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CK-12 Foundation and Leadership Public Schools, College Access Reader: Geometry (Being Reviewed)

CK-12 Foundation and Leadership Public Schools, College Access Reader: Geometry

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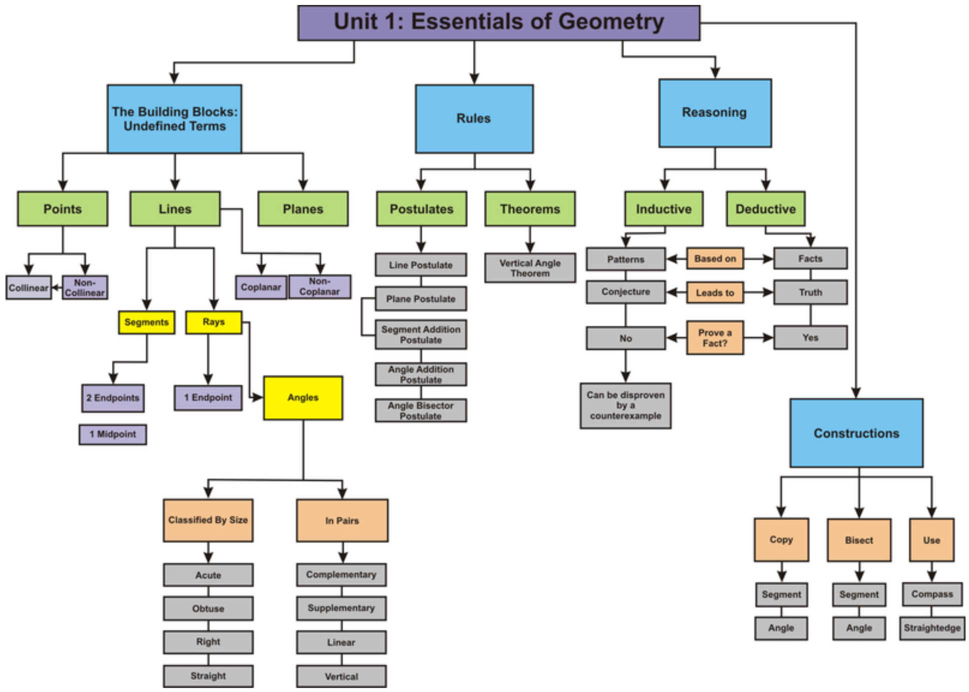
1.1 Concept Map



1.2 Vocabulary Self-Rating

TABLE 1.1: Rating Guide: DK: I am sure I don't know it K: I am sure I know it ? I'm not sure

Word	Before Lesson/Unit	Definition	After Lesson/Unit
Point			
Line			
Plane			
Dimension			
Space			
Collinear			
Coplanar			
Segment			
Endpoint			
Ray			
Intersection			
Postulate			
Theorem			
Congruent			
Midpoint			
Bisector			
Construction			
Compass			
Straightedge			
Angle			
Vertex			
Side			
Right angle			
Perpendicular			
Acute			
Obtuse			
Straight Angle			
Complementary angles			
Supplementary angles			
Linear pair			
Vertical angles			
Polygon			
Concave			
Convex			
Diagonals			
Equiangular			
Equilateral			
Regular			
Inductive reasoning			
Deductive reasoning (logic)			
Conditional/if-then statements			

TABLE 1.1: (continued)

Word	Before Lesson/Unit	Definition	After Lesson/Unit
Hypothesis			
Conclusion			
Conjecture			
Counterexample			

1.3 Word Part Log

TABLE 1.2:

Word Part	Meaning	Examples
Prefixes		
Bi-	*Two*	*Bisect (cut in 2 pieces)* *Bicycle (2 wheels)*
Trans-	*Across*	*Transversal (a line that cuts across)* *Transport (to carry across)*
Roots		
Scribe	*To write*	*Inscribe (drawn inside)* *Scribble (write messily)*
Sect	*To cut*	*Bisect (cut in two)* *Dissect (cut open)*
Suffixes		
-ion	*Result of*	*Transformation (result of being transformed)* *Education (result of being educated)*
-oid	*Resembling*	*Spheroid (resembles a sphere)* *Humanoid (resembles a human)*

1.4 Introduction

Welcome to the exciting world of geometry! Ahead of you lie many exciting discoveries that will help you learn more about the world.

Geometry is used in many areas—from art to science. For example, geometry plays a key role in construction, fashion design, architecture, and computer graphics. This course focuses on the main ideas of geometry that are the foundation of applications of geometry used everywhere.

In this chapter, you will study the basic elements of geometry. Later you will prove things about geometric shapes using the vocabulary and ideas in this chapter—so make sure that you completely understand each of the concepts presented here before moving on.

1.5 Points, Lines, and Planes

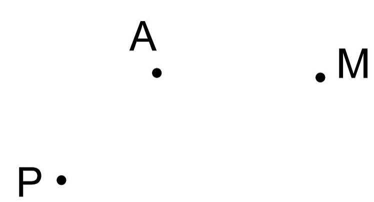
Learning Objectives

- Identify the undefined terms *point, line,* and *plane*.
- Name points, lines, and planes
- State important characteristics of points, lines, and planes

Undefined Terms: Points, Lines, and Planes

The three basic building blocks of geometry are **points, lines** and **planes**. These are **undefined terms**. While we cannot define these terms precisely, we can get an idea of what they are by looking at examples and models.

A **point** is a location that has no size. We use dots to represent points. Points are named and labeled with a single capital letter, as shown in the image below:



- A **point** is named and ______________________________ with a single capital letter.

A **line** is an infinite series of points in a row. It is a one-dimensional object. A line has direction and location, but still does not take up space. Lines are sometimes referred to by one italicized letter, but they can also be identified by two points that are on the line.

In the image below, the same line has several names. It can be called "line g".$\overleftrightarrow{PQ}$, or $\overleftrightarrow{QP}$:

- A **line** is a ______________-dimensional object that extends infinitely.

What does infinite mean? Let's break it down.

The prefix "in-" means "not". So something that is infinite is "not" finite.

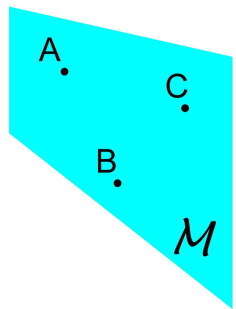
Finite means having an end. It is related to the word "final."

So something that is infinite does "not" "have an end." Something that is infinite goes on and on forever and ever.

- The word "*infinite*" means "does not have an ____________________."

A **plane** is a surface which has infinite width and length. It is a two-dimensional object. Planes are named using a single capital letter or by naming three points contained in the plane. You already know one plane from your algebra class—the xy coordinate plane.

The plane below can be called plane M or "the plane defined by points A, B, and C."

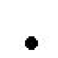
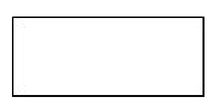
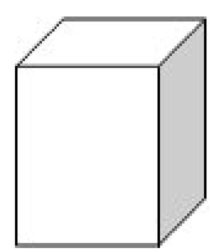


- A **plane** is a two-__________________________________ object that has infinite length and width.

Dimension

An object's **dimensions** refer to how the object extends into space. In geometry, we are going to focus on four types of objects:

TABLE 1.3:

Zero dimensions:	**One dimension:**	**Two dimensions:**	**Three dimensions:**
Does not extend into space. In other words, it extends in zero dimensions	Extends into space in only one way. For example, it has a length.	Extends into space in two ways. For example, it has a length **and** a width.	Extends into space in three ways. For example, it has a length, a width, and a height.
			

Reading Check:

1. *What are the three undefined terms in Geometry?*

____

____

____

2. *Draw a line that could be named "line GJ."*

3. *Which undefined term in Geometry has two dimensions?*

____

4. *What does the word "infinite" mean in your own words?*

5. *Fill in the blanks:*

A point has ________ dimensions, a line has ________ dimension, and a plane has ________ dimensions.

Graphic Organizer for Lesson 2

TABLE 1.4: Undefined Terms

	What is it?	**How many dimensions?**	**Draw a picture.**	**How can you name it?**	**Give a real-life example.**
Point					
Line					
Plane					

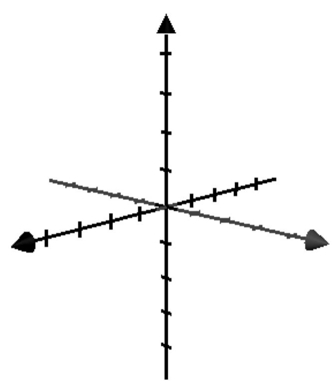
1.6 Defined Terms

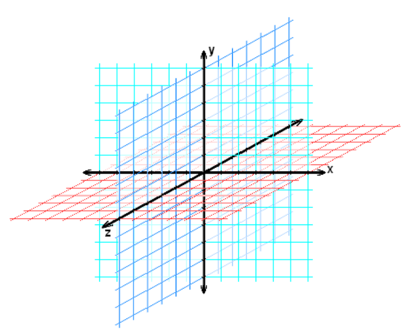
Learning Objectives

- Identify points/lines that are *collinear, coplanar, non-collinear,* and *non-coplanar*.
- Define and name *segments* and *rays*

Defined Terms: Points, Lines, and Planes

Now we can use **point, line,** and **plane** to define new terms. One word that has already been used is **space**. **Space** is the set of all points expanding in three dimensions.





- **Space** is the set of all ______________________________ expanding infinitely in three dimensions.

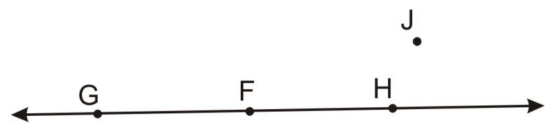
The prefix "co-" means "with".

Some other words that use "co-" are: co-pilot (pilot something ***with*** *someone else), cooperate (operate* ***with*** *someone else), and co-exist (exist* ***with*** *someone else).*

Can you think of other words that use the prefix "co-"?

Collinear and Non-Collinear

Points are said to be **collinear** if they lie along the same line. The picture below shows points F, G, and H are collinear. Point J is **non-collinear** with the other three since it does not lie in the same line:

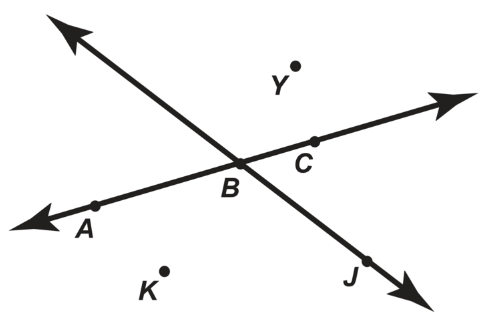


- **Collinear** points lie on the ________________________ line.
- **Non-collinear** points do not lie on the same ________________________.

Reading Check:

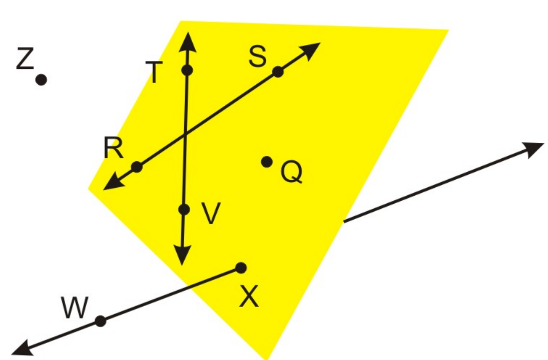
1. Which points are collinear in the picture below?

2. Which points are non-collinear?



Coplanar and Non-Coplanar

Similarly, points and lines can be **coplanar** if they lie within the same plane. The diagram below shows two lines ($\overleftrightarrow{RS}$ and $\overleftrightarrow{TV}$) and one point (Q) that are **coplanar**. It also shows line $\overleftrightarrow{WX}$ and point Z that are non-coplanar with $\overleftrightarrow{RS}$ and Q:



- Points and lines are **coplanar** when they lie in the same ____________________.

Reading Check:

1. *Look around the space you are in right now.*

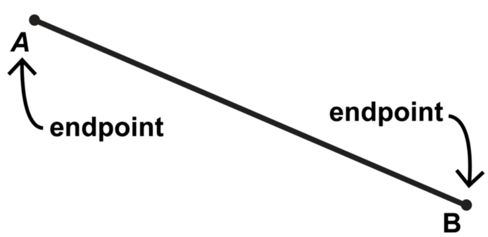
What planes do you see? What points lie in those planes?

2. *The ground or floor underneath you is a plane.*

What is a point that is non-coplanar with the ground or the floor?

Segments, Rays, and Endpoints

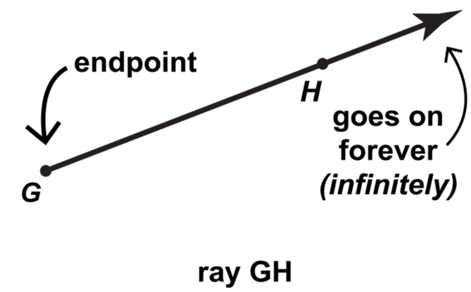
A **segment** designates a portion of a line that has two *endpoints*. Segments are named by their endpoints. Segments can also be named in any order, so the segment below could be named $\overline{AB}$ or $\overline{BA}$:



line segment AB or line segment BA

- A line **segment** has two ______________________________, which is how the segment is named.

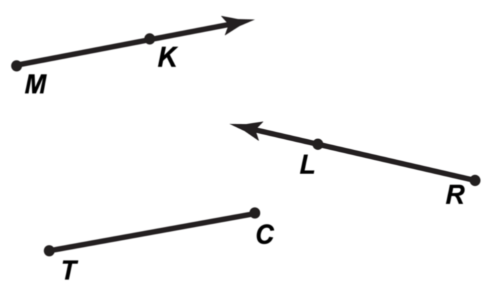
A **ray** is a portion of a line that has only one endpoint and extends *infinitely* in the other direction. Rays are named by their endpoints and another point on the line. The endpoint always comes *first* in the name of a ray, so we would write $\overrightarrow{GH}$ for the figure below:



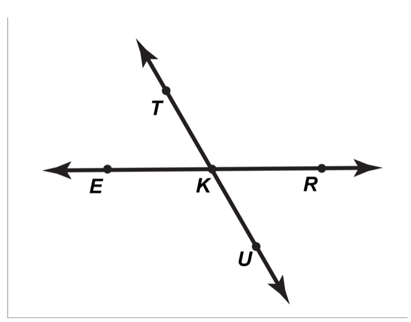
- A **ray** has one ______________________ and extends ________________________ in the other direction.
- A **ray** is named by two points on it, with the endpoint ________________________.

Reading Check

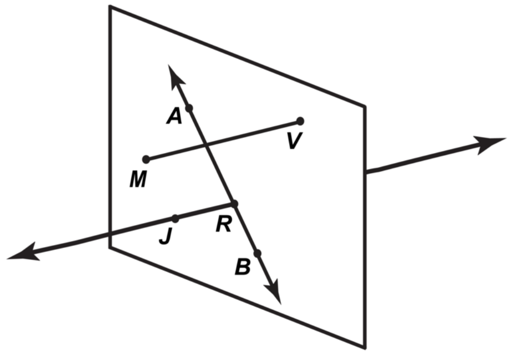
1. *Name each of the following figures:*



2. *Which points are collinear with point T in the figure below?*



3. *Which lines, segments, or rays are coplanar with line $\overleftrightarrow{AB}$ in the figure below?*


A
V
M
J
R
B

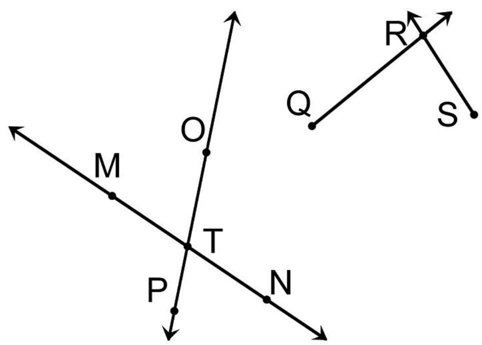
1.7 How Lines and Planes Intersect

Learning Objectives

- Explain how lines, planes, and a line a plane intersect.
- Explain the difference between *postulates* and *theorems*.
- Identify and apply basic postulates of points, lines, and planes.

Intersection

An **intersection** is the point or set of points where lines, planes, segments, or rays *cross* each other.



In the image above, R is the point of intersection of $\overrightarrow{QR}$ and $\overrightarrow{SR}$. T is the intersection of $\overleftrightarrow{MN}$ and $\overleftrightarrow{PO}$.

- An **intersection** is a point where objects ____________________ each other.

Basic Postulates and Theorems

Now that we have some basic vocabulary, we can talk about the rules of geometry. Logical systems like geometry start with basic rules, and we call these basic rules **postulates**. We *assume* that a postulate is *true* and by definition a **postulate** is a statement that does not have to be proven.

- A **postulate** is a statement that does not have to be __________________________.

A **theorem** is a statement that can be proven true using postulates, definitions, logic, and other theorems we have already proven.

- A **theorem** is a statement that must be ____________________________.

This section introduces a few basic postulates that you must understand as you move on to learn other theorems. Some postulates and theorems have names; others do not.

Reading Check:

1. *True or False:* When a line and a plane intersect, they meet at a point.

Why did you choose True or False? Explain your reasoning:

2. *True or False:* It is possible to have a plane with just three collinear points.

Why did you choose True or False? Explain your reasoning:

3. *Fill in the blanks:*

A ___________________________ is a statement that is accepted without proof.

Graphic Organizers for Lesson 4

TABLE 1.5: Postulate or Theorem

Postulate	Theorem
Description • Assumed to be true • Doesn't have to be proven	Description • Can't assume that it's true • Must be proven
Examples • A person can only be in one place at one time. • A fish cannot live without water. • A line must contain at least two points.	Examples • Lisa was at home at 8:00 last night. • Henry drinks eight glasses of water each day. • The angles in a triangle add up to 180°.

TABLE 1.5: (continued)

Postulate	Theorem
Add your own examples here. What are some things that are so obvious that they don't have to be proven? • ______________________________ ______________________________ ______________________________ • ______________________________ ______________________________ ______________________________ • ______________________________ ______________________________ ______________________________	Add your own examples here. What are some things you know are true, but you'd have to prove them to someone else? • ______________________________ ______________________________ ______________________________ • ______________________________ ______________________________ ______________________________ • ______________________________ ______________________________ ______________________________

TABLE 1.6:

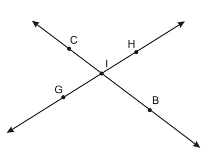
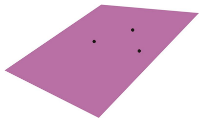
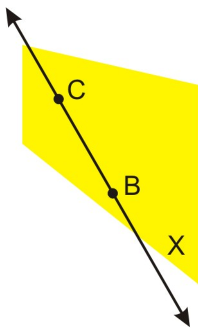
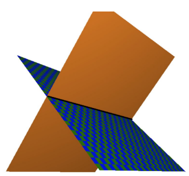
Line Postulate and Plane Postulates	Try to disprove it with a picture. You can't do it!
Line Postulate: There is exactly one line through any two points. 	
Postulate: Any line contains at least two points. 	
Postulate: The intersection of any two distinct lines will be a single point. 	

TABLE 1.6: (continued)

Line Postulate and Plane Postulates	Try to disprove it with a picture. You can't do it!
Plane Postulate: There is exactly one plane that contains any three non-collinear points.  **Postulate:** Any plane contains at least three non-collinear points. **Postulate:** A line connecting points in a plane also lies within the plane.  **Postulate:** The intersection of two planes is a line. 	

1.8 Congruent Segments, Midpoints, and Bisectors

Learning Objectives

- Define *midpoints, congruent,* and *bisectors*.
- Use the *Segment Addition Postulate* to find lengths of segments.
- Use midpoints to find the lengths of segments.

Congruent Line Segments

One of the most important words in geometry is **congruent**. This term refers to geometric objects that have exactly the same size and shape. Two segments are **congruent** if they have the same length.

- **Congruent** objects have the _______________________ size and shape.

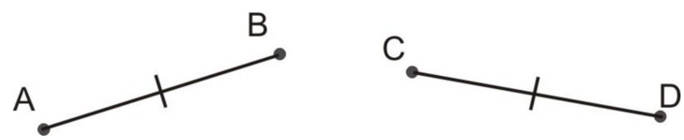
Notation Notes:

1. When two things are **congruent** we use the symbol $\cong$.

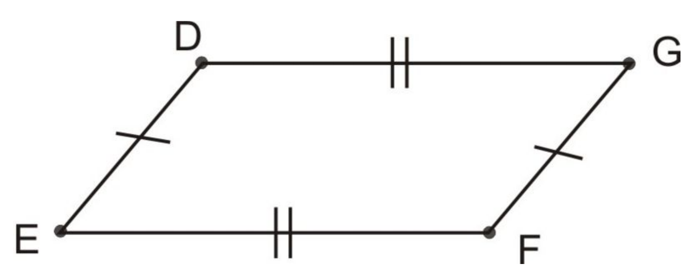
For example if $\overline{AB}$ is congruent to $\overline{CD}$, then we would write $\overline{AB} \cong \overline{CD}$.

2. When we draw congruent segments, we use *tic marks* to show that two segments are congruent.

3. If there are multiple pairs of congruent segments (which are not congruent to each other) in the same picture, use two tic marks for the second set of congruent segments, three for the third set, and so on. See the two following illustrations:



$\overline{AB}$ and $\overline{CD}$ are **congruent** line segments.



$\overline{ED}$ and $\overline{FG}$ are one set of **congruent** line segments, and $\overline{DG}$ and $\overline{EF}$ are another set of **congruent** line segments.

- The geometric symbol for **congruent** is ____________.

The length of segment $\overline{AB}$ can be written in two ways: $m\overline{AB}$ or simply AB. This might be a little confusing at first, but it will make sense the more you use this notation.

Let's say we used a ruler and measured $\overline{AB}$ and we saw that it had a length of 5 cm. Then we could write $m\overline{AB} = 5\ cm$, or $AB = 5\ cm$.

If we know that $\overline{AB} \cong \overline{CD}$, then we can write $m\overline{AB} = m\overline{CD}$ or simply $AB = CD$.

Reading Check:

Line segment $\overline{CD}$ has the same length as line segment $\overline{RQ}$.

How could you write this statement three different ways?

____

____

____

Segment Midpoints

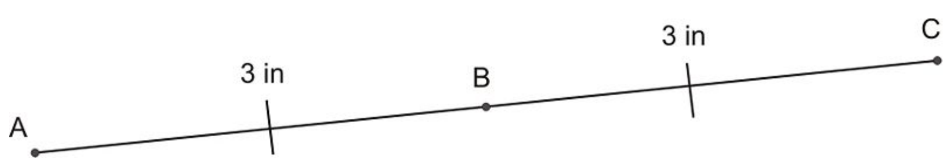
Now that you understand congruent segments, there are a number of new terms and types of figures you can explore.

A **segment midpoint** is a point on a line segment that divides the segment into two **congruent** segments. So, each segment between the midpoint and an endpoint will have the *same* length.

- A **segment midpoint** splits a segment into two ____________________ parts.

In the diagram below,

point B is the **midpoint** of segment $\overline{AC}$ since $\overline{AB}$ is **congruent** to $\overline{BC}$:



There is even a special *postulate* dedicated to **midpoints**.

Segment Midpoint Postulate

Any line segment will have exactly one midpoint—no more, and no less.

Reading Check:

1. *Why can there only be one midpoint?*

2. *How would you explain your answer above to a person who thought there could be three midpoints on one line segment?*

Because a segment midpoint divides the segment into two **congruent** segments, you can use the **midpoint** to find the *measure* of a line segment.

In the picture below, point M is the **midpoint** of segment $\overline{LN}$.

If $\overline{LM} = 4\ cm$, then $\overline{MN}$ must also be equal to 4 cm, because the **midpoint** divides the segment into two **congruent** segments.

What would be the length of the entire segment $\overline{LN}$? The entire segment is equal to 8 cm, because each *half* is equal to 4 cm, and $4\ cm + 4\ cm = 8\ cm$.

If $\overline{LM} =$ ______________ and $\overline{MN} =$ ______________ then $\overline{LN} =$ ______________.

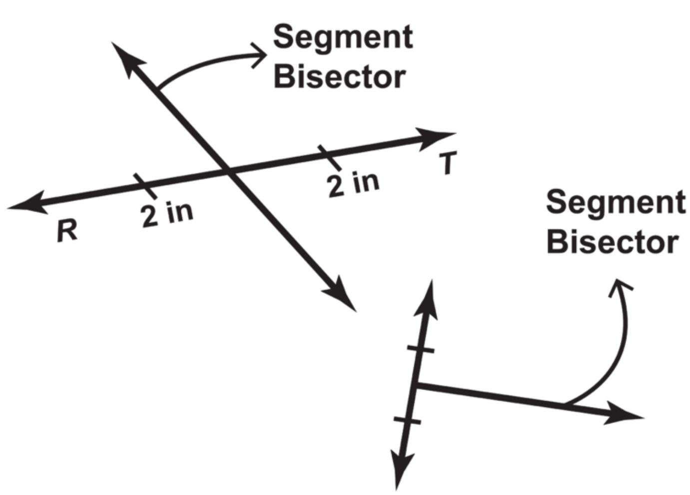
Segment Bisectors

Now that you know how to find midpoints of line segments, you can explore **segment bisectors.**

A **bisector** is a line, segment, or ray that passes through a **midpoint** of another segment.

You probably know that the prefix "*bi*-" means *two* (think about the two wheels of a **bi**cycle). So, a **bisector** cuts a line segment into two *congruent* parts.

- A **bisector** cuts a segment into two ___________________________ parts.
- A **bisector** passes through a line segment's ___________________________.

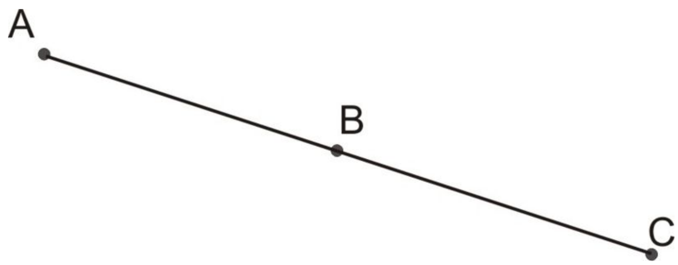


Segment Addition Postulate

The measure of any line segment can be found by *adding* the measures of the smaller segments that comprise it.

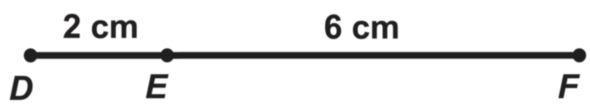
That may seem like a lot of confusing words, but the logic is quite simple:

In the diagram below, if you add the lengths of $\overline{AB}$ and $\overline{BC}$, you will have found the length of $\overline{AC}$. In geometric symbols: $AB + BC = BC$.



- You can ________________ together all parts of a line segment to get the length of the entire segment.

For example, in the picture below, $m\overline{DE} = 2\ cm$ and $m\overline{EF} = 6\ cm$. What is the total length of line segment $\overline{DF}$?



Because of the **Segment** __________________________ **Postulate**, you can simply add the two parts of the segment together to get the total length of the line segment.

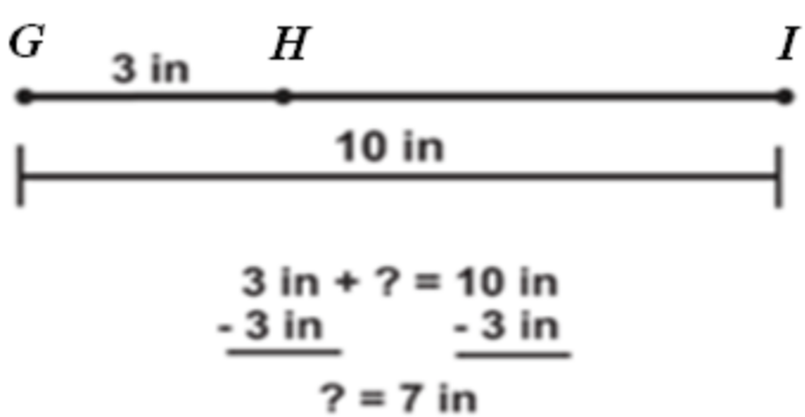
The length of line segment $\overline{DF}$ is 8 cm, because $2\ cm + 6\ cm = 8\ cm$.

Is E the **midpoint** of line segment $\overline{DF}$?

It is *not* the midpoint of $\overline{DF}$ because it does not divide the line segment into two congruent parts. $\overline{DE}$ is shorter than $\overline{EF}$, so $\overline{DE}$ and $\overline{EF}$ are *not* congruent.

You can also use the **Segment Addition Postulate** to find missing measures of line segments within a larger line segment.

In the example below, $m\overline{GH} = 3\ in.$ and $m\overline{GI} = 10\ in.$ You can write an equation to help you find the length of $\overline{HI}$:



You can subtract GH from GI. Since $GI =$ _______ and $GH =$ _______, $GI - GH = 10 - 3 = 7$

So the length of $\overline{HI} = 7$ inches. You will also notice that $3\ in. + 7\ in. = 10\ in.$

Reading Check:

1. *Is C the midpoint of line segment $\overline{XY}$ below?* ____________________

Why or why not? Explain.

2. *Use the Segment Addition Postulate to find the length of segment $\overline{XY}$ in the figure above.*

1.9 Constructions and Copying a Line Segment

Learning Objectives

- Define geometric construction
- Copy a given line segment using only a compass and a straightedge

Introduction to Constructions: Compass and Straightedge

The word **construction** in geometry has a very specific meaning: the drawing of geometric items such as lines and circles using only a **compass** and **straightedge**. A **compass** is a drawing tool that can be used to measure distances and draw arcs and circles. An **arc** is a piece of a circle.

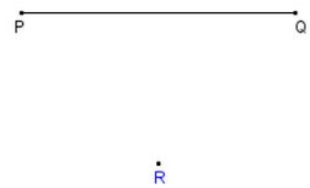
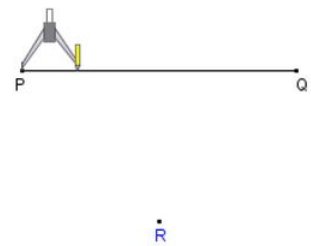
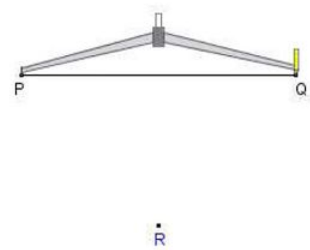
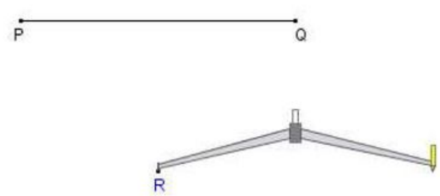
We make **constructions** in geometry using a **compass** and a _________________________.

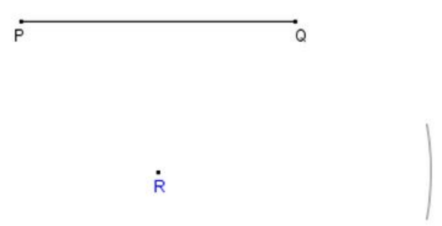
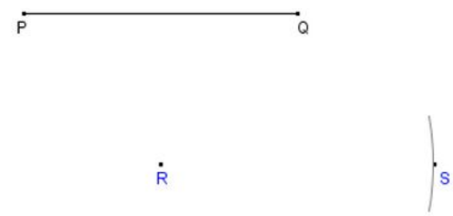
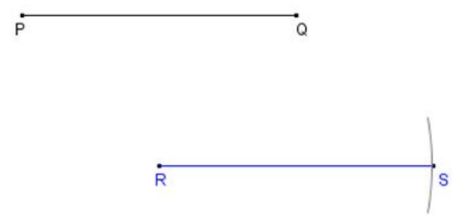
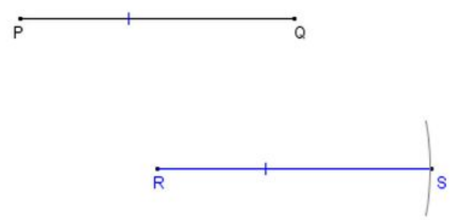
Compasses come in different forms. These are some examples and drawings of compasses. You can also see an arc in the last two pictures:

A **straightedge** can be anything with straight edge, like an index card. The important thing to remember with constructions is that you are *not* allowed to measure angles with a protractor or measure lengths with a ruler.

Construction: How to Copy a Line Segment

http://www.mathopenref.com/constcopysegment.html

	After doing this	Your work should look like this
	Start with a line segment PQ that we will copy.	
Step 1	Mark a point R that will be one endpoint of the new line segment.	
Step 2	Set the compass point on the point P of the line segment to be copied.	
Step 3	Adjust the compass width to the point Q. The compass width is now equal to the length of the line segment PQ.	
Step 4	Without changing the compass width, place the compass point on the the point R on the line you drew in step 1	

Step 5	Without changing the compass width, Draw an arc roughly where the other endpoint will be.	
Step 6	Pick a point S on the arc that will be the other endpoint of the new line segment.	
Step 7	Draw a line from R to S.	
Step 8	Done. The line segment RS is equal in length (congruent to) the line segment PQ.	

Reading Check:

1. *Fill in the blank:* A construction is a geometric drawing made with only a ________________________________-__ and a ___________________________________.

2. *Copy the line segment seen below using the steps you learned to make a construction. If you don't have the tools to do it, write the steps you would take to copy it.*

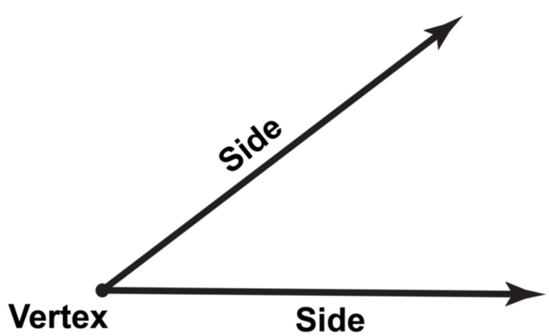
1.10 Congruent Segments, Midpoints, and Bisectors

Learning Objectives

- Be able to properly name, draw, and label *angles*.
- Understand and apply the *Angle Addition Postulate*.
- Classify angles as *acute, obtuse,* or *straight* based on their measurements.

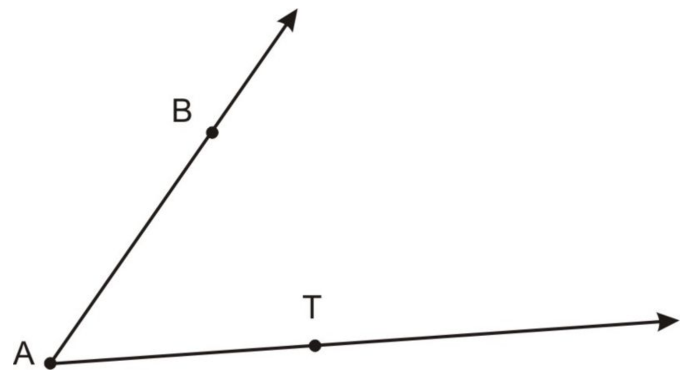
Angle, Vertex, and Sides

An **angle** is formed when two **rays** share a *common* endpoint. That common endpoint is called the **vertex** and the two rays are called the **sides** of the angle:



- Two **rays** that share a common endpoint are called an _______________________.
- The ________________________ is the shared common endpoint of an **angle**.
- In an **angle**, the two **rays** are called ______________________.

In the diagram below, $\overrightarrow{AB}$ and $\overrightarrow{AT}$ form an angle, $\angle BAT$, or $\angle TAB$, or $\angle A$. You can use one letter to name this angle since point A is the **vertex** and there is only *one* angle at point A. The symbol $\angle$ is used for naming angles.



The same basic definition for angle also holds when lines, segments, or rays intersect.

- The symbol for naming **angles** is ___________.

Naming Angles

If two or more angles share the *same* **vertex**, you MUST use *three* letters to name the angle.

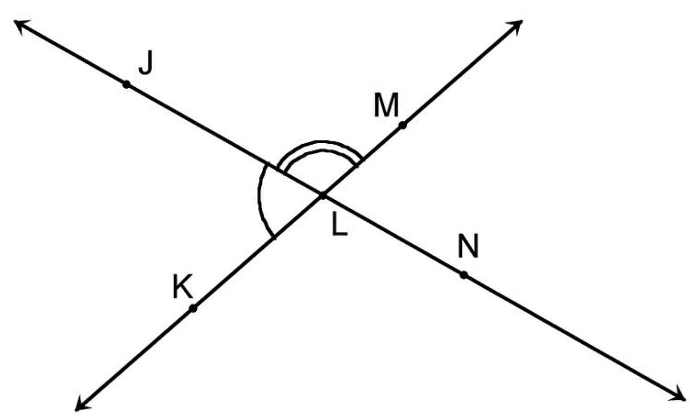
The **vertex** letter is always in the *middle* of the three naming letters.

- To name an angle, use __________________ letters with the vertex letter in the ________________________.

For example, in the image below it is unclear which angle is referred to by $\angle L$.

To talk about the angle with *one* arc, you would write $\angle KLJ$.

For the angle with *two* arcs, you would write $\angle JLM$.



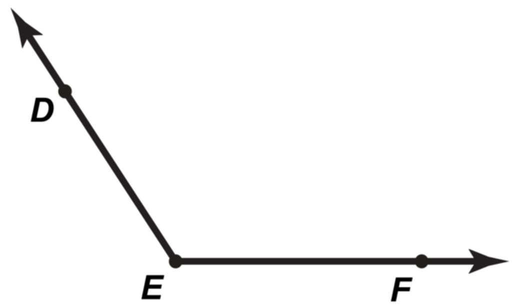
Reading Check:

Name the following angle three different ways:

____

____

____

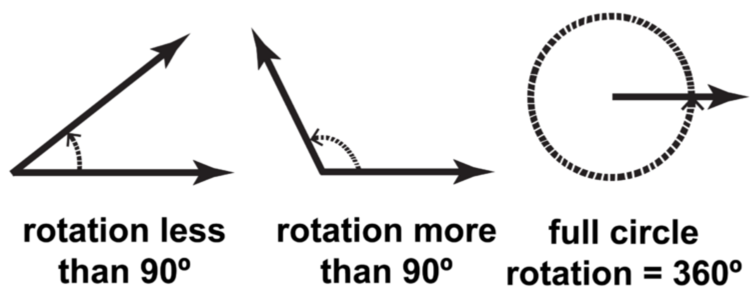


Measuring Angles

We use a ruler to measure segments by their *length*. But how do we measure an angle?

The length of the sides does not change how wide an angle is "open." Instead of using length, the size of an angle is measured by the amount of *rotation* from one side to another.

By definition, a full turn is defined as 360 degrees. Use the symbol $^\circ$ for **degrees**.



- One full rotation is defined as ________________ degrees.
- The symbol for **degrees** looks like __________.

Right Angles and Perpendicular Lines

The angle that is made by rotating through one-fourth of a full turn is very special. It measures $\frac{1}{4} \cdot 360^\circ = 90^\circ$ and we call this a **right angle**.

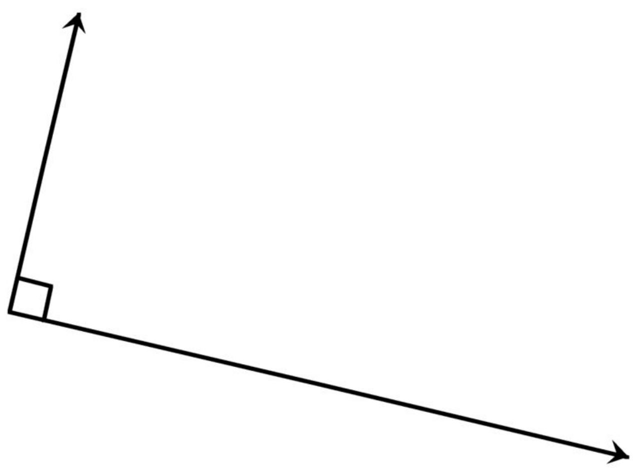
- A **right angle** measures one-quarter of 360°, or ______________.

Right angles are easy to identify, as they look like the corners of most buildings, or a corner of a piece of paper.

A **right angle** measures exactly 90°.

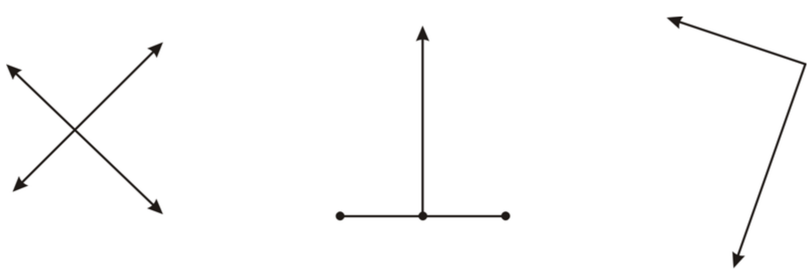
Right angles are usually marked with a small square. When two lines, two segments, or two rays intersect at a right angle, we say that they are **perpendicular**. The symbol ⊥ is used for two **perpendicular** lines.

Below is an example of two **perpendicular** rays. The small square inside the **vertex** shows that the rays meet at a **right angle**:



- **Right angles** are marked with a small ___________________ in the **vertex**.
- **Perpendicular** lines intersect at a ____________________ angle.
- The symbol for **perpendicular** looks like __________.

Reading Check:



Mark the right angles in the three pictures above.

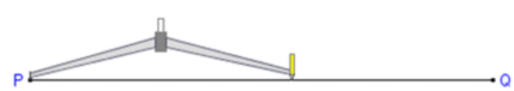
Construction: Perpendicular Bisector

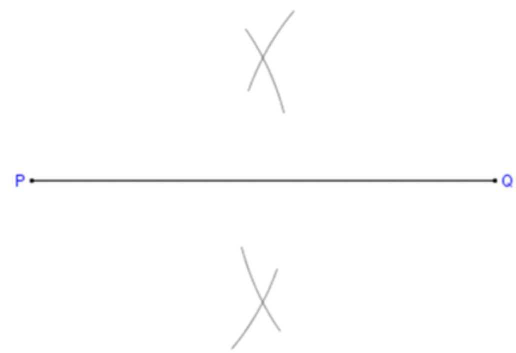
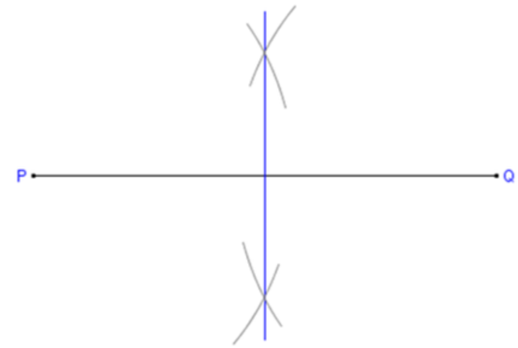
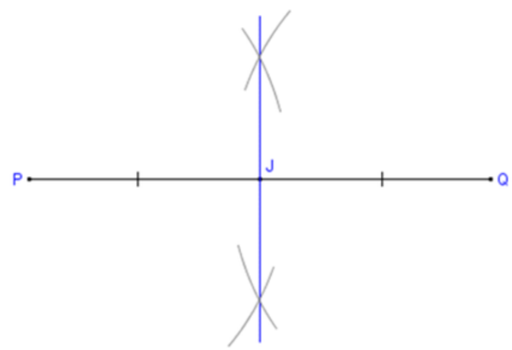
The following steps show you how to construct a line that is both...

- **Perpendicular** to the given line segment (in other words, it intersects the segment at a 90° angle) and
- A **bisector** (in other words, it cuts the line segment exactly in **half**)

Remember, for a geometric construction, you only use a **compass** and a **straightedge**. No rulers or protractors are allowed!

http://www.mathopenref.com/constbisectline.html

	After doing this	Your work should look like this
	Start with a line segment PQ.	
1	Place the compass on one end of the line segment.	
2	Set the compass width to a approximately two thirds the line length. The actual width does not matter.	
3	Without changing the compass width, draw an arc above and below the line.	

4	Again without changing the compass width, place the compass point on the the other end of the line. Draw an arc above and below the line so that the arcs cross the first two.	
5	Using a straightedge, draw a line between the points where the arcs intersect.	
6	Done. This line is perpendicular to the first line and bisects it (cuts it at the exact midpoint of the line).	

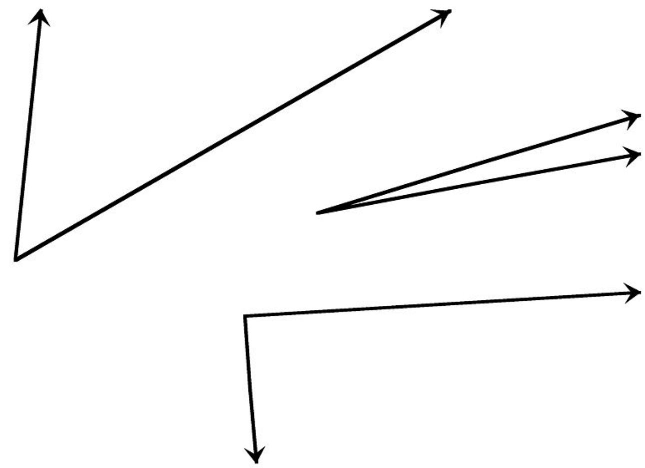
Reading Check:

1. *True or False:* A **perpendicular bisector** intersects the line segment at a 180° angle.

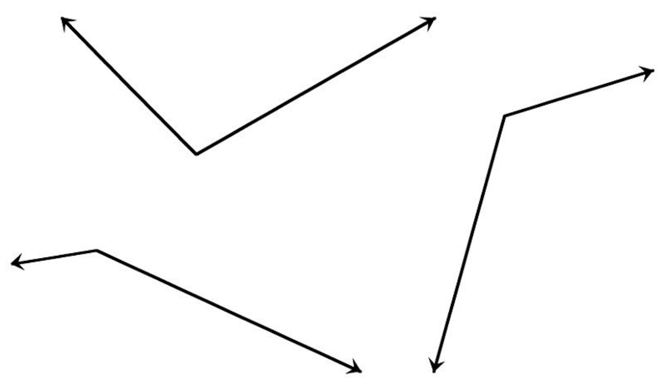
2. *Fill in the blank:* When you **bisect** a line segment, you cut it exactly at its ________________________________-_______________.

Other Types of Angles: Acute, Obtuse, and Straight Angles

An **acute angle** measures between 0° and 90°. These are some examples of **acute** angles:

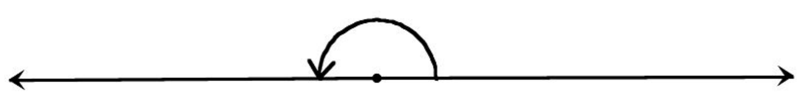


An **obtuse angle** measures between 90° and 180°. Below are examples of **obtuse** angles:



- An ______________________________ angle measures between 0° and 90°.
- An ______________________________ angle measures between 90° and 180°.

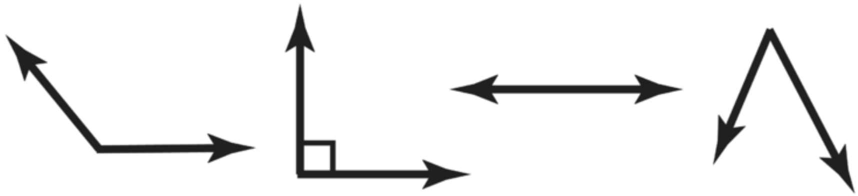
A **straight angle** measures exactly 180°. These angles are easy to spot since they look like straight lines.



You can use this information to classify any angle you see.

Reading Check:

*Label each angle as **acute, obtuse, right,** or **straight**:*

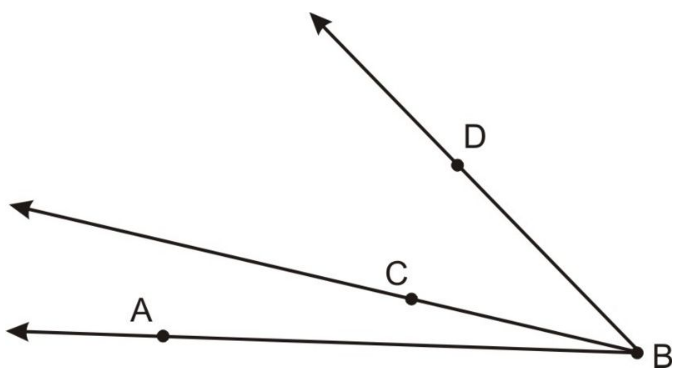


______________ ______________ ______________ ______________

Angle Addition Postulate

The measure of any angle can be found by *adding* the measures of the smaller angles that comprise it.

In the diagram below, if you *add* $m\angle ABC$ and $m\angle CBD$, you will get $m\angle ABD$:

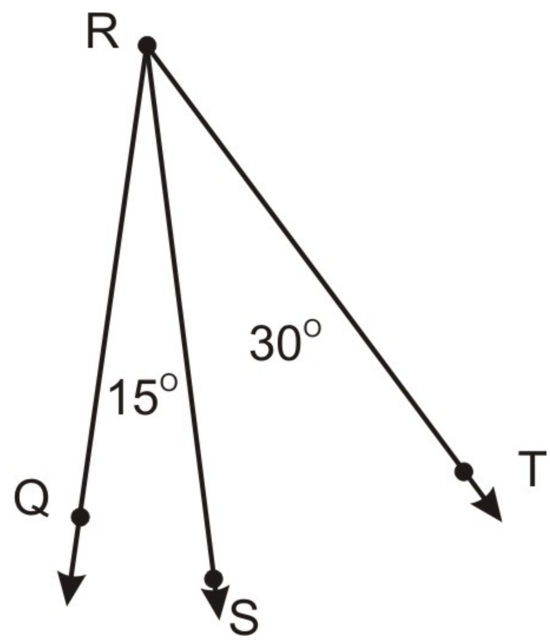


- The **Angle Addition Postulate** tells us to ________________ the measures of the smaller angles that make up a larger angle.

Use the **Angle Addition Postulate** just as you did the **Segment Addition Postulate** to identify the way different angles combine.

Example 1

What is $m\angle QRT$ *in the diagram below?*



You can see that $m\angle QRS$ is 15°. You can also see that $m\angle SRT$ is 30°. Using the **Angle Addition Postulate**, you can add these values together to find the total measure of $\angle QRT$:

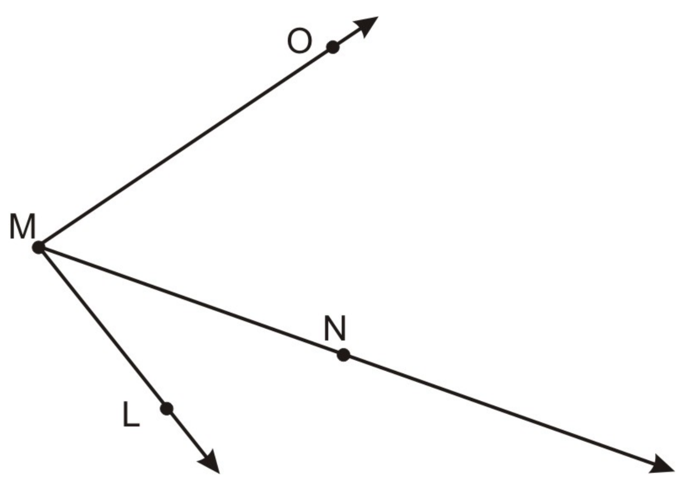
$m\angle QRS =$ ___________ and $m\angle SRT =$ ___________

So $m\angle QRS + m\angle SRT = 15° + 30° = 45°$

Therefore, $m\angle QRT$ is 45°.

Example 2

What is $m\angle LMN$ *in the diagram below given* $m\angle LMO = 85°$ *and* $m\angle NMO = 53°$*?*



To find $m\angle LMN$, you must *subtract* $m\angle NMO$ from $m\angle LMO$:

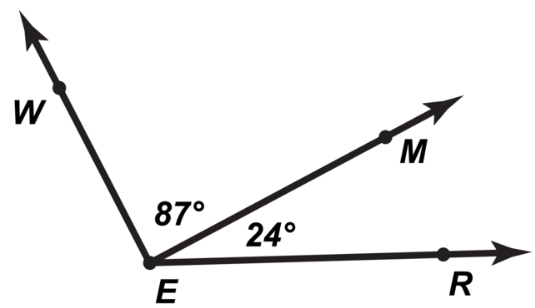
If $m\angle LMO =$ ___________ and $m\angle NMO =$ ___________,

Then $m\angle LMO - m\angle NMO = 85° - 53° = 32°$

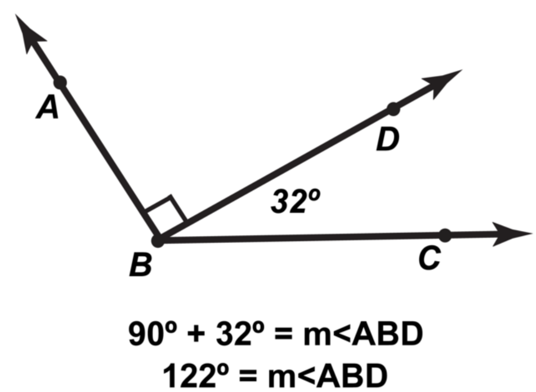
So $m\angle LMN = 32°$.

Reading Check:

1. *Find the measure of angle* $\angle WER$ *in the diagram below:*



2. *Samantha is trying to find the measure of angle* $\angle ABD$ *in the figure below. Can you identify Samantha's mistake?*



Congruent Angles

You already know that congruent line segments have exactly the same length. You can also apply the concept of congruence to other geometric figures: when **angles** are **congruent**, they have exactly the *same measure*. They may point in different directions, have different side lengths, have different names or other attributes, but their *measures* will be *equal*.

- **Congruent angles** have the ______________________ measure.

When writing that two angles are **congruent**, we use the congruent symbol:

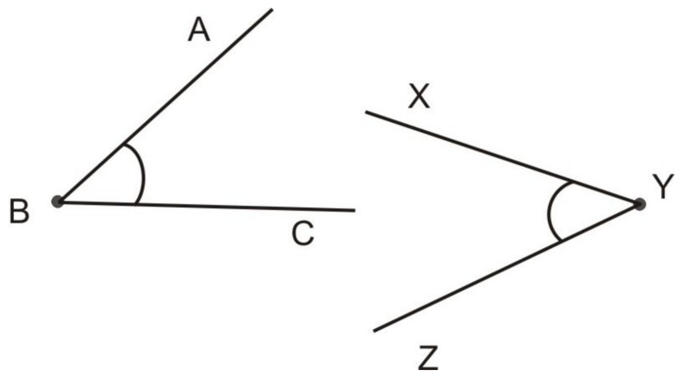
$$\angle ABC \cong \angle XYZ$$

Alternatively, the symbol $m\angle ABC$ refers to the *measure* of $\angle ABC$, so we could write $m\angle ABC = m\angle XYZ$ and that has the *same* meaning as $\angle ABC \cong \angle XYZ$.

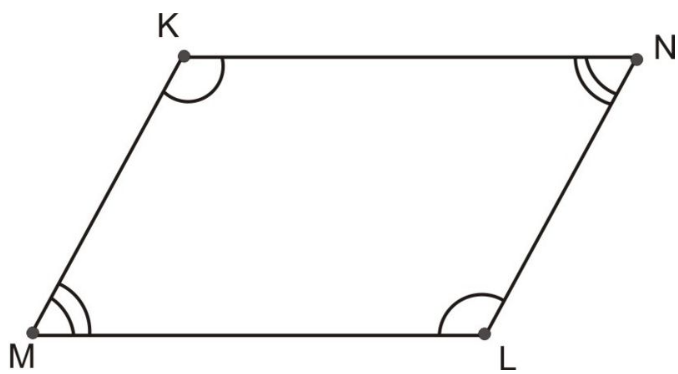
You may notice then, that *numbers* (such as measurements) are *equal* while *objects* (such as angles and segments) are *congruent*.

- Numbers are equal while objects are ______________________________.

When drawing *congruent angles*, you use an **arc** in the middle of the angle to show that two angles are congruent. If two different pairs of angles are congruent, use *one* set of arcs for one pair, then *two* for the next pair and so on:



$\angle ABC \cong \angle XYZ$ (one arc)

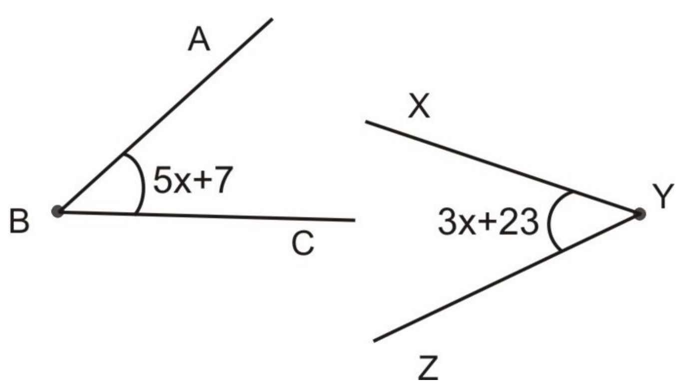


$\angle K \cong \angle L$ (one arc)

$\angle M \cong \angle N$ (two arcs)

Example 3

The two angles shown below are congruent:



What is the measure of each angle?

Start by setting the angles equal to each other:

$$\begin{aligned} 5x+7 &= 3x+23 \\ 5x-3x &= 23-7 \\ 2x &= 16 \\ x &= 8 \end{aligned}$$

So, the value of x is 8. You are not done, however, because the question asks you for the *measure* of each angle. Use the value of x to find the measure of *one* of the angles in the problem (you can choose either angle because we know the angles are **congruent**):

$$\begin{aligned} m\angle ABC &= 5x+7 \\ &= 5(8)+7 \\ &= 40+7 \\ &= 47 \end{aligned}$$

Since we know $m\angle ABC = m\angle XYZ$, both angles measure 47°.

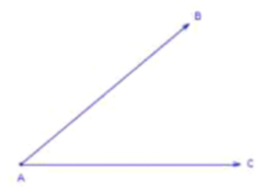
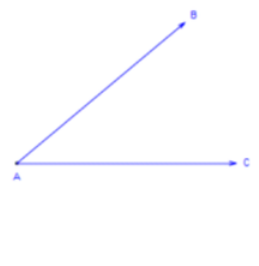
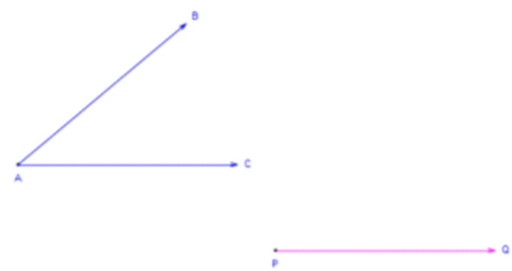
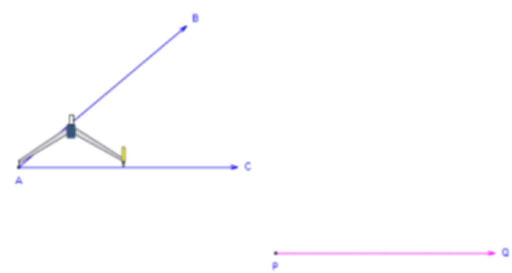
<u>Note:</u> If you wanted to *check* your work, try substituting the value of x into the *other* angle:

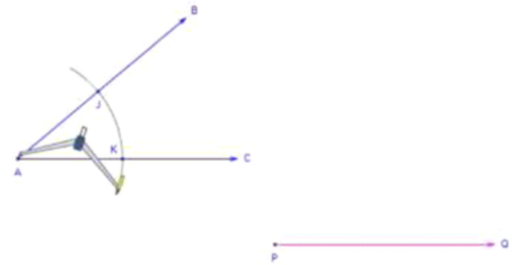
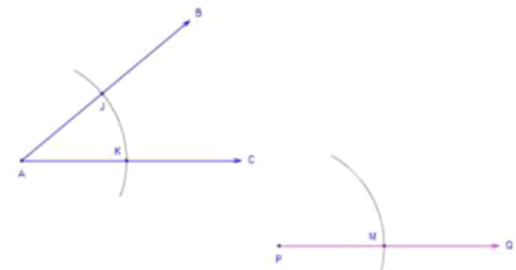
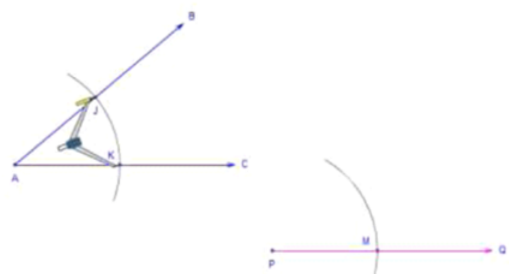
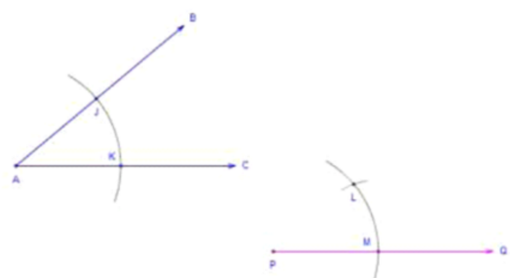
$$\begin{aligned} m\angle XYZ &= 3x+23 \\ &= 3(\underline{\qquad\quad})+23 \\ &= \underline{\qquad\quad}+23 \\ &= \underline{\qquad\quad} \end{aligned}$$

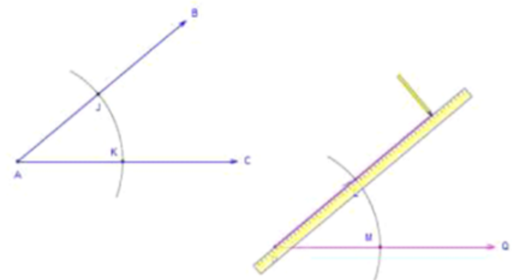
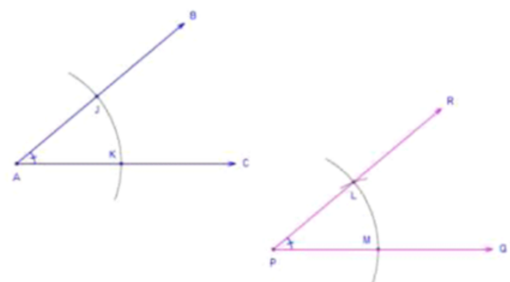
Did you get the same angle measure?

Construction: Copying A Congruent Angle

http://www.mathopenref.com/constcopyangle.html

After doing this	Your work should look like this
Start with a angle BAC that we will copy.	
1. Make a point P that will be the vertex of the new angle.	
2. From P, draw a ray PQ. This will become one side of the new angle. • This ray can go off in any direction. • It does not have to be parallel to anything else. • It does not have to be the same length as AC or AB.	
3. Place the compass on point A, set to any convenient width.	

After doing this	Your work should look like this
4. Draw an arc across both sides of the angle, creating the points J and K as shown.	
5. Without changing the compass width, place the compass point on P and draw a similar arc there, creating point M as shown.	
6. Set the compass on K and adjust its width to point J.	
7. Without changing the compass width, move the compass to M and draw an arc across the first one, creating point L where they cross.	

After doing this	Your work should look like this
8. Draw a ray PR from P through L and onwards a little further. The exact length is not important.	
Done. The angle ∠RPQ is congruent (equal in measure) to angle ∠BAC.	

Reading Check:

1. *Circle the correct answer choice:* When two angles are **congruent**, they have the same (direction / degree / measure).

2. *Fill in the blank:* When you are constructing a **congruent** angle, you use the _____________ to measure how open the angle is.

Angle Bisectors

An **angle bisector** divides an angle into two congruent angles, each having a measure of exactly half of the original angle.

Angle Bisector Postulate

Every angle has exactly one bisector.

Do you remember what a ***bisector*** *is?*

A bisector cuts something in half.

So an ***angle bisector*** *cuts an angle in half into two* ***congruent*** *angles.*

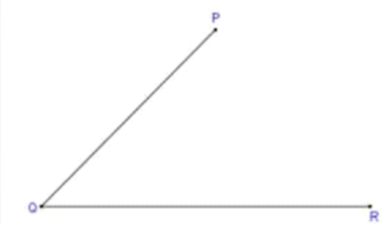
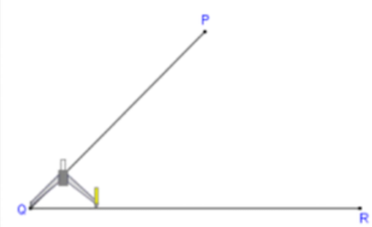
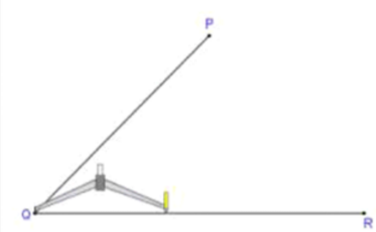
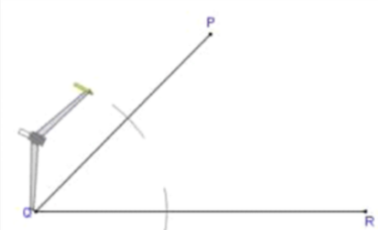
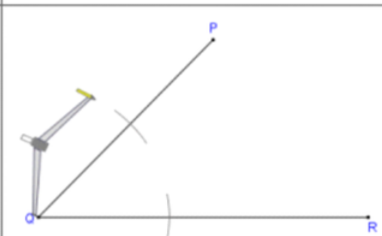
Reading Check:

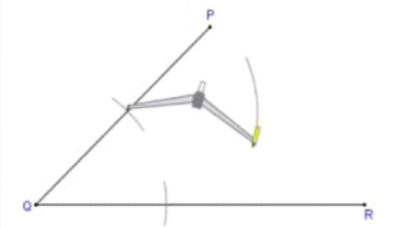
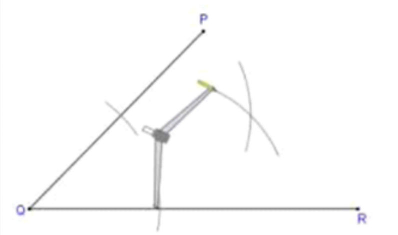
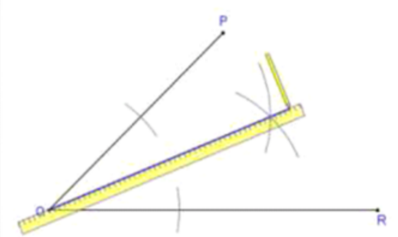
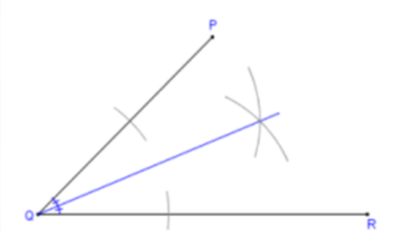
1. *How many angle bisectors can one angle have?*

2. *In your own words, describe the angles that are formed by an angle bisector.*

Construction: Angle Bisector

http://www.mathopenref.com/constbisectangle.html

After doing this	Your work should look like this
Start with angle PQR that we will bisect.	
1. Place the compass point on the angle's vertex Q.	
2. Adjust the compass to a medium wide setting. The exact width is not important.	
3. Without changing the compass width, draw an arc across each leg of the angle.	
4. The compass width can be changed here if desired. Recommended: leave it the same.	

After doing this	Your work should look like this
5. Place the compass on the point where one arc crosses a leg and draw an arc in the interior of the angle.	
6. Without changing the compass setting repeat for the other leg so that the two arcs cross.	
7. Using a straightedge or ruler, draw a line from the vertex to the point where the arcs cross	
Done. This is the bisector of the angle ∠PQR.	

<u>**Reading Check:**</u>

True or false: The first mark you make when **bisecting** an angle is an arc that crosses both rays of the angle.

1.11 Angle Pair Relationships

Learning Objectives

- Define *complementary angles, supplementary angles, adjacent angles, linear pairs*, and *vertical angles*.
- Use angle pair relationships to write and solve equations
- Apply the *Linear Pair Postulate* and the *Vertical Angles Theorem*.

Complementary Angles

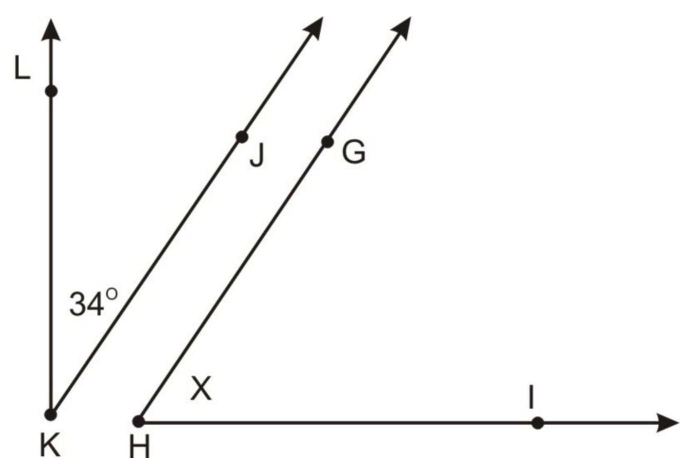
Two angles are **complementary angles** if the *sum* of their measures is 90°.

Complementary angles do not have to be congruent to each other. Rather, their only defining quality is that the *sum* of their measures is equal to the measure of a right angle: 90°. If the *outer* rays of two adjacent angles form a right angle, then the angles are **complementary**.

- The measures of **complementary angles** add up to ___________.

Example 1

The two angles below are complementary. $m\angle GHI = x$. *What is the value of* x*?*



Since you know that the two angles must *sum* to 90°, you can create an equation, then solve for the variable. In this case, the variable is x:

Since the angles are **complementary**, $m\angle LKJ + m\angle GHI = 90^\circ$

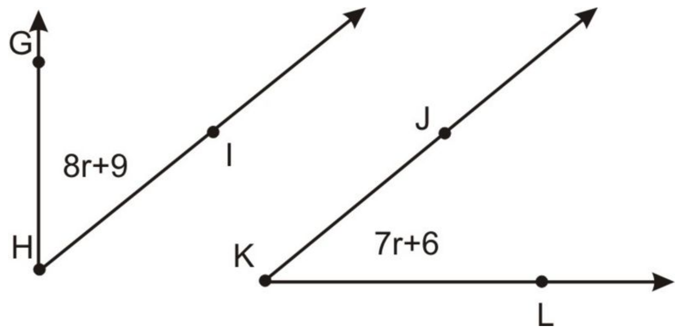
Substitute:

$$\begin{aligned} 34^\circ + x &= 90^\circ \\ \text{[U+0080][U+0093]}34^\circ \quad & \quad -34^\circ \\ x &= 56^\circ \end{aligned}$$

The measure of $\angle GHI = 56^\circ$.

Example 2

The two angles below are complementary. What is the measure of each angle?



If the angles are **complementary**, then their measures add up to __________.

The best way to solve this problem is to set up an equation where the two angle expressions *sum* to 90°. Then solve the equation for r.

$$(\underline{\qquad\qquad}) + (\underline{\qquad\qquad}) = 90^\circ$$

To find the measure of each angle, you must *substitute* the value for r back into the original expressions to find the value of each angle.

$$\begin{aligned} (8r+9)+(7r+6) &= 90 \\ 15r+15 &= 90 \\ -15 &\quad -15 \\ 15r &= 75 \\ r &= 5 \end{aligned}$$

The value of r is 5. Now substitute this value back into both angle expressions to find the measures of the two angles in the diagram:

$8r+9$	$7r+6$
$8(5)+9$	$7(5)+6$
$40+9$	$35+6$
49	41

$m\angle GHI = 49^\circ$ and $m\angle JKL = 41^\circ$. You can check to make sure these numbers are accurate by verifying that they are **complementary** (add up to 90°):

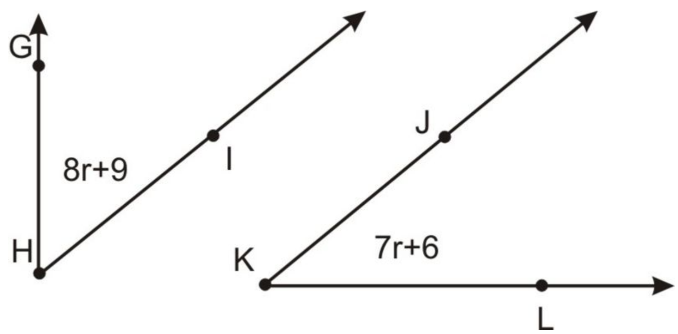
$49 + 41 = 90$

Since these two angle measures sum to 90°, they *are* **complementary**.

Reading Check:

1. *When angles are complementary, what does their sum equal?*

2. *If you know that the two angles below are complementary, how would you solve for r? Describe the steps you would use in words. You do not have to solve the problem!*



Supplementary Angles

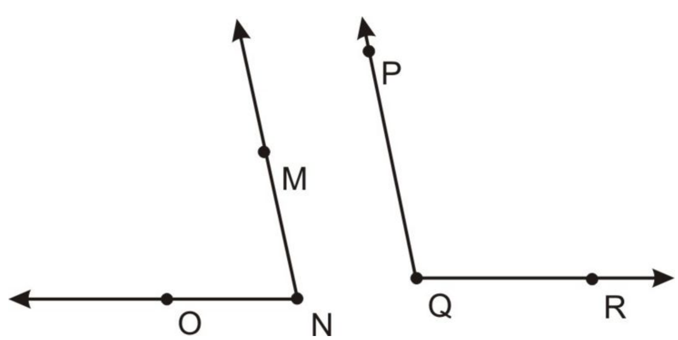
Two angles are **supplementary** if their measures sum to 180°.

Just like complementary angles, **supplementary** angles need not be congruent, or even touching. Their defining quality is that when their measures are *added* together, the sum is 180°. You can use this information just as you did with complementary angles to solve different types of problems.

- The measures of **supplementary angles** add up to ________________.

Example 3

The two angles below are supplementary. If $m\angle MNO = 78°$*, what is* $m\angle PQR$*?*



Use a variable for the unknown angle measure and then solve for the variable. In this case, let's substitute y for $m\angle PQR$.

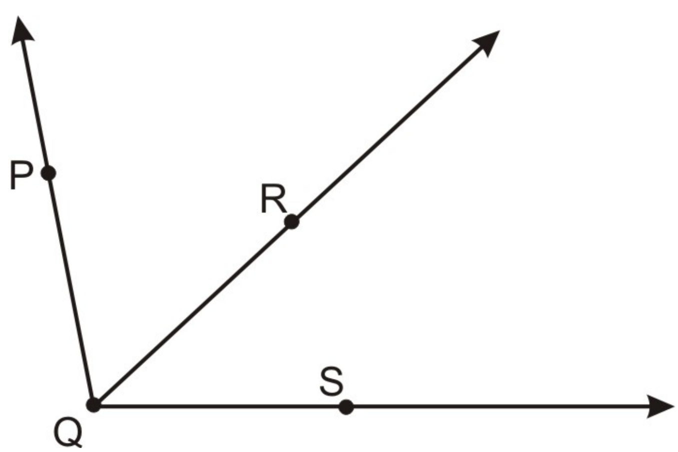
If the angles are **supplementary**, then their measures add up to ____________.

$$amp;m\angle MNO+m\angle PQR=180^\circ$$
$$78+y=180$$
$$-78 \qquad -78$$
$$y=102$$

So, the measure of $y=102$ and thus $m\angle PQR=102^\circ$.

Linear Pairs

Before we talk about a special pair of angles called **linear pairs**, we need to define **adjacent angles**. Two angles are **adjacent** if they *share* the *same vertex* and *one side*, but they do not overlap. In the diagram below, $\angle PQR$ and $\angle RQS$ are **adjacent**:



However, $\angle PQR$ and $\angle PQS$ are *not* **adjacent** since they *overlap* (i.e. they share common points in the interior of the angle).

- ______________________________ angles are next to each other: they share the same vertex and one side.

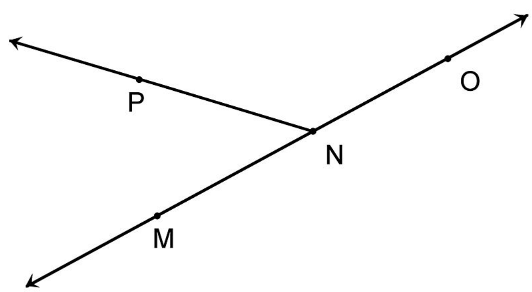
Adjacent *is a word meaning "next to."*

Things that are adjacent are usually touching, and they share a border.

Oregon, Nevada, and Arizona are adjacent to California because they share a border with California.

Now we are ready to talk about linear pairs. A **linear pair** is two angles that are **adjacent** and whose *non-common sides* form a *straight line*.

In the diagram below, $\angle MNP$ and $\angle PNO$ are a **linear pair**. Note that $\overleftrightarrow{MO}$ is a line.



- **Linear pairs** are angles that are next to each other along a _________________ line.

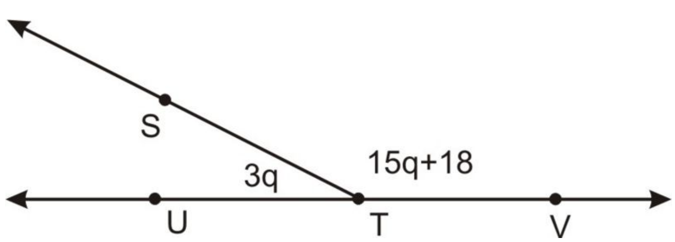
Linear pairs are so important in geometry that they have their own postulate.

Linear Pair Postulate

If two angles are a **linear pair**, then they are **supplementary**.

Example 4

The two angles below form a linear pair. What is the value of each angle?



We just learned that **linear pairs** are ____, so we know that they add up to 180°.

The best way to solve this problem is to set up an equation where the two angle expressions *sum* to 180°. Then solve the equation for q.

$$\begin{aligned} m\angle UTS \quad + \quad m\angle STV \quad &= 180^\circ \\ (\_\_\_\_\_\_\_\_\_\_\_) + (\_\_\_\_\_\_\_\_\_\_\_) &= 180^\circ \end{aligned}$$

To find the measure of each angle, you must *substitute* the value for q back into the original expressions to find the value of each angle.

$$\begin{aligned} (3q) + (15q + 18) = &\ 180 \\ 18q + 18 = &\ 180 \\ -18 \quad &-18 \\ 18q = &\ 162 \\ q = &\ 9 \end{aligned}$$

The value of q is 9. Now substitute this value back into both angle expressions to find the measures of the two angles in the diagram:

$3q$	$15q + 18$
$3(9)$	$15(9) + 18$
27	$135 + 18$
	153

$m\angle UTS = 27^\circ$ and $m\angle STV = 53^\circ$. You can check to make sure these numbers are accurate by verifying that they are **supplementary** (add up to 180°):

$27 + 53 = 180$

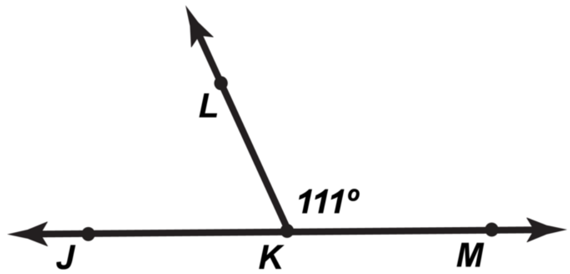
Since these two angle measures sum to 180°, they *are* **supplementary**.

Reading Check:

1. *Fill in the blank:*

Linear pairs add up to 180°. In other words, they are __________________________.

2. *Find the measure of angle $\angle JKL$ in the picture below:*

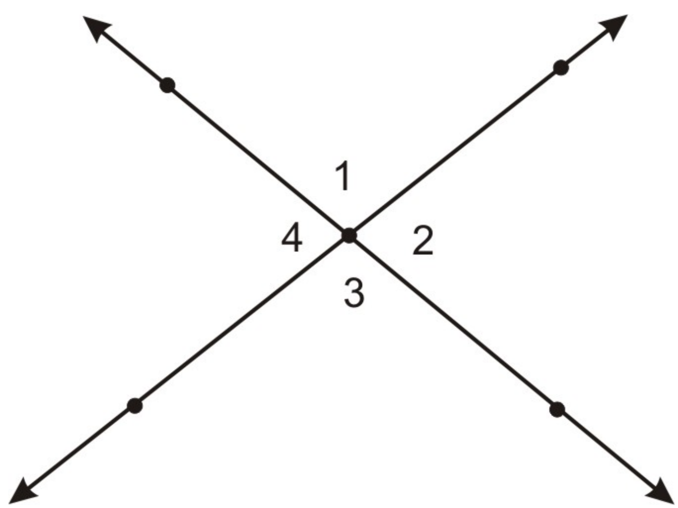


Vertical Angles

Now that you understand **supplementary** and **complementary** angles, you can examine more complicated situations.

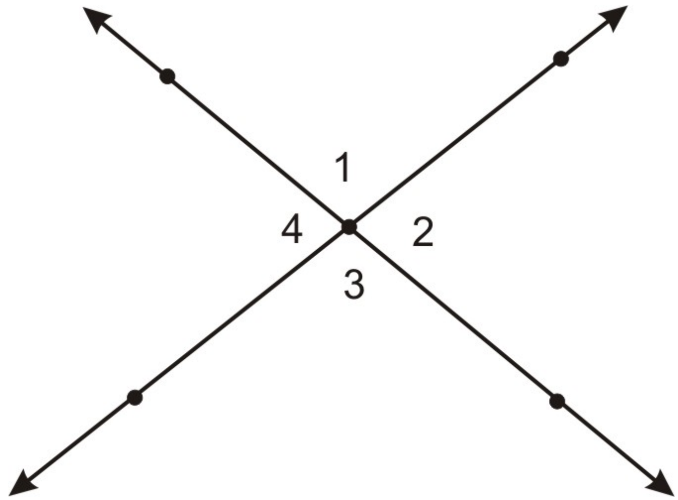
Special angle relationships are formed when two lines *intersect*, and you can use your knowledge of **linear pairs** of angles to explore each angle further.

Vertical angles are defined as two *non-adjacent* angles formed by *intersecting* lines. In the diagram below, $\angle 1$ and $\angle 3$ are **vertical angles**. Also, $\angle 4$ and $\angle 2$ are **vertical angles**.



Vertical angles are non-________________________________, which means they are *not* next to each other.

Vertical angles are formed by _________________________________ lines, and as you can see in the diagram below, they are always directly across from each other at the intersection:



Suppose that you know $m\angle 1 = 100^\circ$. You can use that information to find the measurement of *all* of the other angles.

For example, $\angle 1$ and $\angle 2$ must be **supplementary** since they are a **linear pair**.

So, to find $m\angle 2$, subtract 100° from 180°:

$$\begin{aligned} m\angle 1 + m\angle 2 &= 180^\circ \\ 100^\circ + m\angle 2 &= 180^\circ \\ -100^\circ \qquad &\ \ -100^\circ \\ m\angle 2 &= 80^\circ \end{aligned}$$

So $\angle 2$ measures 80°. Knowing that angles $\angle 2$ and $\angle 3$ are also **supplementary** means that $m\angle 3 = 100^\circ$, since the sum of 100° and 80° is 180°.

If angle $\angle 3$ measures 100°, then the measure of angle $\angle 4$ must be 80°, since $\angle 3$ and $\angle 4$ are also **supplementary**.

Notice that angles $\angle 1$ and $\angle 3$ are **congruent** (100°) and $\angle 2$ and $\angle 4$ are **congruent** (80°).

Vertical Angles Theorem

The **Vertical Angles Theorem** states that if two angles are **vertical angles** then they are *congruent*.

- **Vertical angles** are ______________________________ to each other.

Introduction to Proof: Proving the Vertical Angle Theorem

We can prove the **Vertical Angles Theorem** using a process just like the one we used above. There was nothing special about the given measure of $\angle 1$.

Here is proof that **vertical angles** will *always* be **congruent**:

Since $\angle 1$ and $\angle 2$ form a **linear pair**, we know that they are **supplementary**:

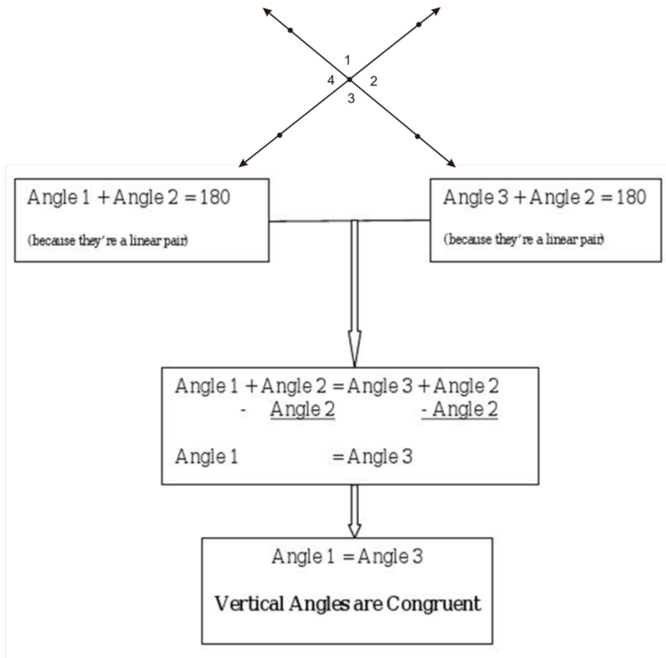
$m\angle 1 + m\angle 2 = 180^\circ$

For the same reason, $\angle 2$ and $\angle 3$ are **supplementary:** $m\angle 2 + m\angle 3 = 180^\circ$

Using a substitution (they both $= 180^\circ$), we can write $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$.

Finally, subtracting $m\angle 2$ on both sides gives us $m\angle 1 = m\angle 3$.

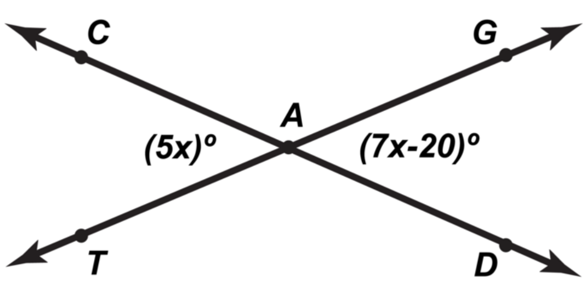
Or, by the definition of congruent angles: $\angle 1 \cong \angle 3$.



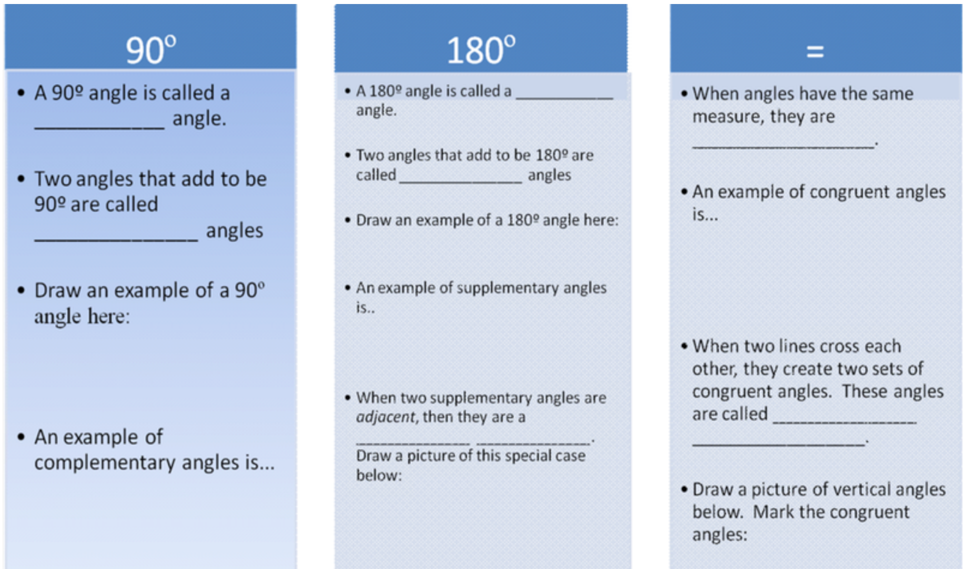
<u>**Reading Check:**</u>

1. *Fill in the blank:* The reason that **vertical angles** are *congruent* is that each pair of **adjacent angles** is a ______-______________________ pair.

2. *Find the measure of angle $\angle CAT$ in the picture below*.



Graphic Organizer for Lesson 8



90°

- A 90º angle is called a ___________ angle.
- Two angles that add to be 90º are called _____________ angles
- Draw an example of a 90° angle here:
- An example of complementary angles is...

180°

- A 180º angle is called a __________ angle.
- Two angles that add to be 180º are called _____________ angles
- Draw an example of a 180º angle here:
- An example of supplementary angles is..
- When two supplementary angles are *adjacent*, then they are a _____________ _____________.

  Draw a picture of this special case below:

=

- When angles have the same measure, they are ___________________.
- An example of congruent angles is...
- When two lines cross each other, they create two sets of congruent angles. These angles are called ________________ _________________.
- Draw a picture of vertical angles below. Mark the congruent angles:

1.12 Polygons

Learning Objectives

- Define *polygon*.
- Identify polygons as *convex* or *concave*.
- Classify polygons by number of sides.
- Determine if a polygon is regular or not.

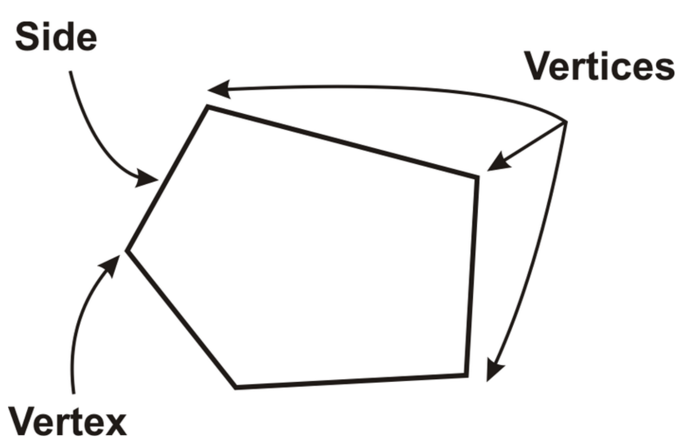
Defining Polygons

A **polygon** is any *closed* planar figure that is made entirely of line segments that intersect at their endpoints. Polygons can have three or more sides and angles.

- A ___________________________ is a closed figure made of straight line segments that intersect at their endpoints.
- **Polygons** can have ___________________ or more sides and angles.

The segments are called the **sides** of the polygons, and the points where the segments intersect are called **vertices**. [Note that the *singular* of vertices is **vertex**.]

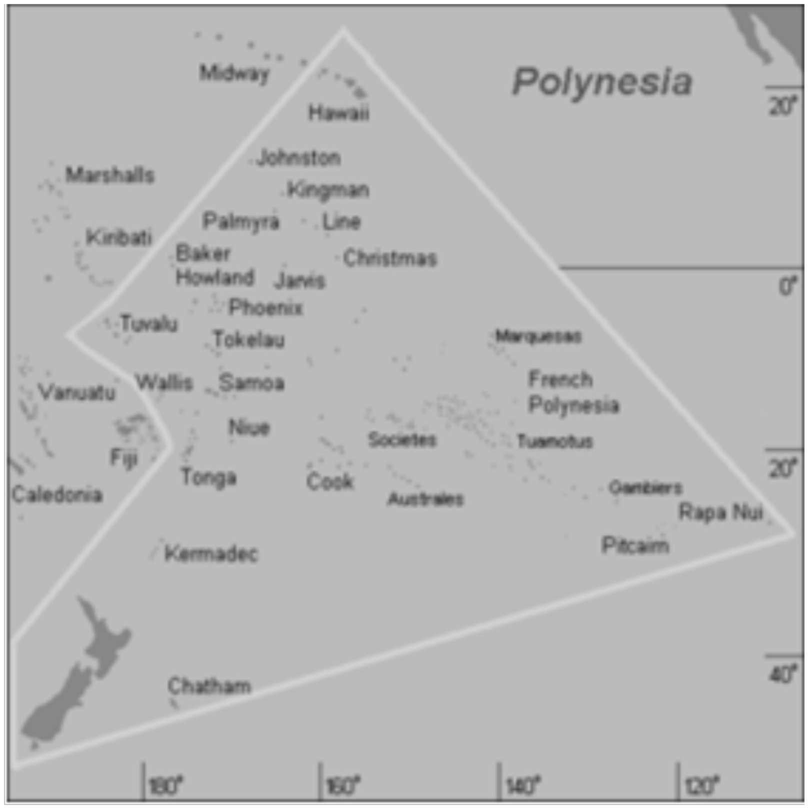
The following shape is a **polygon**:



- The segments in a **polygon** are called ______________________.
- Points where the **sides** of a **polygon** intersect are called _________________________.
- The *plural* form of **vertex** is ___________________________________.

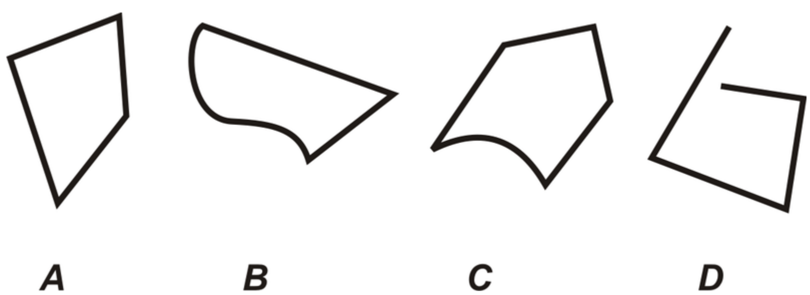
The prefix "poly" *means "many."*

For example, the region of the world known as "Polynesia" is made of many islands.



Example 1

Which of the figures below is a polygon?



The easiest way to identify the polygon is to identify which shapes are *not* polygons.

Choices **B** and **C** each have at least one curved side. So they *cannot* be polygons because polygons must have *straight* sides.

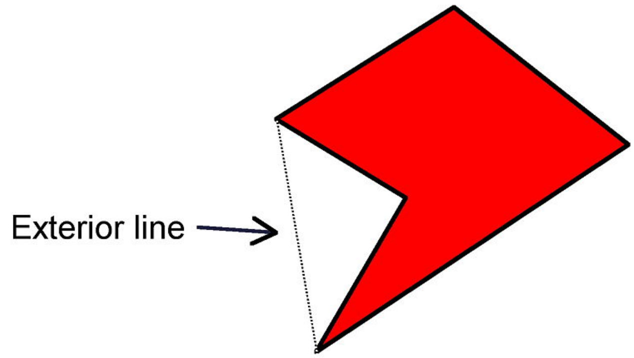
Choice **D** has all straight sides, but one of the vertices is *not* at the endpoints of the two adjacent sides, so it is *not* a polygon.

Choice **A** is composed entirely of line segments that intersect at their endpoints. So, it is a **polygon**. The correct answer is **A**.

Convex and Concave Polygons

Now that you know how to identify polygons, you can begin to practice classifying them.

The first type of classification to learn is whether a polygon is **convex** or **concave**. Think of the term **concave** as referring to a cave, or an *interior* space. A **concave** polygon has a section that "points inward" toward the middle of the shape. In any **concave** polygon, there are at least two vertices that can be connected *without* passing through the *interior* of the shape. The polygon below is **concave** and demonstrates this property:



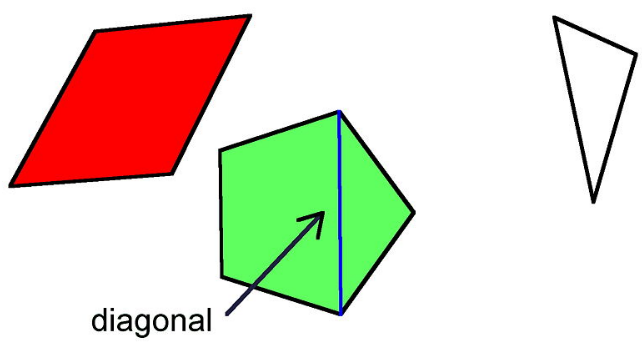
- A _________________________ polygon has a section that points inward.

Diagonals

A **convex** polygon does not share this property. Any time you connect the vertices of a **convex** polygon, the segments between non-adjacent vertices will travel through the *interior* of the shape.

- In a **convex** polygon, the vertices are only connected in the ______________________ of the shape.

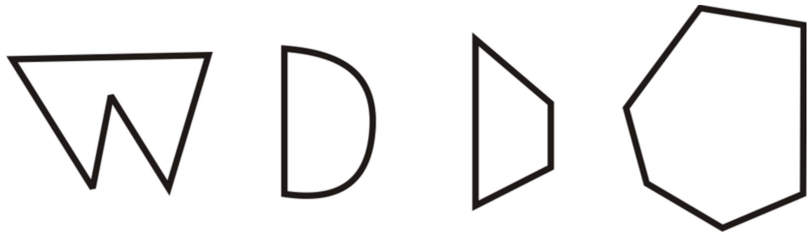
Lines segments that *connect to vertices* traveling only on the interior of the shape are called **diagonals**.



- A **diagonal** is a line segment in a _______________________ polygon that connects the vertices only on the interior of the shape.

Reading Check:

Identify each polygon below as concave or convex. If you do not think that the shape is a polygon, identify it as "not a polygon."



________ ________ ________ ________

Classifying Polygons by their Numbers of Sides

The most common way to *classify* a **polygon** is by the *number of sides*. Regardless of whether the polygon is convex or concave, it can be named by the number of sides.

The *prefix* in each name reveals the number of sides. Refer to the polygon chart at the end of this lesson to name and classify samples of polygons.

Regular, Equiangular and Equilateral Polygons

Polygons that have **congruent sides** are *equilateral.*

Polygons that have **congruent angles** are *equiangular.*

If a polygon is both *equilateral* and *equiangular*, it is called a **regular** polygon.

A square is an example of a **regular** polygon because it has *four congruent angles* and *four congruent sides.*

- **Equilateral polygons** have *congruent* ________________.
- **Equiangular polygons** have *congruent* ________________.
- A **regular polygon** is ________________ and ________________.

You can probably figure out that "equi" means "equal" and "angular" means "angles."

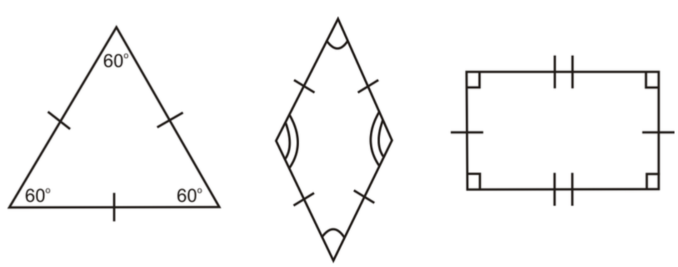
But what about "lateral"? Lateral is related to the Spanish word "lado," which means "side."

What do you think the word "quadrilateral" means when you break it down?

"Regular" in this case means "consistent," as in "the song has a regular rhythm" or "her regular bedtime is 10:00."

Reading Check:

Identify each polygon below as equiangular, equilateral, or regular.



________ ________ ________

Graphic Organizer for Lesson 9

Classifying Polygons by their Numbers of Sides

The most common way to classify a polygon is by the number of sides. Regardless of whether the polygon is convex or concave, it can be named by the number of sides. The prefix in each name reveals the number of sides. The chart below shows names and samples of polygons.

TABLE 1.7:

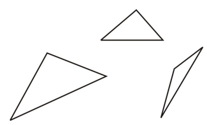
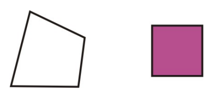
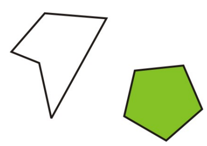
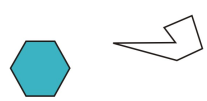
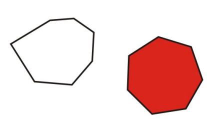
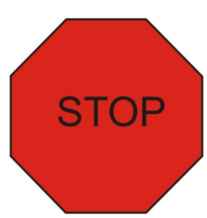
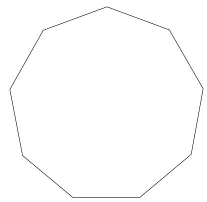
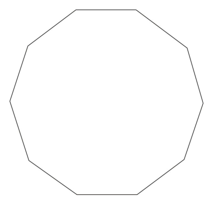
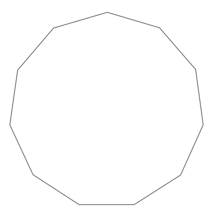
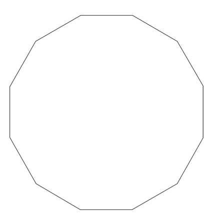
Polygon Name	Number of Sides	Sample Drawings	Draw your own example	Can you think of any other words that begin with...
Triangle	3			...tri? • Tricycle • Triple
Quadrilateral	4			...qua? • Quarter • Quadruple
Pentagon	5			...penta? • Pentameter
Hexagon	6			
Heptagon	7			
Octagon	8			...octo? • Octopus

TABLE 1.7: (continued)

Polygon Name	Number of Sides	Sample Drawings	Draw your own example	Can you think of any other words that begin with...
Nonagon	9			
Decagon	10			...deca? • Decade
Undecagon	11			
Dodecagon	12			
$n-$***gon***	n (where $n > 3$)			

1.13 Inductive and Deductive Reasoning

Learning Objectives

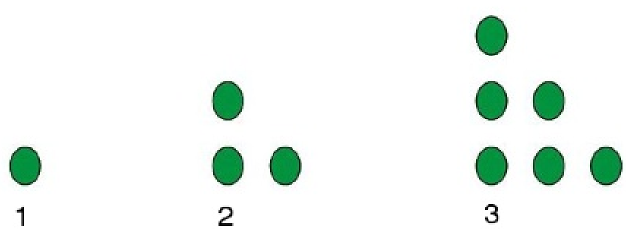
- Use *inductive* reasoning to draw a conclusion from a series of examples.
- Use *deductive* reasoning to draw a conclusion from facts or accepted statements.
- Compare and contrast inductive and deductive reasoning.

Inductive Reasoning

One method of reasoning is called **inductive reasoning**. This means drawing conclusions based on *examples*.

- **Inductive reasoning** is based on ______________________________.

For example, a dot pattern is shown below. Look at the first three pictures. Do you see the pattern?



How many dots would there be in the bottom row of a fourth pattern?

There will be 4 dots. There is one *more* dot in the bottom row of each figure than in the previous figure. Also, the number of dots in the bottom row is the *same* as the figure number.

What would the total number of dots be in the bottom row if there were 6 patterns?

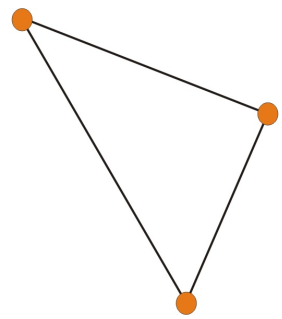
There would be a total of 21 dots. The rows would contain 1, 2, 3, 4, 5, and 6 dots. The total number of dots is $1+2+3+4+5+6=21$.

This dot pattern is an example of ______________________________ **reasoning** because we are basing our conclusions about what comes next on the *examples* we have seen so far.

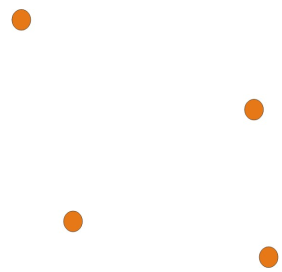
Now look at a pattern of points and line segments.

For two points, there is one line segment with those points as endpoints:

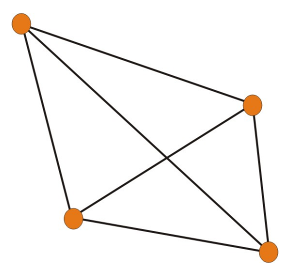
For three non-collinear points (points that do *not* lie on a single line), there are *three* line segments with those points as endpoints:



For four points, no three points being collinear, how many line segments with those points as endpoints are there?



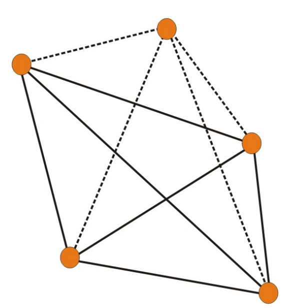
The 6 segments are shown below:



For five points, no three points being collinear, how many line segments with those points as endpoints are there?

Answer: 10.

When we add a 5th point, there is a new segment from that point to each of the other four points. We can draw the four new dashed segments shown on the next page. Together with the 6 segments for the 4 points above, this makes $6+4=10$ *segments.*



5 points have 10 segments that connect them, as you see above.

Again, this is another problem where we used inductive reasoning: we used examples to make conclusions.

Inductive reasoning about patterns is a natural way to study new material. But there is a serious limitation to inductive reasoning: no matter how many examples we have, examples alone do not prove anything. To *prove* relationships, we will learn to use **deductive** reasoning, also known as *logic*.

- **Deductive reasoning**, also known as ______________________, proves relationships.

Can you think of a time in your life when you did something different from a pattern?

Have you ever changed the pattern of events in your life?

Deductive Reasoning

We all use logic—whether we call it that or not—in our daily lives. And as adults we use logic in our work as well as in making the many decisions a person makes every day.

Deductive reasoning begins with *accepted facts* or statements which we know are *true*. Then, we draw conclusions based on those facts.

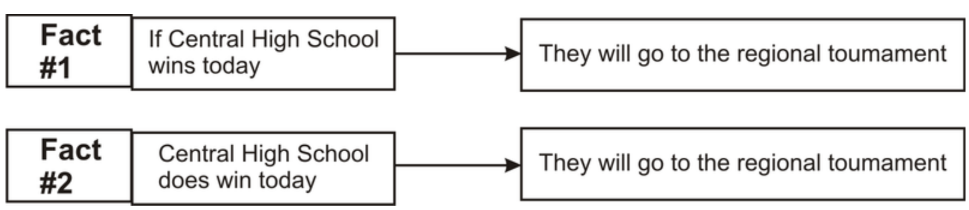
- **Deductive reasoning** starts with accepted ________________________ which we know are ______________-_________.

Example 1

Suppose Bea makes the following statements, which are known to be true.

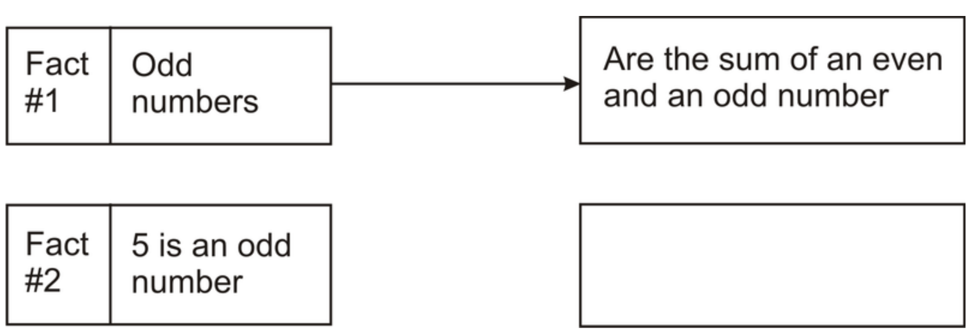
- If Central High School wins today, they will go to the regional tournament.
- Central High School does win today.

Common sense tells us that there is an obvious logical conclusion if these two statements are true: Central High School will go to the regional tournament.



Reading Check:

You are given two facts in the boxes below. Fill in the last box with the conclusion.



5 is the sum of an even and an odd number. (This is true, since $5 = 2+3$)

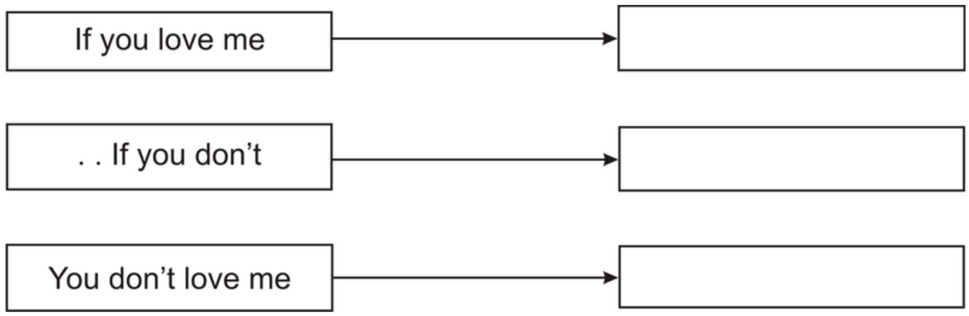
The diagrams on this page are example of _______________________________ **reasoning** because we start with facts that we have accepted to be true.

Suppose the following two statements are true:

If you love me let me know. If you don't then let me go. (A country music classic. Lyrics by John Rostill.)

You don't love me.

Fill in the boxes and find the logical conclusion:



You have now worked with both **inductive** and **deductive reasoning**. They are different but not opposites. In fact, they will work together as we study geometry and other mathematics.

Inductive reasoning means reasoning from *examples*.

You may look at a few examples, or many. Enough examples might make you *suspect* that a relationship is true always, or might even make you *sure* of this. But until you go beyond the inductive stage, you cannot be absolutely certain that it is *always* true.

- ______________________________ **reasoning** is reasoning from examples.

That's where **deductive reasoning** enters and takes over. We have a suggestion arrived at inductively. We then apply rules of *logic* to *prove*, beyond any doubt, that the relationship is *true* always.

- ______________________________ **reasoning** uses accepted facts and logic to prove that something is true.

Inductive means "lead to."

Inductive reasoning can "lead" you in the right direction to a conclusion, but you won't know for certain that you have arrived at the correct conclusion.

Deductive means "lead from."

(Think of the Spanish word "de," which means "from.") Because deductive reasoning is coming from facts, you know you have arrived at a true conclusion.

- **Inductive** means __________________ to.
- **Deductive** means lead ________________.

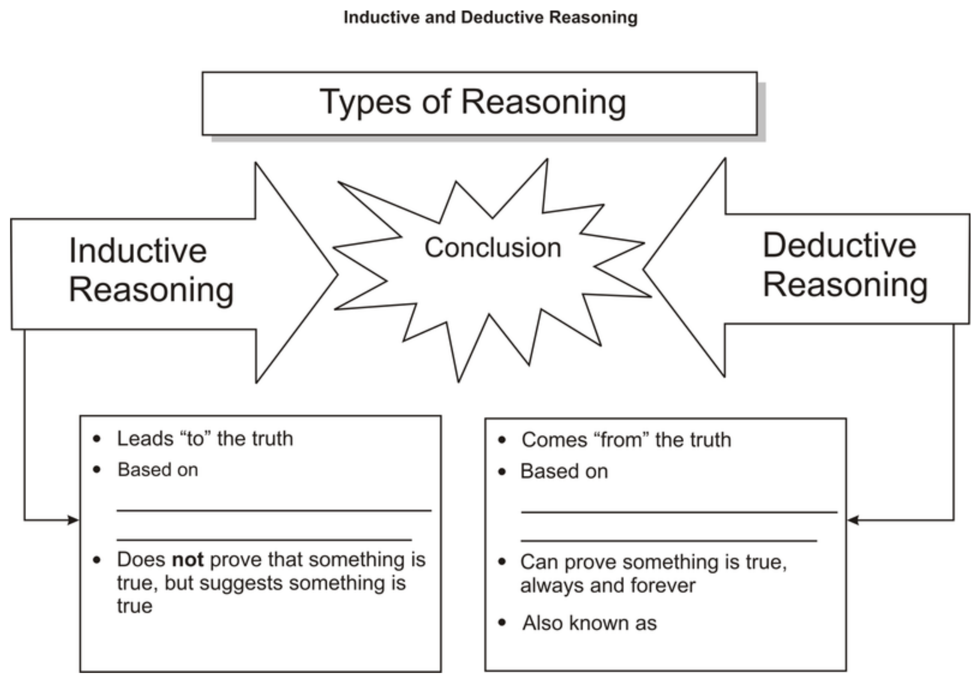
Reading Check:

1. *Fill in the blank:*

Deductive reasoning uses ______________________________ to draw a conclusion.

2. *Which type of reasoning, inductive, or deductive, is used to prove statements are true?*

Graphic Organizer for Lesson 10



1.14 Conditional Statements

Learning Objectives

- Write definitions as "if-then" statements.
- Identify the *hypothesis* and *conclusion* of an if-then statement.
- Write a *counterexample* to disprove a conjecture.

Conditional Statements, or If-Then Statements

In geometry, and in ordinary life, we often make **conditional**, or **if-then**, statements.

- Statement 1: **If** 2 divides evenly into x, **then** x is an even number.
- Statement 2: **If** the weather is nice, I will wash the car. ("Then" is implied even if not stated.)
- Statement 3: **If** a triangle has three congruent sides, it is an equilateral triangle. ("Then" is implied; this is a definition.)
- Statement 4: All *equiangular* triangles are *equilateral*. ("If" and "then" are both implied.)

A conditional statement and an ________-________ statement are the same thing.

Hypothesis and Conclusions

An **if-then** statement has two parts:

- The "if" part is called the **hypothesis**.
- The "then" part is called the **conclusion**.

The "if" part of an "if-then" statement is called the ______________________________.

The "then" part of an "if-then" statement is called the ______________________________.

For example, in statement 1 above:

The **hypothesis** is "2 divides evenly into x."

The **conclusion** is "x is an even number."

What does it mean to *<u>imply</u>* *something?*

What does it mean if something is *<u>implied</u>*?

To "imply" is to say something without saying it directly.

If something is implied, it isn't said directly, but we assume it is there. The meaning is suggested, but it isn't actually there.

- To "imply" means to state something without stating it ________________________.

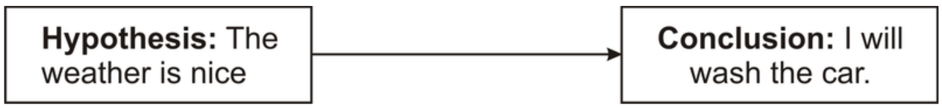
Look at statement 2 from the previous page:

Even though the word "then" is not actually present, the statement could be rewritten as:

If the weather is nice, *then* I will wash the car.

This is the *meaning* of statement 2. The word "then" is *implied*.

The **hypothesis** is "the weather is nice." The **conclusion** is "I will wash the car."



Statement 4 is a little more complicated:

- Statement 4: All *equiangular* triangles are *equilateral*. ("If" and "then" are both implied.)

"If" and "then" are both *implied* without being stated.

Statement 4 can be rewritten as: *If* a triangle is *equiangular, then* it is *equilateral*.

The **hypothesis** is: ____

The **conclusion** is: ____

Reading Check:

Identify the hypothesis and conclusion in each of these statements.

1. If a polygon has three sides, then it is a triangle.

- Hypothesis:____

____

- Conclusion:____

____

2. A segment has two endpoints.

(∗Hint: First, change this into an if-then statement: If it is a ___________________, then it is a _____________________.)

- Hypothesis:____

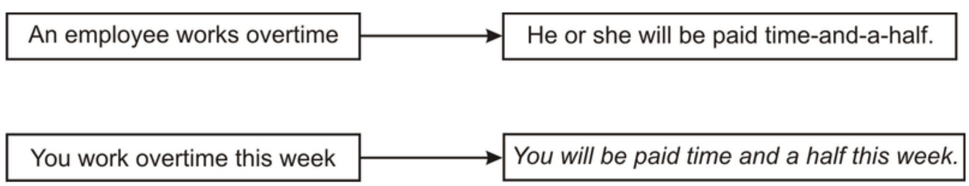
____

- Conclusion:____

What is meant by an if-then statement? Suppose your friend makes the following statements:

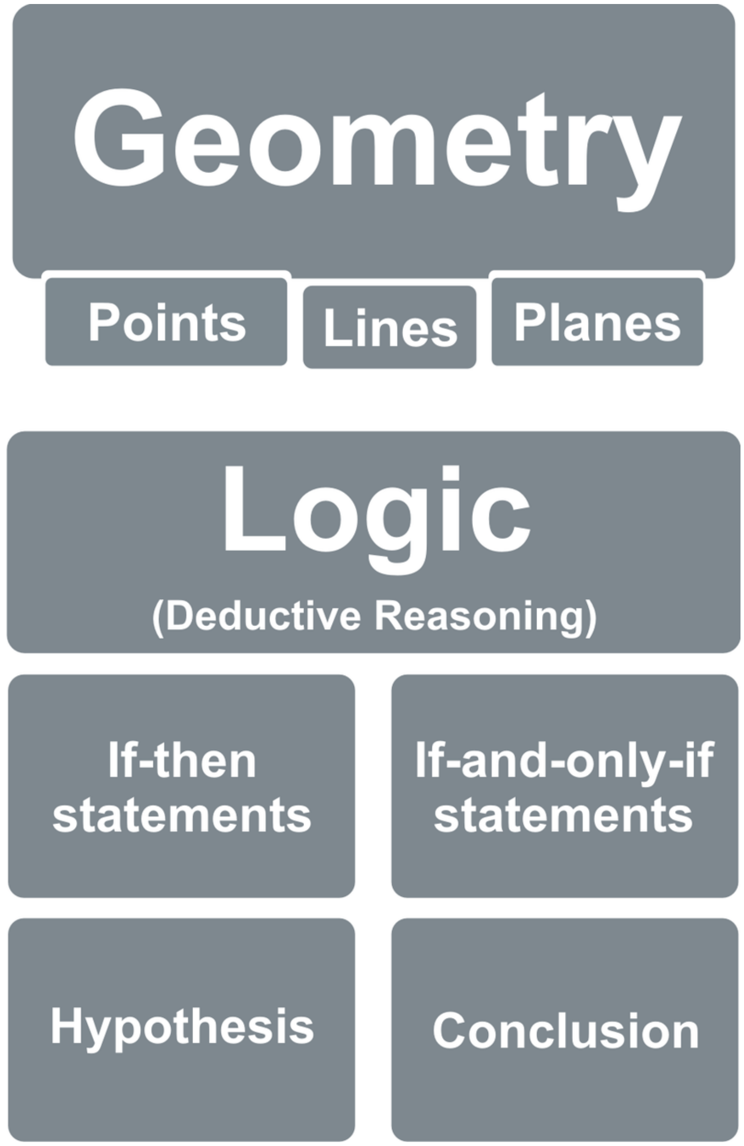
- If an employee works overtime, then he or she will be paid time-and-a-half.
- You worked overtime this week.

If we accept these statements, what other fact *must* be true? Combining these two statements, we can state with no doubt: You will be paid time-and-a-half this week.



We called points, lines, and planes the *building blocks* of geometry. We will soon see that **hypothesis, conclusion**, as well as **if-then** and **if-and-only-if** statements are the *building blocks* that **deductive reasoning**, or *logic*, is built on. This type of reasoning will be used throughout your study of geometry. In fact, once you understand logical reasoning you will find that you apply it to other studies and to information you encounter all your life.

- **Deductive reasoning** is also know as ______________________.



Conjectures and Counterexamples

A **conjecture** is an "educated guess" that is often based on *examples* in a *pattern*. Examples suggest a relationship, which can be stated as a possible rule, or **conjecture**, for the pattern. Numerous examples may make you strongly believe the conjecture. However, no number of examples can *prove* the conjecture. It is always possible that the next example would show that the conjecture does not work.

- A **conjecture** is an educated ____________________ based on examples in a pattern.

Example 1

Here are three numbers in a sequence:

1, 2, 3

What do you think the next number in the sequence will be?

Your answer is a **conjecture**. It is an educated guess. You do not know for sure what the next number will be.

Here is the same sequence with a few more numbers included:

1, 2, 3, 5, 8, 13...

If your first **conjecture** was that the next number would be 4, then your conjecture was wrong. The numbers in this sequence are found by *adding* the two *previous* numbers:

$$1+2=3$$
$$2+3=5$$
$$3+5=8$$
$$5+8=13 \text{ and so on}\ldots$$

Reading Check:

Ramona studied *positive even* numbers. She broke some positive even numbers down as follows:

$\mathbf{8}=3+5$ $\mathbf{14}=5+9$ $\mathbf{36}=17+19$ $\mathbf{82}=39+43$

What conjecture might be suggested by Ramona's results?

Ramona made this conjecture:

Every positive even number is the sum of two different positive odd numbers.

If you can find just *one* example that makes this **conjecture** *false*, then you have disproven the entire statement. A statement that makes the conjecture false is called a **counterexample**. *The prefix "counter" means "opposite" or "against."*

A counterexample is an example that goes "against" your conjecture.

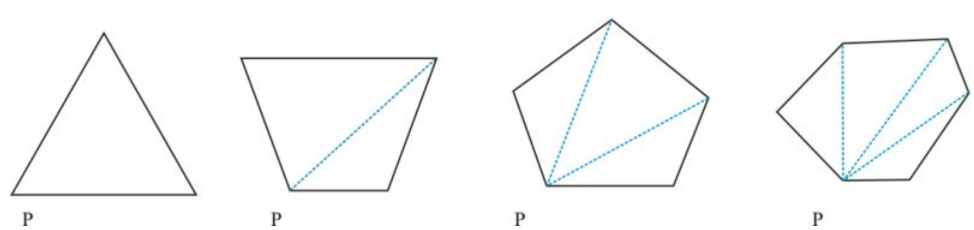
Can you think of a counterexample to this statement? "All shapes have straight sides."

A **counterexample** is an example that goes ___________________________ your conjecture.

In other words, a **counterexample** is an example that proves your **conjecture** is *false*.

Reading Check:

Arthur is making figures for a graphic art project. He drew polygons and some of their diagonals:



Based on these examples, Arthur made this conjecture:

If a convex polygon has n sides, then there are $n-3$ diagonals from any given vertex of the polygon.

1. *Is Arthur's conjecture correct?*

2. *Can you find a counterexample to the conjecture?*

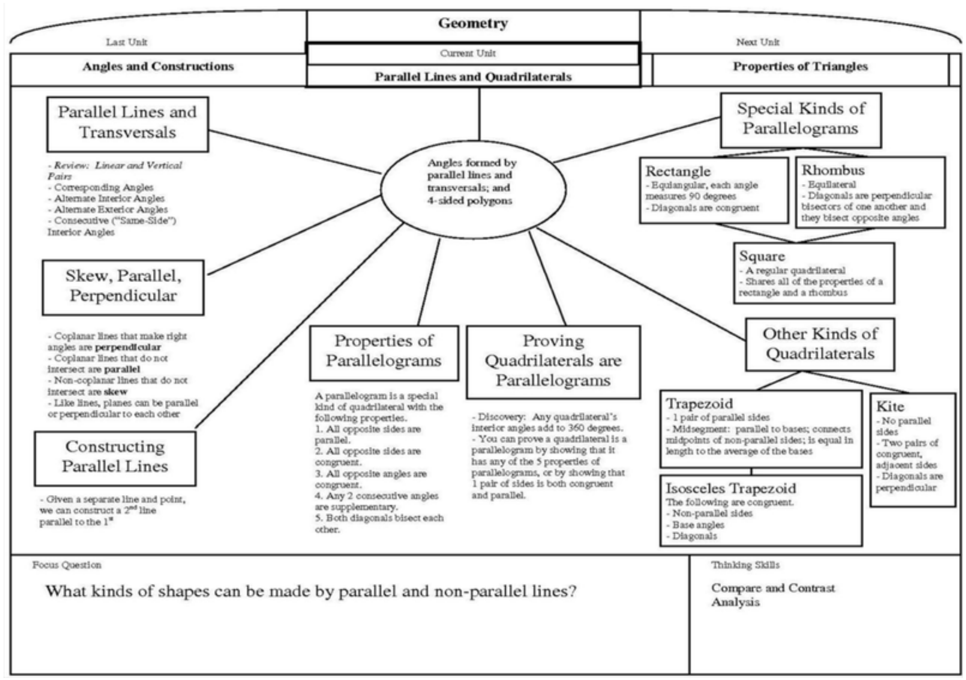
The conjecture appears to be correct. If Arthur draws other polygons, in every case he will be able to draw n[U+0080][U+0093]3 diagonals if the polygon has n sides.

Notice that we have not *proved* Arthur's conjecture. Many examples have (almost) convinced us that it is true.

CHAPTER 2 Parallel Lines and Quadrilaterals

Chapter Outline

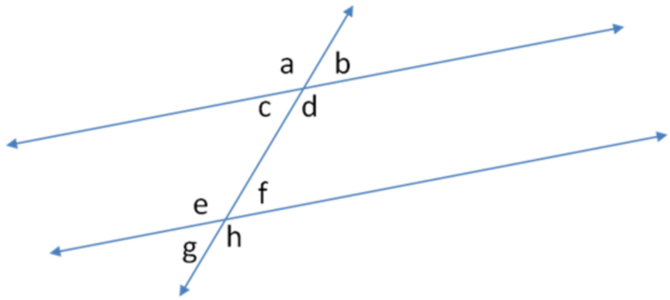
2.1 Concept Map



2.2 Anticipation Guide

Try to answer each of the questions as best you can.

1. In the picture below, which angles appear to be congruent? Circle all that apply.



a. a and e

b. a and g

c. c and f

d. f and d

2. Using the picture in question number 1, match the angle pairs with their angle pair type.

a. __________ corresponding angles – E. b and g

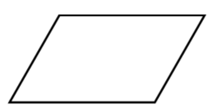
b. __________ alternate interior angles – F. b and f

c. __________ alternate exterior angles – G. c and f

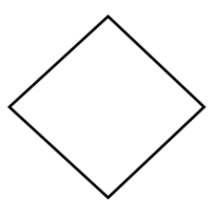
d. __________ same-side interior angles – H. c and e

3. Can you match each shape with its name?

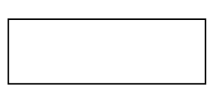
Rhombus



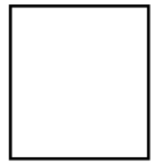
Trapezoid



Parallelogram

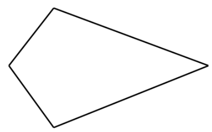


Kite

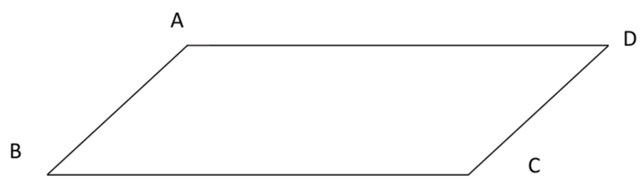


Rectangle

Square



4. Look at the picture of the parallelogram below. Which sides appear to be congruent? Circle all that apply.



a. *AD* and *AB*

b. *AD* and *BC*

c. *AB* and *CD*

d. *CD* and *BC*

5. Look at the picture of the parallelogram in number 4. Which angles appear to be congruent? Circle all that apply.

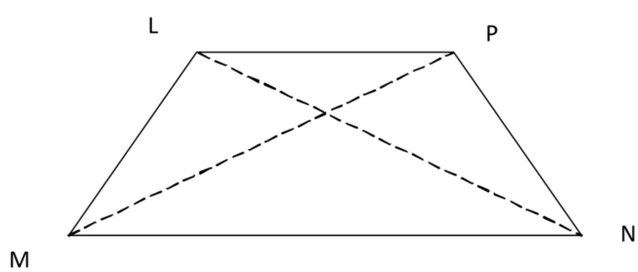
a. *A* and *B*

b. *B* and *C*

c. *A* and *C*

d. *B* and *D*

6. Look at the isosceles trapezoid below. Which of the following statements appear to be true? Circle all that apply.



a. LP is congruent to MN

b. LM is congruent to PN

c. Angle L is congruent to angle N

d. Angle L is congruent to angle M

e. Angle L is congruent to angle P

f. LP is parallel to MN

g. LM is parallel to PN

h. LN is congruent to MP

i. LN is perpendicular to MP

2.3 Vocabulary Self-Rating

TABLE 2.1: Rating Guide: DK: I am sure I don't know it K: I am sure I know it ? I'm not sure

Word	Before Lesson/Unit	After Lesson/Unit
Transversal		
Interior		
Exterior		
Corresponding angles		
Alternate interior angles		
Alternate exterior angles		
Consecutive/same-side interior angles		
Parallel lines		
Perpendicular lines		
Skew lines		
Parallelogram		
Converse statement		
Rectangle		
Square		
Rhombus		
Biconditional statement		
Counterexample		
Trapezoid		
Isosceles trapezoid		
Base angles		
Median		
Kite		
Vertex angle		
Non-vertex angle		
Venn Diagram		

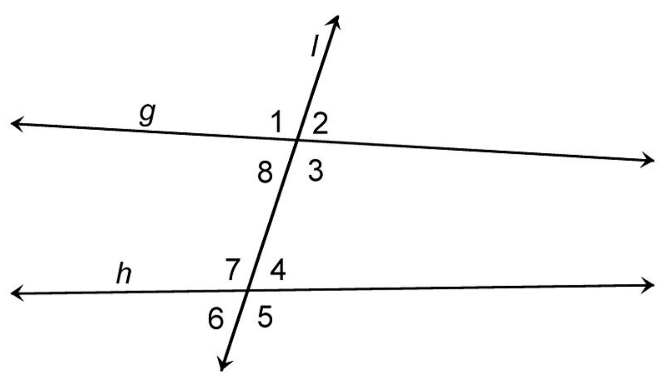
2.4 Non-parallel Lines and Transversals: Identifying Angle Pairs, Part 1

Learning Objectives

- Identify angles made by transversals: *corresponding, alternate interior, alternate exterior and same-side/consecutive interior angles.*

Angles and Transversals

Many geometry problems involve the intersection of three or more lines. Examine the diagram below:



In the diagram, lines g and h are crossed by line l. We have quite a bit of vocabulary to describe this situation:

- Line l is called a **transversal** because it intersects two other lines (g and h). The intersection of line l with g and h forms eight angles as shown.
- The area between lines g and h is called the **interior** of the two lines. The area not between lines g and h is called the **exterior**.
- Angles $\angle 1$ and $\angle 2$ are a **linear pair** of angles. We say they are **adjacent** because they are *next to* each other along a straight line: they share a side and do not overlap.
  - There are many linear pairs of angles in this diagram. Some examples are $\angle 2$ and $\angle 3$, $\angle 6$ and $\angle 7$, and $\angle 8$ and $\angle 1$.

A ______________________________ is a line that cuts across two or more lines.

The area *in between* lines g and h is called the ________________________ of the lines.

The area *above* line g and *below* line h is called the ______________________ of the lines.

The word _____________________________ means "next to."

A linear ____________________ is a set of two angles that are adjacent to each other along a straight line.

The prefix "trans" means "across."

A ***transversal*** cuts *across* two or more lines.

Other words that have the prefix *"trans"* are *trans*port (to carry across) and *trans*mit (to send across).

Can you think of some other words that begin with the prefix "trans"?

Reading Check:

What do you remember about ***linear pairs*** *from Unit 1?*

- They are **adjacent**, which means they are right next to each other and they share a side.
- They are **supplementary**, which means they add up to 180°.

Supplementary angles sum to ____________________.

- $\angle 1$ and $\angle 3$ in the diagram are **vertical angles**. They are *nonadjacent* angles (angles that are *not* next to each other) made by the intersection of two lines.
  - Other pairs of vertical angles in the diagram on the previous page are $\angle 2$ and $\angle 8$, $\angle 4$ and $\angle 6$, and $\angle 5$ and $\angle 7$.

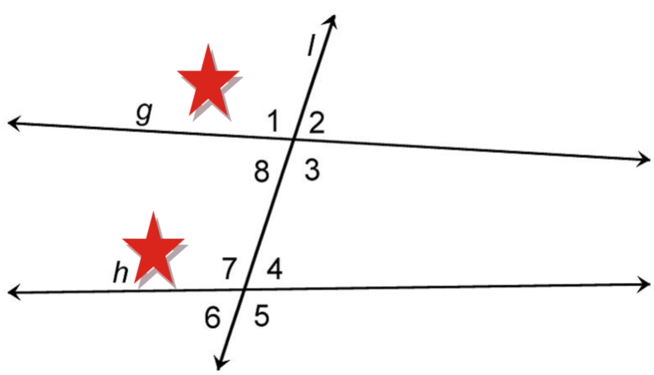
Pairs of ________________________________ angles are *across* from each other at an intersection of two lines.

Reading Check:

What do you remember about ***vertical angles*** *from Unit 1?*

- **Vertical** angles get their name because they have the same **vertex**.
- You learned in Unit 1 that vertical angles are **congruent**. In other words, vertical angles have the same measure.

Corresponding angles are in the *same position* relative to both lines crossed by the transversal. $\angle 1$ is on the upper left corner of the intersection of lines g and l. $\angle 7$ is on the upper left corner of the intersection of lines h and l. So we say that $\angle 1$ and $\angle 7$ are **corresponding angles**.



Corresponding angles are in the ________________________ position at each intersection.

$\angle 3$ and $\angle 5$ are _________________________________ angles because they are both in the bottom right corner of their intersections.

The word "corresponding" means "matching" or "similar."

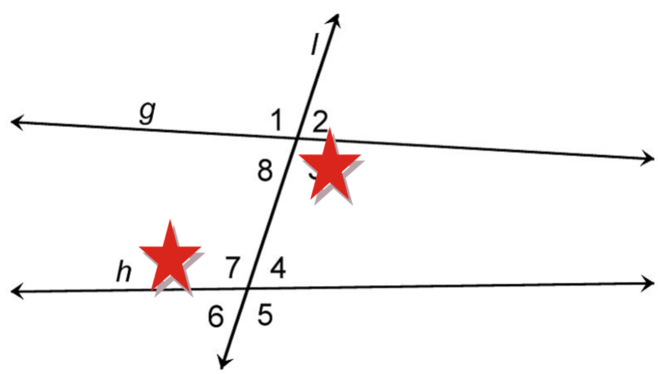
For example, Mexico City corresponds with Washington, D.C., because they are both the capitals of their countries.

Reading Check:

Fill in the blanks with the angle that corresponds to each of the following angles in the diagram above. The first one has been done for you.

1. Angle 1 and angle ______.
2. Angle 8 and angle ______.
3. Angle 2 and angle ______.
4. Angle 5 and angle ______.

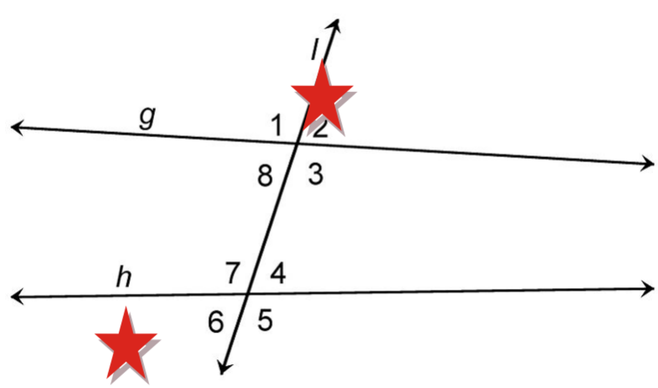
- ∠3 and ∠7 are called **alternate interior angles**. They are in the *interior* region of the lines *g* and *h* and are on *opposite* sides of the transversal.



Alternate __________________________ angles are *inside* lines *g* and *h* and on *opposite* sides of line *l*, the transversal.

∠4 and ∠8 are another example of __________________________ interior angles.

- Similarly, ∠2 and ∠6 are **alternate exterior angles** because they are on *opposite* sides of the transversal, and in the *exterior* of the region between *g* and *h*.



Alternate exterior angles are on opposite sides of the ______________________________.

∠1 and ∠5 are another example of ____________________ ___________________ angles.

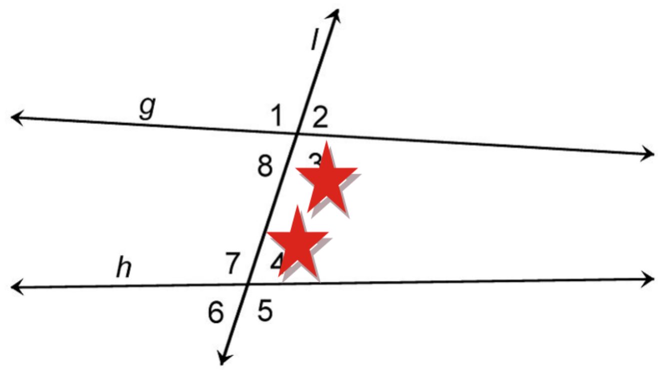
*If you **alternate** between two things, you <u>switch</u> between them.*

*Do you see how **alternate** angles switch sides?*

One angle is to the <u>right</u> of the transversal and the other angle is to the <u>left</u> of the transversal.

To _______________________________ means "to switch."

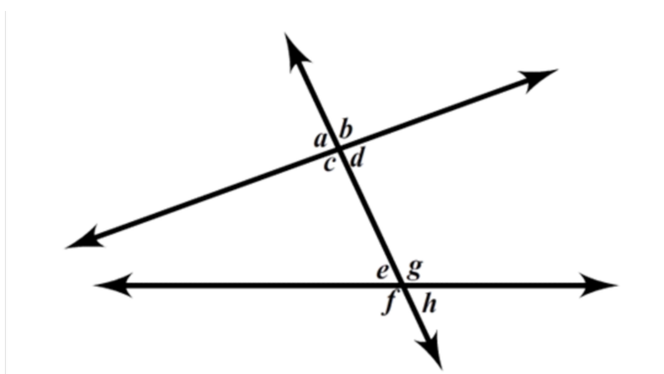
- Finally, $\angle 3$ and $\angle 4$ are **consecutive interior angles**. These are also known as **same-side interior angles**. They are on the *interior* of the region between lines g and h and are on the *same* side of the transversal. $\angle 8$ and $\angle 7$ are also **consecutive interior angles**.



Consecutive interior angles are the same thing as _______________ - ________________ interior angles.

Reading Check:

Look at the picture of the lines being cut by a transversal below. Then, circle the type of angle pair represented by the angles given.



1. a and h corresponding alternate interior alternate exterior same-side interior
2. b and g corresponding alternate interior alternate exterior same-side interior
3. d and h corresponding alternate interior alternate exterior same-side interior
4. d and g corresponding alternate interior alternate exterior same-side interior

2.5 Parallel, Perpendicular, and Skew Lines

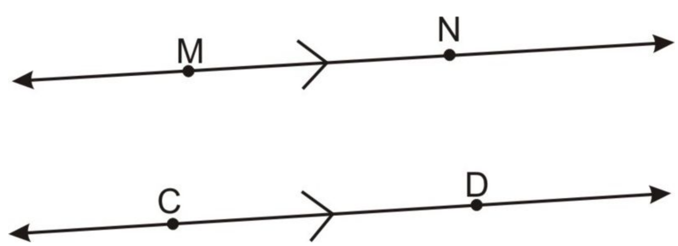
Learning Objectives

- Identify *parallel lines, skew lines* and *perpendicular lines.*

Parallel Lines and Planes

Parallel lines are **coplanar** (they lie in the *same plane*) and never *intersect*.

Below is an example of two **parallel** lines:



Parallel lines *never* ______________________________ each other.

Coplanar lines are in the same __________________________.

We use the symbol || for **parallel**, so we describe the figure above by writing $\overleftrightarrow{MN} \parallel \overleftrightarrow{CD}$.

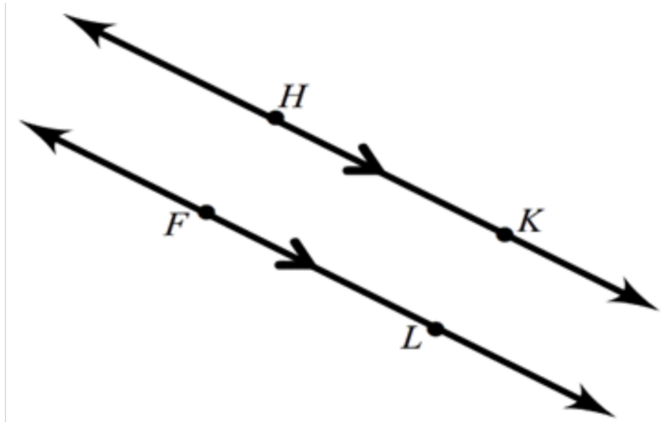
When we draw a pair of parallel lines, we use an arrow mark (>) on the lines to show that the lines are **parallel**. Just like with congruent segments, if there are two (or more) pairs of parallel lines, we use one arrow (>) for one pair and two (or more) arrows ($\gg$) for the other pair.

There are two types of symbols to show that lines are **parallel**:

- In a geometric *statement*, the symbol _____________ is put in between two lines ("line XY" is written as $\overleftrightarrow{XY}$) to give the *notation* for parallel lines.
- In a *picture*, we draw the symbol _____________ on both lines to show that the lines are parallel to each other.

Reading Check:

What symbols let you know that the lines below are parallel?

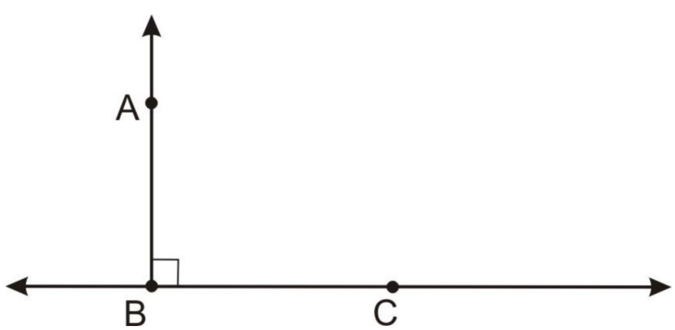


Fill in the blanks to make a symbolic statement that the two lines are parallel.

_____ || _____

Perpendicular Lines

Perpendicular lines intersect at a 90° right angle. This intersection is usually shown by a small square box in the 90° angle.

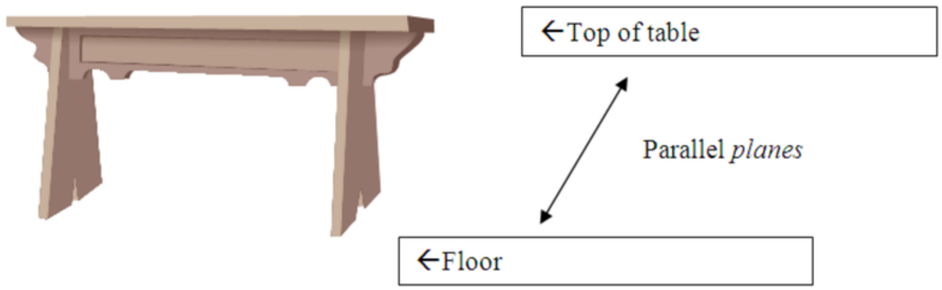


Perpendicular lines meet at a ________________ angle.

The symbol $\perp$ is used to show that two lines, segments, or rays are **perpendicular**. In the picture above, we could write $\overrightarrow{BA} \perp \overleftrightarrow{BC}$. (Notice that $\overrightarrow{BA}$ is a ray while $\overleftrightarrow{BC}$ is a line.)

Note that although **"parallel"** and **"perpendicular"** are defined in terms of lines, the same definitions apply to rays and segments with the minor adjustment that two segments or rays are parallel (or perpendicular) if the lines that contain the segments or rays are parallel (or perpendicular).

If you think about a table, the top of the table and the floor below it are usually in **parallel** planes.

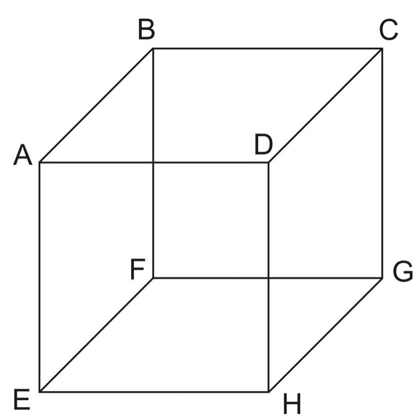


Skew Lines

The other of relationship you need to understand is **skew** lines. Skew lines are lines that are *non-coplanar* (they do *not* lie in the same plane) and *never intersect*.

Skew lines are in *different* ______________________ and *never* _______________________.

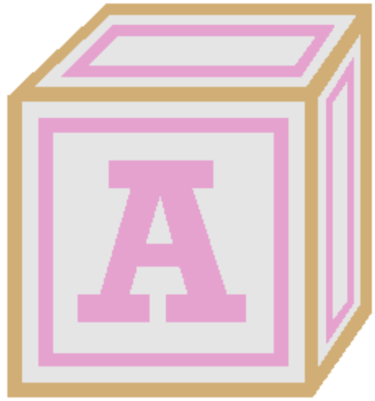
Segments and rays can also be **skew**. In the cube below, segment $\overline{AB}$ and segment $\overline{CG}$ are **skew**:



Reading Check:

In the picture to the right...

- Put arrows on two line segments to show they are **parallel**
- Put a small square box at the intersection of two **perpendicular** segments
- Circle two line segments that are **skew**.


A

2.6 Parallel Lines and Transversals: Identifying Angle Pairs, Part 2

Learning Objectives

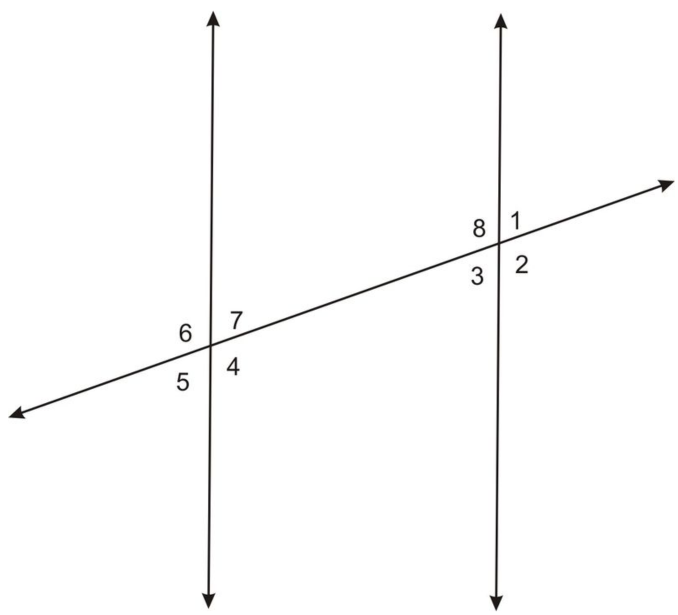
- Identify angles formed by two parallel lines and a non-perpendicular transversal.
- Identify and use the *Corresponding Angles Postulate, Alternate Interior Angles Theorem, Alternate Exterior Angles Theorem,* and *Same-Side Interior Angles Theorem.*

Parallel Lines with a Transversal — Review of Terms

As a quick review, it is helpful to practice identifying different categories of angles.

Example 1

In the diagram below, two vertical parallel lines are cut by a transversal. Identify the pairs of corresponding angles, alternate interior angles, alternate exterior angles, and consecutive interior angles.



Corresponding angles: $\angle 5$ and $\angle 3$, ____________, ____________, and ____________.

- **Corresponding angles** are formed on *different* lines, but in the *same* relative *position* to the transversal — in other words, they face the same direction.
  - There are four pairs of corresponding angles in this diagram — $\angle 6$ and $\angle 8$, $\angle 7$ and $\angle 1$, $\angle 5$ and $\angle 3$, and $\angle 4$ and $\angle 2$.

(Using the same diagram on the previous page):

Alternate interior angles: $\angle 7$ and $\angle 3$, and ____________.

- These angles are on the **interior** of the lines crossed by the transversal and are on **opposite** sides of the transversal.
  - There are two pairs of **alternate interior angles** in this diagram — ∠7 and ∠3, and ∠8 and ∠4.

Alternate exterior angles: ∠6 and∠2, and ____________.

- These are on the **exterior** of the lines crossed by the transversal and are on **opposite** sides of the transversal.
  - There are two pairs of **alternate exterior angles** in this diagram — ∠6 and ∠2, and ∠5 and ∠1.

Consecutive (same-side) interior angles: ∠7 and ∠8, and ____________.

- Consecutive interior angles are in the **interior** region of the lines crossed by the transversal, and are on the **same side** of the transversal.
  - There are two pairs of **consecutive interior angles** in this diagram — ∠7 and ∠8, and ∠3 and ∠4.

Angle Postulates and Theorems

By now you have had lots of practice and should be able to identify relationships between angles.

*Do you remember the difference between a **postulate** and a **theorem**?*

The difference is...

*A **postulate** does not have to be proven. It is self-evident, or obvious. A **theorem** is not obvious. It has to be proven.*

A **postulate** does not have to be ______________________________.

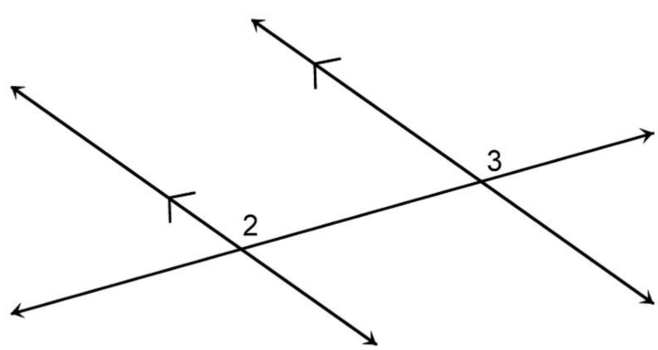
A ______________________________ must be proven.

We will explore a number of **postulates** and **theorems** that involve the different types of angle relationships you just learned.

Corresponding Angles Postulate

If the lines crossed by a transversal are *parallel*, then **corresponding angles** will be *congruent*.

Examine the following diagram:



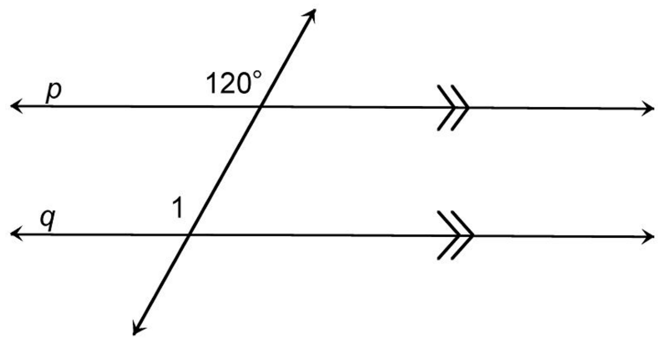
You already know that ∠2 and ∠3 are **corresponding angles** because they are formed by two lines crossed by a transversal and have the same relative placement next to the transversal.

The **Corresponding Angles Postulate** says that because the lines are *parallel* to each other (which we can tell because of the similar arrows on them), the **corresponding angles** will be *congruent*.

- We know the two lines in the diagram above are **parallel** because they are both marked with ____________-______________.
- The **Corresponding Angles Postulate** tells us that corresponding angles are __________________________-_________ if the transversal crosses **parallel** lines.

Reading Check:

In the diagram below, lines p and q are parallel. What is the measure of $\angle 1$?

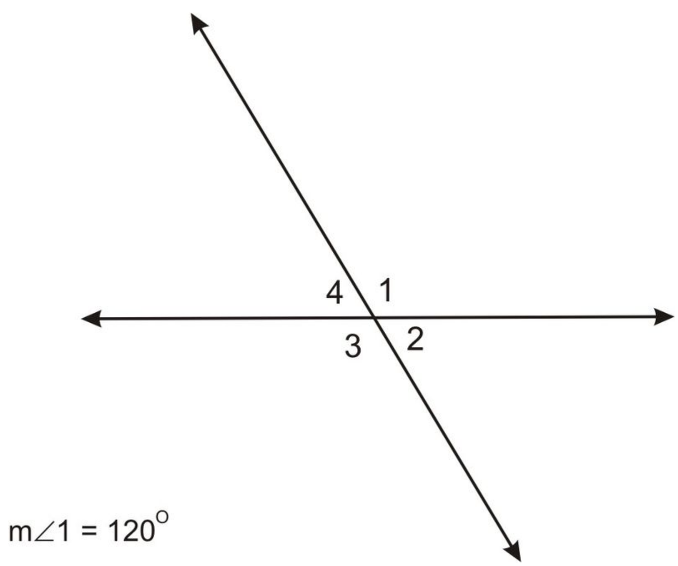


Because lines p and q are parallel, the 120° angle and $\angle 1$ are **corresponding angles**. We know by the **Corresponding Angles Postulate** that they are *congruent*. Therefore, $m\angle 1 = 120^\circ$.

Alternate Interior Angles Theorem

You can use the Corresponding Angles Postulate to derive the relationships between all other angles formed when two lines are crossed by a transversal.

Look at the diagram below and the chart on the following page:



If you know that the measure of angle 1 is 120°, *you can find the measures of the other angles in the picture.*

What are the measures of angles 2, 3, and 4?

Try to figure them out and then check your work in the chart on the next page.

TABLE 2.2:

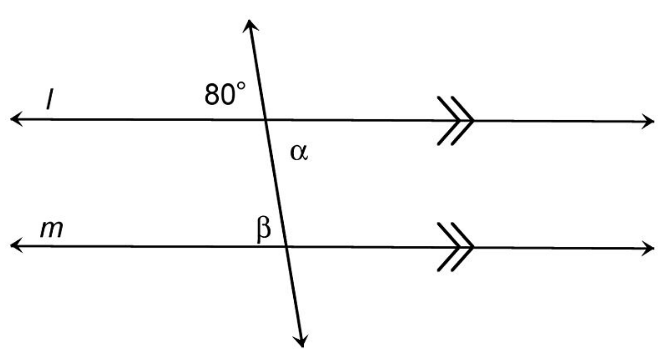
Angle Measure	How We Know
$\angle 2 = 60^\circ$	Because it makes a **linear pair** with $\angle 1$. Remember, linear pairs are *supplementary*. That means they add to be 180° $120 + 60^\circ = 180^\circ$
$\angle 3 = 120^\circ$	There are two ways to find the measure of $\angle 3$. First, $\angle 3$ and $\angle 1$ are **vertical angles**, and vertical angles are *congruent*. $120^\circ = 120^\circ$ Or, you can find the measure of Angle 3 using **linear pairs**. $\angle 2$ and $\angle 3$ make a linear pair. This means they are *supplementary* (add to 180°). We already know that $\angle 2 = 60^\circ$, so $\angle 3$ must be 120°. $60^\circ + 120^\circ = 180^\circ$
$\angle 4 = 60^\circ$	There are two ways to find the measure of $\angle 4$. First, $\angle 4$ and $\angle 2$ are **vertical angles**, and vertical angles are congruent. We know that $\angle 2 = 60^\circ$ $60^\circ = 60^\circ$ Or, you can find the measure of Angle 4 using **linear pairs**. $\angle 3$ and $\angle 4$ make a linear pair. So do $\angle 1$ and $\angle 4$. This means they are *supplementary* (add to 180°). We already know that $\angle 3 = 120^\circ$, so $\angle 4$ must be 60°. $60^\circ + 120^\circ = 180^\circ$

As you can see from this chart, there are some very important relationships we already know that can help us find the measure of missing angles! In this example, the ones we used multiple times are:

- **Linear pairs** are *supplementary* and
- **Vertical angles** are *congruent*

Example 2

Lines l and m in the diagram below are parallel. What are the measures of angles α *and* β*?*



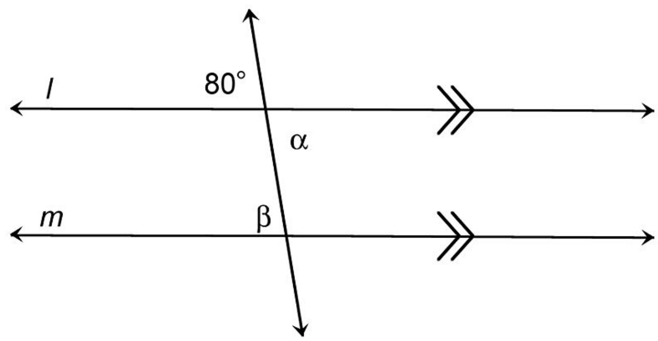
In this problem, you need to find the angle measures of two **alternate interior angles** given an exterior angle. Use what you know.

- There is one angle that measures 80°. Angle β and the 80° angle are **corresponding angles**. So by the Corresponding Angles Postulate, $m\angle\beta = 80^\circ$.

- Now, because $\angle\alpha$ is made by the same intersecting lines and is *opposite* the 80° angle, these two angles are **vertical angles**. Since you already learned that vertical angles are *congruent*, we conclude $m\angle\alpha = 80^\circ$.
- Finally, compare angles α and β. They both measure 80°, so they are *congruent*. This will be true any time two parallel lines are cut by a transversal.

Alternate Interior Angles Theorem

Alternate interior angles formed by two *parallel* lines and a transversal are always *congruent*.



- When two *parallel* lines are crossed by a transversal, the **alternate interior angles** are __________________-_________________.

Alternate Exterior Angles Theorem

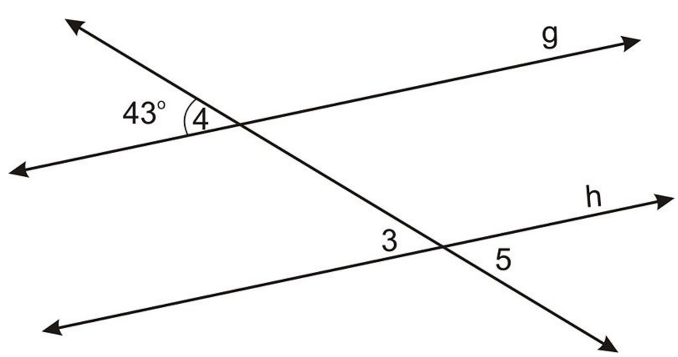
Now you know that pairs of...

- **corresponding,**
- **vertical,** and
- **alternate interior angles** are *congruent.*

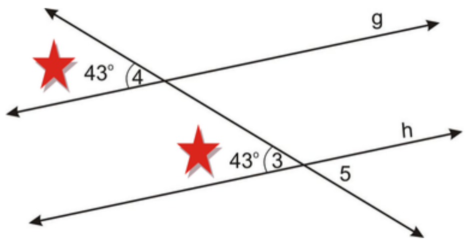
We will use logic to show that **Alternate Exterior Angles** are *congruent* — when two parallel lines are crossed by a transversal, of course.

Example 3

Lines g and h in the diagram below are parallel. If $m\angle 4 = 43^\circ$, what is the measure of $\angle 5$?

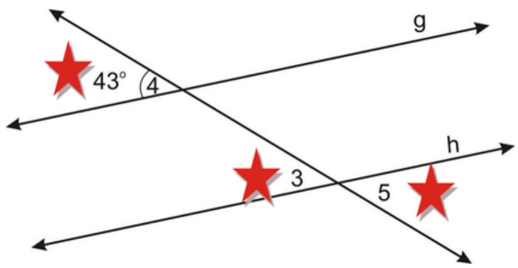


You know from the problem that $m\angle 4 = 43^\circ$. That means that $\angle 4$'s *corresponding* angle, which is $\angle 3$, will measure 43° as well:



← **Corresponding angles** are *congruent*

The corresponding angle you just filled in is also *vertical* to $\angle 5$. Since **vertical angles** are *congruent*, you can conclude that $m\angle 5 = 43^\circ$:



← **Vertical angles** are *congruent*

So, $\angle 4$ is *congruent* to $\angle 5$. In other words, the **alternate exterior angles** are *congruent*.

Alternate Exterior Angles Theorem

If two *parallel* lines are crossed by a transversal, then **alternate exterior angles** are *congruent*.

In Example 3 on the previous page, we proved the **Alternate Exterior Angles Theorem**. We figured this out because:

- When lines are *parallel*, **corresponding angles** are ____________________________.
- Then, all **vertical angles** are ________________________________.

These led us to our conclusion that:

- When two *parallel* lines are cut by a transversal, the **alternate exterior angles** formed are ______________-__________________.

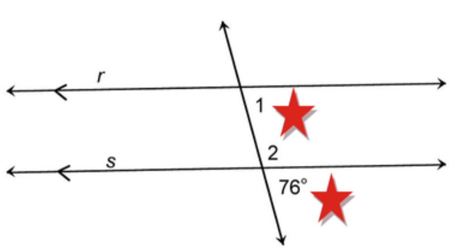
Consecutive (Same-Side) Interior Angles Theorem

The last category of angles to explore in this lesson is **consecutive interior angles**. They fall on the *interior* of the parallel lines and are on the *same* side of the transversal. Use your knowledge of corresponding angles to identify their mathematical relationship.

Example 4

Lines r and s in the diagram below are parallel.

If the angle corresponding to $\angle 1$ measures 76°, what is $m\angle 2$?



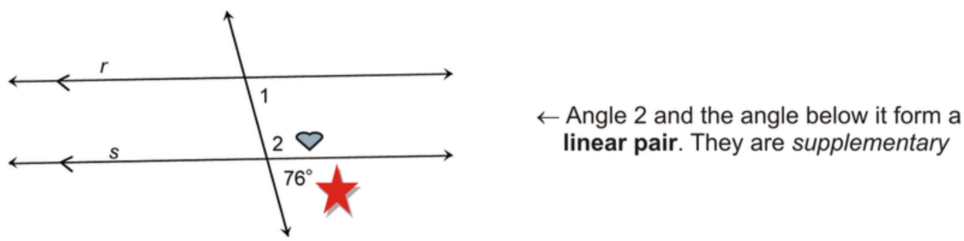
← **Corresponding angles** are *congruent*

This process should now seem familiar. The given $76°$ angle is *adjacent* to $\angle 2$ and they form a **linear pair**. Therefore, the angles are *supplementary*.

Since *supplementary* angles add up to __________, find $m\angle 2$ by subtracting $76°$ from $180°$:

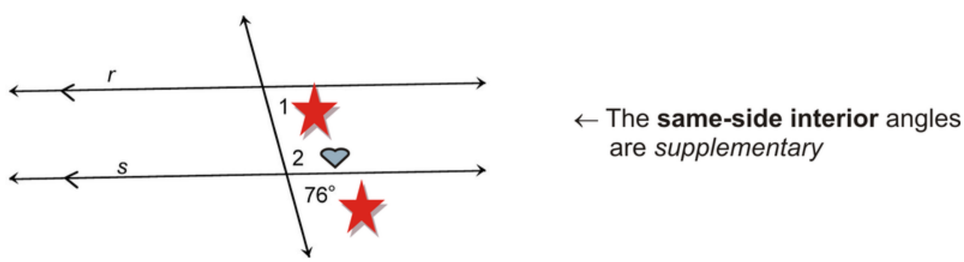
$m\angle 2 = 180° - 76°$

$m\angle 2 = 104°$



Example 4 on the last page shows that if two *parallel* lines are cut by a transversal, the **consecutive interior angles** are *supplementary*; they sum to $180°$.

This is called the **Consecutive Interior Angles Theorem**. We restate it below for clarity.



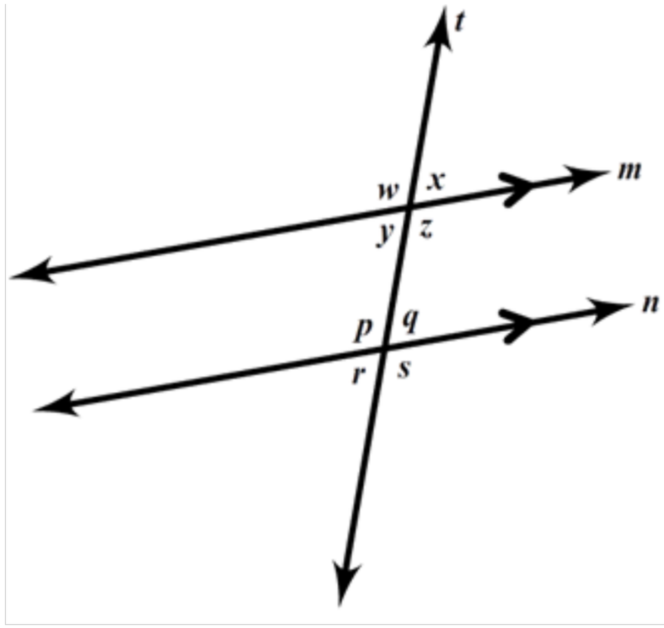
Consecutive (Same-Side) Interior Angles Theorem

If two *parallel* lines are crossed by a transversal, then **consecutive interior angles** are *supplementary*.

- Consecutive Interior Angles and Same-Side Interior Angles are the ______________.
- When two *parallel* lines are crossed by a transversal, **same-side interior angles** are ___________________-____________________.

Reading Check:

In the diagram below, lines m and n are being cut by transversal t.



For questions 1-3, circle the best answer.

1. Which postulate or theorem explains why angles x and q are congruent?

a. Corresponding Angles Postulate

b. Alternate Interior Angles Theorem

c. Alternate Exterior Angles Theorem

d. Consecutive (Same-Side) Interior Angles Theorem

2. If $m\angle w = 70^\circ$, what will be the measure of $\angle s$? What is the reason for your answer?

a. 70°; Corresponding Angles Postulate

b. 70°; Alternate Exterior Angles Theorem

c. 110°; Corresponding Angles Postulate

d. 110°; Alternate Exterior Angles Theorem

3. Which postulate or theorem states that an angle pair is **supplementary** (not **congruent**)?

a. Corresponding Angles Postulate

b. Alternate Interior Angles Theorem

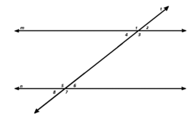
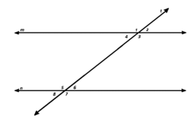
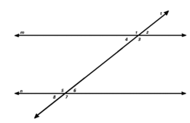
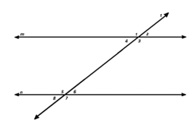
c. Alternate Exterior Angles Theorem

d. Consecutive (Same-Side) Interior Angles Theorem

4. Label all of the angle measures in the diagram above.

Graphic Organizer for Lesson 3

TABLE 2.3: Parallel Lines and Angle Pairs

Angle Pairs	Congruent or Supplementary?	Picture
Corresponding		
Alternate Interior		
Alternate Exterior		
Same-side/Consecutive Interior		

2.7 Construction: Parallel Lines

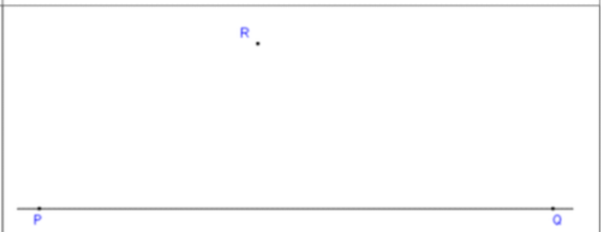
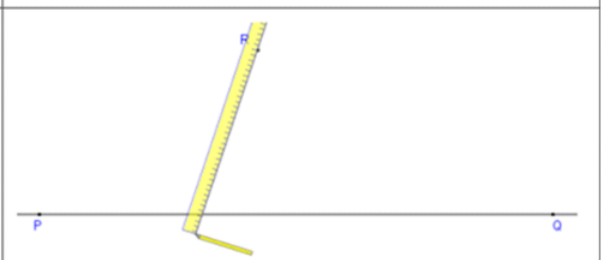
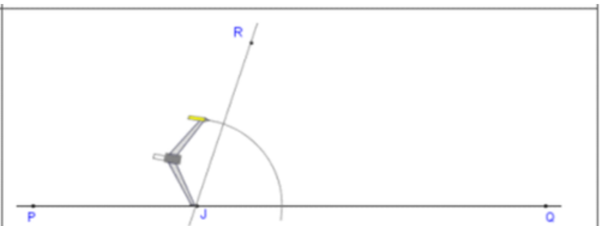
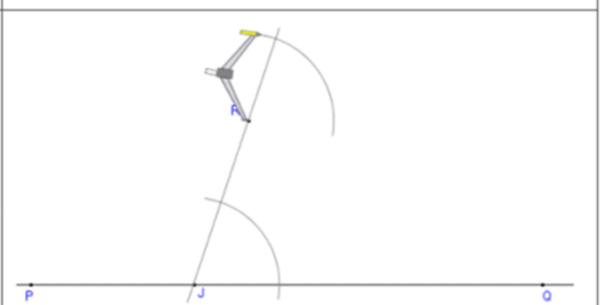
Learning Objectives

- To construct a line parallel to a given line through a given point.

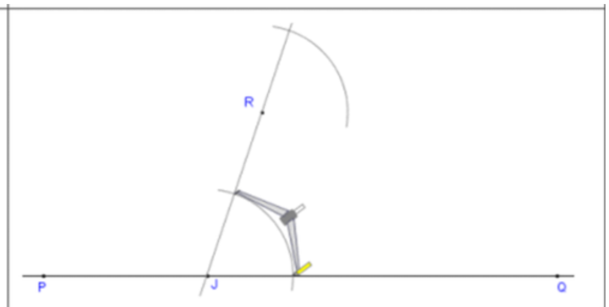
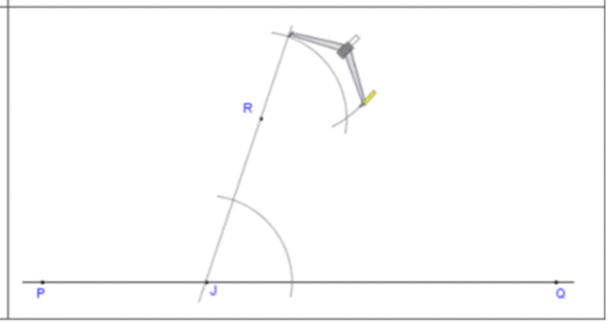
Construction – Parallel Through a Point

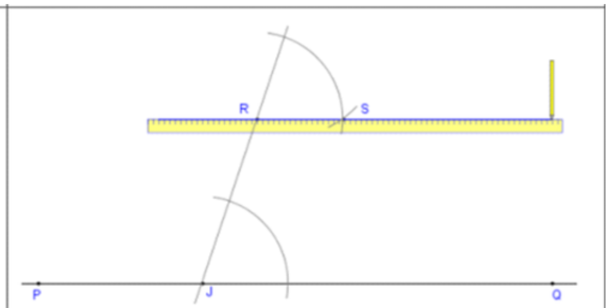
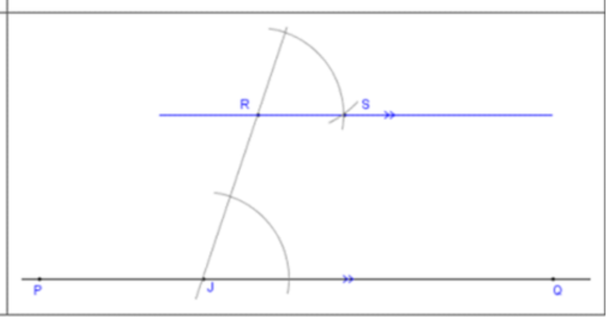
http://www.mathopenref.com/constparallel.html

http://www.youtube.com/watch?v=V1zoNpkiCCI&feature=related

After doing this	Your work should look like this
Start with a line PQ and a point R off the line.	
1. Draw a transverse line through R and across the line PQ at an angle, forming the point J where it intersects the line PQ. The exact angle is not important.	
2. With the compass width set to about half the distance between R and J, place the point on J, and draw an arc across both lines.	
3. Without adjusting the compass width, move the compass to R and draw a similar arc to the one in step 2.	

At this point, you are using the same steps you used to **copy an angle**. You are copying the angle at the bottom of the picture and making the same angle at the top of the picture. When you do this, you are constructing *congruent* **corresponding angles**.

4. Set compass width to the distance where the lower arc crosses the two lines.	
5. Move the compass to where the upper arc crosses the transverse line and draw an arc across the upper arc, forming point S.	

6. Draw a straight line through points R and S.	
Done. The line RS is parallel to the line PQ	

<u>Reading Check</u>

1. Look at the last picture in the table above. Which type of congruent angles do you construct when you construct parallel lines?

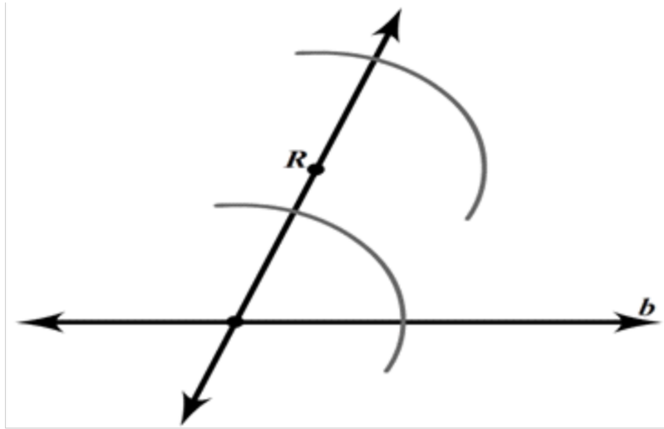
a) Corresponding angles

b) Alternate interior angles

c) Alternate exterior angles

d) Vertical angles

2. *Fill in the blank:* Maria is performing the construction in the picture below. She is constructing a line parallel to line ________ through point _________.



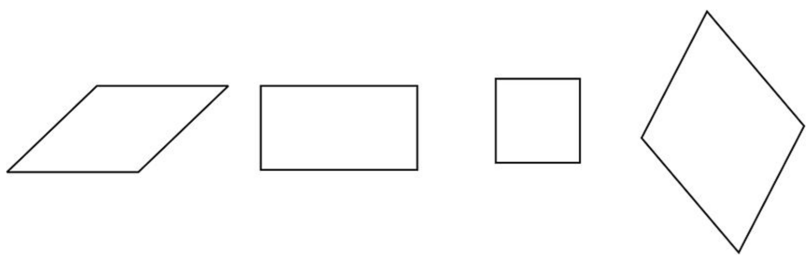
2.8 Properties of Parallelograms

Learning Objectives

- Identify properties of parallelograms.
- Describe the relationships between opposite sides in a parallelogram.
- Describe the relationship between opposite angles in a parallelogram.
- Describe the relationship between consecutive angles in a parallelogram.
- Describe the relationship between the two diagonals in a parallelogram.
- Apply parallelogram properties to solve problems.

Parallelograms

A **parallelogram** is a **quadrilateral** with two pairs of *parallel* sides. Each of the shapes shown below is a parallelogram:



Do you remember what a ***quadrilateral*** *is?*

It is a polygon with four sides.

Some other words that have the same prefix as ***quadrilateral*** *are:*

- ***quad****ruple – to multiply by 4*
- ***qua****rter – one fourth*
- ***quad****ruplets – four brothers and sisters born at the same time*

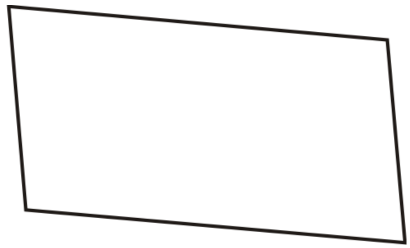
As you can see, **parallelograms** come in a variety of shapes. The *only* defining feature is that *opposite sides are parallel*. But, once we know that a figure is a parallelogram, we have very useful theorems we can use to solve problems involving parallelograms. A **parallelogram** is a ________________________________ with two pairs of *parallel* sides.

A **quadrilateral** is a polygon with ____________ sides.

Reading Check:

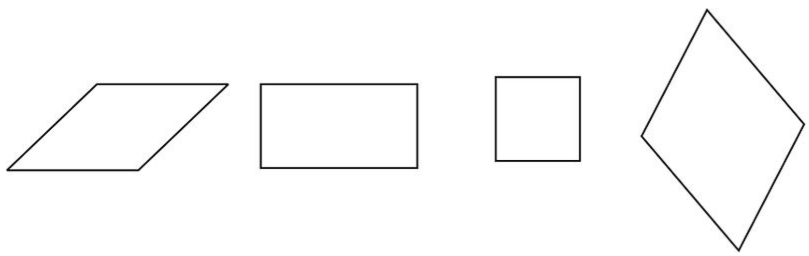
1. What makes a quadrilateral a parallelogram?

2. Mark the parallel lines with arrows (>and ») to show the pairs of *parallel* lines in the picture below:



<u>**Opposite Sides in a Parallelogram**</u>

There are many types of parallelograms. Opposite sides are always parallel.



One of the most important things to know, however, is that *opposite sides* in a parallelogram are also *congruent*.

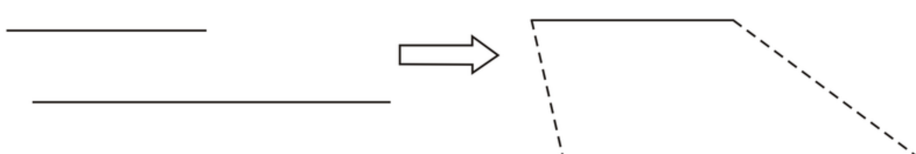
Opposite Sides of Parallelogram Theorem

The opposite sides of a parallelogram are congruent.

Look at the pair of parallel line segments here:

If you want to connect the endpoints of the line segments, you must draw in two **parallel, congruent** line segments.

If the line segments are not congruent, you cannot make a parallelogram:



Not a parallelogram – opposite sides are *not* parallel.

So, even though parallelograms are *defined* by their parallel opposite sides, one of their *properties* is that opposite sides be *congruent*.

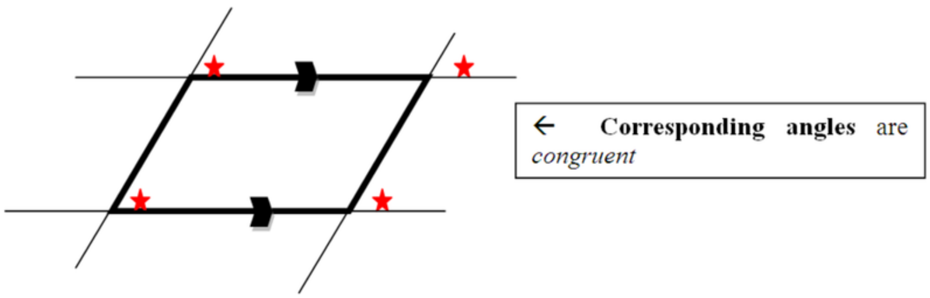
Opposite Angles in a Parallelogram

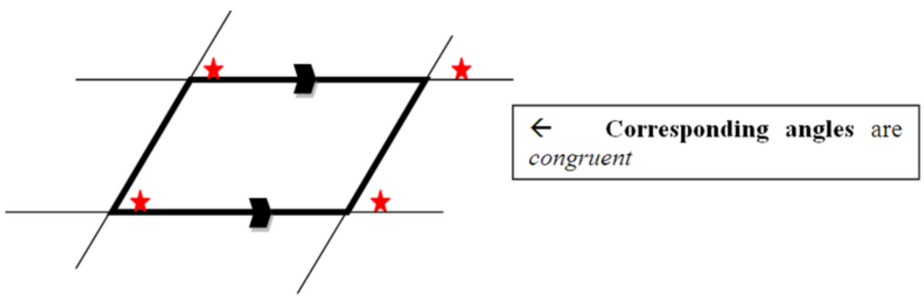
Not only are *opposite sides* in a parallelogram *congruent; opposite angles* are also *congruent*.

Opposite Angles in Parallelogram Theorem

The *opposite angles* of a parallelogram are *congruent*.

You have learned that when lines are *parallel*, their **corresponding angles** are *congruent*.

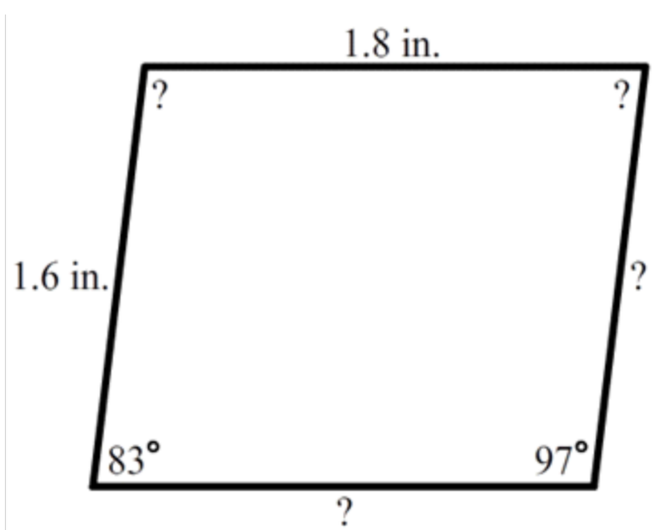




Opposite sides and opposite ____________________ in a parallelogram are *congruent*.

Reading Check:

Mark as many measurements as you can in the picture of the parallelogram below.



Consecutive Angles in a Parallelogram

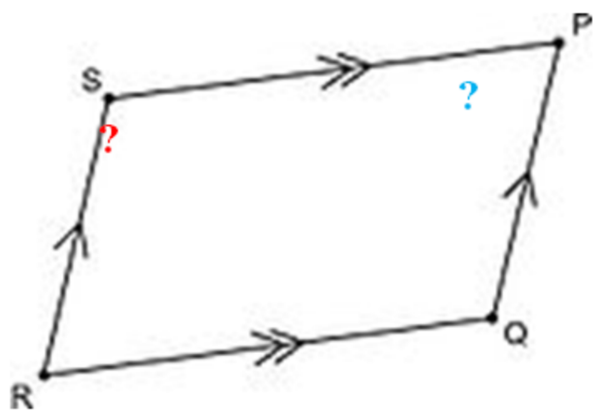
At this point, you understand the relationships between opposite sides and opposite angles in parallelograms.

In a **parallelogram...**

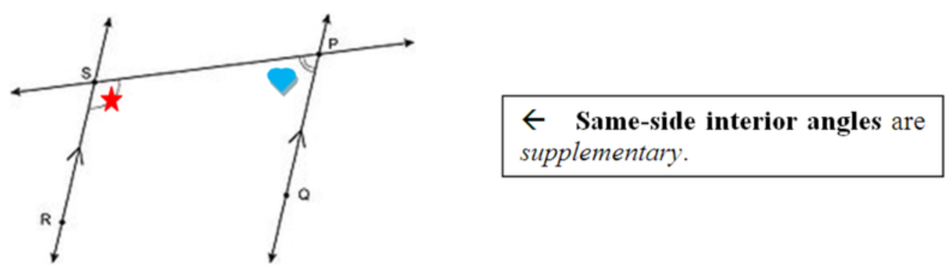
- Opposite sides are *parallel* (this is the definition of a parallelogram)
- Opposite sides are *congruent*
- Opposite angles are *congruent*

Think about the relationship between *consecutive* angles in a parallelogram. You have studied this scenario before, but you can apply what you have learned to parallelograms.

Examine the parallelogram below:



Imagine that you are trying to find the relationship between $\angle SPQ$ and $\angle PSR$ in the diagram on the previous page. To help you understand the relationship, extend all of the segments involved with these angles and remove $\overline{RQ}$ like we have below:



What you should notice is that $\overleftrightarrow{PQ}$ and $\overleftrightarrow{SR}$ are two *parallel* lines cut by transversal $\overleftrightarrow{SP}$.

- $\overleftrightarrow{PQ}$ and $\overleftrightarrow{SR}$ are _______________________________ lines.
- $\angle SPQ$ and $\angle PSR$ are same-side _______________________________ angles.

Earlier in this lesson, you learned that in this scenario, two **consecutive interior angles** are *supplementary*; they sum to 180°. The same is true within the parallelogram. Any two **consecutive angles** inside a parallelogram are *supplementary*.

Consecutive Angles in Parallelogram Theorem

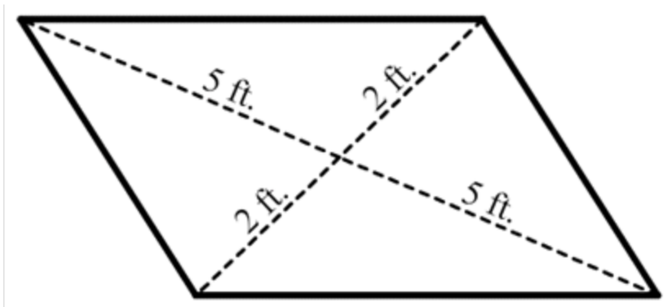
Any two *consecutive* angles of a parallelogram are *supplementary*.

Diagonals in a Parallelogram

There is one more relationship to examine within parallelograms. When you draw the two **diagonals** inside parallelograms, they *bisect* each other. This can be very useful information for examining larger shapes that may include parallelograms.

Diagonals in a Parallelogram Theorem

The *diagonals* of a parallelogram *bisect* one another.



Bisecting diagonals means that the ______________________________ cut each other in half.

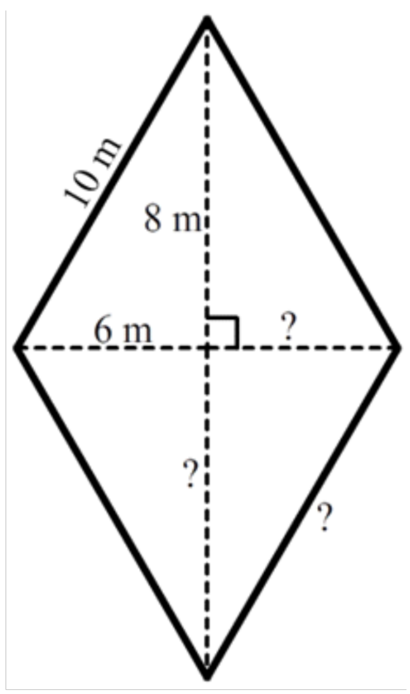
Reading Check:

1. *True or False:* Opposite angles in a parallelogram are congruent.

2. *True or False:* Consecutive angles in a parallelogram are congruent.

3. *True or False:* Opposite sides in a parallelogram are both congruent and parallel.

4. *Find the measure of the missing segments in the picture below.*



2.9 Proving Quadrilaterals are Parallelograms

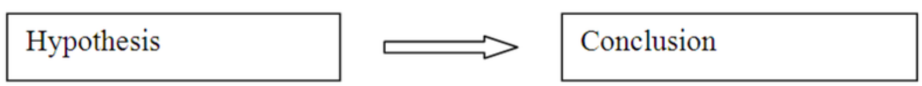
Learning Objectives

- Define *converse* and write the converse of given statements.
- Evaluate a converse statement.
- Prove a quadrilateral is a parallelogram given: two pairs of parallel opposite sides; or, two pairs of congruent opposite sides; or, two pairs of congruent opposite angles; or, two pairs of supplementary consecutive angles; or, diagonals that bisect each other.
- Also, prove a quadrilateral is a parallelogram if one pair of sides is both congruent and parallel.

Converse Statements

A **converse** statement reverses the order of the hypothesis and conclusion in an if-then conditional statement, and is only *sometimes* true.

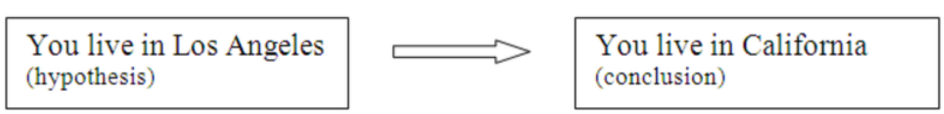
Conditional (if-then) Statement:



Converse Statement:

A **converse** reverses the ________________________ and the ________________________.

Consider the statement: "If you live in Los Angeles, then you live in California."



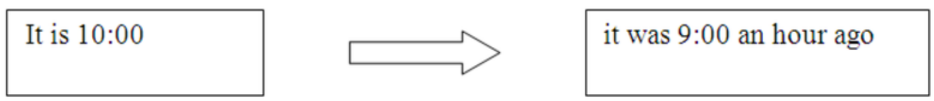
To write the converse of this statement, switch the hypothesis and conclusion:

The converse statement would be "If you live in California, then you live in Los Angeles." Can you see that this statement is **not true**? There are lots of people who live in California who do not live in Los Angeles. Some **counterexamples** to this statement are: people who live in Sacramento, people who live in San Jose, people who live in Oakland, etc.

A **counterexample** is an example that proves that a statement is ____________________.

An example of a statement that is *true* and whose **converse** is also *true* is as follows:

If it is 10:00, then it was 9:00 an hour ago.



The **converse** of this statement is "If it was 9:00 an hour ago, then it is 10:00."

This converse is *true*.

<u>All geometric definitions have true converses.</u>

For example, the *definition* of a triangle could be written this way:

If a shape is a <u>triangle</u>, then it is a <u>three-sided polygon</u>.

Because this statement is a *definition*, its **converse** is also *true*.

If a shape is a <u>three-sided polygon</u>, then it is a <u>triangle</u>.

- Geometric ______________________________ have **converses** that are *true*.

<u>Reading Check:</u>

1. *Write the converse of the following statement:*

If an animal is a cat, then it has a tail.

2. Is the converse you wrote in question #1 true? If not, provide a **counterexample** that proves it is false.

Proving a Quadrilateral is a Parallelogram

There are 6 ways to prove that a **quadrilateral** is a **parallelogram**.

Five of these ways are converses of statements you have already learned.

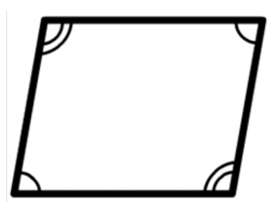
Fill in the table below with the converses of the definition and theorems you have already learned about parallelograms.

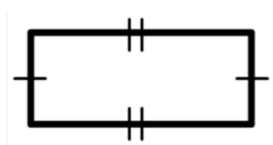
TABLE 2.4:

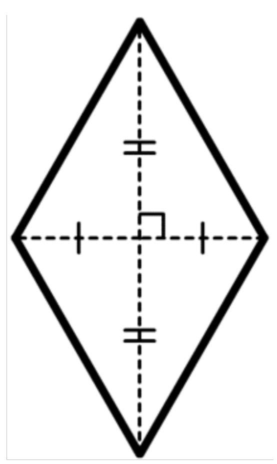
	Statements About Parallelograms	Converse (used to prove a quadrilateral is a parallelogram)
Definition	If a quadrilateral is a parallelogram, then both pairs of opposite sides are parallel.	
Theorems about parallelograms	If a quadrilateral is a parallelogram, then both pairs of opposite sides are congruent.	
Theorems about parallelograms	If a quadrilateral is a parallelogram, then two pairs of opposite angles are congruent.	
Theorems about parallelograms	If a quadrilateral is a parallelogram, then two pairs of consecutive angles are supplementary.	
Theorems about parallelograms	If a quadrilateral is a parallelogram, then diagonals bisect each other.	
New theorem	If a quadrilateral is a parallelogram, then one pair of sides is both parallel and congruent.	

Reading Check:

Which statement could be used to prove that each of the following shapes is a parallelogram?







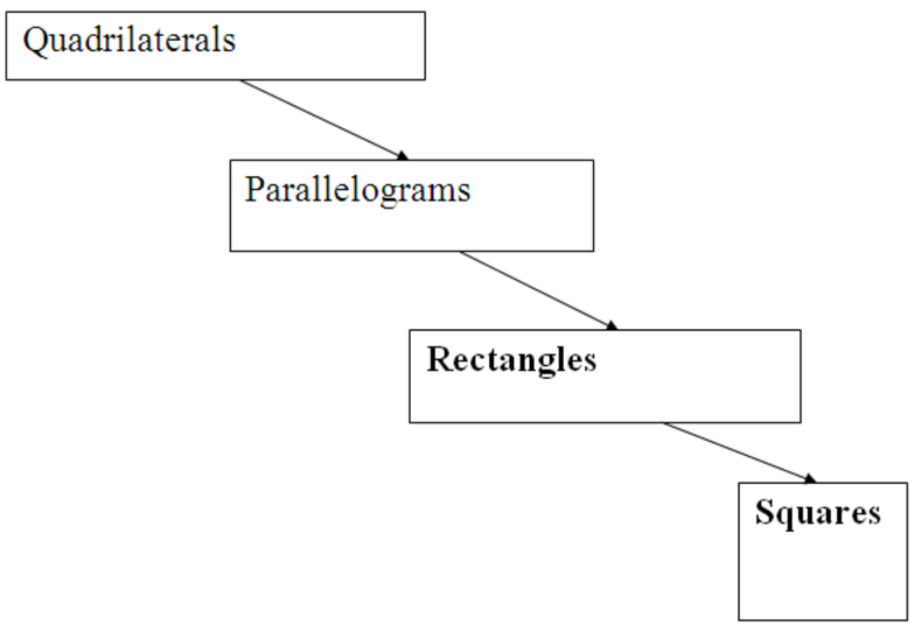
2.10 Properties of Rectangles and Squares

Learning Objectives

- Identify and classify a rectangle.
- Identify and classify a square.
- Identify the relationship between the diagonals in a rectangle.

Rectangles and Squares

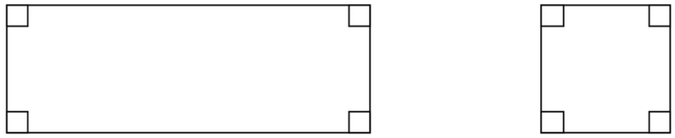
Now that you have a much better understanding of parallelograms, you can begin to look more carefully into certain types of parallelograms. This lesson explores two very important types of parallelograms—**rectangles and squares**.



Rectangles

A **rectangle** is *equiangular:*

- Each angle in a rectangle has the same measure.
- Each angle in a rectangle measures 90°
- In other words, a rectangle has four right angles.
- A **square** is a special kind of **rectangle** and shares all of the properties of rectangles.



Rectangles have four ____________________________ angles.

A **square** is a rectangle with _________________________________ sides.

Diagonals in a Rectangle

Remember that all of the rules that apply to parallelograms still apply to rectangles and squares.

There is one additional property that is specific to rectangles:

The *diagonals* of a **rectangle** are *congruent*.

Do you remember the properties of parallelograms?

These apply to all parallelograms, including rectangles and squares:

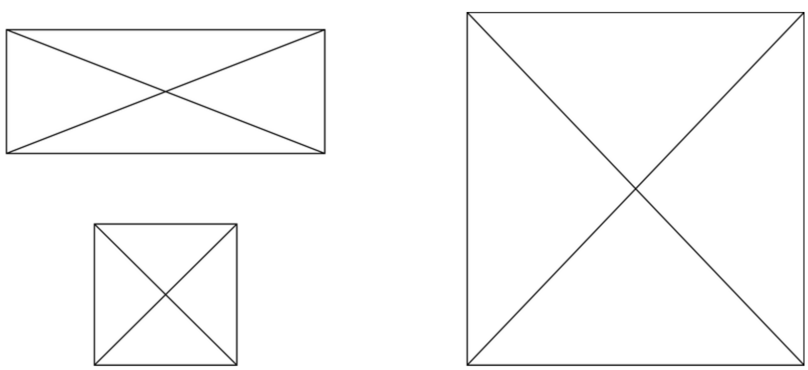
- *opposite sides are parallel*
- *opposite sides are congruent*
- *opposite angles are congruent*
- *consecutive angles are supplementary*
- *diagonals bisect each other*

The *diagonals* in a **rectangle** *and* a **square** are ________________________________.

In a **parallelogram**, *opposite sides* are ____________________ and ____________________.

Theorem for Rectangle Diagonals

The diagonals of a rectangle are congruent.



Reading Check:

1. *True or False:* All the angles in a rectangle are congruent.

2. *True or False:* The diagonals in a rectangle bisect each other.

3. *True or False:* The diagonals in a rectangle are congruent, but the diagonals in a square are not congruent.

2.11 Rhombus Properties

Learning Objectives

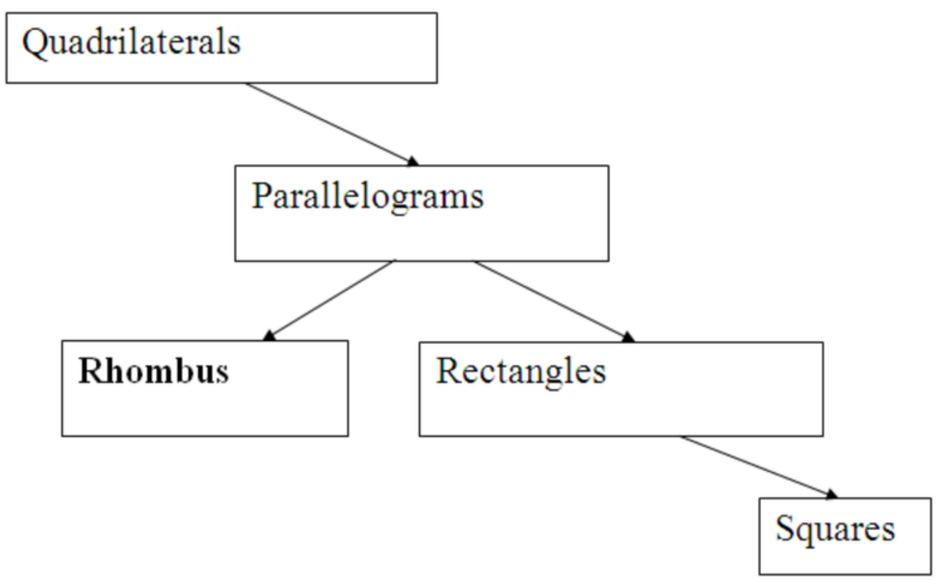
- Identify and classify a *rhombus*.
- Identify the relationship between diagonals in a rhombus.
- Identify the relationship between diagonals and opposite angles in a rhombus.
- Identify and explain *biconditional* statements.

Perpendicular Diagonals in Rhombi

Rhombi (plural of *rhombus*) are *equilateral*.

- All four *sides* of a **rhombus** are *congruent*.
- Also, a **square** is a special kind of rhombus and shares all of the properties of a rhombus.

The diagonals of a rhombus not only bisect each other (because they are parallelograms), they do so at a right angle. In other words, the **diagonals** are *perpendicular*. This can be very helpful when you need to measure angles inside rhombi or squares.

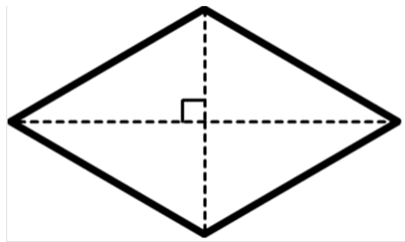


A **rhombus** has four ________________________________ sides.

The *diagonals* of a **rhombus** are the same _______________________ and meet at a _________ angle, meaning they are perpendicular.

Theorem for Rhombus Diagonals

The diagonals of a rhombus are *perpendicular bisectors* of each other.

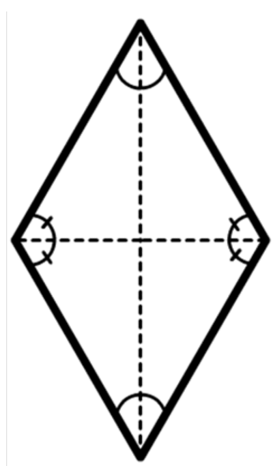


Diagonals as Angle Bisectors

Since a rhombus is a parallelogram, *opposite angles* are *congruent*. One property unique to rhombi is that in any rhombus, the *diagonals* will *bisect* the *interior angles*.

Theorem for Rhombus Diagonals

The diagonals of a rhombus bisect the interior angles.



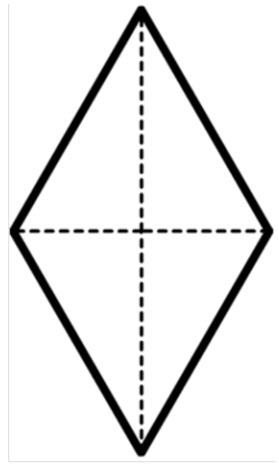
The *diagonals* of a **rhombus** are ______________________________ bisectors of each other.

The *diagonals* of a **rhombus** also ____________________________ the interior angles.

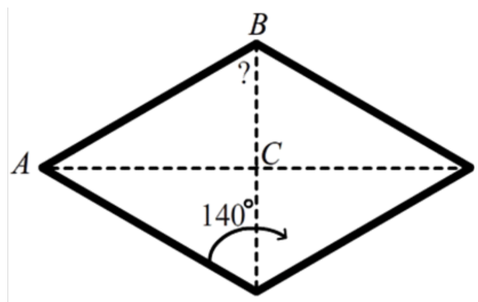
Reading Check:

1. *Fill in the blank:* A rhombus is a parallelogram with congruent _________________________.

2. Label the right angles in the picture below:



3. What is the measure of angle ABC in the rhombus below?



Biconditional Statements

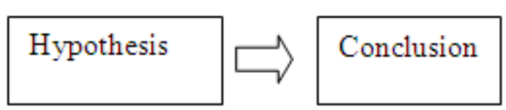
A **biconditional statement** is a conditional statement that also has a true converse.

For example, a true biconditional statement is, "If a quadrilateral is a square then it has exactly four congruent sides and four congruent angles." This statement is *true*, as is its converse: "If a quadrilateral has exactly four congruent sides and four congruent angles, then that quadrilateral is a square."

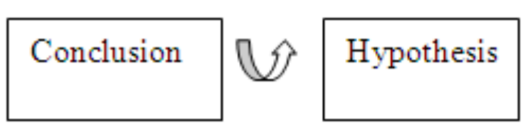
A **biconditional statement** is a *true* if-then statement whose ___________________________ is also *true*.

Remember...

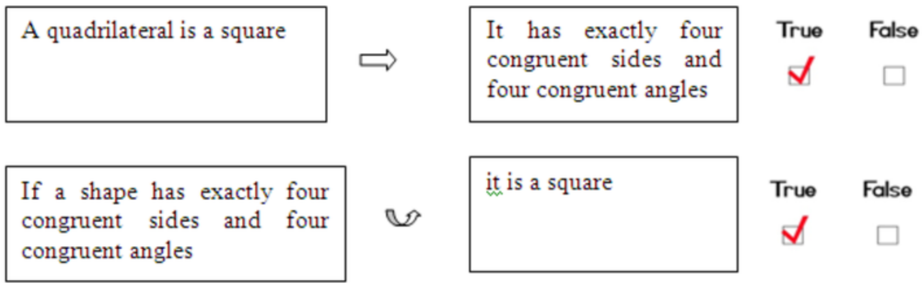
A conditional statement is an "if-then" statement.



A converse is a statement in which the hypothesis and conclusion are reversed.



Sometimes converses are true and sometimes they are not.



When a conditional statement can be written as a **biconditional**, then we use the term **"if and only if."** In the previous example, we could say: "A quadrilateral is a square **if and only if** it has four congruent sides and four congruent angles."

Example 1

Which of the following is a true biconditional statement?

A. A polygon is a square if and only if it has four right angles.

B. A polygon is a rhombus if and only if its diagonals are perpendicular bisectors.

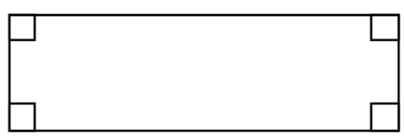
C. A polygon is a parallelogram if and only if its diagonals bisect the interior angles.

D. A polygon is a rectangle if and only if its diagonals bisect each other.

Examine each of the statements to see if it is *true*:

A. A polygon is a square if and only if it has four right angles.

- It is true that if a polygon is a square, it has four right angles. However, the converse statement is not necessarily true. A rectangle also has four right angles, and a rectangle is not necessarily a square. Providing an example that shows something is not true is called a **counterexample**.

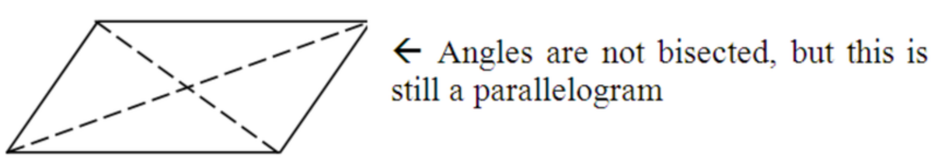


B. A polygon is a rhombus if and only if its diagonals are perpendicular bisectors.

- The second statement seems correct. It is true that rhombi have diagonals that are perpendicular bisectors. The same is also true in converse—if a figure has perpendicular bisectors as diagonals, it is a rhombus.

C. A polygon is a parallelogram if and only if its diagonals bisect the interior angles.

- The third statement is *not* necessarily true. Not all parallelograms have diagonals that bisect the interior angles. This is true only of rhombi, not all parallelograms.



D. A polygon is a rectangle if and only if its diagonals bisect each other.

- This is *not* necessarily true. The diagonals in a rectangle do bisect each other, but parallelograms that are not rectangles also have bisecting diagonals. Choice D is not correct.

So, after analyzing each statement carefully, only B is true. Choice B is the correct answer.

Reading Check:

1. *Write the following biconditional statement as an "if and only if" statement:*

The sun is the star at the center of our solar system. ____-______ if and only if

____.

2. *Is the following statement a true biconditional statement? If not, provide a counterexample.*

A polygon is a quadrilateral if and only if it has four sides.

Graphic Organizer for Lesson 8

TABLE 2.5: Logic Statements

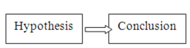
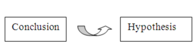
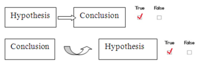
Type of Statement	Description	Example
Conditional Statement 	• "_______________" statement	*If a shape is a polygon, then it has straight sides.*
Converse 	• The hypothesis and conclusion are ________________________ • Not always true • Can be disproven with a ____________________	*If a shape has straight sides, then it is a polygon.*

TABLE 2.5: (continued)

Type of Statement	Description	Example
Biconditional Statement 	• Both the statement and its ____________________ are true. • Good definitions are biconditional.	*If a polygon has three sides, then it is a triangle.* • Statement is true. • Converse (if it's a triangle, then it's a polygon with three sides) is also true.

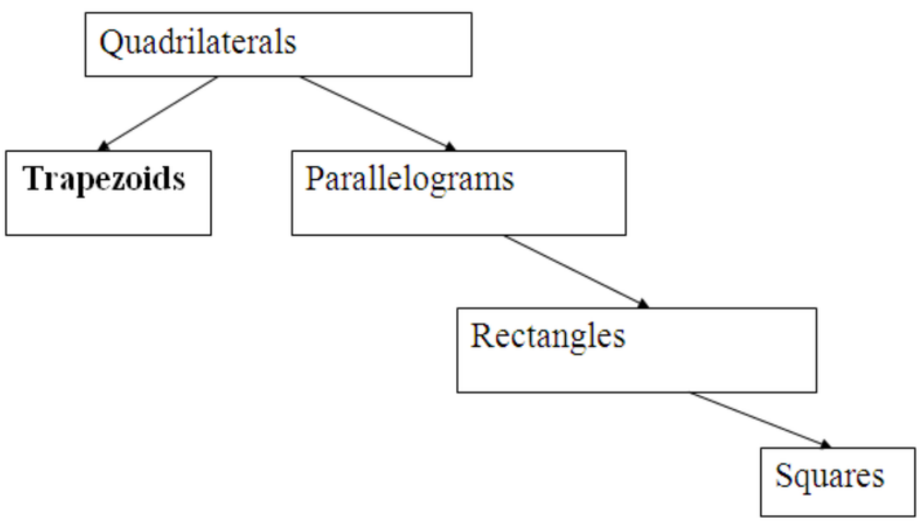
2.12 Trapezoid Properties

Learning Objectives

- Identify and classify a *trapezoid*.
- Identify and classify an isosceles trapezoid.
- State that the base angles of isosceles trapezoids are congruent.
- State that if base angles in a trapezoid are congruent, it is an isosceles trapezoid.
- State that the diagonals in an isosceles trapezoid are congruent.
- State that if the diagonals in a trapezoid are congruent, the trapezoid is isosceles.
- Identify the median of a trapezoid and use its properties.

Trapezoids

Trapezoids are particularly unique figures among quadrilaterals. They have **exactly one pair of parallel sides** so unlike rhombi, squares, and rectangles, they are **not** parallelograms.

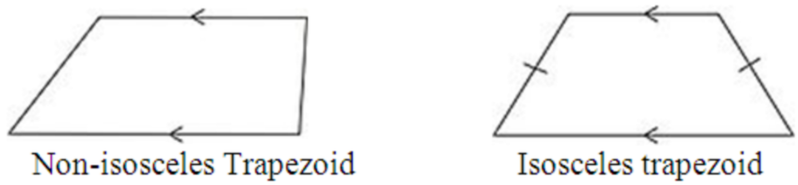


Trapezoids have exactly one set of ______________________ sides.

Trapezoids are quadrilaterals but ____________ parallelograms.

There are special relationships in trapezoids, particularly in isosceles trapezoids.

Below is an example of the difference between isosceles and non-isosceles trapezoids:



The word "isosceles" means "equal legs."

Isosceles trapezoids have non-parallel sides that are of the same lengths.

These equal sides are sometimes called the "legs."

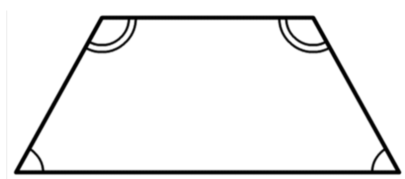
Isosceles trapezoids have *non-parallel sides* (called __________________) that are the same __________________-__________.

Base Angles in Isosceles Trapezoids

The two angles along the same *base* in an isosceles triangle will be *congruent*. Thus, this creates two pairs of congruent angles—one pair along each base.

Theorem for Isosceles Trapezoid

The *base angles* of an **isosceles trapezoid** are *congruent*.

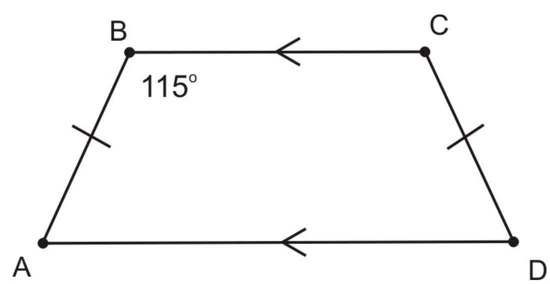


Each set of *base angles* in an **isosceles trapezoid** are ________________________________.

There are ______________ pairs of *base angles* in an **isosceles trapezoid**.

Example 1

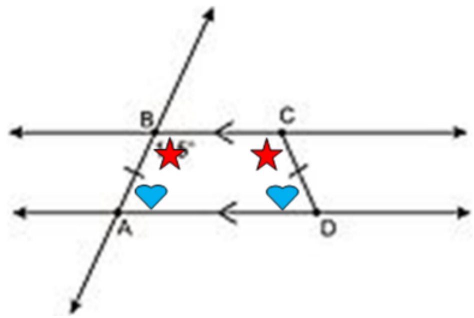
Examine trapezoid ABCD below. What is the measure of angle ADC?



This problem requires two steps to solve.

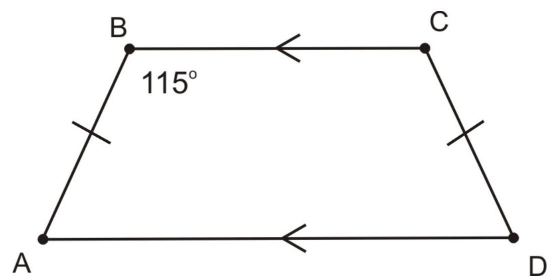
Step 1: You already know that *base angles* in an **isosceles trapezoid** will be *congruent*, but you need to find the relationship between *adjacent* angles as well.

Imagine extending the parallel segments $\overline{BC}$ and $\overline{AD}$ on the trapezoid and the transversal $\overline{AB}$. You will notice that the angle labeled 115° is a *consecutive interior* angle with $\angle BAD$.



← Between two *parallel* lines, *consecutive interior angles* are **supplementary** (they add to 180°)

Consecutive interior angles along two parallel lines will be supplementary. You can find $m\angle BAD$ by subtracting 115° from 180°.



$$m\angle BAD + 115^\circ = 180^\circ$$
$$m\angle BAD = 65^\circ$$

So, $\angle BAD$ measures 65°.

Step 2: Since $\angle BCD$ is *adjacent* to the same *base* as $\angle ADC$ in an **isosceles trapezoid**, the two angles must be *congruent*. Therefore, $m\angle ADC = 65^\circ$.

Identify Isosceles Trapezoids with Base Angles

You previously learned about **biconditional statements** and **converse** statements. You just learned that *if a trapezoid is an isosceles trapezoid then base angles are congruent.*

A trapezoid is an isosceles trapezoid ⇒ Base angles are congruent

The **converse** of this statement is also *true*. If a trapezoid has two congruent angles along the same base, then it is an isosceles trapezoid.

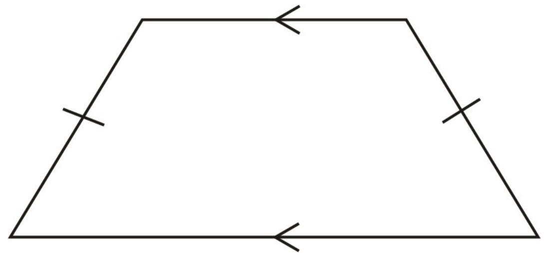
Base angles are congruent ⇒ A trapezoid is an isosceles trapezoid

Theorem for Trapezoid

If two angles along one base of a trapezoid are *congruent*, then the trapezoid is an **isosceles trapezoid.**

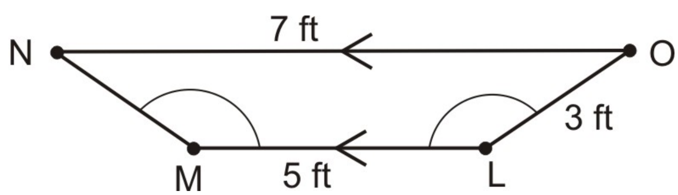
If base angles are *congruent*, then the trapezoid is ___________________________.

You can use this fact to identify lengths in different trapezoids. An **isosceles trapezoid** has one pair of *congruent* sides:



Example 2

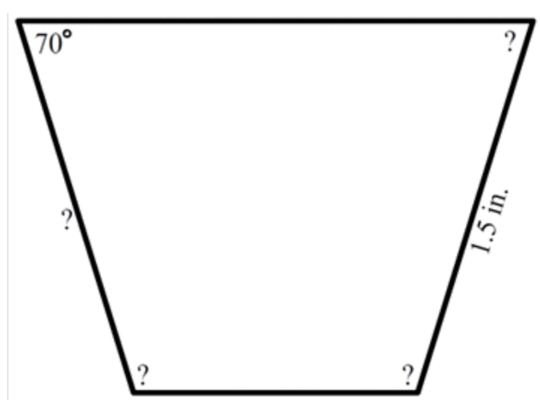
What is the length of $\overline{MN}$ in the trapezoid below?



Notice that in trapezoid $LMNO$, two base angles are marked as *congruent*. So, the trapezoid is *isosceles*. That means that the two *non-parallel* sides have the same length. Since you are looking for the length of $\overline{MN}$, it will be congruent to $\overline{LO}$. So, $MN = 3$ feet.

Reading Check:

Label as much information on the following isosceles trapezoid as you can.

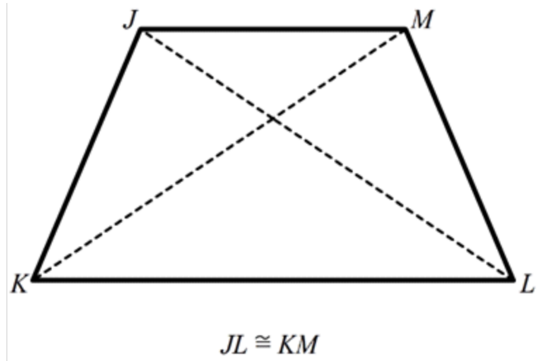


Diagonals in Isosceles Trapezoids

The angles in **isosceles trapezoids** are important to study. The diagonals, however, are also important. The diagonals in an isosceles trapezoid will not necessarily be perpendicular as in rhombi and squares. They are, however, *congruent*. Any time you find a trapezoid that is isosceles, the two *diagonals* will be *congruent*.

Theorem for Trapezoids Diagonals

The *diagonals* of an **isosceles trapezoid** are *congruent*.



The *diagonals* in an **isosceles trapezoid** are ______________________________________.

Identifying Isosceles Trapezoids with Diagonals

The **converse** statement of the theorem stating that diagonals in an isosceles triangle are congruent is also *true*. If a trapezoid has *congruent diagonals*, it is an *isosceles* trapezoid. If you can prove that the diagonals are congruent, then you can identify the trapezoid as isosceles.

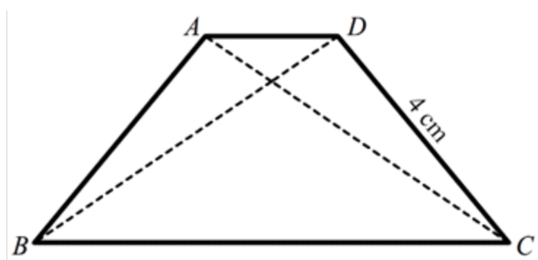
Theorem for Trapezoid Diagonals

If a trapezoid has *congruent diagonals*, then it is an **isosceles trapezoid**.

If a trapezoid has *diagonals* that are *congruent*, then it is ___________________________.

Example 3

In the figure below, $DB = AC$. What is the length of $\overline{AB}$?

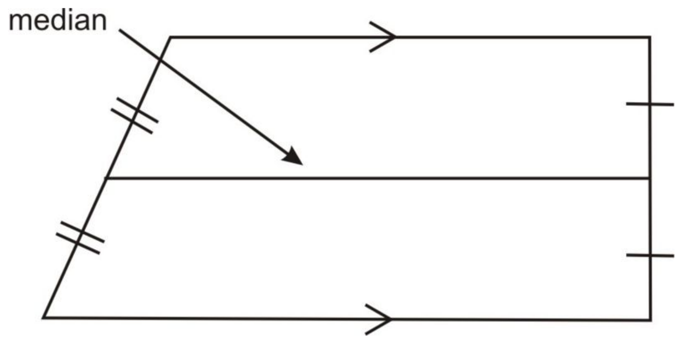


Because DB and AC are *diagonals* of trapezoid $ABCD$, and DB and AC are *congruent*, we know that this trapezoid is **isosceles**.

- Isosceles trapezoids have two congruent sides.
- Since $CD = 4\ cm$, AB must also be equal to 4 cm.

Trapezoid Medians

Trapezoids can also have segments drawn in called **medians**. The **median** of a trapezoid is a segment that *connects the midpoints* of the non-parallel sides in a trapezoid. The median is located *half way* between the *bases* of a trapezoid.

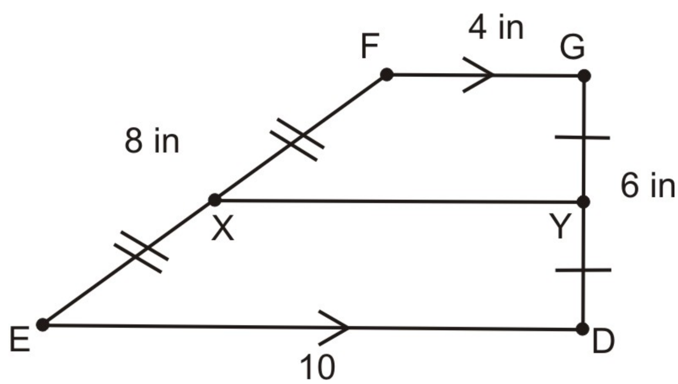


A **median** connects the ________________________ of the *non-parallel* sides in a trapezoid.

A trapezoid's **median** is half way between its _________________________.

Example 4

In trapezoid $DEFG$ below, segment XY is a median. What is the length of $\overline{EX}$?



The **median** of a trapezoid is a segment that is *equidistant* between both *bases*. In other words, it divides the sides into two *congruent* parts.

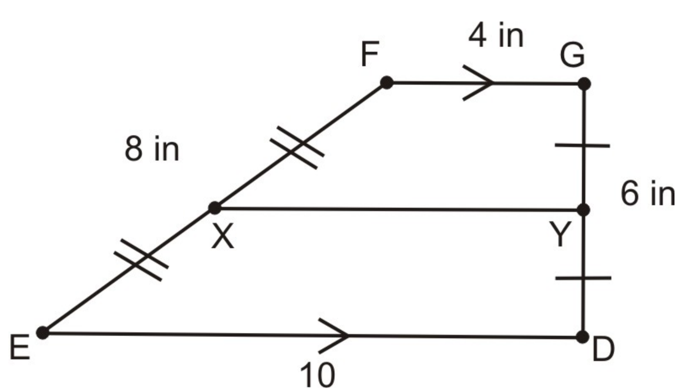
- So, the length of $\overline{EX}$ will be equal to *half* the length of $\overline{EF}$.
- Since you know that $EF = 8$ inches, you can divide that value by 2. Therefore, EX is 4 inches.

Theorem for Trapezoid Medians

The length of the median of a trapezoid is equal to *half* of the *sum of the lengths of the bases*.

In other words, to find the length of the median, *average* the two bases.

Remember, the average is the *sum* of both numbers (bases) *divided* by 2.



This theorem can be illustrated in the example above,

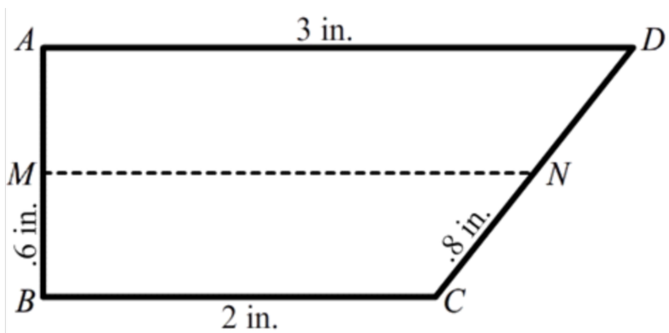
$$XY = \frac{FG + ED}{2}$$
$$XY = \frac{4 + 10}{2}$$
$$XY = 7$$

Therefore, the measure of segment XY is 7 inches.

- The length of a trapezoid's ________________________ is the *average* of its *bases*.

Reading Check:

Find the following measures in trapezoid ABCD below:



$MN =$ _____

$MA =$ _____

$BA =$ _____

$CD =$ _____

$ND =$ _____

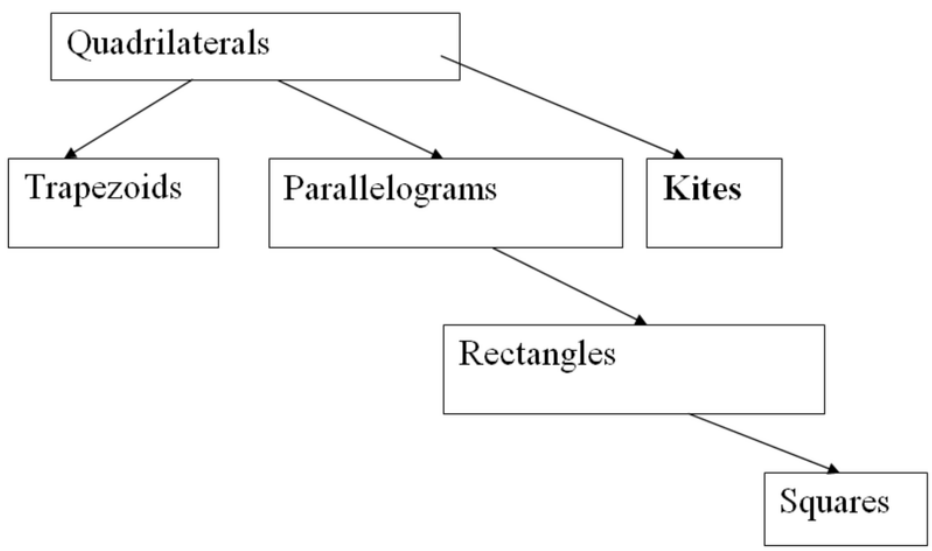
2.13 Kite Properties

Learning Objectives

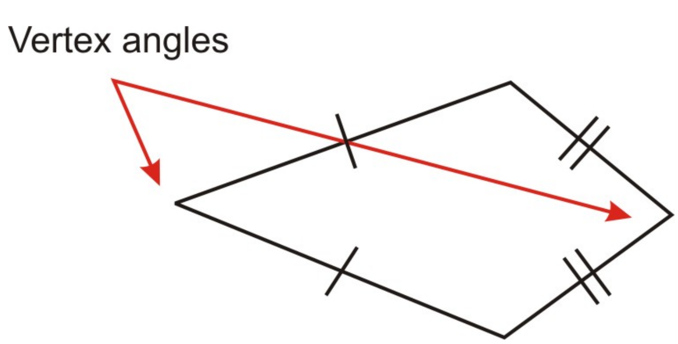
- Identify and classify a *kite*.
- Identify the relationship between diagonals in kites.
- Identify the relationship between opposite angles in kites.

Kites

Among all of the quadrilaterals you have studied so far, **kites** are probably the most unusual.

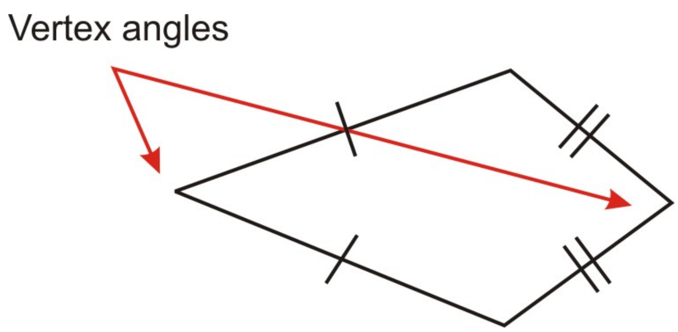


Kites have *no parallel sides*, but they do have *congruent* sides. **Kites** are defined by two pairs of congruent sides that are *adjacent* to each other, instead of opposite each other.



Kites have two pairs of *congruent* sides that are ________________________ to each other.

A **vertex angle** is between two *congruent* sides and a **non-vertex angle** is between sides of different lengths:



The **vertex angle** of a **kite** is between the two ____________________________ sides.

The **non-vertex angle** of a **kite** is between the sides of _________________________ lengths.

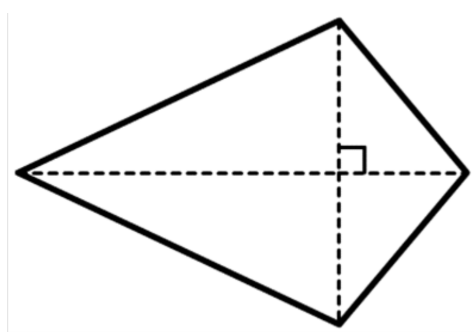
Kites have a few special properties that can be proven and analyzed just as the other quadrilaterals you have studied. This lesson explores those properties.

Diagonals in Kites

The relationship of diagonals in kites is important to understand. The diagonals are *not congruent*, but they are always *perpendicular*. In other words, the diagonals of a kite will always intersect at right angles.

Theorem for Kite Diagonals

The *diagonals* of a **kite** are *perpendicular*.

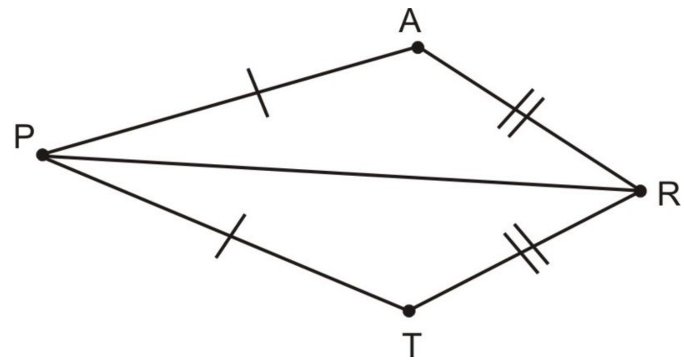


The *diagonals* of a **kite** are ____.

The *diagonals* of a **kite** intersect each other at _____________ angles.

Opposite Angles in Kites

In addition to the bisecting property, one other property of kites is that the **non-vertex angles** are *congruent*.



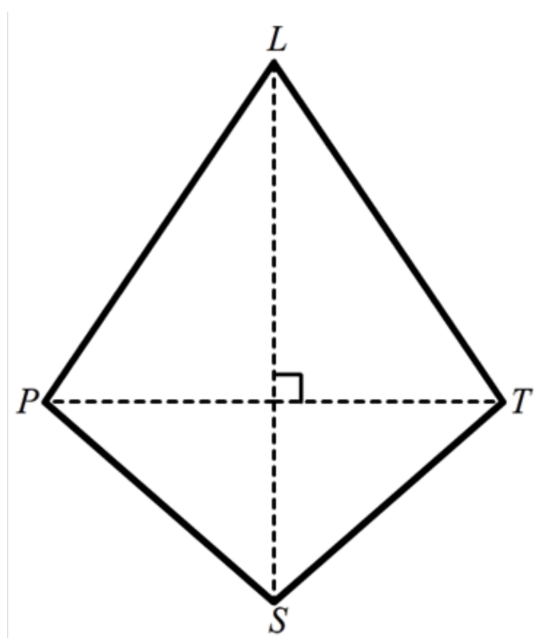
So, in the kite *PART* above, $\angle PAR \cong \angle PTR$.

The **non-vertex angles** in a **kite** are ________________________________.

<u>Reading Check:</u>

On the diagram below, mark the following...

- *Two pairs of congruent sides*
- *One pair of congruent angles*
- *Right angles*



2.14 Classifying Quadrilaterals

Learning Objectives

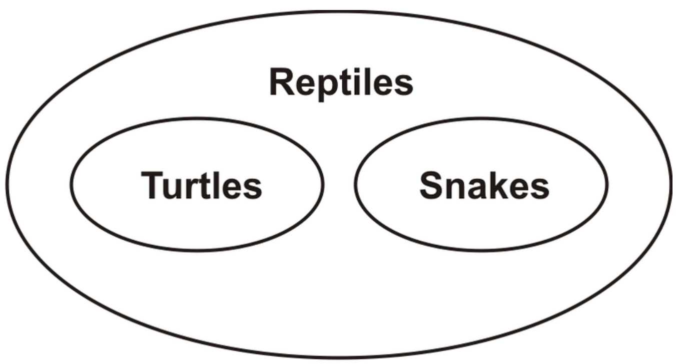
- Use a Venn diagram to classify quadrilaterals.

Using a Venn Diagram for Classification

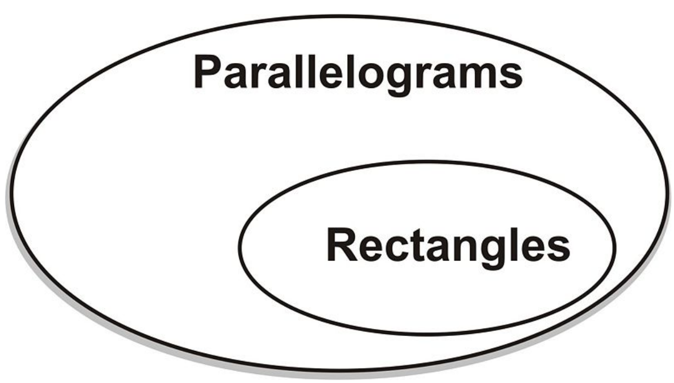
You have just explored many different rules and classifications for quadrilaterals. There are different ways to collect and understand this information, but one of the best methods is to use a **Venn Diagram**. Venn Diagrams are a way to classify objects according to their properties.

A _______________ Diagram is a visual way to classify objects according to their properties.

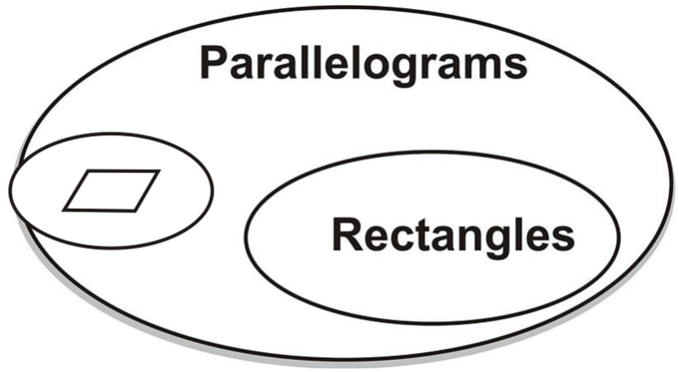
Think of a snake, which is one type of reptile. All snakes are reptiles, but there are also some reptiles that are not snakes (like turtles and lizards).



Now let's apply the use of a **Venn Diagram** to Geometry. Think of a rectangle. A *rectangle* is a type of *parallelogram*, but not all *parallelograms* are *rectangles*. Here's a simple **Venn Diagram** of that relationship:



Notice that *all rectangles are parallelograms,* but *not all parallelograms are rectangles.*



If an item falls into *more* than one category, it is placed in the *overlapping* section between the appropriate classifications. For example, it is possible for an animal to both have legs and be a reptile. An example of an animal that both has legs and is a reptile is a turtle.

There are some reptiles that do not have legs, like snakes.

There are also some animals with legs that are not reptiles, like a cat.

Finally, there are also some animals that *neither* have legs, *nor* are reptiles. These animals would be placed *outside* of the circles. An example of an animal that is neither a reptile nor has legs is a jellyfish.

To begin organizing the information for a **Venn diagram**, you can analyze the quadrilaterals we have discussed thus far by three characteristics: *parallel sides, congruent sides,* and *congruent angles*.

Below is a table that shows how each quadrilateral fits these characteristics:

TABLE 2.6:

Shape	Number of pairs of parallel sides	Number of pairs of congruent sides	Four congruent angles
Parallelogram	2	2	No
Rhombus	2	2	No
Rectangle	2	2	Yes
Square	2	2	Yes
Kite	0	2	No
Trapezoid	1	0	No
Isosceles trapezoid	1	1	No

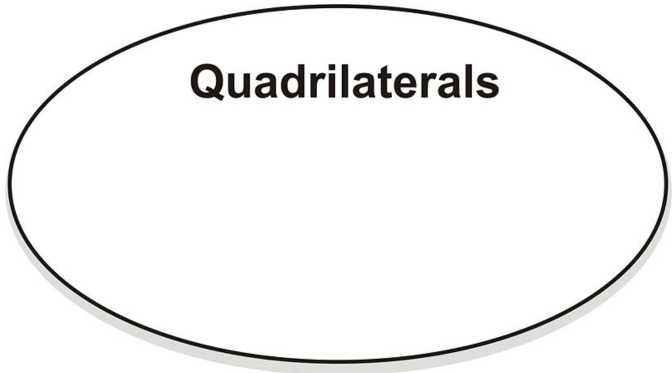
We can choose any characteristics we want to make our **Venn Diagram**. Let's make a Venn Diagram based on the *number of pairs of parallel sides* a quadrilateral has. There will be three main categories:

- Has **two pairs of parallel sides** (parallelogram, rhombus, rectangle, square)
- Has **one pair of parallel sides** (trapezoid, isosceles trapezoid)
- Has **no pairs of parallel sides** (kite)

Example 1

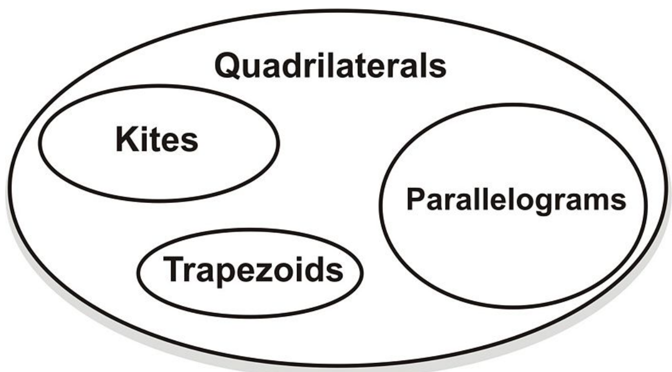
Organize the classification information in the table above in a Venn Diagram.

To begin a Venn Diagram, you must first draw a large ellipse representing the biggest category. In this case, that will be quadrilaterals.



Now, we can add in classes of quadrilaterals.

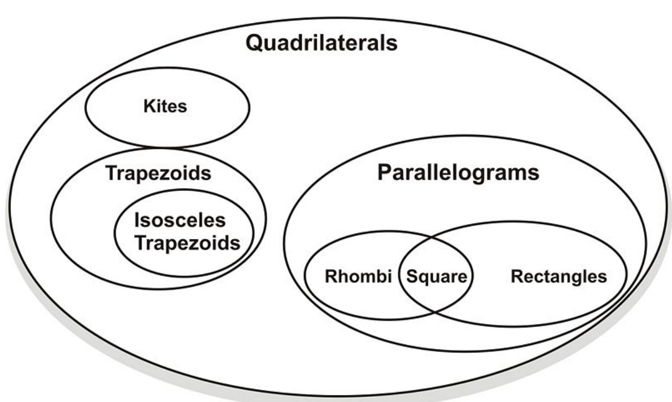
- The first class of quadrilaterals is the one with two pairs of parallel sides: **parallelograms**.
- The second class will be quadrilaterals with one pair of parallel sides: **trapezoids**.
- Finally, the third class will be quadrilaterals with no parallel sides: **kites**.



Zoom in on the parallelogram oval. There are several types of parallelograms:

- **Squares, rectangles,** and **rhombi**are all types of parallelograms.
  - Some rhombuses are rectangles. These rhombuses are called squares.
- Also, under the category of **trapezoids** we need to add **isosceles trapezoids.**

The completed Venn diagram is like this:



You can use this Venn Diagram to quickly answer questions. For instance, is every square a rectangle? (Yes.) Is every rhombus a square? (No, but some are.)

Reading Check:

Use the Venn Diagram above to answer the following questions.

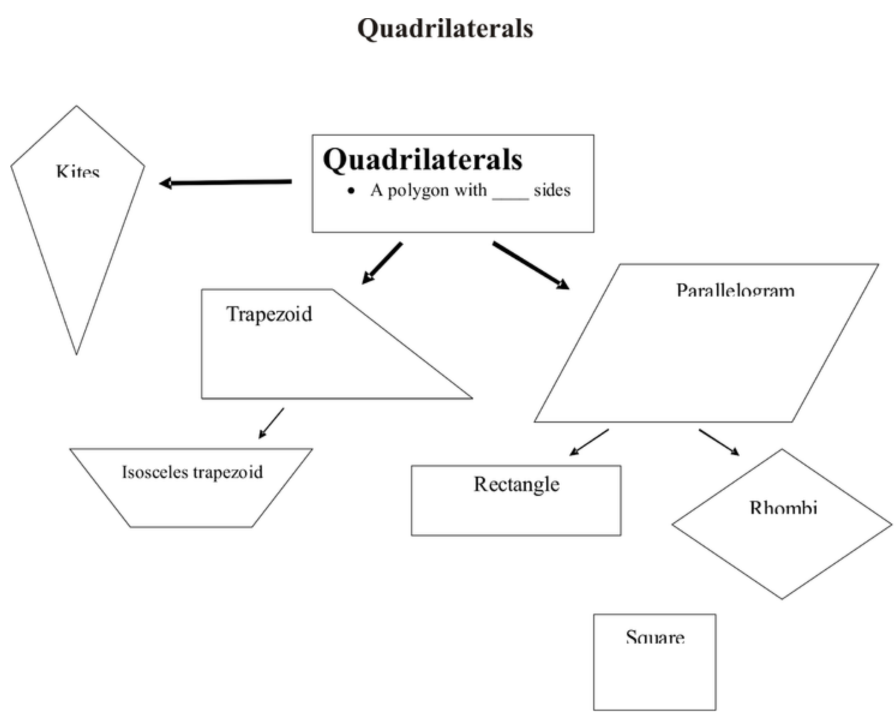
1. *True or False: Rhombuses are parallelograms.*

2. *True or False: Kites are trapezoids.*

3. *True or False: Some rectangles are also rhombuses.*

4. *True or False: Some trapezoids are **not** isosceles.*

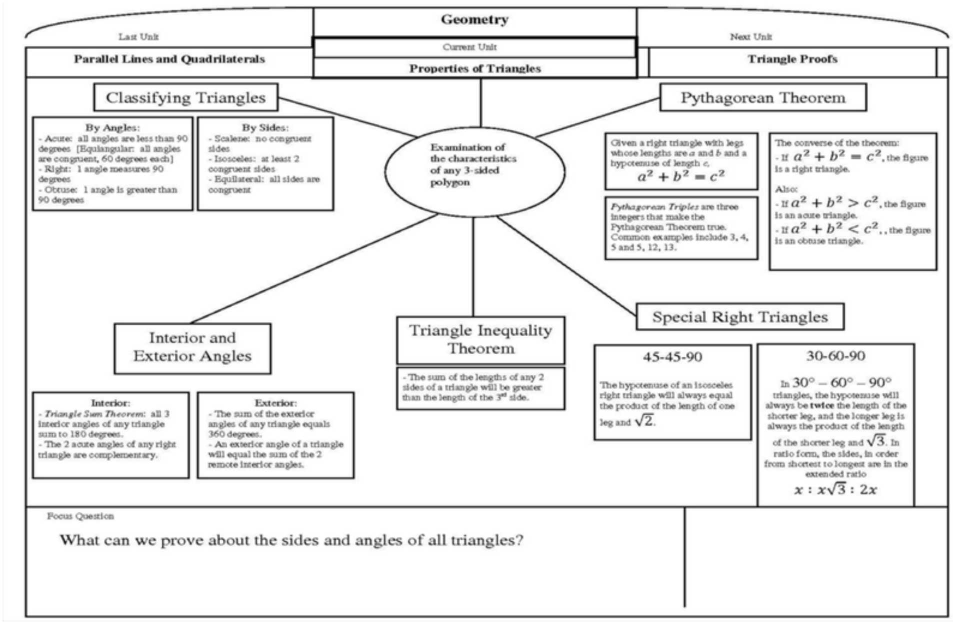
Graphic Organizer for Lesson 11



Chapter 3 Properties of Triangles

Chapter Outline

3.1 Concept Map



3.2 Vocabulary Self-Rating

TABLE 3.1: Rating Guide: DK: I am sure I don't know it K: I am sure I know it ? I'm not sure

Word	Before Lesson/Unit	After Lesson/Unit
Triangle		
Vertex		
Vertices		
Right triangle		
Acute triangle		
Obtuse triangle		
Equiangular		
Scalene		
Isosceles		
Equilateral		
Interior angles in a triangle		
Exterior angles of a triangle		
Remote interior angles		
Pythagorean Theorem		
Hypotenuse		
Legs		
Pythagorean triple		
Converse		
Radical		
Perfect square		
Rationalize the denominator		
Altitude		

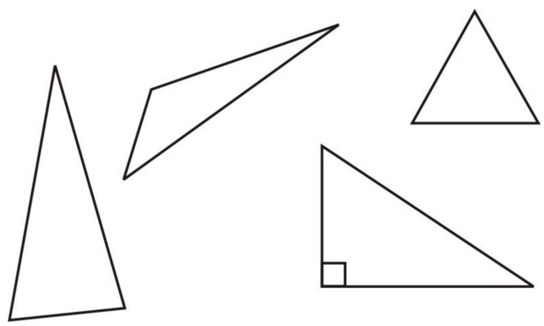
3.3 Triangle Basics

Learning Objectives

- Define triangles.
- Classify triangles as acute, right, obtuse, or equiangular.
- Classify triangles as scalene, isosceles, or equilateral.

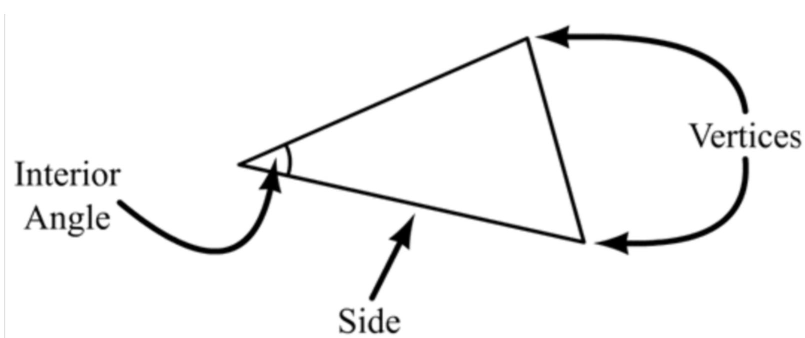
Defining Triangles

The first shape to examine is the **triangle.** Though you have probably heard of triangles before, it is helpful to review the formal definition. A **triangle** is any closed figure made by three line segments intersecting at their endpoints. All of the following shapes are triangles:



A **triangle** is a closed shape made by _____________ line segments that ___________________ at their endpoints.

Every triangle has three **vertices** (points at which the segments meet), three **sides** (the segments themselves), and three **interior angles** (formed at each **vertex**).



A triangle has three _______________________ angles formed at each vertex.

A triangle has three ___________________, which are the line segments that make up the shape.

A triangle has three _________________________, which are the points where the sides meet.

The singular version of the word *vertices* is *vertex.*

Vertices is the plural version.

This means you can have *two vertices* but only *one vertex.*

The plural form of the word **vertex** is _______________________________.

The singular form of the word **vertices** is ____________________________.

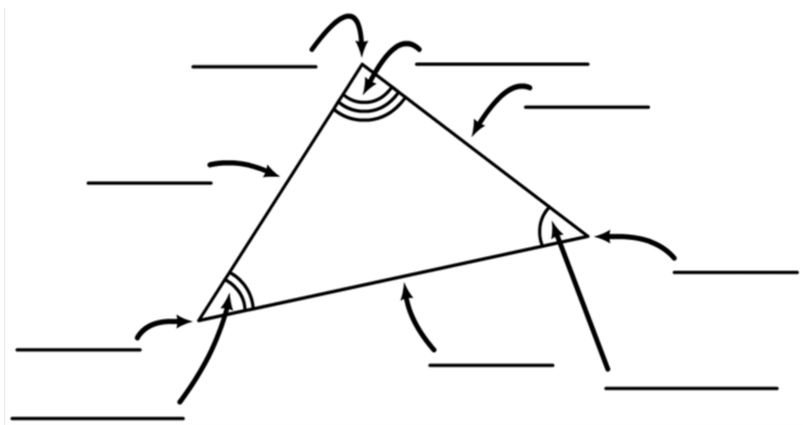
You may have learned in the past that the sum of the **interior angles** in a triangle is always 180°. Later we will prove this property, but for now you can use this fact to find missing angles.

All three **interior angles** in a triangle add up to _______________.

Reading Check:

Label all parts of the following triangle using the vocabulary words:

vertex, interior angle, *or* **side**



Classifications by Angles

Earlier, you learned how to classify angles as **acute, obtuse** or **right.** Now that you know how to identify triangles, we can classify them as well. One way to classify a triangle is by the *measure of its angles.*

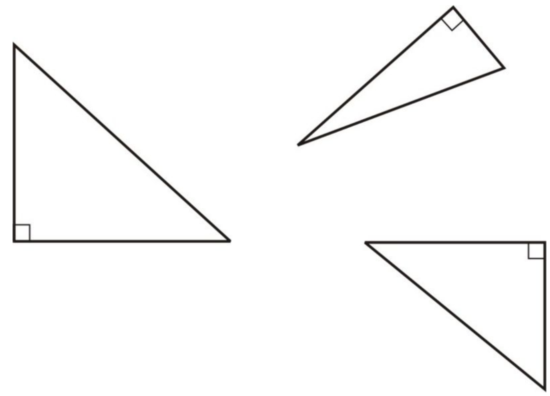
In any triangle, two of the angles will always be acute. The third angle, however, can be acute, obtuse, or right.

In all triangles, at least two of the **interior angles** are ___________________________.

Reading Check:

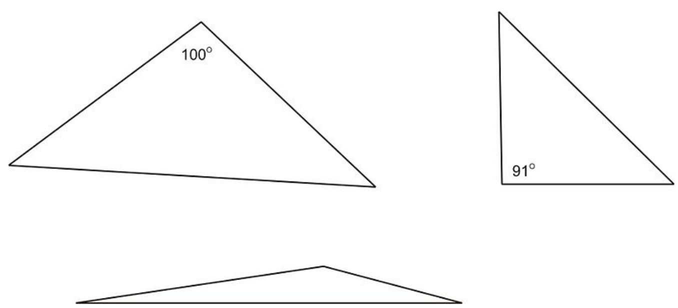
Why do you think two angles in a triangle will always be acute?

This is how triangles are classified. If a triangle has one right angle, it is called a **right triangle.** Below are some pictures of **right triangles:**



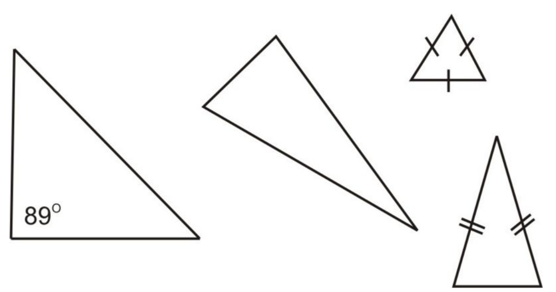
If a triangle has *one right angle* (or one 90° angle), it is called a ____________________ triangle.

If a triangle has one obtuse angle, it is called an **obtuse triangle.** Some pictures of **obtuse triangles** are shown:



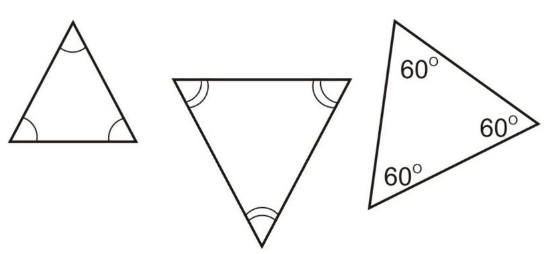
A triangle with *one obtuse angle* (or one angle greater than 90°), is called an _________________ triangle.

If all of the interior angles are acute, it is called an **acute triangle,** such as these:



A triangle with *all acute angles* (or angles smaller than 90°), is called an ___________________ triangle.

A special type of **acute triangle** occurs when *all angles are congruent.* This triangle is called an **equiangular triangle.**



All angles are congruent in an ____________________________________ triangle.

Reading Check:

1. *True or false:* All triangles have two acute angles.

2. *True or false:* A right triangle has two right angles.

3. *True or false:* An equiangular triangle is also, by definition, an obtuse triangle.

4. *Draw a picture of each of the following types of triangles, labeling all the parts:*

a. *equiangular:*

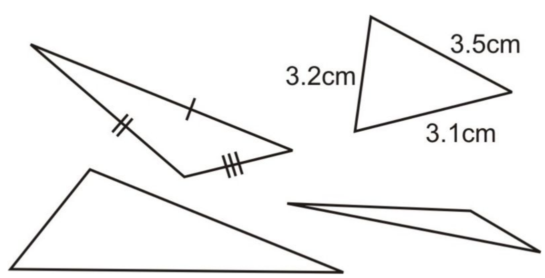
b. *acute:*

c. *obtuse:*

Classifications by Side Lengths

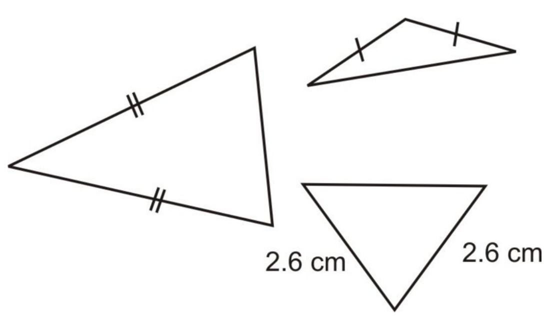
There are more types of triangle classes that are not based on angle measure. Instead, these classifications have to do with the sides of the triangle and their relationships to each other.

When a triangle has all *sides of different lengths,* it is called a **scalene triangle.** The triangles below are all **scalene:**



When all sides of a triangle are different lengths, it is called a _____________________ triangle.

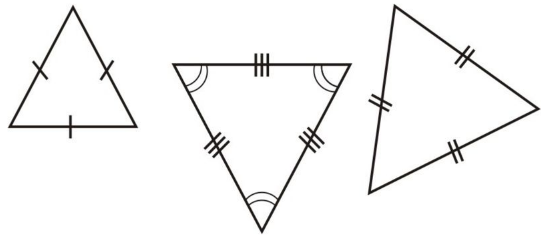
When *at least two sides* of a triangle are *congruent,* the triangle is said to be an **isosceles triangle.**



If a triangle has at least two congruent sides, it is called an _____________________ triangle.

Finally, when a triangle has sides that are *all congruent,* it is called an **equilateral triangle.**

[Note that by the definitions, an equilateral triangle is also an isosceles triangle.]



A triangle with all three sides congruent is called an ____________________________ triangle.

An **equilateral triangle** is the same as an ______________________________ triangle.

Equiangular and *equilateral* are words with similar parts. Let's analyze the words:

The prefix *"equi-"* means *"equal"*

"angular" means *"angled"*

so *"equiangular"* means *"having equal angles"*

"lateral" means *"side to side"*

so *"equilateral"* means *"having equal sides"*

Reading Check:

Give an example of the side lengths of each of the following triangles:

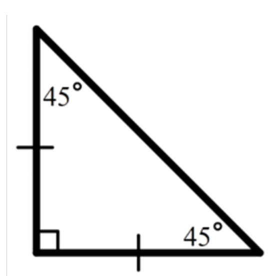
1. *equilateral* ________, ________, and ________
2. *isosceles* ________, ________, and ________
3. *scalene* ________, ________, and ________

Congruent Sides and Congruent Angles

For any triangle, the number of congruent sides will always *equal* the number of congruent angles.

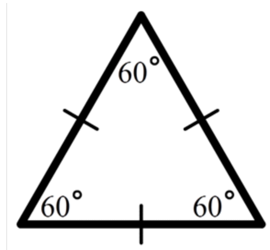
The number of congruent sides is the *same* as the number of congruent ____________________ in all triangles.

For example, a **right isosceles** triangle will have one **right** angle, which means the other two angles must add up to 90° (since all three angles add up to 180° and $180^\circ - 90^\circ = 90^\circ$). If the triangle is **isosceles,** these two other angles must be equal to each other. Therefore, a **right isosceles** triangle has two 45° angles (since $90^\circ \div 2 = 45^\circ$) and two congruent sides.



An **isosceles** triangle has two 45° angles and two ____________________ sides.

Also, any **equilateral** triangle is also **equiangular:** because all three interior angles will sum to 180°, each one of them will measure 60° (since $180^\circ \div 3 = 60^\circ$).



Every **equiangular** triangle is also ____________________.

It is important to have these concepts solidified in your mind as you explore other topics of geometry and mathematics.

Reading Check:

1. *Why is an equilateral triangle always an isosceles triangle?*

2. *Why is an equilateral triangle also equiangular?*

3. *True or false:* Every isosceles right triangle has two 45° angles and two congruent sides.

4. *True or false:* Every equilateral triangle has three 60° angles.

Graphic Organizer for Lesson 1

TABLE 3.2: **Triangle Classification by Angle and Side Length**

Type of Triangle	Draw a picture	Lists some characteristics of this triangle in your own words
Right		
Acute		
Obtuse		
Equiangular		
Scalene		
Isosceles		
Equilateral		

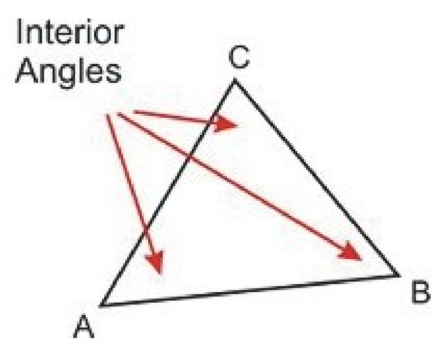
3.4 Triangle Sum and Exterior Angle Theorems

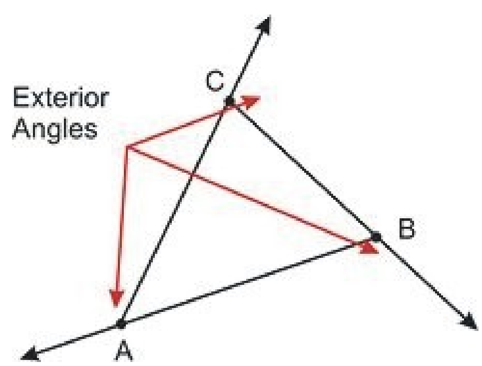
Learning Objectives

- Identify interior and exterior angles in a triangle.
- Understand and apply the Triangle Sum Theorem.
- Utilize the complementary relationship of acute angles in a right triangle.
- Identify the relationship of the exterior angles in a triangle.

Interior and Exterior Angles

The terms interior and exterior help when you need to identify the different angles in triangles. The three angles inside the triangles are called **interior angles.** On the outside, **exterior angles** are the angles formed by extending the sides of the triangle. The exterior angle is the angle formed by one side of the triangle and the extension of the other.





The angles on the inside of a triangle are called __________________ angles.

The angles on the outside of a triangle are called _________________ angles.

You can see that the words *"interior"* and *"exterior"* share the same 6 letters at the end *"-terior."* However, they have different prefixes:

“in-” means “inside” and “ex-” means “outside.”

Can you think of some other pairs of words that share an ending but have opposite prefixes *“in”* and *“ex”*?

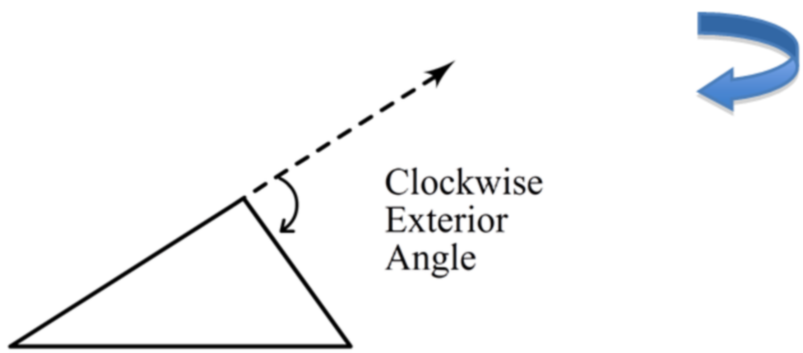
internal and external

include and exclude

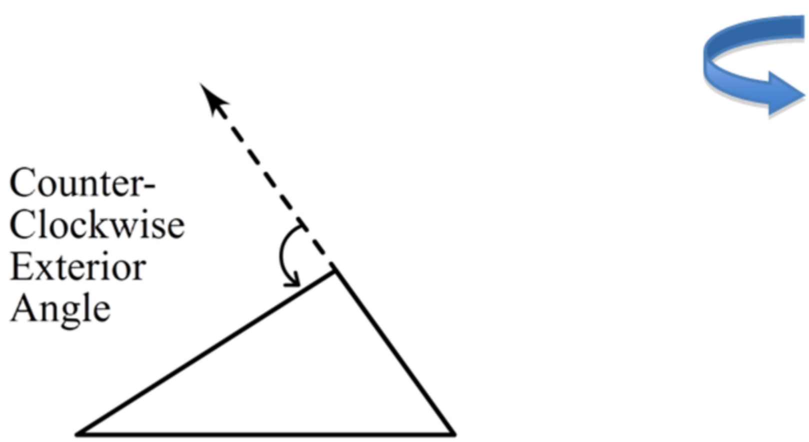
... ?

Note: In triangles and other polygons there are TWO sets of exterior angles, one “going” clockwise, and the other “going” counterclockwise.

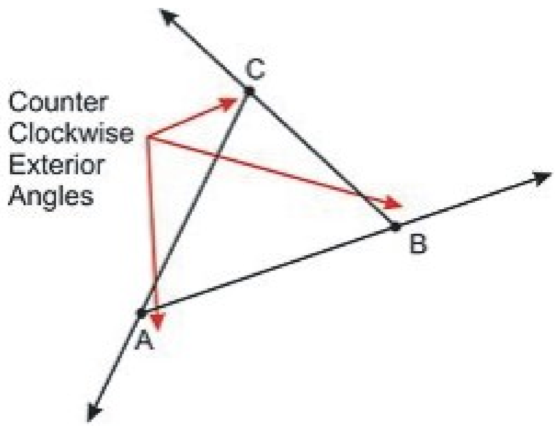
Clockwise goes around a circle in the same direction as a clock tells time:

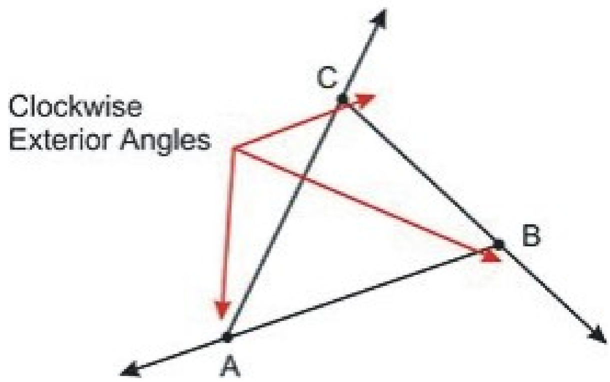


Counterclockwise goes around a circle in the opposite direction as a clock tells time:



The following diagram helps you see the difference between counterclockwise exterior angles and clockwise exterior angles on the same triangle:

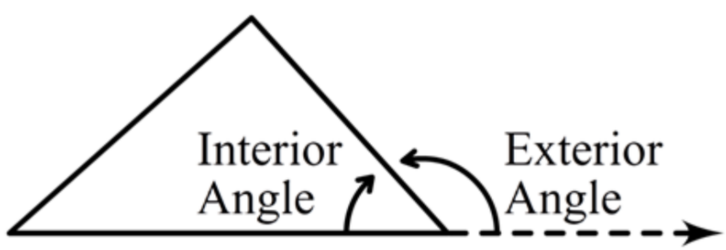




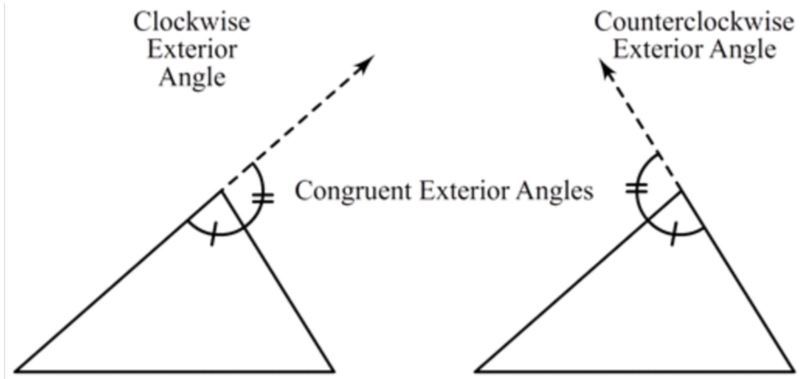
Exterior angles can be measured in two directions, _______________ and _________________ - _______________-_____.

Linear Pair Postulate

If you look at one vertex of the triangle, you will see that the interior angle and an exterior angle form a linear pair. Based on the *Linear Pair Postulate, we can conclude that interior and exterior angles at the same vertex will always be supplementary.*



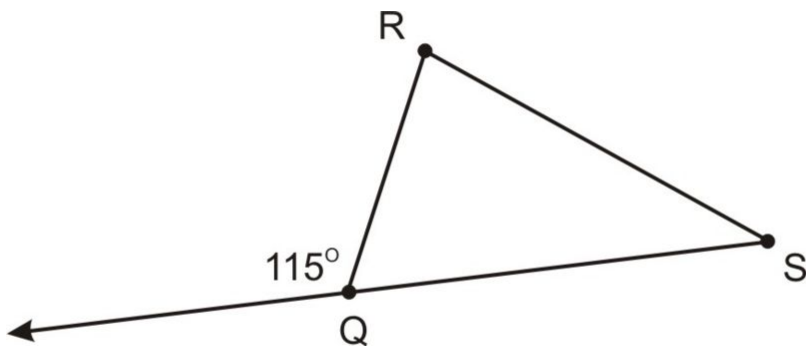
This tells us that the two exterior angles at the same vertex are congruent.



The two **exterior angles** at the same vertex are _____________________.

Example 1

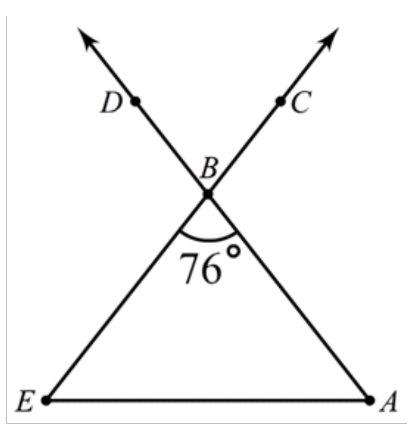
What is $m\angle RQS$ in the triangle below?



$$\begin{aligned} \text{interior angle} + \text{exterior angle} &= 180^\circ \\ m\angle RQS + 115 &= 180 \\ -115 \quad & -115 \\ m\angle RQS &= 65 \\ \text{Thus, } m\angle RQS &= 65^\circ \end{aligned}$$

<u>Reading Check:</u>

1. *What is $m\angle ABC$ in the triangle below?*



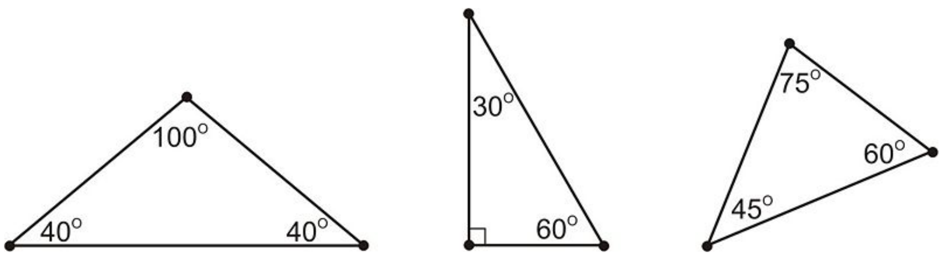
2. *What other angle in the triangle above would be congruent to $\angle ABC$? Why?*

Triangle Sum Theorem

The sum of the measures of the interior angles in a triangle is 180°.

- All three angles in a triangle add up to ___________________.

Regardless of whether the triangle is right, obtuse, acute, scalene, isosceles, or equilateral, the *interior angles will always add up to* 180°. Examine each of the triangles shown below.



Notice that each of the triangles has an angle that sums to 180°.

$$100^\circ + 40^\circ + 40^\circ = 180^\circ$$
$$90^\circ + 30^\circ + 60^\circ = 180^\circ$$
$$45^\circ + 75^\circ + 60^\circ = 180^\circ$$

Reading Check:

1. *True or false:* The interior angles of only right triangles sum to 180°.

2. *True or false:* The interior angles of scalene triangles are equiangular and sum to 180°.

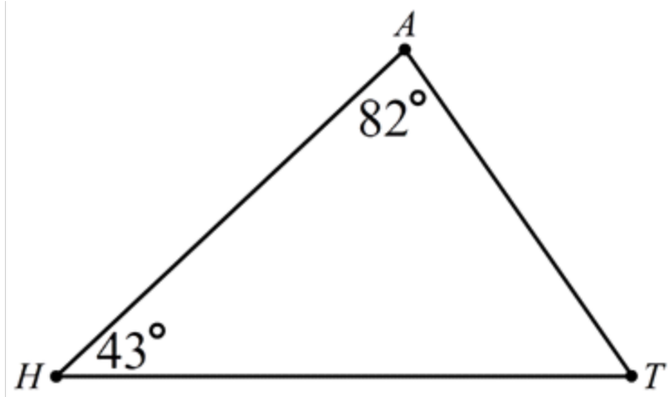
3. *Draw a triangle below and label each of the three interior angles. Show your work to make sure your angles add up to* 180°.

________ + ________ + ________ $= 180^\circ$

You can also use the *Triangle Sum Theorem* to find a missing angle in a triangle. Set the sum of the angles equal to 180° and solve for the missing value.

Example 2

What is $m\angle T$ in the triangle below?

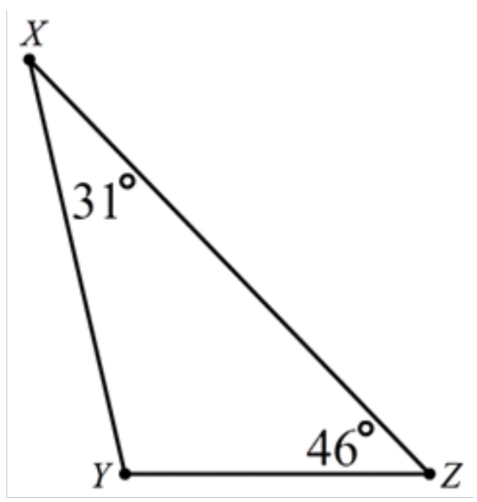


Set up an equation where the three angle measures sum to 180°. Then, solve for $m\angle T$.

$$\begin{aligned} 82^\circ + 43^\circ + m\angle T &= 180^\circ \\ 125 + m\angle T &= 180 \\ -125 \quad & \; -125 \\ m\angle T &= 55 \\ \text{Thus, } m\angle T &= 55^\circ \end{aligned}$$

Reading Check:

What is $m\angle Y$ in the triangle below?



Acute Angles in a Right Triangle

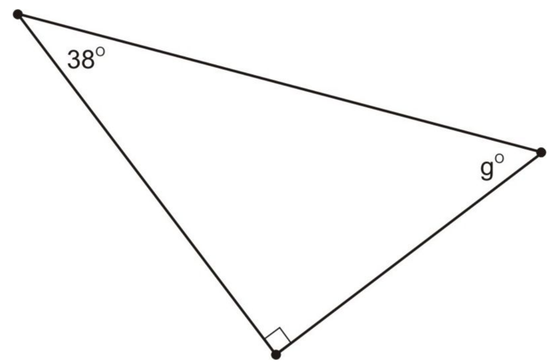
Expanding on the *Triangle Sum Theorem,* you can find more specific relationships. For example, in any right triangle, by definition, one of the angles will measure 90°. This means that the *sum of the other two angles* will always be 90°, resulting in a total sum of 180° (since $90 + 90 = 180$).

Therefore, the two acute angles in a right triangle will always be complementary.

- In a right triangle, the two acute angles will add up to _______________.

Example 3

What is the measure of the missing angle g in the triangle below?



Since the triangle above is a **right triangle,** the *two acute angles* must be *complementary,* which means their sum will be 90°. We will represent the missing angle with the variable g and write an equation.

The *two acute angles* are ___________________ and ___________________ so:

$$38^\circ + g = 90^\circ$$

Now we can use inverse operations to isolate the variable, and then we will have the measure of the missing angle.

$$\begin{aligned} 38 + g &= 90 \\ -38 \quad & \quad -38 \\ g &= 52 \end{aligned}$$

The measure of the missing angle g is 52°

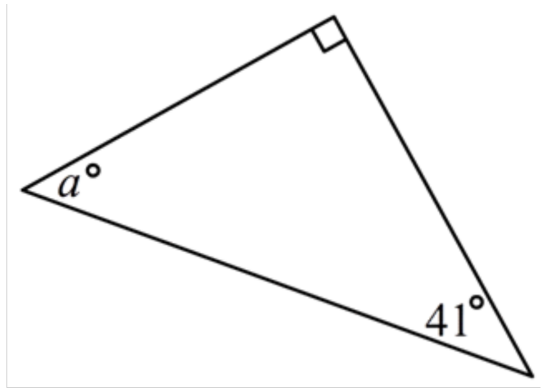
Reading Check:

1. *True or false:*

In a right triangle, the right angle is 90° and the other two angles are equiangular.

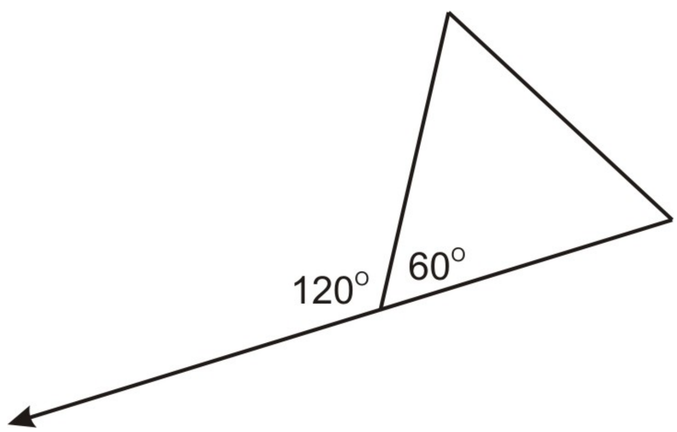
2. *Why are the two acute angles in a right triangle complementary but not supplementary? Explain.*

3. *What is the measure of the missing angle a in the triangle below?*



Exterior Angles in a Triangle

Recall that the **exterior** and **interior** angles around a single vertex create a **linear pair,** so they add up to 180° as shown below.

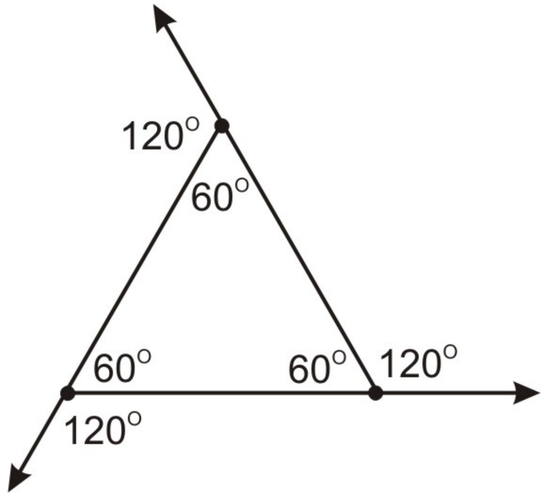


$$120^\circ + 60^\circ = 180^\circ$$

A linear pair describes two angles that add up to ________________.

Imagine an **equilateral** triangle and the exterior angles it forms, like in the diagram below.

Since each **interior** angle measures 60° in an equilateral triangle, each **exterior** angle will measure 120°.



- Every **interior** angle in an equilateral (or equiangular) triangle equals ______________.
- Every **exterior** angle in an equilateral (or equiangular) triangle equals ______________.

What is the sum of the three **exterior** angles? Add them to find out:

$$120^\circ + 120^\circ + 120^\circ = 360^\circ$$

The sum of these three exterior angles is 360°.

The sum of the exterior angles in *any* triangle will always be equal to 360°**.**

- For *any* triangle, all **exterior** angles will add up to ________________.

You can use this information just as you did the *Triangle Sum Theorem* to find missing angles and measurements.

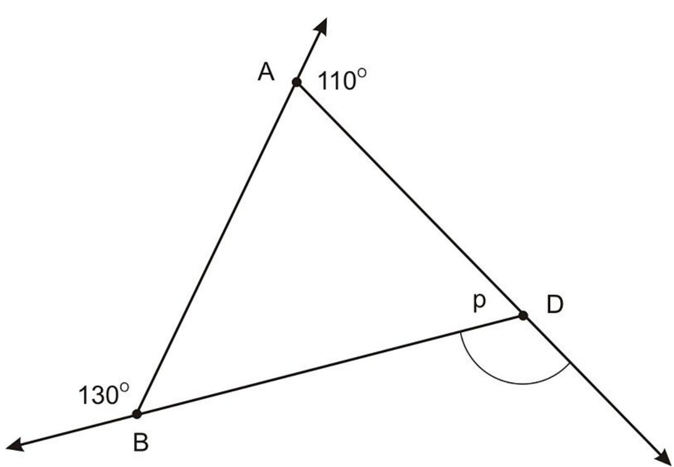
As a review,

The *Triangle Sum Theorem* says that all _________________ angles in a triangle add up to 180°.

We just discovered that all __________________________ angles in a triangle add up to 360°.

Example 4

What is the value of the angle P in the triangle below?



You can set up an equation relating the three exterior angles to 360°. Notice that p is an *interior* angle in the triangle, not an exterior angle, so be careful with how you set up this equation! Solve for the value of the *exterior* angle. Let's call the measure of the exterior angle e.

Label e in the diagram above.

Now, set up an equation with the sum of the *exterior* angles equal to 360°:

$$\begin{aligned} 130^\circ + 110^\circ + e &= 360^\circ \\ 240^\circ + e \quad &= 360^\circ \\ -240^\circ \qquad &\quad -240^\circ \\ e &= 120^\circ \end{aligned}$$

The missing exterior angle measures 120°. However, this is not your final answer! You have one more step to find the value of p.

You can use e (that you found above) to find p because the interior and exterior angles (angle e and angle p) form a **linear pair.**

Linear pairs add up to _______________ so:

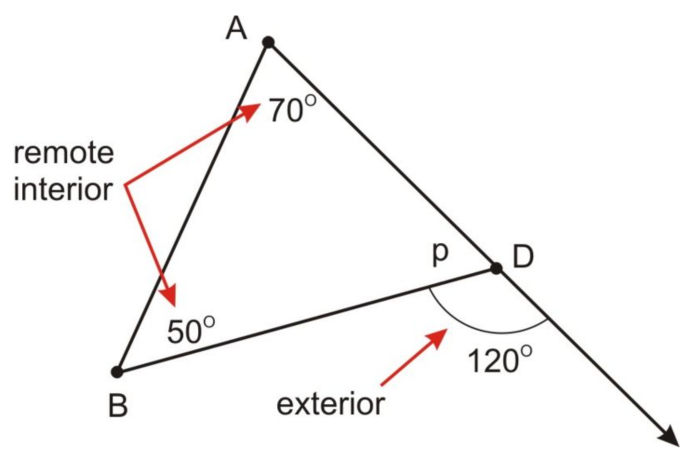
$$\begin{aligned} 120^\circ + p &= 180^\circ \\ -120^\circ \qquad &\quad -120^\circ \\ p &= 60^\circ \end{aligned}$$

Your final answer for the measure of angle p is 60°!

Exterior Angles in a Triangle Theorem

In a triangle, the measure of an exterior angle is equal to the sum of the remote interior angles.

Look at the diagram from the previous example for a moment. If we focus on the exterior angle at angle D, then the interior angles at angles A and B are called **remote interior angles.** Every exterior angle has 2 remote interior angles that correspond to it:



The 120° exterior angle at $\angle D$ above has 2 remote ____________________________ angles, one at angle ______-____ and the other at angle __________.

Notice that the exterior angle at point D measured 120°. At the same time, the interior angle at point A measured 70° and the interior angle at point B measured 50°.

The sum of interior angles $m\angle A + m\angle B = 70^\circ + 50^\circ = 120^\circ$

Notice the measures of the **remote interior angles** *sum* to the measure of the **exterior angle** at point D.

This relationship is always true, and it is a result of the *Linear Pair Postulate* and the *Triangle Sum Theorem.*

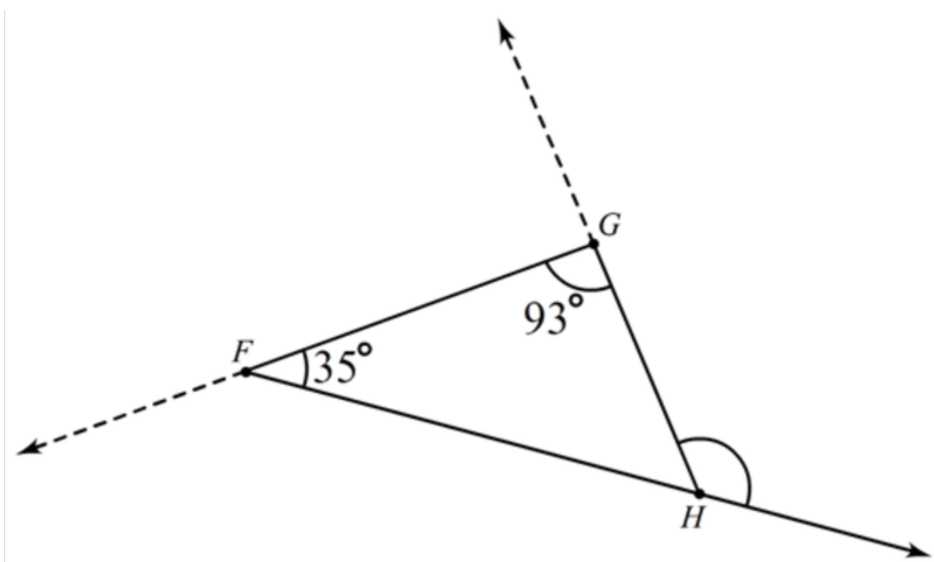
- The measure of an exterior angle is equal to the ______________ of the measures of its ________________-______ interior angles.

Reading Check:

1. *To what angle measure do all three interior angles in a triangle sum?*

2. *To what angle measure do all three exterior angles in a triangle sum?*

3. *Name the two remote interior angles to the exterior angle at point H in the diagram below.*



4. *Find the measure of the exterior angle at point H in the diagram above.*

5. *When used as an adjective, the word "remote" means "removed in space, time, or relation." Based on this definition, why do you think that word is used to identify a particular set of interior angles in a triangle?*

6. *Can you think of another way to find the measure of the exterior angle at point H? Describe your step-by-step method:*

Graphic Organizer for Lesson 2

TABLE 3.3: Angle Sums in a Triangle

Angles	Draw a picture of the angles	What do these angles add up to?
All three interior angles in a triangle		
All three clockwise exterior angles of a triangle		
All three counter clockwise exterior angles of a triangle		
The two acute angles in a right triangle		
One interior angle in a triangle and the exterior angle at the same vertex		
Two remote interior angles in a triangle		

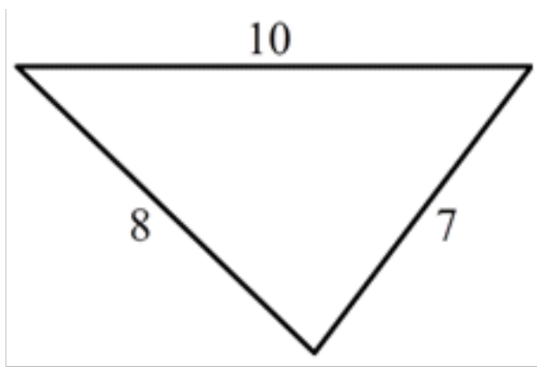
3.5 Triangle Inequality Theorem

Learning Objectives

- Apply the Triangle Inequality Theorem to solve problems.

Triangle Inequality Theorem

For any triangle, for ANY sides you choose: when you add one side to another side, your answer is more than the third side. Try this for any combination of sides using the triangle below:



One side + **Another side** > **Thethird side**		
	$7+8 > 10$	
	$15 > 10$	yes, that's true!
	$8+10 > 7$	
	$18 > 7$	yes, that's true!
and	$7+10 > 8$	
	$17 > 8$	yes, that's true too!

This relationship is called the *Triangle Inequality Theorem*

Triangle Inequality Theorem

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

- The sum of two sides of a triangle must be ______________________ than the third side.

Example 1

Can you have a triangle with sides having lengths 4, 5, and 10?

Without a drawing we can still answer this question — the answer is NO. It is an impossible situation; we cannot have such a triangle. By the Triangle Inequality Theorem, the sum of the lengths of any two sides of the triangle must be greater than the length of the third side. However, when you add 4 and 5 together, your result is NOT more than 10!

$4+5=9$, but $9>10$ is a FALSE statement

Example 2

Can you have a triangle with sides having lengths 6, 6, and 11?

Again, we can answer this question without drawing a picture using the **Triangle Inequality Theorem.** Though this is an *isosceles* triangle, the theorem still works (and it actually makes it easier for us because we only have to test 2 pairs of sides because one repeats itself.) Let's add up each pair of sides and compare the sum with the third side:

$$\begin{aligned} 6+6 &> 11 & \\ 12 &> 11 & \text{yes, that's true!} \\ \text{and} \quad 6+11 &> 6 & \\ 17 &> 6 & \text{yes, that's true too!} \end{aligned}$$

We do not need to test $6+11$ twice, since it is the same for the third pair of sides. Based on the Triangle Inequality Theorem, this triangle DOES exist!

Reading Check:

1. *Create a triangle that has side lengths that SATISFY the Triangle Inequality Theorem. Draw the triangle, label the sides, and show your work to prove that your triangle exists.*

2. *Create a triangle that has side lengths that DO NOT SATISFY the Triangle Inequality Theorem. Draw the triangle, label the sides, and show your work to prove that your triangle DOES NOT exist.*

3. *Can you have a triangle with side lengths of 13, 8, and 5? Show your work to support your answer.*

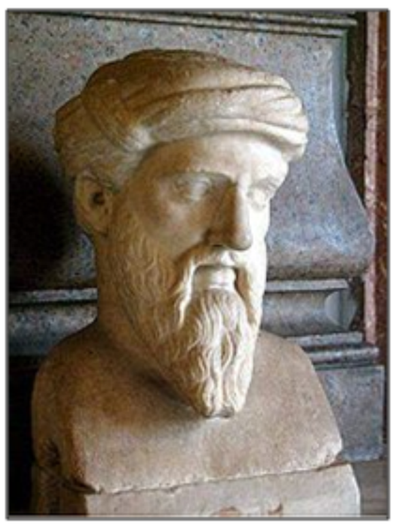
3.6 Pythagorean Theorem, Part 1: Proof & Finding a Missing Side

Learning Objectives

- Identify and employ the Pythagorean Theorem when working with right triangles.

Introduction to Pythagorean Theorem

If you are interested, here is a bit of history to begin our lesson:



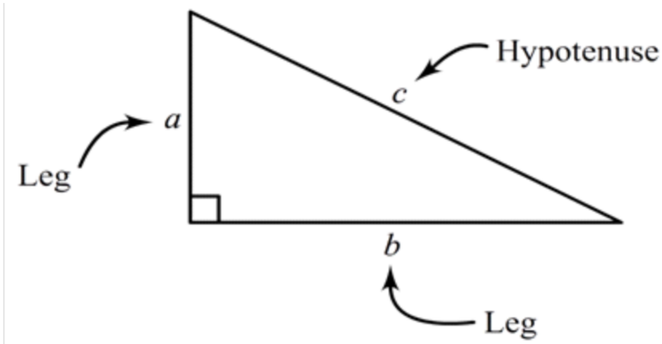
Pythagoras of Samos (c. 570-c. 495 BC) was an Ionian Greek philosopher and founder of the religious movement called Pythagoreanism. Most of our information about Pythagoras was written down centuries after he lived, thus very little reliable information is known about him. He was born on the island of Samos, and may have travelled widely in his youth, visiting Egypt and other places seeking knowledge. Around 530 BC, he moved to Croton, a Greek colony in southern Italy, and there set up a religious sect. His followers pursued the religious rites and practices developed by Pythagoras, and studied his philosophical theories.

Pythagoras made influential contributions to philosophy and religious teaching in the late 6th century BC. He is often revered as a great mathematician, mystic and scientist, and he is best known for the **Pythagorean Theorem** which bears his name. It was said that he was the first man to call himself a *philosopher,* or *lover of wisdom,* and Pythagorean ideas exercised a marked influence on Plato, and through him, all of Western philosophy.

Source: http://en.wikipedia.org/wiki/Pythagoras

The Pythagorean Theorem

The triangle below is a right triangle:



The sides labeled a and b are called the **legs** of the triangle and meet at the right angle.

The third side, labeled c is called the **hypotenuse.** The hypotenuse is the side *opposite* (or *across from*) the right angle. The hypotenuse of a right triangle is also the longest side of the triangle.

- The longest side of a right triangle is called the ______________________.
- The hypotenuse is across from the ___________________ angle in a right triangle.
- The two shorter sides of a right triangle are called __________________.
- The legs of a right triangle form a vertex at the _____________________ angle.

The Pythagorean Theorem states that there is a relationship between the three sides of ANY right triangle. When you square the length of the hypotenuse, it will equal the sum of the squares of the lengths of the two legs. In the triangle on the previous page, the sum of the squares of the legs is $a^2 + b^2$ and the square of the hypotenuse is c^2.

The Pythagorean Theorem

Given a right triangle with legs whose lengths are a and b and a hypotenuse of length c,

$$a^2 + b^2 = c^2$$

Be careful when using this theorem!

You must make sure that the legs are correctly labeled a and b and the hypotenuse is correctly labeled c to use this equation. A more accurate way to write the Pythagorean Theorem is:

$$(leg_1)^2 + (leg_2)^2 = hypotenuse^2$$

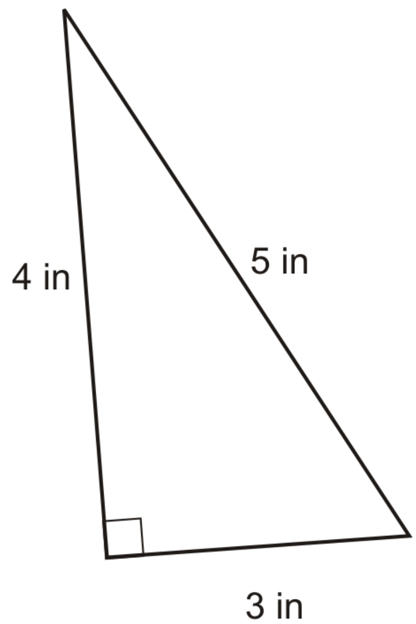
When using the **Pythagorean Theorem:**

- You must make sure that the **hypotenuse** is side _____________ in the triangle
- and the **legs** are sides ____________ and _____________.

The ________________________________ Theorem is only true for **right** triangles!

Example 1

Use the side lengths of the following triangle to test the Pythagorean Theorem.



The legs of the triangle above are *3 inches* and *4 inches*. The hypotenuse is *5 inches.* So, using the Pythagorean Theorem, let's make $a = 3, b = 4$, and $c = 5$. We can substitute these values into the formula for the Pythagorean Theorem to verify that the relationship works:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= 5^2 \\ 9 + 16 &= 25 \\ 25 &= 25 \end{aligned}$$

Since both sides of the equation equal 25, the equation is true. Therefore, the Pythagorean Theorem works on this right triangle.

Reading Check:

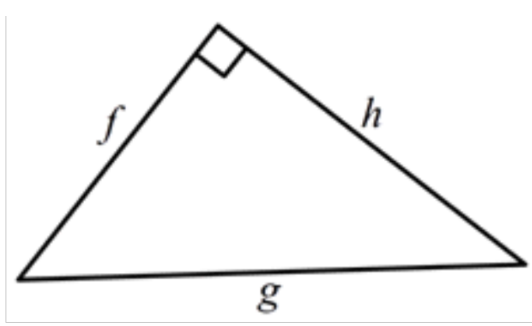
1. *True or false:* The Pythagorean Theorem works for all triangles.

2. *True or false:* The hypotenuse of a right triangle is the shortest side of the triangle.

3. *What are 2 characteristics of the* ***hypotenuse*** *of a triangle?*

1) 2)

4. *In the triangle to the right,*



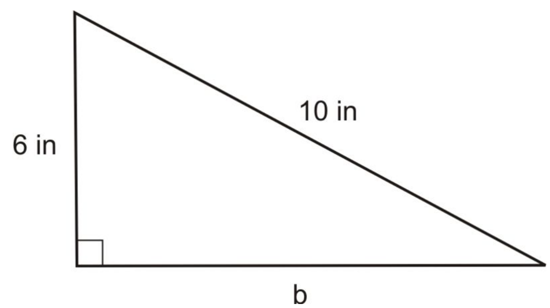
Which side is a leg? _________________________

Which side is the other leg? ___________________

Which side is the hypotenuse? _________________

Example 2

What is the length of b in the triangle below?



Use the **Pythagorean Theorem** to find the length of the missing **leg**, b. Set up the equation $a^2 + b^2 = c^2$, letting $a = 6$ and $c = 10$:

$$\begin{aligned}
a^2 + b^2 &= c^2 \\
6^2 + b^2 &= 10^2 \\
36 + b^2 &= 100 \\
-36 \quad &\; -36 \\
b^2 &= 64 \\
\sqrt{b^2} &= \sqrt{64} \\
b &= \pm 8 \\
b &= 8
\end{aligned}$$

In algebra you learned that $\sqrt{x^2} = \pm x$ because, for example, $(5)^2 = (-5)^2 = 25$.

However, in this case (and in much of geometry), we are only interested in the *positive* solution to $b = \sqrt{64}$ because geometric lengths are positive (having a side with a negative length does not make sense.)

So in Example 2, we can disregard the solution $b = -8$ and our final answer is $b = 8$ inches.

In Example 2 above,

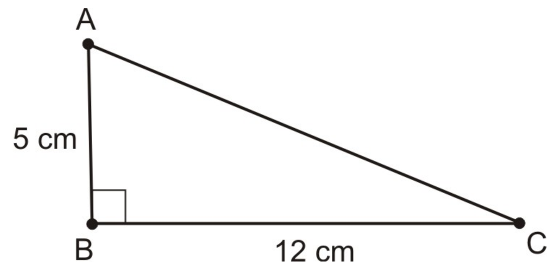
- The **hypotenuse** is side _____________, which is _____________ inches long.
- One **leg** is side _____________, which is _____________ inches long.
- The other **leg** is side _____________, which is _____________ inches long.

When using the **Pythagorean Theorem** and taking the *square root* of a number,

- We only care about the ______________________________ answer because lengths cannot be negative.

Example 3

Find the length of the missing side in the triangle below.



Use the **Pythagorean Theorem** to set up an equation and solve for the missing side, which in this problem is the **hypotenuse.**

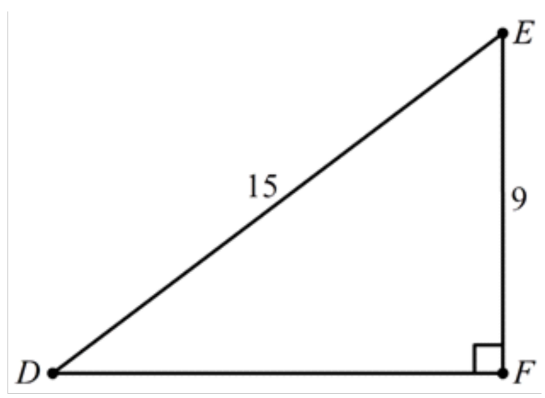
Let $a = 5$ and $b = 12$. We do not know c:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 5^2 + 12^2 &= c^2 \\ 36 + 144 &= c^2 \\ 169 &= c^2 \\ \sqrt{169} &= \sqrt{c^2} \\ 13 &= c \end{aligned}$$

So, the length of the missing side, the ______________________________, is 13 centimeters.

Reading Check:

Find the length of the missing side in the triangle below.



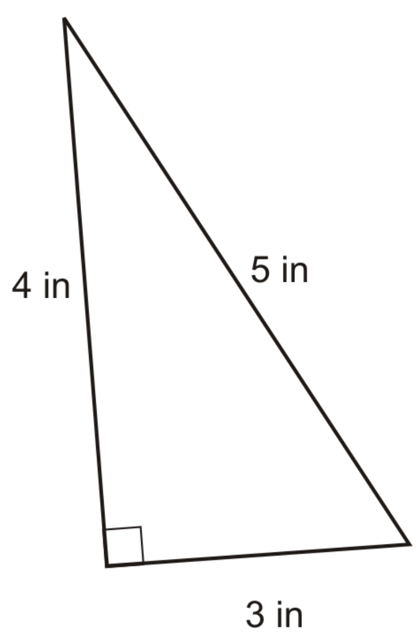
3.7 Pythagorean Theorem, Part 2: Applications & Triples

Learning Objectives

- Identify common Pythagorean triples.

Using Pythagorean Triples

Review the example problems from the previous lesson.

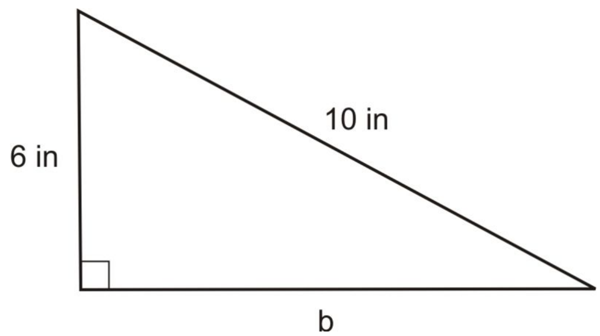


This is the diagram from Example 1:

In Example 1, the sides of the triangle are *3, 4,* and *5*. This combination of numbers is referred to as a **Pythagorean triple.** A Pythagorean triple is three *integers* (whole numbers with no decimal or fraction part) that make the Pythagorean Theorem true.

- A **Pythagorean triple** is a group of three ______________________________ that satisfy the Pythagorean Theorem.

Throughout this chapter, you will learn other Pythagorean triples as well.

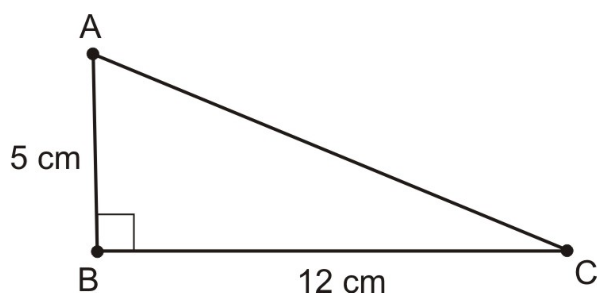


This is the diagram from Example 2:

Using the Pythagorean Theorem equation $a^2+b^2=c^2$, and letting $a=6$ and $c=10$, we calculated that $b=8$ inches.

The triangle in Example 2 is proportional to the same ratio of 3 : 4 : 5. If you divide the lengths of the triangle (6, 8, and 10) by 2, you find the same proportion — 3 : 4 : 5 (because $6 \div 2 = 3, 8 \div 2 = 4$, and $10 \div 2 = 5$).

Whenever you find a **Pythagorean triple,** you can apply these ratios with greater factors as well.



Finally, look at the side lengths of the triangle in Example 3:

The two **legs** are 5 cm and 12 cm and the length of the missing side (the **hypotenuse**) is 13 cm. The side lengths make a ratio of 5 : 12 : 13. This, too, is a **Pythagorean triple**. You can infer that this ratio, multiplied by greater factors, will also yield numbers that satisfy the Pythagorean Theorem.

There are infinitely many Pythagorean triples, but a few of the most common ones and their multiples are in the chart below:

TABLE 3.4:

Pythagorean triple	×2	×3	×4
3 – 4 – 5	6 – 8 – 10	9 – 12 – 15	12 – 16 – 20
5 – 12 – 13	10 – 24 – 26	15 – 36 – 39	20 – 48 – 52
7 – 24 – 25	14 – 48 – 50	21 – 72 – 75	28 – 96 – 100
8 – 15 – 17	16 – 30 – 34	24 – 45 – 51	32 – 60 – 68

Reading Check:

1. *What is a Pythagorean triple?*

2. *Which of the following is NOT a Pythagorean triple? Show your work.*

a. 15 – 36 – 39

b. 15 – 20 – 25

c. 16 – 30 – 35

d. 25 – 60 – 65

3. *Give 2 examples of Pythagorean triples that are NOT in the chart above.*

4. *Why is it helpful to know common Pythagorean triples?*

3.8 Pythagorean Theorem, Part 3: Converse of the Pythagorean Theorem

Learning Objectives

- Understand the converse of the Pythagorean Theorem.
- Identify acute triangles from side measures.
- Identify obtuse triangles from side measures.
- Classify triangles in a number of different ways.

Converse of the Pythagorean Theorem

Could you use the Pythagorean Theorem to prove that a triangle contained a right angle if you did not have an accurate diagram?

You have learned about the Pythagorean Theorem and how it can be used. As you recall, it states that the sum of the squares of the legs of any right triangle will equal the square of the hypotenuse. If the lengths of the legs are labeled a and b, and the hypotenuse is c, then we get the familiar equation:

$$a^2 + b^2 = c^2$$

The converse of the Pythagorean Theorem is also true.

*What is a **converse**?*

*A **converse** is an if-then statement where the hypothesis and the conclusion are switched.*

For example, if we start with the if-then statement: If it is raining, then the street is wet.

*The **converse** of our statement is: If the street is wet, then it is raining.*

The part of the sentence after the "if" (called the hypothesis) switches places with the part of the sentence after "then" (called the conclusion.) You may remember these words from the previous chapters.

Statements and their converses are not always both true! In the case of the Pythagorean Theorem, BOTH the Theorem and its converse are true.

A **converse** is an if-then statement where the hypothesis and the ____________________________ switch places.

Converse of the Pythagorean Theorem

Given a triangle with side lengths a, b and c (where c is the longest side), if the equation $a^2 + b^2 = c^2$ is true, then the triangle is a right triangle.

With this converse, you can use the Pythagorean Theorem to prove that a triangle is a **right** triangle, even if you do not know any of the triangle's angle measurements.

This means that if you know the three side lengths of a triangle, you can substitute them into the equation $a^2 + b^2 = c^2$.

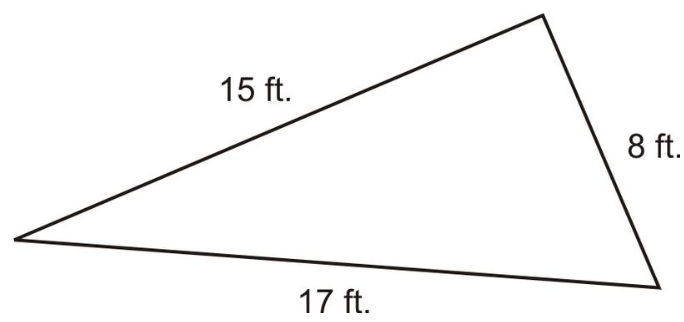
- If you find a *true* statement (such as $100 = 100$), then the Pythagorean Theorem *works* in that case; your triangle *is* a **right** triangle.
- If you find a *false* statement (such as $91 = 100$), then, in that case, the Pythagorean Theorem *does not work;* your triangle is *not* a right triangle.

If $a^2 + b^2 = c^2$ produces a *true* statement, then the triangle is a ___________ triangle.

If $a^2 + b^2 = c^2$ produces a *false* statement, then the triangle is ___________ a right triangle.

Example 1

Does the triangle below contain a right angle?



This triangle does not have any right angle marks or measured angles, so you cannot assume you know whether the triangle is acute, right, or obtuse just by looking at it.

To see if the triangle might be right, try substituting the side lengths into the Pythagorean Theorem to see if they make the equation true. *The hypotenuse is always the longest side,* so *17* should be substituted for c. The other two values can represent a and b and the order is not important.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 8^2 + 15^2 &= 17^2 \\ 64 + 225 &= 289 \\ 289 &= 289 \end{aligned}$$

Since both sides of the equation are equal $(289 = 289)$, these values satisfy the Pythagorean Theorem. Therefore, the triangle described in the problem is a right triangle!

Reading Check:

1. *Write the converse of this if-then statement:*

If it is sunny outside, then the weather must be warm.

Converse:

2. *BONUS! Write a true if-then statement that also has a true converse:*

Statement:

Converse:

3. *A triangle has side lengths 5, 7, and 9. Is this a right triangle? Show your work to defend your answer.*

Identifying Acute Triangles

If the sum of the squares of the two shorter sides of a triangle is **greater than** the square of the longest side, then the triangle is **acute** (all angles in the triangle are less than 90°.)

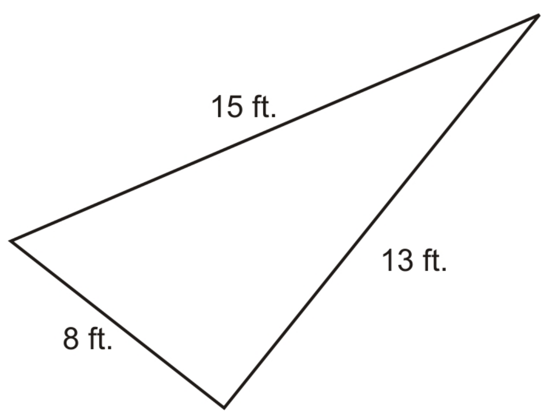
If $a^2 + b^2 > c^2$ then the triangle is **acute**.

You can use this rule the same way you used the Pythagorean Theorem on the last page. Substitute the side lengths for a, b, and c making sure that the longest side is c. After you have simplified both sides of the equation, compare your answers: which side is a larger number? If the $a^2 + b^2$ side is bigger than the c^2 side, then the triangle is **acute**.

- If $a^2 + b^2$ is larger than c^2, then the triangle is ______________________.

Example 2

Is the triangle below acute or right?



The two shorter sides of the triangle are *8* and *13.* The longest side of the triangle is *15.*

Since the legs are the *shorter* sides, first find the sum of the squares of the two shorter legs by substituting the smaller numbers for a and b:

$$8^2 + 13^2 = c^2$$
$$64 + 169 = c^2$$
$$233 = c^2$$

The sum of the squares of the two shorter legs is *233.* Compare this to the square of the longest side, *15.*

$$15^2 = 225$$

The square of the longest side is _____________.

Since $8^2 + 13^2 = 233$ and $233 \neq 225 = 15^2$, this triangle is not a right triangle.

- $a^2 + b^2 \neq c^2$ so the triangle cannot be a ____________________ triangle.

Compare the two values to identify which is greater.

$$233 > 225$$

The sum of the squares of the shorter sides $(a^2 + b^2)$ is greater than the square of the longest side (c^2). Therefore, this is an **acute** triangle.

- Because c^2 is smaller than $a^2 + b^2$, this is an ____________________ triangle.

Reading Check:

1. *Fill in the blanks:*

When the square of the __________________________ side is less than the sum of the squares of the __________-_______________ sides, the triangle is an acute triangle.

2. *A triangle has side lengths 4, 7, and 8. Is this triangle acute or right? Show your work to defend your answer.*

Identifying Obtuse Triangles

You can prove a triangle is **obtuse** (meaning it has one angle larger than 90°) by using a similar method.

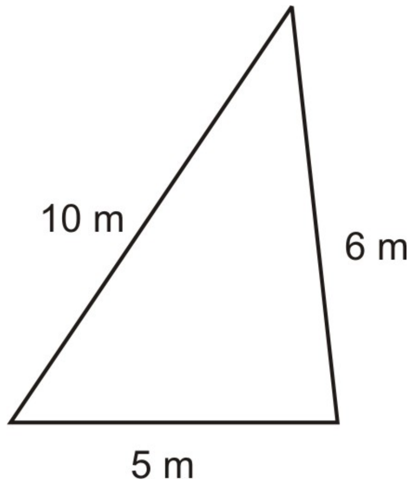
Find the sum of the squares of the two shorter sides in a triangle. If this value is **less than** the square of the longest side, the triangle is **obtuse.**

If $a^2 + b^2 < c^2$, then the triangle is **obtuse.**

- **Obtuse** triangles have one angle _____________________________ that is than 90°.
- If $a^2 + b^2$ is smaller than c^2, then the triangle is ______________________.

Example 3

Is the triangle below obtuse or right?



You can solve this problem in a manner almost identical to Example 2.

The two shorter sides of the triangle are *5* and *6.* The longest side of the triangle is *10.*

First find the sum of the squares of the two shorter legs by substituting the smaller numbers for a and b.

$$\begin{aligned} a^2+b^2 &= 5^2+6^2 \\ &= 25+36 \\ &= 61 \end{aligned}$$

The sum of the squares of the two shorter legs is *61.* Compare this to the square of the longest side, *10.*

$$10^2 = 100$$

The square of the longest side is *100.*

Since $5^2+6^2 = 61$ and $61 \neq 100 = 10^2$, this triangle is not a right triangle.

Compare the two values to identify which is greater.

$$\begin{gathered} 61 < 100 \\ (\text{sum of shorter sides})^2 < (\text{longest side})^2 \end{gathered}$$

Since the sum of the square of the shorter sides (a^2+b^2) is less than the square of the longest side (c^2), this is an **obtuse** triangle.

- Because c^2 is larger than a^2+b^2, this is an ____________________ triangle.

Reading Check:

1. *Fill in the blanks:*

When the square of the ________________________ side is greater than the sum of the squares of the ___________-_______________ sides, the triangle is an obtuse triangle.

2. *True or false:* When the square of the longest side equals the sum of the squares of the shorter sides, the triangle is a right triangle.

3. *A triangle has side lengths* 5, 8, and 10. *Is this triangle acute, obtuse, or right? Show your work to defend your answer.*

Triangle Classification

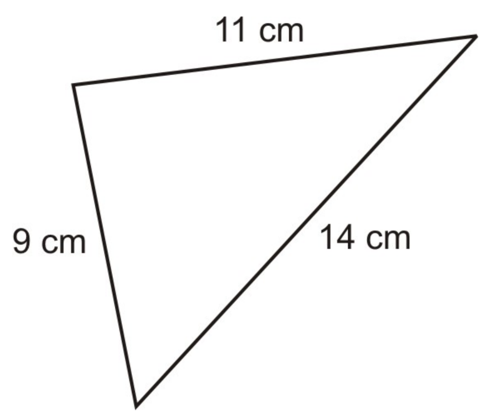
Now that you know the ideas in this lesson, you can classify any triangle as right, acute, or obtuse given the length of the three sides. Be sure to use the *longest* side for the **hypotenuse.**

Remember:

- If $a^2+b^2=c^2$, the figure is a **right** triangle.
- If $a^2+b^2>c^2$, the figure is an **acute** triangle.
- If $a^2+b^2<c^2$, the figure is an **obtuse** triangle.

Example 4

Classify the triangle below as right, acute, or obtuse.



The two shorter sides of the triangle are *9* and *11*. The longest side of the triangle is *14*. First find the sum of the squares of the two shorter legs by substituting the smaller numbers for *a* and *b*.

$$\begin{aligned} a^2+b^2 &= 9^2+11^2 \\ &= 81+121 \\ &= 202 \end{aligned}$$

The sum of the squares of the two shorter legs is *202*. Compare this to the square of the longest side, *14*.

$$14^2=196$$

The square of the longest side is *196*. So the two values are <u>not equal</u> ($202 \neq 196$ or $a^2+b^2 \neq c^2$) and this triangle is <u>not a right triangle</u>.

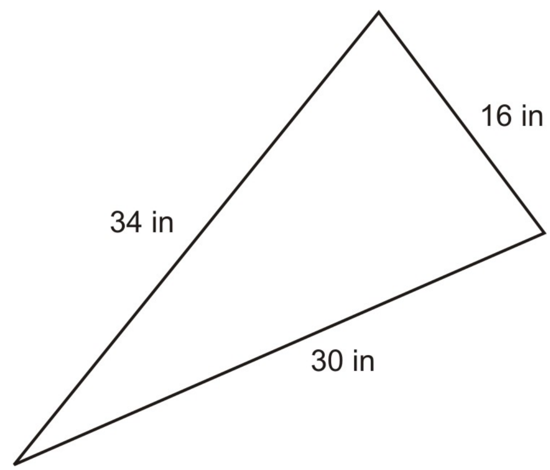
Since you can eliminate the right triangle from your choices, now you can compare the two values, a^2+b^2 and c^2 to identify which is greater:

$$\begin{gathered}202 > 196 \\ (\text{sum of shorter sides})^2 > (\text{longest side})^2\end{gathered}$$

Since the sum of the square of the shorter sides is <u>greater</u> than the square of the longest side (in symbols $a^2+b^2 > c^2$), this is an **acute** triangle.

Example 5

Classify the triangle below as right, acute, or obtuse.



The two *shorter* sides of the triangle are _____________ and ______________.

The *longest* side of the triangle is ________________.

First, set up an equation to find the sum of the squares of the two shorter legs by substituting the smaller numbers for a and b.

$$\begin{aligned}a^2+b^2 &= 16^2+30^2 \\ &= 256+900 \\ &= 1156\end{aligned}$$

The sum of the squares of the two legs is 1156.

Compare this to the square of the longest side, *34*.

$$c^2 = 34^2 = 1156$$

The square of the longest side is *also* 1156.

Since the two values you found are equal $(a^2+b^2=c^2)$, this is a **right** triangle.

- In this example, the Pythagorean Theorem is ____________________.
- When $a^2 + b^2 = c^2$, you have a ____________________ triangle!

Reading Check:

1. *Fill in the blanks:*

In an **acute** triangle, the sum of the squares of the shorter sides is ______________________ (greater than / less than / equal to) the square of the longest side.

In a **right** triangle, the sum of the squares of the shorter sides is ________________________ (greater than / less than / equal to) the square of the longest side.

In an **obtuse** triangle, the sum of the squares of the shorter sides is ______________________ (greater than / less than / equal to) the square of the longest side.

2. *A triangle has side lengths 8, 9, and 13. Classify the triangle as right, acute, or obtuse. Show your work to defend your answer.*

Graphic Organizer for Lesson 6

TABLE 3.5: Triangle Classification and the Pythagorean Theorem

Type of Triangle	**Draw a picture**	**How can you use the Pythagorean Theorem to compare the sides?**	**Give an example of 3 side lengths for this triangle and show work to prove your classification**
Right		$a^2 + b^2 = c^2$	
Acute			
Obtuse			

3.9 Synthesis Day (Day 1 of 2 in Unit 3)

Classroom activity

Note: Please refer to Mike and Rose's lesson plans within the College Access Geometry materials for this lesson.

3.10 Operations with Radicals Review

Learning Objectives

- Simplify Radicals
- Multiply Radicals
- Rationalize the Denominator

Simplifying Radicals

You learned how to simplify radicals in Algebra class, but we will review this topic to get ready for the final lessons in this chapter.

A **radical** is a number under a square root symbol (a square root symbol looks like this: $\sqrt{\ }$).

- When a number is under a square root symbol $\sqrt{\ }$, it is called a ____________________.

Remember, when we *simplify* a **radical**, we use the property $\sqrt{ab} = \sqrt{a}\sqrt{b}$. The key is to break the number inside the radical into 2 factors, one of which is a **perfect square.**

- *Simplify* a **radical** by splitting the number under the $\sqrt{\ }$ into ______________________.

Remember, a ***perfect square*** *is the result when any number is squared.*

For instance, these are examples of perfect squares:

$1^2 = 1$ *(1 is a perfect square)*

$2^2 = 4$ *(4 is a perfect square)*

$3^2 = 9$ *(9 is a perfect square)*

$4^2 = 16$ *(16 is a perfect square)*

$5^2 = 25$ *(25 is a perfect square) and so on. . .*

Some more perfect squares are: 36, 49, 64, 81, 100, etc.

Example 1

Simplify the radical $\sqrt{75}$.

Break up the number 75 into 2 factors, one of which is a perfect square, and simplify:

$$\begin{aligned} \sqrt{75} &= \sqrt{25 \times 3} && (25 \times 3 \text{ is } 75, \text{ and } 25 \text{ is a perfect square}) \\ &= \sqrt{25}\sqrt{3} && \left(\text{because of the property } \sqrt{ab} = \sqrt{a}\sqrt{b}\right) \\ &= 5\sqrt{3} && \left(\text{because } \sqrt{25} = 5\right) \end{aligned}$$

You can check on your calculator that this is true: $\sqrt{75} = 5\sqrt{3} \approx 8.66$

Note: In Example 1, we decided to split 75 into the factors 25 and 3. We *could* have picked the factors 15 and 5, since those also multiply to be 75. However, since neither 15 nor 5 is a perfect square, this choice would not have helped us simplify the radical.

Example 2

Simplify the radical $\sqrt{180}$.

Break up the number 180 into 2 factors, one of which is a perfect square, and simplify:

$$\begin{aligned}180 = & \sqrt{9 \times 20} && (9 \times 20 \text{ is } 180, \text{ and } 9 \text{ is a perfect square}) \\ = & \sqrt{9}\sqrt{20} && \left(\text{because of the property } \sqrt{ab} = \sqrt{a}\sqrt{b}\right) \\ = & 3\sqrt{20} && \left(\sqrt{9} = 3\right)\end{aligned}$$

We are looking good so far. However, $3\sqrt{20}$ is not our final answer because we can keep simplifying the radical!

$$\begin{aligned}180 &= 3\sqrt{4 \times 5} && (4 \times 5 \text{ is } 20, \text{ and } 4 \text{ is } \textit{another} \text{ perfect square}) \\ &= 3\sqrt{4}\sqrt{5} \\ &= 3 \times 2\sqrt{5} && \left(\sqrt{4} = 2\right) \\ &= 6\sqrt{5} && (3 \times 2 = 6)\end{aligned}$$

This is our final answer because $\sqrt{5}$ does not simplify any more.

You can check on your calculator that this is true: $\sqrt{180} = 6\sqrt{5} \approx 13.416$

We learn from Example 2 that *after* you simplify, you should always check if you can simplify *again.* We also learn that you should try to pull out the *largest* perfect square possible for your first step. What if we had chosen different factors?

$$\begin{aligned}180 = & \sqrt{36 \times 5} && (36 \times 5 \text{ is } 180, \text{ and } 36 \text{ is a perfect square}) \\ = & \sqrt{36}\sqrt{5} && \left(\text{because of the property } \sqrt{ab} = \sqrt{a}\sqrt{b}\right) \\ = & 6\sqrt{5} && \left(\sqrt{36} = 6\right)\end{aligned}$$

As you can see, we got the same answer, but in only one step! By picking the *largest* perfect square that goes into 180, we had a shorter problem.

- When simplifying radicals, pick the ____________________________ perfect square possible as one of your factors.

Reading Check:

1. *Fill in the blanks:*

When we simplify a radical, break the number inside the radical into _________ factors, one of which is a ______-____________ ____________________.

2. *Simplify the radical* $\sqrt{80}$.

3. *Simplify the radical* $\sqrt{98}$.

Multiplying Radicals

Just like you simplified a radical by splitting it into parts, the opposite is true as well: two radicals can be multiplied together by combining them underneath a single radical.

- When you multiply two radicals, combine them under a single ______________________.

We know the property $\sqrt{ab} = \sqrt{a}\sqrt{b}$

Since this is an equation, we can switch both sides of the equal sign and the property is still true:

$$\sqrt{a}\sqrt{b} = \sqrt{ab}$$

We can use this to multiply (or combine) two radical expressions.

Example 3

Multiply the radicals $\sqrt{3}\sqrt{5}$.

Multiply the two numbers under a single radical sign:

$$\begin{aligned}\sqrt{3}\sqrt{5} &= \sqrt{3 \times 5}\\ &= \sqrt{15}\end{aligned}$$

Is the radical in your answer fully simplified?

Yes, we cannot simplify $\sqrt{15}$ any further. Therefore, $\sqrt{3}\sqrt{5} = \sqrt{15}$

Example 4

Multiply the radicals $\sqrt{8}\sqrt{48}$.

There are two different ways to do this problem. You can either:

- simplify the individual radicals first and then multiply (*solution #1* below) OR
- multiply the radicals first and then simplify the final radical (*solution #2* below)

Solution #1:

Simplify each radical first:

$$\begin{aligned}\sqrt{8} &= \sqrt{4\times 2}\\ &= \sqrt{4}\sqrt{2}\\ &= 2\sqrt{2}\end{aligned} \qquad \begin{aligned}\sqrt{48} &= \sqrt{16\times 3}\\ &= \sqrt{16}\sqrt{3}\\ &= 4\sqrt{3}\end{aligned}$$

Now multiply your results by multiplying what is outside the radical and then multiplying what is inside the radical:

$$\begin{aligned}\left(2\sqrt{2}\right)\left(4\sqrt{3}\right) &= \left(2\times 4\sqrt{2\times 3}\right)\\ &= 8\sqrt{6}\end{aligned}$$

Solution #2:

Multiply the radicals first:

$$\begin{aligned}\sqrt{8}\sqrt{48} &= \sqrt{8\times 48}\\ &= \sqrt{348}\end{aligned}$$

Now simplify the radical:

$$\begin{aligned}\sqrt{348} &= \sqrt{64\times 6}\\ &= \sqrt{64}\sqrt{6}\\ &= 8\sqrt{6}\end{aligned}$$

Reading Check:

1. *Which of the two approaches to multiplying radicals in Example 4 on the previous page do you think is the simplest? Explain.*

2. *Multiply the radicals and simplify:* $\sqrt{3}\sqrt{6}$

3. *Multiply the radicals and simplify:* $\sqrt{8}\sqrt{2}$

Rationalizing the Denominator

It is always possible to express a fraction with no **radicals** in the denominator. This is called **rationalizing the denominator.**

- Rationalize the denominator means remove all ______________________ from the denominator of a fraction.

Removing radicals from the denominator is especially useful when you are adding fractions. The trick for doing this is based on the basic rule of fractions: *if you multiply the top and the bottom of a fraction by the same number, the fraction is unchanged.* This rule allows us to say, for instance, that $\frac{2}{3}$ is exactly the same number as $\frac{2\times3}{3\times3} = \frac{6}{9}$.

To **rationalize the denominator**, you can multiply the top and the bottom of a fraction by *whatever* is in the denominator, even if it is a radical. So, in a case like $\frac{1}{\sqrt{2}}$, you can multiply both the top and the bottom by $\sqrt{2}$.

$$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2\times2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}$$

- When you rationalize the denominator, multiply the numerator *and* the denominator by the __________ that is in the denominator.

Example 5

Rationalize the denominator of the fraction $\frac{4}{\sqrt{3}}$

The **radical** in the denominator is ____________.

To rationalize, you multiply both the top and the bottom by the denominator, $\sqrt{3}$.

$$\frac{4}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{4\sqrt{3}}{\sqrt{3\times3}} = \frac{4\sqrt{3}}{\sqrt{9}} = \frac{4\sqrt{3}}{3}$$

As you can see, your final answer has no radicals in the _______________.

Example 6

Rationalize the denominator of the fraction $\frac{6}{\sqrt{2}}$

To rationalize, you multiply both the top and the bottom by the denominator, $\sqrt{2}$.

$$\frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{6\sqrt{2}}{\sqrt{2\times2}} = \frac{6\sqrt{2}}{\sqrt{4}} = \frac{6\sqrt{2}}{2}$$

Now, be sure to *simplify* your final fraction by reducing the numbers outside of the radical:

$$\frac{6\sqrt{2}}{2} = 3\sqrt{2}$$

Reading Check:

1. *True or false:*

Rationalize the denominator means get all radicals out of the denominator of a fraction.

2. *Rationalize the denominator and simplify:* $\frac{3}{\sqrt{2}}$

3.11 Special Right Triangles, 45–45–90

Learning Objectives

- Identify and use the ratios involved with right isosceles triangles.

Right Isosceles Triangles

What happens when you draw a diagonal across a square? Try it in the margin. →

You get two **isosceles right triangles.** Since a square has 4 **right angles** inside, 2 of them stay complete when you make the diagonal and the other 2 are cut in half. Each of the half angles are 45° (because $90° \div 2 = 45°$.)

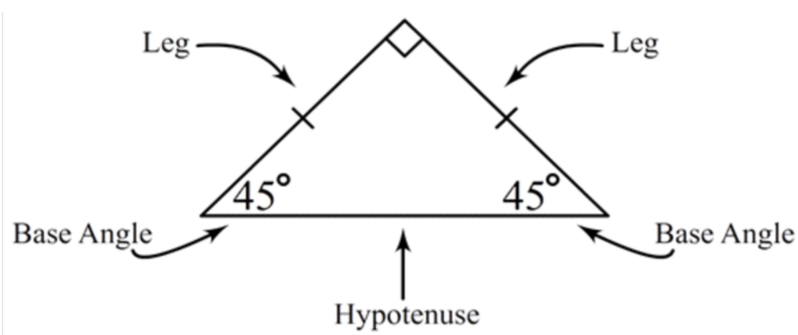
- Each triangle has the angles $45°, 45°$ (from the two angles cut in half), and ___________.

The diagonal becomes the **hypotenuse** of each isosceles right triangle because it is across from the right angle. Since a square has 4 congruent sides, each triangle is **isosceles** where the **legs** are the congruent sides of the square.

- The diagonal of the square becomes the __________________________ of each triangle.

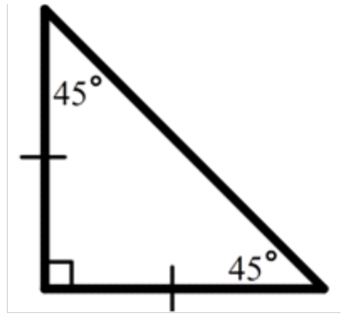
Each of these right triangles is a *special right triangle* called the $45° - 45° - 90°$ right triangles (because the angles inside the triangle are $45°, 45°$, and $90°$.)

As you know, isosceles triangles have two sides that are the same length. Additionally, the base angles of an isosceles triangle are congruent. An **isosceles right triangle** will always have base angles that each measure 45° and a vertex angle that measures 90°.



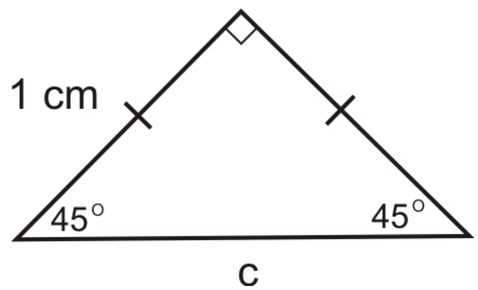
In the diagram above, the ______________ and the base ________________ are each congruent.

Don't forget that the base angles are the angles that are *opposite* the congruent sides. They don't have to be on the bottom of the figure, like in the picture below:



Example 1

The isosceles right triangle below has legs measuring 1 centimeter.



Use the Pythagorean Theorem to find the length of the hypotenuse.

Since the triangle is isosceles, the legs are *1 centimeter* each. Substitute *1* for both a and b, and solve for c:

$$\begin{aligned}
a^2+b^2 &= c^2\\
1^2+1^2 &= c^2\\
1+1 &= c^2\\
2 &= c^2\\
\sqrt{2} &= \sqrt{c^2}\\
c &= \sqrt{2}
\end{aligned}$$

In this example, $c = \sqrt{2}\ cm$.

What if each leg in the example above was 5 cm? Then we would have:

$$\begin{aligned}
a^2+b^2 &= c^2\\
5^2+5^2 &= c^2\\
25+25 &= c^2\\
50 &= c^2\\
\sqrt{50} &= \sqrt{c^2}\\
c &= 5\sqrt{2}
\end{aligned}$$

If each leg is 5 cm, then the hypotenuse is $5\sqrt{2}\ cm$.

When the length of each leg was *1*, the hypotenuse was $1\sqrt{2}$.

When the length of each leg was 5, the hypotenuse was $5\sqrt{2}$.

Is this a coincidence? No. Recall that the legs of all $45^\circ - 45^\circ - 90^\circ$ triangles are **proportional.**

What does proportional mean?

You may recognize the word "proportion," which means "ratio" or "fraction."

"Proportional" describes a relationship between 2 values where you can multiply one of the values by some number and get the second value.

For instance, 3 and 6 have the same "proportional" relationship as 4 and 8, because you need to multiply the first number by 2 to get the second number in both cases.

Another pair of numbers with the same proportional relationship is ________ and _________.

Another example is the sentence: "Punishment should be proportional to the crime".

This means that the worse a crime is, the harsher the punishment should be.

As we discovered in the examples on the previous page,

The hypotenuse of an **isosceles right triangle** will always equal the *product* of the length of one leg and $\sqrt{2}$.

This means that if a leg has a length of x, then you can *multiply* the leg by the number $\sqrt{2}$ to get the hypotenuse. So the hypotenuse has a length of $x\sqrt{2}$.

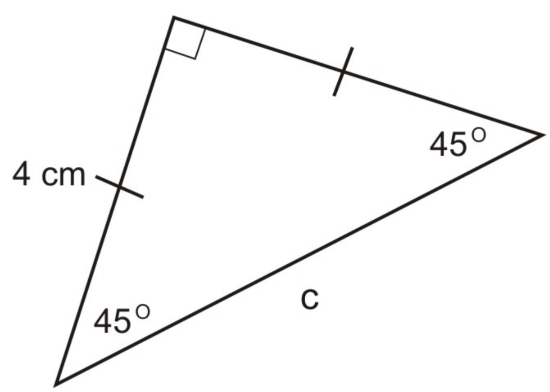
In all $45^\circ - 45^\circ - 90^\circ$ triangles:

- The length of the ___________________________ equals $\sqrt{2}$ times the length of a leg.

This relationship is very important to know!

Example 2

What is the length of the hypotenuse in the triangle below?



We just learned a relationship between the leg and the hypotenuse of a $45^\circ - 45^\circ - 90^\circ$ triangle, so this problem is much easier than using the Pythagorean Theorem again like in Example 1!

First, we must determine *which* side of the triangle is the **hypotenuse.**

Since the **hypotenuse** is the longest _____________ and it is across from the _______________ angle, it must be side c.

This makes the **legs** the other two sides, which have a length of _________________.

Since the length of the hypotenuse is the *product* of one leg and $\sqrt{2}$, you can calculate this length (c) by multiplying the **leg** by $\sqrt{2}$.

One leg is 4 inches, so the hypotenuse (c) will be $4\sqrt{2}$ inches, or about 5.66 inches.

Reading Check:

1. *Every isosceles right triangle has 3 special interior angles. What are they?*

___________ , ___________ , and ___________

2. *If an isosceles right triangle has legs that are 3 inches long, how long is its hypotenuse?*

a. *Draw a picture of the triangle here:*

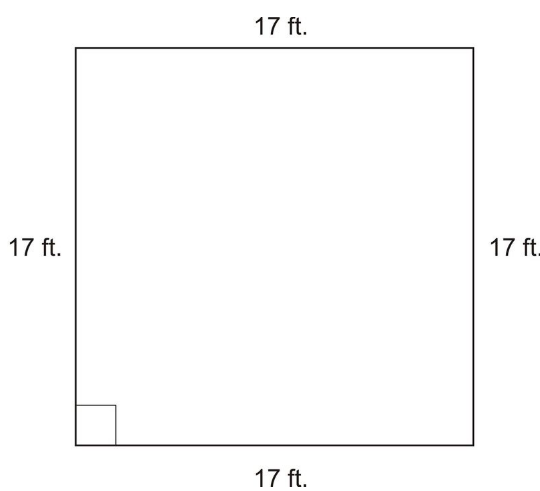
b. *Use the Pythagorean Theorem to find the length of the hypotenuse (like in Example 1):*

c. *Use the special proportional relationship to find the length of the hypotenuse (like in Example 2):*

d. *Are your answers to (b.) and (c.) above the same?*

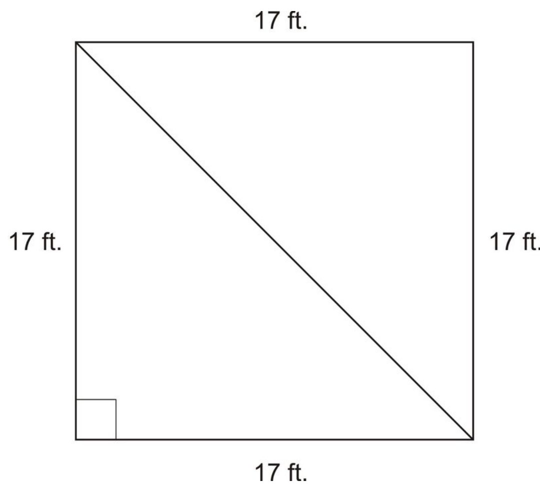
Example 3

Antonio built a square patio in his backyard.



He wants to make a water pipe for flowers that goes from one corner to another, diagonally. How long will that pipe be?

The first step in a word problem like this is to add important information to the drawing. Because the problem asks you to find the length from one corner to another, you should draw a diagonal line segment (from one corner of the square to the opposite corner) into your patio picture:



Once you draw the diagonal path, you can see how triangles help answer this question.

Because both legs of the triangle have the same measurement (17 feet), this is an **isosceles right triangle**. The angles in an isosceles right triangle are $45^\circ, 45^\circ$, and 90°

In an isosceles right triangle, the **hypotenuse** is always equal to the *product* of the length of one **leg** and $\sqrt{2}$. Just multiply these values together!

So, the length of Antonio's water pipe will be the product of 17 and $\sqrt{2}$, or $17\sqrt{2} \approx 17 \cdot (1.414)$ $feet$. This value is approximately equal to 24.04 feet. Therefore, his diagonal water pipe should be 24.04 feet long.

Reading Check:

You cook a grilled cheese sandwich. To make it easier to eat, you cut the sandwich in half diagonally. If each slice of bread (before it is cut) measures 14 cm by 14 cm, how long is the diagonal of your sandwich?

(Hint: draw yourself a picture to start this problem! If you are stuck, look at Example 3 to help you.)

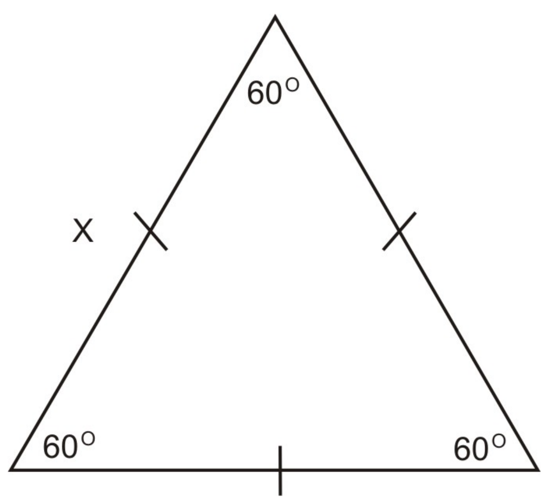
3.12 Special Right Triangles, 30-60-90

Learning Objectives

- Identify and use the ratios involved with $30^\circ - 60^\circ - 90^\circ$ triangles.
- Identify and use ratios involved with equilateral triangles.

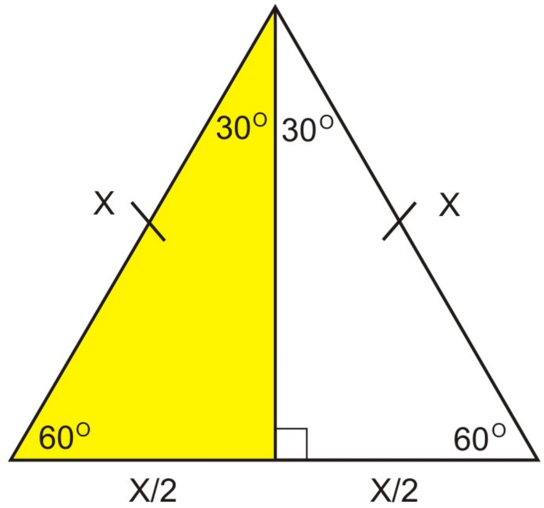
Equilateral Triangles

Remember that an **equilateral** triangle has sides that all have the same length. **Equilateral** triangles are also **equiangular** — all angles have the same measure. In an equilateral triangle, all angles measure exactly 60°.



Equilateral triangles are also ____________________.

Notice what happens when you divide an equilateral triangle in half:



This equilateral triangle is divided into 2 equal parts using an **altitude,** which is a line that is *perpendicular* to the base of the triangle. Since the altitude is perpendicular to the base, it makes a 90° angle with the base.

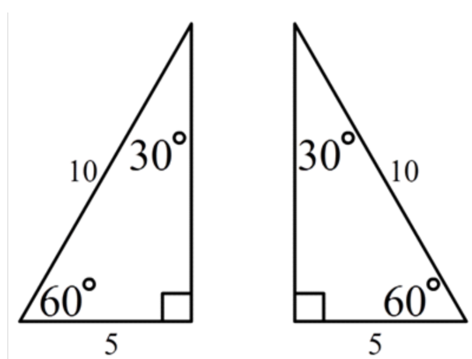
- An altitude is a line that is ______________________________ to the base of a triangle.

The altitude also splits the top 60° angle in the picture in half. Therefore, the angles on either side of the altitude are 30° (because $60^\circ \div 2 = 30^\circ$).

Each resulting right triangle created is a $30^\circ - 60^\circ - 90^\circ$ triangle:

- The hypotenuse of the resulting triangle is the side of the original, and the shorter leg is half of an original side.
- The **altitude** makes a 90° angle at the base and splits the 60° angle into two __________ angles.

This is why the **hypotenuse** is always *twice the length* of the *shorter* **leg** in a $30^\circ - 60^\circ - 90^\circ$ triangle, like in the picture below (where the original equilateral triangle had a side of length 10):



Again, the **hypotenuse** of a $30^\circ - 60^\circ - 90^\circ$ triangle is *always* double the length of the side *opposite* the 30° angle. You can use this information to solve problems about equilateral triangles.

In a $30^\circ - 60^\circ - 90^\circ$ triangle, the_____________ is *double* the shortest side.

The ________________ is the longest side of any right triangle.

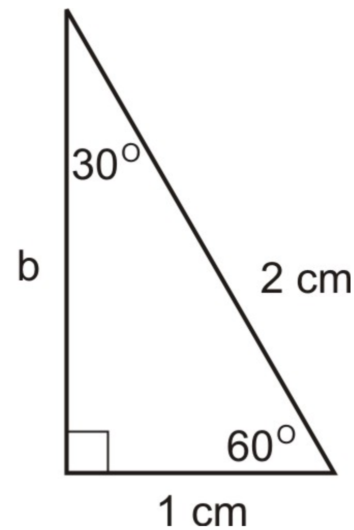
In a $30^\circ - 60^\circ - 90^\circ$ triangle, the side _______________ the 30° angle is the shortest side.

This important type of right triangle has angles measuring $30^\circ, 60^\circ$, and 90°.

Just as you found a constant ratio between the sides of an isosceles right triangle, you can find constant ratios here as well. Use the Pythagorean Theorem to discover these important relationships.

Example 1

Find the length of the missing leg in the following triangle. Use the Pythagorean Theorem to find your answer.



Just like you did for$45^\circ - 45^\circ - 90^\circ$ triangles, use the **Pythagorean Theorem** to find the missing side.

In this diagram, you are given two measurements:

- The **hypotenuse** (which is side ___________) is *2 cm* and
- The shorter **leg** (which is side ___________) is *1 cm*

Substitute these values into the Pythagorean Theorem $(a^2 + b^2 = c^2)$ to find the length of the missing leg (b):

$$\begin{aligned} a^2 + b^2 =& \ c^2 \\ 1^2 + b^2 =& \ 2^2 \\ 1 + b^2 =& \ 4 \\ -1 \qquad & -1 \\ b^2 =& \ 3 \\ b =& \ \sqrt{3} \end{aligned}$$

You can leave the answer in radical form as shown, or use your calculator to find the approximate value of $b \approx 1.732$ *cm*.

We can try this again using a hypotenuse of 6 feet.

Recall that since the $30^\circ - 60^\circ - 90^\circ$ triangle comes from an **equilateral** triangle, you know that the length of the *shorter* **leg** is *half* the length of the **hypotenuse.**

So the **hypotenuse**, c, is 6 feet.

Therefore, the shorter leg, a in the diagram on the previous page, is 3 feet (half of 6.)

$$\begin{aligned} a^2+b^2 &= c^2 \\ 3^2+b^2 &= 6^2 \\ 9+b^2 &= 36 \\ -9 \qquad &\quad -9 \\ b^2 &= 27 \\ b &= \sqrt{27} \\ &= \sqrt{9}\times\sqrt{3} = 3\sqrt{3}\ ft \approx 5.196\ ft \end{aligned}$$

The special relationship is as follows:

In all $30^\circ-60^\circ-90^\circ$ triangles,

- the hypotenuse will always be **twice** the length of the shorter leg,
- and the longer leg is always the product of the length of the shorter leg and $\sqrt{3}$.

In ratio form, the sides, in order from *shortest* to *longest* are in the extended ratio

$$x : x\sqrt{3} : 2x$$

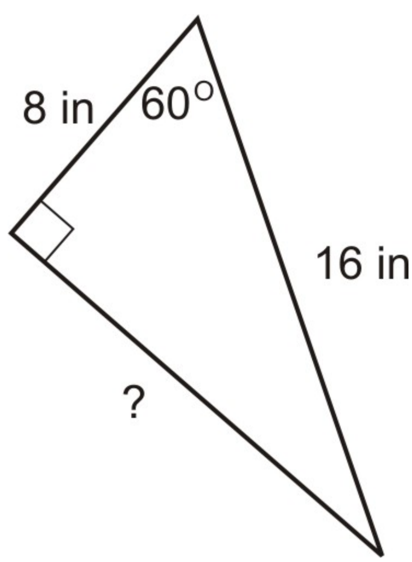
If we use ratio form, where the ratio is $x : x\sqrt{3} : 2x$

We can substitute any number in for x and the sides of the triangle will relate to each other in the same proportion.

For instance, if $x=4$, *the ratio is* $4:4\sqrt{3}:2(4)$ *or* $4:4\sqrt{3}:8$ *and if* $x=7$, *the ratio is* $7:7\sqrt{3}:2(7)$ *or* $7:7\sqrt{3}:14$

Example 2

What is the length of the missing leg in the triangle below?



You could use the Pythagorean Theorem for this problem (like in Example 1), but using the special proportional relationship in a $30^\circ-60^\circ-90^\circ$ triangle that you just learned is a much easier way!

The special relationship is:

- the **hypotenuse** is _____________________ the length of the *shorter* **leg**, and
- the *longer* **leg** is the _______________________ of the length of the *shorter* **leg** and $\sqrt{3}$

First, you know that the **hypotenuse** is 16 because it is *across* from the **right** angle.

Therefore, the other 2 sides are the **legs** in this triangle.

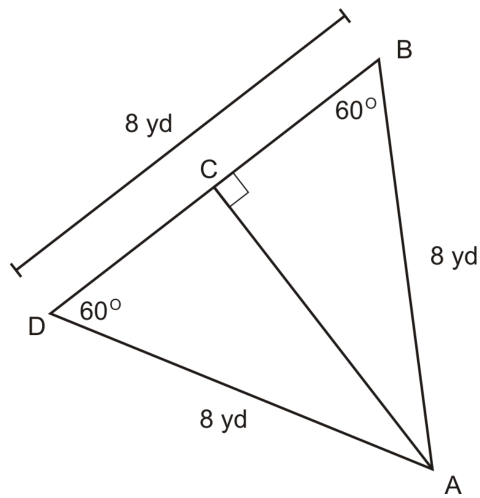
You also know that the *longer* **leg** is across from the 60° angle.

Since the length of the *longer* leg is the *product* of the *shorter* leg and $\sqrt{3}$, you can easily calculate this length:

The short leg is *8 inches,* so the longer leg will be $8\sqrt{3}$ *inches* or about 13.86 inches.

Example 3

What is AC below?



To find the length of segment $\overline{AC}$, identify its relationship to the rest of the triangle. Since it is an **altitude**, it forms two congruent triangles with angles measuring $30°, 60°$, and $90°$.

So, AC will be the product of BC (the shorter leg) and $\sqrt{3}$:

$$\begin{aligned} AC &= BC\sqrt{3} \\ &= 4\sqrt{3} \end{aligned}$$

$AC = 4\sqrt{3}$ yards, or approximately 6.93 yards.

Reading Check:

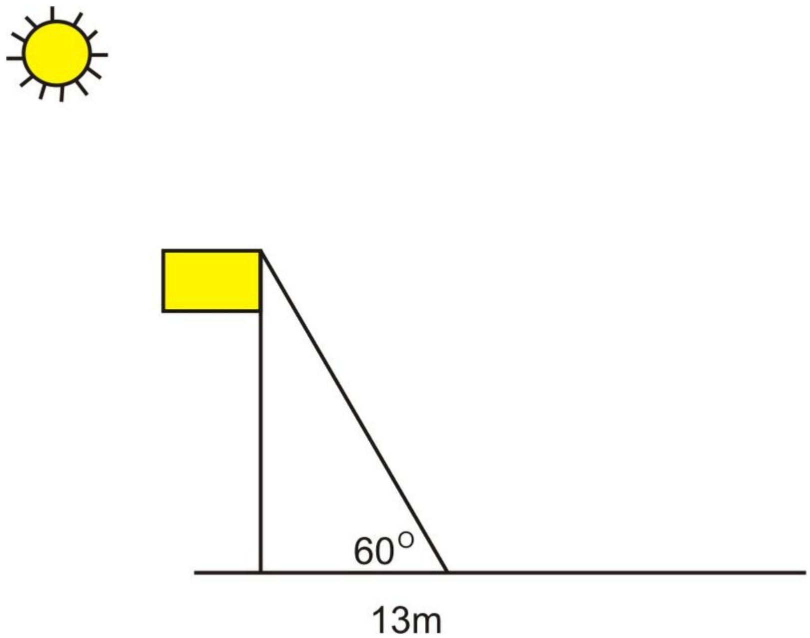
1. *Draw a* $30° - 60° - 90°$ *triangle and label its angle measures.*

2. *From shortest to longest sides, what ratio does every* $30° - 60° - 90°$ *triangle follow?*

________ : ________ : ________

Example 4

The diagram below shows the shadow a flagpole casts at a certain time of day.



If the length of the shadow cast by the flagpole is 13m, what is the height of the flagpole and what is the length of the hypotenuse of the right triangle shown?

The picture shows that this triangle has angles of $30^\circ, 60^\circ$, and 90° (This assumes that the flagpole is perpendicular to the ground, but that is a safe assumption). Although the 30° angle is not written into the picture, you can tell that the top angle is 30° because $180 - (90 + 60) = 30$.

The height of the flagpole is the *longer* leg in the triangle, so use the special right triangle ratios (along with the given height of the base of the triangle) to find the length of the missing sides, the flagpole height and the hypotenuse.

The *longer* leg is the *product* of the *shorter* leg and $\sqrt{3}$.

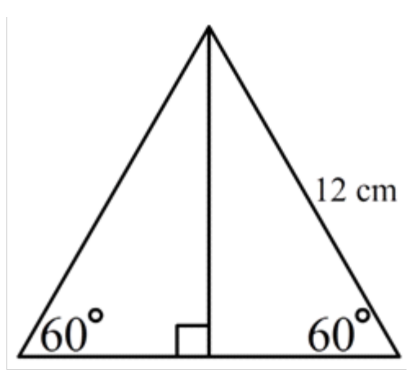
The length of the shorter leg is given as 13 meters, so the height of the flagpole is $13\sqrt{3}\ m$.

To find the length of the *hypotenuse,* use the hypotenuse of a $30^\circ - 60^\circ - 90^\circ$ triangle.

It will always be **twice** the length of the shorter leg, so it will equal $13 \cdot 2$, or 26 meters.

Reading Check:

What is the length of the ***altitude*** *in the triangle below?*

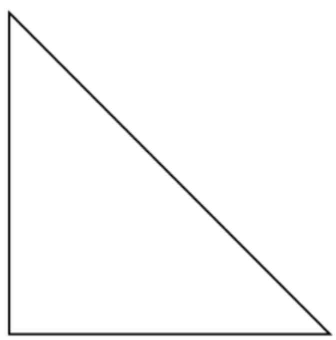


Graphic Organizer for Lessons 9 and 10

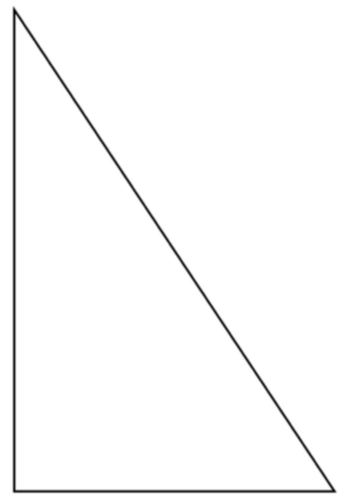
Special Triangles

Label all angles and side lengths for the triangles below. If the shortest side of each triangle is length x, what are the other sides?

45 – 45 – 90



30 – 60 – 90



3.13 Synthesis Day (Day 2 of 2 in Unit 3)

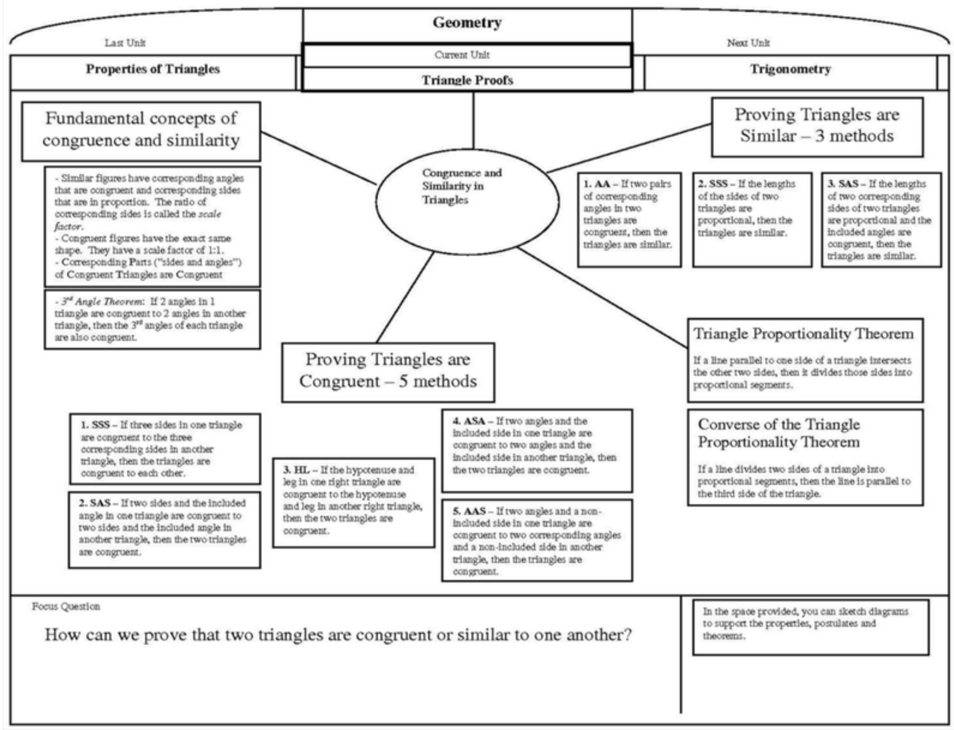
Classroom activity

Note: Please refer to Mike and Rose's lesson plans within the College Access Geometry materials for this lesson.

CHAPTER 4 Triangle Proofs

Chapter Outline

4.1 Concept Map



4.2 Vocabulary Self-Rating

TABLE 4.1: **Rating Guide: DK: I am sure I don't know it K: I am sure I know it ? I'm not sure**

Word	Before Lesson/Unit	After Lesson/Unit
Corresponding angles		
Corresponding sides		
Triangle part		
CPCTC		
Postulate		
Theorem		
Hypotenuse		
Leg		
Reflexive Property		
Symmetric Property		
Transitive Property		
Proof		
Deductive reasoning		
Flow proof		
Two-column proof		
Ratio		
Proportion		
Cross multiplication		
Reciprocal		
Proportional		
Counterexample		
Similar		
Scale factor		
Midsegment		
Parallel lines		
Transversal		

4.3 Corresponding Parts (CPCTC) and Identifying Minimal Conditions

Learning Objectives

- Define congruence in triangles.
- Create accurate congruence statements.
- Understand that if two angles of a triangle are congruent to two angles of another triangle, the remaining angles will also be congruent.

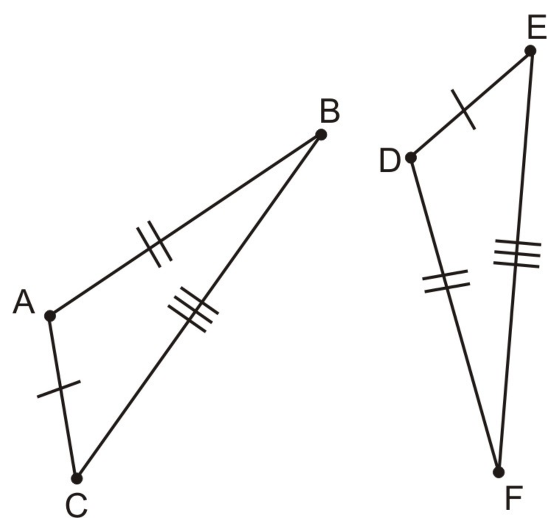
Defining Congruence in Triangles

Two figures are **congruent** if they have exactly the same size and shape. Another way of saying this is that the two figures can be perfectly aligned when one is placed on top of the other—but you may need to rotate or reflect (flip) the figures to make them line up.

When figures have exactly the same size and shape, they are ___________.

When that alignment is done, the angles that are matched are called **corresponding angles**, and the sides that are matched are called **corresponding sides**.

In congruent figures, the angles that match up are called ___________ angles and the matching sides are called ___________ sides.



Though the two triangles above may not look the same at first, when you rotate and flip triangle *DEF* on the right, the sides with the same number of tic marks line up!

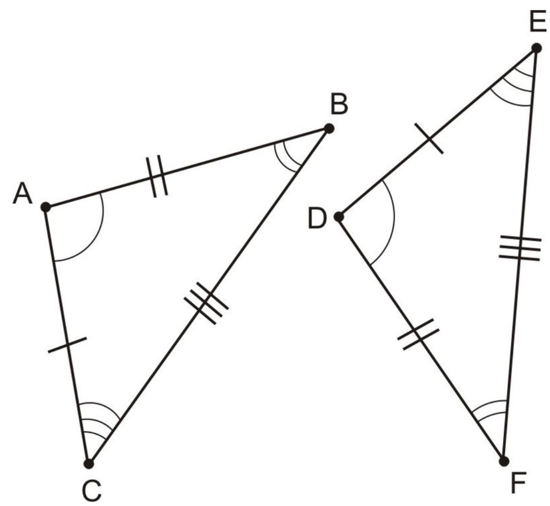
In the diagram above,

- Sides $\overline{AC}$ and $\overline{DE}$ each have one tic mark, indicating that they have the same length. Since they have the same length and are in matching positions in the triangle, they are **corresponding sides**.

- Sides $\overline{BA}$ and $\overline{DF}$ each have two tic marks, showing that they are also congruent and thus, **corresponding sides**.
- Finally, as you can see, $\overline{BC} \cong \overline{EF}$ because they each have three tic marks.

Each of these pairs corresponds because they are congruent to each other.

When two triangles are congruent, the three pairs of **corresponding angles** are also congruent. Notice the tic marks in the triangles below.



In congruent triangles, all three pairs of corresponding angles are ____________.

We use arcs inside the angle to show congruence in angles just as tic marks show congruence in sides. You can see that 1, 2, or 3 arcs inside each angle show which angles are congruent and corresponding. From the markings in the angles we can see:

$\angle A \cong \angle D, \angle B \cong \angle F,$ and $\angle C \cong \angle E.$

Which angle is congruent to $\angle F$? ___________________

Which angle is congruent to $\angle E$? ___________________

Which angle is congruent to $\angle D$? ___________________

Which side is congruent to $\overline{AC}$? ___________________

Which side is congruent to $\overline{DF}$? ___________________

Which side is congruent to $\overline{BC}$? ___________________

A term used to describe sides and angles is *part*. Since a triangle has three sides and three angles, we say that it has six *parts*.

Angles and sides are also called ___________________________ of a triangle.

By definition, if two triangles are congruent, then you know that all pairs of corresponding sides are congruent and all pairs of corresponding angles are congruent. We can therefore say that the **c**orresponding **p**arts (sides and angles) of **c**ongruent **t**riangles are **c**ongruent. This is often called **CPCTC**.

CPCTC

Corresponding **P**arts of **C**ongruent **T**riangles are **C**ongruent.

Reading Check:

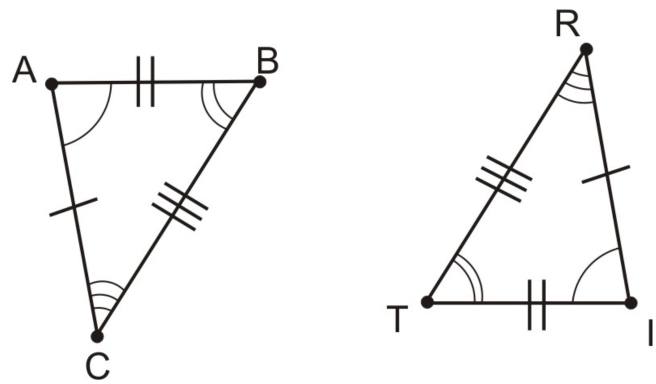
1. *Give 2 examples of a* ***part*** *of a triangle.*

2. *Fill in the blanks: If two triangles are congruent, this means that all of its corresponding _________ and _________ are congruent.*

3. *What do the letters **CPCTC** stand for?*

Example 1

Are the two triangles below congruent?



Begin by examining the sides:

- $\overline{AC}$ and $\overline{RI}$ both have one tic mark, so they are congruent.
- $\overline{AB}$ and $\overline{TI}$ both have two tic marks, so they are congruent as well.
- $\overline{BC}$ and $\overline{RT}$ have three tic marks each.

So *each pair of sides is congruent.*

Next you must check each angle:

- $\angle I$ and $\angle A$ both have one arc, so they are congruent.
- $\angle T \cong \angle B$ because they each have two arcs.
- Finally, $\angle R \cong \angle C$ because they have three arcs.

We can check that each angle in the first triangle matches with its corresponding angle in the second triangle by examining the sides. $\angle B$ corresponds with $\angle T$ because they are formed by the sides with two and three tic marks. Since all pairs of corresponding sides and angles are congruent in these two triangles, we conclude that YES, *the two triangles are congruent*.

Creating Congruence Statements

We have already been using the congruence sign $\cong$ when talking about congruent sides and congruent angles.

When writing congruence statements involving angles or triangles, use other symbols:

- The symbol $\overline{BC}$ means "segment *BC*"
- The symbol $\angle B$ means "angle *B*"
- Similarly, the symbol ΔABC means "triangle *ABC*"

In words, the symbol $\cong$ means _________________________________.

In words, the symbol ΔLMN means _________________________________.

In words, the symbol $\angle N$ means __________________________________.

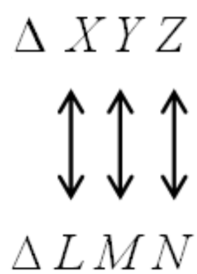
In words, the symbol $\overline{LM}$ means __________________________________.

When you are creating a congruence statement of two triangles, *the order of the letters is very important*. **Corresponding parts must be written in order**. This means that the angle at the first letter of the first triangle corresponds with the angle at the first letter of the second triangle, the angles at the second letter correspond, and the angles at the third letter correspond. If the angles are not matched up between the triangles, the parts will not correspond.

When writing congruence statements, __________________________________ parts must be written in order.

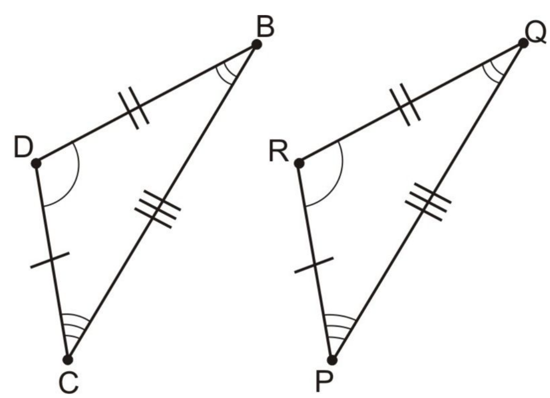
You can use *either* congruence statements or triangle pictures to tell if corresponding parts are in order.

For instance, in the congruence statement $\Delta XYZ \cong \Delta LMN$, the letters that match up tell you which angles in the triangles are congruent:



You can see that angles X and L match up, so $\angle X \cong \angle L$ angles Y and M match up, so $\angle Y \cong \angle M$ and angles Z and N match up, so $\angle$_____ $\cong \angle$ _____

Likewise, in the picture of the triangles below, you can match up the marked angles (or sides) to see what parts correspond:



If you are writing a congruence statement, you could NOT say that $\Delta BCD \cong \Delta PQR$ because the order of the letters does not match up to corresponding congruent angles.

If you look at $\angle B$, it does *not* correspond to $\angle P$. $\angle B$ corresponds to $\angle Q$ instead (indicated by the two arcs in the angles).

$\angle C$ corresponds to $\angle$ ______ (three arcs), and $\angle D$ corresponds to $\angle$ ______ (one arc).

Remember, you must compose the congruent statement so that the vertices are lined up for congruence, which is noted by the number of arcs inside the angles. The statement below is correct:

$$\Delta BCD \cong \Delta QPR$$

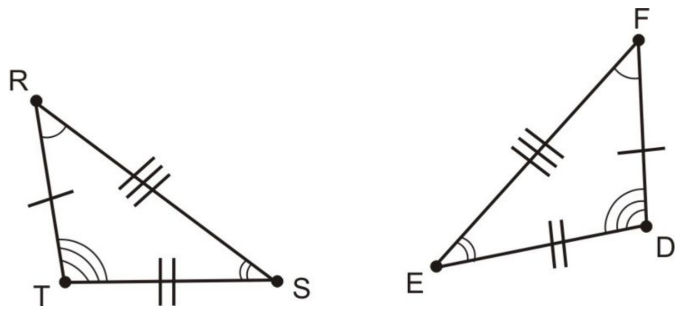
Reading Check:

Use the congruence statement $\Delta ABC \cong \Delta FGH$ to name all congruent angles in the two triangles:

$$\angle\_\_\_\_\_ \cong \angle\_\_\_\_\_$$
$$\angle\_\_\_\_\_ \cong \angle\_\_\_\_\_$$
$$\angle\_\_\_\_\_ \cong \angle\_\_\_\_\_$$

Example 2

Compose a congruence statement for the two triangles below.



To write an accurate congruence statement, you must be able to identify the corresponding pairs in the triangles above. Notice that:

- $\angle R$ and $\angle F$ each have one arc mark.
- Similarly, $\angle S$ and $\angle E$ each have two arcs, and
- $\angle T$ and $\angle D$ have three arcs.

Additionally, you can see from the tic marks on each side that:

- $RS = FE$ (or $\overline{RS} \cong \overline{FE}$),
- $ST = ED$, and
- $RT = FD$.

So, the two triangles are congruent, and to make the most accurate statement, this should be expressed by matching corresponding vertices. You can spell the first triangle in alphabetical order, for example, and then align the second triangle so its angles match with the angles in the first one:

$$\Delta RST \cong \Delta FED$$

Notice in example 2 that you don't need to write the angles in alphabetical order, as long as the **corresponding parts match**.

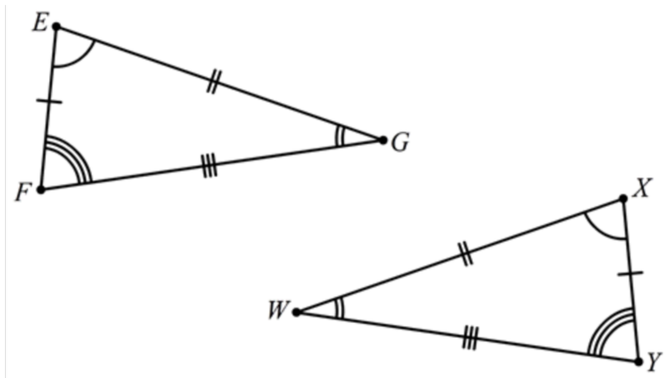
There are six ways to name any triangle by its vertices. You can start at any of the three vertices and then name the triangle's other vertices by progressing clockwise or counter-clockwise around the diagram. This process would give six different possible names for a triangle. For the diagram in Example 2, we could also express the congruence statement as follows:

$$\Delta DEF \cong \Delta TSR$$

Observe that this time we named the triangle on the left of the diagram first. The order does not matter. Both of these congruence statements are accurate because corresponding sides and angles are aligned within the statement.

Reading Check:

1. *In the diagram below, the two triangles are congruent. Create a congruence statement (using the geometry symbols* Δ *and* $\cong$ *) for the diagram. Remember to be careful with corresponding angles and sides!*



2. *For the same diagram above, create another congruence statement that is also true. Make sure your angles match!*

3. *For the same diagram above, create a third congruence statement that is also true. How many more true congruence statement can you write?*

The Third Angle Theorem

Previously, you studied the Triangle Sum Theorem, which states that the sum of the measures of the interior angles in a triangle will always be equal to 180°. This information is useful when showing congruence.

The Triangle Sum Theorem says that the measures of all three angles inside a triangle add up to ____________.

As you practiced, if you know the measures of two angles within a triangle, there is only one possible measurement of the third angle. Thus, if you can prove two corresponding angle pairs congruent, the third pair is also guaranteed to be congruent.

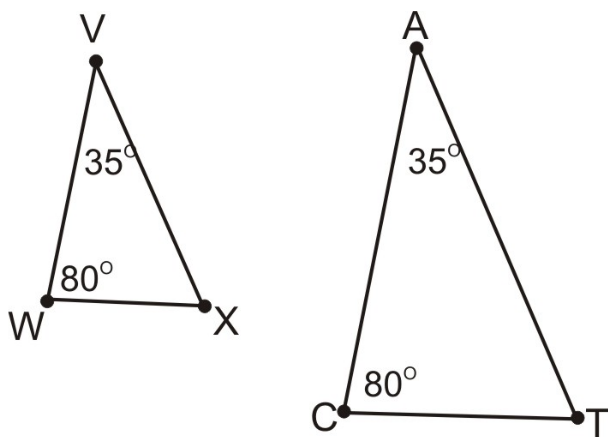
Third Angle Theorem

If two angles in one triangle are congruent to two angles in another triangle, then the third pair of angles are also congruent.

This means that once you know two congruent angle pairs, then the last angle pair is also ____________________-____________.

Example 3

Identify whether or not the missing angles in the triangles below are congruent.



One triangle looks bigger than the other. Does that mean all of its angles are bigger? To identify whether or not the third angles are congruent, you must first find their measures.

Start with the triangle on the left. Since you know two of the angles in the triangle, you can use the triangle sum theorem to find the missing angle. In ΔWVX we know:

$$\begin{aligned} m\angle W + m\angle V + m\angle X &= 180^\circ \\ 80^\circ + 35^\circ + m\angle X &= 180^\circ \\ 115^\circ + m\angle X &= 180^\circ \\ m\angle X &= 65^\circ \end{aligned}$$

The missing angle of the triangle on the left measures 65°. Repeat this process for the triangle on the right:

$$\begin{aligned} m\angle C + m\angle A + m\angle T &= 180^\circ \\ 80^\circ + 35^\circ + m\angle T &= 180^\circ \\ 115^\circ + m\angle T &= 180^\circ \\ m\angle T &= 65^\circ \end{aligned}$$

Since the measure of both angles is 65°, $\angle X \cong \angle T$.

Reading Check:

1. *Fill in the blanks:*

*The **Third Angle** ____________ says that if two angles in one triangle are ____________ to two angles in another triangle, then the third pair of angles are also ____________.*

2. *In Example 3 (on the previous page), create three true congruence statements for the two triangles.*

4.4 Triangle Congruence using SSS

Learning Objectives

- Understand and apply the SSS Congruence Postulate.

Proving Triangles are Congruent

In the last section you learned that if two triangles are congruent, then the three pairs of corresponding sides are congruent and the three pairs of corresponding angles are also congruent.

In symbols, $\Delta CAT \cong \Delta DOG$ means:

- $\angle C \cong \angle D$
- $\angle A \cong \angle O$
- $\angle T \cong \angle G$

and

- $\overline{CA} \cong \overline{DO}$
- $\overline{AT} \cong \overline{OG}$
- $\overline{CT} \cong \overline{DG}$

Indeed, one triangle congruence statement contains six different congruence statements about sides and angles!

If two triangles are congruent, then they have three pairs of ______________________________ sides and three pairs of congruent ________________________.

In this section we show that proving two triangles are congruent does not necessarily require showing all six congruence statements are true. There are shortcuts for showing two triangles are congruent using certain combinations of three parts of two different triangles.

Side-Side-Side (SSS) Triangle Congruence Postulate

If three sides in one triangle are congruent to the three corresponding sides in another triangle, then the triangles are congruent to each other.

*Do you remember the difference between a **postulate** and a **theorem**?*

From Unit 1:

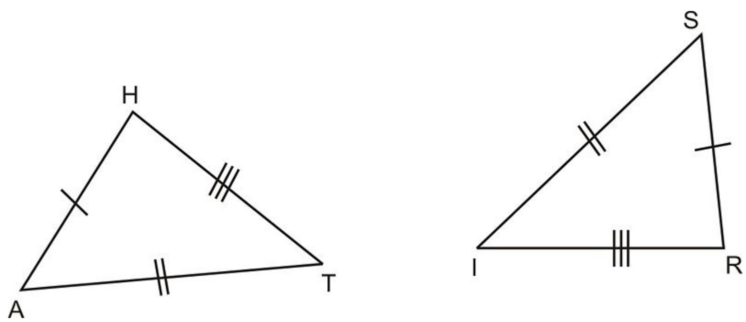
*A **postulate** is a basic rule that we accept without proof.*

*A **theorem** is a statement that can be proven true using postulates, definitions, logic, and other theorems we've already proven to be true.*

*You have already learned many **postulates** and **theorems**; the ones in this lesson and the next few lessons are particular to congruent triangles.*

Example 1

Write a triangle congruence statement based on the diagram below:



We can see from the tic marks that there are three pairs of corresponding congruent sides:

$\overline{HA} \cong \overline{RS}, \overline{AT} \cong \overline{SI}$, and $\overline{TH} \cong \overline{IR}$

Matching up the corresponding sides, we can write the congruence statement:

$$\Delta HAT \cong \Delta RSI$$

Don't forget that ORDER MATTERS when writing triangle congruence statements. Here, we lined up the sides with one tic mark, then the sides with two tic marks, and finally the sides with three tic marks.

Reading Check:

1. *What do the letters **SSS** (in the SSS Congruence Postulate) stand for?*

2. *Use the congruence statement $\Delta ABC \cong \Delta FGH$ to name all congruent sides in the two triangles:*

$$\_\_\_\_\_\_ \cong \_\_\_\_\_\_$$
$$\_\_\_\_\_\_ \cong \_\_\_\_\_\_$$
$$\_\_\_\_\_\_ \cong \_\_\_\_\_\_$$

3. *In your own words, describe the difference between a postulate and a theorem.*

4.5 Triangle Congruence using SAS, HL & ASA

Learning Objectives

- Understand and apply the SAS Congruence Postulate.
- Identify the distinct characteristics and properties of right triangles.
- Understand and apply the HL Congruence Theorem.
- Understand and apply the ASA Congruence Postulate.

SAS Congruence

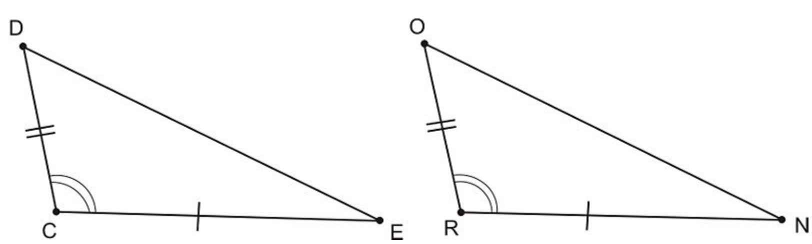
One more way to show two triangles are congruent is by the **SAS Congruence Postulate**.

Side-Angle-Side (SAS) Triangle Congruence Postulate

If two sides and the included angle in one triangle are congruent to two sides and the included angle in another triangle, then the two triangles are congruent.

SAS stands for the ___________________ - ___________________ - ___________________ Triangle Congruence Postulate.

The order of the letters is very significant. You must have *the angles between the two sides* for the SAS postulate to be valid.



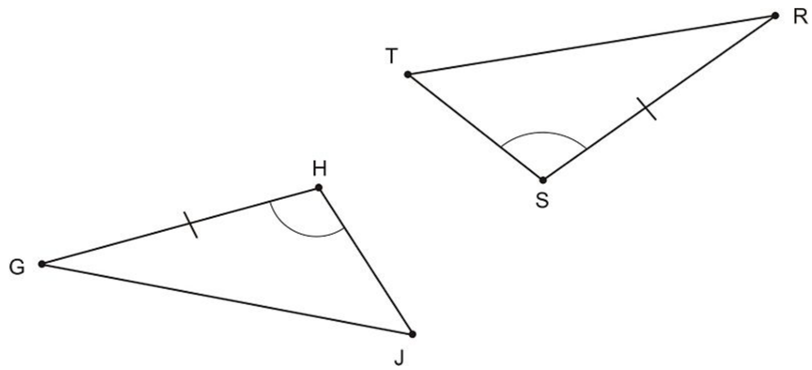
Notice in the diagram above, the congruent angles are *in between* the pair of congruent sides. First, we see that the left sides (the first **S** in **SAS**) of the triangle are congruent because of the double tic marks: $\overline{DC} \cong \overline{OR}$. Next moving around the triangle in a counter-clockwise direction, we see that the angles (**A** in **SAS**) are congruent because of the double arcs: $\angle C \cong \angle R$. Finally, the bottom sides (next part in the counter-clockwise direction) of the triangle (the last **S** in **SAS**) are congruent because of the single tic mark: $\overline{CE} \cong \overline{RN}$.

The congruent side-angle-side pairs in these two triangles satisfy the **SAS Triangle Congruence Postulate**.

In **SAS**, the angle must be _________________________________ the two sides.

Example 1

What information would you need to prove that these two triangles were congruent using the SAS postulate?



A. the measures of $\angle HJG$ and $\angle STR$

B. the measures of $\angle HGJ$ and $\angle SRT$

C. the measures of $\overline{HJ}$ and $\overline{ST}$

D. the measures of sides $\overline{GJ}$ and $\overline{RT}$

If you are to use the **SAS** postulate to establish congruence, you need to have the measures of two sides and the angle in *between* them for both triangles.

So far, you have one side and one angle. So, you must use the other side adjacent to the same angle. In ΔGHJ, that side is $\overline{HJ}$. In triangle ΔRST, the corresponding side is $\overline{ST}$.

So, the correct answer is C.

Reading Check:

1. *What do the letters* ***SAS*** *(in the SAS Congruence Postulate) stand for?*

2. *Where must the angle be (in relation to the sides) for the* ***SAS Congruence Postulate*** *to work?*

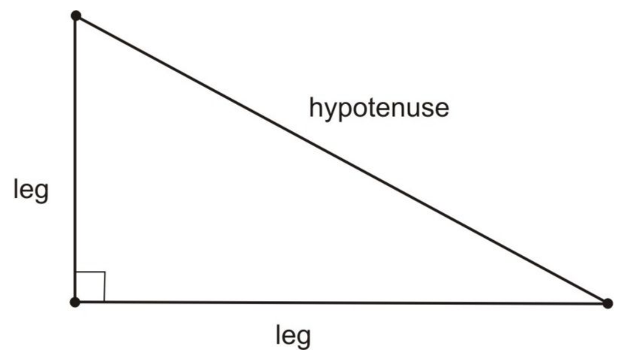
3. *True/False: Two right triangles are congruent if the first triangle has legs that are 6 inches and 8 inches in length, and the second triangles has legs that are 8 inches and 6 inches in length. (Hint: draw a picture to help you!)*

Right Triangles

So far, the congruence postulates we have examined work on any triangle you can imagine. As you know, there are a number of types of triangles:

- **Acute triangles** have all angles measuring less than 90°.
- **Obtuse triangles** have one angle measuring between 90° and 180°.
- **Equilateral triangles** have congruent sides, and all angles measure 60°.
- **Right triangles** have one angle measuring exactly 90°.

In **right triangles**, the sides have special names. The two sides *adjacent* to (or next to) the right angle are called **legs** and the side *opposite* the right angle is called the **hypotenuse**.

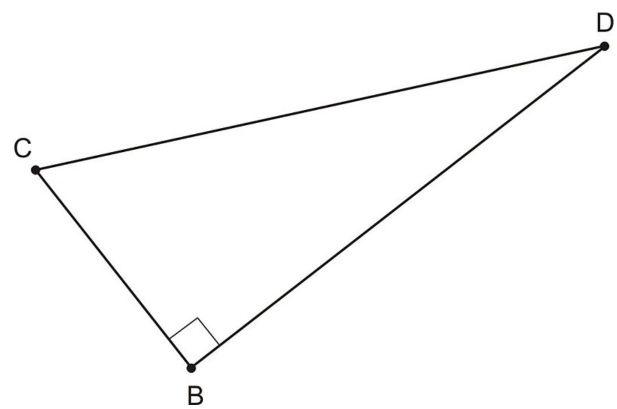


In a right triangle, the side opposite the right angle is called the ___________________________.

In a right triangle, the sides next to the right angle are called ______________________.

Example 2

Which side of right triangle BCD is the hypotenuse?



Looking at ΔBCD, you can identify $\angle CBD$ as a **right angle** (remember the little square tells us the angle is a right angle).

By definition, the **hypotenuse** of a right triangle is *opposite the right angle*. So, side $\overline{CD}$ is the hypotenuse.

Reading Check:

1. *How many right angles are in a right triangle?*

2. *Which side of a right triangle is across from the right angle?*

3. *Which side of a right triangle is next to the right angle?*

4. *In your own words, describe an acute triangle.*

5. *In your own words, describe an obtuse triangle.*

HL Congruence

The "**H**" and "**L**" stand for **hypotenuse** and **leg** of right triangles.

HL Congruence Theorem

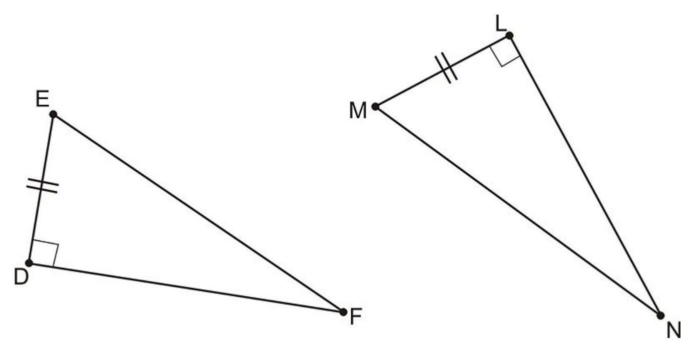
If the hypotenuse and leg in one right triangle are congruent to the hypotenuse and leg in another right triangle, then the two triangles are congruent.

HL stands for ______________________________ - ____________________ Congruence Theorem.

The **HL Congruence Theorem** only works for _________________________ triangles.

Example 3

What information would you need to prove that these two triangles were congruent using the HL theorem?



A. the measures of sides $\overline{EF}$ and $\overline{MN}$

B. the measures of sides $\overline{DF}$ and $\overline{LN}$

C. the measures of angles $\angle DEF$ and $\angle LMN$

D. the measures of angles $\angle DFE$ and $\angle LNM$

Since these are right triangles, you only need *one leg* and the *hypotenuse* to prove congruence. Legs $\overline{DE}$ and $\overline{LM}$ are congruent, so you need to find the lengths of the *hypotenuses*. The hypotenuse of ΔDEF is $\overline{EF}$. The hypotenuse of ΔLMN is $\overline{MN}$. So, you need to find the measures of sides $\overline{EF}$ and $\overline{MN}$.

The correct answer is A.

Reading Check:

1. *What do the letters* ***HL*** *(in the HL Congruence Theorem) stand for?*

2. *What other congruence postulate is the HL Congruence Theorem very similar to?*

(Hint: think about the angle in between the hypotenuse and leg of a right triangle. What do these three parts spell?)

ASA Congruence

One of the other ways you can prove congruence between two triangles is the **ASA Congruence Postulate**. To use the ASA Postulate to show that two triangles are congruent, you must identify two angles and the included side (the side in between them).

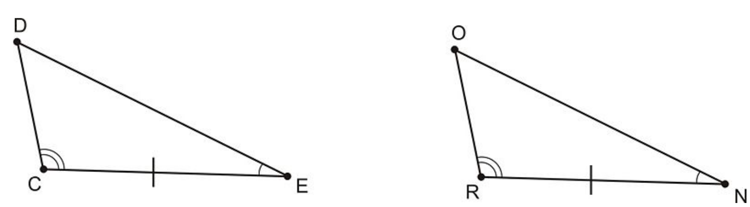
Angle-Side-Angle (ASA) Congruence Postulate

If two angles and the included side in one triangle are congruent to two angles and the included side in another triangle, then the two triangles are congruent. **ASA** stands for the ____________________ - ____________________ - ____________________ Congruence Postulate.

In the **ASA Postulate**, you must use two angles and the side in _______________________ them.

Notice also that by picking two of the angles of the triangle, you have determined the measure of the *third* by the **Triangle Sum Theorem**. So, in reality, you have defined the whole triangle; you have identified all of the angles in the triangle, and by picking the length of one side, you defined the scale.

So, no matter what, if you have two angles and the side in between them, you have described the whole triangle.

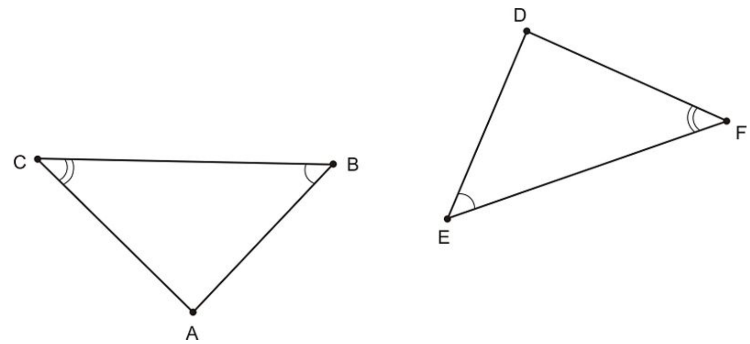


The diagram above shows two congruent triangles using parts from the **ASA Congruence Postulate**. Moving counter-clockwise, the markings on the triangles show congruent matching left angles, then bottom sides, then right angles:

$$\angle DCE \cong \_\_\_\_\_\_$$
$$\overline{CE} \cong \_\_\_\_\_\_$$
$$\angle CED \cong \_\_\_\_\_\_$$

Example 4

What information would you need to prove that these two triangles are congruent using the ASA postulate?



A. the measures of the missing angles

B. the measures of sides $\overline{AB}$ and $\overline{BC}$

C. the measures of sides $\overline{BC}$ and $\overline{EF}$

D. the measures of sides $\overline{AC}$ and $\overline{DF}$

If you are to use the **ASA Postulate** to prove congruence, you need to have two pairs of congruent angles and the *included* side, the side in *between* the pairs of congruent angles.

The side in between the two marked angles in ΔABC is side $\overline{BC}$. The side in between the two marked angles in ΔDEF is side $\overline{EF}$. You would need the measures of sides $\overline{BC}$ and $\overline{EF}$ to prove congruence.

In total, you would then have the following:

$$\angle ABC \cong \_\_\_\_\_\_$$
$$\overline{BC} \cong \_\_\_\_\_\_$$
$$\angle ACB \cong \_\_\_\_\_\_$$

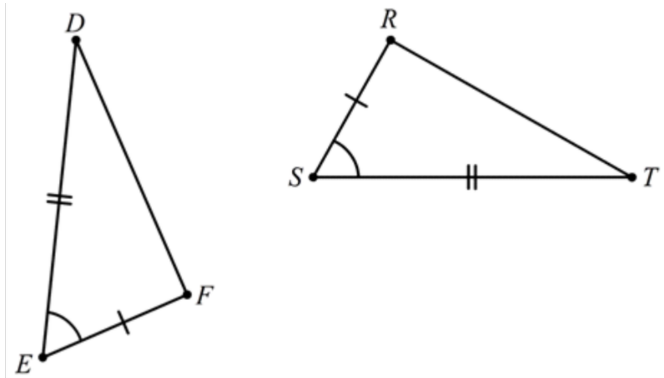
The correct answer is C.

<u>**Reading Check:**</u>

1. *What do the letters* ***ASA*** *(in the ASA Congruence Postulate) stand for?*

2. *Why are the Third Angle Theorem and the ASA Congruence Postulate similar?*

3. *In the diagram below, which triangle congruence postulate would you use to prove that the two triangles are congruent?*


D
E
F
R
S
T

4.6 Triangle Congruence using AAS

Learning Objectives

- Understand and apply the AAS Congruence Theorem.

AAS Congruence

Another way you can prove congruence between two triangles is by using two angles and the *non-included* side.

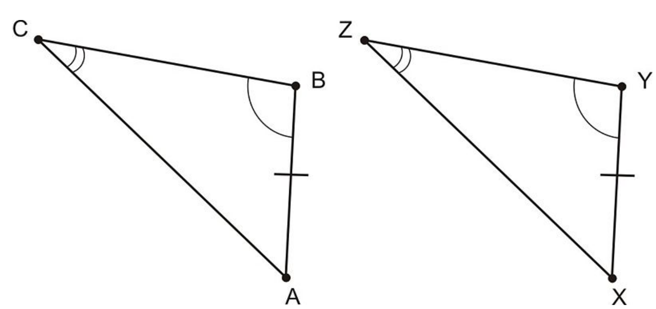
Angle-Angle-Side (AAS) Congruence Theorem

If two angles and a non-included side in one triangle are congruent to two corresponding angles and a non-included side in another triangle, then the triangles are congruent.

In the **AAS Theorem**, you use two _______________ and a _____________________________ side to prove congruence.

This is a *theorem* because it can be *proven*. First, we will do an example to see why this theorem is true, then we will prove it formally. Like the **ASA Postulate**, the **AAS Theorem** uses *two angles* and a *side* to prove triangle congruence. However, the order of the letters (and the angles and sides they stand for) is different.

The **AAS Theorem** is *equivalent* to the **ASA Postulate** because when you know the measure of two angles in a triangle, you also know the measure of the *third* angle. The pair of congruent sides in the triangles will determine the size of the two triangles. We will explore this further in the last section of this lesson.

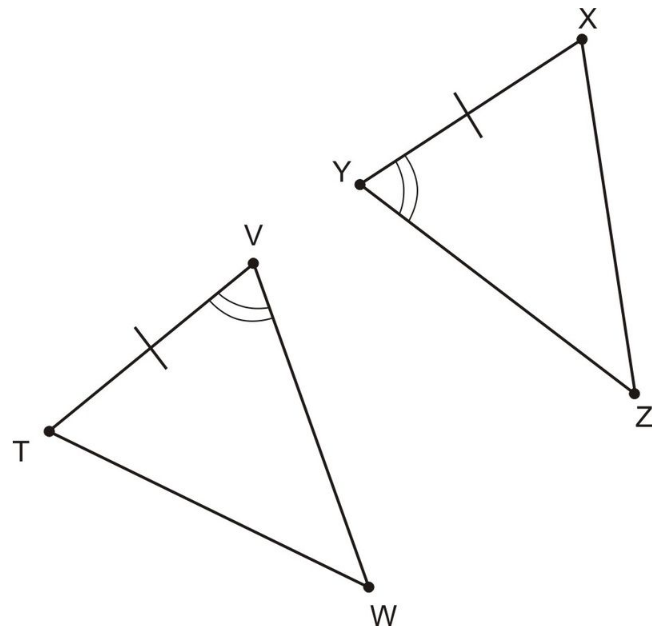


Notice that when you look at the congruent triangles in a clockwise direction (beginning at $\angle C$ and $\angle Z$), the congruent parts spell A-A-S, but when you look at them in a counter-clockwise direction, they spell S-A-A.

The **AAS Theorem** is similar to the ____ because they both use two angles and a side to prove congruence.

Example 1

What information would you need to prove that these two triangles were congruent using the AAS Theorem?



A. the measures of sides $\overline{TW}$ and $\overline{XZ}$

B. the measures of sides $\overline{VW}$ and $\overline{YZ}$

C. the measures of $\angle VTW$ and $\angle YXZ$

D. the measures of angles $\angle TWV$ and $\angle XZY$

If you are to use the AAS Theorem to prove congruence, you need to know that pairs of two angles are congruent and the pair of sides adjacent to one of the given angles are congruent.

You already have one side and its adjacent angle, but you still need another angle. It needs to be the angle not touching the known side, rather than adjacent to it. Therefore, you need to find the measures of $\angle TWV$ and $\angle XZY$ to prove congruence.

Then you would have:

$$\angle TWV \cong \_\_\_\_\_\_ (A)$$
$$\angle WVT \cong \_\_\_\_\_\_ (A)$$
$$\overline{VT} \cong \_\_\_\_\_\_ (S)$$

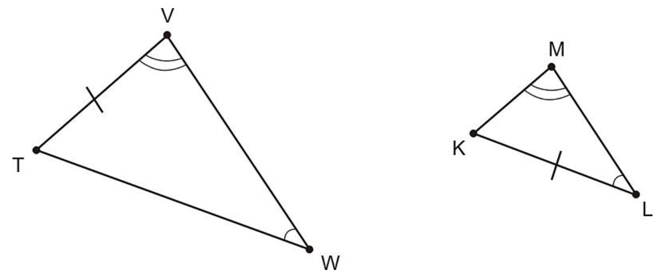
These spell A-A-S.

The correct answer is D.

When you use **AAS** (or any triangle congruence postulate) to show that two triangles are congruent, you need to make sure that the *corresponding pairs of angles and sides actually align*.

When using triangle congruence postulates, it is important for ______________________________ angles and sides to match up.

For instance, look at the diagram below:



Even though two pairs of angles and one pair of sides are congruent in the triangles, these triangles are NOT congruent. Why?

Notice that the marked side in ΔTVW is $\overline{TV}$, which is between the *unmarked angle* and the *angle with two arcs*.

However in ΔKML, the marked side is between the *unmarked angle* and the *angle with one arc*.

Since the corresponding parts DO NOT match up, you CANNOT use AAS to say these triangles are congruent.

If you want to prove that two triangles are ________________________________, you must be careful to make sure that ________________________________ parts of the triangles match up!

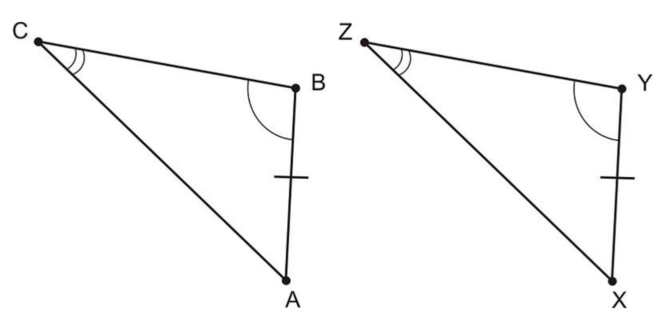
Reading Check:

1. *In the space below, sketch two congruent triangles and mark the parts that are congruent by the* ***AAS Theorem.***

2. *In the space below, sketch two congruent triangles and mark the parts that are congruent by the* ***ASA Postulate.***

AAS and ASA Congruence

The **AAS Triangle Congruence Theorem** is logically the *exact same* as the **ASA Triangle Congruence Postulate**. Look at the following diagrams to see why:



You can see the following in the figure on the previous page:

- $\angle C \cong \angle Z$ because both angles have two arcs
- $\angle B \cong \angle Y$ because both angles have one arc

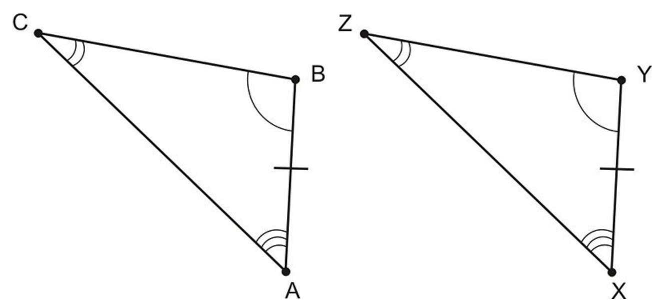
The congruent parts in the figure spell A-A-S, so based on these markings you see, the triangles are congruent because of the **AAS Theorem**.

ALSO,

Since $\angle C \cong \angle Z$ and $\angle B \cong \angle Y$, we can conclude from the **Third Angle Theorem** that $\angle A \cong \angle X$. This is because the sum of the measures of the three angles in each triangle is 180° and if we know the measures of two of the angles, then the measure of the third angle is already determined. We therefore know that *all three angles in both triangles are congruent*.

We know that $\angle A \cong \angle X$ because all three angles in a triangle add up to _______________.

Marking $\angle A \cong \angle X$, the diagram becomes this:



Now we can see that:

- $\angle A \cong \angle X$ (A)
- $\overline{AB} \cong \overline{XY}$ (S)
- and $\angle B \cong \angle Y$ (A)

which shows that $\Delta ABC \cong \Delta XYZ$ is *also true* by the **ASA Postulate**.

Reading Check:

1. *True/False: When you use the AAS Congruence Theorem OR the ASA Congruence Postulate, you can prove that all angles in both triangles are congruent with the Third Angle Theorem.*

2. *True/False: The AAS Theorem and the ASA Postulate are logically completely different.*

Graphic Organizer for Lessons 3-5

TABLE 4.2: Proving Triangle Congruence – Postulates and Theorems

Type of Congruency	*Letters stand for...*	*Postulate or Theorem?*	*Draw a picture*	*Describe the corresponding congruent parts*
SSS				
SAS				
ASA				
AAS				
HL				

4.7 Triangle Congruence Proofs

Learning Objectives

- Explore properties of triangle congruence.
- Understand and practice flow proofs.
- Understand and practice two-column proofs.

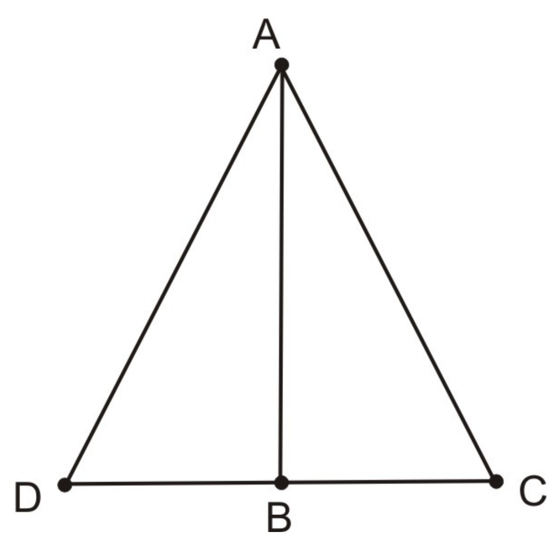
Congruence Properties

In earlier mathematics courses, you have learned concepts like the commutative or associative properties. These concepts help you solve many types of mathematics problems. There are a few properties relating to congruence that will help you solve geometry problems as well. These are especially useful in two-column proofs, which you will learn later in this lesson!

The Reflexive Property of Congruence

The **reflexive property of congruence** states that *any shape is congruent to itself*. This may seem obvious, but in a geometric proof, you need to identify every possibility to help you solve a problem. If two triangles share a line segment, you can prove congruence by the reflexive property.

The **Reflexive Property** says that any shape is ________________________________ to itself.



In the diagram above, you can say that the shared side of the triangles $(\overline{AB})$ is congruent because of the reflexive property. Or in other words, $\overline{AB} \cong \overline{AB}$.

*One way to remember the **Reflexive Property** is that the word "reflexive" has the same root as "reflection."*

"Reflection" should make you think of a mirror.

When you look in the mirror, you see yourself!

Likewise, the reflexive property says that something is equal to itself.

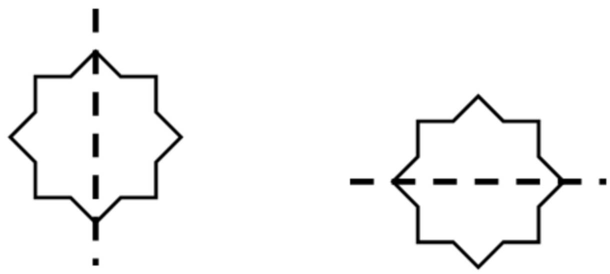
The Symmetric Property of Congruence

The **symmetric property of congruence** states that *congruence works frontwards and backwards*, or in symbols, if $\angle ABC \cong \angle DEF$ then $\angle DEF \cong \angle ABC$. As long as the congruent angles match up between the triangles (like you learned in Lessons 1 and 2), it does not matter which order you write the triangle name.

The **Symmetric Property** says that frontwards congruence and ________________________ congruence are the same.

One way to remember the ***symmetric property*** *is that the word "symmetric" means "the same front to back."*

The picture of the shape below is "symmetric" because the left side is the same as the right side (but backwards) and the top is the same as the bottom (but backwards):



The dotted lines down the middle of the shapes can act like a folding line: if you fold the shape over the line, the two sides will be on top of each other. Each half is the same, but backwards, of its other half.

The Transitive Property of Congruence

The **transitive property of congruence** states that *if two shapes are congruent to a third, they are also congruent to each other*.

In other words, if $\Delta ABC \cong \Delta JLM$, and $\Delta JLM \cong \Delta WYZ$, then $\Delta ABC \cong \Delta WYZ$.

This property is very important in identifying congruence between different shapes.

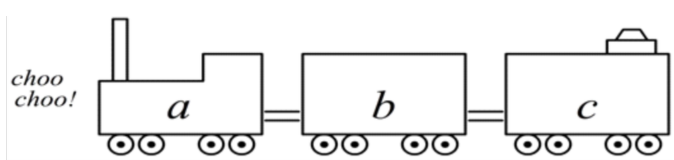
The **Transitive Property** says that if a first shape is congruent to a second, and if the second is congruent to a third, then the ________________ shape is congruent to the _______________.

One way to remember the ***transitive property*** *is that the word "transitive" is similar to the word "train."*

A train has cars that connect to each other. In the transitive property, the equal (or congruent) sign acts like the connecting piece between the cars:

If $a = b$, and $b = c$, then $a = c$.

On a train, this means that the first car (a) is connected to the second car (b), and the second car (b) is connected to the last car (c). So, the first car (a) is connected to the last car (c)!



Graphic Organizer for Lesson 6

TABLE 4.3: Properties of Congruence

Name of Property	***Give an example: Write a statement of congruence using this property***	***In your own words, how can you recognize this property?***	***What is a good way to remember this property?***
Reflexive			
Symmetric			
Transitive			

Reading Check:

Name the property used in the following geometric statements:

1. $\angle MLK \cong \angle KLM$

2. *If* $\angle PQR \cong \angle BCD$ *and* $\angle BCD \cong \angle XYZ$, *then* $\angle PQR \cong \angle XYZ$

3. $\angle FGH \cong \angle FGH$

Proving Triangles Congruent

In geometry we use **proofs** to show something is true. You have seen a few proofs already—they are a special form of argument in which you have to *justify every step of the argument with a reason*. Valid reasons are definitions, properties, postulates, theorems or results from other proofs.

Proofs show that something is ______________________.

In a proof, you must provide a ________________________________ for each step.

Proofs use **deductive reasoning**. You will begin with statements that are accepted as facts, which will lead to other statements based on these facts. Each statement and its corresponding reason will be a step in your proof.

Proofs use ___________________________________ reasoning.

Do you remember ***deductive reasoning*** *from Unit 2?*

Deductive reasoning is also known as logic.

Deductive reasoning starts with accepted facts or statements which we know are true. Then, we draw conclusions based on those facts.

Why do you think deductive reasoning is the best method for proofs?

Graphic Organizer: List of Properties, Postulates, and Definitions

List of Properties, Postulates, and Definitions that often come up in proofs:

TABLE 4.4:

Name of Property, Postulate, etc.	*Give an example or explain in your own words*
Vertical Angles Theorem	
Corresponding Angles Postulate	
Definition of Segment Bisector	
Definition of Angle Bisector	
Definition of Perpendicular Bisector	
Transitive Property	

Flow Proofs

There are many different ways of solving problems in geometry. We already wrote a **paragraph proof** in an earlier lesson that simply described, step by step, the rationale behind an assertion (when we showed why AAS is logically equivalent to ASA). The two-column style (which you will learn next) is generally most popular, easy to read, and organizes ideas clearly. Some students, however, prefer flow proofs. **Flow proofs** show the relationships between ideas more explicitly by using a chart that shows how one idea will lead to the next. Like two-column proofs, it is helpful to always remember the end goal so you can identify what it is you need to prove. Sometimes it is easier to work backwards!

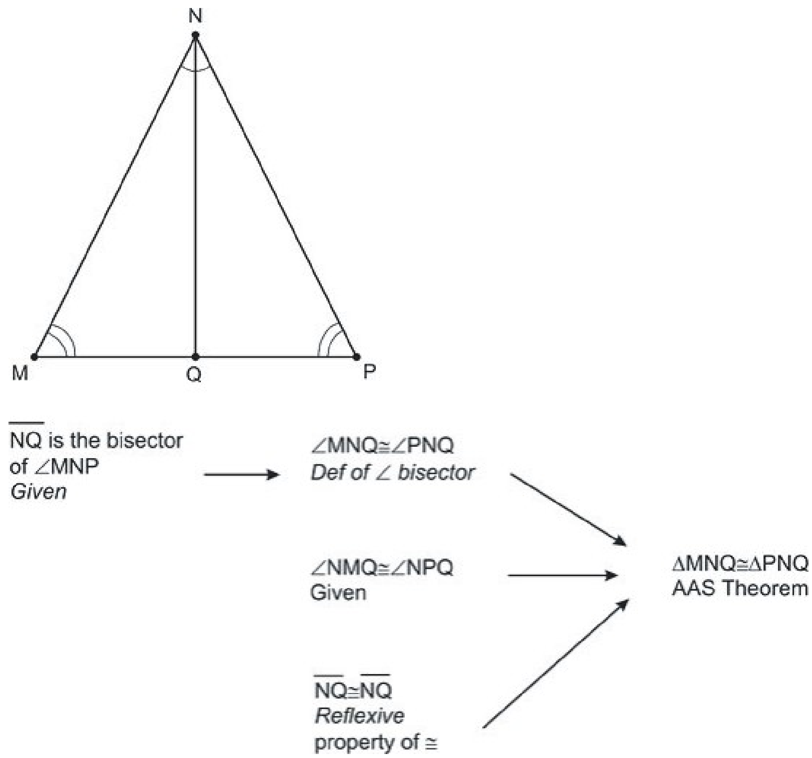
The example below is displayed in a flow style (rather than the **two-column proof** you will see next) and provides a good way to organize your information, where one idea flows into the next.

Example 1

Create a flow proof for the statement below.

Given: $\overline{NQ}$ is the bisector of $\angle MNP$ and $\angle NMQ \cong \angle NPQ$

Prove: $\Delta MNQ \cong \Delta PNQ$



You can see the given information on the left side of the flow proof, and the arrows lead to each additional statement based on the given fact. The reason for each statement is written below the statement. The final statement that you are trying to prove is on the right side of the proof.

Two-Column Proofs

One way to organize your thoughts when writing a proof is to use a **two-column proof**. This is probably the most common kind of proof in geometry, and it has two columns with a specific format:

- In the left column you write statements that lead to what you want to prove.
- In the right hand column, you write reasons for each step you take.
- Most proofs begin with the "given" information, and
- The conclusion is the statement you are trying to prove.

A two-column proof often looks like this:

TABLE 4.5:

Statements	**Reasons**
1. Put the "*given*" statement here. The "*given*" statement is always first.	1. Given *(always first!)*
2. List each step that you need to create your proof. Each separate statement in your proof gets its own line.	2. Each statement has a corresponding reason to explain *why* you are allowed to make the statement. Each reason gets its own line as well.
3. More statements...	3. More reasons that justify your statements...
4. (You may need more...)	4.
5. Put the final "*prove*" statement here.	5. Put the corresponding reason here.

A two-column proof has ________________________________ in the *left* column.

In the *right* column of a two-column proof, list ________________________________ that correspond to each statement.

The most common proofs in geometry are ____________________________________ proofs.

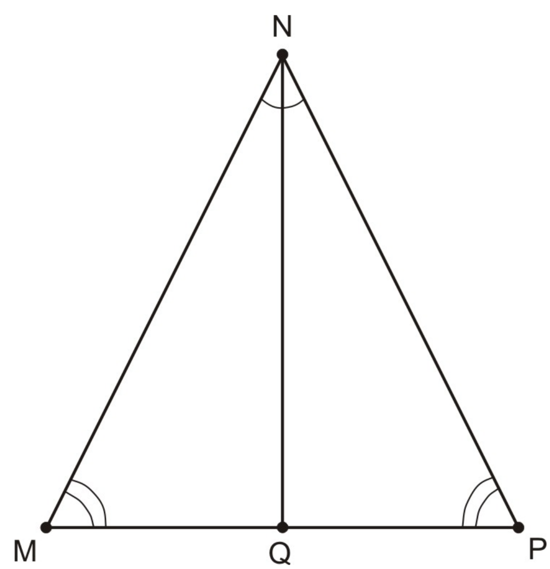
The example on the following page repeats the same proof as Example 1, but it is in the more traditional two-column style instead of the flow style.

Example 2

Create a two-column proof for the statement below.

Given: $\overline{NQ}$ is the bisector of $\angle MNP$ and $\angle NMQ \cong \angle NPQ$

Prove: $\Delta MNQ \cong \Delta PNQ$



Remember that each step in a proof must be clearly explained. You should formulate a *strategy* before you begin the proof. Since you are trying to prove the two triangles congruent, you should look for congruence between the sides and angles. You know that if you can prove SSS, SAS, HL, ASA, or AAS, you can prove congruence.

Since the given information provides *two pairs of congruent angles*, you will most likely be able to show the triangles are congruent using the **ASA Postulate** or the **AAS Theorem**.

Notice that both triangles *share* one side. We know that every side is congruent to itself ($\overline{NQ} \cong \overline{NQ}$), and now you have pairs of *two congruent angles* and *non-included sides*. You can use the **AAS Congruence Theorem** to prove the triangles are congruent.

Remember, geometric statements go on the left and reasons go on the right. We *always start* each column with the given information.

TABLE 4.6:

Statement	Reason
1. $\overline{NQ}$ is the bisector of $\angle MNP$	1. Given
2. $\angle MNQ \cong \angle PNQ$	2. Definition of an angle bisector (a bisector divides an angle into two congruent angles)
3. $\angle NMQ \cong \angle NPQ$	3. Given
4. $\overline{NQ} \cong \overline{NQ}$	4. Reflexive Property

TABLE 4.6: (continued)

Statement	Reason
5. $\Delta MNQ \cong \Delta PNQ$	5. AAS Congruence Theorem (if two pairs of angles and the corresponding non-included sides are congruent, then the triangles are congruent)

Q.E.D.

Q.E.D. *is an acronym of the Latin phrase* ***q****uod* ***e****rat* ***d****emonstrandum, which means "that which was to be demonstrated."*

The phrase is traditionally placed in its abbreviated form at the end of a mathematical proof signaling the completion of the proof.

Notice how the markings in the triangles help in the proof. Whenever you do proofs, use arcs in the angles and tic marks to show congruent angles and sides.

As you can see from the two different styles of proofs of the theorem, there are many different ways of expressing the same information. It is important that you become familiar with proving things using all of these styles because you may find that different types of proofs are better suited for different theorems.

Reading Check:

1. *Fill in the blanks:*

_____________ *go on the left side of a two-column proof and* _____________ *go on the right side.*

2. *Fill in the blank:*

The _____________ *statement always goes first in a two-column proof.*

3. *What type of reasoning are proofs?*

4. *Which style of proofs do you like better, flow proofs or two-column proofs? Why?*

4.8 Proofs with CPCTC

Learning Objectives

- Apply various triangle congruence postulates and theorems.
- Know the ways in which you can prove parts of a triangle congruent.

As you can see, there are many different ways to prove that two triangles are congruent. It is important to know all of the different ways that can prove congruence, and it is important to know which combinations of sides and angles do *not* prove congruence.

Congruence Theorem Review

As you have studied in the previous lessons, there are five theorems and postulates that provide different ways in which you can prove two triangles congruent without checking all of the angles and all of the sides. It is important to know these five rules well so that you can use them in practical applications.

TABLE 4.7:

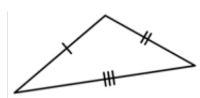
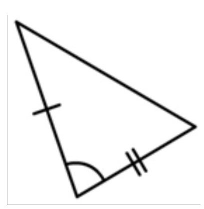
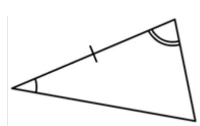
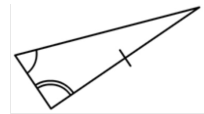
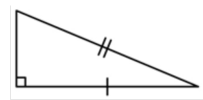
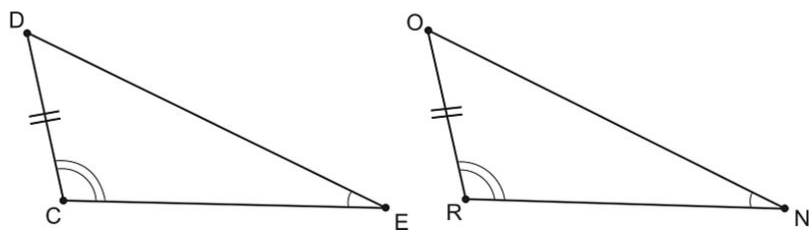
Name	Corresponding congruent parts	Picture	Does it prove congruence?
SSS	Three sides		Yes
SAS	Two sides and the angle between them		Yes
ASA	Two angles and the side between them		Yes

TABLE 4.7: (continued)

Name	Corresponding congruent parts	Picture	Does it prove congruence?
AAS	Two angles and a side *not* between them		Yes
HL	The hypotenuse and a leg in a right triangle		Yes

Example 1

What rule can prove that the triangles below are congruent?



A. SSS

B. SAS

C. ASA

D. AAS

The two triangles in the picture have two pairs of congruent angles ($\angle DCE \cong \angle ORN$ and $\angle CED \cong \angle RNO$) and one pair of corresponding congruent sides ($\overline{DC} \cong \overline{OR}$).

So, the triangle congruence postulate you choose must have two A's (for the angles) and one S (for the side). You can eliminate choices A and B for this reason.

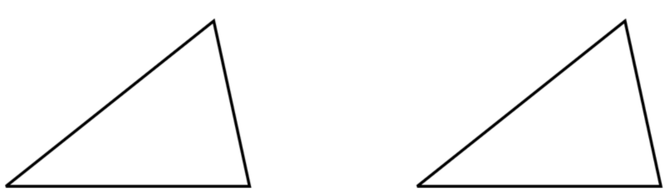
Now that you are deciding between choices C and D, you need to identify where the side is located in relation to the given angles. It is adjacent to one angle, but it is *not* in between them.

Therefore, you can prove congruence using **AAS**.

The correct answer is D.

Reading Check:

1. *The two triangles below are congruent because of the* ***SAS postulate****. Mark the* ***SAS*** *congruent parts with tic marks and arcs:*



2. *True/False: **SSS, SAS, ASA, AAS,** and **HL** (in a right triangle) are 5 different ways to prove that triangles are congruent.*

Proving Parts Congruent

It is one thing to identify congruence when all of the important identifying information is provided, but sometimes you will have to identify congruent parts on your own. This may take a bit of thought, and you must use some **deductive reasoning** (*finding conclusions based on facts*) to find the missing parts.

When you were creating proofs, you also used the **reflexive property of congruence**. This property states that *any segment or angle is congruent to itself*. While this may sound obvious, it can be very helpful in proofs, as you saw in those examples.

Recall that, if two triangles are congruent, then *all pairs of corresponding sides and all pairs of corresponding angles are congruent.*

We say that the **C**orresponding **P**arts (sides and angles) of **C**ongruent **T**riangles are **C**ongruent, or **CPCTC**.

Reading Check:

1. *What is a part of a triangle?*

2. *What does it mean for parts to be corresponding?*

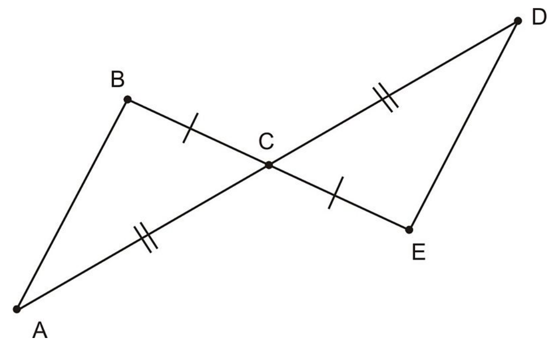
3. *Explain in your own words what **CPCTC** means.*

How do we use the concept of **CPCTC** in a proof?

Check out the example on the following page...

Example 2

How could you prove that segment AB is congruent to segment DE in the diagram below?



We can see that $\overline{BC} \cong \overline{CE}$ and $\overline{AC} \cong \overline{CD}$ because of the tic marks in the figure.

Think of all of the postulates and theorems (in the chart at the beginning of Lesson 7 and in your graphic organizer) that have 2 *S's* (SSS and SAS). We may be able to use SSS or SAS to show the triangles are congruent. However, to use SSS, we would need $\overline{AB} \cong \overline{DE}$ and we cannot yet make this assumption.

Can we show that two of the *angles* are *congruent*?

Notice that $\angle BCA$ and $\angle ECD$ are **vertical angles** (non-adjacent angles made by the intersection of two lines—i.e., angles on the opposite sides of the intersection).

The **Vertical Angle Theorem** states that *all vertical angles are also congruent*.

So, this tells us that $\angle BCA \cong \angle ECD$.

We now have two congruent *sides* and a congruent *angle* between the sides!

- The congruent sides are:

___________ $\cong$ ___________ and ___________ $\cong$ ___________

- The congruent angles are:

_______________ $\cong$ _______________

By putting all of this information together, you can confirm that $\Delta ABC \cong \Delta DEC$ by the **SAS Postulate**.

Finally, if the two triangles are congruent, then their *corresponding parts are congruent*.

Therefore segment *AB* is congruent to segment *DE* by **CPCTC**.

Reading Check:

1. *In the space below, draw a picture of two triangles that are* <u>*congruent*</u> *because of the* ***ASA postulate****. Make sure to mark the congruent parts with tic marks and arcs!*

2. *If you know that 2 pairs of angles and 1 pair of sides are congruent in your picture above, what other parts of the triangles are congruent? (Hint: there are 3 answers!)*

3. *Your answers to question #2 above are true because of* ***CPCTC****. The letters* ***CPCTC*** *stand for: (Fill in the blanks)* ______________________ ______________________ *of* ________________________ ____________________-____ *are* _________________________ .

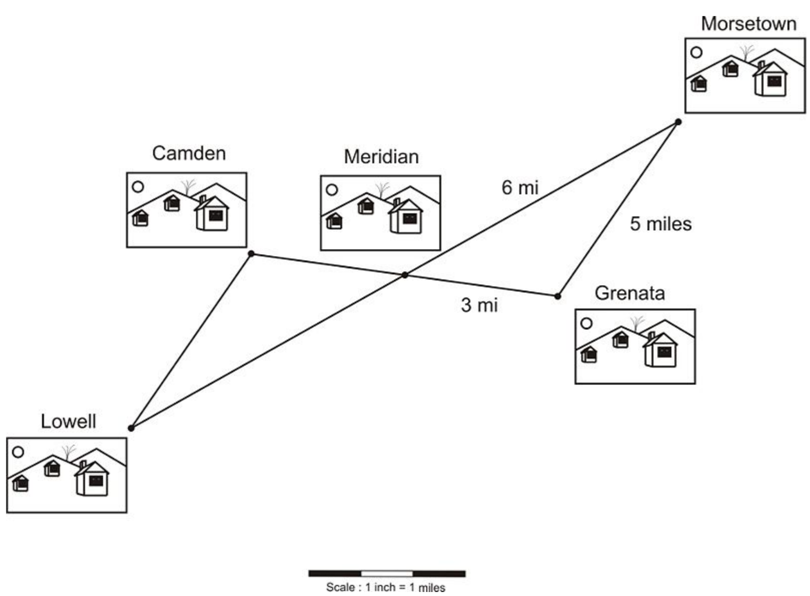
Finding Distances (a real-word application)

One way to use congruent triangles is to help you find distances in real life—usually using a map or a diagram as a model.

When using congruent triangles to identify distances, be sure you always match up corresponding sides. The most common error on this type of problem involves matching two sides that are *not* corresponding.

Example 3

The map below shows five different towns. The town of Meridian is exactly halfway between two pairs of cities: it is halfway between Camden and Grenata AND it is halfway between Lowell and Morsetown.



Using the information in the map, what is the distance between Camden and Lowell?

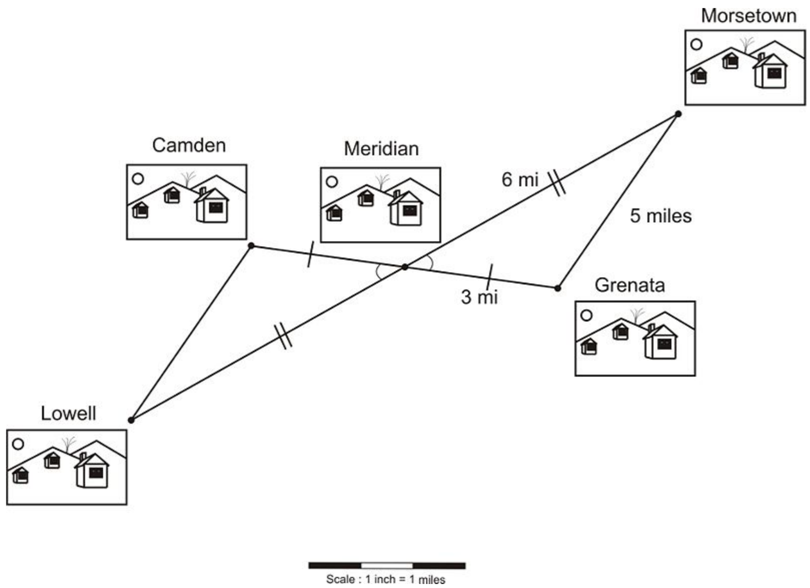
The first step in this problem is to identify whether or not the triangles drawn on the map are congruent.

Since you know that the distance from Camden to Meridian is the same as Meridian to Grenata (since Meridian is halfway between those cities), those two sides of the triangles are *congruent*. You can make a tic mark on these sides.

Similarly, since the distance from Lowell to Meridian is the same as Meridian to Morsetown (again because Meridian is halfway between), those two sides are also a *congruent* pair and you can give them congruent tic marks.

Finally, you can tell that the angles between these lines (at the intersection where Meridian is) are also *congruent* because they are **vertical angles.**

With all of your congruent sides and angles marked, your map will look like this:



So, by the **SAS postulate**, these two triangles are congruent.

This allows us to find the distance between Camden and Lowell by identifying its corresponding side on the other triangle. Because they are both *opposite the vertical angle*, the side connecting Camden and Lowell *corresponds* to the side connecting Morsetown and Grenata.

Since the triangles are congruent, these corresponding sides will also be congruent to each other. Therefore, the distance between Camden and Lowell is 5 miles.

This use of the definition of congruent triangles is one of the most powerful tools you will use in geometry class! It is often abbreviated as **CPCTC**, meaning **C**orresponding **P**arts of **C**ongruent **T**riangles are **C**ongruent.

4.9 Ratios and Proportions

Learning Objectives

- Write and simplify ratios.
- Formulate proportions.
- Use ratios and proportions in problem solving.

Ratios

A **ratio** is a *fraction* (and can be simplified just like a fraction). Usually a ratio is a fraction that *compares* two parts. "The ratio of x to y" can be written in several ways:

$$\frac{x}{y} \qquad \text{or} \qquad x : y \qquad \text{or} \qquad x \quad \text{to } y$$

A ratio is another word for ________________________________.

Proportions

A **proportion** is an *equation*. The two sides of the equation are **ratios** that are *equal* to each other. A proportion looks like this:

$$\frac{a}{b} = \frac{c}{d}$$

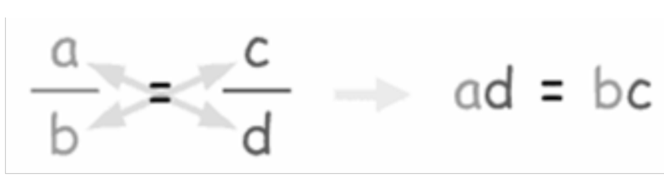
A proportion is an ___________________________________ with equal fractions.

Cross Multiplication Theorem

Let a, b, c, and d be real numbers, with $b \neq 0$ and $d \neq 0$.

If $\frac{a}{b} = \frac{c}{d}$ then $ad = bc$.

When you have a proportion, you can solve the equation by ***cross multiplying****. This means multiplying the parts of the fractions across the equal sign (top left times bottom right and bottom left times top right) like the arrows show:*



Solve a proportion by cross ____.

Reading Check:

1. *True/False: Another name for a fraction is a* ***ratio****.*

2. *True/False: Every* ***proportion*** *always has an equal sign in between the fractions.*

Properties of Proportions

Here are a few more ways of looking at the Cross Multiplication Theorem:

If $a \neq 0, b \neq 0, c \neq 0$, and $d \neq 0$, and $\frac{a}{b} = \frac{c}{d}$, then all of the following are true:

a. $\frac{a}{c} = \frac{b}{d}$ Notice that b and c have changed positions.
b. $\frac{d}{b} = \frac{c}{a}$ Notice that a and d have changed positions.
c. $\frac{b}{a} = \frac{d}{c}$ Notice that we have the **reciprocals** of the original ratios.

A ***reciprocal*** *is a fraction whose numerator and denominator are flipped from the original fraction.*

In other words, when you flip the top and bottom of a fraction, your result will be the ***reciprocal****.*

An easy way to remember the word ***reciprocal*** *is that it sounds like "re-flip-rocal" which reminds you to flip the fraction!*

When the numerator and denominator are flipped, it is called the ________________________.

Reading Check:

What is the reciprocal of the fraction $\frac{5}{6}$?

What does it mean to be proportional?

As you will learn in the next lesson, sometimes parts of triangles are **proportional** to one another. When two values are **proportional**, you can set up a **proportion** to describe their relationship.

When two things are **proportional**, you can set up a ________________________________ between them to describe how they are related.

You may remember from Unit 3 that ***proportional*** *describes a relationship between two values where you can multiply one of the values by some number to get the second value.*

In the example on the following page, we will create a **proportion** between the lengths of the sides of the two triangles. Until the next lesson, you don't need to worry about why this is. The more important thing is that you understand *how* to solve the **proportion.**

Example 1

Look at the diagram below:

Suppose we're given that $\frac{10}{6} = \frac{15}{9} = \frac{x}{y}$.

We know $\frac{10}{6} = \frac{15}{9}$, since $10 \cdot 9 = 6 \cdot 15 = 90$ (from the Cross Multiplication Theorem)

Here are some other proportions that must also be true based on our given information:

$$\frac{15}{x} = \frac{9}{y} \qquad \frac{y}{6} = \frac{x}{10}$$

$$\frac{15}{9} = \frac{x}{y} \qquad \frac{15}{10} = \frac{9}{6} \qquad \frac{x}{15} = \frac{y}{9}$$

There are two more true statements related to the **Cross Multiplication Theorem** below:

The "if" part of these is the same as above.

- If $a \neq 0, b \neq 0, c \neq 0$, and $d \neq 0$, and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$. (This is the equivalent of adding 1 to each ratio.)

And another, nearly the same as the previous,

- If $a \neq 0, b \neq 0, c \neq 0$, and $d \neq 0$, and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{b} = \frac{c-d}{d}$. (This is the equivalent of subtracting 1 from each ratio.)

Reading Check:

1. *True/False: To solve a proportion, use cross multiplication.*

2. *True/False: The reciprocal of a fraction is when both the numerator and the denominator are multiplied by each other.*

3. *There are many true mathematical statements that you can make about a proportion. If you have the proportion $\frac{5}{15} = \frac{x}{3}$, what is an example of another true statement you can make?*

4. *Complete the sentence:*

When two values are proportional,...

4.10 Triangle Similarity using AA and SSS

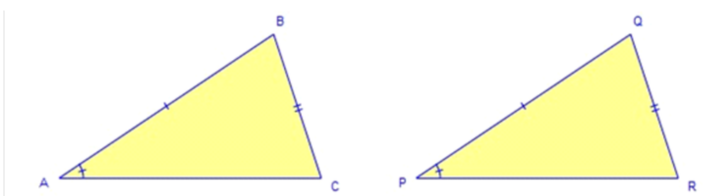
Learning Objectives

- Understand that SSA does not necessarily prove triangles are congruent.
- Understand that AA does not necessarily prove triangles are congruent.
- Determine whether triangles are similar.
- Identify corresponding angles and sides of similar polygons from a statement of similarity.
- Calculate and apply scale factors.
- Understand and apply the AA Similarity Postulate.
- Understand and apply the SSS Similarity Postulate.
- Solve problems about similar triangles.

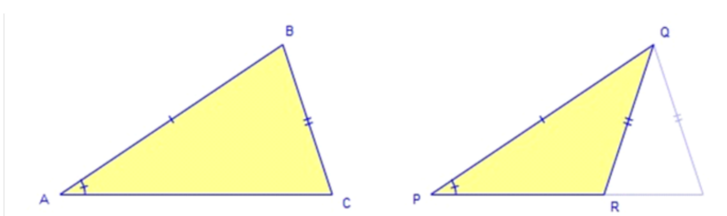
Proving Triangles are Congruent – Why SSA does NOT always work.

SSA relationships do not *necessarily* prove congruence. In other words, if you have two sides and a non-included angle (an angle that is not between them), then you cannot prove congruence.

Observe the triangles below. Two sides and a non-included angle from triangle ABC are congruent to the corresponding sides and non-included angle of triangle PQR. In fact, the triangles *appear* to be congruent.



However, below is a diagram that serves as a **counterexample** to the notion that SSA might work. Again, we see two sides and a non-included angle of triangle ABC which are congruent to their corresponding parts in triangle PQR, yet clearly the two triangles are not congruent to one another.

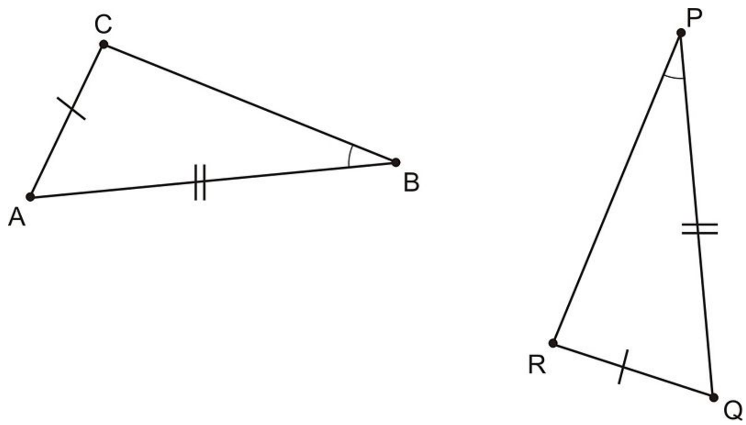


SSA does not prove that two triangles are ________________________________.

A **counterexample** proves that a statement is not ________________________.

Example 1

Can you prove that the two triangles below are congruent?



Note: Figure is not to scale.

The two triangles above *look* congruent, but are labeled, so you cannot assume that how they look means that they are congruent!

There are two sides labeled congruent, as well as one angle. Since the angle is *not* between the two sides, however, this is a case of SSA. *You cannot prove that these two triangles are congruent.* Also, it is important to note that although two of the angles appear to be right angles, they are not marked that way, so you cannot assume that they are right angles.

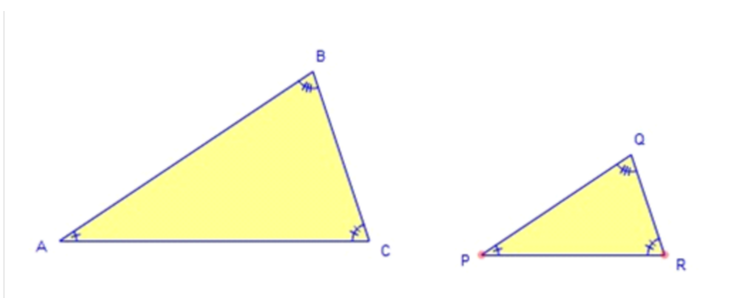
Reading Check:

1. *True/False: SSA is another way of proving that two triangles are congruent.*

2. *True/False: You CANNOT always trust the drawings, so even if triangles look congruent, you CANNOT assume that they are congruent!*

Proving Triangles are Congruent – Why AA does NOT always work.

Based on the marks in the diagram below, you can see that the three angles of triangle *ABC* are congruent to the three corresponding angles of *PQR*, yet the triangles are not congruent to one another. Triangles like this that are the same shape but different sizes are called **similar** triangles.

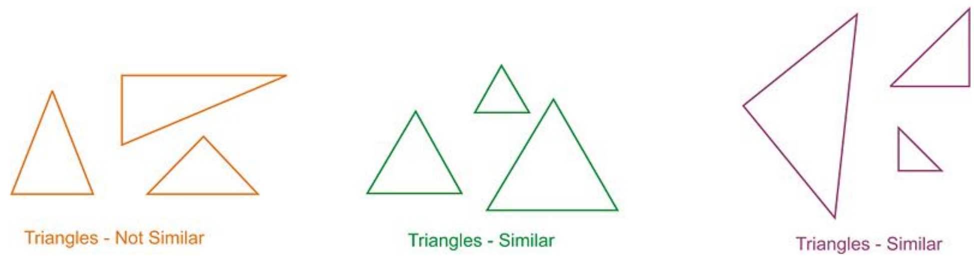


Similar triangles have the ________________ shape but ___________________________ sizes.

Similar Polygons

Similar figures have the same shape, but they may have different sizes.

Look at the triangles below:



- The triangles on the left are *not* similar because they are *not* the same shape.
- The triangles in the middle *are* **similar**. They are all the same shape, no matter what their sizes.
- The triangles on the right *are* **similar**. They are all the same shape, no matter how they are turned or what their sizes.

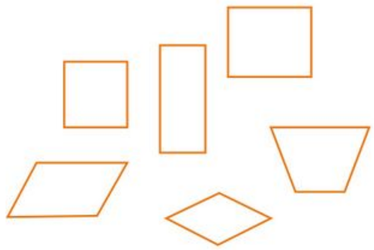
Reading Check:

1. *Fill in the blanks:* ***Similar*** *figures have the same* _________________________ *but different* _________________________.

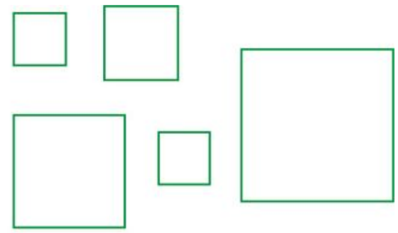
2. *True/False: If two triangles are similar, then their corresponding angles are congruent.*

3. *Can you think of some similar shapes in the real world?*

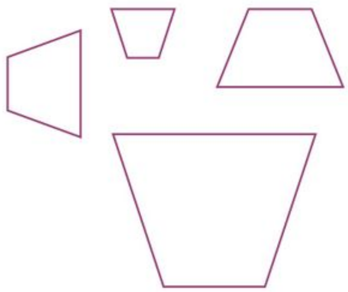
Look at the quadrilaterals below:



Quadrilaterals - None are similar to each other



Quadrilaterals - All similar.



Quadrilaterals - All similar.

- The quadrilaterals in the upper left are *not* similar because they are *not* the same shape.
- The quadrilaterals in the upper right *are* **similar**. They are all the same shape, no matter what their sizes.
- The quadrilaterals in the lower left *are* **similar**. They are all the same shape, no matter how they are turned or what their sizes.

Two polygons are **similar** if and only if:

- they have the same number of sides
- for each angle in either polygon there is a corresponding angle in the other polygon that is congruent
- the lengths of all corresponding sides in the polygons are **proportional**

Reminder: Just as we did with congruent figures, we name similar polygons according to corresponding parts. The symbol $\sim$ is used to represent "is similar to."

Reading Check:

1. *Fill in the blanks: Two polygons are **similar** if and only if:*

They have the same number of ________________________.

Corresponding angles are ________________________________.

Corresponding side lengths are ________________________________.

2. *What does the symbol* $\sim$ *mean?*

3. *In the space below, draw two **similar** rectangles.*

Example 2

Suppose $\Delta ABC \sim \Delta JKL$. Based on this statement, which angles are congruent and which sides are proportional? Write true congruence statements and proportions.

In **similar** triangles, *all matching angles are congruent.*

First, list the angles that match up in the names of the triangles:

$\angle A \cong \angle J, \angle B \cong \angle K$, and $\angle C \cong \angle L$

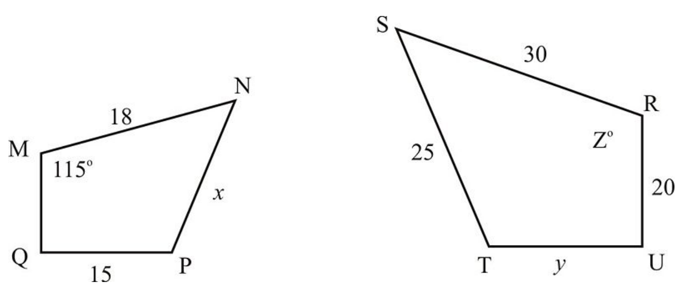
Next, match up the side segments by the order of the letters in the similar triangles:

$$\frac{AB}{JK} = \frac{BC}{KL} = \frac{AC}{JL}$$

Remember that there are many equivalent ways to write a proportion. The answer above is not the only set of true proportions you can create based on the given similarity statement.

Example 3

Given: $MNPQ \sim RSTU$



What are the values of x, y, and z in the diagram above?

In similar figures, corresponding side lengths are *proportional.*

Side x in $MNPQ$ corresponds to the side with length 25 in $RSTU$.

Set up a proportion to solve for x:

$$\frac{x}{25} = \frac{18}{30} \quad \text{you can reduce } \frac{18}{30} \text{ to be\_\_\_\_\_\_}$$
$$\frac{x}{25} = \frac{3}{5}$$
$$\text{Cross multiply to get } 5x = 75 \quad \text{so} \quad x = 15$$

Side y in $RSTU$ corresponds to the side with length 15 in $MNPQ$.

Set up a proportion to solve for y:

$$\frac{y}{15} = \frac{30}{18} \quad \text{you can reduce } \frac{30}{18} \text{ to be\_\_\_\_\_\_}$$
$$\frac{y}{15} = \frac{5}{3}$$
$$\text{Cross multiply to get } 3y = 75 \quad \text{so} \quad y = 25$$

Finally, since Z is an angle, we are looking for $m\angle R$:

$$Z = m\angle R = m\angle M = 115^\circ$$

Because corresponding angles are **congruent** in similar figures.

Example 4

$ABCD$ is a rectangle with length 12 and width 8.

$UVWX$ is a rectangle with length 24 and width 18.

A. *Are corresponding angles in the rectangles congruent?*

Yes. Since both are rectangles, all four angles in both figures are congruent right angles.

B. *Are the lengths of the sides of the rectangles proportional?*

No. The ratio of the lengths is $12 : 24 = 1 : 2$.

The ratio of the widths is $8 : 18 = 4 : 9$ which $\neq 1 : 2$.

Therefore, the lengths of the sides are not proportional.

C. *Are the rectangles similar?*

No. Although the corresponding angles are congruent, the lengths of corresponding sides are not proportional.

Scale Factors

If two polygons are **similar**, we know that the lengths of corresponding sides are *proportional*.

Similar polygons have ________________________________ corresponding sides.

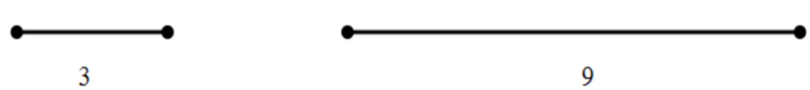
If k is the length of a side in one polygon, and m is the length of the corresponding side in the other polygon, then the ratio $\frac{k}{m}$ is called the **scale factor** relating the first polygon to the second.

Another way to say this is:

The length of every side of the first polygon is $\frac{k}{m}$ times the length of the corresponding side of the other polygon.

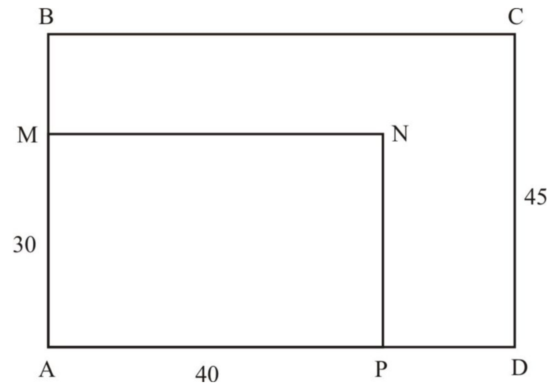
- $\frac{k}{m}$ represents the ________________ factor of similar shapes.

If two corresponding segments are shown below, where the segment on the left has a length of 3 and segment on the right has a length of 9, the **scale factor** relating them is $\frac{3}{9}$ which reduces to $\frac{1}{3}$



Example 5

Look at the diagram below, where ABCD and AMNP are similar rectangles.



A. *What is the scale factor?*

Since $ABCD \sim AMNP$, then AM and AB are corresponding sides. Since $ABCD$ is a rectangle, you know that $AB = DC = 45$.

The scale factor is the ratio of the lengths of any two corresponding sides.

So the scale factor (relating $ABCD$ to $AMNP$) is $\frac{45}{30} = \frac{3}{2}$ or 1.5.

We now know that the length of each side of $ABCD$ is **1.5 times** the length of the corresponding side in $AMNP$.

- *Comment*: We can turn this relationship around "backwards" and talk about the scale factor relating $AMNP$ to $ABCD$. This scale factor is just $\frac{30}{45} = \frac{2}{3}$, which is the **reciprocal** of the scale factor relating $ABCD$ to $AMNP$.

B. *What is the ratio of the perimeters of the rectangles?*

Find each perimeter by adding up the lengths of the sides of the rectangle:

$ABCD$ is a 45 by 60 rectangle. Its perimeter is $45+60+45+60 = 210$.

$AMNP$ is a 30 by 40 rectangle. Its perimeter is $30+40+30+40 = 140$.

The ratio of the perimeters of $ABCD$ to $AMNP$ is $\frac{210}{140} = \frac{3}{2}$.

- *Comment*: You see from this example that the **ratio of the perimeters of the rectangles is the same as the scale factor**. This relationship for the perimeters holds true in general for any similar polygons.

The **scale** factor of similar shapes is the ratio of corresponding sides and the ratio of the ______________________-__________ of the shapes.

The AA Rule for Similar Triangles

Suppose that the triangles ΔABC and ΔMNP have two pairs of congruent angles, say $\angle A \cong \angle M$ and $\angle B \cong \angle N$.

But we know that if triangles have *two* pairs of congruent angles, then the *third* pair of angles are also congruent (by the **Triangle Sum Theorem**).

The AAA rule for similar triangles reduces to the **AA triangle similarity postulate**.

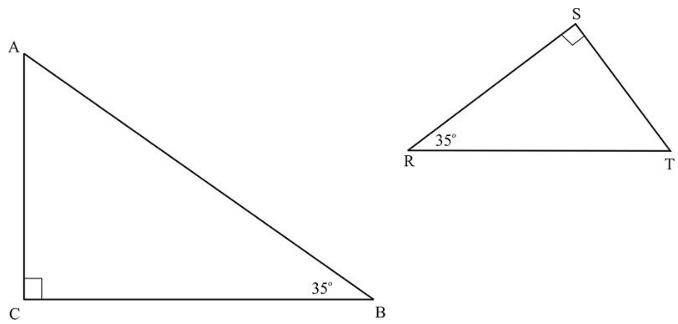
The AA Triangle Similarity Postulate

If two pairs of corresponding angles in two triangles are congruent, then the triangles are similar.

- Two triangles are ______________________________ if *two* pairs of corresponding angles are congruent.

Example 6

Look at the diagram below:



A. *Are the triangles similar? Explain your answer.*

Yes. They both have congruent right angles, and they both have a 35° angle.

These are *two* pairs of congruent corresponding angles so the triangles are similar by **AA**.

You need _________ pairs of corresponding congruent angles to prove similarity by AA.

B. *Write a similarity statement for the triangles.*

$\Delta ABC : \Delta TRS$ or equivalent

C. *Name all pairs of congruent angles.*

Remember, congruent angles match up within the names of the triangles (first letter first letter, and so on...)

$\angle A \cong \angle T, \angle B \cong \angle R$, and $\angle C \cong \angle S$

D. *Write equations stating the proportional side lengths in the triangles.*

$\frac{AB}{TR} = \frac{BC}{RS} = \frac{AC}{TS}$ or equivalent

<u>Reading Check:</u>

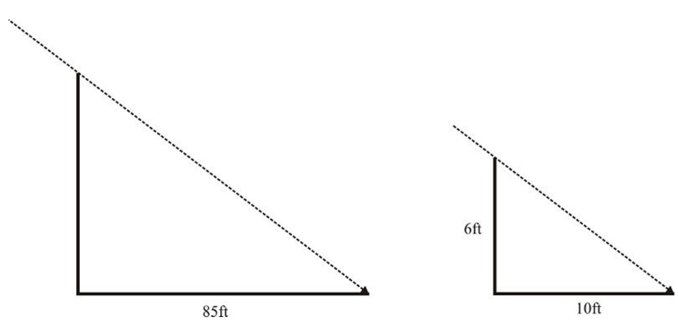
1. *True/False:* ***Scale factor*** *is a relationship that is expressed as a ratio or fraction.*

2. *True/False: The ratio of the perimeters of two polygons is the same as the scale factor of the polygons.*

Example 7

Flo wants to measure the height of a windmill. She held a 6 foot vertical pipe with its base touching the level ground, and the pipe's shadow was 10 feet long. At the same time, the shadow of the windmill tower was 85 feet long. How tall is the windmill?

Draw a diagram:



Note: It is safe to assume that the sun's rays hit the ground at the same angle. It is also proper to assume that the windmill tower is vertical (perpendicular to the ground).

The diagram shows *two similar right triangles*. They are similar because each has a right angle, and the angle where the sun's rays hit the ground is the same for both objects. Because they are similar triangles, their corresponding side lengths are *proportional*.

We can write a proportion with only one unknown, x, the height of the windmill tower:

$$\begin{aligned} \frac{x}{85} &= \frac{6}{10} \\ \text{Cross multiply to get : } 10x &= 85 \cdot 6 \\ 10x &= 510 \\ x &= 51 \end{aligned}$$

Thus, the tower is 51 feet tall.

Note: This method is considered *indirect* measurement because it would be difficult to *directly* measure the height of a tall windmill tower. Imagine how difficult it would be to hold a tape measure up to something that is 51 feet tall!

SSS for Similar Triangles

The SSS Triangle Similarity Postulate

If the lengths of the sides of two triangles are proportional, then the triangles are similar.

- Two triangles with __________________________________ sides are similar.

Example 8

Imagine a diagram with two triangles. (You may want to draw these in the margin). One triangle has sides 6-8-10. The other has sides 9-12-15.

Are the two triangles similar?

What do you notice? All three side lengths in the two triangles are proportional:

$$\frac{6}{9} = \frac{8}{12} = \frac{10}{15} \left(= \frac{2}{3} \right)$$

Yes, the two triangles are similar!

Reading Check:

1. *The following statement is TRUE. Why? Explain:*

All equilateral triangles are similar.

2. *The following statement is FALSE. Why? Explain in words (but you may also draw a picture to show an example):*

All isosceles triangles are similar.

4.11 Triangle Similarity using SAS

Learning Objectives

- Understand and apply the SAS Similarity Postulate.

SAS for Similar Triangles

SAS (Side-Angle-Side) Similarity Postulate

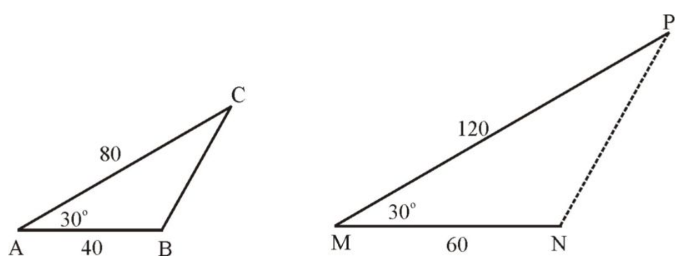
If the lengths of two corresponding sides of two triangles are *proportional* and the included angles are *congruent*, then the triangles are similar.

Two triangles are similar if two pairs of corresponding sides are ___________________________ and the included angles are ______________________________.

Example 1

Cheryl made the diagram below to investigate similar triangles more.

She drew ΔABC first, with $AB = 40,\ AC = 80$, and $m\angle A = 30^\circ$.



Then Cheryl did the following:

She drew $\overline{MN}$, and made $MN = 60$.

Then she carefully drew $\overline{MP}$, making $MP = 120$ and $m\angle M = 30^\circ$.

At this point, Cheryl had drawn two segments ($\overline{MN}$ and $\overline{MP}$) with lengths that are *proportional* to the lengths of the corresponding sides of ΔABC, and she had made the included angle, $\angle M$, congruent to the included angle $(\angle A)$ in ΔABC.

Then Cheryl measured angles. She found that:

$\angle B \cong \angle N$ and $\angle C \cong \angle P$

What could Cheryl conclude? Here again we have automatic results. The other angles are automatically *congruent*, and the triangles are *similar* by AA.

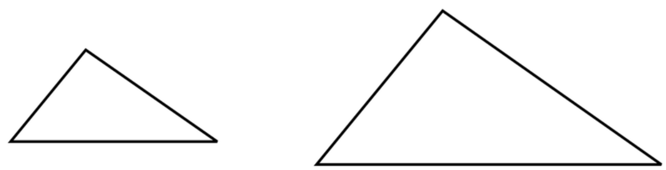
Cheryl's work supports the **SAS for Similar Triangles Postulate.**

Reading Check:

1. *In the* ***SAS for similar triangles postulate****, which parts are* <u>*congruent*</u> *in the similar triangles?*

2. *In the* ***SAS for similar triangles postulate****, which parts are* <u>*proportional*</u> *in the similar triangles?*

3. *The two triangles below are* <u>*similar*</u> *because of the* ***SAS for similar triangles postulate****. Mark the* ***SAS*** <u>*congruent*</u> *parts with tic marks and/or arcs. Create numbers that are* <u>*proportional*</u> *for the* <u>*similar*</u> *parts.*



Similar Triangles Summary

We've explored similar triangles extensively in several lessons. Let's summarize the conditions we've found that guarantee that two triangles are similar.

Two triangles are **similar** if and only if:

- the angles in the triangles are congruent.
- the lengths of corresponding sides in the polygons are proportional.

AA for Similar Triangles

It two pairs of corresponding angles in two triangles are *congruent*, then the triangles are *similar*.

SSS for Similar Triangles

If the lengths of the sides of two triangles are *proportional*, then the triangles are *similar*.

SAS for Similar Triangles

If the lengths of two corresponding sides of two triangles are *proportional* and the included angles are *congruent*, then the triangles are *similar*.

You can use the graphic organizer on the next page to keep all of this information in one place.

Graphic Organizer for Lessons 9-10

TABLE 4.8: Proving Similar Triangles

Type of Similarity	*What do the letters stand for? What does this mean?*	*Draw a picture of two similar triangles and label parts*	*Describe corresponding congruent parts*	*Describe corresponding proportional parts*
AA				
SSS				

TABLE 4.8: (continued)

Type of Similarity	*What do the letters stand for? What does this mean?*	*Draw a picture of two similar triangles and label parts*	*Describe corresponding congruent parts*	*Describe corresponding proportional parts*
SAS				

4.12 Missing Side Lengths and Similarity or the "Side-Splitting Theorem"

Learning Objectives

- Identify proportional segments when two sides of a triangle are cut by a segment parallel to the third side.
- Divide a segment into any given number of congruent parts.

Dividing Sides of Triangles Proportionally

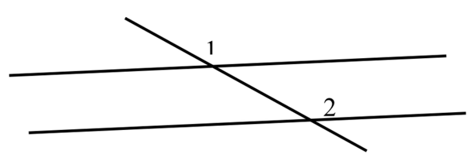
A **midsegment** of a triangle:

- is *parallel* to one side of a triangle and
- divides the other two sides into *congruent* halves (because it connects the *midpoints* of those two sides).

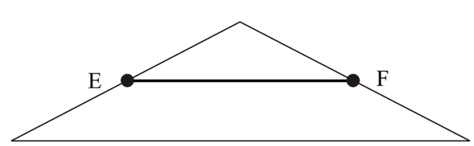
A **midsegment** is ______________________________ to one side of a triangle and connects the ______________-_________________ of the other two sides.

Therefore, the **midsegment** divides those two "other" sides proportionally.

Remember, When two parallel lines are cut by a transversal, corresponding angles are congruent (or $\angle 1 \cong \angle 2$ in the diagram below):



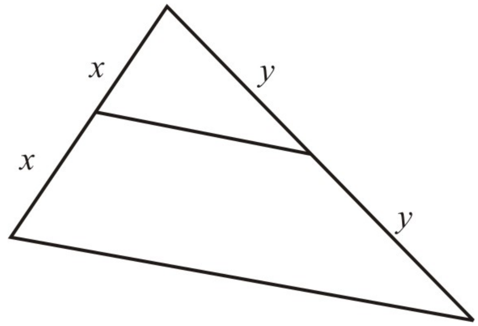
A **midsegment** may look like $\overline{EF}$ in the following picture:



Example 1

Explain the meaning of "the midsegment divides the sides of a triangle proportionally."

Suppose each half of one side of a triangle is x units long, and each half of the other side is y units long:



- One side of the triangle is divided in the ratio $x : x$ and the other side in the ratio $y : y$.
- Both of these ratios are equivalent to 1 : 1 and to each other.

We see that a **midsegment** divides two sides of a triangle proportionally. But what about some other segment?

Triangle Proportionality Theorem

If a line *parallel* to one side of a triangle intersects the other two sides, then it divides those sides into proportional segments.

→ This is also called the **"Side-Splitting" Theorem** because the midsegment *splits* the *sides* that it intersects.

The diagram above demonstrates this theorem as well as the definition of a **midsegment**.

- If a line *intersects* two sides of a triangle and is *parallel* to the third side, then it splits the intersected sides into ____ segments.
- This is called the Side- __________________________ Theorem because it *splits* two sides of a triangle.

Check out the proof of the **Side-Splitting Theorem** on the following page.

Proof:

Given: ΔABC with $\overline{DE} \parallel \overline{AC}$

Prove: $\frac{AD}{DB} = \frac{CE}{EB}$

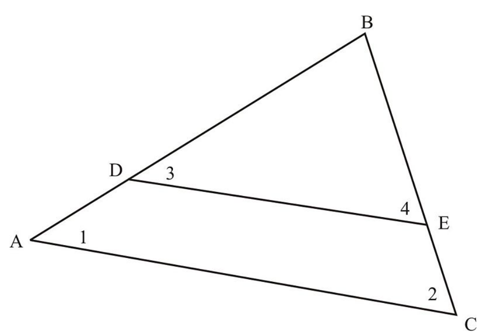


TABLE 4.9:

Statement	Reason
1. $\overline{DE} \parallel \overline{AC}$	1. Given
2. $\angle 1 \cong \angle 3, \angle 2 \cong \angle 4$	2. Corresponding angles are congruent
3. $\Delta ABC : \Delta DBE$	3. AA Similarity Postulate
4. $AD + DB = AB, CE + EB = CB$	4. Segment Addition Postulate
5. $\frac{AB}{DB} = \frac{CB}{EB}$	5.Corresponding side lengths in similar triangles are proportional

TABLE 4.9: (continued)

Statement	Reason
6. $\frac{AD+DB}{DB} = \frac{CE+EB}{EB}$	6. Substitution
7. $\frac{AD+DB}{DB} = \frac{AD}{DB} + \frac{DB}{DB} = \frac{AD}{DB} + 1$ and $\frac{CE+EB}{EB} = \frac{CE}{EB} + \frac{EB}{EB} = \frac{CE}{EB} + 1$	7. Algebra
8. $\frac{AD}{DB} + 1 = \frac{CE}{EB} + 1$	8. Substitution
9. $\frac{AD}{DB} = \frac{CE}{EB}$	9. Subtraction Property of Equality

Can you see why we wrote the proportion in step #6 this way, rather than as $\frac{DB}{AD+DB} = \frac{EB}{CE+EB}$, which is also a true proportion?

It is because $\frac{x+y}{z} = \frac{x}{z} + \frac{y}{z}$, but there is no similar way to simplify $\frac{z}{x+y}$.

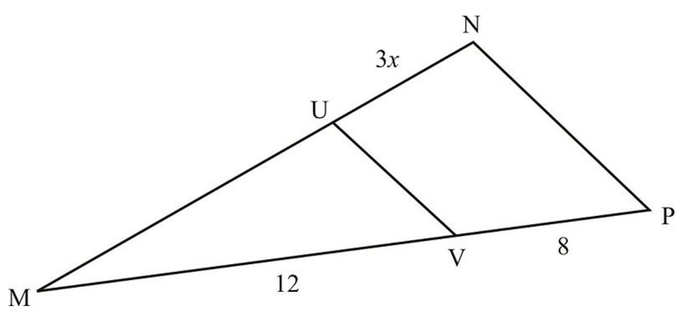
Note: The converse of this theorem is also true.

Converse of the Triangle Proportionality Theorem

If a line divides two sides of a triangle into *proportional* segments, then the line is *parallel* to the third side of the triangle.

Example 2

In the diagram below, $UV : NP = 3 : 5$.



What is an expression in terms of x for the length of $\overline{MN}$?

According to the **Triangle Proportionality Theorem**,

$\frac{3}{5} = \frac{MU}{MU+3x}$

Use cross multiplication so

$$\begin{aligned} 3(MU+3x) &= 5(MU) \\ 3MU + 9x &= 5MU \\ 2MU &= 9x \\ MU &= \frac{9x}{2} = 4.5x \end{aligned}$$

Now find the length of $\overline{MN}$:

$$\begin{aligned} MN &= MU + UN = 4.5x + 3x \\ MN &= 7.5x \end{aligned}$$

Reading Check:

1. *Fill in the blanks:*

*A **midsegment** is _________________________ to one side of a triangle and divides the other two sides of the triangle into ________________________________ halves.*

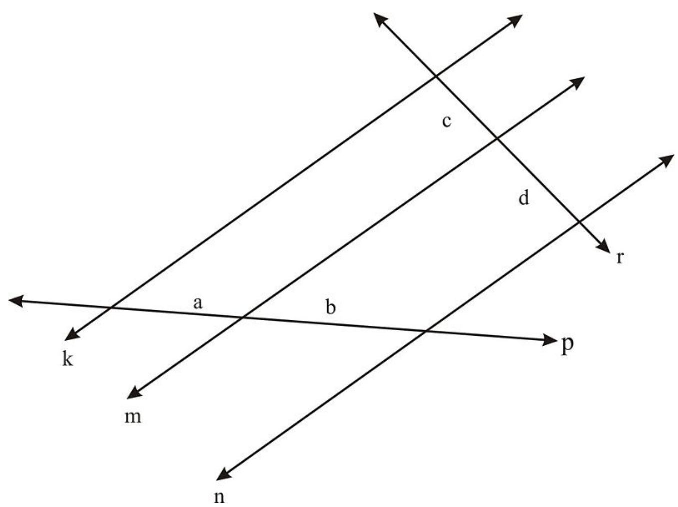
2. *True/False: A midsegment connects the midpoints of all three sides of a triangle.*

3. *True/False: If a line divides two sides of a triangle into proportional segments, then that line must be parallel to the third side of the triangle.*

Parallel Lines and Transversals

Example 3

Look at the diagram below:



k, m, n, p, and r are labels for lines

a, b, c, and d are lengths of segments

k, m, and n are parallel but not equally spaced

We're given that lines k, m, and n are **parallel**. We can see that the parallel lines cut lines p and r (which are **transversals**).

Lines k, m, and n are ___________________________________ to each other.

Lines p and r are _______________________________________.

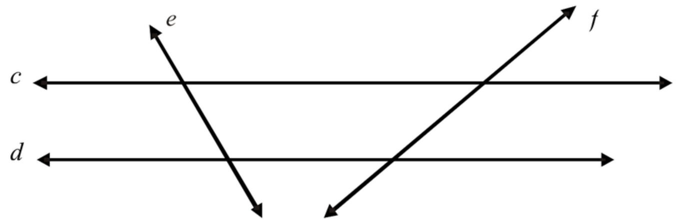
Following the **Triangle Proportionality Theorem**, it is also true that:

The segment lengths on one transversal are *proportional* to the segment lengths on the other transversal.

The transversals have ___________________________________ segment lengths, so we can set up the following proportions:

$\frac{a}{b} = \frac{c}{d}$ and $\frac{a}{c} = \frac{b}{d}$

Reading Check:



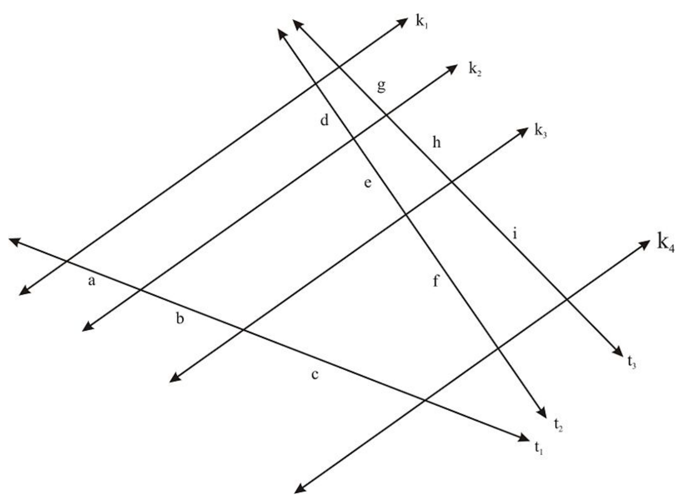
In the diagram above, which lines are parallel?

which lines are transversals?

Example 4

What we discovered in Example 3 can be broadened to any number of parallel lines that cut any number of transversals. When this happens, all corresponding segments of the transversals are proportional!

The diagram below shows several parallel lines, k_1, k_2, k_3, *and* k_4, *that cut several transversals,* t_1, t_2, *and* t_3.



The k lines are all parallel.

Now we have lots of proportional segments.

For example: $\frac{a}{b} = \frac{d}{e}, \frac{a}{c} = \frac{g}{i}, \frac{b}{h} = \frac{a}{g}, \frac{c}{f} = \frac{b}{e}$, and many more.

Reading Check:

In the space below, draw a picture (it can be a shape or a group of lines) that shows that the ***Triangle Proportionality Theorem*** *(or* ***Side-Splitting Theorem****) is true. Be sure to label all the segments in your picture that you will use to make your proportion. Clearly write the proportion next to your picture.*

CHAPTER 5 Trigonometry

Chapter Outline

5.1 Vocabulary Self-Rating

TABLE 5.1: Rating Guide:DK:I am sure I don't know it K: I am sure I know it ? I'm not sure

Word	Before Lesson/Unit	After Lesson/Unit
Trigonometry		
Hypotenuse		
Adjacent		
Opposite		
Complementary		
Trigonometric Functions		
Sine		
Cosine		
Tangent		
SOHCAHTOA		
Ratio		
Isosceles		

5.2 Identifying Trigonometric Ratios

Learning Objectives

- Identify the different parts of right triangles.
- Identify complementary angles in right triangles.
- Become familiar with the basic trigonometric ratios of sine, cosine, and tangent.

What is Trigonometry?

Trigonometry is the study of triangles and the relationships between their side lengths and the angles in between their sides.

*The word **Trigonometry** has two parts: "trig" means triangle and "metry" means measure*

Where is Trigonometry used?

There are many applications of **trigonometry**. Of particular value is the technique of *triangulation,* which is used in astronomy to measure the distance to nearby stars, in geography to measure distances between landmarks, and in satellite navigation systems. GPSs (global positioning systems) would not be possible without trigonometry. Other fields which make use of trigonometry include:

- astronomy (and navigation - on the oceans, in aircraft, and in space)
- music theory and acoustics
- analysis of financial markets
- electronics
- probability theory and statistics
- medical imaging (CAT scans and ultrasound)
- pharmacy, chemistry, and biology
- number theory (and cryptology)
- land surveying and geodesy
- architecture
- various types of engineering (electrical, mechanical, and civil)
- computer graphics
- cartography (the study of maps)

In your own words, **Trigonometry is:**

What are the three most interesting applications of trigonometry for you?

Parts of a Triangle

In trigonometry, there are a number of different labels attributed to different sides of a right triangle. They are usually in relation to a specific angle. The **hypotenuse** of a triangle is always the same, but the terms **adjacent** and **opposite** depend on *which angle* you are referencing.

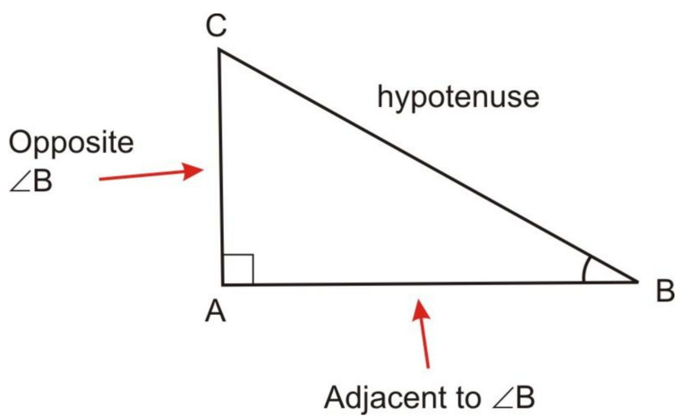
A side **adjacent** to an angle is *the leg of the triangle that helps form the angle.*

A side **opposite** to an angle is *the leg of the triangle that does not help form the angle.*

*What does the word **adjacent** mean?*

***Adjacent** means "next to," so an adjacent side is next to the angle in question.*

*Examine the picture below: Segment AB is next to, or **adjacent** to, angle B. Notice that it is also the leg of the triangle that helps to form angle B.*



In the triangle shown above, segment $\overline{AB}$ is **adjacent** to $\angle B$, and segment $\overline{AC}$ is **opposite** to $\angle B$.

Similarly, $\overline{AC}$ is **adjacent** to $\angle C$, and $\overline{AB}$ is **opposite** to $\angle C$.

The **hypotenuse** is always $\overline{BC}$.

In the picture above,

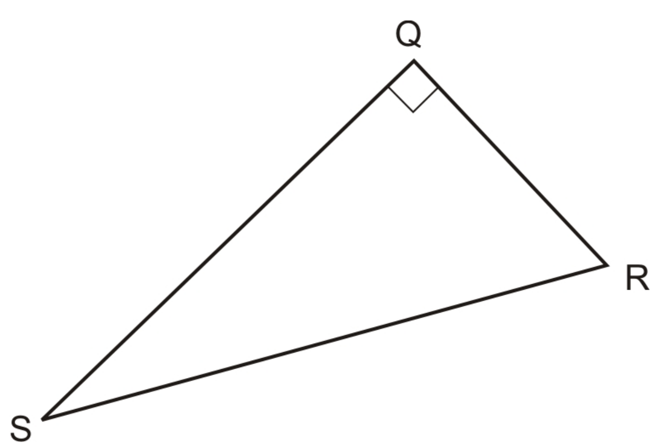
Segment AC is __________ angle B**and** __________ angle C.

Segment AB is __________ angle B**and** __________ angle C.

Because angle A is 90°, Segment CB is always the __________.

Example 1

Examine the triangle in the diagram below.



*Identify which leg is **adjacent** to $\angle R$, **opposite** to $\angle R$, and the **hypotenuse**.*

The first part of the question asks you to identify the leg **adjacent** to $\angle R$. Since an adjacent leg is the one that helps to form the angle and is *not* the hypotenuse, it must be $\overline{QR}$.

The next part of the question asks you to identify the leg **opposite** $\angle R$.

Since an opposite leg is the leg that *does not* help to form the angle, it must be $\overline{QS}$.

The **hypotenuse** is always *opposite the right angle*, so in this triangle the hypotenuse is segment $\overline{RS}$.

Complementary Angles in Right Triangles

Recall that in all triangles, the sum of the measures of all angles must be 180°. Since a right angle has a measure of 90°, the remaining two angles in a right triangle must be **complementary**. Remember, **complementary** angles have a sum of 90°. This means that if you know the measure of one of the smaller angles in a right triangle, you can easily find the measure of the other. Subtract the known angle from 90° and you'll have the measure of the other angle.

Reading Check:

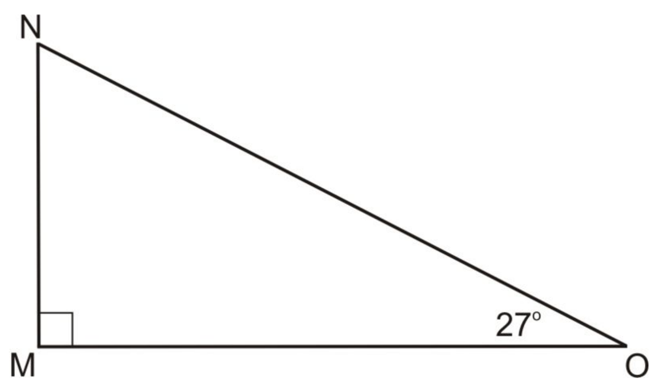
1. *What is the sum of all three angles in a triangle?*

2. *What is the sum of any two complementary angles?*

3. *What is the study of triangles called?*

Example 2

What is the measure of $\angle N$ *in the triangle below?*



To find $m\angle N$, you can subtract the measure of $\angle N$ from 90°.

$$\begin{aligned} m\angle N + m\angle O &= 90 \\ m\angle N &= 90 - m\angle O \\ m\angle N &= 90 - 27 \\ m\angle N &= 63 \end{aligned}$$

So, the measure of $\angle N$ is 63° since $\angle N$ and $\angle O$ are complementary.

If angle N and angle O are **complementary**, that means they add up to ____________.

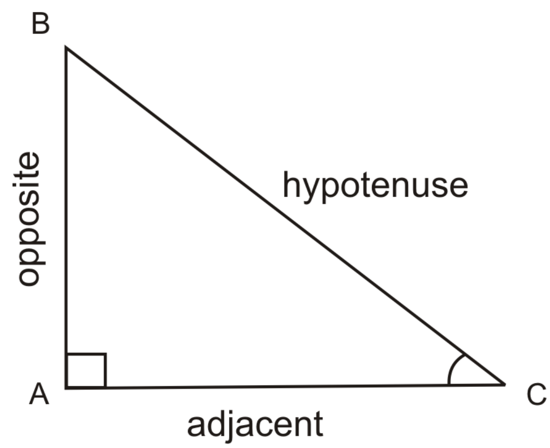
Trigonometric Ratios

The fundamentals of trigonometry are the **trigonometric functions**. There are three basic trigonometric functions: **sine, cosine** and **tangent**. These are abbreviated to: **sin, cos,** and **tan**:

$$\text{sine} = \sin$$
$$\text{cosine} = \cos$$
$$\text{tangent} = \tan$$

These functions are defined from a right-angled triangle.

Consider a right-angled triangle:



θ is a Greek letter.

In trigonometry, it is very common to see the letter θ used as a variable to represent an angle.

You can think of θ just like x – a variable that stands for a number. In the case of an angle, θ stands for a number in degrees.

In the right-angled triangle, we refer to the lengths of the three sides according to how they are placed in relation to the angle θ.

The Greek letter _____ is used to represent an angle in trigonometry.

The side *opposite to* θ is labeled **opposite**, the side *next to* θ is labeled **adjacent** and the side *opposite the right-angle* is labeled the **hypotenuse**.

We define:

$$\sin\theta = \frac{opposite}{hypotenuse}$$
$$\cos\theta = \frac{adjacent}{hypotenuse}$$
$$\tan\theta = \frac{opposite}{adjacent}$$

These functions relate the lengths of the sides of a triangle to its interior angles.

One way to remember the definitions is to use the first letter of each word. Notice the bold letters in the definitions above spell out:

SOHCAHTOA

This stands for the trigonometric functions and which sides correspond to each of them:

Sine is **O**pposite over **H**ypotenuse **(SOH)**

Cosine is **A**djacent over **H**ypotenuse **(CAH)**

Tan is **O**pposite over **A**djacent **(TOA)**

IMPORTANT: The definitions of opposite, adjacent and hypotenuse only make sense when you are working with right-angled triangles! Always check to make sure your triangle has a right-angle before you use them; otherwise you will get the wrong answer.

Graphic Organizer: SOHCAHTOA

TABLE 5.2:

Trigonometric Function	**Trig Function Abbreviation**	**Letters stand for:**	**Ratio of Side Lengths**	**Draw a picture**	**Write yourself some notes to help you remember!**
Sine		**S** **O** **H**	$sin\ \theta =$ ______- ________		
Cosine		**C** **A** **H**	$cos\ \theta =$ ______- ________		
Tangent		**T** **O** **A**	$tan\ \theta =$ ______- ________		

5.3 Tangent Ratio

Learning Objectives

- Identify and use the tangent ratio in a right triangle.
- Understand tangent ratios in special right triangles.

The Tangent Ratio

The first ratio to examine when studying right triangles is the **tangent**.

The tangent of an angle is the *ratio* of the length of the **opposite** side to the length of the **adjacent** side. The **hypotenuse** is not involved in the tangent at all.

Recall that a *ratio* is the same as a *fraction,* so the tangent is the *fraction* of the opposite side over the adjacent side.

This means that tangent is: the ____________________ side divided by the ____________________ side.

Be sure when you find a **tangent** that you find the **opposite** and **adjacent** sides *relative to the angle in question.*

You must be careful that the **opposite** side is *across from* the angle you are taking the tangent of, and the **adjacent** side is *next to* that same angle!

For an acute angle measuring θ, we define:

$$\tan\theta = \frac{opposite}{adjacent}$$

Like always, be sure to reduce the fraction in your final answer!

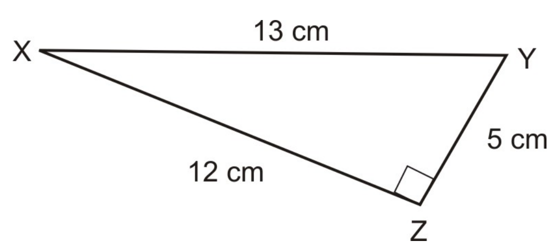
Reading Check:

1. *Fill in the blank: Another word for ratio is* _______________________.

2. *Which side of a right triangle is **NOT** used in the tangent ratio?*

Example 1

What are the tangents of $\angle X$ *and* $\angle Y$ *in the triangle below?*



To find these ratios, first identify the sides **opposite** and **adjacent** to each angle:

For angle X:

The side **opposite** angle X is the segment ___________, which is _______ cm long.

The side **adjacent** to angle X is the segment ___________, which is _______ cm long.

For angle Y:

The side **opposite** angle Y is the segment ___________, which is _______ cm long.

The side **adjacent** to angle Y is the segment ___________, which is _______ cm long.

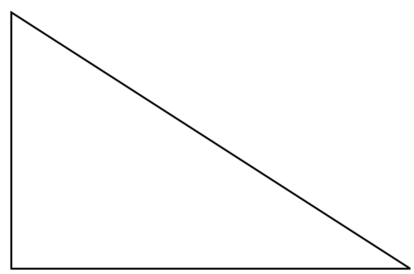
$$\tan \angle X = \frac{opposite}{adjacent} = \frac{5}{12}$$
$$\tan \angle Y = \frac{opposite}{adjacent} = \frac{12}{5}$$

So, the tangent of$\angle X$ is $\frac{5}{12}$ and the tangent of $\angle Y$ is $\frac{12}{5}$.

Notice that the tangent is different for different angles because of which sides are **opposite** and which sides are **adjacent** to each angle in the triangle.

It is common to write $\tan X$ instead of $\tan \angle X$. In this text we will use both notations.

Reading Check:



On the blank triangle above,

1. *Label the right angle.*
2. *Label the triangle* ΔCAT, *where the right angle is at angle* A.
3. *Label the hypotenuse.*
4. *On the side opposite angle* C, *label "opposite* $\angle C$"
5. *On the side adjacent to angle* C, *label "adjacent* $\angle C$"
6. *On the side opposite angle* T, *label "opposite* $\angle T$"
7. *On the side adjacent to angle* T, *label "opposite* $\angle T$"

(Note: some sides will have more than one label!)

Tangents of Special Right Triangles

It may help you to learn some of the most common values for tangent ratios. The list below shows you values for angles in special right triangles.

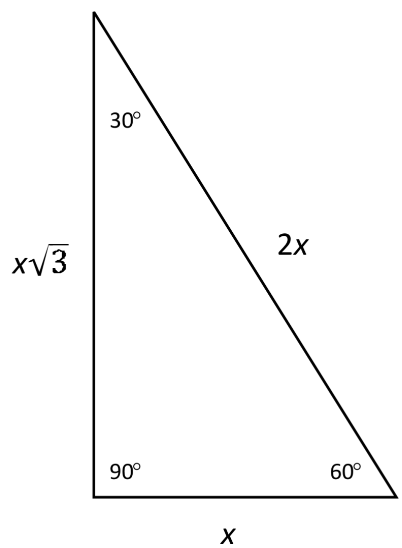
Tangent 30°: $\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ this is approximately equal to (or ≈) 0.577

Tangent 45°: $\frac{1}{1} = 1$

Tangent 60° : $\frac{\sqrt{3}}{1} = \sqrt{3} \approx 1.732$

Notice that you can derive these ratios from the $30^\circ - 60^\circ - 90^\circ$ special right triangle. We will see this on the following page.

The triangle below is labeled with the side lengths that correspond to a $30^\circ - 60^\circ - 90^\circ$ triangle, the shortest side being x (as you saw in Unit 3):



If we start with the 30° angle at the top of the triangle: the **opposite** side is x and the **adjacent** side is $x\sqrt{3}$.

Therefore, tangent of 30° $= \frac{x}{x\sqrt{3}} = \frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$

Next we will use the 60° angle on the right of the triangle: the **opposite** side is $x\sqrt{3}$ and the **adjacent** side is x.

Therefore, tangent of 60° $= \frac{x\sqrt{3}}{x} = \frac{\sqrt{3}}{1}$ or $\sqrt{3}$

In order to figure out tangent of the 45°, we must consider a right **isosceles** triangle, which has one angle that is 90° and the other two that are the same measure (remember the definition of an isosceles triangle!)

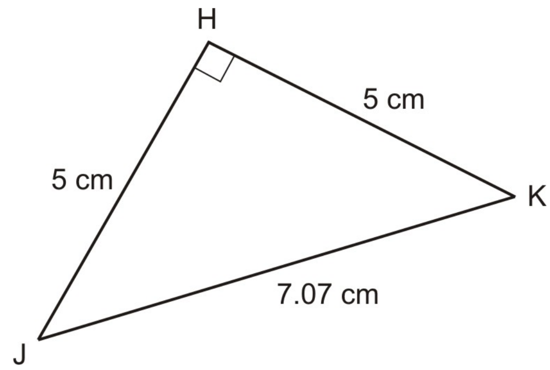
This means that a *right isosceles triangle* has one angle that is _______ and the other two that equal _______.

Likewise, both *legs* of the isosceles triangle are the *same* length! When you take the tangent of the opposite leg over the adjacent leg, the value is 1.

You can use these ratios to identify angles in a triangle. Work backwards from the ratio. If the ratio equals one of these values, you can identify the measurement of the angle.

Example 2

What is$m\angle J$ *in the triangle below?*



Find the tangent of $\angle J$ and compare it to the values of tangent for special right triangles.

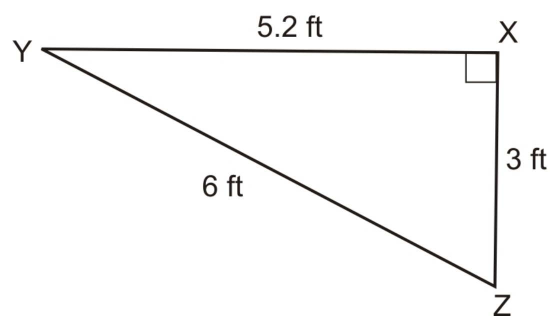
$$\begin{aligned}\tan\ J &= \frac{opposite}{adjacent}\\ &= \frac{5}{5}\\ &= 1\end{aligned}$$

So, the tangent of $\angle J$ is 1. If you look in the list of tangent values, you can see that an angle that measures 45° has a tangent of 1. So, $m\angle J = 45^\circ$.

(You may also notice that the triangle in this example is a right **isosceles** triangle, so the measure of both angles J and K must be 45°.)

Example 3

What is $m\angle Z$ in the triangle below?



Find the tangent of $\angle Z$ and compare it to the values of tangent for special right triangles.

$$\begin{aligned}tan\ Z &= \frac{opposite}{adjacent}\\ &= \frac{5.2}{3}\\ &= 1.7\overline{3}\end{aligned}$$

So, the tangent of $\angle Z$ is about 1.73. If you look at the values of tangent for special triangles a few pages back, you can see that an angle that measures 60° has a tangent of 1.732.So,$m\angle Z \approx 60^\circ$.

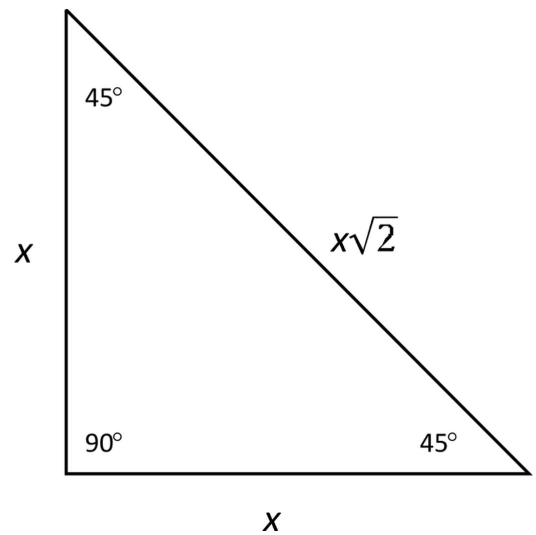
Another interesting thing to notice in this example is that ΔXYZ is a $30^\circ - 60^\circ - 90^\circ$ triangle. You will remember from Unit 3 that this means that the sides of the triangle have a special relationship as well. You can use this fact to see that:

$$XY = 5.2 \approx 3\sqrt{3}$$

and if $XZ = 3$ then of course $XY = 3\sqrt{3}$!

Reading Check (Challenge):

Below is a $45^\circ - 45^\circ - 90^\circ$ triangle.



1. *Which side length is the hypotenuse?*

2. *Show your work to find the* ***tangent*** *of a* 45° *angle. (Notice that it does NOT matter which* 45° *angle you choose!)*

5.4 Sine Ratio

Learning Objectives

- Review the different parts of right triangles.
- Identify and use the sine ratio in a right triangle.

Review: Parts of a Triangle

The sine and cosine ratios relate **opposite** and **adjacent** sides to the **hypotenuse**. You already learned these terms in the previous lesson, but they are important to review and commit to memory.

The **hypotenuse** of a triangle is always *opposite the right angle* and is the longest side of a right triangle.

The terms **adjacent** and **opposite** depend on *which angle you are referencing:*

A side **adjacent** to an angle is the *leg of the triangle that helps form the angle.*

A side **opposite** to an angle is the *leg of the triangle that does not help form the angle.*

In your own words,

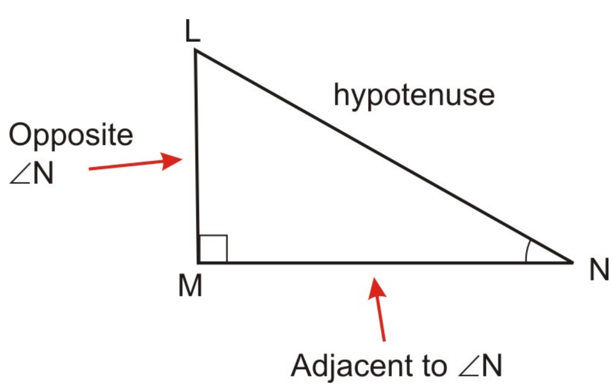
The **hypotenuse** is ____.

The **opposite** side is ____.

The **adjacent** side is ____.

Example 1

Examine the triangle in the diagram below.



Identify which leg is adjacent to angle N, which leg is opposite to angle N, and which segment is the hypotenuse.

The first part of the question asks you to identify the leg **adjacent** to $\angle N$. Since an **adjacent** leg is the one that *helps to form the angle* and is not the hypotenuse, it must be $\overline{MN}$.

The next part of the question asks you to identify the leg **opposite** $\angle N$. Since an **opposite** leg is the leg that *does not help to form the angle,* it must be $\overline{LM}$.

The **hypotenuse** is *always opposite the right angle,* so in this triangle it is segment $\overline{LN}$.

Reading Check:

1. *Which side of a right triangle is the longest side?* ______________________________

2. *Describe where the side you answered in #1 above is in relation to the right angle:*

3. *Which side of a right triangle does not help to make the right angle?* ______________________________

4. *Which side of a right triangle helps to make the right angle and is NOT the hypotenuse?* ___________________-___________

The Sine Ratio

Another important trigonometric ratio is **sine.** A sine ratio must always refer to a particular angle in a right triangle. The **sine** of an angle is the *ratio* of the length of the leg **opposite** the angle to the length of the **hypotenuse.**

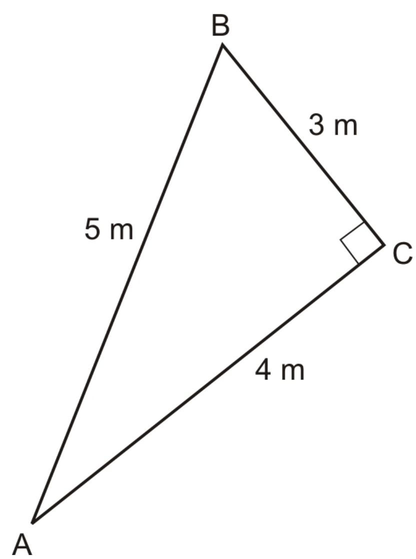
This means that the **sine** ratio is: the ____________________ side divided by the ________________________.

Remember that in a ratio, you list the first item on top of the fraction and the second item on the bottom. So, the ratio of the sine will be:

$$\sin \theta = \frac{opposite}{hypotenuse}$$

Example 2

What are $\sin A$ and $\sin B$ *in the triangle below?*



To find the solutions, you must identify the sides you need and build the ratios carefully. In the **sine** ratio, we will need the **opposite** side and the **hypotenuse.**

Remember, the **hypotenuse** of a right triangle is across from the right angle. The **opposite** side depends on which angle we are using.

The **hypotenuse** is the segment ___________, which is _______ cm long.

For angle A:

The side **opposite** angle A is the segment ___________, which is _______ cm long.

For angle B:

The side **opposite** angle B is the segment ___________, which is _______ cm long.

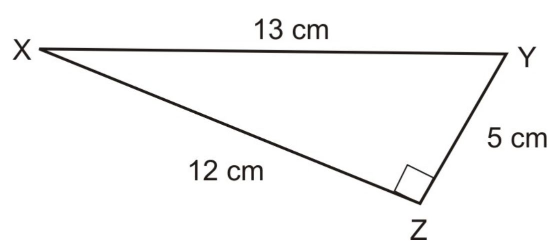
$$\sin A = \frac{opposite}{hypotenuse} = \frac{3}{5}$$
$$\sin B = \frac{opposite}{hypotenuse} = \frac{4}{5}$$

So, $\sin A = \frac{3}{5}$ and $\sin B = \frac{4}{5}$.

Reading Check:

Your friend did the following problem and asked you if it was correct:

Find $\sin X$***using the triangle below.***



$$\sin X = \frac{opposite}{hypotenuse} = \frac{5}{12}$$

1. *Is your friend's work correct? YES / NO (Circle your answer)*

2. *If not, where is the mistake in the problem? Describe the mistake in words and explain to your friend how she should have done the problem correctly.*

5.5 Cosine Ratio

Learning Objectives

- Identify and use the cosine ratio in a right triangle.
- Understand sine and cosine ratios in special right triangles.

The Cosine Ratio

The next ratio to examine is called the **cosine.** The cosine is the *ratio* of the **adjacent** side of an angle to the **hypotenuse.**

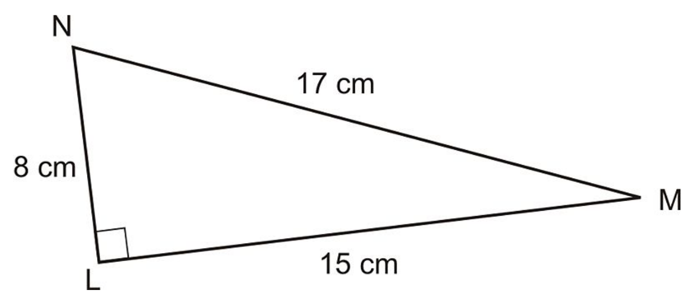
This means that the **cosine** ratio is: the ____________________ side divided by the ________________________.

$$\cos \theta = \frac{adjacent}{hypotenuse}$$

Use the same techniques you used to find sines to find cosines.

Example 1

What are the cosines of $\angle M$ and $\angle N$ in the triangle below?



To find these ratios, identify the sides **adjacent** to each angle and the **hypotenuse**. Remember, an **adjacent** side is the one that *creates the angle* and is not the hypotenuse.

The **hypotenuse** is the segment ___________, which is _______ cm long.

The side **adjacent** to angle M is the segment ___________, which is _______ cm long.

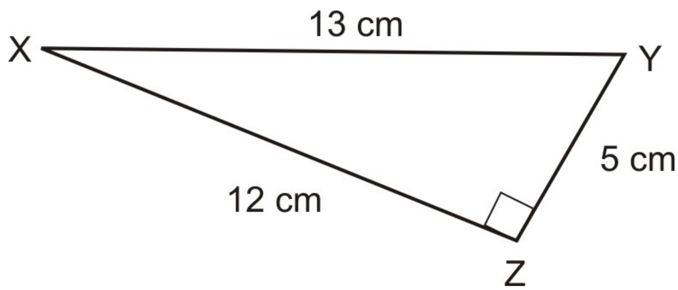
The side **adjacent** to angle N is the segment ___________, which is _______ cm long.

$$\cos M = \frac{adjacent}{hypotenuse} = \frac{15}{17}$$
$$\cos N = \frac{adjacent}{hypotenuse} = \frac{8}{17}$$

So,the cosine of $\angle M$ is $\frac{15}{17}$ and the cosine of $\angle N$ is $\frac{8}{17}$.

Note that ΔLMN on the previous page is NOT one of the special right triangles, but it is a right triangle whose sides are a Pythagorean triple.

Reading Check:



1. *In the triangle above, which side is the hypotenuse?* ____________________

And which side is adjacent to angle Y? ____________________

2. *Fill in the blanks and reduce all fractions:*

$\cos Y =$ ______________ $=$ ___________ $=$

Sines and Cosines of Special Right Triangles

It may help you to learn some of the most common values for sine and cosine ratios. The table below shows you values for angles in special right triangles:

TABLE 5.3:

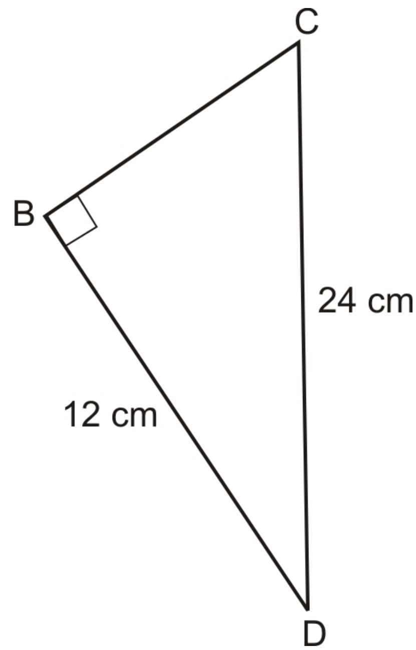
	30°	45°	60°
Sine	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.707$	$\frac{\sqrt{3}}{2} \approx 0.866$
Cosine	$\frac{\sqrt{3}}{2} \approx 0.866$	$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.707$	$\frac{1}{2}$

These values, like the tangent values, are derived from the $30^\circ - 60^\circ - 90^\circ$ and the $45^\circ - 45^\circ - 90^\circ$ triangles. To find them, use your triangles and set up ratios for sine and cosine using the side lengths. Make sure to simplify your fractions! Some of the decimal values are given as well in case you need them in the following examples.

You can use these ratios to identify angles in a triangle. Work backwards from the ratio. If the ratio equals one of these values, you can identify the measurement of the angle.

Example 2

What is the measure of $\angle C$ in the triangle below?



Note: Figure is not to scale.

Find the sine of $\angle C$ and compare it to the values in the table above.

Since we are using angle C and **sine,** we can see that the **opposite** side is segment _________, which has a length of ________ cm.

We also know that the **hypotenuse** in this triangle is segment _________, which has a length of ________ cm.

$$\begin{aligned} \sin C &= \frac{opposite}{hypotenuse} \\ &= \frac{12}{24} \\ (reduce\,!) &= \frac{1}{2} \end{aligned}$$

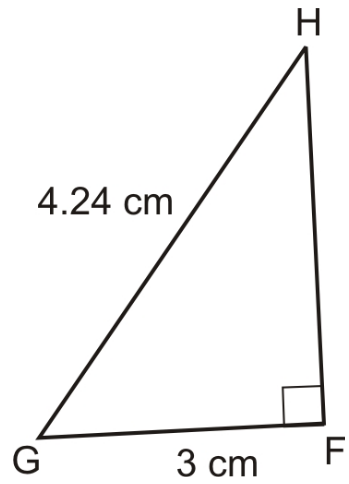
So, the sine of $\angle C$ is $\frac{1}{2}$.

If you look in the table on the previous page, you can see that an angle that measures 30° has a sine of $\frac{1}{2}$. So, $m\angle C = 30^\circ$.

(Note: in the table there are *two* values that equal $\frac{1}{2}$,but only one of them is for sine! The other value is for cosine, which we do not need in this example.)

Example 3

What is the measure of $\angle G$ in the triangle below?



Find the cosine of $\angle G$ and compare it to the values in the table.

Since we are using angle G and **cosine**, we can see that the **adjacent** side is segment _________, which has a length of ________ cm.

We also know that the **hypotenuse** in this triangle is segment _________, which has a length of ________ cm.

$$\begin{aligned}\cos\ G &= \frac{adjacent}{hypotenuse}\\ &= \frac{3}{4.24}\\ &= 0.708\end{aligned}$$

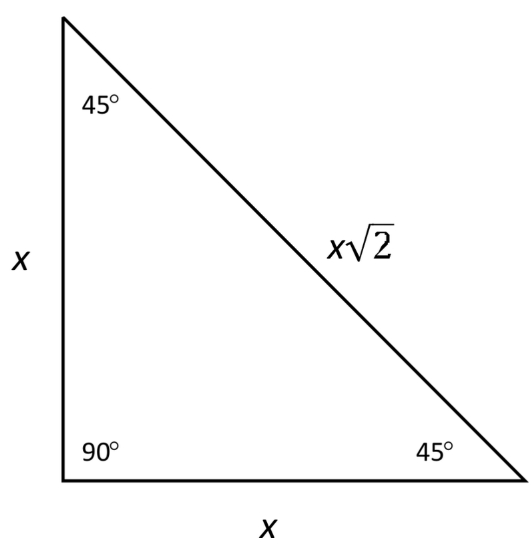
So, the cosine of $\angle G$ is about 0.708.

If you look in the table, you can see that an angle that measures 45° has a cosine of 0.707. So, $\angle G$ measures about 45°.

This example is a $45^\circ - 45^\circ - 90^\circ$ right triangle.

Reading Check (Challenge):

Below is a $45^\circ - 45^\circ - 90^\circ$ triangle.



1. *Which side length is the hypotenuse?*

2. *Show your work to find the* ***sine*** *of a* 45° *angle. (Notice that it does NOT matter which* 45° *angle you choose!)*

3. *Show your work to find the* ***cosine*** *of a* 45° *angle. (Notice that it does NOT matter which* 45° *angle you choose!)*

4. *Are your answers to #2 and #3 above the same as the values in the table?*

5.6 Choose a Trig Ratio; Word Problems

Note: CK12 does not offer text to support these Lessons. MR created customized (and excellent!) real world application of trig problems. Their lesson plans are full of quality sample problems.

CHAPTER 6 Area of Polygons

Chapter Outline

6.1 Vocabulary Self-Rating

TABLE 6.1: Rating Guide: DK: I am sure I don't know it K: I am sure I know it ?: I'm not sure

Word	Before Lesson/Unit	After Lesson/Unit
Perimeter		
Circumference		
Area		
Base		
Height		
Altitude		
Trapezoid		
Rhombus		
Kite		
Diagonal		
Polygon		
Regular		
Apothem		
Interior Angles		
Triangle Sum Theorem		
Vertex		
Exterior Angles		
Vertical Angles		
Linear Pair		
Supplementary		

6.2 Perimeter

Learning Objectives

- Calculate the perimeter of regular and non-regular polygons.

Perimeter

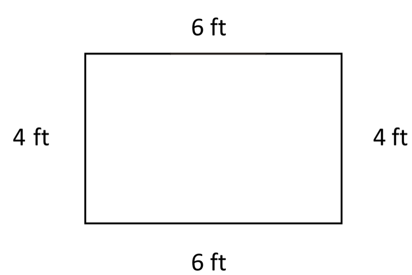
A **perimeter** is a path that surrounds an area.

The word **perimeter** comes from the Greek *peri* (around) and *meter* (measure).

This means that the **perimeter** is the *measurement around* a shape.

Perimeter can be thought of as the length of the outline of a shape. The word may be used either for the path or to describe its length.

To find the **perimeter** of a shape, you need to add up each segment that makes the border of the shape. For instance, in the diagram below, the length of each side of the rectangle is given:



The **perimeter** is the *sum* of all four _____________ of the rectangle, so it would be: $6\ ft + 4\ ft + \_\_\_\ ft + \_\_\_\ ft = 20\ feet$

The perimeter of a circle is called **circumference**.

Reading Check

1. *Fill in the blank: To find the* __________________ *of a shape, add up all of its sides.*

2. *Describe* ***perimeter*** *in your own words:*

6.3 Area of Parallelograms

Learning Objectives

- Understand the basic concepts of the meaning of area.
- Use formulas to find the area of parallelograms, including rectangles.

The **area** of a shape is the space *inside* the perimeter.

You can think of the **perimeter** as a line you may draw with a pen of the outer border of a shape. The **area** is what you would paint with a paintbrush in order to fill in the entire shape, painting inside the perimeter line.

For example,

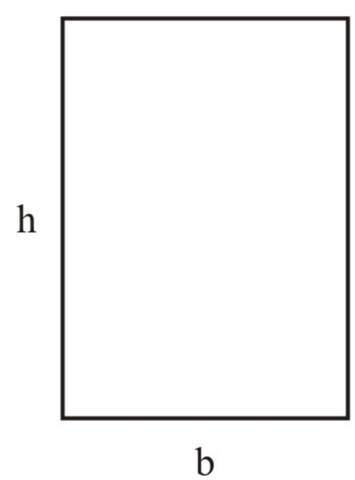
In a fenced field, the *fence* would be the **perimeter** and the *grass in the field* would be the **area**.

In a basketball court, the *sidelines* would be the **perimeter** and the *wooden court surface* would be the **area**.

In a pool, the _______________ would be the **perimeter** and the ___________ would be the **area**.

Area of a Rectangle

If a rectangle has **base** b units and **height** h units, then the **area**, A, is bh square units.



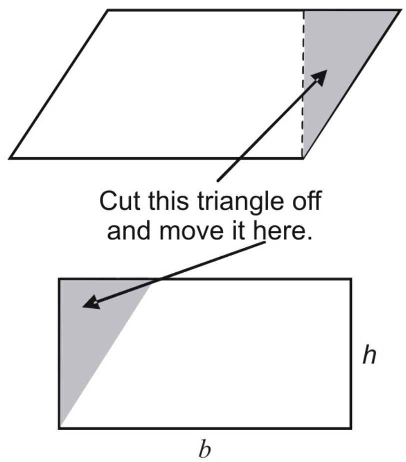
$$\text{Area} = \text{base} \cdot \text{height}$$
$$A = bh$$

Area of a Parallelogram

Example 1

How could we find the area of this parallelogram?

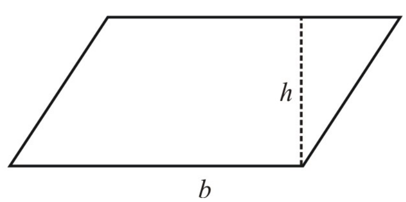
Make it into a rectangle by moving the triangular part:



The rectangle is made of the same parts as the parallelogram, so their areas are the same. The area of the rectangle is bh, so the area of the parallelogram is also bh.

Warning: Notice that the height h of the parallelogram is the *perpendicular distance between two parallel sides of the parallelogram,* **not** a side of the parallelogram (unless the parallelogram is also a rectangle, of course).

If a parallelogram has base b units and height h units, then the area, A, is bh square units.



$$\text{Area} = \text{base} \cdot \text{height}$$
$$A = bh$$

6.4 Area of Triangles

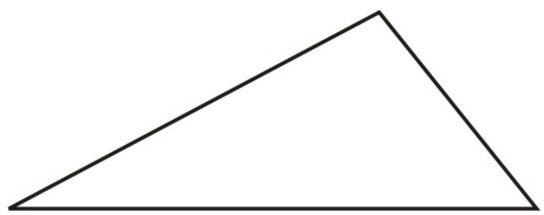
Learning Objectives

- Use formulas to find the area of triangles.

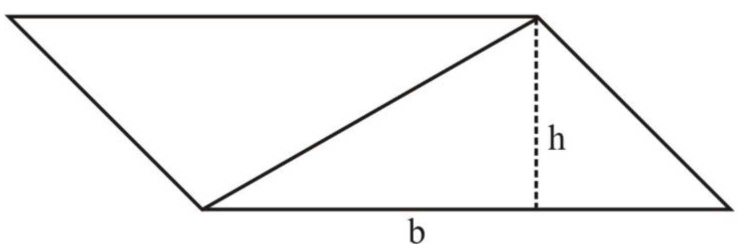
Triangles

Example 1

How could we find the area of this triangle?



Make it into a parallelogram. This can be done by making a copy of the original triangle and putting the copy together with the original. As you can see in the diagram below, one triangle looks the same as the one above, and the other is upside down.



When you put the 2 triangles together, they make a parallelogram. The area of the parallelogram is bh, and since we used 2 triangles to make the parallelogram, the area of the triangle is half of bh : $\frac{bh}{2}$ or $\frac{1}{2}$ bh.

How many triangles are used to make up the parallelogram above? __________

Warning: Notice that the height h (also often called the **altitude**) of the triangle is the *perpendicular distance between a vertex and the opposite side of the triangle*. This means that the **altitude** h meets the base b at a 90° angle.

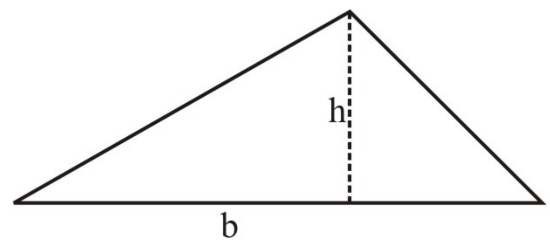
Reading Check

1. *In a triangle, the height and the base in a must be* ____________________________ *because they form a* 90° *angle*.

2. *Another name for the height of a triangle is the* ___________________.

Area of a Triangle

If a triangle has base b units and altitude h units, then the area, A, is $\frac{bh}{2}$ or $\frac{1}{2}bh$ square units.



$$\text{Area} = \frac{1}{2} \cdot \text{base} \cdot \text{height}$$
$$A = \frac{bh}{2} \text{ or } A = \frac{1}{2}bh$$

Reading Check

1. *True or False: In a triangle, the base and the height intersect at an obtuse angle.*

2. *True or False: Multiplying by* $\frac{1}{2}$ *and dividing by 2 are the same thing.*

3. *Fill in the blanks:*

In area formulas, the letter _____ usually stands for the base and the letter _____ usually stands for the height.

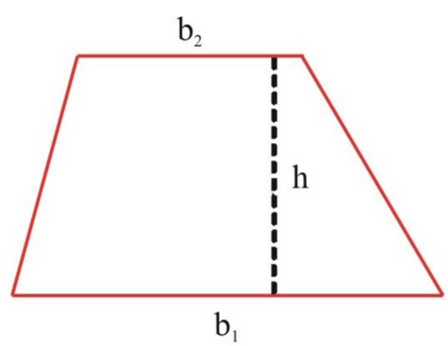
6.5 Area of Trapezoids

Learning Objectives

- Use formulas to find the area of trapezoids – quadrilaterals with exactly one pair of parallel sides.

Area of a Trapezoid

Recall that a **trapezoid** is a quadrilateral with *one pair of parallel sides*. The lengths of the parallel sides are the **bases**. The perpendicular distance between the parallel sides is the height, or **altitude**, of the trapezoid.



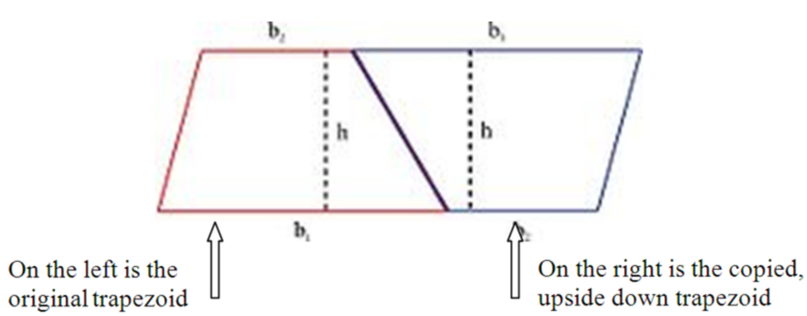
Trapezoid with bases b_1 and b_2 and altitude h

In the trapezoid pictured above, the bases b_1 and b_2 are ______________________ to each other.

The altitude (or height) is _________________________ to both bases.

To find the area of the trapezoid, turn the problem into one about a *parallelogram*. Why? Because you already know how to compute the area of a parallelogram!

- Make a copy of the trapezoid.
- Rotate the copy 180°.
- Put the two trapezoids together to form a parallelogram.



Two things to notice:

1. The parallelogram has a base that is equal to $b_1 + b_2$.

Look at the length of the base in the picture: the left side has a length of _______ and the right side has a length of _______ so the total length of the base of the parallelogram is $b_1 + b_2$.

2. The altitude (h) of the parallelogram is the same as the altitude (h) of the trapezoid.

Now to find the **area** of the **trapezoid**:

The area of the *parallelogram* is: $A = \text{base} \cdot \text{altitude}$.

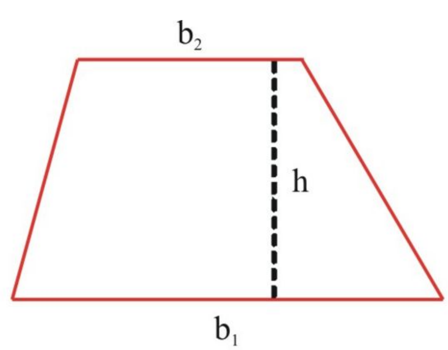
Since the length of the base = ____________ and the altitude = _____ ,

$$\text{Area of the parallelogram} = (b_1 + b_2) \cdot h$$

The parallelogram is made up of *two congruent trapezoids,* so the area of each trapezoid is *half* the area of the parallelogram.

The area of the trapezoid is *half* of $(b_1 + b_2) \cdot h$:

Area of Trapezoid with Bases $b_1$and $b_2$and Altitude h



Trapezoid with bases b_1 and b_2 and altitude h

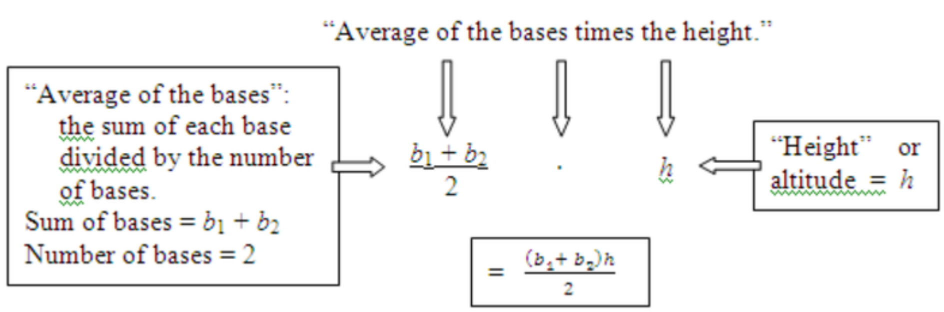
The bases of the trapezoid in the figure above are _______ and ________.

The altitude (or height) of the trapezoid is ______.

For a trapezoid with bases b_1 and b_2 and altitude h,

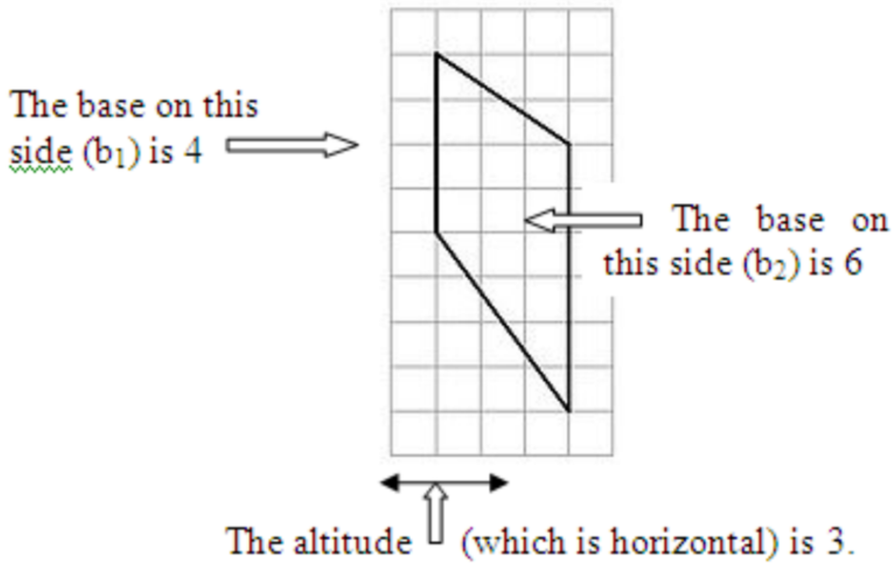
$$A = \frac{1}{2}(b_1 + b_2)h \text{ or } A = \frac{(b_1 + b_2)h}{2}$$

Notice that the formula for the area of a trapezoid could also be written as the "Average of the bases times the height."



Example 1

What is the area of the trapezoid below?



Since this trapezoid is sideways compared to the ones you have seen so far in this lesson, the bases (which are parallel to each other) are on the right and left side of the shape. The altitude is horizontal instead of vertical.

Using the graph paper that the trapezoid above is on, you can count the boxes to determine the length of each important part of the shape.

You can see that the bases of the trapezoid are _____ and _____. The altitude is _____.

To find the area, multiply half of the sum of both bases by the altitude (or height):

$$\begin{aligned}A = \frac{1}{2}(b_1 + b_2)h &= \frac{1}{2}(4+6)\cdot 3\\ &= \frac{1}{2}(10)\cdot 3 = 5\cdot 3 = 15\end{aligned}$$

The area of the trapezoid is 15 square units.

Reading Check

1. *How many bases does a trapezoid have?*

2. *True or false: The bases of a trapezoid are perpendicular to each other.*

3. *In your own words, describe the altitude of a trapezoid:*

4. *True or false: The altitude of a trapezoid is parallel to the bases.*

5. *True or false:*

The average of the bases is the same as the sum of the bases divided by 2.

6. *Draw a picture of a trapezoid in the space below:*

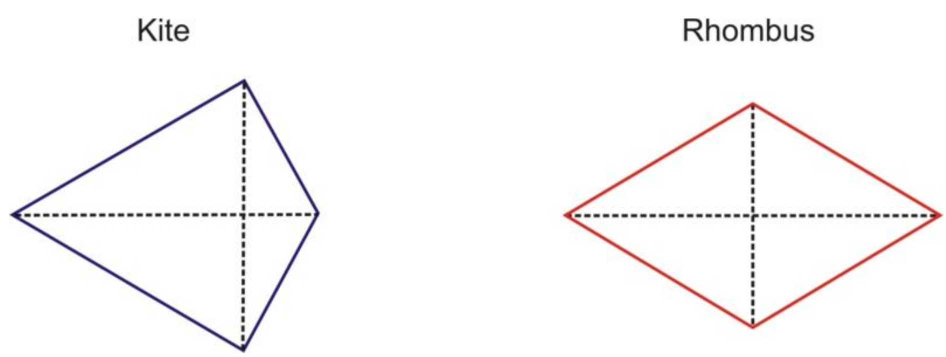
6.6 Area of Rhombus and Kite

Learning Objectives

- Use formulas to find the area of rhombuses and kites – quadrilaterals with perpendicular diagonals.

Area of a Rhombus or Kite

First let's start with a review of some of the properties of a **kite** and a **rhombus**.

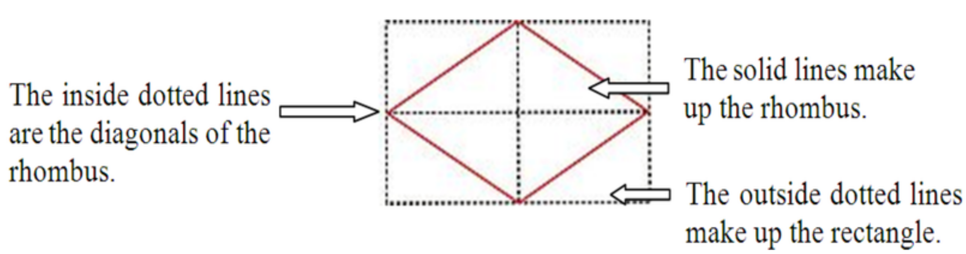


A reminder: the **diagonals** of both a kite and a rhombus are the dotted lines in the figures above. **Diagonals** connect opposite vertices.

TABLE 6.2:

Does it have...?	Kite	Rhombus
Congruent sides	Yes, 2 pairs	Yes, all 4
Opposite angles congruent	1 pair yes. 1 pair maybe	Both pairs yes
Perpendicular diagonals	Yes	Yes
Diagonals bisected	1 yes. 1 maybe	Both yes

Now you are ready to develop area formulas. For both a bite and a rhombus, we will "Frame it in a rectangle." Here's how you can frame a **rhombus** in a rectangle:



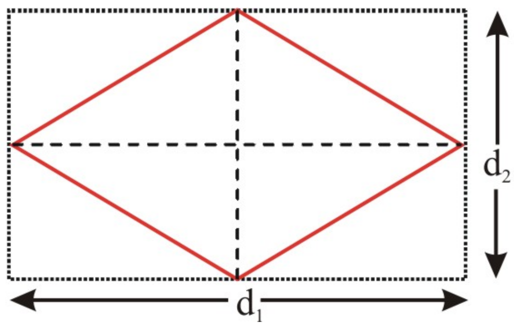
Notice that:

- The *base* and *height* of the **rectangle** are the same as the lengths of the two *diagonals* of the **rhombus**.
- The rectangle is divided into 8 congruent triangles; 4 of the triangles fill the rhombus, so the **area** of the rhombus is *half of the area of the rectangle*.

Area of a Rhombus with Diagonals d_1 and d_2

The diagonals of the rhombus in the figure below are labeled ______ and ______.

These are the same as the sides of the _________________________.



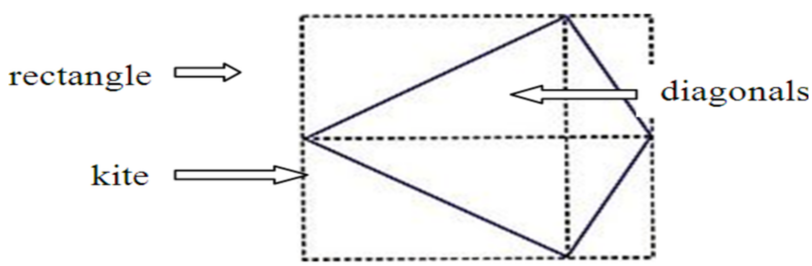
$A = \frac{1}{2}d_1d_2 = \frac{d_1d_2}{2}$

As you learned on the previous page: The area of a **rhombus** is half of the area of the ___________________ it is framed in.

The area of the rectangle is d_1d_2 so half the area of the rectangle is _______________.

Therefore, the **area** of the **rhombus** is __________________.

Next, we will examine the kite. We will follow the same rule: "Frame it in a rectangle." Here's how you can frame a **kite** in a rectangle:



Notice that:

- The *base* and *height* of the **rectangle** are the same as the lengths of the two *diagonals* of the **kite**.
- The rectangle is divided into 8 triangles; 4 of the triangles fill the kite. For every triangle inside the kite, there is a *congruent* triangle outside the kite. So, the **area** of the kite is *half of the area of the rectangle*.

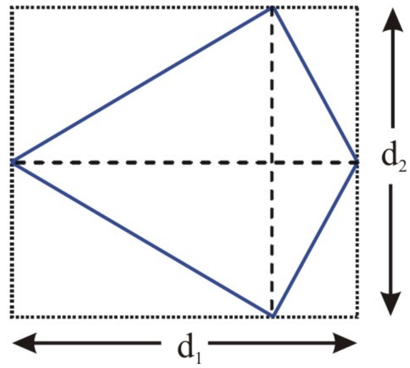
Just like a rhombus:

The area of a **kite** is half of the area of the ___________________ it is framed in.

Area of a Kite with Diagonals d_1 and d_2

The diagonals of the kite in the figure below are labeled ______ and ______.

These are the same as the sides of the _________________________.



$$A = \frac{1}{2}d_1d_2 = \frac{d_1d_2}{2}$$

The area of the rectangle is d_1d_2 so half the area of the rectangle is _______________.

Therefore, the **area** of the **kite** is __________________.

Reading Check

1. *Fill in the blank:*

A kite is a quadrilateral with ______________________________ *diagonals.*

2. *Fill in the blank:*

A rhombus is a quadrilateral with perpendicular ___________________________.

3. *What is a diagonal? Describe it in your own words.*

4. *True or false: All 4 sides of a rhombus are congruent.*

5. *True or false: No interior angles in a kite are congruent.*

6. *True or false: When you frame either a kite or a rhombus in a rectangle, the diagonals of the kite or rhombus are the **same** as the base and height of the rectangle.*

6.7 Area of Regular Polygons

Learning Objectives

- Recognize and use the terms involved in developing formulas for regular polygons.
- Calculate the area and perimeter of a regular polygon.

You already know how to find areas and perimeters of some figures – triangles, parallelograms, and other quadrilaterals. Not surprisingly, the new formulas in this lesson will build on those basic figures – in particular, the triangle.

Parts and Terms for Regular Polygons

Do you remember the names of different polygons from Chapter 1?

First of all, "poly-" means "many" and "-gon" refers to the sides of a shape.

A regular polygon is a shape whose many sides are all congruent.

Regular polygons also have congruent interior angles and congruent central angles (which you will learn about on the next page.)

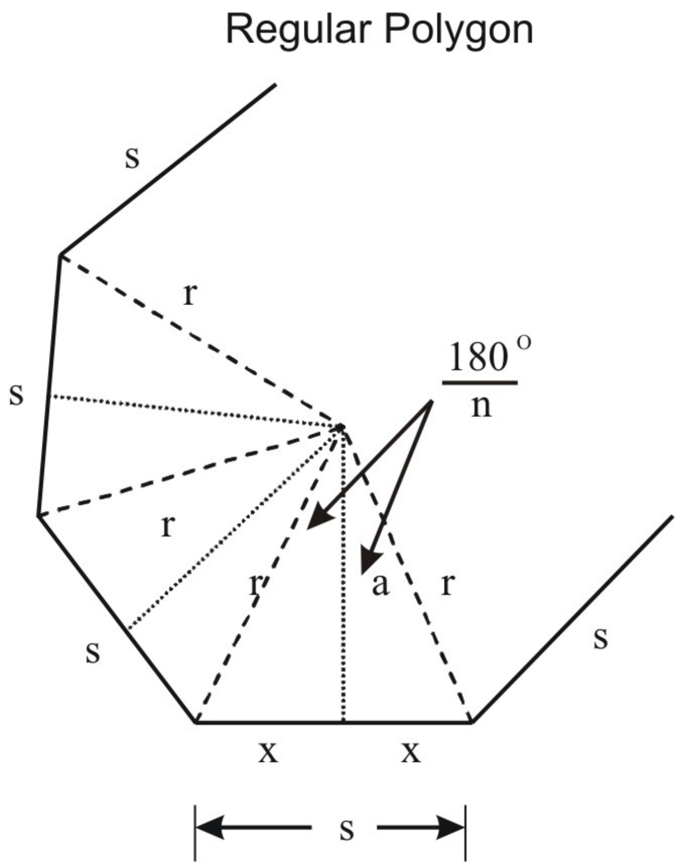
Polygons are classified by how many sides they have.

Here are a few names to review:

- *A pentagon has 5 sides.*
- *A hexagon has 6 sides.*
- *A heptagon has 7 sides.*
- *An octagon has ______ sides. (Hint: how many legs does an octopus have?)*
- *A nonagon has 9 sides.*
- *A decagon has ______ sides. (Hint: how many years are in a decade?)*

Let's start with some background on regular polygons.

Here is a general regular polygon with n sides, where n stands for some number. Some of its sides are shown in the diagram:



In the diagram, here is what each variable represents:

- s is the length of each side of the polygon.
- r is the length of a "radius" of the polygon, which is a segment from the center of the polygon to a vertex (or corner).
- x is the length of one-half of a side of the polygon (so $x = \frac{1}{2}s$ or $2x = s$).
- a is the length of a segment called the **apothem** — a segment from the center to a side of the polygon, perpendicular to the side. (Notice that a is the **altitude** of each of the triangles formed by two radii and a side.)

Think about it:

A triangle would have $n =$ ___ sides.

A square would have $n =$ ___ sides.

An octagon would have $n =$ ___ sides.

The angle between two consecutive radii measures $\frac{360^\circ}{n}$ because n congruent central angles are formed by the radii from the center to each of the n vertices of the polygon.

We can figure this out because an entire circle is 360°, and you can think of the center of the polygon as having a circle of angles around it. If there are n central angles (all equivalent), each central angle between each radius is $\frac{360^\circ}{n}$.

An **apothem** divides each of these central angles into two congruent halves; each of these half angles measures $\frac{1}{2} \cdot \frac{360^\circ}{n} = \frac{360^\circ}{2n} = \frac{180^\circ}{n}$.

Perimeter of a Regular Polygon

We continue with the regular polygon diagrammed on the previous page. Let P be the perimeter. Remember that _____ is the number of sides in the polygon and _____ is the length of each side. In simplest terms,

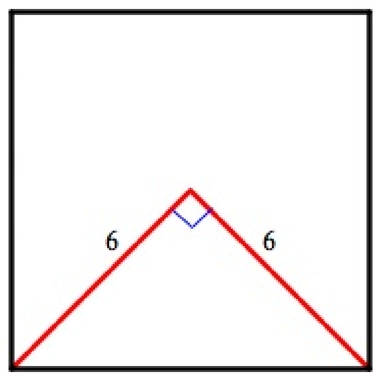
$$P = ns$$

We know this because the perimeter of a shape is the sum of ______________________. Another way to express perimeter is the number of sides times the length of each side.

Example 1

A square has a radius of 6 inches. What is the perimeter of the square?

Notice that a side and two radii make an isosceles right triangle:



- The triangle is *isosceles* because the legs of the triangle are each a radius of the square. Each radius is _____ inches long and both are the same length, so the triangle is *isosceles* because *its legs are congruent.*

- The triangle has a *right* angle because the central angle is $\frac{360^\circ}{n}$ and the square has 4 sides (which means $n = 4$) so each central angle is $\frac{360^\circ}{4} = 90^\circ$.

Not only is this an isosceles right triangle, but it is also a **45–45–90** triangle!

You may remember that if the legs are each _____ inches long, then the hypotenuse of the triangle is $6\sqrt{2}$ inches long. Notice that the hypotenuse is also a side of the square.

To find the perimeter, use the formula $P = ns$. We know $n = 4$ and $s = 6\sqrt{2}$.

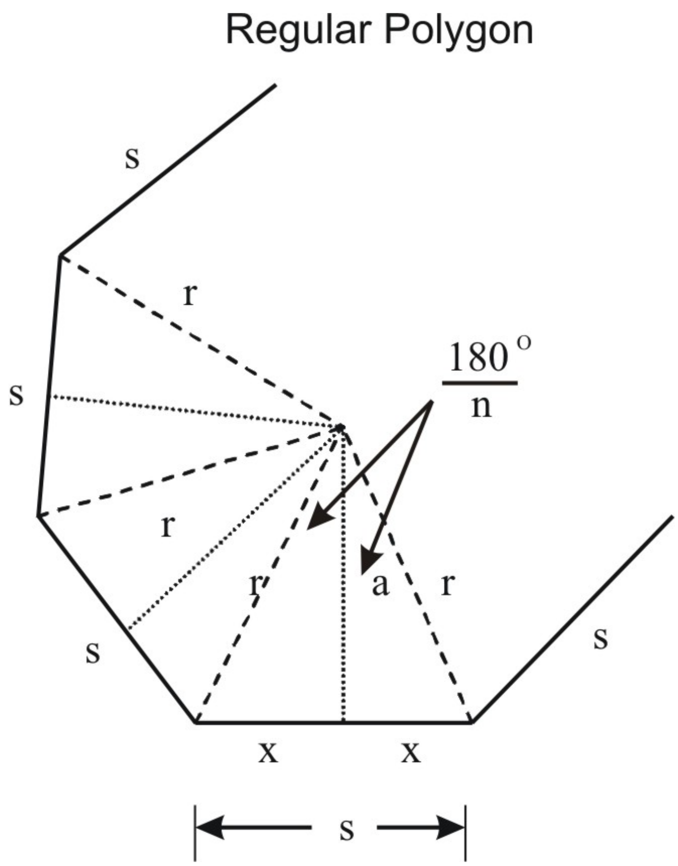
$$\begin{aligned} P &= ns \\ &= 4 \cdot 6\sqrt{2} = 24\sqrt{2} \text{ inches} \quad \text{(on a calculator, this length} \approx 33.9 \text{ inches)} \end{aligned}$$

The perimeter of the square is $24\sqrt{2}$ inches.

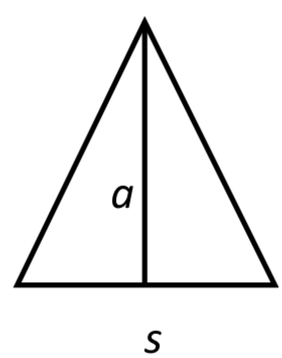
Area of a Regular Polygon

The next logical step is to complete our study of regular polygons by developing area formulas.

Take another look at the regular polygon figure below (it is the same one you saw earlier in this lesson.) Here's how we can find its area, A.



Two radii and a side make a triangle with base s and altitude a:



There are n of these triangles in the polygon.

The area of each *triangle* is: $\frac{1}{2} base \cdot height = \frac{1}{2} sa$

The entire area of the *polygon* is:

$$A = \text{number of triangles} \cdot \text{area of each triangle}$$
$$A = n\left(\frac{1}{2}sa\right) = \frac{1}{2}(ns)a = \frac{1}{2}(Pa) \text{ because perimeter } P = ns$$

Therefore, the Area of a regular polygon with perimeter P and apothem a:

$$A = \frac{1}{2}Pa$$

Reading Check

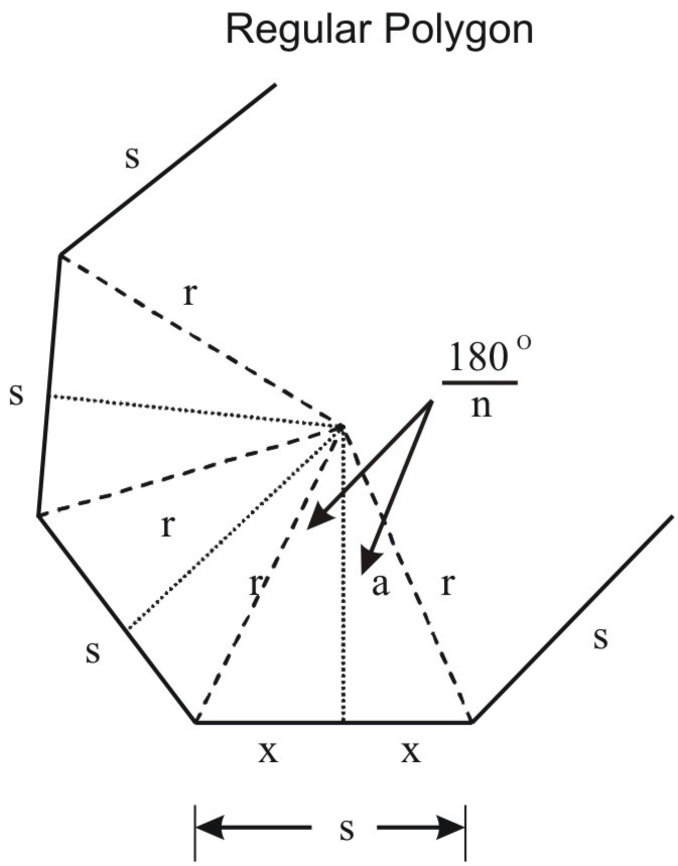
1. *How many sides does a pentagon have?*

2. *True or false: All sides of a regular polygon are the same length.*

3. *If you know the length of one side of a regular pentagon, can you find its* ***perimeter****? How? Explain the steps you would use.*

4. *True or false: A regular polygon that has n sides also has n vertices.*

5. *In the figure below (you have seen it a few times already!),*



a. *What does a stand for?* ___________________________

Describe what this is:

b. *What does r stand for?* ___________________________

c. *What does s stand for?* ___________________________

Graphic Organizer for Lessons 2 – 6: Area

TABLE 6.3:

Shape	Draw a Picture	Area Formula	What does each letter in the Area Formula stand for?
Parallelogram			
Triangle			
Trapezoid			
Rhombus			
Kite			
Regular Polygon			

6.8 Area of Shaded and Composite Figures

Learning Objectives

- Use formulas to find the area of specific types of two-dimensional shapes by analyzing them as the sum or difference of smaller polygons.

Congruent Areas

Before we find the area of more complicated figures, we must know that:

If two figures are congruent, they have the same area.

This means that:

If two shapes are the same, the area inside them is also _____________________.

This is obvious because congruent figures have the same amount of space inside them. However, two figures with the same area are not necessarily congruent.

Area of a Whole is the Sum of Parts

If a figure is composed of two or more parts that do not overlap each other, then the *area of the figure is the sum of the areas of the parts.*

This means that:

You can break down a figure into parts and ___________ the areas of those parts together to get the area of the whole figure.

This is the familiar idea that a whole is the sum of its parts. In practical problems you may find it helpful to break a figure down into parts.

Reading Check

1. *True or false: If two shapes are congruent, then they have the same area.*

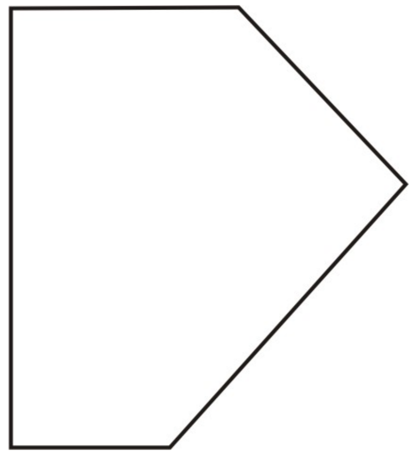
2. *True or false: If two shapes have the same area, then they are congruent.*

3. *In your own words, describe what the phrase means:*

"A whole is the sum of its parts."

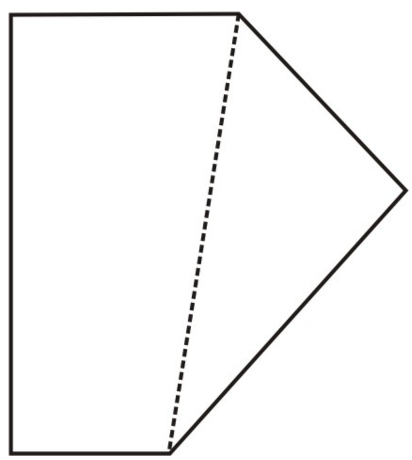
Example 1

Find the area of the figure below.



Luckily, you do not have to learn a special formula for an irregular pentagon (which this figure is because its sides are not congruent.)

Instead, you can break the figure down into a trapezoid and a triangle like the dotted line does below. Use the area formulas for these parts to find the area of the whole figure.



The shape created on the *left* of the dotted line is a ________________________.

The shape created on the *right* of the dotted line is a ________________________.

Without numbers, we can review our formulas in describing the steps:

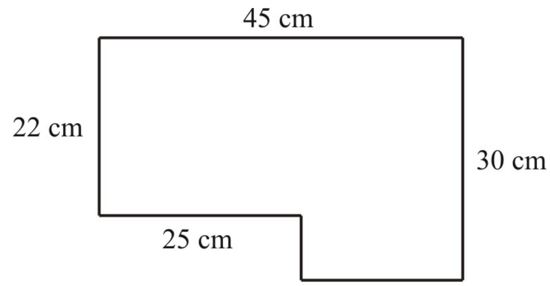
To find the area of the trapezoid, use the formula: $A =$ __________.

To find the area of the triangle, use the formula: $A =$ __________.

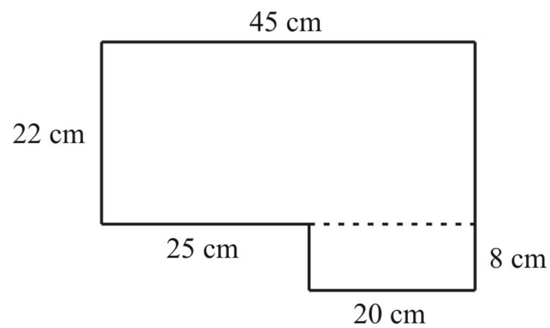
Because the area of the whole is the sum of its parts, we simply add together the areas of the trapezoid and the triangle to find the area of the entire figure!

Example 2

What is the area of the figure shown below?



Break the figure down into two rectangles:



First, look at the *top rectangle*:

The larger rectangle has a base of 45 cm and a height of 22 cm.

Since the area of a rectangle is base $\cdot$ height,

$$A = \_\_\_\_\_\_ \; cm \cdot \_\_\_\_\_\_ \; cm = 990 \; cm^2$$

Second, look at the *bottom rectangle:*

The smaller rectangle has a base of 20 cm and a height of 8 cm.

$$A = \_\_\_\_\_\_ \; cm \cdot \_\_\_\_\_\_ \; cm = 160 \; cm^2$$

Area of the whole figure is the sum of its parts, so:

$$\text{Area} = \text{area of top rectangle} + \text{area of bottom rectangle}$$
$$A = \_\_\_\_\_\_ \; cm^2 + \_\_\_\_\_\_ \; cm^2 = 1150 \; cm^2$$

The area of the entire figure is 1150 cm^2.

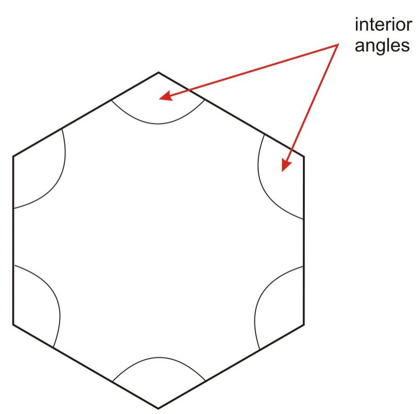
6.9 Sum of the Interior Angles of a Polygon

Learning Objectives

- Identify the interior angles of convex polygons.
- Find the sums of interior angles in convex polygons.

Interior Angles in Convex Polygons

The **interior angles** are the angles on the *inside* of a polygon:



As you can see in the image, a polygon has the *same* number of **interior angles** as it does *sides*.

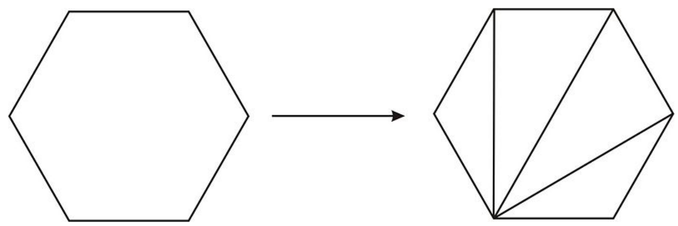
- This means that if a polygon has 3 sides, then it is called a __________________ and it has ______ interior angles.
- If a polygon has 6 sides, then it is called a ________________________ and it has ______ interior angles.
- If a polygon has 8 sides, then it is called a ________________________ and it has ______ interior angles.

Summing Interior Angles in Convex Polygons

You have already learned the **Triangle Sum Theorem**. It states that the *sum* of the measures of the **interior angles** in a triangle will always be 180°. What about other polygons? Do they have a similar rule?

We can use the **Triangle Sum Theorem** to find the sum of the measures of the angles for *any* polygon. The first step is to *cut the polygon into triangles by drawing* ***diagonals*** *from one* ***vertex***. When doing this you must make sure none of the triangles overlap. Check out the diagram on the next page.

This shape below is a *hexagon* because it has _______ sides.



The *hexagon* above is divided into 4 triangles.

You can see that we have picked a single **vertex** (or *corner*) of the hexagon and drawn a line to each vertex *across* the hexagon. You cannot draw a diagonal to the two vertices next to the starting point because those would be sides.

Since each *triangle* has internal angles that sum to 180°, you can find the sum of the interior angles in the hexagon by adding the triangle angles!

The measure of each angle in the hexagon is a *sum of angles from the triangles*.

Since none of the triangles overlap, we can obtain the TOTAL measure of **interior angles** in the hexagon by summing all of the triangles' interior angles. There are 4 triangles, so add 180° 4 times:

$$180^\circ + 180^\circ + 180^\circ + 180^\circ = 720^\circ$$

Or, multiply the *number* of triangles by 180°:

There are ______ triangles in a *hexagon*, so this is:

$$4 \cdot 180^\circ = 720^\circ$$

The sum of the **interior angles** in the *hexagon* is 720°.

- To find the total measure of __________________________ angles in a polygon, multiply the number of ___________________________ you can draw inside the polygon by 180°.

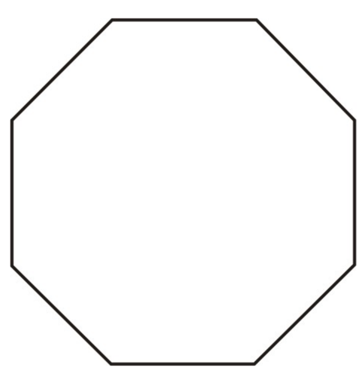
Reading Check

1. *True or false: Interior angles are the angles inside a polygon.*

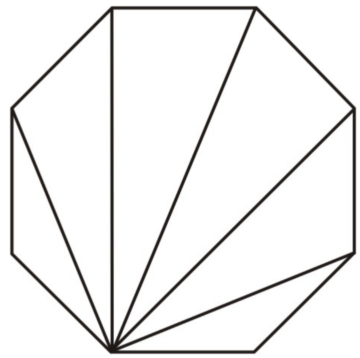
2. *True or false: A polygon has the same number of interior angles as it has sides.*

Example 1

What is the sum of the interior angles in the polygon below?



The shape in the diagram is an *octagon* because it has 8 sides. Draw triangles on the interior using the same process:



The *octagon* can be divided into six triangles.

So, the sum of the **interior angles** will be equal to the sum of the angles in the six triangles (and each triangle is 180°):

$$6 \cdot 180^\circ = 1080^\circ$$

So, the sum of the interior angles is 1080°.

What you may have noticed from these examples is that for *any* polygon, the number of triangles you can draw will be *2 less than the number of sides* (or the number of vertices).

This means that if a polygon has 5 sides, then you can draw $(5-2)$ or 3 triangles.

- Or, if a polygon has 7 sides, then you can draw (_____ – 2) or _____ triangles.
- Or, if a polygon has 10 sides, then it is called a ____________________ and you can draw (_____ – 2) or _____ triangles.

You can create an expression for the sum of the interior angles of any polygon using *n* for the number of sides of the polygon:

The **sum of the interior angles** of a polygon with *n* sides is:

$$\text{Angle Sum} = 180^\circ(n-2)$$

Example 2

What is the sum of the interior angles of a nonagon?

To find the sum of the interior angles in a *nonagon*, use the expression on the previous page. Remember that a *nonagon* has 9 sides, so *n* will be equal to 9.

$$\begin{aligned}\text{Angle Sum} &= 180^\circ(n-2)\\ &= 180^\circ(9-2)\\ &= 180^\circ(7)\\ &= 1260^\circ\end{aligned}$$

So, the sum of the interior angles in a *nonagon* is 1260°.

Reading Check

1. *What is the relationship between the number of sides in a polygon and the number of triangles you can draw inside it with diagonals? Describe in your own words.*

2. *Fill in the blanks:*

The sum of ___________________ angles in any polygon is equal to (the number of ____________________ in the polygon minus _________) times 180°.

3. *Challenge:*

*A **regular** polygon has congruent sides and congruent angles. If you know how to find the TOTAL measure of interior angles in a polygon, how would you find the measure of ONE interior angle in a regular polygon? Describe.*

6.10 Sum of the Exterior Angles of a Polygon

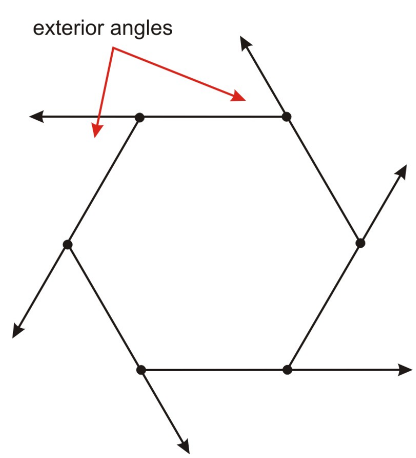
Learning Objectives

- Identify the exterior angles of convex polygons.
- Find the sums of exterior angles in convex polygons.

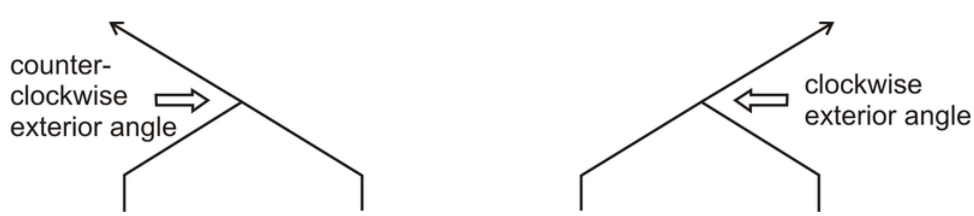
This lesson focuses on the **exterior angles** in a polygon. There is a surprising feature of the *sum of the exterior angles* in a polygon that will help you solve problems about **regular** polygons.

Exterior Angles in Convex Polygons

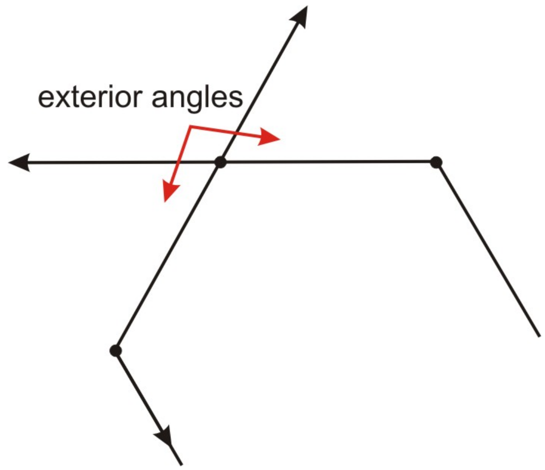
Recall that **interior** means *inside* and that **exterior** means *outside*. So, an **exterior angle** is an angle on the *outside* of a polygon. An **exterior angle** is formed by extending a side of the polygon:



There are two possible **exterior angles** for any given vertex on a polygon. In the figure above, one set of exterior angles is shown, the set in the *counter-clockwise* direction. The other set of **exterior angles** would be formed by extending each side of the polygon in the opposite *(clockwise)* direction. However, it does not matter which exterior angles you use because their measurement will be the *same* on each vertex. Look closely at one vertex below, where we draw both of the *possible* **exterior angles:**



In the above diagrams, both **exterior angles** are drawn separately. On the next page, both exterior angles on a single vertex are drawn *together*:



As you can see, the two **exterior angles** at the same vertex are **vertical angles**.

Since *vertical angles are congruent*, the two exterior angles possible around a single vertex are *congruent*.

The clockwise exterior angle and the counter-clockwise exterior angle at the *same* vertex are __________________-_______________ .

Additionally, because the exterior angle will be a **linear pair** with its *adjacent interior angle*, it will always be **supplementary** to that interior angle.

As a reminder, **supplementary** angles have a sum of 180°.

This means the exterior angle and interior angle at the same vertex will sum to ______°.

Reading Check

1. *True or False: Vertical angles are supplementary.*

2. *True or False:*

Exterior angles are on the outside of a polygon and are formed when you extend one side of the polygon.

3. *Name the 2 sets of exterior angles:*

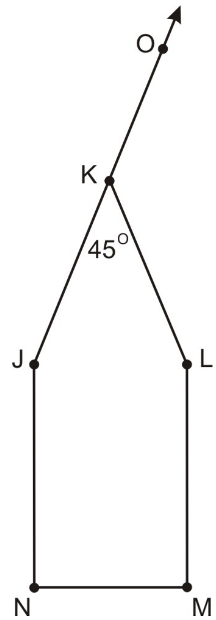
4. *Fill in the blank: Angles that form a linear pair add up to* ________°.

5. *Complete the sentence:*

Since the interior angle and the exterior angle at the same vertex of a polygon form a linear pair, ...

Example 1

What is the measure of the exterior angle $\angle OKL$ *in the diagram below?*



The **interior angle** $\angle JKL$ is labeled as 45°.

Notice that the **interior angle** and the **exterior angle** form a **linear pair**, meaning the two angles are **supplementary** – they add up to 180°.

So, to find the measure of the **exterior angle**, subtract 45° from 180°:

$$180^\circ - 45^\circ = 135^\circ$$

The measure of $\angle OKL$ is 135°.

Summing Exterior Angles in Convex Polygons

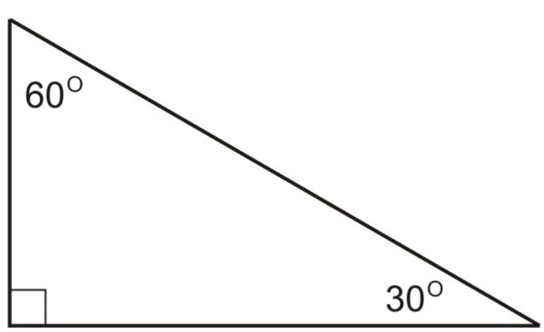
By now you might expect that if you add up various angles in polygons, there will be some sort of pattern or rule.

For example, you already know that the sum of the **interior angles** in a triangle will always be 180°.

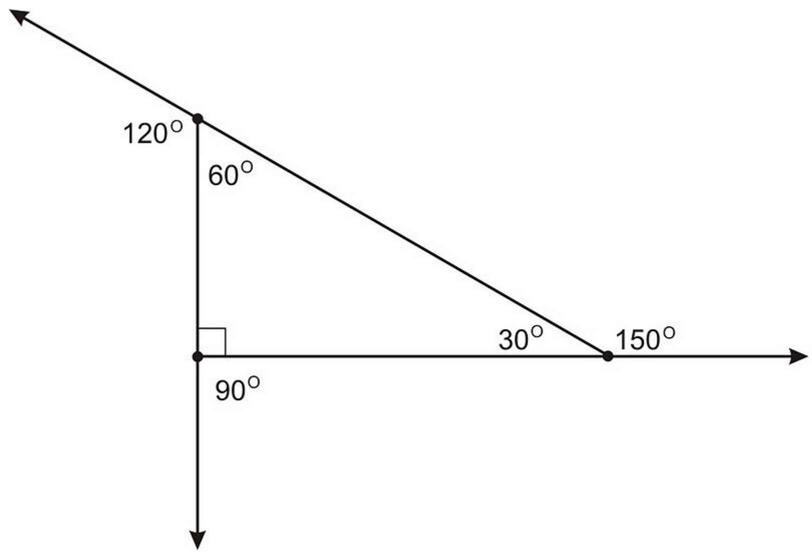
From that fact, you learned that you can find the *sums* of the interior angles of any polygon with n sides using the expression $180^\circ(n-2)$.

There is also a rule for **exterior angles** in a polygon.

Let's begin by looking at a triangle:



To find the **exterior angles** at each vertex, extend the segments and find angles supplementary to the interior angles:



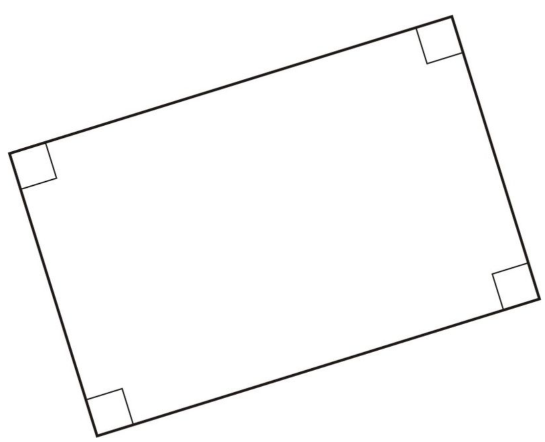
The *sum* of these three **exterior angles** is:

$$150^\circ + 120^\circ + 90^\circ = 360^\circ$$

So, the **exterior angles** in this triangle will sum to 360°.

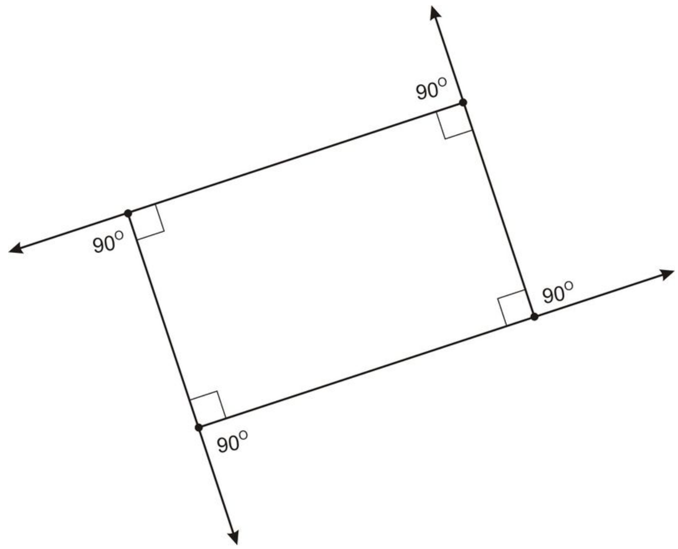
Let's see what happens with another shape.

To compare, examine the exterior angles of a rectangle:



In a rectangle, each **interior angle** measures ______ °.

Since **exterior angles** are **supplementary** to **interior angles**, all exterior angles in a rectangle will *also* measure ______ °.



Find the sum of the 4 **exterior angles** in a rectangle:

$$90^\circ + 90^\circ + 90^\circ + 90^\circ = 360^\circ$$

So, the *sum* of the **exterior angles** in a rectangle is *also* 360°.

In fact, the sum of the exterior angles in **any** convex polygon will *always* be 360°.

It does not matter how many sides the polygon has, the sum will *always* be 360°.

Exterior Angle Sum

The sum of the exterior angles of any convex polygon is 360°.

No matter how many sides a polygon has, the sum of its ___________________________ angles is equal to ______-__ °.

We can prove this using algebra and the facts that at any vertex the sum of the interior and one of the exterior angles is always 180°, and the sum of all interior angles in a polygon is $180^\circ(n-2)$. The proof is on the next page.

Proof:

At any vertex of a polygon the exterior angle and the interior angle sum to 180°. So, adding up all of the exterior angles and interior angles gives a total of 180° times the number of vertices:

$$(\text{Sum of Exterior Angles}) + (\text{Sum of Interior Angles}) = 180^\circ n$$

On the other hand, we already saw that the sum of the interior angles was:

$$\begin{aligned}(\text{Sum of Interior Angles}) &= 180^\circ(n-2)\\ &= 180^\circ n - 360^\circ \ (\text{using the Distributive Property})\end{aligned}$$

Putting these together we have:

$$\begin{aligned}180^\circ n &= (\text{Sum of Exterior Angles}) + (\text{Sum of Interior Angles})\\ &= (180^\circ n - 360^\circ) + (\text{Sum of Exterior Angles})\end{aligned}$$

Subtract $(180°n - 360°)$ from both sides and:

$$360° = (\text{Sum of Exterior Angles})$$

Reading Check

1. *True or False:*

In a polygon, an interior angle and one of the exterior angles at the same vertex are supplementary.

2. *True or False: Supplementary angles add up to* $90°$.

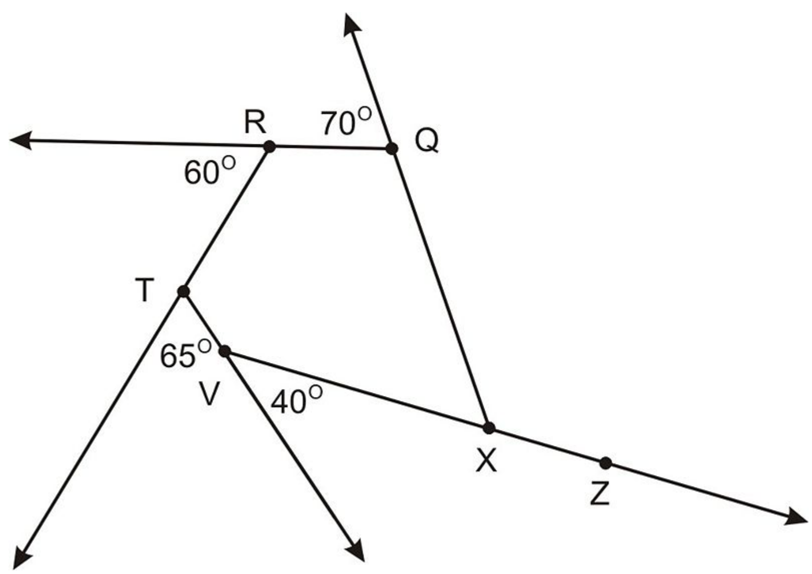
3. *Fill in the blanks:*

The sum of all interior __________________ *in a polygon with n sides is equal to* _______$° \cdot (n-2)$.

4. *What is the rule for the sum of exterior angles in a polygon? Describe.*

Example 2

What is $m\angle QXZ$ *in the diagram below?*



$\angle QXZ$ in the diagram is an **exterior angle**. So, we need to find the measure of one exterior angle on a polygon given the measures of all of the others.

4 of the 5 exterior angles on this polygon have their measurements labeled:

______°, ______°, ______°, and ______°

We know that the *sum* of the exterior angles on a polygon must be equal to 360°, regardless of how many sides the shape has. So, we can set up an equation where we set all of the exterior angles shown (including $m\angle QXZ$) summed and equal to 360°:

$$\begin{aligned} 70^\circ + 60^\circ + 65^\circ + 40^\circ + m\angle QXZ &= 360^\circ \\ 235^\circ + m\angle QXZ &= 360^\circ \\ -235^\circ \qquad\qquad &\quad -235^\circ \\ m\angle QXZ &= 125^\circ \end{aligned}$$

The measure of the missing exterior angle ($\angle QXZ$) is 125°.

We can check that our answer is reasonable by inspecting the diagram and checking whether the angle in question is acute, right, or obtuse. Since the angle should be obtuse, 125° is a reasonable answer (assuming the diagram is accurate).

Reading Check

The sum of the exterior angles of any convex polygon is equal to ________°.

6.11 Classifying a Polygon Using Sum Theorems

Graphic Organizer for Unit 6, Lesson 10: Polygons

TABLE 6.4:

Name of Polygon	**Number of Sides** $(n =)$	**Draw a Picture**	**Sum of Interior Angles** $180^\circ \cdot (n-2) =$	**Sum of Exterior Angles =**	**Measure of Each Interior Angle of a Regular** $n-$**gon =**	**Measure of Each Exterior Angle of a Regular** $n-$**gon =**
Triangle						
Quadrilateral						
Pentagon						
Hexagon						
Heptagon						
Octagon						
Nonagon						
Decagon						

CHAPTER 7 Surface Area and Volume – Nets to Prisms

Chapter Outline

7.1 Vocabulary Self-Rating

TABLE 7.1: Rating Guide: DK: I am sure I don't know it K: I am sure I know it ?: I'm not sure

Word	Before Lesson/Unit	After Lesson/Unit
Polygon		
Polyhedron		
Polyhedra		
Face		
Edge		
Vertex *or* Vertices		
Prism		
Pyramid		
Euler's Formula		
Regular		
Convex		
Semi-regular		
Platonic solids		
Net		
Base		
Right prism		
Oblique prism		
Surface area		
Lateral sides		
Lateral area		
Perimeter		
Apex		
Slant height		
Altitude		
Apothem		
Volume		
Cubic		
Ratio		
Similar		

7.2 Area of Nets; Nets to Prisms

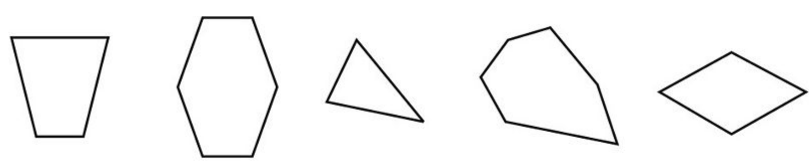
Learning Objectives

- Identify and understand the properties of polyhedra.
- Identify regular Platonic polyhedra.
- Identify, draw, and construct nets for solids.

Introduction

You have already learned that a **polygon** is a 2-dimensional figure that is made of three or more points joined together by line segments. Examples of polygons include triangles, quadrilaterals, pentagons, or octagons.

In general, an $n-$gon is a polygon with n sides. A triangle is a 3-gon or 3-sided polygon, and a pentagon is a 5-gon or 5-sided polygon. Below are examples of polygons:



polygons

You can use polygons to construct a 3-dimensional figure called a **polyhedron**. The plural of **polyhedron** is **polyhedra**.

A **polyhedron** is a 3-dimensional figure that is made up of polygon **faces**. A **face** is an outer side (or boundary) of a polyhedron. A cube is an example of a **polyhedron**, and its **faces** are squares (quadrilaterals).

Reading Check:

1. *Fill in the blank:*

A 3-dimensional figure made of polygon faces is called a ________________________.

2. *Fill in the blank:*

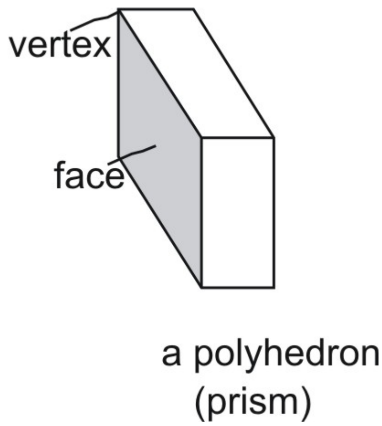
The plural form of polyhedron is ______________________________.

3. *What shape is an example of a polyhedron?*

Polyhedron or Not

A **polyhedron** has the following properties:

- It is a 3-dimensional figure.
- It is made of polygons and only polygons. Each polygon is called a **face** of the polyhedron.
- Polygon faces join together along segments called **edges**.
- Each edge joins exactly two faces.
- Edges meet in points called **vertices** (plural form of **vertex**).
- There are no gaps between edges or vertices.

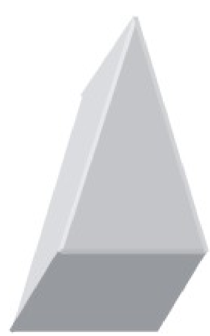


To review:

A **polyhedron** is a ____-dimensional shape that is made up of ____________________ called faces of the polyhedron. The faces are joined together at ________________ and the edges meet at ________________________.

Example 1

Is the figure a polyhedron?



Yes. A figure is a polyhedron if it has all of the properties of a polyhedron. This figure:

- Is 3-dimensional.
- Is constructed entirely of flat polygons (triangles and rectangles).
- Has **faces** that meet in **edges** and **edges** that meet in **vertices**.
- Has no gaps between edges.
- Has no non-polygon **faces** (like curves).
- Has no concave faces.

Since the figure has all of the properties of a polyhedron, it is a polyhedron.

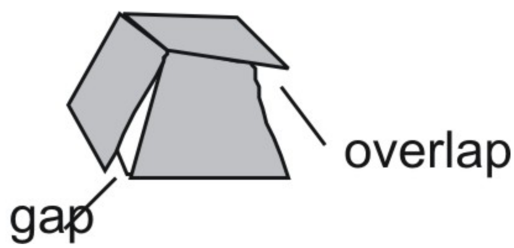
Example 2

Is the figure a polyhedron?

No. This figure has faces, edges, and vertices, but all of its surfaces are *not* flat polygons. Look at the end surface marked *A*. It is flat, but it has a curved **edge** so it is not a polygon. Surface *B* is *not* flat (or planar).

Example 3

Is the figure a polyhedron?



No. The figure is made up of polygons and it has faces, edges, and vertices.

But the faces do not fit together — the figure has gaps. The figure also has an overlap that creates a concave surface. For these reasons, the figure is *not* a polyhedron.

Reading Check:

1. *If a figure has curved edges, is it a polyhedron?* __________

2. *If a figure has gaps or overlapping faces, is it a polyhedron?* __________

3. *List 3 characteristics of polyhedra:*

Face, Vertex, Edge, Base

As you have learned, a **polyhedron** joins **faces** together along **edges**, and **edges** together at **vertices**. The following statements are *true* of any polyhedron:

- Each **edge** joins exactly two **faces**.

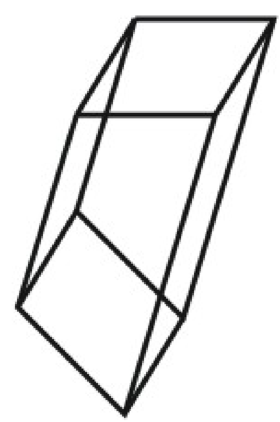
- Each **edge** connects exactly two **vertices**.

Again,

Each **edge** joins two _____________________.

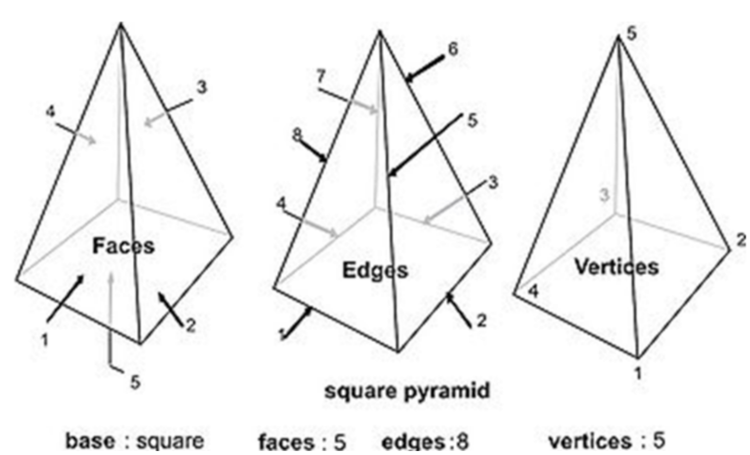
Each **edge** connects two ___________________________.

To see why this is true, take a look at this **prism**. Each of its **edges** joins two **faces** along a single line segment. Each of its **edges** connects exactly two **vertices**.

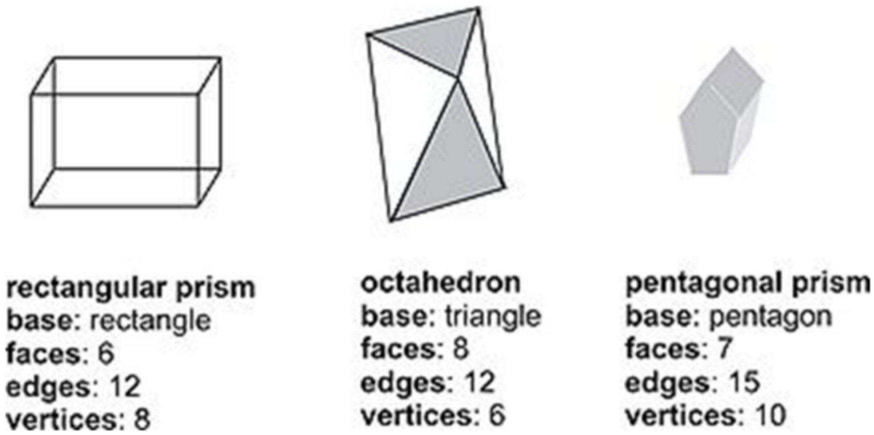


Let's count the number of **faces, edges,** and **vertices** in a few typical **polyhedra**.

The figures below are square **pyramids**. The square pyramid gets its name from its **base**, which is a square. It has 5 faces, 8 edges, and 5 vertices:



Other figures have a different number of faces, edges, and vertices:



If we make a table that summarizes the data from each of the figures we get:

TABLE 7.2:

Figure	**Vertices**	**Faces**	**Edges**
Square pyramid	5	5	8
Rectangular prism	8	6	12
Octahedron	6	8	12
Pentagonal prism	10	7	15

Calculate the *sum* of the number of **vertices** and **edges**. Then compare that sum to the number of edges. *Fill in the empty boxes with the numbers from above!*

TABLE 7.3:

Figure	Vertices	Faces	Edges	Vertices + Faces
	v	f	e	$v+f$
Square pyramid	5		8	$5+5=10$
Rectangular prism		6	12	$8+6=14$
Octahedron	6	8		$6+8=14$
Pentagonal prism	10		15	$10+7=17$

Do you see the pattern? When you add 2 to the number of edges, you get the sum of vertices and faces! The formula that summarizes this relationship is named after mathematician Leonhard Euler. **Euler's formula** says, for any polyhedron:

Euler's Formula for Polyhedra:

$$\text{Vertices} + \text{Faces} = \text{Edges} + 2$$
$$\text{or}$$
$$amp; v + f = e + 2$$

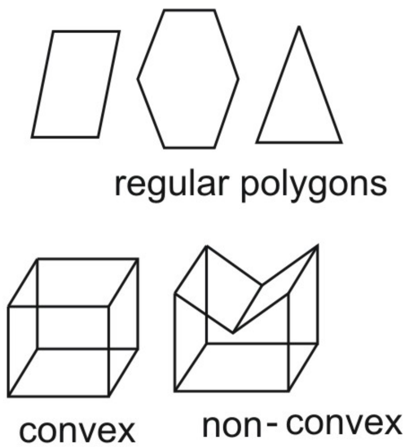
Use **Euler's formula** to find the number of **edges, faces,** or **vertices** in a **polyhedron**.

Regular Polyhedra

Polyhedra can be named and classified in a number of ways — by side, by angle, by base, by number of faces, and so on. Perhaps the most important classification is whether or not a polyhedron is **regular** or not. You will recall that a **regular polygon** is a polygon whose sides and angles are all *congruent*.

A **polyhedron** is **regular** if it has the following characteristics:

- All faces are the same.
- All faces are congruent regular polygons.
- The same number of faces meet at every vertex.
- The figure has no gaps or holes.
- The figure is **convex** — it has no indentations.



Reading Check:

1. *True/False:* A regular polyhedron has faces that are all different.

2. *Fill in the blanks:*

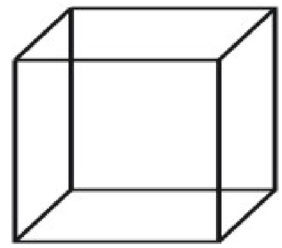
In a regular polyhedron, the same number of ____________________ meet at each ________________________.

3. *True/False:* A donut is not a regular polyhedron because it has a hole in the middle.

4. *In your own words, describe a shape that is not convex:*

Example 4

Is a cube a regular polyhedron?



- All **faces** of a cube are *regular polygons* — squares.
- The cube is **convex** because it has no indented surfaces.
- The cube is simple because it has *no gaps.*

Therefore, a cube *is* a regular polyhedron.

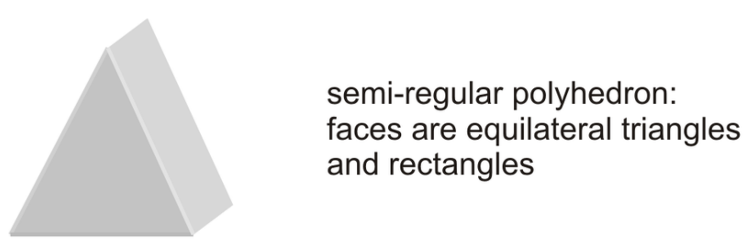
A polyhedron is **semi-regular** if all of its faces are regular polygons and the same number of faces meet at every vertex. **Semi-regular** polyhedra often have two different kinds of faces, both of which are regular polygons.

A **semi-regular polyhedron:**

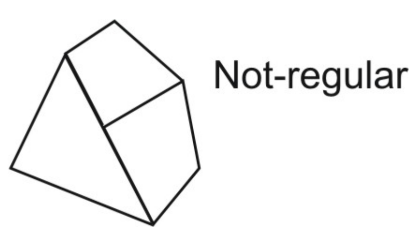
- has faces that are regular ______________________
- has the same number of ______________________ that meet at each vertex
- may have different kinds of __________________

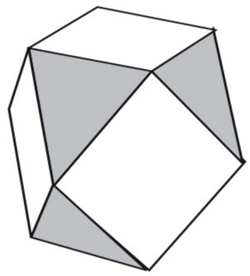
Prisms with a regular polygon base are one kind of semi-regular polyhedron.

Not all semi-regular polyhedra are prisms. An example of a non-prism is shown below:



Completely **irregular** (or not regular) polyhedra also exist. They are made of different kinds of regular and irregular polygons.





So now a question arises. Given that a polyhedron is regular if all of its faces are congruent regular polygons, it is convex and contains no gaps or holes. How many *regular* polyhedra actually exist?

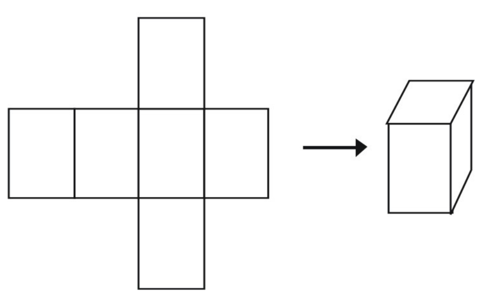
In fact, you may be surprised to learn that *only* ***5 regular polyhedra*** can be made. They are known as the **Platonic** (or noble) **solids**.

How many **Platonic solids** are there? __________

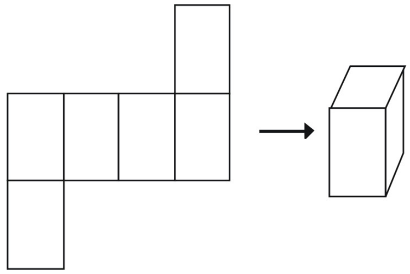
Note that no matter how you try, you can't construct any other regular polyhedra besides the ones above.

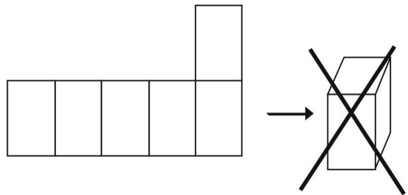
Representing Solids: Nets

One way to represent a solid is to use a **net**. A **net** is a 2-dimensional picture that can be used to create a 3-dimensional solid. If you *cut out* a **net** you can *fold it along the lines* into a 3-D model of a figure. Here is an example of a net for a cube:



There is more than one way to make a net for a single figure. Here is another for a cube:





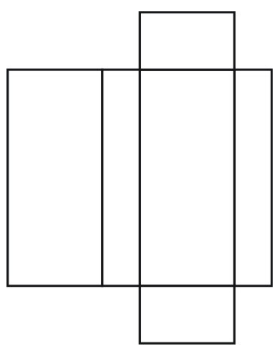
However, not all arrangements will create a cube:

Reading Check:

In your own words, what is a net?

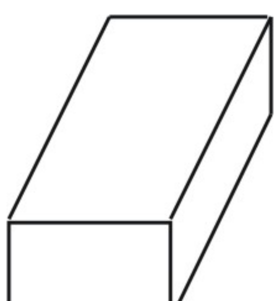
Example 5

What kind of figure does the net create? Draw the figure.



Remember, you can *cut out* the **net** and *fold it* along every line to make a 3-D figure.

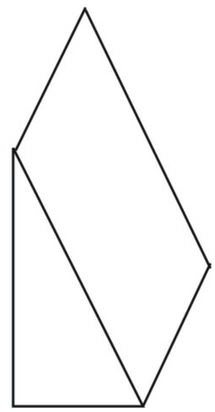
The net above creates a box-shaped rectangular prism as shown below:



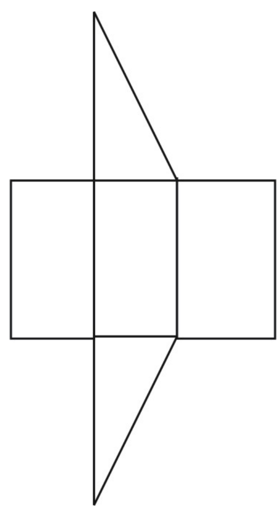
How many faces does the prism have? ________ *(Hint: if you count the number of boxes in the net, it is the same as the number of faces in the 3-D figure!)*

Example 6

What kind of net can you draw to represent the figure shown? Draw the net.



A net for the prism is shown. Other nets are also possible.



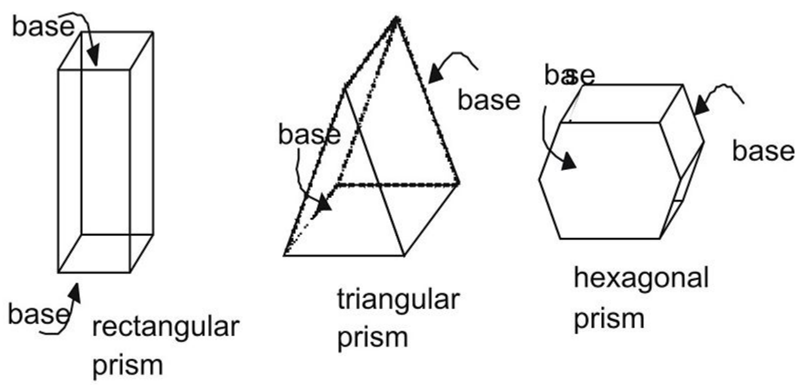
7.3 Base, Lateral and Surface Areas of Prisms

Learning Objectives

- Use nets to represent prisms.
- Find the surface area of a prism.

Prisms

A **prism** is a 3-dimensional figure with a pair of *parallel* and *congruent* ends, or **bases**. The *sides* of a prism are **parallelograms**. **Prisms** are *identified* by their **bases**.



Look at the *names* of the prisms above. Each type of prism is named after its **base**. In the figure on the left, the base is a *rectangle*, so it is called a *rectangular prism*. In the middle, the base is a *triangle*, so the shape is a *triangular prism*. On the right, the base is a *hexagon* (6-sided figure), so the prism is called a *hexagonal prism*.

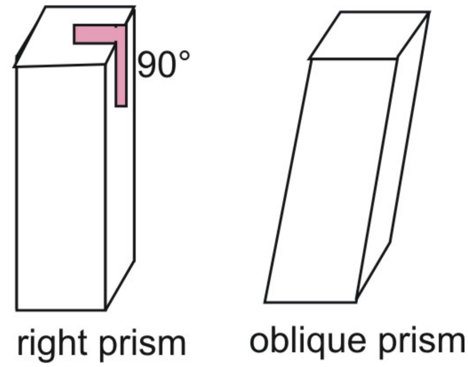
As you can see, the base is not always on the "bottom" of the prism!

Prisms are named by their ________________________.

If a prism is called a *pentagonal prism*, then its base is a ____________________________.

Surface Area of a Prism Using Nets

The **prisms** in the picture above are **right prisms**. In a **right prism**, the lateral sides are *perpendicular* to the **bases** of prism. In the picture below, compare a right prism to an **oblique prism**, in which sides and bases are *not perpendicular*:



Reading Check:

1. *Fill in the blanks:*

A **prism** is a _____-dimensional figure with parallel and congruent ________________ and sides that are ______-____________________________.

2. *In a right prism, what is the relationship between the sides and the bases?*

3. *What is the difference between a right prism and an oblique prism?*

Surface Area

Surface area is the exposed area of a 3-dimensional or solid figure. This means that **surface area** is a measurement of the *outside* area of an object. For example, the surface area of a soccer ball is the outside part of the ball that touches the air. The inside of the ball is not included in surface area. **Surface area** is, as its name describes, the *area* of the *surface* of a solid object.

Surface area is the area of the _________________________ of a 3-dimensional object.

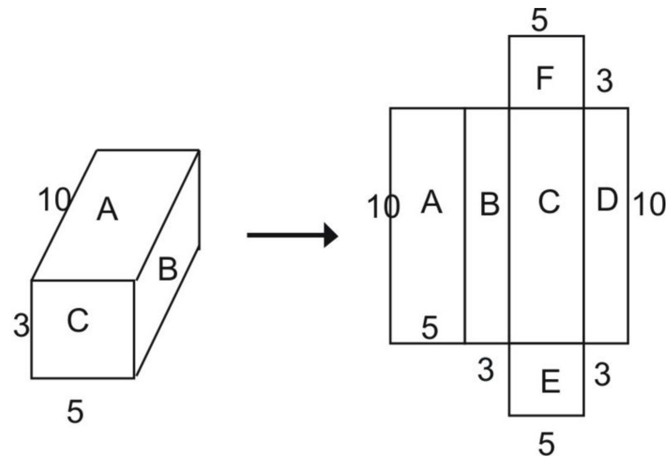
Area Addition Postulate

The **surface area** of a 3-dimensional figure is the sum of the areas of all of its non-overlapping parts.

This means that we find the **surface area** by adding up all of the face areas of the figure. Since the **faces** are on the *surface* of the object, if we add up all of the faces, we will have the entire surface area!

To find the **surface area** of a shape, you add the areas of its _______________________.

You can use a **net** and the **Area Addition Postulate** to find the **surface area** of a **right prism**:



From the **net** of the **prism**, you can see that that the **surface area** of the entire prism equals the *sum* of the shapes that make up the net. Since there are 6 faces of the prism and 6 shapes in the net, there are 6 areas that you add together:

$$\text{Total surface area} = \text{area } A + \text{area } B + \text{area } C + \text{area } D + \text{area } E + \text{area } F$$

$\rightarrow$ You may notice that the faces labeled E and ____ are called the **bases** of the prism, and the faces labeled A, B, ____ , and ____ are called the **sides** of the prism. The **surface area** is the sum of the areas of the **bases** and the areas of the **sides**.

To find the areas of shapes in the **net**, we must use the formula for the area of a rectangle:

$$\text{Area} = \text{length} \cdot \text{width}$$
$$\text{Area of shape } A = 10 \cdot 5 = 50 \text{ square units}$$

Find the areas of the other rectangles in the **net**:

Area of shape $B = 3 \cdot 10 =$ _____ square units

Area of shape $C =$ _____ $\cdot$ _____ $=$ _____ square units

Area of shape $D =$ _____ $\cdot$ _____ $=$ _____ square units

Area of shape $E = 3 \cdot 5 =$ _____ square units

Area of shape $F =$ _____ $\cdot$ _____ $=$ _____ square units

Then insert these areas back into the **surface area** equation above:

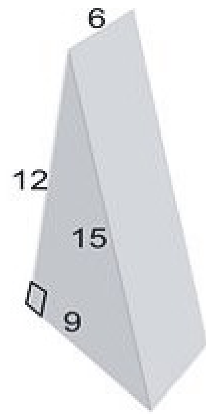
$$\text{Total surface area} = \text{area } A + \text{area } B + \text{area } C + \text{area } D + \text{area } E + \text{area } F$$
$$\text{Total surface area} = 50 + \_\_\_\_ + \_\_\_\_ + \_\_\_\_ + \_\_\_\_ + \_\_\_\_$$
$$\text{Total surface area} = 190 \text{ square units}$$

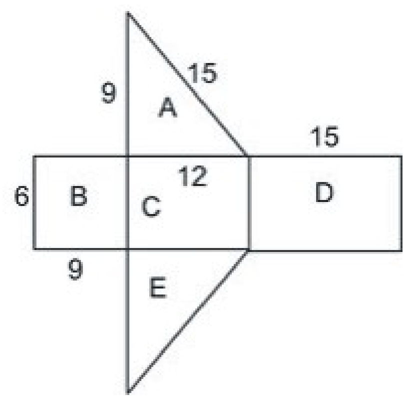
If 2 polygons (or plane figures) are congruent, then their areas are congruent.

This means that if 2 shapes are the same, then they have the same area! This may seem simple, and it is!

Example 1

Use a net to find the surface area of the prism.





The area of the **net** is equal to the **surface area** of the figure:

$$\text{Surface area} = \text{area } A + \text{area } B + \text{area } C + \text{area } D + \text{area } E$$

To find the area of the triangles (shapes A and E), we use the formula:

Area $= \frac{1}{2}\,bh$, where b is the base of the triangle and h is its height

Since triangles A and E are *congruent*, their areas are the *same*:

$$\text{Area of shape } A = \frac{1}{2}(12 \cdot 9) = \frac{1}{2}(108) = 54$$

So the area of shapes A is 54 square units *and* the area of shape E is 54 square units.

For the areas of shapes B, C, and D, we use the formula for the area of a rectangle.

$$\begin{aligned}
&\text{Area of shape } B = 6 \cdot 9 = 54 \text{ square units}\\
&\text{Area of shape } C = _____ \cdot _____ = _____ \text{ square units}\\
&\text{Area of shape } D = _____ \cdot _____ = _____ \text{ square units}
\end{aligned}$$

Now insert the areas you found back into the **surface area** equation on the previous page:

$$\text{Surface area} = \text{area } A + \text{area } B + \text{area } C + \text{area } D + \text{area } E$$
$$\text{Surface area} = 54 + 54 + \_\_\_\_ + \_\_\_\_ + 54$$
$$\text{Surface area} = 324 \text{ square units}$$

Reading Check:

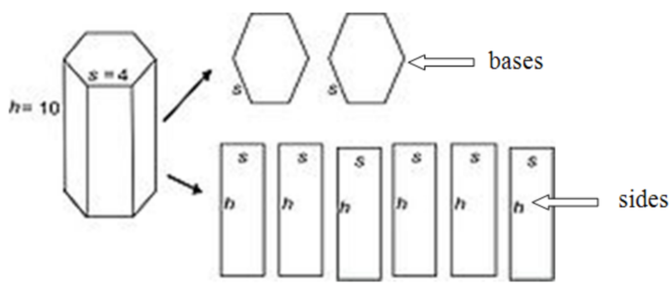
1. *True/False: The surface area of a 3-D figure is the same as the area of its net.*

2. *True/False: You can figure out the area of a net by finding the area of each of the shapes in it (rectangles, triangles, etc.) and multiplying them together.*

Surface Area of a Prism Using Perimeter

The hexagonal prism below has 2 regular hexagons for **bases** and 6 **sides**. Since all sides of the hexagon are *congruent*, all of the rectangles that make up the **lateral sides** of the 3-dimensional figure are also *congruent*.

You can break down the figure like this:



The word "lateral" means "on the side."

For example, in football, a lateral pass is when the quarterback throws the ball sideways to a receiver: the ball is passed to the side instead of forwards.

***Lateral sides** of a solid are the faces on the sides of the shape (not the bases).*

***Lateral area**, which you will learn next, is the area of the sides of the shape.*

The **surface area** of the *rectangular sides* of the figure is called the **lateral area** of the figure. The **lateral area** does *not* include the area of the **bases**. To find the **lateral area**, you can add up all of the areas of the 6 rectangles:

$$\begin{aligned}\text{Lateral area} &= 6 \cdot (\text{area of one rectangle})\\ &= 6 \cdot (s \cdot h)\\ &= 6sh\end{aligned}$$

You can also see that the **perimeter** of the **base** of the hexagonal prism on the previous page is $(s+s+s+s+s+s)$ or $6s$.

Another way to find the **lateral area** of the figure is to *multiply* the perimeter of the base by h, which is the height of the figure:

$$\begin{aligned}\text{Lateral area} &= 6sh\\ &= (6s)\cdot h\\ &= (\text{perimeter of base})\cdot h\\ &= Ph\end{aligned}$$

Substituting P, the **perimeter** of the **base**, for $6s$, we get the formula for *any* **lateral area** of a **right prism:**

$$\text{Lateral area of a prism} = Ph$$

The **lateral area** is the surface area of the rectangular ____________________ of a right prism.

The **lateral area** does not include the area of the _____________________.

Find the **lateral area** of a right prism by multiplying the ________________________ of the base by the _______-_________________ of the prism.

We can use the **lateral area** formula to calculate the *total* **surface area** of the prism. Remember, lateral area does *not* include the area of the bases, but we must include the bases to find the *total* surface area of the prism!

Using P for the **perimeter** of the base and B for the **area** of the base:

$$\begin{aligned}\text{Total surface area} &= \text{Lateral area} + \text{area of 2 bases}\\ &= (\text{perimeter of base } \cdot \text{ height}) + 2\cdot(\text{area of base})\\ &= Ph + 2B\end{aligned}$$

You can use this formula $A = Ph + 2B$ to find the **surface area** of *any* **right prism.**

Reading Check:

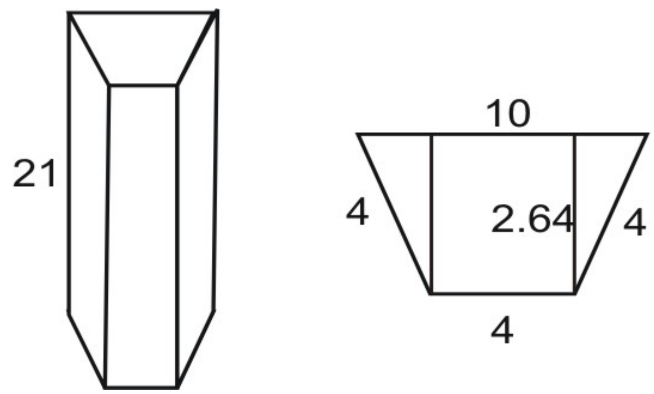
1. *True/False: The lateral area includes the area of the sides and the area of the bases.*

2. *True/False: The perimeter of the base and the area of the base are the same thing.*

3. *How could you find the total surface area of a 3-dimensional shape if you know the height of the shape and the perimeter of its base? What other information would you need to know? Explain.*

Example 2

Use the formula to find the total surface area of the trapezoidal prism below.



The dimensions of the trapezoidal base are shown in the diagram, but we should list our information so it is easy to see:

Height of the entire prism (we will call this H) $= 21$

Base 1 of the trapezoid $(b_1) = 4$

Base 2 of the trapezoid $(b_2) = 10$

height of the trapezoid (call this h so it is not confused with H above) $= 2.64$

Use the formula for **total surface area** from the previous page:

$$\text{Total surface area} = PH + 2B$$

(where P is *perimeter of the base, H is height,* and B is *area of the base*)

Find the **area** of each trapezoidal base. Do this with the formula for area of a trapezoid. Remember that the height of the trapezoid is small h: $A_B = \frac{1}{2}h(b_1 + b_2)$

$$\begin{aligned} A_B &= \frac{1}{2} \cdot 2.64(\_\_\_\_ + \_\_\_\_) \\ A_B &= 18.48 \text{ square units} \end{aligned}$$

Now find the **perimeter** of the base:

$$\begin{aligned} P &= 10 + 4 + 4 + 4 \\ &= 22 \end{aligned}$$

Use these values to find the total surface area of the solid:

$$\begin{aligned} \text{Total surface area} &= PH + 2B \\ &= (\_\_\_\_\_)(21) + 2(\_\_\_\_\_) \\ &= 462 + 36.96 \\ &= 498.96 \text{ square units} \end{aligned}$$

7.4 Base, Lateral and Surface Areas of Pyramids

Learning Objectives

- Identify pyramids.
- Find the surface area of a pyramid using a net or a formula.

Pyramids

A **pyramid** is a 3-dimensional figure with a *single* base and 3 or more *non-parallel* sides that meet at a *single point* above the base called the **apex**. The sides of a pyramid are **triangles**.

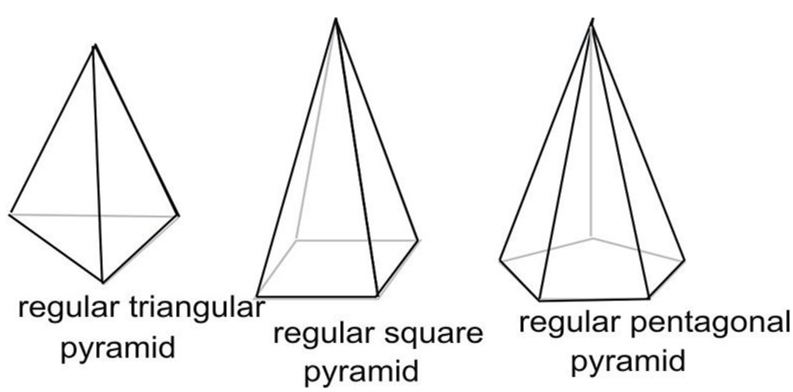
A **pyramid** has only one ____________________________, while a **prism** has 2.

The sides of a **pyramid** are ______________________________.

A **pyramid** has 3 or more _________________________.

What is the name of the point on the very top of the pyramid? _______________

Below are examples of **pyramids**. Notice that each pyramid is named after the shape of its **base** (so the pyramid with a *square* **base** is called a *square pyramid*, and so on):



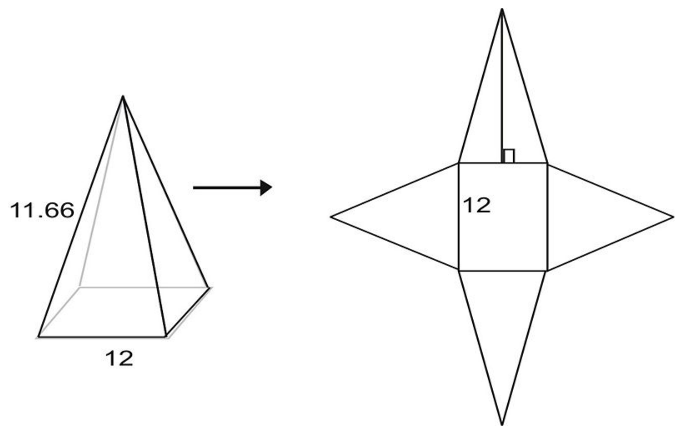
A **regular pyramid** is a pyramid that has a *regular polygon* for its **base** and whose **sides** are all *congruent triangles.*

A **regular pyramid** has sides that are ___________________________ triangles.

The base of a **regular pyramid** is a regular ________________________.

Surface Area of a Pyramid Using Nets

You can deconstruct a pyramid into a net:



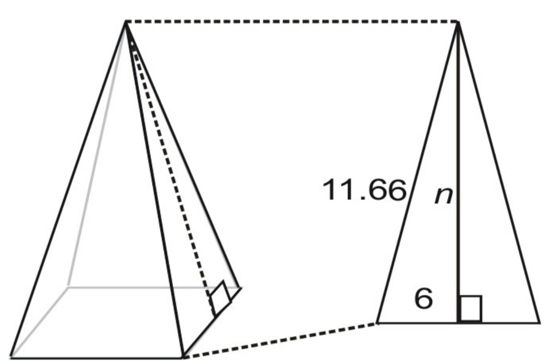
As you can see in the diagram, the easiest way to construct a net of a pyramid is to draw the base in the center, with each triangular side extending outwards from the base edges.

To find the **surface area** of the figure using the net, first find the *area* of the **base**. In this case, the base is a square:

$$\begin{aligned} A &= s^2 \\ &= (12)(12) \\ &= 144 \text{ square units} \end{aligned}$$

Now find the *area* of each **side** of the pyramid. Each side is an *isosceles triangle.* Use the Pythagorean Theorem to find the *height* of the triangles.

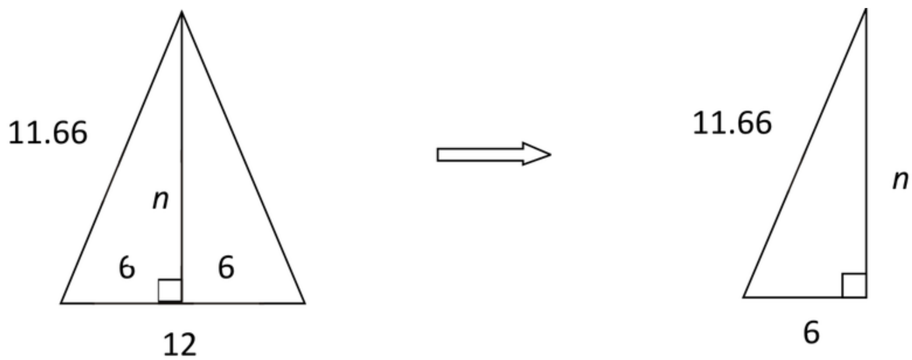
The *height of each triangle* is called the **slant height** of the pyramid. The **slant height** of the pyramid is the **altitude** of one of the triangles.



The height of the isosceles triangle side of the pyramid is labeled *n* above, so we will call the **slant height** *n* for this problem.

The **slant height** is the same as the __________________________ of the triangular side.

The triangular **side** of the pyramid is drawn again below:



We use the Pythagorean Theorem (with only half of the triangle, as shown on the right above) to find the *height* of the triangle:

$$\begin{aligned} 6^2+n^2 &= (11.66)^2 \\ 36+n^2 &= 136 \\ n^2 &= 100 \\ n &= 10 \end{aligned}$$

Now find the **area** of each *triangular side* (using the formula for area of a triangle):

$$\begin{aligned} A &= \frac{1}{2}bh \qquad \text{where } b=12 \text{ and } h=n=10 \\ &= \frac{1}{2}(12)(10) \\ &= 60 \text{ square units} \end{aligned}$$

Since there are 4 triangular sides of the pyramid:

$$\begin{aligned} \text{Area of all 4 triangles} &= 4(60) \\ &= 240 \text{ square units} \end{aligned}$$

Finally, *add* the *total area of the 4 triangles* to the *area of the base:*

$$\begin{aligned} \text{Total area} &= \text{Area of the triangular sides} + \text{Area of the base} \\ &= 240+144 \\ &= 384 \text{ square units} \end{aligned}$$

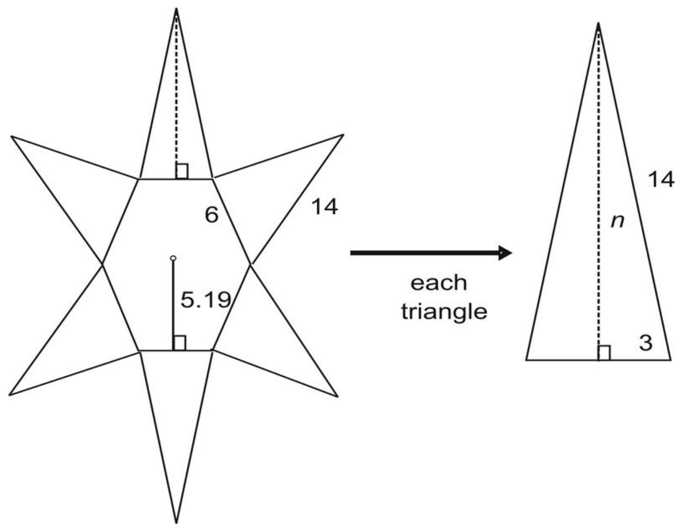
Review the steps for finding the **surface area of a regular pyramid:**

a. Find the area of the **base** of the pyramid.
b. Find the **slant height** of the triangular side of the pyramid.
c. Find the **area** of each triangular **side** and multiply it by the number of sides.

d. Add the area of the sides to the area of the base for total area!

Example 1

Use the net below to find the total area of the regular hexagonal pyramid with an apothem of 5.19. The dimensions are given in centimeters.



The **base** of the pyramid is a **hexagon**, which is a 6-sided figure. The **area** of the hexagonal base is given by the formula for the area of a regular polygon.

Since each **side** of the hexagon measures 6 cm, the perimeter is $6 \cdot 6$ or 36 cm.

The **apothem**, or perpendicular distance to the center of the hexagon, is 5.19 cm.

(*Step #1*)

$$\begin{aligned}\text{Area of base} &= \frac{1}{2}(\text{perimeter})(\text{apothem}) \\ &= \frac{1}{2}(\_\_\_\_\_)(\_\_\_\_\_) \\ &= 93.42 \text{ square cm}\end{aligned}$$

Use the Pythagorean Theorem to find the **slant height** of each **lateral triangle**

(*Step #2*)

$$\begin{aligned}3^2 + n^2 &= (14)^2 \\ 9 + n^2 &= 196 \\ n^2 &= 187 \\ n &\approx 13.67cm\end{aligned}$$

Now find the **area** of *each triangle:*

(*Step #3*)

$$
\begin{aligned}
A &= \frac{1}{2}bh \qquad \text{where } b = 6 \text{ and } h = n = 13.67 \\
&= \frac{1}{2}(\_\_\_\_\_)(\_\_\_\_\_) \\
&= 41 \ \text{ square cm}
\end{aligned}
$$

Together, the **area** of all 6 triangles that make up the **lateral sides** of the pyramid are:

$$
\begin{aligned}
\text{Area} &= 6 \cdot (\text{area of each triangle}) \\
&= 6 \cdot 41 \\
&= 246 \text{ square cm}
\end{aligned}
$$

Add the *area of the 6 lateral triangular sides* to the *area of the hexagonal base*
(*Step #4*)

$$
\begin{aligned}
\text{Total area} &= \text{Area of the triangular sides} + \text{Area of the base} \\
&= 246 + 93.42 \\
&= 339.42 \text{ square cm}
\end{aligned}
$$

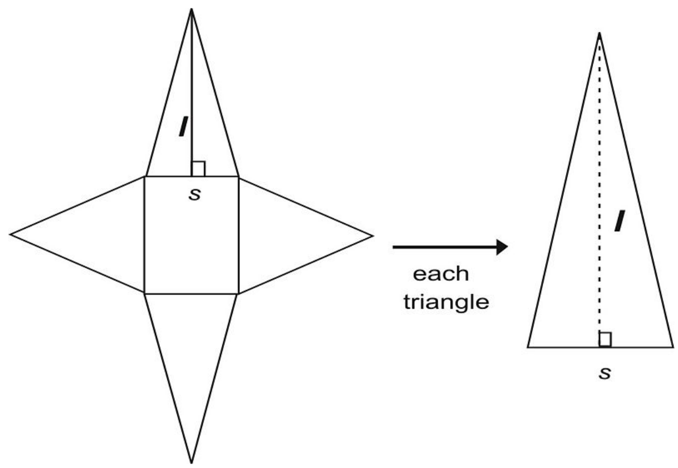
Reading Check:

1. *What is the slant height of a pyramid?*

2. *What are the 4 steps for finding the surface area of a regular pyramid? Describe them in your own words.*

Surface Area of a Regular Pyramid

To get a general formula for the **area** of a **regular pyramid**, look at the net for this square pyramid:



The **slant height** of each **lateral triangle** is labeled l (the lowercase letter L), and the side of the regular polygon base is labeled s. For each **lateral triangle**, the area is:

$$A = \frac{1}{2}sl$$

There are n triangles in a regular pyramid (for example, $n = 3$ for a triangular pyramid, $n = 4$ for a square pyramid, $n = 5$ for a pentagonal pyramid, and so on.)

The total area, L, of the lateral triangles is:

$$L = n \cdot (\text{area of each lateral triangle})$$
$$L = n \cdot \left(\frac{1}{2}sl\right)$$

If we re-arrange the above equation we get:

$$L = \frac{1}{2}nsl$$

Notice that $(n \cdot s)$ is just the perimeter, P, of the regular polygon base. So we can substitute P into our equation to get the following postulate:

$$L = \frac{1}{2}Pl$$

To get the *total* area of the pyramid, add the area of the base, B, to the equation above:

$$A = \frac{1}{2}Pl + B$$

Area of a Regular Pyramid

The **surface area** of a **regular pyramid** is:

$$A = \frac{1}{2}Pl + B$$

Where P is the **perimeter** of the regular polygon that forms the pyramid's **base**, l is the **slant height** of the pyramid, and B is the **area** of the **base**.

Example 2

A tent without a bottom has the shape of a hexagonal pyramid with a slant height l of 30 feet. The sides of the hexagonal perimeter of the figure each measure 8 feet. Find the surface area of the tent that exists above ground.

For this problem, B is zero because the tent has no bottom. So simply calculate the **lateral area** of the figure:

$$\begin{aligned}A &= \frac{1}{2}\,Pl + B\\ &= \frac{1}{2}\,Pl + 0\end{aligned}$$

Now substitute the given information to find the perimeter and the given slant height:

$$\begin{aligned}A &= \frac{1}{2}(30)(6 \cdot 8)\\ &= 720 \text{ square feet}\end{aligned}$$

In the formula for the **surface area** of a **regular pyramid:**

$$A = \frac{1}{2}\,Pl + B$$

A stands for the ______________________ _________________ of the pyramid,

P stands for the ______________________________ of the base,

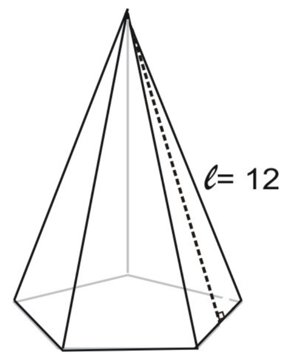
l stands for the _________________ ___________________ of the pyramid, and

B stands for the ____________________ of the __________________.

Example 3

A pentagonal pyramid has a slant height l of 12 cm. The sides of the pentagonal perimeter of the figure each measure 9 cm. The apothem of the figure is 6 cm.

Find the surface area of the figure.



If the **base** is a *pentagon* (which has 5 sides), and each side is 9 cm, then the **perimeter** of the base is $(5 \cdot 9)$ or 45 cm.

First find the **lateral area** of the figure:

$$\begin{aligned} L &= \frac{1}{2}\,Pl \\ &= \frac{1}{2}(45)(12) \\ &= 270\ cm^2 \end{aligned}$$

Now use the formula for the area of a regular polygon to find the **area** of the **base**:

$$\begin{aligned} A &= \frac{1}{2}\ \text{(perimeter)(apothem)} \\ &= \frac{1}{2}(45)(6) \\ &= 135\ cm^2 \end{aligned}$$

Finally, add these values together to find the *total* **surface area:**

$$\begin{aligned} \text{Total area} &= \text{Lateral area} + \text{Base area} \\ &= \_\_\_\_\_cm^2 + \_\_\_\_\_cm^2 \\ &= 405\ cm^2 \end{aligned}$$

Total surface area of a pyramid $=$ Lateral_____ + _____ area

7.5 Volume of Prisms

Learning Objectives

- Find the volume of a prism.

Volume of a Right Rectangular Prism

Volume is a measure of how much space a 3-dimensional figure occupies.

This means that the **volume** tells you how much a 3-dimensional figure can hold.

What does volume represent?

Volume is the space inside a 3-dimensional solid.

One way to understand volume is to compare it to surface area.

We can use real-life examples of objects to compare volume to surface area:

A fish tank:

Surface area = the glass used to build the outside of the tank

Volume = the water inside the tank

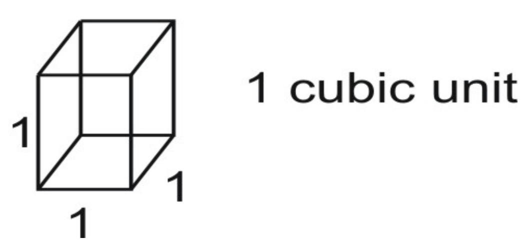
A pillow:

Surface area = the fabric used to make the pillowcase

Volume = the feathers or stuffing inside the pillow

Can you think of other examples?

The basic unit of volume is the **cubic** unit — cubic centimeter, cubic inch, cubic meter, cubic foot, and so on. Each basic cubic unit has a measure of 1 for its length, width, and height:



_______________________ is the measure of space inside a solid object.

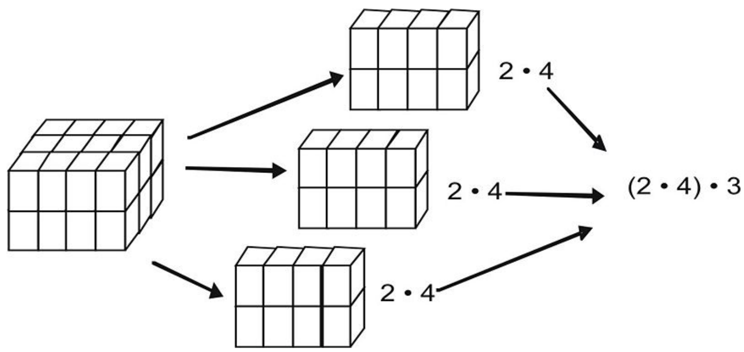
The basic unit of volume is a _________________________ unit.

In calculating volume, it is important to know that if 2 polyhedrons (or solids) are *congruent*, then their **volumes** are *congruent* also.

A **right rectangular prism** is a prism with *rectangular* **bases** and the *angle* between each base and its rectangular lateral sides is also a *right angle*. You can recognize a right rectangular prism by its "box" shape, like in the diagram below.

The **volume** of a solid is the *sum* of the volumes of all of its non-overlapping parts. Using this, we can find the volume of a **right rectangular prism** by counting boxes.

The box below measures 2 units in height, 4 units in width, and 3 units in depth. Each layer has $(2 \cdot 4)$ cubes or 8 cubes.



Together, you get 3 groups of $(2 \cdot 4)$ so the total volume is:

$$\begin{aligned} V &= 2 \cdot 4 \cdot 3 \\ &= 24 \end{aligned}$$

The volume is 24 cubic units.

This same pattern is true for *any* **right rectangular prism.**

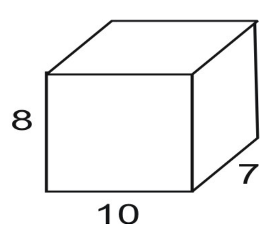
Volume is given by the formula:

$$\begin{aligned} \text{Volume} &= \text{length} \cdot \text{width} \cdot \text{height} \\ V &= l \cdot w \cdot h \end{aligned}$$

You can calculate the **volume** of any **right rectangular prism** by multiplying the ________________________ of the solid, the ________________________ , and its ________________________ .

Example 1

Find the volume of this prism:



Use the formula for **volume** of a **right rectangular prism:**

$$V = l \cdot w \cdot h$$
$$V = \_\_\_\_\_\_ \cdot \_\_\_\_\_\_ \cdot \_\_\_\_\_\_$$
$$V = 560$$

So the **volume** of this rectangular prism is 560 cubic units.

Reading Check:

1. *In your own words, what is volume?*

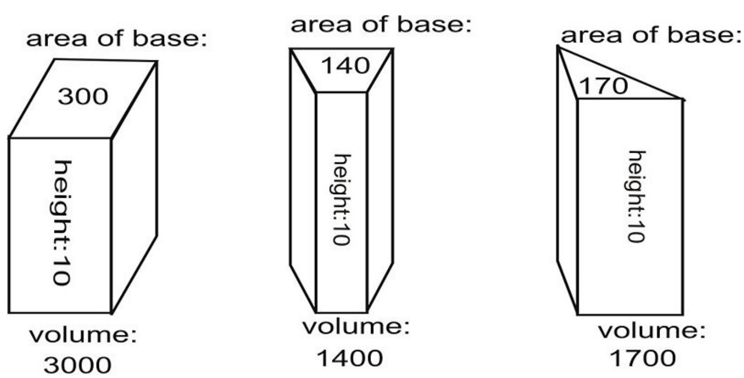
2. *True/False: An appropriate unit for the answer to a volume problem is cubic inches.*

3. *True/False: If 2 solids are congruent, then their volumes are the same.*

4. *True/False: Volume is calculated by taking the sum of the length, the width, and the height of a solid.*

Volume of a Right Prism

Looking at the **volume** of **right prisms** with the *same* height and *different* bases, you can see a pattern. The computed area of each base is given below. The height of all 3 solids is the same, 10.



Putting the data for each solid into a table, we get:

TABLE 7.4:

Solid	Height	Area of base	Volume
Rectangle	10	300	3000
Trapezoid	10	140	1400
Triangle	10	170	1700

The relationship in each case is clear: when you multiply the height of the solid by the area of its base, you get the volume. This relationship can be proven to establish the following formula for any right prism.

The **volume** of a **right prism** is:

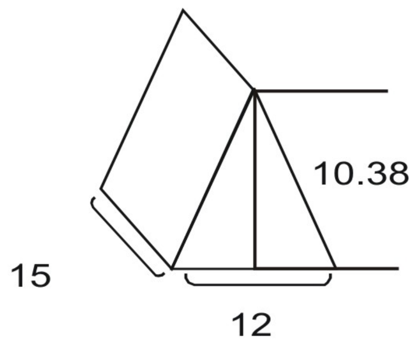
$$V = Bh$$

where B is the **area** of the **base** of the 3-dimensional figure and h is the prism's **height** (also called **altitude**)

To find the **volume** of a **right prism**, you ______________________________ the area of its ________________-________ by the __________________________ of the prism.

Example 2

Find the volume of the prism with a triangular equilateral base and the dimensions shown in centimeters.



To find the **volume**, first find the **area** of the **base**. In this diagram, the base is actually facing forwards instead of on the bottom. The **base** is an equilateral triangle as the directions say, so we use the area of a triangle formula:

$$A = \frac{1}{2}\,bh$$

The **height** (or **altitude**) of the triangle is 10.38 cm. Each side of the triangle measures 12 cm. So the triangle has the following area:

$$\begin{aligned}A &= \frac{1}{2}\,bh\\ &= \frac{1}{2}(12)(10.38)\\ &= 62.28\ cm^2\end{aligned}$$

Now use the formula for the volume of the prism, $V = Bh$, where B is the area of the **base** (the area of the triangle) and h is the **height** of the prism.

Remember that the "height" of the prism is the distance between the bases, so in this case the height of the prism is 15 cm. Imagine that the prism is lying on its side.

$$\begin{aligned}V &= Bh\\ &= (62.28)(15)\\ &= 934.2\end{aligned}$$

Thus, the volume of the prism is 934.2 cm^3 (or cubic centimeters).

7.6 Volume of Pyramids

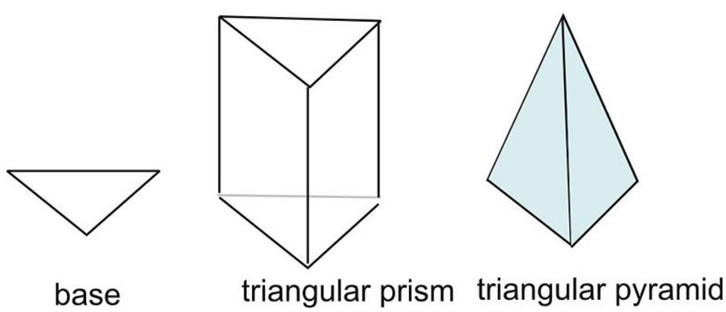
Learning Objectives

- Find the volume of a pyramid.

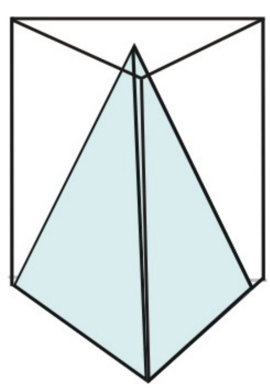
Estimate the Volume of a Pyramid and Prism

Which has a greater volume, a prism or a pyramid, if both solids have the *same* **base** and **height**? To find out, compare prisms and pyramids that have congruent bases and the same height.

On the left is a base for a triangular prism *and* a triangular pyramid. Both figures have the same height. Compare the two figures. Which one appears to have a greater volume?



The **prism** may appear to be greater in volume, but how can you *prove* that the volume of the prism is *greater* than the volume of the pyramid? You can put one figure *inside* of the other. The figure that is *smaller* will fit *inside* of the other figure.



This is shown in the diagram above. Both figures have *congruent* **bases** and the *same* **height**, so when you put the pyramid *inside* the prism, their bases overlap exactly. Since the shapes are the same height, the top of the pyramid touches the top of the prism.

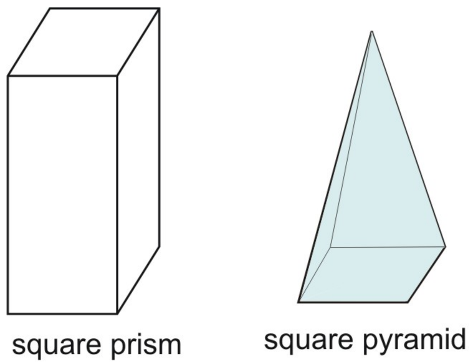
The pyramid clearly fits *inside* of the prism, so the volume of the pyramid must be smaller.

Given a **prism** and a **pyramid** with *congruent bases* and the *same height*, which has a *bigger* **volume?** __________-
____

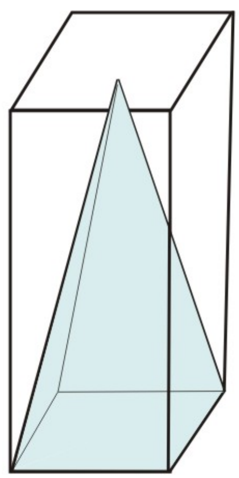
Example 1

Show that the volume of a square prism is greater than the volume of a square pyramid.

Draw or make a square prism and a square pyramid that have congruent bases and the same height.



Now place one figure *inside* of the other. The pyramid fits inside of the prism:



When two figures have the *same* **height** and the *same* **base**, the **volume** of the **prism** is *greater.*

In general, when you compare 2 figures that have *congruent* **bases** and are *equal* in **height**, the **prism** will have a *greater* **volume** than the **pyramid**.

The reason should be obvious. At the "bottom," both figures start out the same — with a square base. But the pyramid quickly slants inward, "cutting away" large amounts of material while the prism does not slant.

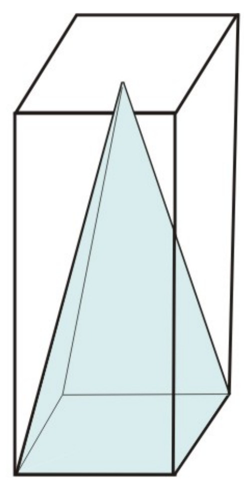
Reading Check:

1. *True/False: Prisms and pyramids with congruent bases have the same volume.*

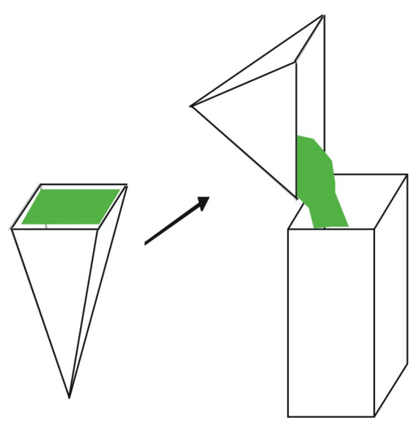
2. *True/False: A pyramid will fit inside a prism with the same base and height.*

Find the Volume of a Pyramid and Prism

Given the figure on the previous page, in which a square pyramid is placed inside of a square prism, we now ask: *how many* of these pyramids would fit inside of the prism?



To find out, obtain a square prism and square pyramid that are both hollow, both have no bottom, and both have the same height and congruent bases.



Now turn the figures upside down. Fill the pyramid with liquid. How many full **pyramids** of liquid will fill the **prism** up to the top?

In fact, it takes *exactly* 3 full pyramids to fill the prism. Since the volume of the prism is:

$$V = Bh$$

where B stands for the **area** of the **base** and h is the **height** of the prism, we can write:

$$
\begin{gathered}
3 \cdot (\text{volume of a square pyramid}) = \text{volume of a square prism} \\
\text{or :} \\
\text{volume of a square pyramid} = \frac{1}{3}\,(\text{volume of a square prism})
\end{gathered}
$$

And, since the volume of a square prism is Bh, the **volume** of a square **pyramid** is:

$$V = \frac{1}{3}Bh$$

Volume of a Pyramid

Given a **right pyramid** with a **base** that has **area** B and **height** h:

$$V = \frac{1}{3}\,Bh$$

Example 2

Find the volume of a pyramid with a right triangle base with sides that measure 5 cm, 12 cm, and 13 cm. The height of the pyramid is 15 cm.

First find the **area** of the **base**:

Since the base is a right triangle and the 3 sides measure 5 cm, 12 cm, and 13 cm, the longest side (13) must be the hypotenuse. The 2 shorter sides (5 and 12) are the legs of the right triangle. Use the leg lengths as the base and height of the triangle:

$$
\begin{aligned}
A &= \frac{1}{2}\,bh \\
&= \frac{1}{2}(5)(12) \\
&= 30 \text{ square cm}
\end{aligned}
$$

Now use the formula above for the **volume** of a **pyramid**:

$$
\begin{aligned}
V &= \frac{1}{3}\,Bh \\
&= \frac{1}{3}(30)(15) \\
&= 150 \text{ cubic cm}
\end{aligned}
$$

Reading Check:

1. *True/False: Since the volume of a prism is 3 times the volume of a pyramid, the volume of a pyramid is half the volume of a prism.*

2. *Fill in the blanks:* In the formula for the volume of a right pyramid, $V = \frac{1}{3}Bh$,

V stands for the __________________________,

B stands for the _______________________ of the ______________________ and

h stands for the __________________________ of the pyramid.

3. *If you were to fill a pyramid with liquid and pour that liquid into a prism with the same base as the pyramid, how many full pyramids would fit into the prism?*

Graphic Organizer for Unit 7

TABLE 7.5: SURFACE AREA and VOLUME of SOLIDS

Shape and Picture	**Surface Area Formula**	**What do I need to know?**	**Volume Formula**	**What do I need to know?**
Prism				
Pyramid				

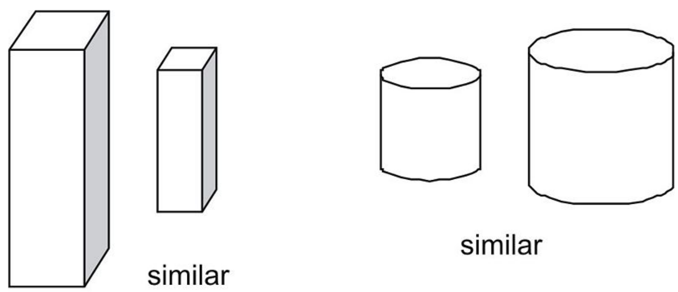
7.7 Change of Dimensions

Learning Objectives

- Determine the Surface Area and Volume of similar polyhedra.

Similar or Not Similar

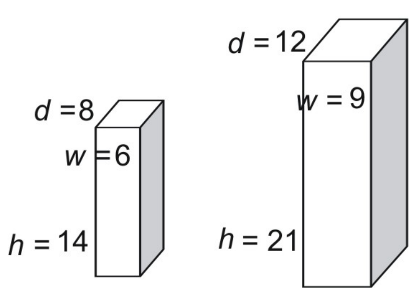
Two solids of the *same type* with **equal ratios** of corresponding linear measures are called **similar solids**. *Corresponding linear measures* means matching measurements, such as heights or widths or radii. Equal **ratios** of these measurements means that the *fraction* relating both heights is *the same as* the *fraction* relating both widths.



To be similar, figures need to have corresponding linear measures that are in proportion to one another. If these linear measures are not in proportion, the figures are not similar.

Example 1

Are these two figures similar?



If the figures are **similar**, all *ratios* for *corresponding measures* must be the *same*.

The ratios are:

$$\text{width}(w) = \frac{6}{9} = \frac{2}{3}$$
$$\text{height}(h) = \frac{14}{21} = \frac{2}{3}$$
$$\text{depth}(d) = \frac{8}{12} = \frac{2}{3}$$

Since the 3 ratios are *equal*, you can conclude that the figures *are* **similar**.

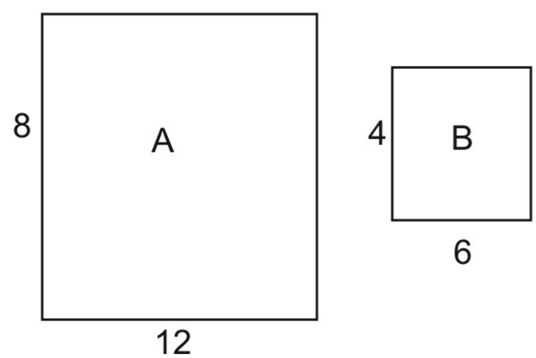
A **ratio** is the same a ____________________________.

_________________________ solids have equal ratios of corresponding linear measures.

Compare Surface Areas and Volumes of Similar Figures

When you compare **similar** 2-dimensional figures, area changes as a function of the *square* of the ratio of corresponding linear measures.

For example, take a look at the areas of these two similar figures:



The ratio between *corresponding* **sides** is:

$$\frac{length\ (A)}{length\ (B)} = \frac{12}{6} = \frac{2}{1}$$

The ratio between the **areas** of the 2 figures is the *square* of the ratio of the linear measurement:

$$\frac{area\ (A)}{area\ (B)} = \frac{12 \cdot 8}{6 \cdot 4} = \frac{96}{24} = \frac{4}{1} \text{ or } \left(\frac{2}{1}\right)^2$$

This relationship holds for solid figures as well:

The ratio of the **areas** of 2 **similar** figures is equal to the **square** of the ratio between the *corresponding* linear **sides**.

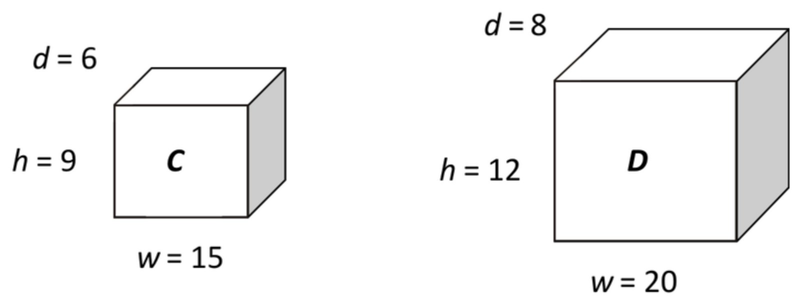
Reading Check:

1. *True/False: If 2 solid shapes are similar, then the ratios of all of their linear measurements (such as height, width, and depth) are the same.*

2. *True/False: If 2 solid shapes are similar, then the ratio of their areas is the square of the ratio of their side lengths.*

Example 2

Find the ratio of the volume between the two similar figures C and D below.



As with surface area, since the 2 figures are **similar** you can use the height, depth, or width of the figures to find the linear ratio. In this example we will use the *widths* of the 2 figures:

$$\frac{width\ (C)}{width\ (D)} = \frac{15}{20} = \frac{3}{4}$$

The ratio between the **volumes** of the 2 figures is the **cube** of the ratio of the linear measurements:

$$\frac{volume\ (C)}{volume\ (D)} = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$$

Does this cube relationship agree with the actual measurements? Compute the **volume** of each figure and compare:

$$\frac{volume\ (C)}{volume\ (D)} = \frac{6 \cdot 9 \cdot 15}{8 \cdot 12 \cdot 20} = \frac{810}{1920} = \frac{27}{64}$$

As you can see, the ratio holds. We can summarize the information in this lesson here:

If 2 solid figures A and B are **similar** and the *ratio* of their linear measurements is $\frac{a}{b}$, then the *ratio* of their **surface areas** is:

$$\frac{surface\ area\ (A)}{surface\ area\ (B)} = \left(\frac{a}{b}\right)^2$$

and the *ratio* of their **volumes** is:

$$\frac{volume\ (A)}{volume\ (B)} = \left(\frac{a}{b}\right)^3$$

Reading Check:

1. *When something is squared, what power is it raised to?*

2. *When something is cubed, what power is it raised to?*

3. *If 2 similar polyhedra have heights that are in a ratio of* $\frac{2}{5}$, *what is the ratio of their surface areas?*

4. *If 2 similar polyhedra have depths that are in a ratio of* $\frac{1}{4}$, *what is the ratio of their volumes?*

5. *True/False: The ratio of side lengths of similar solids is equal to the ratio of their surface areas, and it is also equal to the ratio of their volumes.*

CHAPTER 8 Surface Area and Volume – Cylinders, Cones, and Spheres

Chapter Outline

8.1 Vocabulary Self-Rating

TABLE 8.1: Rating Guide: DK: I am sure I don't know it K: I am sure I know it ?: I'm not sure

Word	Before Lesson/Unit	After Lesson/Unit
Circle		
Center		
Radius		
Interior		
Diameter		
Circumference		
Perimeter		
Inscribe		
Pi (π)		
Area		
Central angle		
Arc		
Semicircle		
Minor arc		
Major arc		
Chord		
Secant		
Degree measure of an arc		
Linear measure of an arc		
Arc length		
Sector		
Cylinder		
Right cylinder		
Surface area		
Lateral area		
Volume		
Cone		
Right circular cone		
Apex		
Altitude		
Sphere		
Hemisphere		
Tangent		

8.2 Circle Basics, Area, and Perimeter

Learning Objectives

- Distinguish between *radius* and *diameter* of a circle and between *interior* and *exterior* points.
- Examine inscribed polygons.
- Calculate the *circumference* of a circle.
- Calculate the *area* of a circle.

Circle, Center, Radius

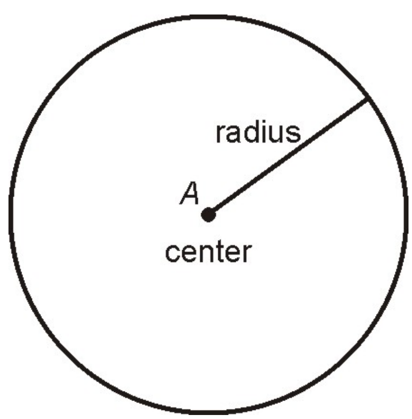
A **circle** is defined as the set of all points that are the same distance away from a specific point called the **center** of the circle. Note that the **circle** consists of only the curve but *not* of the area inside the curve. The distance from the **center** to the **circle** is called the **radius** of the circle.

The ______________________ of a **circle** is the point in the *middle* of the circle.

A _______________________ is the set of all points equidistant from the **center.**

The _______________________ is the distance from the **center** to the outer edge of the **circle.**

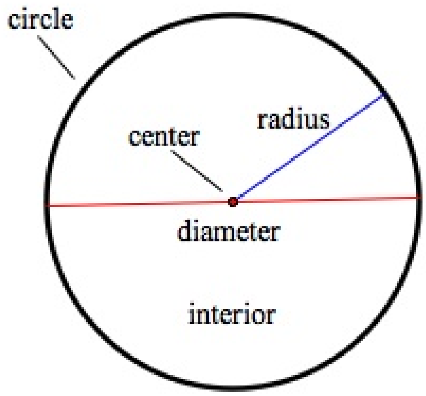
We often label the center with a capital letter and we refer to the circle by that letter. For example, the circle below is called circle A or $\odot A$.



The Parts of a Circle

A **circle** is the set of all points in a plane that are a given distance from another point called the **center**. Flat round things, like a bicycle tire, a plate, or a coin remind us of a circle.

Can you think of some other examples of **circles** *in the real world?*



The diagram above and the list below review the names for the "parts" of a circle.

- **Center:** the point in the middle of the circle
- **Circle:** the points that are a given distance from the **center** (which does *not* include the center or interior)
- The **interior:** all the points (including the center) that are *inside* the circle
- **Circumference:** the distance *around* a circle (exactly the same as **perimeter**)
- **Radius:** any segment from the **center** to a point on the **circle** (sometimes "radius" is used to mean the *length* of the segment and it is usually written as r)
- **Diameter:** any segment from a point on the circle, *through* the center, to another point on the circle (sometimes "diameter" is used to mean the *length* of the segment and it is usually written as d)

Reading Check:

Fill in the blanks using the word bank below:

diameter	**circle**	**circumference**
radius	**interior**	**center**

1. The __________________________ of a __________________________ is the point in the middle of it.

2. The ____________________________ is a special word for the perimeter of a circle.

3. The distance from the center to a point on the circle is called the __________________.

4. When you draw a line from one edge of the circle to another, and it goes through the center, it is called a ___________________________.

5. The __________________________of a circle is the space inside the entire circle.

A **diameter** is formed by two *collinear* radii. This means that when two **radius** segments fall in the same line, the line is called the **diameter.** Likewise, a radius is half the length of a diameter:

$$d = 2r \qquad \text{or} \qquad r = \frac{d}{2}$$

The **diameter** is twice (or two times) the __________________________.

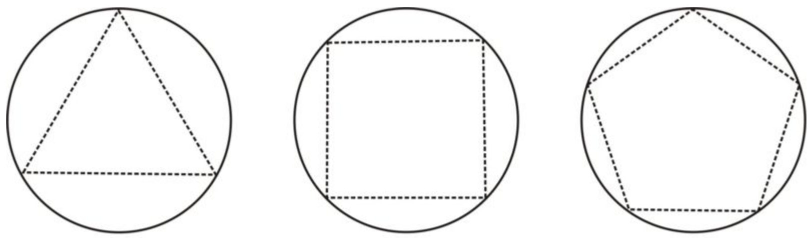
Circumference Formula

The formula for the **circumference** of a **circle** is a classic. It has been known, in rough form, for thousands of years. We will look at one way to derive this formula.

Start with a circle with a **diameter** of 1 unit. *Inscribe* a regular polygon in the circle.

→**Inscribe** means to draw the shape *inside* the circle, where each vertex touches the outer edge of the circle.

We will inscribe regular polygons with more and more sides and see what happens. For each inscribed regular polygon, the perimeter will be given in decimal form (don't worry about how we got the number):



What do you notice?

- The more sides there are, the closer the polygon is to the circle itself.
- The **perimeter** of the inscribed polygon *increases* as the number of sides increases.
- The more sides there are, the closer the **perimeter** of the polygon is to the **circumference** of the circle.

You can see that as we add more sides to the polygon inside the circle, it looks like the polygon is closer to "matching up" with the **circumference!** You will also notice that the **perimeter** number below each picture is *increasing.*

Now imagine that we continued inscribing polygons with more and more sides. It would become nearly impossible to tell the polygon from the circle. The table on the next page shows the results if we did this.

Regular Polygons Inscribed in a Circle with a Diameter of 1 unit:

TABLE 8.2:

Number of sides of polygon	*Perimeter of polygon*
3	2.598
4	2.828
5	2.939
6	3.000
8	3.062
10	3.090
20	3.129
50	3.140
100	3.141
500	3.141

As the number of sides of the inscribed regular polygon *increases,* the **perimeter** seems to approach a "limit." This limit, which is the **circumference** of the circle, is approximately 3.14159. . . This is the famous and well-known number π.

π is the Greek letter **pi** (pronounced "pie") that is often used with circles.

π is an *irrational number.* This means that it is an endlessly non-repeating decimal number.

We often use $\pi \approx 3.14$ as a value for π in calculations, but remember that this is only an approximation.

- The number π approximates to ______________________.

Conclusion: The **circumference** of a circle with **diameter** 1 is π.

Suppose a circle has a diameter of d units.

The *scale factor* of this circle (of diameter d) and the one with diameter 1, is:

$$d:1, \qquad \frac{d}{1}, \qquad \text{or just} \qquad d$$

.

You already learned how a *scale factor* affects linear measures, which include perimeter and circumference. Remember: multiply any linear measures by the scale factor. If the scale factor is d, then the **perimeter** is d times as much.

This means that if the circumference of a circle with diameter 1 is π, then we *multiply* the *scale factor,* d, by the **circumference:**

The **circumference** of a **circle** with **diameter** d is (π times d) or πd.

Circumference Formula

Let d be the **diameter** of a circle, and C be the **circumference.**

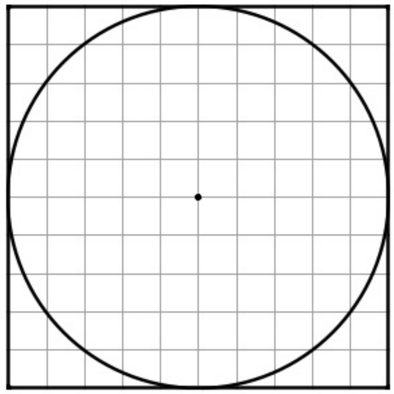
$$C = \pi d$$

Since we know that $d = 2r$, another way to write the circumference is:

$$C = 2\pi r$$

Example 1

A circle is inscribed in a square. Each side of the square is 10 cm long. What is the circumference of the circle?



You can see in the picture above that the length of a side of the square is *also* the **diameter** of the circle. Use the formula for circumference, $C = \pi d$, where $d = 10$:

$$C = \pi d$$
$$C = \pi(10) = 10\pi \approx 31.4\ cm$$

Note that there are two ways to leave your answer:

a. Sometimes we use the decimal approximation $\pi \approx 3.14$. In this example, the circumference is 31.4 cm using that approximation.
b. An exact answer can be given in terms of π (leaving the symbol π in the answer instead of multiplying it out.) Here, the exact circumference is $10\pi\ cm$.

Reading Check:

1. *True/False:* The circumference is the measurement of the outside edge (or perimeter) of a circle.

2. *True/False:* Writing the circumference as $C = \pi d$ is the same as writing it as $C = 2\pi r$ because the diameter is twice the radius.

3. *What is the difference between an approximated answer and an exact answer?*

(*Hint: In an approximated answer, we use the number ___________ for π, while in an exact answer, we ...)

Area of a Circle

The **area** of a circle is a measurement of the space in the **interior** of the circle.

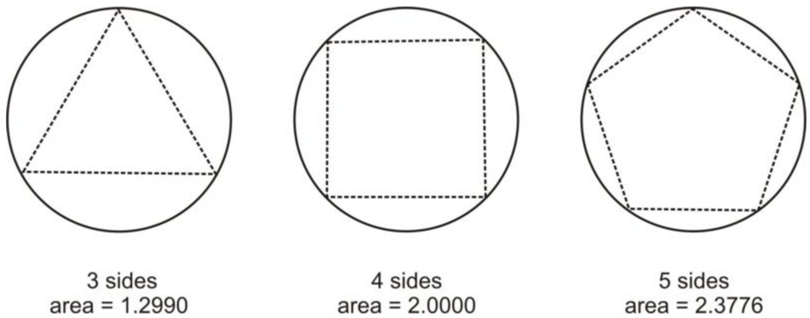
The **area** of a circle measures the _______________________ of the circle.

The big idea:

- Find the areas of regular polygons with radius 1.
- Let the polygons have more and more sides.
- See if a limit shows up in the data.
- Use similarity to generalize the results.

The details:

Begin with polygons having 3, 4, and 5 sides, inscribed in a circle with a *radius* of 1:



Now imagine that we continued inscribing polygons with more and more sides. It would become nearly impossible to tell the polygon from the circle. The table below shows the results if we did this:

Regular Polygons Inscribed in a Circle with a Radius of 1 unit:

TABLE 8.3:

Number of sides of polygon	*Area of polygon (rounded to the nearest ten thousandth)*
3	1.2990
4	2.0000
5	2.3776
6	2.5981
8	2.8284
10	2.9389
20	3.0902
50	3.1333
100	3.1395
500	3.1415
1000	3.1416
2000	3.1416

In looking carefully at the chart on the previous page, you will notice that as the number of sides of the inscribed regular polygon *increases,* the **area** seems to approach a "limit." This limit is approximately 3.1416, which is π.

Conclusion: The **area** of a circle with **radius** 1 is π. Therefore, the **area** of a circle with **radius** r is πr^2.

Area of a Circle Formula

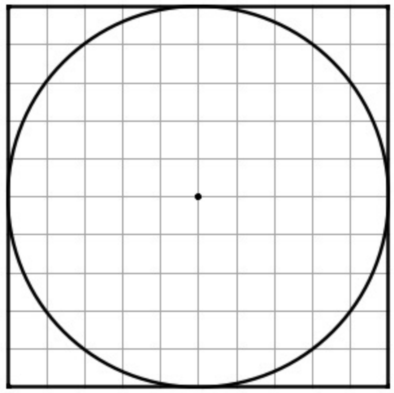
Let r be the **radius** of a circle, and A the **area**.

$$A = \pi r^2$$

You probably noticed that the reasoning about **area** here is very similar to the reasoning we used earlier in this lesson when we explored the *perimeter* of polygons and the **circumference** of circles. What sort of reasoning is this if we based our conclusions on examples? It is **inductive reasoning!**

Example 2

A circle is inscribed in a square. Each side of the square is 10 cm long. What is the area of the circle?



Like in Example 1, the length of a side of the square is *also* the **diameter** of the circle.

Use the formula for area of a circle $A = \pi r^2$, where the radius r is 5 cm:

$$A = \pi r^2$$
$$A = \pi(5^2) = 25\pi \approx 78.5$$

The **area** of the circle is exactly $25\pi\ cm^2$ or approximately $78.5\ cm^2$.

Reading Check

1. *True/False: The area of a circle is the same as the circumference.*

2. *Why did you answer true or false? Explain.*

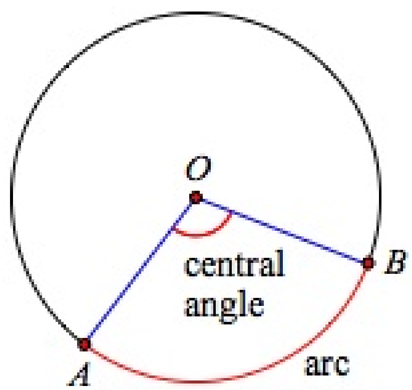
8.3 Arcs, Semi-Circles, and Central Angles

Learning Objectives

- Measure central angles.
- Measure *arcs* of circles.
- Find relationships between minor arcs, semicircles, and major arcs.

Arc, Central Angle

In a circle, the **central angle** is formed by two radii of the circle with its *vertex* at the *center* of the circle. An **arc** is a section of the circle.



A circle's central angle is an angle with a vertex at the ______________________ of the circle.

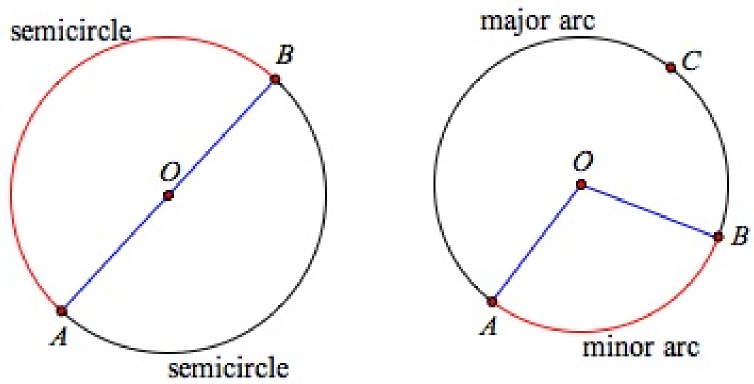
Each *side* of a central angle is also a ________________________ of the circle.

An ________________ is a section of the outer edge of the circle.

Minor and Major Arcs, Semicircle

A **semicircle** is half of a circle.

A **major arc** is *longer* than a semicircle and a **minor arc** is *shorter* than a semicircle.



A ______________________________ is half of a circle.

A __________________ arc is *longer* than a **semicircle** and a __________________ arc is *shorter* than a **semicircle.**

An **arc** can be measured in *degrees* or in a linear measure (cm, ft, etc.). In this lesson we will concentrate on *degree* measure.

The measure of the **minor arc** is the *same* as the measure of the **central angle** that corresponds to it.

This means that the *measure* of the **arc** in between the sides of the **central angle** (from point A to point B below) is the *same* degree measure as the **central angle,** like in the picture below:

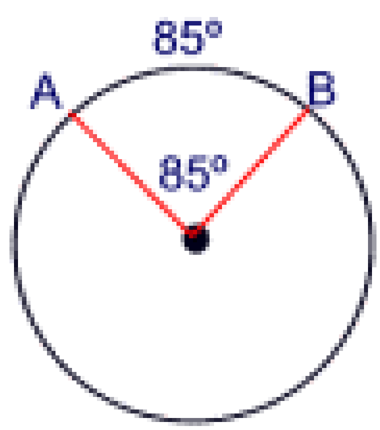


image from http://www.regentsprep.org/Regents/math/geometry/GP15/CircleArcs.htm

The measure of the **major arc** is equal to 360° *minus* the measure of the **minor arc.**

In the example above, the *larger* **arc** from point A to point B (in a counter-clockwise direction) is $360^\circ - 85^\circ = 275^\circ$. **Minor arcs** are named with *two* letters—the letters that denote the *endpoints* of the arc. Below, the **minor arc** corresponding to the **central angle** $\angle AOC$ is called $\widehat{AC}$.

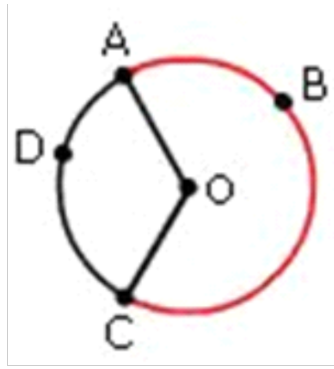


image from http://www.icoachmath.com/SiteMap/MajorArc.html

In order to prevent confusion, **semicircles** and **major arcs** are named with *three* letters—the letters that denote the *endpoints* of the arc and *any other point on* the **major arc.** In the figure above, the **major arc** corresponding to the central angle $\angle AOC$ is called $\widehat{ABC}$ because point B is *on* the **major arc.**

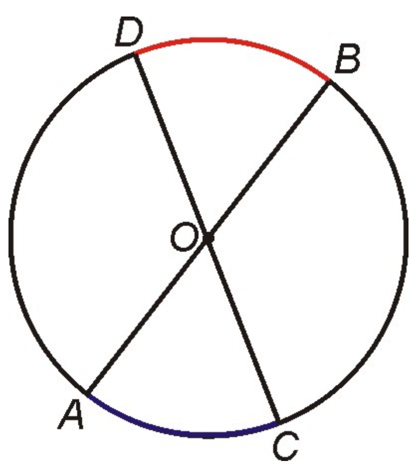
Minor arcs are named with ________ letters, which represent the __________________ of the arc.

Major arcs are named with ________ letters, which represent the endpoints of the arc *and* a _________________ *on* the major arc.

Congruent Arcs

Two **arcs** that correspond to *congruent* **central angles** will also be *congruent.*

In the figure below, $\angle AOC \cong \angle BOD$ because they are **vertical angles.** This also means that $\widehat{AC} \cong \widehat{DB}$:

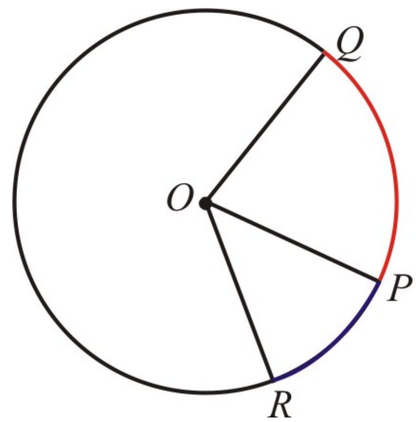


If central angles are *congruent*, their corresponding arcs are also __________________.

Arc Addition Postulate

The measure of the arc formed by two adjacent arcs (or two arcs next to each other) is the *sum* of the measures of the two arcs.

In other words, $m\widehat{RQ} = m\widehat{RP} + m\widehat{PQ}$:



Reading Check:

1. *True/False:* A central angle in a circle creates two arcs along its circumference: the minor arc is the smaller arc and the major arc is the larger arc.

2. *True/False:* If two central angles in a circle are congruent, then their corresponding arcs are supplementary.

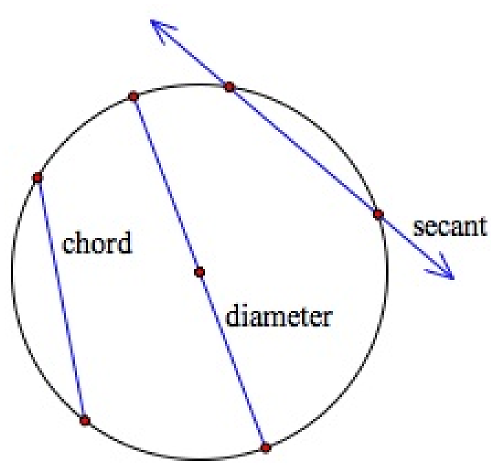
3. *For question #2 above, give an example that proves why this statement is either true or false. Explain your reasoning in words.*

Chord, Diameter, Secant

A **chord** is defined as a line segment starting at one point on the circle and ending at another point on the circle.

A **chord** that goes through the **center** of the circle is called the **diameter** of the circle. Notice that the **diameter** is *twice* as long as the **radius** of the circle.

A **secant** is a line that cuts through the circle and continues infinitely in both directions.

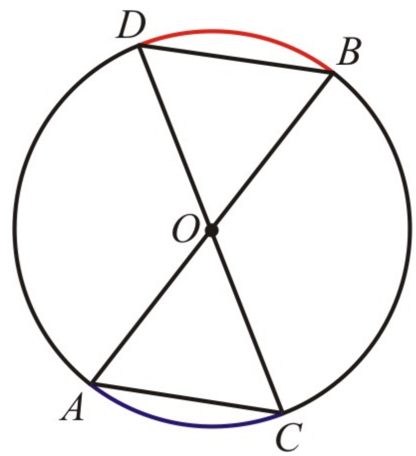


- A line segment from one point on a circle to another point on the circle is called a ______________________.

- When a chord goes through the center of a circle, it is a ___________________.
- A chord that is a *line* extending in both directions is a _____________________.

Congruent Chords Have Congruent Minor Arcs

In the same circle or congruent circles, congruent chords have congruent minor arcs.



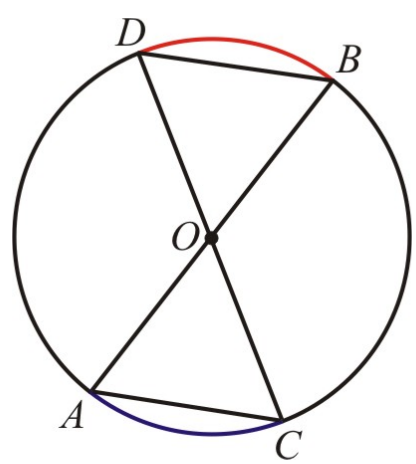
Proof. Draw the diagram above, where the chords $\overline{DB}$ and $\overline{AC}$ are congruent. Construct ΔDOB and ΔAOC by drawing the 4 radii from the center O to points A, B, C, and D.

Then, $\Delta AOC \cong \Delta BOD$ by the SSS Postulate.

This means that central angles $\angle AOC \cong \angle BOD$ (by CPCTC), which leads to the conclusion that $\widehat{AC} \cong \widehat{DB}$.

Congruent Minor Arcs Have Congruent Chords and Congruent Central Angles

In the same circle or congruent circles, congruent minor arcs have congruent chords and congruent central angles.



Proof. Draw the following diagram, in which $\widehat{AC} \cong \widehat{DB}$. In the diagram, $\overline{DO}, \overline{OB}, \overline{AO}$, and $\overline{OC}$ are each a radius of the circle.

Since $\widehat{AC} \cong \widehat{DB}$, this means that the corresponding **central angles** are also *congruent*:

$$\angle AOC \cong \angle BOD$$

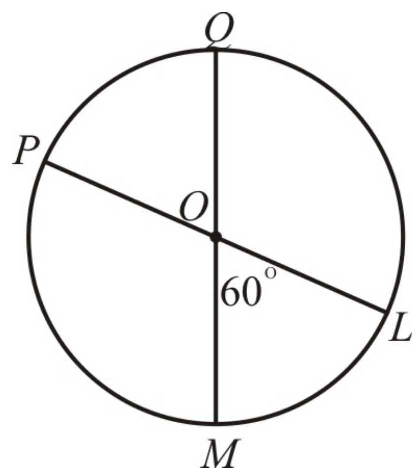
Therefore, $\Delta AOC \cong \Delta BOD$ by the SAS Postulate.

We conclude that $\overline{DB} \cong \overline{AC}$.

Here are some examples in which we apply the concepts and theorems we discussed in this lesson.

Example 1

Find the measure of each arc:



A. $m\widehat{ML}$

B. $m\widehat{PM}$

C. $m\widehat{LMQ}$

Solutions:

A. $m\widehat{ML}$ is equal to $m\angle LOM$ (the central angle) $= 60^\circ$

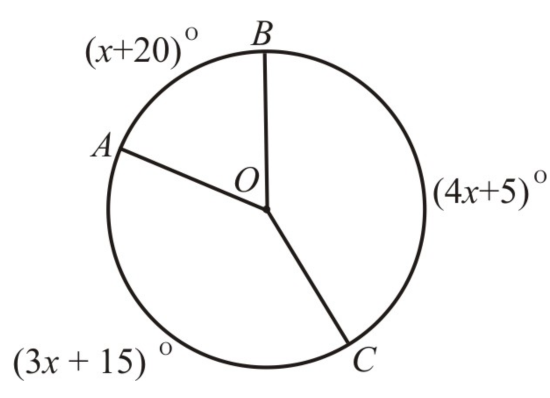
B. $m\widehat{PM}$ is equal to $m\angle POM$ (which is supplementary to the angle $\angle LOM$)

$$180^\circ - 60^\circ = 120^\circ$$

C. $m\widehat{LMQ} = m\widehat{ML} + m\widehat{PM} + m\widehat{PQ}$
$= 60^\circ + 120^\circ + 60^\circ = 240^\circ$

Example 2

Find $m\widehat{AB}$ in circle O. The measures of all three arcs shown must add to 360°.



All three arcs must add to 360° because all three central angles add to 360° (since they complete a circle.)

Fill in the arc measurements based on the picture above:

$m\angle AOB = m\widehat{AB} =$ ________________

$m\angle BOC = m\widehat{BC} =$ ________________

$m\angle AOC = m\widehat{AC} =$ ________________

We can add all three arcs together:

$$\begin{aligned} m\widehat{AB} \quad + m\widehat{BC} \quad + m\widehat{AC} \quad &= 360^\circ \\ x+20+4x+5+3x+15 &= 360 \\ 8x+40 &= 360 \\ 8x &= 320 \\ x &= 40 \end{aligned}$$

We are looking for $m\widehat{AB}$ so we substitute $x = 40$ back into the arc measurement:

$$m\widehat{AB} = x+20 = 40+20 = 60^\circ$$

Reading Check:

What other postulate is similar to the Arc Addition Postulate? Describe.

Graphic Organizer for Lessons 1 – 2: PARTS OF A CIRCLE

TABLE 8.4:

Circle Part	***Draw a Picture***	***What is it?***	***Is there a formula I need to know for this part?***
Center			
Radius			
Diameter			
Circumference			
Area			
Central Angle			
Semicircle			
Minor Arc			
Major Arc			
Chord			
Secant			

8.4 Arc Length

Learning Objectives

- Calculate the length of an arc of a circle.

Arc Length

Arcs are measured in two different ways:

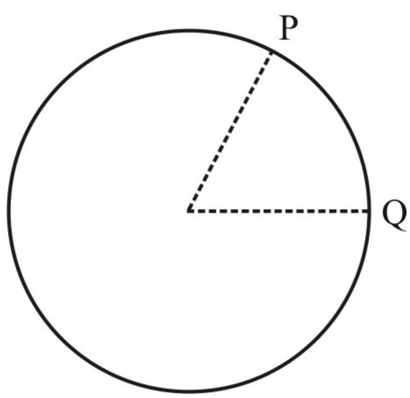
Degree measure: The degree measure of an arc is the fractional part of a 360° complete circle that the arc is.

Linear measure: This is the length, in units such as centimeters and feet, if you traveled from one end of the arc to the other end.

- Arcs can be measured in __________ ways, ______________________________ measure and ___________-________________ measure.

Example 1

Find the length of $\widehat{PQ}$ *if* $m\widehat{PQ} = 60°$. *The radius of the circle is 9 inches.*



Remember, 60° is the measure of the **central angle** associated with $m\widehat{PQ}$. This is the **degree measure** of the arc.

To find the **linear measure** of the arc, or $m\widehat{PQ}$, we use the fact that it is $\frac{60}{360} = \frac{1}{6}$ of an entire circle.

The circumference of the circle is: $C = \pi d = 2\pi r = 2\pi(9) = 18\pi$ *inches*

The length of the arc, in this case, is $\frac{1}{6}$ of the entire circumference of the circle.

The **arc length** of $\widehat{PQ}$ is: $\frac{1}{6} \cdot 18\pi = \frac{18\pi}{6} = 3\pi$ inches or 9.42 inches

In this lesson we study the second type of arc measure—the **linear measure** of an arc, or the arc's length. **Arc length** is directly related to the **degree measure** of an arc.

Suppose a circle has:

- circumference C
- diameter d
- radius r

Also, suppose an **arc** of the circle has degree measure m.

Realize that $\frac{m}{360}$ is the fractional part of the circle that the arc represents.

Arc length

$$Arc\ Length = \frac{m}{360} \cdot C = \frac{m}{360} \cdot \pi d = \frac{m}{360} \cdot 2\pi r$$

Reading Check:

1. *In your own words, describe the linear measure of an arc:*

2. *True/False:* The **degree measure** of an arc is exactly the same as the **linear measure** of an arc.

3. *How could you correct the statement in #2 above to make it true?*

4. *Why do we use the fraction* $\frac{m}{360}$ *to calculate arc length? Describe.*

8.5 Sector Area

Learning Objectives

- Calculate the area of a *sector.*

Area of a Sector

A **sector** is a section of a circle. Think of it like a pie slice: it is a section bounded on two sides by radii and one side by an arc. The tip of a sector is at the center of the circle:

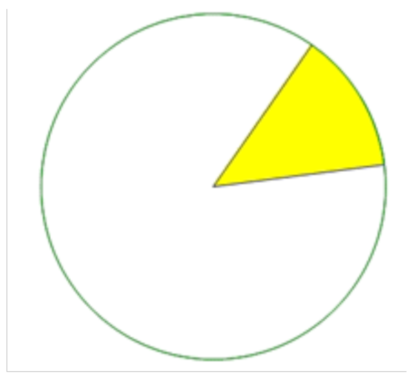


image from http://en.wikibooks.org/wiki/Geometry/Circles/Sectors

- A **sector** is just like a slice of ________________ or pizza.

The *area* of a **sector** is simply an appropriate fractional part of the area of the circle.

Suppose a **sector** of a circle with radius r and circumference C has an arc with a degree measure of m° and an arc length of s units.

- The sector is $\frac{m}{360}$ of the circle.
- The sector is also $\frac{s}{C} = \frac{s}{2\pi r}$ of the circle.

To find the *area* of the **sector,** just find one of these fractional parts of the *area* of the *circle.* We know that the *area* of the *circle* is πr^2. Let A be the *area* of the **sector:**

$$A = \frac{m}{360} \cdot \pi r^2$$

Also,

$$A = \frac{s}{C} \cdot \pi r^2 = \frac{s}{2\pi r} \cdot \pi r^2 = \frac{1}{2}sr$$

Area of a Sector

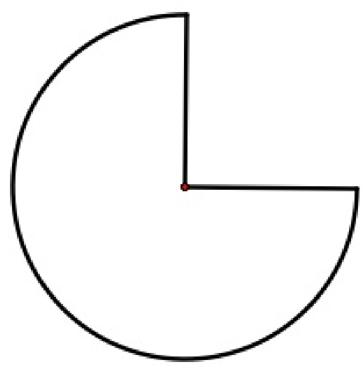
A circle has radius r. A sector of the circle has an arc with degree measure m° and arc length s units. The area of the sector is A square units:

$$A = \frac{m}{360} \cdot \pi r^2 = \frac{1}{2}sr$$

This means that the *area* of a **sector** is one-half of the product of the _________________ and the arc length.

Example 1

Mark drew a sheet metal pattern made up of a circle with a sector cut out. The pattern is made from an arc of a circle and two perpendicular 6 inch radii like so:



How much sheet metal does Mark need for the pattern?

The cut-out **sector** has a **degree measure** of 90° because the radii are *perpendicular*.

The measure of the arc of the metal piece is (the entire circle) – (the cut-out sector) or :

$$360^\circ - 90^\circ = 270^\circ$$

Using the values radius $r =$ __________ and arc degree measure $m =$ _____$^\circ$,

The *area* of the sector

$$\begin{aligned} A &= \frac{m}{360} \cdot \pi r^2 \\ &= \frac{270}{360} \cdot \pi(6)^2 = \frac{3}{4} \cdot 36\pi = 27\pi \text{ square inches} \\ &\approx 84.8\ in^2 \end{aligned}$$

<u>**Reading Check:**</u>

Explain the following statement in your own words:

The area of a sector is a fraction of the area of a circle, and the fraction is calculated by the degree measure of the sector divided by 360°.

8.6 Cylinder: Base Area, Lateral Area, Surface Area and Volume

Learning Objectives

- Find the base area, lateral area, and total surface area of *cylinders.*
- Find the volume of cylinders.

Cylinders

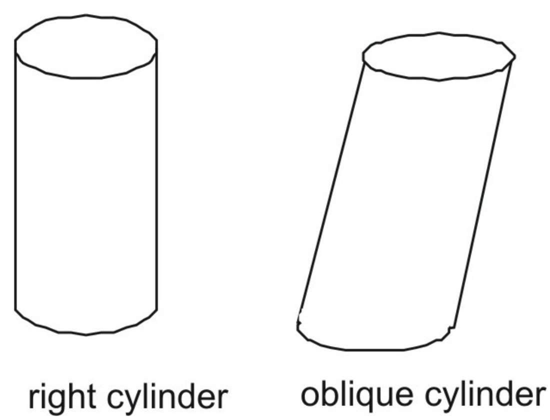
A **cylinder** is a three-dimensional figure with a pair of *parallel* and *congruent* circular ends, or bases. A cylinder has a single curved side that forms a rectangle when laid out flat.

- A **cylinder** has bases in the shape of ___________________________.
- When flattened, the side of a **cylinder** is a _______________________________.

As with prisms, cylinders can be *right* or *oblique.*

The side of a **right cylinder** is *perpendicular* to its circular bases.

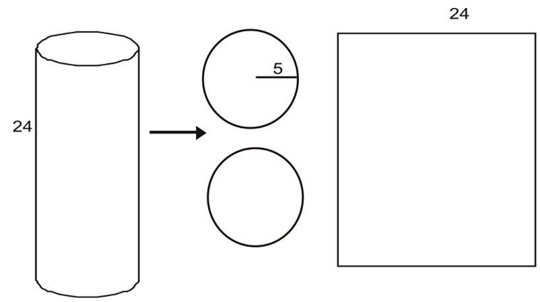
The side of an **oblique cylinder** is *not* perpendicular to its bases.



- **Right cylinders** have a side that is _____________________________to its bases. (In this lesson, we will focus on **right cylinders** only.)

Surface Area of a Cylinder Using Nets

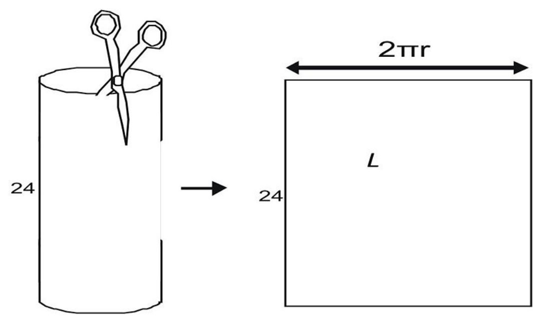
You can deconstruct a cylinder into a net:



The area of each **base** is given by the area of a **circle:**

$$\begin{aligned}A &= \pi r^2 \\ &= \pi(5)^2 \\ &= 25\pi \quad \text{or} \\ &\approx (25)(3.14) = 78.5\end{aligned}$$

The area of the rectangular lateral area L is given by the *product* of a *width* and *height.* The height is given as 24. You can see that the *width* of the area is equal to the **circumference** of the circular **base:**



- The *width* of the lateral area is the same as the _________________ of the circular base.

To find the *width,* imagine taking a can-like cylinder apart with a scissors. When you cut the lateral area, you see that it is equal to the circumference of the can's top. The **circumference** of a circle is given by $C = 2\pi r$ so the lateral area L is $L = 2\pi rh$:

$$\begin{aligned}L &= 2\pi rh \\ &= 2\pi(5)(24) \\ &= 240\pi \quad \text{or} \\ &\approx (240)(3.14) = 753.6\end{aligned}$$

Now we can find the area of the *entire* cylinder using:

Area = (area of both bases) + (area of lateral side)

Using exact numbers (with π in them), we get:

$$\begin{aligned} A &= 2(25\pi) + 240\pi \\ &= 50\pi + 240\pi \\ &= 290\pi \end{aligned}$$

Using decimal approximations, we get:

$$\begin{aligned} A &= 2(78.5) + 753.6 \\ &= 157 + 753.6 \\ &= 910.6 \end{aligned}$$

The formula we used to find the *total* **surface area** can be used for *any* **right cylinder.**

Area of a Right Cylinder

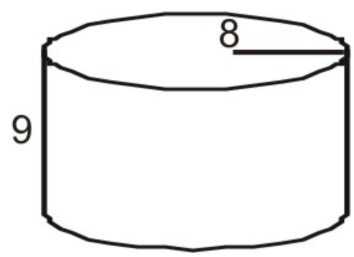
The **surface area** of a **right cylinder,** with radius r and height h is given by

$$A = 2B + L$$

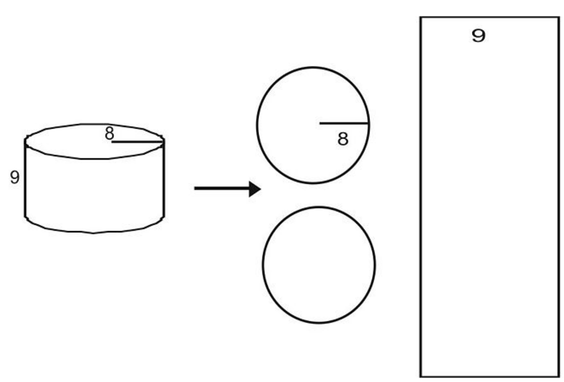
where B is the area of each base of the cylinder and L is the lateral area of the cylinder.

Example 1

Use a net to find the surface area of the cylinder:



First draw and label a net for the figure to help you visualize the pieces of the cylinder:



Calculate the *area* of each **base:**

$$\begin{aligned} A &= \pi r^2 \\ &= \pi(8)^2 \\ &= 64\pi \quad \text{or} \quad \approx (64)(3.14) = 200.96 \end{aligned}$$

Calculate the **lateral area** L:

$$\begin{aligned} L &= 2\pi rh \\ &= 2\pi(8)(9) \\ &= 144\pi \quad \text{or} \quad \approx (144)(3.14) = 452.16 \end{aligned}$$

Find the **area** of the *entire* cylinder:

$$\text{Area} = (\text{area of both bases}) + (\text{area of lateral side})$$

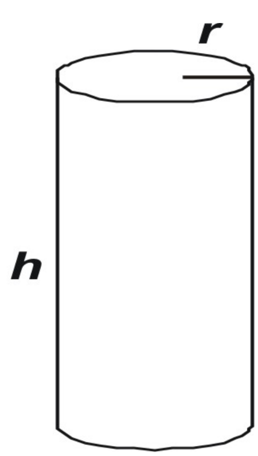
$$\begin{aligned} A &= 2(64\pi) + 144\pi \\ &= 128\pi + 144\pi \\ &= 272\pi \end{aligned} \qquad or \qquad \begin{aligned} A &= 2(200.96) + 452.16 \\ &= 401.92 + 452.16 \\ &= 854.08 \end{aligned}$$

Thus, the total surface area is $272\pi\ units^2$ or approximately 854.08 square units.

Surface Area of a Cylinder Using a Formula

You have seen how to use nets to find the total surface area of a cylinder. The postulate can be broken down to create a general formula for all right cylinders.

$$A = 2B + L$$



Notice that the base B of any cylinder is:

$$B = \pi r^2$$

The lateral area L for any cylinder is:

$$\begin{aligned} L &= \text{width of lateral area} \cdot \text{height of cylinder} \\ &= \text{circumference of base} \cdot \text{height of cylinder} \\ &= 2\pi rh \end{aligned}$$

Putting the two equations together we get:

Area = 2 (area of the ______________________) + ______________________ area

$$\begin{aligned} A &= 2B + L \\ &= 2(\pi r^2) + 2\pi rh \end{aligned}$$

Factoring out $2\pi r$ from the equation gives:

$$A = 2\pi r(r + h)$$

The Surface Area of a Right Cylinder

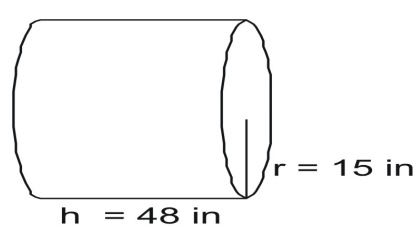
A **right cylinder** with radius r and height h can be expressed as:

$$A = 2\pi r^2 + 2\pi rh \qquad or \qquad A = 2\pi r(r + h)$$

You can use the formulas to find the **area** of any **right cylinder.**

Example 2

Use the formula to find the surface area of the cylinder:



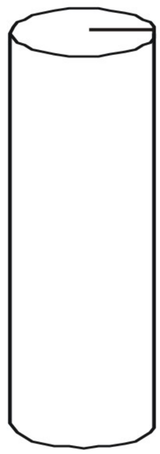
Write the formula, substitute in the values given above, and then solve:

$$\begin{aligned} A &= 2\pi r^2 + 2\pi rh \\ &= 2\pi(\_\_\_\_\_\_\_\_\_^2) + 2\pi(\_\_\_\_\_\_\_\_\_)(\_\_\_\_\_\_\_\_\_) \\ &= 450\pi + 1440\pi \\ &= 1890\pi \ in^2 \approx (1890)(3.14) = 5934.6 \text{ square inches} \end{aligned}$$

The exact area of the cylinder is $1890\pi\ in^2$ and the approximate area is $5934.6\ in^2$.

Example 3

Find the surface area of the cylinder:



Write the formula, substitute in the values given above, and then solve: (This time we will try using the other formula for area)

$$\begin{aligned}A &= 2\pi r(r+h)\\ &= 2\pi(\underline{\qquad\qquad})\cdot(\underline{\qquad\qquad}+\underline{\qquad\qquad})\\ &= 1.5\pi\cdot 6.75\\ &= 10.125\pi\ in^2 \approx (10.125)(3.14) = 31.7925\ in^2\end{aligned}$$

The exact area of the cylinder is $10.125\pi\ in^2$ and the approximate area is $31.7925\ in^2$.

Example 4

Find the height of a cylinder that has radius of 4 cm *and surface area of* $72\pi\ cm^2$.

Write the formula with the given information – you will solve for h:

$$A = 2\pi r(r+h)$$

Fill in what you know (A and r):

$$72\pi = 2\pi(4)\cdot(4+h)$$

Simplify and solve for h:

$$\begin{aligned}\frac{72\pi}{8\pi} &= \frac{8\pi(4+h)}{8\pi}\\ 9 &= 4+h\\ 5 &= h\end{aligned}$$

The height of the cylinder is 5 cm.

Reading Check:

1. *What shapes are the bases of a cylinder?*

2. *When flattened, what shape is the lateral area of a cylinder?*

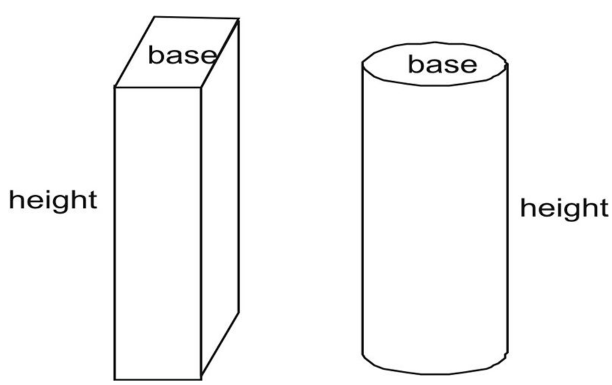
3. *Describe in words how to find the total surface area of a right cylinder. What information do you need to know? How do you use this information to find the area?*

Volume of a Right Cylinder

You have seen how to find the volume of any **right prism:**

$$V = Bh$$

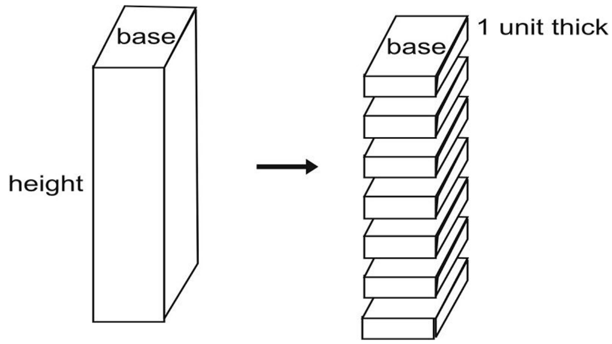
where B is the *area* of the prism's *base* and h is the *height* of the prism:



As you might guess, right prisms and right cylinders are very similar with respect to volume. A cylinder is just a "prism with round bases."

- You can think of a **cylinder** as a **prism** with bases that are __________________.

One way to develop a formula for the volume of a cylinder is to compare it to a prism. Suppose you divided the prism above into slices that were 1 unit thick:



The **volume** of each individual slice would be given by the *product* of the *area* of the *base* and the *height.* Since the height for each slice is 1, the volume of a single slice would be:

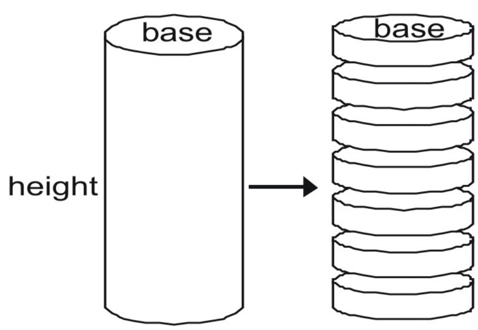
$$\begin{aligned} V(\text{single slice}) &= \text{area of base} \cdot \text{height} \\ &= B \cdot 1 \\ &= B \end{aligned}$$

Now it follows that the **volume** of the entire **prism** is equal to the *area* of the *base* multiplied by the *number* of slices. If there are h slices, then:

$$\begin{aligned} V(\text{whole prism}) &= B \cdot \text{number of slices} \\ &= Bh \end{aligned}$$

Of course, you already know this formula for **volume** of a **prism.** But now you can use the same idea to obtain a formula for the **volume** of a **cylinder.**

Look at a **cylinder** in slices:



Since the *height* of each slice of the cylinder is 1, each slice has a **volume** of $B \cdot 1$, or B. Since the *base* has an **area** of πr^2, each slice has a **volume** of πr^2 and:

$$\begin{aligned} V(\text{whole cylinder}) &= B \cdot \text{number of slices} \\ &= Bh \\ &= \pi r^2 h \end{aligned}$$

This leads to a postulate for the **volume** of any **right cylinder.**

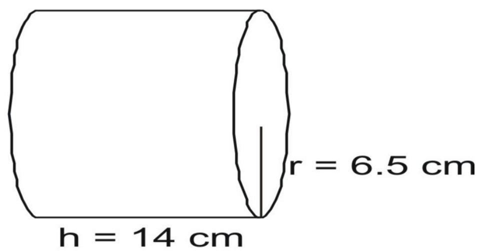
Volume of a Right Cylinder

The **volume** of a **right cylinder** with radius r and height h can be expressed as:

$$Volume = \pi r^2 h$$

Example 5

Use the postulate to find the volume of the cylinder:



Write the formula from the postulate. Then substitute in the given values and solve:

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(6.5^2)(14) \\ &= 591.5\pi \ cm^3 \text{ or} \\ &\approx (591.5)(3.14) = 1857.31 \text{ cubic cm} \end{aligned}$$

The exact volume of the given cylinder is $591.5\pi \ cm^3$ and the approximate volume is $1857.31 \ cm^3$.

Example 6

What is the radius of a cylinder with a height of 10 cm *and a volume of* $250\pi \ cm^3$?

Write the formula for volume – you will solve for r:

$$V = \pi r^2 h$$

Fill in what you know (V and h):

$$250\pi = \pi r^2 (10)$$

Simplify and solve for r:

$$\begin{aligned} \frac{250\pi}{10\pi} &= \frac{10\pi(r^2)}{10\pi} \\ 25 &= r^2 \\ 5 &= r \end{aligned}$$

The radius of the cylinder is 5 cm.

Reading Check:

1. *What is the difference between the area of a cylinder and the volume of a cylinder? Describe in your own words.*

2. *What is the difference between finding the exact value of the volume of a cylinder and the approximate volume? How will your answers be different?*

8.7 Cone: Base Area, Lateral Area, Surface Area and Volume

Learning Objectives

- Find the surface area of a *cone* using a net or a formula.
- Find the volume of a cone.

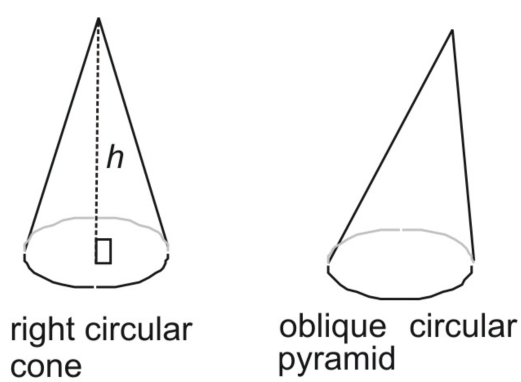
Cones

A **cone** is a three-dimensional figure with a single curved base that tapers to a single point called an **apex.** The base of a cone can be a circle or an oval of some type. In this chapter, we will only use *circular* **cones.**

- The ______________________ is the point on top of a **cone.**
- The *base* of the **cones** we will study is in the shape of a ____________________.

You can remember the name **"cone"** of this shape because it looks like an upside-down ice cream **cone.**

The **apex** of a **right cone** lies above the center of the cone's circle. In an **oblique cone,** the **apex** is *not* in the center:



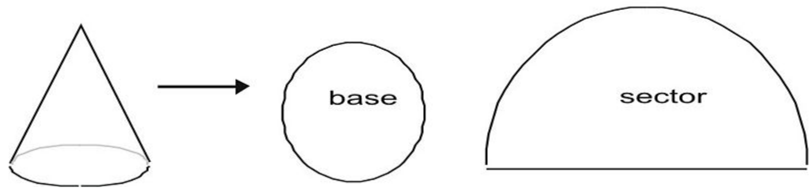
- The **apex** of a **right cone** is the point directly above the _____________________ of the cone's circular base.

The height of a **cone** h is the *perpendicular distance* from the *center* of the cone's *base* to its **apex.**

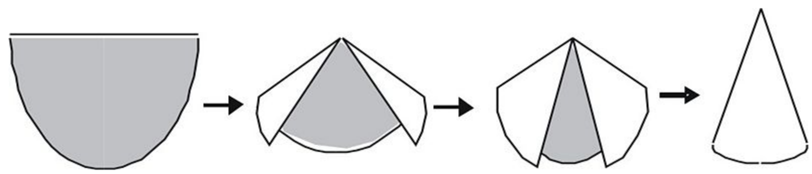
- The height of a cone is just like an altitude: a _______________________ line from the center of the circular base to the apex.

Surface Area of a Cone Using Nets

Most three-dimensional figures are easy to deconstruct into a net. The cone is different in this regard. Can you predict what the net for a cone looks like? In fact, the net for a cone looks like a small circle and a **sector**, or part of a larger circle.



The diagram below shows how the half-circle sector folds to become a cone:



Note that the circle that the sector is cut from is much larger than the base of the cone.

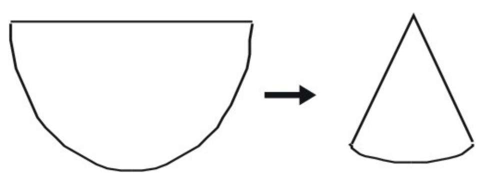
- The net for a **cone** is a circular base plus a ____________________.

Example 1

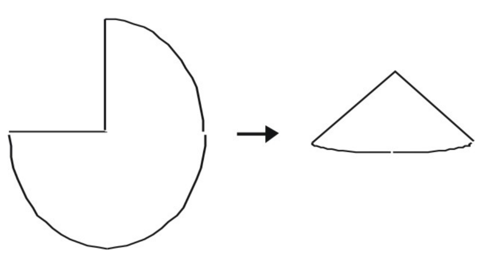
Which sector will give you a taller cone—a half circle or a sector that covers three-quarters of a circle? Assume that both sectors are cut from congruent circles.

Make a model of each sector:

1. The half circle makes a cone that has a height that is about equal to the radius of the semi-circle.



2. The three-quarters sector gives a cone that has a *wider base* (greater diameter) but its height as *not* as great as the half-circle cone.



Example 2

Predict which will be greater in height—a cone made from a half-circle sector or a cone made from a one-third-circle sector. Again, assume that both sectors are cut from congruent circles.

The relationship in Example #1 on the previous page holds true—the *greater* (in degrees) the **sector**, the *smaller* the *height* of the cone.

In other words, the fraction $\frac{1}{3}$ is less than $\frac{1}{2}$, so a one-third sector will create a cone with *greater height* than a one-half sector.

- The *larger* the **sector,** the ____________________________ the *height* of its **cone.**

Example 3

Predict which will be greater in diameter—a cone made from a half-circle sector or a cone made from a one-third-circle sector. Assume that the sectors are cut from congruent circles.

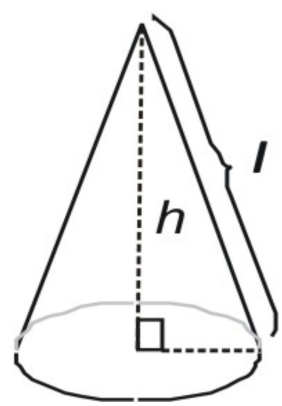
Here you have the opposite relationship—the *larger* (in degrees) the **sector**, the *greater* the *diameter* of the cone.

In other words, $\frac{1}{2}$ is greater than $\frac{1}{3}$, so a one-half sector will create a cone with *greater diameter* than a one-third sector.

- The *larger* the **sector,** the __________________________ the *diameter* of its **cone.**

Surface Area of a Regular Cone

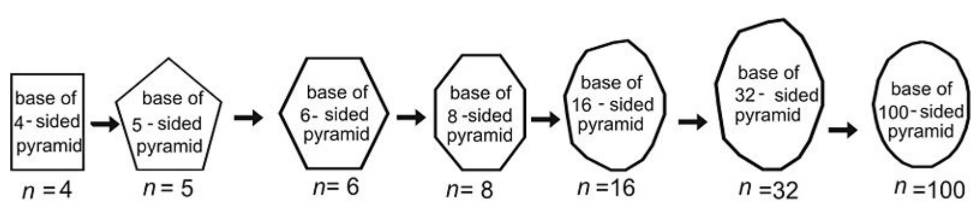
The **surface area** of a **regular pyramid** is given by:



$$A = \left(\frac{1}{2}lP\right) + B$$

where l is the *slant height* of the figure, P is the *perimeter* of the *base*, and B is the *area* of the *base.*

Imagine a series of **pyramids** in which n, the number of *sides* of each figure's *base,* increases.



As you can see, as n increases, the figure more and more resembles a *circle*.

You can also think of this as: a *circle* is like a polygon with an *infinite* number of sides that are infinitely small.

Similarly, a **cone** is like a **pyramid** that has an *infinite* number of *sides* that are infinitely small in length.

As a result, the formula for finding the total **surface area** of a **cone** is similar to the **pyramid** formula. The only difference between the two is that the **pyramid** uses P, the *perimeter* of the base, while a cone uses C, the *circumference* of the base.

$$A(\text{pyramid}) = \frac{1}{2}lP + B \qquad \textit{and} \qquad A(\text{cone}) = \frac{1}{2}lC + B$$

Since the *circumference* of a circle is $2\pi r$:

$$A(\text{cone}) = \frac{1}{2}lC + B = \frac{1}{2}l(2\pi r) + B = \pi rl + B$$

You can also express B as πr^2 to get:

Surface Area of a Right Cone

$$\begin{aligned} A(\text{cone}) &= \pi rl + B \\ &= \pi rl + \pi r^2 \\ &= \pi r(l + r) \end{aligned}$$

Any of these forms of the equation can be used to find the **surface area** of a **right cone.**

Reading Check:

There are a few different formulas to find the surface area of a cone. Pick one formula and describe what every variable represents.

Example 4

Find the total surface area of a right cone with a radius of 8 cm *and a slant height of* 10 cm.

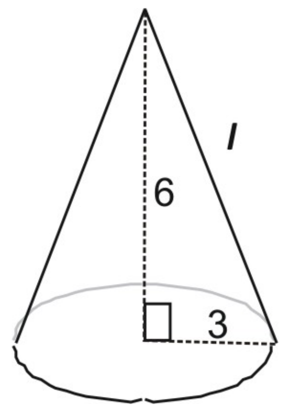
Use the formula:

$$\begin{aligned} A(\text{cone}) &= \pi r(l + r) \\ &= \pi(8) \cdot (10 + 8) \\ &= 8\pi \cdot 18 \\ &= 144\pi \ cm^2 \quad \text{or} \\ &\approx (144)(3.14) = 452.16 \ cm^2 \end{aligned}$$

The exact area of the cone is $144\pi\ cm^2$ and the approximate area is $452.16\ cm^2$.

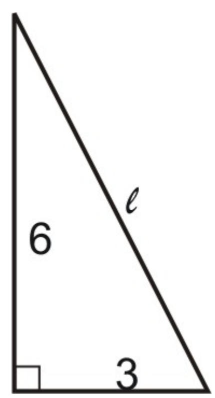
Example 5

Find the total surface area of a right cone with a radius of 3 feet *and an altitude (not slant height) of* 6 feet.



Use the Pythagorean Theorem to find the slant height l:

$$\begin{aligned} r^2 + h^2 &= l^2 \\ 3^2 + 6^2 &= l^2 \\ 9 + 36 &= l^2 \\ 45 &= l^2 \\ \sqrt{45} &= l \\ 3\sqrt{5} &= l \end{aligned}$$



Now use the area formula:

$$\begin{aligned} A(\text{cone}) &= \pi r(l + r) \\ &= \pi(3) \cdot (3\sqrt{5} + 3) \\ &= 3\pi(3\sqrt{5} + 3) \end{aligned}$$

If we leave this as an *exact* answer, we cannot simplify anymore. This would be an ideal time to use a decimal approximation with a calculator:

$$3\pi(3\sqrt{5}+3) \approx 3(3.14)(3\sqrt{5}+3) = 91.45\ cm^2$$

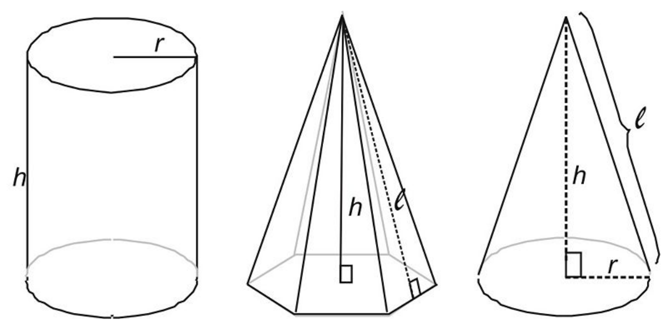
The surface area of the cone is approximately 91.45 cm^2.

Volume of a Cone

Which has a *greater volume,* a **pyramid, cone,** or **cylinder** if the figures have bases with the same "diameter" (i.e., distance across the base) and the same altitude?

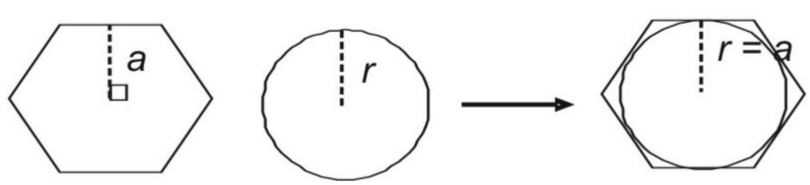
To find out, compare **pyramids, cylinders,** and **cones** that have *bases* with *equal diameters* and the *same altitude.*

Here are three figures that have the same dimensions—cylinder, a right regular hexagonal pyramid, and a right circular cone. Which figure appears to have a greater volume?



It seems obvious that the volume of the **cylinder** is *greater* than the other two figures. This is because the pyramid and cone taper off to a single point, while the cylinder's sides stay the same width.

Determining whether the **pyramid** or the **cone** has a *greater volume* is not so obvious. If you look at the *bases* of each figure you see that the **apothem** of the hexagon is *congruent* to the **radius** of the circle. You can see the relative size of the two bases by superimposing one onto the other:



From the diagram you can see that the hexagon is slightly *larger* in *area* than the circle.

Therefore, the **volume** of the right hexagonal regular **pyramid** would be *greater* than the **volume** of a right circular **cone.** It is, but only because the *area* of the **base** of the *hexagon* is slightly *greater* than the *area* of the **base** of the *circular cone.*

- When comparing the **volumes** of a **cylinder,** a **pyramid,** and a **cone,** the ____________________________ has the *largest* volume and the ____________________________ has the *smallest* volume. The _______________-_____________ has a volume in between the other two shapes.

The formula for finding the **volume** of each figure is virtually identical. Both formulas follow the same basic form:

$$V = \frac{1}{3}Bh$$

Since the *base* of a circular **cone** is, by definition, a **circle**, you can substitute the area of a circle, πr^2 for the *base* of the figure. This is expressed as a volume postulate for cones.

- Instead of using B for base area, we use the area of a ____________________, πr^2, in the formula for volume of a cone.

Volume of a Right Circular Cone

Given a right circular **cone** with height h and a base that has radius r:

$$V = \frac{1}{3}Bh$$
$$V = \frac{1}{3}\pi r^2 h$$

Example 6

Find the volume of a right cone with a radius of 9 cm *and a height of* 16 cm.

Use the formula: $V = \frac{1}{3}\pi r^2 h$

Substitute the values for $r =$ ___________ and $h =$ _____________ :

$$V = \frac{1}{3}\pi(9^2)(16)$$
$$V = \frac{1296\pi}{3} = 432\pi\ cm^3 \quad \text{or}$$
$$\approx (432)(3.14) = 1356.48\ cm^3$$

The cone has an exact volume of 432π cubic centimeters and an approximate volume of 1356.48 cubic centimeters.

By now, you have seen the units cm^2 or in^2 and cm^3 or in^3 in the examples.

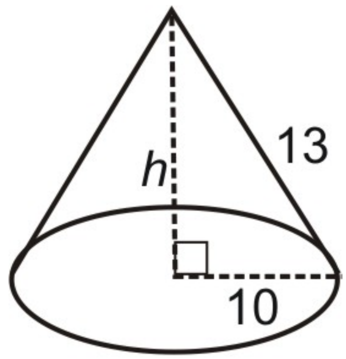
When we calculate area, we use a "square" unit, such as cm^2 (square centimeters) or in^2 (square inches)

When we calculate volume, we use a "cubic" unit, such as cm^3 (cubic centimeters) or in^3 (cubic inches)

Example 7

Find the volume of a right cone with a radius of 10 feet *and a slant height of* 13 feet.

Use the Pythagorean theorem to find the height h:



$$\begin{aligned}
r^2 + h^2 &= l2 \\
10^2 + h^2 &= 132 \\
100 + h^2 &= 169 \\
h^2 &= 169 - 100 = 69 \\
h &= \sqrt{69} \approx 8.31\ ft
\end{aligned}$$

Now use the volume formula: $V = \frac{1}{3}\pi r^2 h$

Substitute the values for $r =$ ____________ and $h =$ ______________ :

$$\begin{aligned}
V &= \frac{1}{3}\pi(10^2)(8.31) \\
V &= \frac{831\pi}{3} = 277\pi\ ft^3 \quad \text{or} \\
&\approx (277)(3.14) = 869.78\ ft^3
\end{aligned}$$

The cone's volume can be written as $277\pi\ ft^3$ or $869.78\ ft^3$.

Reading Check:

1. *What type of units are used to express volume? What type for area?*

2. *What shape is the base of a right circular cone?*

3. *When calculating the volume of a cone, what information do you need?*

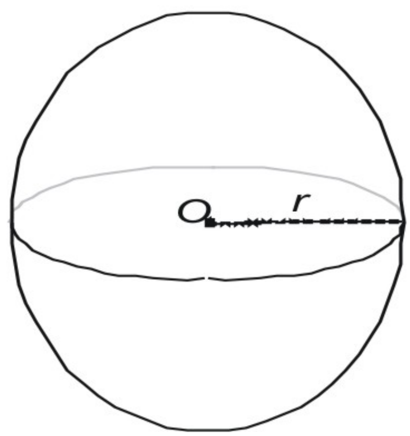
8.8 Spheres and Hemispheres: Surface Area and Volume

Learning Objectives

- Find the surface area of a *sphere*.
- Find the volume of a sphere.

Spheres

A **sphere** is a three-dimensional figure that has the shape of a ball:



Spheres can be characterized in three ways:

1. A sphere is the set of all points that lie a fixed distance r from a single center point O.

2. A sphere is the surface that results when a circle is rotated about any of its diameters:

3. A sphere results when you construct a *polyhedron* with an *infinite* number of faces that are infinitely small. To see why this is true, recall regular polyhedra.

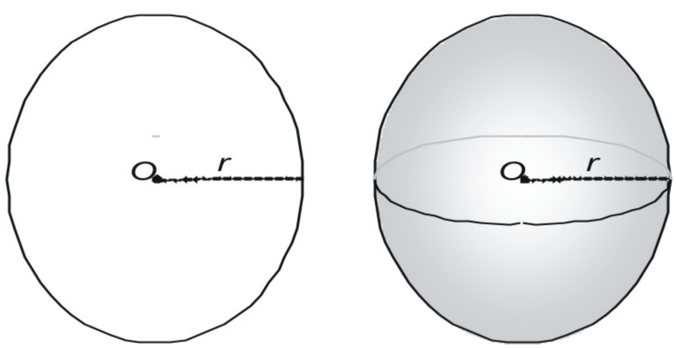
As the number of faces on the figure *increases,* each face gets *smaller* in *area* and the figure comes more to resemble a **sphere.** When you imagine a figure with an *infinite* number of faces, it would look like a **sphere.**

- A 3-D figure that looks like a ball is called a ______________________________.

Parts of a Sphere

As described above, a **sphere** is the surface that is the set of all points a fixed distance from a **center** point O. The words used for spheres are similar to those used for circles:

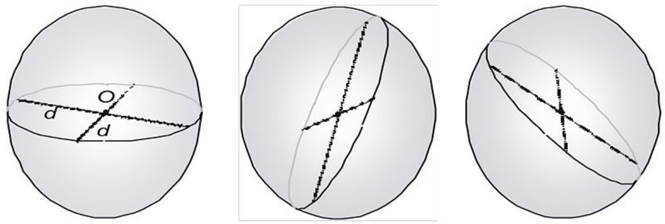
The distance from O to the surface of the sphere is r, the **radius:**



- The _________________________ of a sphere is the distance from the center to the surface of the sphere.

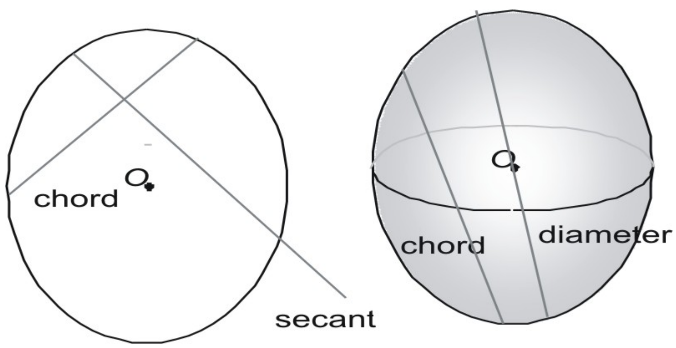
The **diameter** d, of a sphere is the length of the segment connecting any two points on the sphere's surface and passing through the center O.

You can find a **diameter** (actually an infinite number of diameters) on any plane within the sphere. Two diameters are shown in each sphere below:



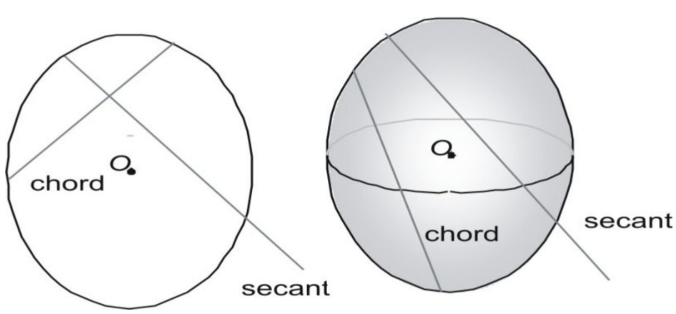
- The ________________________ of a sphere connects two points on the surface of the sphere and goes through the center.

A **chord** for a sphere is similar to the chord of a circle except that it exists in *three* dimensions. Keep in mind that a **diameter** is a kind of **chord**—a special chord that intersects the **center** of the circle or sphere.



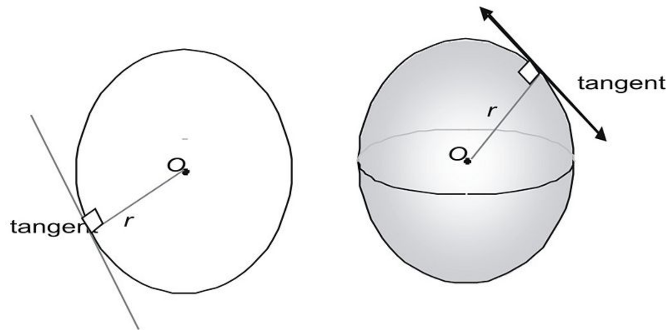
- A _________________________ connects two points on the surface of a sphere.

A **secant** is a line, ray, or line segment that intersects a circle or sphere in two places and extends outside of the circle or sphere.



- A ______________________ is a **chord** that extends in either or both directions.

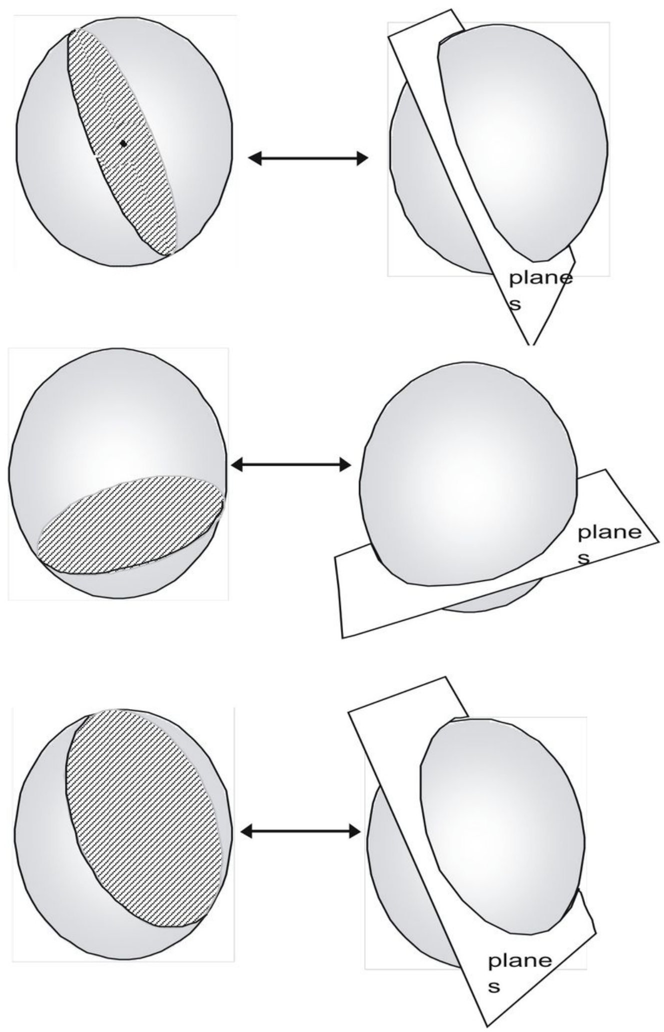
A **tangent** intersects the circle or sphere at only one point.



In a circle, a **tangent** is *perpendicular* to the radius that meets the point where the tangent intersects with the circle. The same thing is true for the sphere. All **tangents** are *perpendicular* to the radii that intersect with them.

- A **tangent** is ______________________________ to the radius at the point where the two intersect, which is on the edge of the circle or sphere.

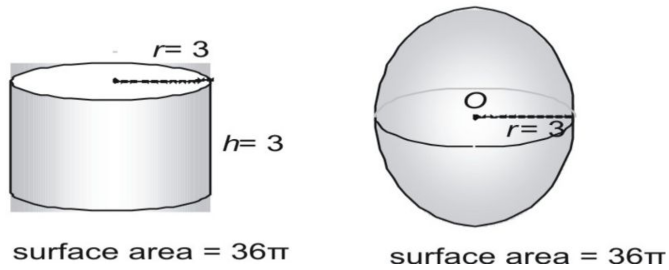
Finally, a **sphere** can be sliced by an infinite number of different planes. Some planes include point *O*, the **center** of the sphere. Other points do not include the center. Look at the diagrams below:



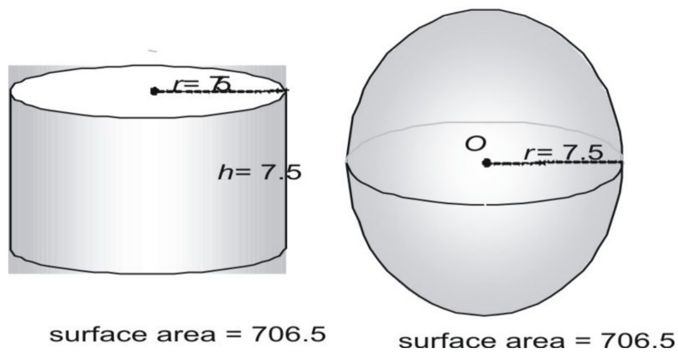
The parts of a sphere are very similar to the parts of a circle: they have the same names and definitions, except the parts of a sphere are in ____________________ dimensions while the parts of a circle are in two dimensions.

Surface Area of a Sphere

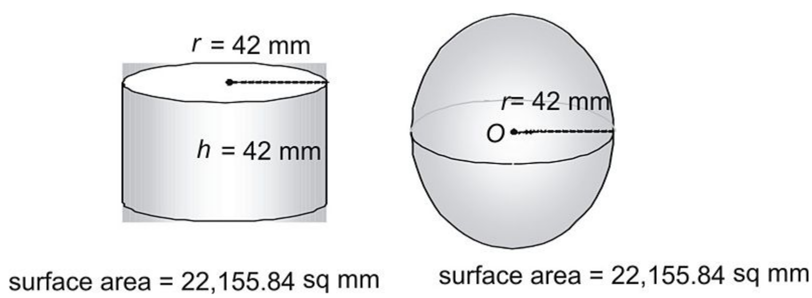
You can figure out the formula for the **surface area** of a **sphere** by taking measurements of **spheres** and **cylinders.** Here we show a **sphere** with a *radius* of 3 and a right **cylinder** with both a *radius* and a *height* of 3, and we express the *area* in terms of π.



Now try a larger pair, expressing the surface area in decimal form:



Look at a third pair:



Is it a coincidence that a **sphere** and a **cylinder** whose *radius* and *height* are *equal* to the *radius* of the **sphere** have the *exact same surface area?* Not at all! In fact, the ancient Greeks used a method that showed that the following formula can be used to find the surface area of any sphere (or any cylinder in which $r = h$).

The Surface Area of a Sphere is given by:

$$A = 4\pi r^2$$

- If a **sphere** has a *radius* that is *equal* to both the *radius* and the *height* of a **cylinder,** then their **surface areas** are the ___________________________!

Example 1

Find the surface area of a sphere with a radius of 14 feet.

Use the formula:

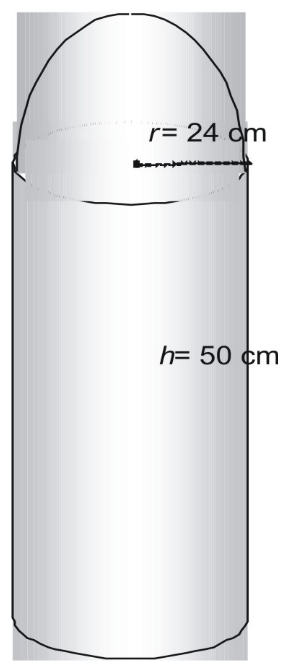
$$\begin{aligned} A &= 4\pi r^2 \\ &= 4\pi(14^2) \\ &= 784\pi\ ft^3 \quad \text{or} \\ &\approx (784)(3.14) = 2461.76\ ft^3 \end{aligned}$$

The sphere has an exact surface area of $784\pi\ ft^3$, which is approximately $2461.76\ ft^3$.

Example 2

Find the surface area of the following figure in terms of π.

(This means give an exact answer, not an approximate answer.)



The figure is made of one-half of a sphere, called a **hemisphere**, and one cylinder *without* its top.

$$\begin{aligned} A(\text{hemisphere}) &= \frac{1}{2}A(\text{sphere}) \\ &= \frac{1}{2}(4\pi r^2) \\ &= 2\pi(24^2) \\ &= 1152\pi\ cm^2 \end{aligned}$$

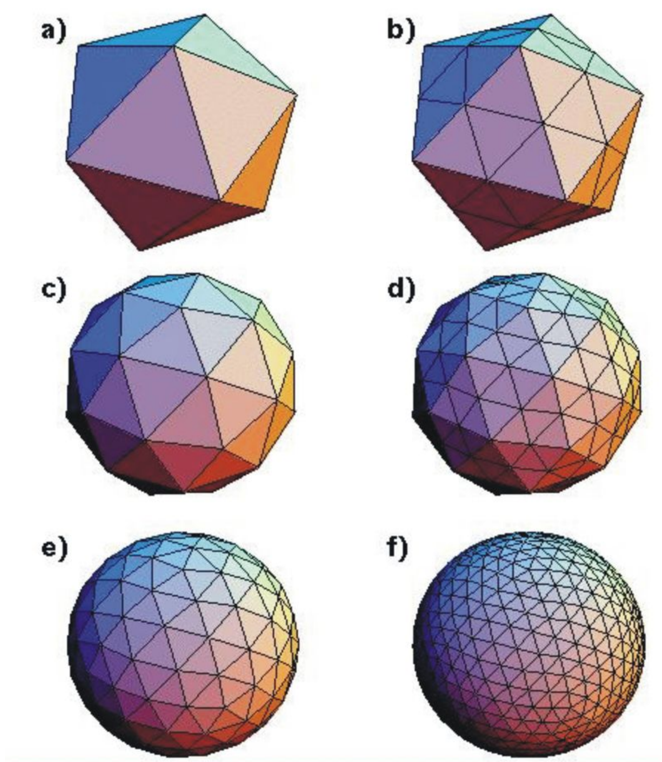
Now find the area of the **cylinder** *without* its top:

$$\begin{aligned}
A(\text{topless cylinder}) &= A(\text{cylinder}) - A(\text{top})\\
&= (2\pi r^2 + 2\pi rh)\,\text{[U+0080] [U+0093]}\,\pi r^2\\
&= \pi r^2 + 2\pi rh\\
&= \pi(24^2) + 2\pi(24)(50)\\
&= 576\pi + 2400\pi\\
&= 2976\pi\ cm^2
\end{aligned}$$

So the *total* surface area is: $1152\pi\ cm^2 + 2976\pi\ cm^2 = 4128\pi\ cm^2$

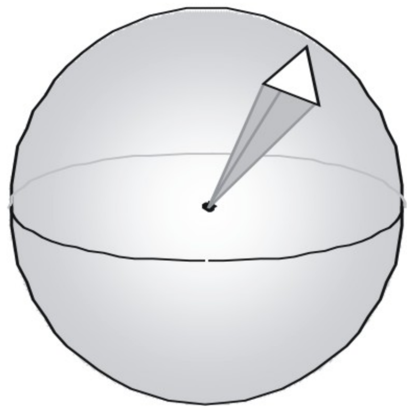
Volume of a Sphere

A **sphere** can be thought of as a *regular polyhedron* with an *infinite* number of congruent polygon faces. A series of polyhedra with an increasing number of faces is shown:



As n, the number of *faces* increases to an infinite number, the figure gets closer and closer to becoming a sphere. You can see that the shape in figure a has sharp edges, while the shape in figure f (with more faces) looks *rounder.*

So a **sphere** can be thought of as a **polyhedron** with an *infinite* number faces. Each of those faces is the *base* of a **pyramid** whose **vertex** is located at O, the *center* of the sphere. This is shown below:



Each of the pyramids that make up the sphere would be congruent to the pyramid shown above.

The **volume** of this **pyramid** is given by:

$$V(\text{each pyramid}) = \frac{1}{3}Bh$$

To find the **volume** of the **sphere,** you simply need to add up the volumes of an infinite number of infinitely small pyramids:

$$\begin{aligned} V(\text{all pyramids}) &= V_1 + V_2 + V_3 + \ldots + V_n \\ &= \frac{1}{3}B_1h + \frac{1}{3}B_2h + \frac{1}{3}B_3h + \ldots + \frac{1}{3}B_nh \\ &= \frac{1}{3}h(B_1 + B_2 + B_3 + \ldots + B_n) \end{aligned}$$

The *sum* of all of the *bases* of the **pyramids** is simply the **surface area** of the **sphere.**

Since you know that the **surface area** of the **sphere** is $4\pi r^2$, you can substitute this quantity into the equation above for the *sum* of the *bases:*

$$\begin{aligned} V(\text{all pyramids}) &= \frac{1}{3}h(B_1 + B_2 + B_3 + \ldots + B_n) \\ &= \frac{1}{3}h(4\pi r^2) \end{aligned}$$

Finally, as n increases and the surface of the figure becomes more "rounded," h, the height of each pyramid becomes equal to r, the radius of the sphere.

So we can substitute r for h. This gives:

$$\begin{aligned} V(\text{all pyramids}) &= \frac{1}{3}r(4\pi r^2) \\ &= \frac{4}{3}\pi r^3 \end{aligned}$$

We can write this as a formula.

Volume of a Sphere

Given a sphere that has radius r

$$V = \frac{4}{3}\pi r^3$$

Example 3

Find the volume of a sphere with a radius of 6 meters.

Use the volume formula above with $r =$ _________ :

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(6^3) = \frac{864\pi}{3} = 288\pi\ m^3$$

The sphere has an exact volume of $288\pi\ m^3$.

How would you compute the approximate volume? (__________)(__________)

Reading Check:

1. *In your own words, describe what a sphere is:*

2. *What is half of a sphere called?*

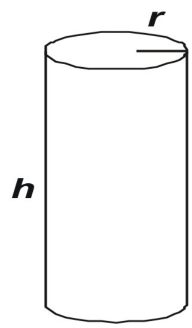
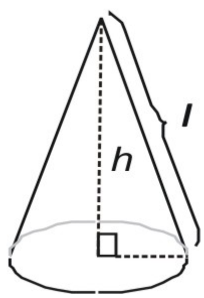
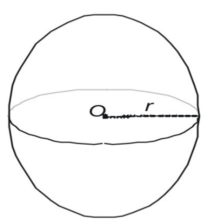
3. *True/False:* The parts of a circle have similar names and definitions to the parts of a sphere.

4. *If you answered true for #3 above, provide an example. If you answered false to #3 above, provide a non-example.*

5. *In your own words, describe what a tangent is:*

Graphic Organizer for Lessons 5 – 7: 3-D Surface Area and Volume

TABLE 8.5:

3-D Figure	Picture	Surface Area Formula	Volume Formula
Cylinder			
Cone			
Sphere			

8.9 Composite Area and Change of Dimensions

Editor's note: This lesson has very limited material supported by CK-12. Please refer to the Geometry Access Reader for additional materials.

Learning Objectives

- Use similarity to generalize the results.

Similarity

We know that *all circles are similar to each other.*

Suppose a circle has a radius of r units.

- The *scale factor* of this circle (with radius r) and the circle with radius 1 is:

$$r : 1, \quad \frac{r}{1}, \quad \text{or just} \quad r$$

.

- You know how a *scale factor* affects area measures:

If the scale factor is r, then the area is r^2 times as much.

Reading Check:

1. *Explain how we know that all circles are similar to each other.*

(Hint: Think about similar measurements that circles may have.)

2. *If the scale factor of a circle is r, how does this relate to a circle's area?*

CHAPTER 9 Properties of Circles

Chapter Outline

9.1 Vocabulary Self-Rating

TABLE 9.1: Rating Guide: DK: I am sure I don't know it K: I am sure I know it ?: I'm not sure

Word	**Before Lesson/Unit**	**After Lesson/Unit**
Inscribed angle		
Intercept		
Intercepted arc		
Central angle		
Corollary		
Chord		
Average		
Mean		
Vertical angles		
Linear pair		
Tangent		
Point of tangency		
Proof by contradiction		
Radius		
Converse		
Secant		
Perpendicular bisector		
Diameter		
Equidistant		
Segment		
AA Similarity Postulate		
Inscribed polygon		
Circumscribed polygon		

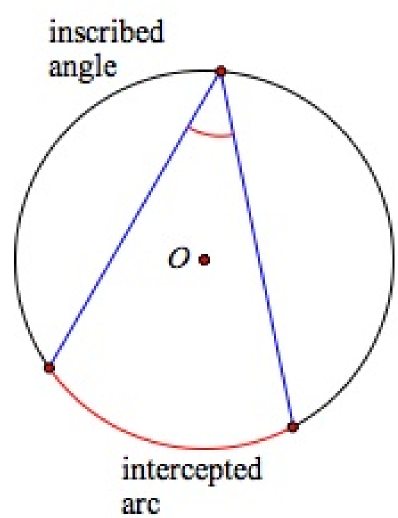
9.2 Inscribed Angles

Learning Objectives

- Find the measure of inscribed angles and the arcs they intercept.

Inscribed Angle, Intercepted Arc

An **inscribed angle** is an angle whose *vertex is on the circle* and whose *sides contain chords* of the circle.



- The ___________________________ of an **inscribed angle** is *on* the circle.
- The *sides* of an **inscribed angle** are _________________________ of the circle.

An **inscribed angle** is said to **intercept** an *arc* of the circle. In the diagram above, the **intercepted arc** is shown between the two sides of the angle.

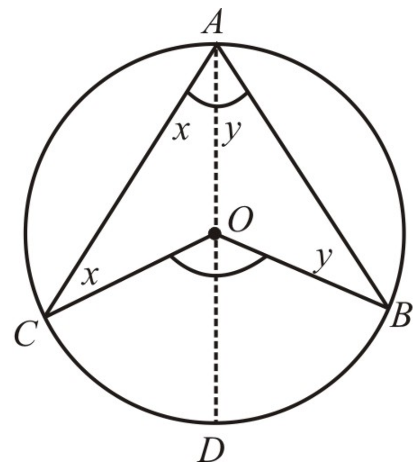
- The sides of an **inscribed angle intercept** an _________________ of the circle.

We will prove shortly that the *measure* of an **inscribed angle** is *half* of the *measure* of the **arc** it **intercepts**.

Notice that the *vertex* of the inscribed angle can be anywhere on the circumference of the circle; it does not need to be diametrically opposite the intercepted arc.

Measure of Inscribed Angle

The measure of a **central angle** is *twice* the measure of the **inscribed angle** that intercepts the *same arc*.



Proof

$\angle COB$ and $\angle CAB$ both intercept $\widehat{CB}$. $\angle COB$ is a **central angle** and $\angle CAB$ is an **inscribed angle**.

We draw the *diameter* of the circle through points A, O, and D, and let $m\angle CAO = x^\circ$ and $m\angle BAO = y^\circ$, as labeled in the diagram above.

We see that ΔAOC is *isosceles* because $\overline{AO}$ and $\overline{AC}$ are *radii* of the circle and are therefore *congruent*. You can make congruent marks on these segments in the diagram.

From this we can conclude that $m\angle ACO = x^\circ$.

Similarly, we can conclude that $m\angle ABO = y^\circ$.

We use the property that the sum of angles inside a triangle equals 180° to find that: $m\angle AOC = 180^\circ - 2x^\circ$ and $m\angle AOB = 180^\circ - 2y^\circ$.

Then, $m\angle COD = 180^\circ - m\angle AOC = 180^\circ - (180^\circ - 2x^\circ) = 2x^\circ$ and $m\angle BOD = 180^\circ - m\angle AOB = 180^\circ - (180^\circ - 2y^\circ) = 2y^\circ$

Therefore: $m\angle COB = 2x^\circ + 2y^\circ = 2(x^\circ + y^\circ) = 2(m\angle CAB)$

This proves that the measure of a **central angle** is *two times* the measure of the **inscribed angle** that intercepts the *same arc*.

Another way of looking at this is that the measure of the **inscribed angle** is *half* the measure of the _____________-_______________ **angle** that intercepts the *same arc*.

*Do you remember what a **corollary** is?*

*A **corollary** is a statement that follows a theorem.*

If we know that a theorem is true, we can often make other statements that are true. These are called corollaries.

- A **corollary** is a statement that follows a _____________________________.

Inscribed Angle Corollaries

The theorem we just proved has several corollaries, which you can prove on your own:

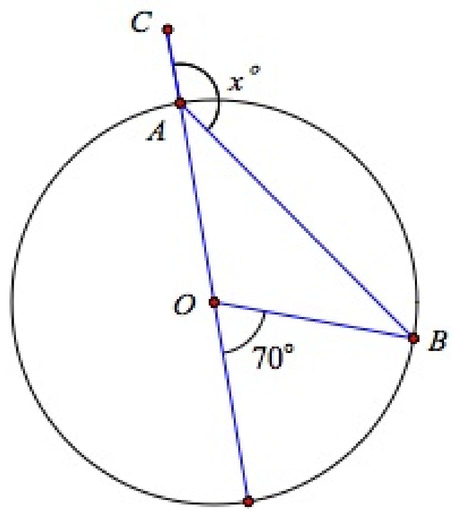
a. **Inscribed angles** intercepting the same arc are *congruent*.
b. *Opposite angles* of an inscribed quadrilateral are *supplementary*.
c. An angle inscribed in a *semicircle* is a *right angle*.

d. An **inscribed right angle** intercepts a *semicircle*.

- A __________________________ angle is inscribed in a semicircle.
- Opposite angles of an inscribed quadrilateral are ___________________________.

Example 1

Find the angle marked x in the circle:



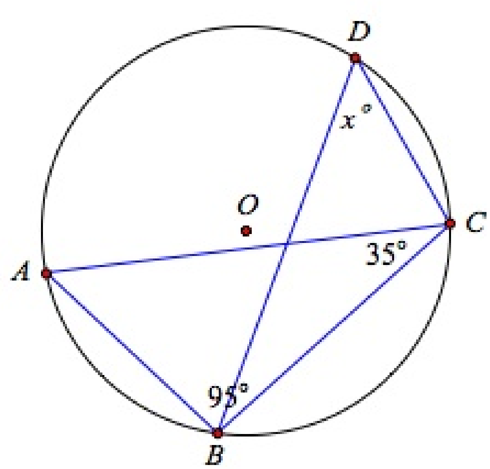
$m\angle EOB$ is *twice* the measure of the **inscribed angle** $\angle OAB$ because $\angle EOB$ is a **central angle** and both angles intercept the *same arc*, $\widehat{EB}$.

So $m\angle OAB = 35^\circ$.

This means that the angle marked $x = 180^\circ - 35^\circ = 145^\circ$.

Example 2

Find the angle marked x in the circle:



We know that an **arc** has the *same* measure as the **central angle** that intercepts it and *double* the measure of the **inscribed angle** that intercepts it.

- An **arc** has the ________________ measure as the **central angle** that intercepts it.
- An **arc** has ________________ the measure of the **inscribed angle** that intercepts it.

Therefore,

$$\begin{aligned} m\widehat{ADC} &= 2 \cdot 95^\circ = 190^\circ \\ \text{so} \quad m\widehat{ABC} &= 360^\circ - 190^\circ = 170^\circ \\ \text{and} \quad m\widehat{AB} &= 2 \cdot 35^\circ = 70^\circ \\ \text{so} \quad m\widehat{BC} &= 170^\circ - 70^\circ = 100^\circ \end{aligned}$$

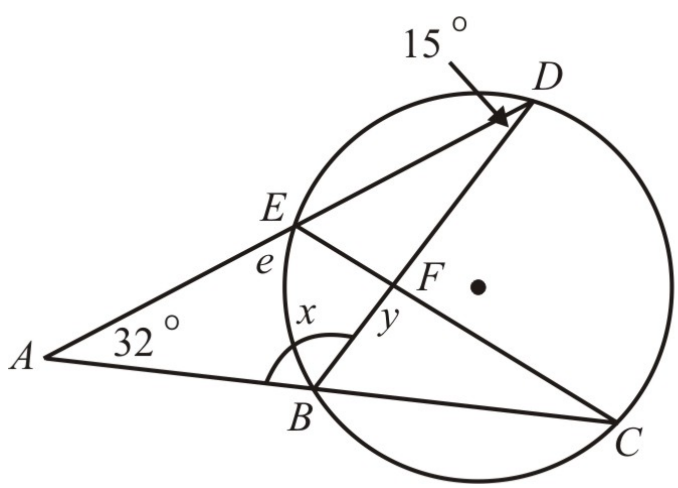
$m\angle BDC$ (which is the angle marked x) is the **inscribed angle** that intercepts the arc $\widehat{BC}$ so x must be *half* of the measure of the arc:

$$\begin{aligned} m\angle BDC &= \frac{1}{2}(m\widehat{BC}) \\ m\angle BDC &= \frac{1}{2}(\_\_\_\_\_\_\_\_\_\_\_\_) = \_\_\_\_\_\_\_\_\_\_\_\_ \end{aligned}$$

Therefore, $x = 50^\circ$.

Example 3

Find the angles marked x and y in the circle:



First we use ΔABD to find the measure of angle x, since all three angles in the triangle add up to 180°:

$$\begin{aligned} x + 15^\circ + 32^\circ &= 180^\circ \\ x + 47^\circ &= 180^\circ \\ x &= 133^\circ \end{aligned}$$

So $m\angle CBD = 180^\circ - 133^\circ = 47^\circ$ because it is a linear pair with $\angle ABD$ or x.

$m\angle BCE = m\angle BDE$ because they are both **inscribed angles** and both intercept the *same arc*, $\widehat{EB}$. This makes $m\angle BCE = 15^\circ$.

Using ΔBFC to find the measure of angle y:

$$\begin{aligned} y + 47^\circ + 15^\circ &= 180^\circ \\ y + 62^\circ &= 180^\circ \\ y &= 118^\circ \end{aligned}$$

Therefore, $x = 133^\circ$ and $y = 118^\circ$.

Reading Check:

1. *What was the most important thing you learned about inscribed angles in this lesson?*

2. *How does a central angle relate to the inscribed angle that shares the same intercepted arc?*

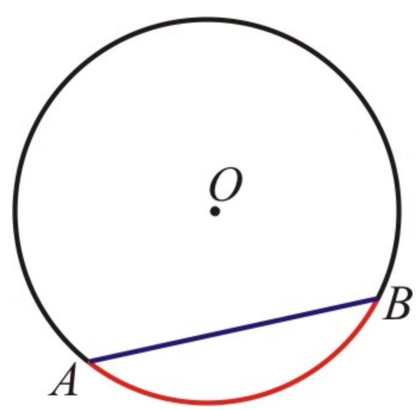
9.3 Angles of Chords

Learning Objectives

- Find the measures of angles formed by chords in a circle.

Chords

Chords are line segments whose *endpoints* are both on a circle. The figure below shows an arc $\widehat{AB}$ and its related **chord** $\overline{AB}$.

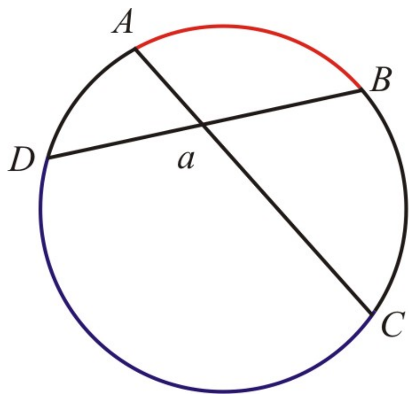


- A **chord** is a line segment whose ______________________ are both on a circle.

Angles Inside a Circle

The measure of the angle formed by two **chords** that *intersect* inside a circle is equal to *half* the sum of the measure of their intercepted arcs. In other words, the measure of the angle is the **average (mean)** of the measures of the intercepted arcs.

- Two intersecting ______________________ make 4 angles in a circle. Each angle is the *average* of the corresponding **intercepted arcs**.
- Another word for the **average** is the ______________________. The **average** of two numbers is *half* their *sum*.



In the figure above, $m\angle a = \frac{1}{2}(m\widehat{AB} + m\widehat{DC})$ and $m\angle b = \frac{1}{2}(m\widehat{AD} + m\widehat{BC})$

Proof

Draw a segment to connect points B and C:

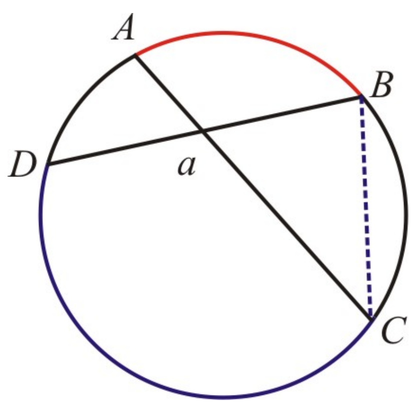


TABLE 9.2:

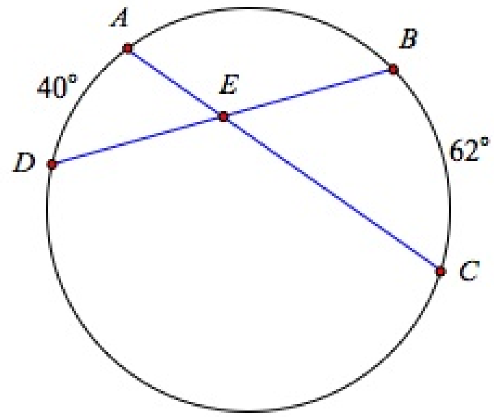
Statements	Reasons
1. $m\angle DBC = \frac{1}{2}m\widehat{DC}$	1.Inscribed angle
2. $m\angle ACB = \frac{1}{2}m\widehat{AB}$	2.Inscribed angle
3. $m\angle a = m\angle ACB + m\angle DBC$	3. The measure of an exterior angle in a triangle is equal to the sum of the measures of the remote interior angles.
4. $m\angle a = \frac{1}{2}m\widehat{DC} + \frac{1}{2}m\widehat{AB}$	4. Substitution
5. $m\angle a = \frac{1}{2}(m\widehat{DC} + m\widehat{AB})$	5. Factor and simplify

The intersection of two **chords** makes two pairs of **vertical angles**. Since **vertical angles** are *congruent*, you can compute the measure of *either* angle of each pair using the same calculation: take the *average* of the two **intercepting arcs**.

- When two **chords** intersect in a circle, two pairs of ________________________ angles are created.
- Each angle measure is equal to the ______________________________ of the two **intercepting arc** measures.
- The *reason* in steps 1 and 2 in the proof above says that an ___________________ angle is *half* of the measure of its **intercepted arc**.

Example 1

Find $m\angle DEC$.



We know that an angle formed by two intersecting **chords** is the *average* of the **intercepted arc** measures.

For $\angle DEC$, the **intercepted arcs** are $\widehat{DC}$ and $\widehat{AB}$, but instead we are given the measures of the arcs $\widehat{AD}$ and $\widehat{BC}$. We can use this information to find what we need:

$$m\angle AED = \frac{1}{2}(m\widehat{AD} + m\widehat{BC}) \text{ so } m\angle AED = \frac{1}{2}(40^\circ + 62^\circ)$$
$$m\angle AED = \frac{1}{2}(102^\circ) = 51^\circ$$

Since $\angle AED$ and $\angle DEC$ are a **linear pair** (and thus *supplementary*):

$$m\angle DEC = 180^\circ - m\angle AED$$
$$m\angle DEC = 180^\circ - 51^\circ = 129^\circ$$

Reading Check:

1. *True or false:* Chords that intersect inside a circle create 4 angles that all have different measures.
2. *Why is the statement in question #1 true or why is it false? Defend your choice.*

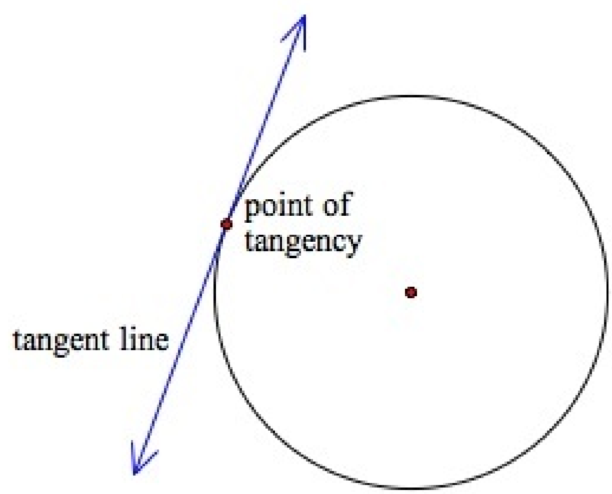
9.4 Angles of Secants and Tangents

Learning Objectives

- Find the relationship between a radius and a tangent to a circle.
- Find the relationship between two tangents drawn from the same point.
- Find the measures of angles formed by secants and tangents.

Tangent to a Circle

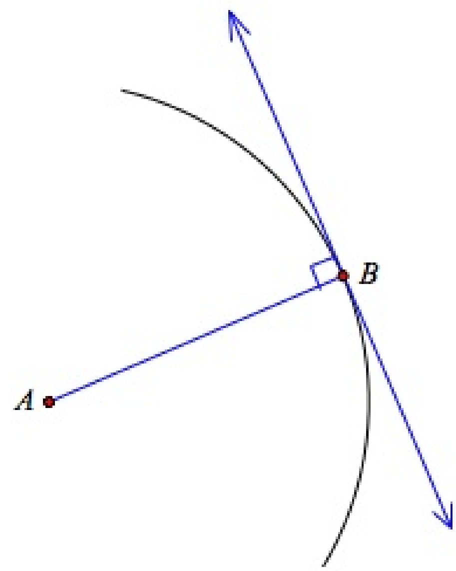
A **tangent to a circle** is a line that intersects the circle at exactly *one point*. This *intersection point* is called the **point of tangency**.



- A ______________________________ line intersects a circle exactly once.
- A point of ____________________________________ is a point where a **tangent line** intersects a circle.

Tangent to a Circle Theorem

A **tangent line** is always at a *right angle* to the **radius** of the circle at the point of tangency.

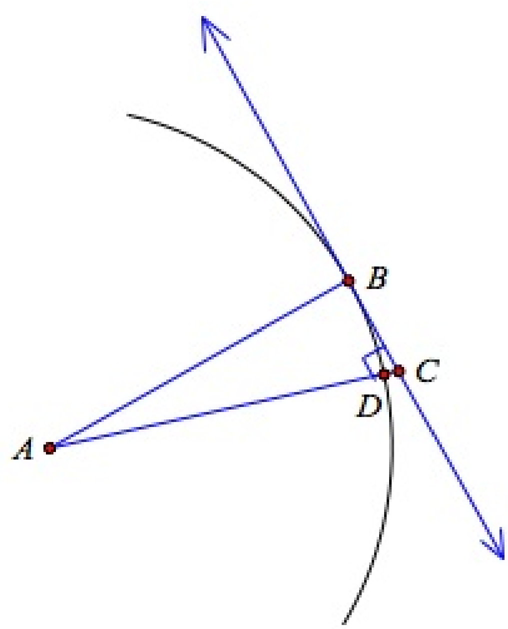


- A **tangent line** and a ___________________________ meet at a *right angle*.

Proof

We will prove the theorem on the previous page by **contradiction** (which starts by assuming the *opposite* of what you are trying to prove).

Start by making a drawing. $\overline{AB}$ is a **radius** of the circle. A is the *center* of the circle and B is the point of *intersection* between the **radius** and the **tangent line**. Another name for point B is the **point of tangency**.



Assume that the tangent line is *not* perpendicular to the radius.

There must be another point C on the **tangent line** such that $\overline{AC}$ is *perpendicular* to the **tangent line**. Therefore, in the right triangle $\Delta ACB, \overline{AB}$ is the *hypotenuse* and $\overline{AC}$ is a *leg* of the triangle.

However, this is not possible because $AC > AB$.

(Note that $AC =$ the length of the radius $+DC$)

Since our *assumption* led us to a contradiction, this means that our assumption was incorrect. Therefore, the **tangent line** must be *perpendicular* to the **radius** of the circle.

Since the **tangent** to a circle and the **radius** of the circle make a *right angle* with each other, we can often use the Pythagorean Theorem to find the length of missing segments.

Complete the sentence:

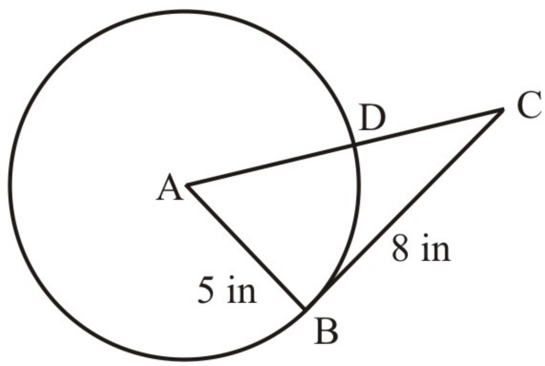
This proof tells us that every **tangent line** is *perpendicular* to

____

____.

Example 1

In the figure, $\overline{CB}$ is tangent to the circle. Find the length of $\overline{CD}$.



Since $\overline{CB}$ is **tangent** to the circle, then $\overline{CB} \perp \overline{AB}$.

This means that ΔABC is a *right triangle* and we can apply the Pythagorean Theorem to find the length of $\overline{AC}$. In this triangle, $\overline{AC}$ is the *hypotenuse*:

$$(AC)^2 = (AB)^2 + (BC)^2$$
$$(AC)^2 = 5^2 + 8^2 = 25 + 64 = 89$$
$$AC = \sqrt{89} \approx 9.43\ in$$

$\overline{AD}$ is congruent to $\overline{AB}$ because they are each a **radius** of the circle!

$\overline{CD}$ is part of $\overline{AC}$: $CD = AC - AD \approx 9.43 - 5 \approx 4.43\ in$.

Reading Check:

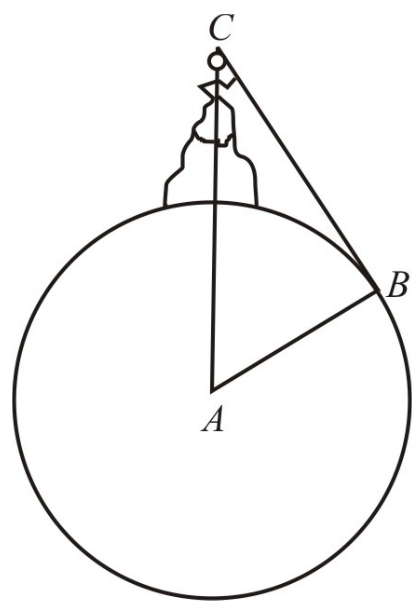
1. *True or flase:* A tangent line touches a circle at exactly one point, and it is perpendicular to the radius at that point.

2. *Draw a picture that shows whether the statement in #1 above is true or false.*

3. *What is the name of the point where the tangent line intersects the radius?*

Example 2

Mark is standing at the top of Mt. Whitney, which is 14,500 feet tall. The radius of the Earth is approximately 3,960 miles. (There are 5,280 feet in one mile.) How far can Mark see to the horizon?



We start by drawing the figure above. Without a drawing, this problem can be tricky! With a drawing, you can see that this is a simple tangent and radius problem.

The distance to the horizon is given by the line segment $\overline{CB}$.

First, we must convert the height of the mountain from *feet* into *miles*:

$$14500\ feet \cdot \frac{1\ mile}{5280\ feet} = 2.75\ miles$$

Since $\overline{CB}$ is **tangent** to the Earth, ΔABC is a *right triangle* and we can use the Pythagorean Theorem. What are the lengths of each side of the triangle?

$\overline{AB}$ is the **radius** and a *leg* of the triangle, and $AB =$ _________

$\overline{AC}$ is the *hypotenuse* of the triangle.

$AC =$ radius $+$ height of Mt. Whitney $=$ _________ $+$ _________

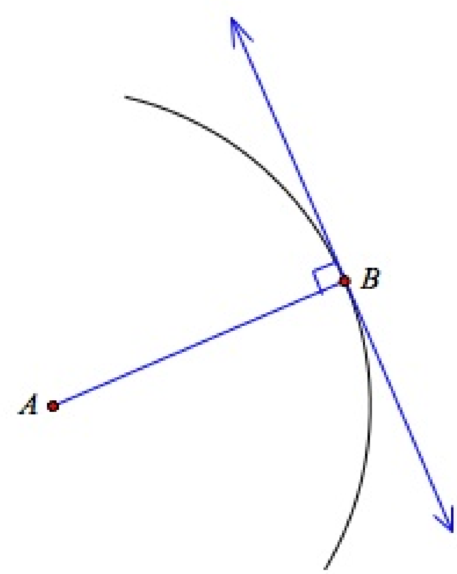
$\overline{CB}$ is the **tangent**, which is the *missing* side of the triangle.

$$(CB)^2 + (AB)^2 = (AC)^2$$
$$(CB)^2 + (3960)^2 = (3960 + 2.75)^2$$
$$(CB)^2 = (3960 + 2.75)^2 - (3960)^2 = 21787.56$$
$$CB = \sqrt{21787.56} \approx 147.6\ miles$$

Mark can see about 147.6 miles to the horizon.

Converse of a Tangent to a Circle Theorem

If a line is *perpendicular* to the **radius** of a circle at its outer endpoint, then the line is **tangent** to the circle.



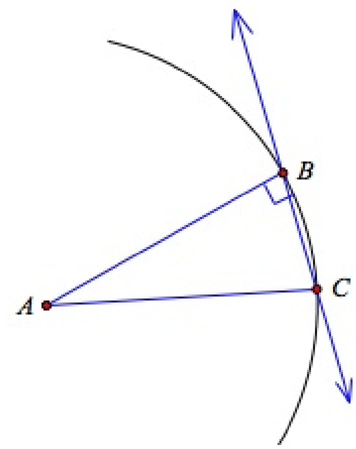
Remember, a ***converse*** *is an if-then statement where the* <u>*hypothesis*</u> *(part after "if") and the* <u>*conclusion*</u> *(part after "then") are switched.*

Proof

We will prove this theorem by *contradiction*.

Since the line is *perpendicular* to the **radius** at its outer endpoint it must touch the circle at point B. For this line to be **tangent** to the circle, it must only touch the circle at this point and no other.

Assume that the line *also* intersects the circle at point C.



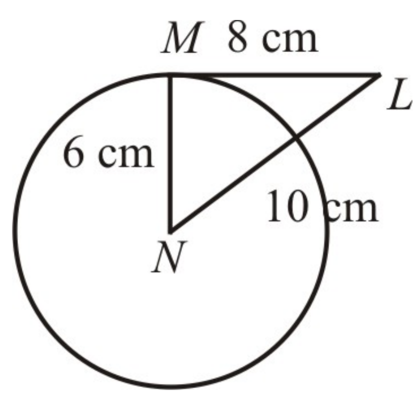
Since both $\overline{AB}$ and $\overline{AC}$ are **radii** of the circle, $\overline{AB} \cong \overline{AC}$, and ΔABC is *isosceles*.

This means that $m\angle ABC = m\angle ACB = 90^\circ$.

However, it is *impossible* to have *two* right angles in the same triangle. We arrived at a *contradiction* so our assumption must be incorrect. We conclude that line $\overline{BC}$ is **tangent** to the circle at point B.

Example 3

Determine whether $\overline{LM}$ is tangent to the circle:



$\overline{LM}$ is **tangent** to the circle if $\overline{LM} \perp \overline{MN}$.

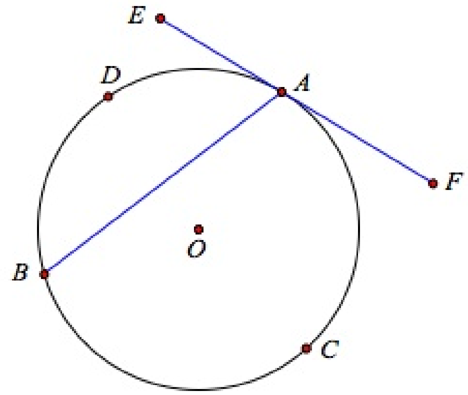
To show that ΔLMN is a *right triangle* we use the Converse of the Pythagorean Theorem, which means that we show the Pythagorean Theorem *works* with the given numbers:

$$(LM)^2 + (MN)^2 = (LN)^2$$
$$\_\_\_\_\_\_\_\_\_\_^2 + \_\_\_\_\_\_\_\_\_\_^2 = \_\_\_\_\_\_\_\_\_\_^2$$
$$64 + 36 = 100 \text{ yes, this is correct!}$$

The lengths of the sides of the triangle satisfy the Pythagorean Theorem, so $\overline{LM}$ is *perpendicular* to $\overline{MN}$ and is therefore **tangent** to the circle.

Measure of Tangent-Chord Angle Theorem

The measure of an angle formed by a **chord** and a **tangent** that *intersect* on the circle is *half* the measure of the intercepted arc.



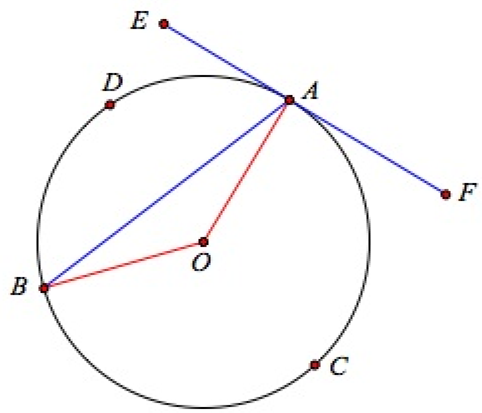
- When a **chord** intersects a **tangent** on a circle, the angle formed is _____________ the measure of the intercepted arc.

In other words:

$m\angle FAB = \frac{1}{2}m\widehat{ACB}$ and $m\angle EAB = \frac{1}{2}m\widehat{ADB}$

Proof

Draw the radii of the circle to points A and B:



ΔAOB is *isosceles* because $\overline{BO}$ and $\overline{OA}$ are both radii, so:

$$\begin{aligned} m\angle BAO = m\angle ABO &= \frac{1}{2}(180^\circ - m\angle AOB) \\ &= 90^\circ - \frac{1}{2}m\angle AOB \end{aligned}$$

We also know that $m\angle BAO + m\angle EAB = 90^\circ$ because $\overline{FE}$ is **tangent** to the circle.

$$(90^\circ - \frac{1}{2}m\angle AOB) + m\angle EAB = 90^\circ$$

Which means that $m\angle EAB = \frac{1}{2}m\angle AOB$

Since $\angle AOB$ is a *central angle* that corresponds to $\widehat{ADB}$ then, $m\angle EAB = \frac{1}{2}m\widehat{ADB}$.

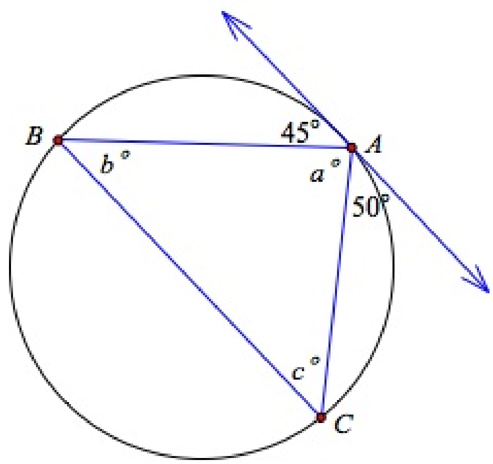
This completes the proof.

Reading Check:

In your own words, describe the angle that is formed when a chord intersects a tangent at a point on a circle.

Example 4

Find the values of a, b and c:



First we find angle a:

$$\begin{aligned}50^\circ + 45^\circ + m\angle a &= 180^\circ \\ 95^\circ + m\angle a &= 180^\circ \\ m\angle a &= 85^\circ\end{aligned}$$

Next we find angles b and c.

Using the Measure of the Tangent-Chord Theorem (which says that the intercepted arc is *double* the angle formed by the __________________ and the ___________________), we conclude that:

$m\widehat{AB} = 2(45^\circ) = 90^\circ$ and $m\widehat{AC} = 2(50^\circ) = 100^\circ$

Therefore,

$$\begin{aligned}m\angle b &= \frac{1}{2}(m\widehat{AC}) = \frac{1}{2}(100^\circ) = 50^\circ \quad \text{and} \\ m\angle c &= \frac{1}{2}(m\widehat{AB}) = \frac{1}{2}(90^\circ) = 45^\circ\end{aligned}$$

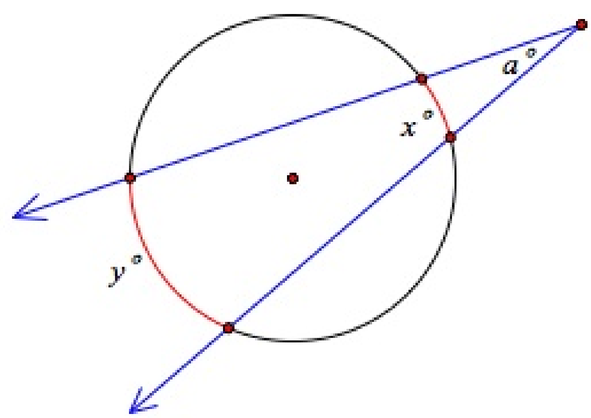
We can confirm this because all three angles in a triangle add up to 180°:

$$amp;m\angle a+m\angle b+m\angle c=180^\circ$$
$$\_\_\_\_\_\_\_\_+\_\_\_\_\_\_\_\_+\_\_\_\_\_\_\_\_=180^\circ$$

And yes, $180^\circ=180^\circ$!

Angles Outside a Circle Theorem

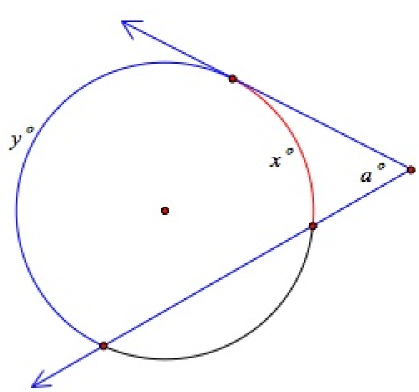
The measure of an angle formed by two **secants** drawn from a point *outside* the circle is equal to *half* the *difference* of the measures of the intercepted arcs.

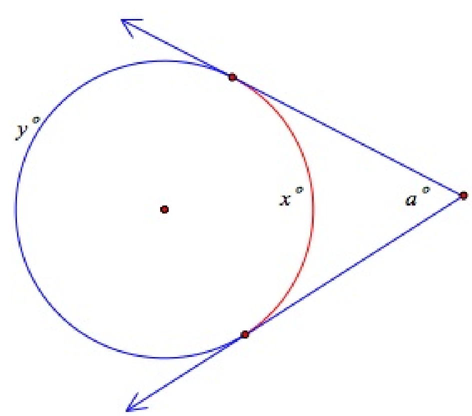


In other words, $m\angle a=\frac{1}{2}(y^\circ-x^\circ)$

- To find the measure of $\angle a$ in the diagram above, first take the arc with a measure of __________ and *subtract* the arc with a measure of ___________. Then take *half* of that difference.

This theorem also applies for an angle formed by two **tangents** to the circle drawn from a point *outside* the circle and for an angle formed by a **tangent** and a **secant** drawn from a point *outside* the circle:





The diagram above is an example of the angle formed by: a _______________________ and a _______________-________

The diagram above is an example of the angle formed by: a _______________________ and a _______________-________

Proof

Draw a line to connect points A and B:

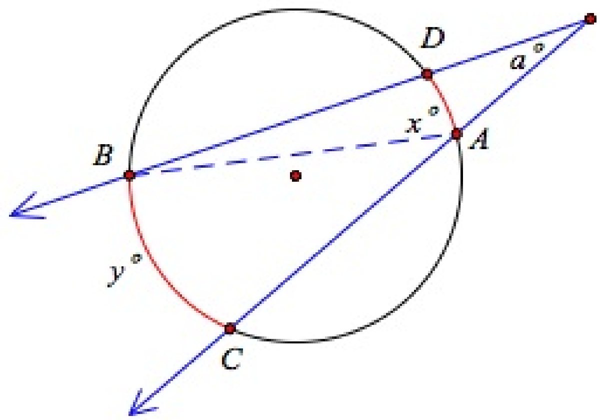
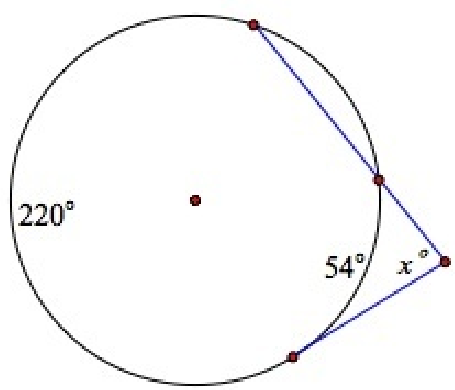


TABLE 9.3:

Statements	Reasons
1. $m\angle DBA = \frac{1}{2}x^\circ$	1. Inscribed angle
2. $m\angle BAC = \frac{1}{2}y^\circ$	2. Inscribed angle
3. $m\angle BAC = m\angle DBA + m\angle a$	3. The measure of an exterior angle in a triangle is equal to the sum of the measures of the remote interior angles.
4. $\frac{1}{2}y^\circ = \frac{1}{2}x^\circ + m\angle a$	4. Substitution
5. $m\angle a = \frac{1}{2}(y^\circ - x^\circ)$	5. Subtract, simplify, and factor.

Example 5

Find the measure of angle x:



$m\angle x = \frac{1}{2}(220^\circ - 54^\circ) = 83^\circ$

Reading Check:

1. *When two secants, two tangents, or a tangent and a secant intersect outside of a circle, the angle formed makes the same relationship with the intercepted arcs. What is this relationship? Describe.*

2. *Draw a picture of two secants that intersect outside of a circle:*

3. *Draw a picture of two tangents that intersect outside of a circle:*

4. *Draw a picture of a secant and a tangent that intersect outside of a circle:*

5. *True or false:* The measure of an angle formed by two secants (*or* two tangents *or* a secant and a tangent) outside a circle is the same as the average of the intercepted arcs.

9.5 Similar Triangles Review

[*Editor's note: This day is set aside for a quiz and a review of similar triangles.]

Learning Objectives

- Review the concepts of angle congruence and segment proportionality in similar triangles.

Chords

Remember from lesson 2: a **chord** is a line segment that has both *endpoints* on a circle.

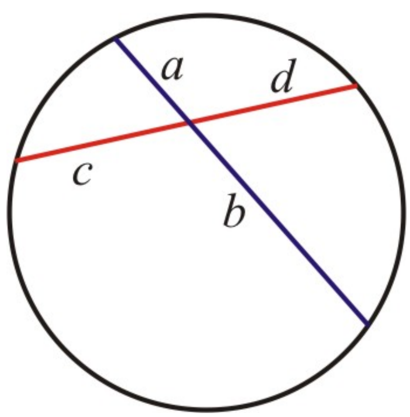
- Line segments whose endpoints are both on a circle are called _______________.

Segments of Chords Theorem

If two **chords** intersect *inside* a circle so that one chord is divided into segments of lengths a and b and the other into segments of lengths c and d, then the segments of the **chords** satisfy the following relationship:

$$ab = cd$$

This means that the *product* of the segment lengths of one chord *equals* the *product* of the segment lengths of the second chord:



- The intersection point splits each ______________________ into two segments.
- The *product* of both segment lengths of one **chord** is ____________________ to the *product* of both segment lengths of the other **chord**.

We prove this theorem on the following page.

Proof

We connect points A and C and points D and B to make ΔAEC and ΔDEB:

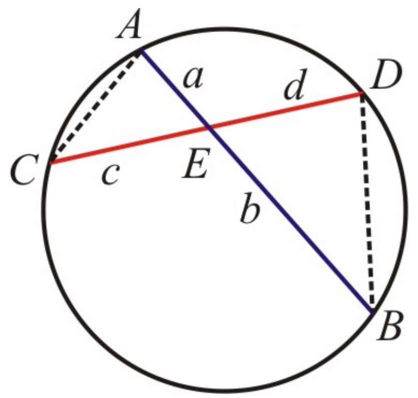
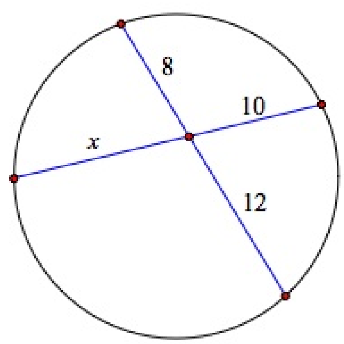


TABLE 9.4:

Statements	Reasons
1. $\angle AEC \cong \angle DEB$	1. Vertical angles are congruent
2. $\angle CAB \cong \angle BDC$	2. Inscribed angles intercept the same arc
3. $\angle ACD \cong \angle ABD$	3. Inscribed angles intercept the same arc
4. $\Delta AEC \cong \Delta DEB$	4. AA similarity postulate
5. $\frac{c}{b} = \frac{a}{d}$	5. In similar triangles, the ratios of corresponding sides are equal.
6. $ab = cd$	6.Cross multiplication

Example 1

Find the value of the variable x:



Use the *products* of the segment lengths of each **chord**:

$$\begin{aligned}10x &= 8 \cdot 12\\10x &= 96\\x &= 9.6\end{aligned}$$

Reading Check:

1. *In your own words, define a chord.*

2. *True or false:* When two chords intersect inside a circle, the sum of the segment lengths of one chord is equal to the sum of the segment lengths of the other chord.

3. How *could you change the statement in #2 above to make it true?*

4. *In the space below, make up your own problem with two chords that intersect inside a circle, and then solve your problem.*

(Hint: if you are having trouble, look at Example 1 on the previous page and model your problem after that one.)

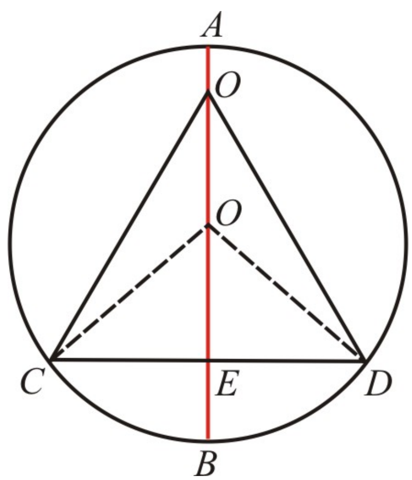
9.6 Segments of Chords

Learning Objectives

- Find the lengths of chords in a circle.
- Find the measure of arcs in a circle.

Perpendicular Bisector of a Chord

- The **perpendicular bisector** of a **chord** is a **diameter**.



- The ___________________________ of a circle is the **perpendicular bisector** of any **chord**.

Proof

Draw two chords, $\overline{AB}$ and $\overline{CD}$ such that $\overline{AB}$ is the **perpendicular bisector** of $\overline{CD}$.

The diagram above is a good visual example of this.

We can see that $\Delta COE \cong \Delta DOE$ for any point O on chord $\overline{AB}$.

The *congruence* of the triangles can be proven by the SAS (Side-Angle-Side) Postulate:

- $\overline{CE} \cong \overline{ED}$ (it gets *bisected* by the **perpendicular bisector**)
- $\overline{OE} \cong \overline{OE}$ (itself)
- $\angle OEC$ and $\angle OED$ are right angles (*perpendicular* makes right angles)

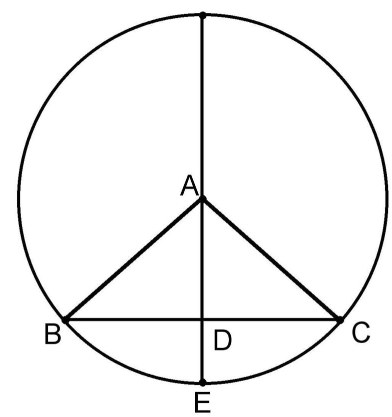
This means that $\overline{CO} \cong \overline{DO}$ because of CPCTC.

Any point that is *equidistant* from C and D lies along $\overline{AB}$ by the Perpendicular Bisector Theorem. Since the *center* of the circle is one such point, it must lie along $\overline{AB}$ so $\overline{AB}$ is a **diameter**.

If O is the *midpoint* of $\overline{AB}$ then $\overline{OC}$ and $\overline{OD}$ are *radii* of the circle and $\overline{AB}$ is a **diameter** of the circle.

Perpendicular Bisector of a Chord Bisects Intercepted Arc

The **perpendicular bisector** of a **chord** *bisects* the **arc** *intercepted* by the chord.



- The __________________________ bisector of a chord also bisects the ________ that is intercepted by that chord.

Proof

We can see that $\Delta CAD \cong \Delta BAD$ because of the SAS Postulate:

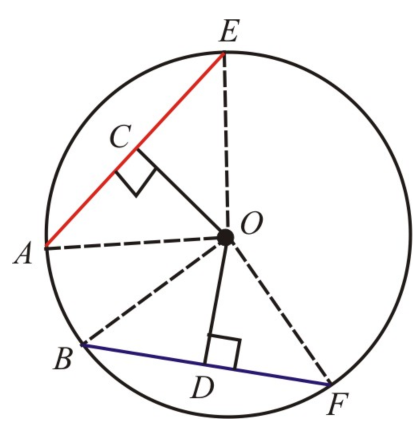
- $\overline{BD} \cong \overline{DC}$ (it gets *bisected* by the **perpendicular bisector**)
- $\overline{DA} \cong \overline{DA}$ (itself)
- $\angle ADB$ and $\angle ADC$ are right angles (*perpendicular* makes right angles)

This means that $\angle DAB \cong \angle DAC$ because of CPCTC, which means that the *intercepted arcs* are *congruent* as well: $\widehat{BE} \cong \widehat{CE}$.

The final congruency statement proves that arc $\widehat{BC}$ is *bisected*, so the proof is complete.

Congruent Chords Equidistant from Center

Congruent **chords** in the same circle are **equidistant** (or *equal distance*) from the *center* of the circle.



Recall that the definition of distance from a *point* to a *line* is the *length* of the *perpendicular* segment drawn to the line from the point. CO and DO are these distances, and we must prove that they are *equal*. Check out the proof on the next page.

Proof

$\Delta AOE \cong \Delta BOF$ because of the SSS Postulate:

- $\overline{AE} \cong \overline{BF}$ (it is given that these are congruent chords)
- $\overline{AO} \cong \overline{BO}$ (both are radii of the circle)
- $\overline{EO} \cong \overline{FO}$ (both are radii of the circle)

Since the *triangles* are *congruent*, their corresponding *altitudes* $\overline{CO}$ and $\overline{DO}$ must also be *congruent*: $\overline{DO} \cong \overline{CO}$.

Therefore, $\overline{AE} \cong \overline{BF}$ and they are *equidistant* from the center.

Converse of Congruent Chords Theorem

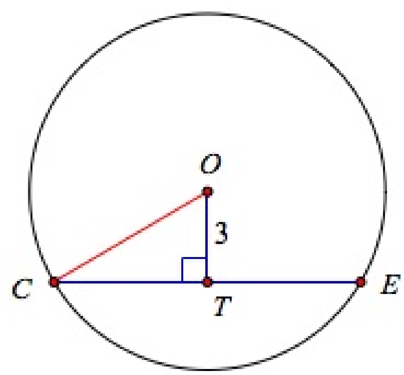
Two **chords** *equidistant* from the center of a circle are *congruent*.

- The *converse* of what we just proved is also *true*: if the **chords** are *equidistant* from the center, then they are ______________________________.

Example 1

$CE = 12$ inches, and is 3 inches from the center of circle O.

Find the radius of the circle.



Draw the radius $\overline{OC}$.

$\overline{OC}$ is the *hypotenuse* of the right triangle ΔCOT.

$OT =$ __________ inches and $CT =$ *half* of 12 inches $=$ __________ inches.

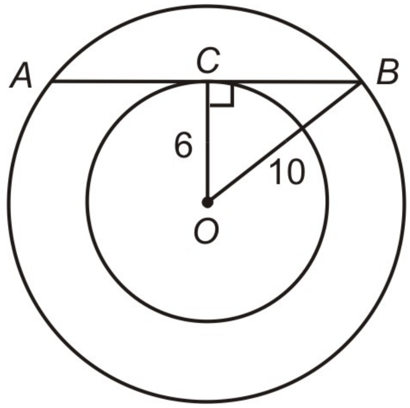
Apply the Pythagorean Theorem:

$$(OC)^2 = (OT)^2 + (CT)^2$$
$$(OC)^2 = 3^2 + 6^2 = 9 + 36 = 45$$
$$OC = \sqrt{45} = 3\sqrt{5} \approx 6.7 \text{ } inches$$

The radius of circle O is about 6.7 inches long.

Example 2

Two concentric circles have radii of 6 inches and 10 inches. A segment tangent to the smaller circle is a chord of the larger circle. What is the length of the segment?



Start by drawing a figure that represents the problem like the one above.

$OC =$ __________ inches and $OB =$ __________ inches

ΔCOB is a *right triangle* because the **radius** $\overline{OC}$ of the *smaller* circle is *perpendicular* to the **tangent** $\overline{AB}$ at point C.

Apply the Pythagorean Theorem:

$$\begin{aligned}
(OC)^2 + (BC)^2 &= (OB)^2 \\
6^2 + (BC)^2 &= 10^2 \\
36 + (BC)^2 &= 100 \\
(BC)^2 &= 100 - 36 = 64 \\
BC &= \sqrt{64} = 8 \textit{ inches}
\end{aligned}$$

You learned earlier in this lesson that because $\overline{OC}$ is a radius (which is *part* of a diameter), it is also a **perpendicular bisector** of the chord $\overline{AB}$, which means that BC is *half* of AB, or $AB = 2BC$.

Therefore, $AB = 2(8) = 16$ inches.

Reading Check:

1. *True or false:* The diameter of a circle is perpendicular to every chord in the circle.
2. *True or false:* The perpendicular bisector of a chord also bisects the arc that is intercepted by the chord.
3. *True or false:* Two chords that are the same distance away from the center of a circle are also the same length.

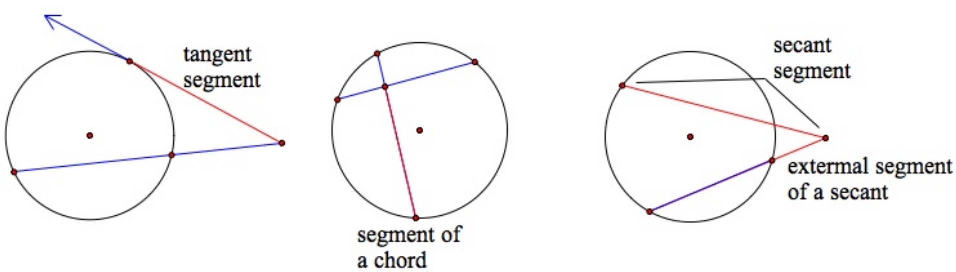
9.7 Segments of Secants and Tangents

Learning Objectives

- Find the lengths of segments of secants and tangents.

Secant and Tangent Segments

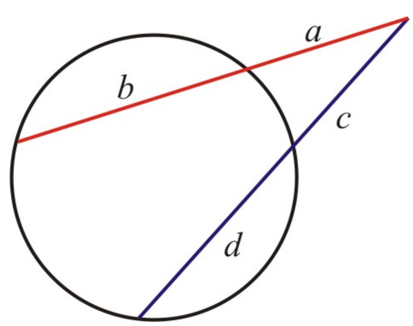
In this section we will discuss **segments** (or *parts of lines*) associated with circles, and the angles formed by these segments. The figures below give the names of segments associated with circles:



Segments of Secants Theorem

If two **secants** are drawn from a common point *outside* a circle and the segments are labeled as below, then the **segments** of the **secants** satisfy the following relationship:

$a(a+b) = c(c+d)$



This means that the *product* of the *outside segment* of one **secant** and its *whole length* equals the *product* of the *outside segment* of the other **secant** and its *whole length*.

- Multiply the outside part of one _________________________ by the whole length of that secant, and it will equal the product of the outside part of the other secant and its whole length.

Proof

We connect points A and D and points B and C to make ΔBCN and ΔADN.

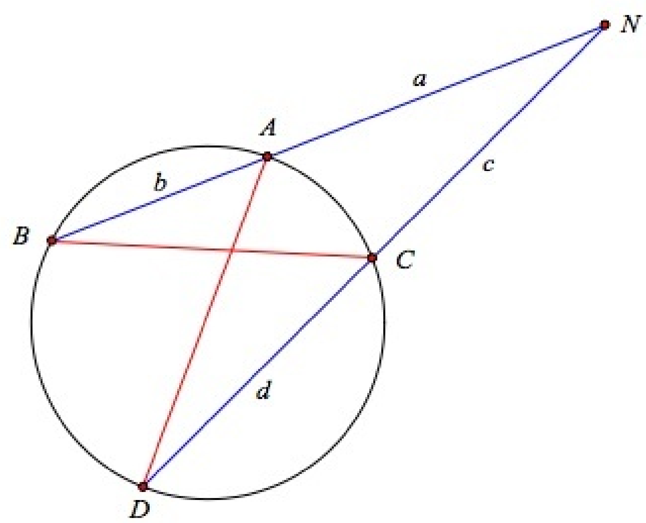
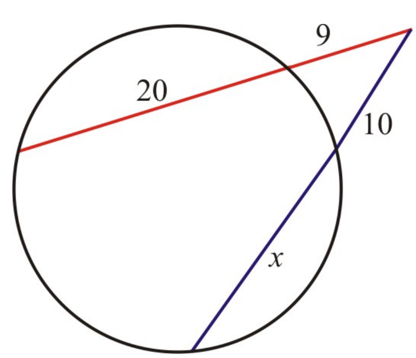


TABLE 9.5:

Statements	Reasons
1. $\angle BNC \cong \angle DNA$	1. These are the same angle.
2. $\angle NBC \cong \angle NDA$	2. Both inscribed angles intercept the same arc, so the angles are congruent.
3. $\Delta BCN \sim \Delta DAN$	3. AA Similarity Postulate
4. $\frac{a}{c} = \frac{c+d}{a+b}$	4. In similar triangles, the ratios of corresponding sides are equal.
5. $a(a+b) = c(c+d)$	5. Cross multiplication

Example 1

Find the value of the variable x:



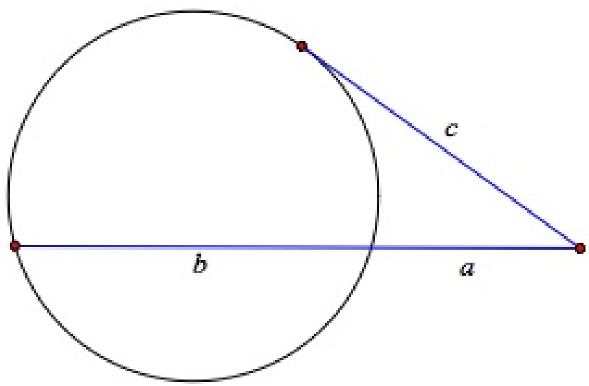
Use the *product* of **secant segments**:

$$10(10+x) = 9(9+20)$$
$$100+10x = 9(29)$$
$$100+10x = 261$$
$$10x = 161$$
$$x = 16.1$$

Segments of Secants and Tangents Theorem

If a **tangent** and a **secant** are drawn from a point *outside* the circle then the **segments** of the **secant** and the **tangent** satisfy the following relationship:

$$a(a+b) = c^2$$



This means that the *product* of the *outside segment* of the **secant** and its *whole length* equals the *square* of the **tangent segment**.

- The ____________________________ segment squared is the same as the product of the outside part of the secant and the secant's whole length.

Before we prove this theorem, let's review the two types of **segment** relationships you just learned. Complete the following table:

TABLE 9.6:

Segments	Draw a picture	Relationship
Secant – Secant (intersection *outside* of a circle)		
Tangent – Secant (intersection *outside* of a circle)		

Proof

We connect points C and A and points B and C to make ΔBCD and ΔCAD.

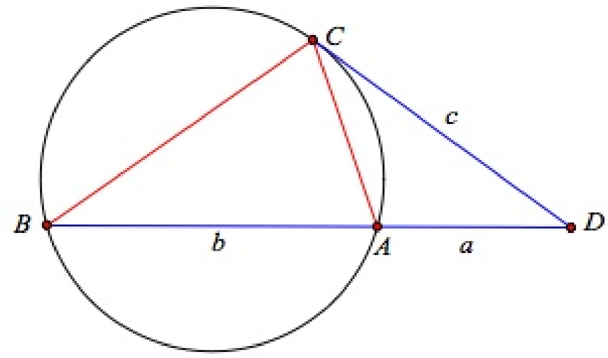


TABLE 9.7:

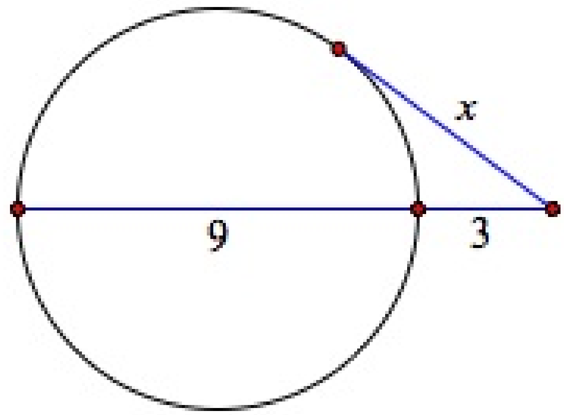
Statements	Reasons
1. $m\angle CDB = m\angle BAC - m\angle DBC$	1. The measure of an angle outside a circle is equal to half the difference of the measures of the intercepted arcs (or their corresponding angles).
2. $m\angle BAC = m\angle ACD + m\angle CDB$	2. The measure of an exterior angle in a triangle is equal to the sum of the measures of the remote interior angles.
3. $m\angle CDB = m\angle ACD + m\angle CDB - m\angle DBC$	3. Substitution
4. $m\angle DBC = m\angle ACD$	4. Subtract and simplify
5. $\Delta BCD \sim \Delta CAD$	5. AA Similarity Postulate
6. $\frac{c}{a+b} = \frac{a}{c}$	6. In similar triangles, the ratios of corresponding sides are equal.
7. $a(a+b) = c^2$	7. Cross multiplication

This proof reviewed some postulates and theorems that you learned earlier in this unit and some that you learned in past units:

- The measure of an angle *outside* a circle is equal to _________________ the *difference* of the measures of the intercepted arcs, which are the same as the arcs' corresponding angles.
- The measure of an _________________________ angle in a triangle is equal to the *sum* of the measures of the *remote interior angles*.
- The **AA Similarity Postulate** says that two triangles are ____________________ if they have two congruent corresponding angles.

Example 2

Find the value of the variable x assuming that it represents the length of a tangent segment:



The **tangent segment** *squared* is equal to the *product* of the **secant segments**:

$$x^2 = 3(9+3)$$
$$x^2 = 3(12) = 36$$
$$x = 6$$

Reading Check:

In the space below, make up a problem that involves two segments that intersect outside of a circle. Your problem can use two secants or a secant and a tangent.

Draw a clear picture and label each segment. Make sure to include a variable for the missing segment. Then, solve your problem.

Graphic Organizer for Unit 9

TABLE 9.8: MORE PARTS of a CIRCLE

Circle Part(s)	Draw a Picture	What is it?	Is there a *relationship* I need to know?	
Inscribed Angle Intercepted Arc				
Central Angle				
2 Chords that intersect inside a circle			Angles/Arcs	Segments

TABLE 9.8: (continued)

Circle Part(s)	**Draw a Picture**	**What is it?**	**Is there a *relationship* I need to know?**	
Tangent				
2 Secants that intersect outside a circle			**Angles/Arcs**	**Segments**
2 Tangents that intersect outside a circle				
A Tangent and a Secant that intersect outside a circle			**Angles/Arcs**	**Segments**

9.8 Inscribed and Circumscribed Polygons

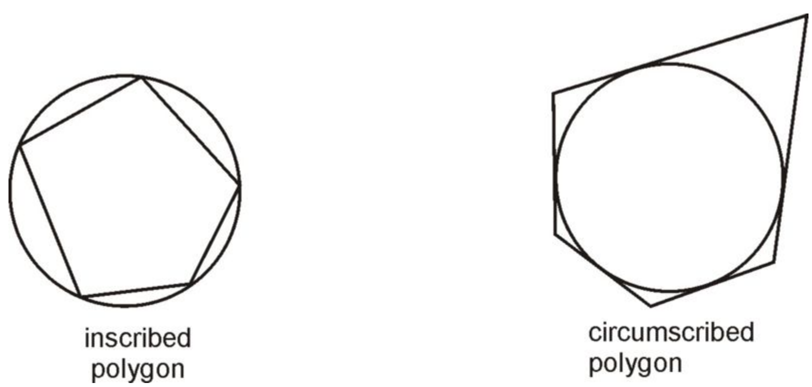
Learning Objectives

- Recognize characteristics of inscribed polygons formed by chords and circumscribed polygons formed by tangent segments.

Inscribed and Circumscribed Polygons

- A convex polygon whose *vertices* all touch a circle is said to be an **inscribed polygon**.
- A convex polygon whose *sides* all touch a circle is said to be a **circumscribed polygon**.

The figures below show examples of **inscribed** and **circumscribed polygons:**



- $A(n)$ ______________________________ polygon looks like it is *inside* a circle.
- $A(n)$ ______________________________ polygon looks like it is *outside* a circle.
- When all of the _________________________ of a convex polygon touch a circle, the polygon is **inscribed**.
- When all of the _________________________ of a convex polygon touch a circle, the polygon is **circumscribed**.

Reading Check:

1. *Which type of polygon, inscribed or circumscribed, has sides that are* chords *of the circle?*

2. *Which type of polygon, inscribed or circumscribed, has sides that are* tangents *of the circle?*

CHAPTER 10 Coordinate Geometry

Chapter Outline

10.1 Vocabulary Self-Rating

TABLE 10.1: Rating Guide: DK: I am sure I don't know it K: I am sure I know it ?: I'm not sure

Word	Before Lesson/Unit	After Lesson/Unit
Distance		
Overbar		
Estimation		
Estimate		
Ruler		
Absolute value		
$x-y$ coordinate plane		
$x-$coordinate		
$y-$coordinate		
Right triangle		
Pythagorean Theorem		
Distance Formula		
Midpoint		
Equidistant		
Slope		
Undefined		
Parallel		
Perpendicular		
Slope-intercept form		
$y-$intercept		
Circle		
Center		
Radius		
Leg		
Hypotenuse		
Concentric		
Translation		
Prime		
Preimage		
Image		
Isometry		
Reflection		
Perpendicular bisector		
Rotation		
Angle of rotation		
Arc		
Central angle		
Diameter		
Acute		
Complementary		

10.2 Distance and Midpoint

Learning Objectives

- Derive the *Distance Formula* using the Pythagorean Theorem.
- Use the *Distance Formula* to find the length of a line segment with known endpoints.
- Use the *Midpoint Formula* to calculate the coordinates of the midpoint of a line segment given both endpoints, or to determine the coordinates of one endpoint given the midpoint and the other endpoint.

Measuring Distances

There are many different ways to identify measurements. This lesson will present some that may be familiar, and probably a few that are new to you.

Before we begin to examine distances, however, it is important to identify the meaning of **distance** in the context of geometry. The **distance** between two points is defined by the *length of the line segment that connects them.*

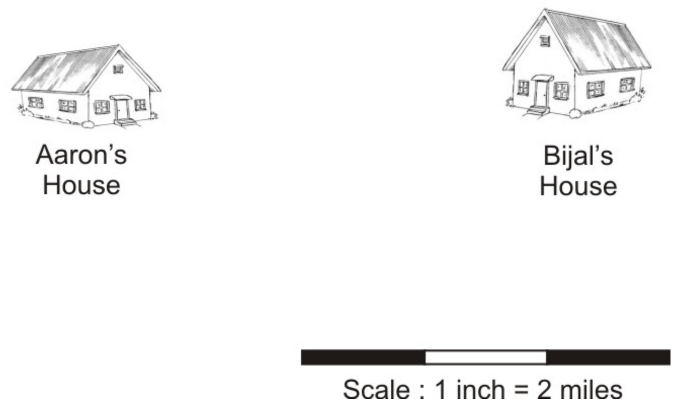
- The **distance** between two points is the __________________________ of the line segment that connects them.

The most common way to *measure* **distance** is with a ruler. Also, distance can be estimated using scale on a map.

Notation Notes: When we name a segment we use the endpoints and an **overbar** (a bar or line above the letters) with no arrows. For example, "segment AB" is written $\overline{AB}$. The *length* of a segment is named by giving the endpoints *without* using an **overbar.** For example, the length of $\overline{AB}$ is written AB. In some books you may also see $m\overline{AB}$, or *measure* of $\overline{AB}$, which means the same as AB, that is, it is the *length* of the segment with endpoints A and B.

Example 1

Use the scale to estimate the distance between Aaron's house and Bijal's house. Assume that the first third of the scale in black represents one inch.



You need to find the **distance** between the two houses in the map. The scale shows a sample distance. Use the scale to estimate the distance. You will find that *approximately* 3 segments of the length of the scale fit between the two points. Be careful — 3 is not the answer to this problem! As the scale shows 1 inch equal to 2 miles, you must multiply 3 units by 2 miles:

$$3 \ inches \cdot \frac{2 \ miles}{1 \ inch} = 6 \ miles$$

The **distance** between the houses is *about* six miles.

You can also use **estimation** to identify measurements in other geometric figures. Remember to include words like *approximately, about,* or *estimation* whenever you are finding an estimated answer.

The word "estimation" means using a non-exact guess of what a number is. Another similar word is "approximation."

Both of these words are nouns. The verb forms are: "to estimate" or "to approximate."

We use these words when we are not sure of the exact measurement of a distance, length, or other number, but when we can make an educated guess.

- To estimate (or to ________________________________) a number means to give a non-exact but educated guess of what it is.

Rulers

You have probably been using **rulers** to measure distances for a long time and you know that a **ruler** is a *tool with measurement markings.*

- A **ruler** is a tool with ________________________________ markings.

Using a ruler: If you use a ruler to find the distance between two points, the distance will be the **absolute value** of the *difference between the numbers* shown on the ruler.

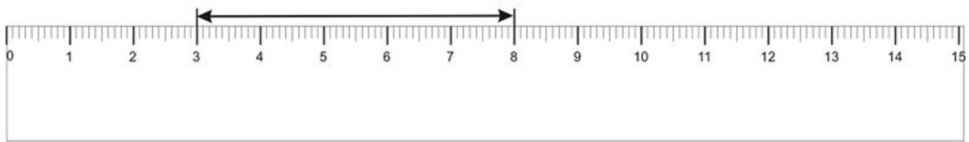
This means that you do not need to start measuring at the zero mark, as long as you use *subtraction* to find the distance.

Note: We say **absolute value** here since distances in geometry must always be *positive*, and subtraction can give a negative result.

- You do not need to measure from zero on a ruler; just ______________________ the start number from the end number to find the distance!
- The *distance* on a ruler is the ____________________________ value of the *difference* between the numbers.
- Absolute value is always a ____________________________ number.

Example 2

What distance is marked on the ruler in the diagram below? Assume that the scale is marked in centimeters.



The way to use the ruler is to find the **absolute value** of the *difference* between the numbers shown. This means you *subtract* the numbers and then make sure your answer is *positive.* The line segment spans from 3 cm to 8 cm:

$$|3-8| = |-5| = 5$$

The **absolute value** of the *difference* between the two numbers shown on the ruler above is 5 cm. So the line segment is 5 cm long.

Remember, we use vertical bars around an expression to show absolute value: $|x|$

Distances on a Grid

In algebra you most likely worked with graphing lines in the $x-y$**coordinate plane.** Sometimes you can find the distance between points on a **coordinate plane** using the values of the coordinates:

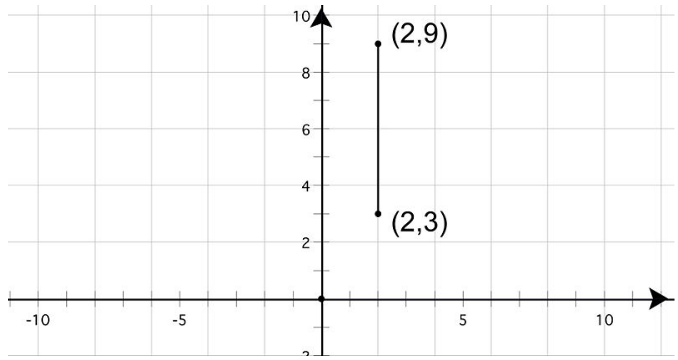
- If the two points line up *horizontally,* look at the change of value in the x-**coordinates.**
- If the two points line up *vertically,* look at the change of value in the y-**coordinates.**

The *change in value* will show the *distance* between the points. Remember to use **absolute value,** just like you did with the **ruler.** Later you will learn how to calculate distance between points that do not line up horizontally or vertically.

- When points line up *horizontally,* they have the *same* _______-coordinate. This means their _______-coordinates are *different* so we take their *difference* to find the distance between the points.
- When points line up *vertically,* they have the *same* _______-coordinate. This means their _______-coordinates are *different* so we take their *difference* to find the distance between the points.

Example 3

What is the distance between the two points shown below?



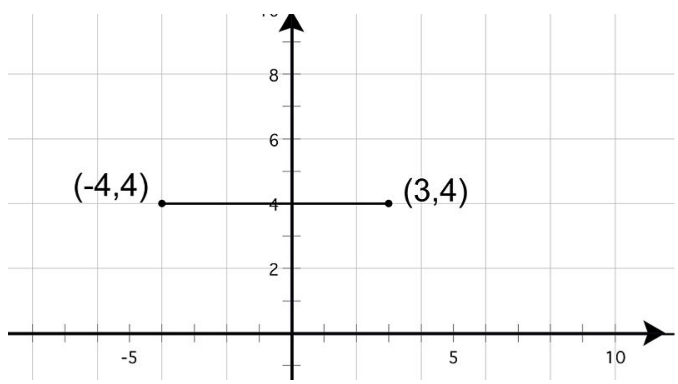
The two points shown on the grid are at (2, 9) and (2, 3). These points line up *vertically* (meaning they have the same $x-$**coordinate** of 2), so we can look at the *difference* in their $y-$**coordinates:**

$$|9-3| = |6| = 6$$

So, the distance between the two points is 6 units.

Example 4

What is the distance between the two points shown below?



The two points shown on the grid are at (–4, 4) and (3, 4). These points line up *horizontally* (meaning they have the same $y-$**coordinate** of 4), so we can look at the *difference* in their $x-$**coordinates.** Remember to take the **absolute value** of the *difference* between the values to find the distance:

$$|-4-3| = |-7| = 7$$

The distance between the two points is 7 units.

Reading Check:

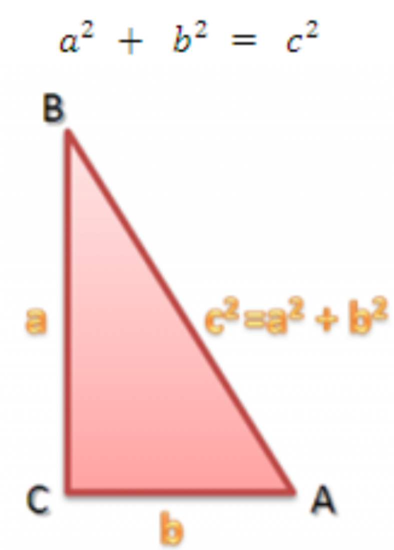
1. *What is absolute value? Explain in your own words.*

2. *When 2 points line up vertically, what value do they have in common?*

3. *When 2 points line up horizontally, what value do they have in common?*

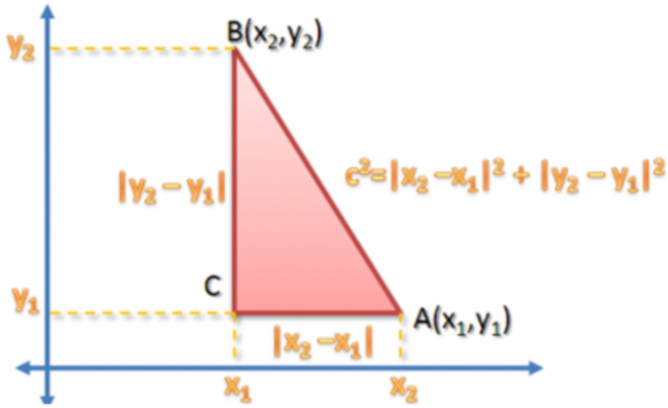
The Distance Formula

We have learned that a **right triangle** with sides of lengths a and b and hypotenuse of length c has a special relationship called the **Pythagorean Theorem.** The sum of the squares of a and b is equal to the square of c. Placing this in equation form we have:



If we put this triangle in a **coordinate plane** so A has coordinates of (x_1, y_1) and B has coordinates of (x_2, y_2), we can find the lengths of the legs of the triangle using what we just learned about points that line up *horizontally* or *vertically:*

the length of AC is $|x_2 - x_1|$ and the length of BC is $|y_2 - y_1|$



We are finding the *length*, which means that we want a *positive* value; the **absolute value** bars guarantee that our answer is always *positive*. But in the final equation,

$$c^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

the **absolute value** bars are *not* needed since we squared all three terms, and squared numbers are always *positive.*

Getting the square root of both sides we have,

$$c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We say that c is the distance between the points A and B, and we call the formula above the ***Distance Formula.***

Reading Check:

1. *On which famous theorem is the Distance Formula based?*

2. *How can you find the distance of one of the legs of a right triangle like the one in the diagram on the previous page? Pick one leg and explain in your own words.*

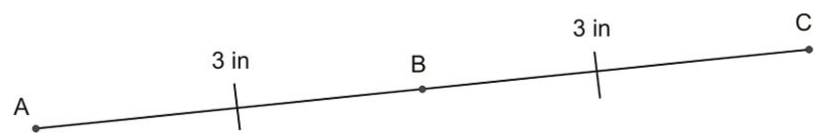
Segment Midpoints

Now that you understand congruent segments, there are a number of new terms and types of figures you can explore.

A **segment midpoint** is a point on a line segment that divides the segment into *two congruent segments.* So, each segment between the midpoint and an endpoint will have the same length.

- A **midpoint** divides a segment into two ___________________________ parts.

In the diagram below, point B is the **midpoint** of $\overline{AC}$ since $\overline{AB}$ is *congruent* to $\overline{BC}$:



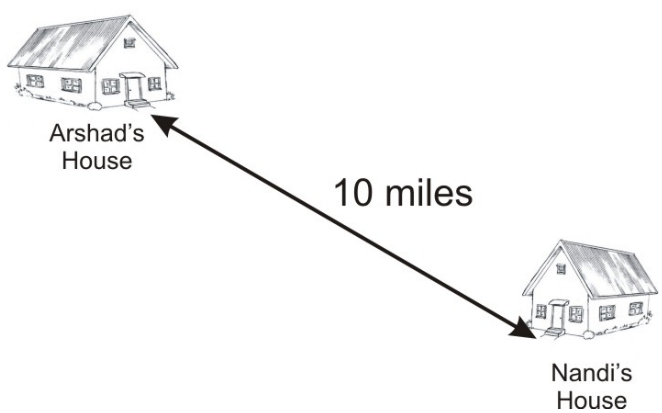
There is even a special postulate dedicated to midpoints:

Segment Midpoint Postulate

Any line segment will have exactly one midpoint—no more, and no less.

Example 5

Nandi and Arshad measure and find that their houses are 10 miles apart. If they agree to meet at the midpoint between their two houses, how far will each of them travel?



The easiest way to find the distance to the **midpoint** of the imagined segment connecting their houses is to divide the *length* (which is 10 miles) by 2:

$$10 \div 2 = 5$$

Each person will travel five miles to meet at the **midpoint** between their houses.

The Midpoint Formula

The **midpoint** is the *middle* point of a line segment. It is **equidistant** (*equal distances*) from both *endpoints.*

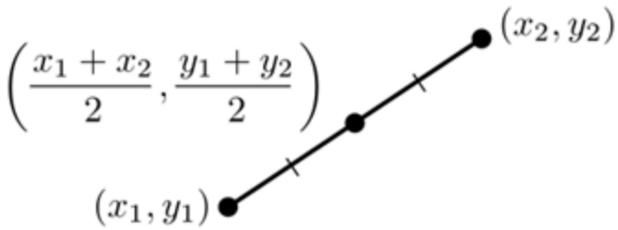
The formula for determining the **midpoint** of a segment in a **coordinate plane** is the *average* of the $x-$**coordinates** and the $y-$**coordinates**. Remember, to find the *average* of 2 numbers, you take the *sum* of the numbers and then *divide* by 2.

If a segment has endpoints (x_1, y_1) and (x_2, y_2):

- the *average* of the $x-$**coordinates** is: $\frac{x_1+x_2}{2}$

- and the *average* of the y-**coordinates** is: $\frac{y_1+y_2}{2}$

Therefore, the **midpoint** is at:



Reading Check:

1. *What is an average? Explain in your own words.*

2. *Where is a midpoint located on a line segment? Describe.*

3. *What does the word equidistant mean?*

4. *How many midpoints can a line segment have?*

5. *In the space below, draw a line segment. Then draw and label its midpoint.*

10.3 Parallel and Perpendicular

Learning Objectives

- Identify and compute *slope* in the coordinate plane.
- Use the relationship between slopes of *parallel* lines.
- Use the relationship between slopes of *perpendicular* lines.
- Identify equations of *parallel* lines.
- Identify equations of *perpendicular* lines.

Slope in the Coordinate Plane

If you look at a graph of a line, you can think of the **slope** as the *steepness* of the line.

- **Slope** is the measure of the ______________________________ of a line.

Mathematically, you can calculate the **slope** using two different points on a line. Given two points (x_1, y_1) and (x_2, y_2) the **slope** is:

$$slope = \frac{y_2 - y_1}{x_2 - x_1}$$

You may have also learned that **slope** equals "rise over run."

This means that:

- The numerator (top) of the fraction is the "rise," or how many units the slope goes *up* (positive) or *down* (negative).

$\rightarrow$ *Up* or *down* is how the **slope** moves along the $y-$axis.

- The denominator (bottom) of the fraction is the "run," or how many units the slope goes to the *right* (positive) or *left* (negative).

$\rightarrow$ *Right* or *left* is how the **slope** moves along the $x-$axis.

You can remember "rise" as up or down because an elevator "rises" up or down.

"Rise" (up/down) is in the y direction.

You can remember "run" as moving right or left because a person "runs" with her right and left feet.

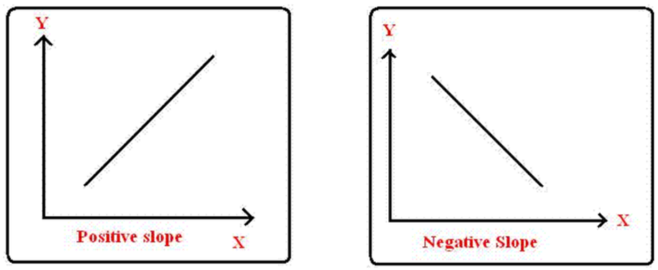
"Run" (right/left) is in the x direction.

- The numerator of the **slope** represents the change in the _________ direction.

- The **slope's** ________________________ represents the change in the x direction.

In other words, first calculate the distance that the line travels *up* (or *down*), and then divide that value by the distance the line travels *left to right.*

A line that goes *up* from *left to right* has *positive* **slope,** and a line that goes *down* from *left to right* has *negative* **slope:**



images from http://www.tutorvista.com/math/positive-and-negative-slope

- A line that goes *up* from left to right has a __________________________ slope.
- A line that goes *down* from left to right has a _________________________ slope.

Example 1

What is the slope of a line that goes through the points (2, 2) *and* (4, 6)?

You can use the **slope** formula on the previous page to find the **slope** of this line. When substituting values, (x_1, y_1) is (2, 2) and (x_2, y_2) is (4, 6):

$x_1 =$ _________, $y_1 =$ _________, and $x_2 =$ _________, $y_2 =$ _________

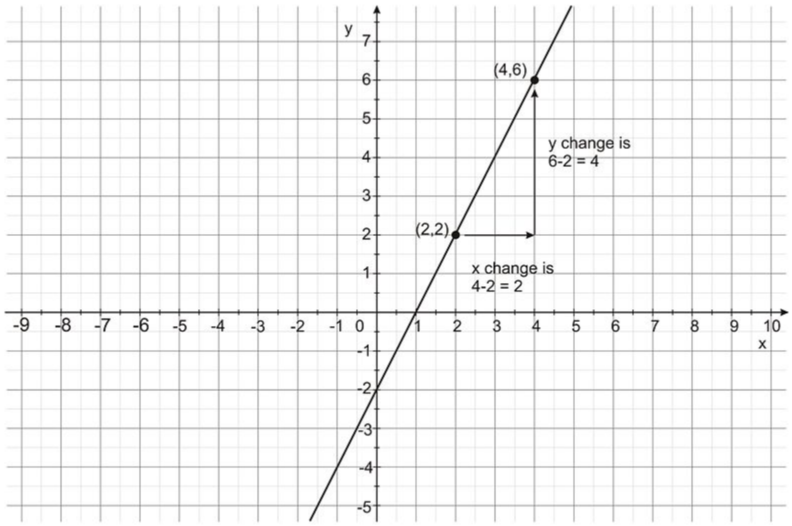
$$slope = \frac{y_2 - y_1}{x_2 - x_1}$$
$$slope = \frac{6-2}{4-2} = \frac{4}{2} = 2$$

The **slope** of this line is 2.

→ What does that mean graphically?

Look at the graph on the next page to see what the line looks like.

Notice: If the **slope** is *positive,* the line should go *up* from left to right. Does it?



You can see that the line "rises" 4 units as it "runs" 2 units to the right. So, the "rise" (numerator) is 4 units and the "run" (denominator) is 2 units. Since $4 \div 2 = 2$, the **slope** of this line is 2.

As you read on the previous page, the **slope** of this line is 2, a *positive* number.

- Any line with a *positive* **slope** will go ____________________ from *left* to *right.*
- Any line with a *negative* **slope** will go ____________________ from *left* to *right.*

Example 2

What is the slope of the line that goes through the points (1, 9) *and* (3, 3)?

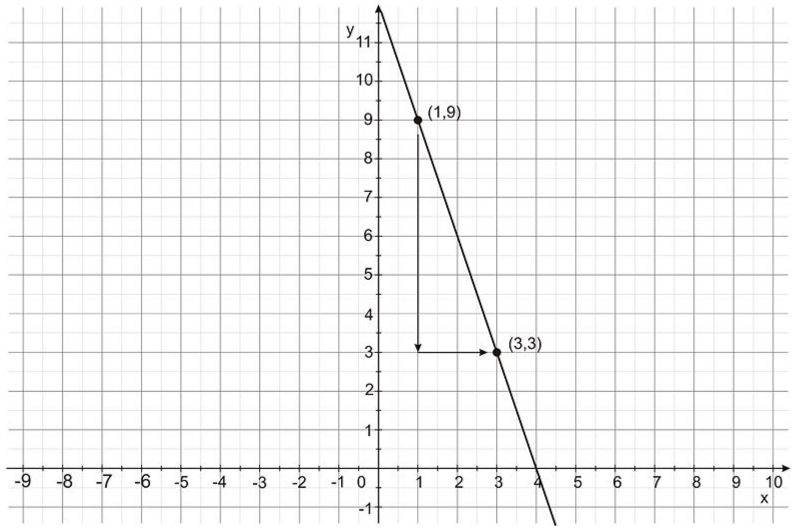
Again, use the formula to find the **slope** of this line:

$x_1 =$ _________, $y_1 =$ _________, and $x_2 =$ _________, $y_2 =$ _________

$$slope = \frac{y_2 - y_1}{x_2 - x_1}$$
$$slope = \frac{3-9}{3-1} = \frac{-6}{2} = -3$$

The **slope** of this line is –3.

Because the **slope** of the line in example 2 is *negative,* it will go *down* to the *right.* The points and the line that connects them are shown below:



Some types of lines have special **slopes.** Check out following examples to see what happens with *horizontal* and *vertical* lines.

Example 3

What is the slope of a line that goes through the points (4, 4) *and* (8, 4)?

Use the formula to find the **slope** of this line:

$x_1 = \_\_, y_1 = \_\_$, and $x_2 = \_\_, y_2 = \_\_$

$$slope = \frac{y_2 - y_1}{x_2 - x_1}$$
$$slope = \frac{4-4}{8-4} = \frac{0}{8} = 0$$

The **slope** of this line is 0.

Every line with a **slope** of 0 is *horizontal.*

- A ____ line has a **slope** equal to zero.

Example 4

What is the slope of a line that goes through the points (3, 2) *and* (3, 6)?

Use the formula to find the **slope** of this line:

$x_1 = \_\_\_\_\_\_\_\_\_, y_1 = \_\_\_\_\_\_\_\_\_$, and $x_2 = \_\_\_\_\_\_\_\_\_, y_2 = \_\_\_\_\_\_\_\_\_$

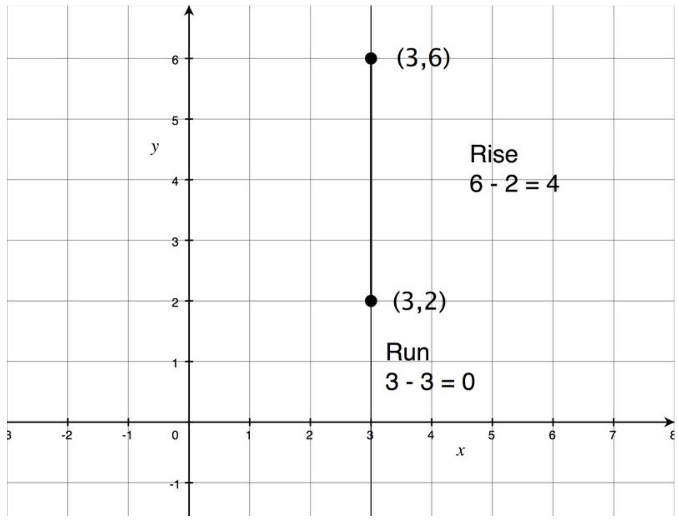
$$slope = \frac{y_2 - y_1}{x_2 - x_1}$$
$$slope = \frac{6-2}{3-3} = \frac{4}{0}$$

Zero is *not* allowed to be in the denominator of a fraction! Therefore, the **slope** of this line is **undefined.**

Every line with an **undefined slope** is *vertical.*

- All *vertical* lines have **slopes** that are ______________________.

The line in example 4 is *vertical* and its slope is undefined:

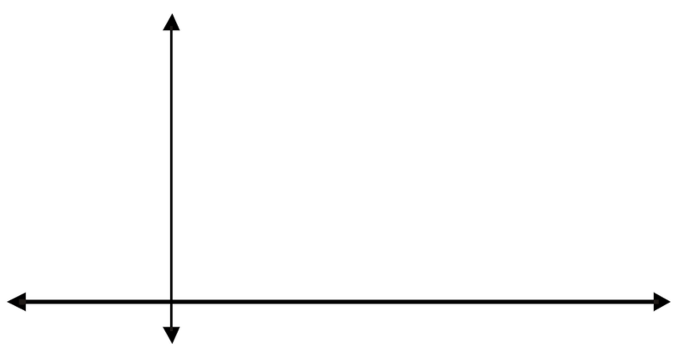


In review, if you look at a graph of a line from *left to right,* then:

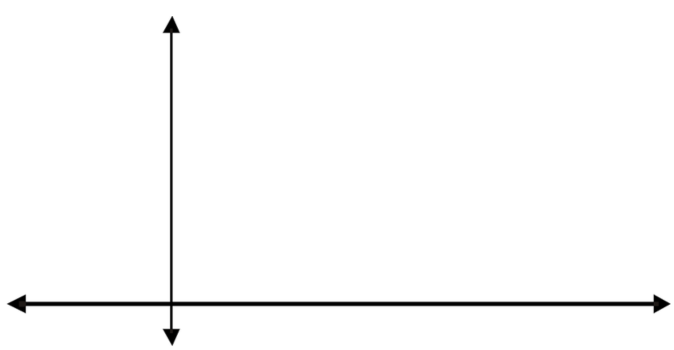
- Lines with *positive* slopes point *up* to the *right.*
- Lines with *negative* slopes point *down* to the *right.*
- *Horizontal* lines have a slope of *zero.*
- *Vertical* lines have *undefined* slope.

Reading Check:

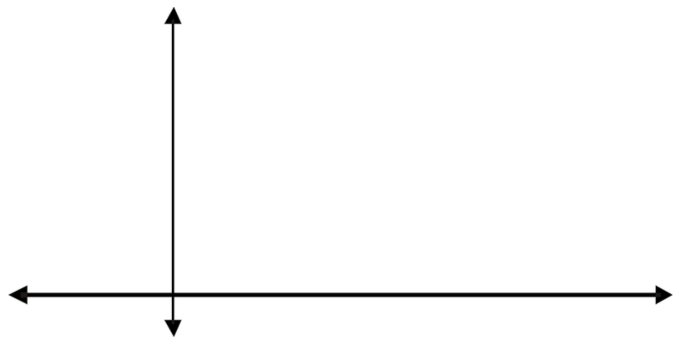
1. *On the coordinate plane below, draw a line with a positive slope.*



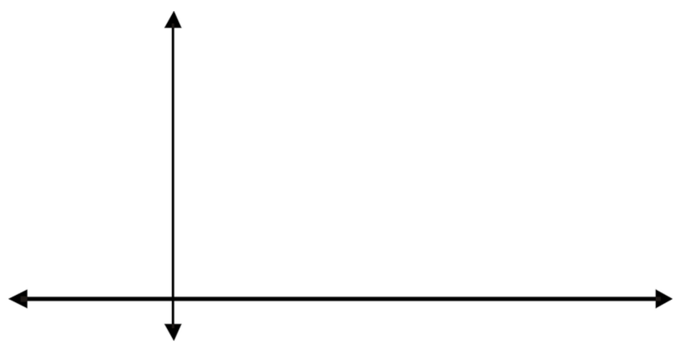
2. *On the coordinate plane below, draw a line with a negative slope.*



3. *On the coordinate plane below, draw a line with a slope of zero.*



4. *On the coordinate plane below, draw a line with an undefined slope.*



Slopes of Parallel Lines

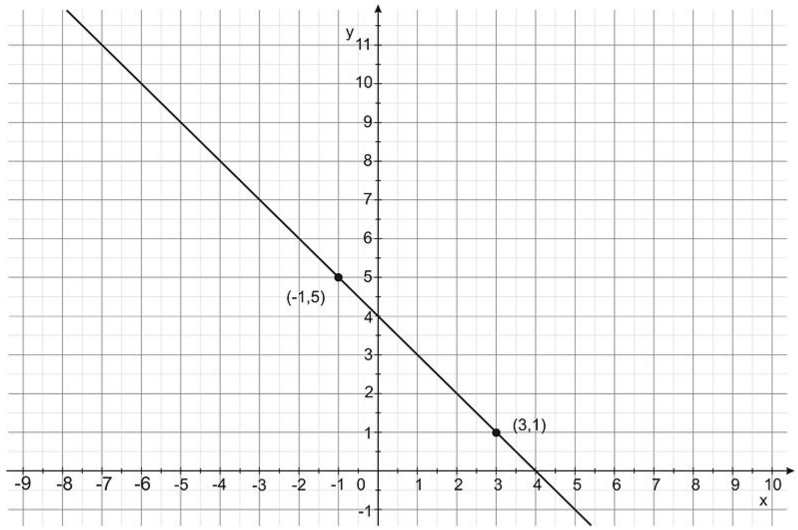
Now that you know how to find the **slope** of lines using $x-$ and $y-$coordinates, you can think about how lines are related to their **slopes.**

If two lines in the coordinate plane are **parallel,** then they will have the *same* slope. Conversely, if two lines in the coordinate plane have the *same* slope, then those lines are **parallel.**

- **Parallel** lines have the ______________________________ **slope.**

Example 5

Which of the following answers below could represent the slope of a line parallel to the one shown on the graph?



A. – 4

B. – 1

C. $\frac{1}{4}$

D. 1

Since you are looking for the **slope** of a **parallel** line, it will have the *same* **slope** as the line on the graph. First find the **slope** of the given line, and then choose the answer with that *same* **slope.** To do this, pick any two points on the line and use the **slope** formula.

For example, for the points (–1, 5) and (3, 1) :

$x_1 =$ _________, $y_1 =$ _________, and $x_2 =$ _________, $y_2 =$ _________

$$slope = \frac{y_2 - y_1}{x_2 - x_1}$$
$$slope = \frac{1-5}{3-(-1)} = \frac{-4}{3+1} = \frac{-4}{4} = -1$$

The **slope** of the line on the graph is –1. The answer is *B*.

Slopes of Perpendicular Lines

Parallel lines have the *same* **slope.** There is also a mathematical relationship for the **slopes** of **perpendicular** lines.

The **slopes** of **perpendicular** lines will be the *opposite reciprocal* of each other.

Opposite here means the *opposite sign.*

If a **slope** is *positive,* then its *opposite* is *negative.*

If a **slope** is *negative,* then its *opposite* is *positive.*

A *reciprocal* is a fraction with its numerator and denominator *flipped.*

The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$. The reciprocal of $\frac{1}{2}$ is 2. The reciprocal of 4 is $\frac{1}{4}$.

- The *opposite* of 5 is _______________.

- The *reciprocal* of 5 is _______________.

The *opposite reciprocal* can be found in two steps:

1. First, find the *reciprocal* of the given **slope.** If the **slope** is a fraction, you can simply *switch* the numbers in the numerator and the denominator to find the *reciprocal.* If the **slope** is not a fraction, you can make it into a fraction by putting a 1 in the denominator. Then find the *reciprocal* by flipping the numerator and denominator.

2. The second step is to find the *opposite* of the given number. If the value is *positive,* make it *negative.* If the value is *negative,* make it *positive.*

The *opposite reciprocal* of $\frac{5}{4}$ is $-\frac{4}{5}$ and the *opposite reciprocal* of - 3 is $\frac{1}{3}$.

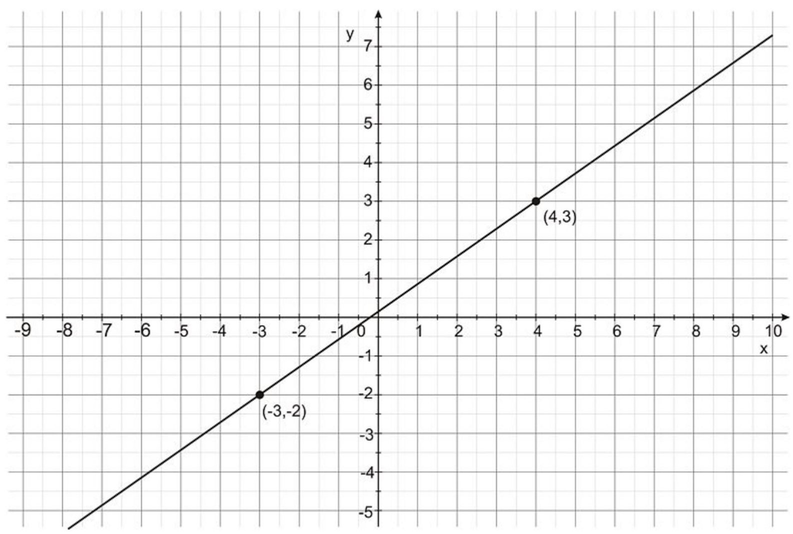
- The *opposite reciprocal* of 5 is _______________.

Another way to check if lines are **perpendicular** is to *multiply* their **slopes:** if the **slopes** of two lines *multiply* to be –1, then the two lines are **perpendicular.**

- The slopes of ____________________________ lines multiply to be –1.

Example 6

Which of the following numbers could represent the slope of a line perpendicular to the one shown below?



A. $\frac{-7}{5}$

B. $\frac{7}{5}$

C. $\frac{-5}{7}$

D. $\frac{5}{7}$

Since you are looking for the **slope** of a **perpendicular** line, it will be the *opposite reciprocal* of the **slope** of the line on the graph. First find the **slope** of the given line, then find its *opposite reciprocal.* You can use the slope formula to find the original line's slope. Pick two points on the line.

For example, for the points (–3, –2) and (4, 3) :

$x_1 =$ _________, $y_1 =$ _________, and $x_2 =$ _________, $y_2 =$ _________

$$slope = \frac{y_2 - y_1}{x_2 - x_1}$$
$$slope = \frac{3-(-2)}{4-(-3)} = \frac{3+2}{4+3} = \frac{5}{7}$$

The **slope** of the line on the graph is $\frac{5}{7}$. Now find the *opposite reciprocal* of that value. First *switch* the numerator and denominator in the fraction, then find the *opposite sign.* The *opposite reciprocal* of $\frac{5}{7}$ is $\frac{-7}{5}$. The answer is A.

Slope-Intercept Equations

The most common type of linear equation to study is called **slope-intercept form,** which uses both the **slope** of the line and its $y-$**intercept.** A $y-$**intercept** is the point where the line crosses the vertical $y-$axis. This is the value of y when x is equal to 0.

- **Slope-intercept form** is an equation that uses the ______________________ and the ______________-______________ of a line.
- The $y-$**intercept** is the point where the line intersects the ________________.
- At the $y-$**intercept,** x equals ____________.

The formula for an equation in **slope-intercept form** is:

$$y = mx + b$$

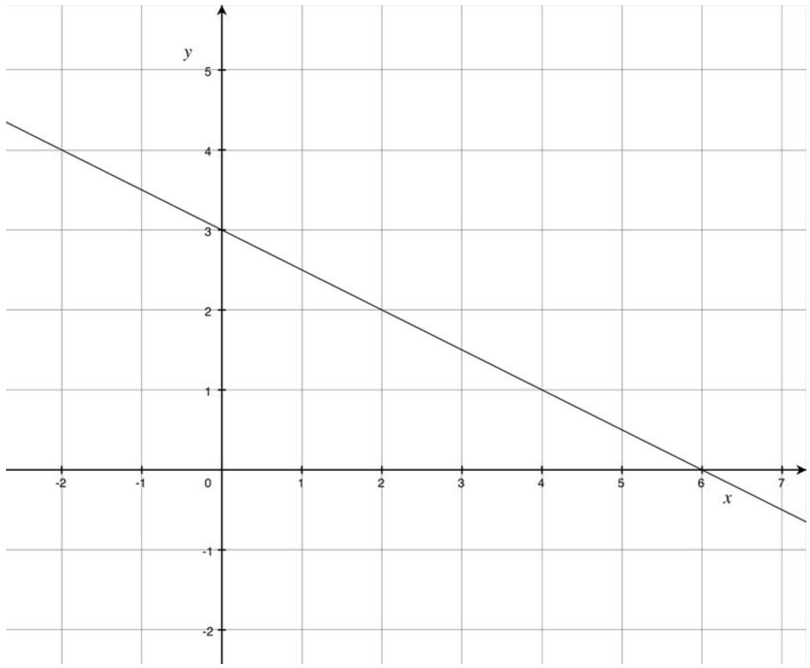
In this equation, y and x remain as variables, m is the **slope** of the line, and b is the $y-$**intercept** of the line. For example, if you know that a line has a **slope** of 4 and it crosses the $y-$axis at (0, 8), then its equation in **slope-intercept form** is: $y = 4x + 8$.

- In **slope-intercept form,** m represents the ____________.
- In **slope-intercept form,** b represents the ____________.

This form is especially useful for finding the equation of a line given its graph. You already know how to calculate the **slope** by finding two points and using the slope formula. You can find the $y-$**intercept** by seeing where the line crosses the $y-$axis on the graph. The value of b is the $y-$coordinate of this point.

Example 7

Write an equation in slope-intercept form that represents the following line:



First find the **slope** of the line. You already know how to do this using the slope formula. There are no points given on the line, so you have to pick your own points. See where the line goes right through an intersection (corner point) on the graph paper. You can use the two points (0, 3) and (2, 2) :

$x_1 =$ _________, $y_1 =$ _________, and $x_2 =$ _________, $y_2 =$ _________

$$slope = \frac{y_2 - y_1}{x_2 - x_1}$$
$$slope = \frac{2-3}{2-0} = \frac{-1}{2} = -\frac{1}{2}$$

The **slope** of the line is $\frac{-1}{2}$. This will replace m in the **slope-intercept** equation.

Now you need to find the $y-$**intercept.** On the graph, find where the line intersects the $y-$axis. It crosses the $y-$axis at (0, 3) so the $y-$**intercept** is 3. This will replace b in the **slope-intercept** equation, so now you have all the information you need.

The equation for the line shown in the graph is: $y = -\frac{1}{2}x + 3$.

- In the slope-intercept equation, the slope is represented by the letter _________.
- In the slope-intercept equation, the $y-$intercept is the letter _________.

Equations of Parallel Lines

You studied **parallel** lines and their graphical relationships, so now you will learn how to easily identify equations of **parallel** lines. When looking for **parallel** lines, look for equations that have the *same* **slope.**

As long as the $y-$**intercepts** are *not* the same and the **slopes** are *equal,* the lines are **parallel.** If the $y-$**intercept** and the **slope** are *both* the *same,* then the two equations are for the *same* exact line, and a line cannot be parallel to itself.

- **Parallel** lines have the ________________________ **slope.**

Reading Check:

1. *True or false:*

The **reciprocal** of a fraction is when you flip the numerator and the denominator.

2. *Make up an example that supports the statement in #1 above.*

3. *What is the slope-intercept form of an equation?*

4. *What do the letters m and b stand for in the slope-intercept equation?*

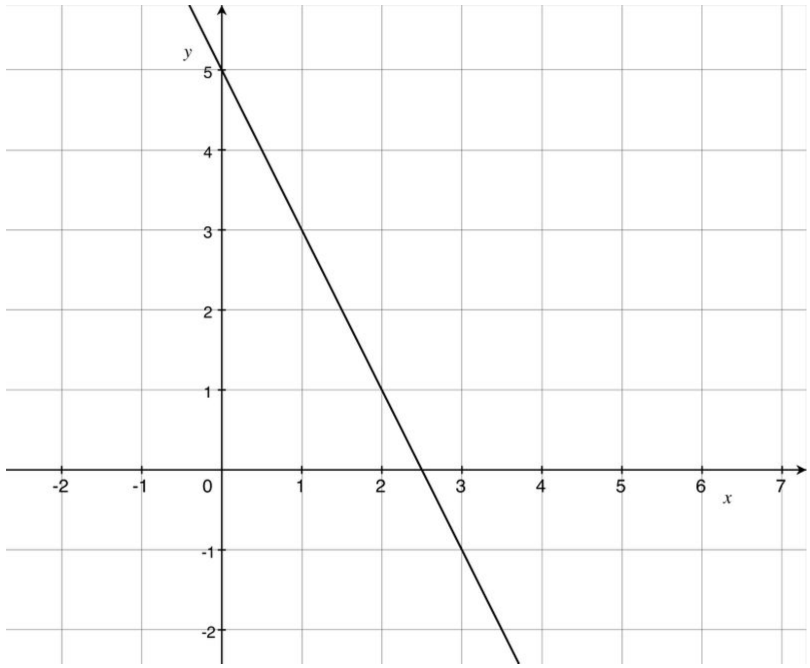
m :

b :

5. *How are the slopes of parallel lines related?*

Example 8

Juan drew the line below:



Which of the following equations could represent a line parallel to the one Juan drew?

A. $y = -\frac{1}{2}x - 6$

B. $y = \frac{1}{2}x + 9$

C. $y = -2x - 18$

D. $y = 2x + 1$

If you find the **slope** of the line in Juan's graph, you can find the **slope** of a **parallel** line because it will be the *same.* Pick two points on the graph and find the **slope** using the slope formula. Use the points (0, 5) and (1, 3) :

$x_1 =$ __________, $y_1 =$ __________, and $x_2 =$ __________, $y_2 =$ __________

$$slope = \frac{y_2 - y_1}{x_2 - x_1}$$
$$slope = \frac{3-5}{1-0} = \frac{-2}{1} = -2$$

The **slope** of Juan's line is –2. Look at your four answer choices: which equation has a slope of –2? All other parts of the equation do not matter. The only equation that has a slope of –2 is choice C, so that is the correct answer.

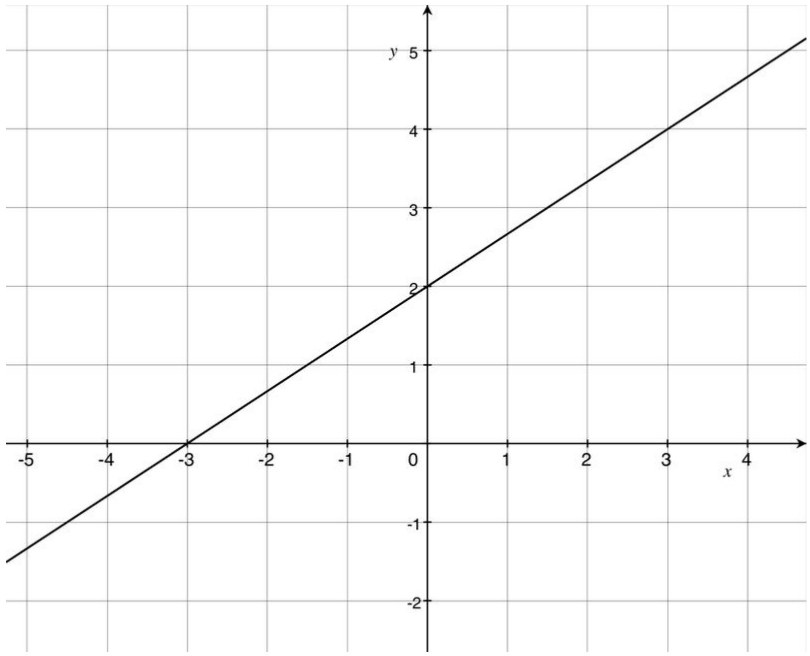
Equations of Perpendicular Lines

You also studied **perpendicular** lines and their graphical relationships: remember that the **slopes** of **perpendicular** lines are *opposite reciprocals.* To easily identify equations of **perpendicular** lines, look for equations that have **slopes** that are *opposite reciprocals* of each other.

Here, it does not matter what the $y-$**intercept** is; as long as the **slopes** are *opposite reciprocals,* the lines are **perpendicular.**

Example 9

Kara drew the line in this graph:



Which of the following equations could represent a line perpendicular to the one Kara drew above?

A. $y = \frac{3}{2}x + 10$

B. $y = -\frac{3}{2}x + 6$

C. $y = \frac{2}{3}x - 4$

D. $y = -\frac{2}{3}x - 1$

First find the **slope** of the line in Kara's graph. Then find the *opposite reciprocal* of this **slope.** To begin, pick two points on the graph and calculate the **slope** using the slope formula. Use the points (0, 2) and (3, 4) :

$x_1 =$ _________, $y_1 =$ _________, and $x_2 =$ _________, $y_2 =$ _________

$$slope = \frac{y_2 - y_1}{x_2 - x_1}$$
$$slope = \frac{4 - 2}{3 - 0} = \frac{2}{3}$$

The **slope** of Kara's line on the graph is $\frac{2}{3}$.

Find the *opposite reciprocal:* the *reciprocal* of $\frac{2}{3}$ is $\frac{3}{2}$, and the *opposite* of $\frac{3}{2}$ is $-\frac{3}{2}$.

So, $-\frac{3}{2}$ is the *opposite reciprocal* of (or **perpendicular slope** to) $\frac{2}{3}$.

Now look in your answer choices for the equation that has a slope of $-\frac{3}{2}$.

The only equation that has a slope of $-\frac{3}{2}$ is choice B, so that is the correct answer.

Reading Check:

1. *How are the slopes of perpendicular lines related to each other?*

2. *In the context of perpendicular slope values, what does opposite mean?*

3. *True or false: On a graph, perpendicular lines intersect at an angle of* 45°.

4. *Correct the statement in #3 above to make it true.*

10.4 Equation of a Circle

Learning Objectives

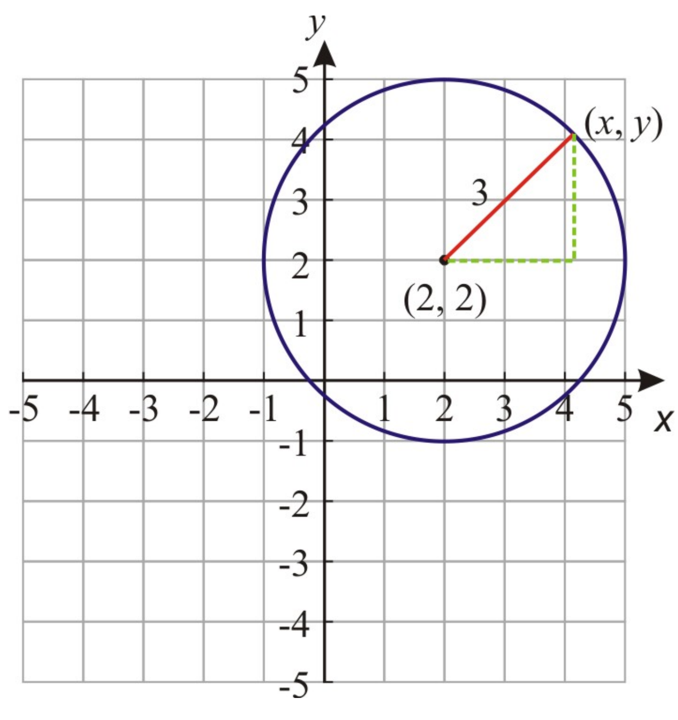
- Write the equation of a *circle.*

Equations and Graphs of Circles

A **circle** is defined as the set of all points that are the same distance from a single point called the **center.** This definition can be used to find an equation of a circle in the coordinate plane.

- A **circle** is the set of all points *equidistant* from the ________________________.

Look at the **circle** shown below. As you can see, this circle has its **center** at the point (2, 2) and it has a **radius** of 3.



All of the points (x,y) on the **circle** are a *distance* of 3 units away from the **center** of the **circle.**

We can express this information as an equation with the help of the **Pythagorean Theorem.** The *right triangle* shown above has **legs** of lengths $(x-2)$ and $(y-2)$, and **hypotenuse** of length 3. We can write:

$$(x-2)^2+(y-2)^2=3^2 \qquad \text{or}$$
$$(x-2)^2+(y-2)^2=9$$

We can *generalize* this equation for a **circle** with **center** at point (x_0, y_0) and **radius** r:

$$(x-x_0)^2+(y-y_0)^2=r^2$$

Example 1

Find the center and radius of the following circles:

A. $(x-4)^2+(y-1)^2=25$

B. $(x+1)^2+(y-2)^2=4$

A. We rewrite the equation as: $(x-4)^2+(y-1)^2=5^2$. Compare this to the standard equation. The **center** of the **circle** is at the point (4, 1) and the **radius** is 5.

B. We rewrite the equation as: $(x-(-1))^2+(y-2)^2=2^2$. The **center** of the **circle** is at the point (–1, 2) and the **radius** is 2.

Example 2

Graph the following circles:

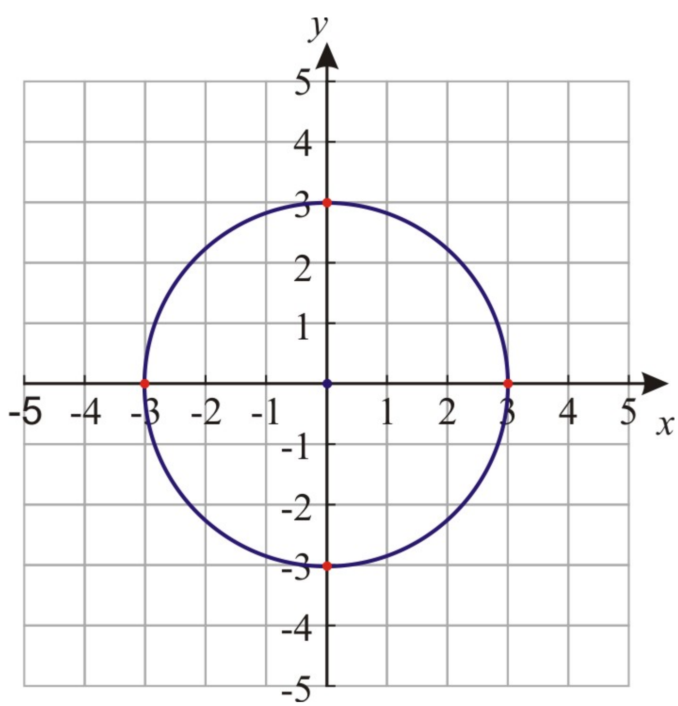
A. $x^2+y^2=9$

B. $(x+2)^2+y^2=1$

In order to graph a **circle,** we first graph the **center** point and then draw points that are the *length* of the **radius** *away* from the **center** in the directions up, down, right, and left. Then connect the outer points in a smooth circle!

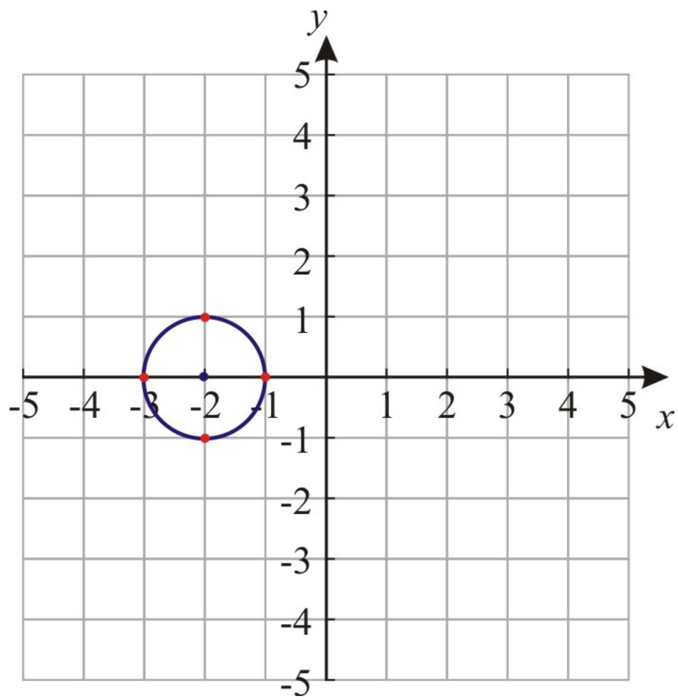
A. We rewrite the equation as: $(x-0)^2+(y-0)^2=3^2$. The center of the circle is at the point (0, 0) and the radius is 3.

Plot the center point and a point 3 units up at (0, 3), 3 units down at (0, –3), 3 units right at (3, 0) and 3 units left at (–3, 0) :



B. We rewrite the equation as: $(x-(-2))^2+(y-0)^2=1^2$. The center of the circle is at the point (–2, 0) and the radius is 1.

Plot the center point and a point 1 unit up at (–2, 1), 1 unit down at (–2, –1), 1 unit right at (–1, 0) and 1 unit left at (–3, 0) :



Reading Check:

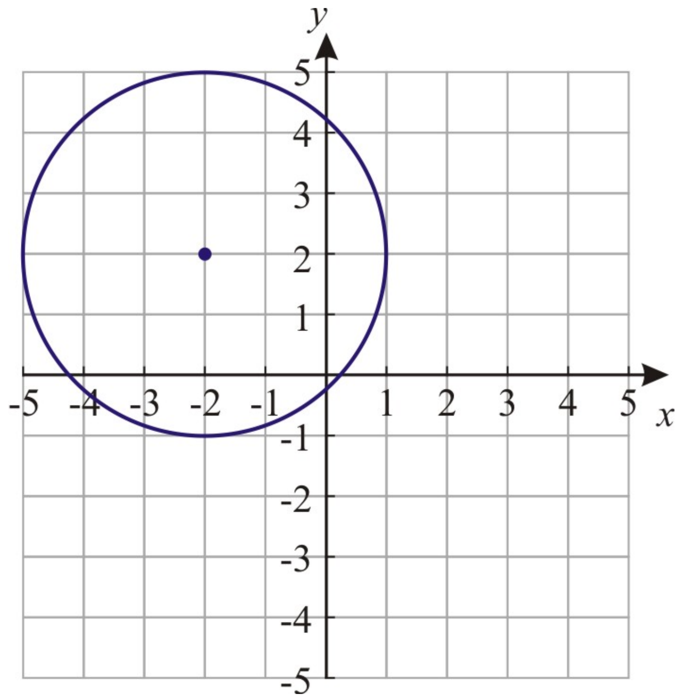
1. *In your own words, describe the radius of a circle.*

2. *In the general equation of a circle* $(x-x_0)^2+(y-y_0)^2=r^2$, *the variables* x_0 *and* y_0 *stand for a special point. What point is this?*

3. *How can you find the radius of a circle from its equation? What do you need to do to the right side of the equation?*

Example 3

Write the equation of the circle in the graph below:



From the graph, we can see that the **center** of the **circle** is at the point (–2, 2) and the **radius** is 3 units long, so we use these numbers in the standard circle equation:

$$(x+2)^2+(y-2)^2 = 3^2$$
$$(x+2)^2+(y-2)^2 = 9$$

Example 4

Determine if the point $(1,3)$ *is on the circle given by the equation:*

$$(x-1)^2+(y+1)^2 = 16$$

In order to find the answer, we simply plug the point (1, 3) into the equation of the **circle** given.

Substitute the number _________ for *x* and the number _________ for *y*:

$$(1-1)^2+(3+1)^2 = 16$$
$$(0)^2+(4)^2 = 16$$
$$16 = 16$$

Since we end up with a *true* statement, the point (1, 3) satisfies the equation. Therefore, the point <u>is</u> on the circle.

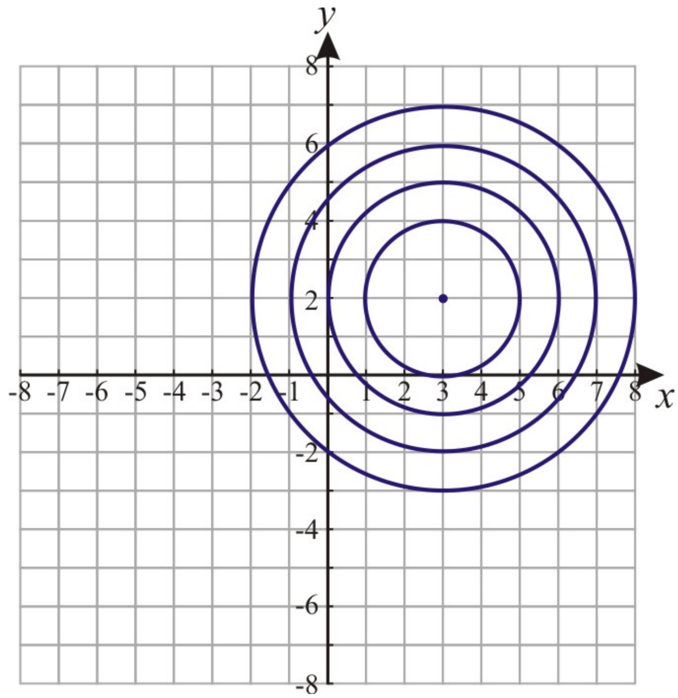
Concentric Circles

Concentric circles are circles of *different* **radii** that share the *same* **center** point.

- **Circles** with the *same* ___________________ but different _________________ are called **concentric circles.**

Example 5

Write the equations of the concentric circles shown in the graph:



All 4 circles have the *same* **center** point at (3, 2) so we know the equations will all be:

$$(x-3)^2+(y-2)^2$$

Since the circles have *different* **radius** lengths, the right side of the equations will all be different numbers.

The *smallest* **circle** has a radius of 2:

$$(x-3)^2+(y-2)^2 = 2^2 \quad \text{or}$$
$$(x-3)^2+(y-2)^2 = 4$$

The *next larger* **circle** has a radius of 3: $(x-3)^2+(y-2)^2 = 9$

The *next larger* **circle** has a radius of 4: $(x-3)^2+(y-2)^2 = 16$

The *largest* **circle** has a radius of 5: $(x-3)^2+(y-2)^2 = 25$

Look at the word concentric:

In Spanish, the word "con" means "with."

The second part of the word, "-centric" looks very similar to the word "center."

When we put these two parts together, "concentric" means "with" the same "center."

Reading Check:

1. *If you are given a point and an equation of a circle, how can you tell if the given point is* on *the circle? Describe what you would do.*

2. *What are concentric circles?*

3. *If you are given two equations of two different circles, how can you tell if the circles are concentric? Describe what the two equations would have to have in common.*

10.5 Translating and Reflecting

Learning Objectives

- Graph a *translation* in a coordinate plane.
- Recognize that a *translation* is an *isometry.*
- Find the *reflection* of a point in a line on a coordinate plane.
- Verify that a *reflection* is an *isometry.*

Translations

A **translation** moves *every* point a given *horizontal* distance and/or a given *vertical* distance.

- When a point is moved a certain distance horizontally and/or vertically, the move is called a _____________-__________________.

For example, if a **translation** moves point $A(3,7)$ 2 units to the *right* and 4 units *up* to $A'(5,11)$, then this **translation** moves *every* point in a larger figure the *same way.*

*The symbol next to the letter A' above is called the **prime** symbol.*

*The **prime** symbol looks like an apostrophe like you may use to show possessive, such as, "that is my brother's book."*

(The apostrophe is before the s in brother's)

*In math, we use the **prime** symbol to show that two things are related.*

*In the **translation** above, the original point is related to the translated point, so instead of renaming the translated point, we use the **prime** symbol to show this.*

The *original* point (or figure) is called the **preimage** and the *translated* point (or figure) is called the **image.** In the example given above, the **preimage** is point $A(3,7)$ and the **image** is point $A'(5,11)$. The **image** is *designated* (or *shown*) with the **prime** symbol.

- Another name for the *original* point is the __________________________.
- Another name for the *translated* point is the __________________________.
- The *translated* point uses the _______________ symbol next to its naming letter.

Example 1

*The point $A(3,7)$ in a **translation** becomes the point $A'(2,4)$. What is the **image** of $B(-6,1)$ in the same **translation**?*

Point A moved 1 unit to the *left* and 3 units *down* to get to A'. Point B will also move 1 unit to the *left* and 3 units *down*.

We *subtract* 1 from the $x-$coordinate and 3 from the $y-$coordinate of point B:

$$B' = (-6-1, 1-3) = (-7,-2)$$

$B'(-7,-2)$ is the **image** of $B(-6,1)$.

Using the **Distance Formula,** you can notice the following:

$$AB = \sqrt{(-6-3)^2+(1-7)^2} = \sqrt{(-9)^2+(-6)^2} = \sqrt{117}$$
$$A'B' = \sqrt{(-7-2)^2+(-2-4)^2} = \sqrt{(-9)^2+(-6)^2} = \sqrt{117}$$

Since the endpoints of $\overline{AB}$ and $\overline{A'B'}$ moved the *same* distance horizontally and vertically, both segments have the *same length.*

Translation is an Isometry

An **isometry** is a transformation in which *distance* is "preserved." This means that the *distance* between any two points in the **preimage** (*before* the **translation**) is the *same* as the *distance* between the points in the **image** (*after* the **translation**).

- An **isometry** is when ______________________________ is *preserved* from the **preimage** to the **image.**

As you saw in Example 1 above:

The **preimage** $AB =$ the **image** $A'B'$ (since they are both equal to $\sqrt{117}$)

Would we get the same result for any other point in this translation? The answer is yes. It is clear that for any point X, the distance from X to X' will be $\sqrt{117}$. Every point moves $\sqrt{117}$ units to its **image.**

This is true in general:

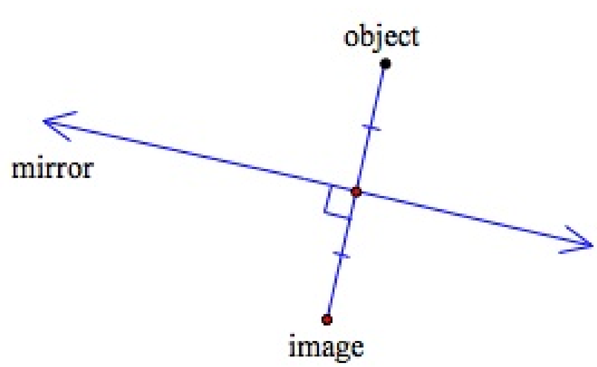
Translation Isometry Theorem

Every **translation** in the coordinate plane is an **isometry.**

- Every translation in an $x-y$ coordinate plane is an ________________________.

Reflection in a Line

A **reflection** in a line is as if the line were a *mirror:*



- When an object is **reflected** in a line, the line is like a ______________________.

An object **reflects** in the mirror, and we see the **image** of the object.

- The **image** is the *same* distance behind the mirror line as the object is in front of the mirror line.
- The "line of sight" from the *object* to the *mirror* is **perpendicular** to the mirror line itself.
- The "line of sight" from the *image* to the *mirror* is also **perpendicular** to the mirror line.

Reflection of a Point in a Line

Point P' is the **reflection** of point P in line k if and only if line k is the **perpendicular bisector** of $\overline{PP'}$.

- The mirror line is a perpendicular ____________________________ of the line that connects the *object* to its reflected image.

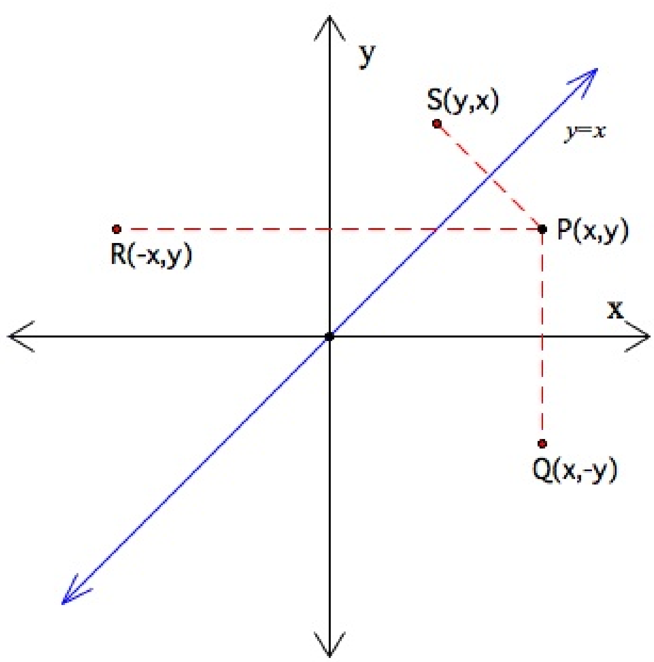
Reflections in Special Lines

In a coordinate plane there are some "special" lines for which it is relatively easy to create **reflections:**

- the $x-$axis
- the $y-$axis
- the line $y = x$ (this line makes a 45° angle between the $x-$axis and the $y-$axis)
- The _________-axis, the _________-axis, and the line _________ = _________ are "special" lines to use as *mirrors* when finding **reflections** of figures.

We can develop simple formulas for reflections in these lines.

Let $P(x,y)$ be a point in the coordinate plane:



We now have the following reflections of $P(x,y)$:

- Reflection of P in the $x-$axis is $Q(x,$ [U+0080][U+0093]$y)$

[the $x-$coordinate stays the *same,* and the $y-$coordinate is *opposite*]

- Reflection of P in the $y-$axis is $R(-x, y)$

[the $x-$coordinate is *opposite,* and the $y-$coordinate stays the *same*]

- Reflection of P in the line $y = x$ is $S(y, x)$

[switch the $x-$coordinate and the $y-$coordinate]

Look at the graph above and you will be convinced of the first two **reflections** in the axes. We will prove the third **reflection** in the line $y = x$ on the next page.

- Reflections in the $x-$axis have the same ________-coordinate, but the $y-$coordinate has the ______________-____________ value.
- Reflections in the $y-$axis have an ____________________________ $x-$coordinate, and the $y-$coordinate stays the _____________________.
- For reflections in the $y = x$ line, ___________________ the $x-$ and $y-$coordinates.

Example 2

Prove that the reflection of point $P(h, k)$ in the line $y = x$ is the point $S(k, h)$.

Here is an "outline" proof:

First, we know the **slope** of the line $y = x$ is 1 because $y = 1x + 0$.

Next, we will investigate the **slope** of the line that connects our two points, $\overline{PS}$. Use the slope formula and the values of the points' coordinates given above:

Slope of $\overline{PS}$ is $\frac{k-h}{h-k} = \frac{-1(h-k)}{h-k} = -1$

Therefore, we have just shown that $\overline{PS}$ and $y = x$ are **perpendicular** because the *product* of their **slopes** is –1.

Finally, we can show that $y = x$ is the **perpendicular bisector** of $\overline{PS}$ by finding the **midpoint** of $\overline{PS}$:

Midpoint of $\overline{PS}$ is $\left(\frac{h+k}{2}, \frac{h+k}{2}\right)$

We know the **midpoint** of $\overline{PS}$ is on the line $y = x$ because the $x-$coordinate and the $y-$coordinate of the **midpoint** are the same.

Therefore, the line $y = x$ is the **perpendicular bisector** of $\overline{PS}$.

Conclusion: The points P and S are **reflections** in the line $y = x$.

Example 3

Point $P(5, 2)$ is reflected in the line $y = x$. The image is P'. P' is then reflected in the $y-$axis. The image is P''. What are the coordinates of P''?

We find one reflection at a time:

- Reflect P in the line $y = x$ to find P' :

For reflections in the line $y = x$ we ____________________ coordinates.

Therefore, P' is (2, 5).

- Reflect P' in the $y-$axis:

For reflections in the $y-$axis, the $x-$coordinate is ____________________ and the $y-$coordinate stays the ______-_________________.

Therefore, P'' is (–2, 5).

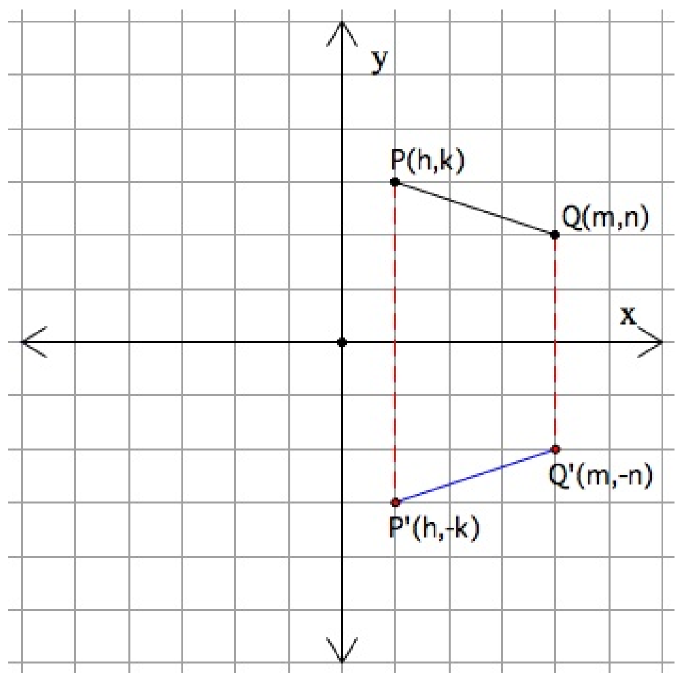
Reflections Are Isometries

Like a **translation,** a **reflection** in a line is also an **isometry.** Distance between points is "preserved" (stays the same).

- A reflection in a line is an _________________________, which means that *distance* is *preserved.*

We will verify the **isometry** for **reflection** in the $x-$axis. The proof is very similar for **reflection** in the $y-$axis.

The diagram below shows $\overline{PQ}$ and its reflection in the $x-$axis, $\overline{P'Q'}$:



Use the Distance Formula:

$$\begin{aligned} PQ &= \sqrt{(m-h)^2+(n-k)^2} \\ P'Q' &= \sqrt{(m-h)^2+(-n-(-k))^2} = \sqrt{(m-h)^2+(k-n)^2} \\ &= \sqrt{(m-h)^2+(n-k)^2} \end{aligned}$$

So $PQ = P'Q'$

Conclusion: When a segment is **reflected** in the $x-$axis, the image segment has the *same length* as the *original* **preimage** segment. This is the meaning of **isometry.** You can see that a similar argument would apply to **reflection** in *any* line.

Reading Check:

1. *True or false:* Both translations and reflections are isometries.

2. *What is the meaning of the statement in #1 above?*

3. *If a translation rule is* $(x+3, y-1)$*, in which directions is a point moved?*

4. *When a point or figure is reflected in a line, that line acts as a mirror.*

a. *How does the* $x-$*axis change a point that is reflected? What do you do to the coordinates of the point in this type of reflection?*

b. *How does the* $y-$*axis change a point that is reflected? What do you do to the coordinates of the point in this type of reflection?*

c. *How does the line* $y = x$ *change a point that is reflected? What do you do to the coordinates of the point in this type of reflection?*

10.6 Rotating

Learning Objectives

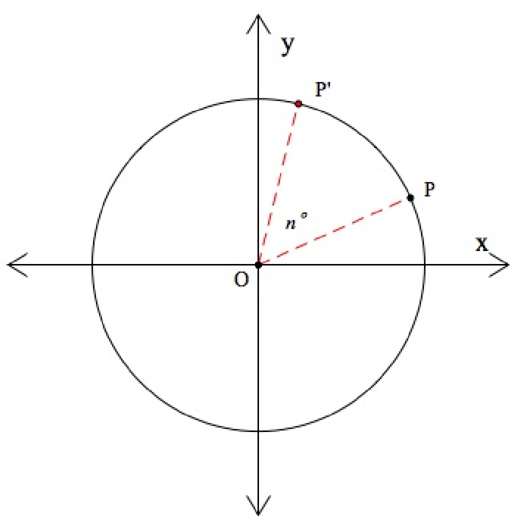
- Find the image of a point in a *rotation* in a coordinate plane.
- Recognize that a *rotation* is an *isometry.*

Sample Rotations

In this lesson we will study **rotations** *centered* at the *origin* of a coordinate plane. We begin with some specific examples of **rotations.** Later we will see how these **rotations** fit into a general formula.

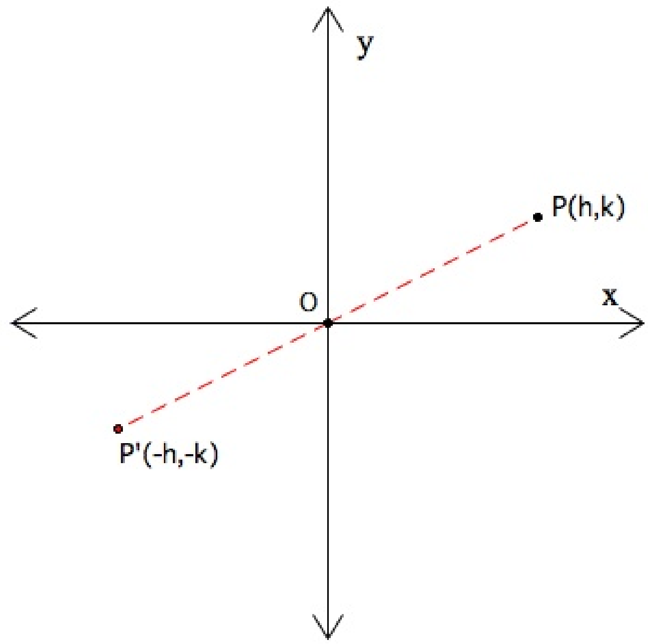
We define a **rotation** as follows: In a **rotation** centered at the *origin* with an **angle of rotation** of n°, a point moves *counterclockwise* along an **arc** of a circle. The **central angle** of the circle measures n°.

The *original* **preimage** point is one *endpoint* of the **arc**, and the **image** of the *original* point is the *other endpoint* of the **arc**:



- **Rotations** centered at the origin move points ________________________ along an **arc** of a circle.
- For a **rotation** of n°, the **central angle** of the circle measures _____________.
- The **preimage** point is one endpoint of the __________________ and the **image** is the other endpoint.

Our first example is **rotation** through an angle of 180°:



In a 180°**rotation,** the **image** of $P(h,k)$ is the point $P'(-h,-k)$.

Notice:

- P and P' are the *endpoints* of a **diameter** of a circle.

→ This means that the distance from the point P to the *origin* (or the distance from the point P' to the *origin*) is a **radius** of the circle.

The distance from P to the origin *equals* the distance from ________ to the origin.

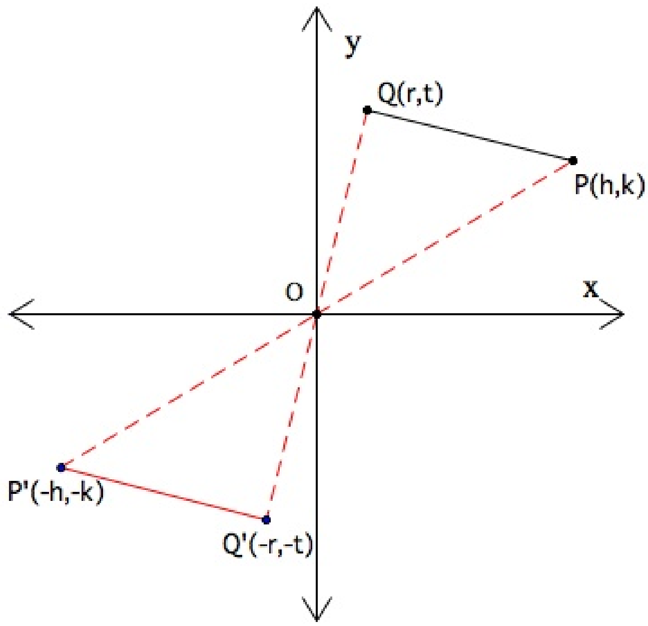
- The **rotation** is the *same* as a **"reflection** in the *origin*."

→This means that if we use the *origin* as a mirror, the point P is directly across from the point P'.

A 180° ______________________________ is also a **reflection** in the *origin.*

In a **rotation** of 180°, the $x-$coordinate and the $y-$coordinate of the __________________ become the *negative* versions of the values in the **image**.

A 180°**rotation** is an **isometry**. The **image** of a segment is a *congruent* segment:



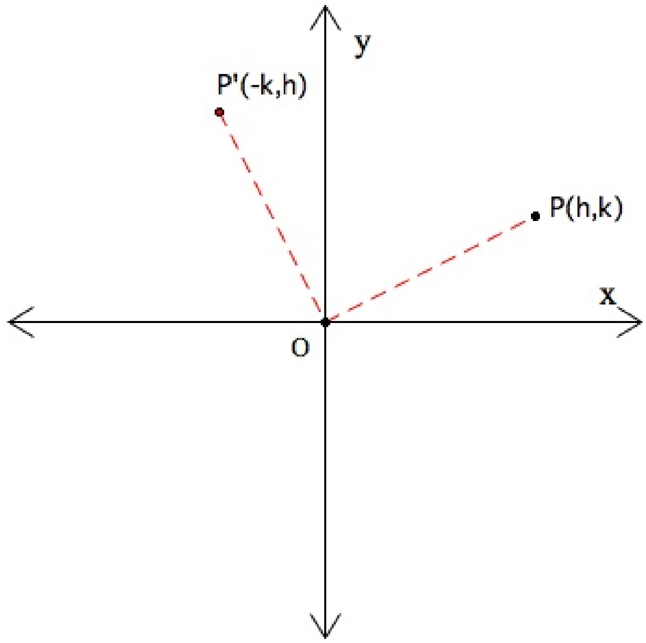
Use the Distance Formula:

$$\begin{aligned} PQ &= \sqrt{(k-t)^2+(h-r)^2} \\ P'Q' &= \sqrt{(-k-(-t))^2+(-h-(-r))^2} = \sqrt{(-k+t)^2+(-h+r)^2} \\ &= \sqrt{(t-k)^2+(r-h)^2} \\ &= \sqrt{(k-t)^2+(h-r)^2} \end{aligned}$$

So $PQ = P'Q'$

- A 180°**rotation** is an ______________________________, so distance is preserved.
- When a segment is **rotated** 180° (or **reflected** in the *origin*), its **image** is a _____________________________-________ segment.

The next example is a **rotation** through an angle of 90°. The **rotation** is in the *counterclockwise* direction:



In a 90°**rotation,** the **image** of $P(h,k)$ is the point $P'(-k,h)$.

Notice:

- $\overline{PO}$ and $\overline{P'O}$ are both **radii** of the same circle, so $PO = P'O$.

If PO and $P'O$ are both **radii**, then they are the *same* ______________________.

- $\angle POP'$ is a *right* angle.
- The **acute** angle formed by $\overline{PO}$ and the $x-$axis and the **acute** angle formed by $\overline{P'O}$ and the $x-$axis are **complementary** angles.

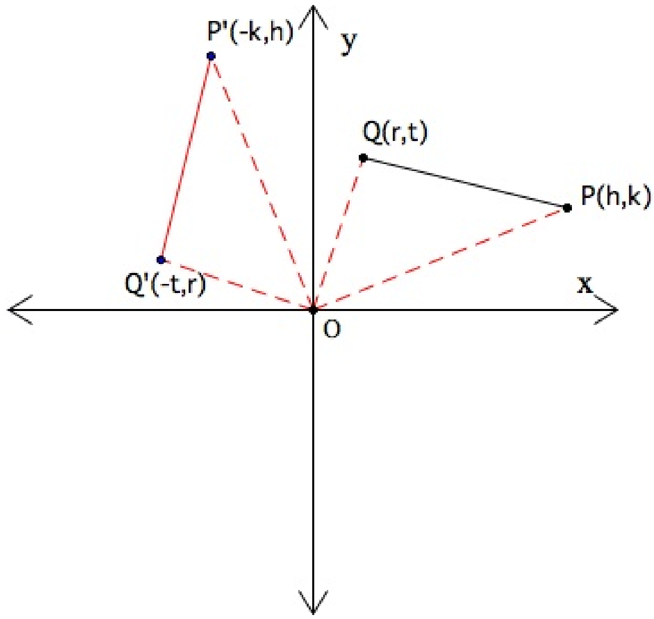
Remember, **complementary** angles add up to ________________°.

You can see by the coordinates of the **preimage** and **image** points, in a 90°**rotation:**

- the $x-$ and $y-$coordinates are *switched* AND
- the $x-$coordinate is *negative.*

In a 90°**rotation,** *switch* the _________- and _________-coordinates and make the new $x-$coordinate ___________-________________.

A 90°**rotation** is an **isometry.** The **image** of a segment is a *congruent* segment.



Use the Distance Formula:

$$PQ = \sqrt{(k-t)^2 + (h-r)^2}$$
$$P'Q' = \sqrt{(h-r)^2 + (-k-(-t))^2} = \sqrt{(h-r)^2 + (t-k)^2}$$
$$= \sqrt{(k-t)^2 + (h-r)^2}$$

So $PQ = P'Q'$

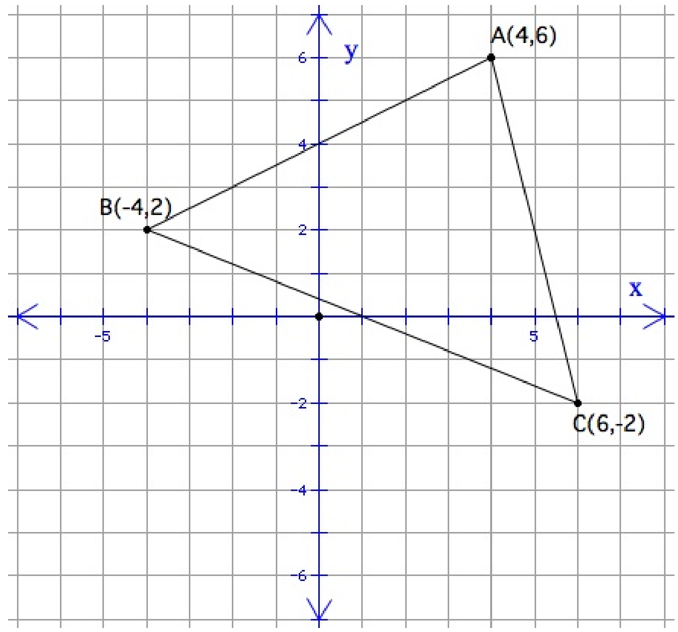
Reading Check:

Which of the following are isometries? Circle all that apply:

$30^\circ rotation$	$45^\circ rotation$	$60^\circ rotation$
$90^\circ rotation$	$150^\circ rotation$	$180^\circ rotation$
$Reflection$	$Translation$	$Bisection$

Example 1

What are the coordinates of the vertices of ΔABC in a rotation of 90°?



Point *A* is (4, 6), *B* is (–4, 2), and *C* is (6, –2).

In a 90°**rotation,** the $x-$coordinate and the $y-$coordinate are *switched* AND the new $x-$coordinate is made *negative:*

- *A* becomes A' : switch *x* and *y* to (6, 4) and make *x* negative (–6, 4)
- *B* becomes B' : switch *x* and *y* to (2, –4) and make *x* negative (–2, –4)
- *C* becomes C' : switch *x* and *y* to (–2, 6) and make *x* negative $(-(-2),6) = (2,6)$

So the vertices of $\Delta A'B'C'$ are (–6, 4), (–2, –4), and (2, 6).

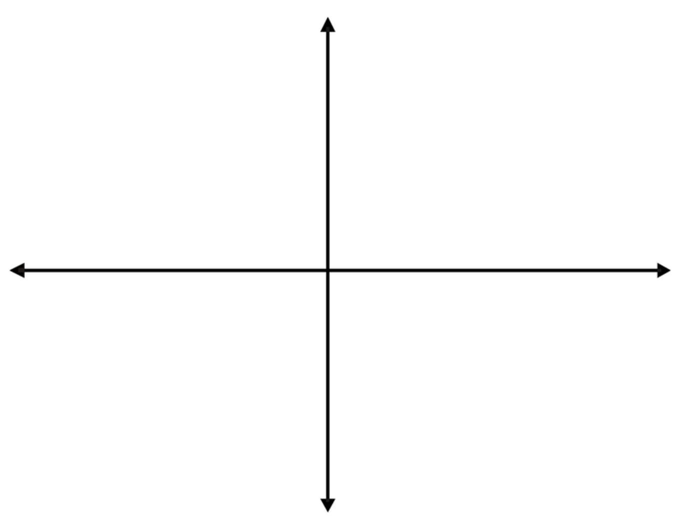
Plot each of these points on the coordinate plane above and draw in each side of the new **rotated** triangle. Can you see how ΔABC is **rotated** 90° to $\Delta A'B'C'$?

Reading Check:

1. *True or false:* A rotation is always in the counterclockwise direction.

2. *On the coordinate plane below, create a point anywhere you like, and label it P.*

Then draw a second point W that is the image of point P rotated 180°.



3. *On the coordinate plane above, draw a third point R that is the* image *of your original point P*rotated 90°.

4. *Is a* 90° *rotation an isometry? Explain.*

5. *Is a* 180° *rotation an isometry? Explain.*

6. *What type of rotation is the same as a reflection in the origin?*