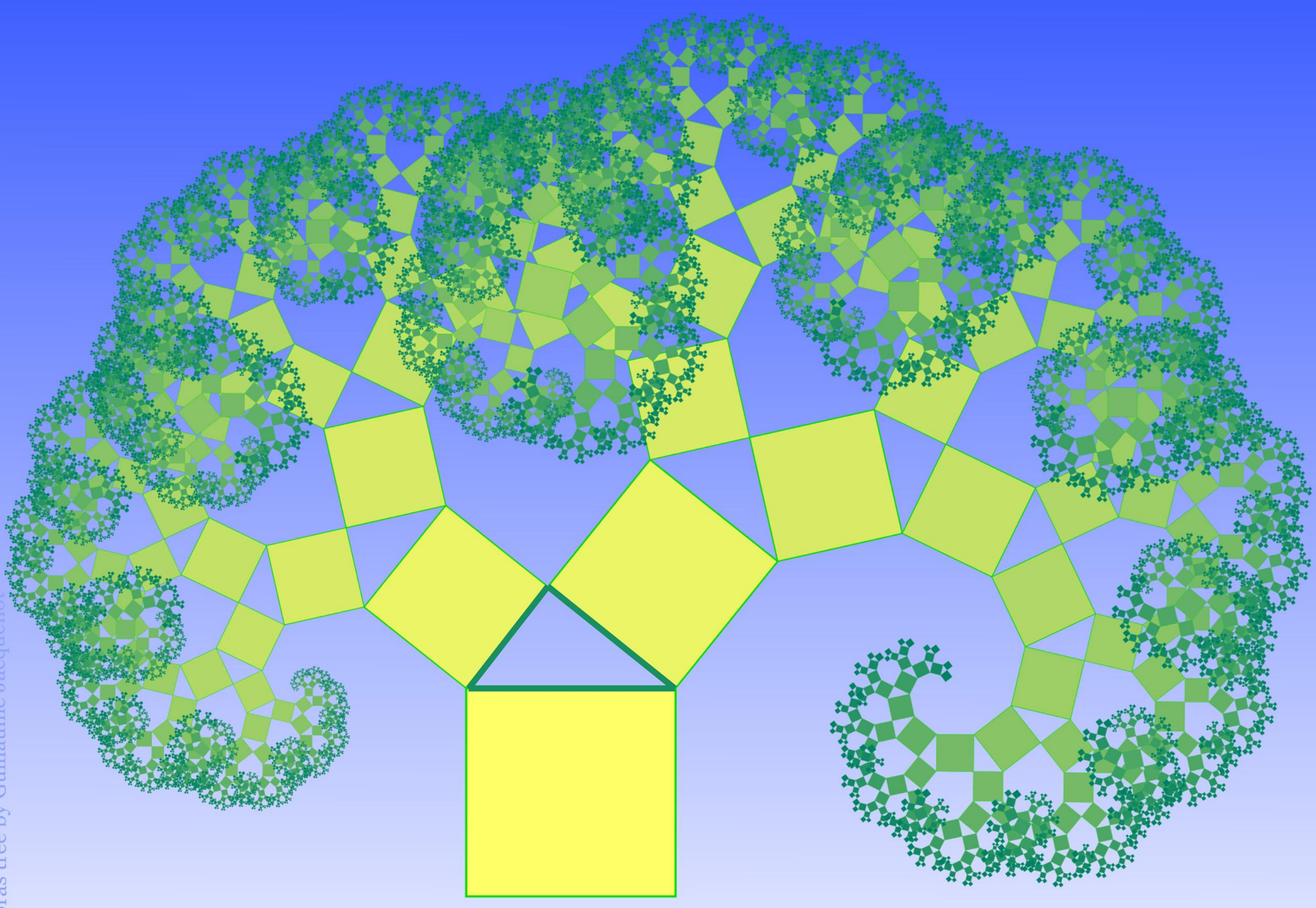
Pythagorean Theorem

History, Applications, and Proofs



ythagoras tree by Guillaume Jacquenot

Geometry - Pythagorean Theorem

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CHAPTER 1 History of the Pythagorean Theorem

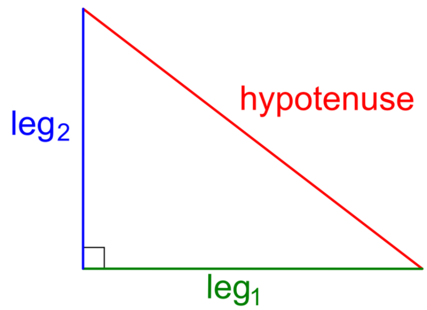
Chapter Outline

1.1 Pythagoras and the Pythagoreans

More than 2,500 years ago, around 530 BCE, a man by the name of Pythagoras founded a school in modern southeast Italy. Members of the school, which was actually more of a brotherhood, were bound by a pledge of allegiance to their master Pythagoras and took an oath of silence to not divulge secret discoveries. Pythagoreans shared a common belief in the supremacy of numbers, using them to describe and understand everything from music to the physical universe. Studying a wide range of intellectual disciplines, Pythagoreans made a multitude of discoveries, many of which were attributed to Pythagoras himself. No records remain of the actual discoverer, so the identity of the true discoverer may never be known. Perhaps the most famous of the Pythagoreans' contributions to knowledge is proving what has come to be known as the **Pythagorean Theorem**.

1.2 Pythagorean Theorem

The Pythagorean Theorem allows you to find the lengths of the sides of a **right triangle**, which is a triangle with one 90° angle (known as the right angle). An example of a right triangle is depicted below.



A right triangle is composed of three sides: two legs, which are labeled in the diagram as leg_1 and leg_2, and a **hypotenuse**, which is the side opposite to the right angle. The hypotenuse is always the longest of the three sides. Typically, we denote the right angle with a small square, as shown above, but this is not required.

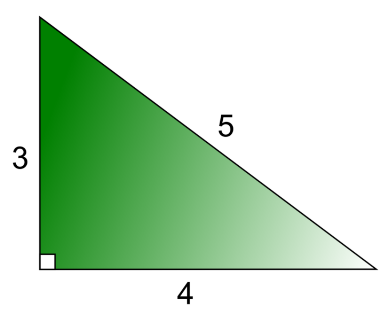
The Pythagorean Theorem states that the length of the hypotenuse squared equals the sum of the squares of the two legs. This is written mathematically as:

$$(leg_1)^2 + (leg_2)^2 = (hypotenuse)^2$$

To verify this statement, first explicitly expressed by Pythagoreans so many years ago, let's look at an example.

Example 1

Consider the right triangle below. Does the Pythagorean Theorem hold for this triangle?



Solution

As labeled, this right triangle has sides with lengths 3, 4, and 5. The side with length 5, the longest side, is the hypotenuse because it is opposite to the right angle. Let's say the side of length 4 is leg_1 and the side of length 3 is leg_2.

Recall that the Pythagorean Theorem states:

$$(leg_1)^2 + (leg_2)^2 = (hypothenuse)^2$$

If we plug the values for the side lengths of this right triangle into the mathematical expression of the Pythagorean Theorem, we can verify that the theorem holds:

$$\begin{aligned}(4)^2 + (3)^2 &= (5)^2 \\ 16 + 9 &= 25 \\ 25 &= 25\end{aligned}$$

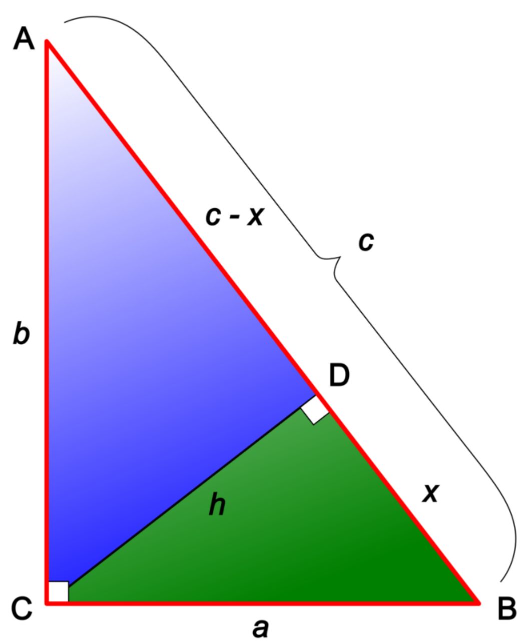
Although it is clear that the theorem holds for this specific triangle, we have not yet proved that the theorem will hold for all right triangles. A simple proof, however, will demonstrate that the Pythagorean Theorem is universally valid.

Proof Based on Similar Triangles

The diagram below depicts a large right triangle (triangle *ABC*) with an altitude (labeled *h*) drawn from one of its vertices. An **altitude** is a line drawn from a vertex to the side opposite it, intersecting the side perpendicularly and forming a 90° angle.

In this example, the altitude hits side *AB* at point *D* and creates two smaller right triangles within the larger right triangle. In this case, triangle *ABC* is similar to triangles *CBD* and *ACD*. When a triangle is **similar** to another triangle, corresponding sides are proportional in lengths and corresponding angles are equal. In other words, in a set of similar triangles, one triangle is simply an enlarged version of the other.

Similar triangles are often used in proving the Pythagorean Theorem, as they will be in this proof. In this proof, we will first compare similar triangles *ABC* and *CBD*, then triangles *ABC* and *ACD*.



Comparing Triangles *ABC* and *CBD*

In the diagram above, side AB corresponds to side CB. Similarly, side BC corresponds to side BD, and side CA corresponds to side DC. It is possible to tell which side corresponds to the appropriate side on a similar triangle by using angles; for example, corresponding sides AB and CB are both opposite a right angle.

Because corresponding sides are proportional and have the same ratio, we can set the ratios of their lengths equal to one another. For example, the ratio of side AB to side BC in triangle ABC is equal to the ratio of side CB to corresponding side BD in triangle CBD:

$$\frac{\text{length of } AB}{\text{length of } BC} = \frac{\text{length of } CB}{\text{length of } BD}$$

Written with variables, this becomes:

$$\frac{c}{a} = \frac{a}{x}$$

Next, we can simplify this equation by multiplying both sides of the equation by $a$and x:

$$x \times a \times \frac{c}{a} = \frac{a}{x} \times x \times a$$

With simplification, we obtain:

$$cx = a^2$$

Comparing Triangles ABC and ACD

Triangle ABC is also similar to triangle ACD. Side AB corresponds to side CA, side BC corresponds to side CD, and side AC corresponds to side DA.

Using this set of similar triangles, we can say that:

$$\frac{\text{length of } CA}{\text{length of } DA} = \frac{\text{length of } AB}{\text{length of } AC}$$

Written with variables, this becomes:

$$\frac{b}{c-x} = \frac{c}{b}$$

Similar to before, we can multiply both sides of the equation by $c-x$and b:

$$b \times (c-x) \times \frac{b}{c-x} = \frac{c}{b} \times b \times (c-x)$$
$$b^2 = c(c-x)$$
$$c^2 = cx + b^2$$

Earlier, we found that $cx = a^2$. If we replace cx with a^2, we obtain $c^2 = a^2 + b^2$. This is just another way to express the Pythagorean Theorem. In the triangle ABC, side c is the hypotenuse, while sides a and b are the two legs of the triangle.

1.3 A Debate about True Origins

Although the theorem has been attributed to and named after Pythagoras and his community of scholars, it is believed that the concepts behind the theorem were known long before the Pythagoreans proved it. Among historians, an ongoing debate ensues about the possibilities that the ideas behind the Pythagorean Theorem were independently discovered by different groups at different times. A wide variety of theories exist, but there is substantial evidence that various civilizations used the Pythagorean Theorem, or were at least aware of the main principles of the theorem, to find the side lengths of right triangles.

Babylonians

In 1800 BCE, more than a thousand years before Pythagoras founded his school, a group of people living in Mesopotamia (located in present-day Iraq) already understood the relationship between the side lengths of a right triangle. These people, called Babylonians, were the first known group to demonstrate a conceptual understanding of the Pythagorean Theorem.

Historians have gained an understanding of the Babylonians from studying the ancient clay tablets they have left behind. These tablets were used throughout Mesopotamia to record a variety of information about commerce, culture, and daily life. Two of these clay tablets have particular relevance to the Pythagorean Theorem. On one of these tablets, which has been named YCB (short for Yale Babylonian Collection) 7289 since its discovery, there is an illustration of a tilted square with its two diagonals drawn in. In their own numeration system, Babylonians labeled the sides of the square as having a length equivalent to the value of 1 in our number system and the diagonal with a length equivalent to 1.414213. This decimal is a miraculously accurate approximation of $\sqrt{2}$, which proves that the Babylonians had very refined methods of calculation.

The Pythagorean Theorem was never explicitly written on any of the recovered clay tablets, but the engravings on tablet YBC 7289 display an early understanding of the Pythagorean Theorem because the diagonal of the square can be thought of as the hypotenuse of a right triangle. The legs of this right triangle, which are simply the sides of the square, each have a length of 1. By the Pythagorean Theorem, of which the Babylonians must have had some understanding, the diagonal must have a length of $\sqrt{1^2+1^2}$ (because $(leg_1)^2+(leg_2)^2=(hypotenuse)^2$). This is simply $\sqrt{2}$ or, as the Babylonians approximated, 1.414213.

A second tablet (shown in **Figure** 1.1), named Plimpton 322 after the collection to which it belongs, reveals the Babylonians' advanced understanding of right triangles. Inscribed in this tablet is a table of **Pythagorean triples**, which are sets of three positive integers (a, b, c) that would satisfy the Pythagorean Theorem ($a^2+b^2=c^2$). One example of a Pythagorean triple is the set (3, 4, 5), as seen in **Example 1**. We will explore Pythagorean triples more fully in the chapter "Applying the Pythagorean Theorem."

Egyptians

Like Mesopotamia, Egypt was a great ancient civilization whose inhabitants were very commercially and culturally advanced. The Egyptians never explicitly expressed the Pythagorean Theorem as we know it today, but they must have used it in constructing their pyramids. It is known that, when building the pyramids, Egyptians used a knotted rope as an aid in making right angles. This rope had twelve evenly spaced knots (similar to the diagram below) that could be formed into a 3-4-5 right triangle with one angle of 90°. The ropes were used as a model for the much larger right triangles used in the pyramids, which were built during a period of 1,500 years as a way to honor the pharaohs.

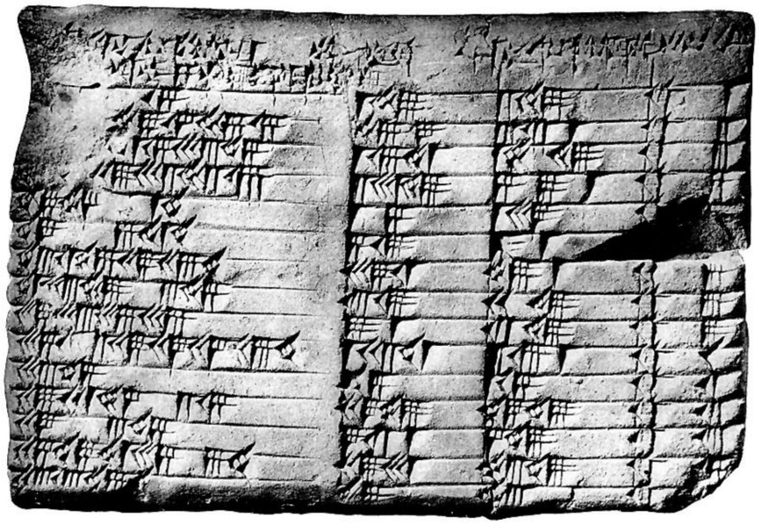
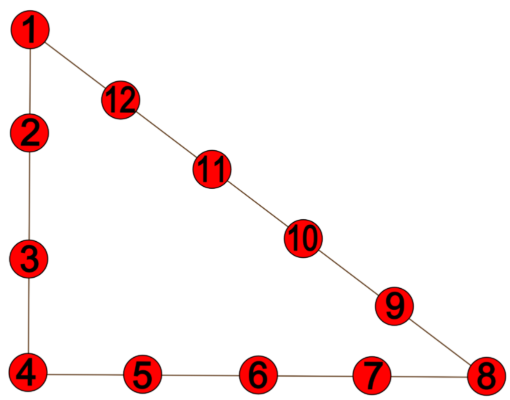


FIGURE 1.1
Plimpton 322 tablet with engravings of Pythagorean triples.



Activity 1

Try what the Egyptians did yourself! Cut a 12-inch piece of yarn and mark every inch by tying a knot. Next, try to construct a right triangle with this piece of yarn so that each side has an integer length, just like the Egyptians did.

Other Ancient Civilizations

It is believed that other ancient civilizations, such as China and India, also understood the Pythagorean Theorem before Pythagoras himself proved it. The debate about whether the theorem was discovered in one place at one time or in many places at different times still lingers today.

1.4 The Theorem's Significance

Although the Babylonians may be the first to understand the concepts of the Pythagorean Theorem and the Pythagoreans were the first to explicitly prove it, Euclid of Alexandria, active around 300 BCE, was the man responsible for popularizing the theorem. Euclid, head of the department of mathematics at a school in Alexandria, took it upon himself to compile all knowledge about mathematics known at his point in history. The result was a book called *Elements*, which included two of Euclid's own proofs of the Pythagorean Theorem.

The propagation of this theorem is significant because the theorem is applicable to a variety of fields and situations. Though the theorem is fundamentally geometric, it is useful in many branches of science and mathematics, and you are likely to encounter it often as you continue to study more advanced topics.

The Pythagorean Theorem, however, is also relevant to a variety of situations in everyday life. Architecture, for instance, employs the concepts behind the Pythagorean Theorem. Measuring and computing distances will also often involve using this theorem. Televisions, when advertised, are measured diagonally; for example, a television may be listed as "a 40-inch," meaning that its diagonal is 40 inches long. The length of the television, the width of the television, and the Pythagorean Theorem were used to get this measurement.

Look out for ways that you can use this theorem in your everyday life—there may be more than you expected!

1.5 References

1. . Image ofthePlimpton 322 tablet. Public Domain

CHAPTER 2 Applying the Pythagorean Theorem

Chapter Outline

The Pythagorean Theorem is useful in a variety of mathematical situations because it can be applied to solve many different types of problems. As we have seen, the most basic application of the theorem is finding the length of one side of a right triangle when the lengths of the other two sides are known. In this chapter, we'll expand on other applications of the theorem.

2.1 Pythagorean Triples

As mentioned earlier in the "History of the Pythagorean Theorem" chapter, the Babylonians demonstrated an understanding of the Pythagorean Theorem by listing Pythagorean triples on a clay tablet. Pythagorean triples are sets of three **integers**—positive whole numbers—that make a right triangle. Pythagorean triples are frequently used in examples and problems, making it worthwhile to memorize some of the more common triples. Pythagorean triples are frequently used in examples and problems, making it worthwhile to memorize some of the more common triples.

The most common Pythagorean triples are (3, 4, 5) and (5, 12, 13). Multiples of these triples—such as (6, 8, 10)—are also Pythagorean triples. Pythagorean triples are frequently used in examples and problems, making it worthwhile to memorize some of the more common triples. Though these triples are the most common, there is an infinite number of combinations of integers that satisfy the Pythagorean Theorem. Table 2.1 lists all primitive triples with a hypotenuse length less than 100. Note that the set (6, 8, 10) is not listed in the table because it is a multiple of the primitive triple (3, 4, 5).

TABLE 2.1: A list of primitive Pythagorean triples with a hypotenuse length less than 100.

leg_1	leg_2	$hypotenuse$
3	4	5
5	12	13
7	24	25
8	15	17
9	40	41
11	60	61
12	35	37
13	84	85
16	63	65
20	21	29
28	45	53
33	56	65
36	77	85
39	80	89
48	55	73
65	72	97

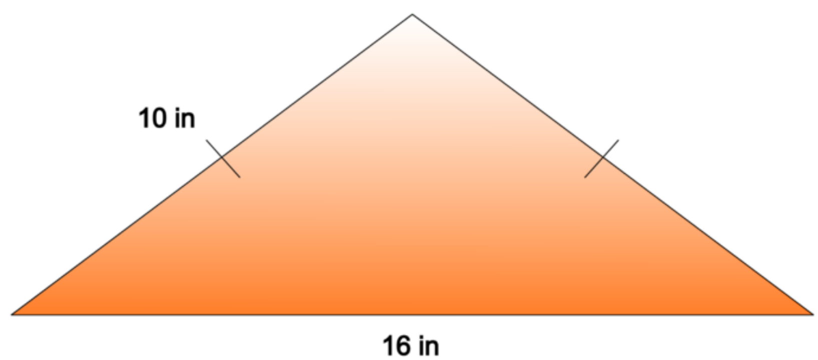
2.2 Area of an Isosceles Triangle

Thus far we have been using the Pythagorean Theorem to find the side lengths of a right triangle, but the theorem can also be used to calculate the area of a non-right triangle. For an isosceles triangle, you can determine its height by constructing an altitude. The length of the altitude can then be found by applying the Pythagorean Theorem.

To practice using the Pythagorean Theorem in this way, let's look at an example.

Example 2

Find the area of the triangle below.

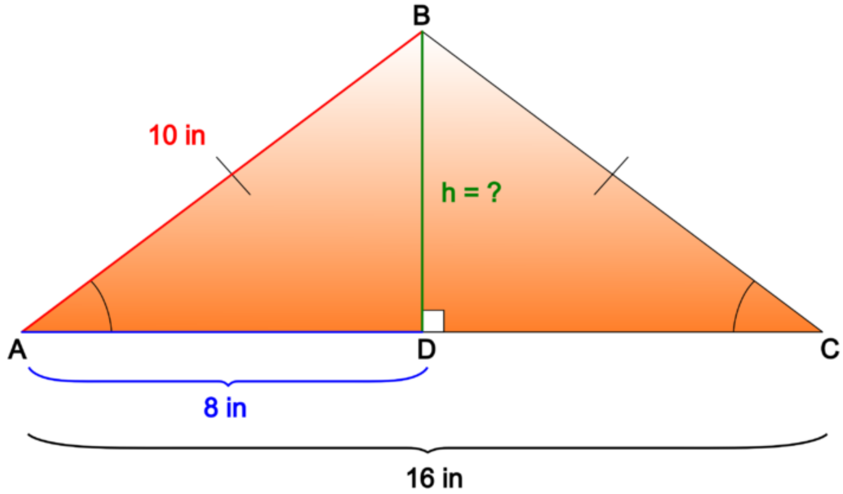


Solution

To find the area of a triangle, use the equation:

$$Area = \frac{1}{2} \times b \times h$$

In this example, we need to determine a b (base) and an h (height). Let's use the side with a length of 16 inches as the base for this triangle. We can then construct the "height" by drawing an altitude through the vertex opposite the base (angle B), as seen in the illustration below. The altitude creates two 90° angles with the base, making it **perpendicular** to the base and thus creating the two right triangles ABD and CBD. These two right triangles are actually identical—they each have a 90° angle, a hypotenuse of length 10 inches, and the same angle at A and C. The altitude actually **bisects** the base of length 16 inches, which means that the lengths on either side of the point where the altitude intersects the base are both equal to 8 inches.



Therefore, we have created two identical right triangles with one leg that is 8 inches long (half of the base) and one hypotenuse that is 10 inches long. Using the Pythagorean Theorem, we can determine the length of the third side, which in this case is the altitude.

The Pythagorean Theorem states:

$$(leg_1)^2 + (leg_2)^2 = (hypotenuse)^2$$

In this case, we let the 8-inches long leg be leg_1 and h be leg_2. We can plug in known lengths and then solve for the unknown h:

$$\begin{aligned}
(8 \text{ in})^2 + (h)^2 &= (10 \text{ in})^2 \\
64 \text{ in}^2 + (h)^2 &= 100 \text{ in}^2 \\
(h)^2 &= 100 \text{ in}^2 - 64 \text{ in}^2 \\
(h)^2 &= 36 \text{ in}^2 \\
h &= \sqrt{36 \text{ in}^2} \\
h &= 6 \text{ in}
\end{aligned}$$

The length of the altitude, or h, is 6 inches. The side lengths of the triangle are 6, 8, and 10 inches. You may have noticed that this combination of side lengths is a multiple of the common 3-4-5 Pythagorean triple. If you recognized this before we used the Pythagorean Theorem to calculate the altitude, you could have immediately identified the length of the altitude as 6 inches.

Now that we know the altitude—or what we call the "height" for this example—of the triangle is 6 inches and the base is 16 inches, we can find the area of the entire isosceles triangle by plugging these values into the equation for the area of a triangle.

$$\begin{aligned}
Area &= \frac{1}{2} \times b \times h \\
Area &= \frac{1}{2} \times 16 \text{ in} \times 6 \text{ in} \\
Area &= 48 \text{ in}^2
\end{aligned}$$

Note that the base of the triangle is 16 inches, not 8 inches. It is very common for students to mistakenly use 8 inches as the base b because we used that as the side length for the right triangle. If, however, a student did assume that the base had a length of 8 inches, he or she would find that the area would be half of the actual area, or 24 square inches. This makes sense because drawing the altitude divided the triangle into two smaller, equivalent triangles with an area of 24 square inches. Adding the areas of each of the smaller triangles will give you 48 square inches, the entire area of the larger isosceles triangle.

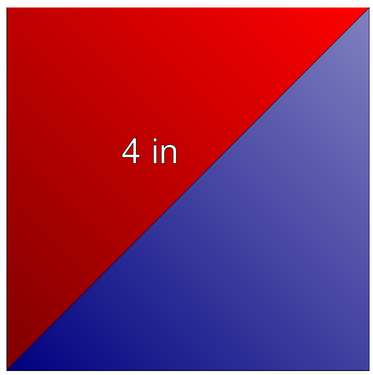
2.3 Side Lengths of Shapes other than Triangles

The Pythagorean Theorem, though most directly related to determining lengths of right triangles, can be used to find the side lengths of many other shapes. As shown in the example above, it is possible to construct right triangles within the original shapes and then use the Pythagorean Theorem to determine side lengths.

Look to create right triangles in rectangles, squares, and even circles. Let's look at a couple of examples to see how the Pythagorean Theorem can be applied to other shapes.

Example 3

Find the perimeter of the square below.



Solution

The diagonal of the square has a length of 4 inches and can be thought of as the hypotenuse of a right triangle. The legs of this right triangle are two adjoining sides of the square; because by definition all sides of a square are equal in length, the two legs of the right triangle are equal as well.

The definition of the Pythagorean Theorem, $(leg_1)^2 + (leg_2)^2 = (hypotenuse)^2$, can then be simplified because $leg_1 = leg_2$:

$$\begin{aligned}(leg_1)^2 + (leg_2)^2 &= (hypotenuse)^2\\ 2 \times (leg_1)^2 &= (hypotenuse)^2\end{aligned}$$

Now we can plug in the value of the hypotenuse and solve for the length of leg_1:

$$\begin{aligned}2 \times (leg_1)^2 &= (4 \text{ in})^2\\ \frac{2 \times (leg_1)^2}{2} &= \frac{16 \text{ in}^2}{2}\\ (leg_1)^2 &= 8 \text{ in}^2\\ \sqrt{(leg_1)^2} &= \sqrt{8 \text{ in}^2}\\ leg_1 &= \sqrt{8 \text{ in}^2} \approx 2.828 \text{ in}\end{aligned}$$

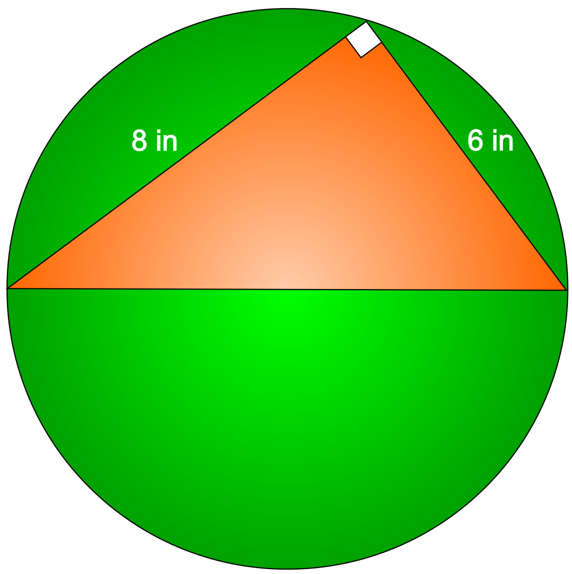
Because we have defined the side of the square as leg_1, the length of each side of the square is $\sqrt{8 \text{ in}^2}$, or about 2.828 inches. The perimeter of the square would be four times the length of each side:

$$4 \times \sqrt{8 \text{ in}^2} \approx 11.314 \text{ in}$$

The perimeter of the square is about 11.314 inches.

Example 4

Find the circumference of the circle below.



Solution

The triangle inscribed in the circle is a right triangle, as indicated by the square drawn in the corner. Using the Pythagorean Theorem, we can find the length of the hypotenuse of the right triangle, which is also the diameter of the circle.

$$\begin{aligned}
(leg_1)^2 + (leg_2)^2 &= (hypotenuse)^2 \\
(8 \text{ in})^2 + (6 \text{ in})^2 &= (hypotenuse)^2 \\
64 \text{ in}^2 + 36 \text{ in}^2 &= (hypotenuse)^2 \\
100 \text{ in}^2 &= (hypotenuse)^2 \\
\sqrt{100 \text{ in}^2} &= \sqrt{(hypotenuse)^2} \\
10 \text{ in} &= hypotenuse
\end{aligned}$$

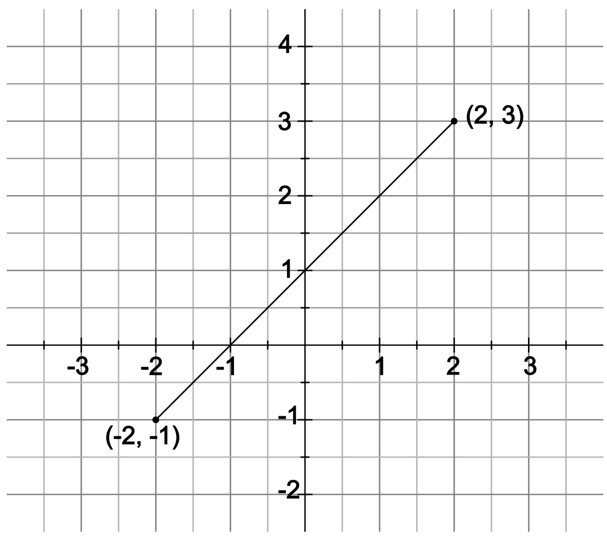
The hypotenuse, which is equal to the circle's diameter, is 10 inches. To find the circumference of the circle, we should use the equation $circumference = 2\pi r$, in which r is the radius of the circle. The radius of the circle, which is half the diameter, is 5 inches. Therefore, the $circumference = 2\pi(5 \text{ inches}) \approx 31.416$ inches.

2.4 Finding Distances on a Coordinate Grid

You can also apply the Pythagorean Theorem to find the distance between two points on a coordinate grid. Let's look at several examples to see how this is done.

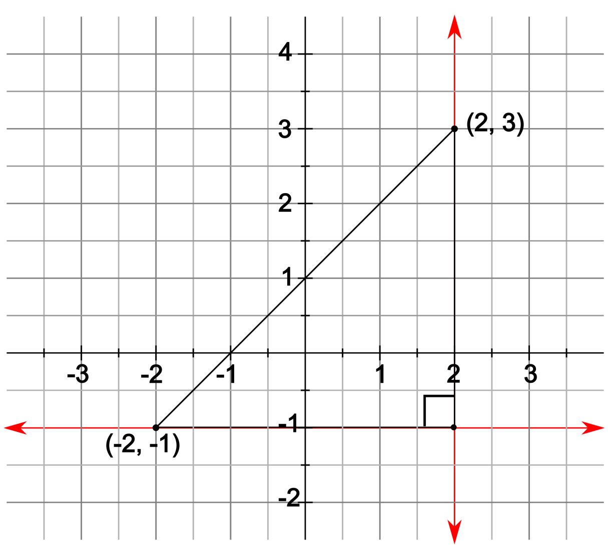
Example 5

Find the distance between the two points on the coordinate system below.



Solution

The line segment connecting these two points is neither horizontal nor vertical, so it is not possible to simply count the number of spaces on the grid. It is, however, possible to think of the line segment connecting the two points as the hypotenuse of a right triangle. Drawing a vertical line at $x = 2$ and a horizontal line at $y = -1$ will create a right triangle, as shown in the figure below.



It is easy to determine the lengths of the legs of this right triangle because they run parallel to the x- and y-axes. You can simply count on the grid how long each leg is. The horizontal leg has a length of 4 and the vertical leg also has a length of 4. Using the Pythagorean Theorem, you can find the length of the diagonal line, which is also the hypotenuse of the right triangle with two legs of length 4.

$$\begin{aligned}(4)^2+(4)^2 &= (distance)^2\\ 16+16 &= (distance)^2\\ 32 &= (distance)^2\\ \sqrt{32} &= distance\\ 5.657 &\approx distance\end{aligned}$$

The line segment connecting the points $(1,5)$ and $(5,2)$ is about 5.657 units long.

2.5 The Distance Formula

We can generalize the process used in **Example 5** to apply to any situation where you want to find the distance between two coordinates. Using points (x_1, y_1) and (x_2, y_2), we can derive a general distance formula. Similar to **Example 5**, we will let the line segment connecting two coordinates be the hypotenuse of a right triangle.

Let's start by finding the length of the horizontal leg by finding the difference in the x-coordinates. The difference in the x-coordinates would be $|x_2 - x_1|$. The absolute value brackets are used to indicate that the length of the horizontal leg must be a positive value because a negative distance does not have any physical meaning. We can also find the length of the vertical leg by finding the difference in y-coordinates. The difference in the y-coordinates would be $|y_2 - y_1|$. Once again, absolute value brackets are used because lengths cannot be negative.

Now that we have found the lengths of the legs of the right triangle we have created, we can plug them into our equation for the Pythagorean Theorem.

$$(leg_1)^2 + (leg_2)^2 = (hypotenuse)^2$$
$$(|x_2 - x_1|)^2 + (|y_2 - y_1|)^2 = (distance)^2$$

At this point, it is not necessary to use the absolute value brackets because any value squared will be positive. For example, even if $x_2 - x_1$ were a negative value, it would become a positive value when squared. Therefore, we can rewrite the equation above without the absolute value brackets, and the expression will remain the same:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (distance)^2$$

To solve for the distance, we can take the roots of both sides and obtain the **distance formula**:

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = distance$$

To check this formula with an example, let's plug in the coordinates from **Example 5** and see if we get the same distance as the answer. The coordinates from **Example 5** were (-2,-1) and (2,3), and we determined that the distance between these two points was about 5.657 units.

$$\sqrt{(2-(-2))^2 + (3-(-1))^2} = distance$$
$$\sqrt{(4)^2 + (4)^2} = distance$$
$$\sqrt{16+16} = distance$$
$$\sqrt{32} = distance$$
$$5.657 \approx distance$$

As you can see, the equation holds for this example.

Though the distance formula is different from the Pythagorean Theorem, the formula is simply another form of the theorem and is not an entirely different concept. It is not necessary to memorize this equation because you can simply think about how the Pythagorean Theorem applies to a line segment on a coordinate system. Nonetheless, it is helpful to know that this equation holds for any two coordinates.

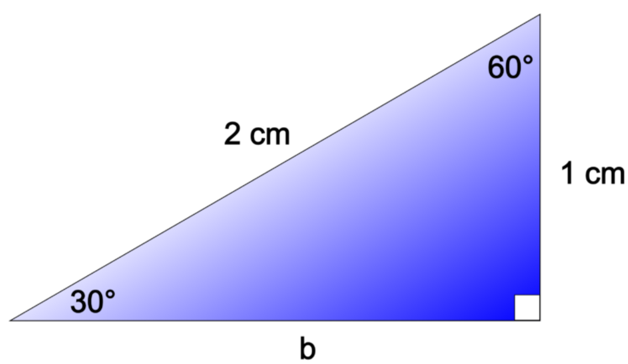
2.6 The 30°-60°-90° Triangle

There is an infinite number of variations on the right triangle. While there must be one 90° angle, the other angle measurements and the side lengths can be any combination that satisfies the Pythagorean Theorem. However, there are some angle combinations that create "special triangles" whose side lengths have special ratios. We will explore two kinds of special right triangles in this section.

Let's first look at a 30°-60°-90° triangle and use the Pythagorean Theorem to find the general ratio between the sides of the triangle.

Example 6

Find the length of the unknown side, b.



Solution

Use the Pythagorean Theorem to find the length of the third side:

$$(leg_1)^2 + (leg_2)^2 = (hypotenuse)^2$$

We know that the side with the length of 2 centimeters is the hypotenuse because it is opposite the right angle and looks like the longest side.

$$\begin{aligned}(b)^2 + (1\text{ cm})^2 &= (2\text{ cm})^2\\ b^2 + 1\text{ cm}^2 &= 4\text{ cm}^2\\ b^2 &= 3\text{ cm}^2\\ b &= \sqrt{3}\text{ cm}\end{aligned}$$

In this case, the length of the shorter leg is simply 1 centimeter, the length of the hypotenuse is 2 centimeters, and the length of the longer leg is $\sqrt{3}$ centimeters. However, this ratio can be applied to all similar triangles, or triangles with the same angle measurements of 30°, 60°, and 90°. No matter what the length is for the shorter leg, the hypotenuse is always double that length, and the longer leg is always $\sqrt{3}$ times the length of the shorter leg.

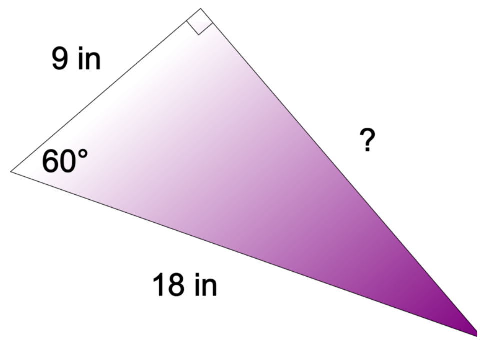
Note that the shorter leg is opposite the 30° angle and the longer leg is opposite the 60° angle: the smallest angle always faces the shortest side, the second smallest angle always faces the second shortest side, and the largest angle always faces the longest side.

To understand this concept, you can think of an alligator's mouth: the angle and length represent how wide the mouth is open, or the distance between the two teeth. The wider the mouth is opened (or the larger the angle), the greater the distance between the teeth (or the longer the side opposite to the angle). Similarly, the less the mouth is opened (or the smaller the angle), the smaller the distance between the teeth (or the shorter the side opposite the angle).

Let's look at another example of a 30°-60°-90° triangle.

Example 7

Find the length of the missing leg in the triangle below.



Solution

The triangle above is a 30°-60°-90° triangle. The hypotenuse, or the side opposite the 90° angle, has a length of 18 inches, and the shorter leg, or the side opposite the 30° angle, has a length of 9 inches.

We determined in **Example 6** that the hypotenuse is double the length of the shorter leg, and the longer leg is $\sqrt{3}$ times the length of the shorter leg. Therefore, the longer leg in this example has a length of $9\sqrt{3}$ inches, or approximately 15.588 inches.

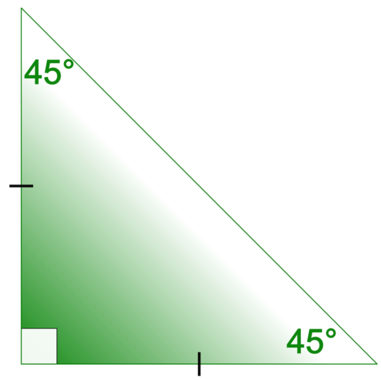
We can check our answers with the Pythagorean Theorem:

$$\begin{aligned}(9 \text{ in})^2 + (9\sqrt{3} \text{ in})^2 &= (18 \text{ in})^2 \\ 81 \text{ in}^2 + (81 \times 3) \text{ in}^2 &= 324 \text{ in}^2 \\ 324 \text{ in}^2 &= 324 \text{ in}^2\end{aligned}$$

The answer holds.

2.7 The 45°-45°-90° Triangle

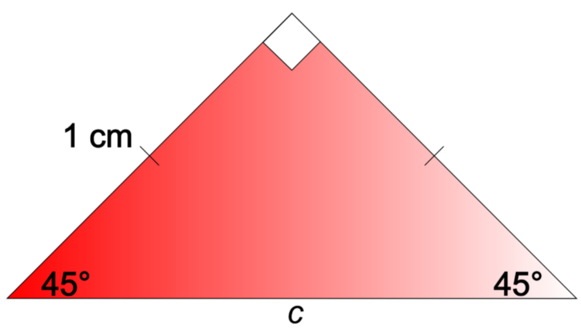
Next, let's look at another special right triangle: the 45°-45°-90° triangle. This type of triangle is also known as a right isosceles triangle because it has two sides of the same length. Below is an image of this special triangle:



Let's look at a simple example to determine the general ratio between the sides of this type of triangle.

Example 8

Find the length of side c.



Solution

We can use the Pythagorean Theorem to find the length of side c by treating c as the hypotenuse.

$$\begin{aligned}(1\text{ cm})^2+(1\text{ cm})^2&=(c)^2\\ 1\text{ cm}^2+1\text{ cm}^2&=c^2\\ 2\text{ cm}^2&=c^2\\ \sqrt{2}\text{ cm}&=c\end{aligned}$$

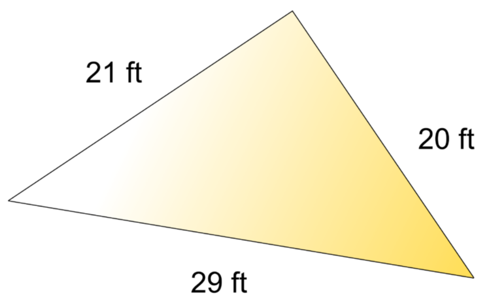
This example shows that, with a 45°-45°-90° triangle, the hypotenuse is always $\sqrt{2}$ times the length of the legs.

2.8 Converse Pythagorean Theorem

The converse of the Pythagorean Theorem is that if the three side lengths of a triangle satisfy the equation $(leg_1)^2 + (leg_2)^2 = (hypotenuse)^2$, they represent the sides of a right triangle. This converse allows you to prove whether a triangle with known side lengths is or is not a right triangle without actually knowing the angle measurements.

Example 9

Does the triangle below contain a right angle?



Solution

Looking at this triangle, it is not clear whether or not there is a right angle; there is no right angle mark, but it seems that if there were a right angle, it would be opposite to the side with the longest length of 29 feet.

To test whether or not this triangle is or is not a right triangle, we can use the Pythagorean Theorem and see if $(leg_1)^2 + (leg_2)^2 = (hypotenuse)^2$ holds true. Because the side of 29 feet is the longest, it would be the hypotenuse if the triangle were a right triangle, leaving the legs to be the sides of length 20 feet and length 21 feet. Let's plug these values into the Pythagorean Theorem equation. If the equation holds, the triangle is a right triangle; if it does not hold, this triangle is not a right triangle.

$$\begin{aligned}(leg_1)^2 + (leg_2)^2 &= (hypotenuse)^2 \\ (20 \text{ ft})^2 + (21 \text{ ft})^2 &= (29 \text{ ft})^2 \\ 400 \text{ ft}^2 + 441 \text{ ft}^2 &= 841 \text{ ft}^2 \\ 841 \text{ ft}^2 &= 841 \text{ ft}^2\end{aligned}$$

We can conclude that the triangle is a right triangle because both sides of the equation are equal. You may also want to note that (20, 21, 29) is one of the common Pythagorean triples mentioned in Table 2.1.

2.9 Determining Right, Acute, or Obtuse Triangles

If the triangle does not contain a right angle, you can still find out more about the triangle, specifically whether it is acute or obtuse. In an **acute** triangle, all angles are less than 90°. An **obtuse** triangle, on the other hand, has one angle that is greater than 90°.

We can use the Converse Pythagorean Theorem to determine if a triangle is right, acute, or obtuse. If the sum of the squares of the two shorter sides of a triangle is greater than the square of the longest side, the triangle is acute. However, if the sum of the squares of the two shorter sides of a triangle is smaller than the square of the longest side, the triangle is obtuse. Written with inequalities, this becomes:

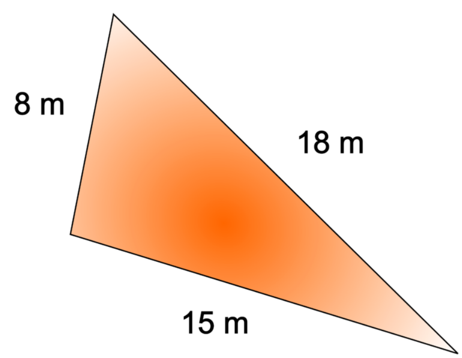
- if $a^2+b^2>c^2$, the triangle is acute,
- if $a^2+b^2=c^2$, the triangle is a right triangle,
- if $a^2+b^2<c^2$, the triangle is obtuse,

where a and b are the lengths of the two shorter sides of the triangle and c is the length of the longest side.

Let's apply the Converse Pythagorean Theorem to an example problem.

Example 10

Is the triangle below acute, obtuse, or right?



Solution

The two shorter sides of the triangle are 8 meters and 15 meters, and the longest side is 18 meters. Therefore, if this triangle were a right triangle, the 18 meter side would be the hypotenuse. The 8 meter and the 15 meter sides take the places of a and b while the 18 meter side takes the place of c:

$$a^2+b^2=(8\text{ m})^2+(15\text{ m})^2=64\text{ m}^2+225\text{ m}^2=289\text{ m}^2$$

$$c^2=(18\text{ m})^2=324\text{ m}^2$$

Since $289\text{ m}^2<324\text{ m}^2$, $a^2+b^2<c^2$. Consequently, the triangle is obtuse.

Architects use the Converse Pythagorean Theorem to determine whether the right angles (such as corners) in their buildings are truly right angles. This is similar to how the Egyptians used knotted ropes to measure 3-4-5 right triangles. For more details, see the chapter “History of the Pythagorean Theorem.”

CHAPTER 3 Proving the Pythagorean Theorem

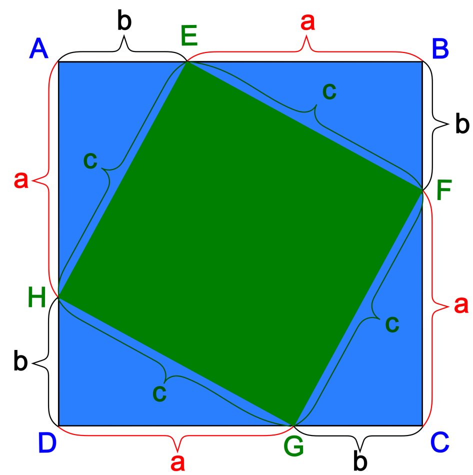
Chapter Outline

There are over 400 known proofs of the Pythagorean Theorem, and each proof demonstrates the validity of the theorem in a unique way. Proofs are especially helpful when the concept being proven is not obvious, as in the case of the Pythagorean Theorem. Although it has been established in the chapter "History of the Pythagorean Theorem" that the Pythagorean Theorem is true, working through the proofs in this chapter can help you gain a deeper understanding of the theorem and under what conditions it holds.

Writing proofs requires organization, and we recommend using the following format for every proof you do. For each proof, you should include a section listing the "**given**," or the proof's set-up, which may involve shapes, labels, or angle measurements. Next, you should include a statement of what you are going to prove in a "**prove**" section. In the next section, the "**proof**" section, work through the proof systematically and explain each step you take. If a proof requires many steps, it may be helpful to number each step as a way to stay organized. At the end of your proof, it's always a good idea to have a "**conclusion**" section that states what your proof demonstrates. We will use this format for the following proofs.

3.1 "Chinese Proof"

The following proof is included in the *Zhou Bi Suan Jing*, one of the oldest Chinese mathematical works known to scholars. Surviving copies of this text date back to the Han Dynasty (206-221 BCE). Although it is one of the oldest recorded proofs of the Pythagorean Theorem, it is also one of the most elegant.



Given: $ABCD$ is a square with sides of length $a+b$. Another square, square $EFGH$, is inscribed inside $ABCD$. Each side of square $EFGH$ has a length c. There are also four right triangles inside square $ABCD$.

Prove: $a^2+b^2=c^2$

Proof:

1. The area of square $ABCD = (a+b)^2$
2. The area of each right triangle is $\frac{1}{2}ab$
3. The area of square $EFGH$ is c^2
4. The area of square $ABCD$ is equal to the sum of the area of square $EFGH$ and the areas of the four right triangles:
    a. area of square $ABCD =$ area of square $EFGH + 4(\text{area of } AHE)$
    b. $(a+b)^2 = c^2 + 4(\frac{1}{2}ab)$
5. Simplify the expression:
    a. Expand:
        i. $(a+b)(a+b) = c^2+2ab$
        ii. $a^2+ab+ab+b^2 = c^2+2ab$
    b. Simplify completely by subtracting out the ab terms:$a^2+b^2=c^2$

Conclusion: This proof proves that $a^2+b^2=c^2$ when a and b are legs of a right triangle and c is the hypotenuse. In other words, this proof proves the Pythagorean Theorem.

Activity 2

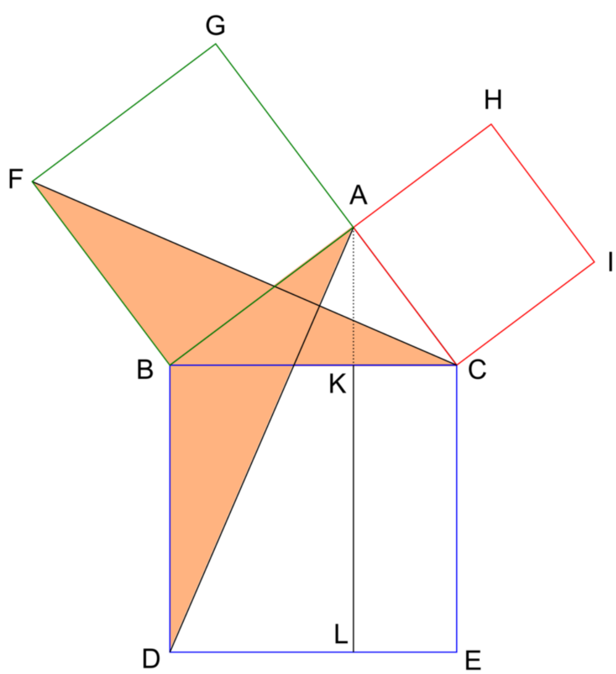
Check out this applet to play around with the same shapes used in the Chinese proof:

- http://www.ies.co.jp/math/java/geo/pythafor/pythafor.html

3.2 Euclid's Proof

FIGURE 3.1
Euclid of Alexandria.

As mentioned in the "History of the Pythagorean Theorem" chapter, Euclid of Alexandria popularized the Pythagorean Theorem through two proofs included in his book *Elements*. We will go through one of his proofs here; it is one of the most well-known proofs, but it is also rather complicated.

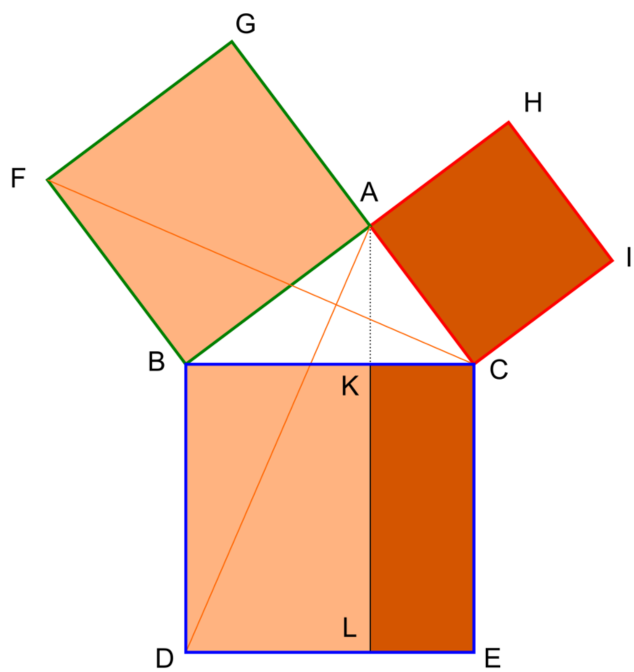


Given: Let ABC be a right triangle with $\angle CAB = 90^\circ$. A square is formed at each side of the triangle, giving us squares $CBDE$, $BAGF$, and $ACIH$. We then create triangles BCF and BDA by connecting C to F and A to D. A line parallel to BD and CE is drawn from point A, perpendicularly intersecting lines BC and DE at K and L, respectively. $\angle CAB$ and $\angle BAG$ are both right angles; therefore C, A, and G are **collinear**, or lying on the same line. The same is true for B, A, and H.

Prove: (length of $AB)^2 +$ (length of $AC)^2 =$ (length of $BC)^2$

Proof:

1. $\angle CBD$ and $\angle FBA$ are right angles (because each angle in a square is 90°); therefore, $\angle ABD = \angle FBC$, since both are the sum of a right angle and $\angle ABC$.
2. Since AB and BD are equal to FB and BC, respectively (due to the fact that all sides a square are equal), triangle ABD must be congruent to triangle FBC.
3. Because A is collinear with K and L, rectangle $BDLK$ must be twice the area of triangle ABD.
   a. Area of $BDLK =$ length of $BD \times$ length of BK.
   b. Area of $ABD = \frac{1}{2} \times$ length of BD (base) $\times$ length of BK (height, which is the length of the altitude from A).
   c. Thus, the area of $ABD = \frac{1}{2} \times$ area of $BDLK$.
4. Since C is collinear with A and G, square $BAGF$ must be twice the area of triangle FBC.
   a. Area of $BAGF =$ (length of $AB) \times$ (length of $FB) =$ (length of $AB)^2$.
   b. Area of $FBC = \frac{1}{2} \times$ length of FB (base—length of $FB =$ length of AB) $\times$ length of AB (height, which is the length of the altitude from C).
   c. Thus the area of $FBC = \frac{1}{2} \times$ area of $BAGF$.
5. Since triangle ABD is congruent to triangle FBC, rectangle $BDLK$ must have the same area as square $BAGF$.
   a. Area of $BDLK =$ area of $BAGF =$ (length of $AB)^2$.
6. Similarly, it can be shown that the area of $CKLE =$ area of $ACIH =$ (length of $AC)^2$.



7. Adding these two results, you obtain (length of $AB)^2 +$ (length of $AC)^2 =$ (length of $BD) \times$ (length of $BK) +$ (length of $KL) \times$ (length of KC).
   a. From this point on, we will write these expression as: $AB^2 + AC^2 = (BD \times BK) + (KL \times KC)$.
8. Because length of $BD =$ length of KL:
   a. Substitute BD for KL: $(BD \times BK) + (KL \times KC) = (BD \times BK) + (BD \times KC)$.
   b. Factor out the BDs: $(BD \times BK) + (BD \times KC) = BD \times (BK + KC)$.

c. Looking at the figure above, we see that:
   i. $BD \times BK =$ area of rectangle $BDLK$.
   ii. $BD \times KC =$ area of rectangle $CKLE$.
   iii. Area of rectangle $BDLK +$ area of rectangle $CKLE =$ area of square $CBDE$.
   iv. Therefore, $BD \times (BK + KC) = BD \times BC = BC^2$.

9. Rewritten, this is $AB^2 + AC^2 = BC^2$, which is what we are trying to prove.

Conclusion: (length of $AB)^2 +$ (length of $AC)^2 =$ (length of $BC)^2$. Because AB and AC are the legs of a right triangle, this proof demonstrates that the Pythagorean Theorem is valid.

Activity 3

Show that this proof works using a real life example. Draw a right triangle on a piece of construction paper and then cut out three squares with lengths equal to the sides of the right triangle you drew. Try to fit the two smaller squares (the squares with the lengths of the legs of the right triangle) into the largest square (with the length of the hypotenuse of the right triangle). You may have to cut up the smaller squares to make them fit nicely, but, by the Pythagorean Theorem, they should.

Activity 4

Check out this applet to see an animated version of this proof:

- http://www.ies.co.jp/math/java/geo/pythasx/pythasx.html

3.3 An American President's Proof

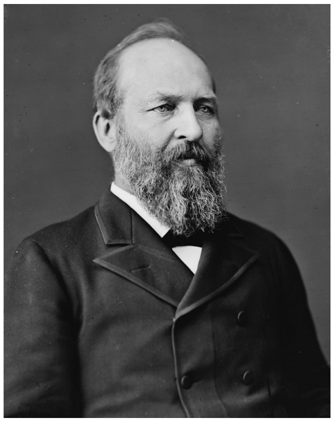
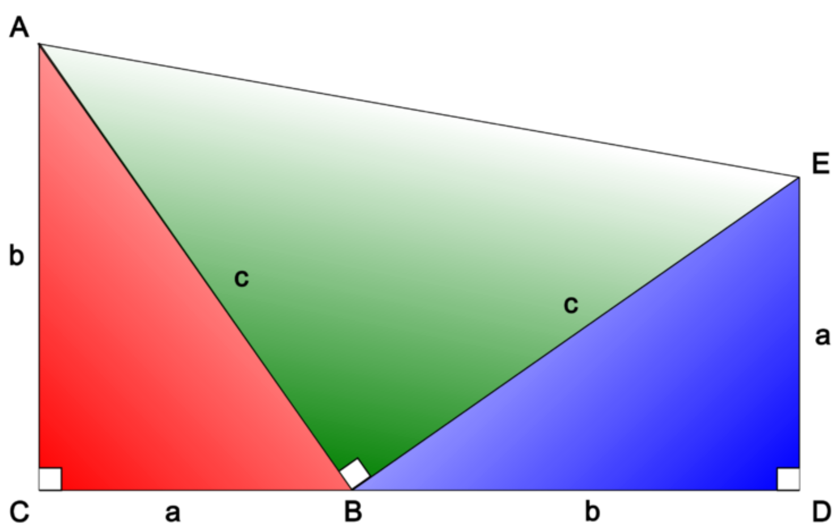


FIGURE 3.2
Twentieth president of the United States, James A. Garfield.

This proof is famous because James Garfield, twentieth president of the United States, proposed it in 1876, four years before he was elected. It is a rather elegant proof that uses trapezoids and right triangles to demonstrate the validity of the Pythagorean Theorem.



Given: Triangle ABC is a right triangle with legs of lengths a and b and hypotenuse of length c. We extend CB to a point D so that $BD = AC = b$. Next, we draw a line DE perpendicular to BD so that $DE = CB = a$. Connecting point E to point B with a diagonal line will create a second right triangle identical to triangle ABC. In other words, triangles ABC and BED are **congruent**. Consequently, $\angle ABC$ and $\angle EBD$ are **complementary**, meaning that they add up to 90°. Since CD is a line, $\angle ABC$, $\angle EBD$, and $\angle ABE$ add up to 180°. Therefore, $\angle ABE$ is a right angle, and triangle ABE is a right isosceles triangle with two legs each of length c.

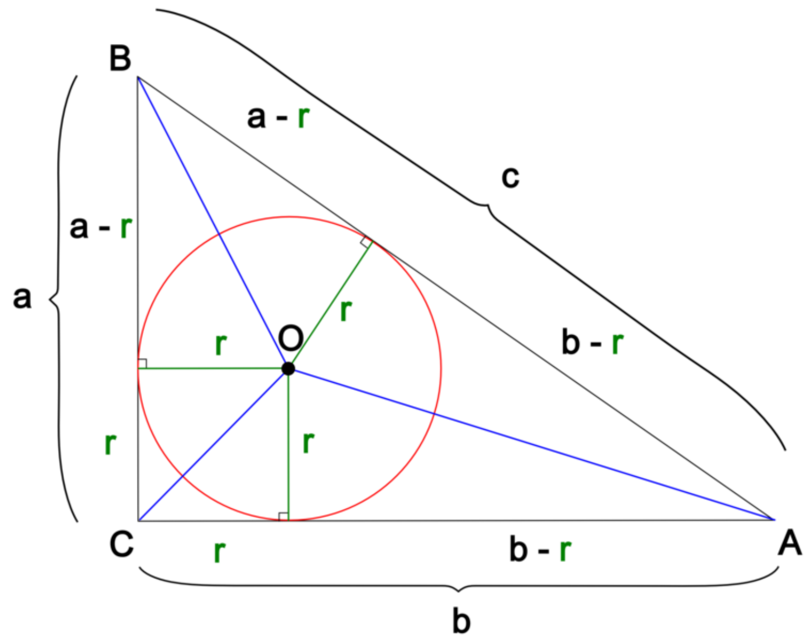
Prove: $a^2 + b^2 = c^2$

Proof:

1. The area of a trapezoid = altitude × average width.
   a. For trapezoid $ACDE$, let the altitude = length of CD and the average width = length of $\frac{(AC+ED)}{2}$.
   b. As a result, the area of trapezoid $ACDE = (b+a) \times \frac{(a+b)}{2} = \frac{(a+b)^2}{2}$.
2. The area of trapezoid $ACDE$ is also equal to the area of triangle ABE plus the area of triangles ABC and BED. In other words, the area of the trapezoid is equal to the sum of the areas of the three triangles within it.
   a. The area of the three triangles $= \frac{c^2}{2} + 2\left(\frac{ab}{2}\right) = \frac{c^2}{2} + ab$.
3. We can set the expression for the area of trapezoid $ACDE$ equal to the area of the three triangles: $\frac{(a+b)^2}{2} = \frac{c^2}{2} + ab$.
4. We can now simplify the expression:
   a. Multiply both sides by 2 to get rid of the denominator:
      i. $2 \times \left(\frac{(a+b)^2}{2}\right) = \left(\frac{c^2}{2} + ab\right) \times 2$
      ii. $(a+b)^2 = c^2 + 2ab$
   b. Expand:
      i. $(a+b)(a+b) = c^2 + 2ab$
      ii. $a^2 + ab + ab + b^2 = c^2 + 2ab$
   c. Simplify completely by subtracting out the ab terms: $a^2 + b^2 = c^2$

Conclusion: This proof proves that $a^2 + b^2 = c^2$ when a and b are legs of a right triangle and c is the hypotenuse. In other words, this proof proves the Pythagorean Theorem.

3.4 Proof Using a Circle



Given: A circle is inscribed in the right triangle ABC with legs of lengths a and b and a hypotenuse of length c. The circle has center O and radius r, and three radii are drawn perpendicular to the sides of the triangle, dividing each side into the parts shown.

Prove: $a^2+b^2=c^2$

Proof:

1. The line drawn from the center of the circle intersects perpendicularly to the hypotenuse of the right triangle, c, and splits the hypotenuse into two parts: $c=(a-r)+(b-r)=a+b-2r$
2. Rearranging this equation to solve for r, we get:
   a. $c=a+b-2r$
   b. $-2r=c-a-b$
   c. $r=\frac{c-a-b}{-2}$
   d. $r=\frac{a+b-c}{2}$
3. The area of triangle $ABC=\frac{1}{2}\times \text{base}\times \text{height}=\frac{1}{2}ba$.
4. The area of triangle ABC is also equal to the sum of three triangles: AOB, BOC, and COA, each of which have altitude r.
   a. area of triangle $AOB=\frac{1}{2}rc$
   b. area of triangle $BOC=\frac{1}{2}ra$
   c. area of triangle $COA=\frac{1}{2}rb$
   d. area of triangle $ABC=$ sum of triangles AOB, BOC, and $COA=\frac{1}{2}rc+\frac{1}{2}ra+\frac{1}{2}rb$
5. Set the two expressions for the area of triangle ABC equal to one another: $\frac{1}{2}ab=\frac{1}{2}rc+\frac{1}{2}ra+\frac{1}{2}rb$
6. Substitute the expression for r from step 2: $\frac{1}{2}ab=\frac{1}{2}\left(\frac{a+b-c}{2}\right)c+\frac{1}{2}\left(\frac{a+b-c}{2}\right)a+\frac{1}{2}\left(\frac{a+b-c}{2}\right)b$
   a. Multiply both sides by 2:

i. $2\times\left(\frac{1}{2}ab\right)=\left(\frac{1}{2}\left(\frac{a+b-c}{2}\right)c+\frac{1}{2}\left(\frac{a+b-c}{2}\right)a+\frac{1}{2}\left(\frac{a+b-c}{2}\right)b\right)\times 2$
ii. $ab=\left(\frac{a+b-c}{2}c\right)+\left(\frac{a+b-c}{2}a\right)+\left(\frac{a+b-c}{2}b\right)$

b. Factor out $\left(\frac{a+b-c}{2}\right)$: $ab=\left(\frac{a+b-c}{2}\right)(c+a+b)$
c. Multiply both sides by 2:
   i. $2\times(ab)=\left(\left(\frac{a+b-c}{2}\right)(c+a+b)\right)\times 2$
   ii. $2ab=(a+b-c)(c+a+b)$
d. Distribute: $2ab=ac+a^2+ab+bc+ba+b^2-c^2-ca-cb$
e. Simplify:
   i. $2ab=a^2+2ab+b^2-c^2$
   ii. $c^2=a^2+b^2$

Conclusion: This proof proves that $a^2+b^2=c^2$ when a and b are legs of a right triangle and c is the hypotenuse. In other words, this proof proves the Pythagorean Theorem.

3.5 References

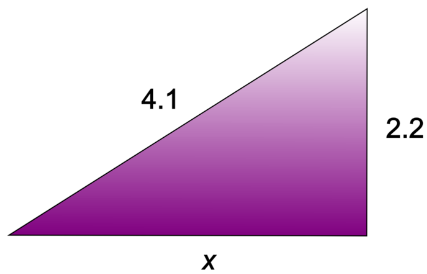
1. . Illustrationof Euclid of Alexandria. Public Domain
2. . Photograph of James Abram Garfied, twentiethpresident of the UnitedStates. Public Domain

CHAPTER 4 Exercises

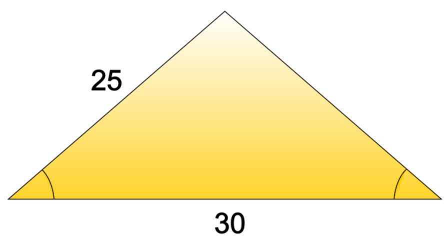
Chapter Outline

Note: A calculator may be necessary for some of these problems.

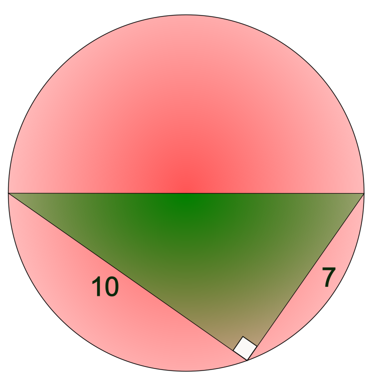
1. What is the distance between $(1, 3)$ and $(8, 8)$?
2. What is the length of x?



3. Do the numbers 9, 18, and 24 make up a Pythagorean triple?
4. What is the area of the triangle below?

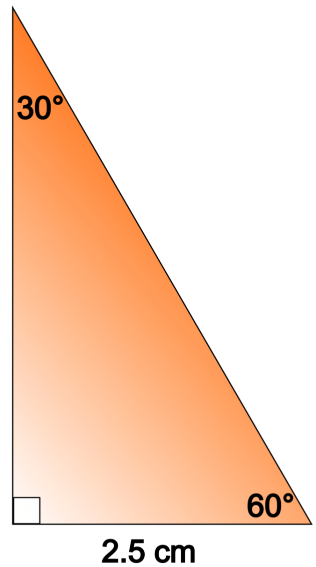


5. What is the circumference of the circle below?

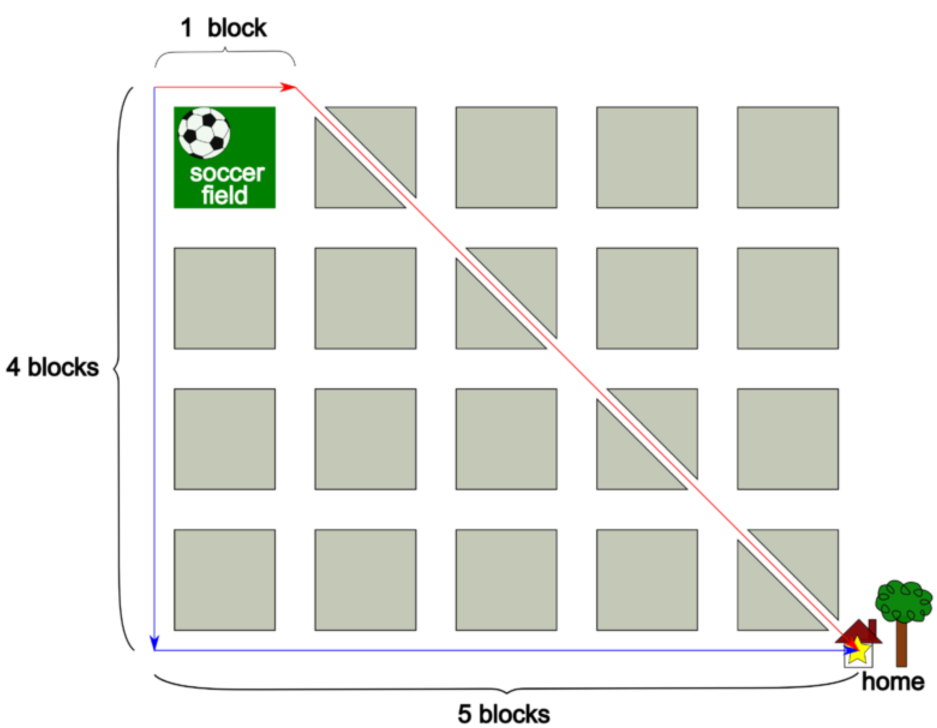


6. What is the distance between $(-5, -8)$ and $(1, 2)$?
7. Is a triangle with side lengths of 8, 14, and 15 acute, obtuse, or right?
8. A baseball diamond includes four bases—first, second, third, and home—that are all 90 feet apart. The path a baserunner takes involves right angles at each of the bases. Imagine a player at bat with a runner on first. The pitcher throws a wild pitch and the player on first base goes for the steal to second. Once the catcher (at home base) gets a hold of the ball, he decides to throw it over the pitcher's head to second base. How far does the catcher have to throw the ball to get the player out?

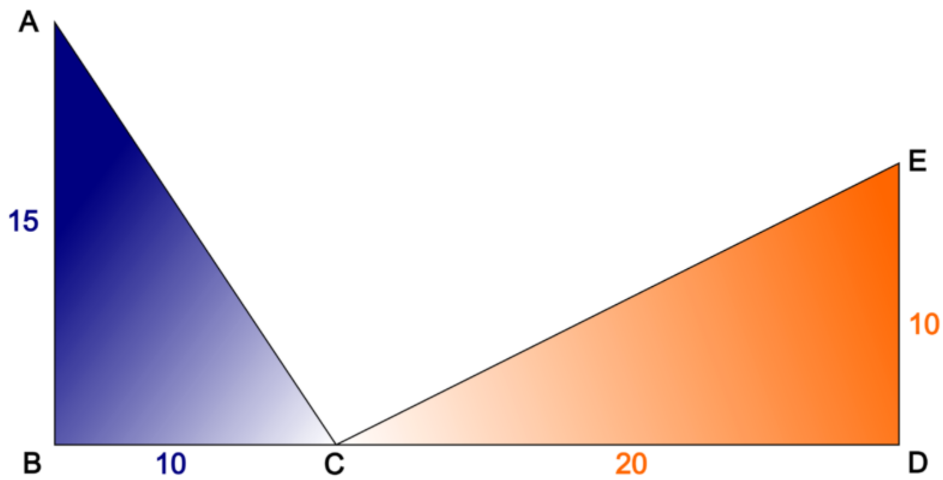
9. Imagine that you get to put a TV in your room. However, there is only enough space for a television that is 17 inches long and 15 inches high. You've already purchased a 20-inch television. Will it fit inside the space you have prepared? (Recall that televisions are measured diagonally.)
10. Find the area of the triangle.



11. Erin hurt her leg at her soccer game but still has to walk home. To keep her leg from feeling worse, she wants to take the shortest possible route. Looking at the two possible routes below, determine how much shorter the red route is than the blue. Each block has a length of 100 meters.



12. Find two points on the line $y = 2$ that are 10 units from $(2, -4)$.
13. In the figure below, the angles at B and D are 90°. Find the distance of $AC + CE$.



14. Instead of walking along two sides of a rectangular field, Ryan walked along the diagonal. By taking this shortcut, he saved a distance equal to half the length of the long side of the field. Find the length of the long side of the field given that the length of the shorter side is 78 meters.

4.1 Solutions

- Solutions available upon request. **Please send an email to teachers-requests@ck12.org to request solutions.**

1. 8.602
2. 3.460
3. No
4. 300 units^2
5. 38.348
6. 11.662
7. Acute
8. 127.279 feet
9. Yes
10. 5.413 cm^2
11. 234.315 meters
12. $(-6, 2)$ and $(10, 2)$
13. 40.388
14. 104 meters