7)=0$$

$$x=\frac{2}{7} \text{ and } x=\frac{7}{2}$$

Notice there are two answers for x, but they are really parts of the same solution. One answer represents the number and the other answer represents its reciprocal.

Check:

$\frac{2}{7}+\frac{7}{2}=\frac{4+49}{14}=\frac{53}{14}$. **The answer checks out.**

Work problems are problems where two people or two machines work together to complete a job. Work problems often contain rational expressions. Typically we set up such problems by looking at the part of the task completed by each person or machine. The completed task is the sum of the parts of the tasks completed by each individual or each machine.

To determine the part of the task completed by each person or machine we use the following fact:

$$\text{Part of the task completed} = \text{rate of work} \times \text{time spent on the task}$$

It's usually useful to set up a table where we can list all the known and unknown variables for each person or machine and then combine the parts of the task completed by each person or machine at the end.

Example C

Mary can paint a house by herself in 12 hours. John can paint a house by himself in 16 hours. How long would it take them to paint the house if they worked together?

Solution

Define variables:

Let $t =$ the time it takes Mary and John to paint the house together.

Construct a table:

Since Mary takes 12 hours to paint the house by herself, in one hour she paints $\frac{1}{12}$ of the house.

Since John takes 16 hours to pain the house by himself, in one hour he paints $\frac{1}{16}$ of the house.

Mary and John work together for t hours to paint the house together. Using

$$Part\ of\ the\ task\ completed = rate\ of\ work \cdot time\ spent\ on\ the\ task$$

we can write that Mary completed $\frac{t}{12}$ of the house and John completed $\frac{t}{16}$ of the house in this time.

This information is nicely summarized in the table below:

TABLE 12.7:

Painter	Rate of work (per hour)	Time worked	Part of task
Mary	$\frac{1}{12}$	t	$\frac{t}{12}$
John	$\frac{1}{16}$	t	$\frac{t}{16}$

Set up an equation:

In t hours, Mary painted $\frac{t}{12}$ of the house and John painted $\frac{t}{16}$ of the house, and together they painted 1 whole house. So our equation is $\frac{t}{12} + \frac{t}{16} = 1$.

Solve the equation:

$$\text{Find the lowest common denominator:} \qquad \text{LCM} = 48$$

$$\text{Multiply all terms in the equation by the LCM:} \qquad 48 \cdot \frac{t}{12} + 48 \cdot \frac{t}{16} = 48 \cdot 1$$

$$\text{Cancel common factors in each term:} \qquad 4 \cdot \frac{t}{1} + 3 \cdot \frac{t}{1} = 48 \cdot 1$$

$$\text{Simplify:} \qquad 4t + 3t = 48$$

$$7t = 48 \Rightarrow t = \frac{48}{7} = 6.86\ hours$$

Check: The answer is reasonable. We'd expect the job to take more than half the time Mary would take by herself but less than half the time John would take, since Mary works faster than John.

Watch this video for help with the Examples above.

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CK-12 Foundation: Solve Applicationsby Adding and Subtracting

Vocabulary

- The total resistance can be found using the expression: $R_{tot} = \frac{R_1 R_2}{R_1 + R_2}$

Guided Practice

Suzie and Mike take two hours to mow a lawn when they work together. It takes Suzie 3.5 hours to mow the same lawn if she works by herself. How long would it take Mike to mow the same lawn if he worked alone?

Solution

Define variables:

Let $t =$ the time it takes Mike to mow the lawn by himself.

Construct a table:

TABLE 12.8:

Painter	Rate of work (per hour)	Time worked	Part of Task
Suzie	$\frac{1}{3.5} = \frac{2}{7}$	2	$\frac{4}{7}$
Mike	$\frac{1}{t}$	2	$\frac{2}{t}$

Set up an equation:

Since Suzie completed $\frac{4}{7}$ of the lawn and Mike completed $\frac{2}{t}$ of the lawn and together they mowed the lawn in 2 hours, we can write the equation: $\frac{4}{7} + \frac{2}{t} = 1$

Solve the equation:

Find the lowest common denominator: $\text{LCM} = 7t$

Multiply all terms in the equation by the LCM: $7t \cdot \frac{4}{7} + 7t \cdot \frac{2}{t} = 7t \cdot 1$

Cancel common factors in each term: $t \cdot \frac{4}{1} + 7 \cdot \frac{2}{1} = 7t \cdot 1$

Simplify: $4t + 14 = 7t$

$$3t = 14 \Rightarrow t = \frac{14}{3} = 4\frac{2}{3} \text{ } hours$$

Check: The answer is reasonable. We'd expect Mike to work slower.

Practice

For 1-5, perform the indicated operation. Leave the denominator in factored form.

1. $\frac{4x}{x+1} - \frac{2}{2(x+1)}$
2. $\frac{10}{21} + \frac{9}{35}$
3. $\frac{2x}{x-4} + \frac{x}{4-x}$
4. $\frac{5}{2x+3} - 3$
5. $\frac{5x+1}{x+4} + 2$

For 6-8, find the missing resistance.

6. $R_1 = 4, R_2 = 6, R_{tot} = ?$
7. $R_1 = 1, R_2 = ?, R_{tot} = \frac{2}{3}$
8. $R_1 = ?, R_2 = 12, R_{tot} = \frac{36}{15}$

Solve the following work problems.

9. Andrea can wash the windows on their house in 30 minutes and Jorge can wash the windows on their house in 40 minutes. How long will it take for them to wash the windows together?
10. A pool can be filled by one pipe in 5 hours and by a different pipe in 7 hours. How long will it take the pool to fill using both pipes?

12.12 Rational Equations Using Proportions

- Here you'll learn how to use cross products and lowest common denominators to solve rational equations.

What if you had a rational equation like $\frac{x+2}{x}-2=\frac{1}{x+3}$? How could you solve it for x? After completing this Concept, you'll be able to solve rational equations like this one using cross products and lowest common denominators.

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CK-12 Foundation: 1212S Solutions ofRational Equations

Guidance

A **rational equation** is one that contains rational expressions. It can be an equation that contains rational coefficients or an equation that contains rational terms where the variable appears in the denominator.

An example of the first kind of equation is: $\frac{3}{5}x+\frac{1}{2}=4$.

An example of the second kind of equation is: $\frac{x}{x-1}+1=\frac{4}{2x+3}$.

The first aim in solving a rational equation is to eliminate all denominators. That way, we can change a rational equation to a polynomial equation which we can solve with the methods we have learned this far.

Solve Rational Equations Using Cross Products

A rational equation that contains just one term on each side is easy to solve by **cross multiplication.** Consider the following equation:

$$\frac{x}{5}=\frac{x+1}{2}$$

Our first goal is to eliminate the denominators of both rational expressions. In order to remove the 5 from the denominator of the first fraction, we multiply both sides of the equation by 5:

$$\begin{aligned}5\cdot\frac{x}{5}&=5\cdot\frac{x+1}{2}\\x&=\frac{5(x+1)}{2}\end{aligned}$$

Now, we remove the 2 from the denominator of the second fraction by multiplying both sides of the equation by 2:

$$\begin{aligned}2\cdot x &= 2\cdot\frac{5(x+1)}{2}\\ 2x &= 5(x+1)\end{aligned}$$

Then we can solve this equation for x.

Notice that this equation is what we would get if we simply multiplied each numerator in the original equation by the denominator from the opposite side of the equation. It turns out that we can always simplify a rational equation with just two terms by multiplying each numerator by the opposite denominator; this is called **cross multiplication.**

Example A

Solve the equation $\frac{2x}{x+4}=\frac{5}{x}$.

Solution

Cross-multiply. The equation simplifies to: $2x^2=5(x+4)$

Simplify: $2x^2=5x+20$

Move all terms to one side of the equation: $2x^2-5x-20=0$

Solve using the quadratic formula: $x=\frac{5\pm\sqrt{185}}{4}\Rightarrow \underline{\underline{x=-2.15}}$ or $\underline{\underline{x=4.65}}$

It's important to plug the answer back into the original equation when the variable appears in any denominator of the equation, because the answer might be an excluded value of one of the rational expressions. If the answer obtained makes any denominator equal to zero, that value is not really a solution to the equation.

Check: $\frac{2x}{x+4}=\frac{5}{x}\Rightarrow\frac{2(-2.15)}{-2.15+4}\overset{?}{=}\frac{5}{-2.15}\Rightarrow\frac{-4.30}{1.85}\overset{?}{=}-2.3\Rightarrow -2.3=-2.3$. The answer checks out.

$\frac{2x}{x+4}=\frac{5}{x}\Rightarrow\frac{2(4.65)}{4.65+4}\overset{?}{=}\frac{5}{4.65}\Rightarrow\frac{9.3}{8.65}\overset{?}{=}1.08\Rightarrow 1.08=1.08$. The answer checks out.

Solve Rational Equations Using Lowest Common Denominators

Another way of eliminating the denominators in a rational equation is to multiply all the terms in the equation by the lowest common denominator. You can use this method even when there are more than two terms in the equation.

Example B

Solve $\frac{3}{x+2}-\frac{4}{x-5}=\frac{2}{x^2-3x-10}$.

Solution

Factor all denominators: $\frac{3}{x+2}-\frac{4}{x-5}=\frac{2}{(x+2)(x-5)}$

Find the lowest common denominator: $\text{LCD}=(x+2)(x-5)$

Multiply all terms in the equation by the LCD:

$$(x+2)(x-5)\cdot\frac{3}{x+2}-(x+2)(x-5)\cdot\frac{4}{x-5}=(x+2)(x-5)\cdot\frac{2}{(x+2)(x-5)}$$

The equation simplifies to: $3(x-5)-4(x+2)=2$

Simplify: $3x-15-4x-8=2$

$$\underline{\underline{x=-25}}$$

Check: $\frac{3}{x+2}-\frac{4}{x-5}=\frac{2}{x^2-3x-10}\Rightarrow\frac{3}{-25+2}-\frac{4}{-25-5}\overset{?}{=}\frac{2}{(-25)^2-3(-25)-10}\Rightarrow .003=.003$. The answer checks out.

Example C

Solve $\frac{2x}{2x+1}+\frac{x}{x+4}=1$.

Solution

Find the lowest common denominator: $\text{LCD}=(2x+1)(x+4)$

Multiply all terms in the equation by the LCD:

$$(2x+1)(x+4)\cdot\frac{2x}{2x+1}+(2x+1)(x+4)\cdot\frac{x}{x+4}=(2x+1)(x+4)$$

Cancel all common terms. $2x(x+4)+x(2x+1)=(2x+1)(x+4)$

The simplified equation is:

Eliminate parentheses: $2x^2+8x+2x^2+x=2x^2+9x+4$

Collect like terms: $2x^2=4$

$$x^2=2\Rightarrow\underline{\underline{x=\pm\sqrt{2}}}$$

Check: $\frac{2x}{2x+1}+\frac{x}{x+4}=\frac{2\sqrt{2}}{2\sqrt{2}+1}+\frac{\sqrt{2}}{\sqrt{2}+4}=0.739+0.261=1$. The answer checks out.

$\frac{2x}{2x+1}+\frac{x}{x+4}=\frac{2\left(-\sqrt{2}\right)}{2\left(-\sqrt{2}\right)+1}+\frac{-\sqrt{2}}{-\sqrt{2}+4}=1.547-0.547=1$. The answer checks out.

Watch this video for help with the Examples above.

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CK-12 Foundation: Solving Rational Equations

Vocabulary

- For a **quadratic equation** in standard form, $ax^2+bx+c=0$, the **quadratic formula** looks like this:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Guided Practice

Solve the following rational equations :

1. $\frac{3}{5}x+\frac{1}{2}=4$.

2. $\frac{x}{x-1}+1=\frac{1}{x+3}$.

Solutions:

1.

Start with the originl equation: $\frac{3}{5}x+\frac{1}{2}=4$

Multiply by the LCD: $10\cdot\left(\frac{3}{5}x+\frac{1}{2}\right)=10\cdot 4$

Simplify: $6x+5=40$

Isolate x first by subtracting 5 from each side: $6x+5-5=40-5$

Simplify: $6x=35$

Isolate x by dividing by 6: $\frac{6x}{6}=\frac{35}{6}$

Simplify: $x=5\frac{5}{6}$

2.

Start with the originl equation: $\frac{x}{x-1}+1=\frac{1}{x+3}.$

Multiply by the LCD: $(x-1)(x+3)\cdot\left(\frac{x}{x-1}+1\right)=(x-1)(x+3)\cdot\frac{1}{2x+3}.$

Simplify: $x(x+3)+1(x-1)(x+3)=1(x-1)$

Distribute: $x^2+3x+x^2+3x-1x-3=x-1$

Combine like terms: $2x^2+5x-3=x-1$

Set one side equal to 0: $2x^2+4x+-2=0$

Now we have a quadratic equation. Since it's not factorable (check for yourself!), we must use the quadratic formula.

Start with the quadratic formula. $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Substitute in the appropriate values. $x=\frac{-4\pm\sqrt{4^2-4(2)(-2)}}{2(2)}$

Simplify. $x=\frac{-4\pm\sqrt{32}}{4}=\frac{-4\pm4\sqrt{2}}{4}=-1\pm\sqrt{2}$

This means that $x=-1+\sqrt{2}\approx0.4$ or $x=-1-\sqrt{2}\approx-2.4$

Practice

Solve the following equations.

1. $\frac{2x+1}{4} = \frac{x-3}{10}$
2. $\frac{4x}{x+2} = \frac{5}{9}$
3. $\frac{5}{3x-4} = \frac{2}{x+1}$
4. $\frac{7}{x+3} = \frac{x+1}{2x-3}$
5. $\frac{7x}{x-5} = \frac{x+3}{x}$
6. $\frac{2}{x+3} - \frac{1}{x+4} = 0$
7. $\frac{3}{2x-1} + \frac{2}{x+4} = 2$
8. $\frac{2x}{x-1} - \frac{x}{3x+4} = 3$
9. $\frac{x+1}{x-1} + \frac{x-4}{x+4} = 3$
10. $\frac{x}{x-2} + \frac{x}{x+3} = \frac{1}{x^2+x-6}$
11. $\frac{2}{x^2+4x+3} = 2 + \frac{x-2}{x+3}$
12. $\frac{1}{x+5} - \frac{1}{x-5} = \frac{1-x}{x+5}$
13. $\frac{x}{x^2-36} + \frac{1}{x-6} = \frac{1}{x+6}$
14. $\frac{2x}{3x+3} - \frac{1}{4x+4} = \frac{2}{x+1}$
15. $\frac{-x}{x-2} + \frac{3x-1}{x+4} = \frac{1}{x^2+2x-8}$

12.13 Applications Using Rational Equations

Here you'll learn how to solve distance problems and other real-world applications that involve rational equations.

What if you took an airplane trip? The current of the jet stream is 100 miles per hour. It took you the same amount of time to travel 3000 miles without the jet stream as it did to travel 2000 miles with the jet stream. How could you determine the speed of the plane in calm air? After completing this Concept, you'll be able to solve real-world applications like this one using rational equations.

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CK-12 Foundation: 1213S Solve Applications UsingRational Equations

Guidance

A motion problem with no acceleration is described by the formula $distance = speed \times time$. These problems can involve the addition and subtraction of rational expressions.

Example A

Last weekend Nadia went canoeing on the Snake River. The current of the river is three miles per hour. It took Nadia the same amount of time to travel 12 miles downstream as it did to travel 3 miles upstream. Determine how fast Nadia's canoe would travel in still water.

Solution

Define variables:

Let $s =$ speed of the canoe in still water

Then, $s+3 =$ the speed of the canoe traveling downstream

$s-3 =$ the speed of the canoe traveling upstream

Construct a table:

TABLE 12.9:

Direction	Distance (miles)	Rate	Time
Downstream	12	$s+3$	t
Upstream	3	$s-3$	t

Write an equation:

Since $distance = rate \times time$, we can say that $time = \frac{distance}{rate}$.

The time to go downstream is: $t = \frac{12}{s+3}$

The time to go upstream is: $t = \frac{3}{s-3}$

Since the time it takes to go upstream and downstream are the same, we have: $\frac{3}{s-3} = \frac{12}{s+3}$

Solve the equation:

Cross-multiply: $3(s+3) = 12(s-3)$

Simplify: $3s+9 = 12s-36$

Solve: $s = 5\ mi/h$

Check: Upstream: $t = \frac{12}{8} = 1\frac{1}{2}\ hour$; downstream: $t = \frac{3}{2} = 1\frac{1}{2}\ hour$. **The answer checks out.**

Example B

Peter rides his bicycle. When he pedals uphill he averages a speed of eight miles per hour, when he pedals downhill he averages 14 miles per hour. If the total distance he travels is 40 miles and the total time he rides is four hours, how long did he ride at each speed?

Solution

Define variables:

Let $d =$ distance Peter bikes uphill at 8 miles per hour.

Construct a table:

TABLE 12.10:

Direction	Distance (miles)	Rate (mph)	Time (hours)
Uphill	d	8	t_1
Downhill	$40-d$	14	t_2

Write an equation:

We know that $time = \frac{distance}{rate}$.

The time to go uphill is: $t_1 = \frac{d}{8}$

The time to go downhill is: $t_2 = \frac{40-d}{14}$

We also know that the total time is 4 hours: $\frac{d}{8} + \frac{40-d}{14} = 4$

Solve the equation:

Find the lowest common denominator: $\text{LCD} = 56$

Multiply all terms by the common denominator: $7d + 160 - 4d = 224$

Solve: $d = 21.3\ mi$

Check: Uphill: $t = \frac{21.3}{8} = 2.67\ hours$; downhill: $t = \frac{40-21.3}{14} = 1.33\ hours$. **The answer checks out.**

Example C

A group of friends decided to pool together and buy a birthday gift that cost $200. Later 12 of the friends decided not to participate any more. This meant that each person paid $15 more than their original share. How many people were in the group to begin with?

Solution

Define variables:

Let $x =$ the number of friends in the original group.

Make a table:

TABLE 12.11:

	Number of people	Gift price	Share amount
Original group	x	200	$\frac{200}{x}$
Later group	$x-12$	200	$\frac{200}{x-12}$

Write an equation:

Since each person's share went up by $15 after 12 people refused to pay, we write the equation $\frac{200}{x-12} = \frac{200}{x} + 15$

Solve the equation:

Find the lowest common denominator: $\text{LCD} = x(x-12)$

Multiply all terms by the LCD: $$x(x-12) \cdot \frac{200}{x-12} = x(x-12) \cdot \frac{200}{x} + x(x-12) \cdot 15$$

Cancel common factors and simplify: $$200x = 200(x-12) + 15x(x-12)$$

Eliminate parentheses: $$200x = 200x - 2400 + 15x^2 - 180x$$

Get all terms on one side of the equation: $$0 = 15x^2 = 180x - 2400$$

Divide all terms by 15 : $$0 = x^2 - 12x - 160$$

Factor: $$0 = (x-20)(x+8)$$

Solve: $$x = 20, x = -8$$

The answer that makes sense is $x =$ **20people.**

Check: Originally \$200 shared among 20 people is \$10 each. After 12 people leave, \$200 shared among 8 people is \$25 each. So each person pays \$15 more. **The answer checks out.**

Watch this video for help with the Examples above.

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CK-12 Foundation: Solving Applications UsingRational Equations

Vocabulary

- Since $distance = rate \times time$, we can say that $time = \frac{distance}{rate}$.

Guided Practice

Carrie is a runner. When she runs uphill she averages a speed of 2 miles per hour, when she runs downhill she averages 5 miles per hour. If she is running through the hilly streets of San Francisco and the total distance she travels is 13.5 miles and the total time she run 4.5 hours, how long did she run uphill and how long did she run downhill?

Solution

Define variables:

Let $t_1 =$ time Carrie runs uphill.

Let $t_2 =$ time Carrie runs downhill.

Let $d =$ the distance Carrie runs uphill.

Construct a table:

TABLE 12.12:

Direction	**Distance (miles)**	**Rate (mph)**	**Time (hours)**
Uphill	d	2	t_1
Downhill	$13.5-d$	5	t_2

Write an equation:

We know that $time = \frac{distance}{rate}$.

The time to go uphill is: $$t_1 = \frac{d}{2}$$

The time to go downhill is: $$t_2 = \frac{13.5-d}{5}$$

We also know that the total time is 4.5 hours: $$\frac{d}{2} + \frac{13.5-d}{5} = 4.5$$

Solve the equation:

Find the lowest common denominator: $\text{LCD} = 2 \cdot 5 = 10$

Multiply all terms by the common denominator: $5d + 27 - 2d = 45$

Solve: $d = 6\ mi$

Since $d = 6$ is the distance Carrie to ran uphill, then $13.5 - d = 13.5 - 6 = 7.5$ is the distance Carrie ran downhill.

Check: Uphill: $t = \frac{6}{2} = 3\ hours$; downhill: $t = \frac{13.5-6}{5} = 1.5\ hours$. **The answer checks out.**

Practice

For 1-4, solve for x.

1. $\frac{3x^2+2x-1}{x^2-1} = -2$
2. $x + \frac{1}{x} = 2$
3. $-3 + \frac{1}{x+1} = \frac{2}{x}$
4. $\frac{1}{x} - \frac{x}{x-2} = 2$

For 5-10, solve the following applications.

5. Juan jogs a certain distance and then walks a certain distance. When he jogs he averages 7 miles/hour and when he walks he averages 3.5 miles per hour. If he walks and jogs a total of 6 miles in a total of 1.2 hours, how far does he jog and how far does he walk?

6. A boat travels 60 miles downstream in the same time as it takes it to travel 40 miles upstream. The boat's speed in still water is 20 miles per hour. Find the speed of the current.
7. Paul leaves San Diego driving at 50 miles per hour. Two hours later, his mother realizes that he forgot something and drives in the same direction at 70 miles per hour. How long does it take her to catch up to Paul?
8. On a trip, an airplane flies at a steady speed against the wind and on the return trip the airplane flies with the wind. The airplane takes the same amount of time to fly 300 miles against the wind as it takes to fly 420 miles with the wind. The wind is blowing at 30 miles per hour. What is the speed of the airplane when there is no wind?
9. A debt of $420 is shared equally by a group of friends. When five of the friends decide not to pay, the share of the other friends goes up by $25. How many friends were in the group originally?
10. A non-profit organization collected $2250 in equal donations from their members to share the cost of improving a park. If there were thirty more members, then each member could contribute $20 less. How many members does this organization have?

Summary

This chapter begins by distinguishing between three variation models: direct variation, inverse variation, and joint variation. It then moves on to graphing and solving rational functions, with particular attention paid to asymptotes. Next, it addresses simplifying rational expressions by factoring and dividing. The four mathematical operations of multiplication, division, addition, and subtraction are then performed on rational expressions. The chapter concludes with real-life applications of rational equations and methods for solving them.

CHAPTER 13 Probability and Statistics

Chapter Outline

13.1 MEASUREMENT OF PROBABILITY
13.2 EMPIRICAL PROBABILITY
13.3 PERMUTATIONS
13.4 PROBABILITY AND PERMUTATIONS
13.5 COMBINATIONS
13.6 PROBABILITY AND COMBINATIONS
13.7 MUTUALLY EXCLUSIVE EVENTS
13.8 INDEPENDENCE VERSUS DEPENDENCE
13.9 MEASURES OF CENTRAL TENDENCY AND DISPERSION
13.10 MEASURES OF SPREAD/DISPERSION
13.11 STEM-AND-LEAF PLOTS AND HISTOGRAMS
13.12 BOX-AND-WHISKER PLOTS
13.13 SAMPLING METHODS
13.14 PLANNING AND CONDUCTING SURVEYS

Introduction

What if you wanted to know the likelihood of a certain event like the likelihood that you would win a bike raffle? How could you determine your winning chances, or odds? After completing this chapter, you'll be able to use probability to answer such questions. You'll also be introduced to statistical ways of measuring a set of data. For example, how could you describe the data set $\{13, 5, 8\ 9, 20\}$? In addition, you'll use graphical outputs like stem-and-leaf plots, histograms, and box-and-whisker plots to visually display and analyze such data.

13.1 Measurement of Probability

Here you'll learn how to find the sample space of possible outcomes for an event. You'll also find the theoretical probability of and the odds for and against events.

What if you were playing a board game in which you rolled two dice simultaneously? You need to roll exactly 11 on your next turn to win the game. How could you determine the probability that you would roll an 11? After completing this Concept, you'll be able to find the theoretical probability and calculate the odds of events like this one.

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CK-12 Foundation: Theoretical Probability

Guidance

A sample space is the set of all possible outcomes for an event. In tossing a coin, the sample space consists of 2 **outcomes** – getting heads, and getting tails. Each of these outcomes (*heads* and *tails*) could be considered an **event**. Each event has 1 matching element in the **sample space**.

For example, the roll of a single die has 6 possible outcomes: the die can show any number from 1 to 6. We can say that the **sample space** for rolling a single die contains 6 outcomes: $\{1, 2, 3, 4, 5, 6\}$. If we say we are interested in rolling a six, then rolling a six is our **event** and this event has 1 matching element in the sample space: $\{6\}$. If, on the other hand, we are interested in rolling an even number, then rolling an even number is our event, and this event has 3 matching elements in the sample space: $\{2, 4, 6\}$.

Example A

A pair of standard, 6-sided dice are rolled, and the total of the numbers that come up determines a player's score. Find the sample space of possible outcomes, and determine how many outcomes result in a score of 5.

Solution

The scores a player can get are those in the following set: $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. But the sample space isn't just this set of 11 events. For example, there's only one way to score 12: the player has to roll a six on each of the dice. But to score 5, the player can roll a 1 and a 4, or a 2 and a 3. Furthermore, there are 2 possibilities for each of these combinations (imagine one die is red, and 1 die is green: we could roll 1 on the red die and 4 on the green or we could roll 4 on the red die and 1 on the green). Even though the dice we actually use may appear identical, they're still separate entities, and it does make a difference which one rolls which number. So there are 4 ways a player can score 5:

$$(1\&4) \quad (2\&3) \quad (3\&2) \quad (4\&1)$$

To find the full sample space, we must consider all possible outcomes. The best way to do this is with a table (the outcomes that give a score of 5 are highlighted):

		Die number 1 result					
		1	**2**	**3**	**4**	**5**	**6**
Die number 2 result	**1**	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

The sample space (shown above) has 36 outcomes, 4 of which result in a score of five.

Notice that the number of outcomes in the sample space when you roll 2 dice is the product of the number of outcomes when you roll one die and the number of outcomes when you roll the other die. That is, there are 6 possible outcomes for one die and 6 outcomes for the other die, so there are $6 \times 6 = 36$ possible outcomes when you roll both dice together. This property will be important later.

Find Theoretical Probability of an Event

The theoretical probability of an event is a measure of how likely a given outcome (the *event*) is for a particular experiment, such as tossing a coin. If the experiment were carried out a nearly infinite number of times, the probability of a particular event would be the ratio of how many times a particular outcome occurred to how many times the experiment was performed.

We write the probability that a particular event, E, occurs as:

$$P(E)$$

For example, when tossing a coin we may only be interested in getting heads. We could denote that probability as:

$$P(Heads) \quad \text{or simply} \quad P(H)$$

We know that the sample space for tossing a single coin has two elements: *Heads* and *Tails* (or H and T). Each is as likely as the other to occur, so we know that:

$$P(H) = \frac{1}{2}$$

To find the probability of a particular event, we look at how many possible outcomes would contribute to that event, and divide that number by the total number of outcomes in the sample space.

$$P(E) = \frac{EventSpace}{SampleSpace}$$

When we roll a single 6-sided die, we know that the ***chances*** of rolling a 3 are 1 in 6. This is, in effect, the probability of rolling 3:

$$P(3) = \frac{1}{6}$$

Example B

Four coins are tossed simultaneously. What is the probability of getting three or more tails?

Solution

We'll start by listing all the possible outcomes in a table. But how many elements will the table have?

Remember that when we rolled 2 dice together, we found the number of outcomes by multiplying the number of outcomes for one die by the number of outcomes for the other die. So if we're flipping four coins, it makes sense to multiply the number of outcomes for each of the four coins together. There are 2 outcomes for each coin, so when all four coins are flipped there should be $2 \times 2 \times 2 \times 2 = 16$ outcomes. Let's organize them as follows:

HHHH	HHHT	HHTH	HHTT
HTHH	HTHT	HTTH	HTTT
THHH	THHT	THTH	THTT
TTHH	TTHT	TTTH	TTTT

Once we fill out the table, we see that there are indeed 16 possible outcomes, and 5 of those outcomes match our **event**. So the probability of getting three or more tails is $P(3 \text{ or more tails}) = \frac{5}{16}$.

Find Odds For and Against an Event

When we talk of **probability**, we generally think (as we've seen in this lesson) of the ratio of the number of times our **event** occurs to the number of times the experiment was carried out.

Another way to talk about the chances of an event occurring is with **odds**. You may have heard of the phrases "fifty-fifty" or "even odds" to describe an unpredictable situation like the chances of getting heads when you toss a coin. The phrase means that the coin is as likely to come up tails as it is heads (each event occurring 50% of the time). The **odds** of an event are given by the ratio of the number of times the event occurs to the number of times the event *does not* occur. In *sample space* terms it means:

$$\text{Odds} = \frac{\text{number of matching events in sample place}}{\text{number of non-matching events in sample place}}$$

whereas we would describe probability as

$$\text{Probability} = \frac{\text{number of matching events in sample place}}{\text{number of total events in sample place}}$$

To avoid confusion with probability, odds are usually left as a ratio such as 1:5, which would be read as "one **to** five". When *probability* is read as a ratio, it's usually written as a fraction like $\frac{1}{5}$, which would usually be read as "one **in** five."

Example C

Find the odds of the following events:

a) *Tossing a coin and getting heads.*

b) *Rolling a die and getting a 3.*

c) *Tossing 4 coins and getting exactly 3 tails.*

Solution

The key to finding odds is looking at how many outcomes result in the event and how many **do not**:

a) The sample space consists of 2 outcomes: 1 heads, 1 not-heads. The odds of getting **heads** are 1 : 1 (*one to one*, or ***even***).

b) The sample space consists of 6 outcomes: 1 three and 5 not-three. The odds of getting **three** are 1 : 5 (*one to five*).

c) Look back at example 4, where we found the sample space for tossing 4 coins. The sample space consists of 4 outcomes where exactly 3 tails came up and 12 outcomes when they did not. So the odds of getting **3 tails** are 3 : 12 = 1 : 4 (*one to four*).

Look carefully at part b above. This illustrates the need to avoid confusion between odds and probability. We know that the *probability* of getting a 3 is $P(3) = \frac{1}{6}$ or *"one in six,"* but the *odds* describes the same event with the ratio 1 : 5 or *"one to five"*.

Watch this video for help with the Examples above.

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CK-12 Foundation: Theoretical Probability

Vocabulary

- A **sample space** is the set of all possible outcomes for an event.
- The **odds** of an event are given by the ratio of the number of times the event occurs to the number of times the event *does not* occur. In *sample space* terms it means:

$$\text{Odds} = \frac{\text{number of matching events in sample place}}{\text{number of non-matching events in sample place}}$$

- whereas we would describe **probability** as

$$\text{Probability} = \frac{\text{number of matching events in sample place}}{\text{number of total events in sample place}}$$

Guided Practice

Determine the probability of scoring 5 as the combined score of a roll of 2 dice.

Solution

We have just seen (in example 2) that there are 4 ways we can score 5 if we roll 2 dice: {(1 & 4), (2 & 3), (3 & 2), (4 & 1)}. The sample space consists of $6 \times 6 = 36$ elements, so the probability of rolling a 5 with 2 dice is $P(5) = \frac{4}{36} = \frac{1}{9}$

Practice

For 1-4, find the number of outcomes in the sample space of:

1. Tossing 3 coins simultaneously.
2. Rolling 3 dice and summing the score.
3. Rolling 3 dice and interpreting the result as a 3 digit number.
4. Pulling a card from a standard 52-card deck.

For 5-8, find the theoretical **probability:**

5. Tossing 3 coins simultaneously and getting 2 or more heads.
6. Rolling 3 dice, summing the score and getting 17.
7. Rolling 3 dice and interpreting the result as a 3 digit number, and getting 333.
8. Pulling a club from a standard 52-card deck.

For 9-12, find the **odds:**

9. Tossing 3 coins simultaneously and getting 2 or more heads.
10. Rolling 3 dice, summing the score and getting 12.
11. Rolling 3 dice and interpreting the result as a 3 digit number, and **not** getting 333.
12. Pulling a club from a standard 52-card deck.

13.2 Empirical Probability

Here you'll learn how to perform a probability simulation to get an estimate of an outcome's true probability. You'll also find the experimental probability of events.

What if your friend were performing a magic trick with a deck of cards? Every time he randomly pulls out a card it is the queen of hearts so you suspect he is cheating. How could you perform an experiment to determine if the queen of hearts is more likely to show up than others? After completing this Concept, you'll be able to perform a probability simulation and find the experimental probability of an event like this one.

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CK-12 Foundation: Experimental Probability

Guidance

A **probability simulation** is an experiment designed to determine a probability from many trials. By looking at the number of favorable outcomes and dividing by the number of trials, we can get an estimate of the true probability. The more trials we can do, the better our estimate of the true probability will be, but given that we can't do infinitely many trials, the result we get will *always* be just an estimate of the true probability. Often we perform probability simulations because we can't determine theoretical probability from looking at the sample space.

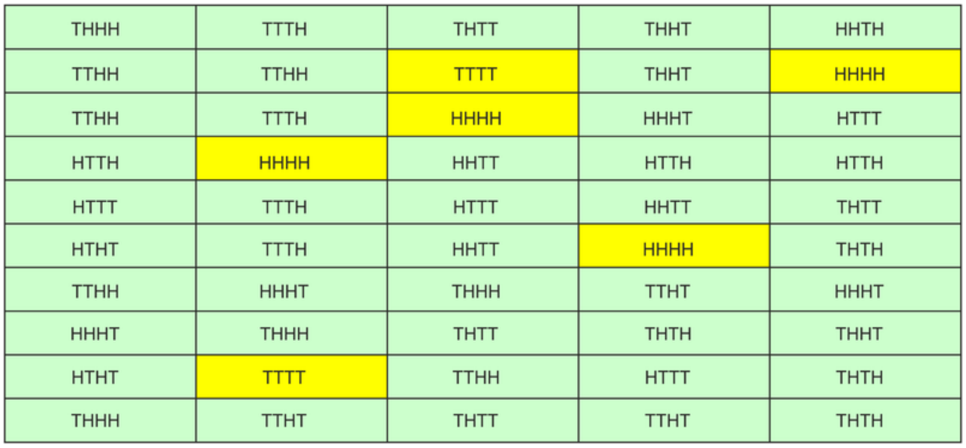
Example A

A cable TV company sends out a repair technician to replace faulty receiver boxes. The company has boxes made by ***Panasonic*** *and* ***Scientific Atlanta****, both in equal quantities. The technician carries 3 of each type of box in his van, and always replaces a box with one of the same brand. If the technician visits 4 homes before returning back to the depot, determine the probability that he will not have enough of one box type to make all the needed replacements.*

Solution

This is a situation that we can model far more easily than we could conduct a real-life experiment. Since there are equal numbers of both boxes, we need to set up a model with a probability of $\frac{1}{2}$ for each element, like a coin toss. Visiting four houses where each house has an equal chance of needing one type of box or the other is like flipping a coin four times. Getting four heads or four tails is like needing four of one type of box, which is the only situation where the technician would not have enough of one type.

So let's suppose we flip four coins 50 times and record the results, and they look like this:



THHH	TTTH	THTT	THHT	HHTH
TTHH	TTHH	TTTT	THHT	HHHH
TTHH	TTTH	HHHH	HHHT	HTTT
HTTH	HHHH	HHTT	HTTH	HTTH
HTTT	TTTH	HTTT	HHTT	THTT
HTHT	TTTH	HHTT	HHHH	THTH
TTHH	HHHT	THHH	TTHT	HHHT
HHHT	THHH	THTT	THTH	THHT
HTHT	TTTT	TTHH	HTTT	THTH
THHH	TTHT	THTT	TTHT	THTH

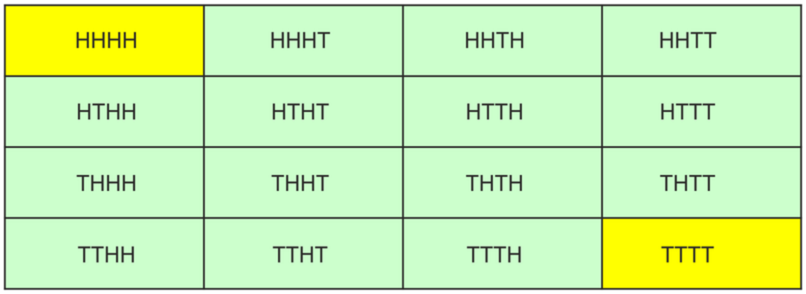
Out of 50 trials, it's as if the technician required four of the same type of box 6 times. So we can say that the probability that the technician will run out of one box type is approximately $\frac{3}{25}$ or **0.12**.

Example B

The problem in Example A can also be solved by finding the theoretical probability. Let's see how to do that:

Solution:

Notice that instead of actually flipping the coins a lot of times, we could have used our earlier knowledge of theoretical probability and the sample space for four coin flips:



HHHH	HHHT	HHTH	HHTT
HTHH	HTHT	HTTH	HTTT
THHH	THHT	THTH	THTT
TTHH	TTHT	TTTH	TTTT

Here, we can see that we would expect the technician to run out of one type of box about 2 times out of 16, so the probability is about $\frac{1}{8}$ or **0.125**. But if we hadn't known for sure what the odds of getting heads or tails on each flip were, we wouldn't have been able to calculate the odds of getting four heads or tails this way, so we would have had to find out by experiment instead.

We also can check probability calculations like this against actual experimental data to see for ourselves whether something happens as often as we'd expect it to. If it doesn't, something might be going on that we need to investigate.

Find Experimental Probability of an Event

In the previous examples, we saw how theoretical probability can be approximated by doing an experiment. We used a results table to approximate the probability of a certain **event** occurring (the technician running out of either type of box). We can approximate the probability of an event by using:

$$P(E) \approx \frac{\text{number of matching events}}{\text{total number of trials}}$$

Randomness in the results will mean that we always get an approximation of the true probability, but the more trials we do, the more accurately our **experimental probability** will match the theoretical probability.

Example C

Nadia and Peter are playing dice, but Peter keeps winning and Nadia suspects him of cheating. She is suspicious about the number of times Peter rolls a six, and so she conducts the following experiment: She rolls the suspect die 100 times, writing down the result each time. The results are:

$$\begin{gathered}4,1,4,5,3,6,2,5,1,6,2,6,4,5,1,6,4,3,6,3,2,1,1,3,4,\\5,5,2,3,1,1,2,3,1,2,2,1,6,6,3,4,6,3,6,6,2,2,3,4,6,\\1,6,6,2,6,4,3,3,2,5,3,3,2,6,6,6,6,6,1,4,1,2,6,6,6,\\3,6,4,5,6,3,5,4,6,6,4,6,6,6,6,6,2,6,6,1,1,5,1,4,6.\end{gathered}$$

Organize the data in a table and determine if 6 is more likely to come up then the other numbers.

Solution

Here's what we get if we tally up all the results in a table:

TABLE 13.1:

Number	Tally	Total	P(*number*)
1	𝍸 𝍸 𝍸	15	$P(1) = \frac{15}{100} \approx 0.15$
2	𝍸 𝍸 ||||	14	$P(2) = \frac{14}{100} \approx 0.14$
3	𝍸 𝍸 𝍸	15	$P(3) = \frac{15}{100} \approx 0.15$
4	𝍸 𝍸 |||	13	$P(4) = \frac{13}{100} \approx 0.13$
5	𝍸 ||||	9	$P(5) = \frac{9}{100} \approx 0.09$
6	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 ||||	34	$P(6) = \frac{34}{100} \approx 0.34$

It's clear looking at the table that something strange is going on with the die in question – 6 occurs approximately twice as often as the other numbers, so we could *reasonably* assume that the die is weighted unfairly. However, we still can't be 100% certain that the results we are seeing are not just due to chance. We must therefore talk only in terms of **likelihood**, and not **certainty**.

Watch this video for help with the Examples above.

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CK-12 Foundation: Experimental Probability

Vocabulary

- A **probability simulation** is an experiment designed to determine a probability from many trials. By looking at the number of favorable outcomes and dividing by the number of trials, we can get an estimate of the true probability.

- We can calculate the **experimental probability** of an event using the formula:

$$P(E) \approx \frac{\text{number of matching events}}{\text{total number of trials}}$$

Guided Practice

Juan suspects that his lucky coin is actually weighted so that he gets heads more often than tails in a coin toss. The reason he is suspicious is because he seems to get several heads in a row when he tosses it. He conducts a probability simulation and gets the following results:

$HTTHHHHTHHTTTHHHHHH$

$HHTTHTHHTHTTTHHTHTTHT$

What is the experimental probability of getting heads with Juan's coin in this case?

Solution:

First, find the number of total tosses, the number of heads, and the number of tails:

40 tosses

23 heads

17 tails

This means that the experimental probability of getting heads with Juan's coin is:

$$P(E) \approx \frac{\text{number of matching events}}{\text{total number of trials}} = \frac{26}{40} = 0.65$$

For the coin to be fair, we would expect heads around 50% of the time. 65% is higher than 50%, but this difference may be due to random chance. Further experimentation would help to get a more accurate answer.

Practice

1. Peter and Andrew each visit the hardware store in the high street every week. The store is open 6 days a week (it is closed on Sundays) and Peter and Andrew visit the store on random days when it is open.

   a. Use a pair of dice to simulate what day Andrew and Peter each visit the store, and determine experimentally the probability that they both visit the store on the same day.
   b. What would you expect the theoretical probability to be?

2. Find experimentally both the probability and odds for the next car passing a stoplight being red if the previous 25 car colors were: red, blue, white, blue, silver, red, black, black, white, red, green, red, black, blue, white, red, silver, white, red, black, white, blue, silver, red, black.

For 3-13, determine whether you could calculate the theoretical probability of the given event based on your knowledge of the possible outcomes, or whether you would have to do a test (or get more real-world information some other way) to find the experimental probability:

3. Flipping a coin three times in a row and getting three heads.
4. Pulling a nickel from your pocket when you know you have three nickels and five dimes in your pocket.
5. Pulling a nickel from your pocket when you know you have ten coins in your pocket but can't remember what they are.
6. Guessing the right answer on a multiple-choice question.
7. Guessing the right answer on a free-response question.
8. Getting a perfect score on a twenty-question multiple-choice test.
9. Getting a perfect score on a test that has ten multiple-choice questions and ten free-response questions.
10. Guessing a randomly chosen high school student's age correctly.
11. Sharing a birthday with one of your three best friends.
12. Getting a flat tire while driving home.
13. Having your left front tire be the one that goes flat, whenever you *do* get your next flat tire.

13.3 Permutations

Here you'll learn a method of calculating the number of ways in which objects can be arranged, called a permutation. You'll also use factorial notation in a formula to find the number of permutations.

What if you were picked to judge the pie contest at your local fair? You can pick three to move on to the finals from 10 entries. How many ways could you pick your three favorites? After completing this Concept, you'll be able to calculate permutations like this one using factorials.

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CK-12 Foundation: Permutations

Guidance

In this lesson we'll be looking at ways of arranging things. To illustrate what we mean by this, let's look at a simple example. Think about choosing your favorite color from the following list of choices; *red, blue, green, yellow, pink, purple, orange, brown, black*. Clearly there are nine different colors, so there are nine possible choices you could make.

Now think about choosing your **top three** colors in order of preference. There are many different ways you can choose the top three. You might choose red as your favorite, followed by black, then green. Someone else might choose the same three colors as you, but in a different order! When you are choosing items from a list and the **order in which you choose them is important**, the arrangement is called a ***permutation***. How many different permutations do you think there are in this situation?

In this lesson, we'll use counting methods to determine how many permutations a given situation has. We'll also discover a formula to calculate permutations when counting alone is impractical.

Counting Permutations

In simple cases, sometimes it's easiest to calculate permutations by just listing all the possibilities and counting them. Let's examine a situation where it is relatively straightforward to do that.

Example A

Nadia and Peter are going to watch two movies on a rainy Saturday. Nadia will choose the first movie, and Peter gets to choose the second. The four movies they have to choose from are The Lion King, Aladdin, Toy Story and Pinocchio. Given that Peter will choose a different movie than Nadia, how many permutations are there for the movies they watch?

Solution

Since the order in which they watch the movies is important, and they don't plan to choose the same movie twice, we can list all the different possibilities in a table:

TABLE 13.2:

First Movie	**Second Movie**
Lion King	*Aladdin*
Lion King	*Toy Story*
Lion King	*Pinocchio*
Aladdin	*Lion King*
Aladdin	*Toy Story*
Aladdin	*Pinocchio*
Toy Story	*Lion King*
Toy Story	*Aladdin*
Toy Story	*Pinocchio*
Pinocchio	*Lion King*
Pinocchio	*Aladdin*
Pinocchio	*Toy Story*

You can see that this table contains all the possibilities for the situation. There are **four** movies for Nadia to choose from. For every movie that Nadia chooses first, Peter has **three** choices left for his movie. By simply counting the rows in the table you can see that there are **12 permutations** in this situation.

Example B

I have 5 cards with the numbers 1 through 5 on them. I take three cards and arrange them to form a 3-digit number. How many 3-digit numbers can I make?

Since the numbers we can make fit a numerical ordering pattern, we can list the possibilities in increasing order:

$$\begin{array}{l} 123\ 124\ 125\ 132\ 134\ 135\ 142\ 143\ 145\ 152\ 153\ 154 \\ 213\ 214\ 215\ 231\ 234\ 235\ 241\ 243\ 245\ 251\ 253\ 254 \\ 312\ 314\ 315\ 321\ 324\ 325\ 341\ 342\ 345\ 351\ 352\ 354 \\ 412\ 413\ 415\ 421\ 423\ 425\ 431\ 432\ 435\ 451\ 452\ 453 \\ 512\ 513\ 514\ 521\ 523\ 524\ 531\ 532\ 534\ 541\ 542\ 543 \end{array}$$

By arranging the table this way, you can see how the number of remaining choices decreases as we choose numbers. There are **five** choices for the first number, **four** choices for the second number and **three** choices for the third number. Counting the table entries gives a total of **60 permutations**.

If we look closely at the last two examples we can see a pattern start to appear. Mathematicians love patterns—they tend to lead to formulas, which make life much easier! After all, why spend hours counting possibilities when a formula can calculate them in seconds?

In example A, Nadia had four choices and Peter had three. The number of permutations was $4 \times 3 = 12$.

In example B there were 5 choices for the first digit, followed by 4 for the second digit, and then 3 for the third digit. The total number of permutations was $5 \times 4 \times 3 = 60$.

In the introduction, we thought about picking our top three choices from a list of nine colors. You should now be able to do that. Even without listing all the possibilities, you can see that you have 9 choices for your favorite, 8 choices for your second favorite, and 7 choices for your third. The number of permutations is thus $9 \times 8 \times 7 = 504$.

Factorial Notation

Look again at the color list in the introduction, and think this time about writing down **every color** in order of preference. You would have 9 choices for your favorite, followed by 8 choices for your second favorite, then 7, then 6, then 5, and so on. To determine the number of permutations for any possible list, we would perform the following calculation:

Color Permutations $= 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

This sort of pattern crops up a great deal in statistics, probability and number theory. It's so common that it has its own notation: $4 \times 3 \times 2 \times 1$ is written as **4!** and is called **four factorial**. So the number of color permutations above is nine factorial = 9! = 362,880.

So what happens when we only want the first few terms in a factorial? For example, the number of permutations for arranging ALL the colors is 362,880, but the number of permutation for the ***top three*** is 504.

One way to get this result is to divide one factorial by another. Look at nine factorial divided by six factorial:

$$\frac{9!}{6!} = \frac{9 \times 8 \times 7 \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = 9 \times 8 \times 7 = 504$$

The terms in six factorial cancel out all but the first three terms in nine factorial. You should see that if we wanted the first **four** terms we would divide by **5!**, or for only the first **two** terms we would divide by **7!**. In general, however many terms we want to keep, we divide by the factorial of the quantity:

$$(number\ of\ items\ in\ list) - (number\ of\ items\ we\ are\ choosing)$$

So to get the first **five terms** in **twelve factorial** we would use the formula $\frac{12!}{(12-5)!} = \frac{12!}{7!}$.

Formulas like this are useful if you have a calculator that can handle factorials: you can just type in $\frac{12!}{7!}$ instead of $12 \times 11 \times 10 \times 9 \times 8$. However, some factorials are too big for some calculators to handle, so in those cases you would need to simplify the fraction and do the multiplication by hand.

Example C

How many ways can Dale choose his favorite 5 songs from the current Billboard Hot 100^{TM}*?*

Solution

To find the answer, consider how many choices he has at each stage. For his first choice he has 100 songs to choose from, then 99, then 98 and so on. We need the first 5 terms only, so our calculation is:

$$\begin{aligned}\text{Permutations} = \frac{100!}{(100-5)!} = \frac{100!}{95!} &= 100 \times 99 \times 98 \times 97 \times 96 \\ &= 9,034,502,400\end{aligned}$$

Notice that that's a pretty big number – far too large to count in a table! This is why we need formulas to help us count permutations.

Finding Permutations Using a Formula

We've just seen that a formula for determining the number of permutations for choosing 3 objects from a list of 9 objects is:

$$\frac{9!}{6!} = \frac{9 \times 8 \times 7 \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = 9 \times 8 \times 7 = 504 \text{ permutations}$$

Now we're ready to come up with a general formula for determining permutations. When we are choosing r ordered items from a group of n items, the number of permutations is given by the first r terms in $n!$ We use the notation ${}_nP_r$ for this, and the general form for calculating permutations is:

$${}_nP_r = \frac{n!}{(n-r)!}$$

Example D

How many ways are there to choose a 5-song mix from a CD containing 12 tracks?

Solution

Choosing 5 from 12: ${}_{12}P_5 = \frac{12!}{(12-5)!} = \frac{12!}{7!} = 12 \times 11 \times 10 \times 9 \times 8 = 95,040 \ ways$

Watch this video for help with the Examples above.

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CK-12 Foundation: Permutations

Vocabulary

- A **permutation** is when we are choosing r ordered items from a group of n items, where the order chosen matters. The number of permutations is given by the formula:

$${}_nP_r = \frac{n!}{(n-r)!}$$

Guided Practice

How many 3-letter "words" can be made from the letters in "computer"? (The words do NOT need to be real, or even pronounceable – for example "rtp" ***would*** *count as a word)*

Solution

Choosing 3 from 8: ${}_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 8 \times 7 \times 6 = 336 \ words$

Practice

1. In how many ways can the letters a, b, c, d, e be arranged?

2. In how many ways can the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 be arranged?
3. From a collection of 12 books, 5 are to be selected and placed in a particular order on a shelf. How many arrangements are possible?
4. 3 cards are taken at random from a deck of 52 cards and laid in a row. How many possible outcomes are there for the card arrangements?
5. How many distinct 3-letter permutations can you make from the letters in the word HEXAGON?
6. How many distinct 2-letter permutations can you make from the letters in the word GEESE?
7. A jukebox has 50 songs on it. If $1.00 pays for three songs, how many permutations are there for choosing 3 different songs?

For problems 8-16, evaluate the following:

8. ${}_3P_1$
9. ${}_7P_1$
10. ${}_6P_2$
11. ${}_8P_8$
12. ${}_9P_3$
13. ${}_7P_3$
14. ${}_{19}P_7$
15. ${}_{99}P_3$
16. ${}_3P_0$

13.4 Probability and Permutations

Here you'll learn how to find the probability of events that involve permutations.

What if you were asked to pick the final three winners in a talent contest? In how many ways could you pick a first, second, and third place winner from 20 contestants? After completing this Concept, you'll be able to calculate probabilities of events like this one using permutations.

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CK-12 Foundation: Probability and Permutations

Guidance

When we talk about probability, we are talking about chance. We may use words like 'possible' to indicate something has a low to medium chance of happening, or 'probable' to indicate something has a high chance of happening.

When we perform calculations in probability we are generally looking at a situation to find the number of **favorable outcomes** in relation to the total number of all possible outcomes. In mathematics, a **favorable outcome** simply means the outcome we are looking to solve for. Over the course of 100 days a child may get ice-cream on 25 of those days. A mathematician would look at that data and conclude that the fraction of days when the child gets ice-cream is one-fourth. He may also conclude that on any day the child has a "one in four chance" of receiving ice-cream. This is one way to think about probability: the number of times something favorable happens divided by the total number of outcomes. We have no way of knowing for sure if the child is going to get ice-cream today, but we can estimate the chance that he will by looking at the number of times he did get ice-cream and the number of times he did not.

Example A

In a previous Concept, we saw an example where Nadia and Peter are going to watch two movies on a rainy Saturday. Nadia will choose the first movie, and Peter gets to choose the second. The four movies they have to choose from are The Lion King, Aladdin, Toy Story and Pinocchio. Given that Peter will choose a different movie than Nadia, what is the probability that Aladdin is the second movie they will watch?

Solution

Previously, we found that there are a total of 12 possible outcomes. Out of those 12, Aladdin was the second movie 3 times. So the probability is as follows:

$P(\text{Aladdin being second}) = \frac{3}{12} = \frac{1}{4} =$ **one in four** or 25%.

Example B

The card game "21 hearts" consists of dealing two cards to a player and counting the points as follows: face cards (King, Queen, Jack) are worth 10 points, and number cards are worth their face value ***except for aces****, which are worth 11. The maximum score is 21. If the game is played only with the 13 cards in the hearts suit, what is the probability that a player will score 21?*

Solution

To find the answer, we need to know two pieces of information: 1) the total number of permutations it's possible to get in the game and 2) the number of permutations that will score 21.

To find the total number of permutations for the game, use the formula ${}_nP_r$ with $n = 13$ and $r = 2$:

$$_{13}P_2 = \frac{13!}{(13-2)!} = \frac{13!}{11!} = 13 \times 12 = 156 \text{ permutations}$$

Now we need to determine how many of those permutations score 21. We can do that by simply listing all ways to score 21. Remember, of course, that as we are talking about permutations, the combinations (ace, king) and (king, ace) each count as separate hands! The winning permutations are:

$$\begin{array}{cccc} (\text{ace, king}) & (\text{king, ace}) & (\text{ace, queen}) & (\text{queen, ace}) \\ (\text{ace, jack}) & (\text{jack, ace}) & (\text{ace}, 10) & (10, \text{ace}) \end{array}$$

There are 8 winning hands, so the probability of scoring 21 is given by:

$P(21) = \frac{8}{132} = \frac{2}{33}$ =**two in thirty-three** or approximately 6%.

Sometimes when looking at probability (especially in games of chance), there are too many wining permutations to calculate directly. In such circumstances it can be useful to remember that a player must either win or lose – it's impossible to do both! In these circumstances remember that:

$$P(\text{winning}) + P(\text{losing}) = 1$$

Since it's certain that you'll either win or lose, the probability that you'll win and the probability that you'll lose add up to 1.

So to calculate the probability of losing we can use:

$$P(\text{winning}) = 1 - P(\text{losing})$$

Example C

A funfair game consists of throwing three darts at a board with 16 numbered squares in a 4×4 grid. The squares are numbered 1 through 16 and no number is repeated or omitted. In order to win a player needs to score 9 or more. If a dart hits a square that has already been taken or if a dart misses the board the player must throw the dart again. What are the chances of winning the game?

There are too many winning permutations to list easily, but there are only a few **losing permutations**. If you need a score of 9 or more to win, then you'll lose with a score of 8 or less. There are only four combinations of numbers that add up to 8 or less (remember that you can't repeat a number), and they can occur in any of the following orders:

$\mathbf{1,2,3(=6)}$	$1,3,2$	$2,1,3$	$2,3,1$	$3,1,2$	$3,2,1$
$\mathbf{1,2,4(=7)}$	$1,4,2$	$2,1,4$	$2,4,1$	$4,1,2$	$4,2,1$
$\mathbf{1,2,5(=8)}$	$1,5,2$	$2,1,5$	$2,5,1$	$5,1,2$	$5,2,1$
$\mathbf{1,3,4(=8)}$	$1,4,3$	$3,1,4$	$3,4,1$	$4,1,3$	$4,3,1$

So there are 24 permutations totaling less than 9.

The *total* number of ways to choose 3 numbers out of 16 is

$$_{16}P_3 = \frac{16!}{16-3!} = \frac{16!}{13!} = 16 \times 15 \times 14 = 3360 \text{ permutations}$$

So the probability of losing the game is $\frac{24}{3360}$, or $\frac{1}{140}$. That means the probability of winning is:

$P(\text{winning}) = 1 - P(\text{losing}) = 1 - \frac{24}{3360} = 1 - \frac{1}{140} = \frac{139}{140}$ or approximately **99.3%**.

Vocabulary

- The **probability** of a set of events is found by

$$\text{Probability} = \frac{\text{number of matching events in sample place}}{\text{number of total events in sample place}}$$

- A **permutation** is when we are choosing r ordered items from a group of n items, where the order chosen matters. The number of permutations is given by the formula:

$$_nP_r = \frac{n!}{(n-r)!}$$

Guided Practice

What is the probability that a randomly generated 3-letter arrangement of the letters M, M, and O will result in spelling the word MOM?

Solution

To find the probability, we first must describe the sample space. It might be tempting to think that the only possibilities are MMO, MOM and OMM. However, we must treat each M as an distinct letter. Let's think of them as M_1 and M_2. Since we have 3 distinct objects and we want to order all three of them, this is:

$_3P_3 = 3! = 3 \cdot 2 \cdot 1 = 6$

Thus, there are 6 ways to order the 3 letters. Let's see what that looks like:

$$\begin{aligned}
&M_1M_2O\\
MOM = {}&M_1OM_2\\
&M_2M_1O\\
MOM = {}&M_2OM_1\\
&OM_1M_2\\
&OM_2M_1
\end{aligned}$$

2 of these 6 orderings spell MOM, which means the probability of spelling MOM is:

$P(MOM) = \frac{2}{6} \approx 0.67.$

Practice

For 1-5, find the number of permutations.

1. ${}_5P_2$
2. ${}_9P_4$
3. ${}_{11}P_5$
4. How many ways can you plant a rose bush, a lavender bush and a hydrangea bush in a row?
5. How many ways can you pick a president, a vice president, a secretary and a treasurer out of 28 people for student council?

For 6-10, find the probabilities.

6. What is the probability that a randomly generated 3-letter arrangement of the letters A,E,L, Q and U will result in spelling the word EQUAL?
7. What is the probability that a randomly generated 3-letter arrangement of the letters in the word SPIN ends with the letter N?
8. A bag contains eight chips numbered 1 through 8. Two chips are drawn randomly from the bag and laid down in the order they were drawn. What is the probability that the 2-digit number formed is divisible by 3?
9. A prepaid telephone calling card comes with a randomly selected 4-digit PIN, using the digits 1 through 9 without repeating any digits. What is the probability that the PIN for a card chosen at random does not contain the number 7?
10. Janine makes a playlist of 8 songs and has her computer randomly shuffle them. If one song is by Little Bow Wow, what is the probability that this song will play first?

13.5 Combinations

Here you'll learn another method of calculating the number of ways in which objects can be arranged, one in which the order of the objects does not matter, called a combination. You'll also use factorial notation in a formula to find the number of combinations.

What if you were ordering an ice cream sundae? You can pick three ice cream flavors from 12 total flavors to make it. How many combinations of ice cream do you have to choose from? After completing this Concept, you'll be able to calculate combinations like this one using a formula.

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CK-12 Foundation: Combinations

Guidance

In the previous lesson we looked at situations where the order of an arrangement is important. For example, when looking at 3-digit numbers, the number 123 is very different from the number 312, even though it contains the same 3 digits. But in some situations, the order is not important – for example, when looking at cards in a hand of poker, or choosing toppings to put on a pizza. In these situations, when **order is not important**, we are looking at ***combinations*** of items. For example, if you are a poker player you might want to know the probability of being dealt four aces in a hand of five cards. You don't care in which order you receive the ace cards – only the fact that you have four of them is important.

Counting Combinations

Just as with permutations, it's sometimes easiest to calculate combinations by listing all the possibilities available, and counting them. The only difference is, when we list one combination, we automatically exclude a larger number of permutations. For example a poker hand that is (ace, ace, ace, ace, king♣) is identical to (ace, ace, ace, king♣, ace), (ace, ace, king♣, ace, ace), (ace, king♣, ace, ace, ace) and (king♣, ace, ace, ace, ace). So we must be careful to use a listing method that includes all combinations without repeating ones that are really the same. Let's examine a situation where it's relatively straightforward to do that.

Example A

Anne wishes to knit herself a striped sweater. She has 4 colors of yarn available; red, blue, green and yellow. How many different combinations of two colors does she have to choose from?

Solution

When we just choose color pairs, there will be fewer combinations than we would have if we were counting permutations as in the previous lesson. For example ***red and blue*** is equivalent to ***blue and red***, and we should only count one as a unique pairing. We start by listing the color pairs but we will also write down equivalent pairings at the same time. This will help prevent us from repeating combinations:

TABLE 13.3:

Pairing	**Equivalent pairings (do not count)**
Red & blue	blue & red
Red & green	green & red
Red & yellow	yellow & red
Green & blue	blue & green
Green & yellow	yellow & green
Yellow & blue	blue & yellow

So there are 6 distinct combinations. There are also 6 "repeat" pairings – for every pair of colors we choose there is 1 combination but 2 permutations. **Anne can choose from six distinct color pairs for her sweater**.

Example B

Triominoes Pizza Company specializes in 3-topping pizzas. If the available toppings are cheese, pepperoni, mushroom, pineapple and olives, how many different 3-topping combinations can customers choose from?

Solution

We'll start by making a table and list first choice, second choice and third choice:

TABLE 13.4:

1^{st} **topping**	2^{nd} **topping**	3^{rd} **topping**
cheese	pepperoni	mushroom
cheese	pepperoni	pineapple
cheese	pepperoni	olives
cheese	mushroom	pineapple
cheese	mushroom	olives
cheese	pineapple	olives
pepperoni	mushroom	pineapple
pepperoni	mushroom	olives
pepperoni	pineapple	olives
mushroom	pineapple	olives

Note that as we progress through the choices for first topping, the number of combinations we have for the second and third toppings get fewer. This is because some combinations have already been used, in a different order.

By counting the table entries we see that there are **10 possibilities** for a 3-topping pizza.

Determining Combinations by Looking at Permutations

You can see that there are always fewer combinations than permutations in a given situation – but you should also see that knowing which combinations have already been used is important to avoid counting combinations twice. One combination can give rise to several permutations of the same objects.

Another way to calculate combinations is this: If we know how many **permutations** there are in a system and how many permutations there are for **each combination**, then we can **divide the number of permutations by the number of permutations for each combination to get the number of combinations**.

To illustrate this, look again at the *Triominoes Pizza* menu. If we were to look at **permutations** of toppings, we could quickly calculate that there are $5 \times 4 \times 3 = 60$ permutations. But for every 3-topping choice there are several permutations which are all equivalent. For example, the following all give identical pizzas:

Cheese, pepperoni & mushroom	Cheese, mushroom & pepperoni
Pepperoni, cheese & mushroom	Pepperoni, mushroom & cheese
Mushroom, cheese & pepperoni	Mushroom, pepperoni & cheese

So for every different combination there are 6 distinct permutations. Since we have a formula for counting permutations, we can use it to find out how many total permutations there are and simply divide that number by 6:

$$\text{Combinations} = \frac{1}{6} \cdot {}_5P_3 = \frac{1}{6} \cdot \frac{5!}{2!} = \frac{1}{6} \cdot \frac{5 \times 4 \times 3 \times \cancel{2} \times \cancel{1}}{\cancel{2} \times \cancel{1}} = \frac{60}{6} = 10$$

Find Combinations Using a Formula

If you look back at examples 1 and 2 you can see that the number of permutations is a simple multiple of the number of combinations. In example 1 there are two times as many permutations as combinations. In example 2 there are six times as many permutations as combinations. One question you might be asking is: "How do I know how the number of combinations is related to the number of permutations?"

A more important question might be "where did the numbers two and six come from?" If you think carefully you should realize that any time you've chosen 2 objects, there are only 2 ways of ordering them, while if you've chosen 3 objects there are 3! ways of ordering them (6 ways). Similarly, if you choose 7 objects, there will be 7! ways of arranging them. We can make use of this fact when calculating combinations. The number of combinations is the number of permutations divided by the number of ways of arranging the items you choose. If you choose r objects from a collection of n objects there are $r!$ ways of arranging what you choose. In equation form, the number of combinations is therefore:

$${}_nC_r = \frac{n!}{(n-r)!\,r!}$$

In other words, the number of combinations is equal to the number of permutations divided by $r!$ (because $r!$ is the number of permutations for each combination). We can use this new formula to quickly calculate combinations without listing them all.

Example C

Andrew is packing for a vacation. He owns twelve shirts and wants to pack five of them. How many combinations of shirts does he have to choose from?

Solution

Since the order he packs his shirts in is not important, we are looking at a combination. He is choosing five shirts from a total of twelve:

Choosing 5 from 12: ${}_{12}C_5 = \frac{12!}{(12-5)!5!} = \frac{12!}{7!5!} = \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1} = 792$ combinations.

Watch this video for help with the Examples above.

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CK-12 Foundation: Combinations

Vocabulary

- A **combination** is when we are choosing r ordered items from a group of n items, where the order chosen does not matter. The number of combinations is given by the formula:

$$_nC_r = \frac{n!}{(n-r)!r!}$$

Guided Practice

At Summerfield High School, the student council has 8 students, 3 of whom are needed to be on the prom committee. How many ways can the prom committee be chosen?

Solution:

There are:

$$\begin{aligned} _nC_r &= \frac{n!}{(n-r)!r!} \\ _8C_3 &= \frac{8!}{(8-3)!3!} \\ _8C_3 &= \frac{8!}{5!3!} \\ _8C_3 &= \frac{8 \cdot 7 \cdot 6}{3!} \\ _8C_3 &= 8 \cdot 7 = 56 \end{aligned}$$

56 ways to chosen the prom committee.

Practice

For 1-4, how many combinations are possible in the following situations?

1. Buying a hot-dog with **two** of the following toppings: ketchup, mustard, chili, cheese, pickles.
2. Choosing 5 CDs from a selection of 8.
3. Selecting 3 games from a box containing Scrabble, Twister, Connect-4, Snap Mousetrap.
4. What do you notice about the answers to parts a and c? How might this be explained by the formula for finding numbers of combinations?

For 5-14, evaluate the following:

5. ${}_{3}C_{1}$
6. ${}_{7}C_{1}$
7. ${}_{6}C_{2}$
8. ${}_{8}C_{8}$
9. ${}_{9}C_{3}$
10. ${}_{9}C_{6}$
11. ${}_{7}C_{3}$
12. ${}_{17}C_{4}$
13. ${}_{30}C_{11}$
14. ${}_{10}C_{0}$

13.6 Probability and Combinations

Here you'll learn how to find the probability of events that involve combinations.

What if you had a bag that contained 5 red marbles, 4 blue marbles, and one white marble? You randomly pick three marbles from the bag. What is the probability that one of those marbles is the white one? After completing this Concept, you'll be able to calculate the probability of events like this one involving combinations.

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CK-12 Foundation: Probability and Combinations

Guidance

Just as with permutations, combinations crop up frequently in probability. In many card games the objective is to acquire a winning hand. In order to do that, it's useful for players to know how likely a given hand is to arise, and also the probability that another player has a better hand. Mathematicians have been studying such games of chance for centuries.

Example A

A word game requires players to select 4 tiles from a bag containing 26 tiles, each with one of the letters A through Z written on it. If each letter appears once and only once, what is the probability that a player will be able to spell CATS with his tiles?

Solution

Since a player needs a C, an A, a T and an S in **any order**, we are looking at a **combination** calculation. We first need to determine how many combinations there are, choosing 5 letters from a total of 26:

Choosing 4 from 26:

$$\begin{aligned} {}_{26}C_4 &= \frac{26!}{(26-4)!4!} = \frac{26!}{22!4!} \\ &= \frac{26 \times 25 \times 24 \times 23}{4 \times 3 \times 2 \times 1} \\ &= 59,800 \text{ combinations.} \end{aligned}$$

Since only one combination allows a player to spell CATS, the probability of getting that combination is $\frac{1}{59,800}$.

Example B

A funfair game consists of pulling numbered chips from a bag. The game starts with nine chips numbered 1 through 9, and players are allowed to pull out three chips. A player wins by drawing the number 7 chip. What is the probability that a player will win?

Solution

To find the probability for winning this game we need to know two pieces of information: 1) the total number of combinations for the game and 2) the number of combinations that contain a 7.

To find the total number of combinations for the game, use the formula ${}_nC_r$ with $n = 9$ and $r = 3$:

$${}_9C_2 = \frac{9!}{(9-3)!3!} = \frac{9!}{6!3!} = \frac{9\times 8\times 7}{3\times 2\times 1} = 84 \text{ combinations}$$

Now we need to determine how many combinations contain a 7. We can figure this out by reasoning as follows: given that there MUST be a 7 in the list, the number of combinations that contain a seven is the same as the number of combinations of choosing any *two* numbers out of the eight chips that *don't* include the 7. In other words, if we imagine that we got to pick the 7 chip on purpose, how many ways would there be of picking the other two chips?

To find that number, we use the formula ${}_nC_r$ with $n = 8$ and $r = 2$:

$${}_8C_2 = \frac{8!}{(8-2)!2!} = \frac{8!}{6!2!} = \frac{8\times 7}{2\times 1} = 28 \text{ combinations}$$

So the probability is given by:

$$P(\text{getting a } 7) = \frac{84}{28} = \frac{1}{3} = \text{or one in three.}$$

Example C

Calculate the probability of being dealt four aces in a five card poker hand.

Solution

The first thing we need to know to solve this problem is the total number of unique hands. Since players can arrange their cards however they wish, the order of the cards is unimportant. So we will calculate, using the formula, the number of combinations for choosing 5 cards from a 52 card deck.

Choosing 5 from 52: ${}_{52}C_5 = \frac{52!}{(52-5)!5!} = \frac{52!}{47!5!} = \frac{52\times 51\times 50\times 49\times 48}{5\times 4\times 3\times 2\times 1} = 2,598,960$ unique hands

Next we need to calculate how many hands there are that contain four aces. This sounds difficult, but we can think about it like this:

- If a hand contains four aces, it must also contain exactly ONE other card.
- Since the four aces are accounted for, there are 48 (that's 52 – 4) cards left in the deck.

Since a unique hand is independent of the order in which the cards are dealt, there must be 48 unique hands that contain four aces (one unique hand for every non-ace card in the deck).

There are 48 possible hands that contain four aces. So the probability of being dealt four aces in poker is:

$$P(\text{four aces}) = \frac{48}{2,598,690} = \frac{1}{54,145}$$

Watch this video for help with the Examples above.

Multimedia

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CK-12 Foundation: Probability and Combinations

Vocabulary

- The **probability** of a set of events is found by

$$\text{Probability} = \frac{\text{number of matching events in sample place}}{\text{number of total events in sample place}}$$

Guided Practice

Practice

For 1-3, calculate the number of combinations:

1. ${}_8C_4$
2. ${}_{11}C_5$
3. ${}_{20}C_2$

For 4-8, a town lottery requires players to choose three different numbers from the numbers 1 through 36.

4. How many different combinations are there?
5. What is the probability that a player's numbers match all three numbers chosen by the computer?
6. What is the probability that two of a player's numbers match the numbers chosen by the computer?
7. What is the probability that one of a player's numbers matches the numbers chosen by the computer?
8. What is the probability that none of a player's numbers match the numbers chosen by the computer?

9. A bag contains 13 dominoes. Each domino has a different number of dots on it, and the numbers of dots are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. Peter selects 2 dominos at random from the bag. What is the probability that the total number of dots on the two dominoes he selects is 7?
10. Looking at the odds that you came up with in question 4, devise a sensible payout plan for the lottery—in other words, how big should the prizes be for players who match 1, 2, or all 3 numbers? Assume that tickets cost \$1. Don't forget to take into account the following:
    a. The town uses the lottery to raise money for schools and sports clubs.
    b. Selling tickets costs the town a certain amount of money.
    c. If payouts are too low, nobody will play!

13.7 Mutually Exclusive Events

Here you'll learn how to find the probability of two or more events (called mutually exclusive events) in which the probability of both of them happening together is zero. You'll also find the probability of two or more events (called overlapping events) in which the probability of both of them happening together is not zero.

What if you rolled a pair of dice? How could you find the probability that you will roll either a 1 or a 12? After completing this Concept, you'll be able to find the probability of mutually exclusive events like this one.

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CK-12 Foundation: Mutually ExclusiveEvents

Guidance

Imagine you are going to see a movie. Your friend has just bought the tickets, and you are not sure which movie you are seeing. There are 4 movies playing. Harry Potter (which you have already seen, but your friend has not) is one of them.

- What are the chances you will be seeing Harry Potter?
- What are the chances you will NOT be seeing Harry Potter?

This is an easy example of a **mutually exclusive** event: you will either see Harry Potter, or you will not. You cannot do both!

Finding the probability of mutually exclusive events is easy; what's not as easy is finding the probability of events that can overlap depend on each other. In this lesson, you'll learn how to find the probability of any two events that can be related to each other.

Find the Probability of Mutually Exclusive Events

In probability, when two events are mutually exclusive, the probability of **both** happening together is zero.

Examples of mutually exclusive events in probability include:

- Flipping a coin and:
  - getting heads
  - getting tails
- Picking a **single** card from a deck and:
  - getting an ace
  - getting a 7

  - getting a queen
  - etc...
- Picking a **single** colored marble from a bag and:
  - getting a red marble
  - getting a blue marble
  - getting a green marble
  - etc..

What this means mathematically is two-fold. If our two mutually exclusive events are A and B:

- $P(A \text{ and } B) = 0$. There is no possibility of **both** events happening.
- $P(A \text{ or } B) = P(A) + P(B)$. To find the probability of **either** event happening, sum the individual probabilities.

Example A

There are 7 marbles in a bag: 4 green, 2 blue and 1 red. Peter reaches into the bag and blindly picks a single marble. The following letters refer to these events:

- *A – the marble is red*
- *B – the marble is blue*
- *C – the marble is green*

Find the following:

a. $P(A)$

b. $P(B)$

c. $P(C)$

d. $P(B \text{ or } A)$

e. $P(C \text{ or } A)$

f. $P(C \text{ or } B \text{ or } A)$

g. $P(A \text{ and } C)$

Solution

Look at the 3 events A, B and C. They must be mutually exclusive: if we pick a single marble from the bag then it must be **either** red **or** blue **or** green. There is **no possibility** of it being both blue *and* green!

a. There are 7 marbles and only one is red, so $P(A) = \frac{1}{7}$.

b. There are 7 marbles and 2 are blue, so $P(B) = \frac{2}{7}$.

c. There are 7 marbles and 4 are green, so $P(C) = \frac{4}{7}$.

d. The events are mutually exclusive, therefore $P(B \text{ or } A) = P(B) + P(A) = \frac{2}{7} + \frac{1}{7} = \frac{3}{7}$.

e. The events are mutually exclusive, therefore $P(C \text{ or } A) = P(C) + P(A) = \frac{4}{7} + \frac{1}{7} = \frac{5}{7}$.

f. The events are mutually exclusive, therefore $P(C \text{ or } B \text{ or } A) = P(C) + P(B) + P(A) = \frac{4}{7} + \frac{2}{7} + \frac{1}{7} = \frac{7}{7} = 1$.

g. The events are mutually exclusive, so $P(A \text{ and } C) = 0$.

The last two results make sense: $P(C \text{ or } B \text{ or } A)$ means the probability of the marble being green blue or red. It **must** be one of these. And $P(A \text{ and } C)$ is the probability of the marble being both red **and** green. There are no such marbles in the bag!

Earlier we learned about permutations and combinations. We often have to use these calculations when determining the probability of mutually exclusive events.

Example B

There are 7 marbles in a bag: 4 green, 2 blue and 1 red. Peter reaches into the bag and blindly picks out 4 marbles. Find the probability that he removes at least 3 green marbles.

Solution

There are 2 distinct ways this could occur:

a) Peter picks 3 green marbles and 1 other.

b) Peter picks 4 green marbles.

These events are mutually exclusive – he cannot remove (three green and one other) and (four green) at once. If we find $P(A)$ and $P(B)$ we know that the total probability $P(A \text{ or } B) = P(A) + P(B)$.

We are choosing 4 marbles from a bag containing 7 marbles. The total number of marble combinations there are is ${}_7C_4 = \frac{7!}{4!3!} = 35$ possible combinations.

a) The number of combinations that contain 3 green marbles +1 other:

First, the 3 greens. We are choosing 3 green marbles out of a total of 4: ${}_4C_3 = \frac{4!}{3!1!} = 4$

Now to choose 1 other – we're picking 1 marble out of the 3 non-green ones: ${}_3C_1 = \frac{3!}{2!1!} = 3$

The total number of combinations of 3 greens and 1 other = ${}_4C_3 \times {}_3C_1 = 4 \times 3 = 12$

So $P(A) = \frac{12}{35}$.

b) The number of combinations that contain 4 green balls:

We are choosing 4 from 4 possible green balls: ${}_4C_4 = 1$ possible combination

So $P(B) = \frac{1}{35}$

$P(A \text{ or } B) = P(A) + P(B)$, so the probability of getting at least 3 green balls is $\frac{12}{35} + \frac{1}{35} = \frac{13}{35}$ or **slightly better than 1 in 3**.

Find the Probability of Overlapping Events

Sometimes we wish to look at overlapping events. In essence, this means two events that are NOT mutually exclusive. For instance, if you pick a card at random from a standard deck, what is the probability that you'll get a card which is either a seven or a diamond? Let's use this as our next example:

Example C

If you pick a card at random from a standard 52-card deck, what is the probability that you get a card which is either a seven or a diamond?

Solution

One thing we can say for certain is that these two events are not mutually exclusive (it is possible to get a card which both a seven and a diamond). First of all, let's look at the information we have:

- There are 52 cards – the chances of picking any **particular** one is $\frac{1}{52}$.
- There are 4 sevens (diamond, heart, club, spade) – the chances of picking a seven is $\frac{4}{52} = \frac{1}{13}$.
- There are 13 diamonds (ace through king) – the chances of picking a diamond is $\frac{13}{52} = \frac{1}{4}$.
- The chances of picking the **seven of diamonds** is $\frac{1}{52}$.

So there are 4 sevens, 13 diamonds and 1 card that is **both** (the seven of diamonds). That means we can't just add up the number of sevens and the number of diamonds, because if we did that, we'd be counting the seven of diamonds twice. Instead, we have to add up the number of sevens and the number of diamonds, and then *subtract* the number of cards that fit in both categories. That means there are $(4+13-1)=16$ cards that are seven **or** diamond. The probability of getting a seven or a diamond is therefore $\frac{16}{52}=\frac{4}{13}$.

We can check this by listing all the cards in the deck and highlighting those that are sevens or diamonds:

A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦
A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣
A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠

You can see that there are 16 cards that fit, not the 17 we'd get if we just added up the sevens and the diamonds.

Look again at the numbers: the number of cards that are seven **or** diamond is (number of sevens) plus (number of diamonds) minus (number of seven **and** diamond). In probability terms we can write this as:

$$P(\text{seven or diamonds}) = P(\text{seven}) + P(\text{diamonds}) - P(\text{seven and diamonds})$$

This leads to a general formula:

$$\text{For overlapping events}: P(A \; or \; B) = P(A) + P(B) - P(A \; and \; B)$$

Watch this video for help with the Examples above.

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CK-12 Foundation: Mutually ExclusiveEvents

Vocabulary

- If our two events A and B are **mutually exclusive**:

$P(A \; and \; B) = 0$. There is no possibility of **both** events happening.

$P(A \; or \; B) = P(A) + P(B)$. To find the probability of **either** event happening, sum the individual probabilities.

- If our two events A and B are **overlapping**:

$P(A \; and \; B) \neq 0$

$$P(A \; or \; B) = P(A) + P(B) - P(A \; and \; B)$$

Guided Practice

A cooler contains 6 cans of Sprite, 9 cans of Coke, 4 cans of Dr Pepper and 7 cans of Pepsi. If a can is selected at random, calculate the probability that it is either Pepsi or Coke.

Solution:

Selecting a Pepsi or selecting a Coke are mutually exclusive events. This means we can use the formula $P(A \text{ or } B) = P(A) + P(B)$.

Finding the probability of each event separately:

$$P(\text{Pepsi}) = \frac{\text{Cans of Pepsi}}{\text{Total cans of soda}} = \frac{7}{26} \approx 0.269$$

$$P(\text{Coke}) = \frac{\text{Cans of Coke}}{\text{Total cans of soda}} = \frac{9}{26} \approx 0.346$$

This means that the probability of selecting a Pepsi or a Coke is:

$P(\text{Pepsi or Coke}) = P(\text{Pepsi}) + P(\text{Coke}) = 0.269 + 0.346 = 0.615.$

There is a 61.5% probability of selecting a Pepsi or a Coke.

Practice

For 1-6, determine whether the following pairs of events are mutually exclusive or overlapping:

1. The next car you see being red; the next car you see being a Ford.
2. A train being on time; the train being full.
3. Flipping a coin and getting heads; flipping a coin and getting tails.
4. Selecting 3 cards and getting an ace; selecting 3 cards and getting a king.
5. Selecting 3 cards and getting 2 aces; selecting 3 cards and getting 2 kings.
6. A person's age is an even number; a person's age is a prime number.

For 7-10, a card is selected at random from a standard 52 card deck. Calculate the probability that:

7. The card is **either** a red card **or** an even number (2, 4, 6, 8 or 10).
8. The card is **both** a red card **and** an even number.
9. The card is red **or** even but **not both**.
10. The card is black **or** red but **not an ace**.

13.8 Independence versus Dependence

Here you'll learn how to find the probability of two events (called independent events) in which one event has no bearing on the probability of the second event. You'll also learn how to find the probability of two events (called dependent events) in which one event does have a bearing on the probability of the second event.

What if you were playing a game in which you picked a card from a standard deck and then rolled a die? How could you find the probability that you would pick an ace and roll a six? After completing this Concept, you'll be able to find the probabilities of independent events like these.

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CK-12 Foundation: Independent Events

Guidance

If the result of one event has no bearing on the probability of the second event, we call them **independent events**. For example, if you flip a coin 3 times and get heads 3 times, what is the probability that the next flip will result in tails? Many people think that the previous run of heads somehow influences the flip to make tails more likely, but in reality the previous flips have no bearing on the outcome of the new flip – how could they? The coin doesn't have a brain or a memory.

(The idea that tails is more likely after a run of heads is called the Gambler's Fallacy, and it probably arises from people getting confused about something called *prior probability*. Now that you've learned a little about probability, you know that getting three heads and one tail is a little more likely than getting four heads on four coin flips, so *before* you flip the coin, you'd expect that it's more likely you'll get three heads than four heads. But *after* you've already flipped the coin three times, the chances of getting heads on the first three flips don't matter any more, because you *already got* those three heads; the probability of getting those first three heads has gone from 12.5% to 100%! So the only probability that still matters is the probability of getting heads on the one flip remaining, which is just the same as it always is on a single coin flip: 50%.)

Because one flip of the coin has no effect on the outcome of any other flips, each flip of the coin counts as an **independent event**.

To find the probability of multiple independent events happening together, we multiply the individual probabilities:

$$\text{For independent events} : P(A \text{ and } B) = P(A) \cdot P(B)$$

Example A

Find the probability of rolling a 5 on a 6-sided die and getting heads if you flip a coin at the same time.

Solution

Clearly the outcome of rolling a die has no effect on flipping a coin, so the two events are **independent**. So $P(5 \text{ and heads}) = P(5) \cdot P(\text{heads}) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$.

Example B

Out of the 480 students in a school, 40 have art first period; also, 96 students have math first period. Find the probability that a student picked at random will either have math or art in first period.

Solution

A student cannot take both math and art during the same period, so the events are not overlapping. If event A is having art first period and event B is having math first period, $P(A \text{ and } B) = 0$. We want to find $P(A \text{ or } B)$.

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$P(A \text{ or } B) = \frac{40}{480} + \frac{96}{480} - 0$

$P(A \text{ or } B) = \frac{96}{480}$

$P(A \text{ or } B) = \frac{1}{5} = 20\%$

Find the Probability of Dependent Events

If the result of one event influences the probability of the second, we call them **dependent events**. For example, if you pick two cards from a deck, the chances of getting an ace on the first pick is $\frac{4}{52} = \frac{1}{13}$. If you keep that ace and draw again, the chance of getting another ace on your second pick is less: there are now only 3 aces left in the deck (of 51 cards), so the chance of getting an ace is $\frac{3}{51} = \frac{1}{17}$. To find the probability of getting two aces we multiply the two individual probabilities: $\frac{1}{13} \times \frac{1}{17} = \frac{1}{221}$.

Example C

Three cards are picked from a standard 52 card deck. The cards are not replaced. Find the probability of picking 3 queens.

Solution

There are 52 cards and 4 of them are queens, so the chance of getting a queen on the first pick is $\frac{4}{52} = \frac{1}{13}$.

Assuming you get a queen on the first pick, there are 51 cards remaining of which 3 are queens, so the chance of getting a queen on the second pick is $\frac{3}{51} = \frac{1}{17}$.

If you were successful on the second pick, there will be 50 cards remaining of which 2 are queens, so the chance of getting a queen on the third pick is $\frac{2}{50} = \frac{1}{25}$.

The probability of picking 3 queens in a row is $\frac{1}{13} \times \frac{1}{17} \times \frac{1}{25} = \frac{1}{5525}$ or **1 in 5,525**.

Watch this video for help with the Examples above.

Multimedia

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CK-12 Foundation: Independent Events

Vocabulary

- If the result of one event has no bearing on the probability of the second event, we call them **independent events**.
- If the result of one event influences the probability of the second, we call them **dependent events**.

Guided Practice

100 raffle tickets were sold and Peter bought 4 of them. There are 3 prizes, and winners are selected randomly from a hat containing all the numbers. Find the probability that Peter wins all 3 prizes.

Solution

For the first draw, Peter's numbers account for 4 tickets out of 100: $\frac{4}{100} = \frac{1}{25}$

For the second draw, Peter's remaining numbers (assuming he won the first draw) account for 3 tickets out of 99: $\frac{3}{99} = \frac{1}{33}$

For the first draw, Peter's remaining numbers (assuming he won the first two draws) account for 2 tickets out of 98: $\frac{2}{98} = \frac{1}{49}$

The probability that Peter wins all 3 prizes is $\frac{1}{25} \times \frac{1}{33} \times \frac{1}{49} = \frac{1}{40425}$ or **1 in 40, 425**.

Practice

For 1-6, determine whether the events are dependent or independent.

1. Driving at night and falling asleep at the wheel.
2. Visiting the zoo and seeing a giraffe.
3. The next 2 cars you see are both red.
4. A coin tossed twice comes up heads both times.
5. Being dealt 4 aces in a hand of poker.
6. It is your birthday and it is a windy day.

For 7-10, a bag contains 10 colored marbles – 4 red, 4 blue and 2 green. Calculate the probability of:

7. Removing 2 green marbles in a row if you replace the marble each time.
8. Removing 2 green marbles in a row if you do not replace the marble each time.
9. Removing 3 marbles without replacing and getting all blue.
10. Removing 4 marbles without replacing and getting **exactly** 3 blue.

13.9 Measures of Central Tendency and Dispersion

Here you'll learn how to find the measures of central tendency (the mean, the median, and the mode) for a set of data.

What if you polled 20 adults and asked them how much money they save for retirement each year? You record the results. How could you numerically describe the average amount your survey participants are saving annually? After completing this Concept, you'll be able to calculate and compare measures of central tendency to describe a data set like this one.

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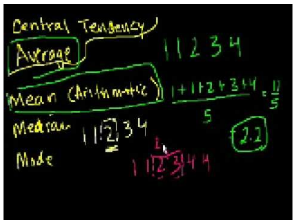
CK-12 Foundation: Measures of Central Tendency

Watch This

The following video is an introduction to the mean, median, and mode.



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Click image to the left for more content.

KhanAcademy: Statistics: The Average

The narrator models finding the mean, median, and mode of a set of numbers. While this is similar to some of the content below, you may find it to be a helpful comparison of what the three measures of central tendency show.

Guidance

The word "average" is often used to describe the general characteristics of a group of unequal objects. Mathematically, an average is a single number which can be used to summarize a collection of numerical values. In mathematics, there are several types of "averages" with the most common being the **mean**, the **median** and the **mode**.

Mean

The **arithmetic mean** of a group of numbers is found by dividing the sum of the numbers by the number of values in the group. In other words, we add all the numbers together and divide by the number of numbers.

Example A

Find the mean of the numbers 11, 16, 9, 15, 5, 18.

Solution

There are six separate numbers, so the mean $= \frac{11+16+9+15+5+18}{6} = \frac{74}{6} = 12\frac{1}{3}$.

The arithmetic mean is what most people automatically think of when the word average is used with numbers. It's generally a good way to take an average, but it can be misleading when a small number of the values lie very far away from the rest. A classic example would be when calculating average income. If one person (such as former Microsoft Corporation chairman Bill Gates) earns a great deal more than everyone else who is surveyed, then that one value can sway the mean significantly away from what the majority of people earn.

Example B

The annual incomes for 8 professions are shown below. Form the data, calculate the mean annual income of the 8 professions.

TABLE 13.5:

Profession	**Annual Income**
Farming, Fishing, and Forestry	\$19,630
Sales and Related	\$28,920
Architecture and Engineering	\$56,330
Healthcare Practitioners	\$49,930
Legal	\$69,030
Teaching & Education	\$39,130
Construction	\$35,460
Professional Baseball Player*	\$2,476,590

(*Source:* Bureau of Labor Statistics, except (*)-The Baseball Players' Association (playbpa.com)).

Solution

There are 8 values listed, so the mean is

$\frac{19630+28920+56330+49930+69030+39130+35460+2476590}{8} = \$346,877.50$

As you can see, the *mean* annual income is substantially larger than the income of 7 out of the 8 professions. The effect of the single **outlier** (the baseball player) has a dramatic effect on the mean, so the mean is not a good method for representing the 'average' salary in this case.

Median

The median is another type of average. It is defined as the value in the middle of a group of numbers. To find the median, we must first list all the numbers *in order from least to greatest*.

Example C

Find the median of the numbers 11, 21, 6, 17, 9.

Solution

We first list the numbers in ascending order: 6, 9, **11**, 17, 21.

The median is the value *in the middle* of the set (in bold).

The median is 11. There are two values higher than 11 and two values lower than 11.

If there is an even number of values, then the median is the arithmetic mean of the two numbers in the middle (in other words, the number halfway between them).

The median is a useful measure of average when the data set is highly skewed by a small number of points that are extremely large or extremely small. Such outliers will have a large effect on the mean, but will leave the median relatively unchanged.

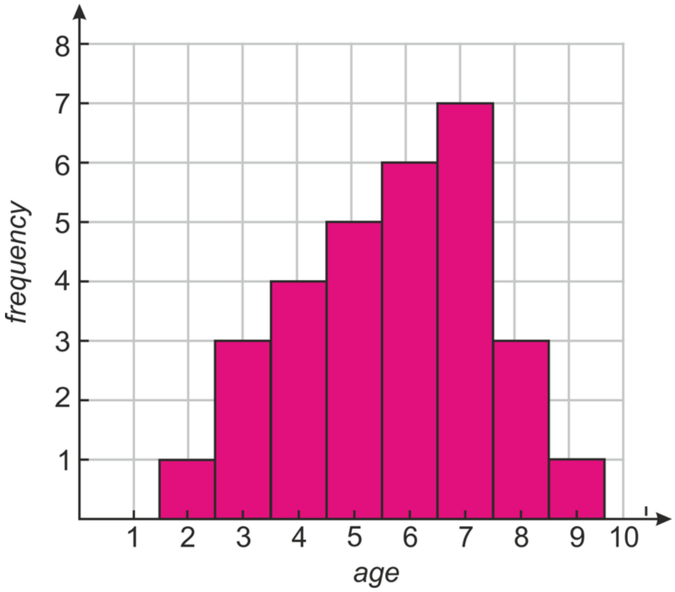
Mode

The mode can be a useful measure of data when that data falls into a small number of categories. It is simply a measure of the most common number, or sometimes the most popular choice. The mode is an especially useful concept for data sets that contains non-numerical information, such as surveys of eye color or favorite ice-cream flavor.

Of course, a data set can contain more than one mode; when it does, it is called **multimodal**. In fact, every value in a data set could be a mode, if every value appears an equal number of times. However, this situation is quite rare. You might encounter data sets with two or even three modes, but more than that would be unlikely unless you are working with very small sample sets.

Example D

Jim is helping to raise money at his church bake sale by doing face painting for children. He collects the ages of his customers, and displays the data in the graph below. Find the mean, median and mode for the ages represented.



Solution

By reading the graph we can see that there was one 2-year-old, three 3-year-olds, four 4-year-olds, etc. In total, there were $1+3+4+5+6+7+3+1=30$ customers.

The mean age is found by adding up all the ages multiplied by the number of times each age appears, and then dividing by 30:

$$\frac{2(1)+3(3)+4(4)+5(5)+6(6)+7(7)+8(3)+9(1)}{30}=\frac{170}{30}=5\frac{2}{3}$$

Since there are 30 children, the median is half way between the 15^{th} and 16^{th} oldest (that way there will be 15 younger and 15 older than the median age). Both the 15^{th} and 16^{th} oldest fall in the 6-year-old range, therefore the median is 6.

The mode is given by the age group with the highest frequency. Reading directly from the graph, we see that the mode is 7; there are more 7-year-olds than any other age.

Vocabulary

- The **arithmetic mean** of a group of numbers is found by dividing the sum of the numbers by the number of values in the group. In other words, we add all the numbers together and divide by the number of numbers.

- The **median** is another type of average. It is defined as the value in the middle of a group of numbers. To find the median, we must first list all the numbers *in order from least to greatest*. A useful formula for finding the middle value is as follows:

if there are n values in the data set, the median is the $\frac{n+1}{2}$th value.

- The **mode** is the most frequent number(s). If no number repeats, there is is no mode. There can be more than one mode.

Guided Practice

Find the mean, median and mode of the numbers 2, 17, 1, -3, 12, 8, 12, 16.

Solution:

Mean $= \frac{2+17+1+(-3)+12+8+12+16}{9} = 7.\overline{22}$

We first list the numbers in ascending order: -3, 1, 2, 8, 12, 12, 16, 17.

The median is the value *in the middle* of the set, so the median lies between 8 and 12. Halfway between 8 and 12 is 10, so **10 is the median**.

The **mode** is the most frequent number or numbers. The only number that repeats is 12, so **12 is the mode.**

Practice

1. Find the **median** and **mode** of the salaries given in Example A.
2. Find the **median** and **mode** of the salaries given in Example B.
3. Find the mean, median and mode of the data set: $14,9,3,14,2,7,13,6$.
4. Find the mean, median and mode of the data set: $5,3,5,0,1,5,3,4,4,4$.
5. Find the mean, median and mode of the data set: $8,5,10,4,4,10,6,4,7,8,2,8,10,9,2,1,6,10,5,3$.
6. Find the mean, median and mode of the following numbers. Which of these will give the best *average*? 15, 19, 15, 16, 11, 11, 18, 21, 165, 9, 11, 20, 16, 8, 17, 10, 12, 11, 16, 14
7. Ten house sales in Encinitas, California are shown in the table below. Find the mean, median and standard deviation for the sale prices. Explain, using the data, why the **median house price** is most often used as a measure of the house prices in an area.

TABLE 13.6:

Address	Sale Price	Date Of Sale
643 3RD ST	$1,137,000	6/5/2007

TABLE 13.6: (continued)

Address	Sale Price	Date Of Sale
911 CORNISH DR	$879,000	6/5/2007
911 ARDEN DR	$950,000	6/13/2007
715 S VULCAN AVE	$875,000	4/30/2007
510 4TH ST	$1,499,000	4/26/2007
415 ARDEN DR	$875,000	5/11/2007
226 5TH ST	$4,000,000	5/3/2007
710 3RD ST	$975,000	3/13/2007
68 LA VETA AVE	$796,793	2/8/2007
207 WEST D ST	$2,100,000	3/15/2007

For 8-10, determine which average measure of center (mean, median or mode) would be most appropriate for the following.

8. The life expectancy of store-bought goldfish.
9. The age in years of audience for a kids TV program.
10. The weight of potato sacks that a store labels as “5 pound bag.”

13.10 Measures of Spread/Dispersion

Here you'll learn how to find the measures of dispersion (the range, the variance, and the standard deviation) to determine how spread out the data is.

What if you went to your local used car lot and wrote down the sticker price of all the cars there? How could you describe how spread out your data is? After completing this Concept, you'll be able to measure the dispersion of a collection of data like this.

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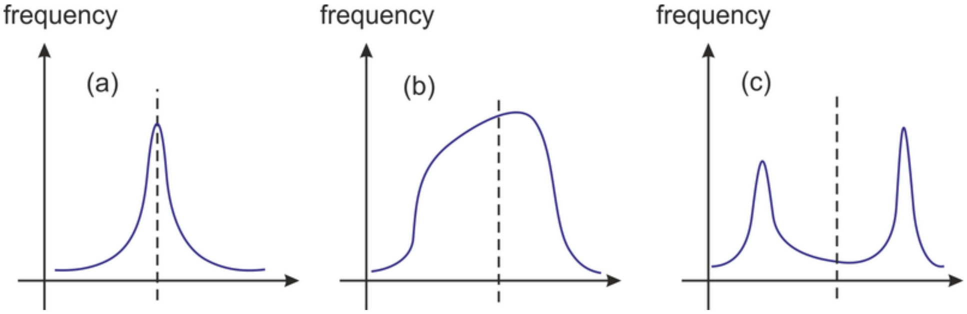
CK-12 Foundation: Measures of Dispersion

Guidace

Look at the graphs below. Each represents a collection of many data points and shows how the individual values (solid line) compare to the mean of the data set (dashed line). You can see that even though all three graphs have a common mean, the *spread* of the data differs from graph to graph. In statistics we use the word **dispersion** as a measure of how spread out the data is.



Range

Range is the simplest measure of dispersion. It is simply the total spread in the data, calculated by subtracting the smallest number in the group from the largest number.

Example A

Find the range and the median of the following data:

223, 121, 227, 433, 122, 193, 397, 276, 303, 199, 197, 265, 366, 401, 222

Solution

The first thing to do in this case is to order the data, listing all values in ascending order:

121, 122, 193, 197, 199, 222, 223, 227, 265, 276, 303, 366, 397, 401, 433

Note: It is extremely important to make sure that you don't skip any values when you reorder the list. Two ways to do this are (i) cross out the numbers in the original list as you write them in the second list, and (ii) count the number of values in both lists when you are done. In this example, both lists contain 15 values, so we can be sure we didn't miss any (as long as we didn't count any twice!)

The range is found by subtracting the lowest value from the highest: $433-122=311$.

And now that the list is ordered, we can see that the median is the 8th value: **227**.

Variance

The range is not a particularly good measure of dispersion, as it does not eliminate points that have unusually high or low values when compared to the rest of the data (the **outliers**). A better method involves measuring the distance each data point lies from a central average.

Look at the following data values:

11, 13, 14, 15, 19, 22, 24, 26

The mean of these values is 18; of course, the values all differ from 18 by varying amounts. Here's a list of the values' **deviations** from the mean:

-7, -5, -4, -3, 1, 4, 6, 8

If we take the mean of these deviations, we find that it is zero:

$$\frac{-7-5-4-3+1+4+6+8}{8}=\frac{0}{8}=0$$

This comes as no surprise. You can see that some of the values are positive and some are negative, as the mean lies somewhere near the middle of the range. You can use algebra to prove (try it!) that the sum of the deviations will always be zero, no matter what numbers are in the list. So, the sum of the deviations is not a useful tool for measuring variance.

But if we *square* the differences, all the negative differences become positive, and then we can tell how great the average deviation is. If we do that for this data set, we get the following list:

49, 25, 16, 9, 1, 16, 36, 64

The sum of those squares is 216, so their average is $\frac{216}{8}=27$.

We call this averaging of the square of the differences from the mean (the mean squared deviation) the **variance**. The variance is a measure of the dispersion, and its value is lower for tightly grouped data than for widely spread data. In the example above, the variance is 27.

What does it mean to say that tightly grouped data will have a low variance? You can probably already imagine that the size of the variance also depends on the size of the data itself. Mathematicians have tried to standardize the definition of variance in various ways; the **standard deviation** is one of the most commonly used.

Standard Deviation

You can see from the previous example that using variance gives us a measure of the spread of the data (you should hopefully see that tightly grouped data would have a smaller mean squared deviation and so a smaller variance) but it is not immediately clear what a number like 27 actually refers to. Since it is the *mean of the squares* of the deviation, however, it seems logical that taking its square root would be a better way to make sense of it. The root mean square (i.e. square root of the variance) is called the **standard deviation**, and is given the symbol s.

Example B

Find the mean, the variance and the standard deviation of the following values.

121, 122, 193, 197, 199, 222, 223, 227, 265, 276, 303, 366, 397, 401, 433

Solution

The mean will be needed to find the variance, and from the variance we can determine the standard deviation. The sum of all fifteen values is 3945, so their mean is $\frac{3945}{15} = 263$.

The variance and standard deviation are often best calculated by constructing a table. Using this method, we enter the deviation and the square of the deviation for each separate data point.

TABLE 13.7:

Value	Deviation	Deviation^2
121	–142	20,164
122	–141	19,881
193	–70	4,900
197	–66	4,356
199	–64	4,096
222	–41	1,681
223	–40	1,600
227	–36	1,296
265	2	4
276	13	169
303	40	1,600
366	103	10,609
397	134	17,956
401	138	19,044
433	170	28,900
sum:	0	136,256

The variance is the mean of the squares of the deviations, so it is $\frac{136{,}256}{15} = 9083.733$. The standard deviation is the square root of the variance, or approximately 95.31.

If you look at the second column of the table, you can see that the standard deviation is a good measure of the spread. It looks to be a reasonable estimate of the average distance that each point lies from the mean.

Calculate and Interpret Measures of Central Tendency and Dispersion for Real-World Situations

Example C

A number of house sales in a town in Arizona are listed below. Calculate the mean and median house price. Also calculate the standard deviation in sale price.

TABLE 13.8:

Address	Sale Price
518 CLEVELAND AVE	\$117, 424
1808 MARKESE AVE	\$128, 000
1770 WHITE AVE	\$132, 485
1459 LINCOLN AVE	\$77, 900
1462 ANNE AVE	\$60, 000

TABLE 13.8: (continued)

Address	Sale Price
2414 DIX HWY	\$250, 000
1523 ANNE AVE	\$110, 205
1763 MARKESE AVE	\$70, 000
1460 CLEVELAND AVE	\$111, 710
1478 MILL ST	\$102, 646

Solution

The sum of all ten values is \$1,160,370, so their mean is **\$116,037**.

The median is halfway between the 5^{th} and 6^{th} highest values. Those two middle values (if we reorder the list by price) are \$110,205 and \$111,710, so the median is **\$110,957.50**.

Now we can rewrite the table with the deviations and their squares added in:

TABLE 13.9:

Value (\$)	Deviation	Deviation^2
60,000	-56037	3140145369
70,000	-46037	2119405369
77,900	-38137	1454430769
102,646	-13391	179318881
110,205	-5832	34012224
111,710	-4327	18722929
117,424	1387	1923769
128,000	11963	14311369
132,485	16448	270536704
250,000	133963	17946085369
	SUM:	**25178892752**

The variation is $\frac{25178892752}{10} = 2517889275.2$, and the square root of that is about 50179. So the standard deviation is **\$50,179**.

In this case, the mean and the median are close to each other, indicating that the house prices in this area of Mesa are spread fairly symmetrically about the mean. Although there is one house that is significantly more expensive than the others, there are also a number that are cheaper to balance out the spread.

Watch this video for help with the Examples above.

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CK-12 Foundation: Measures of Dispersion

Vocabulary

- In statistics we use the word **dispersion** as a measure of how spread out the data is.

- **Range** is the simplest measure of dispersion. It is simply the total spread in the data, calculated by subtracting the smallest number in the group from the largest number.

- We call this averaging of the square of the differences from the mean (the mean squared deviation) the **variance**.

- The root square root of the variance is called the **standard deviation**, and is given the symbol s.

Guided Practice

James and John both own fields in which they plant cabbages. James plants cabbages by hand, while John uses a machine to carefully control the distance between the cabbages. The diameters of each grower's cabbages are measured. James's cabbages have an average (mean) diameter of 7.10 inches with a standard deviation of 2.75 inches; John's have a mean diameter of 6.85 inches with a standard deviation of 0.60 inches.

John claims his method of machine planting is better. James insists it is better to plant by hand. Use the data to provide a reason to justify ***both sides*** *of the argument.*

Solution

- James's cabbages have a larger mean diameter, so on average they are larger than John's. The larger standard deviation also means that there will be a number of cabbages which are significantly bigger than most of John's.
- John's cabbages are smaller on average, but only by a little bit (one quarter inch). Meanwhile, the smaller standard deviation means that the sizes of his cabbages are much more predictable. The spread of sizes is much less, so they all end up being closer to the mean. While he may not have many extra large cabbages, he will not have any that are excessively small either, which may be better for any stores to which he sells his cabbages.

Practice

1. Two bus companies run services between Los Angeles and San Francisco. Inter-Cal Express takes a mean time of 9.5 hours to make the trip, with a standard deviation of 0.25 hours. Fast-Dog Travel takes 8.75 hours on average, with a standard deviation of 2.5 hours. If Samantha needs to travel between the cities, which company should she choose if:
   a. She needs to be on time for a meeting in San Francisco.
   b. She travels weekly to visit friends who live in San Francisco and wishes to minimize the time she spends on a bus over the entire year.

For problems 2-6, suppose you have a collection of data points for which you have already found the mean, median, mode, range, variance, and standard deviation. Then, you collect two new data points—one that is higher than any of the values in the original set, and one that is lower than any of the values in the original set.

2. Based on just this information, can you tell what will happen to the mean value of the data set when these new points are added? (In other words, can you say anything at all about whether the mean will or won't increase, decrease, or stay the same, or do you not have enough information to tell—and if not, what additional information would you need?)
3. Can you tell what will happen to the median value?
4. Can you tell what will happen to the mode? (Assume the original data set has only one mode.)
5. Can you tell what will happen to the range?

6. Can you tell what will happen to the variance and standard deviation?

For problems 7-11, suppose that instead of collecting two new values for your data set above, you have only collected one new value—one that is higher than all the values in the original set.

7. Now can you tell what will happen to the mean value?
8. Can you tell what will happen to the median value?
9. Can you tell what will happen to the mode?
10. Can you tell what will happen to the range?
11. Can you tell what will happen to the variance and standard deviation?

Finally, for problems 12-16, suppose that instead of being higher than all the values in the original data set, your new value is somewhere in the middle of the original data set. Specifically, suppose it is higher than the mean, lower than the median, and equal to the mode.

12. Now can you tell what will happen to the mean?
13. Can you tell what will happen to the median?
14. Can you tell what will happen to the mode?
15. Can you tell what will happen to the range?
16. Can you tell what will happen to the variance and standard deviation?

13.11 Stem-and-Leaf Plots and Histograms

Here you'll learn various ways to graphically display a data set, including stem-and-leaf plots and histograms. You'll also learn how to interpret such displays.

What if you wanted to kill time while waiting for your connecting flight at the airport? You start counting the number of people who pass by you in one minute. You do this for 15 consecutive minutes. How could you graphically display your data? After completing this Concept, you'll be able to make stem-and-leaf plots and histograms to display data like this.

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Click image to the left for more content.

CK-12 Foundation: Stem-and-Leaf Plots and Histograms

Guidance

Imagine asking a class of 20 algebra students how many brothers and sisters they had. You would probably get a range of answers from zero on up. Some students would have no siblings, but most would have at least one. The results might look like this:

1, 4, 2, 1, 0, 2, 1, 0, 1, 2, 1, 0, 0, 2, 2, 3, 1, 1, 3, 6

We could organize this information in many ways. The first way might just be to create an ordered list, relisting all the numbers in order, starting with the smallest:

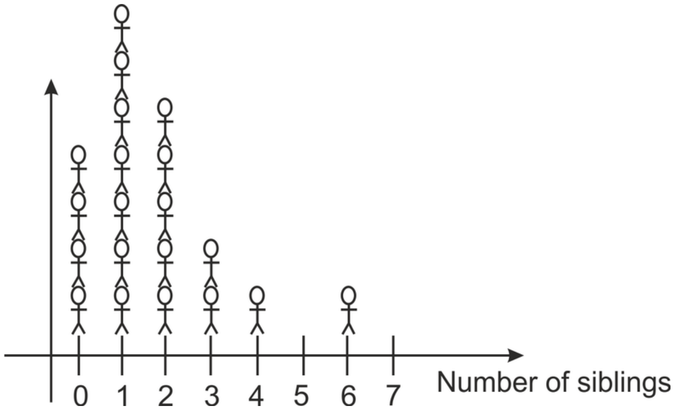
0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 4, 6

Another way to list the results is in a table:

TABLE 13.10:

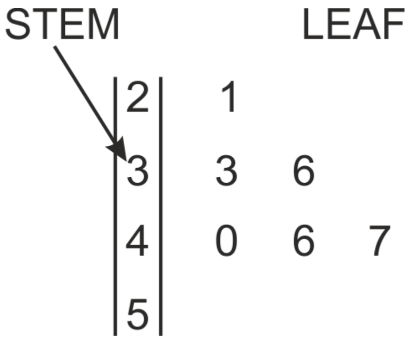
Number of siblings	Number of matching students
0	4
1	7
2	5
3	2
4	1
5	0
6	1

We could also make a visual representation of the data by making categories for the number of siblings on the $x-$axis, and stacking representations of each student above the category marker. We could use crosses, stick-men or even photographs of the students to show how many students are in each category.



Make and Interpret Stem-and-Leaf Plots

Another useful way to display data is with a **stem-and-leaf plot**. Stem-and-leaf plots are especially useful because they give a visual representation of how the data is clustered, but preserve all of the numerical information. A stem-and-leaf plot consists of a vertical "stem" containing the first digit of each number, with the rest of each number written to the right of the stem like a "leaf." In the stem and leaf plot below, the first number represented is 21. It is the only number with a stem of 2, so that makes it the only number in the 20's. The next two numbers have a common stem of 3. They are 33 and 36. The next numbers are 40, 46 and 47.



Stem-and-leaf plots have a number of advantages over simply listing the data in a single line.

- They show how data is distributed, and whether it is symmetric around the center.
- They can be used as the data is being collected.
- They make it easy to determine the median and mode.

Stem-and-leaf plots are not ideal for all situations; in particular they are not practical when the data is too tightly clustered. For example, with the data above about students' siblings, all the data points would occupy the same stem (zero). In that case, no additional information could be gained from a stem-and-leaf plot.

Example A

While traveling on a long train journey, Rowena collected the ages of all the passengers traveling in her carriage. The ages for the passengers are shown below. Arrange the data into a stem-and-leaf plot, and use the plot to find the median and mode ages.

35, 42, 38, 57, 2, 24, 27, 36, 45, 60, 38, 40, 40, 44, 1, 44, 48, 84, 38, 20, 4, 2, 48, 58, 3, 20, 6, 40, 22, 26, 17, 18, 40, 51, 62, 31, 27, 48, 35, 27, 37, 58, 21

Solution

The first step is to determine a sensible ***stem***. Since all the values fall between 1 and 84, the stem should represent the tens column, and run from 0 to 8 so that the numbers represented can range from 00 (which we would represent by placing a leaf of 0 next to the 0 on the stem) to 89 (a leaf of 9 next to the 8 on the stem). We then go through the data and fill out our plot:

0	2	1	4	2	3	6					
1	7	8									
2	4	7	0	0	2	6	7	7	1		
3	5	8	6	8	8	1	5	7			
4	2	5	0	0	4	4	8	8	0	0	8
5	7	8	1	8							
6	0	2									
7											
8	4										

You can see immediately that the interval with the most number of passengers is the 40-49 group. In order to correctly determine the median and the mode, it is helpful to construct a second, **ordered stem and leaf plot**, placing the leaves on each branch in ascending order

0	1	2	2	3	4	6					
1	7	8									
2	0	0	1	2	4	6	7	7	7		
3	1	5	5	6	7	8	8	8			
4	0	0	0	0	2	4	4	5	8	8	8
5	1	7	8	8							
6	0	2									
7											
8	4										

The mode is now apparent—there are 4 zeros in a row on the 4-branch, so the mode is 40. The median is the middle value; since there are 43 data points, the median is the 22^{nd} value. (Using our formula from earlier, $\frac{43+1}{2} = 22$.) So the median is 37.

Make and Interpret Histograms

Look again at the example of the algebra students and their siblings. The data was collected in the following list.

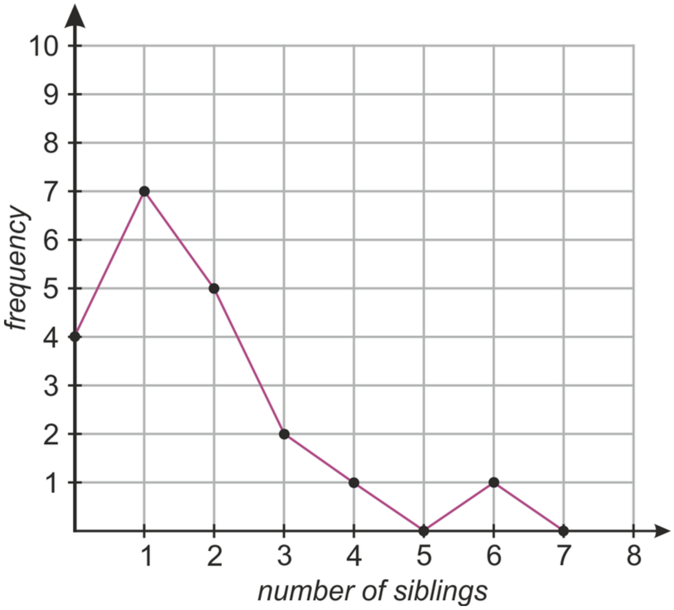
1, 4, 2, 1, 0, 2, 1, 0, 1, 2, 1, 0, 0, 2, 2, 3, 1, 1, 3, 6

We were able to organize the data into a table. Here is the table again, but this time we will use the word *frequency* as a header to indicate the number of times each value occurs in the list.

TABLE 13.11:

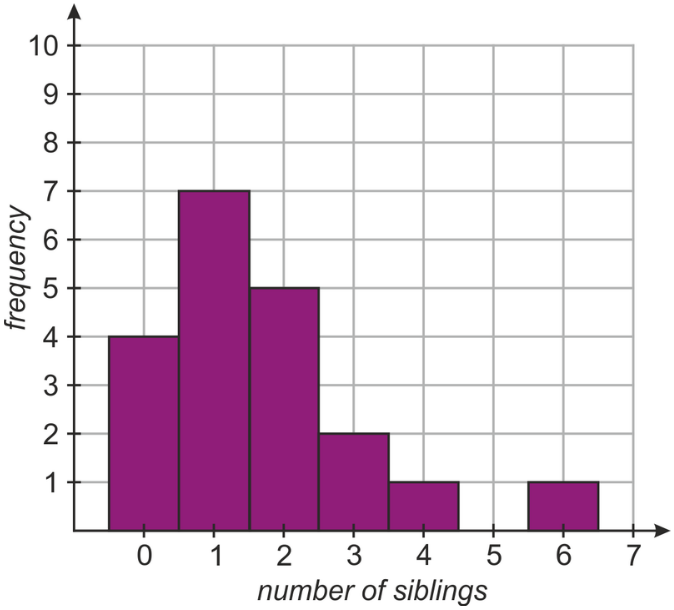
Number of siblings	**Frequency**
0	4
1	7
2	5
3	2
4	1
5	0
6	1

Now we could use this table as an (x, y) coordinate list to plot a line diagram like this one:



While this diagram does indeed show the data, it is somewhat misleading. For example, the continuous line joining the number of students with one and two siblings makes it look like we know something about how many students have 1.5 siblings (which of course, is impossible). In this case, where the data points are all integers, it's wrong to suggest that the function is continuous between the points!

When the data we are representing falls into well defined categories (such as the integers 1, 2, 3, 4, 5 & 6) it is more appropriate to use a ***histogram*** to display that data. A histogram for this data is shown below.



Each number on the $x-$axis has an associated column, whose height shows how many students have that number of siblings. For example, the column at $x = 2$ is 5 units high, indicating that there are 5 students with 2 siblings.

The categories on the $x-$axis are called **bins**. Histograms differ from bar charts in that they don't necessarily have fixed widths for the bins. They are also useful for displaying **continuous data** (data that varies continuously rather than in integer amounts). To illustrate this, here are some examples.

Example B

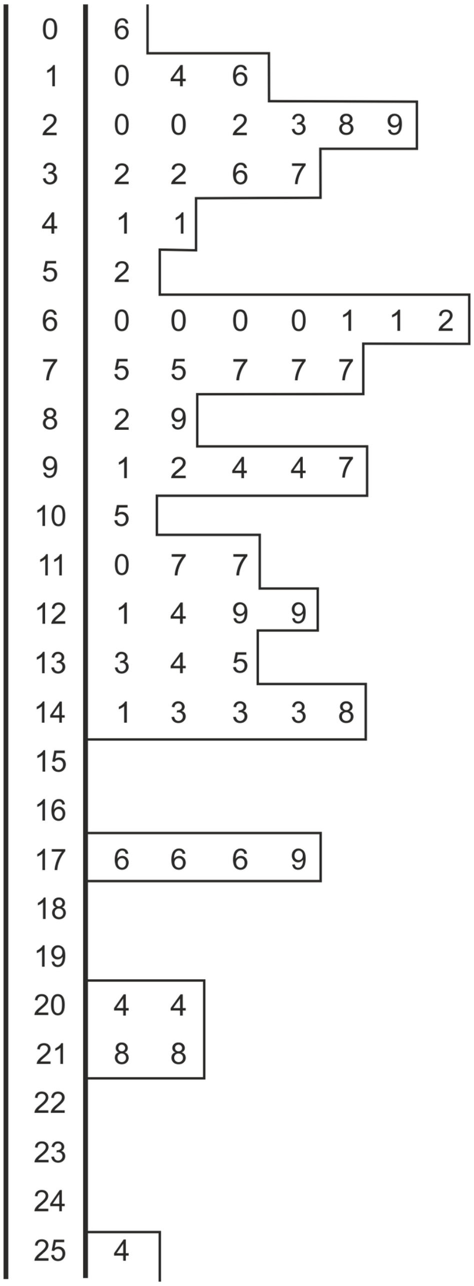
Monthly rainfall (in millimeters) for Beaver Creek Oregon was collected over a five year period, and the data is shown below. Display the data in a histogram.

41.1, 254.7, 91.6, 60.9, 75.6, 36.0, 16.5, 10.6, 62.2, 89.4, 124.9, 176.7, 121.6, 135.6, 141.6, 77.0, 82.8, 28.9, 6.7, 22.1, 29.9, 110.0, 179.3, 97.6, 176.8, 143.5, 129.8, 94.9, 77.0, 60.8, 60.0, 32.5, 61.7, 117.2, 194.5, 208.6, 176.8, 143.5, 129.8, 94.9, 77.0, 60.8, 20.0, 32.5, 61.7, 117.2, 194.5, 208.6, 133.1, 105.2, 92.0, 60.7, 52.8, 37.8, 14.8, 23.1, 41.3, 75.7, 134.6, 148.8

Solution

Notice the similarity between histograms and stem-and-leaf plots. A stem-and-leaf plot resembles a histogram on its side. We could start by making a stem-and-leaf plot of our data.

For our data above our stem would be the tens, and run from 1 to 25. Instead of rounding the decimals in the data, we **truncate** them, meaning we simply remove the decimal. For example, 165.7 would have a stem of 16 and a leaf of 5, and we would just leave out the seven tenths.



By outlining the numbers on the stem and leaf plot, we can see what a histogram with a bin-width of 10 would look like. You can see that with so many bins, the histogram looks random, with no clear pattern visible. In a situation like this we need to reduce the number of bins. We will increase the bin width to 25 and collect the data in a table:

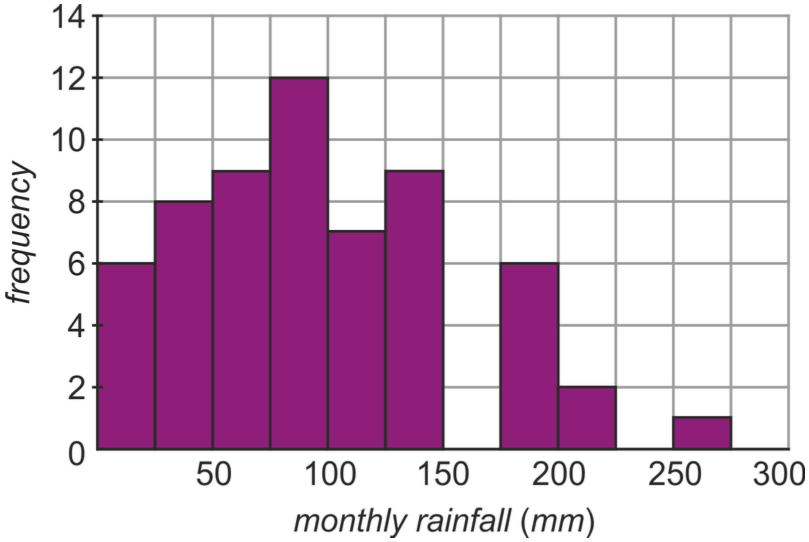
TABLE 13.12:

Rainfall (mm)	**Frequency**
$0 \leq x < 25$	7
$25 \leq x < 50$	8
$50 \leq x < 75$	9
$75 \leq x < 100$	12
$100 \leq x < 125$	6
$125 \leq x < 150$	9
$150 \leq x < 175$	0
$175 \leq x < 200$	6

TABLE 13.12: (continued)

Rainfall (mm)	Frequency
$200 \leq x < 225$	2
$225 \leq x < 250$	0
$250 \leq x < 275$	1

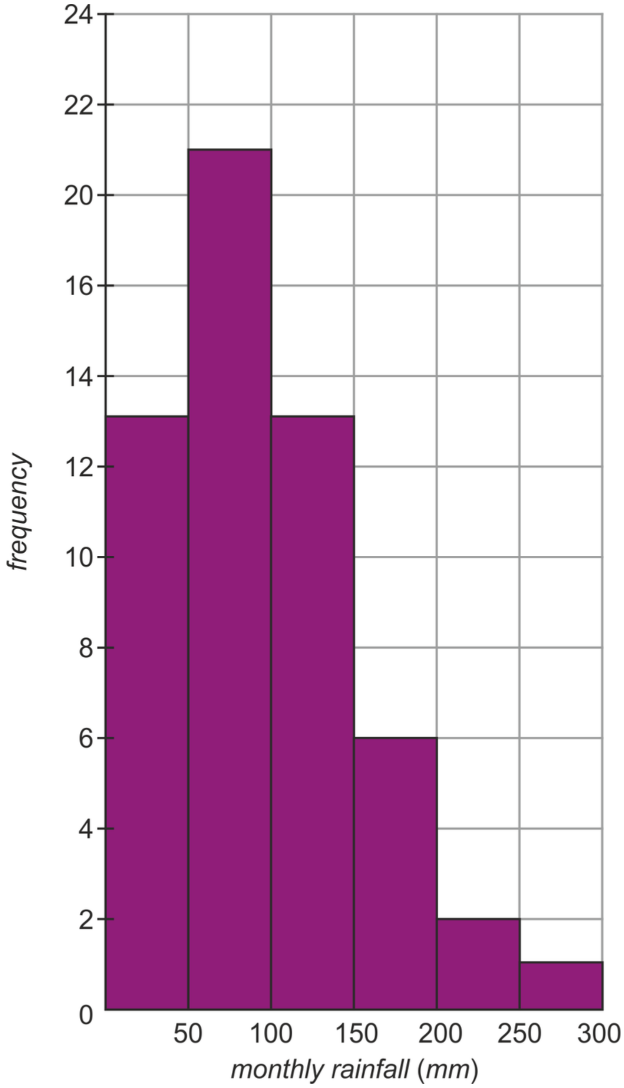
The histogram associated with this bin width is below.



The pattern in the distribution is far more apparent with fewer bins. So let's look at what the histogram would look like with even fewer bins. We will combine bins by pairs to give 6 bins with a bin-width of 50. Our table and histogram now looks like this.

TABLE 13.13:

Rainfall (mm)	Frequency
$0 \leq x < 50$	15
$50 \leq x < 100$	21
$100 \leq x < 150$	15
$150 \leq x < 200$	6
$200 \leq x < 250$	2
$250 \leq x < 300$	1



The pattern is much clearer now. The normal monthly rainfall is around 75 mm, but sometimes it will be a very wet month and be higher (even much higher).

You can see that although it may be counter-intuitive, sometimes you can see more information by reducing the number of intervals (or bins) in a histogram. It's a bit like zooming out on a picture; you can't see as many of the details, but the overall shape of what you are looking at may become clearer.

Make Histograms Using a Graphing Calculator

Look again at the data from Example 1. We've seen how to manipulate raw data to give a stem-and-leaf plot and a histogram. Now let's take some of the tedious sorting work out of the process by using a graphing calculator to automatically sort our data into bins.

Example C

The following unordered data represents the ages of passengers on a train carriage.

35, 42, 38, 57, 2, 24, 27, 36, 45, 60, 38, 40, 40, 44, 1, 44, 48, 84, 38, 20, 4, 2, 48, 58, 3, 20, 6, 40, 22, 26, 17, 18, 40, 51, 62, 31, 27, 48, 35, 27, 37, 58, 21.

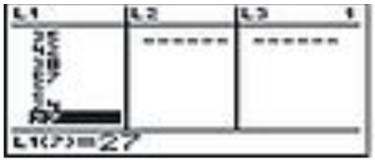
Use a graphing calculator to display the data as a histogram with bin-widths of 10, 5 and 20.

Solution

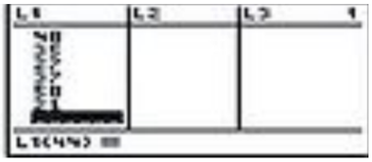
Input the data in your calculator:

Press **[START]** and choose the **[EDIT]** option.

Input all 43 data points into the table in column L_1.



```
L1     L2     L3     1
L1(7)=27
```



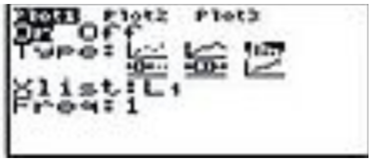
```
L1     L2     L3     1
L1(44) =
```

Select plot type:

Bring up the **[STATPLOT]** option by pressing **[2nd]**, **[Y=]**.

Highlight **1:Plot1** and press **[ENTER]**. This will bring up the plot options screen. Highlight the histogram and press **[ENTER]**. Make sure the **Xlist** is the list that contains your data.

```
STAT PLOTS
1:Plot1…On
    L1   1
2:Plot2…Off
    L1   L2
3:Plot3…Off
    L1   L2
4↓PlotsOff
```



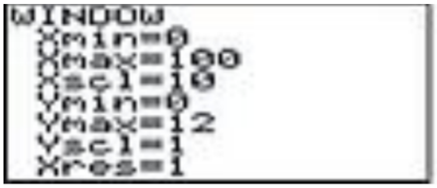
```
Plot1 Plot2 Plot3
On Off
Type:
Xlist:L1
Freq:1
```

Select bin widths and plot:

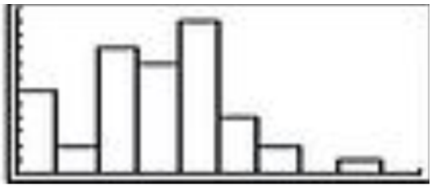
Press **[WINDOW]** and ensure that **Xmin** and **Xmax** allow for all data points to be shown. The **Xscl** value determines the bin width.

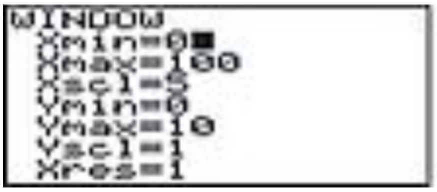
Press **[GRAPH]** to display the histogram.

You can change bin widths and see how the histogram changes, by varying **Xscl**. Below are histograms with bin widths of 10, 5 and 20. (In this example $Xmin = 0$ and $Xmax = 100$ will work whatever bin width we choose, but notice that to display the histogram correctly we need to use a different **Ymax** value for each.)

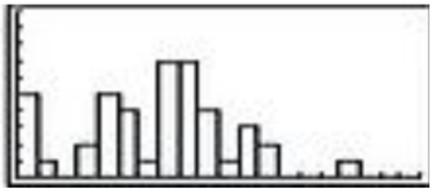


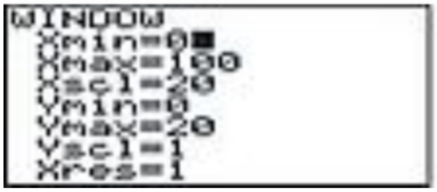
```
WINDOW
 Xmin=0
 Xmax=100
 Xscl=10
 Ymin=0
 Ymax=12
 Yscl=1
 Xres=1
```



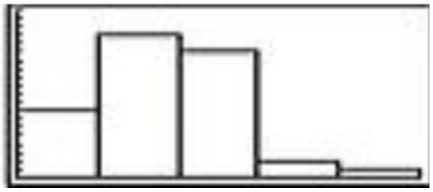


```
WINDOW
 Xmin=0
 Xmax=100
 Xscl=5
 Ymin=0
 Ymax=10
 Yscl=1
 Xres=1
```





```
WINDOW
 Xmin=0
 Xmax=100
 Xscl=20
 Ymin=0
 Ymax=20
 Yscl=1
 Xres=1
```



Watch this video for help with the Examples above.

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CK-12 Foundation: Stem and Leaf Plots and Histograms

Vocabulary

- A **stem-and-leaf plot** consists of a vertical “stem” containing the first digit of each number, with the rest of each number written to the right of the stem like a “leaf.”
- A **frequency table** is one that records the frequency of each value in a data set.
- When the data we are representing falls into well defined categories (such as the integers 1, 2, 3, 4, 5 6) it is more appropriate to use a **histogram** to display that data. A histogram is a graph of a frequency table, where the frequency of each value is represented by the height of a bar.

Guided Practice

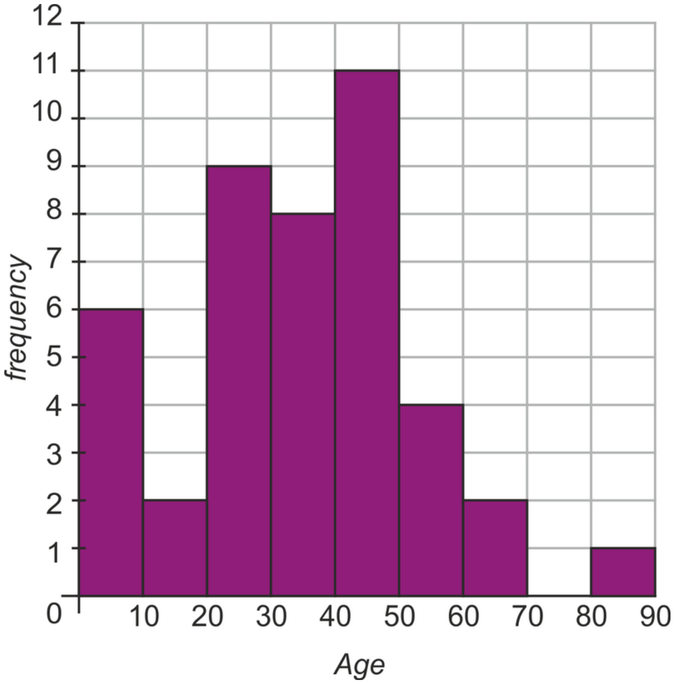
Rowena made a survey of the ages of passengers in a train carriage, and collected the results in a frequency table. Display the results as a histogram.

TABLE 13.14:

Age range	Frequency
0 – 9	6
10 – 19	2
20 – 29	9
30 – 39	8
40 – 49	11
50 – 59	4
60 – 69	2
70 – 79	0
80 – 89	1

Solution

Since the data is already collected into intervals we will use these as our bins for the histogram. Even though the top end of the first interval is 9, the bin on our histogram will extend to 10. This is because, as we move to continuous data, we have a range of numbers that goes right up to the lower end of the following bin, even if it doesn’t include that number. The range of values for the first bin would therefore be $0 \leq x < 10$, and all the other bins would have similarly described ranges.



Practice

1. Create a stem-and-leaf plot for the following data. Use the first digit (**hundreds**) as the stem, and the second (**tens**) as the leaf. Truncate any **units** and **decimals**. Order the plot to find the median and the mode. **data:** 607.4, 886.0, 822.2, 755.7, 900.6, 770.9, 780.8, 760.1, 936.9, 962.9, 859.9, 848.3, 898.7, 670.9, 946.7, 817.8, 868.1, 887.1, 881.3, 744.6, 984.9, 941.5, 851.8, 905.4, 810.6, 765.3, 881.9, 851.6, 815.7, 989.7, 723.4, 869.3, 951.0, 794.7, 807.6, 841.3, 741.5, 822.2, 966.2, 950.1.
2. Make a frequency table for the data in Question 1. Use a bin width of 50.
3. Plot the data from Question 1 as a histogram with a bin width of

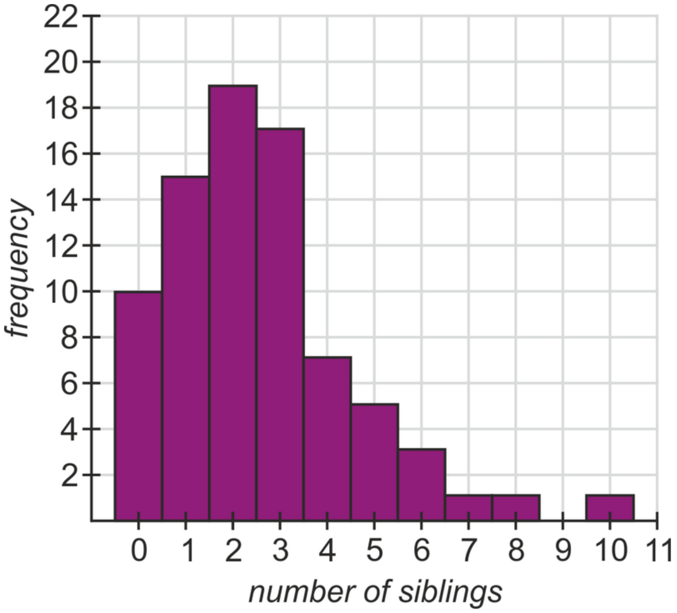
    a. 50
    b. 100

For 4-6, use the following stem-and-leaf plot which shows data collected for the speed of 40 cars in a 35 mph limit zone in Culver City, California.

2	6 7 8 8 9
3	0 0 1 1 1 2 2 2 2 3 3 4 5 5 5 6 6 7 7 9 9 9
4	0 2 1 1 2 4 4 5 6 8 9
5	0 7

4. Find the mean, median and mode speed.
5. Complete the frequency table, starting at 25 mph with a bin width of 5 mph.
6. Use the table to construct a histogram with the intervals from your frequency table.

For 7-11, the histogram shown below displays the results of a larger scale survey of the subjects' number of siblings.



Use it to find:

7. The median of the data.
8. The mean of the data.
9. The mode of the data.
10. The number of people who have an odd number of siblings.
11. The percentage of the people surveyed who have 4 or more siblings.

13.12 Box-and-Whisker Plots

Here you'll learn another way to graphically display a data set, called a box-and-whisker plot. You'll also learn how to interpret such displays and how to determine the effect of outliers on a data set.

What if your teacher recorded each of her student's scores on the last math test? How could she display that data in such a way that it was broken up into four distinct segments? After completing this Concept, you'll be able to make and interpret box-and-whisker plots for data such as this.

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CK-12 Foundation: Box-and-Whisker Plots

Guidance

Consider the following list of numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

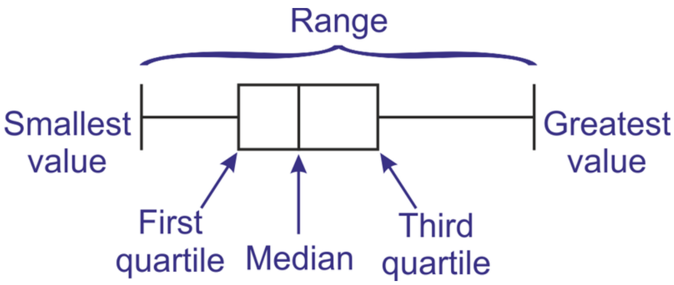
The median is the $\left(\frac{n+1}{2}\right)$ th value. There are 10 values, so the median lies halfway between the 5^{th} and the 6^{th} value. The median is therefore 5.5. This splits the list cleanly into two halves.

The lower list is: 1, 2, 3, 4, 5

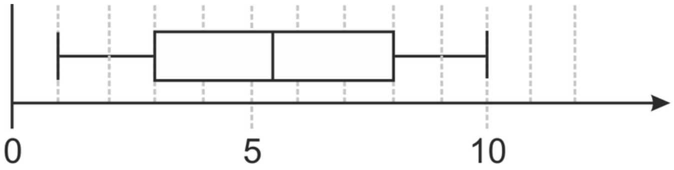
And the upper list is: 6, 7, 8, 9, 10

The median of the lower half is 3. The median of the upper half is 8. These numbers, together with the median, cut the list into four quarters. We call the division between the lower two quarters the **first quartile**. The division between the upper two quarters is the **third quartile** (the **second quartile** is, of course, the median).

A **box-and-whisker plot** is formed by placing vertical lines at five positions, corresponding to the smallest value, the first quartile, the median, the third quartile and the greatest value. (These five numbers are often referred to as the **five number summary**.) A **box** is drawn between the position of the first and third quartiles, and horizontal line segments (the **whiskers**) connect the box with the two extreme values.



The box-and-whisker plot for the integers 1 through 10 is shown below.



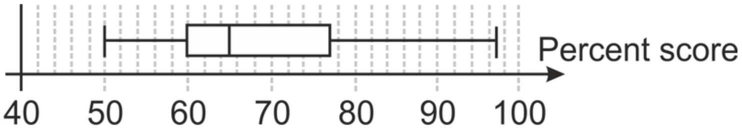
With a box-and-whisker plot, a simple measure of dispersion can be gained from the distance from the first quartile to the third quartile. This **inter-quartile range** is a measure of the spread of the middle half of the data.

Example A

Forty students took a college algebra entrance test and the results are summarized in the box-and-whisker plot below. How many students would be allowed to enroll in the class if the pass mark was set at

a) 65%

b) 60%



Solution

From the plot, we can see the following information:

Lowest score = 50%

First quartile = 60%

Median score = 65%

Third quartile = 77%

Highest score = 97%

Since the pass marks given in the question correspond with the median and the first quartile, the question is really asking how many students there are in: a) the upper half and b) the upper 3 quartiles.

a) Since there are 40 students, there are 20 in the upper half; that is, **20 students** scored above 65%.

b) Similarly, there are 30 students in the upper 3 quartiles, so **30 students** scored above 60%.

Example B

Harika is rolling 3 dice and adding the numbers together. She records the total score for each of 50 rolls, and the scores she gets are shown below. Display the data in a box-and-whisker plot, and find both the ***range*** *and the* ***inter-quartile range****.*

9, 10, 12, 13, 10, 14, 8, 10, 12, 6, 8, 11, 12, 12, 9, 11, 10, 15, 10, 8, 8, 12, 10, 14, 10, 9, 7, 5, 11, 15, 8, 9, 17, 12, 12, 13, 7, 14, 6, 17, 11, 15, 10, 13, 9, 7, 12, 13, 10, 12

Solution

First we'll put the list in order. Since there are 50 data points, $\left(\frac{n+1}{2}\right) = 26.5$, so the median will be the mean of the 25^{th} and 26^{th} values. The median will split the data into two lists of 25 values; we can write them as two distinct lists.

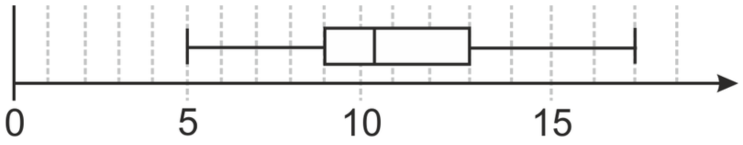
$$5,6,6,7,7,7,8,8,8,8,8,9,\boxed{9},9,9,9,10,10,10,10,10,10,10,10,\boxed{10},\boxed{11},11,$$
$$11,11,12,12,12,12,12,12,12,12,12,\boxed{13},13,13,13,14,14,14,15,15,15,17,17$$

Since each sub-list has 25 values, the first and third quartiles of the entire data set can be found from the median of each smaller list. For 25 values, $\left(\frac{n+1}{2}\right) = 13$, and so the quartiles are given by the 13^{th} value from each smaller sub-list.

From the ordered list we can see the five number summary:

- The lowest value is 5
- The first quartile is 9
- The median is 10.5
- The third quartile is 13
- The highest value is 17.

The box-and-whisker plot therefore looks like this:



The **range** is given by subtracting the smallest value from the largest value: $17-5=12$.

The **inter-quartile range** is given by subtracting the first quartile from the third quartile: $13-9=4$.

Representing Outliers in a Box-and-Whisker Plot

Box-and-whisker plots can be misleading if we don't take outliers into account. An **outlier** is a data point that does not fit well with the other data in the list. For box-and-whisker plots, we can define which points are outliers by how far they are from the *box* part of the diagram. Defining which data are outliers is somewhat arbitrary, but many books use the norm that follows. Our basic measure of distance will be the inter-quartile range (IQR).

- A **mild outlier** is a point that falls more than 1.5 times the IQR outside of the box.
- An **extreme outlier** is a point that falls more than 3 times the IQR outside of the box.

When we draw a box-and-whisker plot, we don't include the outliers in the "whisker" part of the plot; instead, we draw them as separate points.

Example C

Draw a box-and-whisker plot for the following ordered list of data:

$$1,2,5,\boxed{9},10,10,\boxed{11, 12},13,13,\boxed{14},19,25,30$$

Solution

From the ordered list we see:

- The lowest value is 1.

- The first quartile (Q_1) is 9.
- The median is 11.5.
- The third quartile (Q_3) is 14.
- The highest value is 30.

Before we start to draw our box-and-whisker plot, we can determine the IQR:

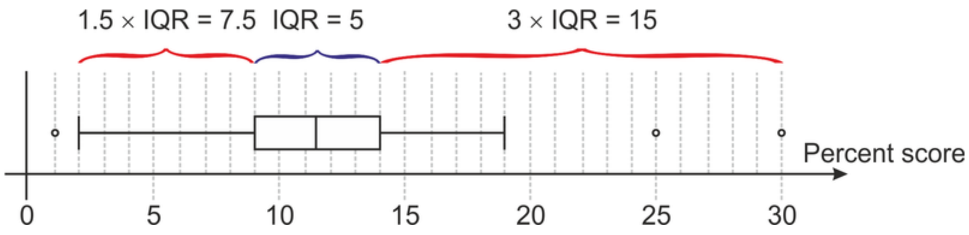
$$IQR = Q_3 - Q_1 = 14 - 9 = 5$$

Outliers are points that fall more than 1.5 times the IQR outside of the box—in other words, values that are more than 7.5 units less than 9 or greater than 14. So any values less than 1.5 or greater than 21.5 are outliers.

Looking back at the data we see:

- The value of 1 is less than 1.5, so it is a **mild outlier**.
- The value 2 is the **lowest value that falls within the included range**.
- The value 30 is greater than 21.5. In fact, it's not just more than 7.5 units outside the box, it's more than twice that far outside the box. Since it falls more than 3 times the IQR above the third quartile, it's an **extreme outlier**.
- The value 25 is also greater than 21.5, so it is a **mild outlier**.
- The value 19 is the **highest value that falls within the included range**.

So when we draw our box-and-whisker plot, the whiskers will only go out as far as 2 and 19 respectively. The points outside of that range are all outliers. Here is the plot:



Making Box-and-Whisker Plots Using a Graphing Calculator

Graphing calculators make analyzing large lists of data easy. They have built-in algorithms for finding the median and the quartiles, and can be used to display box-and-whisker plots.

Example D

The ages of all the passengers traveling in a train carriage are shown below.

35, 42, 38, 57, 2, 24, 27, 36, 45, 60, 38, 40, 40, 44, 1, 44, 48, 84, 38, 20, 4, 2, 48, 58, 3, 20, 6, 40, 22, 26, 17, 18, 40, 51, 62, 31, 27, 48, 35, 27, 37, 58, 21

Use a graphing calculator to:

a) *obtain the 5 number summary for the data.*

b) *create a box-and-whisker plot.*

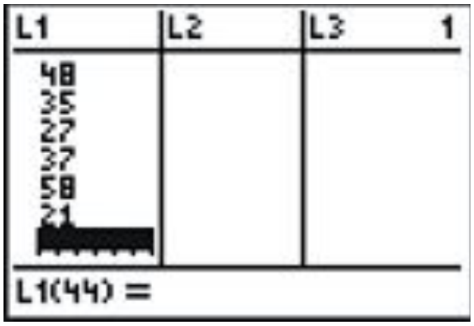
c) *determine if any of the points are outliers.*

Solution

Enter the data in your calculator:

Press **[START]** then choose **[EDIT]**.

Enter all 43 data points in list L_1.



```
L1      |L2     |L3      1
  48
  35
  27
  37
  58
  21
L1(44) =
```

Find the 5 number summary:

Press **[START]** again. Use the right arrow to choose **[CALU]**.

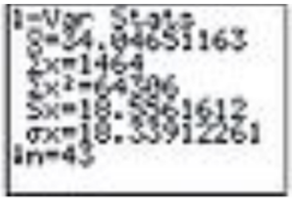
Highlight the **1-Var Stats** option. Press **[EDIT]**.

The single variable statistics summary appears.

Note the **mean** ($\bar{x}$) is the first item given.

Use the down arrow to bring up the data for the five *number summary*. n is the number of data points, and the final fie numbers in the screen are the numbers we require.

```
EDIT CALC TESTS
1:1-Var Stats
2:2-Var Stats
3:Med-Med
4:LinReg(ax+b)
5:QuadReg
6:CubicReg
7↓QuartReg
```

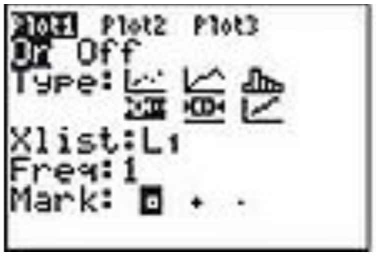


```
1-Var Stats
 x̄=34.04651163
 Σx=1464
 Σx²=64706
 Sx=18.5561612
 σx=18.33912261
↓n=43
```

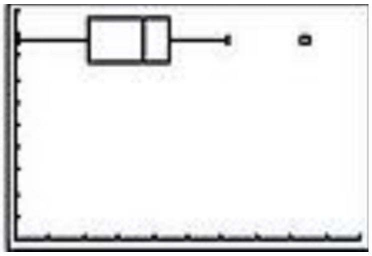
```
1-Var Stats
↑n=43
 minX=1
 Q1=21
 Med=37
 Q3=45
 maxX=84
```

TABLE 13.15:

	Symbol	Value
Lowest Value	**minX**	1
First Quartile	Q_2	21
Median	**Med**	37
Third Quartile	Q_3	45
Highest Value	**maxX**	84



```
Plot1 Plot2 Plot3
On Off
Type:
Xlist:L1
Freq:1
Mark: □ + ·
```



Display the box-and-whisker plot:

Bring up the **[STARTPLOT]** option by pressing **[2nd]. [Y=]**.

Highlight **1:Plot1** and press **[ENTER]**.

There are two types of box-and-whisker plots available. The first automatically identifies outliers. Highlight it and press **[ENTER]**.

Press **[WINDOW]** and ensure that **Xmin** and **Xmax** allow for all data points to be shown. In this example, $Xmin = 0$ and $Xmax = 100$.

Press **[GRAPH]** and the box-and-whisker plot should appear.

The calculator will automatically identify outliers and plot them as such. You can use the **[TRACE]** function along with the arrows to identify outlier values. In this case there is one outlier: 84.

Watch this video for help with the Examples above.

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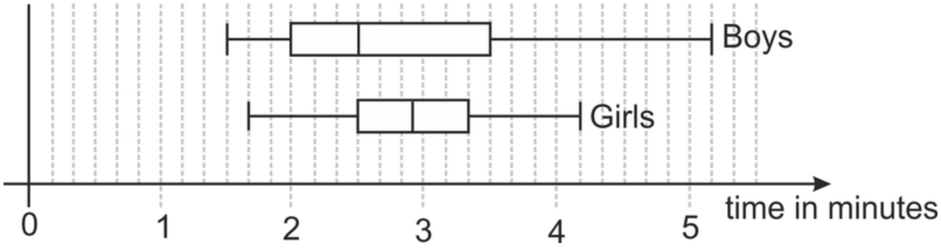
CK-12 Foundation: Box and Whisker Plots

Vocabulary

- We call the division between the lower two quarters the **first quartile**. The division between the upper two quarters is the **third quartile** (the **second quartile** is, of course, the median).
- A **box-and-whisker plot** is formed by placing vertical lines at five positions, corresponding to the smallest value, the first quartile, the median, the third quartile and the greatest value. (These five numbers are often referred to as the **five number summary**.) A **box** is drawn between the position of the first and third quartiles, and horizontal line segments (the **whiskers**) connect the box with the two extreme values.

Guided Practice

The box-and-whisker plots below represent the times taken by a school class to complete an obstacle course. The times have been separated into boys and girls. The boys and the girls each think that they did best. Determine the five number summary for both the boys and the girls and give a convincing argument for each of them.



Solution

Comparing two sets of data with a box-and-whisker plot is relatively straightforward. For example, you can see that the data for the boys is more spread out, both in terms of the range and the inter-quartile range.

The five number summary for each is shown in the table below.

TABLE 13.16:

	Boys	Girls
Lowest value	1:30	1:40
First Quartile	2:00	2:30
Median	2:30	2:55
Third Quartile	3:30	3:20
Highest value	5:10	4:10

Here are some points each side could use in their argument:

Boys:

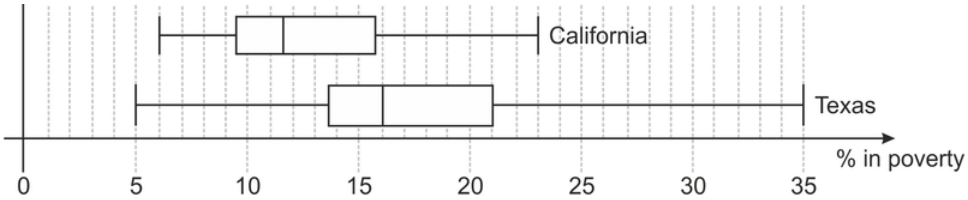
- The boys had the fastest time (1 minute 30 seconds), so the fastest individual was a boy.
- The boys also had the smaller median (2 minutes 30 seconds), meaning half of the boys were finished when only one fourth of the girls were finished (since the girls' first quartile is also 2:30). In other words, the boys' average time was faster.

Girls:

- The boys had the slowest time (5 minutes 10 seconds), so by the time all the girls were finished there was still at least one boy completing the course.
- The girls had the smaller third quartile (3 min 20 seconds), meaning that even without taking the slowest fourth of each group into account, the girls were still quickest.

Practice

1. Draw a box-and-whisker plot for the following unordered data: 49, 57, 53, 54, 57, 49, 67, 51, 57, 56, 59, 57, 50, 49, 52, 53, 50, 58
2. A simulation of a large number of runs of rolling 3 dice and adding the numbers results in the following 5-number summary: **3, 8, 10.5, 13, 18**. Make a box-and-whisker plot for the data and comment on the differences between it and the plot in example B.
3. The box-and-whisker plots below represent the percentage of people living below the poverty line by county in both Texas and California. Determine the 5-number summary for each state, and comment on the spread of each distribution.



4. The 5-number summary for the average daily temperature in Atlantic City, NJ^1 (given in $^\circ F$) is: **31, 39, 52, 68, 76**. Draw the box-and-whisker plot for this data and use it to determine which of the following, if any, would be considered outliers if they were included in the data:
   a. January's record high temperature of 78°
   b. January's record low temperature of -8°
   c. April's record high temperature of 94°
   d. The all time record high of 106°
5. In 1887 Albert Michelson and Edward Morley conducted an experiment to determine the speed of light. The data for the first 10 runs (5 results in each run) is given below. Each value represents how many kilometers

per second over 299,000 km/s was measured. Create a box-and-whisker plot of the data. Be sure to identify outliers and plot them as such. *850, 740, 900, 1070, 930, 850, 950, 980, 980, 880, 960, 940, 960, 940, 880, 800, 850, 880, 900, 840, 880, 880, 800, 860, 720, 720, 620, 860, 970, 950, 890, 810, 810, 820, 800, 770, 760, 740, 750, 760, 890, 840, 780, 810, 760, 810, 790, 810, 820, 850*

6. Is it possible to have outliers on both ends of a data set? Explain.
7. Is it possible for more than half the values in a data set to be outliers? Explain.
8. Is it possible for more than a quarter of the values in a data set to be outliers? Explain.
9. Is it possible for either of the whiskers in a box-and-whisker plot to be of zero length? Explain.
10. Is it possible for either of the whiskers in a box-and-whisker plot to be longer than the box? Explain.
11. Is it possible for either of the whiskers in a box-and-whisker plot to be twice as long as the box? Explain.

[1] Information taken from data published by Rutgers University Climate Lab (http://climate.rutgers.edu)

13.13 Sampling methods

Here you'll learn methods for collecting information called sampling methods. You'll also learn how to identify biased and unbiased sample sizes and survey questions.

What if you wanted to find the percent of people in your neighborhood who smoke? You decide to conduct an experiment. Would polling people coming out of the local movie theater or the local cigarette shop give you more accurate results? After completing this Concept, you'll be able to identify biased samples and survey questions.

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Click image to the left for more content.

CK-12 Foundation: Sampling Methods

Guidance

One of the most important applications of statistics is collecting information. Statistical studies are done for many purposes: A government agency may want to collect data on weather patterns. An advertising firm might seek information about what people buy. A consumer group could conduct a statistical study on gas consumption of cars, or a biologist might study primates to find out more about animal behaviors. All of these applications and many more rely on the collection and analysis of information.

One method to collect information is to conduct a **census**. In a census, information is collected on all the members of the population of interest. For example, when voting for a class president at school **every person** in the class votes, so this is an example of a census. With this method, the whole population is polled.

It's sensible to include everyone's opinion when the population is small, like that of a high school. But conducting a census on a very large population can be very time-consuming and expensive. An alternate method for collecting information is by using a **sampling method**. This means that information is collected from a small sample that represents the population with which the study is concerned. The information from the sample is then extrapolated to the population—that is, we assume the results for the whole population would be about the same as the results for the sample.

Sampling Methods

The word **population** in statistics means the group of people we wish to study, as opposed to the population at large. When we use sampling to conduct a statistical study, first we need to decide how to choose the sample population. It is essential that the sample is a **representative sample** of the population we are studying. For example, if we are trying to determine the effect of a drug on teenage girls, it would make no sense to include males or older women in our sample population.

There are several ways to choose a population sample from a larger group. The two main types of sampling are **random sampling** and **stratified sampling**.

Random Sampling

This method simply involves picking people at random from the population we wish to poll. However, this doesn't mean we can simply ask the first fifty people who walk by in the street. For instance, if you were conducting a survey on people's eating habits, you'd get different results if you were standing in front of a fast-food restaurant than if you were standing in front of a health food store. In a true random sample, everyone in the population must have the same chance of being chosen. Calling people on the phone, for example, might be a better way of getting a random sample for a survey about eating habits.

Stratified Sampling

This method of sampling actively seeks to poll people from many different backgrounds. The population is first divided into different categories (or **strata**) and the number of members in each category is determined. Gender and age groups are commonly used strata, but others could include salary, education level or even hair color. Then, a sample is made up by picking members from each category in the same proportion as they are in the population. For example, imagine you are conducting a survey that calls for a sample size of 100 people. If you know that 10% of the population you're studying are males between the ages of 10 and 25, then you would seek 10 males in that age group to be part of your sample. Once those 10 have responded, no more males between 10 and 25 may take part in the survey.

Sample Size

In order for sampling to work well, the sample size must be large enough to lessen the effect of a biased sample. For example, if you randomly sample 6 children, there is a fairly good chance that most or all of them will be boys. If you randomly sample 6000 children, it's far more likely that they will be approximately equally spread between boys and girls. Even in stratified sampling (when we would likely poll equal numbers of boys and girls) it's important to have a large enough sample to include other kinds of different viewpoints.

The sample size is determined by the precision desired for the population. The larger the sample size is, the more precise the estimate is. However, the larger the sample size, the more expensive and time consuming the statistical study becomes. In more advanced statistics classes you'll learn how to use statistical methods to determine the best sample size for a given survey.

Example A

For a class assignment you have been asked to find if students in your school are planning to attend university after graduating high-school. Students can respond with "yes", "no" or "undecided". How will you choose which students to interview if you want your results to be reliable?

Solution

The best method for obtaining a representative sample would be stratified sampling. Students in the upper grades might be more sure of their post-graduation plans than students in the lower grades, so it makes sense to divide your sample by grade level. You'll need to find out what proportion of the total student population is included in each grade, then interview the same proportion of students from each grade when conducting the survey.

Identify Biased Samples

Once we have identified our population, it is important that the sample we choose accurately reflect the spread of people present in the population. If the sample we choose ends up with one or more sub-groups that are either over-represented or under-represented, then the sample is **biased**. The results of a **biased sample** might not really represent the entire population, so we want to avoid selecting one. Stratified sampling helps, but it doesn't always eliminate bias in a sample. Even with a large sample size, we may be consistently picking one group over another.

Some samples may deliberately seek a biased sample in order to bolster a particular viewpoint. For example, if a group of students were trying to petition the school to allow eating candy in the classroom, they might try to show that a lot of students support this idea by surveying students immediately before lunchtime when they are all hungry. The practice of polling only those who you believe will support your cause is sometimes referred to as **cherry**

picking.

Many surveys may have a **non-response bias**. For example, if researchers simply hand out questionnaires on a street corner and ask people to fill them out and then mail them in, most people will just throw the questionnaires away. Only people who are really interested in the subject will bother to send them in, and those might also be the people who are more likely to answer the questions a certain way. (Imagine if the questionnaire asked "Do you care a lot about surveys?" People who cared about surveys would answer it, people who didn't care wouldn't bother, and a researcher just looking at the surveys that got sent in would conclude that everybody cares about surveys, because everybody who actually answered the survey said yes!)

Non-response bias may be reduced by conducting face-to-face interviews. When you talk to people in person, you can get them to agree to answer a question before you tell them what it is, and then the people you get answers from won't just be the people who care a lot about the question.

Self-selected respondents tend to have stronger opinions on subjects than others and are more motivated to respond. For this reason, *phone-in* and *online* polls also tend to be poor representations of the overall population. Even if it looks like both sides are responding, the poll may disproportionately represent extreme viewpoints from both sides, while ignoring more moderate opinions which may, in fact, be the majority view. Self-selected polls are generally regarded as unscientific.

A classic example of a biased sample occurred in the 1948 Presidential Election. On Election night, the Chicago Tribune printed the headline DEWEY DEFEATS TRUMAN, which turned out to be mistaken. The reason the paper was mistaken is that their editor trusted the results of a phone survey. Telephones were still relatively new at the time, so the people who had them tended to be wealthier than average; therefore, a sample of people who had telephones was not a representative sample of the population at large.

(The above text is adapted from Wikipedia http://www.wikipedia.org/wiki/Biased_sample.)

Example B

Identify each sample as biased or unbiased. If the sample is biased explain how you would improve your sampling method.

a) *Asking people shopping at a farmer's market if they think locally grown fruit and vegetables are healthier than supermarket fruits and vegetables.*

b) *You want to find out public opinion on whether teachers get paid a sufficient salary by interviewing the teachers in your school.*

c) *You want to find out if your school needs to improve its communications with parents by sending home a survey written in English.*

Solution

a) This would be a biased sample because people who shop at farmer's markets are more likely than the average person to think that locally grown produce is better. The study could be improved by interviewing an equal number of people coming out of a supermarket, or by interviewing people in a more neutral environment such as the post office.

b) This is a biased sample because teachers probably would think they should get a higher salary, but that doesn't mean everybody else would agree. A better sample could be obtained by constructing a stratified sample with people in different income categories.

c) This is a biased sample because only English-speaking parents would understand the survey, and parents who don't speak English would be more likely to find that the school doesn't communicate with them well. The study could be improved by sending different versions of the survey written in languages spoken at the students' homes.

Identify Biased Questions

When you are creating a survey, you must think very carefully about the questions you should ask, how many questions are appropriate and even the order in which the questions should be asked. A **biased question** is a question that is worded in such a way (whether intentional or not) that it causes a swing in the way people answer it. Biased questions can lead even a representative, non-biased population sample to answer in a way that does not accurately reflect the larger population.

While biased questions are a bad way to judge the overall mood of a population, they are sometimes used by politicians or advertising companies to falsely suggest that a product or policy is more or less popular than it really is.

Example C

There are several ways to spot biased questions:

- **They may use polarizing language, words and phrases that people associate with emotions:**
  - Is it right that farmers murder animals to feed people?
  - How much of your time do you waste on TV every week?
  - Should we be able to remove a person's freedom of choice over cigarette smoking?
- **They may refer to a majority or to a supposed authority:**
  - Would you agree with the American Heart and Lung Association that smoking is bad for your health?
  - The president believes that criminals should serve longer prison sentences. Do you agree?
  - Do you agree with 90% of the public that the car on the right looks better?
- **The question may be phrased so as to suggest the person asking the question already knows the answer:**
  - It's OK to smoke so long as you do it on your own, right?
  - You shouldn't be forced to give your money to the government, should you?
  - You wouldn't want criminals free to roam the streets, would you?
- **The question may be phrased in ambiguous way (often with double negatives) which may confuse people:**
  - Do you reject the possibility that the moon landings never took place?
  - Do you disagree with people who oppose the ban on smoking in public places?

In addition to biased questions, the overall design of a survey can be biased in other ways. In particular, **question order** can play a role. For example, a survey may contain several questions on people's attitudes to cigarette smoking. Then, if the question "What, in your opinion, are the three biggest threats to public health today?" is asked at the end of the survey, people will be more likely to give "smoking" as one of their answers than they would be if that question had been asked as part of a different survey, or if it had been placed at the beginning of this survey instead of at the end.

Watch this video for help with the Examples above.

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CK-12 Foundation: Sampling Methods

Vocabulary

- One method to collect information is to conduct a **census**. In a census, information is collected on all the members of the population of interest.
- A **sampling method** means that information is collected from a small sample that represents the population with which the study is concerned. The information from the sample is then extrapolated to the population—that is, we assume the results for the whole population would be about the same as the results for the sample.
- The word **population** in statistics means the group of people we wish to study, as opposed to the population at large.
- When we use sampling to conduct a statistical study, first we need to decide how to choose the sample population. It is essential that the sample is a **representative sample** of the population we are studying.
- **Random sampling** is a method that simply involves picking people at random from the population we wish to poll. However, this doesn't mean we can simply ask the first fifty people who walk by in the street. For instance, if you were conducting a survey on people's eating habits, you'd get different results if you were standing in front of a fast-food restaurant than if you were standing in front of a health food store. In a true random sample, everyone in the population must have the same chance of being chosen. Calling people on the phone, for example, might be a better way of getting a random sample for a survey about eating habits.
- **Stratified Sampling** is a method of sampling that actively seeks to poll people from many different backgrounds. The population is first divided into different categories (or **strata**) and the number of members in each category is determined. Gender and age groups are commonly used strata, but others could include salary, education level or even hair color. Then, a sample is made up by picking members from each category in the same proportion as they are in the population.
- If the sample we choose ends up with one or more sub-groups that are either over-represented or under-represented, then the sample is **biased**. The results of a **biased sample** might not really represent the entire population, so we want to avoid selecting one.
- Many surveys may have a **non-response bias**. For example, if researchers simply hand out questionnaires on a street corner and ask people to fill them out and then mail them in, most people will just throw the questionnaires away. Only people who are really interested in the subject will bother to send them in, and those might also be the people who are more likely to answer the questions a certain way.
- A **biased question** is a question that is worded in such a way (whether intentional or not) that it causes a swing in the way people answer it. Biased questions can lead even a representative, non-biased population sample to answer in a way that does not accurately reflect the larger population.

Guided Practice

Suppose you are interested in learning how popular the internet music program Spotify is at your school. You select a random sample of your friends. Is this sample likely to be representative of your school?

Solution:

By selecting a random sample of your friends, not everyone in your school has an equal chance to be selected, in fact, students who are not your friends do not have a chance of being selected at all. Therefore, this is not a random sample of students at your school. Your sample may be biased because your circle of friends is likely to represent similar interests, and not represent all interests of the students at the school. At best, this sample could represent how popular Spotify is with your friends.

Practice

For 1-6, comment on the way the following samples have been chosen. For the unsatisfactory cases, suggest a way to improve the sample choice.

1. You want to find whether wealthier people have more nutritious diets by interviewing people coming out of a five-star restaurant.
2. You want to find if there is there a pedestrian crossing needed at a certain intersection by interviewing people walking by that intersection.
3. You want to find out if women talk more than men by interviewing an equal number of men and women.
4. You want to find whether students in your school get too much homework by interviewing a stratified sample of students from each grade level.
5. You want to find out whether there should be more public busses running during rush hour by interviewing people getting off the bus.
6. You want to find out whether children should be allowed to listen to music while doing their homework by interviewing a stratified sample of male and female students in your school.

For 7-10, a university wants to know if its statistics course challenging enough for students. Every semester, the university offers several sections of the course. Explain the type(s) of bias most evident in each sampling technique and/or what sampling method is most evident. Be sure to justify your choice.

7. The first 30 students to buy the textbook at the beginning of the next semester.
8. The name of a color is selected at random, and on a given day, all statistics professors ask students wearing that color their opinion on the statistics course.
9. A flier is passed out on campus, asking students who have taken statistics at the university to reply by mail.
10. Five students are selected at random from each section of the statistics course during a given semester.
11. There are 35 students taking statistics in your school, and you want to choose 10 of them for a survey about their impressions of the course. Use your calculator to select a SRS of 10 students. (Seed your random number generator with the number 10 before starting.) Assuming the students are assigned numbers from 1 to 35, which students are chosen for the sample?
12. For a class assignment, you have been asked to find out how students get to school. Do they take public transportation, drive themselves, get a ride from their parents, carpool, walk, or bike? You decide to interview a sample of students. How will you choose those you wish to interview if you want your results to be reliable?

13.14 Planning and Conducting Surveys

Here you'll learn how to design a survey, conduct a survey, and graphically represent the results of that survey. You'll also learn how to analyze and interpret statistical survey data.

What if you polled the students in your class and asked them who their favorite math teacher is in your school? How could you graphically display the results? After completing this Concept, you'll be able conduct a survey like this, display your results in a pie or bar chart, and analyze and interpret the results.

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CK-12 Foundation: Designing,Conducting and Analyzing Surveys

Guidance

A survey is a way to ask a lot of people a few well-constructed questions. The survey is a series of unbiased questions that the subject must answer. Some advantages of surveys are that they are efficient ways of collecting information from a large number of people, they are relatively easy to administer, a wide variety of information can be collected and they can be focused (researchers can stick to just the questions that interest them.) Some disadvantages of surveys arise from the fact that they depend on the subjects' motivation, honesty, memory and ability to respond. Moreover, answer choices to survey questions could lead to vague data. For example, the choice "moderately agree" may mean different things to different people or to whoever ends up interpreting the data.

Conducting a Survey

There are various methods for administering a survey. It can be done as a face-to face interview or a phone interview where the researcher is questioning the subject. A different option is to have a self-administered survey where the subject can complete a survey on paper and mail it back, or complete the survey online. There are advantages and disadvantages to each of these methods.

The advantages of face-to-face interviews include fewer misunderstood questions, fewer incomplete responses, higher response rates, and greater control over the environment in which the survey is administered; also, the researcher can collect additional information if any of the respondents' answers need clarifying. The disadvantages of face-to-face interviews are that they can be expensive and time-consuming and may require a large staff of trained interviewers. In addition, the response can be biased by the appearance or attitude of the interviewer.

The advantages of self-administered surveys are that they are less expensive than interviews, do not require a large staff of experienced interviewers and can be administered in large numbers. In addition, anonymity and privacy encourage more candid and honest responses, and there is less pressure on respondents. The disadvantages of self-administered surveys are that responders are more likely to stop participating mid-way through the survey and respondents cannot ask them to clarify their answers. In addition, there are lower response rates than in personal interviews, and often the respondents who bother to return surveys represent extremes of the population – those people who care about the issue strongly, whichever way their opinion leans.

Designing a Survey

Surveys can take different forms. They can be used to ask only one question or they can ask a series of questions. We can use surveys to test out people's opinions or to test a hypothesis.

When designing a survey, the following steps are useful:

1. Determine the goal of your survey: What question do you want to answer?
2. Identify the sample population: Whom will you interview?
3. Choose an interviewing method: face-to-face interview, phone interview, self-administered paper survey, or internet survey.
4. Decide what questions you will ask in what order, and how to phrase them. (This is important if there is more than one piece of information you are looking for.)
5. Conduct the interview and collect the information.
6. Analyze the results by making graphs and drawing conclusions.

Example A

Martha wants to construct a survey that shows which sports students at her school like to play the most.

a) *List the goal of the survey.*

b) *What population sample should she interview?*

c) *How should she administer the survey?*

d) *Create a data collection sheet that she can use to record her results.*

Solution

a) The goal of the survey is to find the answer to the question: "Which sports do students at Martha's school like to play the most?"

b) A sample of the population would include a random sample of the student population in Martha's school. A good strategy would be to randomly select students (using dice or a random number generator) as they walk into an all-school assembly.

c) Face-to-face interviews are a good choice in this case. Interviews will be easy to conduct since the survey consists of only one question which can be quickly answered and recorded, and asking the question face to face will help eliminate non-response bias.

d) In order to collect the data to this simple survey Martha can design a data collection sheet such as the one below:

TABLE 13.17:

Sport	Tally
baseball	
basketball	
football	
soccer	
volleyball	
swimming	

This is a good, simple data collection sheet because:

- Plenty of space is left for the tally marks.
- Only one question is being asked.

- Many possibilities are included, but space is left at the bottom in case students give answers that Martha didn't think of.
- The answer from each interviewee can be quickly collected and then the data collector can move on to the next person.

Once the data has been collected, suitable graphs can be made to display the results.

Example B

Raoul wants to construct a survey that shows how many hours per week the average student at his school works.

a) *List the goal of the survey.*

b) *What population sample will he interview?*

c) *How would he administer the survey?*

d) *Create a data collection sheet that Raoul can use to record his results.*

Solution

a) The goal of the survey is to find the answer to the question "How many hours per week do you work?"

b) Raoul suspects that older students might work more hours per week than younger students. He decides that a stratified sample of the student population would be appropriate in this case. The strata are grade levels 9^{th} through 12^{th}. He would need to find out what proportion of the students in his school are in each grade level, and then include the same proportions in his sample.

c) Face-to-face interviews are a good choice in this case since the survey consists of two short questions which can be quickly answered and recorded.

d) In order to collect the data for this survey Raoul designed the data collection sheet shown below:

TABLE 13.18:

Grade Level	Number of Hours Worked	Total number of students
9^{th} grade		
10^{th} grade		
11^{th} grade		
12^{th} grade		

This data collection sheet allows Raoul to write down the actual numbers of hours worked per week by students as opposed to just collecting tally marks for several categories.

Display, Analyze, and Interpret Statistical Survey Data

In the previous section we considered two examples of surveys you might conduct in your school. The first one was designed to find the sport that students like to play the most. The second survey was designed to find out how many hours per week students worked.

For the first survey, students' choices fit neatly into separate categories. Appropriate ways to display the data might be a pie chart or a bar graph. Let's revisit this example.

Example C

In Example A Martha interviewed 112 students and obtained the following results.

TABLE 13.19:

Sport	Tally	
baseball	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 |	31
basketball	𝍸 𝍸 𝍸 ||	17
football	𝍸 𝍸 ||||	14
soccer	𝍸 𝍸 𝍸 𝍸 𝍸 |||	28
volleyball	𝍸 ||||	9
swimming	𝍸 |||	8
gymnastics	|||	3
fencing	||	2
		Total: 112

a) *Make a bar graph of the results showing the percentage of students in each category.*

b) *Make a pie chart of the collected information, showing the percentage of students in each category.*

Solution

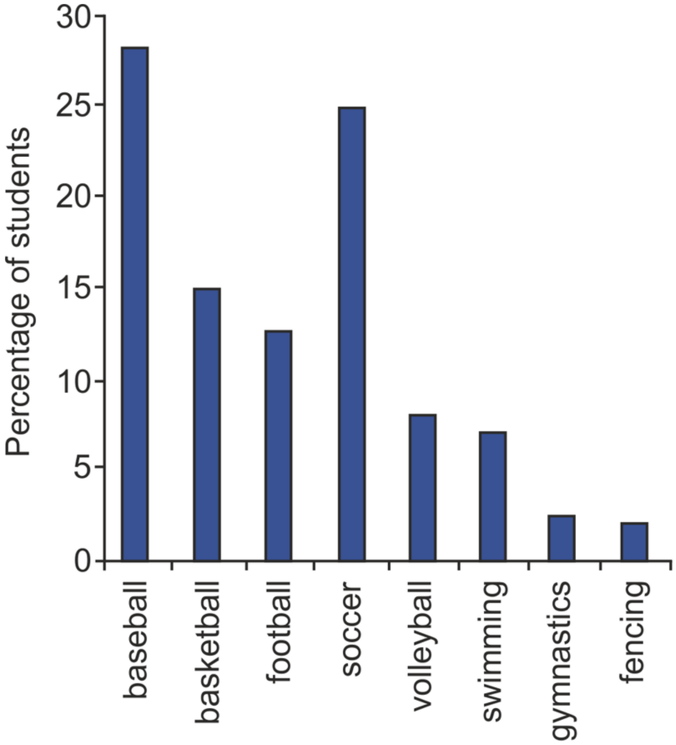
a) To make a bar graph, we list the sport categories on the $x-$axis and let the percentage of students be represented by the $y-$axis.

To find the percentage of students in each category, we divide the number of students in each category by the total number of students surveyed:

TABLE 13.20:

Sport	Percentage
baseball	$\frac{31}{112} = .28 = 28\%$
basketball	$\frac{17}{112} = .15 = 15\%$
football	$\frac{14}{112} = .125 = 12.5\%$
soccer	$\frac{28}{112} = .25 = 25\%$
volleyball	$\frac{9}{112} = .08 = 8\%$
swimming	$\frac{8}{112} = .07 = 7\%$
gymnastic	$\frac{3}{112} = .025 = 2.5\%$
fencing	$\frac{2}{112} = .02 = 2\%$

Now we can make a graph where the height of each bar represents the percentage of students in each category:

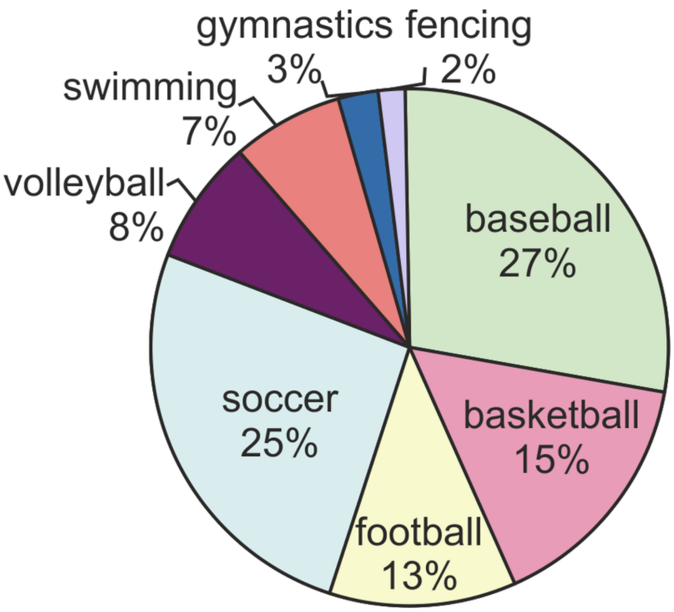


b. To make a pie chart, we find the percentage of the students in each category by dividing the number of students in each category as in part a. The central angle of each slice of the pie is found by multiplying the percentage of students in each category by 360 degrees (the total number of degrees in a circle). To draw a pie-chart by hand, you can use a protractor to measure the central angles that you find for each category.

TABLE 13.21:

Sport	Percentage	Central angle
baseball	$\frac{31}{112} = .28 = 28\%$	$.28 \times 360^\circ = 101^\circ$
basketball	$\frac{17}{112} = .15 = 15\%$	$.15 \times 360^\circ = 54^\circ$
football	$\frac{14}{112} = .125 = 12.5\%$	$.125 \times 360^\circ = 45^\circ$
soccer	$\frac{28}{112} = .25 = 25\%$	$.25 \times 360^\circ = 90^\circ$
volleyball	$\frac{9}{112} = .08 = 8\%$	$.08 \times 360^\circ = 29^\circ$
swimming	$\frac{8}{112} = .07 = 7\%$	$.07 \times 360^\circ = 25^\circ$
gymnastics	$\frac{3}{112} = .025 = 2.5\%$	$.025 \times 360^\circ = 9^\circ$
fencing	$\frac{2}{112} = .02 = 2\%$	$.02 \times 360^\circ = 7^\circ$

Here is the pie-chart that represents the percentage of students in each category:



For the second survey, actual numerical data can be collected from each student. In this case we can display the data using a stem-and-leaf plot, a frequency table, a histogram, or a box-and-whisker plot.

Watch this video for help with the Examples above.

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CK-12 Foundation: Designing,Conducting and Analyzing Surveys

Vocabulary

- A **stem-and-leaf plot** consists of a vertical "stem" containing the first digit of each number, with the rest of each number written to the right of the stem like a "leaf."
- A **frequency table** is one that records the frequency of each value in a data set.
- When the data we are representing falls into well defined categories (such as the integers 1, 2, 3, 4, 5 6) it is more appropriate to use a **histogram** to display that data. A histogram is a graph of a frequency table, where the frequency of each value is represented by the height of a bar.
- A **box-and-whisker plot** is formed by placing vertical lines at five positions, corresponding to the smallest value, the first quartile, the median, the third quartile and the greatest value. (These five numbers are often referred to as the **five number summary**.) A **box** is drawn between the position of the first and third quartiles, and horizontal line segments (the **whiskers**) connect the box with the two extreme values.
- A **pie chart** is a circular graph where the proportion of data in each category is represented by an equivalent proportion of the circle.

Guided Practice

In Example B Raoul found that that 30% of the students at his school are in 9^{th} grade, 26% of the students are in the 10^{th} grade, 24% of the students are in 11^{th} grade and 20% of the students are in the 12^{th} grade. He surveyed a total of 60 students using these proportions as a guide for the number of students he interviewed from each grade. Raoul recorded the following data:

TABLE 13.22:

Grade Level	Number of hours worked	Total number of students
9^{th} grade	0, 5, 4, 0, 0, 10, 5, 6, 0, 0, 2, 4, 0, 8, 0, 5, 7, 0	18
10^{th} grade	6, 10, 12, 0, 10, 15, 0, 0, 8, 5, 0, 7, 10, 12, 0, 0	16
11^{th} grade	0, 12, 15, 18, 10, 0, 0, 20, 8, 15, 10, 15, 0, 5	14
12^{th} grade	22, 15, 12, 15, 10, 0, 18, 20, 10, 0, 12, 16	12

a) *Construct a stem-and-leaf plot of the collected data.*

b) *Construct a frequency table with bin size of 5.*

c) *Draw a histogram of the data.*

d) *Find the five number summary of the data and draw a box-and-whisker plot.*

Solution

a) The ordered stem-and-leaf plot looks as follows:

0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 2 4 4 5 5 5 5 5 6 6 7 7 8 8 8
1	0 0 0 0 0 0 0 0 2 2 2 2 2 5 5 5 5 5 5 6 8 8
2	0 0 2

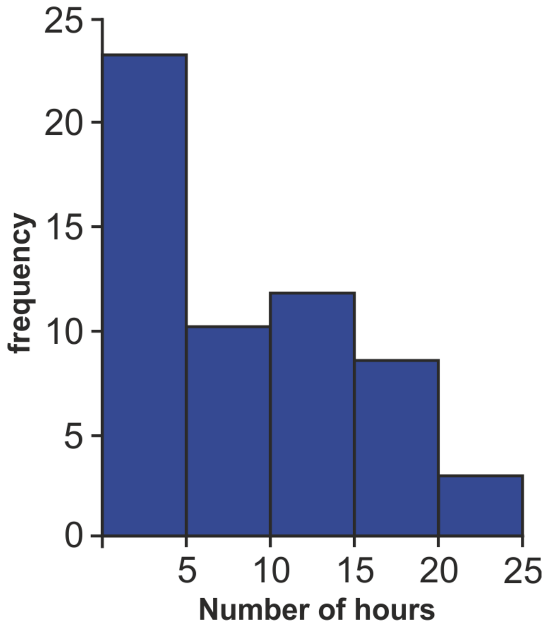
We can easily see from the stem-and-leaf plot that the mode of the data is 0. This makes sense because many students do not work in high school.

b) We construct the frequency table by counting how many students fit in each category.

TABLE 13.23:

Hours worked	Frequency
$0 \leq x < 5$	23
$5 \leq x < 10$	12
$10 \leq x < 15$	13
$15 \leq x < 20$	9
$20 \leq x < 25$	3

c) The histogram associated with this frequency table is shown below.



d) The five number summary is as follows:

smallest number = 0

largest number = 22

Since there are 60 data points, $\left(\frac{n+1}{2}\right) = 30.5$. The median is the mean of the 30^{th} and the 31^{st} values:

median = 6.5

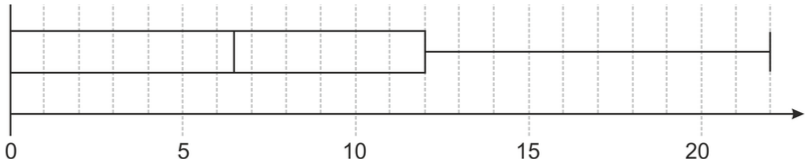
Since each half of the list has 30 values in it, then the first and third quartiles are the medians of each of the smaller lists. The first quartile is the mean of the 15^{th} and 16^{th} values:

first quartile = 0

The third quartile is the mean of the 45^{th} and 46^{th} values:

third quartile = 12

The associated box-and-whisker plot is shown below.



Practice

1. Make a pie chart for the problem in the Guided Practice.
2. Melissa conducted a survey to answer the question "What sport do high school students like to watch on TV the most?" She collected the following information on her data collection sheet.

TABLE 13.24:

Sport	**Tally**	
baseball	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 ||	32
basketball	𝍸 𝍸 𝍸 𝍸 𝍸 |||	28
football	𝍸 𝍸 𝍸 𝍸 ||||	24
soccer	𝍸 𝍸 𝍸 |||	18
gymnastics	𝍸 𝍸 𝍸 ||||	19
figure skating	𝍸 |||	8
hockey	𝍸 𝍸 𝍸 |||	18
	Total:	147

a) Make a pie-chart of the results showing the percentage of people in each category.

b) Make a bar-graph of the results.

3. Samuel conducted a survey to answer the following question: "What is the favorite kind of pie of the people living in my town?" By standing in front of his grocery store, he collected the following information on his data collection sheet:

TABLE 13.25:

Type of pie	**Tally**	
apple	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 ||	37
pumpkin	𝍸 𝍸 |||	13
lemon meringue	𝍸 ||	7
chocolate mousse	𝍸 𝍸 𝍸 𝍸 |||	23
cherry	||||	4

TABLE 13.25: (continued)

Type of pie	Tally	
chicken pot pie	𝍸𝍸𝍸𝍸𝍸𝍸 |	31
other	𝍸 ||	7
	Total:	122

a) Make a pie chart of the results showing the percentage of people in each category.

b) Make a bar graph of the results.

4. Myra conducted a survey of people at her school to see "In which month does a person's birthday fall?" She collected the following information in her data collection sheet:

TABLE 13.26:

Month	Tally	
January	𝍸𝍸𝍸 |	16
February	𝍸𝍸 |||	13
March	𝍸𝍸 ||	12
April	𝍸𝍸 |	11
May	𝍸𝍸 |||	13
June	𝍸𝍸 ||	12
July	𝍸 ||||	9
August	𝍸 ||	7
September	𝍸 ||||	9
October	𝍸 |||	8
November	𝍸𝍸 |||	13

TABLE 13.26: (continued)

Month	**Tally**	
December	𝍸 𝍸 |||	13
	Total:	136

a) Make a pie chart of the results showing the percentage of people whose birthday falls in each month.

b) Make a bar graph of the results.

5. Nam-Ling conducted a survey that answers the question “Which student would you vote for in your school’s elections?” She collected the following information:

TABLE 13.27:

Candidate	9^{th} **graders**	10^{th} **graders**	11^{th} **graders**	12^{th} **graders**	**Total**
Susan Cho	𝍸 𝍸	||			𝍸		19
Margarita Martinez	𝍸 ||	𝍸 𝍸 𝍸 𝍸	||||		31
Steve Coogan	𝍸	||||	||	𝍸	16
Solomon Duning	𝍸				𝍸 𝍸 𝍸 |||			26
Juan Rios	𝍸 |||	|||	𝍸	𝍸 𝍸 ||	28
Total	36	30	30	24	120

a) Make a pie chart of the results showing the percentage of people planning to vote for each candidate.

b) Make a bar graph of the results.

6. Graham conducted a survey to find how many hours of TV teenagers watch each week in the United States. He collaborated with three friends that lived in different parts of the US and found the following information:

TABLE 13.28:

Part of the country	**Number of hours of TV watched per week**	**Total number of teens**
West Coast	10, 12, 8, 20, 6, 0, 15, 18, 12, 22, 9, 5, 16, 12, 10, 18, 10, 20, 24, 8	20
Mid West	20, 12, 24, 10, 8, 26, 34, 15, 18, 6, 22, 16, 10, 20, 15, 25, 32, 12, 18, 22	20
New England	16, 9, 12, 0, 6, 10, 15, 24, 20, 30, 15, 10, 12, 8, 28, 32, 24, 12, 10, 10	20

TABLE 13.28: (continued)

Part of the country	Number of hours of TV watched per week	Total number of teens
South	24, 22, 12, 32, 30, 20, 25, 15, 10, 14, 10, 12, 24, 28, 32, 38, 20, 25, 15, 12	20

a) Make a stem-and-leaf plot of the data.

b) Decide on an appropriate bin size and construct a frequency table.

c) Make a histogram of the results.

d) Find the five-number summary of the data and construct a box-and-whisker plot.

In exercises 7-10, consider the following survey questions.

7. “What do students in your high-school like to spend their money on?”
   a. Which categories would you include on your data collection sheet?
   b. Design the data collection sheet that can be used to collect this information.
   c. Conduct the survey. This activity is best done as a group with each person contributing at least 20 results.
   d. Make a pie chart of the results showing the percentage of people in each category.
   e. Make a bar graph of the results.
8. “What is the height of students in your class?”
   a. Design the data collection sheet that can be used to collect this information.
   b. Conduct the survey. This activity is best done as a group with each person contributing at least 20 results.
   c. Make a stem-and-leaf plot of the data.
   d. Decide on an appropriate bin size and construct a frequency table.
   e. Make a histogram of the results.
   f. Find the five-number summary of the data and construct a box-and-whisker plot.
9. “How much allowance money do students in your school get per week?”
   a. Design the data collection sheet that can be used to collect this information,
   b. Conduct the survey. This activity is best done as a group with each person contributing at least 20 results.
   c. Make a stem-and-leaf plot of the data.
   d. Decide on an appropriate bin size and construct a frequency table.
   e. Make a histogram of the results.
   f. Find the five-number summary of the data and construct a box-and-whisker plot.
10. “What time do students in your school get up in the morning during the school week?”
    a. Design the data collection sheet that can be used to collect this information.
    b. Conduct the survey. This activity is best done as a group with each person contributing at least 20 results.
    c. Make a stem-and-leaf plot of the data.
    d. Decide on an appropriate bin size and construct a frequency table.
    e. Make a histogram of the results.
    f. Find the five-number summary of the data and construct a box-and-whisker plot.

Summary

This chapter begins by differentiating between theoretical and experimental probabilities. It then moves into ways of arranging a subset of a set of items called permutations and combinations. The probabilities of such arrangements are also covered. Next, it distinguishes between mutually exclusive and independent events. From there, the chapter discusses ways of measuring a data set, called measures of central tendency and dispersion. Such measures include the mean, median, mode, range, variance, and standard deviation. Finally, the chapter concludes with methods of graphically representing a data set and statistical means of collecting and analyzing data.