Calculus

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Preface

This book on calculus has been written in accordance with the syllabi of B.Sc. Math honors students. The book can also be used by students with little or no background of calculus. The subject matter has been presented in a way that the students will not find any difficulty in understanding the various concepts included in the various volume. The book contains nine chapters.

The initial chapter is devoted to the various facts, especially as they appear to a beginner, of the nature of mathematics in general and of calculus in particular. The next two chapters deal systematically with the standard topics of limit and continuity and differentiability of the functions. Chapter four deals with the successive differentiation, in chapter five we have discussed the various aspects of the calculus which are generally called the backbone of the calculus. Chapter six contains on introduction of polar coordinates and conic sections, chapter seven has been devoted to the some properties of the integration. Chapter eight is hyperbolic function and last chapter nine cover the introductory knowledge of vectors.

Each chapter contains a good number of examples have taken from the question papers of different university examinations. Nearly all exercises require some thinking.

It is very much hoped that the book in its present form will help to make the study of the subject more interesting, relevant, and meaningful.

I am thankful to the publisher for their keen interest in the book.

I acknowledge with pleasure the assistance of many friends and the colleagues.

Thanks are due also to Mr. Khurram Irfan for their sincere help and interest in the computational work.

It gives me a special pleasure to express my gratitude to my wife Huma and my children Hiba and Abdul Ahad for the many ways in which they have contributed.

Suggestion for improvement will be thankfully acknowledged.

Mohammed Arif

Contents

CHAPTER

1

Preliminaries

1.1 REAL NUMBERS

In this chapter we present basic information that you will need for your study of calculus. We begin by discussing the real number system. The system of real numbers has evolved as a result of a process of successive extensions of the system of **natural numbers** [1, 2, 3, 4, …]. If we add two natural numbers, we get a natural number for example 6 + 2 = 8, but the inverse operation of subtraction is not always possible for example 2 – 5 is meaningless in so far as the natural numbers is concerned. Natural numbers are also referred to as positive integers. In order that the operation of subtraction performed without any restriction the natural numbers enlarge by introducing the negative integers and a number zero [0]. Thus to every positive integer [n] correspond a unique negative integer [$-n$] (called the additive inverse of n) so the relation between n, $-n$ and 0 as $n + (-n) = 0$, and $n + 0 = n$ for every natural number n. Hence the positive integers (natural numbers), the negative integers and the number zero together constitute what is known as the system of **integers** [0, ±1, ±2, ±3, ±4 …]. A **rational number** is a number that can be written as quotient of two integers, where the integer in the denominator is not zero: $r = \dfrac{m}{n}$ where $n \neq 0$ $\left[\dfrac{1}{3}, \dfrac{1}{2}, \dfrac{-3}{2}, \dfrac{0}{2}, 321, \ldots\right]$. Every rational number can be written as a repeating decimal for example $\dfrac{1}{3}$ = .33333 …, 3/11 = 0.272727 …. The rational numbers can be represented geometrically as points on a number line. The number line can be used to give us sense of order. We put a number

m to the right of the number n if m *is* greater than n. We then write this inequality as

$$m > n$$

Similarly if n is greater than m, then m is to the left of n, and we write the inequality as

$$m < n$$

If m is less than or equal to n, that is $m < n$ or $m = n$ then we use the notation as $m \leq n$ we write $m \geq n$ to indicate that m is greater than or equal to n Hence every rational number can be represented by a point of a line, "Is the converse true?" "Is it possible to assign a rational number to every point on the number line?" The answer is no.

If we construct a square with one side of unit length, Fig. 1.1 and take a point on the number line such that OP is equal in length to the diagonal of this square. It will now be shown that the point P cannot correspond to a rational number.

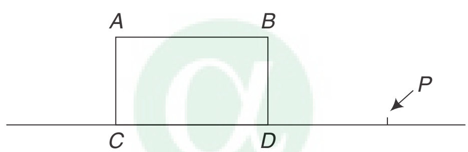


Fig. 1.1

Hence we see that there are so many number of points on the number line which do not correspond to any rational number. If we want to measure the length OP It is necessary to extend our system of numbers further by the introduction of *irrational numbers.* Thus any number that is not a rational number is called **irrational number. (the ratio between the circumference and diameter of a circle is also an irrational number)** Examples of irrational numbers are $\sqrt{2} = 1.41421356 \ldots$, $\pi = 3.141592$ (not repeated). Rational numbers and irrational numbers together constitute what is known as the system of **real numbers.**

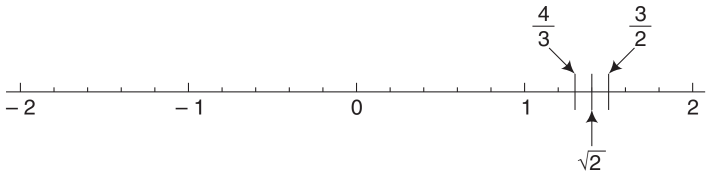


Fig. 1.2

Between any two real numbers, there is a rational number and an irrational number.

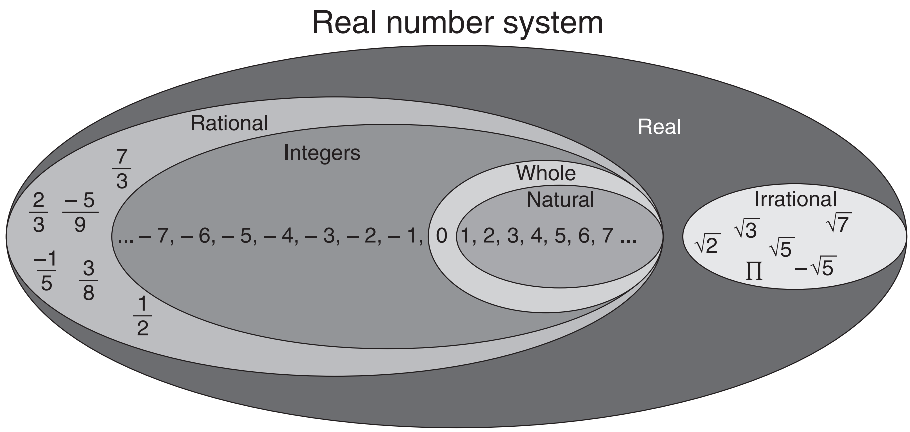


Fig. 1.3

1.2 SET

A set of objects is any well defined collection of objects, and these objects are the elements of the set. If S is a set, the notation $m \in S$ means that m is an element of S, and $m \notin S$ means that m is not an element of S. The empty set, denoted by ϕ, is the set containing no elements. If S and T are two sets then the union of S and T, denoted by $S \cup T$ is the set of elements in S or T or both. That is

$$S \cup T = \{m : m \in S \text{ or } m \in T \text{ or both}\}$$

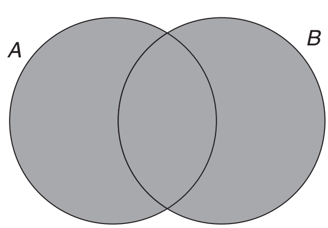


Fig. 1.4 Shaded region is A ∪ B

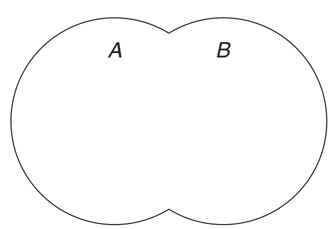


Fig. 1.5 $A \cup B$

In Boolean Logic, following UNION is represented by the intersection of two or more circles.

The intersection of S and T denoted by $S \cap T$ is the set of elements both in S and T

$$S \cap T = \{m : m \in S \text{ and } m \in T\}$$

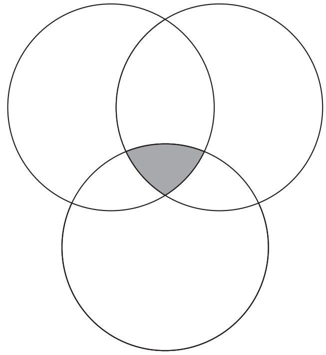


Fig. 1.6

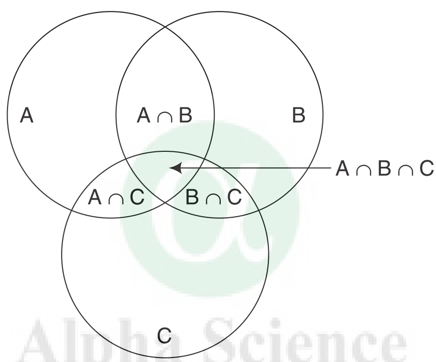


Fig. 1.7

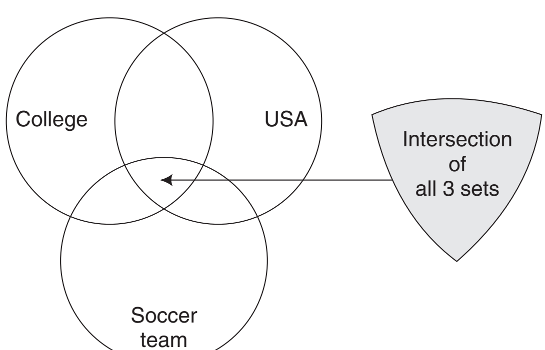


Fig. 1.8

Some typical sets

The students of class B.Sc. (Hons.) Math 1st Year in Z.H.C. New Delhi is a set.

The set of all islands in Micronesia

The set of all atolls in Yap State

The set of all cars on Mokil

The students of class B.Sc. (Hons.) 1st Year in Z.H.C. New Delhi is not a set (Why).

All big cities in India is not a set (why).

N: The set of natural numbers. $\{1, 2, 3 \ldots\}$

I: The set of integers. $\{0, \pm 1, \pm 2, \pm 3 \ldots\}$

Q: The set of rational numbers. $\left\{1, 2, \frac{3}{5}, \pm\frac{7}{2}, 0 \ldots\right\}$

R: The set of real numbers. $\left\{0, 1, \pm 2, \pm 3, \sqrt{2}, \pi, \frac{3}{4} \ldots\right\}$

Subset

If S and T are two sets such that each element of S is also an element of T then S is called a subset of T and denoted as $S \subseteq T$. i.e. the set of natural numbers is the subset of the set of integers.

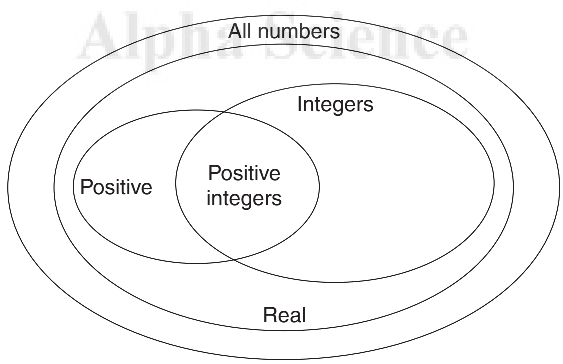


Fig. 1.9

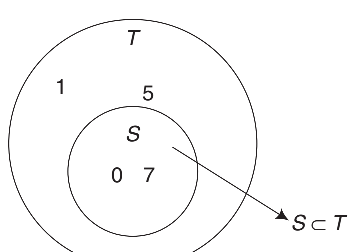


Fig. 1.10

Equality of sets

Two sets are said to be equal when they consist of exactly same elements. Thus, sets S and T are equal ($S = T$) if every element of S is an element of T and every element of T is also an element of S. Thus $\{a, b, c\} = \{b, c, a\}$.

Fig. 1.11 Six wine glasses divided into two equal sets of three

1.3 INTERVALS

A subset S_1 of R is called an **interval** if S_1 contains at least two distinct elements and every element lies between any two members of S_1.

The set of all nonzero real numbers between –1 and 1 is not an interval; since 0 is absent, the set fails to contain every real number between –1 and 1. Fig. 1.12.

Fig. 1.12

The open interval $]m, n[$ is the set of real numbers between m and n, not including the numbers m and n. i.e. $]–1, 4[= \{x: –1 < x < 4\}$ hence $m = –1$, $n = 4$. Fig. 1.13.

Fig. 1.13

The closed interval $[m, n]$ is the set of numbers between m and n, including the numbers m and n.

i.e. $$[0, 6] = \{x: 0 \leq x \leq 6\} \quad m = 0, n = 6.$$

Fig. 1.14

The half open interval $[m, n]$ is given by

$$]0, 6] = \{x : 0 < x \leq 6\} \quad m = 0, n = 6.$$

Fig. 1.14(a)

Interval may be infinite.

i.e. $[m, \infty[\; = \{x : x \geq m\}$

i.e. $]m, \infty[\; = \{x : x > m\}$

i.e. $]-\infty, m] = \{x : x \leq m\}$

i.e. $]-\infty, \infty[\; = R$

The symbol and $-\infty$ denoting infinity and minus infinity, respectively are not real numbers and do not obey the usual laws of algebra, but they can be used for notational convenience.

Solving inequalities

By which process we find the interval or intervals of numbers that satisfy an inequality in x is called solving the inequality.

Example 1 Solve the following inequalities

(i) $2x - 3 < x + 4$ (ii) $-\frac{x}{5} < 3x + 2$

(iii) $\frac{5}{x+1} \geq 5$ (iv) $\frac{6}{x-1} \geq 5$

Solution

(i) $2x - 3 < x + 4$	(ii) $-\frac{x}{5} < 3x + 2$	(iii) $\frac{5}{x+1} \geq 5$	(iv) $\frac{6}{x-1} \geq 5$
$2x < x + 7$	$-x < 15x + 10$	$5 \geq 5x + 5$	$6 \geq 5x - 5$
$x < 7$	$0 < 16x + 10$	$0 \geq 5x$	$11 \geq 5x$
	$-\frac{5}{8} < x$	$0 \geq x$	$\frac{11}{5} \geq x$

The term "absolute value" has been used in this sense since at least 1806 in French and 1857 in England the notation $|m|$ was introduced by Karl Weierstrass in 1841. Other names for *absolute value* include "the numerical value" and "the magnitude".

1.4 ABSOLUTE VALUE

The absolute value (or **modulus**) of a number m is the distance from that number to zero and is written $|m|$. Hence 3 is 3 units from zero, so that $|3| = 3$. The number -2 is 2 units from zero, so that $|-2| = 2$ or $-(-2) = 2$, Fig. 1.15.

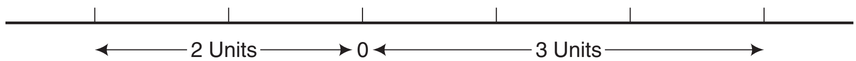


Fig. 1.15

$$|m| = m \text{ if } m \geq 0$$

$$|m| = -m \text{ if } m < 0$$

$$|m| = \begin{cases} m & m \geq 0 \\ 0 & m = 0 \\ -m & m < 0 \end{cases}$$

$$(-m)^2 = m^2 = |m|^2 \text{ or } |m| = \sqrt{m^2}$$

i.e. $$|-4| = \sqrt{(-4)^2} = 4,\ |4| = \sqrt{(4)^2} = 4,\ |-3| = \sqrt{-(3)^2} = 3$$

Properties for absolute value:

(i) $|-m| = m$

(ii) $|mn| = |m||n|$

(iii) $\left|\dfrac{m}{n}\right| = \dfrac{|m|}{|n|}$

(iv) $|m + n| \leq |m| + |n|$ (Triangle inequality) i.e. $|-4 + 7| = |3| < |-4| + |7| = 11$

i.e. $|4 + 7| = |11| = |4| + |7|$

i.e. $|-4 - 7| = |-11| = 11 = |-4| + |-7|$

(v) $|m - n| = 0 \Leftrightarrow m = n$

(vi) $|x| = m$ if and only if $x = \pm m$

(vii) $|x| < m$ if and only if $-m < x < m$

i.e. $|x| < 2 \Rightarrow -2 < x < 2$, Fig. 1.16

Fig. 1.16

(viii) $|x| > m$ if and only if $x > m$ or $x < -m$

i.e. $|x| > 2 \Rightarrow x > 2$ or $x < -2$, Fig. 1.17

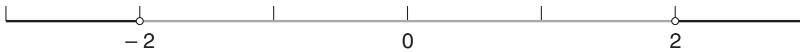


Fig. 1.17

(ix) $|x| \leq m$ if and only if $-m \leq x \leq m$

(x) $|x| \geq m$ if and only if $x \geq m$ or $x \leq -m$

(xi) $|x - m| < l \Leftrightarrow m - l < x < m + l$

Example 2 Solve the inequality $|x + 3| \geq 7$.

Solution $|x + 3| \geq 7$

$x + 3 \geq 7$ or $x + 3 \leq -7$

Hence either $x \geq 4$ or $x \leq -10$

Example 3 Solve the inequality $|2x - 1| \leq 3$

Solution $|2x - 1| \leq 3$

$-3 \leq 2x - 1 \leq 3 \Rightarrow -2 \leq 2x \leq 4 \Rightarrow -1 \leq x \leq 2.$

1.5 THE CARTESIAN PLANE

The Cartesian plane is named of the great French mathematician Rene Descartes. In the section 1.1 we identified the points on the line with real numbers by assigning those coordinates. Now the points in the plane can be identified with ordered pairs of real numbers. Let OX and OY be two fixed straight line perpendicular to each other. The line OX is called the *x-axis* while OY is called the *y-axis*. Both of them together are called the coordinates axes. The point O is termed as the origin of coordinates. Let P be any point in the plane, to reach this point let us draw a straight line from P, parallel to OY to meet OX in M. The distance OM is called *x-coordinate* (**abscissa**) and distance MP is called *y-coordinate* (**ordinate**) of the point P This ordered pair with abscissa as first member, is called the coordinate of P If $OM = x$, $MP = y$ then (x, y) are coordinate of P. This coordinate system is called the rectangular coordinate system or Cartesian coordinate system. The coordinate axes of this Cartesian plane divide the plane into four regions called quadrants Fig. 1.18.

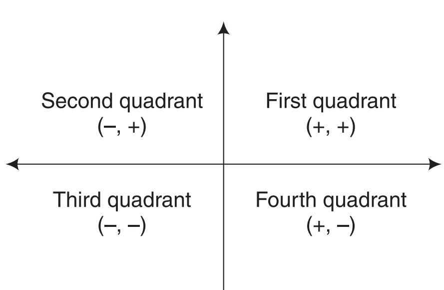


Fig. 1.18

We will usually refer to the Cartesian plane as the *xy-plane*.

If (x_1, y_1) and (x_2, y_2) are the two points in the *xy-plane* then the distance between these two points is the length of the line segment *PQ*, Fig. 1.19.

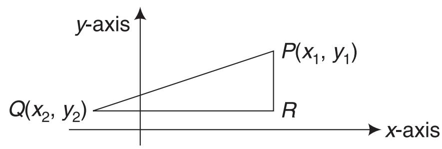


Fig. 1.19

Since *PQR* is a right triangle and by the Pythagoras theorem, $\overline{PQ}^2 = \overline{PR}^2 + \overline{QR}^2$

But $\overline{PR} = |y_1 - y_2|$, $QR = |x_1 - x_2|$ so that

$$(PQ)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

i.e. $(x_1, y_1) \Rightarrow (2, 3)$ and $(x_2, y_2) \Rightarrow (-3, 4)$ then

$$PQ = \sqrt{(-3-2)^2 + (4-3)^2} = \sqrt{26}$$

1.6 LINE

The line plays a very important rule to study the calculus. Two distinct points (x_1, y_1) and (x_2, y_2) determine a line (in Fig. 1.19 *PQ* is a line). The slope of a line tell us the direction and steepness of a line measure of the relative rate of change of the *x-coordinate* and *y-coordinate* points on the line as we move along the line.

The slope m of a line passing throw the points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \tan\theta \text{ (if } x_1 \neq x_2) \text{ (Fig. 1.20)}$$

The speed of the cart A will be faster than the cart B because the **slope** m of the line *OA* > line *OB*, Fig. (1.21).

In Fig. 1.20 if $\Delta x = (x_2 - x_1) = 0$ and $y_1 \neq y_2$ then the line is vertical and the slope is said to undefined. {Fig. 1.24 vertical line}.

In Fig. 1.20 if $\Delta y = (y_2 - y_1) = 0$ and $x_1 \neq x_2$ then the line is horizontal and the slope is zero. {Fig. 1.24 horizontal line}.

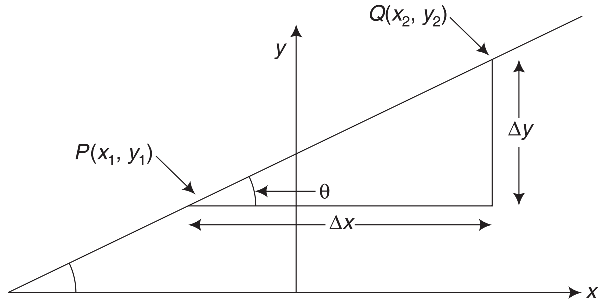


Fig. 1.20

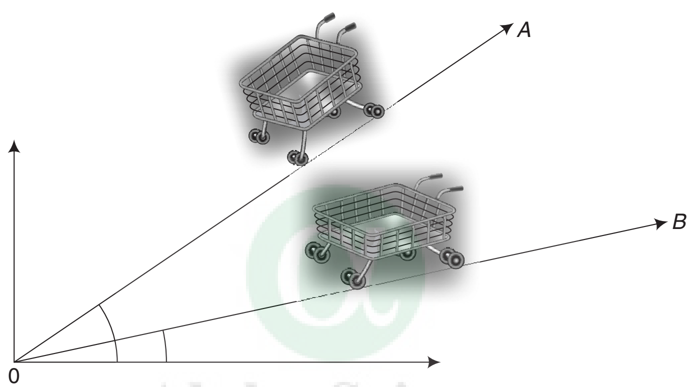


Fig. 1.21

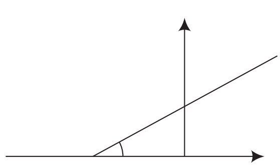


Fig. 1.22 Positive slope $m > 0$

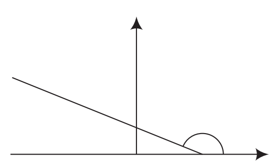


Fig. 1.23 Negative slope $m < 0$

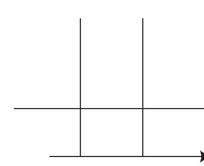
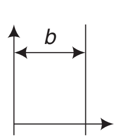


Fig. 1.24

Some properties of straight lines

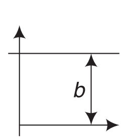
Equation of line — *Description of line*

$\boldsymbol{x = b}$



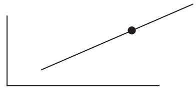
No slope, x intercept is ***b*,** line is vertical

$\boldsymbol{y = b}$



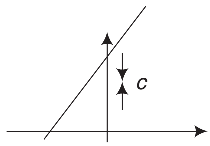
Slope is 0, y intercept is ***b*,** and line is horizontal

$\boldsymbol{y - y_1 = m(x - x_1)}$



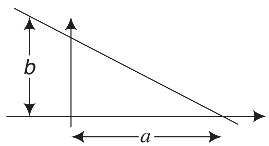
slope m, line pasing through a point (x_1, y_1).

$\boldsymbol{y = mx + c}$



slope m, y intercept is c, and x intercept is $\frac{-c}{m}$, $m \neq 0$

$\frac{x}{a} + \frac{y}{b} = 1$



slope is $\left(-\frac{b}{a}\right)$, y intercept is b, and x intercept is a.

Two lines are parallel if and only if they have same slope Fig. 1.25

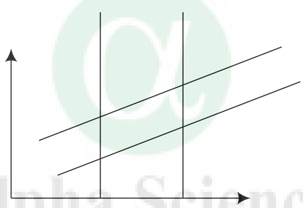


Fig. 1.25

If two lines L_1 and L_2 have the slopes m_1 and m_2 *respectivaly* and both lines are perpendicular to each other then, Fig. 1.26.

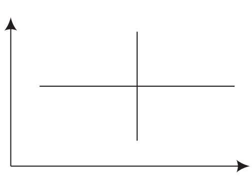


Fig. 1.26

$$m_1\, m_2 = -1 \quad (m_1 \neq 0)$$

If $m > 0$ then $0° < \theta < 90°$ Fig. 1.20.

If $m < 0$ then $90° < \theta < 180°$ as we move from P to Q in Fig. 1.27, y decreases and $\Delta y < 0$. Thus the length $P\,R$ is $-\Delta y$, and we have

$$\tan(\pi - \theta) = \frac{\Delta y}{\Delta x} = -m$$

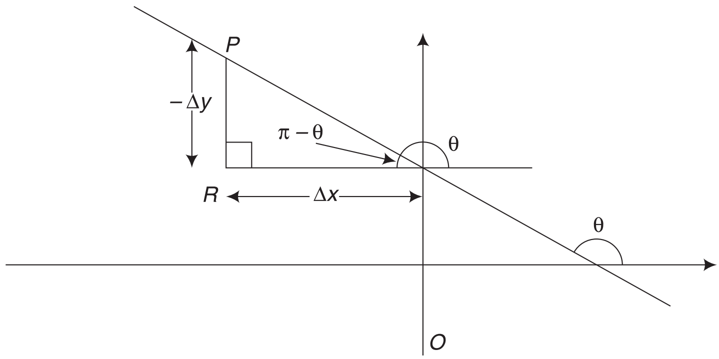


Fig. 1.27

1.7 FUNCTION AND THEIR GRAPH

We know that the area $A = \pi r^2$ of a circle depends on the radius of the circle so here the radius r of the circle is an independent variable and the area A is dependent variable.

In the above example the value of one variable quantity, which we might call y, (*area A*) depends on the value of another variable quantity, which we might call x, (*radius r*) Since the value of y is completely determined by the value of x, we say that y is a function of x. A symbolic way to say "y is a function of x" is by writing.

$$y = f(x).$$

The function notation $y = f(x)$ was first used by the Leonhard Euler in 1734-1735.

In writing $y = f(x)$, the symbol f which stands for the function rule, and the symbol $f(x)$, which is the value of the function takes on for a given independent number x in the domain of f. Here $f(x)$ is a number in the range of f.

Definition A function from a set X to a set Y is a rule that assigns a unique element $f(x) \in Y$ to each element $x \in X$. The set X is called the domain of f, and the set of all values of $f(x)$ as X varies through X is called the range of the function f.

Definition A function is a set of ordered pairs where for any x value in the set, there is only one y value.

i.e. $\{(0, 0), (1, 1), (2, 8), (3, 27)\} \Rightarrow$ Function

i.e. $\{(0, 0), (1, 1), (1, -1) (3, 27)\} \Rightarrow$ Not a Function

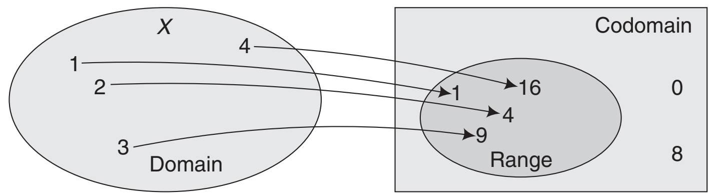


Fig. 1.28

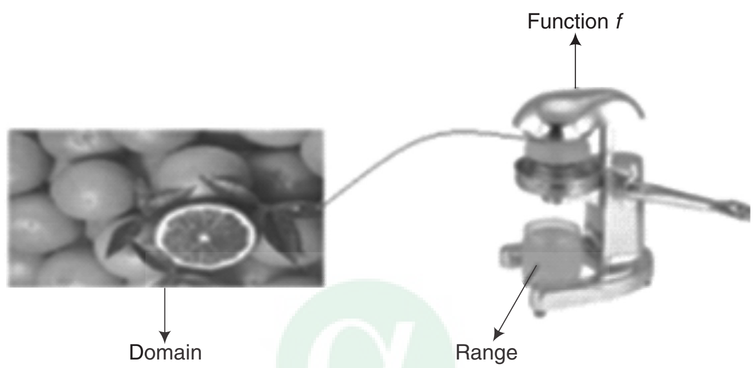


Fig. 1.29

To find the domain of a function we need to ask the question "for what values of x does the rule $y = f(x)$ make sense"? For example the area of a circle whose radius r defined as $A(r) = \pi r^2$, since the radius of the circle always will be positive real number except zero so the domain of the area of the circle is the set of all positive real numbers except zero. This set can be written as $R^+ - \{0\}$.

To find the range of a function we must ask "what values do we obtain for y as x takes on all values in domain f" ? For example

$$f(x) \Rightarrow A(r) = \pi r^2 \Rightarrow f$$

Hence the domain of f is $R^+ - \{0\}$, the range of f is also $R^+ - \{0\}$ {Area is neither zero nor negative}.

Example 4 Find the domain of the following functions.

(i) $y = \dfrac{3}{x}$, (ii) $y = 2x - 1$, (iii) $y = \sqrt{(1 - x^2)}$, (iv) $\dfrac{1}{\sqrt{(1 - x)}}$

(v) $y = \dfrac{1}{\sqrt{(1 - x)(x - 2)}}$ (vi) $y = \sqrt{x + 2}$ (vii) $y = \dfrac{1}{1 - \sin x}$

Solution

(i) $y = \frac{3}{x}$ The function not define at $x = 0$

so the domain is $R - \{0\}$ or $]-\infty, 0[\cup]\ 0, \infty[$

(ii) The domain of the function $y = 2x - 1$ is all real numbers.

(iii) We know that there is no real number whose square is negative,

so far as real number are concerned, the square is negative number does not exist.

Now $1 - x^2 > 0$, if and only if x satisfies the relation

$$-1 \leq x \leq 1.$$

So the domain of the function $y = \sqrt{(1 - x^2)}$ is the interval $[-1, 1]$.

(iv) The domain of the function $y = \frac{1}{\sqrt{(1-x)}}$ is the interval $]-\infty, 1[$, because at $x = 1$, y is not define and $1 - x$ is negative for $x > 1$.

(v) The domain of the function $y = \frac{1}{\sqrt{(1-x)(x-2)}}$ is the open interval $]1, 2[$.

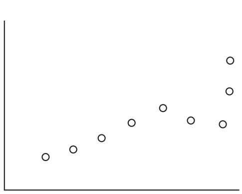
(vi) The function $y = \sqrt{x+2}$ has meaning if and only if $x + 2$ is non negative, therefore the domain is $[-2, \infty[$.

(vii) The function $y = \frac{1}{1 - \sin x}$ has the domain if and only if $\sin x \neq 1$ so $x \neq \left(2n + \frac{1}{2}\right)\pi$ where $n = 0, \pm 1, \pm, 2 \ldots$

Graph of the function

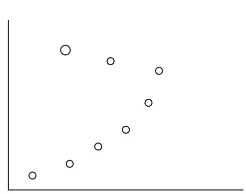
The commercial and Political Atlas was the one of the first book to use graph for representing numerical data published in 1786 by the Scottish political economist William Play-fair.

Types of graphs



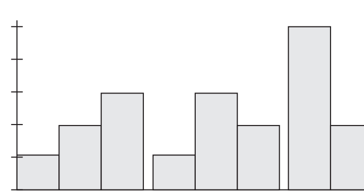
Scatter plot

Fig. 1.30



Line graph

Fig. 1.31



Bar graph

Fig. 1.32

Definition The graph of the function f is the set of order pairs $[\{x, f(x)\}: x \in$ domain f]. Hence a point lies on the graph of f if and only if its coordinates satisfy the equation $y = f(x)$ For example the equation $y = 2x + 1$ can be thought of as a function since for every real number x there is a unique real number y that is equal to $2x + 1$. The domain and range of f is R.

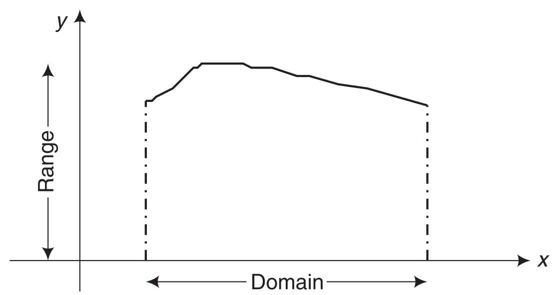


Fig. 1.33

The vartical line test

A curve in the xy – *plan* is the graph of a function if and only if it intersect the vertical line only once.

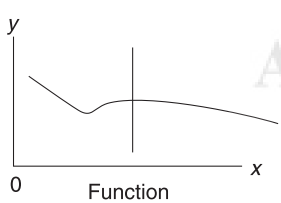


Fig. 1.34

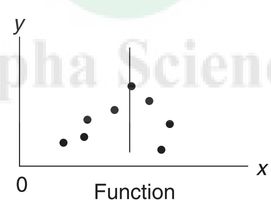


Fig. 1.35

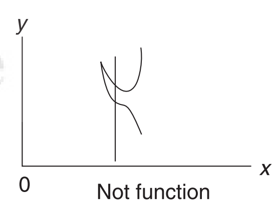


Fig. 1.36

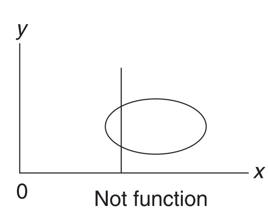


Fig. 1.37

The graph of a cricle $x^2 + y^2 = 4$ given in Fig. 1.38. In the open interval]–1, 1[for every real number x there are two values of y given by $y \pm \sqrt{(4 - x^2)}$ Hence we do not have a function. We can obtain two seprate functions by defining $y_1 = \sqrt{(4 - x^2)}$ and $y_2 = -\sqrt{(4 - x^2)}$.

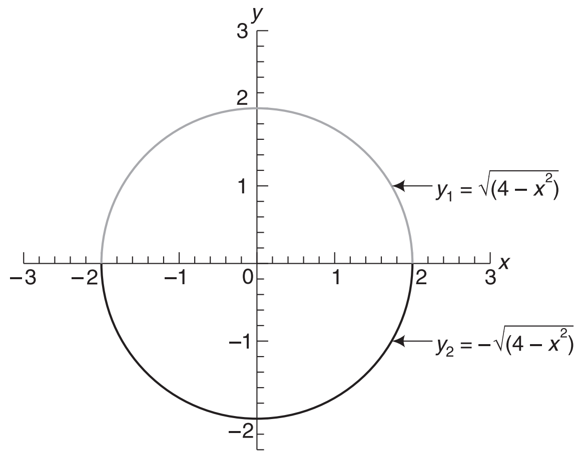


Fig. 1.38

Types of functions

One-one Function: A function $f: X \to Y$ is said to be *one-one (injective or univalent)* if distinct element of X have distinct element in Y, Fig. 1.39.

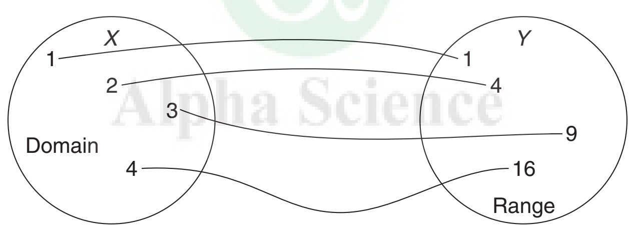


Fig. 1.39

Onto Function: A function $f: X \to Y$ is said to be *Onto (or subjective)* if the range set of f coincides with its co-domain. In symbols $f(x) = Y$. Fig. 1.39.

One-one onto Function: A function $f: X \to Y$ is said to be *one-one onto* (*or bijective)* function if it is both-*one-one and onto*, Fig. 1.39. i.e. Let X be the set of all even integers and Y be the set of all odd integers. Then f defined as $f(x) = x + 1$ is one – one onto.

Many one Function: A function $f: X \to Y$ is said to be a *many-one* function if it is not *one-one*. i.e. $f(x_1) = f(x_2)$ even $x_1 \neq x_2$ for $x_1, x_2 \in X$ Fig. 1.40.

Let R be the set of all real numbers and let $f: R \to R$ be defined as

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is a rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$$

Then, f is a *many-one* function.

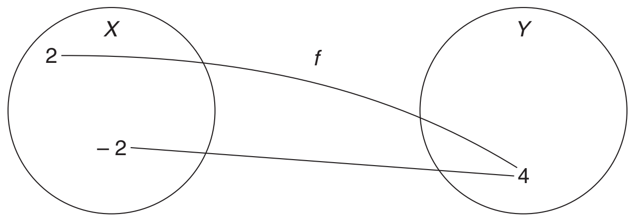


Fig. 1.40

Constant Function: A function f: $X \rightarrow Y$ is said to be a *constant* function if each element of X is associated with the same element of Y. i.e. $f(x) = 1 \ \forall \ x \in R$, Fig. 1.41.

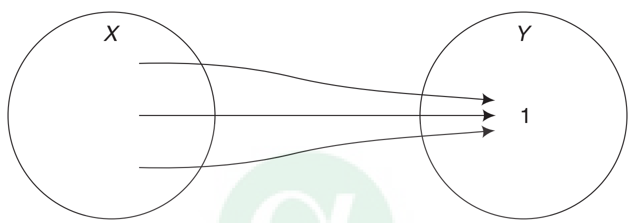


Fig. 1.41

Identity Function: A function f:$X = Y$ is said to an *identity* function if each element $x \in X$ is associated with x itself. In symboles $\forall \ x \in X$ i.e. if $X = \{1, 2, 3\}$ then $f(1) = 1$, $f(2) = 2$, $f(3) = 3$ is an *identity* function.

Linear Function: is the function of the form $y = mx + c$, where m and c are constant.

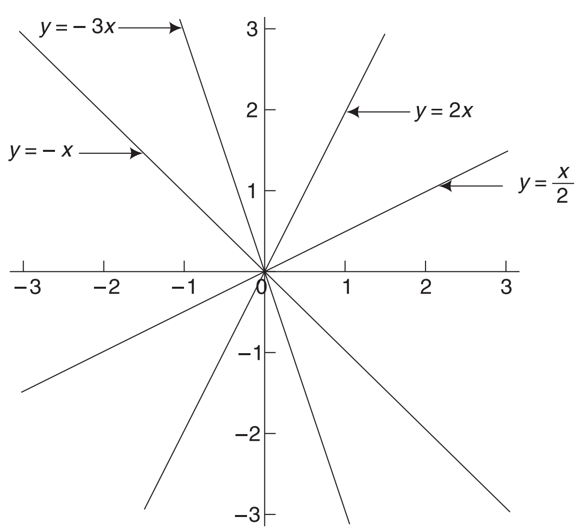


Fig. 1.42

Power Function: is a function of the form $f(x) = x^a$, where a is constant.

(i) When $a = n$ is a positive integer 1, 2, 3, 4, 5 ….

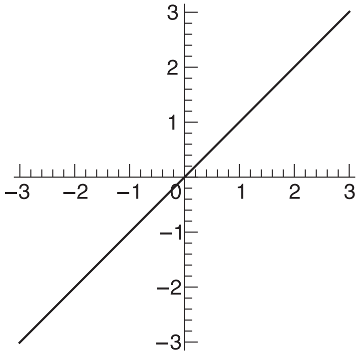


Fig. 1.43 $f(x) = x$

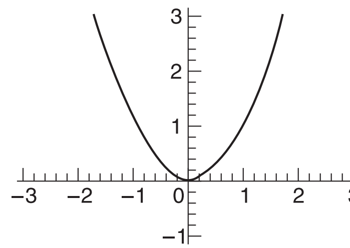


Fig. 1.44 $f(x) = x^2$

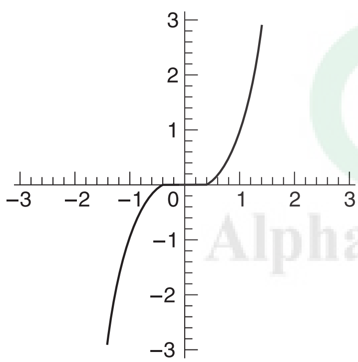


Fig. 1.45 $f(x) = x^3$

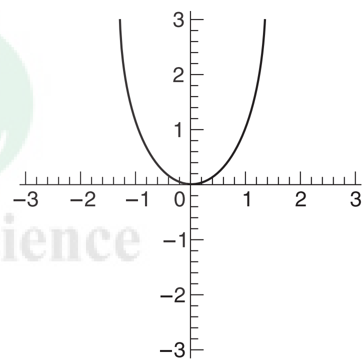


Fig. 1.46 $f(x) = x^4$

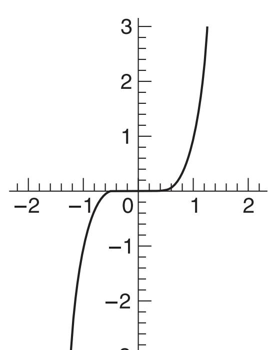


Fig. 1.47 $f(x) = x^5$

(ii) When a is -1 or -2

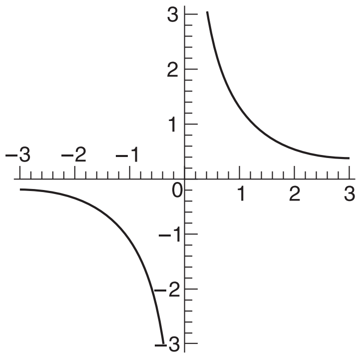


Fig. 1.48 $f(x) = \frac{1}{x}$

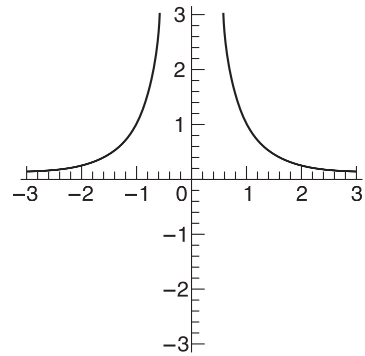


Fig. 1.49 $f(x) = \frac{1}{x^2}$

(iii) When a is $\frac{1}{2}, \frac{3}{2}, \ldots.$

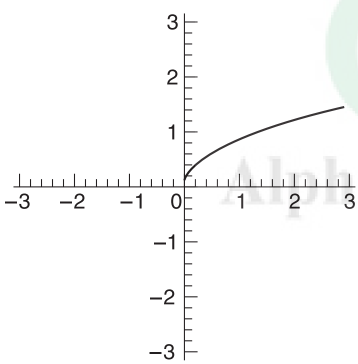


Fig. 1.50 $f(x) = \sqrt{x}$

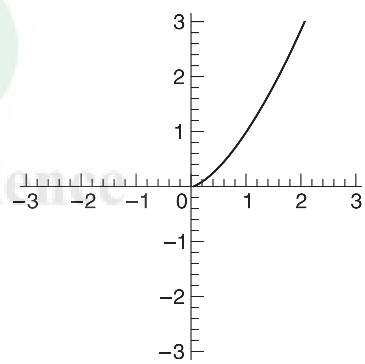


Fig. 1.51 $f(x) = x^{\frac{3}{2}}$

Polynomial Function

is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + a_{n-3} x^{n-3} + a_1 x^2 + a_1 x + a_0$$

Where n is a positive integer and a_n, a_{n-1} … a_2, a_1, a_0 are constants. All polinomials have domain $(-\infty, \infty)$. If the leading coefficient $a_n \neq 0$ and $n > 0$, then n is called the degree of the polynomials.

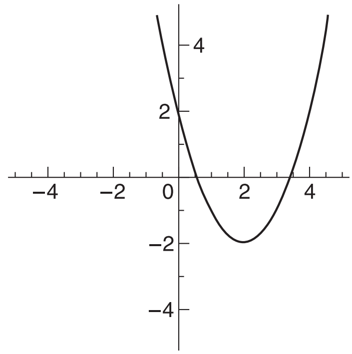


Fig. 1.52(a) $f(x) = x^2 - 4x + 2$ (degree 2)

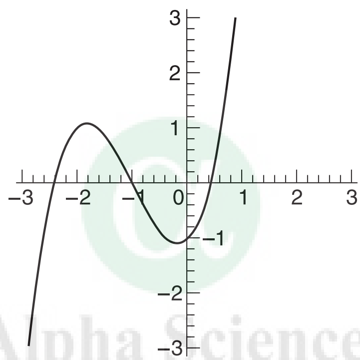


Fig. 1.52(b) $f(x) = x^3 + 3x^2 + x - 1$ (degree 3)

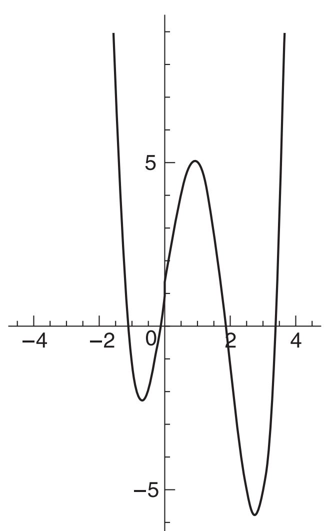


Fig. 1.52(c) $f(x) = x^4 - 4x^3 + 7x + 1$ (degree 4)

Constant Function: is zero degree polinomial $f(x) = a$. i.e. $f(x) = \frac{1}{2}$

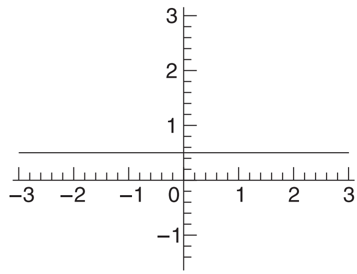


Fig. 1.53 $f(x) = \frac{1}{2}$

Rational Function: is a ratio of two polinomial functions p and q defined as $f(x) = \frac{p(x)}{q(x)}$. The domain of this function is the set of real numbers for which $q(x) \neq 0$ i.e. $f(x) = \frac{3x^2 - 4}{5x + 3}$

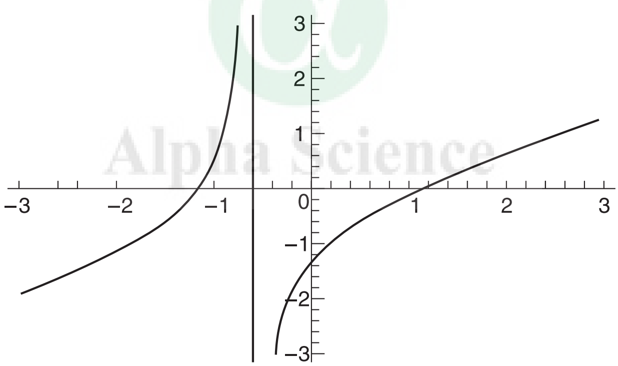


Fig. 1.54 $f(x) = \frac{3x^2 - 4}{5x + 3}$

Algebraic Functions: A function is called algebraic function if it can be constructed from the polinomials using a finite number of algebraic operations (addition, subtraction, multiplication, division and taking roots) i.e. $f(x) = x^{\frac{1}{3}}(x - 2)$.

Function that are not algebraic are called transcendental.

The following are transcendental functions,

Trignometric Functions: are the functions sine, cosine, tangent, secant, cosecent and contangent.

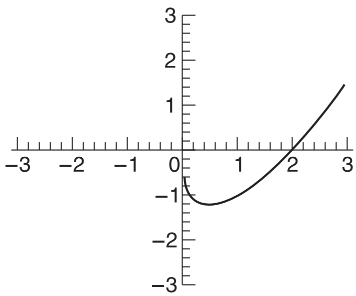


Fig. 1.55 $f(x) = x^{\frac{1}{3}}\,(x - 2)$

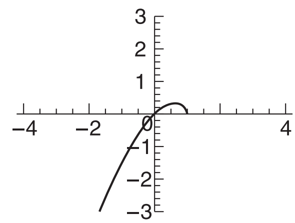


Fig. 1.56 $f(x) = x(1 - x)^{\frac{3}{5}}$

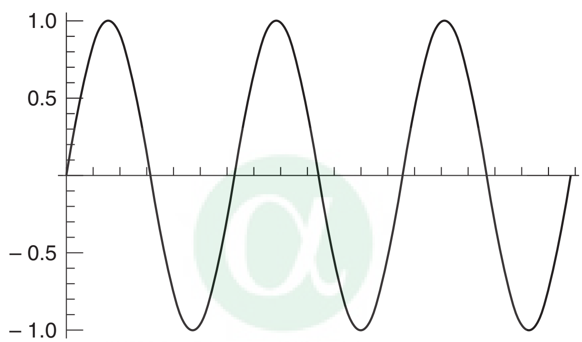


Fig. 1.57 $f(x) = \sin x$

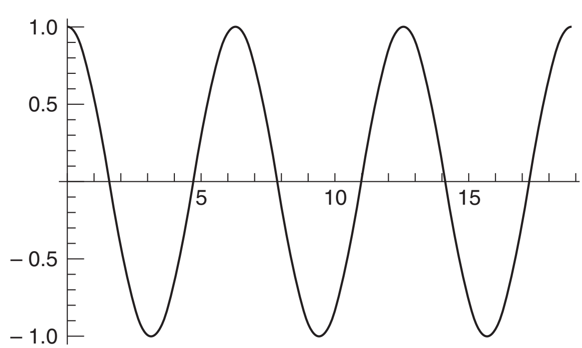


Fig. 1.58 $f(x) = \cos x$

Exponential Functions: are functions of the form $f(x) = a^x$, where base $a > 0$ is a positive constant and $a \neq 1$. The domain of exponential functions is $]-\infty, \infty[$ and range $]0, \infty[$ i.e. $f(x) = 5^x$, $f(x) = 10^x$.

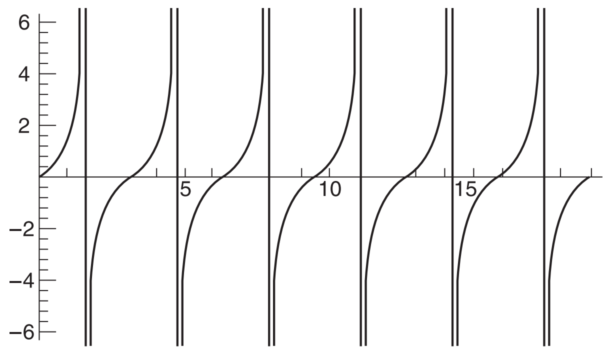


Fig. 1.59 $f(x) = \tan x$

The natural exponential function $f(x) = e^x (e \approx 2.71828182)$ is some times denoted as exp (x).

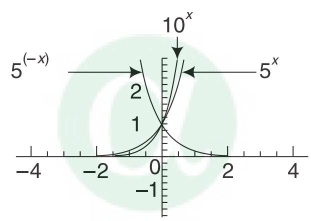


Fig. 1.60

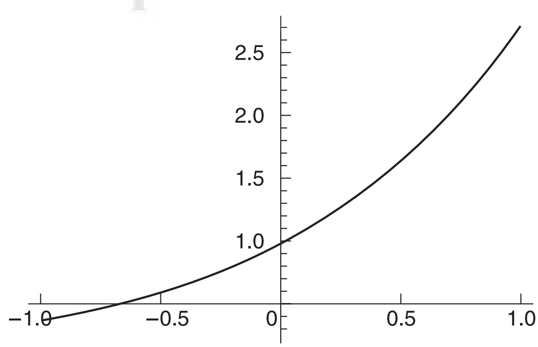


Fig. 1.61 $f(x) = e^x$

Logarithmic Function: are functions of the form $f(x) = \log_b x$, where b is positive constant, $b \neq 1$.

i.e. $$f(x) = \log_3(2x + 3)$$

The natural logarithm, $\log_e x$, is denoted by $\ln x$. The common logarithm $\log_{10} x$, denoted by $\log x$

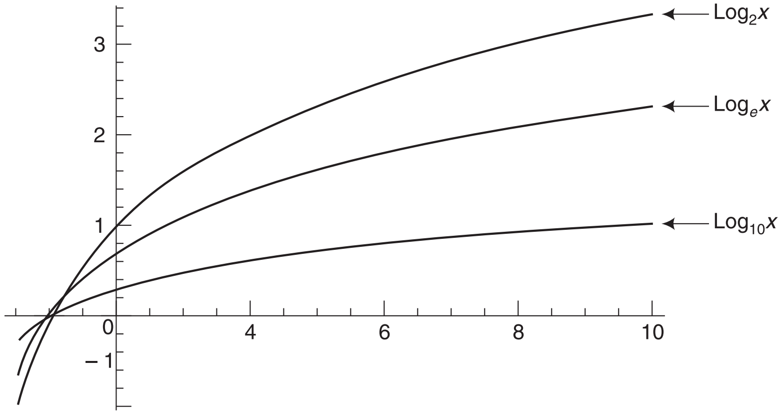


Fig. 1.62

Basic properties of the natural logarithm

$\log 1 = 0$, $\log e = 1$, $e^{\log x} = x$, $\forall\ x > 0$, $\log e^y = y$, $\forall\ y > 0$, $b^x = e^{x\log b}$ for any $b > 0$ $(b \neq 1)$

$$\log_b x = \frac{\log x}{\log b} \text{ for any } b > 0 (b \neq 1),\ a = \log b \text{ if and only if } b = 10^a.$$

Even and odd Function: The function $y = f(x)$ is symmetric about y–axis if $f(x) = f(-x)$ and this function is called the even function. The symmetry with respect to the origin occurs when $f(-x) = -f(x)$ for all x and this function is called the odd function. Fig. 1.44 and Fig. 1.46 denoted the even functions while the Fig. 1.45 and Fig. 1.47 are the example of odd functions.

Composite Function: If f and g are two functions, then the composite function fog is defined by $(fog)\ (x) = f(g(x))$

$$\text{Dom}(fog) = \{x: x \in \text{ dom } g \text{ and } g(x) \in \text{ dom } f\} \Leftrightarrow x \to g \to g(x) - f \to f(g(x))$$

Let $f(x) = \sqrt{x}$ and $g(x) = 2x^2 + 3$, then $(fog)\ (x) = f(g(x)) = f(2x^2 + 3)$ $= \sqrt{2x^2 + 3}$

$\text{Dom}(fog) = \{x: x \in g(x) = 2x^2 + 3 \in \text{ dom } f\}$ But $2x^2 + 3 > 0$, $2\ x^2 + 3 \in$ dom f for every real x so Dom $(fog) = R$.

And $\quad (gof)\ (x) = g(f(x)) = g\ \sqrt{x} = 2x + 3.$

[To make an ice-cream from orange juice without any other material is an example of composite function *fog*. Fig. 1.63.]

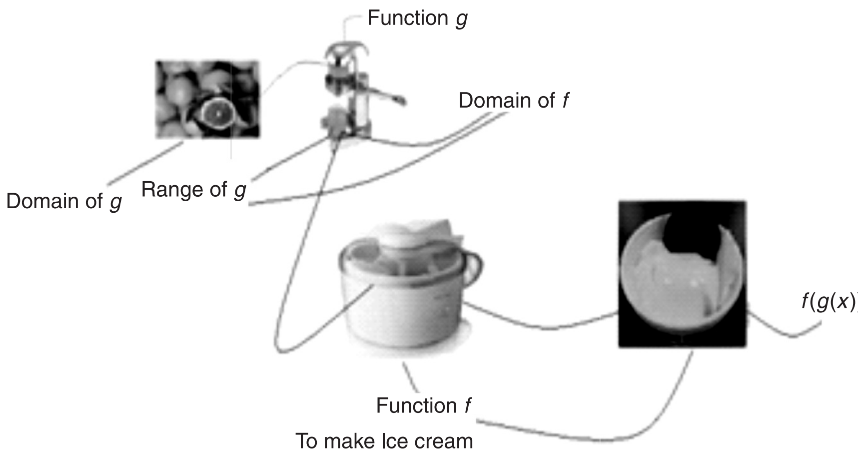


Fig. 1.63 Composite function

Some general rules for $y = f(x)$

The graph of $y = f(x) + c$, shift the graph of $y = f(x)$ **up** c units if $c \geq 0$ and **down** $|c|$ units if $c < 0$, Fig. 1.64.

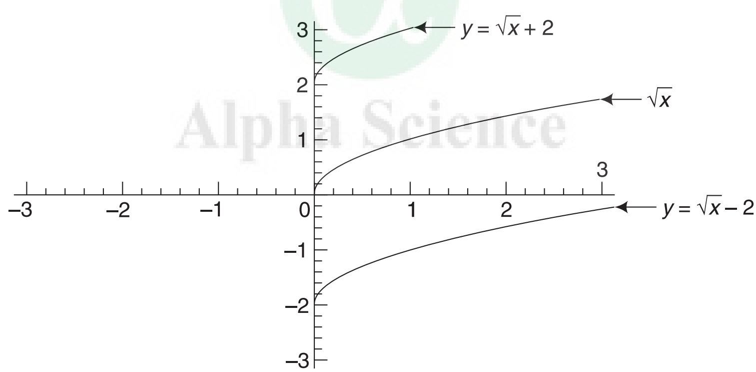


Fig. 1.64

The graph of $y = f(x - c)$, shift the graph of $y = f(x)$ to the **right** if $c > 0$ and to the **left** $|c|$ units if $c < 0$, Fig. 1.65.

Inverse Function: Let f be a function with domain X and range is Y. Then the function $f^{-1}$with domain Y and range X is the inverse of f if

$$f^{-1}[f(x)] = x \ \forall \ x \text{ in } X$$

$$f[f^{-1}(y)] = y \ \forall \ y \text{ in } Y.$$

And it is possible if and only if the function f is one-to-one, Fig. 1.66.

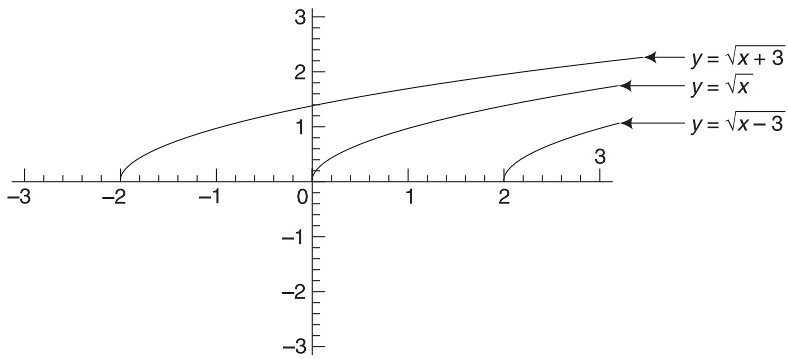


Fig. 1.65

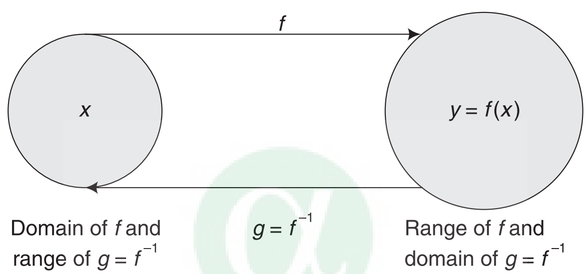


Fig. 1.66

Let $f(x) = 2x + 1$ to find f^{-1} let $y = 2x + 1$ then $x = 2y + 1$ (interchange the x and y and solve) $y = \dfrac{(x-1)}{2}$, Hence $f^{-1} = \dfrac{(x-1)}{2}$. For verification

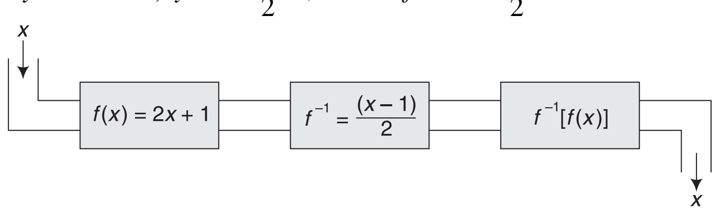


Fig. 1.67

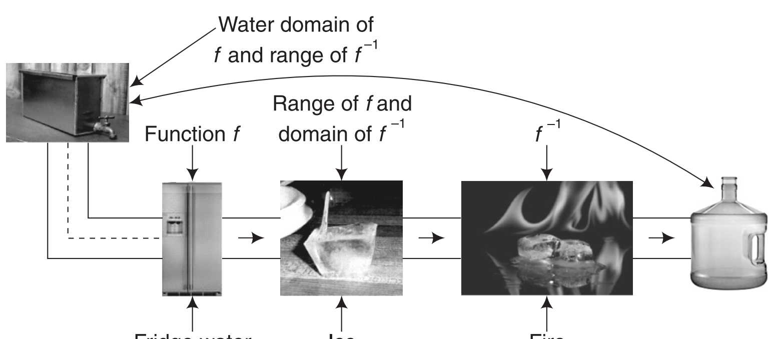


Fig. 1.68

Horizontal line test: A function f has an inverse f^{-1} if and only if a horizontal line intersect the graph of $y = f(x)$ not more than one point.

A graph of a function that has inverse function, Fig. 1.69. A graph of a function that has no inverse function, Fig. 1.70.

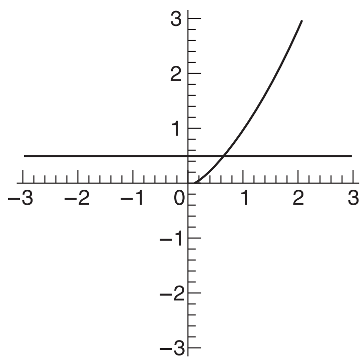


Fig. 1.69 A graph of a function that has inverse function

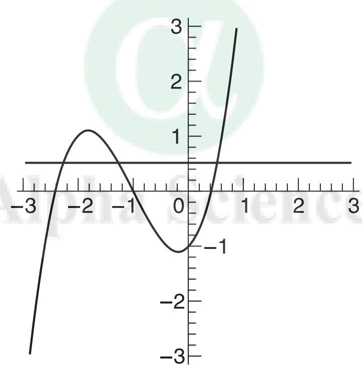


Fig. 1.70 A graph of a function that has no inverse function

Example 5 Find the inverse functions of the following functions.

(i) $f(x) = \dfrac{x+6}{2x-3}$ (ii) $f(x) = \sqrt{2x-1}$

(iii) $f(x) = \sqrt[7]{3x+4}$ (iv) $f(x) = \begin{cases} 3x, & x \le 0 \\ x^2, & x > 0. \end{cases}$

(v) $f(x) = \{(0, 1), (1, 4), (2, 7), (3, 10)\}$

Solution

(i) Let $y = f(x) = \dfrac{x+6}{2x-3}$ (ii) Let $y = f(x) = \sqrt{2x-1}$

$x = \dfrac{y+6}{2y-3}$, (interchange x and y) $y^2 = 2x - 1$

$\Rightarrow 2yx - 3x - y - 6 = 0$ $x^2 = 2y - 1$ (interchange x and y)

$\Rightarrow y = \dfrac{3x+6}{2x-1}$ Hence $f^{-1}(x) = \dfrac{3x+6}{2x-1}$ $\Rightarrow y = \dfrac{x^2+1}{2}$, hence $f^{-1}(x) = \dfrac{x^2+1}{2}$

(iii) Let $y = f(x) = \sqrt[7]{3x+4}$.

$y^7 = 3x + 4$, $x^7 = 3y + 4$, (interchange x and y) $\Rightarrow y = \dfrac{x^7-4}{3}$, Hence $f^{-1}(x) = \dfrac{x^7-4}{3}$

(*iv*) $y = f(x) = \begin{cases} 3x, & x \le 0 \\ x^2 & x > 0 \end{cases}$. $x = f(y) = \begin{cases} 3y, & y \le 0 \\ y^2, & y > 0 \end{cases}$.

Hence $f^{-1}(x) = \begin{cases} \dfrac{x}{3}, & x \le 0 \\ \sqrt{x}, & x > 0 \end{cases}$.

(v) $f(x) = \{(0, 1), (1, 4), (2, 7), (3, 10)\}$, $f^{-1}(x) = \{(1, 0), (4, 1), (7, 2), (10, 3)\}$

Example 6 Use the inverse function solve log $(x - 4) = 3$.

Solution Since logarithmic and exponential functions are inverse of each other, use the property of logarithmic (page 1.25).

$$(x - 4) = 10^3 \Rightarrow x = 1004.$$

Example 7 A camera is to take a series of photographs of a hot air balloon rising vertically. The distance between the camera at P and the launching point of balloon Q is 200 meters. The camera must keep the balloon one sight and therefore its angle of elevation α must change with the hight h of the ballon.

(i) Find the angle α as a function of the hight h.

(ii) Find the angle α in degree when h is 600 meters, Fig. 1.71.

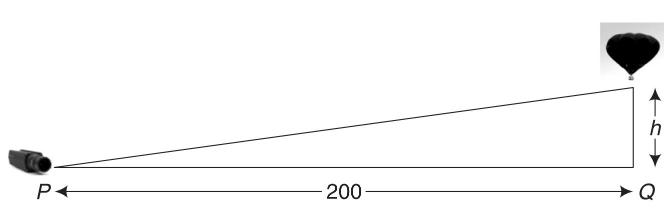


Fig. 1.71

Solution

(i) From Fig. 1.71 $\tan \alpha = \dfrac{h}{200}$

And we know that $\tan^{-1}(\tan\alpha) = \alpha$

$$\alpha = \tan^{-1}\left(\frac{h}{200}\right)$$

(ii) $\alpha = \tan^{-1}\left(\frac{600}{200}\right)$

$\alpha = \tan^{-1}(3)$

Piecewise Function: An example of Piecewise Function is a absolute function $f(x) = |x|$, in the sense that the formula for f changes, depending on the value of x, Fig. 1.72.

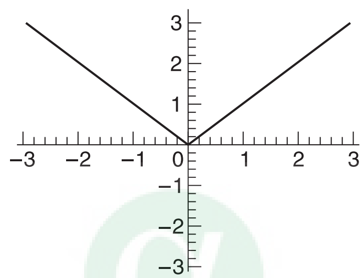


Fig. 1.72

Example 8 Sketch the graph of the following function

$$f(x) = \begin{cases} 0, & x \le -2 \\ \sqrt{4 - x^2}, & -2 < x < 2 \\ x, & x \ge 2 \end{cases}$$

Solution

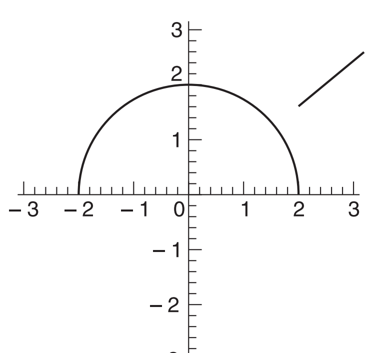


Fig. 1.73

Note: In geometric problems to preserve the true shape of a graph we must use the equal length of the units on both axes. For example if we sketch the graph of a circle in a coordinate system in which 1 unit in the y-axis is smaller than 1 unit in the x-axis, then the cricle will be change in the form of the ellips, Fig. 1.74.

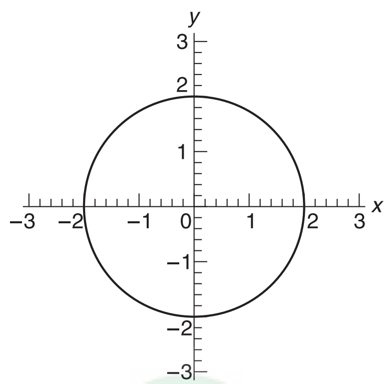


Fig. 1.74(a) Units equal in both axes

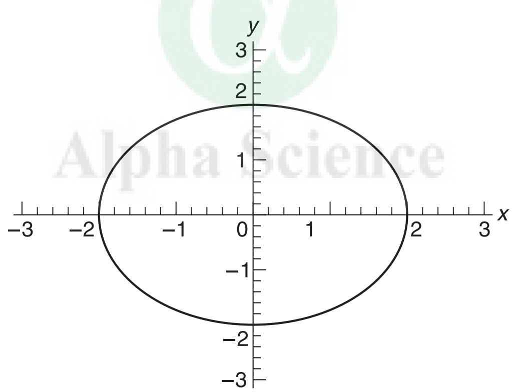


Fig. 1.74(b) 1 unit in the y-axis is smaller than 1 unit in the x-axis

In many cases it is not possible to display a graph by using the units of equal length. For example if we want to sketch the portion of the graph of the function $y = 2x^2$ over the interval $-2 \le x \le 2$, then there is no problem using units of equal length. However, If we want to sketch the portion of the graph over the interval $-6 \le x \le 6$, then there is a problem keeping the units equal in length in this case the value of y varies between 0 and 72. To solve this problem the easy way is to the compress the unit of length along the y-axis shown in Fig. 1.75.

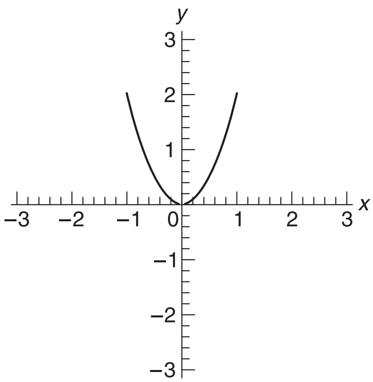


Fig. 1.75(a)

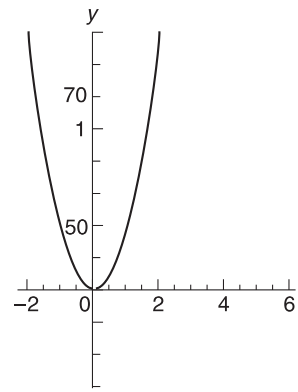


Fig. 1.75(b)

Example 9 Solve $e^x - e^{-x} = 2$ for x.

Solution $e^x - \dfrac{1}{e^x} = 2$ or $e^{2x} - 2e^x - 1 = 0$

Let $v = e^x$ then we have $v^2 - 2v - 1 = 0$

Solving for v we obtain $v = \dfrac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$ or, since $v = e^x$

$e^x = 1 \pm \sqrt{2}$ but e^x cannot be negative, hence

$e^x = 1 + \sqrt{2}$

$\log e^x = \log(1 + \sqrt{2}) \Rightarrow x = \log(1 + \sqrt{2}) \approx .881$

Exercises

1. Solve the following inequalities

(i) $-3x > 6$, (ii) $9x - 3 \le 5 - 3x$,

(iii) $-\left(\dfrac{x-4}{2}\right) \ge 5x - \dfrac{7}{2}$, (iv) $\dfrac{4}{5}(x-2) < \dfrac{1}{3}(x-6)$,

(v) $-3 < \dfrac{(7-2x)}{3} \le 4$, (vi) $-\dfrac{x}{3} < 2x + 1$,

(vii) $2x - \dfrac{1}{2} \ge 7x + \dfrac{7}{6}$, (viii) $|x - 4| < 5$,

(ix) $|x - 1| \le 3$, (x) $|5x - 9| < 6$,

(xi) $\left|5 - \dfrac{2}{x}\right| < 1$, (xii) $\left|\dfrac{x}{5} - 1\right| \le 1$,

(xiii) $\left|3 - \dfrac{1}{x}\right| < \dfrac{1}{2}$, (xiv) $|2x - 3| \le 4$,

2. Find the domain of the following functions

(i) $y = \dfrac{1}{x^3 - x}$, (ii) $y = \sqrt{\left(\dfrac{1}{x} - 2\right)}$,

(iii) $y = \sqrt{\left[\dfrac{1}{\sqrt{x}} - \sqrt{x+1}\right]}$, (iv) $y = \sqrt{1 + 2\sin x}$,

(v) $y = \dfrac{x^2 + 3}{x^2 + |x|}$, (vi) $y = 3\sqrt{1 - x}$,

(vii) $y = \sqrt{\dfrac{x}{1 - x}}$, (viii) $y = \sqrt{2x - 6}$,

(ix) $y = \sqrt{1 - |x|}$, (x) $y = \dfrac{1}{\sqrt{-x^2 + 6x - 9}}$,

3. Find the domain and range of the following functions

(i) $y = x^3$, (ii) $y = x^{2/3}$,

(iii) $y = \dfrac{1}{x^2}$, (iv) $y = \sqrt{4 - x}$,

(v) $y = \sqrt{1 - x^2}$, (vi) $y = \begin{cases} 1, & x \geq 0 \\ 2, & x < 0 \end{cases}$

4. Find fog and gof then determine the domain of each.

(i) $f(x) = x + 1$, $g(x) = 2x$, (ii) $f(x) = 2 - \sqrt{x}$, $g(x) = (x + 1)^2$

(iii) $f(x) = \dfrac{1 + x}{1 - x}$, $g(x) = \dfrac{x}{1 - x}$, (iv) $f(x) = \dfrac{|x|}{x}$, $g(x) = x^2$,

(v) $f(x) = \dfrac{x}{1 + x^2}$, $g(x) = \dfrac{1}{x}$.

5. Find the inverse (if it exist) of each of the following functions

(i) $y = \dfrac{2x - 6}{3x + 3}$, (ii) $y = \sqrt[7]{2x + 1}$, $x \geq -\dfrac{1}{2}$,

(iii) $y = \dfrac{ax + b}{cx + d}$, (iv) $y = x^2 - 2x$, $x \geq 1$,

(v) $y = e^{x-1} + 3$, (vi) $y = \log(x + 2) - 3$,

(vii) $y = \sqrt[3]{x + 1}$.

6. Two right circular cones, with the same height $h = 50$ cm, are to be constructed. The volumes of these cones are to be 200 and 400 cm^3, find the radius of the base of the each cone.

7. The population of a certain city increase according to the formula $P = 200{,}000\, e^{0.01t}$ where P is the population and t the number of years,

with $t = 0$ corresponding to the year 2000, when will the population be 300,000 and 500,000.

8. Suppose that ΔABC has an obtuse angle γ. Draw BD perpendicular to AC, forming right triangles ΔABC and ΔBDC (with right angles D) show that $\frac{\sin \alpha}{a} = \frac{\sin \gamma}{c}$.
9. If $\sin \alpha + \cos \alpha = s$ and $\sin \alpha - \cos \alpha = t$ where α is accute angle, show that $\alpha = \tan^{-1} \frac{s + t}{s - t}$.
10. Express the following function in piecewise form without using absolute value.

    (i) $f(x) = 4 + |2x - 3|$, (ii) $f(x) = 3\,|x - 2| - |x + 1|$.
11. Find x such that

    (i) $\log x = \sqrt{3}$ (ii) $\log (x + 1) = 6$

    (iii) $5^x = 7$ (iv) if $6 = 75\, e^{\frac{-s}{125}}$ then find s.

Answers

1. (i) $x < -2$, (ii) $x \le \frac{2}{3}$,

   (iii) $]-\infty, 1]$ (iv) $x < -\frac{6}{7}$,

   (v) $-\frac{5}{2} \le x < 8$, (vi) $x > -\frac{3}{7}$,

   (vii) $x \le -\frac{1}{3}$, (viii) $-1 < x < 9$,

   (ix) $-2 \le x \le 4$, (x) $\frac{3}{5} < x < 3$,

   (xi) $\frac{1}{3} < x < \frac{1}{2}$, (xii) $0 \le x \le 10$,

   (xiii) $\frac{2}{7} < x < \frac{2}{5}$, (xiv) $-\frac{1}{2} \le x \le \frac{7}{2}$.

2. (i) $R - [-1, 0, 1]$, (ii) $\left]0, \frac{1}{2}\right]$,

   (iii) $\left[0, \frac{1}{2}(-1 + \sqrt{5})\right]$, (iv) $\left[\left(2n - \frac{1}{6}\right)\pi, \left(2n + \frac{7}{6}\right)\pi\right]$,

   (v) R, (vi) $]-\infty, 1]$,

   (vii) $[0, 1]$, (viii) $[3, \infty[$

   (ix) $[-1, 1]$, (x) For no value of x.

3. (i) $]-\infty, \infty[$; $]-\infty, \infty[$, (ii) $]-\infty, \infty[$; $[0, \infty[$

(iii) $]-\infty; 0[\cup]0, \infty[$; $]0, \infty[$ (iv) $]-\infty, 4]$; $[0, \infty[$

(v) $[-1, 1]$; $[0, 1]$, (vi) R; $[1, 2]$.

4. (i) $2x + 1, R$; $2x + 2, R$ (ii) $2 - |x + 1|, R$; $(3 - \sqrt{x})^2, [0, \infty[$,

(iii) $\dfrac{1}{1-2x}, x \neq \dfrac{1}{2}, 1; -\dfrac{1}{2x} - \dfrac{1}{2}, x \neq 0, 1,$

(iv) $1, R - \{0\}$; $1, R - \{0\}$ (v) $\dfrac{x}{x^2+1}, x \neq 0; \dfrac{1}{x} + x, x \neq 0$

5. (i) $y = \dfrac{3x+6}{2-3x}$, (ii) $y = \dfrac{1}{2}(x^7 - 1)$,

(iii) $y = \dfrac{b - bx}{cx - d}$, (iv) $y = 1 + \sqrt{x+1}$,

(v) $y = \log(x - 3) + 1$, (vi) $y = e^{x+3} - 2$,

(vii) $y = x^3 - 1$.

6. (i) 1.95 cm, (ii) 2.76 cm.

7. 2041, 2092.

10. (i) $f(x) = \begin{cases} 7 - 2x, & x < \dfrac{3}{2} \\ 1 + 2x, & x \geq \dfrac{3}{2} \end{cases}$

(ii) $f(x) = \begin{cases} 7 - 2x, & x < -1 \\ 5 - 4x, & -1 \leq x < 2 \\ 2x - 7, & x \geq 2 \end{cases}$

11. (i) 53.95, (ii) 402.42, (iii) 1.21 (iv) 315.71.

2 CHAPTER

Limit and Continuity

2.1 INTRODUCTION

The calculus was invented independently in England by Sir Isaac Newton and in Germany by Gottfried Wilhelm Leibnitz in the last quarter of the seventeenth century though it was to some extent the answer to the four major problems already tackled by the Greeks. The first problem was, given the formula for the distance a body covers as a function of the time, to find the velocity and acceleration at any instant; and conversely, given the formula describing the acceleration of a body as a function of the time, to find the velocity and the distance traveled. The second type of the problem was to find the tangent to a curve. The third problem was that of finding the maximum or minimum value of a function, and the fourth problem was finding the lengths of curves, such as the distance covered by a planet in a given period of time or to find the area enclosed by a given curve.

We know that the average speed of an object can be fined by the formula

$$\frac{\Delta y}{\Delta t} = \frac{y_1 - y_0}{t_1 - t_0}$$

(y_1 is the position of the object at a time t_1 and y_0 is the position at a time t_0).

Let a ball falls from the top of a 100 ft cliff then the average speed during the first 2 sec of fall is

$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32 \text{ ft./sec}$$

(Physically experiments a solid object dropped from rest to fall freely near the earth surface modeled by the function $y = 16t^2$ ft. during the first t sec.).

The average speed from 1 second to 2 second

$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{2 - 1} = 48 \text{ ft./sec}$$

We can also calculate the average speed of the ball over a time interval (t_0, t_0, + h where $h = \Delta t$ is the length of the interval

$$\frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16(t_0)^2}{h}$$

This formula cannot be use for calculate the "instantaneous" sped at t_0 when $h = 0$. But it can be use to calculate average speeds over increasingly short time intervals starting for different value of t_0, for example when $t_0 = 1$ we use this formula as

Length of the intervals	*Average speed over interval of length h starting at $t_0 = 1$*
1	48
.5	40
.1	33.6
.01	32.16
.001	32.016
.0001	32.0016S

So the average speed on intervals starting at $t_0 = 1$ seems to approach a limiting value of 32 as the length of the interval decrease and tends to zero.

In an another case let us examine the behavior of the function $y = f(x) = x + 3$

x	*x + 3*	*x*	*x + 3*
3	6	1	4
2.5	5.5	1.5	4.5
2.2	5.2	1.9	4.9
2.01	5.01	1.99	4.99
2.001	5.001	1.999	4.999

What happens to $f(x)$ as x gets close to the value 2. Keeping in mind that x can get close to 2 from the right of 2 and from the left of 2 along the x-axis. We describe this by seeing that as x gets close to 2 from the either side the $f(x)$ gets close to 5. In mathematically symbol, we write

$$\lim_{x \to 2} (x + 3) = 5$$

Observe that in our investigation of $\lim_{x \to 2}$ we are only concerned with the value of $f(x)$ close $x = 2$ and not the value of $f(x)$ at $x = 2$. In above two examples we have seen that, when a variable get closer and closer a particular value [*in first example t close to zero and in second x close to 2*], the value of the function [*speed tends to 32ft. and f*(x) *tends to 5*] tends to a number.

The above description leads us to introduce the idea of the limit of the function as follows.

2.2 INFORMAL DEFINITION OF LIMIT

Let a function $f(x)$ be defined on an open interval I about x_0, except possibly at x_0 itself. If function $f(x)$ gets arbitrarily close to a number L for all x sufficiently close to x_0 we say that f approaches the limit L as x approaches x_0, and we write

$$\mathbf{Lim}_{x\to x_0} f(x) = L$$

The existence of a limit as x tends to x_0 does not depend on how the function may be defined at x_0.

Let $\lim\limits_{x\to 2} f(x) = \dfrac{x^2 - 4}{x - 2}$ and $\lim\limits_{x\to 2} g(x) = x + 2$. The function $f(x)$ has limit 4 as x tends to 2 even though f has not defined at $x = 2$, Fig. 2.1. The function $g(x)$ also has the limit 4, Fig. 2.2. This is a special kind of equality of limits.

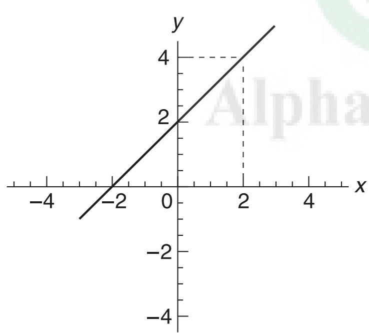


Fig. 2.1

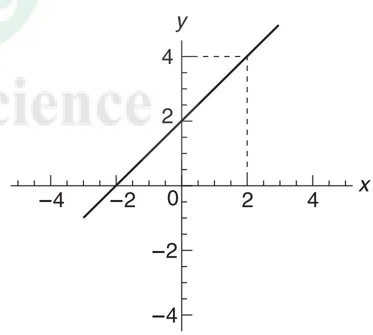


Fig. 2.2

The above informal definition of the limit is commonly called a two sided limit because it requires the values of the function to get closer and closer to a number L as values of x approaches x_0 from either sides. However, some function exhibit different behaviours on the two sides as x approaches x_0. For example consider the function $f(x) = \dfrac{|x|}{x}$. The value of this function $f(x)$ close to 1 as x approaches 0 from the right and the value of $f(x)$ close to –1 as x approach 0 from the left, Fig. 2.3.

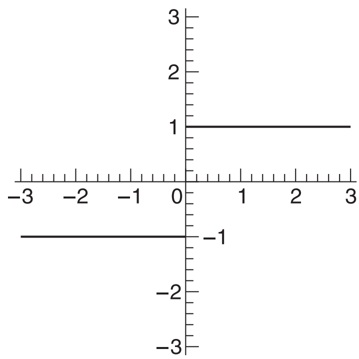


Fig. 2.3

The above statements can be describe as "the limit of $f(x) = \frac{|x|}{x}$ is 1 as x approaches 0 from the right" and that "the limit of $f(x) = \frac{|x|}{x}$ is –1 as x approaches 0 from the left". We denote these limits by writing

$$\lim_{x\to 0^+} \frac{|x|}{x} = 1 \text{ and } \lim_{x\to 0^-} \frac{|x|}{x} = -1$$

[0^+ indicate a limit from the right of the zero and 0^- limit from the left of the zero]

This leads the following idea of the limit.

2.3 INFORMAL DEFINITION OF RIGHT-HAND LIMIT (R.H.L.) AND LEFT-HAND LIMIT (L.H.L.)

Let $f(x)$ be a function defined on an interval (a, b) $[a < b]$ and $f(x)$ approaches M as x approaches x_0, $x_0 \in [a, b]$ from the right in the interval (a, b) then we say that f has **Right-hand limit** M at x_0, and we have

$$\lim_{x\to x_0^+} f(x) = M$$

Similarly $f(x)$ approaches N as x approaches x_0 from the left in the interval (a, b), then we say that f has **Left-hand limit** N at x_0, and we have

$$\lim_{x\to x_0^-} f(x) = N$$

Relation between one sided and two sided limit

The two sided limit of a function $f(x)$ exist at x_0 if and only if the Right-hand limit (R.H.L.) and Left-hand limit (L.H.L.) exist at x_0 and have the same value; that is

$$\lim_{x\to x_0} f(x) = L \text{ if and only if } \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x)$$

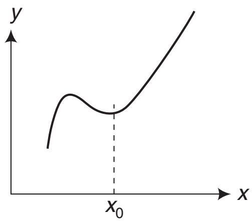


Fig. 2.4 Limit exist

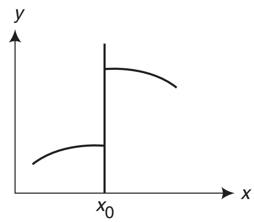


Fig. 2.5 Limit does not exist

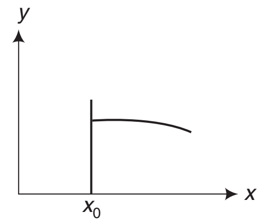


Fig. 2.6 Limit does exist

Example 1 Calculate $\lim_{x\to 0} |x|$.

Solution We have $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

If $x > 0$ then $|x| = x$, which tends to zero as x tends to 0 from the right of 0. If $x < 0$ then $|x| = -x$, which again tends to zero as x tends to 0 from the left of 0. Hence $\lim_{x\to 0} |x| = 0$ limit exist (Both limit exist and equal) Fig. 2.7.

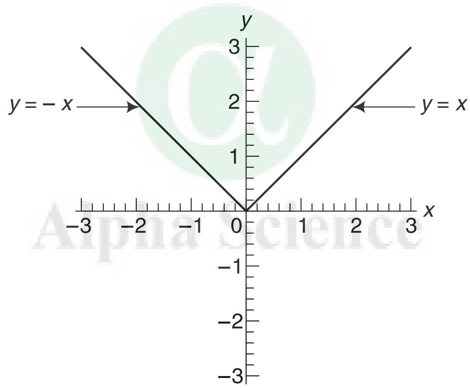


Fig. 2.7

Example 2 Calculate the limit of the following function $f(x)$ *a* $x = 0$, *where*

$$f(x) = \begin{cases} x + 2, & x > 0 \\ x - 2, & x < 0 \end{cases}$$

Solution $\text{Lim}_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x + 2) = 2$ and $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x - 2) = -2$

Since the limits are different, hence limit does not exist (both limit exist but not equal) Fig. 2.8.

Example 3 Calculate the limit of the function $f(x) = [x]$ *a* $x = 2$, where $[x]$ is greatest integer function.

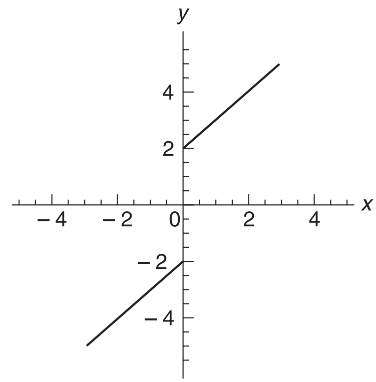


Fig. 2.8

Solution We know that

$$\lim_{x\to n^+} [x] = n \text{ and } \lim_{x\to n^-} [x] = n - 1, \text{ therefore}$$

$$\lim_{x\to 0^+} [x] = 2 \text{ and } \lim_{x\to 2^-} [x] = 1. \text{ Hence the limit does not exist}$$

Example 4 *Calculate* $\lim_{x\to 0} f(x) = \sqrt{x}$.

Solution $\operatorname{Lim}_{x\to 0^+} \sqrt{x} = 0$ but $\lim_{x\to 0^-} \sqrt{x} =$ not defined. Hence the limit does not exist.

Example 5 Calculate the limit of the following function $f(x)$ at $x = 0$, where

$$f(x) = \begin{cases} 0, & x \le 0 \\ \sin \dfrac{1}{x}, & x > 0 \end{cases}$$

Solution The function oscillate too much between –1 and +1 when x tends to 0 from the right side so the limit does not exist, Fig. 2.9.

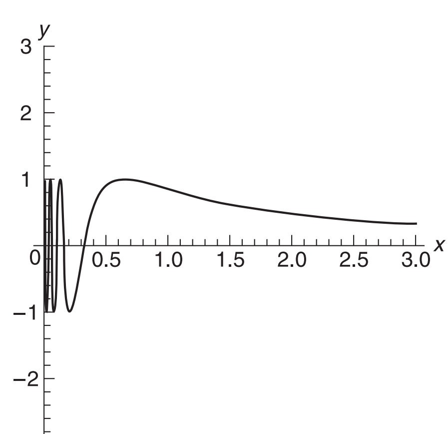


Fig. 2.9

2.4 INFINITE LIMITS

The value of $\frac{1}{x}$ for x near 0 are given in Table 2.1 and in Fig. 2.10. We see that as x approaches to 0 from the right $f(x) = \frac{1}{x}$ grows without bounded in the positive direction but if x approaches 0 from the left $f(x) = \frac{1}{x}$ grows without bounded in the negative direction. Since the behaviour of $\frac{1}{x}$ depends on the way in which x approaches 0 and we can write as

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \text{ and } \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Table 2.1

x	$f(x) = \frac{1}{x}$	x	$f(x) = \frac{1}{x}$
−1	−1	1	1
−.01	−100	.01	100
−.001	−1000	.001	1000
−.0001	−10000	.0001	10000
−.00001	−100000	.00001	100000

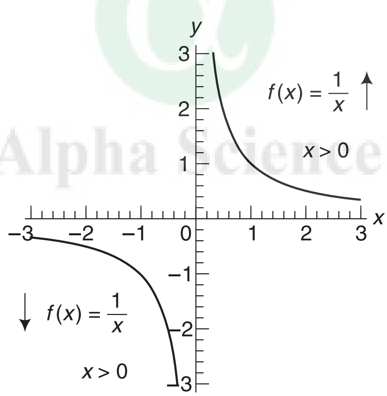


Fig. 2.10

Definition

(i) If $f(x)$ grows without bound in the positive direction as x approaches x_0 from either side, then we say that $f(x)$ tends to infinity as x_0 and we write

$$\lim_{x \to x_0} \frac{1}{x} = -\infty. \text{ i.e, } \lim_{x \to 0} \frac{1}{x^2} = \infty.$$

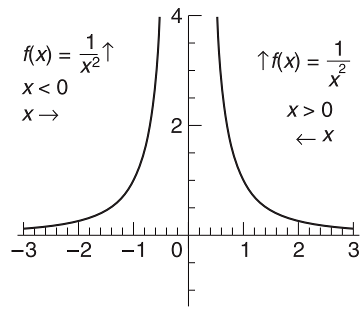


Fig. 2.11

(ii) If $f(x)$ grows without bound in the negative direction as x approaches x_0 from either side, then we say that $f(x)$ tends to minus infinity as x approaches x_0 and we write

$$\lim_{x \to x_0} \frac{1}{x} = -\infty \text{ i.e, } \lim_{x \to 0} \frac{-1}{(x-1)^2} = -\infty.$$

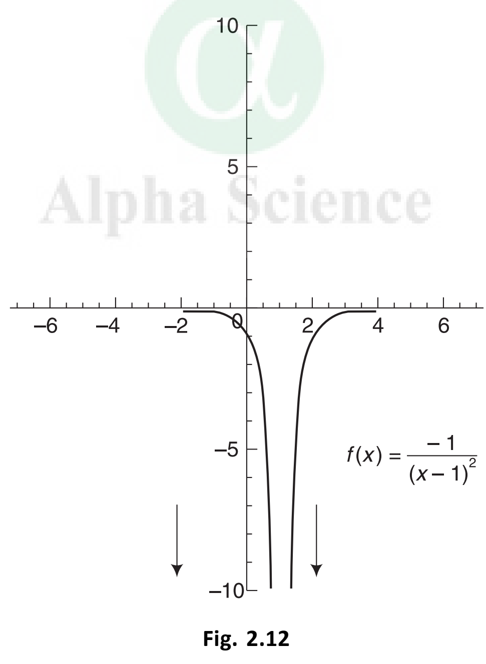


Fig. 2.12

2.5 LIMITS AT INFINITY

The limit as x infinity of $f(x)$ is L, written

$$\lim_{x \to \infty} f(x) = L$$

If $f(x)$ is defined for all large values of x and gets close to L as x increases without bound then it's called the limit at infinity, and if $f(x)$ is defined for all values of x that are large in the negative direction and if $f(x)$ gets close to L as x increases in the negative direction without bound then it's called the limit at minus infinity.

Example 6 Calculate $\lim_{x\to\infty} \dfrac{1}{x^2}$.

Solution From Table 2.2 we have $\lim_{x\to\infty} \dfrac{1}{x^2} = 0$.

Table 2.2

x	$\frac{1}{x^2}$
1	1
10	0.01
100	0.001
1000	0.0001
10000	0.00001

Example 7 Calculate $\lim_{x\to\infty} \dfrac{x^3 + 2x + 3}{x^5 + 3x^2 + 4}$

Solution $$\lim_{x\to x_0} \frac{x^3 + 2x + 3}{x^5 + 3x^2 + 4} = \lim_{x\to\infty} \frac{1 + \frac{2}{x^2} + \frac{3}{x^3}}{x^2\left(1 + \frac{3}{x^3} + \frac{4}{x^5}\right)} = 0$$

2.6 THEOREMS ON LIMITS

Let f and g be two functions such that

$$\lim_{x\to x_0} f(x) = L_1 \text{ and } \lim_{x\to x_0} g(x) = L_2$$

Then:

(i) $\lim_{x\to x_0} [f(x) + g(x)] = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) = L_1 + L_2$

(ii) $\lim_{x\to x_0} [f(x) - g(x)] = \lim_{x\to x_0} f(x) - \lim_{x\to x_0} g(x) = L_1 - L_2$

(iii) $\lim_{x\to x_0} [f(x)\, g(x)] = \lim_{x\to x_0} f(x) \lim_{x\to x_0} g(x) = L_1L_2$

(iv) $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \dfrac{L_1}{L_2}$

(v) $\lim_{x\to x_0} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to x_0} f(x)} = \sqrt[n]{L_1}$ Provided $L_1 > 0$ if n is even.

(vi) limits of polynomials:

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \ldots a_2x^2 + a_1x + a_0.$

Then $\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + a_{n-2}c^{n-2} + a_{n-3}c^{n-3} + \ldots a_2c^2 + a_1c + a_0.$

(vii) Limits of Rational functions:

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then $\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$

The Sandwich Theorem:

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing a number c, except at possibly $x = c$ itself. Suppose also that $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} f(x) = L$.

Example 8 Applying the Sandwich Theorem given that

$$2 - \frac{x^2}{4} \leq f(x) \leq 2 + \frac{x^2}{3} \text{ find } \lim_{x\to 0} f(x)$$

Solution $\lim_{x\to 0}\left(2 - \frac{x^2}{2}\right) = 2$ and $\lim_{x\to 0}\left(2 + \frac{x^2}{3}\right) = 2$

The Sandwich Theorem implies $\lim_{x\to 0} f(x) = 2$.

Example 9 Find the limits of the following functions

(i) $\lim_{x\to -1}\left(\frac{x^3 + 2x^2 + 3}{x^2 + 4}\right)$, (ii) $\lim_{x\to 4}\left(\frac{2x - 6}{x^2 + x - 12}\right)$,

(iii) $\lim_{x\to 0}\left(\frac{x}{\sqrt{x+1} - 1}\right)$, (iv) $\lim_{x\to 0}\left(\frac{3 - \sqrt{x+9}}{x}\right)$,

(v) $\lim_{x\to a}\left(\frac{x^n - a^n}{x - a}\right)$, (vi) $\lim_{x\to a}\left(\frac{\sin x - \sin a}{x - a}\right)$,

(vii) $\lim_{x\to 0}\left(\frac{e^{\sin x} - 1}{x}\right)$, (viii) $\lim_{x\to 0}\left(\frac{\tan x - \sin x}{x^3}\right)$,

(ix) $\lim_{x\to 0}\left(\frac{3x + |x|}{7x - 5|x|}\right)$, (x) $\lim_{x\to a}\left(\frac{\frac{1}{e^x} - \frac{-1}{e^x}}{\frac{1}{e^x} + \frac{-1}{e^x}}\right)$,

(xi) Let $f(x) = \begin{cases} \frac{1}{(x+2)}, & x < -2 \\ x^2 - 5, & -2 \leq x \leq 3 \\ \sqrt{x+13}, & x > 3 \end{cases}$ At $\lim_{x\to -2} f(x)$, $\lim_{x\to 0} f(x)$ and $\lim_{x\to 3} f(x)$,

(xii) $\lim\limits_{x\to\infty}\left(\sqrt{(x^2+1)}-\sqrt{(x+1)}\right)$, (xiii) $\lim\limits_{x\to-\infty}(6x^3+2x^2+3x+5)$.

Solution

(i) $$\lim_{x\to-1}\left(\frac{x^3+2x^2+3}{x^2+4}\right)=\left(\frac{(-1)^3+2(-1)^2+3}{(-1)^2+4}\right)=\frac{4}{3}.$$

(ii) $$\lim_{x\to4}\left(\frac{2x-6}{x^2+x-12}\right)=\lim_{x\to4}\left(\frac{2(x-3)}{(x-3)\,(x+4)}\right)=\frac{1}{4}.$$

(iii) $$\lim_{x\to0}\left(\frac{x}{\sqrt{x+1}-1}\right)=\lim_{x\to0}\left(\frac{x\sqrt{x+1}+1}{(\sqrt{x+1}-1)\,(\sqrt{x+1}+1)}\right)$$

$$=\lim_{x\to0}\left(\frac{x\sqrt{x+1}+1}{x+1-1}\right)=2.$$

(iv) $$\lim_{x\to0}\frac{3-\sqrt{x+9}}{x}\ \lim_{x\to0}\left(\frac{(3-\sqrt{x+9})\,(3+\sqrt{x+9})}{x(3+\sqrt{x+9})}\right)$$

$$=\lim_{x\to0}\left(\frac{9-9-x}{x(3+\sqrt{x+9})}\right)=\frac{-1}{6}.$$

(v) $$\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)=\lim_{x\to a}\left(\frac{(x-a)\,(x^{n-1}+x^{n-2}a+\ldots+a^{n-1}}{x-a}\right)$$

$$=\lim_{x\to a}(x^{n-1}+x^{n-2}a+\ldots+a^{n-1}=a^{n-1})\lim_{x\to a}(x^{n-1})+\lim_{x\to a}(x^{n-2}a)+\ldots+a^{n-1}$$

$$=na^{n-1}.$$

(vi) $$\lim_{x\to a}\left(\frac{\sin x-\sin a}{x-a}\right)=\lim_{x\to a}\left(\frac{2\cos\dfrac{(x+a)}{2}\sin\dfrac{(x-a)}{2}}{x-a}\right)$$

$$=\lim_{x\to a}\left(\frac{\sin\dfrac{(x-a)}{2}}{\dfrac{x-a}{2}}\right)\lim_{x\to a}\cos\frac{(x+a)}{2}=1\cdot\cos\frac{(a+a)}{2}=\cos a.$$

(vii) $$\lim_{x\to0}\left(\frac{e^{\sin x}-1}{x}\right)=\lim_{x\to0}\left(\frac{e^{\sin x}-1}{\sin x}\right)\lim_{x\to0}\frac{\sin x}{x}=1.$$

$$\left\{\lim_{x\to0}\frac{e^x-1}{x}=1\text{ and }\lim_{x\to0}\frac{\sin x}{x}=1\right\}$$

(viii) $\lim_{x\to 0}\left(\frac{\tan x - \sin x}{x^3}\right) = \lim_{x\to 0}\left(\frac{\frac{\sin x}{\cos x} - \sin x}{x^3}\right) = \lim_{x\to 0}\left(\frac{\sin x\,(1-\cos x)}{x^3 \cos x}\right)$

$$= \lim_{x\to 0}\frac{\tan x}{x}\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2} = \lim_{x\to 0}\frac{\tan x}{x}\frac{2}{4}\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}} = \frac{1}{2}.$$

(ix) $\lim_{x\to 0}\left(\frac{3x+|x|}{7x-5|x|}\right)$, L.H.L $\lim_{x\to 0^-}\frac{3x-x}{7x-(-5x)} = \frac{1}{6}$.

And R.H.L $\lim_{x\to 0}\frac{3x+x}{7x-(5x)} = 2$. Hence limit does not exit

(x) $\lim_{x\to 0}\left(\frac{e^{\frac{1}{x}} - e^{\frac{-1}{x}}}{e^{\frac{1}{x}} + e^{\frac{-1}{x}}}\right)$, L.H.L $\lim_{x\to 0^-}\left(\frac{e^{\frac{-1}{x}}\left(e^{\frac{2}{x}}-1\right)}{e^{\frac{-1}{x}}\left(e^{\frac{2}{x}}+1\right)}\right) = \lim_{h\to 0}\left(\frac{\left(e^{\frac{2}{0-h}}-1\right)}{\left(e^{\frac{2}{0-h}}+1\right)}\right) = -1,$

and R.H.L. $\lim_{x\to 0^+}\left(\frac{e^{\frac{1}{x}}\left(1-e^{\frac{-2}{x}}\right)}{e^{\frac{1}{x}}\left(1+e^{\frac{-2}{x}}\right)}\right) = \lim_{h\to 0}\left(\frac{\left(1-e^{\frac{2}{0-h}}\right)}{\left(1+e^{\frac{2}{0-h}}\right)}\right) = 1,$

Hence limit does not exist.

(xi) $f(x) = \begin{cases} \frac{1}{(x+2)}, & x < -2 \\ x^2 - 5, & -2 \le x \le 3 \\ \sqrt{x+3}, & x > 3 \end{cases}$ At $\lim_{x\to -2} f(x)$

L.H.L. $\lim_{x\to -2^-} f(x) =. \lim_{x\to -2^-}\frac{1}{(x+2)} = -\infty$ and R.H.L. $\lim_{x\to -2^+} f(x)$

$$= \lim_{x\to -2^+} x^2 - 5 = -1.$$

Hence limit does not exist.

At $\lim_{x\to 0} f(x)$

L.H.L. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x^2-5) = -5$ and R.H.L. $\lim_{x\to 0^+} f(x)$

$$= \lim_{x\to 0^+}(x^2-5) = -5.$$

L.H.L. = R.H.L. = –5. Hence limit exist.

At $\lim_{x\to 3} f(x)$

L.H.L. $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(x^2-5) = 4$ and R.H.L. $\lim_{x\to 3^+} f(x)$

$$= \lim_{x\to 3^+}\sqrt{x+3} = \sqrt{6}.$$

Hence limit does not exist.

(xii) $\lim_{x\to\infty}\left(\sqrt{(x^2+1)}-\sqrt{(x+1)}\right)=\lim_{x\to\infty}\dfrac{(x^2+1)-(x+1)}{\left(\sqrt{(x^2+1)}+\sqrt{(x+1)}\right)}$

$$=\lim_{x\to\infty}\frac{(x-1)}{\left(\sqrt{\left(1+\frac{1}{x^2}\right)}+\sqrt{\left(\frac{1}{x}+\frac{1}{x^2}\right)}\right)}=\infty.$$

(xiii) $\lim_{x\to-\infty}(6x^3+2x^2+3x+5)=\lim_{x\to-\infty}x^3\left(6+\frac{2}{x}+\frac{3}{x^2}+\frac{5}{x^3}\right)=-\infty.$

Example 10 For the function in Fig. 2.13 and Fig. 2.14 find the limit at $x = c$

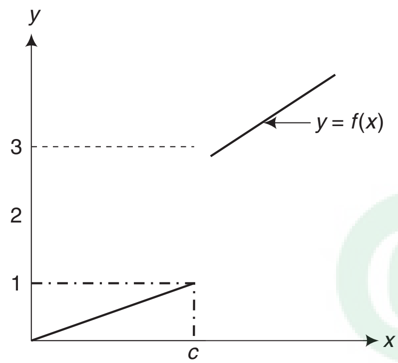


Fig. 2.13

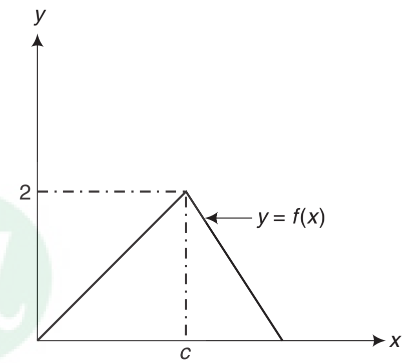


Fig. 2.14

Solution. In Fig. 2.13 $\lim_{x\to c^-} f(x) = 1$ and $\lim_{x\to c^+} f(x) = 3$. limit does not exist

In Fig. 2.14 $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = 2$. *limit exist*

2.7 FORMAL DEFINITION OF LIMITS

In this section we give a formal definition of the limit. To do this we first examine how to control the input of a function to ensure that the output is kept within preset bounds. Let us consider the example in section 2.1 on page 2, in this example we have seen that as x approaches 2 the value of the function $y = f(x) = x + 3$ get closer to 5. Now see this example in an another way, how close to $x_0 = 2$ does x have to be so that $y = x + 3$ differs from 5 by say, less than 2 units or what values of x is $|x - 5| = 2$? to find out, we solve the inequality

$$|x + 3 - 5| < 2$$
$$|x - 2| < 2$$
$$-2 < x - 2 < 2$$
$$0 < x < 4$$

Keeping x within 2 units of $x_0 = 2$ keep y with 2 units of $y_0 = 2$ Fig. 2.15.

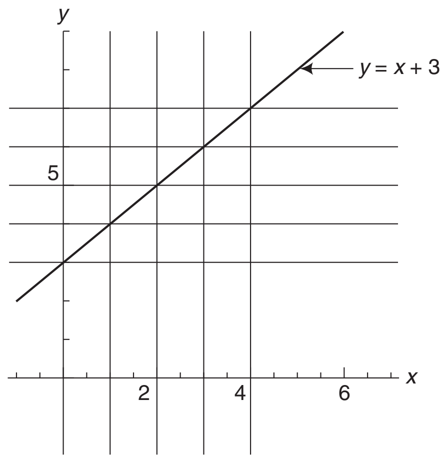


Fig. 2.15

Formal Definition

The statement $\lim_{x \to x_0} f(x) = L$ means

If given any number $\varepsilon > 0$ we may find a number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ if

$$0 < |x - x_0| < \delta$$

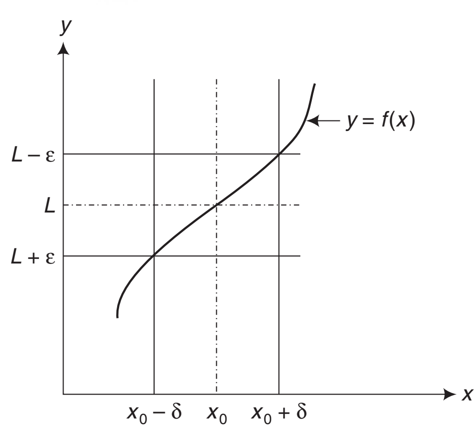


Fig. 2.16

Fig. 2.16(a) Limit exist

Fig. 2.16(b) Limit does not exist

Example 11 Show that $\lim\limits_{x\to 3}(2x-1)=5$

Solution $\underbrace{|2x-1}_{f(x)} - \underbrace{5|}_{L} < \varepsilon$ *if* $0 < |x - \underbrace{3|}_{x_0} < \delta$

The aim of this example is to prove that the limit of the function $f(x) = (2x - 1)$ is 5 when x approaches to 3. We have $|2x - 1 - 5| = 2|x - 3|$

And this must be less than ε whenever $|x - 3| < \delta$, for a given > 0, choose $\delta = \frac{\varepsilon}{2}$.

$|f(x) - L| = 2|x - 3| < 2\delta = 2\,\frac{\varepsilon}{2} = \varepsilon$ Fig. 2.17 $\left\{\text{If we choose } \varepsilon = 1 \text{ then } \delta = \frac{1}{2}\right\}$

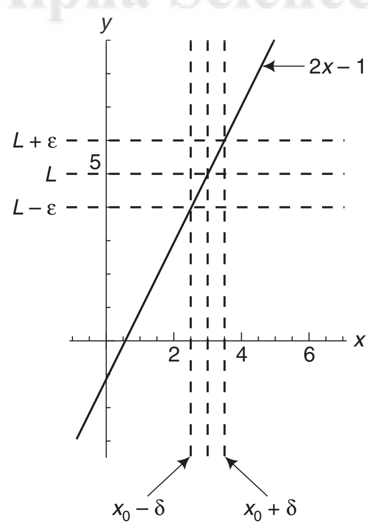


Fig. 2.17

Example 12 *Show that* $\lim_{x\to 5} x^2 = 25$

Solution By definition

$$|f(x) - L| = |x^2 - 25| = |(x - 5)\ (x + 5)|$$

$$|(x - 5)\ (x + 5)| \qquad (2.1)$$

This must be less than ε whenever $|x - 5| < \delta$

Using the triangular inequality, we say that

$$|x + 5| = |(x - 5) + 10| \le |x - 5| + |10| \qquad (2.2)$$

From (2.1) and (2.2)

$$|x + 5|\ |x - 5| \le (|x - 5| + |10|)\ |x - 5|$$

Therefore if $|x - 5| < \delta$, then

$$|x + 5|\ |x - 5| \le (\delta + 10)\delta$$

For a given $\varepsilon > 0$ choose δ

$$(\delta + 10)\ \delta \le \varepsilon$$

Let $\delta \le 1$ then

$$(\delta + 10)\ \delta \le 11\delta$$

Hence $11\delta \le \varepsilon$

So when $\delta = \min\left(\frac{\varepsilon}{11}, 1\right)$ then $\lim_{x\to 5} x^2 = 25.$

Example 13 Show that $\lim_{x\to 0} x \sin \frac{1}{x} = 0$

Solution $|f(x) - L| = |x \sin 1/x - 0|$

$$= \left|x \sin \frac{1}{x}\right| \le |x|$$

This must be less than ε whenever $|x - 0| < \delta$

For a given $\varepsilon > 0$ choose $\delta = \varepsilon$, then

$$|f(x) - L| = \left|x \sin \frac{1}{x}\right| \le |x| = \varepsilon$$

Hence when $\delta = \varepsilon$, then $\lim_{x\to 0} \sin \frac{1}{x} = 0$

Example 14 Show that $\lim_{x\to 0^+} \frac{1}{1 + e^{\frac{-1}{x}}} = 1$

Solution $|f(x) - L| = \left|\frac{1}{1 + e^{\frac{-1}{x}}} - 1\right|$

This must be less than ε whenever $|x - 0| < \delta$

When $\frac{1}{1 + e^{\frac{1}{x}}} < \varepsilon$ or $e^{\frac{1}{x}} + 1 > \frac{1}{\varepsilon}$

Or $$\frac{1}{x} > \log\left(\frac{1}{\varepsilon} - 1\right)$$

$\Rightarrow$ $$0 < x < 1/\log\left(\frac{1}{\varepsilon} - 1\right)$$

For a given $\varepsilon > 0$ we choose $\delta = \dfrac{1}{\log\left(\frac{1}{\varepsilon} - 1\right)}$

Thus for any $\varepsilon > 0$ and $\delta > 0$

$$\left|\frac{1}{1 + e^{\frac{-1}{x}}} - 1\right| < \varepsilon \text{ when } \delta = \frac{1}{\log\left(\frac{1}{\varepsilon} - 1\right)}.$$

Example 15 Show that $\lim\limits_{x\to 0} \dfrac{1}{x}$ does not exist

Solution Let L be any number then

$$|f(x) - L| < \varepsilon$$

$\Rightarrow$ $$\left|\frac{1}{x} - L\right| < \varepsilon$$

$\Rightarrow$ $$L - \varepsilon < \frac{1}{x} < L + \varepsilon \text{ If } \varepsilon = 1, \text{ then}$$

$$\left|\frac{1}{x}\right| < |L| + 1$$

$$|x| > \frac{1}{|L| + 1}$$

Hence for any $\varepsilon > 0$, and $\delta > 0$, there will always be a number x in the interval $0 < |x - 0| < \delta$ such that $\dfrac{1}{|x|} > |L| + \varepsilon$ so the limit does not exist.

Limits as $x \to \pm\infty$

In section 2.5 we discussed the limits

$$\lim_{x\to\infty^+} f(x) = L \text{ and } \lim_{x\to\infty^-} f(x) = L$$

These limit can be defined more precisely as

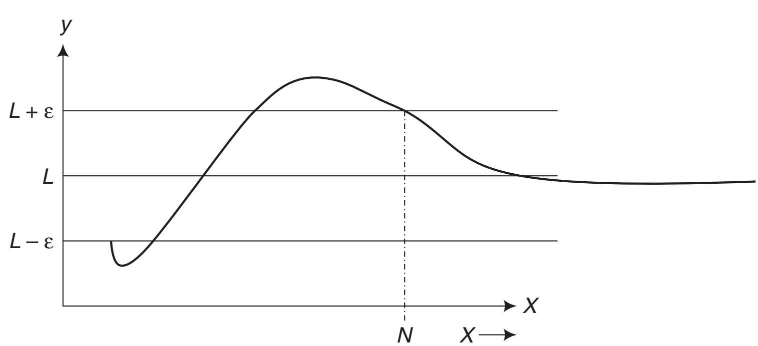


Fig. 2.18

The expression $\lim_{x\to\infty^+} f(x) = L$ means given any $\varepsilon > 0$ there exists a positive number N such that

$$|f(x) - L| < \varepsilon, \text{ when } x > N.$$

The expression $\lim_{x\to\infty^-} f(x) = L$ means given any $\varepsilon > 0$ there exists a negative number N such that

$$|f(x) - L| < \varepsilon, \text{ when } x < -N$$

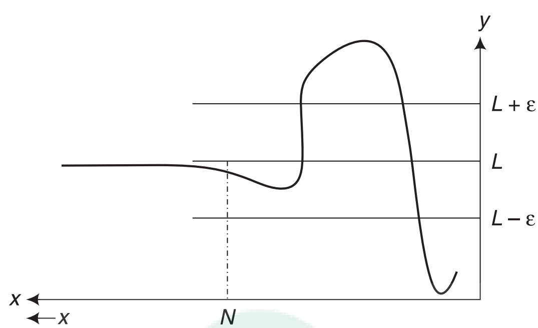


Fig. 2.19

Example 16 Show that $\lim_{x\to\infty}\left(3 + \frac{1}{x}\right) = 3.$

Solution Applying the definition, we have

$$\left|3 + \frac{1}{x} - 3\right| < \varepsilon \text{ when } x > N$$

Because $x \to \infty$ we assume that $x > 0$ so

$$\frac{1}{x} < \varepsilon \text{ when } x > N$$

Or $$x > \frac{1}{\varepsilon} \text{ when } x > N$$

It is clear that $N = \frac{1}{\varepsilon}$ satisfies the requirement and according the definition there is a positive number $\frac{1}{\varepsilon}$ when $\lim_{x\to\infty}(3 + 1/x) = 3$, Fig. 2.20.

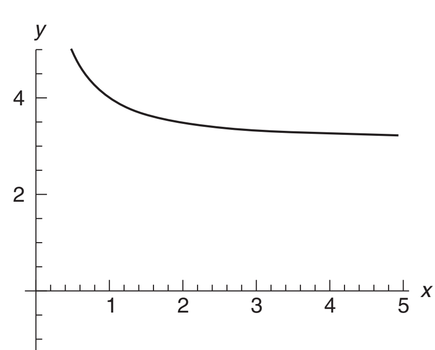


Fig. 2.20

Infinite limits

In section 2.4 we discussed the limits $\lim_{x \to x_0} f(x) = \infty$ and $\lim_{x \to x_0} f(x) = -\infty$

The expression $\lim_{x \to x_0} f(x) = +\infty$ means, given any positive number $N > 0$ (however large), there exists some $\delta > 0$ such that $f(x) > N$, when $0 < |x - x_0| < \delta$, Fig. 2.21.

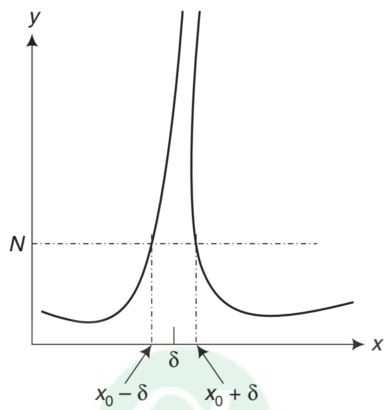


Fig. 2.21

The expression $\lim_{x \to x_0} f(x) = -\infty$ means, given any negative number N there exists some $\delta > 0$ such that $f(x) < -N$, when $0 < |x - x_0| < \delta$, Fig. 2.22.

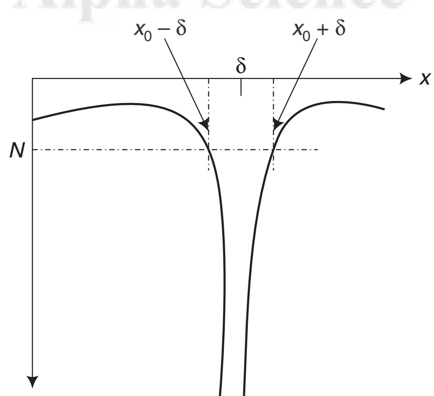


Fig. 2.22

Example 17 Show that $\lim_{x \to 0} \frac{1}{x^2} = \infty$

Solution Given a positive number N, we find $\delta > 0$ such that $0 < |x - 0| < \delta \Rightarrow \frac{1}{x^2} > N$

Now $\frac{1}{x^2} > N$ if and only if $|x| = \frac{1}{\sqrt{N}}$,

Hence, when $\delta = \frac{1}{\sqrt{N}}$ then $|x| < \delta \Rightarrow \frac{1}{x^2} > \frac{1}{\delta^2} \geq N$

Therefore, by definition $\lim_{x \to 0} \frac{1}{x^2} = \infty$, Fig. 2.23. $\left(\text{If } N = 2 \text{ then } \delta = \pm \frac{1}{\sqrt{2}}\right)$

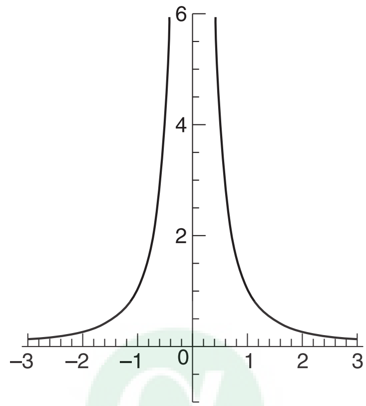


Fig. 2.23

Exercises

Find the limits of the following.

1. $\lim_{x \to 3} \frac{(x-1)(x-3)}{(x+2)}$,
2. $\lim_{x \to -1} \frac{x^2 + 6x + 5}{x^2 - 3x - 4}$,
3. $\lim_{x \to 4^-} \frac{3 - x}{x^2 - 2x - 8}$,
4. $\lim_{x \to 5^-} \sqrt{\frac{x+2}{x+1}}$,
5. $\lim_{x \to 0^+} \frac{\sqrt{x^2 + 4x + 5} - \sqrt{5}}{x}$,
6. $\lim_{x \to 0} \frac{3}{\sqrt{3x+1} + 1}$,
7. $\lim_{x \to 0} \frac{\sqrt{3x+1} - 1}{x}$,
8. $\lim_{x \to 2} \frac{\sqrt{x^2 + 12} - 4}{x - 2}$,
9. $\lim_{x \to 0^+} \frac{x}{\sqrt{1 - \cos x}}$,
10. $\lim_{x \to 0} \frac{3 \sin^{-1} x}{4x}$,
11. $\lim_{x \to 0} \frac{\sin(1+x) - \sin(1-x)}{x}$,
12. $\lim_{x \to 0} \frac{\operatorname{cosec}^2 x}{\cot^2 x}$,

13. $\lim_{x\to 2^+} \frac{1}{|2-x|}$, 14. $\lim_{x\to 2} \frac{|x+2|\,(x+3)}{(x+2)}$,

15. $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1} x}$, 16. $\lim_{x\to 0} \frac{e^x+e^{-x}-2}{x^2}$,

17. If $f(x) = \begin{cases} x-1, & x \le 3 \\ 3x-7, & x > 3 \end{cases}$ then find $\lim_{x\to 3} f(x)$,

18. If $f(x) = \begin{cases} 3-2x, & x \le 2 \\ x^2-5, & x > 2 \end{cases}$ then find $\lim_{x\to 2} f(x)$,

19. If $\sqrt{6-2x^2} \le f(x) \le \sqrt{6-x^2}$ for $-1 \le x \le 1$ use the sandwich theorem show that $f(x) = 6$.

20. If $1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$ holds for all values of x approaches to zero then find $\lim_{x\to 0} \frac{x \sin x}{2 - 2\cos x}$.

21. For the function $f(x)$ graphed in Fig. 2.24, find

(a) $\lim_{x\to 2^+} f(x)$, (b) $\lim_{x\to 2^-} f(x)$, (c) $\lim_{x\to 2} f(x)$, (d) $\lim_{x\to \infty^+} f(x)$,

(e) $\lim_{x\to \infty^-} f(x)$, (f) $f(2)$.

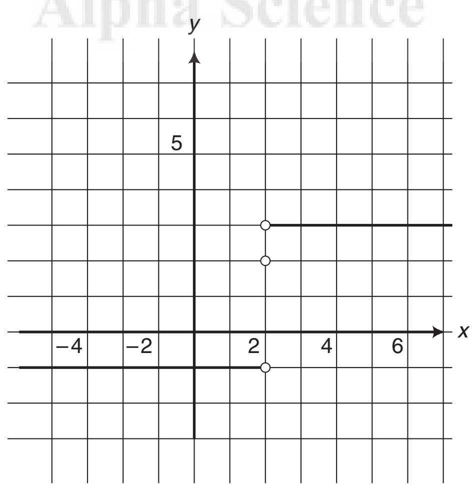


Fig. 2.24

22. For the function $f(x)$ graphed in Fig. 2.25 find

(a) $\lim_{x\to 0^+} f(x)$, (b) $\lim_{x\to 0^-} f(x)$, (c) $\lim_{x\to 0} f(x)$, (d) $\lim_{x\to \infty^+} f(x)$,

(e) $\lim_{x\to \infty^-} f(x)$.

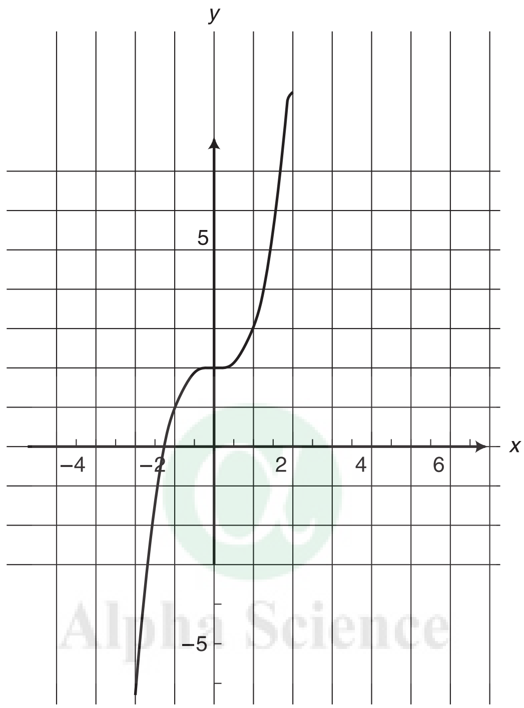


Fig. 2.25

23. For the function $f(x)$ graphed in Fig. 2.26, find

(a) $\lim_{x\to 2^+} f(x)$, (b) $\lim_{x\to 2^-} f(x)$, (c) $\lim_{x\to 0} f(x)$, (d) $\lim_{x\to \infty^+} f(x)$,

(e) $\lim_{x\to \infty^-} f(x)$.

24. For the function $f(x)$ graphed in Fig. 2.27, find

(a) $\lim_{x\to 0^+} f(x)$, (b) $\lim_{x\to 0^-} f(x)$ (c) $\lim_{x\to 0} f(x)$ (d) $\lim_{x\to \infty^+} f(x)$,

(e) $\lim_{x\to \infty^-} f(x)$, (f) $f(0)$.

25. Show that $\lim_{x\to 2} (2x + 4) = 8$.

26. Show that the function $f(x) = \sin \frac{1}{x}$, whenever $x \neq 0$. Defined on $R - \{0\}$ does not approaches 0 as $x \to 0$.

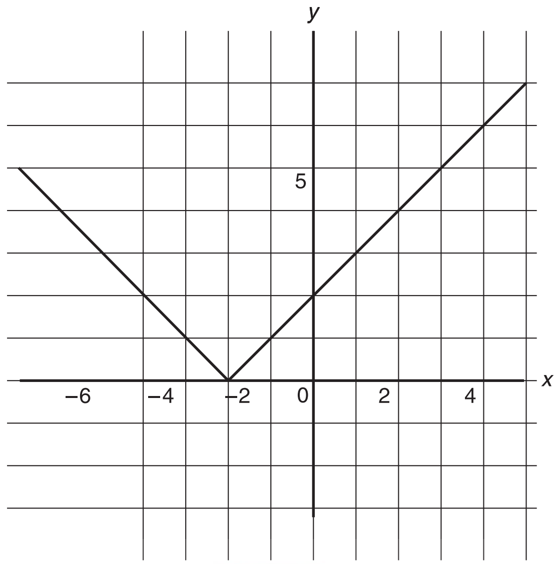


Fig. 2.26

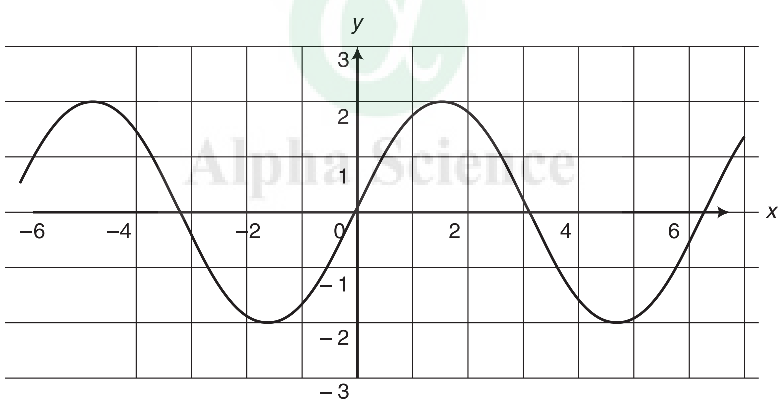


Fig. 2.27

27. Show that the limit of the function $f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 2 \end{cases}$ when $x \to 4$ does not exist.

28. Show that the limit $\lim_{x \to 1} 2^{\frac{1}{(x-1)}}$ does not exist.

29. Show that the limit $\lim_{x \to 0^+} \sqrt{x}$ does not exist.

30. For the limit $\lim_{x \to 1} \sqrt{2x - 1} = 1$, find a $\delta > 0$ that works for $\varepsilon = 1$.

31. Prove that $\lim_{x \to 2} f(x) = 4$ if $f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$.

32. By using the formal definition of the limit find the values of $\delta > 0$ of the following.

(i) $\lim_{x\to -1} (7x + 5) = -2$; when $\varepsilon = .01$,

(ii) $\lim_{x\to 0} \sqrt{x+1} = 1$; when $\varepsilon = .1$,

(iii) $\lim_{x\to 0} \sqrt{1-5x} = 4$; when $\varepsilon = .5$,

(iv) $\lim_{x\to -5} \frac{1}{x} = \frac{1}{5}$; when $\varepsilon = .05$,

(v) $\lim_{x\to -2} x^2 = 4$; when $\varepsilon = .5$,

33. Obtain the limit of the function $f(x) = \dfrac{x^2-4}{(x-2)^2(x-4)}$ when x approaches to 2, –7, $-\infty$, $+\infty$.

34. Given that $f(x) = \dfrac{x^2}{(x-1)(x-2)}$ show that $\lim_{x\to 1^+} f(x) = +\infty$ and $\lim_{x\to 1^-} f(x) = -\infty$.

35. show that $\lim_{x\to\infty} \dfrac{\sqrt{x^2+1}}{x+1} = 1$ and $\lim_{x\to -\infty} \dfrac{\sqrt{x^2+1}}{x+1} = -1$.

36. show that $\lim_{x\to 2} \dfrac{1}{(x-2)^4} = \infty$.

37. Use the definition on page (18) find N when $\lim_{x\to\infty} \dfrac{1}{x^2} = 0$; $\varepsilon = .01$.

Answers

1. 0, 2. $\frac{-4}{5}$, 3. $+\infty$, 4. $\sqrt{3}$,

5. $\frac{2}{\sqrt{5}}$, 6. $\frac{3}{2}$, 7. $\frac{3}{2}$, 8. $\frac{1}{2}$,

9. $\frac{2}{\sqrt{2}}$, 10. $\frac{3}{4}$, 11. 2 cos 1, 12. 1,

13. $+\infty$, 14. 1 and –1, 15. 1, 16. 1,

17. 2, 18. –1, 20. 1,

21. (a) 3, (b) –1, (c) does not exist,
(d) 3, (e) –1, (f) 2,

22. (a) 0, (b) 0, (c) 0, (d) $+\infty$,
(e) $-\infty$,

23. (a) 2, (b) 2, (c) 2, (d) $+\infty$,
(e) $-\infty$,

24. (a) 0, (b) 0, (c) 0, (d) $+\infty$,
(e) $-\infty$, (f) 0,

32. (i) $\frac{1}{700}$, (ii) (– .19, .21), (iii) .75, (iv) 1,
(v) $(-\sqrt{4.5}, -\sqrt{3.5})$,

33. ($-\infty$ and $+\infty$), ($-\infty$ and $+\infty$), 0, 0,

37. 10.

2.8 CONTINUITY

A function is continuous at a point if its defined at that point and its graph moves unbroken through that point, Fig. 2.28.

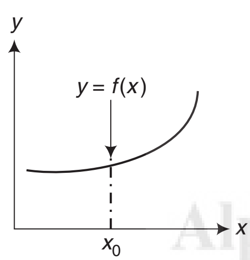


Fig. 2.28(a) Continuous at x_0

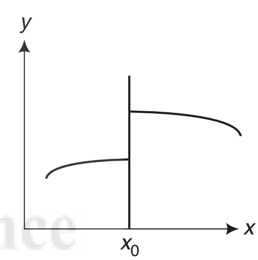


Fig. 2.28(b) Not Continuous at x_0

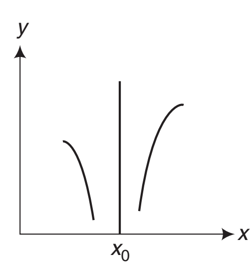


Fig. 2.28(c) Not Continuous at x_0

Continuity at a point: A function $f(x)$ is continuous at an interior point $x = x_0$ of the domain of f if

1. $f(x_0)$ is defined,
2. $\lim_{x \to x_0} f(x)$ exists,
3. $\lim_{x \to x_0} f(x) = f(x_0)$

A function that is not continuous at a point x_0 is said to be discontinuous at that point

The third condition of continuity says that if x approaches x_0 then the function $f(x)$ must be close to $f(x_0)$ and this condition can be defined as:

Let $f(x)$ be a function defined on an interval I. Then the function is continuous at any interior point $x_0 \in I$, if for any $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$, when $0 < |x - x_0| < \delta$, Fig. 2.29.

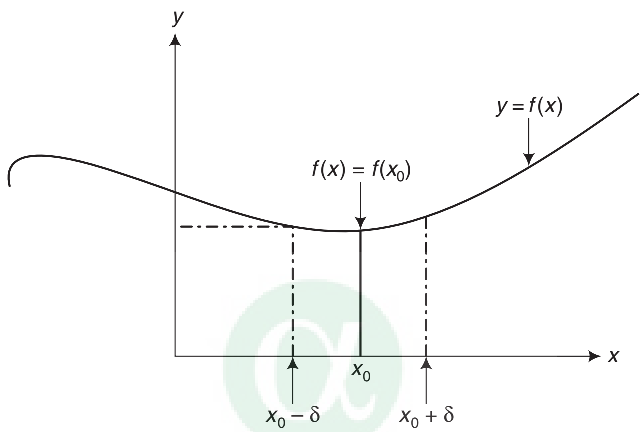


Fig. 2.29

When a function defined in an interval I say (a, b), then the continuity of the function can be defined at three points in which one is interior point and other two points are the end points of the interval.

The continuity at end points is defined by taking one sided limits.

A function $f(x)$ is continuous at a left end point $x = a$ of its domain if $\lim_{x \to a^+} f(x) = f(a)$ and the function $f(x)$ is continuous at b right end point $x = b$ of its domain if $\lim_{x \to b^-} f(x) = f(b)$, Fig. 2.30.

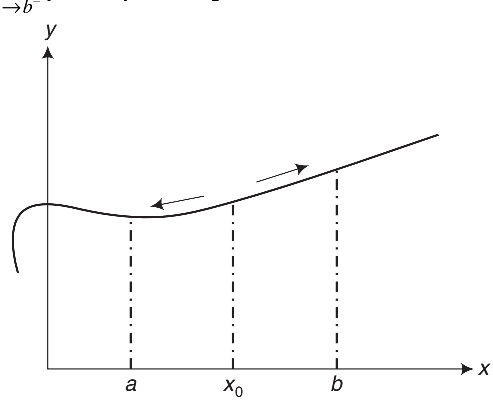


Fig. 2.30

For example the function $f(x) = \sqrt{9 - x^2}$ is continuous at every points of its domain [–3, 3] including $x = -3$, where f is right continuous, and $x = 3$ where f is left continuous, Fig. 2.31.

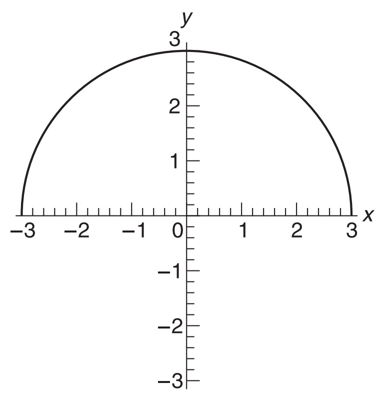


Fig. 2.31

The other function $f(x)$ where $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

Is the right continuous at $x = 0$, but is neither left continuous nor continuous there, Fig. 2.32.

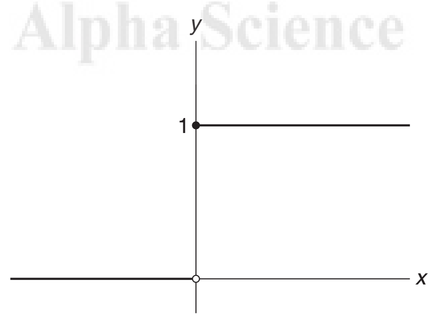


Fig. 2.32

A function f is continuous in the open interval (a, b) if it continuous at every point in that interval (a may be $-\infty$ and/or b may be $+\infty$).

A function f is continuous in closed interval $[a, b]$ if the following conditions hold:

1. The function f is continuous in the open interval (a, b).
2. $f(a)$ and $f(b)$ both exist.
3. $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.

Rotation of the earth about the sun or rotation of the moon about the earth with respect to time is continuous, Fig. 2.32(a).

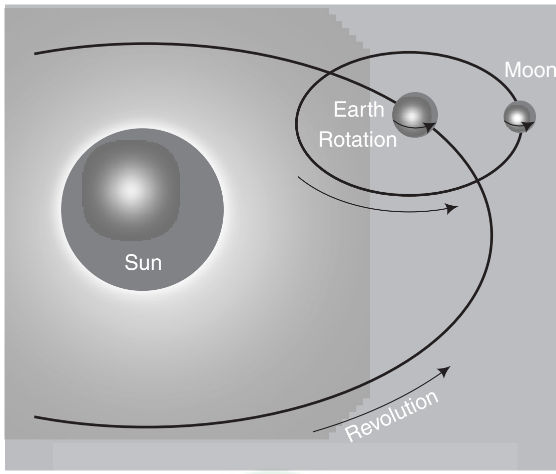


Fig. 2.32(a)

Theorems: Let the function f and g be the continuous at x_0, and c be a constant. Then following functions are continuous at x_0:

1. Sums $f + g$.
2. Difference $f - g$.
3. Product $f \cdot g$
4. Constant multiples $c \cdot f$
5. Quotients $\frac{f}{g}$ provided $g \neq 0$.
6. Powers $f^{\frac{r}{s}}$ provided it is defined on an open interval containing x_0, where r and s integers
7. If f is continuous at a and if $\lim_{x \to x_0} g(x) = a$, then $\lim_{x \to x_0} f(g(x)) = f(a)$.
8. Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots a_2 x^2 + a_1 x + a_0$ be a polynomial, then $\lim_{x \to x_0} P(x) = P(x_0)$ for every real number x_0. Therefore every polynomial function is continuous at every real number.
9. If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous whenever it is defined ($Q(c) \neq 0$) by the quotient rule.
10. **Intermediate value theorem:** Let f be continuous on $[a, b]$, and if c is any number between $f(a)$ and $f(b)$, then there is a number x_0 in (a, b) such that $f(x_0) = c$, Fig. 2.33.

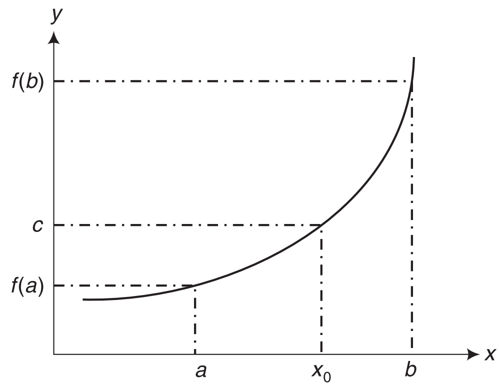


Fig. 2.33

For example the polynomial $P(x) = x^3 + x^2 + x - 2$ is continuous in the interval [0, 1]. Since $P(0) = -2$ and $P(1) = 1$, there must be a number x_0 in (0, 1) such that $P(x_0) = 0$. (since 0 is between –2 and 1). (root of the polynomial), Fig. 2.34.

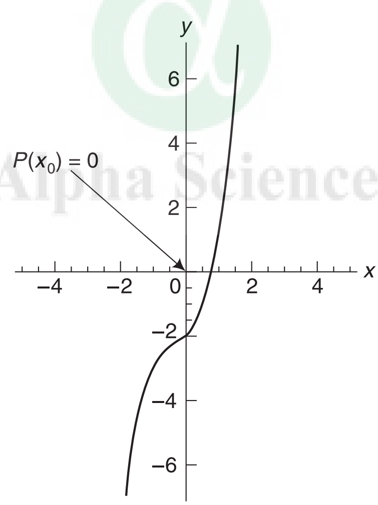


Fig. 2.34

Let $P\left(\frac{1}{2}\right) = -1.125$ and $P\left(\frac{9}{10}\right) = .439$, therefore there is a root between $\frac{1}{2}$ and $\frac{9}{10}$.

Using the calculator, we can narrow the root down further.

11. If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ are none zero and have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b), Fig. 2.35.

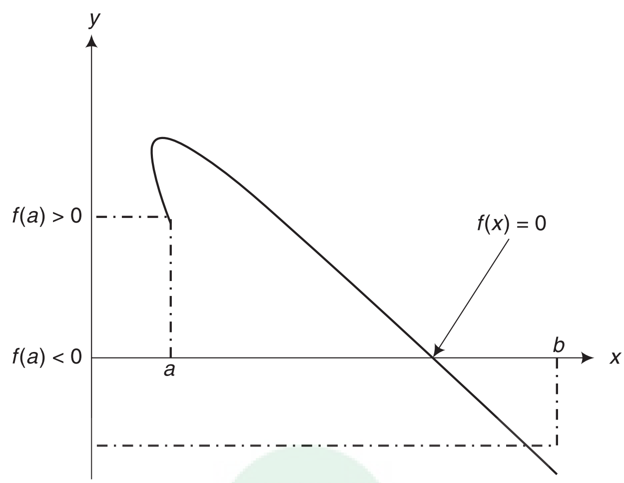


Fig. 2.35

Example 18 Show that $f(x) = 5x + 2$ is continuous for $x = 1$

Solution

1. $f(x) = 5x + 2$ is defined at $x = 1$ (polynomial function)
2. L.H.L. $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} 5x + 2 = 7$ and R.H.L. $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} 5x + 2 = 7$

   Hence $\lim\limits_{x\to 1} f(x)$ exist
3. $\lim\limits_{x\to 1} f(x) = f(1) = 7$. Therefore the function $f(x) = 5x + 2$ is continuous at $x = 1$.
4. Let $\varepsilon > 0$, we have $|f(x) - f(1)| = |5x + 2 - 7| = |5x - 5| = 5|x - 1|$

   Now $5|x - 1| < \varepsilon$ when $0 < |x - 1| < \delta$. Hence $|x - 1| < \varepsilon$ when $\delta = \dfrac{\varepsilon}{5}$

Thus there exist an interval $\left(1 - \dfrac{\varepsilon}{5}, 1 + \dfrac{\varepsilon}{5}\right)$ around 1 such that for every value of $x\varepsilon\left(1 - \dfrac{\varepsilon}{5}, 1 + \dfrac{\varepsilon}{5}\right)$, the numerical value of the difference between $f(x)$ and $f(1)$ is less than a positive number ε.

Hence $f(x)$ is continuous at $x = 1$.

Example 19 Show that $f(x) = |x| + |x - 1|$ is cotinuous at $x = 0$ and $x = 1$

Solution $f(x) = -x - (x - 1) = 1 - 2x$ when $x < 0$

$$f(x) = x - (x - 1) = 1 \text{ when } 0 \le x < 1$$

$$f(x) = x + (x - 1) = 2x - 1 \text{ when } x \geq 1$$

Now, L.H.L. at $x = 0$ $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1 - 2x) = 1$ and R.H.L. $\lim_{x \to 0^+} f(x) = 1$

Also $f(0) = |0| + |0 - 1| = 1$. Therefore $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 1 = f(0)$.

Hence the $f(x)$ is continuous at $x = 0$.

Now L.H.L. at $x = 0$ $\lim_{x \to 1^-} f(x) = 1$ and R.H.L $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x - 1) = 1$

Also $f(1) = |1| + |1 - 1| = 1$. Therefore $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 1 = f(1)$.

Hence the $f(x)$ is also continuous at $x = 1$.

Example 20 Show that $f(x) = |x|$ is continuous everywhere

Solution $|x| = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0 \end{cases}$

Here $|x|$ in the interval $(0, +\infty)$ which is a polynomial and $|x| = -x$ in the interval $(-\infty, 0)$ which is also polynomial. But polynomial are continuous everywhere so $x = 0$ is the only possible discontinuity for $|x|$ to prove the continuity at $x = 0$ we must show $\lim_{x \to 0} |x| = 0$.

L.H.L. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$, and R.H.L. $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} |x| = 0$,

And $\lim_{x \to 0} |x| = 0 = |0|$. Hence the function is continuous at $x = 0$.

Example 21 Show that $f(x) = \sin^2 x$ is continuous for every values of x

Solution Let $\varepsilon > 0$ we have $|f(x) - f(c)| = |\sin^2 x - \sin^2 c|$

$$= |\sin (x + c) \sin (x - c)| \leq |\sin (x + c)| \, |\sin (x - c)|$$

Now $|\sin (x + c)| \leq 1$ for every value of x and c and $|\sin (x - c)| \leq |x - c|$

Hence, we have $|f(x) - f(c)| \leq |x - c|$

$|f(x) - f(c)| < \varepsilon$ when $|x - c| < \varepsilon$

Thus, there exist an interval $(c + \varepsilon, c - \varepsilon)$ around c such that for every value of x

$$|\sin^2 x - \sin^2 c| < \varepsilon$$

Hence, $\sin^2 x$ is continuous when $x = c$, therefore also for every value of x: c being any number.

Example 22 Examine the continuity of the function

$$f(x) = \begin{cases} \dfrac{x}{1 + e^{\frac{1}{x}}}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases} \quad \text{at } x = 0$$

Solution L.H.L. $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{h \to 0} \dfrac{(0 - h)}{1 + e^{\frac{1}{(0-h)}}} = \lim\limits_{h \to 0} \dfrac{-h}{1 + e^{\frac{1}{(0-h)}}} = 0$ and

R.H.L. $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{h \to 0} \dfrac{(0 + h)}{1 + e^{\frac{1}{(0-h)}}} = 0$ and $f(0) = 0$.

Since $\lim\limits_{x \to 0} f(x) = f(0)$. Hence the function is continuous at $x = 0$.

Example 23 Determine whether the following function are continuous at $x = 3$

$$f(x) = \frac{x^2 - 9}{x - 3}, \quad g(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, & x \neq 3 \\ 5, & x = 3 \end{cases}, \quad h(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, & x \neq 3 \\ 6, & x = 3 \end{cases}$$

Solution In above all cases we have $\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} g(x) = \lim\limits_{x \to 3} h(x)$ $= \lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3} = \lim\limits_{x \to 3} (x + 3) = 6$

But the function $f(x)$ is not defined at $x = 3$ so $f(x)$ is not continuous at $x = 3$. Figure 2.36(a). The function $g(x)$ is defined at $x = 3$ and $g(3) = 5$, hence $\lim\limits_{x \to 3} g(x) \neq g(3)$ so the function $g(x)$ is not continuous at $x = 3$ Fig. 2.36(b). In last case The function $h(x)$ is defined at $x = 3$ and $h(3) = 6$ hence $\lim\limits_{x \to 3} h(x) = h(x)$ so the function $h(x)$ is continuous at $x = 3$, Fig. 2.36(c).

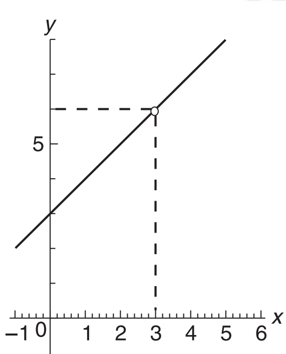


Fig. 2.36(a)

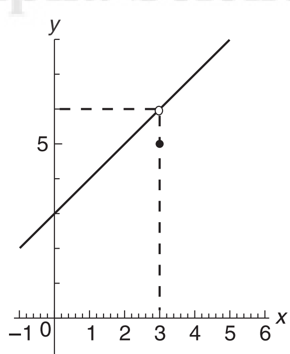


Fig. 2.36(b)

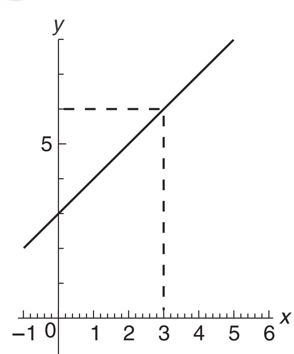


Fig. 2.36(c)

Example 24 Find the value of K if the function f is given by

$$f(x) = \begin{cases} 2x - 3, & x < 1 \\ K, & x = 1 \\ x - 2, & x > 1 \end{cases}$$

is continuous at $x = 1$

Solution L.H.L. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2x - 3) = -1$ and

R.H.L. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x - 2) = -1$ and $f(1) = K$ given

So $\lim_{x\to 1} f(x) = f(1) = -1 = K$. Hence $K = -1$.

2.9 TYPES OF DISCONTINUITY

(*a*) *Removable Discontinuity*: A function $f(x)$ has a removable discontinuity at a point $x = x_0$, if there is any one of the following posibility.

(i) $\lim_{x\to x_0} f(x)$ exists and is finite, but f is not defined at $x = x_0$, Fig. 2.37.

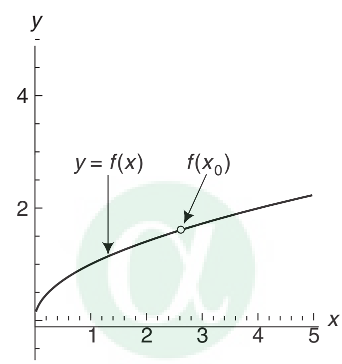


Fig. 2.37

(ii) $\lim_{x\to x_0} f(x)$ exists and is finite, and f is defined at $x = x_0$ but $\lim_{x\to x_0} f(x) \neq f(x_0)$, Fig. 2.38.

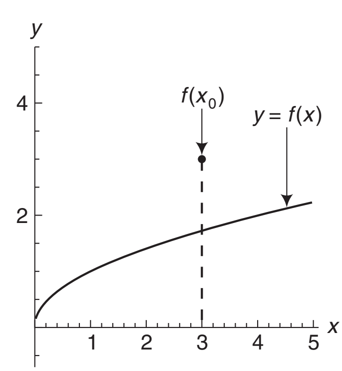


Fig. 2.38

This type of discontinuity can be removed by defining f at $x = x_0$ such that $\lim_{x\to x_0} f(x) = f(x_0)$

Example 25 Show that the function defined as

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & \text{when} \quad x \neq 0 \\ 1, & \text{when} \quad x = 0 \end{cases}$$

Has removable discontinuity at $x = 0$

Solution $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\sin 2x}{x} = \lim\limits_{x\to 0} \left(\dfrac{\sin 2x}{2x}\right) 2 = 2$ but $f(0) = 1$ (given)

Hence $\lim\limits_{x\to 0} f(x) \neq f(0)$

Thus the limit exists but is not equal to the value of the function at the given point.

The function has the removable discontinuity at $x = 0$ and this discontinuity can be removed at this point such as $f(0) = 2$.

(b) Discontinuity of the first kind: A function $f(x)$ is said to have a discontinuity of the first kind if.

$$\lim_{x\to x0^-} f(x) \neq \lim_{x\to x_0^+} f(x) \text{ or } \lim_{x\to x_0^-} f(x) \neq \lim_{x\to x_0^+} f(x) = f(x_0) \text{ or } \lim_{x\to x_0^-} f(x)$$
$$= f(x_0) \neq \lim_{x\to x_0^+} f(x)$$

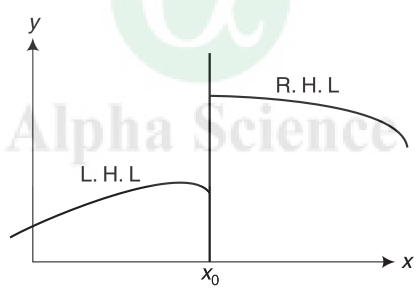


Fig. 2.39

Example 26 Show that the following function has the discontinuity of the first kind

$$f(x) = \begin{cases} \dfrac{x - |x|}{x}, & \text{when} \quad x \neq 0 \\ 2, & \text{when} \quad x = 0 \end{cases}$$

at $\qquad x = 0$

Solution L.H.L. $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{x - |x|}{x} = \lim\limits_{x\to 0^-} \dfrac{x + x}{x} = 2$ and R.H.L. $\lim\limits_{x\to 0^+}$ $f(x) = \lim\limits_{x\to 0^+} \dfrac{x - x}{x} = 0$

And $f(0) = 2$ (given). Hence $\lim\limits_{x\to 0^-} f(x) = f(0) \neq \lim\limits_{x\to 0^+} f(x)$

Thus the function has discontinuity of the first kind from the right at $x = 0$

(c) Discontinuity of the second kind: If neither $\lim_{x \to x_0^-} f(x)$ nor $\lim_{x \to x_0^+} f(x)$ exists, then the discontinuity at $x = x_0$ is said to be of the second kind, Fig. 2.40.

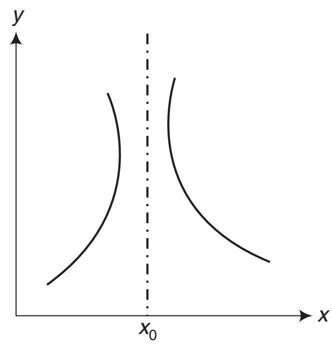


Fig. 2.40

Example 27 Examine the continuity of the function $f(x) = \dfrac{1}{(x - a)} \csc \dfrac{1}{(x - a)}$ at $x = a$

Solution L.H.L. $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} \dfrac{1}{(x - a)} \csc \dfrac{1}{(x - a)} = \lim_{h \to 0} \dfrac{1}{(a - h - a)} \csc \dfrac{1}{(a - h - a)}$

$$= \lim_{h \to 0} \frac{1}{-h} \csc \frac{1}{-h} = \lim_{t \to \infty} t \csc, \left[t = \frac{1}{h} \right]$$

When $t \to \infty$ oscillates between –1 and +1, therefore the L.H.L. of $f(x)$ oscillates between $-\infty$ and ∞. Similarly R.H.L. of the function $f(x)$ does not exist. Hence $f(x)$ is discontinuous of the second kind.

(d) Jump Discontinuity: A function has jump discontinuity when the function "Jumps" from one finite value to another at a point, Fig. 2.41.

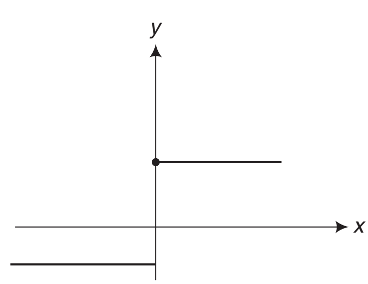


Fig. 2.41

(e) If either $\lim_{x \to 0^-} f(x)$ or $\lim_{x \to 0^+} f(x)$ is infinitely large, then f is said to have an infinite discontinuity at $x = x_0$ [example 17 on page 1.18]

Exercises

1. Find the value of x (if any) at which f is discontinuous of the follwing functions.

   (i) $f(x) = x^2 + 2x + 5,$ (ii) $f(x) = \dfrac{x^2}{2x^2 + 1},$

   (iii) $f(x) = \dfrac{2x}{3x^2 + 2},$ (iv) $f(x) = \dfrac{x - 4}{x^2 - 16},$

   (v) $f(x) = \dfrac{x}{|x| - 3},$ (vi) $fx = f(x)$

2. Examine the continuity of $f(x)$ of the following functions at the indicated points.

   (i) $$f(x) = \begin{cases} \dfrac{1}{2} - x, & 0 \le x < \dfrac{1}{2} \\ 1, & x = \dfrac{1}{2} \\ \dfrac{3}{2} - x, & \dfrac{1}{2} < x < 1 \end{cases}$$

   At $x = \dfrac{1}{2},$

   (ii) $$f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \\ x^2 - 2x, & x > 2 \end{cases}$$

   At $x = 1$ and $x = 2,$

   (iii) $$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$$

   At $x = 0,$

   (iv) $$f(x) = \begin{cases} \dfrac{e^{x^2}}{e^{1/x^2} - 1}, & x \ne 0 \\ 1 & x = 0 \end{cases}$$

   At $x = 0,$

   (v) $$f(x) = \begin{cases} \sin^2 x, & x \ne 0 \\ 1, & x = 0 \end{cases}$$

   At $x = 0,$

(vi) $f(x) = \begin{cases} \dfrac{\tan^2 x}{3x}, & x \neq 0 \\ \dfrac{2}{3}, & x = 0 \end{cases}$

At $x = 0$,

(vii) $f(x) = \begin{cases} x, & x \leq 0 \\ \cos \dfrac{1}{x}, & x > 0 \end{cases}$

At $x = 0$,

(viii) $f(x) = \begin{cases} \dfrac{\sin x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

At $x = 0$,

(ix) $f(x) = \begin{cases} x^m \sin \dfrac{1}{x}, & x \neq 0, m > 0 \\ 0 & x = 0 \end{cases}$

At $x = 0$, what happens when $m = 1$,

(x) $f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & x \neq 0 \\ 2, & x = 0 \end{cases}$

At $x = 0$,

(xi) $f(x) = \begin{cases} x\left[1 + \dfrac{1}{3} \sin (\log x^2)\right], & x \neq 0 \\ 0, & x = 0 \end{cases}$

At $x = 0$,

(xii) $f(x) = \begin{cases} \left(1 - \dfrac{x}{4}\right)^{\frac{1}{x}}, & x \neq 0 \\ e^{\frac{-1}{4}}, & x = 0 \end{cases}$

At $x = 0$,

3. The function $f(x) = \dfrac{[\log(1 + 2x) - \log(1 - 3x)]}{x}$ is not defined at $x = 0$, find $f(0)$ if $f(x)$ is cotinuous at $x = 0$.
4. Show that the function $f(x) = |x| + |x - 1| + |x - 2|$ is continuous at the points $x = 0, 1, 2$.

5. (a) Determine the values of a and b for which the function

$$f(x) = \begin{cases} ax^2 + b, & x \le 0 \\ \dfrac{-3}{x^2+1} + 1, & x > 0 \end{cases}$$

Is continuous at $x = 0$,

(b) Determine the values of a and b for which the function

$$f(x) = \begin{cases} 2x^2 + b. & x \ge 0 \\ 3\sqrt{\dfrac{x^2}{2} + 1} + b, & x < 0 \end{cases}$$

Is continuous at $x = 0$ and $f(1) = 2$,

6. Find the value of K when the following functions are continuous at x everywhere,

(i) $f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2kx, & x \ge 3 \end{cases}$

(ii) $f(x) = \begin{cases} kx^2, & x \le 2 \\ 2x + k, & x > 2 \end{cases}$

(iii) $f(x) = \begin{cases} 7x - 4, & x \le 1 \\ kx^2, & x > 1 \end{cases}$

7. For what values of k would the function $f(x) = \dfrac{x^3 - 6x^2 + 11x - 6}{x - k}$ have removable discontinuity at $x = k$.
8. Use the intermediate value theorem show that the function $f(x) = \cos x - x$, has a root between 0 and 1.
9. Use the intermediate value theorem show that the function $f(x) = \log x - e^{-x}$, has a root between 1 and 2.
10. Show that there is some k with $0 < k < 2$ such that $k^2 + \cos(\pi k) = 4$.
11. Use the intermediate value theorem to show that there is a square with a diagonal length that is between r and $2r$ an area that is half the area of a circle of radius r.
12. Prove that $f(x) = \dfrac{5}{\sqrt{x^4 + 7x^2 + 2}}$ is continuous everywhere.

Answers

1. (i) None. (ii) None.
   (iii) None. (iv) $f(x)$ is not defined at $x = 4$.
   (v) $f(x)$ is not defined at $x = \pm 3$ (vi) None.

2. (i) Discontinuous. (ii) Continuous, Continuous.
   (iii) Continuous. (iv) Discontinuous.
   (v) Discontinuous. (vi) Discontinuous.
   (vii) Discontinuous. (viii) Continuous.
   (ix) Continuous. (x) Continuous.
   (xi) Continuous. (xii) Continuous.
3. $f(0) = 5$.
5. (*a*) $b = -2$ and whatever a. (b) $a = 0$ and $b = -3$.
6. (i) $k = \frac{4}{3}$, (ii) $k = \frac{4}{3}$, (iii) $k = 3$.
7. 1, 2.

3 CHAPTER

Differentiation

3.1 INTRODUCTION

In four major problems which has been described in chapter 2 and one was to find the tangent to a curve. There are so many uses of the tangent to a curve, in optics, the tangent determined the angle at which a ray of light entered a curve lens, Fig. 3.1.

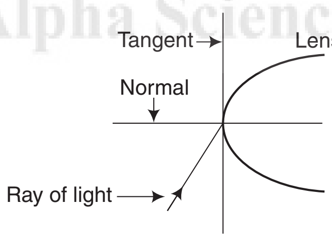


Fig. 3.1

In mechanics, the tangent line determined the direction of motion of an object at every point along its path, Fig. 3.2. In geometry in a general prospect a tangent is a line which touch a curve at a point. In seventeenth century the Greeks knew how to find the tangent line to a circle, it is always perpendicular to the radial line, the Greeks used the definition of the tangent as a line touching a curve at only one point and lying on one side of the curve sufficed. But it was insufficient for some of the complicated curves. To give the definition of a tangent line that applies to a wide variety of curves use the dynamic approach.

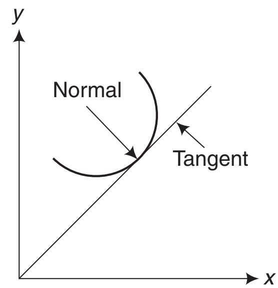


Fig. 3.2

The tangent to a curve $y = (x)$ at a point $P(x_0, y_0)$ is the line through the point P whose slope is the limit of the secant slopes which passes through from P and a point $(x_0 + h, y_0 + h)$, as Q tends to P from either side.

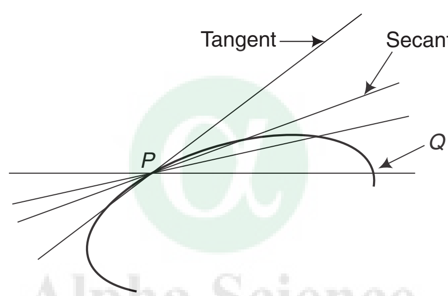


Fig. 3.3

The tangent line to the curve $y = f(x)$ at a point $P(x_0, f(x_0))$ (where $y_0 = f(x_0)$) is a line $y = y_0 + m(x - x_0)$, Fig. 3.4. Where exists:

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\left[m = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta x}{\Delta y} = \frac{f(x_0 + h) - f(x_0)}{x_0 + h - x_0} \right]$$

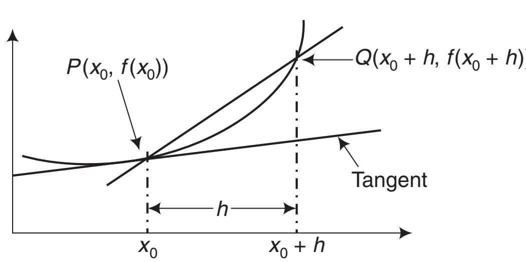


Fig. 3.4

Let $y = \frac{3}{2}x - 3$. Then if we move from the point $P(1, -1.5)$ to Q (3, 1.5) along the line. We see that the as x has changed (increased) 2 unit, y has increased 3 units corresponding to the slope $m = \frac{3}{2}$, Fig. 3.5.

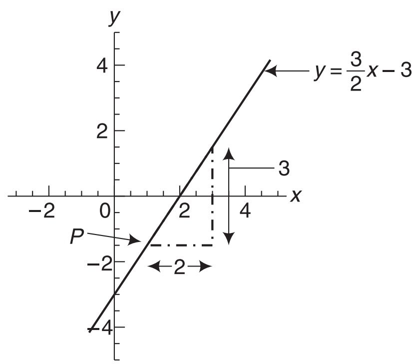


Fig. 3.5

$$m = \frac{1.5 - (-1.5)}{3 - 1} = \frac{3}{2}$$

Example 1 Find the equation of the tangent line to the parabola $y = x^2$ at a point (1, 1)

Solution Given point P is (1, 1) $\Rightarrow x_0 = 1,\ y_0 = 1 = f(x_0)$, and $f(x_0 + h) = (1 + h)^2$

$$m = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h\to 0} \frac{(1 + h)^2 - 1}{h} = \lim_{h\to 0} \frac{h(2 + h)}{h} = 2$$

Hence the equation of the tangent line is $y = 1 + 2(x - 1) = 2x - 1$.

3.2 DEFINITION OF DERIVATIVE

(i) The derivative of a function $y = f(x)$ is the slope of a tangent line to the graph of the function $f(x)$ at the point $P(x_0, f(x_0))$ denoted as $f'(x_0)$.

(ii) Let f be a function defined on an open interval (a, b) and there is point $x_0 \in (a, b)$. Suppose that

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} \text{ exists and finite.}$$

Then f is said to be differentiable or smooth at x_0 and derivative of f at x_0 denoted $f'(x_0)$ is given by

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Consider the graph of the function $y = f(x)$ drawn in Fig. 3.6, we observed that there is a unique tangent line at each point and each tangent has a slope, for example the slope at the point $(x_1, f(x_1))$

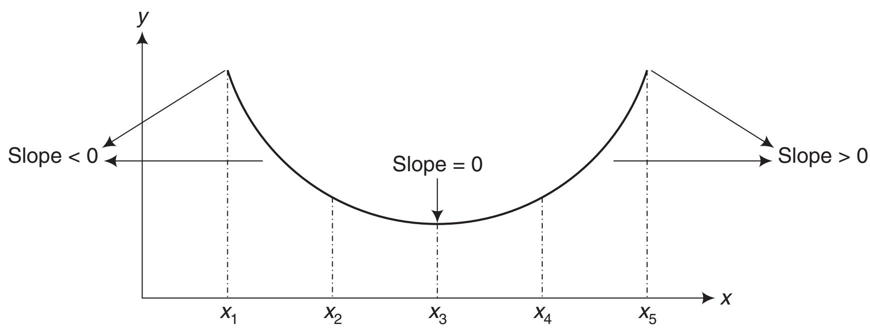


Fig. 3.6

Is negative while the slope at the point $(x_3, f(x_3))$ is zero and the slope at $(x_4, f(x_4))$ is positive.

According to the definition of a **function** in chapter 1, it is a rule that assigns a unique real number to every number in its domain. Here in Fig. 3.6 we have a new **function** f' called the **derivative of** f, that assigns to each number x_0 a new number $f'(x_0)$ (slope of the tangent line at x_0), and the domain of f' is contained in domain f.

One sided derivative

Let $y = f(x)$ be a function and let x_0 be a point in the domain of. The right-hand derivative of f at $x = x_0$ is the limit $\lim_{h \to 0^+} \dfrac{f(x_0 + h) - f(x_0)}{h}$ and Left-hand derivative of f at $x = x_0$ is the limit $\lim_{h \to 0^-} \dfrac{f(x_0 + h) - f(x_0)}{h}$. A function f is differentiable at $x = x_0$ if and only if f has both a right-hand derivative and Left-hand derivative of f at $x = x_0$ and both of these derivatives are equal.

A function $y = f(x)$ is differentiable on a closed interval $[a, b]$ if its differentiable on the (a, b) and if limits $\lim_{h \to 0^+} \dfrac{f(a + h) - f(a)}{h}$ Right-hand derivative at a, and $\lim_{h \to 0^-} \dfrac{f(b + h) - f(b)}{h}$ Left-hand derivative at b exist at end points

Geometrically, if the function $y = f(x)$ is differentiable at $x = x_0$, then the graph of the function has a tangent line at x_0. If the function f defined but is not differentiable at x_0, then the graph of the function at x_0 has **1. A corner,** Fig. 3.7 **2. A cusp,** Fig. 3.8.

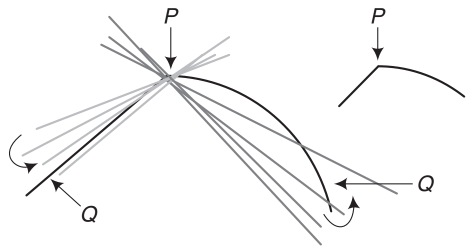


Fig. 3.7 One sided derivative differ

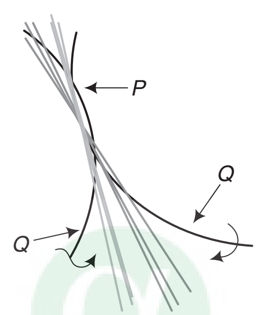


Fig. 3.8 Slope of *PQ* approaches ∞ from one side and –∞ from other

3.3 VERTICAL TANGENT

Figure 3.9 Slope of *PQ* approaches ∞ from both sides or approaches –∞ from both sides

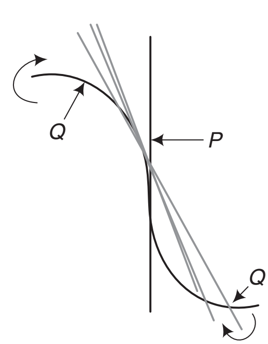


Fig. 3.9

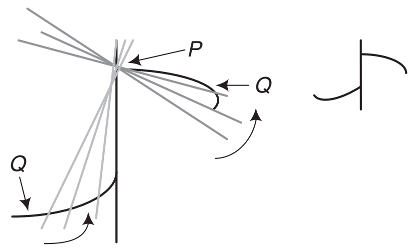


Fig. 3.10

Alternative definition of the derivative

Let $t = x_0 + h$ then $h = t - x_0$ and $h \to 0$ is equivalent $t \to x_0$. Hence

$$f'(x_0) = \lim_{t \to x_0} \frac{f(t) - f(x_0)}{t - x_0}, \quad \text{(Fig. 3.11)}$$

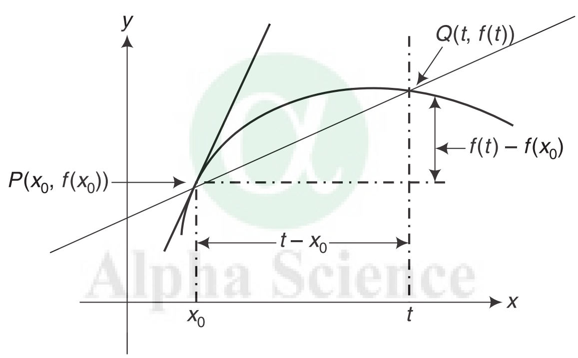


Fig. 3.11

Sometimes this definition more convenient to use in the computation.

Some common alternative notations for the derivative of the function $y = f(x)$, where x is independent and y is dependent variable are

$$y' = f'(x) = \frac{df}{dx} = \frac{dy}{dx} = \frac{df(x)}{dx} = D(f)(x) = D_x f(x)$$

The symbols $\frac{d}{dx}$ and D indicate the operation of differentiation and are called differentiation operators.

The symbols $\frac{dy}{dx}$ is read " the derivative of y with respect to x".

Example 2 Find the derivative of $y = \sqrt{x}$, and show that this function is not differentiable at $x = 0$ and also calculate the slope of the tangent line at the point $(5, \sqrt{5})$, *Solution* We have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{(x+h) - x}{h\left[\sqrt{(x+h)} + \sqrt{x}\right]} = \lim_{h\to 0} \frac{h}{h\left[\sqrt{(x+h)} + \sqrt{x}\right]} = \frac{1}{2\sqrt{x}}$$

Thus $f'(x) = \dfrac{1}{2\sqrt{x}}$ When $x > 0$ (3.1)

Now when $x = 0$

$$f'(0) = \lim_{h\to 0^+} \frac{\sqrt{(0+h)} - \sqrt{0}}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty$$

Since the right-hand derivative is not finite, so the function is not *differentiable at* $x = 0$.

And from (3.1) $f'(5) = \dfrac{1}{2\sqrt{5}}$

Hence the slope at $(5, \sqrt{5})$ is $\dfrac{1}{2\sqrt{5}}$.

Example 3 Find the derivative of $y = \sin x$

Solution

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \sin h \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x\,(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\sin h \cos x}{h}$$

$0 + 1.\cos x = \cos x$.

Example 4 Find the derivative of $y = \log_a x$, *when* $x \in [0, \infty]$, $a > 0$

Solution

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\log_a(x+h) - \log_a x}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \log_a \left(\frac{x+h}{x}\right) = \lim_{h\to 0} \frac{x}{h} \frac{1}{x} \log_a \left(1 + \frac{h}{x}\right)$$

$$= \lim_{h\to 0} \frac{1}{x} \log_a \left(1 + \frac{h}{x}\right)^{\frac{x}{h}} = \frac{1}{x} \log_a e = \frac{1}{x}$$

Here $\lim_{h\to 0} \left(1 + \frac{h}{x}\right)^{\frac{x}{h}} = e.\left[\text{Let } a = e \text{ then } \log_e x = \log x, \frac{df(x)}{dx} = \frac{d \log x}{dx} = \frac{1}{x}\right].$

Example 5 Find the derivative of $y = \frac{2}{(\sqrt{5-7x})^3}$

Solution $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{2[5 - 7(x+h)]^{\frac{-3}{2}} - 2[5 - 7(x)]^{\frac{-3}{2}}}{h}$

$$= \lim_{h\to 0} \frac{2[5-7x]^{\frac{-3}{2}} \left[\left\{1 + \frac{3}{2}\left(\frac{7h}{5-7x}\right) + \ldots\right\} -1\right]}{h} = 21[5-7x]^{\frac{-5}{2}}.$$

Example 6 If $f(x) = \begin{cases} 2x, & x \le 0 \\ 3 + x, & x > 0 \end{cases}$

Then calculate $f'(0)$

Solution $f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}.$

Now $f(h) = \begin{cases} 2h, & h \le 0 \\ 3 + h, & h > 0 \end{cases}$

And $\frac{f(h)}{h} = \begin{cases} 2 & h \le 0 \\ \frac{3+h}{h}, & h > 0 \end{cases}$ But $\frac{3+h}{h} = 1 + \frac{3}{h}$ and $\lim_{h\to 0} \frac{3}{h} = \infty$

Thus $\lim_{h\infty 0} \frac{f(h)}{h} = \infty$. Hence $f'(0)$ does not exist.

Example 7 Show that the function f defined by $f(x) = |x - 1| + |x| + |x + 1|$ is not differentiable

At $x = -1, 0$ and 1

Solution We have

$$f(x) = \begin{cases} -(x-1) - x - (x+1) = -3x, & x < -1 \\ -(x-1) - x + (x+1) = -x + 2, & -1 \le x < 0 \\ -(x-1) + x + (x+1) = x + 2, & 0 \le x < 1 \\ (x-1) + x + (x+1) = 3x, & x \ge 1 \end{cases}$$

By Alternative definition *At* $x = -1$

$$\text{R.H.D.} = \lim_{x\to -1^+} \frac{f(x) - f(-1)}{x - (-1)}$$

$$= \lim_{x \to -1^+} \frac{-x + 2 - 3}{x + 1} = \lim_{x \to -1^+} \frac{-(x+1)}{x+1} = -1$$

$$\text{L.H.D.} = \lim_{x \to -1^-} \frac{f(x) - f(-1)}{x - (-1)}$$

$$= \lim_{x \to -1^-} \frac{-3x - 3}{x + 1} = -3$$

Hence f is not differentiable at $x = -1$.

At $x = 0$

$$\text{R.H.D.} = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - (0)}$$

$$= \lim_{x \to 0^+} \frac{x + 2 - 2}{x} = 1$$

$$\text{L.H.D.} = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - (0)}$$

$$= \lim_{x \to 0^-} \frac{-x + 2 - 2}{x} = -1$$

Hence f is not differentiable at $x = 0$.

At $x = 1$

$$\text{R.H.D.} = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - (1)}$$

$$= \lim_{x \to 1^+} \frac{3x - 3}{x - 1} = 3$$

$$\text{L.H.D.} = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - (1)}$$

$$= \lim_{x \to 1^-} \frac{x + 2 - 3}{x - 1} = 1$$

Hence f is not differentiable at $x = 1$.

3.4 DERIVABILITY IMPLYING CONTINUITY

Theorem If a function $y = f(x)$ is derivable at a point x_0, then $f(x)$ is also continuous at x_0

Proof: Let $y = f(x)$ be differentiable at a point x_0 then

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{3.2}$$

We write $f(x_0 + h) - f(x_0) = \dfrac{f(x_0 + h) - f(x_0)}{h} h$

$$\Rightarrow \quad \lim_{h\to 0} [f(x_0 + h) - f(x_0)] = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} \lim_{h\to 0} h$$

$$= f'(x_0).\ 0 \qquad \text{From (3.2)}$$

$$\Rightarrow \quad \lim_{h\to 0} [f(x_0 + h) - f(x_0)] = 0$$

$$\Rightarrow \quad \lim_{h\to 0} f(x_0 + h) = f(x_0)$$

Hence $f(x)$ is continuous at $x = x_0$.

The converse of this theorem is not necessarily true, i.e. a function may be continuous for a value of x without being differentiable for that value.

Example 8 Show that the function $f(x) = |x|$ is continuous at the origin but not differentiable.

Solution In example 20, we have shown that $f(x) = |x|$ is continuous at the origin, now for derivative

$$\text{R.H.D. } \lim_{h\to 0^+} f'(0) = \lim_{h\to 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = 1$$

$$\text{L.H.D. } \lim_{h\to 0^-} f'(0) = \lim_{h\to 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = -1$$

$$\lim_{h\to 0^+} f'(0) \neq \lim_{h\to 0^-} f'(0)$$

Thus the function $f(x) = |x|$ is not differentiable at $x = 0$, {has a corner at $x = 0$} but differentiable in the interval $]-\infty, 0[U] \infty, 0[$, Fig. 3.12.

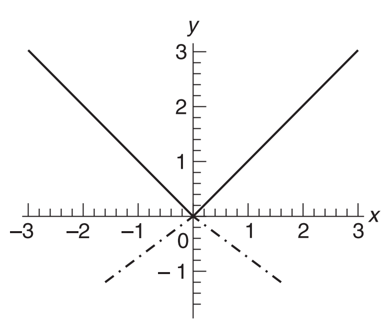


Fig. 3.12

Example 9 Show that the function $(x) = x\dfrac{e^{\frac{1}{x}} - e^{\frac{-1}{x}}}{e^{\frac{1}{x}} + e^{\frac{-1}{x}}}$, $x \neq 0$, $f(0) = 0$ is continuous at the origin but not differentiable.

Solution For continuity L.H.L. $\lim_{x\to 0^-} f(x) = \lim_{h\to 0^-} (0-h)\dfrac{e^{\frac{1}{(0-h)}} - e^{\frac{-1}{(0-h)}}}{e^{\frac{1}{(0-h)}} + e^{\frac{-1}{(0-h)}}}$ $= 0$

R.H.L. $\lim_{h\to 0^+} f(x) = \lim_{h\to 0^+} (0+h)\dfrac{e^{\frac{1}{(0+h)}} - e^{\frac{-1}{(0+h)}}}{e^{\frac{1}{(0+h)}} + e^{\frac{-1}{(0+h)}}} = 0$

And $f(0) = 0$. Hence $\lim_{h\to 0^-} f(x) = \lim_{h\to 0^+} f(x) = f(0) = 0$, so the function is continuous

At $x = 0$

$$\text{L.H.D} = \lim_{h\to 0^-} (0+h)\frac{\left(\dfrac{e^{\frac{1}{h}} - e^{\frac{-1}{h}}}{e^{\frac{1}{h}} + e^{\frac{-1}{h}}}\right) - f(0)}{h} = \lim_{h\to 0^-}\frac{e^{\frac{2}{h}} - 1}{e^{\frac{2}{h}} + 1} = -1$$

$$\text{R.H.D} = \lim_{h\to 0^+} (0+h)\frac{\left(\dfrac{e^{\frac{1}{h}} - e^{\frac{-1}{h}}}{e^{\frac{1}{h}} + e^{\frac{-1}{h}}}\right) - f(0)}{h} = \lim_{h\to 0^-}\frac{1 - e^{-\frac{2}{h}}}{1 + e^{-\frac{2}{h}}} = 1$$

$$\lim_{h\to 0^+} f'(0) \neq \lim_{h\to 0^-} f'(0)$$

Hence f is not differentiable at $x = 0$.

Example 10 Show that $f(x) = \begin{cases} x^2 + 3, & x \le 1 \\ x + 3 & x > 1 \end{cases}$ is continuous but not differentiable *at* $x = 1$.

Solution For continuity L.H.L.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 + 3 = 4$$

R.H.L. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x + 3 = 4$

And $f(1) = 4$. Hence $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 4$, so the function is continuous

At $x = 1$

$$\text{L.H.D.} = \lim_{h\to 0^-}\frac{(1+h)^2 + 3 - f(1)}{h} = \lim_{h\to 0^-}\frac{1 + h^2 + 2h + 3 - 4}{h}$$

$$= \lim_{h\to 0^-}\frac{h(h+2)}{h} = 2$$

$$\text{R.H.D.} = \lim_{h \to 0^+} \frac{1 + h + 3 - f(1)}{h} = \lim_{h \to 0^+} \frac{4 + h - 4}{h} = 1$$

Hence f is not differentiable *at* $x = 1$, Fig. 3.13

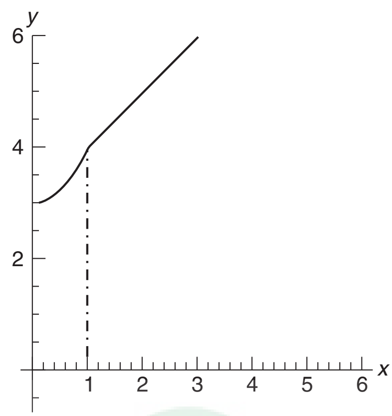


Fig. 3.13

Example 11 Show that $f(x) = \begin{cases} x^n \sin \dfrac{1}{x} & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous and differentiable at $x = 0$.

Solution For continuity

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^n \sin \frac{1}{x} = f(0) = 0$$

so the function is continuous

For derivative

$$f'(0) = \lim_{h \to 0} \frac{(0 + h)^n \sin \dfrac{1}{(0 + h)} - f(0)}{h} = \lim_{h \to} \frac{h^n \sin \dfrac{1}{h}}{h} \qquad (3.3)$$

Equation (3.3) shows that the limit exist when $n \geq 2$, hence it is observed that the given function is continuous when $n > 0$, but differentiable, when $n \geq 2$.

Some differentiation formulas

1. Let $f(x) = c$, a constant function. Then $f'(x) = 0$.
   In Fig. 3.14 a constant function $f(x) = c$ is a horizontal line with a slope of zero.

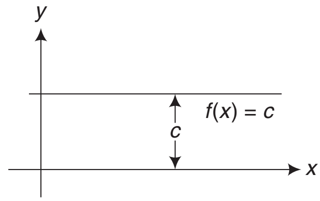


Fig. 3.14

We have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = 0$$

The converse of this also true-namely if $f'(x) = 0$ on an interval then f is constant function on that interval.

2. If n is a positive integer, then x^n is differentiable and $\frac{d(xn)}{dx} = nx^{n-1}$.

Proof: $$f'(x) = \frac{d(x^n)}{dx} = \lim_{t\to x} \frac{f(t) - f(x)}{t - x} = \lim_{t\to x} \frac{t^n - x^n}{t - x}$$

$$\lim_{t\to x} \frac{(t^{n-1} + t^{n-2}x + t^{n-3}x^2 + \ldots tx^{n-2} + x^{n-1})(t - x)}{t - x}$$

Here we use the formula $a^n - b^n = (a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots ab^{n-2} + b^{n-1})(a - b)$

$$x^{n-1} + x^{n-2}x + x^{n-3}x^2 + \ldots xx^{n-2} + x^{n-1} = nx^{n-1}$$

Let $f(x) = x^5$ then $f'(x) = 5x^4$

3. Let c be constant. If f and g are differentiable function then cf, $f + g$ and $f - g$ are also differentiable

(i) $\frac{d(cf)}{dx} = c\frac{df}{dx}$

(ii) $\frac{d(f \pm g)}{dx} = \frac{df}{dx} \pm \frac{dg}{dx}$.

Proof: $$\frac{d(f+g)}{dx} = \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \frac{df}{dx} + \frac{dg}{dx}$$

Let $f(x) = 3x^4 + 2x^3$, then $\frac{df}{dx} = \frac{d(3x^4)}{dx} + \frac{d(2x^3)}{dx} = 12x^3 + 6x^2$

4. **Product rule**

Let f and g be are differentiable function then fg is differentiable, and

$$\frac{d(fg)}{dx} = g\,\frac{df}{dx} + f\,\frac{dg}{dx}$$

Proof: $\frac{d(fg)}{dx} = \lim_{h \to 0} \frac{f(x+h)\, g(x+h) - f(x)\, g(x)}{h}$

$$\frac{d(fg)}{dx} = \lim_{h \to 0} \frac{f(x+h)\, g(x+h) - f(x)\, g(x+h) + f(x)\, g(x+h) - f(x)\, g(x)}{h}$$

$$= \lim_{h \to 0} g(x+h) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + f(x) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= g(x)\,\frac{df}{dx} + f(x)\,\frac{dg}{dx}.$$

Similarly, if $y = f(x)\, g(x)\, h(x)$ then

$$\frac{dy}{dx} = h(x)\, g(x)\,\frac{df}{dx} + h(x)\, f(x)\,\frac{dg}{dx} + g(x)\, f(x)\,\frac{dh}{dx}$$

5. **Quotient rule**

Let f and g be are differentiable function then $\frac{f}{g}$ is differentiable,

and $\left(\frac{f}{g}\right)' = \frac{d\left(\frac{f}{g}\right)}{dx} = \frac{g(x)\,\frac{df}{dx} - f(x)\,\frac{dg}{dx}}{g^2(x)} = \frac{gf' - fg'}{g^2}$

Proof:

$$\frac{d(fg)}{dx} = \lim_{h \to 0} \frac{f(x+h)\, g(x+h) - f(x)\, g(x)}{h}$$

$$\frac{d\left(\frac{f}{g}\right)}{dx} = \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)\, g(x) - f(x)\, g(x+h)}{hg(x+h)\, g(x)}$$

$$= \lim_{h \to 0} \frac{f(x+h)\, g(x) - f(x)\, g(x) + f(x)\, g(x) - f(x)\, g(x+h)}{hg(x+h)\, g(x)}$$

$$= \lim_{h \to 0} \frac{\dfrac{g(x)[f(x+h) - f(x)]}{h} - \lim\limits_{h \to 0} \dfrac{f(x)[g(x+h) - g(x)]}{h}}{\lim\limits_{x \to 0} g(x+h)\, g(x)}$$

$$= \frac{g(x)\dfrac{df}{dx} - f(x)\dfrac{dg}{dx}}{g^2(x)}$$

[Here $g(x)$ is differentiable $\Rightarrow$ $g(x)$ is continuous. Also we have supposed that the value of $g(x)$ is not zero for the value of x under consideration].

Cor. Derivative of $y = x^m$ where $x \neq 0$ and m is any nonzero negative integer $y = x^m = \dfrac{1}{x^n}$ [$m = -n$, $n \in N$, use by above quotient rule where $f = 1$, and $g = x^m$ we show that $\dfrac{d(x^m)}{dx} = -nx^{n-1} = mx^{n-1}$.

6. Derivative of the inverse of an invertible function.

Let $y = f(x)$ be a differentiable function in $[a, b]$ and suppose g is a inverse function of f such that

$$y = f(x) \Leftrightarrow x = g(y)$$

We have to find a relation between $f'(x)$ and $g'(y)$ for the corresponding values of x and y.

Let Δy be the change in y corresponding to change Δx in x, as determined from $y = f(x)$. The change Δx in x corresponding to change Δy in y, as determined from $x = g(y)$. When $f'(x) \neq 0$, we have

$$1 = \frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta y} \Rightarrow \frac{\Delta x}{\Delta y} = \frac{1}{\dfrac{\Delta y}{\Delta x}}.$$

Let $\Delta x \Rightarrow 0$, Therefore $\dfrac{dx}{dy} = \dfrac{1}{\dfrac{dy}{dx}} \Rightarrow \dfrac{dy}{dx} \cdot \dfrac{dx}{dy} = 1$. Hence $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$ are reciprocal to each other.

Cor. Derivative of $y = x^{\frac{1}{n}}$ when $x > 0$, $n \in N$.

We know that the function $y = x^{\frac{1}{n}}$ is the inverse of $y^n = x$ so that $y = x^{\frac{1}{n}} \Leftrightarrow y^n = x$. Now $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1 \Rightarrow \frac{dy}{dx} ny^{n-1} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{n} y^{-(n-1)}$

$= \frac{1}{n} \left[x^{\frac{1}{n}}\right]^{-(n-1)} = \frac{1}{n} x^{\frac{1}{n}-1}, x > 0.$

Hence $\frac{d\left(x^{\frac{1}{n}}\right)}{dx} = \frac{1}{n} x^{\frac{1}{n}-1}$ $x > 0$, $n \in N$.

7. Derivative of a polynomial function.

Let $y = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + a_{n-3} x^{n-3} + \ldots a_2 x^2 + a_1 x + a_0$, then $\frac{dy}{dx} = na_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + (n-2) a_{n-2} x^{n-3} + (n-3)$ $a_{n-3} x^{n-4} + \ldots.$

8. The Derivative of composite function. [The chain rule]

Let $y = f(u)$ is a function of u and $u = g(x)$ is a function of x, so that $y(x) = (fog)(x)$. Then $\frac{du}{dx} = g'(x)$ and $\frac{dy}{du} = f'(u)$. Now what is $\frac{dy}{dx}$? To find $\frac{dy}{dx}$, suppose that a particle is moving in the xy-plane in such a way that $x = 5t$, where t stands for time. Suppose that the particle is moving with a velocity of 5 ft/sec. in the direction of x, that is $\frac{dx}{dt} = 5$. In addition, suppose that for every 1 unit change in the x direction, the particle moves 3 units in the y direction; that is $\frac{dy}{dx} = 3$ $(y = 3x)$. Now we ask, what is the velocity of particle, in feet per second, in the y direction that is what is $\frac{dx}{dt}$? we may write $\frac{dy}{dt}$ as $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ $= 3 \times 5 = 15$ ft/sec.

This result implies that if x changing 5 times as fast as t, and if y is changing 3 times as fast as x then y is changing 15 times as fast as t, Fig. 3.15.

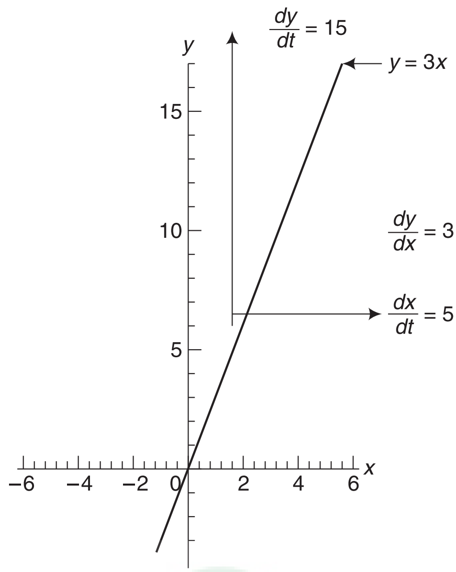


Fig. 3.15

In this example y is a function of x while x is a function of t, this means that the y is a composite function.

Let $y(x) = f(g(x))$. That is y is the composite function fog. Now we can show that

$$\frac{dy}{dx} = \frac{d(\text{fog})\ (x)}{dx} = f'(g(x))\ g'(x)$$

Proof:

$$\frac{df(g(x))}{dx} = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \left[\frac{g(x+h) - g(x)}{g(x+h) - g(x)}\right]$$

$$= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \quad \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

Now we note that as $h \to 0 \Rightarrow g(x + h) \to g(x)$, because g, being differentiable is continuous at x. Then $\Delta g = g(x + h) - g(x)$, we may write $g(x + h) = \Delta g + g(x)$ and $\Delta g \to 0$ as $h \to 0$, thus

$$= \lim_{\Delta g\to 0} \frac{f(\Delta g + g(x)) - f(g(x))}{\Delta g} \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(g(x))\ g'(x)$$

9. if $y = x^r$ where $r = \frac{p}{q}$ (p and q are integer and $q \neq 0$) then $\frac{dy}{dx} = rx^{r-1}$.

Proof: $$\frac{dy}{dx} = \frac{dx^r}{dx} = \frac{dx^{\frac{p}{q}}}{dx} = \frac{d\left(x^{\frac{1}{q}}\right)^p}{dx}$$

Let $u = x^{\frac{1}{q}}$, then $y = u^p$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = pu^{p-1}\frac{1}{q}x^{\frac{1}{q}-1}$$

$$= \frac{P}{q}\left(x^{\frac{1}{Q}}\right)^{P-1} x^{\frac{1}{Q}-1}$$

$$= \frac{p}{q}x^{\frac{p}{q}-1}$$

$$= rx^{r-1}$$

10. Derivative of trigonometry functions.

In example 3 we have show that when $f(x) = \sin x$ then $f'(x) = \cos x$ or $y = \sin x$ then $\frac{dy}{dx} = \cos x$. Similarly we can show that $\frac{d \cos x}{dx} = -\sin x$, $\frac{d \tan x}{dx} = \sec^2 x$, $d \cot x/dx = -\operatorname{cosec}^2 x$, $d \sec x/dx = \sec x \tan x$ And $\frac{d \operatorname{cosec} x}{dx} = -\operatorname{cosec} x \cot x$.

3.5 DIFFERENTIALS, DIFFERENTIAL COEFFICIENT

Let $y = f(x)$ be a differentiable function then the differential dx is an independent variable is called the differential of x, and the differential dy is

$$dy = f'(x)\, dx$$

Where the symbol "dy" is simply the dependent variable of x and dx.

The variable $f'(x)$ being the coefficient of the differential dx is known as differential coefficient, if $dx \neq 0$ then $\frac{dy}{dx}$, Fig. 3.16.

Let $y = x^2$. Since $\frac{dy}{dx} = 2x$ and $dy = 2xdx$, here $2x$ is the **Differential coefficient.** Now if we take $x = 2$ then $dy = 4dx$, this shows that if we move along the tangent line to the curve $y = x^2$ at the point (2, 4) then any change of the dx units in the horizontal direction produces a change of dy units in the vertical direction.

Relation between dx, dy and Δx, Δy.

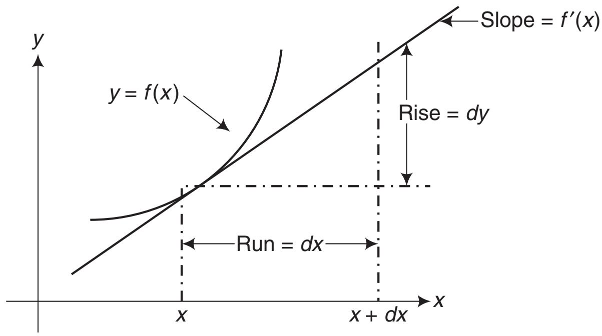


Fig. 3.16

For a given function $y = f(x)$, then change in y from its value at some initial number x to its value at a new number $x + \Delta x$ denoted as $\Delta y = f(x + \Delta x) - f(x)$ represent the change in y that occurs when we start at x and move along the curve $y = f(x)$ until we have moved Δx ($= dx$) units in the direction of x, and dy represents the change in y that occurs if we start at x and move along the tangent line until we have moved dx ($= \Delta x$) units in the direction of x, Fig. 3.17.

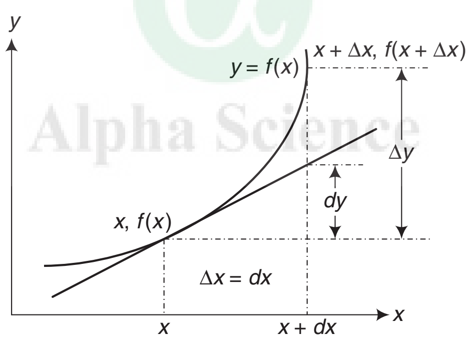


Fig. 3.17 $= x + \Delta x$

Example 12 Let $y = x^2$. Find dy and Δy at $x = 1$ with $dx = \Delta x = 2$.

Solution $\Delta y = f(x + \Delta x) - f(x) = (x + \Delta x)^2 - x^2 = (1 + 2)^2 - 1^2 \approx 8$

If $y = x^2$ the $\dfrac{dy}{dx} = 2x \Rightarrow dy = 2x\,dx$ and $dy = 2 \times 1 \times 2 = 4$.

Above results shows that Δy and dy are generally different, the differential dy will nonetheless be a good approximation for Δy provided $dx = \Delta x$ is approaches to zero to see this

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

It follows that if Δx approaches to zero, then we will have $f'(x) \approx \dfrac{\Delta y}{\Delta x}$ or,

$$\Delta y = f'(x)\ \Delta x = f'(x)\ dx = dy$$

3.6 IMPLICIT DIFFERENTIATION

An equation of the form $y = f(x)$ is said to be define y explicitly as a function of x because the variable y appears alone one side of the equation. For example, for each of functions $y = 2x + 1$, $y = 3x^2$, $y = \sqrt{2x + 1}$, $y = 3 + 2x + x^3$, the variable y appears alone on the left-hand side however, sometimes functions are defined by equation in which y is not alone on one side, for example $x^3 + y^3 = 3xy^4$, $\left(x^{\frac{3}{2}} + y^{\frac{9}{4}}\right)^3 - 6y^3 = 2$. Here x and y are not given separately. In general, we say that x and y are given implicitly if neither one is expressed as an explicitly function of the other. [This is not to say that one variable cannot be solved explicitly in terms of the other]. Let $xy = 2$ or $x^2 + y^2 = 1$, in these cases the variable x and y are given implicitly but it is easy to solve for one variable in terms of the other say $y = \dfrac{2}{x}$ and $y = \pm\sqrt{1 - x^2}$, in general it is not necessary to solve an equation for y in terms of x in order to differentiate the function defined implicitly by the equation.

Now let us consider the equation $xy = 2$ or $y = 2x$ and $\dfrac{dy}{dx} = \dfrac{-2}{x^2}$. Now we calculate the derivative another way

Let $xy = 2$ then $\dfrac{d(xy)}{dx} = \dfrac{d(2)}{dx} \Rightarrow x\dfrac{d(y)}{dx} + y\dfrac{d(x)}{dx} = 0$ or $\dfrac{dy}{dx} = -\dfrac{y}{x}$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{x^2}\ \left[y = \frac{2}{x}\right]$$

This above method of obtaining derivatives is called implicit differentiation.

Example 13 Use implicit differentiation to find $\dfrac{dy}{dx}$ if $x^3 + y^3 = 2xy$.

Solution
$$\frac{d(x^3 + y^3)}{dx} = \frac{d(2xy)}{dx}$$

$$3x^2 + 3y^2\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$$

$$\frac{dy}{dx}[3y^2 - 2x] = 2y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{2y - 3x^2}{3y^2 - 2x}.$$

Example 14 Use implicit differentiation to find $\frac{dy}{dx}$ if $3y^2 + 2 \sin x = \cos y$.

Solution $$\frac{d(3y^2)}{dx} + \frac{d(2 \sin x)}{dx} = \frac{d(\cos y)}{dx}$$

$$6y \frac{dy}{dx} + 2 \cos x = -\sin y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-2 \cos x}{6y + \sin y}.$$

Example 15 Find the $\frac{dy}{dx}$ of the following functions

(i) $y = a^x$, (ii) $y = \log \left[\frac{\sqrt{2+x} + \sqrt{2-x}}{\sqrt{2+x} - \sqrt{2-x}}\right]$,

(iii) $y = \log \left[e^{-2x} \left\{\frac{3x-1}{4x+3}\right\}^{\frac{2}{3}}\right]$, (iv) $y = \log_{\cos x} \tan x$,

(v) $= \frac{\sin x}{1 + \cos x}$, (vi) $y = \sqrt{1 + \sin^4 x}\, \sqrt{x^3}$,

(vii) $y = (x^x)^x + \left(1 + \frac{1}{x}\right)^{x^2}$, (viii) $y = \sin^{-1} \frac{2x + 3\sqrt{1 - x^2}}{\sqrt{13}}$,

(ix) $y = \tan^{-1} \left[\frac{2 \cos x - 3 \sin x}{3 \cos x + 2 \sin x}\right]$, (x) $y = \sqrt{\sin x + \sqrt{\sin x} + \ldots}$,

(xi) $2^x + 2^y = 2^{x+y}$,

(xii) $y = 2 \cos t + 2 \sin t$ and $x = 3 \cos t - \sin 3t$,

Solution (i) Let $y = a^x$, $x \in [-\infty, \infty]$, $a > 0$ we have $y = a^x \Leftrightarrow x = \log_a y$

Now $x = \log_a y \Rightarrow \frac{dx}{dy} = \frac{1}{y} \log_a e$ also $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1 \Rightarrow \frac{dy}{dx} \frac{1}{y} \log_a e = 1$

$$\Rightarrow \quad \frac{dy}{dx} = y \log_e a = a^x \log_e a.$$

Cor. Let $a = e$ so that $y = e^x$ and $\frac{dy}{dx} = e^x \log_e e = e^x$.

(ii) Let $y = \log \left[\frac{\sqrt{2+x} + \sqrt{2-x}}{\sqrt{2+x} - \sqrt{2-x}}\right]$

$$= \log \left[\frac{\sqrt{2+x} + \sqrt{2-x}}{\sqrt{2+x} - \sqrt{2-x}} \times \frac{\sqrt{2+x} + \sqrt{2-x}}{\sqrt{2+x} + \sqrt{2-x}}\right]$$

$$= \log \left[\frac{4 + 2\sqrt{4 - x^2}}{2x}\right]$$

$$= \log\left[2 + \sqrt{4 - x^2}\right] - \log\,[x]$$

$$\frac{dy}{dx} = \frac{1}{\left[2 + \sqrt{4 - x^2}\right]}\ \frac{d\left[2 + \sqrt{4 - x^2}\right]}{dx} - \frac{1}{x}$$

$$= \frac{1}{\left[2 + \sqrt{4 - x^2}\right]}\ \frac{1}{2}\,(-2x)\ [4 - x^2]^{\frac{-1}{2}} - \frac{1}{x}$$

$$= \frac{-x}{\left[2 + \sqrt{4 - x^2}\right]\sqrt{[4 - x^2]}} - \frac{1}{x}.$$

(iii) Let $y = \log\left[e^{-2x}\left\{\frac{3x - 1}{4x + 3}\right\}^{\frac{2}{3}}\right] = -2x + 2/3\ [\log\,(3x - 1) - \log\,(4x + 3)]$

$$\frac{dy}{dx} = -2 + \frac{2}{3}\left[\frac{3}{3x - 1} - \frac{4}{4x + 3}\right]$$

(iv) Let $y = \log_{\cos x} \tan x = \frac{\log[\tan x]}{\log[\cos x]}$

$$\frac{dy}{dx} = \frac{\log\,[\cos x]\ 2\csc 2x + \tan x\,\log\,[\tan x]}{(\log\,[\cos x])^2}.$$

(v) Let $y = \frac{\sin x}{1 + \cos x}$

$$\frac{dy}{dx} = \left[\frac{(1 + \cos x)\cos x - \sin x\,(-\sin x)}{(1 + \cos x)^2}\right] = \frac{1}{(1 + \cos x)}.$$

(vi) Let $y = \sqrt{1 + \sin^4 x}\ \sqrt{x^3}$

$$\frac{dy}{dx} = \frac{3}{2}\,x^{\frac{1}{2}}\,(1 + \sin^4 x)^{\frac{1}{2}} + \frac{1}{2}\,x^{\frac{3}{2}}\,(1 + \sin^4 x)^{\frac{-1}{2}}\ \frac{d(1 + \sin^4 x)}{dx}$$

$$= \frac{3}{2}\,x^{\frac{1}{2}}\,(1 + \sin^4 x)^{\frac{1}{2}} + \frac{1}{2}\,x^{\frac{3}{2}}\,(1 + \sin^4 x)^{\frac{-1}{2}}\ 4\sin^3 x\cos x$$

$$= \left[\frac{3x^{\frac{1}{2}}\,(1 + \sin^4 x) + 4x^{\frac{3}{2}}\,\sin^3 x\cos x}{2\sqrt{1 + \sin^4 x}}\right].$$

(vii) Let $y = (x^x)^x + \left(1 + \frac{1}{x}\right)^{x^2} = e^{x \log x^x} + e^{x^2}\left\{\log\left(1 + \frac{1}{x}\right)\right\}, = e^{x^2} \log x$

$+ e^{x^2} \{\log (1 + x) - \log x\}$ $\frac{dy}{dx} = e^{x^2} \log x \frac{d(x^2 \log x)}{dx} + e^{x^2}\{\log (1 + x)$

$- \log x\} \frac{d[x^2\{\log(1 + x) - \log x\}]}{dx}$

$$= (x^x)^x \left[x^2 \frac{1}{x} + 2x \log x\right] + \left(1 + \frac{1}{x}\right)^{x^2} \left[x^2 \left\{\frac{1}{x + 1} - \frac{1}{x}\right\}\right.$$

$$\left. + 2x \log \left\{1 + \frac{1}{x}\right\}\right]$$

$$= x^{x^2+x} + \log(ex^2) + \left(1 + \frac{1}{x}\right)^{x^2+1} \left[- \frac{1}{x + 1} + 2 \log\left\{1 + \frac{1}{x}\right\}\right].$$

(viii) Let $y = \sin^{-1} \left[\frac{2x + 3\sqrt{1 - x^2}}{\sqrt{13}}\right] = \sin^{-1} \left[\frac{2x}{\sqrt{13}} + \frac{3\sqrt{1 - x^2}}{\sqrt{13}}\right]$

Let $x = \cos \theta$, $\frac{2}{\sqrt{13}} = \sin \alpha$, $\frac{3}{\sqrt{13}} \cos \alpha$ then $\tan \alpha = \frac{2}{3} \Rightarrow \alpha = \tan^{-1} \frac{2}{3}$, hence

$y = \sin^{-1} [\cos \theta \sin \alpha + \cos \alpha \sin \theta] = \sin^{-1}[\sin (\alpha + \theta)] = \alpha + \theta$

$$\Rightarrow \qquad y = \tan^{-1} \frac{2}{3} + \cos^{-1} x$$

$$\frac{dy}{dx} = 0 - \frac{1}{\sqrt{1 - x^2}} = - \frac{1}{\sqrt{1 - x^2}}$$

(ix) Let $y = \tan^{-1} \left[\frac{2 \cos x - 3 \sin x}{3 \cos x + 2 \sin x}\right] = \tan^{-1} \left[\frac{\frac{2}{3} - \tan x}{1 + \frac{2 \tan x}{3}}\right]$, let $\tan \alpha = \frac{2}{3}$

$\Rightarrow \alpha = \tan^{-1} \frac{2}{3}$

$$y = \tan^{-1} \left[\frac{\tan \alpha - \tan x}{1 + \tan \alpha \tan x}\right] = \tan^{-1}[\tan(\alpha - x)] = \alpha - x$$

$$y = \tan^{-1} \frac{2}{3} - x$$

$$\frac{dy}{dx} = 0 - 1 = -1$$

(x) Let $y = \sqrt{\sin x + \sqrt{\sin x} + \dots} = \sqrt{\sin x + y}$

$$y^2 = \sin x + y$$

$$2y \frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\cos x}{2y - 1}.$$

(xi) $2^x + 2^y = 2^{x+y}$

Differentiate each term w.r. to x we have

$$2^x \log 2 + 2^y \log 2 \frac{dy}{dx} = 2^{x+y} \log 2 \left[1 + \frac{dy}{dx}\right]$$

$$[2^y - 2^{x+y}] \frac{dy}{dx} = 2^x(2^y - 1)$$

$$\frac{dy}{dx} = \frac{2^{x-y}(2^y - 1)}{(1 - 2^x)}$$

(xii) $y = 2 \cos t + 2 \sin t$ and $x = 3 \cos t - \sin 3t$

$$\frac{dy}{dt} = -2 \sin t + 2 \cos t$$

$$\frac{dx}{dt} = -3 \sin t - 3 \cos 3t$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2 \sin t + 2 \cos t}{-3 \sin t - 3 \cos 3t}$$

$$\frac{dy}{dx} = \frac{2[\sin t - \cos t]}{3[\sin t + \cos 3t]}.$$

Example 16 (i) Solve $\frac{d\sqrt{x^3 + \csc x}}{dx}$, (ii) Solve $\frac{d[\sin \sqrt{1 + \cos x}]}{dx}$,

Solution (i) *Let* $u = x^3 + \csc x$, then $\frac{d\sqrt{x^3 + \csc x}}{dx} = \frac{d\sqrt{u}}{dx} = \frac{1}{2\sqrt{u}} \frac{du}{dx}$

$$= \frac{1}{2\sqrt{x^3 + \csc x}} \frac{d(x^3 + \csc x)}{dx}$$

$$= \frac{1}{2\sqrt{x^3 + \csc x}} [3x^2 - \csc x \cot x].$$

(ii) Let $u = \sqrt{1 + \cos x}$, then $\dfrac{d[\sin \sqrt{1 + \cos x}]}{dx} = \dfrac{d[\sin u]}{dx} = \cos u \dfrac{du}{dx}$

$$= \cos \sqrt{1 + \cos x} \, \frac{d\sqrt{1 + \cos x}}{dx}$$

$$= \cos \sqrt{1 + \cos x} \, \frac{-\sin x}{2\sqrt{1 + \cos x}}.$$

3.7 THE DERIVATIVE AS A RATE OF CHANGE

Suppose that a particle is moving along x-axis so that we know its position s on that axis as a function of time t is $s = f(t)$. The displacement of the particle from time t to time $(t + \Delta t)$ is

$$\Delta s = f(t + \Delta t) - f(t),$$

Then the average velocity of the particle over that time interval is

$$v_{av} = \frac{\text{displacement}}{\text{time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

To find out the velocity at the exact instant t, we take the limit of the average velocity over the interval from t to $t + \Delta t$ as Δt tends to zero, and defined as

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \tag{3.4}$$

and the speed is $|v(t)| = \left|\dfrac{ds}{dt}\right|$

The acceleration is the derivative of the velocity w. r. to time $\Rightarrow a(t) = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$

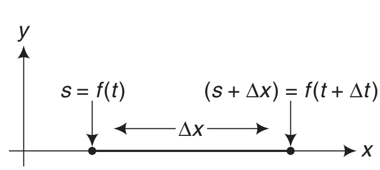


Fig. 3.18

In Fig. 3.19 we have shown that the relation between secant slope, tangent slope, average velocity and the instantaneous velocity of a moving object. The average velocity of the object which moves from P to Q is $\dfrac{500 - 300}{7 - 3} = 50$ m/sec, and its equal to the secant slope which is passes though from P and Q.

The instantaneous velocity of the particle at a time $t = 3$ is about 100 m/sec, and its equal to the slope of the tangent at P.

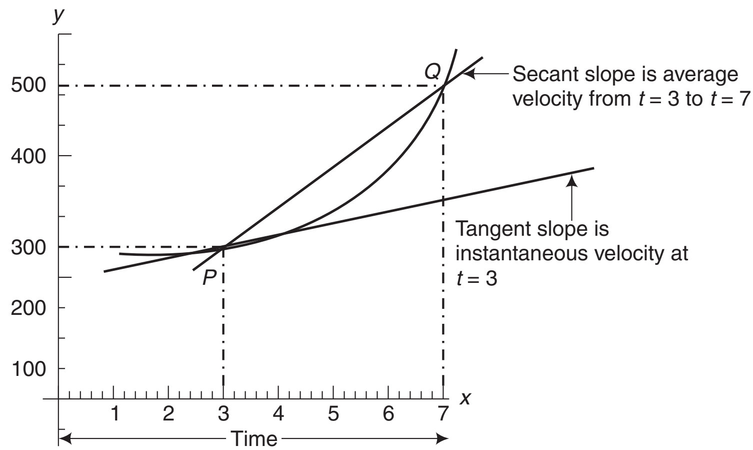


Fig. 3.19

Example 17 The position at a time t of an object moving along a line is given by

$$s(t) = 3t^3 - 40.5t^2 + 162t$$

for t on [0, 5]. Find the initial position, velocity and acceleration for the object.

Solution at $t = 0$, is the initial position so

$$s(0) = 3(0)^3 - 40.5(0)^2 + 162(0) = 0$$

Velocity $$v(t) = \frac{ds}{dt} = \frac{d(3t^3 - 40.5t^2 + 162t)}{dt}$$

From equation (3.4)

$$= 9t^2 - 81t + 162$$

$$= 9(t - 3)(t - 6) \tag{3.5}$$

The initial velocity $v(0) = 9(0)^2 - 81(0) + 162 = 162$

Now, the velocity will be zero when $t = 3$ and $t = 6$, which shows that the object will be at rest when $t = 3$ and $t = 6$. (We can also find the velocity of the object for $t = 1, 2, 4, 5$ *from the equation* (3.5)).

For the acceleration

$$a(t) = \frac{dv}{dt} = 18t - 81$$

$$= 18(t - 4.5)$$

3.8 LINEARIZATION

To solve the certain problems, it may be useful to approximate a nonlinear function by a linear function. For example to describe the motion of a simple pendulum may be greatly simplified by using the fact that if x approaches to 0, then $x \approx x$. The existence of such linear approximation provides us with a geometric interpretation of differentiability at a point x_0 then the tangent line to the graph of the function f passes through the point $P(x_0, f(x_0))$ will very closely approximate the graph of f for values of x near x_0, Fig. 3.4. In an another example, in Fig. 3.20 we can observed that the tangent line to the curve $y = x^2$ lies close to the curve near the point of tangency.

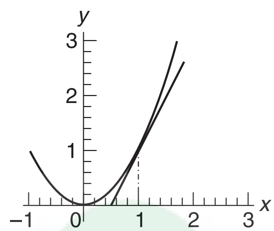


Fig. 3.20

For a small interval to either side the value of the function along the tangent line give good approximation to the value of the function on the curve, Fig. 3.21, in this figure we can see the tangent and the curve very close throughout entire interval. If a function $y = f(x)$ is differentiable at $x = a$, the tangent line at a point $P(a, f(a))$ on the graph $y = f(x)$ has slope $m = f'(a)$ and equation

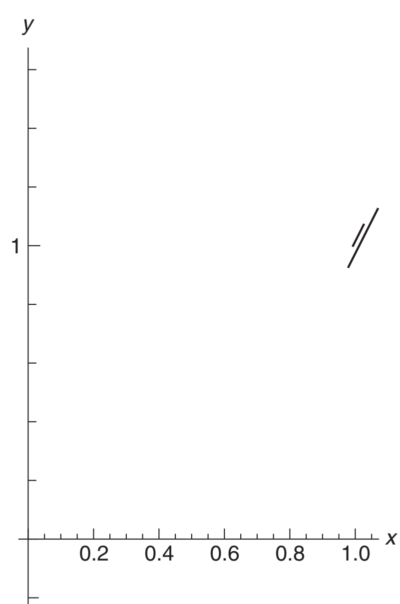


Fig. 3.21

$$\frac{y - f(a)}{x - a} = f'(a)$$

Or $$y = f(a) + f'(a)\,(x - a)$$

Thus, this tangent line is the graph of the linear function

$$L(x) = f(a) + f'(a)\,(x - a)$$

For as long as this line remain close to the graph of f, $L(x)$ gives a good approximation to $f(x)$, and is called a linearization of the function at a point $x = a$.

Example 18 Find the linearization of the following functions at $x = 0$.

(i) $f(x) = \dfrac{1}{2 - x}$, (ii) $f(x) = \sin x$.

Solution (i) $f(x) = \dfrac{1}{2 - x} = (2 - x)^{-1}$, $f'(x) = \dfrac{1}{(2 - x)^2}$, $f(0) = \dfrac{1}{2}$, $f'(0) = \dfrac{1}{4}$

Hence $L(x) = f(0) + f'(0)\,(x - 0) = \dfrac{1}{2} + \dfrac{1}{4}x$

(ii) $f(x) = \sin x$, $f'(x) = \cos x$, $f(0) = 0$, $f'(0) = 1$,

Hence $L(x) = f(0) + f'(0)\,(x - 0) = 0 + 1(x - 0) = x$.

Exercises

1. Show that

$$f(x) = \begin{cases} x^2 + x + 1, & x \le 1 \\ 3x, & x > 1 \end{cases}$$

Is continuous at $x = 1$, determine whether f is differentiable at $x = 1$.

2. Let

$$f(x) = \begin{cases} x^2, & x \le 1 \\ \sqrt{x}, & x > 1 \end{cases}$$

determine whether f is differentiable at $x = 1$. If so, then find the value of the derivative.

3. Discusse the continuity and derivability of the function of

$$f(x) = \begin{cases} 1 + x, & x \le 0 \\ x, & 0 < x < 1 \\ 2 - x, & 1 \le x \le 2 \\ 3x - x^2, & x > 2 \end{cases}$$

At $x = 0, 1, 2$.

4. A function f is defined as

$$f(x) = \begin{cases} \frac{1}{2}(b^2 - a^2), & 0 \le x \le a \\ \frac{1}{2}b^2 - \frac{x^2}{6} - \frac{a^3}{3x}, & a < x \le b \\ \frac{1}{3}\left[\frac{b^3 - a^3}{x}\right], & x > b \end{cases}$$

Prove that f *and* f' are continuous but f'' is discontinuous.

5. Show that

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

Is differentiable at $x = 0$.

6. Show that the function

$$f(x) = \begin{cases} -2x, & x < 1 \\ \sqrt{x} - 3, & x \ge 1 \end{cases}$$

Is continuous but not differentiable at $x = 1$.

7. Find the $\frac{dy}{dx}$ of the following functions

(i) $y = x^{-3} + \frac{1}{x^7}$, (ii) $y = (x^3 + 7x^2 - 8)\left(\frac{2}{x^3} + \frac{1}{x^4}\right)$,

(iii) $y = \left(x + \sqrt{a^2 + x^2}\right)^n$. (iv) $y = \frac{1 + x^2}{x^2 - 3x + 2}$,

(v) $\frac{1 + x^3}{(x^2 - 1)(x^3 - 1)}$, (vi) $y = \sqrt{(1 + x)(1 - x)}$,

(vii) $y = \log\left[\frac{\sqrt{1 - x^2} + x}{\sqrt{1 + x^2} - x}\right]$, (viii) $y = \log\left[\sqrt{a^2 + x^2} + x\right]$,

(ix) $y = \log \tan\left[\frac{x}{2} + \frac{\pi}{4}\right]$, (x) $y = \log_{10}(\sin^{-1} x^2)$,

(xi) $y = \log\left[\frac{a + b\tan\theta}{a - b\tan\theta}\right]$, (xii) $y = (\log x)^x + (\sin^{-1} x)^{\sin x}$,

(xiii) $y = \tan^{-1}\left[\frac{1 + \sin x}{1 - \sin x}\right]^{\frac{1}{2}}$, (xiv) $y = \sec^{-1}\left[\frac{1}{2x^2 - 1}\right]$,

(xv) $y = \cos^{-1}\left[\dfrac{x - x^{-1}}{x + x^{-1}}\right]$, (xvi) $y = \sin^{-1}\left[2ax\sqrt{1 - a^2x^2}\ \right]$,

(xvii) $y = \sin^{-1}\dfrac{x^2}{\sqrt{a^4 + x^4}}$, (xviii) $y = \csc^{-1}(\sin x)$,

(xix) $y = x^3 \sin^2(5x)$, (xx) $y = \cos^3(\sin 2x)$,

(xxi) $y = \left[\dfrac{x - 5}{2x + 1}\right]^3$, (xxii) $y = \cos\sqrt{x^2 + 3x + 4}$,

(xxiii) $y = \dfrac{x\cos x}{\sqrt{1 + x^2}}$,

(xxiv) $y = \sin^{-1}(3x - 4x^3) + \cos^{-1}(4x^3 - 3x)$.

(xxv) $y = |\sin x|$, $(-\pi < x < \pi)$.

8. If $x = 2t + 3$ and $y = t^2 - 1$, find the value of $\dfrac{dy}{dx}$ at $t = 6$.

9. If $x = a\left[\cos t + \log\tan\dfrac{t}{2}\right]$ and $y = a\sin t$, find the value of $\dfrac{dy}{dx}$.

10. If $x = \dfrac{2at^2}{1 + t^2}$ and $y = \dfrac{2at^3}{1 + t^2}$, find the value of $\dfrac{dy}{dx}$.

11. If $x = \sin^{-1}\sqrt{\dfrac{t^2}{1 + t^2}}$ and $y = \cos t\sqrt{\cos 2t}$, find the value of $\dfrac{dy}{dx}$.

12. Find the $\dfrac{dy}{dx}$ when

(i) $\sin(xy) + \dfrac{x}{y} = x^2 - y$, (ii) $(x^2 + y^2)^2 = xy$,

(iii) $x^2y + 3xy^3 - x = 3$, (iv) $\tan^3(y^2x + y) = x$.

(v) $x\sqrt{1 + y} + y\sqrt{1 + x} = 0$, (vi) $x^m y^m = (x + y)^{m+n}$,

(vii) $(\cos x)^2 = (\sin y)^x$, (viii) $x = e^{\tan^{-1}\left[\frac{y - x^2}{x^2}\right]}$,

(ix) $Y = (\cos x)^{(\cos x)^{(\cos x)^{\cdots}}}$,

13. Find the derivative of $\sin^{-1}\left[\dfrac{1 - x}{1 + x}\right]$ with respect to $\sqrt{x}$.

14. Find the derivative of $u = x^{\sin x}$ with respect to $(\sin x)^x$.

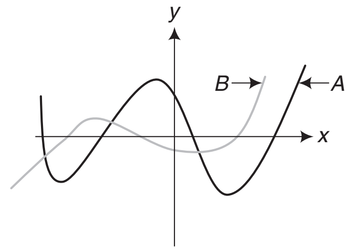
15. Find the derivative of $\tan^{-1}\dfrac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{\sqrt{1 + x^2} + \sqrt{1 - x^2}}$ with respect to $\cos^{-1} x^2$.

16. If $\lambda = \left[\dfrac{bx + c}{dx + e}\right]^7$, find $\dfrac{d\lambda}{dx}$ (b, c, d, e are constant).

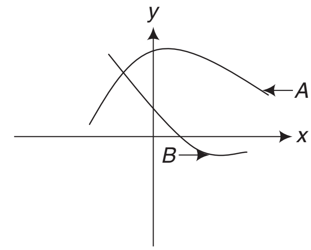
17. Find $\dfrac{d^{95}}{dx^{95}}$ [sin x] and $\dfrac{d^{19}}{dx^{19}}$ [x sin x].

18. In the following figures identify the graph of the function f and the graph of its derivative f'.

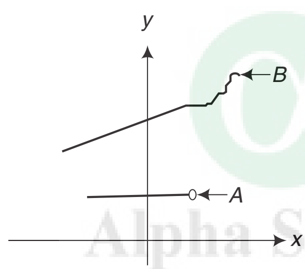
(i)



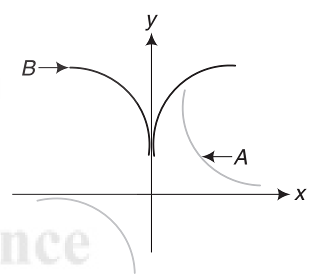
(ii)



(iii)



(iv)



19. If $s = t^3 - 2t^2 + 3t - 4$ is the position of a particle at a time t, then find the velocity and Accerelation of the particle at the end of 1, 2 seconds.

20. Fuel in a rocket burns for 3.5 minutes, in the first t seconds, the rocket reaches a height of $70t^2$ feet above the earth, what is the velocity of the rocket after 3 seconds.

21. An object moves along a straight line so that after t minutes, its position relative to its starting point (in meters) is $s = 10t + \dfrac{t}{e^t}$. At what speed is the object moving at the end of 4 minutes.

22. An object moving on the x-axis has a position $x(t) = t^3 - 9t^2 + 24t + 20$, after t seconds. What is the total distance travelled by the object during the first 8 seconds.

23. Find the linearization of the following functions at the given points.

(i) $f(x) = \sqrt{1 + x}$ at $x = 0$, (ii) $f(x) = x + \dfrac{1}{x}$ at $x = 1$,

(iii) $f(x) = \cos x$ at $x = \frac{\pi}{2}$, (iv) $f(x) = \sec x$ at $x = 0$,

(v) $f(x) = x^3 - 2x + 3$ at $x = 2$.

Answers

1. Differentiable.
2. not differentiable.
3. Continuous at 0,1 but not derivable. Discontinuous at 2.

7. (i) $-3x^{-4} - 7x^{-8}$.

(ii) $-15x^{-2} - 14x^{-3} + 48x^{-4} - 32x^{-5}$.

(iii) $\dfrac{n(x + \sqrt{a^2 + x^2})^n}{\sqrt{a^2 + x^2}}$.

(iv) $\dfrac{-3x^2 + 2x + 3}{(x^2 - 3x + 2)^2}$.

(v) $\dfrac{-2x^7 - 6x^4 + 6x^2 + 2x}{(x^2 - 1)^2 (x^3 - 1)^2}$.

(vi) $\dfrac{1}{(1 + x)^{\frac{1}{2}} (1 - x)^{\frac{3}{2}}}$.

(vii) $\dfrac{2}{\sqrt{1 + x^2}}$.

(viii) $\dfrac{1}{\dfrac{-2}{\sqrt{1 - x^2}} \sqrt{a^2 + x^2}}$.

(ix) $\sec x$.

(x) $\dfrac{2x \log_{10} e}{\sqrt{1 - x^4} \sin^{-1} x^2}$.

(xi) $\dfrac{2ab}{a^2 \cos^2 x - b^2 \sin^2 x}$.

(xii) $(\log x)^{x+1} [2 + \log x] + (\sin^{-1} x)\sin^{-1} x$.

(xiii) $\frac{1}{2}$.

(xiv) $\dfrac{-2}{\sqrt{1 - x^2}}$.

(xv) $\dfrac{-2}{1 + x^2}$.

(xvi) $\dfrac{2a}{\sqrt{1 - a^2 x^2}}$.

(xvii) $\dfrac{2a^2 x}{a^4 + x^4}$.

(xviii) $\dfrac{-\cos x}{\sin x \sqrt{\sin^2 x - 1}}$.

(xix) $10x^3 \sin 5x \cos 5x + 3x^2 \sin^2 5x$.

(xx) $- 6 \cos^2(\sin 2x) \sin(\sin 2x) \cos 2x$.

(xxi) $\dfrac{33(x - 5)^4}{(2x + 1)^4}$.

(xxii) $-\dfrac{1}{2} (2x + 3) (x^2 + 3x + 4)^{\frac{-1}{2}} \sin \sqrt{x^2 + 3x + 4}$.

(xxiii) $\dfrac{\cos x}{(1 + x^2)^{\frac{3}{2}}} - \dfrac{x \sin x}{\sqrt{1 + x^2}}$. (xxiv) 0.

(xxv) $\begin{cases} \cos x, & 0 < x < \pi \\ -\cos x, & -\pi < x < 0 \end{cases}$.

8. 6.

9. $\tan t$.

10. $\dfrac{1}{2} (t^3 + 3t)$.

11. 1.

12. (i) $\dfrac{\left[2x - \dfrac{1}{y} - \cos (xy)\right]}{\left[x \cos(xy) - \dfrac{x}{y^2} + 1\right]}$,

(ii) $\dfrac{y - 4x(x^2 + y^2)}{4y(x^2 + y^2) - x}$,

(iii) $\dfrac{1 - 2xy - 3y^3}{x^2 + 9xy^2}$,

(iv) $\dfrac{1 - 3y^2 \tan^2(xy^2 + y) \sec^2(xy^2 + y)}{3(2xy + 1) \tan^2(xy^2 + y) \sec^2(xy^2 + y)}$;

(v) $\dfrac{-1}{(1 + x)^2}$,

(vi) $\dfrac{y}{x}$,

(vii) $\dfrac{\log \sin y + y \tan x}{\log \cos x - x \cot y}$,

(viii) $2x + x[2 \tan (\log x) + \sec^2 (\log x)]$,

(ix) $\dfrac{y^2 \tan x}{y \log \cos x - 1}$.

13. $\dfrac{-2}{1 + x}$.

14. $\dfrac{x^{\sin x}\left[\dfrac{1}{x} \sin x + \cos x \log x\right]}{(\sin x)^x [x \cot x + \log (\sin x)]}$.

16. $7\left[\dfrac{bx + c}{dx + e}\right]^6 \left[\dfrac{be - cd}{(dx + e)^2}\right]$.

17. $\cos x$ and $-x \cos x - 19 \sin x$.

18. (i) Graph of f is A and of f' is B,
(ii) Graph of f is A and of f' is B,
(iii) Graph of f is B and of f' is A,
(iv) Graph of f is B and of f' is A.

19. 2, 2; 7, 8.

20. 420 ft/sec.

21. 9.91m/min.

22. 136.

23. (i) $1 + \dfrac{x}{2}$, (ii) 2, (iii) $-x + \dfrac{\pi}{2}$, (iv) 1,
(v) $10x - 13$.

4

CHAPTER

Successive Differentiation

4.1 INTRODUCTION

In chapter 3 we have seen that the derivative $\frac{dy}{dx} = f'$ of the function $y = f(x)$ with respect to x is in general also a function of x. This new function f' may also be differentiable, in which case the derivative of the first derivative is called the second derivative of the original function $y = f(x)$ and denoted as $\frac{d^2y}{dx^2} = f''$ or D^2y where $D \equiv \frac{d}{dx}$. Similarly, the derivative of the second derivative f'' is called the third derivative of f and denoted as $\frac{d^3y}{dx^3} = f'''$.

Hence, the successive derivatives of the function f are represented by the symbols,

$$f', f'', f''' \ldots, f^n, \ldots$$

Where each term is the derivative of the preceding one.

The symbols $f^n(c) = \left[\frac{d^n y}{dx^n}\right]_{x=c} = y_n(c)$ denote the value of the nth derivative of the function f at the point c. Thus if

$$y = x^3 + 3x^2 + 2, \text{ then}$$

$$\frac{dy}{dx} = f' = y_1 = 3x^2 + 6x$$

$$\frac{d^2y}{dx^2} = f'' = y_2 = 6x + 6$$

$$\frac{d^3y}{dx^3} = f''' = y_3 = 6.$$

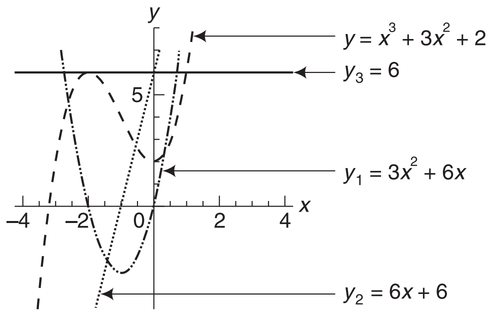


Fig. 4.1

Example 1 If $y = \sin\left[m \log\left\{x + \sqrt{x^2+1}\right\}\right]$ then show that $(x^2+1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + m^2y = 0$

Solution $$y = \sin\left[m \log\left\{x + \sqrt{x^2+1}\right\}\right]$$

We get $$y_1 = \cos\left[m \log\left\{x + \sqrt{x^2+1}\right\}\right] m \frac{\left\{1 + \dfrac{x}{\sqrt{x^2+1}}\right\}}{\left\{x + \sqrt{x^2+1}\right\}}$$

$$= \cos\left[m \log\left\{x + \sqrt{x^2+1}\right\}\right] \frac{m}{\sqrt{x^2+1}}$$

$$\left(\sqrt{x^2+1}\right) y_1 = m \cos\left[m \log\left\{x + \sqrt{x^2+1}\right\}\right]$$

$$(x^2+1)\, y_1^2 = m^2\left[1 - \sin^2\left\{m \log\left(x + \sqrt{x^2+1}\right)\right\}\right]$$

$$(x^2+1)\, y_1^2 = m^2[1 - y^2]$$

Differentiating again, we get

$$(x^2+1)\, 2y_1 y_2 + y_1^2\, 2x = -m^2\, 2yy_1$$

Hence, $$(x^2+1)\, y_2 + xy_1 + m^2y = 0.$$

Example 2 If $y = a\cos(\log x) + b\sin(\log x)$ then show that $x^2y_2 + xy_1 + y = 0$.

Solution $$y = a\cos(\log x) + b\sin(\log x)$$

We get $y_1 = \{-a\sin(\log x)\}\dfrac{1}{x} + \{b\cos(\log x\,)\}\dfrac{1}{x}$

$$\Rightarrow \qquad xy_1 = -a \sin(\log x) + b \cos(\log x)$$

Differentiating again, we get

$$xy_2 + y_1 = \{-a \cos(\log x)\} \frac{1}{x} - \{b \sin(\log x)\} \frac{1}{x}$$

Hence, $x^2 y_2 + xy_1 + y = 0$.

Example 3 If $x = \sin t$, $y = \sin pt$, prove that $(1 - x^2)y_2 - xy_1 + p^2 y = 0$.

Solution $\qquad x = \sin t$

$$\frac{dx}{dt} = \cos t, \ \frac{dy}{dt} = p \cos pt$$

$$\frac{dy}{dx} = y_1 = \frac{p \cos pt}{\cos t}$$

Differentiating again, we get

$$y_2 = \frac{p[\cos t(-p \sin pt) - \cos pt(-\sin t)]}{\cos^2 t} \frac{dt}{dx}$$

Put y, y_1 and y_2 in given equation $(1 - x^2)y_2 - xy_1 + p^2 y = 0$ and we get the result.

Example 4 If third derivative of $f(x) = ax^3 + bx + c$ is 6 then find the value of a.

Solution $\qquad y = ax^3 + bx + c$ (given)

$\therefore \qquad y_1 = 3ax^2 + b$, $y_2 = 6ax$ and $y_3 = 6a$

It is given that $\qquad y_3 = 6$

Thus, $\qquad 6 = 6a \Rightarrow a = 1$.

Example 5 If $ky = \sin(x + y)$ where k is a constant, show that $y_2 = -y(1 + y_1)^3$.

Solution $\qquad ky = \sin(x + y)$

We get $\qquad ky_1 = \cos(x + y)(1 + y_1)$

$$ky_1 = (1 - k^2y^2)^{\frac{1}{2}} (1 + y_1) \tag{4.1}$$

$$\Rightarrow \qquad ky_2 = \frac{-2k^2 yy_1}{2(1 - k^2y^2)^{\frac{1}{2}}} (1 + y_1) + y_2 \sqrt{1 - k^2y^2} \tag{4.2}$$

$$\Rightarrow \qquad ky_2 = -\frac{k^2 yy_1}{\dfrac{ky_1}{(1 + y_1)}} (1 + y_1) + y_2 \frac{ky_1}{(1 + y_1)} \qquad \text{(using (4.1))}$$

$$\Rightarrow \qquad \frac{k}{(1 + y_1)} y_2 = -ky(1 + y_1)^2 \Rightarrow y_2 = -y(1 + y_1)^3.$$

Example 6 If $y = 2\cos(\log x) + 3\sin(\log x)$, then show that $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$.

Solution We have $y = 2\cos(\log x) + 3\sin(\log x)$

$$\Rightarrow \qquad \frac{dy}{dx} = [-2\sin(\log x)]\frac{1}{x} + [3\cos(\log x)]\frac{1}{x}$$

$$\Rightarrow \qquad x\frac{dy}{dx} = -2\sin(\log x) + 3\cos(\log x)$$

Differentiating again w.r.t., we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x}[2\cos(\log x) + 3\sin(\log x)]$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{y}{x} \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

4.2 SOME STANDARD RESULTS OF NTH DERIVATIVES

1. Let

$$y = (ax+b)^m$$

$$y_1 = m(ax+b)^{m-1}\, a$$

$$y_2 = m(m-1)\,(ax+b)^{m-2}\, a^2$$

$$y_3 = m(m-1)\,(m-2)\,(ax+b)^{m-3}\, a^3,$$

$$y_n = m(m-1)\,(m-2)\,\ldots\,(m-n+1)\,(ax+b)^{m-n}a^n,$$

If m is positive integer then y_n can be written as

$$\boldsymbol{y_n = \frac{m!}{(m-n)!}\, a^n(ax+b)^{m-n},}$$

Cor. 1 If $m = -1$, we have

$$y_n = -1(-2)\,(-3)\,\ldots\,(-n)\,(ax+b)^{-1-n}\, a^n$$

$$\Rightarrow \qquad \boldsymbol{\frac{d^n\left(\dfrac{1}{ax+b}\right)}{dx^n} = \frac{(-1)^n\,(n!)\,a^n}{(ax+b)^{n+1}}}.$$

Cor. 2 If $y = \log(ax+b)$, then we have,

$$y_1 = \frac{a}{(ax+b)}\;\; y_2 = \frac{a^2}{(ax+b)^2} \Rightarrow \boldsymbol{y_n = a^n}\,\frac{(-1)^{n-1}\,(n-1)!a^n}{(ax+b)^n}.$$

2. Let $y = a^{mx}$, then

$$y_1 = ma^{mx} \log a,$$

$$y_2 = m^2 a^{mx} (\log a)^2 \ldots \boldsymbol{y_n = m^n a^{mx} (\log a)^n}$$

Cor. If $a = e$, we have $\dfrac{d^n(e^{mx})}{dx^n} = \boldsymbol{m^n e^{mx}}$.

3. Let $y = \sin(ax + b)$, then

$$y_1 = a\cos(ax + b) = a\sin\left(ax + b + \frac{\pi}{2}\right)$$

$$y_2 = a^2\cos\left(ax + b + \frac{\pi}{2}\right) = a^2\sin\left(ax + b + \frac{2\pi}{2}\right)$$

$$y_3 = a^3\cos\left(ax + b + \frac{2\pi}{2}\right) = a^3\sin\left(ax + b + \frac{3\pi}{2}\right).$$

…………………………………………………………

…………………………………………………………

Thus $$\boldsymbol{\frac{d^n\sin(ax + b)}{dx^n} = a^n\sin\left(ax + b + \frac{n\pi}{2}\right).}$$

Similarly,

4. $$\frac{d^n\cos(ax + b)}{dx^n} = a^n\cos\left(ax + b + \frac{n\pi}{2}\right).$$

Example 7 Find the nth derivative of $\dfrac{x^4}{(x-1)(x-2)}$.

Solution Let $y = \dfrac{x^4}{(x-1)(x-2)} = x^2 + 3x + 7 + \dfrac{15x - 14}{(x-1)(x-2)}$

$\Rightarrow$ $$y = x^2 + 3x + 7 + \frac{-1}{(x-1)} + \frac{16}{(x-2)}$$

[By partial fraction]

We use Cor. 1 of 1 in last two terms, we get,

$$y_n = -\frac{(-1)^n\,(n!)}{(x-1)^{n+1}} + 16\,\frac{(-1)n\,(n!)}{(x-2)^{n+1}}$$

(first three terms will be zero after second derivative)

Example 8 Find the nth derivative of $\tan^{-1}\left[\frac{1+x}{1-x}\right]$.

Solution Let $y = \tan^{-1}\left[\frac{1+x}{1-x}\right]$, Put $x = \tan\theta$, we get $y = \tan^{-1}\left[\frac{1+\tan\theta}{1-\tan\theta}\right]$
$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$

$$y = \frac{\pi}{4} + \theta = \frac{\pi}{4} + \tan^{-1} x$$

$$\therefore \quad y_1 = 0 + \frac{1}{1+x^2} = \frac{1}{(x-1)\,(x+1)} = \frac{1}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right], \text{ hence}$$

$$y_n = \frac{(-1)^{n-1}\,(n-1)!}{2ir^n}\,[(x-i)^{-n} - (x+i)^{-n}]$$

[To eliminate the 'i' and express the result in real form] let $x = r\cos\theta$, $1 = r\sin\theta$, so that $r = \sqrt{(1+x^2)}$ and $\theta = \tan^{-1}\left(\frac{1}{x}\right)$

$$y_n = \frac{(-1)^{n-1}\,(n-1)!}{2ir^n}\,[(\cos\theta - i\sin\theta)^{-n} - (\cos\theta + i\sin\theta)^{-n}]$$

$$y_n = \frac{(-1)^{n-1}\,(n-1)!}{2ir^n}\,[2i\sin n\theta] = y_n = \frac{(-1)^{n-1}\,(n-1)!}{r^n}\,[\sin n\theta]$$

[here we used the ***De Moivre's Theorem*** which state that $(\cos\theta \pm i\sin\theta)^n = (\cos n\theta \pm i\sin n\theta)$.

Example 9 Find the nth derivative of $\sin^2 x\cos x$.

Solution Let $y = \sin^2 x\cos x = \frac{(1-\cos 2x)}{2}\cos x$

$$= \frac{1}{4}\,[2\cos x - 2\cos 2x\cos x]$$

$$= \frac{1}{4}\,[2\cos x - (\cos 3x + \cos x)] = \frac{1}{4}\,[\cos x - \cos 3x], \text{ hence}$$

$$y_n = \frac{1}{4}\left[\cos\left(x + \frac{n\pi}{2}\right) - 3^n\cos\left(3x + \frac{n\pi}{2}\right)\right].$$

Example 10 If $y = \sin mx + \cos mx$, show that $y_n = m^n\,[1 + (-1)^n\sin(2\,mx)]$

Solution Given $y = \sin mx + \cos mx$

$$y_n = m^n\left[\sin\left(mx + \frac{n\pi}{2}\right) + \cos\left(mx + \frac{n\pi}{2}\right)\right]$$

$$\therefore\ y_n = m^n\left[\sin^2\left(mx + \frac{n\pi}{2}\right) + \cos^2\left(mx + \frac{n\pi}{2}\right) + 2\sin\left(mx + \frac{n\pi}{2}\right)\cos\left(mx + \frac{n\pi}{2}\right)\right]^{\frac{1}{2}}$$

$$y_n = m^n[1 + \sin(2mx + n\pi)]$$

$$y_n = m^n[1 + \sin 2mx \cos n\pi + \cos 2mx \sin n\pi], \text{ hence}$$

$$y_n = m^n[1 + (-1)^n \sin(2mx)] \ (\sin n\pi = 0 \ \cos n\pi = (-1)^n)$$

Example 11 If $y = x(x + 1) \log (x + 1)^3$, prove that

$$y_n = \frac{3(-1)^{n-1}(n-3)!\,(2x+n)}{(x+1)^{n-1}} \text{ if } n \geq 3.$$

Solution

$$y = x(x + 1) \log (x + 1)^3 \quad \text{(given)}$$

$$y_1 = 3[(x + 1) \log (x + 1) + x \log (x + 1) + x]$$

$$y_2 = 3\left[\log (x + 1) + 1 + \log (x + 1) + 1 + \frac{x}{x+1}\right]$$

$$y_3 = 3\left[\frac{2}{x+1} + \frac{1}{(x+1)^2}\right]$$

Differentiating $(n - 3)$ times, we get

$$y_n = 3\left[\frac{2(-1)^{n-3}(n-3)!}{(x+1)^{n-2}} + \frac{(-1)^{n-3}(n-2)!}{(x+1)^{n-1}}\right] \quad \frac{d^n\left[\frac{1}{(x+1)^2}\right]}{dx^n} = \frac{(-1)^n(n+1)!}{(x+1)^{n+2}}$$

$$y_n = \frac{3(-1)^{n-1}(n-3)!\,(2x+n)}{(x+1)^{n-1}}, \text{ Hence}$$

$$y_n = \frac{3(-1)^{n-1}(n-3)!\,(2x+n)}{(x+1)^{n-1}}.$$

Example 12 Find the nth derivative of $e^{(ax+b)} \sin x$.

Solution Let $y = e^{(ax+b)} \sin x$

$$y_1 = \cos e^{(ax+b)} + a \sin x e^{(ax+b)}$$

$$y_1 = e^{(ax+b)} [\cos x + a \sin x]$$

Put

$$1 = r \sin\theta,\ a = r\cos\theta \text{ so that } r = \sqrt{1 + a^2},\ \theta = \tan^{-1}\left(\frac{1}{a}\right)$$

$$y_1 = e^{(ax+b)} [r \sin\theta \cos x + r \cos\theta \sin x]$$

$$y_1 = re^{(ax+b)} [\sin\theta \cos x + \cos\theta \sin x]$$

$$y_1 = re^{(ax+b)} \sin(\theta + x), \text{ Now}$$

$$y_2 = re^{(ax+b)} [\cos(\theta + x) + a \sin(\theta + x)]$$

$$y_2 = r^2 e^{(ax+b)} [\sin \theta \cos (\theta + x) + \cos \theta \sin (\theta + x)]$$

$$y_2 = r^2 e^{(ax+b)} \sin (x + 2\theta)$$

$$y_3 = r^3 e^{(ax+b)} \sin (x + 3\theta), \text{ hence}$$

$$y_n = r^n e^{(ax+b)} \sin (x + n\theta)$$

$$\mathrm{y}_n = r^n e^{(ax+b)} \sin (x + n\theta).$$

Example 13 Find the nth derivative of $\sin^3 x$.

Solution Let $y = \sin^3 x = \frac{1}{4} [3 \sin x - \sin 3x]$, then

$$y_1 = \frac{1}{4} [3 \cos x - 3 \cos 3x]$$

$$y_1 = \frac{1}{4} \left[3 \sin \left(x + \frac{\pi}{2}\right) - 3 \sin \left(3x + \frac{\pi}{2}\right)\right]$$

$$y_2 = \frac{1}{4} \left[3 \cos \left(x + \frac{\pi}{2}\right) - 3^2 \cos \left(3x + \frac{\pi}{2}\right)\right]$$

$$y_2 = \frac{1}{4} \left[3 \sin \left(x + \frac{2\pi}{2}\right) - 3^2 \sin \left(3x + \frac{2\pi}{2}\right)\right], \text{ hence}$$

$$y_n = \frac{1}{4} \left[3 \sin \left(x + \frac{n\pi}{2}\right) - 3^n \sin \left(3x + \frac{n\pi}{2}\right)\right].$$

Example 14 Find the nth derivative of $e^x \sin^4 x$

Solution Let $y = e^x \sin^4 x = \frac{1}{4} e^x (1 - \cos 2x)^2$

$$= \frac{1}{4} e^x \left[1 - 2 \cos 2x + \frac{1 + \cos 4x}{2}\right]$$

$$= \frac{1}{8} (3e^x - 4e^x \cos 2x + e^x \cos 4x) \text{ hence}$$

$$y_n = \frac{1}{8} \left[3e^x - 4.5^{\frac{n}{2}} e^x \cos (2x + n \tan^{-1} 2) + 17^{\frac{n}{2}} e^x \cos (4x + n \tan^{-1} 4)\right],$$

$$\sqrt{1^2 + 2^2} = 5, \sqrt{1^2 + 4^2} = 17$$

Exercises

1. If $y = (\sin^{-1} x)^2$, then show that $\frac{d^2y}{dx^2} = 2 (1 - x^2)^{-1} [1 + x(1 - x^2)^{-1} \sin^{-1} x]$.
2. If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, then show that $(1 - x^2) \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} - y = 0$.

3. If $y = 2e^{ax} + 3e^{bx}$, then show that $\frac{d^2y}{dx^2} - (a + b)\frac{dy}{dx} + aby = 0$

4. If $y = e^{\frac{-cx}{2}}(a \cos nx + b \sin nx)$, then show that $\frac{d^2y}{dx^2} + c\frac{dy}{dx} + \left(n^2 + \frac{c^2}{4}\right)$ $y = 0$.

5. If $y = e^{nt}(a + bt)$, then show that $\frac{d^2y}{dt^2} - 2n\frac{dy}{dt} + n^2y = 0$.

6. If $p^2 = a^2 \cos^2 x + b^2 \sin^2 x$, then show that $\frac{d^2p}{dx^2} + p = \frac{a^2b^2}{p^3}$

7. If $py = \sin(x + y)$, then show that $\frac{d^2y}{dx^2} = -y\left(1 + \frac{dy}{dx}\right)^3$

8. If $y = \sin(\sin x)$, then show that $\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y \cos^2 x = 0$

9. If $ax^2 + 2hxy + by^2 = 1$, then show that $\frac{d^2y}{dx^2}. = \frac{h^2 - ab}{(hx + by)^3}$.

10. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then show that $\frac{d^2y}{dx^2}$
$= \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^3}$.

11. If $y^3 + x^3 + 3ax^2 = 1$, then show that $\frac{d^2y}{dx^2} = -\frac{2a^2x^2}{y^5}$.

12. If $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ show that $\frac{d^2y}{dx^2} = -\frac{1}{a(1 - \cos\theta)^2}$.

13. If $y = \sec x$, then show that $\left[\frac{d^2y}{dx^2}\right]_{x=\frac{\pi}{4}} = 3\sqrt{2}$.

14. If $x \cos y = y$, then show that $\frac{d^2y}{dx^2} = \frac{-\sin 2y + y(\sin^2 y + 1)}{(x \sin y - x)^3}$.

15. If $2xy - y^2 = 6$, then show that $\frac{d^2y}{dx^2} = \frac{-6}{(y - x)^3}$.

16. If $y = e^{ax}\sin^3 x$, then show that
$$y_n = \frac{3}{4}\left(\sqrt{1 + a^2}\right)^n \sin\left(x + n\tan^{-1}\frac{1}{a}\right) - \frac{1}{4}\left(\sqrt{9 + a^2}\right)^n \sin\left(3x + n\tan^{-1}\frac{1}{a}\right).$$

17. If $y = e^{(ax+b)}\cos x$, then show that $y_n = r^n e^{(ax+b)}\cos(x + n\theta)$. where $r = \sqrt{1 + a^2}$, $\tan^{-1}\left(\frac{1}{a}\right)$

18. Show that the nth derivative of $y = \dfrac{x}{(x-m)(x-l)}$ is

$\dfrac{(-1)^n\, n!}{(m-1)}\left[\dfrac{m}{(x-m)^{n+1}} - \dfrac{l}{(x-l)^{n+1}}\right]$.

19. Show that the nth derivative of $y = \dfrac{x^3}{(x-1)^3(x-2)}$ is

$(-1)^{n+1}\left[\dfrac{4n!}{(x-1)^{n+1}} + \dfrac{3(n+1)!}{(x-1)^{n+2}} + \dfrac{(n+2)!}{2(x-1)^{n+3}} - \dfrac{4n!}{(x-2)^{n+1}}\right]$.

20. Show that the nth derivative of $y = \dfrac{x}{1+x+x^2+x^3}$ is

$\dfrac{1}{2}(-1)^n\, n!\, \sin^{n+1}\theta\left[\sin(n+1)\theta - \cos(n+1)\theta + (\sin\theta + \cos\theta)^{-n-1}\right]$.

21. Show that the nth derivative of $y = \dfrac{x}{1+x+x^2}$ is

$\dfrac{(-1)^n\, n!}{r^{n+1}}\left[\cos(n+1)\theta - \dfrac{\sin(n+1)\theta}{\sqrt{3}}\right]$, where, $r = \sqrt{1+x+x^2}$,

$\theta = \cot^{-1}\dfrac{(2x+1)}{\sqrt{3}}$.

22. Show that the nth derivative of $y = \tan^{-1}\left(\dfrac{x\sin\theta}{1-x\cos\theta}\right)$ is $(-1)^{n-1}$

$(n-1)!\, \sin n\alpha\, \sin^n\alpha\, \csc^n\theta$, where $\cot\alpha = x\csc\theta - \cot\theta$.

23. Show that the nth derivative of $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ is $\dfrac{1}{2}(-1)^{n-1}$

$(n-1)!\, \sin n\theta\, \sin^n\theta$, where $\theta = \cot^{-1} x$.

24. Show that the value of r, when $y = x^r$ satisfies the equation $16x^2y_2 + 24xy_1 + y = 0$, is $r = \dfrac{-1}{4}$.

25. Show that the nth derivative of $y = \dfrac{x}{a^2+x^2}$ is $\dfrac{(-1)^n\, n!}{r^{n+1}}[\cos(n+1)\theta]$,

where $r = \sqrt{a^2+x^2}$, $\theta = \tan^{-1}\dfrac{(a)}{x}$.

26. Show that the nth derivative of $y = e^{2x}\cos x\sin^2 2x$ is

$\dfrac{1}{4}e^{2n}\left[2.5^{\frac{n}{2}}\cos\left(n\tan^{-1}\dfrac{1}{2}+x\right) - 13^{\frac{n}{2}}\cos\left(n\tan^{-1}\dfrac{3}{2}+3x\right) - 29^{\frac{n}{2}}\cos\left(n\tan^{-1}\dfrac{5}{2}+5x\right)\right]$.

27. Show that the nth derivative of $y = \cos^3 x \sin^2 x$ is

$$\frac{1}{16}\left[2\cos\left(x + \frac{n\pi}{2}\right) - 3^n \cos\left(3x + \frac{n\pi}{2}\right) - 5^n \cos\left(5x + \frac{n\pi}{2}\right)\right].$$

4.3 LEIBNITZ'S THEOREM

In this section we are going to extend the product rule of differentiation. If $u = f(x)$ and $v = g(x)$ be two functions of x, possessing derivatives of higher order then by the product rule the first derivative of $(uv)_1 = u_1v + uv_1$. $\left(\text{where the suffixes denote the order of differentiation for example } u_1 = \frac{du}{dx}\right)$.

In 1684 Leibnitz extended this product rule for the nth derivatives and this extension is known as **Leibnitz's Theorem.**

Statement: If u and v be two functions of x, possessing derivatives of higher order, then

$$(uv)_n = {}^nC_0u_nv + {}^nC_1u_{n-1}v_1 + {}^nC_2u_{n-2}v_2 + \ldots {}^nC_ru_{n-r}v_r + \ldots + {}^nC_nv_nu$$

Proof: This theorem will be proved by Mathematical induction.

We have $(uv)_1 = u_1v + uv_1 = {}^1C_0u_1v + {}^1C_1uv_1$

$$(uv)_2 = (u_1v + uv_1)_1 = u_2v + u_1v_1 + u_1v_1 + uv_2 = {}^2C_0u_2v + {}^2C_1u_1v_1 + {}^2C_2v_2u.$$

Thus, the theorem is true for $n = 1, 2$.

Let us assume that the theorem is true for $n = m$, so that we have

$$(uv)_m = {}^mC_0u_mv + {}^mC_1u_{m-1}v_1 + {}^mC_2u_{m-2}v_2 \ldots + \ldots + {}^mC_{r-1}u_{m-r+1}v_{r-1} + {}^mC_ru_{m-r}v_r + \ldots {}^mC_mv_mu.$$

Differentiating both sides, we get

$$(uv)_{m+1} = {}^mC_0u_{m+1}v + mC_0u_mv_1 + {}^mC_1u_mv_1 + {}^mC_1u_{m-1}v_2 + mC_2u_{m-1}v_2$$
$$+ mC_2u_{m-2}v_3 + \ldots {}^mC_{r-1}u_{m-r+2}v_{r-1} +$$
$$+ {}^mC_{r-1}u_{m-r+1}v_r + {}^mC_ru_{m-r+1}v_r + mC_ru_{m-r}v_{r+1} + \ldots + {}^mC_mv_{m+1}u$$
$$= {}^mC_0u_{m+1}v + (1 + {}^mC_1)u_mv_1 + ({}^mC_1 + {}^mC_2)u_{m-1}v_2 + \ldots + ({}^mC_{r-1} + {}^mC_r)u_{m-r+1}v_r + \ldots + {}^mC_mv_{m+1}u.$$

We know that

$$1 + {}^mC_1 = 1 + m = {}^{m+1}C_1,$$
$${}^mC_m = 1 = {}^{m+1}C_{m+1},$$
$${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r,$$

$$\therefore \qquad (uv)_{m+1} = {}^{m+1}C_0 u_{m+1} v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2 + \ldots$$
$$+ {}^{m+1}C_r u_{m-r+1} v_r + \ldots + {}^{m+1}C_{m+1} v_{m+1} u.$$

Thus, the theorem is true for $n = m$, it is certainly true for $n = m + 1$. It is already verified for $n = 1, 2$ and hence the theorem is true for all positive integral values of n.

Example 15 If $y = (\sin^{-1} x)^2$, then show that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - n^2 y_n = 0$

Solution $y = (\sin^{-1} x)^2$

$y_1 = 2 \sin^{-1} x \dfrac{1}{\sqrt{1 - x^2}} \Rightarrow (1 - x^2)y_1^2 = 4(\sin^{-1} x)^2 = 4y$, differentiate again we get,

$(1 - x^2)\, 2y_1 y_2 - 2xy_1^2 = 4\, y_1 \qquad (1 - x^2)\, y_2 - xy_1 - 2 = 0.$

Now differentiate $n$times, by Leibnitz Theorem we get,

$$[(1 - x^2)\, y_{n+2} - {}^nC_1 2xy_{n+1} - {}^nC_2 2\, y_n] - [xy_{n+1} + {}^nC_1 y_n] = 0$$

$$(1 - x^2)\, y_{n+2} - 2xn\, y_{n+1} - \frac{n(n-1)}{2} 2\, y_n - xy_{n+1} - ny_n = 0$$

Hence, $(1 - x^2)y_{n+2} - (2n + 1)\, xy_{n+1} - n^2 y_n = 0.$

Example 16 If $y = e^{m \sin^{-1} x}$, then show that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + m^2)y_n = 0$

Solution $y = e^{m \sin^{-1} x}$ then $y_1 = e^{m \sin^{-1} x} \dfrac{m}{\sqrt{1 - x^2}} \Rightarrow (1 - x^2)y_1^2 = m^2 y^2,$

differentiate again we get, $(1 - x^2)2\, y_1 y_2 - 2xy_1^2 = m^2 2yy_1$

$$(1 - x^2)\, y_2 - xy_1 = m^2 y.$$

Now differentiate $n$times, by Leibnitz Theorem we get,

$$[(1 - x^2)\, y_{n+2} - {}^nC_1 2xy_{n+1} - {}^nC_2 2y_n] - [xy_{n+1} + {}^nC_1 y_n] = m^2 y_n$$

$$(1 - x^2)y_{n+2} - 2xny_{n+1} - \frac{n(n-1)}{2} 2y_n - xy_{n+1} - ny_n = m^2 y_n$$

Hence, $(1 - x^2)\, y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + m^2)y_n = 0$

Example 17 If $y = a \cos (\log x) + b \sin (\log x)$,

then show that $x^2 y_{n+2} + (2n + 1)xy_{n+1} + (n^2 y_n) = 0$

Solution $\qquad y_1 = -a \sin (\log x) \dfrac{1}{x} + b \cos (\log x) \dfrac{1}{x}$

$\Rightarrow \qquad xy_1 = -\, a \sin (\log x) + b \cos (\log x)$, differentiate again

We get,

$$xy_2 + y_1 = -a\cos(\log x)\frac{1}{x} - b\sin(\log x)\frac{1}{x}$$

Now differentiate $n$times, by Leibnitz Theorem we get,

$$[x^2y_{n+2} + {}^nC_1 2xy_{n+1} + {}^nC_2 2y_n] + [xy_{n+1} + {}^nC_1 y_n] + y_n = 0$$

Hence, $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.

Example 18 If $y = \log\left(x + \sqrt{1+x^2}\right)$ then show that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$. Find $y_n(0)$

Solution $y = \log\left(x + \sqrt{1+x^2}\right)$ then $y_1 = \frac{1}{x+\sqrt{1+x^2}}\left[1 + \frac{x}{\sqrt{1+x^2}}\right]$

$= \frac{1}{\sqrt{1+x^2}}$, $\Rightarrow (1+x^2)y_1^2 = 1$ differentiate again we get,

$$(1+x^2)\,2y_1y_2 + 2xy_1^2 = 0$$

$$(1+x^2)\,y_2 + xy_1 = 0$$

Now differentiate $n$times, by Leibnitz Theorem we get,

$$[(1+x^2)y_{n+2} + {}^nC_1 2xy_{n+1} + {}^nC_2 2y_n] + [x\,y_{n+1} + {}^nC_1 y_n] = 0$$

Hence, $(1+x^2)\,y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$ Now to find $y_n(0)$ putting $x = 0$

$$\Rightarrow \qquad y_{n+2}(0) = -n^2y_n(0) \qquad (4.3)$$

We know that $y(0) = 1$, $y_1(0) = 1$, $y_2(0) = 0$

Putting $n = 1, 2, 3, \ldots$ in equation (4.3), we get

$$y_3(0) = -1^2y_1(0) = -1^2.$$

$$y_4(0) = -2^2y_2(0) = 0.$$

$$y_5(0) = -3^2y_3(0) = (-1)^2 3^2.$$

$$y_6(0) = -4^2y_4(0) = 0. \text{ So}$$

$y_n(0) = 0$ If n is even and $(-1)^{\frac{(n-1)}{2}}\, 1^2 \cdot 3^2 \cdot 5^2 \ldots \ldots (n-2)^2$, if n is odd.

Example 19 Find the nth derivative of $(x^2e^{2x}\sin 3x)$

Solution Let $y = (x^2e^2x\sin 3x)$ then

$$y_n = (e^{2x}\sin 3x)_n.\ x^2 + {}^nC_1(e^{2x}\sin 3x)_{n-1}.\ 2x + {}^nC_2(e^{2x}\sin 3x)_{n-2}.2$$

$$= (2^2 + 3^2)^{\frac{n}{2}} e^{2x} \sin\left(3x + n\tan^{-1}\frac{3}{2}\right).\ x^2 + 2xn(2^2 + 3^2)^{\frac{n-1}{2}}$$

$$e^{2x} \sin\left(3\ x + (n-1)\tan^{-1}\frac{3}{2}\right) + 2\ \frac{n(n-1)}{2}\ (2^2 + 3^2)^{\frac{n-2}{2}}$$

$$e^{2x} \sin\left(3x + (n-2)\tan^{-1}\frac{3}{2}\right).$$

$$= (13)^{\frac{n}{2}} e^{2x} \sin\left(3x + n\tan^{-1}\frac{3}{2}\right)\cdot x^2 + 2xn\ (13)^{\frac{n-1}{2}}\ e^{2x}$$

$$\sin\left(3x + (n-1)\tan^{-1}\frac{3}{2}\right) + n(n-1)\ (13)^{\frac{n-2}{2}}\ e^{2x}$$

$$\sin\left(3x + (n-2)\tan^{-1}\frac{3}{2}\right).$$

Example 20 If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$ or $y = \left[x + \sqrt{1+x^2}\right]^m$ then show that $(x^2 - 1)$ $y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

Solution $\quad y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x \Rightarrow y^{\frac{2}{m}} - 2xy^{\frac{-1}{m}} + 1 = 0$

$\therefore$
$$y^{\frac{1}{m}} = \frac{2x \pm \sqrt{4x^2 - 4}}{2} \Rightarrow y = \left(x \pm \sqrt{x^2 - 1}\right)^m, \text{ then}$$

$$y_1 = m\left(x \pm \sqrt{x^2 - 1}\right)^{m-1}\left[1 \pm \frac{x}{\sqrt{x^2 - 1}}\right]$$

$(x^2 - 1)y_1^2 = m^2y$. Differentiate again we get,

$$(x^2 - 1)2y_1y_2 + 2xy_1^2 = 2m^2yy_1$$

$$(x^2 - 1)y_2 + xy_1 - m^2y = 0.$$

Now, differentiate $n$times, by Leibnitz Theorem we get,

$$[(x^2 - 1)y_{n+2} + {}^nC_1 2xy_{n+1} + {}^nC_2 2y_n] + [xy_{n+1} + {}^nC_1 y_n] = m^2y_n$$

Hence, $(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

Example 21 If $x^{2n} = x^n\ x^n$ prove that

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2\cdot 2^2} + \frac{n^2(n-1)^2\ (n-2)^2}{1^2\cdot 2^2\cdot 3^2} + \ldots = \frac{(2n)!}{(n!)^2}.$$

Solution Let $y = x^{2n}$, then

$$y_n = 2n(2n-1)\ (2n-2)\ \ldots\ (n+1)x^{2n-n} = \frac{(2n)!}{n!}\ x^n \quad (4.4)$$

We also know that by Leibnitz theorem

$$y_n = x^n \cdot D^n(x^n) + n_{C_1} D^{n-1}(x^n) \cdot D(x^n) + {}^nC_2 D^{n-2}(x^n) \cdot D^2(x^n) \; \ldots + D^n(x^n) \cdot x^n$$

$$\left(D = \frac{dy}{dx}, \text{ and } D^n = \frac{d^n}{dx^n}\right)$$

$$= x^n(n!) + n \cdot nx^{n-1}(n!x) + \frac{n(n-1)}{2} n(n-1)\,(x^2)x^{n-2}\frac{n!}{2} \ldots + n! \cdot xn,$$

Hence, $x^n(n!)\left[1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2 \cdot 2^2} + \frac{n^2(n-1)^2\,(n-2)^2}{1^2 \cdot 2^2 \cdot 3^2}\right]$ (4.5)

Equating the R.H.S. of (4.4) and (4.5) and dividing both sides by $x^n(n!)$, we get the result.

Exercises

1. Find y_n If

   (i) $y = x^3 e^{bx}$, (ii) $y = \frac{\log x}{x}$,

   (iii) $y = e^{bx}[b^2x^2 - 2nbx + n(n+1)]$.

2. If $y = \sin(m \sin^{-1} x)$ then show that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 - m^2)\, y_n = 0$

3. If $y = e^{m\cos^{-1} x}$, then show that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + m^2)\, y_n = 0$.

4. If $y = e^{\tan^{-1} x}$, then show that $(1 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + n(n + 1)\, y_n = 0$.

5. If $y = (1 - x)^{-n}e^{-bx}$, then show that $(1 - x)y_{n+2} + (2n + 1)xy_{n+1} + nby_{n-1} = 0$.

6. If $y = \log\left(x + \sqrt{1 + x^2}\right)^2$ then show that $(1 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + n^2y_n = 0$.

7. If $= \frac{\sin^{-1} x}{\sqrt{1 - x^2}}$, then show that $(1 - x^2)y_{n+2} - (2n + 3)xy_{n+1} - (n + 1)^2y_n = 0$.

8. If $y = (x^2 - 1)^n$, then show that $(x^2 - 1)y_{n+2} + 2xy_{n+1} - n(n + 1)y_n = 0$.

9. If $= a\left(x + \sqrt{1 + x^2}\right)^m + b\left(x - \sqrt{1 + x^2}\right)^m$, then show that $(1 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0$.

10. If $\cos^{-1}\left(\frac{y}{a}\right) = \log\left(\frac{x}{n}\right)^n$ then show that $x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$.

11. If $y = e^{\frac{x^2}{2}} \cos x$, then show that $(y_{2n+2})_0 - 4n(y_{2n})_0 + 2n(2n-1)(y_{2n-2})_0 = 0$.

12. If $x = \tan(\log y)$, then show that $(1 + x^2)y_{n+1} + (2nx - 1)y_n + n(n-1)y_{n-1} = 0$.

13. If $y = e^{\mathrm{mtan}^{-1}x}$, $a_0 + a_1x + a_2x_2 + \ldots$

    Then show that $(1 + n)a_{n+1} + (n-1)a_{n-1} = ma_n$.

14. If $= Ae^{px} + Be^{qx}$, show that $y_2 - (p+q)y_1 + pqy = 0$.

15. If $y = \sin nx + \cos nx$, Then show that $y_r = n^r[1 + (-1)^r \sin 2n\mathrm{x}]^{\frac{1}{2}}$.

Answers

1. (i) $b^n e^{bx} x^3 + nb^{n-1}3x^2 + n(n-1)b^{n-2}3x + n(n-1)(n-2)\,b^{n-3}$,

   (ii) $\dfrac{(-1)^{n+1}\,n!}{x^{n+1}}\left[\log x - 1 - \dfrac{1}{2} \ldots - \dfrac{1}{n}\right]$,

   (iii) $b^{n+2} \cdot e^{bx} \cdot x^2$

5 CHAPTER

More About Derivative

5.1 INCREASING AND DECREASING FUNCTIONS

Consider the graph of a function in Fig. 5.1. The function increases in the intervals (c, x_{11}), (x_{12}, x_{13}) and (x_{14}, x_{15}) {the function increases if we moves up}, while the function decreases in the intervals (x_{11}, x_{12}) and (x_{13}, x_{14}) {the function decreases if we moves down}. In the interval (x_4, b) the function neither increasing nor decreasing. {Constant function}. Moreover, when $x = x_{11}$, x_{12}, x_{13}, x_{14} the tangent line is horizontal so that the derivative of the function at these points are zeros $(f' = 0)$. The tangent line at the points $x = x_{01}$, x_{03} and x_{05} have positive slope so that the derivative of the function at these points are greater than zero $(f' > 0)$ and the tangent line at the points $x = x_{02}$ and x_{04} have negative slope so that the derivative of the function at these points are less than zero $(f' < 0)$.To be more precise, we have the following definitions.

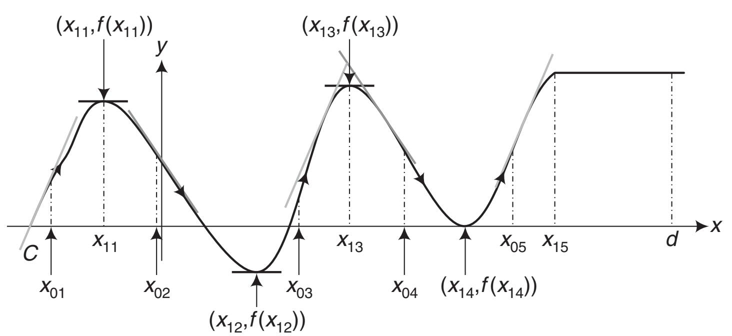


Fig. 5.1

For increasing function

1. The function f defined on an interval $[a, b]$ is said to be **increasing** on that interval if whenever $a \leq x_1 < x_2 \leq b$, we have $f(x_1) < f(x_2)$.
2. Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval $]a, b[$, and if $f'(x) > 0$ for every value of x in $[a, b]$, then function ***f* is increasing** on $[a, b]$.

   {*not need differentiability at the end points for the required result for example in the Fig. 5.3 the function f is increasing on [a, b] but not differentiable at a and b*}.

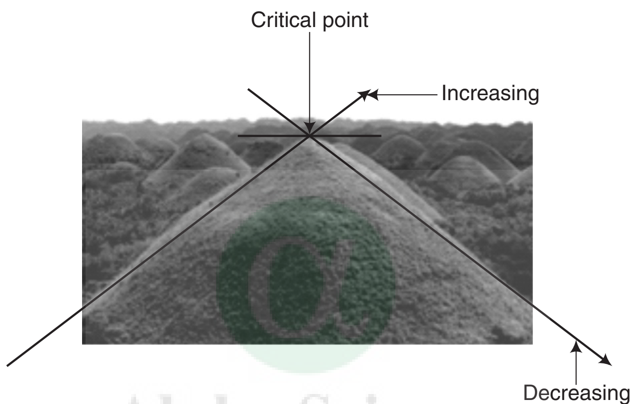


Fig. 5.2

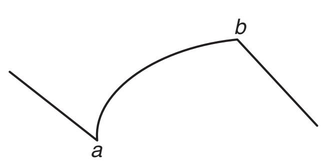
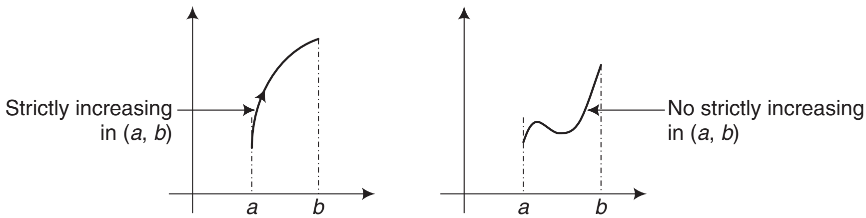


Fig. 5.3

For Decreasing function

1. The function f defined on an interval $[a, b]$ is said to be **decreasing** on that interval if whenever $a \leq x_1 < x_2 \leq b$, we have $f(x_1) > f(x_2)$.
2. Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval $]a, b[$, and if $f'(x) < 0$ for every value of x in $[a, b]$, then function ***f* is decreasing** on $[a, b]$.

   {A function f is said to be (strictly) monotonic on an interval I if it is either strictly increasing on all of I or strictly decreasing on all of I.}



For constant function

Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval $]a, b[$, and if $f'(x) = 0$ for every value of x in $[a, b]$, then function $\boldsymbol{f}$ **is constant** on $[a, b]$.

Critical point: Let f be a function that is continuous on a closed interval $[a, b]$ and let a point x_0 be in (a, b). Then the number x_0 is called a **critical point** of a function f if $\boldsymbol{f'(x_0) = 0}$ **or** $\boldsymbol{f'(x_0)}$ **does not exist**. {By this definition, a critical point x_0 is not an end point of the interval}.

In Fig. 5.1 the points x_{11}, x_{12}, x_{13}, x_{14}, x_{15} are the critical points.

Example 1 Find the value of x for which the function $f(x) = x^3 - 3x^2 - 9x + 2$ increasing and decreasing.

Solution We have $f(x) = x^3 - 3x^2 - 9x + 2$

$$f'(x) = 3x^2 - 6x - 9 = 3(x + 1)(x - 3)$$

For critical number $f'(x) = 3(x + 1)(x - 3) = 0 \Rightarrow x = -1$ and $x = 3$, hence -1 and 3 are the critical point of the function further these critical numbers divide the x-axis into three parts.

$$-\infty < x < -1,\ -1 < x < 3, \text{ and } 3 < x < \infty.$$

It is observed that $f'(x) > 0$ when $-\infty < x < -1$ and $3 < x < \infty$. We have $f'(x) < 0$ when $-1 < x < 3$. Hence the function increasing when $-\infty < x < -1$ and $3 < x < \infty$, and decreasing when $3 < x < \infty$. Fig. 5.4

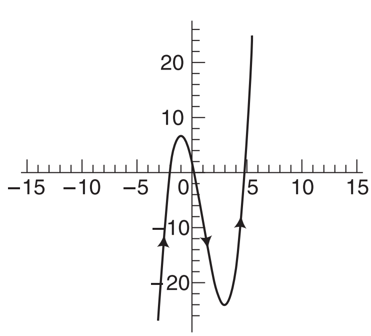


Fig. 5.4

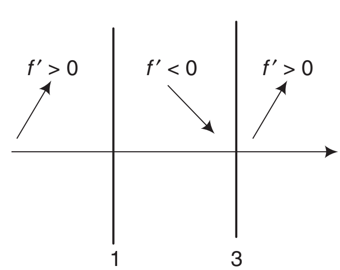


Fig. 5.5 Interval where f is increasing and decreasing

5.2 CONCAVITY

The sign of the derivative of a function f tell us where the graph of f is increasing or decreasing but not tell us about the direction of the curvature (curvature means the rate by which the curve bend or turned), for example in the Fig. 5.6 the graph of the functions are increasing but they are quite different from one another.

In Fig. 5.6(c), on both sides of the point x_0 the graph is increasing, but on the left side it has an upward curvature and on the right side it has downward curvature.

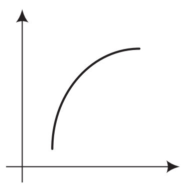


Fig. 5.6(a)

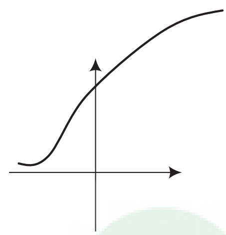


Fig. 5.6(b)

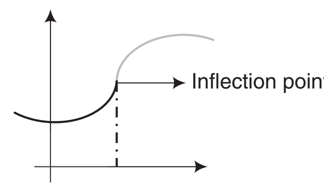


Fig. 5.6(c)

1. On intervals where the graph of f has upward curvature we say that f is **concave up**, and on intervals where the graph of f has downward curvature we say that f is **concave down**.
2. Another way to think of concavity is suggested by Fig. 5.7, the graph of f has upward curvature or **concave up** on intervals where the graph lies above the tangent lines, and it has downward curvature or **concave down** on intervals where it lies below its tangent lines.
3. We know that the derivative of f' is f'', and if $f'' > 0$, then f' has a positive derivative, and so we can conclude that f' is increasing and f is **concave up.** If $f'' < 0$ then f' is decreasing and f is **concave down.**

Theorem Let $f''(x)$ exists on an open interval I.

(i) If $f''(x) > 0$, on I, then f is **concave up** on I.

(ii) If $f''(x) < 0$, on I, then f is **concave down** on I.

Inflection point: A point $(x_0, f(x_0))$ is called a Inflection point of a function f if f changes its direction of concavity at that point x_0. Fig. 5.6(c).

Theorem: f may have a point of inflection at $(x_0, f(x_0))$, if $\boldsymbol{f''(x_0) = 0}$**, or** $\boldsymbol{f''(x_0)}$ **does not exist.**

Note: The converse of above theorem is not true in general. That is, if $f''(x_0) = 0$, then x_0 is not necessarily a point of inflection. For Example, let

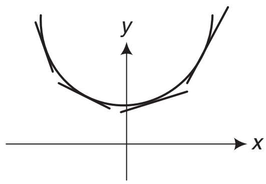


Fig. 5.7(a) Concave up

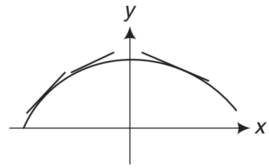


Fig. 5.7(b) Concave down

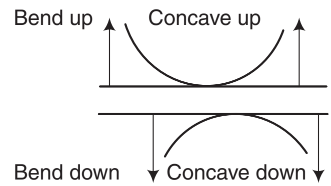


Fig. 5.7(c)

$f(x) = x^4 + 2$, then $f'(x) = 4x^3$, and $f''(x) = 12x^2$, thus $f''(x) = 0$, when $x = 0$, but this is not a point of inflection, Fig. 5.7(d).

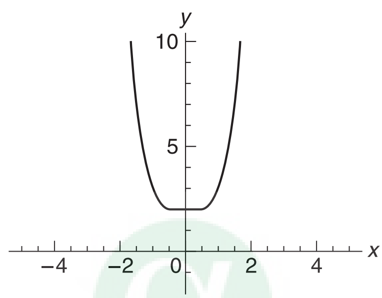


Fig. 5.7(d)

Example 2 Find the intervals on which the following functions are concave up and concave down.

(i) $f(x) = x^3 - x + 6$, (ii) $f(x) = x^3 - 3x^2 + 6$,

(iii) $f(x) = x^2 - 2x + 1$.

Solution (i) We have $f(x) = x^3 - x + 6$ then $f'(x) = 3x^2 - 1$ and $f''(x) = 6x$, Therefore, $f''(x) > 0$ *if* $x > 0$, and $f''(x) < 0$ *if* $x < 0$, hence the graph of f is concave up for $x > 0$, and concave down for $x < 0$, and point of inflection is $x = 0$ Fig. 5.8.

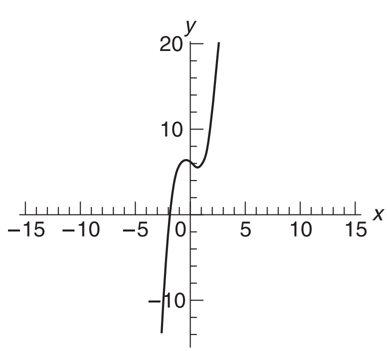


Fig. 5.8

(ii) We have $f(x) = x^3 - 3x^2 + 6$, then $f'(x) = 3x^2 - 6x$ and $f''(x) = 6(x - 1)$ Therefore, $f''(x) > 0$ if $x > 1$, and $f''(x) < 0$ if $x < 1$, hence the graph of f is concave up on $1 < x < \infty$, and concave down on $-\infty < x < 1$ and point of inflection is $x = 1$, Fig. 5.9.

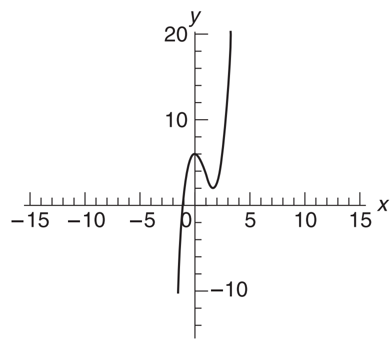


Fig. 5.9

(iii) We have $f(x) = x^2 - 2x + 1$ then $f'(x) = 2x - 2$ and $f''(x) = 2$, Therefore, $f''(x) > 0$ for all x, hence the graph of f is concave up on $-\infty < x < \infty$, Fig. 5.10.

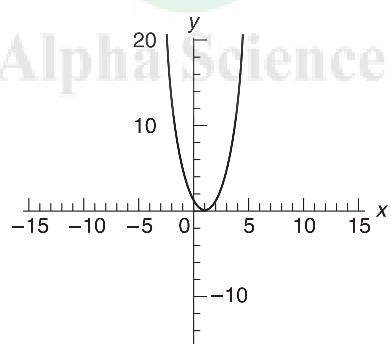


Fig. 5.10

Example 3 Find the intervals on which the function $f(x) = \cos x$ are concave up and concave down.

Solution We have $f(x) = \cos x$, then $f'(x) = -\sin x$ and $f''(x) = -\cos x$ Therefore, $f''(x) > 0$ *if* $\frac{\pi}{2} < x < \frac{3\pi}{2}$, and $f''(x) < 0$ *if* $0 < x < \frac{\pi}{2}$, and $\frac{3\pi}{2} < x < 2\pi$, hence the graph of f is concave up on $\frac{\pi}{2} < x < \frac{3\pi}{2}$ and concave down on $0 < x < \frac{\pi}{2}$, and $\frac{3\pi}{2} < x < 2\pi$, and point of inflection is $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ Fig. 5.11.

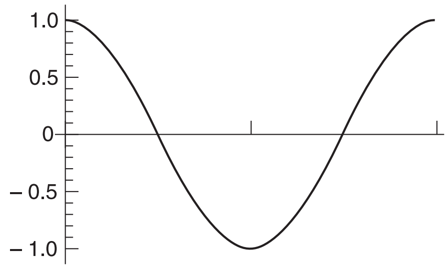


Fig. 5.11

5.3 MAXIMUM OR MINIMUM

The maximum and minimum values of a function f are also known as relatively greatest and least values of the function in that these are the greatest and least values of the function relatively to some neighborhoods of the points in question. Thus, a maximum or minimum value of f may not be the greatest or least values of the function in a finite interval. In fact a function can have several maximum and minimum values even a minimum value may be greater than a maximum value. Figure 5.12 shows that the Points x_0, x_2, x_4 and x_5 are the local or relative maximum and points x_1, x_3 and c are the local or relative minimum values of the corresponding function f.

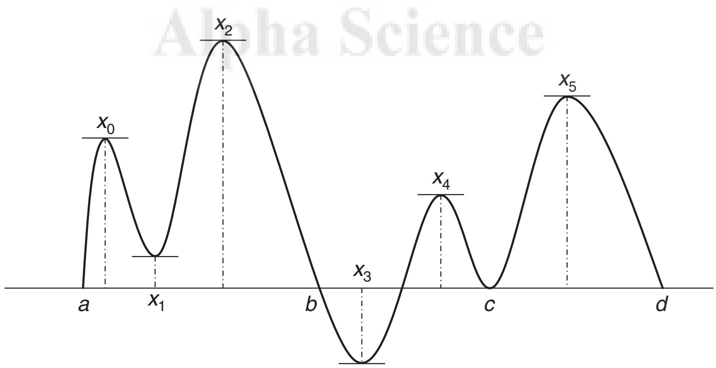


Fig. 5.12

Definition The function f has a **local maximum** at a point x_0 if there is an open interval (a, b) containing x_0 such that $f(x_0) \geq f(x)$ for every $x \in (a, b)$.

The function f has a **local minimum** at a point x_0 if there is an open interval (a, b) containing x_0 such that $f(x_0) \leq f(x)$ for every $x \in (a, b)$.

The function f has a **global maximum** at a point x_0 if $f(x_0) \geq f(x)$ for every x in domain of f.

The function f has a **global minimum** at a point x_0 if $f(x_0) \leq f(x)$ for every x in domain of f.

Note: If a function f has a maximum or minimum at a point x_0, then x_0 is called **extremum**.

In example 2(ii) function has a **local maximum** at a point $x_0 = 0$ and has a **local minimum** at a point $x_0 = 2$, while in example 3 function has a **global minimum** at a point $x_0 = 1$.

Theorem: Suppose the function f has a local maximum or local minimum at a point x_0, then x_0 is a critical point (Example 2(ii) at critical point $f'(x_0) = 0$).

The converse of this theorem is not true, for example let $f(x) = x^3$, then $f'(x) = 3x^2$, now when $x = 0$, $f'(x) = 0$, but at $x = 0$ the function has neither a local maximum nor local minimum, Fig. 5.13.

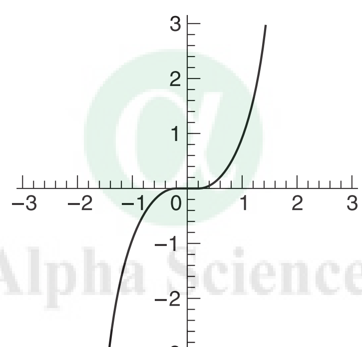
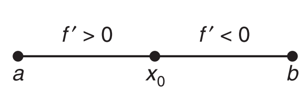


Fig. 5.13

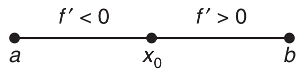
It is observed that at a critical point a function may have a local maximum, a local minimum, or neither.

Following two ways we may determine when a critical point is a local maximum or local minimum.

1. **First derivative test:** Let x_0 be a critical point of a function f with $x_0 \in (a, b)$ suppose that f is continuous on a closed interval $[a, b]$ and differentiable on open interval $]a, b[$, except possible at x_0.
   (i) if $f'(x) > 0$ for $a < x < x_0$ and $f'(x) < 0$ for $x_0 < x < b$, f has a local maximum at x_0.



(ii) if $f'(x) < 0$ for $a < x < x_0$ and $f'(x) > 0$ for $x_0 < x < b$, f has a local minimum at x_0.



(iii) if $f'(x) < 0$ for $a < x < b$ or $f'(x) > 0$ for $a < x < b$, (except possibly at x_0 itself), f has neither a local maximum nor a local minimum.

For example let $f(x) = 2x^2 - 4x + 3$, then $f'(x) = 4x - 4 = 4(x - 1)$. Since $f'(x) < 0$ for $x < 1$, $f'(x) > 0$ for $x > 1$, and $f'(1) = 0$, f has a local minimum at $x = 1$ (which is also a global minimum), Fig. 5.14.

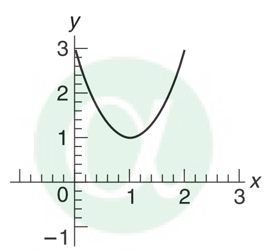


Fig. 5.14

2. **Second derivative test:** Let f be a differentiable function on an open interval I containing x_0 and suppose that $f''(x_0)$ exists.
   (i) If $f'(x_0) = 0$ and $f''(x_0) > 0$, (concave up at x_0) f has a local minimum at x_0.
   (ii) If $f'(x_0) = 0$ and $f''(x_0) < 0$, (concave down at x_0) has a local maximum at x_0.

Note: If $f'(x_0) = 0$ and $f''(x_0) = 0$, then f may have a local minimum, local maximum or neither at x_0.

Example 4 For the following functions (*a*) Find the intervals on which f is increasing, decreasing, (*b*) the open intervals on which f is concave up, concave down, (*c*) Locate the local maximum, local minimum, (*d*) Find all the values of x at which f has an inflection point.

(i) $f(x) = x^4 - 4x^3 + 4x^2 + 1$, (ii) $f(x) = x^{\frac{4}{3}} - x^{\frac{1}{3}}$,

(iii) $f(x) = \cos^2 x - 2\sin x$: $[0, 2\pi]$ (iv) $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

Solution (i) We have $f(x) = x^4 - 4x^3 + 4x^2 + 1$, then $f'(x) = 4x^3 - 12x^2 + 8x = 4x(x-1)(x-2)$ $f'(x) = 0 \Rightarrow 4x(x-1)(x-2) = 0$, $x = 0$, $x = 1$, $x = 2$.

And $f''(x) = 12x^2 - 24x + 8 = 4(3x^2 - 6x + 2)$.

Therefore $f''(x) = 0 \Rightarrow 3x^2 - 6x + 2 = 0$, or $x = \dfrac{6 \pm \sqrt{36 - 24}}{6} = 1 \pm \dfrac{1}{\sqrt{3}}$

(a) For critical number $f'(x) = 4x(x-1)(x-2) = 0$, $\Rightarrow x = 0$, $x = 1$ and $x = 2$, hence when $x = 0$, $f(x) = 1$, $x = 1$, $f(x) = 2$ and $x = 2$, $f(x) = 1$, hence (0, 1), (1, 2) and (2, 1) are the critical point of the function further these critical numbers divide the x-axis into four parts.

$-\infty < x < 0$, $0 < x < 1$, $1 < x < 2$ and $2 < x < \infty$.

It is observed that $f'(x) > 0$ when $0 < x < 1$ and $2 < x < \infty$. We have $f'(x) < 0$ when $-\infty < x < 0$, and $1 < x < 2$. Hence the function increasing when $0 < x < 1$ and $2 < x < \infty$, and decreasing when $-\infty < x < 0$, and $1 < x < 2$.

(b) $f''(x) > 0$ if $x < 1 - \dfrac{1}{\sqrt{3}}$, and $x > 1 + \dfrac{1}{\sqrt{3}}$, $f''(x) < 0$, if $1 - \dfrac{1}{\sqrt{3}} < x < 1 + \dfrac{1}{\sqrt{3}}$, hence the graph of f is concave up for $x < 1 - \dfrac{1}{\sqrt{3}}$, and $x > 1 + \dfrac{1}{\sqrt{3}}$, and concave down when $1 - \dfrac{1}{\sqrt{3}} < x < 1 + \dfrac{1}{\sqrt{3}}$.

(c) When $x = 0$, $f''(0) = 8 \Rightarrow f''(x) > 0$, when $x = 1$, $f''(1) = -4$, $\Rightarrow f''(x) < 0$ and when $x = 2$, $f''(2) = 8 \Rightarrow f''(x) > 0$, hence function have local minimum at (0, 1), (2, 1) and local maximum at (1, 2). (*d*) points of inflection are $\left(1 - \dfrac{1}{\sqrt{3}}, \dfrac{13}{9}\right)$, and $\left(1 + \dfrac{1}{\sqrt{3}}, \dfrac{13}{9}\right)$, Fig. 5.15.

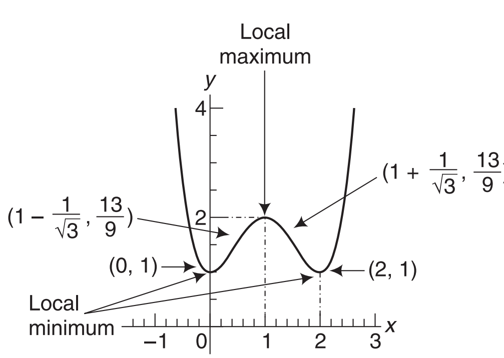


Fig. 5.15

(ii) We have $f(x) = x^{\frac{4}{3}} - x^{\frac{1}{3}}$, then $f'(x) = \dfrac{4\left(x - \frac{1}{4}\right)}{3x^{\frac{2}{3}}}$, $f'(x) = 0 \Rightarrow \left(x - \frac{1}{4}\right) = 0$, or $x = \frac{1}{4}$. And $f''(x) = \dfrac{4\left(x + \frac{1}{2}\right)}{9x^{\frac{5}{3}}}$. Therefore $f''(x) = 0 \Rightarrow \left(x + \frac{1}{2}\right) = 0$, or $x = -\frac{1}{2}$

(a) For critical number $f'(x) = \dfrac{4\left(x - \frac{1}{4}\right)}{3x^{\frac{2}{3}}} = 0 \Rightarrow x = \frac{1}{4}$, hence $\frac{1}{4}$ is the critical point of the function further these critical numbers divide the x-axis into two parts.

$-\infty < x < \frac{1}{4}$ and $\frac{1}{4} < x < \infty$.

It is observed that $f'(x) > 0$, when $\frac{1}{4} < x < \infty$. We have $f'(x) < 0$, when $-\infty < x < \frac{1}{4}$. Hence the function increasing when $\frac{1}{4} < x < \infty$, and decreasing when $-\infty < x < \frac{1}{4}$.

(b) $f''(x) > 0$, *if* $-\infty < x < -\frac{1}{2}$, and $0 < x < \infty$, $f''(x) < 0$, if $-\frac{1}{2} < x < 0$, hence the graph of f is concave up for $-\infty < x < -\frac{1}{2}$, and $0 < x < \infty$, and concave down when $-\frac{1}{2} < x < 0$.

(c) function have neither local minimum nor local maximum.

(d) points of inflection are $x = -\frac{1}{2}$, and $x = 0$.

(iii) We have $f(x) = \cos^2 - 2\sin x$, then $f'(x) = -2\sin x \cos x - 2\cos x = -2\cos x[1 + \sin x]$, $f'(x) = 0 \Rightarrow -2\cos x\, [1 + \sin x] = 0$, or $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

And $f''(x) = 2\sin x\,[1 + \sin x] - 2\cos^2 x = 4[1 + \sin x]\left[\sin x - \frac{1}{2}\right]$. Therefore $f''(x) = 0 \Rightarrow [1 + \sin x]\left[\sin x - \frac{1}{2}\right] = 0$, *or* $x = 0$, $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

(a) For critical number $f'(x) = -2\cos x\,[1 + \sin x] = 0$, or $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$ which are the critical point of the function further these critical numbers divide the x-axis into three parts.

$$0 < x < \frac{\pi}{2},\ \frac{\pi}{2} < x < \frac{3\pi}{2} \text{ and } \frac{3\pi}{2} < x < 2\pi.$$

It is observed that $f'(x) > 0$, when $\frac{\pi}{2} < x < \frac{3\pi}{2}$. We have $f'(x) < 0$ when $0 < x < \frac{\pi}{2}$, and $\frac{3\pi}{2} < x < 2\pi$. Hence the function increasing when $\frac{\pi}{2} < x < \frac{3\pi}{2}$, and decreasing when $0 < x < \frac{\pi}{2}$, and $\frac{3\pi}{2} < x < 2\pi$.

(b) $f''(x) > 0$, if $\frac{\pi}{6} < x < \frac{5\pi}{6}$, $f''(x) < 0$ if $0 < x < \frac{\pi}{6}$, and $\frac{5\pi}{6} < x < 2\pi$, hence the graph of f is concave up for $\frac{\pi}{6} < x < \frac{5\pi}{6}$, and concave down when $0 < x < \frac{\pi}{6}$, and $\frac{5\pi}{6} < x < 2\pi$.

(c) When $x = \frac{\pi}{2}$, $f''\left(\frac{\pi}{2}\right) > 0$, when $x = \frac{3\pi}{2}$, $f''\left(\frac{3\pi}{2}\right) < 0$, hence function have local minimum at $x = \frac{\pi}{2}$, and local maximum at $x = \frac{3\pi}{2}$.

(d) points of inflection are $x = \frac{\pi}{6}$, and $x = \frac{5\pi}{6}$ Fig. 5.16.

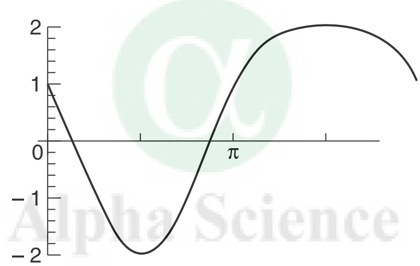


Fig. 5.16

(iv) We have $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, then $f'(x) = \frac{-x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $f'(x) = 0 \Rightarrow x = 0$, (because $e^{-\frac{x^2}{2}}$ always positive).

And $f''(x) = \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}}(x^2 - 1)e^{-\frac{x^2}{2}}$. Therefore $f''(x) = 0$ $\Rightarrow (x^2 - 1)e^{-\frac{x^2}{2}} = 0$, or $x = \pm 1$

(a) For critical number $f'(x) = \frac{-x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = 0 \Rightarrow x = 0$, and $f(0) = \frac{1}{\sqrt{2\pi}} \approx 0.4$ hence (0, 0.4) is the critical point of the function further these critical numbers divide the x-axis into two parts.

$$-\infty < x < 0.4 \text{ and } 0.4 < x < \infty.$$

It is observed that $f'(x) > 0$, when $-\infty < x < 0.4$. We have $f'(x) < 0$, when $0.4 < x < \infty$. Hence the function increasing when $-\infty < x < 0.4$, and decreasing when $0.4 < x < \infty$.

(b) $f''(x) > 0$, if $-\infty < x < -1$, and $1 < x < \infty$, $f''(x) < 0$ if $-1 < x < 1$, hence the graph of f is concave up for $-\infty < x < -1$, and $1 < x < \infty$, and concave down when $-1 < x < 1$.

(c) When $x = 0.4$, $f''(0.4) < 0$, hence function has local maximum at $x = 0.4$.

(d) when $x = 1$, and $x = -1$, $f(1) = f(-1) = 0.24$, points of inflection are $(-1, 0.24)$ and $(1, 0.24)$, Fig. 5.17.

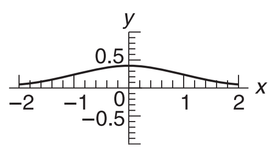


Fig. 5.17

Example 5 Find constants a, b and c that guarantee that the function $f(x) = ax^3 + bx^2 + c$, will have a relative extremum at (2, 9) and an inflection point at (1, 4).

Solution We have $f(x) = ax^3 + bx^2 + c$, then $f(x) = 3ax^2 + 2bx$, and $f''(x) = 6ax + 2b$.

For relative extremum (local maximum or minimum) we know that $f'(x) = 0$, hence

$$3ax^2 + 2bx = 0 = 3a(2)^2 + 2b.\ 2 = 12a + 4b = 4(3a + b) \quad (5.1)$$

Given points (2, 9) and (1, 4) lie on the function $f(x) = ax^3 + bx^2 + c$, we have,

$$9 = 8a + 4b + c \quad (5.2)$$

and

$$4 = a + b + c \quad (5.3)$$

from equations (5.1) and (5.2) we have,

$$9 = -4a + c \quad (5.4)$$

from equations (5.1) and (5.3) we have,

$$4 = -2a + c \quad (5.5)$$

Now from equations (5.4) and (5.5) we have, $a = -\frac{5}{2}$, put this value of a in (5.1) we have, $b = -\frac{15}{2}$, thus, from (5.4), $c = -1$.

Example 6 A rectangular sheet of metal has four equal square portions removed at the corners, and the sides are then turned up so as to form an open rectangular box. Show that when the volume contained in the box is a maximum, then the depth will be $\frac{1}{6}\left[(a + b) - (a^2 + b^2 - ab)^{\frac{1}{2}}\right]$.

Solution Let x be the length of the side of each of the squares removed and V be the volume of the box

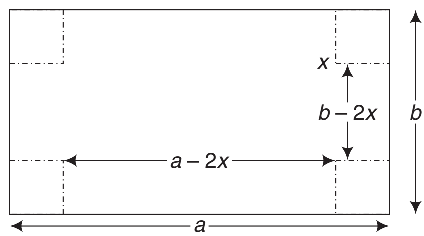


Fig. 5.18

$$\therefore \quad V = (a - 2x)(b - 2x)x$$

(x is the height of the required box shown in Fig. 5.18)

$$= 4x^3 - 2(a + b)x^2 + abx, \text{ hence}$$

$$\frac{dV}{dx} = 12x^2 - 4(a + b)x + ab, \text{ for maximum } V, \text{ have}$$

$$\frac{dV}{dx} = 12x^2 - 4(a + b)x + ab = 0, \text{ or}$$

$$x = \frac{4(a + b) \pm \sqrt{16(a + b)^2 - 48ab}}{24}$$

$$= \frac{(a + b) \pm \sqrt{a^2 + b^2 - ab}}{6}$$

$$\frac{d^2V}{dx^2} = 24x - 4(a + b) = -4\sqrt{\{(a - b)^2 + ab\}}, \text{ when}$$

$$x = \frac{(a + b) - \sqrt{a^2 + b^2 - ab}}{6}$$

Which is negative, hence V maximum when depth is $\frac{(a + b) - \sqrt{a^2 + b^2 - ab}}{6}$.

Example 7 A garden is to be laid out in a rectangular area and protected by *a* wire fence. What is the largest area of the garden if only 200 running feet of wire is available for the fence.

Solution Let x be the length, y be the width and A be the area of the given rectangle.

Then $\quad A = xy$, and $2x + 2y = 200$ (perimeter of the garden = 200).

Now $\quad A = x(100 - x) = 100x - x^2. \qquad (y = 100 - x).$

We know that x cannot be negative, and $0 < x + x \leq 200 \Rightarrow 0 < x \leq 100$.

Now $\dfrac{dA}{dx} = 100 - 2x$, for maximum A, we have

$\dfrac{dA}{dx} = 100 - 2x = 0$, *or* $x = 50$. Thus maximum A occurs when $x = 50$ or $x = 100$.

At $x = 100$, $A = 0$, and at $x = 50$, $A = 2500$ ft^2.

Hence maximum area occurs at $x = 50$.

Example 8 How should we choose the height and radius to minimize the amount of material needed to manufacture a closed cylindrical can of one liter.

Solution Let r be the radius, h be the height and S be the surface area of the given can.

The can consists of two circular disk of radius r, Fig. 5.20, and a rectangular sheet with dimensions $(h \times 2\pi r)$, Fig. 5.21, the surface area will be

$$S = 2\pi rh + \pi r^2 + \pi r^2 = 2\pi rh + 2\pi r^2 \qquad (5.6)$$

$$\text{Volum} = 1000 = \pi r^2 h \text{ or } h = 1000/\pi r^2 \ (1 \text{ liter} = 1000 \text{ cm}^3)$$

Substituting h in (5.6), we have

$$S = 2\pi r^2 + \frac{2000}{r}$$

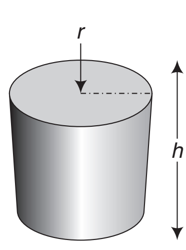


Fig. 5.19

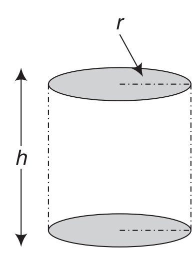


Fig. 5.20

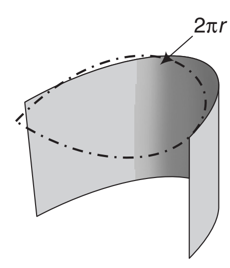


Fig. 5.21

Since S is a continuous function of r on the interval $(0, \infty)$, and for minimum S, we calculate

$$\frac{ds}{dr} = 4\pi r - \frac{2000}{r^2} = 0, \text{ or } r = \frac{10}{\sqrt[3]{2\pi}} \approx 5.4.$$

Hence $r \approx 5.4$ is the only critical number in the interval $(0, \infty)$. For this value of r, we have

$$h = \frac{1000}{\pi(5.4)^2} \approx 10.8.$$

We know that, $\frac{d^2s}{dr^2} = 4\pi + \frac{4000}{r^3} > 0$ fo $r \approx 5.4$ hence S is minimum when $r \approx 5.4$ and $h \approx 10.8$.

Example 9 Show that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}r$.

Solution Suppose O is the centre of the circle and A is the vertex of the triangle Fig. 5.22. Let $AO = x$. Now $AM = \sqrt{x^2 - r^2}$, and in triangle ABD $\frac{BD}{AD} = \tan(BAD) = \frac{OM}{AM} \Rightarrow BD = AD\,\frac{OM}{AM} = (r + x)\,\frac{r}{\sqrt{x^2 - r^2}}$

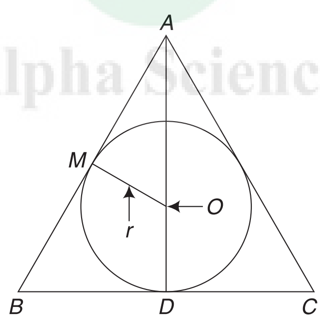


Fig. 5.22

$\therefore$ P = Perimeter of the triangle $= AB + BC + AC = 2AB + 2BD = 2AM + 2BD = 2\sqrt{x^2 - r^2} + 4(r + x)\,\frac{r}{\sqrt{x^2 - r^2}}$, now, $\frac{dP}{dx} = 3(r + x)^{\frac{1}{2}}(x - r)^{\frac{-1}{2}} - (r + x)^{\frac{3}{2}}(x - r)^{\frac{-3}{2}} = 2(r + x)^{\frac{-1}{2}}(x - r)^{\frac{-1}{2}}[x - 2r]$. For least Perimeter $\frac{dP}{dx} = 0$ $\Rightarrow x = 2r$.

We also have $\frac{dP}{dx} > 0$ when $x > 2r$ and $\frac{dP}{dx} < 0$ when $x < 2r$, hence P is minimum when $x = 2r$.

Thus $$P = 2(2r + r)^{\frac{3}{2}} (2r - r)^{\frac{-1}{2}} = 6\sqrt{3}r.$$

5.4 OPTIMIZATION IN BUSINESS, ECONOMICS, AND LIFE SCIENCES

The terms velocity [change in displacement with respect to the time] and acceleration [change in velocity with respect to the time] use to refer to the derivatives of functions describing motion of an object.

The terms rates of change and derivatives also use in Economics and in Economics they called them **marginal.**

In a manufacturing operation, the cost of production $C(x)$ is a function of x, the number of units produced. The **marginal cost of production** is the **rate of change of cost with respect to level of production,** so its $\boldsymbol{\frac{dC}{dx} = C'(x)}$**.**

Suppose to produce x kg. ghee in a week we need $C(x)$ rupees. It costs more to produce $x + h$ units per week, and the cost difference, divided by h, is the average cost of producing each additional kg.

$\frac{C(x+h) - C(x)}{h}$ = Average cost of the additional h kg. of ghee produced.

The limit of this ratio as $h \to 0$ is the marginal cost of producing more ghee per week when the current weekly production is x kg., Fig. 5.23.

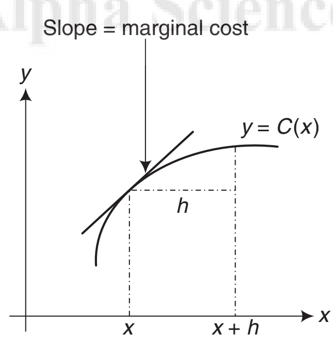


Fig. 5.23

$$\textbf{Marginal cost of production } \frac{dC}{dx} = C'(x) = \lim_{h\to 0} \frac{C(x+h) - C(x)}{h}.$$

Sometimes the marginal cost of production is defined to the extra cost of producing one unit:

$$\frac{\Delta c}{\Delta x} = \frac{C(x+1) - C(\text{x})}{1}$$

Which is approximated by the value of $\frac{dc}{dx}$ at x and this approximation is acceptable if the slope of the graph of C does not change quickly near x. The approximation works best for large value of x.

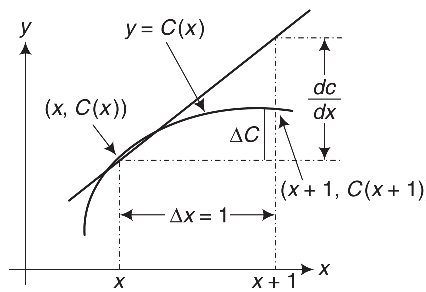


Fig. 5.24

The **demand** function $p(x)$ is defined for the price that consumer will pay for each unit of commodity when x units are brought to the market. Then the **total revenue** $R(x) = xp(x)$ derived from the sale of the x units and the **total profit** is defined as $\boldsymbol{P(x) = R(x) - C(x)}$.

Maximum Profit: Profit is maximized when marginal revenue $R'(x)$ is equals to marginal cost $C'(x)$, Fig. 5.25. To prove this.

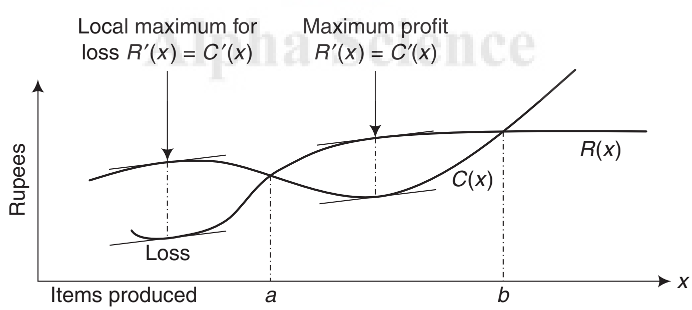


Fig. 5.25

Let $R(x)$ and $C(x)$ be differentiable for all $x > 0$, and if $P(x) = R(x) - C(x)$ has a maximum value, It occurs at a level of production at which

$$P'(x) = 0 \tag{5.7}$$

Differentiate (5.7) with respect to x, we have

$$P'(x) = R'(x) - C'(x) = 0 \Rightarrow R'(x) = C'(x).$$

In an another principle of economics the **average cost** $\boldsymbol{A(x) = \frac{C(x)}{x}}$ is related to the marginal cost as follows.

Minimum Average Cost: Average cost minimized at the level of production when $\boldsymbol{C'(x) = \dfrac{C(x)}{x}}$.

To prove Differentiate $A(x)$ with respect to x, we have

$$A'(x) = \frac{xC'(x) - C(x)}{x^2} = \frac{C'(x) - \dfrac{C(x)}{x}}{x} = \frac{C'(x) - A(x)}{x}.$$

Now for minimum average cost $A'(x) = 0$, hence we have $C'(x) = \dfrac{C(x)}{x}$.

Example 10 We give the Cost price $C(x)$ and the Selling price p when x units of commodity are produced in each of the following cases, find the level of the production that maximizes the profit.

(i) $C(x) = \frac{1}{6}x^2 + 5x + 200$, $p(x) = 40 - x$;

(ii) $C(x) = \frac{1}{5}x + 6$, $p(x) = \dfrac{70 - x}{x + 20}$

Solution (i) The marginal cost is $C'(x) = \frac{1}{3}x + 5$.

The revenue $R(x) = xp(x) = x(40 - x) = 40x - x^2$, and the marginal revenue is $R'(x) = 40 - 2x$. The profit maximized when $R'(x) = C'(x)$, $40 - 2x = \frac{1}{3}x + 5 \Rightarrow x = 15$, hence the price that corresponds the maximum profit is $p(15) = 40 - 15 = 25$.

(ii) The marginal cost is $C'(x) = \frac{1}{5}$.

The revenue $R(x) = xp(x) = x\left(\dfrac{70 - x}{x + 20}\right) = \dfrac{70x - x^2}{x + 20}$, and the marginal revenue is $R'(x) = 1400 - 40x - x^2$. The profit maximized when $R'(x) = C'(x)$, $1400 - 40x - x^2 = \frac{1}{5} \Rightarrow x \approx 22$, hence the price that corresponds the maximum profit is $P(22) \approx 1.14$.

Example 11 A manufacturer can produce a particular commodity at a cost of \$100, if manufacturer sold approximately $s(x) = 2000\ e^{-.01x}$ commodities in a week by x dollars per commodity. At what price should the manufacturer sell the commodity to maximize profit.

Solution The cost price $C(x)$ of the sold commodity in a week is $100.2000e^{-.01x}$, hence $C(x) = 100.2000e^{-.01x} = 200000e^{-.01x}$ and Revenue $R(x) = 2000xe^{-.01x}$ the marginal cost is $C'(x) = -2000e^{-.01x}$.

And the marginal revenue is $R'(x) = 2000e^{-.01x} - 20xe^{-.01x}$.

The profit maximized when

$$-2000e^{-.01x} = 2000e^{-.01x} - 20xe^{-.01x} \Rightarrow x = 200,$$

hence the price that corresponds the maximum profit is $x = 200$.

Example 12 If the total revenue (in dollars) from the sale of x units of a particular commodity is

$$R(x) = -x^2 + 34x - 64$$

Then at what level of sales is the average revenue per unit equal to the marginal revenue.

Solution The average revenue $\frac{R(x)}{x} = -x + 34 - \frac{64}{x}$, and

Marginal revenue $R'(x) = -2x + 34$. Now

$$\frac{R(x)}{x} = R'(x) = -x + 34 - \frac{64}{x} = -2x + 34 \Rightarrow x = 8.$$

Hence the average revenue per unit equal to the marginal revenue when $x = 8$

Example 13 A shopkeeper buys 4000 fish pot from a company for a year. The cost of per fish pot is \$ 5. The shopkeeper pays \$ 2 per fish pot for storage and \$ 40 for ordering fees per shipment. Suppose that the Selling price of fish pot is constant throughout the year and the next shipment arrives when the preceding shipment has been used up. How many fish pot should the shopkeeper order each time to minimize the cost price.

Solution Let x be the number of fish pot order for one shipment. Assume that the same number of fish pot order for each shipment. Suppose that the shopkeeper store the fish pot half of a given order $\left(\frac{x}{2}\right)$, and assume that storage cost remain unchanged throughout the year.

The total cost $C(x)$ = cost of fish pot + cost of storage + order cost

$$C(x) = 4000 \times 5 + \left(\frac{x}{2}\right) \times 2 + 40 \times \left(\frac{4000}{x}\right), \text{ for minimize}$$

$$C'(x) = 1 - 160000x^{-2} = 0 \Rightarrow x = \pm 400$$

Hence $x = -400 \neq \in [0, 4000]$ now we can see $C'(x) < 400$ is when $x < 400$, and $C'(x) > 0$ when $x > 400$.

Thus 400 fish pot should the shopkeeper order each time to minimize the cost price.

Example 14 A mathematical model $P(x) = Ax^s e^{-\frac{sx}{r}}$ developed by A. Lasota where A, s and r are positive constant, x is the number of granulocytes. Find the granulocytes level that maximizes the function $P(x)$.

Solution Given $P(x) = Ax^s e^{-\frac{sx}{r}}$,

$$P'(x) = Asx^{s-1} e^{-\frac{sx}{r}} - Ax^s \frac{s}{r} e^{-\frac{sx}{r}}, \text{for maximum granulocytes level } x$$

$$P'(x) = 0 = Asx^{s-1} e^{-\frac{sx}{r}} - Ax^s \frac{s}{r} e^{-\frac{sx}{r}} \Rightarrow x = r, \text{ now}$$

$$P''(x) = Ase^{-\frac{sx}{r}}\left[(s-1)x^{s-2} - \frac{2s}{r}x^{s-1} + s\frac{x^s}{r^2}\right], \text{ hence}$$

$P''(x) < 0$, when $x = r$

granulocytes level is maximum when $x = r$.

Example 15 Suppose the total cost (in dollars) of manufacturing x units of the particular commodity is

$$C(x) = x^2 + x + 64$$

At what level of production is the average cost per unit, minimum.

Solution The average cost is $A(x) = \dfrac{C(x)}{x} = x + 1 + \dfrac{64}{x}$, and $A(x)$ is minimized when

$$C'(x) = A(x). \text{ Thus } 2x + 1 = x + 1 + \frac{64}{x} \Rightarrow x = \pm 8.$$

So the minimum cost occurs, when $x = 8$.

Example 16 A company sold 6000 pieces of a commodity per month at the rate of \$ 6 per piece while the producing price is \$ 4 per piece. Due to increasing the price of raw material, the company is planning to raise the price \$ 1 per piece and estimates that 600 fewer pieces will be sold each month. At what price should the company sell the commodity to maximize the profit.

Solution Let x be the number of \$1 price increases.

The number of pieces sold = 6000 – 600 (number of \$1 increases)

$$= 6000 - 600x.$$

Now the price per piece is $6 + x$, thus the

Cost price = $(6000 - 600x)4$, and Selling price is $(6000 - 600x)(6 + x)$

Profit $p(x)$ = Revenue – Cost price

$$= (6000 - 600x)(6 + x) - (6000 - 600x)4$$

$$= 600(10 - x)(2 + x) \qquad (5.8)$$

Equation (5.8) shows that the domain of x is [0, 10], now

$$p'(x) = 600(10 - x) - 600(2 + x)$$

$$p'(x) = 1200(4-x) = 0 \Rightarrow x = 4, \text{ for maximum profit, we have}$$

$$p(0) = 12000,\ p(10) = 0 \text{ and } p(4) = 21600.$$

Hence the maximum profit is 21600, which occurs when piece sold for $6 + 4 = 10$ \$.

Exercises

1. For the following functions (a) Find the intervals on which f is increasing, decreasing, (b) The open intervals on which f is concave up, concave down, (c) Locate the local maximum, local minimum, (d) Find all the values of x at which f has an inflection point.

   (i) $f(x) = x^2 - 4x + 8$, (ii) $f(x) = x^4 - 4x^2 + 8$,

   (iii) $f(x) = x^4 - 4x^3 + 10$, (iv) $f(x) = 3x^5 - 5x^3 + 2$,

   (v) $f(x) = \sqrt[3]{x+5}$, (vi) $f(x) = 3x + \cot x$, $[0, \pi]$,

   (vii) $f(x) = \sin x + \cos x$, $[0, 2\pi]$ (viii) $f(x) = x^2 e^{-4x}$.

2. Window in the form of a rectangle surmounted semicircular opening. The total perimeter is 20 meters. Show that the dimension of the windows to admit maximum light through the whole opening is $\dfrac{40}{(\pi+4)} \times \dfrac{20}{(\pi+4)}$.

3. An open box with a square box is to be made of a metal sheet, whose area is 16 m^2. Show that the maximum volume of the box is $\dfrac{32\sqrt{3}}{9}$.

4. Show that the greatest area of a rectangle which is inscribed in a right angle triangle having sides of length 3 in, 4 in and 5 in is 3 in^2.

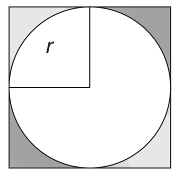
5. A container with square base and vertical sides to be made from a ft^2 of material. Show that the dimensions of the container with maximum volume is $\sqrt{\dfrac{a}{6}} \times \dfrac{\sqrt{6a}}{2}$.

6. Show that a cylinder of given volume, open at the top has minimum total surface area provided its height is equal to the radius of its base.

7. Show that the volume of greater cylinder which can be inscribed in a cone of height h and semi-vertical angle θ *is* $\dfrac{4\pi}{27}h^2\tan^2\theta$.

8. A cone is inscribed in a sphere of radius R, prove that its volume as well as its curved surface is maximum when its altitude is $\dfrac{4R}{3}$.

9. A open top of right circular cylinder of radius r and height h have a volume V. The bottom are cut from square as shown in following figure. If the shaded corners are wasted, but there is no other waste, show that the ratio r/h for the can requiring the least material including waste is $\frac{\pi}{4}$.



10. A cone shaped drinking cup made by a paper is to hold 5 cm^3 of liquid. Show that height $h = \frac{15}{\pi}\sqrt{\frac{\pi^2}{125}}$ cm. and radius $r = 6\sqrt{\frac{125}{\pi^2}}$ cm. of the cup that will require the least paper.
11. A 20 cm. long wire bent into a circle and a square. Show that $\frac{20\pi}{\pi + 4}$ cm. long wire should be used for the circle if the total area enclosed by the figure (s) is to be minimum.
12. A manufacturer estimates that when x units of a particular commodity are produced each month, the total cost (in dollars) will be $C(x) = \frac{1}{8}x^2 + 4x + 200$ and all units can be sold at a price of $p(x) = 49 - x$ dollars per unit. Show that the price that corresponds to the maximum profit is 29 dollars/units .
13. A company sells a commodity at a price of \$50 per unit. If the total production cost of x units per day is $C(x) = .002x^2 + 25x + 10000$ and if the production capacity is at most 5000 units per day, how many units of commodity be manufactured and sold daily to maximize the profit.
14. Suppose a manufacturer produced x units per day at the cost of $C(x) = 10x + 2000$. and sell S dollars per unit where $x = 500 - s$, then show that the maximum profit is 58025.
15. When we should be sold a property whose price after t years from now is $2000e^{\sqrt{t}}$ dollars when the rate of interest is 9% compounded continuously.
16. If the population of a country will be $P(t) = 100e^{0.01t}$ million after 5 years from now, then at what rate will the population be changing with respect to time after 5 years from now.

17. If demand function of a commodity is $p(x) = \dfrac{a - 2x}{b}$ for $0 \le x \le a$, and b are positive constant then find the total revenue function explicitly and determine the intervals in which the revenue function increases and decreases.

18. Determine a, b and c such that the graph of $f(x) = ax^3 + bx^2 + c$ has an inflection point (–1, 2), and slope is 1.

19. Find the numbers a, b, c and d that guarantee that the function $f(x) = ax^3 + bx^2 + cx + d$ will have a relative maximum at (1, –1), and a relative minimum at (–1, 1).

20. If the concentration in the blood at a time t of a drug injected into the body given as

$$C(t) = 544.4(e^{-0.4t} - e^{-0.5t}).$$

What time does the largest concentration occur.

Answers

1. (i) (a) increasing when $1 < x < \infty$ decreasing when $\infty < x < 1$, (b) concave up on $-\infty < x < \infty$, (c) local minimum at $x = 1$, (d) no inflection point.

(ii) (a) increasing when $-2 < x < 0$ and $2 < x < \infty$, decreasing when $-\infty < x < -2$ (b) concave up when $-\infty < x -.8164$ and $.8164 < x < \infty$, concave down when $-.8164 < x < .8164$ (c) local minimum at $x = -2$ and $x = 2$ local maximum at $x = 0$ (d) inflection point at (–.8164, 5.77) and (.8164, 5.77).

(iii) (a) increasing when $3 < x < \infty$, decreasing when $-\infty < x < 0$ and $0 < x < 3$, (b) concave up when $-\infty < x < 0$ and $2 < x < \infty$ concave down when $0 < x < 2$ (c) local minimum at $x = 3$, (d) inflection point at. (0, 10) and (2, –6).

(iv) (a) increasing when $-\infty < x < -1$, and $1 < x < \infty$, decreasing when $-1 < x < 1$, (b) concave up when $-0.7071 < x < 0$ and $0.7071 < x < \infty$, concave down when $-\infty < x < -0.7071$ and $0 < x < 0.7071$ (c) local maximum at $x = -1$ local minimum at $x = 1$, (d) inflection point at. (–0.7071, 3.237), (0, 2) and (0.7071, 0.7625).

(v) (a) f is increasing when $-\infty < x < \infty$, (b) concave up when $-\infty < x < -5$, concave down when $-5 < x < \infty$ (c) no extremum, (d) inflection point at. $x = -5$ **(vi)** (a) increasing when $\dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$, decreasing when $0 < x < \dfrac{\pi}{4}$ and $\dfrac{3\pi}{4} < x <, \pi$ (b) concave up when $0 < x < \dfrac{\pi}{2}$, concave down when $\dfrac{\pi}{2} < x < \pi$ (c) no extremum, (d) inflection point at $x = \dfrac{\pi}{2}$.

(vii) (a) increasing when $0 < x < \frac{\pi}{4}$ and $\frac{5\pi}{4} < x < 2\pi$, decreasing when $\frac{\pi}{4} < x <, \frac{5\pi}{4}$, (b) concave up when $\frac{3\pi}{4} < x < \frac{7\pi}{4}$, concave down when $0 < x < \frac{3\pi}{4}$ and $\frac{7\pi}{4} < x < 3\pi$ (c) local maximum at $x = \frac{\pi}{4}$ and local minimum at $x = \frac{5\pi}{4}$ (d) inflection point at $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$.

(viii) (a) increasing when $0 < x < .5$, decreasing when $-\infty < x < 0$ and $0.5 < x < \infty$, (b) concave up when $-\infty < x < .20$ and $0.89 < x < \infty$ concave down when $0.20 < x < .89$ (c) local minimum at $x = 0$ and local maximum at $x = 0.5$ (d) inflection point at (0.20, 0179) and .89.

13. Manufacture units is 5000 and maximum profit is 65000.
15. 31 years.
16. 1.22 million people/year.
17. Revenue function is $\left\{x\left(\frac{a-2x}{b}\right)\right\}$ increasing in $\left[0, \frac{a}{4}\right]$ and decreasing in $\left[\frac{a}{4}, a\right]$.
18. $a = \frac{-1}{3}, b = -1, c = \frac{8}{3}$.
19. $a = \frac{1}{2}, b = 0, c = -\frac{3}{2}$ and $d = 0$.
20. $t \approx 2.233$.

5.5 ASYMPTOTES

"asymptote" comes from the Greek word asymptotes meaning "nonintersecting".

Definition If the distance between the graph of a function and some fixed line approaches zero as the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an **asymptote** of the graph, Fig. 5.26.

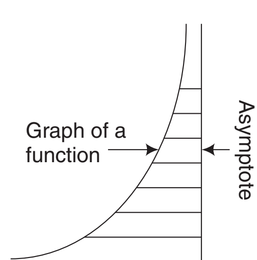


Fig. 5.26

Vertical asymptote: A line $\boldsymbol{x = k}$ is called a vertical asymptote (parallel to y-axis) of the graph of a function $f(x)$ if $f(x)$ tends to $+\infty$ or $-\infty$ as x approaches k from left or right, Fig. 5.27.

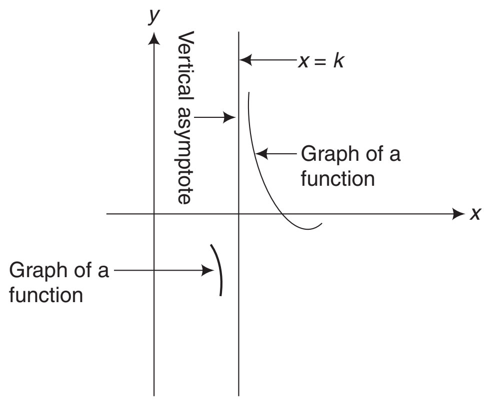


Fig. 5.27

Or

A line $\boldsymbol{x = k}$ is a vertical asymptote of the graph of a function $f(x)$ if either

$$\lim_{x\to k^+} f(x) = \pm\infty \text{ or } \lim_{x\to k^-} f(x) = \pm\infty.$$

Example 17 Find the vertical asymptote of the following functions.

(i) $f(x) = \dfrac{2x^2}{x^2 - 1}$, (ii) $f(x) = \dfrac{5}{16 - x^2}$.

Solution (i) $f(x) = \dfrac{2x^2}{x^2 - 1} = \dfrac{2x^2}{(x-1)(x+1)}$ and $f(x) \to \infty$ as $x \to \pm 1$, hence asymptotes are $x = \pm 1$, Fig. 5.28.

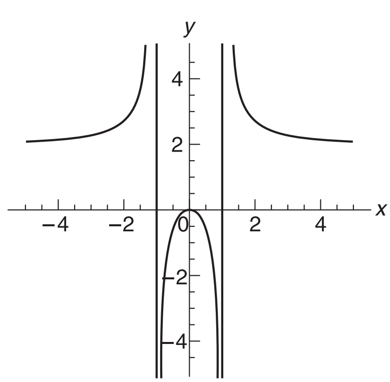


Fig. 5.28

(ii) $f(x) = \dfrac{5}{16 - x^2} = \dfrac{5}{(4 - x)(4 + x)}$ and $f(x) \to \infty$ as $x \to \pm 4$, hence asymptotes are $x = \pm 4$, Fig. 5.29.

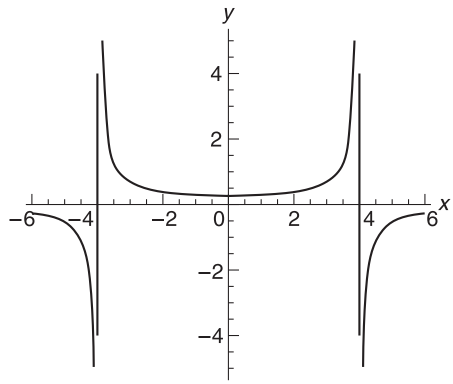


Fig. 5.29

Or

The vertical asymptotes are obtained by equating to zero the real linear factors in the coefficient of the highest power of, y in the algebraic equation of the curve.

Suppose we have an algebraic equation $(x^2 + y^2)x - by^2 = 0$, then the linear factor of the coefficient of the highest power of y is $(x - b)$ so the vertical asymptote is $x = b$.

Horizontal asymptote: A line $\boldsymbol{y = L}$ is called a horizontal asymptote (parallel to x-axis) of the graph of a function $f(x)$ if $f(x)$ tends to L as $x \to +\infty$ or $x \to -\infty$, Fig. 5.30.

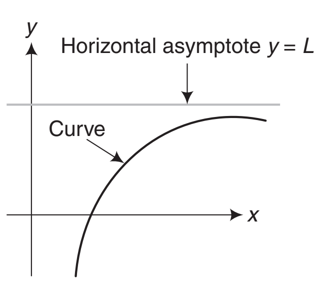


Fig. 5.30(a)

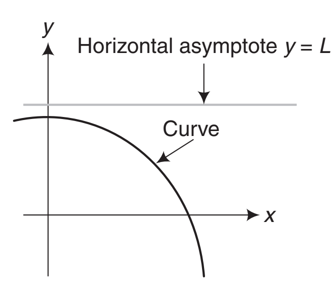


Fig. 5.30(b)

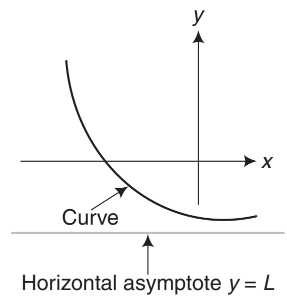


Fig. 5.30(c)

Or

A line $\boldsymbol{y = L}$ is a horizontal asymptote of the graph of a function $f(x)$ if $\lim_{x \to \pm\infty} f(x) = L$.

Example 18 Find the horizontal asymptote of the following functions.

(i) $f(x) = \dfrac{2x}{x - 5}$, (ii) $f(x) = \dfrac{x^2}{x^2 - 1}$,

(iii) $f(x) = \dfrac{3x^2 - 1}{x^2}$

Solution (i) $f(x) = \dfrac{2x}{x - 5}$. $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \dfrac{2x}{x\left(1 - \dfrac{5}{x}\right)} = 2$ and, hence asymptote is $y = 2$.

(ii) $f(x) = \dfrac{x^2}{(x^2 - 1)}$. $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \dfrac{x^2}{x^2\left(1 - \dfrac{1}{x^2}\right)} = 1$, hence asymptote is $y = 1$.

(iii) $f(x) = \dfrac{3x^2 - 1}{x^2}$. $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \dfrac{x^2\left(3 - \dfrac{1}{x^2}\right)}{x^2} = 3$, hence asymptote is $y = 3$

Or

The horizontal asymptotes are obtained by equating to zero the real linear factors in the coefficient of the highest power of, x in the algebraic equation of the curve.

Suppose we have an algebraic equation $x^2y^2 - a^2(x^2 + y^2) = 0$, then the linear factor of the coefficient of the highest power of x are $(y - a)$ and $(y + a)$ so the horizontal asymptote are $y = a$ and $y = -a$.

Example 19 Find constants a and b that guarantee that the graph of the function defined by

$$f(x) = \frac{ax^2 + 6}{bx^2 - 2}$$

Will have a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = -1$.

Solution We know that for vertical asymptote $bx^2 = 2$ and for given $x = 1$, we have $b = 2$, and for horizontal asymptote

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2\left(a + \dfrac{6}{x^2}\right)}{x^2\left(b - \dfrac{2}{x^2}\right)} = -1 \Rightarrow a = -b, \text{ hence } a = -2.$$

Note: Let

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \ldots b_1 x + b} \text{ be a rational function}$$

such that $m > n$ and $b_m \neq 0$, then

$$\lim_{x\to\infty} x^n \frac{(a_n + a_{n-1}x^{-1} + a_{n-2}2x^{-2} + \ldots a_1/x^{n-1} + a_0/x^n)}{x^m(b_m + b_{m-1}x^{-1} + b_{m-2}x^{-2} + \ldots b_1/x^{m-1} + b_0/x^m)}$$

shows that the function has x-axis only horizontal asymptote and if $m = n$ then $y = \dfrac{a_n}{b_m}$ is the only horizontal asymptote and the value of the function tends to infinity as x tends to infinity when $m < n$ which shows that the function has no any horizontal asymptote.

Oblique asymptote: The line $y = mx + c$ $(m \neq 0)$ is called an asymptote of the graph of a function $f(x)$ if the perpendicular distance of any point $P(x, y)$ on the graph from the line approaches zero as $x \to \infty$ or $x \to -\infty$, Fig. 5.31.

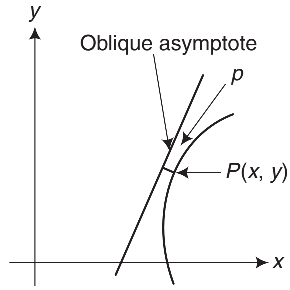


Fig. 5.31

We shell now determine the condition in order that the line $y = mx + c$ is an asymptotes of the graph of the function $f(x)$. If p denote the perpendicular distance of any point $P(x, y)$ on the graph from the line, Fig. 5.31.

$$p = \frac{|y - mx - c|}{\sqrt{1 + m^2}}$$

Now by definition $p \to 0$ as $x \to \pm\infty$

$$\Rightarrow \qquad \lim_{x \to \pm\infty} (y - mx - c) = 0$$

$$\Rightarrow \qquad \lim_{x \to \pm\infty} x\left(\frac{y}{x} - m - \frac{c}{x}\right) = 0 \tag{5.9}$$

Since otherwise the limit in above equation would be

$$\pm\infty \Rightarrow \lim_{x \to \pm\infty} \frac{y}{x} = m. \left(\lim_{x \to \pm\infty} \frac{c}{x} = 0\right)$$

Hence $\qquad m = \lim_{x \to \pm\infty} \frac{y}{x}$ and from (5.9)

$$c = \lim_{x \to \pm\infty} (y - mx) \tag{5.10}$$

We have thus the following method to determine to oblique asymptotes which are not parallel to y-axis.

(i) Determine $\lim_{x \to \pm\infty} \frac{y}{x}$; let $m = \lim_{x \to \pm\infty} \frac{y}{x}$.

(ii) Put this value of m in equation (5.10) and find c

Then $y = mx + c$ is an asymptote.

Example 20 Find the asymptote of the curve $x^3 + y^3 - 3axy = 0$.

Solution $x^3 + y^3 - 3axy = 0$, dividing by x^3, we obtain

$$1 + \left(\frac{y}{x}\right)^3 - 3a\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = 0$$

Taking limit as $x \to \infty$, then $1 + m^3 = 0 \qquad \left(m = \lim_{x \to \infty} \frac{y}{x}\right)$

Or $\qquad (1 + m)(1 + m^2 - m) = 0.$

Thus $m = -1$ is the only real root, now $y - mx = y + x$. (for $m = -1$).

Let $y + x = p$ so that, p is a variable which tends to c when putting $y = p - x$ in given curve, we get

$$x^3 + (p - x)^3 - 3ax(p - x) = 0$$

or $\qquad 3(p + a)\, x^2 - 3(p^2 + ap)x + p^3 = 0,$

or $\qquad 3(p + a) - 3(p^2 + ap)\frac{1}{x} + \frac{p^3}{x^2} = 0.$

Taking limit as $x \to \infty$, we have $3(c + a) = 0 \Rightarrow c = -a$ ($p \to c$ when $x \to \infty$), thus $y + x + a = 0$ is the only require *oblique* asymptote.

Example 21 Find the asymptote of the curve $(x^2 + y^2)\, x - ay^2 = 0$.

Solution $(x^2 + y^2)\, x - ay^2 = 0$, dividing by x^3, we obtain

$$1 + \left(\frac{y}{x}\right)^2 - \left(\frac{a}{x}\right)\left(\frac{y}{x}\right)^2 = 0$$

Taking limit as $x \to \infty$, then $1 + m^2 = 0$ $\left(m = \lim_{x\to\infty} \frac{y}{x}\right)$

Or $1 + m^2 = 0.$

It gives no real value of m. Hence there is no asymptote of the form $y = mx + c$. Further to obtain asymptotes parallel to x-axis. We suppose that $x = my + d$, then

$$m = \lim_{x\to\infty} \frac{x}{y} \text{ and } d = \lim_{x\to\infty} (x - my)$$

Now dividing the given equation by y^3, we get

$$\left\{\left(\frac{x}{y}\right)^2 + 1\right\} \frac{x}{y} - \frac{a}{y} = 0$$

Taking limit as $y \to \infty$, then $(1 + m^2)\; m = 0 \Rightarrow = m = 0$. $\left(m = \lim_{x\to\infty} \frac{x}{y}\right)$

Now in order to find d, we get $x = 0y + d$ in given curve we get,

$$(d^2 + y^2)\, d - ay^2 = 0$$

or $$\left\{\left(\frac{d}{y}\right)^2 + 1\right\} d - a = 0,$$

Taking limit as $y \to \infty$, we have $d = a$ thus $x - a = 0$ is the require asymptote. $y + x + a = 0$ is the only oblique asymptote.

Multiple points

A point on a curve through which two or more branches of the curve pass, is called a **multiple point** or a **singular point** of the curve, Fig. 5.32. If only two branches of the curve pass from a point on the curve then this point is called the **double point** of the curve. The double point will be a **node** if the two tangents to the two branches of the curve are real and distinct, Fig. 5.33.

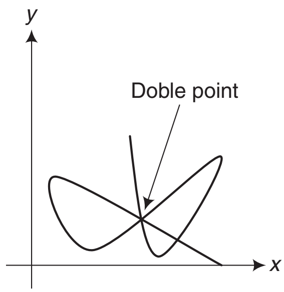


Fig. 5.32

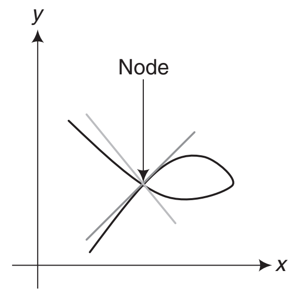


Fig. 5.33

The double point will be a **cusp** if the two tangents to the two branches of the curve are coincident, Fig. 5.34. A double point is called an **isolated point** or **conjugate point** if there no real tangents on the curve in a neighborhood of the point.

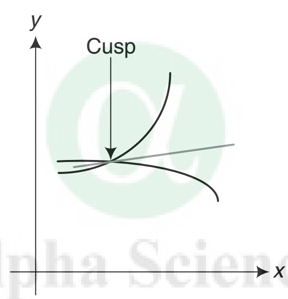


Fig. 5.34

To determine the position and nature of the double point we solve $(f_{xy})^2 - f_{xx} f_{yy}$.

Hence, if f_{xy}, f_{xx} and f_{yy} are not all zero at a double point $P(x, y)$, then this point is a node, cusp or a conjugate point according as

$$(f_{xy})^2 - f_{xx} f_{yy} > 0,\ (f_{xy})^2 - f_{xx} f_{yy} = 0 \text{ or } (f_{xy})^2 - f_{xx} f_{yy} < 0.$$

Example 22 Determine the position and nature of the double points on the following.

(i) $x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$.

(ii) $y^3 = x^3 + 2x^2$.

Solution (i) $f(x, y) = x^3 - y^2 - 7x^2 + 4y + 15x - 13$

$$f_x = 3x^2 - 14x + 15$$

(f_x = Differentiate with respect to x)

$f_y = -2y + 4$ (f_y = Differentiate with respect to y)

$f_x \to 0 \Rightarrow 3x^2 - 14x + 15 = 0 \Rightarrow x = 3, 5/3$ and

$f_y \to 0 \Rightarrow -2y + 4 = 0 \Rightarrow y = 2$, hence

The point (3, 2) lie on the curve and

Now $f_{xx} = 6x - 14$ (f_{xx} = Differentiate with respect to x of f_x)

$f_{yy} = -2$ (f_{yy} = Differentiate with respect to y of f_y)

And $f_{xy} = 0$ (f_{xy} = Differentiate with respect to y of f_x)

Hence, $\{(f_{xy})^2 - f_{xx}f_{yy}\}_{P(3,2)} = \{(0)^2 + 2(6 \times 3 - 14)\}_{P(3,2)} = 8 > 0.$

Thus (3, 2) is a node.

(ii) $y^3 = x^3 + 2x^2$

$f(x, y) = y^3 - x^3 - 2x^2$

$f_x = -3x^2 - 4x$ (f_x = Differentiate with respect to x)

$f_y = 3y^2$ (f_y = Differentiate with respect to y)

$f_x \to 0 \Rightarrow -3x^2 - 4x = 0 \Rightarrow x = 0, -4/3$ and

$f_y \to 0 \Rightarrow 3y^2 = 0 \Rightarrow y = 0$, hence the point (0, 0) lies on the curve and

Now $f_{xx} = -6x - 4$ (f_{xx} = Differentiate with respect to x of f_x)

$f_{yy} = 6y$ (f_{yy} = Differentiate with respect to y of f_y)

And $f_{xy} = 0$ (f_{xy} = Differentiate with respect to y of f_x)

Hence, $\left\{(f_{xy})^2 - f_{xx} fyy\right\}_{P(0,0)} = \left\{(0)^2 - (-6 \times 0 - 4) \times (6 \times 0)\right\}_{P(0,0)}$
$= 0$

Thus (0, 0) is a cusp.

5.6 SKETCHING OF A CARTESIAN CURVE

The following steps are very useful in sketching a Cartesian curve.

1. Symmetry

(i) The curve is symmetrical about the x-axis if all powers of y in the equation are even.

(ii) The curve is symmetrical about the y-axis if all powers of x in the equation are even.

(iii) The curve is symmetrical about the line $y = x$ if the equation of the curve remains unchanged on interchanging x and y.

(iv) The curve is symmetrical in opposite quadrant if the equation of the curve remains unchanged when x and y are replaced by $-x$ and $-y$ respectively.

2. Origin

Find out whether the origin lies on the curve. If it does, find out the tangent or tangents at the origin. In case the origin is a multiple point, then find out its nature.

3. Intersection with coordinate axes

Find out the point of intersection of the curve with the coordinate axes and the tangent at such points.

4. Asymptotes

Find out the asymptotes of the curve.

5. Critical point

Find out the values of x at which $\frac{dy}{dx} = 0$. At such points y generally changes its character from an increasing function of x to a decreasing function of x vice-versa.

6. Point of inflection

Find out the point of inflection $\left(\frac{d^2y}{dx^2} = 0\right)$ and the region where the curve is concave up and concave down.

7. Solving the equation

If possible, solve the equation for y in terms of x observe how y varies as x varies from $-\infty$ to $+\infty$.

8. Region

Find out the regions of the plane in which no part of the curve lies. Such a region is generally obtained on solving the equation for one variable in terms of the other, and find out the set of values of one variable which make the other imaginary.

Example 23 Trace the curve $y = x^3 - 6x^2 + 11x - 6$.

Solution

1. The curve is not symmetric about any line.
2. Origin does not lies on the curve. (when $x = 0 \Rightarrow y = -6$).
3. Point of intersection with coordinate-axis

   When $x = 0$ then point is $(0, -6)$ and when $y = 0 \Rightarrow x^3 - 6x^2 + 11x - 6 = 0 \Rightarrow x = 1, 2$ and 3, hence other point of intersection are $(1, 0)$, $(2, 0)$ and $(3, 0)$.
4. Neither vertical nor horizontal asymptotes lie.

5. $\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 12x + 11 = 0 \Rightarrow x \approx 1.4$ and 2.4

x	$-\infty$		1.4		2.4		∞
$\frac{dy}{dx}$		+	0	–	0	+	
y		Increasing		Decreasing		Increasing	

6. $\frac{d^2y}{dx^2} = 0 \Rightarrow 6x - 12 = 0 \Rightarrow x = 2. \Rightarrow$ Point of inflection is (2, 0).

Also $\frac{d^2y}{dx^2} < 0 \ \forall \ x \in [-\infty, 2] \Rightarrow$ curve concave down and $\frac{d^2y}{dx^2} > 0 \ \forall \ x \in [2, \infty]$ concave up.

7.

x	– 1	– .5	.5	1.5	2.5	4
$y \approx$	–24	–13	–1.8	.37	–.37	6

Hence the curve given as

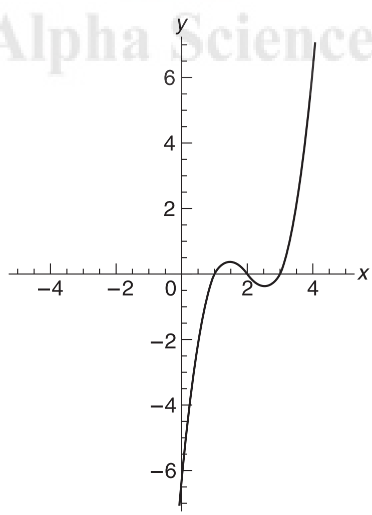


Fig. 5.35

Example 24 Trace the curve $y = x^3 - 9x$.

Solution 1. The curve is not symmetric about any line.

2. Origin lies on the curve. (when $x = 0 \Rightarrow y = 0$).
3. Point of intersection with coordinate-axis

   When $x = 0$, then point is (0, 0) and when $y = 0 \Rightarrow x^3 - 9x = 0 \Rightarrow x = 0$ and ± 3, hence other point of intersection are $(0, -3)$ and $(0, +3)$.
4. Neither vertical nor horizontal asymptotes lie.
5. $\dfrac{dy}{dx} = 0 \Rightarrow 3x^2 - 9 = 0 \Rightarrow x = \pm\sqrt{3} = \pm 1.732$

x	$-\infty$		-1.732		1.732		∞
$\dfrac{dy}{dx}$		+	0	–	0	+	
y		Increasing		Decreasing		Increasing	

6. $\dfrac{d^2y}{dx^2} = 0 \Rightarrow 6x = 0 \Rightarrow = 0$. Point of inflection **is** (0, 0)

   Also $\dfrac{d^2y}{dx^2} < 0 \ \forall \ x \in [-\infty, 0] \Rightarrow$ curve concave down and $\dfrac{d^2y}{dx^2} > 0 \ \forall \ x \in [0, \infty]$ concave up.
7.

x	-2	-1	0	1	2	4
$y \approx$	10	8	0	-8	-10	28

Hence the curve given as

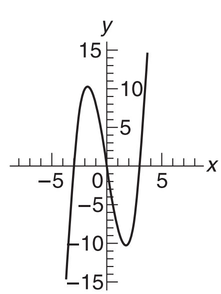


Fig. 5.36

Example 25 Trace the curve $y = \dfrac{9a^3}{x^2 + 3a^2}$

Solution 1. The curve is symmetrical about y-axis. (The power of x is even)

2. Origin does not lie on the curve. (when $x = 0 \Rightarrow y = 3a$).

3. Point of intersection with coordinate-axis

   The curve meets x-axis at (0, 3a).

   Hence (0, 3a) is only the point of intersection.

4. The equation of the given curve can be written as $y(x^2 + 3a^2) = 9a^3 \Rightarrow y = 0$ is the only horizontal asymptote.

5. $\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{-18xa^3}{(x^2 + 3a^2)^2} = 0 \Rightarrow x = 0$

x	$-\infty$		0		∞
$\dfrac{dy}{dx}$		+	0	–	
y		Increasing		Decreasing	

6. There is no Point of inflection.

7.

x	$-2a$	$-a$	a	$2a$
$y \approx$	1.2a	2.25a	02.25a	1.2a

Hence the curve given as

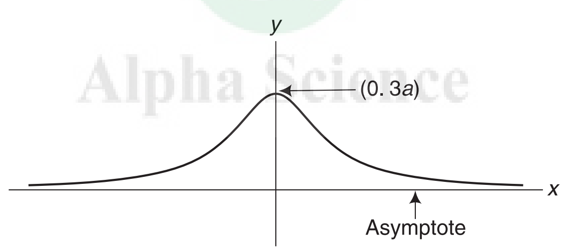


Fig. 5.37

Example 26 Trace the curve $a^2y^2 = x^2(a^2 - x^2)$

Solution

1. The curve is symmetrical about both axes. (The power of x and y both are even)
2. Origin lies on the curve. (when $x = 0 \Rightarrow y = 0$).
3. Point of intersection with coordinate-axis

   When $x = 0$ then point is (0, 0) and when $y = 0 \Rightarrow (a, 0)$ and $(-a, 0)$.
4. Neither vertical nor horizontal asymptotes lies.
5. $\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{(a^2 - 2x^2)}{a\sqrt{a^2 - x^2}} = 0 \Rightarrow x = \pm\dfrac{a}{\sqrt{2}}$

x	0		$\frac{a}{\sqrt{2}}$		a
$\frac{dy}{dx}$		+	0	−	
y		Increasing		Decreasing	

6. There is no Point of inflection.
7. y is real when $x^2 < a^2$ i.e. $-a < x < a$. Thus the entire curve lies between $-a$ and a.
8. $y = \pm\frac{x}{a}\sqrt{(a^2 - x^2)}$

x	0	$\frac{a}{\sqrt{2}}$	a
$y \approx$	0	$\frac{a}{2}$	0

Hence the curve given as

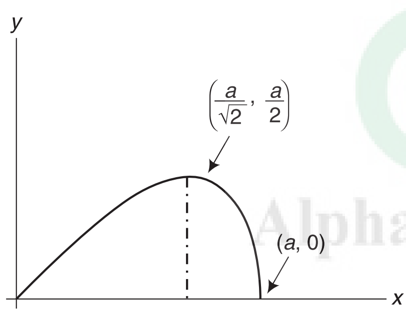


Fig. 5.38

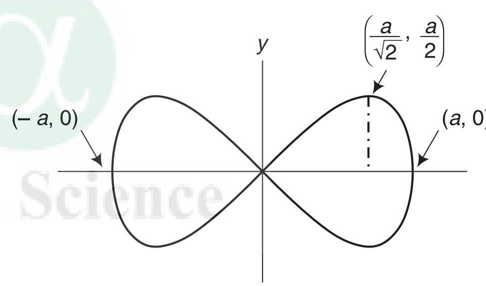


Fig. 5.39

Note: The following equations of the curve also have almost same figures as, Fig. 5.39

(i) $y^2(a^2 + x^2) = x^2(a^2 - x^2)$. (ii) $x^2(y^2 + x^2) = a^2(x^2 - y^2)$.

(iii) $a^4y^2 = x^4(a^2 - x^2)$ (iv) $a^6y^2 = x^6(a^2 - x^2)$.

(v) $x^4 = a^2(x^2 - y^2)$.

Example 27 Trace the curve $x^2y^2 = a^2(y^2 - x^2)$

Solution

1. The curve is symmetrical about both axes. (The power of x and y both are even)
2. Origin lies on the curve. (when $x = 0 \Rightarrow y = 0$).
3. Point of intersection with coordinate-axis

   The curve intersects the coordinate-axis only at (0, 0).

4. There exist only vertical asymptotes $y = \pm a$. {the coefficient of the highest power of y is $(a^2 - x^2)$}.
5. $\frac{dy}{dx} = \frac{a^3}{\sqrt[3]{a^2 - x^2}} \Rightarrow \frac{dy}{dx} \neq 0$
6. There is no Point of inflection.
7. y is real when $x^2 < a^2$ i.e. $-a < x < a$. Thus the entire curve lies between $-a$ and a.
8. $y = \pm \frac{ax}{\sqrt{(a^2 - x^2)}}$

x	0	$\frac{a}{2}$	a
$y \approx$	0	$\pm \frac{a}{\sqrt{3}}$	∞

Hence the curve given as

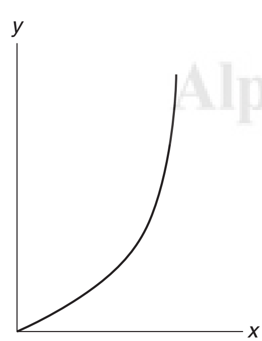


Fig. 5.40(a)

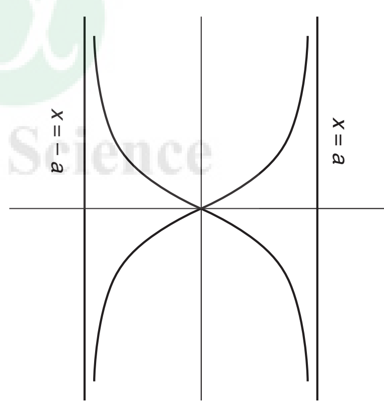


Fig. 5.40(b)

Example 28 Trace the curve $y^2 = (x - 2)(x - 3)(x - 4)$

Solution

1. The curve is symmetrical about x-axis. (The power of y is even)
2. Origin does not lie on the curve. (when $x = 0 \Rightarrow y = \pm\sqrt{24}$).
3. Point of intersection with coordinate-axis

   When $x = 0$ then point is $(0, \pm\sqrt{24})$ and when $y = 0 \Rightarrow (2, 0)$ $(3, 0)$ and $(4, 0)$.
4. Neither vertical nor horizontal asymptotes lies.

5. $\dfrac{dy}{dx} = \pm \dfrac{3x^2 - 18x + 26}{2\sqrt{(x-2)(x-3)(x-4)}} \Rightarrow \dfrac{dy}{dx} = 0 \Rightarrow x \approx 2.4$ and 3.5

x	2		2.4		3
$\dfrac{dy}{dx}$		+	0	–	
y		Increasing		Decreasing	

$\dfrac{dy}{dx} > 0 \ \forall\ x > 4$

6. There is no Point of inflection.
7. y is real when $x > 4$ and when $2 < x < 3$, and imaginary when $0 < x < 2$ and $3 < x < 4$ Thus the curve does not lie between $0 < x < 2$ and $3 < x < 4$
8. $y = \pm\sqrt{(x-2)(x-3)(x-4)}$

x	2.4	4.5
$y \approx$	.6	1.36

Hence the curve given as

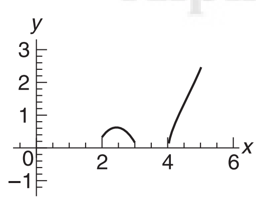


Fig. 5.41(a)

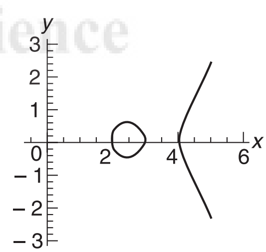


Fig. 5.41(b)

Example 29 Trace the curve $y = (9 - x^2)e^{-\frac{x^2}{6}}$.

Solution

1. The curve is symmetrical about y-axis. (The power of x is even)
2. Origin does not lie on the curve. (when $x = 0 \Rightarrow y = 9$).
3. Point of intersection with coordinate-axis

   When $x = 0$ then point is (0, 9) and when $y = 0 \Rightarrow$ (3, 0) and (–3, 0).

   $\lim_{x \to \pm\infty} \dfrac{(9 - x^2)}{e^{\frac{x^2}{6}}} = 0$ $\left(\frac{\infty}{\infty}\right.$ form this type of limit solve in next section$\left.\right)$.

4. Hence $y = 0$ is the horizontal asymptote.

5. $\frac{dy}{dx} = \frac{x}{3}(-15 + x^2)\, e^{-\frac{x^2}{6}} \Rightarrow \frac{dy}{dx} = 0 \Rightarrow x \approx 0$ and $\pm\sqrt{15}$

x	$-\infty$		$-\sqrt{15}$		0		$\sqrt{15}$		∞
$\frac{dy}{dx}$		$-$	0	$+$	0	$-$	0	$+$	
y	Decreasing			Increasing	Decreasing		Increasing		

6. $\frac{d^2y}{dx^2} = e^{-\frac{x^2}{6}}\left(-5 - \frac{2x^2}{3} - \frac{x^4}{9}\right) = 0 \Rightarrow x = \pm 2$. Hence at point of inflection are (–2, 2.5) and (2, 2.5).

Also $\frac{d^2y}{dx^2} > 0\ \forall x \in [-\infty, -2.5]$ and $\forall x \in [2.5, \infty] \Rightarrow$ curve concave up and $\frac{d^2y}{dx^2} < 0\ \forall\ x \in\ ]-2.5, 2.5[$ concave down.

7.

x	1	4
$y \approx$	6.9	–.7

Hence the curve given as

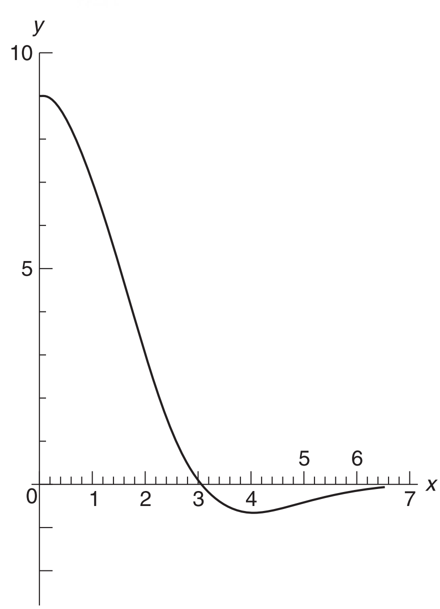


Fig. 5.42(a)

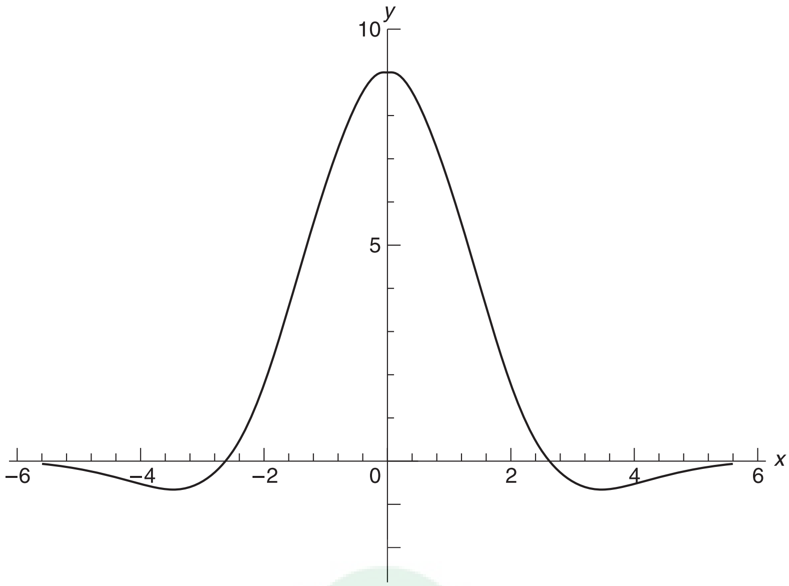


Fig. 5.42(b)

Exercises

1. Find all the vertical and horizontal asymptotes of the graph of the following functions.

   (i) $f(x) = \dfrac{2x + 3}{3 - x}$, (ii) $f(x) = 5 + \dfrac{3x}{4 - x}$,

   (iii) $f(x) = \dfrac{3x^3 + 2}{x^3 - 27}$, (iv) $f(x) = \dfrac{2x^2 - x + 2}{x - 2}$,

   (v) $f(x) = \dfrac{2x + 3}{x^2 - 3x + 2}$, (vi) $f(x) = \dfrac{1}{(x + 3)^3}$,

   (vii) $f(x) = \dfrac{(x - 2)(x + 1)}{(x - 2)^2}$, (viii) $2y^2x - a^2(x + 2a) = 0$,

   (ix) $\dfrac{2a^2}{x^2} + \dfrac{b^2}{y^2} = 1$, (x) $x^2y - 5x^2 + 5xy + 6y + 2 = 0$.

2. Find the Oblique asymptotes of the graph of the following functions

   (i) $x^2y^2 - 4a^2(x^2 + y^2) = 0$, (ii) $x^3 - y^3 = ax^2$,

   (iii) $y^2(x^2 - 5a) = x^2 - a^2$, (iv) $y^2(a^2 - x^2) = 2x^4$,

   (v) $(y - a)^2 (x^2 - a^2) = x^4 - a^4$, (vi) $x(x^2 + y^2) = 4ay^2$

3. Find constants a and b that guarantee that the graph of the function defined by

$$f(x) = \frac{ax + 5}{3 - bx}$$

Will have a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = -3$.

4. Show that, in general the graph of the function

$$f(x) = \frac{ax^2 + bx + c}{rx^2 + sx + t}$$

Will have $y = \frac{a}{r}$ as a horizontal asymptote and that when $br \neq as$ the graph will cross this asymptote at the point where, $x = \frac{at - cr}{br - as}$.

5. Sketch the graph of the following functions.

(i) $y = \frac{2x - 5}{x - 3}$, (ii) $y = \frac{x - 5}{x - 3}$,

(iii) $y = x^3(3x - 5)(x - 2)^2$, (iv) $y = x^3 - 2x + 1$,

(v) $y = \frac{x^2 - 4}{x^2 - 8}$, (vi) $y = \frac{x^2 - 2}{x^3}$,

(vii) $y = \frac{x^2 - 3x}{x - 2}$, (viii) $y = \frac{x^2 + 3x - 3}{(x + 2)^2}$,

(ix) $y = \frac{x^3 - x^2 - 9}{x - 2}$, (x) $y = \frac{x^2 - 2x - 3}{2x^2 + x - 7}$,

(xi) $y = \frac{3x^2 - x - 7}{-12x^2 + 2x + 9}$, (xii) $y^2(x^2 - 4) = 2(x - 1)$,

(xiii) $y^2(x - 4) = x^2(x + 4)$, (xiv) $y^2x^2 = x^2 - 4$,

(xv) $y^2 = 2x^2 \frac{(a - x)}{(a + x)}$, (xvi) $y^2x = 2a^2(a - x)$,

(xvii) $y^2a = x^2(a^2 - x)(x - 2)$, (xviii) $y^2 = 4x^4/(a^2 - x^2)$,

(xix) $y = 4a^2x/(a^2 + x^2)$, (xx) $y^3 = 4x - x^3$,

(xxi) $y(1 + x^2) = 2x^2$, (xxii) $y(1 - x^2) = 2x^2 + 1$,

(xxiii) $y^2 = 4x^2(5 - x)$, (xxiv) $y^2 = (x - 3)^2(x - 1)$.

Answers

1. (i) $x = 3,\ y = -2,$ (ii) $x = 4,\ y = 2,$

   (iii) $x = 3,\ y = 3,$ (iv) $x = 2,$ No Hor. Asymp.,

   (v) $x = 2,\ x = 1,\ y = 0.$ (vi) $x = -1,\ y = 0,$

   (vii) $x = 2,\ y = 1,$ (viii) $x = 0,\ y = \pm\dfrac{a}{\sqrt{2}},$

   (ix) $x = \pm\sqrt{2a},\ y = \pm b,$ (x) $x = 2,\ x = 3,\ y = 5.$

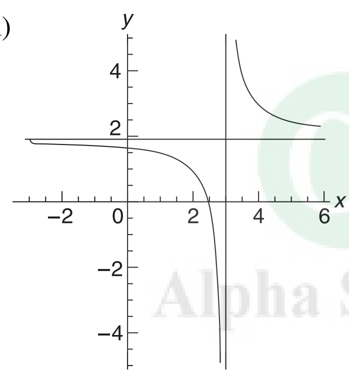
2. (i) $y = \pm 2a,$ (ii) $y = x + \dfrac{a}{3},$

   (iii) $x = \pm\sqrt{5a},$ (iv) $x = \pm a,$

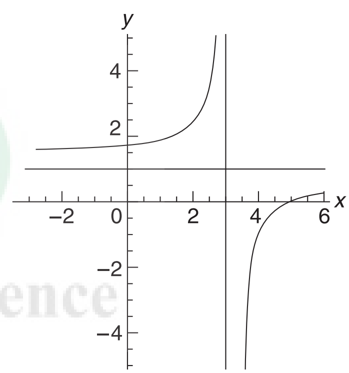
   (v) $y = \pm x + a,$ (vi) $x = 4a,$

3. $a = \dfrac{9}{5},\ b = \dfrac{3}{5},$

5. (i)



(ii)



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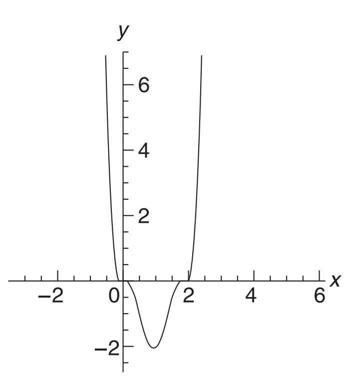

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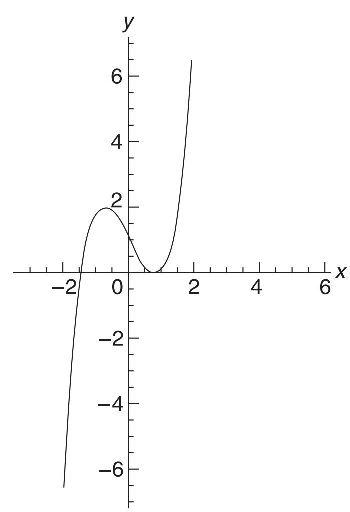

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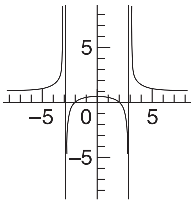
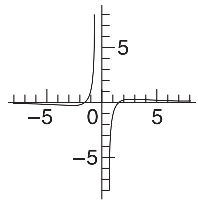


(vi)



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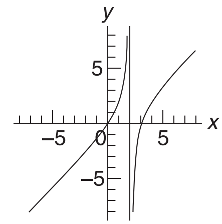

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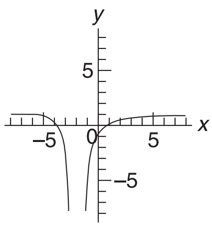

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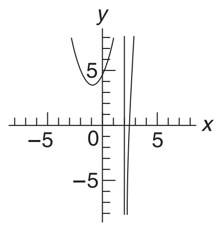
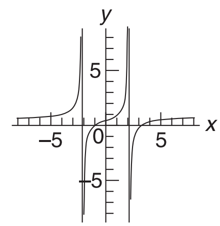


(x)



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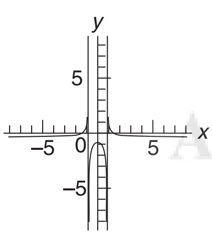

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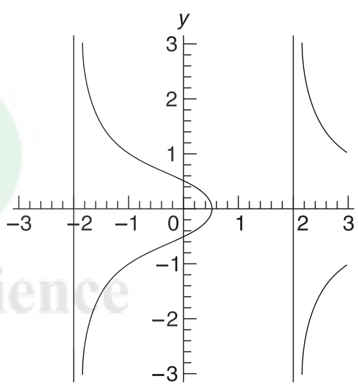

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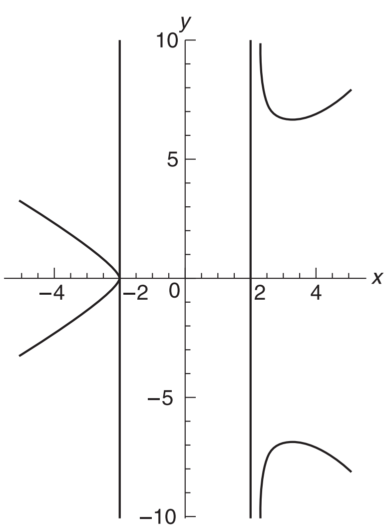

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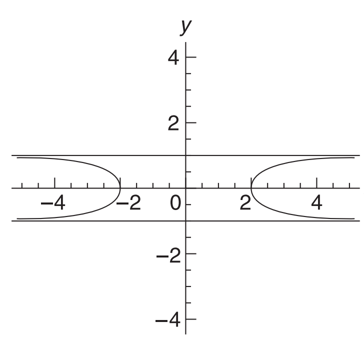

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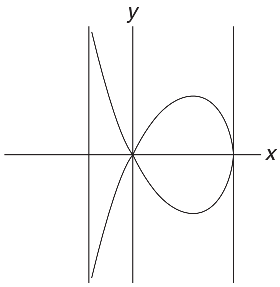
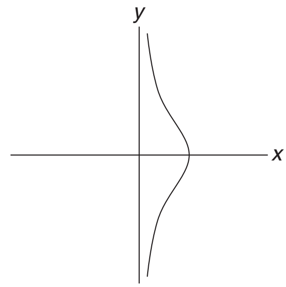
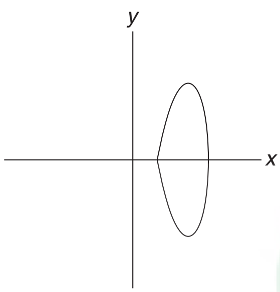


(xvi)



(xvii)



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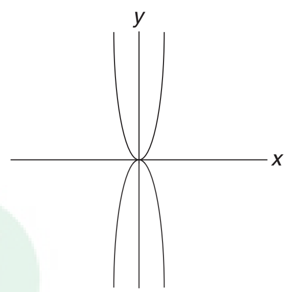

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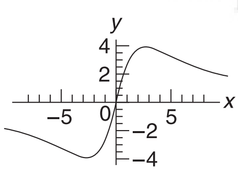

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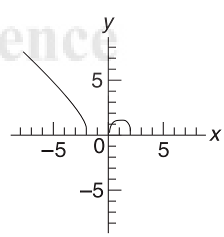
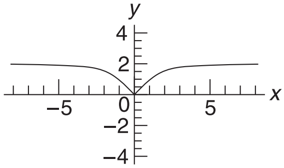
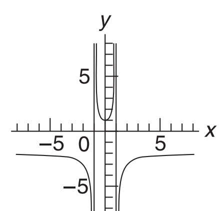


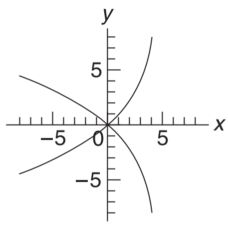
(xxi)



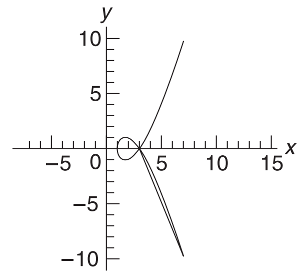
(xxii)



(xxiii)



(xxiv)



5.7 INDETERMINATE FORMS

We know that $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ when $\lim_{x\to c} g(x) \neq 0$ but we have no any information about $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ when $\lim_{x\to c} g(x) = 0$. We also unable to determined the value of a fraction when $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ or when $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$ and this type of limit are called the **Indeterminate form $\frac{0}{0}$ and $\frac{\infty}{\infty}$ for $x = c$** respectively, because their value cannot be determined at $x = c$ without further analysis. In the late seventeenth century, John Bernoulli discovered a rule for calculating the valve of this type of fractions, and this rule is known today as **l'Hopital's rule.**

(i) **Indeterminate form $\frac{0}{0}$:** If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, and suppose that f and g both are differentiable functions on an open interval containing $x = c$ except possibly at $x = c$ with $g'(x) \neq 0$ then $\lim_{x\to c} \dfrac{f'(x)}{g'(x)} = l \Rightarrow \lim_{x\to c} \dfrac{f(x)}{g(x)} = l$, (where l is either a finite number, ∞ or $-\infty$). This statement is also true in the case of a limit as $x \to c^-$, $x \to c^+$, $x \to +\infty$ or $x \to -\infty$.

Proof: Let f and g both are differentiable functions such that $f(c) = g(c) = 0$

$$\frac{f'(c)}{g'(c)} = \frac{\lim_{x\to c} \dfrac{f(x)-f(c)}{x-c}}{\lim_{x\to c} \dfrac{g(x)-g(c)}{x-c}} = \lim_{x\to c} \frac{\dfrac{f(x)-f(c)}{x-c}}{\dfrac{g(x)-g(c)}{x-c}}$$

$$= \lim_{x\to c} \frac{f(x)-f(c)}{g(x)-g(c)} = \lim_{x\to c} \frac{f(x)}{g(x)}$$

To apply the **L'Hopital's rule** use the following steps.

1. Check that the limit of $\dfrac{f(x)}{g(x)}$ is an indeterminate form. If it is not, then L'Hopital's rule cannot be used. For example,

   $\lim_{x\to 0} \dfrac{1-\cos x}{\sec x} = \dfrac{\lim_{x\to 0}(1-\cos x)}{\lim_{x\to 0} \sec x} = \dfrac{0}{1} = 0$, Now if we apply l'Hopital's rule then

   $\lim_{x\to 0} \dfrac{1-\cos x}{\sec x} = \lim_{x\to 0} \dfrac{\sin x}{\sec x \tan x} = \lim_{x\to 0} \dfrac{\cos x}{\sec x} = \dfrac{1}{1} = 1,$

   and this answer is wrong.
2. Differentiate f and g separately.
3. Determined $\lim_{x\to c} \dfrac{f'(x)}{g'(x)}$. If its finite, $+\infty$ or $-\infty$ then its equal of $\lim_{x\to c} \dfrac{f(x)}{g(x)}$. If $\lim_{x\to c} f'(x) = 0$ and $\lim_{x\to c} g'(x) = 0$, then $\lim_{x\to c} \dfrac{f''(x)}{g''(x)} = l \Rightarrow \lim_{x\to c} \dfrac{f'(x)}{g'(x)} = l$, and so on.

Example 30 Evaluate

(i) $\lim_{x\to 0} \dfrac{\sin x}{x}$, (ii) $\lim_{x\to \frac{\pi}{2}} \dfrac{1-\sin x}{2\cos x}$,

(iii) $\lim_{x\to 0^-} \dfrac{2\tan x}{x^2}$, (iv) $\lim_{x\to +\infty} \dfrac{x\left(\frac{5}{3}\right)}{\sin\left(\frac{2}{x}\right)}$,

(v) $\lim_{x\to 2} \dfrac{x^4-16}{x-2}$, (vi) $\lim_{x\to 0} \dfrac{x-1+\cos x}{x^3+2x}$.

(vii) $\lim_{x\to 0} \dfrac{e^{2x}-e^{-2x}-4\log(1+x)}{x\sin x}$,

(viii) $\lim_{x\to 0} \left(\cot x - \dfrac{2}{x}\right)$, *Solution* (i) $\lim_{x\to 0} \frac{\sin x}{x}$. This is the form of $\frac{0}{0}$, $\lim_{x\to 0} \frac{\cos x}{1} = 1$.

(ii) $\lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{2\cos x}$ This is the form of $\frac{0}{0}$, $\lim_{x\to\frac{\pi}{2}} \frac{-\cos x}{2(-\sin x)} = \frac{0}{-2} = 0.$

(iii) $\lim_{x\to 0^-} \frac{2\tan x}{x^2}$. This is the form of $\frac{0}{0}$, $\lim_{x\to 0^-}\frac{2\sec^2 x}{2x} = -\infty$

(iv) This is the form of $\frac{0}{0}$ $\lim_{x\to\infty} \frac{x^{\left(-\frac{5}{3}\right)}}{\sin\left(\frac{2}{x}\right)}$ $\lim_{x\to+\infty} \frac{-\frac{5}{3}x^{\left(-\frac{8}{3}\right)}}{\left(-\frac{2}{x^2}\right)\cos\left(\frac{2}{x}\right)}$

$= \lim_{x\to+\infty} \frac{\frac{5}{3}x^{\left(-\frac{2}{3}\right)}}{\cos\left(\frac{2}{x}\right)} = \frac{0}{1} = 0.$

(v) This is the form of $\frac{0}{0}$, $\lim_{x\to 2} \frac{x^4-16}{x-2}$. $\lim_{x\to 2} \frac{4x^3}{1} = 32.$

(vi) $\lim_{x\to 0} \frac{x-1+\cos x}{x^3+2x}$. This is the form of $\frac{0}{0}$,

$\lim_{x\to 0} \frac{x-1+\cos x}{x^3+2x}$. $\lim_{x\to 0} \frac{1-\sin x}{3x^2+2} = \frac{1}{2}.$

(vii) $\lim_{x\to 0} \frac{e^{2x}-e^{-2x}-4\log(1+x)}{x\sin x}$. This is the form of $\frac{0}{0}$

$\lim_{x\to 0} \frac{e^{2x}-e^{-2x}-4\log(1+x)}{x\sin x}$. $\lim_{x\to 0} \frac{2e^{2x}+2e^{-2x}-\frac{4}{(1+x)}}{\sin x + x\cos x}$

This is again the form of $\frac{0}{0}$, $\lim_{x\to 0} \frac{2e^{2x}+2e^{-2x}-\frac{4}{(1+x)}}{\sin x + x\cos x}$.

$\lim_{x\to 0} \frac{4e^{2x}-4e^{-2x}+\frac{4}{(1+x)^2}}{2\cos x - x\sin x} = \frac{4}{2} = 2.$

(viii) $\lim_{x\to 0}\left(\cot x - \frac{2}{x}\right) = \lim_{x\to 0}\left(\frac{\cos x}{\sin x} - \frac{2}{x}\right)$

$= \lim_{x\to 0}\left(\frac{x\cos x - 2\sin x}{x\sin x}\right)$. This is the form of $\frac{0}{0}$,

$$\lim_{x\to 0}\left(\frac{x\cos x - 2\sin x}{x\sin x}\right), \lim_{x\to 0}\left(\frac{\cos x - x\sin x - 2\cos x}{x\cos x + \sin x}\right) = -\infty$$

(ii) **Indeterminate form $\frac{\infty}{\infty}$:** If $\lim_{x\to c} f(x)\ \infty$ and $\lim_{x\to c} g(x) = \infty$, and suppose that f and g both are differentiable functions on an open interval containing $x = c$ except possibly at $x = c$ then $\lim_{x\to c}\frac{f'(x)}{g'(x)} = l \Rightarrow \lim_{x\to c}\frac{f(x)}{g(x)} = l$, (where l is either a finite number, ∞ or $-\infty$). This statement is also true in the case of a limit as $x \to c^-$, $x \to c^+$, $x \to +\infty$ or $x \to -\infty$.

Example 31 Evaluate

(i) $\lim_{x\to\infty}\frac{x^2}{e^x}$ (ii) $\lim_{x\to 0} a^2 x \log \sin x$.

Solution (i) $\lim_{x\to\infty}\frac{x^2}{e^x}$ this is the form of $\frac{\infty}{\infty}$

$$\lim_{x\to\infty}\frac{x^2}{e^x}$$

$$\lim_{x\to\infty}\frac{2x}{e^x}$$

$$\lim_{x\to\infty}\frac{2}{e^x} = 0$$

(ii) $\lim_{x\to 0} a^2 x \log \sin x = \lim_{x\to 0}\frac{a^2 \log \sin x}{\frac{1}{x}}$ this is the form of $\frac{\infty}{\infty}$

$$\lim_{x\to 0}\frac{a^2 \cot x}{\frac{-1}{x^2}} = a^2 \lim_{x\to 0}(-x).\lim_{x\to 0}\frac{x}{\tan x} = 0$$

(iii) Indeterminate form $0.\infty$ or $\infty - \infty$: In these both cases we may try to make the form as $\frac{0}{0}$ or $\frac{\infty}{\infty}$. For example,

We take $\lim_{x\to c}(f(x).\ g(x))$

When $\lim_{x\to c} f(x) = 0$, $\lim_{x\to c} g(x) = \infty$.

As $f(x).g(x) = \dfrac{g(x)}{\frac{1}{f(x)}}$.

Similarly we take $\lim_{x\to c} (f(x) - g(x))$

When $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \infty.$

As $f(x) - g(x) = \left[\frac{1}{f(x)} - \frac{1}{g(x)}\right] \div \frac{1}{f(x)\cdot g(x)}.$

Example 32 Evaluate

(i) $\lim_{x\to\frac{\pi}{2}} a\left(x-\frac{\pi}{2}\right) \tan x,$ (ii) $\lim_{x\to 0}\left(\frac{1}{\sin 3x} - \frac{1}{3x}\right).$

Solution (i) $\lim_{x\to\frac{\pi}{2}} a\left(x - \frac{\pi}{2}\right) \tan x = \lim_{x\to\frac{\pi}{2}} \frac{a\left(x - \frac{\pi}{2}\right)}{\cot x}$

This is the form of $\frac{0}{0}$

$$\lim_{x\to\frac{\pi}{2}} \frac{a}{-\csc^2 x} = -a.$$

(ii) $\lim_{x\to 0}\left(\frac{1}{\sin 3x} - \frac{1}{3x}\right) = \lim_{x\to 0}\left(\frac{3x - \sin 3x}{3x \sin 3x}\right)$

This is the form of $\frac{0}{0}$

$\lim_{x\to 0}\left(\frac{3 - 3\cos 3x}{3\sin 3x + 9x\cos 3x}\right)$

This is again the form of $\frac{0}{0}$

$\lim_{x\to 0}\left(\frac{9\sin 3x}{9\cos 3x + 9\cos 3x - 27x\sin 3x}\right) = \frac{0}{18} = 0.$

(iv) Indeterminate form 0^0, 1^∞ and ∞^0

We consider $\lim_{x\to c} [\{f(x)\}^{g(x)}]$

When $\lim_{x\to c} f(x) = 0$, $\lim_{x\to c} g(x) = 0$, $\lim_{x\to c} f(x) = 1$, $\lim_{x\to c} g(x) = \infty$

and $\lim_{x\to c} f(x) = \infty$, $\lim_{x\to c} g(x) = 0,$

As $y = [\{f(x)\}^{g(x)}]$

$\Rightarrow \log y = g(x)\cdot \log f(x)$

(it is necessary that $f(x)$ is positive in the neighborhood of c)

Let $\lim \log y = \lim_{x \to c} [g(x) \cdot \log f(x)] = l$ (from (iii))]

$\Rightarrow \lim y = e^l$

$\Rightarrow \lim [\{f(x)\}^{g(x)}] = e^l.$

Example 33 Evaluate

(i) $\lim_{x \to 0} [\{x\}^{\sin 2x}]$ (ii) $\lim_{x \to \infty} \left(1 + \frac{1}{bx}\right)^{ax}$

Solution (i) Let $y = \lim_{x \to 0} [\{x\}^{\sin 2x}]$

$$\Rightarrow \quad \log y = \log \lim_{x \to 0} [\{x\}^{\sin 2x}]$$

$$\log y = \lim_{x \to 0} \log [\{x\}^{\sin 2x}]$$

$$\log y = \lim_{x \to 0} \sin 2x \log x$$

$$\log y = \lim_{x \to 0} = \lim_{x \to 0} \frac{\log x}{\csc 2x}$$

$$\log y = \lim_{x \to 0} \frac{\frac{1}{x}}{-2 \csc 2x \cot 2x}$$

$$\log y = \lim_{x \to 0} \frac{-\sin^2 2x}{2x \cos 2x}.$$

$$\log y = \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\right)\left(-\frac{\sin 2x}{\cos 2x}\right) = 1.0 = 0$$

Hence $y = e^0 = 1$

(ii) $\lim_{x \to \infty} \left(1 + \frac{1}{bx}\right)^{ax}$

Let $y = \lim_{x \to \infty} \left(1 + \frac{1}{bx}\right)^{ax}$

$$\Rightarrow \log y = \lim_{x \to \infty} ax \log \left(1 + \frac{1}{bx}\right)^{ax}$$

$$\log y = \lim_{x \to \infty} a \frac{\log \left(1 + \frac{1}{bx}\right)}{\frac{1}{x}}$$

$$\log y = \lim_{x\to\infty} a \frac{\dfrac{1}{\left(1+\dfrac{1}{bx}\right)}\left(-\dfrac{1}{bx^2}\right)}{\left(\dfrac{-1}{x^2}\right)}$$

$$\log y = \frac{a}{b}\lim_{x\to\infty}\frac{1}{\left(1+\dfrac{1}{bx}\right)}$$

$$\log y = \frac{a}{b}\frac{1}{(1+0)}$$

$$\log y = \frac{a}{b}$$

$$y = e^{\frac{a}{b}}.$$

Some special formulas

(i) $\lim_{x\to+\infty}\dfrac{\log x}{x^n} = 0 = \lim_{x\to+\infty} x^n e^{-ax}$, (ii) $\lim_{x\to 0^+}\dfrac{\log x}{x^n} = -\infty$. $\lim_{x\to+\infty}\dfrac{e^{ax}}{x^n} = +\infty$.
Where a and n are positive

Example 34 Find so that $\lim_{x\to\infty}\left(\dfrac{x+a}{x-2a}\right)^x = 5.$

Solution Let $y = \lim_{x\to+\infty}\left(\dfrac{x+a}{x-2a}\right)^x$

$\Rightarrow$ $$\log y = \log \lim_{x\to+\infty}\left(\frac{x+a}{x-2a}\right)^x$$

$$\log y = \lim_{x\to+\infty}\frac{\log\left(\dfrac{x+a}{x-2a}\right)}{\dfrac{1}{x}}$$

$$= \lim_{x\to+\infty}\frac{\left(\dfrac{x-2a}{x+a}\right)\left[\dfrac{(x-2a)-(x+a)}{(x-2a)^2}\right]}{\left(-\dfrac{1}{x^2}\right)}$$

$$\log y = \lim_{x\to+\infty}\frac{-x^2[-3a]}{x^2-ax-2a^2} = 3a$$

$$\log y = 3a$$

$$y = e^{3a} = 5 \text{ (given)}$$

$\Rightarrow$ $3a = \log 5$, hence

$$a = \frac{\log 5}{3}.$$

Example 35 Find a and b so that $\lim_{x\to 0} \frac{x(1 + 2a\cos x) - b\sin x}{x^3} = 4.$

Solution $\lim_{x\to 0} \frac{x(1 + 2a\cos x) - b\sin x}{x^3}.$

This is the form of $\frac{0}{0}$ for all values of a and b, when $x\to 0$.

$\therefore \lim_{x\to 0} \frac{x(1 + 2a\cos x) - b\sin x}{x^3}$

$$\lim_{x\to 0} \frac{1 + 2a\cos x - 2ax\sin x - b\cos x}{3x^2}.$$

If $x = 0$; the denominator is zero then the fraction will be the form of $\frac{0}{0}$. when

$$1 + 2a - b = 0 \qquad (5.11)$$

Again differentiate, we gate

$$\lim_{x\to 0} \frac{-4a\sin x - 2ax\cos x + b\sin x}{6x} \text{ This is the form of } \frac{0}{0}$$

$$\lim_{x\to 0} \frac{-6a\cos x + 2ax\sin x + b\cos x}{6} = \frac{b - 6a}{6}$$

$$\frac{b - 6a}{6} = 4 \text{ (given)}$$

$$b - 6a = 24 \qquad (5.12)$$

From (5.11) and (5.12) we have $a = -\frac{23}{a}$ and $b = -\frac{21}{2}$.

Exercises

The series expansion of the function also very helpfull to solve the indeterminate form.

For example,

(i) $\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots\right) - x}{x^3} = 6.$

(ii) $\lim_{x\to 0} \dfrac{e^{ax} - e^{bx}}{2x}$

$$= \lim_{x\to 0} \frac{\left(1 + ax + \frac{a^2x^2}{2!} + \dots\right) - \left(1 + bx + \frac{b^2x^2}{2!} + \dots\right)}{2x} = \frac{a-b}{2}.$$

1. Find the limits of the following.

(i) $\lim_{x\to -1} \dfrac{x^4 - 1}{x^2 - 1}$, (ii) $\lim_{x\to 0} \dfrac{\sin 3x}{x}$,

(iii) $\lim_{x\to \frac{\pi}{2}} \dfrac{1 - \sin^2 x}{\cos^3 x}$, (iv) $\lim_{x\to 0} \dfrac{1 - \cos 3x}{1 - \cos 5x}$,

(v) $\lim_{x\to \frac{\pi}{2}} \dfrac{a \sec x}{b + \tan x}$, (vi) $\lim_{x\to \frac{\pi}{2}} \left(\dfrac{\cos x}{\frac{\pi}{2} - x}\right)$,

(vii) $\lim_{x\to \infty} \left(\dfrac{x + \cos x}{x + 2}\right)$, (viii) $\lim_{x\to 0} \dfrac{(2 - \cos x) \sin 3x}{x^3 \cos x}$,

(ix) $\lim_{x\to \infty} x^{\frac{5}{2}} \sin \dfrac{1}{x}$, (x) $\lim_{x\to \frac{\pi}{2}} (2 - \sin x) \tan x$,

(xi) $\lim_{x\to 0} x^3 \log(x^5)$, (xii) $\lim_{x\to 0} \tan x \log(x^3)$

(xiii) $\lim_{x\to 0} \left(\dfrac{2}{x^3} - \log\sqrt{x}\right)$, (xiv) $\lim_{x\to 0} \left(\dfrac{\cos x}{\sin 2x} - \dfrac{2}{x}\right)$,

(xv) $\lim_{x\to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 2x}\right)$, (xvi) $\lim_{x\to \infty} \left(1 + \dfrac{1}{3x}\right)^{5x}$

(xvii) $\lim_{x\to 0} (\tan x)^{\frac{1}{\log x}}$, (xviii) $\lim_{x\to 0} (\cos x)^{\frac{2}{x^2}}$,

(xix) $\lim_{x\to a} \left(2 - \dfrac{x}{a}\right)^{\sec \frac{\pi x}{2a}}$, (xx) $\lim_{x\to 0} \dfrac{(1 + 2x)^{\frac{1}{2x}} - e + \frac{ex}{2}}{x^2}$,

(xxi) $\lim_{x\to \infty} \sqrt{x^2 + 2x} - x$, (xxii) $\lim_{x\to 0} (e^x + x)^{\frac{1}{2x}}$,

(xxiii) $\lim_{x\to 0} \dfrac{a^{\cos x} - a}{\log \cos x}$, (xxiv) $\lim_{x\to 0} \dfrac{e^x - e^{-x} - 2x}{x(1 - \cos x)}$,

(xxv) $\lim_{x \to 1} \frac{x^{2x} - x}{x - 1 - \log x}$, (xxvi) $\lim_{x \to 0} \frac{x - \tan^{-1} 2x}{x^3}$

(xxvii) $\lim_{x \to 0} (2 + x)^{\frac{\log a}{x}}$, (xxviii) $\lim_{x \to \infty} \frac{x(3 + \sin 3x)}{x + 1}$,

(xxix) $\lim_{x \to \infty} \left(\frac{x + 2}{x + 3}\right)^x$, (xxx) $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

(xxxi) $\lim_{x \to \infty} x^4 \left[\cos\left(\frac{1}{x}\right) + \frac{1}{x^2 2!} - 1\right]$.

(xxxii) $\lim_{x \to 0} \left(\frac{1 - \cos x^2}{x^2 \sin x^2}\right)$, (xxxiii) $\lim_{x \to 0} \left(\frac{1}{e^x - 1} - \frac{1}{x}\right)$,

(xxxiv) $\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{x^2} \log(x + 1)\right]$.

2. Find a if $\lim_{x \to 0} \frac{\tan ax + \tan a^2 x}{\sin 4x} = \frac{1}{2}$.

3. Find a if $\lim_{x \to \infty} \left(\frac{x + a}{x - 3a}\right)^{2x} = 2$.

4. Find a and b if $\lim_{x \to 0} \left(\frac{\sin 3x}{x^3} + \frac{2a}{x^2} + b\right) = -2$.

5. Find a if $\lim_{x \to 0} \left(\frac{\sin 3x + a \sin x}{x^3}\right)$ is finite and find the limit.

6. Find a, b and c if $\lim_{x \to 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 1$.

Answers

1. (i) 2, (ii) 3,

(iii) undefined, (iv) $\frac{9}{25}$,

(v) a, (vi) 1,

(vii) 1, (viii) $\frac{3}{2}$,

(ix) undefined, (x) 0,

(xi) 0, (xii) 0,

(xiii) undefined, (xiv) 0,

(xv) $\frac{-16}{3}$, (xvi) $\frac{5}{3}$,

(xvii) e (xviii) $\frac{1}{e}$,

(xix) $e^{\frac{2}{\pi}}$, (xx) $\frac{11e}{24}$,

(xxi) 1, (xxii) e,

(xxiii) $a \log a$, (xxiv) $\frac{4}{3}$,

(xxv) 8, (xxvi) $\frac{4}{3}$,

(xxvii) $e^{\frac{\log a}{2}}$, (xxviii) limit does not exist,

(xxix) -1, (xxx) $\frac{1}{e^{\frac{1}{6}}}$,

(xxxi) $\frac{1}{24}$. (xxxii) 0,

(xxxiii) 1, (xxxiv) $\frac{1}{e}$,

2. 1 and -2.

3. $\frac{\log 2}{8}$,

4. $a = \frac{-3}{2}$, $b = \frac{5}{2}$,

5. $a = -3$, limit is -4,

6. $a = c = \frac{-1}{2}$, $b = 1$.

5.8 THE MEAN VALUE THEOREM

In this section we discuss one of the most important theorems of mathematics, **the mean value theorem**. This theorem can be use to prove some important facts about differentiation. To stating and proving the mean value theorem, we need one preliminary result, which is Rolle's theorem.

Rolle's theorem: Let f be a continuous function on the closed interval $[a, b]$ and differentiable on the open interval $]a, b[$. If $f(a) = f(b)$ or $f(a) = f(b) = 0$. Then there exist at least one number c in $[a, b]$ at which $f'(c) = 0$.

Proof: A function f which is continuous on a closed interval is also bounded therein. Let m and M be the least and greatest value of f respectively then there exist points c and d in $[a, b]$ such that $f(c) = M$ and $f(d) = m$.

There are two possibilities: either $m = M$ or $m \neq M$

If $m = M$ then $f(x) = m = M$, for all $x \in [a, b] \Rightarrow f(x)$ is constant on $[a, b]$

$\therefore$ f is derivable in $]a, b[$ thus $f'(x) = 0$ for all $x \in [a, b]$.

Now when $M \neq m \Rightarrow$ at least one of them must be different from the equal values $f(a)$, $f(b)$ so that

$$f(c) = M \neq f(a) \Rightarrow c \neq a \text{ and } f(c) = M \neq f(b) \Rightarrow c \neq b \ (f(a) = f(b))$$

This means that c lies in the open interval $]a, b[$. Now we shall show that c is the point where $f'(c) = 0$.

Suppose f is differentiable at c then (c is interior point and f is differentiable on $]a, b[$)

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

exists and have the same value when h tends to zero through positive or negative values.

We have $\quad f(c + h) \leq f(c) \ (f(c) = M)$

$$\frac{f(c + h) - f(c)}{h} \leq 0 \text{ for } h > 0 \qquad \Rightarrow f'(c) \leq 0 \tag{5.13}$$

$$\frac{f(c + h) - f(c)}{h} \geq 0 \quad \text{for} \quad h < 0 \qquad \Rightarrow f'(c) \leq 0 \tag{5.14}$$

The relations (5.13) and (5.14) both will be true if, and only if

$$f'(c) = 0.$$

The same conclusion can be proved in the similar manner if

$$f(c) = m \neq f(a) \Rightarrow c \neq a \text{ and } f(c) = m \neq f(b) \Rightarrow c \neq b.$$

Geometrically, Rolle's theorem asserts that there is at least one point lying between $x = a$ and $x = b$, at which the tangent to the curve of the function f is parallel to the x-axis, Fig. 5.43

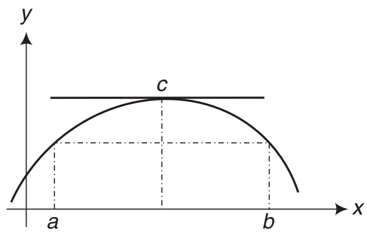


Fig. 5.43(a)

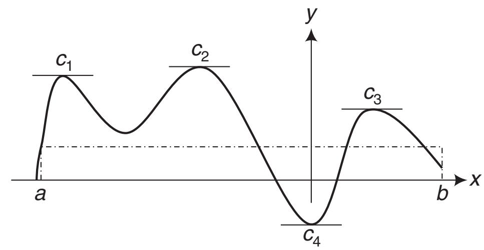


Fig. 5.43(b)

Remark: Rolle's theorem does not hold if

(i) $f(a) \neq f(b)$, Fig. 5.44.

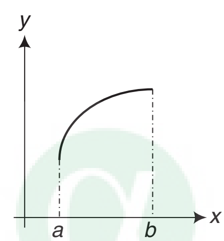


Fig. 5.44

(ii) f is discontinuous at the end point or interior points, Fig. 5.45.

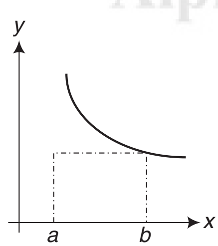


Fig. 5.45(a)

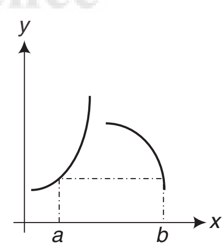


Fig. 5.45(b)

(iii) f is continuous on $[a, b]$ but not differentiable at an interior point, Fig. 5.46.

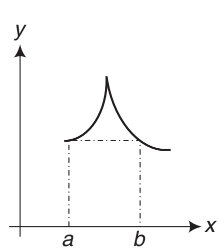


Fig. 5.46

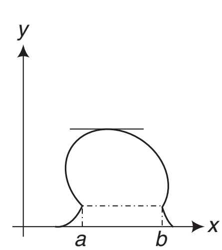


Fig. 5.47

Figure 5.47 shows that for the Rolle's theorem no need of the differentiability at the end points.

A **physical** example of Rolle's theorem is given in Fig. 5.48.

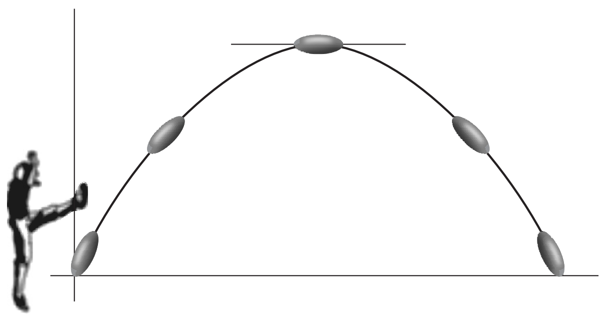


Fig. 5.48

Algebraic: Between two zeros a and b of $f(x)$ there exists at least one zero of $f'(x)$.

Example 36 Verify Rolle's Theorem for the following:

(i) $f(x) = \dfrac{x^3}{2} - 2x$, $[-2, 2]$

(ii) $f(x) = \sqrt{9 - x^2}$, $[-3, 3]$

(iii) $f(x) = |2x|$, $[-2, 2]$

(iv) $f(x) = 2 \sin x - \sin 3x$, $[0, \pi]$

Solution (i) $f(x) = \dfrac{x^3}{2} - 2x$

The given function being a polynomial function is continuous in $[-2, 2]$.

Now $f'(x) = \dfrac{3x^2}{2} - 2$, thus the function is differentiable in $[-2, 2]$.

$$f(-2) = 0 = f(2)$$

Hence the function satisfies all the conditions of the Rolle's Theorem.

So there exist a point $c \in [-2, 2]$ such that $f'(c) = 0$.

Clearly $f'(x) = \frac{3x^2}{2} - 2 = 0 \Rightarrow x = \pm \frac{2}{\sqrt{3}}$, Fig. 5.49

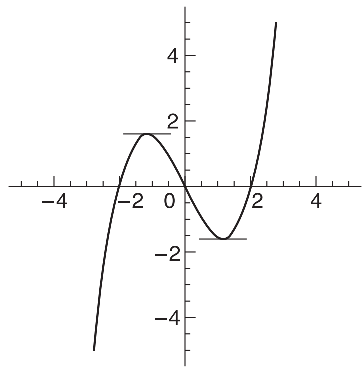


Fig. 5.49

(ii) $f(x) = \sqrt{9 - x^2}$

The given function being a algebraic function is continuous in [–3, 3].

Now $f'(x) = \frac{-x}{\sqrt{9 - x^2}}$, thus the function is differentiable in]–3, 3[.

$$f(-3) = 0 = f(3)$$

Hence the function satisfies all the conditions of the Rolle's Theorem.

So there exist a point $c \in [-3, 3]$ such that $f'(c) = 0$.

Clearly $f'(x) = \frac{-x}{\sqrt{9 - x^2}} = 0 \Rightarrow 0$, Fig. 5.50

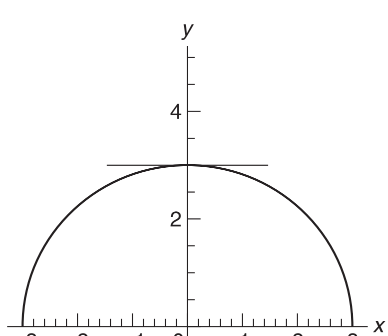


Fig. 5.50

(iii) $f(x) = |2x|, [-2, 2]$

The given function is continuous in $[-2, 2]$ and $f(-2) = 4 = f(2)$ but the function is not differentiable at $x = 0 \in [-2, 2]$. Thus not all the three conditions of the Rolle's Theorem are satisfied.

Clearly the conclusion of Rolle's Theorem is not valid for the given function.

(iv) $f(x) = 2 \sin x - \sin 3x, [0, \pi]$

The given function is continuous in $[0, \pi]$.

Now $f'(x) = 2 \cos x - 3 \cos 3x$, which exist for all $x \in [0, \pi]$ thus the function is differentiable in $[-0, \pi]$.

$$f'(0) = 0 = f(\pi)$$

Hence the function satisfies all the conditions of the Rolle's Theorem

So there exist a point $[0, \pi]$ such that $f'(c) = 0$.

Clearly $f'(x) = 2 \cos x - 3 \cos 3x = 0 \Rightarrow 11 \cos x - 12 \cos^3 x = 0 \Rightarrow \cos x = 0$ and $\cos x \pm \sqrt{\frac{11}{12}}$, Fig. 5.51

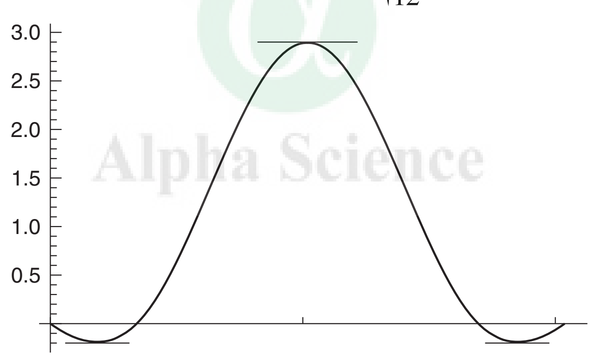


Fig. 5.51

Example 37 show that the $x^3 + x + 2$ has exactly one real solution.

Solution Let $f(x) = x^3 + x + 2$ and $f'(x) = 3x^2 + 1 \Rightarrow f'(x)$ always will be positive and not will be zero. By Rolle's Theorem if a function is continuous in a closed interval $[a, b]$ and differentiable in open interval $]a, b[$ and $f(a) = f(b) = 0$ then there exist at least one point in this interval such that $f'(c) = 0$ but here is $f'(c) \neq 0$. Therefore, f has no more than one zero. By intermediate value theorem the graph of the function crosses the x-axis at $x = -1$ between $(-2, 0)$. Hence the given equation has one real solution $x = -1$, Fig. 5.52.

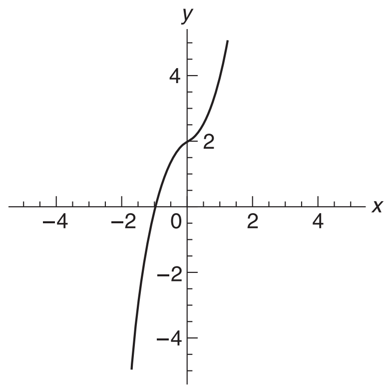
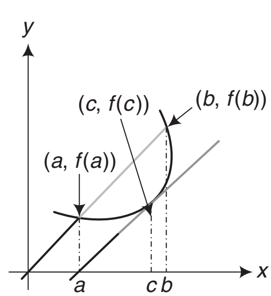


Fig. 5.52

We are ready to state the mean value theorem.

The mean value theorem: Let f be a continuous function on the closed interval $[a, b]$ and differentiable on the open interval $]a, b[$. Then there exist at least one number c in $[a, b]$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. (This is also known **The Lagrange's mean value theorem**)

Geometrical interpretation Fig. 5.53 shows that the expression $\dfrac{f(b) - f(a)}{b - a}$ represent the slope of the secant line whose joining the points $(a, f(a))$ and $(b, f(b))$ and $f'(x)$ is the slope of the tangent line to the curve $y = f(x)$, the mean value theorem states that there is always a number $c \in [a, b]$ such that the slope of the tangent line at the point $(c, f(c))$ is the same as the slope of the secant line (secant and tangent line are parallel to each other).



Fg. 5.53

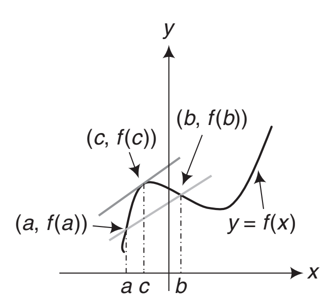


Fig. 5.54

Proof: We know that the equation of the secant is

$$y - f(a) = \frac{f(b) - f(a)}{b - a}\ (x - a), \text{ (Fig. 5.55)}$$

$$\Rightarrow y = f(a) + \frac{f(b) - f(a)}{b - a}\ (x - a)$$

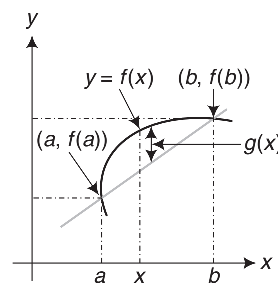


Fig. 5.55

Let $g(x)$ be the difference between the height of the graph of the function $y = f(x)$ and the height of the secant line, then

$$g(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}\ (x - a)\right] \tag{5.15}$$

The function $g(x)$ satisfies all the three condition of the *Rolle's Theorem* on the interval $[a, b]$ ($f(x)$ is continuous on $[a, b]$ and differentiable $]a, b[$ on so is $g(x)$ and $g(a) = g(b) = 0$).

Hence there exist a number $c \in [a, b]$ such that $g'(c) = 0$, thus from equation (5.15)

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

Since $g'(c) = 0 \Rightarrow f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Note: 1. Let x and b be two points such that $x \neq a$ but x is sufficiently close to a then there exists a number c between x and a which depends on x (by Mean-Value theorem) such that

$$f'(c) = \frac{f(x) - f(a)}{x - a}$$

But by definition of derivative

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

And by definition of Mean-Value theorem

$$= \lim_{x \to a} f'(c)$$

$$= \lim_{x \to a} f'(x) \text{ (Since } f'(x) \text{ exists and } c \text{ lies between } x \text{ and } a).$$

2. Let a function f satisfies the condition of The Lagrange's mean value theorem in $[a, b]$.

   Let $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$. We know that

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(c) \text{ where } c \in [x_1, x_2] \quad (5.16)$$

Now if $f'(x) = 0$, for all $x \in [a, b]$ then from equation (5.16) we get $f(x_2) = f(x_1) \Rightarrow$ function is constant and this the converse of the theorem, 'Derivative of a constant function is the zero function'.

Cor. If two functions f and g have the same derivative for all $x \in [a, b]$ then these functions are differ only by a constant. Let $\phi'(x) = f'(x) - g'(x) = 0$, for all $x \in [a, b] \Rightarrow \phi$ is constant.

Physical Interpretation Suppose a car with zero initial velocity covered 400 ft in 5-sec, thus its average velocity in 5-sec is 80 ft/sec. The Mean Value theorem state that the speedometer of the car must reached exactly 80 ft/sec during the cover this 400 ft distance.

Example 38 Verify Lagrange's mean value theorem for the following functions

(i) $f(x) = 5 + 3x - x^2$, $a = 0$, $b = 3$,

(ii) $f(x) = \log x + 2$, $a = 1$, $b = e$

Solution (i) $f(x) = 5 + 3x - x^2$, $a = 0$, $b = 3$

$f(x)$ is continuous in $[0, 3]$ and $f'(x) = 3 - 2x$ exists in $]0, 3[$

Now $f(a) = f(0) = 5$, $f(b) = f(3) = 5$, and $f'(c) = 3 - 2c$, hence

$$3 - 2c = \frac{5 - 5}{3 - 0} = \frac{0}{3} \Rightarrow c = \frac{3}{2} \in [0, 3]$$

(ii) $f(x) = \log x + 2$, $a = 1$, $b = e$

$f(x)$ is continuous in $[1, e]$ and $f'(x) = \frac{1}{x}$ exists in$[1, e]$

Now $f(a) = f(1) = 2$, $f(b) = f(e) = 3$, and $f'(c) = \frac{1}{c}$, hence

$$\frac{1}{c} = \frac{3 - 2}{e - 1} = \frac{1}{e - 1} \Rightarrow c = e - 1 \in [1, e].$$

Example 39 Find a point on the graph of the function $f(x) = x^2 + 4x + 4$ where the tangent is parallel to the chord joining $(-1, -6)$ and $(1, 6)$.

Solution $f(x) = x^2 + 4x + 4 \therefore f'(x) = 2x + 4$, and slope of the given chord is $\dfrac{6-(-6)}{1-(-1)} = 6$,

$\therefore 2x + 4 = 6$. Hence the required point is (1, 9).

Example 40 Show that $\dfrac{a-b}{1+a^2} < \cot^{-1} a < \dfrac{a-b}{1+b^2}$ where $0 < a < b$.

Solution Let $f(x) = \cot^{-1} x,\ a < x < b$

$$\Rightarrow \qquad f'(x) = \frac{-1}{1+x^2}$$

By Lagrange's mean value theorem, there exists a number $c \in [a, b]$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\Rightarrow \qquad \frac{\cot^{-1} b - \cot^{-1} a}{b-a} = \frac{-1}{1+c^2}$$

$$a < c \Rightarrow a^2 < c^2 < \Rightarrow 1 + a^2 < 1 + c^2$$

$$\Rightarrow \qquad \frac{-1}{1+c^2} < \frac{-1}{1+a^2}$$

$$\Rightarrow \qquad \frac{\cot^{-1} b - \cot^{-1} a}{b-a} < \frac{-1}{1+a^2}$$

$$\Rightarrow \qquad \frac{a-b}{1+a^2} < \cot^{-1} b - \cot^{-1} a \tag{5.17}$$

Now

$$c < b \Rightarrow c^2 < b^2 \Rightarrow 1 + c^2 < 1 + b^2$$

$$\Rightarrow \qquad \frac{-1}{1+b^2} < \frac{-1}{1+c^2}$$

$$\Rightarrow \qquad \frac{-1}{1+b^2} < \frac{\cot^{-1} b - \cot^{-1} a}{b-a}$$

$$\Rightarrow \qquad \cot^{-1} b - \cot^{-1} < \frac{a-b}{1+b^2} \tag{5.18}$$

Thus from (5.17) and (5.18) we have

$$\frac{a-b}{1+a^2} < \cot^{-1} b - \cot^{-1} a < \frac{a-b}{1+b^2}.$$

Example 41 Show that $0 < [\log (1 + 2x)]^{-1} - (2x)^{-1} < 1,\ x > 0$

Solution Let $f(x) = \log(1 + 2x)$, in $[0, x]$

$$\Rightarrow \qquad f'(x) = \frac{2}{1+2x}$$

By Lagrange's mean value theorem, there exists a number $c = \alpha x \in [0.x, 1.x]$

$$\Rightarrow < \alpha < 1 \text{ such that } \frac{f(x) - f(0)}{x - 0} = f'(c) = f'(\alpha x)$$

$$\Rightarrow \qquad \log(1 + 2x) = \frac{2x}{1 + 2\alpha x}$$

$$0 < \alpha < 1 \text{ and } x > 0 \Rightarrow 2\alpha x < 2x \Rightarrow 1 + 2\alpha x < 1 + 2x$$

$$\Rightarrow \qquad \frac{1}{1+2x} < \frac{1}{1+2\alpha x}.$$

$$\Rightarrow \qquad \frac{2x}{1+2x} < \log(1 + 2x) \tag{5.19}$$

Now

$$0 < \alpha < 1 \text{ and } x > 0 \Rightarrow 0 < 2\alpha x \Rightarrow 1 < 1 + 2\alpha x$$

$$\Rightarrow \qquad \frac{2x}{1+2\alpha x} < 2x$$

$$\Rightarrow \qquad \log(1 + 2x) < 2x \tag{5.20}$$

Thus, from (5.19) and (5.20) we have

$$\frac{2x}{1+2x} < \log(1 + 2x) < 2x \tag{5.21}$$

Now

$$\Rightarrow \qquad \frac{1+2x}{2x} > \frac{1}{\log(1+2x)} > \frac{1}{2x}$$

$$\Rightarrow \qquad 1 + \frac{1}{2x} > \frac{1}{\log(1+2x)} > \frac{1}{2x}$$

$$\Rightarrow \qquad 1 > \frac{1}{\log(1+2x)} - \frac{1}{2x} > 0. \text{ Hence}$$

$$0 < [\log(1 + 2x)]^{-1} - (2x)^{-1} < 1,\ x > 0$$

Example 42 A car covered 100 km in 2hr. Show that the speedometer reads exactly 50 km/hr at least once during the trip.

Solution Let $s(t)$ be the position function of the car then by Lagrange's mean value theorem there is at least one point c in [0, 2] such that

$$s'(c) = \frac{s(2) - s(0)}{2 - 0} = \frac{100}{2} = 50\text{km/hr}.$$

Cauchy's mean value theorem: Let f and g be two functions continuous on the closed interval $[a, b]$ and differentiable on the open interval $]a, b[$ and $g'(x) \neq 0$ for all $x \in [a, b]$. Then there exist at least one number c in $[a, b]$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Proof: By given condition we know that $g(b) \neq g(a)$. For if $g(b) = g(a)$, then g satisfies the conditions of the Rolle's theorem and so there exists at least one number at which $g'(x)$ vanish and this contradict of given condition $g'(x) \neq 0$.

Let a new function $\phi(x) = f(x) + Ag(x)$ for all $x \in [a, b]$

Where A is a constant to be determined such that

$$\phi(a) = \phi(b)$$

Hence $f(a) + Ag(a) = f(a) + Ag(b)$

$$\Rightarrow \quad A = -\frac{f(b) - f(a)}{g(b) - g(a)}$$

Now $\phi(x)$ is continuous in $[a, b]$ and differentiable on $]a, b[$. ($f(x)$ and $g(x)$ both are continuous and differentiable and A is constant) and as we know that $\phi(a) = \phi(b)$. Thus $\phi(x)$ satisfies all the conditions of the Rolle's theorem

$$\Rightarrow \quad \phi'(c) = 0 = f'(c) + Ag'(c)$$

$$\Rightarrow \quad A = -\frac{f'(c)}{g'(c)}$$

$$\Rightarrow \quad \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Another forms of Cauchy's mean value theorem.

(i) If the functions f and g are differentiable in $[a, a + h]$ and $g'(x) \neq 0$ for any $x \in [a, a + h]$ then there exists at least one number $\theta \in]0, 1[$ such that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)} \quad (0 < \theta < 1).$$

(ii) Let a function f be continuous in $[a, b]$ and differentiable in $]a, b[$ and let $g(x) = x^2$, hence by Cauchy's mean value theorem we have

$$\frac{f'(c)}{2c} = \frac{f(b)-f(a)}{b^2-a^2} \Rightarrow f'(c)\,(b^2 - a^2) = 2c\{f(b) - f(a)\},\ c \in [a, b]$$

Geometrical, Physical Interpretation

The Cauchy's mean value theorem may be written as

$$\frac{f'(c)}{g'(c)} = \frac{\dfrac{\{f(b)-f(a)\}}{(b-a)}}{\dfrac{\{g(b)-g(a)\}}{(b-a)}}$$

Which shows that the ratio of the mean rates of increases of two functions in an interval is equal to the ratio of the actual rates of increases of the functions at some point within the interval.

Example 43 Verify Cauchy's mean value theorem for the following functions

(i) $f(x) = x^2$, $g(x) = x^5$ in $[1, 3]$, (ii) $f(x) = e^{2x}$, $g(x) = e^{-2x}$ in $[0, 1]$

Solution

(i) $f(x) = x^2$, and $g(x) = x^5$

Here $f(x)$ and $g(x)$ both are continuous in $[1, 3]$ and differentiable in $]1, 3[$ $g'(x) = 5x^4$. Further $g'(x) \neq 0$ for all $x \in [1, 3]$ hence by Cauchy's mean value theorem there exists a point c in $]1, 3[$ such that

$$= \frac{9-1}{243-1} = \frac{2c}{5c^4} \Rightarrow c = \sqrt[3]{12.1} \in [1, 3].$$

Hence Cauchy's mean value theorem is verified.

(ii) $f(x) = e^{2x}$, and $g(x) = e^{-2x}$

Here $f(x)$ and $g(x)$ both are continuous in $[0, 1]$ and differentiable in $]0, 1[$ $g'(x) = -2e^{-2x}$. Further $g'(x) \neq 0$ for all $x \in [0, 1]$ hence by Cauchy's mean value theorem there exists a point c in $]0, 1[$ such that

$$= \frac{e^2-1}{e^{-2}-1} = \frac{2e^{2c}}{-2e^{-2c}} \Rightarrow c = \frac{1}{2} \in [0, 1].$$

Hence Cauchy's mean value theorem is verified.

Example 44 Show that $\dfrac{\cos\alpha - \cos\beta}{\sin\beta - \sin\alpha} = \tan\theta,\ 0 < \alpha < \theta < \beta < \dfrac{\pi}{2}$.

Solution Let $f(x) = \cos x$, $g(x) = \sin x$, $x \in [\alpha, \beta]$

Here $f(x)$ and $g(x)$ both are continuous in $[\alpha, \beta]$ and differentiable in $]\alpha, \beta[$ $g'(x) = \cos x$. Further $g'(x) \neq 0$ for all $x \in [\alpha, \beta]$ hence by Cauchy's mean value theorem there exists a point θ in $]\alpha, \beta[$ such that

$$\frac{\cos\beta - \cos\alpha}{\sin\beta - \sin\alpha} = \frac{\sin\theta}{\cos\theta}$$

$$\Rightarrow \quad \frac{\cos\alpha - \cos\beta}{\sin\beta - \sin\alpha} = \tan\theta,\ 0 < \alpha < \theta < \beta < \frac{\pi}{2}.$$

Exercises

1. Verify Rolle's theorem for the following

   (i) $f(x) = |2x|$, $[-1, 1]$

   (ii) $f(x) = 26\cos x - \sin x$, $\left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$,

   (iii) $f(x) = x(x-2)e^{-x}$, $[0, 2]$

   (iv) $f(x) = \log\left[\dfrac{x^2 + ab}{x(a+b)}\right]^2$, $[a, b]$

   (v) $f(x) = (x-a)(x-b)^n$ where n is a positive integer $[a, b]$.

2. Prove that if $a_0, a_1, a_2, \ldots, a_n$ are real numbers such that

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \ldots \frac{a_{n-1}}{2} + a_n = 0$$

   Then there exists at least one real number x between 0 and 1 such that

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots a_n = 0.$$

3. Show that there is no real number t for which the equation $x^2 - 5x + t = 0$ has two distinct root in $[0, 1]$.
4. Prove that if P be any polynomial and P' the derivative of P, then between any two consecutive zeros of P', there lies at the most one zero of P.
5. prove that between any two real roots of $e^x \cos x = x$, there is at least one real root of $\cos x - \sin x = e^{-x}$.
6. If f' is continuous on $[a, a+h]$ and derivable on $]a, a+h[$, then prove that there exists a real number c between a and $a+h$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(c)$$

7. Verify Lagrange's mean value theorem for the following

   (i) $f(x) = x^2 + x - 2$, [0, 2] (ii) $f(x) = \sqrt{16 - x^2}$, [–4, 0]

   (iii) $f(x) = 2\log x$, [1, e] (iv) $f(x) = \dfrac{1}{x-2}$, [3, 6].

8. Show that there is no number c between –1 and 7 such that $f'(c) = \dfrac{f(7) - f(-1)}{8}$ when $f(x) = x^{\frac{2}{3}}$.

9. Use the Lagrange's mean value theorem show that $\dfrac{x}{1 + 4x^2} < \tan^{-1} 2x < x$.

10. If $f(x) = x(2 - \log x)$, $x > 0$, show that $(b - a)(1 - \log c) = b(2 - \log b) - a(2 - \log a)$.

11. Find a point of $f(x) = (x - 2)^2$ where the tangent is parallel to chord joining (2, 0) and (3, 1).

12. Use the Lagrange's mean value theorem show that $|\sin x - \sin y| \le |x - y|$ for all real values of x and y.

13. Verify Cauchy's mean value theorem for the following functions

    (i) $f(x) = x^3 + 3$, $g(x) = x^2 + 4$ in [1, 2],

    (ii) $f(x) = \sin x$, $g(x) = \cos x$ in $\left[0, \dfrac{\pi}{2}\right]$.

14. If f' and g' exists for all $x \in [a, b]$ and if $g'(x) \neq 0$ for all $x \in]a, b[$ then prove that for some $c \in]a, b[$

$$\frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}.$$

15. Prove that $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$, whenever $0 < x < \dfrac{\pi}{2}$.

Answers

1. (i) not verified (ii) verified, c ≈ π

   (iii) verified, $c = 0.59$ (iv) verified, $c = \sqrt{ab}$

   (v) verified, $c = \dfrac{an + b}{1 + n}$.

7. (i) verified, $c = 1$ (ii) verified, $c = \sqrt{8}$

   (iii) verified, $c = \dfrac{e - 1}{2}$ (iv) verified, $c = 4$.

11. $\left(\dfrac{5}{2}, \dfrac{1}{4}\right)$.

13. (i) verified, $c = 1.55$ (ii) not verified.

5.9 TAYLOR POLYNOMIALS AND TAYLOR'S THEOREM

Taylor Polynomials Let f be a function whose first nth derivatves f^n exists on the closed interval $[x_0, x_1]$ and suppose $c \in [x_0, x_1]$ and $x \in [x_0, x_1]$ then the nth degree polynomial $P_n(x)$ of f at c is

$$P_n(x) = f(c) + \frac{f'(c)}{1!}(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \frac{f'''(c)}{3!}(x - c)^3 + \ldots \frac{f^n(c)}{n!}(x - c)^n \quad \ldots (5.22)$$

The **Taylor Polynomials** is a general case of a **Maclaurin Polynomial**. Hence if we take $c = 0$ in equation (5.22) we get Maclaurin Polynomial as

$$P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots \frac{f^n(0)}{n!}x^n \ldots \quad (5.23)$$

Example Let $f(x) = e^x$. Hence

$f'(0) = f''(0) = f'''(0) = f^n(0) = e^0 = 1$. Therefore Maclaurin Polynomial is

$$P_n(x) = \boldsymbol{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!}}$$

Figure 5.56 shows the graphs of $f(x) = e^x$ and the graphs of Maclaurin Polynomials and we have seen that these polynomials are good approximation of e^x for x near 0.

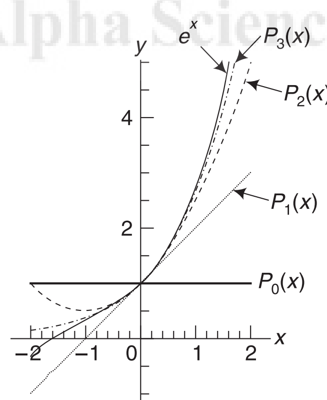


Fig. 5.56

Similarly the Taylor Polynomials for $f(x) = \log x$ about $x = 2$ can be found as

$$P_n(x) = \log 2 + \frac{1}{2}(x - 2) - \frac{1}{8}(x - 2)^2 + \frac{1}{24}(x - 2)^3 + \ldots, \text{ where}$$

$$f(x) = \log x,\ f(2) = \log 2,\ f'(x) = \frac{1}{x},\ f'(2) = \frac{1}{2},\ f''(x) = \frac{-1}{x^2},\ f''(2) = -\frac{1}{4} \ldots$$

Figure 5.57 shows the graphs of $f(x) = \log x$ and the graphs of Taylor Polynomials and we have seen that these polynomials are good approximation of $\log x$ for x near 2.

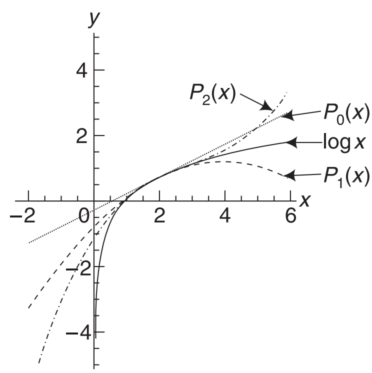


Fig. 5.57

Remainder Term Let $P_n(x)$ be the *nth* degree Taylor Polynomial of function *f*. Then the remainder Term, denoted by $R_n(x)$, is given by

$$R_n(x) = f(x) - P_n(x)$$

Taylor's Theorem (Taylor's Formula with Remainder) Let f be a function whose $(n + 1)^{th}$ derivatves f^{n+1} exists on the closed interval $[x_0, x_1]$ and suppose $a \in [x_0, x_1]$ and $x \in [x_0, x_1]$ then there is a number c in (a, x) or (x, a) such that

$$R_n(x) = \frac{f^{n+1}(c)}{n + 1!}(x - a)^{n+1}$$

This above expression is called Lagrange's form of remainder and use this we can write Taylor's Formula as

$$f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \ldots \frac{f^n(a)}{n!}(x - a)^{\mathrm{n}} + \frac{f^{n+1}(c)}{n + 1!}(x - a)^{n+1}.$$

Or

$$f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3$$

$$+ \ldots \frac{f^{n-1}(a)}{n - 1!}(x - a)^{n-1} + \frac{f^n(a + \theta x - a\theta)}{n!}(x - a)^n \text{ where } 0 < \theta < 1$$

Or

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$+ \ldots \frac{f^{n-1}(a)}{n-1!}(x-a)^{n-1} + \frac{\boldsymbol{f^n(a+\theta x - a\theta)}}{\boldsymbol{n-1!}} \boldsymbol{(x-a)^n (1-\theta)^{n-1}}$$

where $0 < \theta < 1$

(With **Cauchy's remainder**)

Proof: Suppose x be a fixed number in $[a, b]$ and let a function $\varphi(t)$ define as

$$\varphi(t) = f(x) - f(t) - \frac{f'(t)}{1!}(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \frac{f'''(t)}{3!}(x-t)^3$$

$$+ \ldots \frac{f^n(t)}{n!}(x-t)^n - \frac{R_n(x)}{(x-a)^{n+1}}(x-t)^{n+1}. \qquad (5.24)$$

From equation (5.24), we have

$$\varphi(x) = 0 \text{ and } \varphi(a) = f(x) - P_n(x) - R_n(x) = 0.$$

Since f^{n+1} exists $\Rightarrow f^n$ is differentiable so that φ is a differentiable for t in (a, x)

Here function φ is continuous in $[a, x]$ and differentiable in $]a, x[$ and $\varphi(x) = 0 = \varphi(a)$, hence φ hold all the conditions of the Rolle's theorem in $[a, x]$ so that there exists a number $c \in (a, x)$ with $\varphi'(c) = 0$. Therefore from equation (5.24) we have

$$\varphi'(c) = 0 - f'(c) + f'(c) - f''(c)(x-c) + \ldots$$

$$\frac{-f^{n+1}(c)}{n!}(x-c)^n - \frac{(n+1)\, R_n(x)}{(x-a)^{n+1}}(x-c)^{n+1} = 0.$$

$$\Rightarrow \qquad R_n(x) = \frac{f^{n+1}(c)}{n+1!}(x-a)^{n+1}$$

hence proved.

Example 45 Obtain the Maclaurin's expension of

$\sin x$, $\log(1+x)$, $(1+x)^m$

$$f(x) = \sin x \; f(0) = 0$$

$$f'(x) = \cos x \; f'(0) = 1$$

$$f''(x) = -\sin x \; f''(0) = 0 \; f'''(x) = -\cos x \; f'''(0) = -1$$

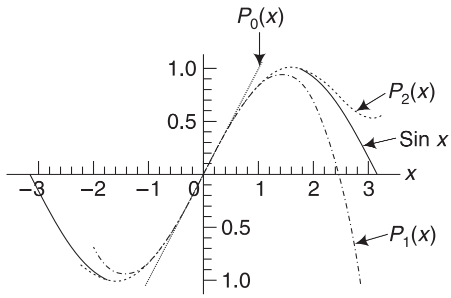


Fig. 5.58

Hence

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$$

Now to obtain Lagrange's form of remainder

$$R_n(x) = \frac{f^{n+1}(c)}{n+1!}(x)^{n+1} \left\{ f^{n+1}(c) = \sin\left(c + \frac{(n+1)\pi}{2}\right)\right\}$$

We have $R_n(x) = \dfrac{(x)^{n+1}}{n+1!}\sin\left((c + \dfrac{(n+1)\pi}{2}\right)$ so that

$$|R_n(x)| = \left|\frac{(x)^{n+1}}{n+1!}\right| \left|\sin\left(c + \frac{(n+1)\pi}{2}\right)\right| \le \left|\frac{(x)^{n+1}}{n+1!}\right|$$

$\Rightarrow R_n(x) \to 0$ as $n \to \infty$ for all $x \in \boldsymbol{R}$.

$$f(x) = \log(1+x)\ f(0) = 0$$

$$f'(x) = \frac{1}{(1+x)}\ f'(0) = 1$$

$$f''(x) = \frac{1}{(1+x)^2}\ f''(0) = -1$$

$$f'''(x) = -\frac{2}{(1+x)^3}\ f'''(0) = -2$$

Hence

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots$$

Now to obtain Lagrange's form of remainder

$$R_n(x) = \frac{(x)^n}{n!} f^n(\theta x) \left\{ f^n(\theta x) = \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n} \right\}$$

Let $0 \leq x \leq 1$

$$= \frac{(x)^n}{n!} \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n}$$

$$= \frac{(-1)^{n-1}}{n}\left(\frac{x}{1+\theta x}\right)^n$$

$$0 \leq \frac{x}{1+\theta x} < 1$$

$$\therefore \left(\frac{x}{1+\theta x}\right)^n \to 0 \text{ as } n \to \infty$$

Also $\frac{1}{n} \to 0$ as $n \to \infty$ Thus $R_n(x) \to 0$ as $n \to \infty$.

Now when $-1 < x < 0$. For this case we consider Cauchy's form of remainder

$$R_n(x) = \frac{(x)^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x)$$

$$\frac{(x)^n}{(n-1)!}(1-\theta)^{n-1} \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n}$$

$$(-1)^{n-1}(x)^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \frac{1}{1+\theta x}$$

Now $0 < 1-\theta < 1+\theta x$ then $\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \to 0$ as $n \to \infty$

Also, $(x)^n \to 0$ as $n \to \infty$

And $1/1 + \theta x < 1/1 - |x|$ $\therefore$ $R_n(x) \to 0$ as $n \to \infty$.

$f(x) = (1+x)^m$. Let m is a real number then $f(x)$ have continuous derivatives of every order when $(1 + x)$ is greater than zero.

$$f(0) = 1$$

$$f'(x) = m(1+x)^{m-1} \qquad f'(0) = m$$

$$f''(x) = m(m-1)(1+x)^{m-2} \qquad f''(0) = m(m-1)$$

$$f'''(x) = m(m-1)(m-2)(1+x)^{m-3} \qquad f'''(0) = m(m-1)(m-2)$$

Hence

$$\mathbf{(1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} \ldots}$$

When $-1 < x < 1$

Now to obtain Cauchy's form of remainder

$$R_n(x) = \frac{(x)^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x)$$

$$f^n(\theta x) = m(m-1)(m-2)\ \dots\ (m-n+1)(1+\theta x)^{m-n}$$

$$R_n(x) = \frac{(x)^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x)$$

$$= \frac{(x)^n}{(n-1)!}(1-\theta)^{n-1}\, m(m-1)(m-2)\ \dots\ (m-n+1)(1+\theta x)^{m-n}$$

Let $|x| < 1$ Now $-1 < x \Rightarrow 1 - \theta < 1 + \theta x \Rightarrow \dfrac{(1-\theta)}{(1+\theta x)} < 1$

Hence, we have

$$0 < \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} < 1$$

Now let $(m - 1)$ be positive

$$0 < 1 + \theta x < 2 \Rightarrow 0 < (1+\theta x)^{m-1} < 2^m$$

Let $(m - 1)$ be negative then we get

$$1 + \theta x \geq 1 - |x| \Rightarrow (1+\theta x)^{m-1} \leq (1-|x|)^{m-1}$$

$$\Rightarrow \lim_{n\to\infty} \frac{m(m-1)(m-2)\ \dots\ (m-n+1)}{(n-1)!} x^n = 0$$

Thus $R_n(x) \to 0$ as $n \to \infty$ if $|x| < 1$.

Example 46 Show that $e^x \cos 2x = 1 + x - x^2 - 5\dfrac{x^3}{3} + \dots$

Solution

$f(x) = e^x \cos 2x \qquad f(0) = 1$

$f'(x) = e^x \cos 2x - 2e^x \sin 2x \qquad f'(0) = 1$

$f''(x) = -2e^x \cos 2x - 4e^x \sin 2x \qquad f''(0) = -2$

$f'''(x) = -2e^x \cos 2x - 8e^x \cos 2x \qquad f'''(0) = -10$

Hence by Maclaurin's expansion we have

$$e^x \cos 2x = 1 + x - 2\frac{x^2}{2!} - 10\frac{x^3}{3!} = 1 + x - x^2 - 5\frac{x^3}{3} + \dots$$

Exercises

1. Show that for all real x

(i) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots,$

(ii) $\log(1 - x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \ldots,$

(iii) $\dfrac{1}{(1 - x)} = 1 + x + x^2 + x^3 + \ldots,$

(iv) $\sin \pi x = 0 + \pi x + \dfrac{(\pi x)^2}{2!} - \dfrac{(\pi x)^3}{3!} + \ldots,$

(v) $\log(1 + \sin x) = 1 - \dfrac{x^2}{2} + \dfrac{x^3}{6} - \dfrac{x^4}{24} - \ldots,$

(vi) $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \ldots,$

(vii) $\tan^{-1} x = \tan^{-1} \dfrac{\pi}{4} + \dfrac{x - \dfrac{\pi}{4}}{1 + \dfrac{\pi^2}{16}} - \dfrac{\pi\left(x - \dfrac{\pi}{4}\right)^2}{4\left(1 + \dfrac{\pi^2}{16}\right)^2} + \ldots,$

(viii) $\sin(e^x - 1) = x + \dfrac{x^2}{2!} + \dfrac{5x^4}{4!} + \ldots,$

6

CHAPTER

Polar Coordinates and Conic Section

6.1 POLAR COORDINATE

We know that a point P in the Cartesian plane (xy-plane) represent by their x (abscissa) and y (ordinate) say coordinates (Fig. 6.1). There are so many ways to represent this point P in the plane, the most important of which is called the **Polar Coordinate system.** Let a fixed point O and a ray that extends in one direction from O and this ray we label OQ. The fixed point O is called the **pole** and the horizontal ray OQ is called the **polar axis** (Fig. 6.2). Suppose P is any other point in the plane, let r be the distance between O and P, and let θ be the angle (in radians) between OQ and OP, measured counterclockwise from OQ to OP. The number r is called the **radial coordinate** of P and number θ is called **angular coordinate** of P. Then every point in the plane, except the pole, can be represented by a pair of number (r, θ), where $r > 0$ and $0 \leq \theta \leq 2\pi$. The pole can be represent as $(0, \theta)$ for any number θ. The representation $P = (r, \theta)$ is called the polar representation of the point, and r and θ are called the **polar coordinates** of P. For example the points (3, 60°), (4, 120°), (2, 225°), located as in the following figures.

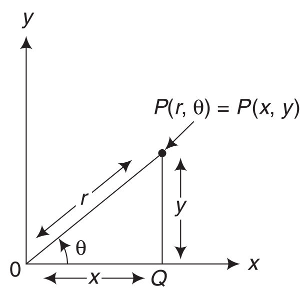


Fig. 6.1

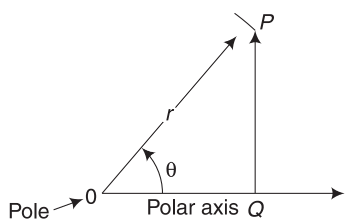


Fig. 6.2

Let (r, θ) be any pair of real numbers and r is positive then to locate the point $P = (r, \theta)$, we first rotate the polar axis through the angle of θ in the counterclockwise direction if θ is positive and in the clockwise direction if θ is negative and if r is negative then the ray OP is extended backward through the pole and the point P is then located $|r|$ units from the pole along this extended ray for example

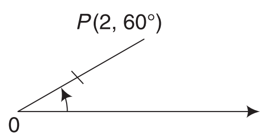


Fig. 6.3

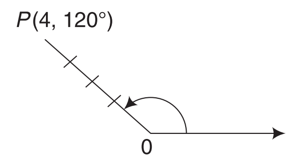


Fig. 6.4

The points (1, 675°), (–2, 45°), (–1, –90°) located as.

The Fig. 6.5 and Fig. 6.7 represent the coordinates of the same point similarly the Fig. 6.6 and Fig. 6.8 also represent the coordinates of another same point. Hence we conclude that the polar coordinates of a point are not unique. In general if (r, θ) represent the coordinate of a point P then $(r, \theta + n.\ 360)$ and $(r, \theta - n.\ 360)$ are also represent the coordinates of the same point P for positive n.

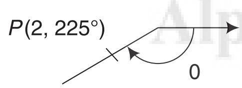


Fig. 6.5

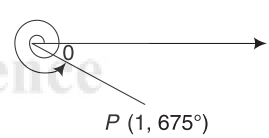


Fig. 6.6

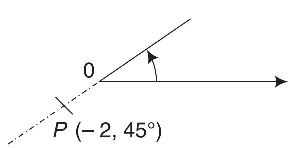


Fig. 6.7

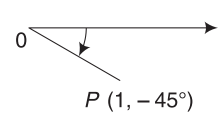


Fig. 6.8

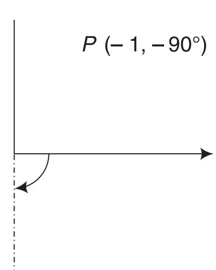


Fig. 6.9

6.2 RELATION BETWEEN POLAR AND RECTANGULAR COORDINATES

To find the relation between polar and rectangular coordinates, we replace the pole by the origin and polar axis along the x-axis in the Cartesian plane as in Fig. 6.1 and Fig. 6.2. Let $P = (x, y) = (r, \theta)$ be a point in the plane. Then from Fig. 6.1 we have

Changing from polar to rectangular coordinates

$$\boldsymbol{x = r\cos\theta \quad \text{and} \quad y = r\sin\theta} \tag{6.1}$$

Changing from rectangular to polar coordinates

We know that $x^2 + y^2 = r^2$, so that if we specify that $r > 0$ we have

$$\boldsymbol{r = \sqrt{x^2 + y^2}} \tag{6.2}$$

And $$\boldsymbol{\tan\theta} = \frac{y}{x} \quad \text{if } x \neq 0 \tag{6.3}$$

To determine θ from equation (6.3) we must be taken of the signs x and y into account. For example, for (2, 2) and (–2, –2), $\tan\theta = 1$. But in the first case $\theta = \dfrac{\pi}{4}$ and in the second case $\theta = \dfrac{5\pi}{4}$, Fig. 6.10.

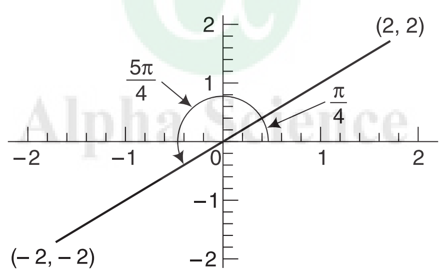


Fig. 6.10

Note: We can find the value of θ from equation (6.3) as

$$\theta = \tan^{-1}\frac{y}{x} \qquad \text{if } x > 0,\ y > 0;$$

$$\theta = \tan^{-1}\frac{y}{x} + 2\pi \qquad \text{if } x > 0,\ y < 0;$$

$$\theta = \tan^{-1}\frac{y}{x} + \pi \qquad \text{if } x < 0,\ y < 0;$$

{in the case $r < 0$, we have $(-r, \theta) = (r, \theta + \pi)$}.

Example 1 Convert from polar to rectangular coordinates:

(i) $\left(2, \dfrac{\pi}{3}\right)$ (ii) $\left(3, -\dfrac{\pi}{4}\right)$

Solution (i) $r = 2$ and $\theta = \frac{\pi}{3}$ hence from (6.1)

$x = r\cos\theta = 2\cos\left(\frac{\pi}{3}\right) = 2\cdot\frac{1}{2} = 1$ and $y = r\sin\theta = 2\sin\left(\frac{\pi}{3}\right)$

$= 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$. So the coordinates $(1, \sqrt{3})$.

(ii) $r = 3$ and $\theta = -\frac{\pi}{4}$ We can take θ as $\theta + n.\ 2\pi = -\frac{\pi}{4} + 2\pi$

$= \frac{7\pi}{4}$, then from (6.1) $x = r\cos\theta = 3\cos\left(\frac{7\pi}{4}\right) = 3\cdot\frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$ and

$y = r\sin\theta = 3\sin\left(\frac{7\pi}{4}\right) = 3\cdot\left(-\frac{1}{\sqrt{2}}\right) = \frac{-3}{\sqrt{2}}$. So the coordinates

$\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$.

Or

$r = 3$ and $\theta = -\frac{\pi}{4}$, then from (6.1)

$x = r\cos\theta = 3\cos\left(-\frac{\pi}{4}\right) = 3\cdot\frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$ $\{\cos(-\theta) = \cos\theta\}$ and

$y = r\sin\theta = 3\sin\left(-\frac{\pi}{4}\right) = 3\cdot\left(\frac{-1}{\sqrt{2}}\right) = \frac{-3}{\sqrt{2}}$ $\{\sin(-\theta) = -\sin\theta\}$.

So the coordinates $= \left(\frac{3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right)$.

Example 2 Convert from rectangular to polar coordinates (with $r > 0$ and $0 \le \theta < 2\pi$)

(i) $(1, \sqrt{3})$ (ii) $(1, -1)$ (iii) $(-2\sqrt{3}, -2)$

Solution (i) from (6.2) $r = \sqrt{x^2 + y^2} = \sqrt{(1)^2 + (\sqrt{3})^2} = 2$ and from (6.3)

$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{1} \Rightarrow \theta = \frac{\pi}{3}$. So the coordinates $\left(2, \frac{\pi}{3}\right)$.

(ii) $r = \sqrt{x^2 + y^2} = \sqrt{(1)^2 + (\sqrt{-1})^2} = \sqrt{2}$ and from (6.3) $\tan\theta = \frac{y}{x}$

$= \frac{-1}{1} = -1$

But here $x > 0$, $y < 0 \Rightarrow \theta = \tan^{-1}(-1) + 2\pi = \frac{-\pi}{4} + 2\pi = \frac{7\pi}{4}$.

So the coordinates $= \left(\sqrt{2}, \frac{7\pi}{4}\right)$.

(iii) $r = \sqrt{x^2 + y^2} = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = 4$ and from (6.3) $\tan\theta = \frac{-2}{-2\sqrt{3}}$

$= \frac{1}{\sqrt{3}}$.

But here $x < 0,\ y < 0 \Rightarrow \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \pi = \frac{\pi}{6} + \pi = \frac{7\pi}{6}$. So the coordinates $= \left(4, \frac{7\pi}{6}\right)$.

6.3 GRAPHING IN POLAR COORDINATES

In the Cartesian coordinates we define the graph of the equation $y = f(x)$ as the set of the points (x, y) whose coordinates satisfies the equation $y = f(x)$. As we have seen in section (6.1) that in polar coordinates each point in the plane has an infinite number of representations.

> The graph of an equation $r = f(\theta)$ $\left(\text{i.e. } r = \sin\theta,\ r = 2\theta,\ = \frac{1}{1 + \sin\theta}\right)$ in polar coordinates r and θ consists of those points P having at least one representation $P = (r, \theta)$ whose coordinates satisfy the equation $r = f(\theta)$.

Rules of symmetry

(i) The graph of the polar equation will be symmetric about the polar axis if θ can be replaced by $-\theta$ without changing the equation, Fig. 6.11.

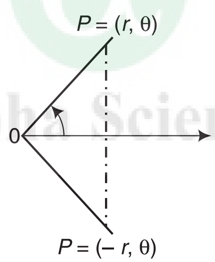


Fig. 6.11

(ii) The graph of the polar equation will be symmetric about the line $\theta = \frac{\pi}{2}$ if θ can be replaced by $(\pi - \theta)$ without changing the equation, Fig. 6.12.

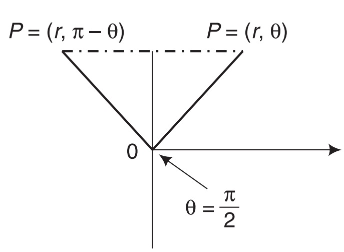


Fig. 6.12

(iii) The graph of the polar equation will be symmetric about the pole if θ can be replaced by $(\pi + \theta)$ or r replaced by $-r$ without changing the equation, Fig. 6.13.

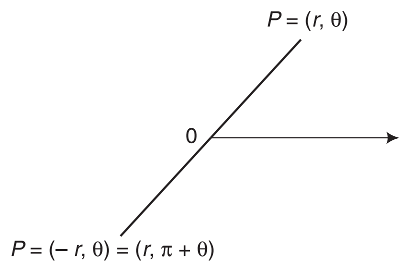


Fig. 6.13

Note: The above rules do not give necessary condition for symmetry. That is, symmetry may be exists in a situation where none of any above rules holds.

Example 3 Sketch the curve $r = \cos\theta$.

Solution Since $\cos(-\theta) = \cos\theta$, hence from rule (i) there is symmetry about polar axis, so we determine the values of the equation only when $0 \le \theta \le \pi$. In Table 6.1 we tabulate r for some values of θ for $0 \le \theta \le \pi$.

Table 6.1

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$r = \cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$\frac{-1}{2}$	$\frac{-\sqrt{3}}{2}$	-1

Therefore the polar coordinates are $(1, 0)$, $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{6}\right)$, $\left(\frac{1}{2}, \frac{\pi}{3}\right)$, $\left(0, \frac{\pi}{2}\right)$, $\left(-\frac{1}{2}, \frac{2\pi}{3}\right)$, $\left(\frac{-\sqrt{3}}{2}, \frac{5\pi}{6}\right)$ and $(-1, \pi)$.

In these coordinates (order pair) first part defines the length of r and second part defines the angle (in degree) in counterclockwise direction from the polar axis. {i.e. let 1 cm. = 1 unit then in (1, 0) r = 1 cm.}

By use these polar coordinates we can plot the curve as

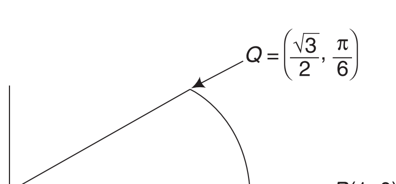


Fig. 6.14

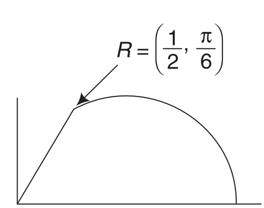


Fig. 6.15

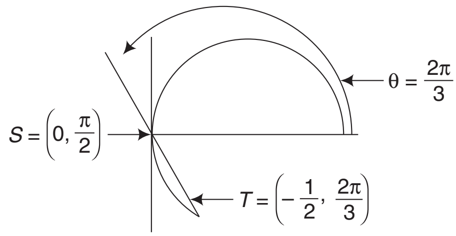


Fig. 6.16

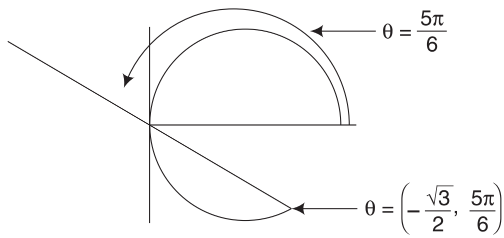


Fig. 6.17

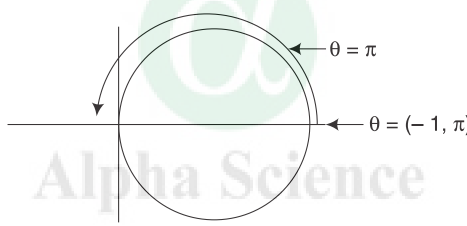


Fig. 6.18

Joined the points P, Q, R … V, we get require graph as follows

Figure 6.19a shows that the graph is a circle of radius $\frac{1}{2}$. Hence the equation of the form $\boldsymbol{r = 2a\cos\theta}$ represent the circle of radius $\boldsymbol{a}$**.** Similarly the equation $\boldsymbol{r = 2a\sin\theta}$ also represent the circle of radius $\boldsymbol{a}$, Fig. 6.19(b).

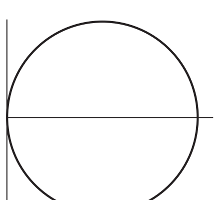


Fig. 6. 19(a) curve $\boldsymbol{r = \cos\theta}$

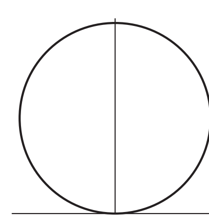


Fig. 6.19(b) curve $\boldsymbol{r = 2a\sin\theta}$

To see this analytically, we convert to Cartesian-coordinates. Since $r = \cos\theta$, we have for $r \neq 0$,

$$r^2 = r\cos\theta \Rightarrow x^2 + y^2 = x \Rightarrow \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$$

Which is a circle centered at the $\left(\frac{1}{2}, 0\right)$ and with radius $\frac{1}{2}$.

The equation $r = \cos\theta$ can also be graphed in Cartesian θr-coordinates system with coordinates $(0, 1)$, $\left(\frac{\pi}{2}, 0\right)$, $(\pi, -1)$ as follows

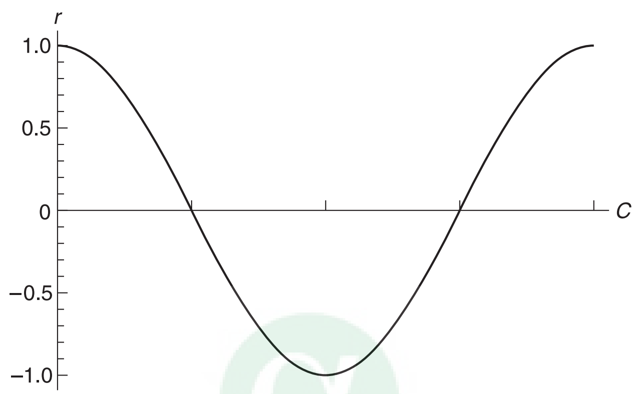


Fig. 6.20

Example 4 Sketch the curve $r = \sin 2\theta$.

Solution The graph plotted as follows

Table 6.2

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
$r = \sin 2\theta$	0	1	0	-1	0	1	0

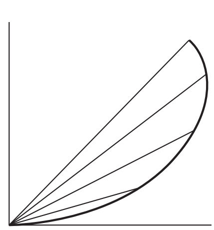


Fig. 6.21 r varies from 0 to 1 as θ varies from 0 to $\frac{\pi}{4}$.

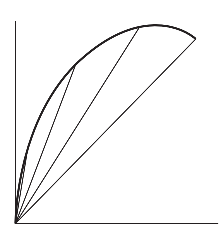


Fig. 6.22 r varies from 1 to 0 as θ varies from $\frac{\pi}{4}$ to $\frac{\pi}{2}$.

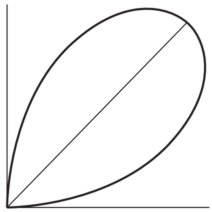


Fig. 6.23 θ varies from 0 to $\frac{\pi}{2}$.

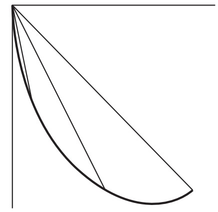


Fig. 6.24 *r* varies from 0 to −1 as θ varies from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$.

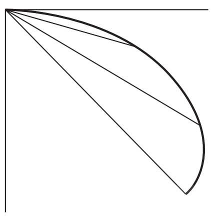


Fig. 6.25 *r* varies from −1 to 0 as θ varies from $\frac{3\pi}{4}$ to π.

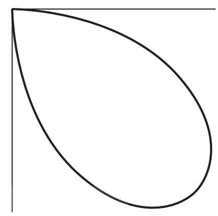


Fig. 6.26 θ varies from $\frac{\pi}{2}$. to π.

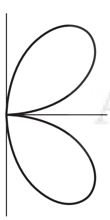


Fig. 6.27 θ varies from 0. to π.

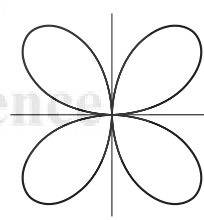


Fig. 6.28 θ varies from 0. to 2π.

The equation $r = \sin 2\theta$ can also be graphed in Cartesian θ*r*-coordinates system with coordinates (0, 0), $\left(\frac{\pi}{4}, 1\right)$ $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{3\pi}{2}, -1\right)$, (π, 0) as follows

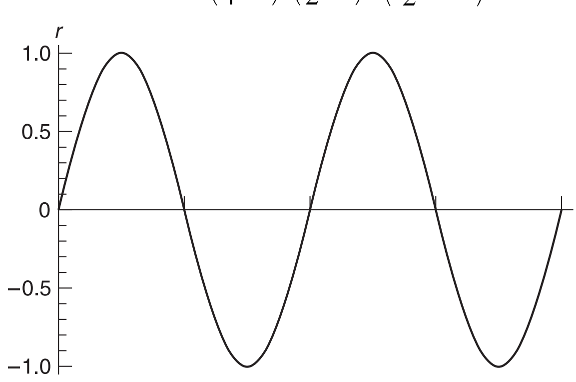


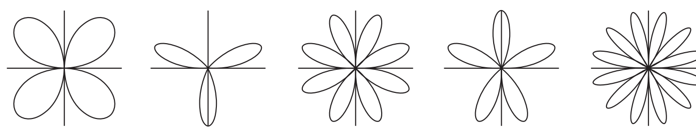
Fig. 6.29

Rose curves families

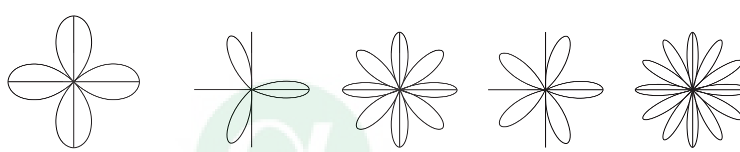
In polar coordinates, equation of the form $r = a \sin n\theta$ and $r = a \cos n\theta$, where $a > 0$ and n is a positive integer represent the flower-shaped curves families called **roses.** If n is odd then the rose contain n equally spaced petals with radius a, and for even n contain $2n$ equally spaced petals of radius a.

Rose curves

$\boldsymbol{r = a \sin n\theta}$ **$n = 2$** **$n = 3$** **$n = 4$** **$n = 5$** **$n = 6$**



$\boldsymbol{r = a \cos n\theta}$



Example 4 Sketch the curve $r = 1 + \sin \theta$.

Solution The graph plotted as follows

Since $\sin(\pi - \theta) = \sin \theta$, the curve is symmetric about the line $\theta = \frac{\pi}{2}$. Now,

Table 6.3

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{5\pi}{3}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
$r = 1 + \sin \theta$	1	$\frac{3}{2}$	$1 + \frac{\sqrt{2}}{2}$	$1 + \frac{\sqrt{3}}{2}$	2	1	$1 - \frac{\sqrt{3}}{2}$	0	$\frac{1}{2}$	1

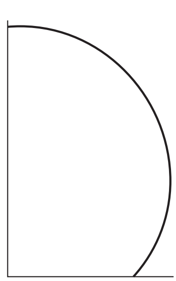


Fig. 6.30 r varies from 1 to 2 as θ varies from 0 to $\frac{\pi}{2}$.

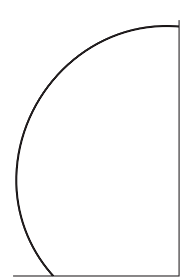


Fig. 6.31 r varies from 2 to 1 as θ varies from $\frac{\pi}{2}$ to π.

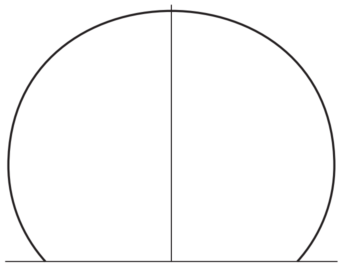


Fig. 6.32 θ varies from θ. to π.

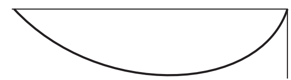


Fig. 6.33 *r* varies from 1 to 0 as θ varies from π to $\frac{3\pi}{2}$.

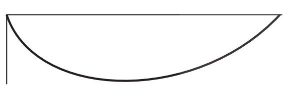


Fig. 6.34 *r* varies from 0 to 1 as θ varies from $\frac{3\pi}{2}$ to. 2π

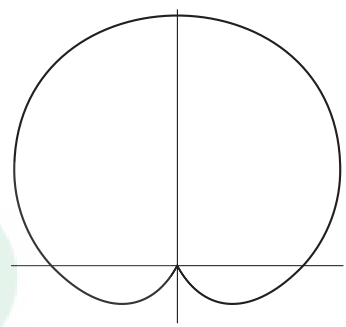


Fig. 6.35 θ varies from 0. to 2π.

The equation $r = 1 + \sin\theta$ can also be graphed in Cartesian θ*r*-coordinates system with coordinates (0, 1), $\left(\frac{\pi}{2}, 2\right)$, (π, 1), $\left(\frac{3\pi}{2}, 0\right)$, (2π, 1), as follows

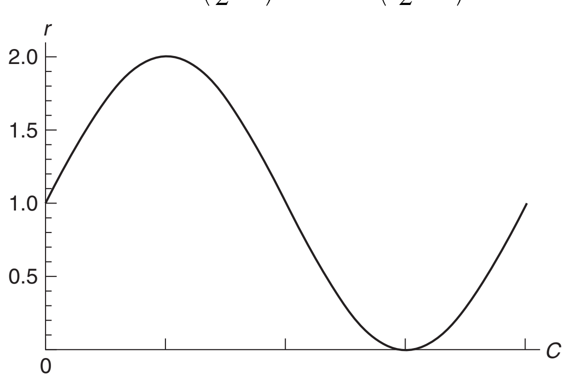


Fig. 6.36

Example 5 Sketch the curve $r = 1 + 2\cos\theta$.

Solution Here $\cos(-\theta) = \cos\theta \Rightarrow$ the curve is symmetric about polar axis, Now,

Table 6.4

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$r = 1 + 2\cos\theta$	3	$1+\sqrt{3}$	$1+\sqrt{2}$	2	1	0	$1-\sqrt{2}$	$1-\sqrt{3}$	-1

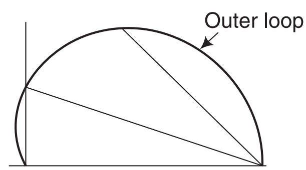


Fig. 6.37 r vaires from 3 to 0 as θ varies from 0 to $\frac{2\pi}{3}$.

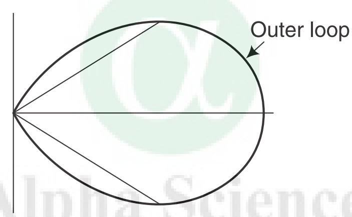


Fig. 6.38 r varies from 0 to 1 and then again 1 to 0 as θ varies from $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$.

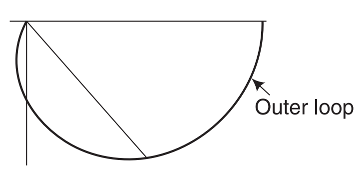


Fig. 6.39 r varies from 0 to 3 as θ varies from $\frac{4\pi}{3}$ to 2π.

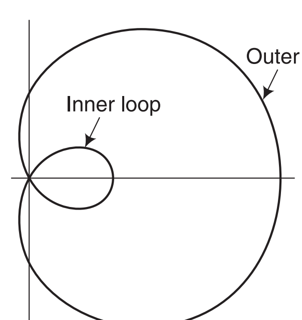


Fig. 6.40 θ varies from 0. to 2π.

Families of cardioids and limacons

The graph of the equation

$$r = a \pm b \sin \theta \quad \text{or} \quad r = a \pm b \cos \theta$$

When $a > 0$ and $b > 0$ is called the limacons.There are four possible shapes

(i) if $\frac{a}{b} < 1$, then we get the limacon with inner loop and the length of this loop is equal to the difference of a and b.

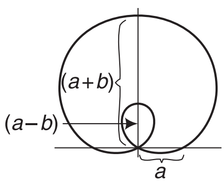


Fig. 6.41 $r = a + b \sin \theta$

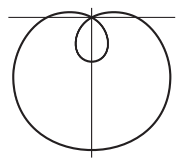


Fig. 6.42 $r = a - b \sin \theta$

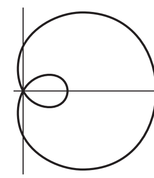


Fig. 6.43 $r = a + b \cos \theta$

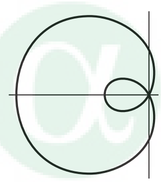


Fig. 6.44 $r = a - b \cos \theta$

(ii) if $\frac{a}{b} = 1$, then we get the limacon which called the **cardioids** as follows

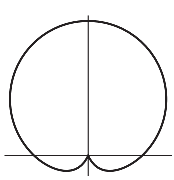


Fig. 6.45 $r = a + b \sin \theta$

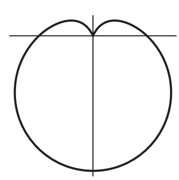


Fig. 6.46 $r = a - b \sin \theta$

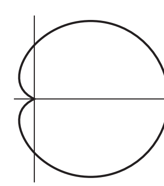


Fig. 6.47 $r = a + b \cos \theta$

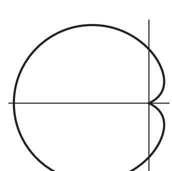


Fig. 6.48 $r = a - b \cos \theta$

(iii) if $1 < \frac{a}{b} < 2$, then we get the limacon which called the **dimpled limacon** as follows

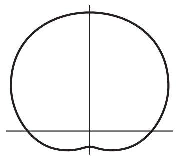


Fig. 6.49 $r = a + b \sin \theta$

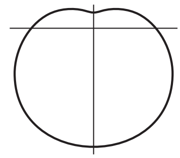


Fig. 6.50 $r = a - b \sin \theta$

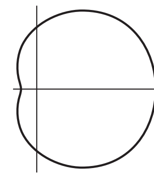


Fig. 6.51 $r = a + b \cos \theta$

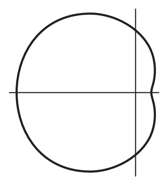


Fig. 6.52 $r = a - b \cos \theta$

(iv) if $\frac{a}{b} \geq 2$, then we get the limacon which called the **convex limacon** as follows

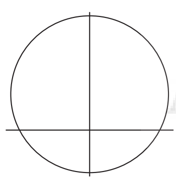


Fig. 6.53 $r = a + b \sin \theta$

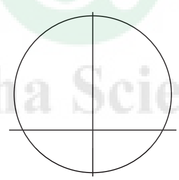


Fig. 6.54 $r = a - b \sin \theta$

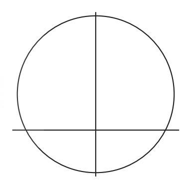


Fig. 6.55 $r = a + b \cos \theta$

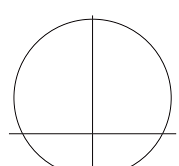


Fig. 6.56 $r = a - b \cos \theta$

Example 6 Sketch the curve $r^2 = 9 \cos 2\theta$.

Solution Here $(-r) = r^2$ which implies the symmetry about the pole we also know that the given equation does not express r as a function of θ. Hence we solve the function as $r = 3\sqrt{\cos 2\theta}$ and $r = -3\sqrt{\cos 2\theta}$ and combine these two graph we get required curve.

Table 6.5

θ	0	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	2π
$r = 3\sqrt{\cos 2\theta}$	3	0	0	3	0	0	3

We know that the given curve not define for $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$ ….

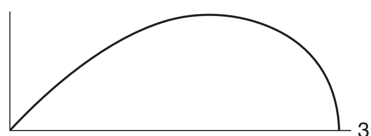


Fig. 6.57 r varies from 3 to 0 as θ varies from 0 to $\frac{\pi}{4}$.

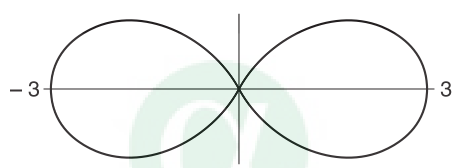


Fig. 6.58

The equation $r^2 = 9 \cos 2\theta$ can also be graphed in Cartesian θr-coordinates system with coordinates $(0, 3)$, $\left(\frac{\pi}{4}, 0\right)$, $\left(\frac{3\pi}{4}, 0\right)$, $(\pi, 3)$, $\left(\frac{5\pi}{4}, 0\right)$, $\left(\frac{7\pi}{4}, 0\right)$, $(2\pi, 3)$ as follows

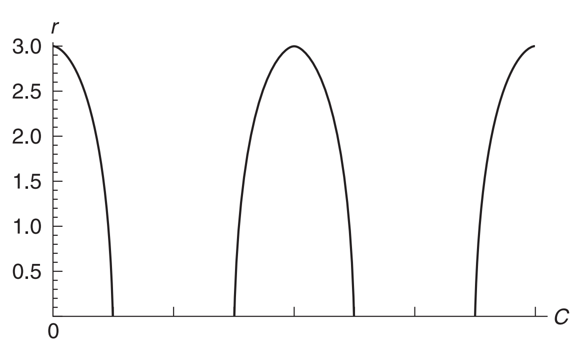


Fig. 6.59(a)

Lemniscates

The curve with the equation
$r^2 = a \cos b\theta$ or $r^2 = a \sin b\theta$ is a lemniscates of the form

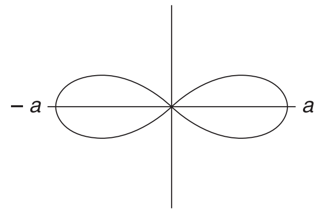


Fig. 6.59(b) $r^2 = a \cos b\theta$

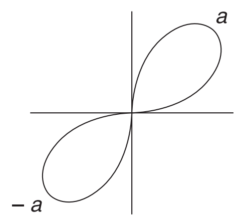


Fig. 6.60 $r^2 = a \sin b\theta$

Exercises

1. Plot the following points in polar coordinates.

   (i) $\left(2, \frac{\pi}{4}\right)$ (ii) $\left(3, \frac{2\pi}{3}\right)$

   (iii) $\left(-1, \frac{9\pi}{4}\right)$ (iv) $\left(-2, \frac{3\pi}{4}\right)$

   (v) $\left(-4, \frac{-\pi}{6}\right)$ (vi) $\left(-2, \frac{-2\pi}{3}\right)$.

2. Find the rectangular coordinates of the following points, whose polar coordinates are

   (i) $\left(2, \frac{\pi}{6}\right)$ (ii) $\left(1, \frac{2\pi}{3}\right)$

   (iii) $\left(-1, \frac{-3\pi}{4}\right)$ (iv) $\left(-2, \frac{\pi}{4}\right)$

   (v) $\left(-1, \frac{-\pi}{6}\right)$ (vi) $\left(2, \frac{9\pi}{4}\right)$

3. Find the polar coordinates of the following points, whose rectangular coordinates are

   (i) $(2, 3)$ (ii) $(4, -3)$

   (iii) $(-2, 2\sqrt{3})$ (iv) $(3, \tan^{-1} 1)$

   (v) $\left(1, \tan^{-1} \frac{1}{\sqrt{3}}\right)$.

4. Sketch the curve in polar coordinates

(i) $\theta = \dfrac{\pi}{3}$ (ii) $\theta = \dfrac{-3\pi}{4}$

(iii) $r = 2$ (iv) $r = 3 \sin \theta$

(v) $r = 2 \cos \theta$ (vi) $r = 3 + 4 \cos \theta$

(vii) $r = -1 - \cos \theta$ (viii) $r = 1 + \cos \theta$

(ix) $r = 3 - 2 \cos \theta$ (x) $r = 3 + 2 \sin \theta$

(xi) $r = 1 - 2 \sin \theta$ (xii) $r = 3(1 - \sin \theta)$

(xiii) $r = 2 \sin 2\theta$ (xiv) $r = 2 \cos 3\theta$

(xv) $r^2 = 16 \cos 2\theta$ (xvi) $r^2 = 4 \sin 2\theta$

5. Express the following equation in polar coordinates

(i) $x = 5$ (ii) $y = -2$

(iii) $x^2 + y^2 = 4$ (iv) $x^2 + y^2 - 8y = 0$

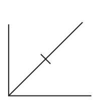
(v) $x^2(x^2 + y^2) = 4y^2$

6. If a and b varies then show that the polar equation $r = 2a \sec \theta$ when $\left(-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}\right)$ describe the lines of family perpendicular to the polar axis and $r = 2b \cos \theta$ when $(0 < \theta < \pi)$ describe the lines of family parallel to the polar axis.

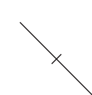
7. If A and B both are not zero then prove that the graph of the equation $r = \dfrac{1}{2}(A \sin \theta + B \cos \theta)$ is a circle of radius $\dfrac{\sqrt{A^2 + B^2}}{4}$.

Answers

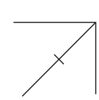
1. (i)



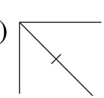
(ii)



(iii)



(iv)



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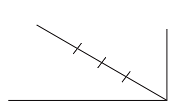


(vi)

2. (i) $(\sqrt{3}, 1)$ (ii) $\left(\dfrac{-1}{2}, \dfrac{\sqrt{3}}{2}\right)$

(iii) $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ (iv) $(-\sqrt{2}, -\sqrt{2})$

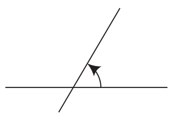
(v) $\left(-\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$ (vi) $(\sqrt{2}, \sqrt{2})$

3. (i) (3.605, .982) (ii) (5,5.639)

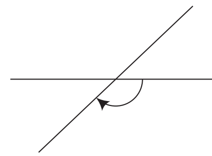
(iii) $\left(4, \dfrac{2\pi}{3}\right)$ (iv) (3.10108, .25605)

(v) (1.1287, .4823).

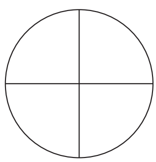
4. (i)



(ii)



(iii)



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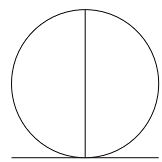

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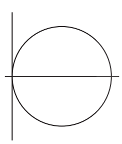

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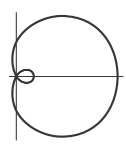

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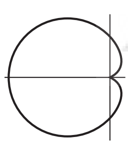

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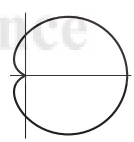

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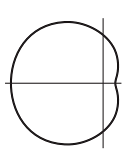
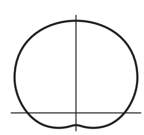
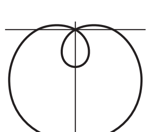


(x)



(xi)



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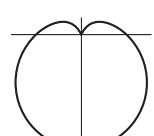

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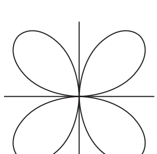

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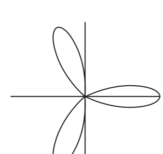

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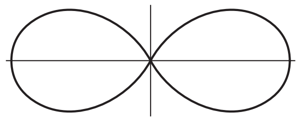

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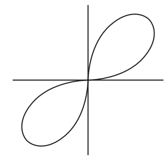


5. (i) $r \cos \theta = 5$ (ii) $r \sin \theta = -2$

(iii) $r = 2$ (iv) $r^2 - 8r \sin \theta = 0,\ r = 8 \sin \theta$

(v) $r^4 \cos^2 \theta = 4r^2 \sin^2 \theta$.

6.4 CONIC SECTIONS

Those sections are obtained from intersection of a plane and double-napped circular cone are called the conic section and the names of these are circles, parabolas, ellipses and hyperbolas

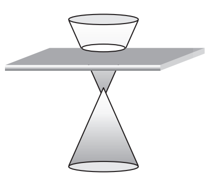


Fig. 6.61(a)

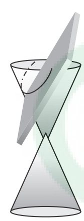


Fig. 6.61(b)

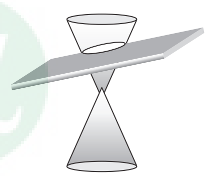


Fig. 6.61(c)

Fig. 6.61(d)

Parabola: It is the locus of a point which moves such that its distance from a fixed point is equal to from a fixed line. Fixed point is called the focus and fixed line is called the directrix of the parabola, Fig. 6.62.

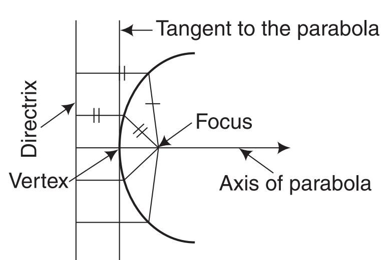


Fig. 6.62

Suppose that the distance between the focus and the vertex is a then by definition the distance between the vertex and the directrix also is equal to a;

consequently the distance between the focus and the directrix is $2a$, (Fig. 6.63). This Figure also shows that the parabola passes through the two corners of a rectangular box that passes through the vertex and the focus of the parabola along the axis of symmetry and extends $2a$ units above and $2a$ units below the axis of symmetry. A line which is passes through the focus and perpendicular to the axis of the parabola is called the latusrectum. If the distance between the vertex and the focus is a then the length of the latusrectum is $4a$, Fig. 6.63.

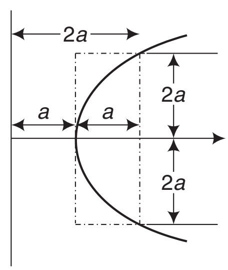


Fig. 6.63

Equation of the parabola: Let S be the focus and ZQ be the directrix of the parabola. Draw PM and PQ perpendicular from P to the axis of the parabola and the directrix of the parabola respectively. Now by definition, Fig. 6.64

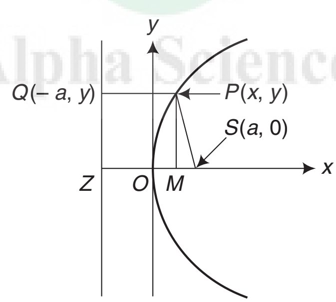


Fig. 6.64

$$SO = OZ = a$$

$$(SP)^2 = (PQ)^2 \quad \text{and} \quad (PQ)^2 = (ZM)^2$$

But $$(SP)^2 = (a - x)^2 + y^2$$

So $$(SP)^2 = (ZM)^2$$

$$(a - x)^2 + y^2 = (a + x)^2$$

$\Rightarrow$ $$\boldsymbol{y^2 = 4ax}$$

This is the standard equation of the parabola.

Note: for this standard equation to find the coordinate of the vertex we put $y^2 = 0 = x$, here vertex at (0, 0), according to the definition equation of the directrix is $x + a = 0$, the coordinate of the focus is $(a, 0)$, to obtain the tangent to the parabola we put coefficient of $4a$ is equal to zero, hence here equation of tangent is $x = 0 \Rightarrow y$-axis is tangent and to find the equation of the axis of the parabola we put $y^2 = 0 \Rightarrow x$-axis is the axis of the parabola.

Other cases of the parabolas

Open right

$y^2 = 4ax$ or $(y - k)^2 = 4a(x - h)$

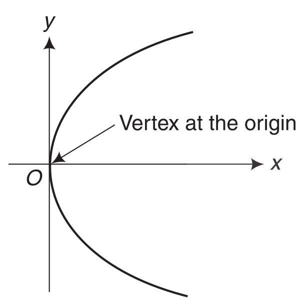


Fig. 6.65

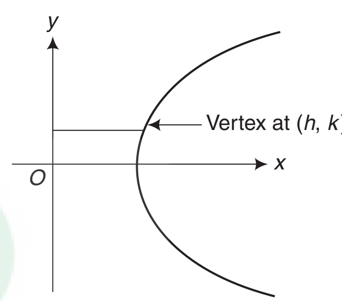


Fig. 6.66

Open left

$y^2 = -4ax$ or $(y - k)^2 = -4a(x - h)$

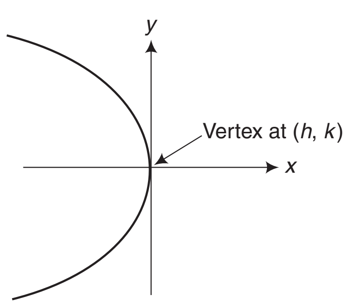


Fig. 6.67

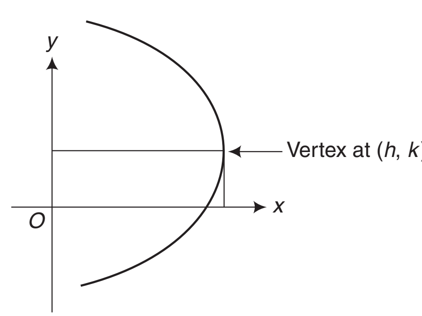


Fig. 6.68

Open up

$x^2 = 4ay$ or $(x - h)^2 = 4a(y - k)$

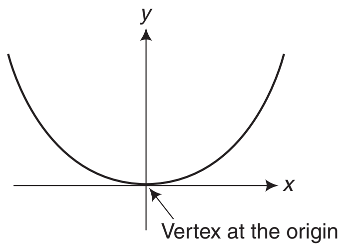


Fig. 6.69

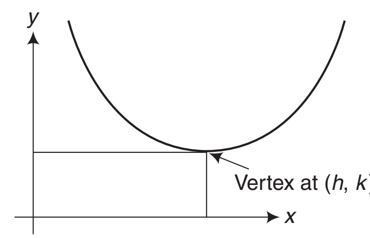


Fig. 6.70

Open down

$x^2 = -4ay$ or $(x - h)^2 = -4a(y - k)$

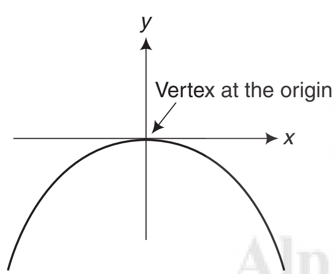


Fig. 6.71

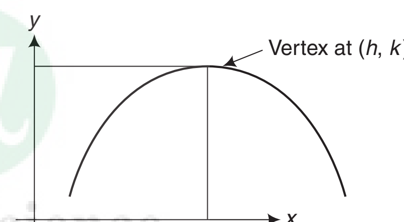


Fig. 6.72

Some useful points for sketching the parabola of standard equation

Suppose we want to sketch the graph of the parabola whose equation is $x^2 = 4y$.

(i) Write the given equation in the standard form $\Rightarrow x^2 = 4(1.)\ y \Rightarrow a = 1$ $\Rightarrow$ (the distance between the focus and vertex is 1)

(ii) Parabola open. The given equation has the form as in Fig. 6.69, therefore the parabola open up.

(iii) Determine the symmetry (the curve has the symmetry about x-axis If the vertex at the origin or the symmetry about a line parallel to x-axis If the vertex not at the origin if equation has y^2 term and symmetry about the y-axis If the vertex at the origin or the symmetry about a line parallel to y-axis If the vertex not at the origin if equation has x^2 term). Now the given equation has x^2 term and to obtain the vertex we put $x = 0 = y$. therefore the centre lies at the origin so the given parabola has the symmetry about the y-axis.

Now we can draw the parabola by use these three points and use the property define in Fig. 6.63 as

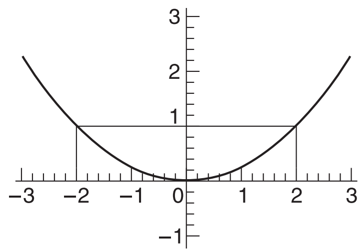


Fig. 6.73

Example 7 Find the vertex, focus, directrix, tangent and axis of the parabola $y^2 - 4y + 8x - 20 = 0$, and also sketch the curve.

Solution (i) Given equation $y^2 - 4y + 8x - 20 = 0$ can be written as $(y - 2)^2 = -4(2)(x - 3) \Rightarrow a = -2$.

(ii) The given equation has the form as in Fig. 6.68, therefore the parabola open left.

To find the vertex we put $(y - 2)^2 = 0 = (x - 3) \Rightarrow$ the vertex lies at (3, 2). Here $a = -2 \Rightarrow$ distance from the vertex to the focus is 2, therefore the coordinate of the focus is (3 – 2 = 1, 2). Now the distance between the vertex and the directrix is 2, so the equation of the directrix is $x = 3 + 2 = 5$. For equation of tangent $(x - 3) = 0$, therefore the equation of the tangent is $x = 3$. For axis of the parabola $(y - 2)^2 = 0$, so the axis of the parabola is $y = 2$.

(iii) the given equation has y^2 term and the vertex at (3, 2) so the given parabola has the symmetry about the line which is parallel to x-axis.

Now we can draw the parabola by use these three points and use the property define in Fig. 6.63 as (Fig. 6.74)

Example 8 Find the vertex, focus, directrix, tangent and axis of the parabola $x^2 - 2x + 12y - 23 = 0$, and also sketch the curve.

Solution (i) Given equation $x^2 - 2x + 12y - 23 = 0$ can be written as $(x - 1)^2 = -4(3)(y - 2) \Rightarrow a = -3$.

(ii) The given equation has the form as in Fig. 6.72, therefore the parabola open down

To find the vertex we put $(x - 1)^2 = 0 = (y - 2) \Rightarrow$ the vertex lies at (1, 2). Here $a = -3 \Rightarrow$ distance from the vertex to the focus is 3, therefore the coordinate of the focus is (1, 2 – 3 = –1). Now the distance between the vertex and the directrix is 3, so the equation of the directrix is $y = 3 + 2 = 5$. For equation of tangent $(y - 2) = 0$, therefore the equation of the tangent is

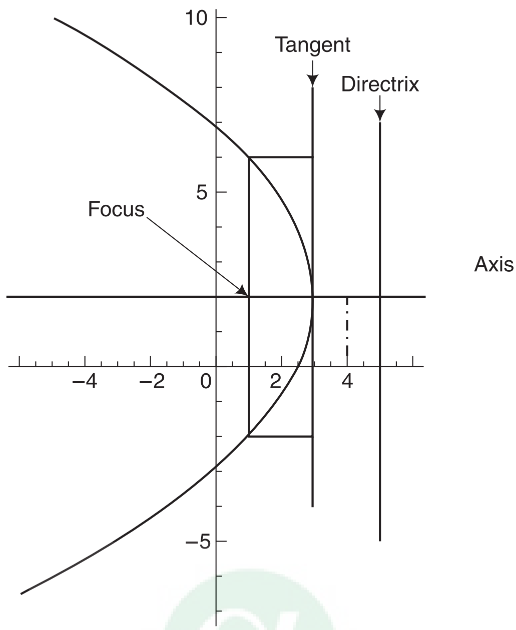


Fig. 6.74

$y = 2$. For axis of the parabola $(x - 1)^2 = 0$, so the axis of the parabola is $x = 1$.

(iii) the given equation has x^2 term and the vertex at (1, 2) so the given parabola has the symmetry about the line which is parallel to x-axis.

Now we can draw the parabola by use these three points and use the property define in Fig. 6.63 as

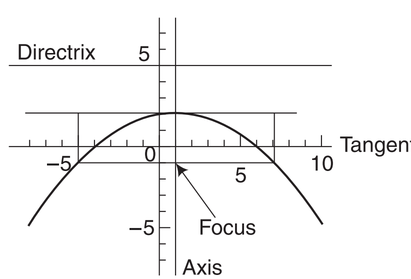


Fig. 6.75

Example 9 Find the equation of the parabola whose satisfies the following conditions

(i) vertex(0, 0); focus(2, 0)

(ii) vertex (1, 2); directrix $y = -1$

(iii) focus (0, – 4) directrix $y = 2$.

Solution (i) vertex at(0, 0) and focus at(2, 0) here distance between the focus and vertex is 2. $\Rightarrow a = 2$, and directrix at $x = -2$, Fig. 6.76.

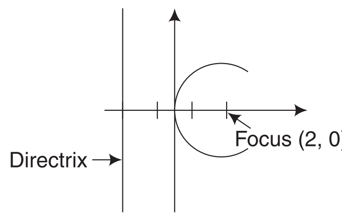


Fig. 6.76

So the equation is $(y - 0)^2 = 8(x - 0) = y^2 = 8x$.

(ii) vertex at (1, 2) and directrix at $y = -1$, therefore distance between the directrix and vertex is 3 $\Rightarrow$ distance between the focus and vertex is 3, but here vertex is at (1, 2), Fig. 6.77.

So the equation is $(x - 1)^2 = 12(y - 2)$.

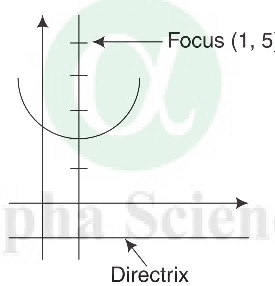


Fig. 6.77

(iii) focus at (0, – 4) and directrix at $y = 2$, we know that the vertex lie between the directrix and focus, therefore the coordinate of the vertex is (0, – 1), and $a = -3$, Fig. 6.78.

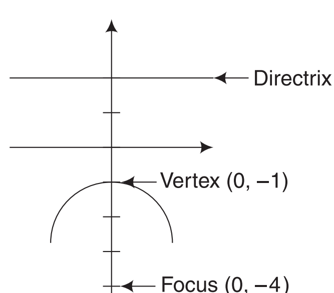


Fig. 6.78

So the equation is $x^2 = -12(y + 1)$

Parametric equation of a parabola:

The equation $y^2 = 4ax$ may be written as

$$\frac{y}{2a} = \frac{2x}{y} = t$$

Then $y = 2at$ and $yt = 2x$ or $x = at^2$, hence the parametric equation of the parabola is

$$\boldsymbol{x = at^2,\ y = 2at.}$$

Equation of the tangent at parabola: Let P and Q be two points on the parabola whose coordinates are (x_1, y_1) and (x_2, y_2) respectively, Fig. 6.79

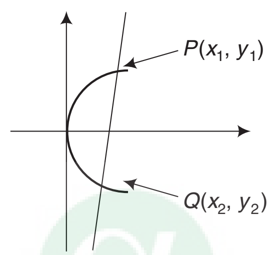


Fig. 6.79

The equation of PQ is

$$(y - y_1) = \frac{(y_2 - y_1)}{(x_2 - x_1)}\ (x - x_1) \tag{6.4}$$

Since P and Q lies on the parabola

$$\therefore \quad y_1^2 = 4ax_1 \tag{6.5}$$

$$\text{and} \quad y_2^2 = 4ax_2 \tag{6.6}$$

From (6.5) and (6.6), we get $y_2^2 - y_1^2 = 4a(x_2 - x_1)$

$$\Rightarrow \quad \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{4a}{y_2 + y_1}$$

Substituting $\frac{(y_2 - y_1)}{(x_2 - x_1)}$ in (6.4), we get

$$(y - y_1) = \frac{4a}{y_2 + y_1}\ (x - x_1) \tag{6.7}$$

The line PQ will be the tangent at P if $Q \to P \Rightarrow x_2 \to x_1$ and $y_2 \to y_1$, then (6.7) becomes

$$(y - y_1) = \frac{2a}{y_1}(x - x_1)$$

$$yy_1 = y_1^2 + 2ax - 2ax_1$$

$$\mathbf{yy_1 = 2a(x + x_1)} \qquad (y_1^2 = 4ax_1)$$

This is the required equation of tangent at a point (x_1, y_1).

The equation of the tangent in parametric form at $(at^2, 2at)$ is which can be obtained from equation (6.4) after substitution $(at^2, 2at)$ and $(at_1^2, 2at_1)$ and $t_1 \to t$.

$$\mathbf{ty = x + at^2}$$

Properties of parabola

1. The tangent at a point P on a parabola makes equal angles with the line through P parallel to the axis of symmetry and the line through P and the focus. (Reflection property) Fig. 6.80(a).

Let the coordinate of P is $(at^2, 2at)$, a tangent PQ at P meet the x-axis at Q, we know that the equation of the tangent PQ is $ty = x + at^2$, so the coordinate of Q is $(-at^2, 0)$, Fig. 6.80(b).

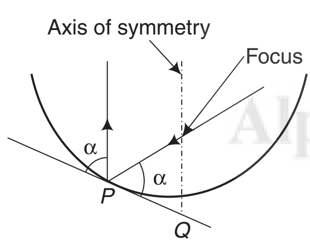


Fig. 6.80(a)

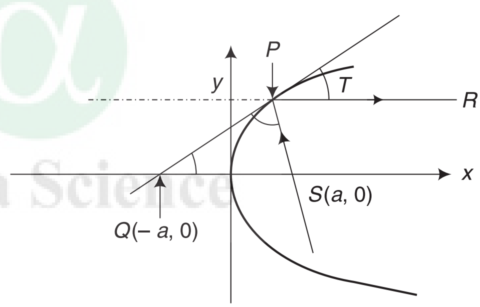


Fig. 6.80(b)

Therefore $SQ = |-at^2 - a| = a(1 + t^2)$

And $PS = \sqrt{(at^2 - a)^2 + (2at - 0)^2} = \sqrt{(at^2 + a)^2} = a(1 + t^2)$

$\Rightarrow PS = SQ \Rightarrow$ in triangle PQS, $\angle SQP = \angle SPQ$

But ray PR is parallel to x-axis, hence $\angle SQP = \angle RPT$.

$$\Rightarrow \qquad \angle QPS = \angle RPT.$$

2. The tangent at a point P on a parabola bisect the angle between the focal chord through P and the perpendicular from P on the directrix. Fig. 6.81.

We know that the equation of tangent PT is $ty = x + at^2$ and the slope of this tangent is $\frac{1}{t}$.

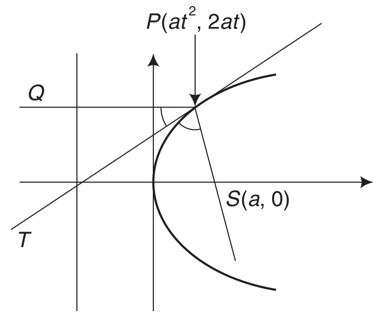


Fig. 6.81

Similarly the slope of *PQ* and *PS* are 0 and $\dfrac{2t}{t^2-1}$ respectively. If θ and ϕ be the angle *QPT* and *SPT* respectivaly, then

$$\tan\theta = \frac{\frac{1}{t} - 0}{1 + \frac{1}{t}\cdot 0} = \frac{1}{t}$$

And

$$\tan\phi = \frac{\frac{2t}{t^2-1} - \frac{1}{t}}{1 + \frac{1}{t}\cdot\frac{2t}{t^2-1}} = \frac{1}{t}$$

Hence, $\tan\theta = \tan\phi \Rightarrow \theta = \phi$.

3. The tangents at the extremities of a focal chord intersect at right angle on the directrix.

Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the extremities of the focal chord, Fig. 6.82.

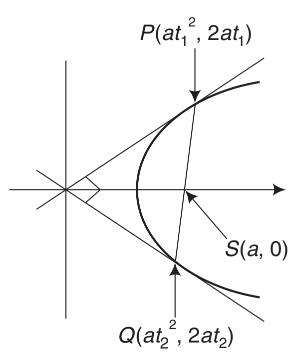


Fig. 6.82

We know that the equation of PQ is

$$(y - 2at_1) = \frac{(2at_2 - 2at_1)}{(at_2^2 - at_1^2)}\ (x - at_1^2) \tag{6.8}$$

Since PQ passes through $(a, 0)$ we get

$$t_1 t_2 = -1 \tag{6.9}$$

Equation of tangents at P and Q are

$$t_1 y = x + at_1^2 \tag{6.10}$$

And
$$t_2 y = x + at_2^2 \tag{6.11}$$

These two tangents intersect when

$$x = a t_1 t_2 \tag{6.12}$$

Therefore from (6.9) and (6.12) we have $x = -a$.

And the slope of (6.10) and (6.11) are $\frac{1}{t_1}$ and $\frac{1}{t_2}$ respectively

Then
$$\frac{1}{t_1} \cdot \frac{1}{t_2} = -1$$

Hence the result.

Example 10 In Fig. 6.83, find the area of rectangle.

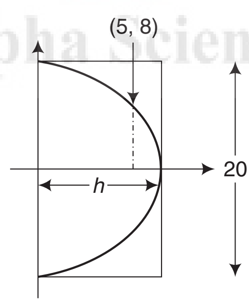


Fig. 6.83

Solution The equation of the given parabola can be written as $y^2 = -4a(x - h)$

Now this parabola Passes through from (0, 10) and (5, 8), hence we have

$$100 = 4ah \tag{6.13}$$

$$64 = -20a + 4ah \tag{6.14}$$

From equations (6.13) and (6.14), we have $h = \frac{125}{9}$

So the area of rectangle is $20 \times \frac{125}{9} \approx 277.77$

Example 11 Suppose a comet moves in parabolic orbit with sun at its focus and that the line from the sun to the comet makes an angle of 60° with the axis of the parabola when the comet is 30 million miles from the centre of the sun. determine the nearest position of the sun to the comet.

Solution From above Fig. 6.84 let the comet at P, then $PS = 30$ and the angle PSM is equal to 60°. Let QN be the directrix then by definition the comet will be the nearest to the sun when it will be at the vertex.

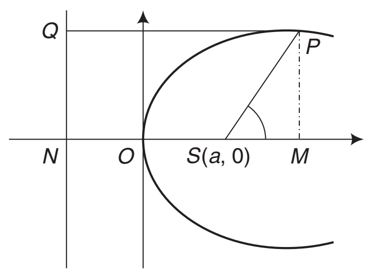


Fig. 6.84

We know that $PQ = MN = 2a + 30 \cos 60 = 2a + 15 = 30$ ($PQ = PS$ by bdefinition), hence $a = 7.5$

The nearest distance between the sun and comet is 7.5 million.

***Ellipse*:** is the locus of a point which moves such that the sum of whose distances from two fixed point is a given positive constant that is greater than the distance between the fixed point, these two fixed points are called the **foci**, midpoint *of* the line segment join the foci is called the **centre**, the line segment through the foci and across the ellipse is called the **major axis,** and the line segment perpendicular to the major axis across the ellipse and through the centre is called the **minor axis.** The end points of the major axis are called the **vertices** of the ellipse, Fig. 6.85.

Or

Ellipse is the locus of a point which moves such that its distance from a fixed point is $e(e < 1)$ times from a fixed line, fixed point is called the focus and fixed line is called the directrix of the ellipse. (*e is called the ecentricity*), Fig. 6.86.

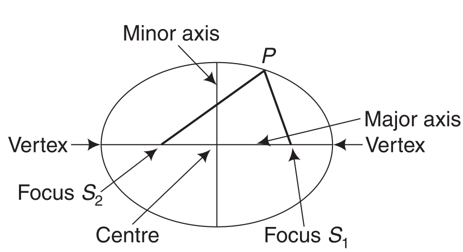


Fig. 6.85 $PS_1 + PS_2$ = constant > distance between S_1 and S_2

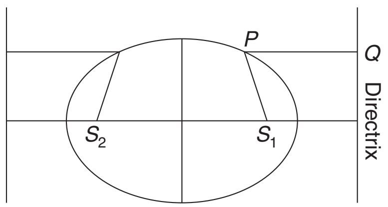


Fig. 6.86(a) $PS_1 = e\ PQ$

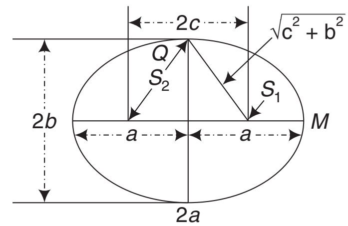


Fig. 6.86(b)

Equation of the ellipse:

suppose the length of the major axis is $2a$, length of the minor axis is $2b$ and distance from the centre to the focus is c, then the relation between a, b and c can be obtained by using the definition of the ellipse as, Fig. 6.87

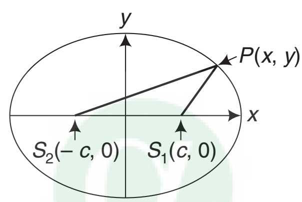


Fig. 6.87

$$QS_1 + QS_2 = MS_1 + MS_2 \Rightarrow 2\sqrt{b^2 + c^2} = (a - c) + (a + c)$$

$$\Rightarrow \qquad \boldsymbol{a = \sqrt{b^2 + c^2}} \qquad (6.15)$$

Where M and Q are the points on the ellipse at the end of the major and minor axis respectively. Equation (i) implies that for all points on the ellipse the sum of the distances to the foci is $\boldsymbol{2a}$**,** which is constant and greater than the distance between the foci, this equation also show that $a \geq b$, and a will be equal to b only when $c = 0$, which implies that the major axis will be the larger from the minor axis and these two axes have the equal length only when the foci coincide, and in this case the ellipse will be a circle.

To obtain the standard equation of the ellipse, let $P(x, y)$ be a point on the ellipse and foci on the x-axis with centre at the origin, Fig. 6.87. Since the sum of the distances from P to the foci is $2a$, hence

$$PS_1 + PS_2 = 2a$$

$$\Rightarrow \sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a$$

$$\Rightarrow (x + c)^2 + y^2 = 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + (x - c)^2 + y^2$$

After simplify, we get

$$\sqrt{(x-c)^2+y^2} = a - \frac{c}{a}x$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$$

With the help of equation (6.15), we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{6.16}$$

Let C be the center at the origin, S_1 be the focus at the x-axis, Z_1M be the directrix, PM and S_1Z_1 are the perpendicular from P and S_1 on the directrix respectively, Fig. 6.88.

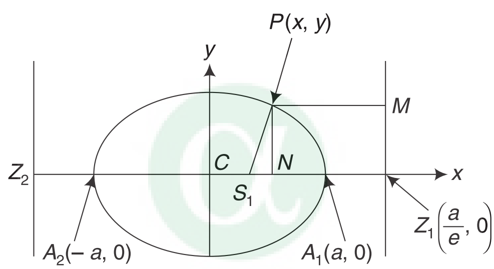


Fig. 6.88

Now by definition

$$S_1A_1 = eA_1Z_1 \tag{6.17}$$

And

$$S_1A_2 = eA_2Z_1 \tag{6.18}$$

Adding (6.17) and (6.18), we get

$$S_1A_1 + S_1A_2 = e(\mathrm{A}_1Z_1 + \mathrm{A}_2Z_1) = e(CZ_1 - CA_1 + CZ_1 + CA_2)$$

$$S_1A_1 + S_1A_2 = 2eCZ_1 \qquad (CA_1 = CA_2)$$

$$\therefore \quad CZ_1 = \frac{a}{e} \qquad (S_1A_1 + S_1A_2 = 2a)$$

So the equation of the directrix is $x = \frac{a}{e}$

Now subtracting (6.17) from (6.18), we get

$$S_1A_2 - S_1A_1 = e(A_2Z_1 - A_1Z_1)$$

$$(A_2C + CS_1) - (CA_1 - CS_1) = 2ae$$

$$\Rightarrow \quad CS_1 = ae$$

Hence the coordinate of the focus is (ae, 0) and the equation of the directrix is $x = a/e$.

Now $$PS_1 = ePM \quad \text{(by definition)}$$

Squaring both sides, we get

$$(PS_1)^2 = e^2(PM)^2$$

$$(x - ae)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2 \quad (PM = (NZ_1)$$

$$(ae)^2 + x^2 - 2aex + y^2 = a^2 + x^2e^2 - 2aex$$

$$\Rightarrow \quad x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

Or

$$\frac{x^2}{a^2} + \frac{y^2}{a^2\,(1 - e^2)} = 1$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad b^2 = a^2(1 - e^2)$$

If we take the major axis at y-axis and minor axis at x-axis, then the equation of ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

Some useful points for sketching the Ellipse of standard equation

Suppose we want to sketch the graph of the Ellipse whose equation is $x^2 + 9y^2 = 9$.

1. Write the equation in the standard form. So the given equation can be written as

$$\frac{x^2}{3^2} + \frac{y^2}{1^2} = 1$$

2. Determine a = length of the semi major axis and b = length of the semi minor axis then draw a rectangle whose sides are $2a$ and $2b$. Hence here $a = 3$ and $b = 1$.
3. Now sketch the ellipse such that the centre of the ellipse (to obtained the centre of the ellipse we put $x^2 = 0 = y^2$) is at the center of the rectangle and touch the sides as, Fig. 6.89

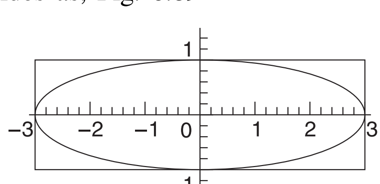


Fig. 6.89

For the coordinates of the foci, we obtain

$$c = \sqrt{a^2 - b^2} = \sqrt{9 - 1} = \sqrt{8}$$

Thus, the coordinates of the foci are $(\sqrt{8}, 0)$ and $(\sqrt{8}, 0)$.

Example 12 Describe the graph of the equation

$9x^2 + 4y^2 - 36x - 24y + 36 = 0$

Solution 1. The given equation

$$9x^2 + 4y^2 - 36x - 24y + 36 = 0$$

Can be written as

$$9(x^2 + 4 - 4x) + 4(y^2 - 6y + 9) = 36$$

Or

$$\frac{(x-2)^2}{4} + \frac{(y-3)^2}{9} = 1$$

2. $a = 3$ and $b = 2$
3. To obtain the coordinates of the centre we put $(x - 2)^2 = 0 \Rightarrow x = 2$ and $(y - 3)^2 = 0 \Rightarrow y = 3$,

So we can draw the ellipse as, Fig. 6.90.

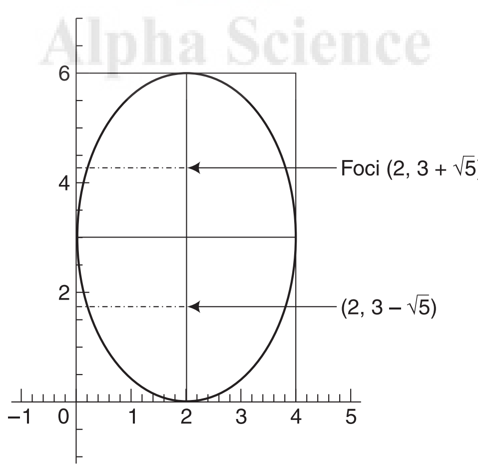


Fig. 6.90

The equation of major axis can be obtain as $x - 2 = 0$, and the equation of minor axis can be obtain as $y - 3 = 0$, for the coordinates of foci we fined $c = \sqrt{b^2 - a^2} = \sqrt{9 - 4} = \sqrt{5}$. Hence the coordinates of the foci are $(2, 3 + \sqrt{5})$

and $(2, 3 - \sqrt{5})$. And the coordinates of end of the major axis are (2, 0) and (2, 6).

***Example* 13** Describe the graph of the equation

$$\frac{(x + 3)^2}{6} + \frac{(y + 2)^2}{5} = 1$$

Solution Here $a^2 = 6$ and $b^2 = 5$

To obtain the coordinates of the centre we put $(x + 3)^2 = 0 \Rightarrow x = -3$ and $(y + 2)^2 = 0 \Rightarrow y = -2$,

So we can draw the ellipse as Fig. 6.91.

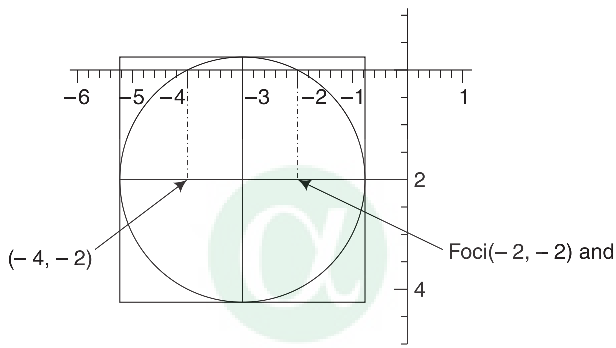


Fig. 6.91(a)

The equation of major axis can be obtain as $x = -3$, and the equation of minor axis can be obtain as $y = -2$, for the coordinates of foci we find $c = \sqrt{a^2 - b^2} = \sqrt{6 - 5} = 1$. Hence the coordinates of the foci are $(-2, -2)$ and $(-4, -2)$. And the coordinates of end of the major axis are $(-3 + \sqrt{6}, -2)$ and $\{-(3 + \sqrt{6}), -2\}$. The coordinates of end of the minor axis are $(-3, \sqrt{5} - 2)$ and $\{-3, -(2 + \sqrt{5})\}$.

***Example* 14** Find the equation of the ellipse whose satisfies the following conditions

(i) Ends of the major axis $(\pm 5, 0)$ and the ends of the minor axis $(0, \pm 3)$,

(ii) Length of major axis 18 and foci $(\pm 2, 0)$,

(iii) Centre at (0, 0); major and minor axes along the coordinate axes and passes through the points (2, 3) and (1, 4).

Solution (i) To write the equation of the ellipse we needs a and b and here $a = 5$ and $b = 3$ (as by rough sketch Fig. 6.91)

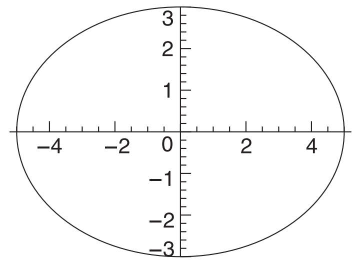


Fig. 6.91(b)

Hence the equation is $\frac{x^2}{25} + \frac{y^2}{9} = 1.$

(ii) Here $a = \frac{18}{2} = 9$, $c = 2$ and $b^2 = a^2 - c^2$,

therefore $b^2 = (9)^2 - (2)^2 = 77$

Hence the equation is $\frac{x^2}{81} + \frac{y^2}{77} = 1.$

(iii) we know that the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

And this ellipse passes through from (2, 3) and (1, 4) hence

$$\frac{4}{a^2} + \frac{9}{b^2} = 1 \quad (6.19)$$

And

$$\frac{1}{a^2} + \frac{16}{b^2} = 1 \quad (6.20)$$

From (6.19) and (6.20), we get $a^2 = \frac{55}{7}$ and $b^2 = \frac{55}{3}$

Hence the equation is $7x^2 + 3y^2 = 55.$

Equation of the tangent at ellipse: Let P and Q be two points on the ellipse whose coordinates are (x_1, y_1) and (x_2, y_2) respectively, Fig. 6.92.

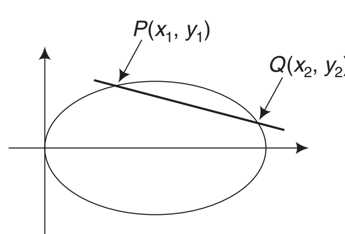


Fig. 6.92

The equation of PQ is

$$(y - y_1) = \frac{(y_2 - y_1)}{(x_2 - x_1)} (x - x_1) \tag{6.21}$$

Since P and Q lies on the ellipse

$$\therefore \quad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \tag{6.22}$$

$$\text{And} \quad \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1 \tag{6.23}$$

From (6.22) and (6.23), we get

$$\frac{(y_2 - y_1)}{(x_2 - x_1)} = -\left(\frac{x_1 + x_2}{y_2 + y_1}\right)\frac{b^2}{a^2}$$

Substituting $\frac{(y_2 - y_1)}{(x_2 - x_1)}$ in (6.21), we get

$$(y - y_1) = -\left(\frac{x_1 + x_2}{y_2 + y_1}\right)\frac{b^2}{a^2}(x - x_1) \tag{6.24}$$

The line PQ will be the tangent at P if $Q \to P \Rightarrow x_2 \to x_1$ and $y_2 \to y_1$, then (6.24) becomes

$$\Rightarrow \quad (y - y_1) = -\frac{b^2 x_1}{a^2 y_1}(x - x_1)$$

$$\Rightarrow \quad \frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = -\frac{xx_1}{a^2} + \frac{x_1^2}{a^2}$$

$$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \qquad \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1\right)$$

$$\boldsymbol{\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1}$$

This is the required equation of tangent at a point (x_1, y_1).

Auxiliary circle: The circle described on the major axis of an ellipse as diameter A_1A_2 is called the Auxiliary circle of the ellipse, Fig. 6.93.

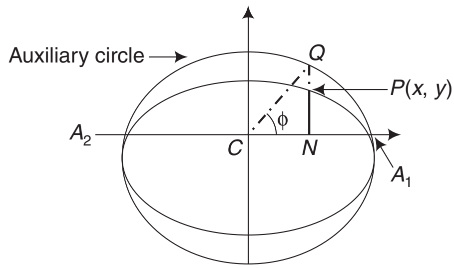


Fig. 6.93

The equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Let $P(x, y)$ be any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Draw PN perpendicular on A_1A_2 and produced PN to meet the auxiliary circle at Q, the point P lies on the ellipse, we have

$$\frac{x^2}{a^2} + \frac{(PN)^2}{b^2} = 1 \tag{6.25}$$

And the point Q lies on the circle, we have

$$x^2 + (QN)^2 = a^2 \tag{6.26}$$

From (6.25) and (6.26), we have

$$\frac{(PN)^2}{b^2} - \frac{(QN)^2}{a^2} = 0$$

$$\Rightarrow \qquad \frac{PN}{QN} = \frac{b}{a}$$

Eccentric angle: The eccentric angle of a point on the ellipse is the angle which the straight line joining the centre to the corresponding point on the auxiliary circle makes with positive side of the major axis in Fig. 6.93 the angle ϕ is the eccentric angle. From Fig. 6.93, we have

$$\frac{CN}{CQ} = \cos\phi \Rightarrow CN = CQ\cos\phi = a\cos\phi$$

And we know that $\frac{PN}{QN} = \frac{b}{a} \Rightarrow PN = QN\,\frac{b}{a}\,a\sin\phi = b\sin\phi$ $(QN = a\sin\phi)$

Hence the coordinate of P is $(a\cos\phi, b\sin\phi)$, and the coordinate of Q is $(a\cos\phi, a\sin\phi)$.

Properties of Ellipse

1. A tangent line at a point P on the ellipse makes equal angles with the line joining the point P to the foci. (Reflection property), Fig. 6.94.

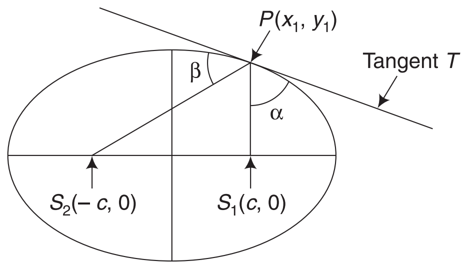


Fig. 6.94

Let the line S_1P makes the angle α with tangent line T and the line S_2P makes the angle β with tangent line T, Fig. 6.94.

Now we want to show that $\alpha = \beta$

The equation of the tangent line T at a point $P(x_1, y_1)$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

then the slope of this line is $\frac{-x_1 b^2}{y_1 a^2} = m_1$, the slope of S_1P is $\frac{y_1}{x_1 - c} = m_2$

and the slope of S_2P is $\frac{y_1}{x_1 + c} = m_3$.

$$\tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\frac{y_1}{x_1 - c} - \left(\frac{-x_1 b^2}{y_1 a^2}\right)}{1 + \left(\frac{-x_1 b^2}{y_1 a^2}\right)\left(\frac{y_1}{x_1 - c}\right)}$$

$$= \frac{y_1^2 a^2 + x_1 b^2 (x_1 - c)}{y_1 a^2 (x_1 - c) - y_1 x_1 b^2} = \frac{-b^2}{y_1 c}$$

$\left(x_1^2 b^2 + y_1^2 a^2 = a^2 b^2, \text{ and } a^2 - b^2 = c^2\right)$. Similarly we can find

$$\tan \beta = \frac{m_1 - m_3}{1 + m_1 m_3} = \frac{\left(\frac{-x_1 b^2}{y_1 a^2}\right) - \frac{y_1}{x_1 + c}}{1 + \left(\frac{-x_1 b^2}{y_1 a^2}\right)\left(\frac{y_1}{x_1 + c}\right)}$$

$$= \frac{-\left\{y_1^2\, a^2 + x_1\, b^2(x_1 + c)\right\}}{y_1 a^2(x_1 + c) - y_1\, x_1 b^2} = \frac{-b^2}{y_1 c}, \text{ hence } \alpha = \beta.$$

2. If the tangent at any point P meets the major axis at Q and the minor axis at T, then

   $\boldsymbol{CN \cdot CQ = a^2, CM \cdot CT = b^2}$**.** Fig. 6.95

   We know that the equation of tangent at P is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$.

   Now at Q, $xx_1 = a^2$, But $CN = x_1$, and $CQ = x$, hence $CN \cdot CQ = a^2$. Similarly $CM \cdot CT = b^2$

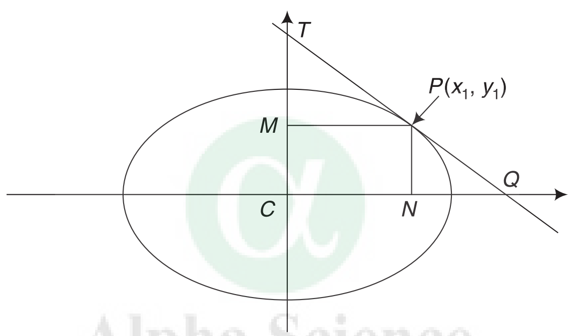


Fig. 6.95

3. If S_1Y_1 and S_2Y_2 be the perpendicular from the foci upon the tangent at any point of the ellipse, then $S_1Y_1 \cdot S_2Y_2$ Fig. 6.96

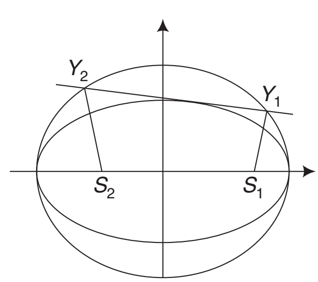


Fig. 6.96

   Let $y = mx + \sqrt{a^2m^2 + b^2}$ be any tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, the coordinates S_1 and S_2 are $(ae, 0)$ and $(-ae, 0)$ respectively.

$$S_1Y_1 = \frac{mae + \sqrt{a^2m^2 + b^2}}{\sqrt{1 + m^2}}$$

(S_1Y_1 is a perpendicular distance from S_1 to the tangent)

And $$S_2Y_2 = \frac{-mae + \sqrt{a^2m^2 + b^2}}{\sqrt{1 + m^2}}$$

(S_2Y_2 is a perpendicular distance from S_2 to the tangent)

$$S_1Y_1 \cdot S_2Y_2 = \frac{a^2m^2 + b^2 - a^2m^2e^2}{1 + m^2}$$

$$S_1Y_1 \cdot S_2Y_2 = \frac{b^2(1 + m^2)}{1 + m^2} = b^2. \quad \{b^2 = a^2(1 - e^2)\}$$

Example 15 Find the equation of the ellipse traced by a point which moves such that the sum of its distances to (5, 1) and (5, 7) is 14.

Solution By definition we know that the given points (5, 1) and (5, 7) are the foci, and the center of the ellipse lies between the foci therefore the coordinate of the center is (5, 4), Fig. 6.97

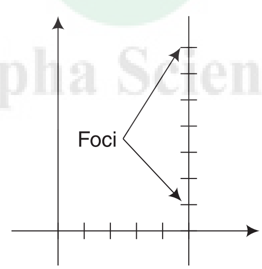


Fig. 6.97

Now $2a = 14 \Rightarrow a = 7$, $c = 3$, $b^2 = a^2 - c^2$, $b^2 = 49 - 9 \Rightarrow b = \sqrt{40}$

So the equation of the ellipse is

$$\frac{(x - 5)^2}{40} + \frac{(y - 4)^2}{49} = 1$$

Example 16 Find the values of k such that the line $x + 3y = k$ touch the ellipse $x^2 + 6y^2 = 10$. Find the point of tangency.

Solution Equation of the ellipse is $x^2 + 6y^2 = 10$, then $\frac{dy}{dx} = -\frac{x}{6y}$, x_0 and y_0 are the points on the ellipse where a line touch the ellipse, hence the slope of the tangent line is $= -\dfrac{x_0}{6y_0}$. Now if given line touch the ellipse at the same point (x_0, y_0), then $-\dfrac{x_0}{6y_0} = -\dfrac{1}{3}$

$$\Rightarrow \quad x_0 = 2y_0 \tag{6.27}$$

This point (x_0, y_0) lies on the ellipse, we have

$$x_0^2 + 6y_0^2 = 10 \tag{6.28}$$

From equations (6.27) and (6.28), we have $x_0 = \pm 2$ and $y_0 = \pm 1$, therefore the point of tangency are (2, 1) and (–2, –1).

Put these values in given equation $x + 3y = k$, we have $k = \pm 5$.

Example 17 Find the equation of the tangent to the ellipse $4x^2 + 3y^2 = 5$ which are parallel to the straight line $3x + 7 = y$.

Solution Let the equation of a line which is parallel to $3x + 7 = y$ is $y = 3x + k$, now to find the value of k we proceed as in example 16, we get $k = \pm\sqrt{\dfrac{155}{12}}$.

***Hyperbola*:** is the locus of a point which moves such that the difference of whose distances from two fixed point is a given positive constant that is less than the distance between the fixed point, these two fixed points are called the **foci**, midpoint *of* the line segment join the foci is called the **center**, the line segment through the foci and across the *hyperbola* is called the transverse axis(focal axis)**,** and the line segment perpendicular to the transverse axis and through the centre is called the **conjugate axis.** The hyperbola intersect the transverse axis at two points called the **vertices,** Fig. 6.98.

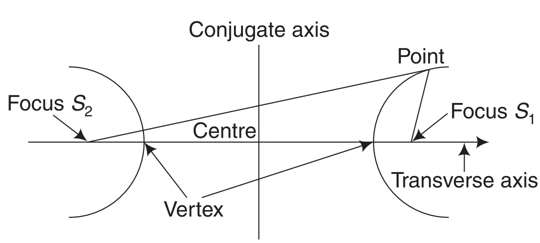


Fig. 6.98

Or

Hyperbola is the locus of a point which moves such that its distance from a fixed point is $e(e > 1)$times from a fixed line, fixed point is called the focus and fixed line is called the directrix of the *hyperbola*. (*e is called the ecentricity*), Fig. 6.99.

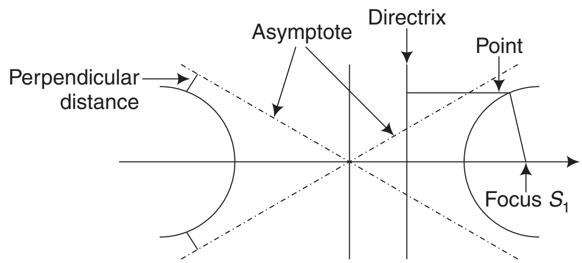


Fig. 6.99

The asymptote: An asymptote is a straight line which meets the hyperbola in two coincident points at infinity but it does not completely lie at infinity.

Or

A pair of straight lines intersect at the centre of the hyperbola and have the property that as a point P moves along the hyperbola away from the centre, the vertical distance between the point P and one of the asymptotes approaches zero in Fig. 6.99.

Equation of the hyperbola: suppose the length of the transverse axis is $2a$, length of the conjugate axis is $2b$ and distance from the center to the focus is c, then the relation between a, b and c can be obtained by using the definition of the hyperbola as, Fig. 6.100(a)

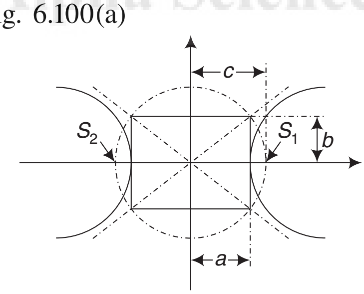


Fig. 6.100(a)

Let V is one vertex of the hyperbola and the distance from V to the further focus minus the distance from V to the closer focus is

$[(c - a) + 2a] - (c - a) = 2a$, Fig. 6.100(b)

From Fig. 6.100(a), we get $b^2 = c^2 - a^2$

To obtain the standard equation of the hyperbola, let $P(x, y)$ be a point on the hyperbola and foci on the x-axis with centre at the origin. Figure 6.100(b). Since the difference of the distances from P to the foci is $2a$, hence

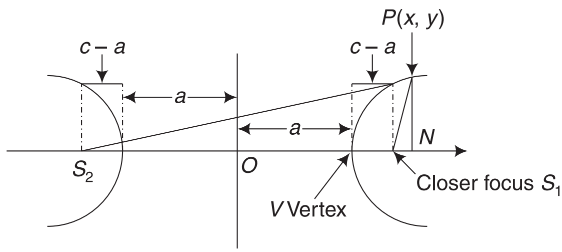


Fig. 6.100(b)

Now again from Fig. 6.100(b)

$$PS_2 - PS_1 = 2a \quad \text{(by definition)}$$

$$\Rightarrow \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$$

$$\Rightarrow (x+c)^2 + y^2 = 4a^2 + 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

After simplify, we get

$$\sqrt{(x-c)^2 + y^2} = \frac{c}{a}x - a$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{(c^2 - a^2)} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ (b^2 = c^2 - a^2)$$

Some useful points for sketching the hyperbola of standard equation

Suppose we want to sketch the graph of the hyperbola whose equation is $x^2 - 9y^2 = 9$.

1. Write the equation in the standard form. So the given equation can be written as

$$\frac{x^2}{3^2} - \frac{y^2}{1^2} = 1$$

2. Determine $2a$ = length of the transverse axis and $2b$ = length of the conjugate axis then draw a rectangle whose sides are $2a$ and $2b$. Hence here $a = 3$ and $b = 1$.
3. Draw the asymptotes along the diagonals of the rectangle.
4. Now sketch the hyperbola such that the centre of the hyperbola (to obtained the centre of the hyperbola we put $x^2 = 0 = y^2$) is at the centre of the rectangle and touch the sides as, Fig. 6.101.

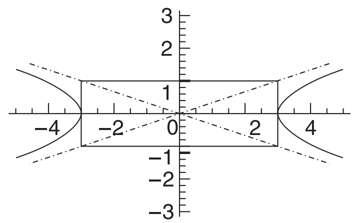


Fig. 6.101

Example 18 Describe the graph of the equation

$$4x^2 - 9y^2 - 24x + 36y = 36 = 0$$

Solution

1. Write the equation in the standard form. So the given equation can be written as

$$\frac{(x-3)^2}{3^2} - \frac{(y-2)^2}{2^2} = 1$$

2. Determine $2a$ = length of the transverse axis and $2b$ = length of the conjugate axis then draw a rectangle whose sides are $2a$ and $2b$. Hence here $a = 3$ and $b = 2$ and $c = \sqrt{a^2 + b^2} = \sqrt{9 + 4} = \sqrt{13}$.
3. Draw the asymptotes along the diagonals of the rectangle.
4. Now sketch the hyperbola such that the center of the hyperbola (to obtained the centre of the hyperbola we put $(x - 3)^2 = 0 \Rightarrow x = 3$ $(y - 2)^2 = 0 \Rightarrow y = 2$) is at the centre of the rectangle and touch the sides as, Fig. 6.102. Equation of the transverse axis is $x - 3 = 0$. and the equation of the conjugate axis is y – 2 = 0. Coordinate of the foci are $(3 + \sqrt{13}, 2)$ and $(3 - \sqrt{13}, 2)$.

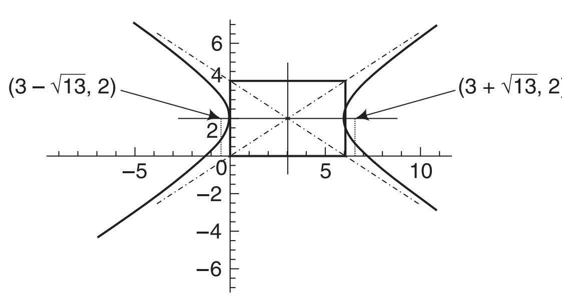


Fig. 6.102

***Example* 19** Describe the graph of the equation

$$\frac{(x+2)^2}{1} - \frac{(y-3)^2}{4} = 1$$

Solution 1. Given equation is

$$\frac{(x+2)^2}{1} - \frac{(y-3)^2}{4} = 1$$

2. Here $a = 2$ and $b = 1$ and $c = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$
3. Draw the asymptotes along the diagonals of the rectangle.
4. Now sketch the hyperbola such that the center of the hyperbola (to obtained the center of the hyperbola we put $(x + 2)^2 = 0 \Rightarrow x = -2$ $(y - 3)^2 = 0 \Rightarrow y = 3$) is at the center of the rectangle and touch the sides as, Fig. 6.103. Equation of the transverse axis is $y - 3 = 0$ and the equation of the conjugate axis is $x + 2 = 0$. Coordinate of the foci are $(-2 + \sqrt{5}, 3)$ and $(-2 - \sqrt{5}, 3)$.

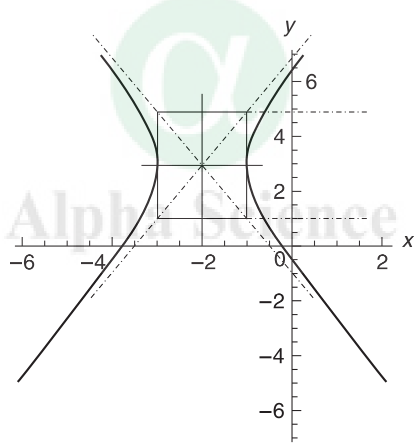


Fig. 6.103

Equation of asymptotes:

Let $$y = mx + c \tag{6.29}$$

be the asymptote to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{6.30}$$

The point of intersection of the asymptote and the hyperbola are given by the equation

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1 \text{ (from (6.29) and (6.30))}$$

$$x^2(b^2 - a^2m^2) - 2a^2mcx - a^2(b^2 + c^2) = 0 \tag{6.31}$$

Since the line $y = mx + c$ will meets the hyperbola (6.30) at infinity if both roots of (6.31) are infinite.

For this the coefficients of x^2 and x are both zero.

$$\Rightarrow \qquad b^2 - a^2m^2 = 0 = -2a^2mc$$

$$\Rightarrow \qquad m = \pm \frac{b}{a} x \text{ and } c = 0$$

Substitutes the value of m and c in (6.29), we have

$$y = \pm \frac{b}{a} x$$

i.e.

$$\frac{x}{a} - \frac{y}{b} = 0 \text{ and } \frac{x}{a} + \frac{y}{b} = 0$$

Now the combined equation of asymptotes is

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 0 \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

Conjugate hyperbola: A hyperbola whose transverse and conjugate axis are respectively the conjugate and transverse axis of a given hyperbola is called conjugate hyperbola of given hyperbola, Fig. 6.104.

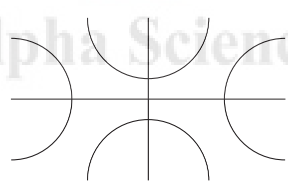


Fig. 6.104

Thus the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{6.32}$$

is conjugate to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \tag{6.33}$$

and the equation to its asymptotes is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \tag{6.34}$$

Here the equation (6.33) differs from equation (6.32) by a constant and the equation (6.34) differs from (6.33) by exactly the same quantity that (6.33) differs from (6.32). Bothe the asymptotes passes through the origin. The two asymptotes are equally inclined to the transverse axis with slope $\frac{b}{a}$.

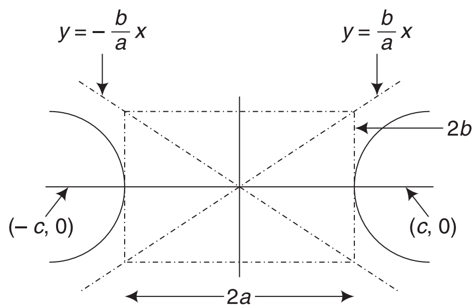


Fig. 6.104(a)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

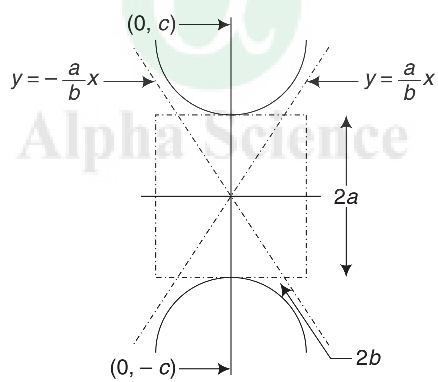


Fig. 6.104(b)

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Example 20 Find the equation of the hyperbola whose satisfies the following conditions

(i) Vertices $(\pm 2, 0)$ and foci $(\pm 3, 0)$,

(ii) Vertices $(0, \pm 6)$ and asymptotes $y = \pm 3x$,

(iii) Foci $(0, \pm 2)$; asymptotes $y = \pm x$.

Solution (i) Here $a = 2$, $c = 3$ and $b^2 = c^2 - a^2$, $b^2 = 9 - 4 = 5$, hence the equation is

$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$

(ii) Since the vertices are on the y-axis, the equation of the hyperbola is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ and the asymptote are $y = \pm \frac{a}{b} x$

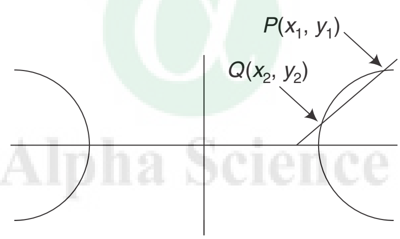
$a = 6$ and $\frac{a}{b} = 3 \Rightarrow b = 2$, hence the equation is

$$\frac{y^2}{36} - \frac{x^2}{4} = 1$$

(iii) $c = 2$, and $\frac{a}{b} = 1$, $a^2 + b^2 = 4 \Rightarrow 2a^2 = 4 \Rightarrow a^2 = b^2 = 2$, hence the equation is

$$\frac{y^2}{2} - \frac{x^2}{2} = 1$$

Equation of the tangent at hyperbola: Let P and Q be two points on the hyperbola whose coordinates are (x_1, y_1) and (x_2, y_2) respectively, Fig. 6.92.



The equation of PQ is

$$(y - y_1) = \frac{(y_2 - y_1)}{(x_2 - x_1)} (x - x_1) \quad (6.35)$$

Since P and Q lie on the hyperbola

$$\therefore \quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1 \quad (6.36)$$

And

$$\frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} = 1 \quad (6.37)$$

From (6.36) and (6.37), we get

$$\frac{(y_2 - y_1)}{(x_2 - x_1)} = \left(\frac{x_1 + x_2}{y_2 + y_1}\right) \frac{b^2}{a^2}$$

Substituting $\frac{(y_2 - y_1)}{(x_2 - x_1)}$ in (6.35), we get

$$(y - y_1) = \left(\frac{x_1 + x_2}{y_2 + y_1}\right) \frac{b^2}{a^2} (x - x_1) \tag{6.38}$$

The line PQ will be the tangent at P if $Q \to P \Rightarrow x_2 \to x_1$ and $y_2 \to y_1$, then (6.38) becomes

$$\Rightarrow \qquad (y - y_1) = \frac{b^2 x_1}{a^2 y_1} (x - x_1)$$

$$\Rightarrow \qquad \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$$

$$\Rightarrow \qquad \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \quad \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1\right)$$

$$\mathbf{\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1}$$

Which is the required equation of tangent at a point (x_1, y_1).

Properties of hyperbola

A tangent line at a point P on the hyperbola makes equal angles with the line joining the point P to the foci. (Reflection property), Fig. 6.106.

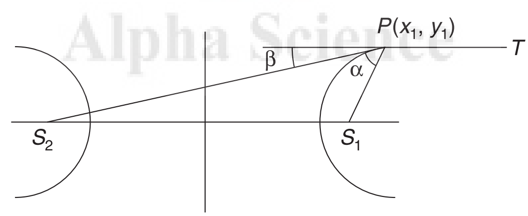


Fig. 6.106

Let the line S_1P makes the angle α with tangent line T and the line S_2P makes the angle β. with tangent line T, Fig. 6.106. Now we want to show that $\alpha = \beta$.

The equation of the tangent line T at a point $P(x_1, y_1)$ is $\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1$ then the slope of this line is $\dfrac{x_1 b^2}{y_1 a^2} = m_1$, the slope of S_1P is $\dfrac{y_1}{x_1 - c} = m_2$ and the slope of S_2P is $\dfrac{y_1}{x_1 + c} = m_3$.

$$\tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\dfrac{y_1}{x_1 - c} - \left(\dfrac{x_1 b^2}{y_1 a^2}\right)}{1 + \left(\dfrac{x_1 b^2}{y_1 a^2}\right)\left(\dfrac{y_1}{x_1 - c}\right)} = \frac{y_1^2 a^2 - x_1 b^2 (x_1 - c)}{y_1 a^2 (x_1 - c) + y_1 x_1 b^2}$$

$= \dfrac{b^2}{y_1 c}$, $(x_1^2 b^2 - y_1^2 a^2 = a^2 b^2$, and $a^2 + b^2 = c^2)$. Similarly we can find

$$\tan \beta = \frac{m_1 - m_3}{1 + m_1 m_3} = \frac{\left(\dfrac{x_1 b^2}{y_1 a^2}\right) - \dfrac{y_1}{x_1 + c}}{1 + \left(\dfrac{x_1 b^2}{y_1 a^2}\right)\left(\dfrac{y_1}{x_1 + c}\right)} = \frac{\{-y_1^2 a^2 + x_1 b^2 (x_1 + c)\}}{y_1 a^2 (x_1 + c) + y_1 x_1 b^2} = \frac{b^2}{y_1 c},$$

hence $\alpha = \beta$.

Example 21 Find the coordinates of all points on the hyperbola $5x^2 - y^2 = 5$ where the two lines that passes through the point and the foci intersect at right angle.

Solution From Fig. 6.106 let the line S_1P and S_2P intersect at right angle at P, the coordinates of S_1 and S_2 are $(\sqrt{6}, 0)$ and $(-\sqrt{6}, 0)$ respectively. Then the slope of S_1P is $\dfrac{y_1}{x_1 - \sqrt{6}}$ and S_1P is $\dfrac{y_1}{x_1 + \sqrt{6}}$, if the line intersect at right angle then $\dfrac{y_1}{x_1 - \sqrt{6}} \cdot \dfrac{y_1}{x_1 + \sqrt{6}} = -1$

$$\Rightarrow \qquad y_1^2 = 6 - x_1^2 \tag{6.39}$$

Now the point P lie on the hyperbola, we have

$$5x_1^2 - y_1^2 = 5 \tag{6.40}$$

From (6.39) and (6.40), we have $\left(\pm\sqrt{\dfrac{11}{6}}, \dfrac{5}{\sqrt{6}}\right)$ and $\left(\pm\sqrt{\dfrac{11}{6}}, \dfrac{-5}{\sqrt{6}}\right)$.

Exercises

1. Sketch the parabola, and label the focus, vertex and directrix of the following equations.

   (i) $y^2 = 8x$, (ii) $y^2 = -6x$,
   (iii) $x^2 = 9y$, (iv) $x^2 = -4y$,
   (v) $(y + 2)^2 = 6(x + 1)$, (vi) $(x - 1)^2 = -4(y + 1)$,
   (vii) $y^2 = -6x - 2y + 3$.

2. Sketch the ellipse, and label the foci, vertices and the ends of the minor axis of the following equations.

   (i) $\frac{x^2}{9} + \frac{y^2}{4} = 1$, (ii) $6x^2 + y^2 = 6$,

   (iii) $\frac{x^2}{9} + \frac{y^2}{25} = 1$, (iv) $9(x - 1)^2 + 16(y - 2)^2 = 144$,

   (v) $(x + 2)^2 + 4(y + 3)^2 - 16 = 0$,

   (vi) $9x^2 + 4y^2 + 6x - 8y - 31 = 0$.

3. Sketch the hyperbola, and label the foci, vertices and the and the asymptotes of the following equations.

   (i) $\frac{x^2}{4} - \frac{y^2}{1} = 1$, (ii) $4x^2 - y^2 = 4$,

   (iii) $\frac{y^2}{25} - \frac{x^2}{9} = 1$, (iv) $4(x - 1)^2 - 9(y - 2)^2 = 36$,

   (v) $(y + 1)^2 - 4(x + 2)^2 - 4 = 0$,

   (vi) $4x^2 - 9y^2 + 16x + 36y - 56 = 0$.

4. Find an equation for the parabola which satisfies the following conditions.

   (i) Vertex (0, 0); focus (2, 0),

   (ii) Vertex (0, 0); directrix $x = 3$,

   (iii) focus (0, –2); directrix $y = 2$,

   (iv) focus (5, 0); directrix $x = -5$,

   (v) Focus (–1, 4) directrix $x = 7$,

   (vi) Axis $y = 0$; passes through (2, 3) and (3, –2).

5. Find an equation for the ellipse which satisfies the following conditions.

   (i) Ends of major axis (± 4, 0); ends of minor axis (0, ± 3),

   (ii) Length of major axis 20; foci (± 3, 0),

   (iii) Foci (± 2, 0); $b = \sqrt{3}$,

   (iv) Foci (± 2, 0); $a = 9$,

   (v) $b = 4$; $c = 3$; centre at the origin; foci on a coordinate axis,

   (vi) Foci (1, 3) and (1, 5); minor axis of length 4,

   (vii) Length of minor axis 8; foci (1, 1) and (1, –3)

6. Find an equation for the hyperbola which satisfies the following conditions.

   (i) Vertices $(\pm 1, 0)$ foci $(\pm 2, 0)$,

   (ii) Vertices $(\pm 2, 0)$; asymptote $y = \pm 3x$,

   (iii) Vertices $(0, \pm 1)$ foci $(0, \pm 3)$,

   (iv) asymptote $y = \pm \frac{3}{2} x$; $b = 6$;

   (v) Vertices (2, 2) and (10, 2); foci 10 units apart,

   (vi) asymptote $y = 3x + 1$ and $y = -3x + 7$; passes through origin,

   (vii) Vertices (–3, –2) and (5, –2); $b = 4$.

7. Show that the equation $y = mx + c$ will touch the

   (i) Parabola $y^2 = 4ax$ if $c = \frac{a}{m}$,

   (ii) Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c = \sqrt{a^2m^2 + b^2}$,

   (iii) Hyperbola $\frac{x_1}{a^2} - \frac{y_2}{b^2} = 1$ if $c = \sqrt{a^2m^2 - b^2}$.

8. Show that the vertex is the closet point on a parabola to the focus.

9. Show that the equation of the parabola traced by a point that moves so that its distance from (1, 2) is the same as its distance to $y = 1$ is $(x - 1)^2 = 2\left(y - \frac{3}{2}\right)$.

10. Show that the equation of the ellipse traced by a point that moves so that the sum of its distance to (5, 1) and (5, 5) is 12 is $\frac{(x-5)^2}{32} + \frac{(y-3)^2}{36} = 1$.

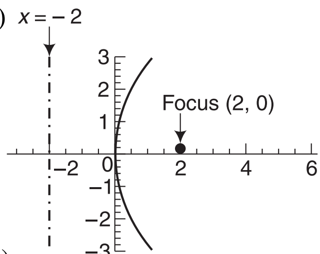
11. Show that the equation of the hyperbola traced by a point that moves so that the difference between its distance to (0, 0) and (2, 2) is 2 is $2xy - 2x - 2y + 1 = 0$.

12. Show that the equation of the chord joining two points on the ellipse whose eccentric angles are θ_1 and θ_2 is $\frac{x}{a} \cos \frac{\theta_1 + \theta_2}{2} + \frac{y}{b} \sin \frac{\theta_1 + \theta_2}{2} = \cos \frac{\theta_1 - \theta_2}{2}$.

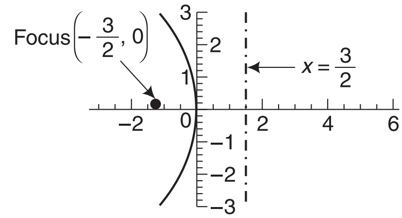
13. If we take the asymptotes as the coordinate axis then prove that the equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be written as $xy = c^2$ where $c = \frac{a^2 + b^2}{4}$.

Answers

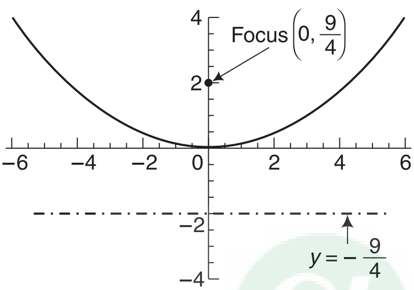
1. (i)



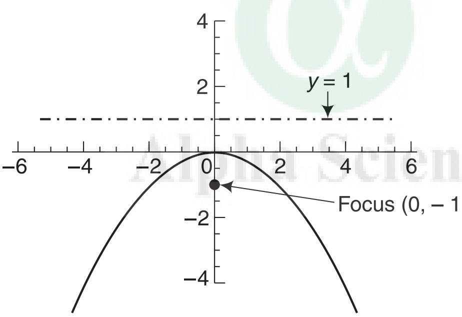
(ii)



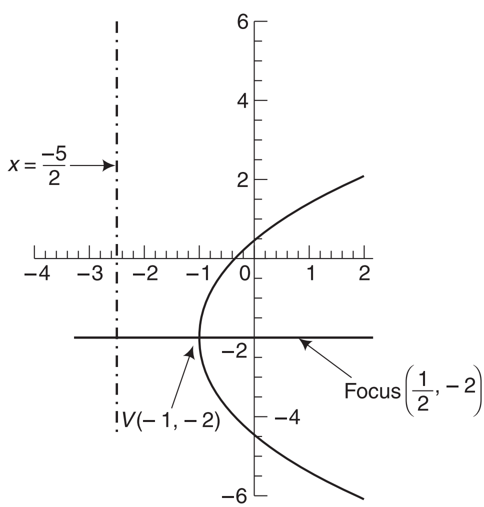
(iii)



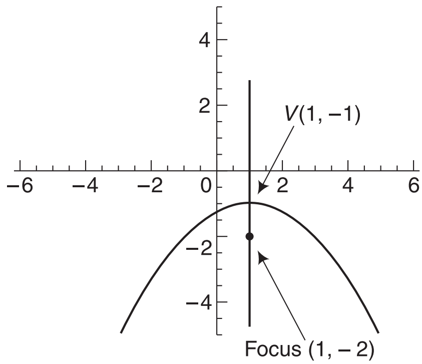
(iv)



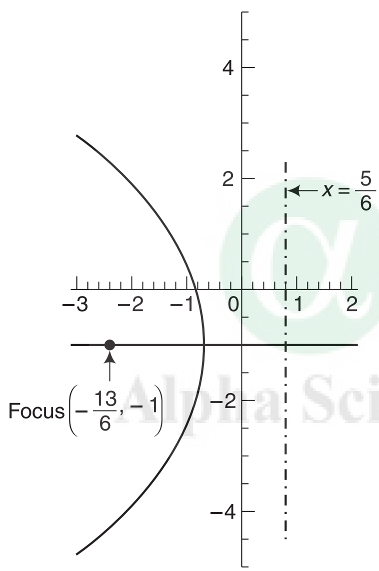
(v)



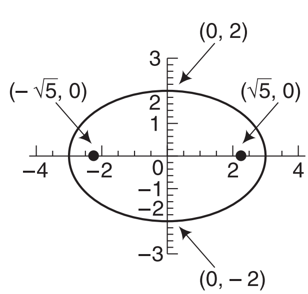
(vi)


V(1, –1)
Focus (1, – 2)

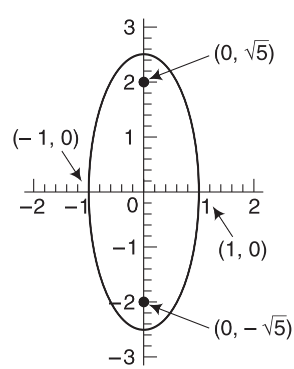
(vii)


$x = \frac{5}{6}$
Focus $\left(-\frac{13}{6}, -1\right)$

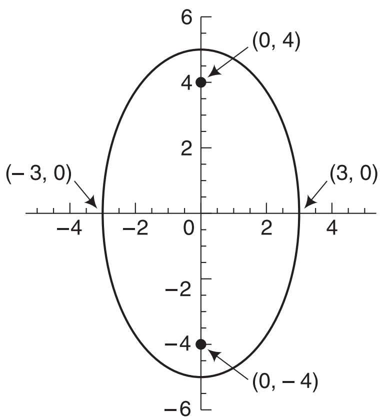
2. (i)


(0, 2)
(– √5, 0)
(√5, 0)
(0, – 2)

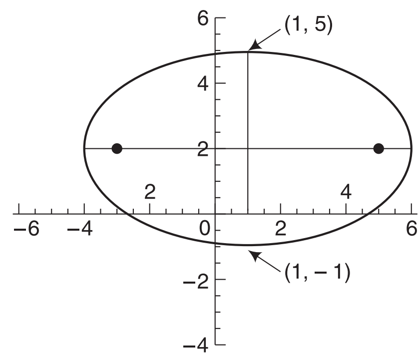
(ii)


(0, √5)
(– 1, 0)
(1, 0)
(0, – √5)

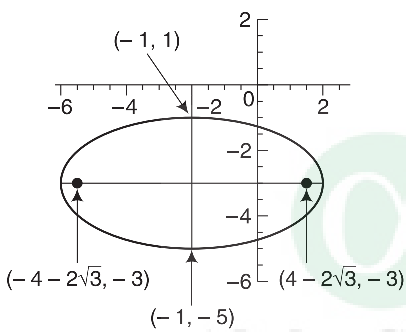
(iii)


(0, 4)
(– 3, 0)
(3, 0)
(0, – 4)

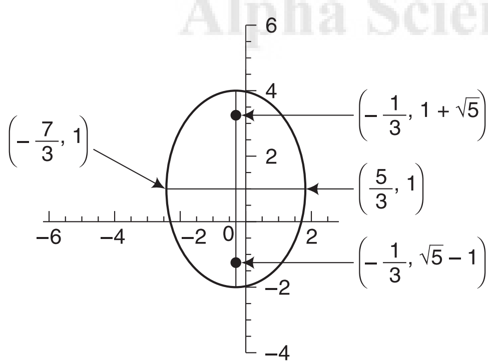
(iv)


(1, 5)
(1, – 1)

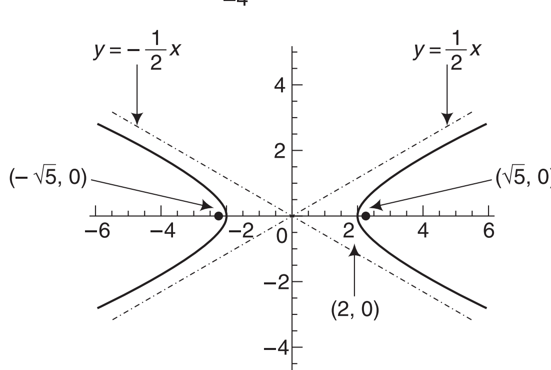
(v)


(– 1, 1)
(– 4 – 2√3, – 3)
(4 – 2√3, – 3)
(– 1, – 5)

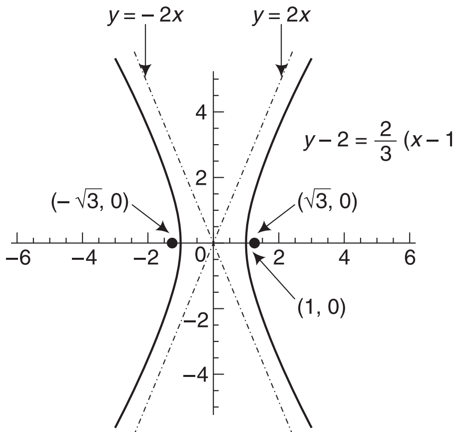
(vi)


$\left(-\frac{1}{3}, 1+\sqrt{5}\right)$
$\left(-\frac{7}{3}, 1\right)$
$\left(\frac{5}{3}, 1\right)$
$\left(-\frac{1}{3}, \sqrt{5}-1\right)$

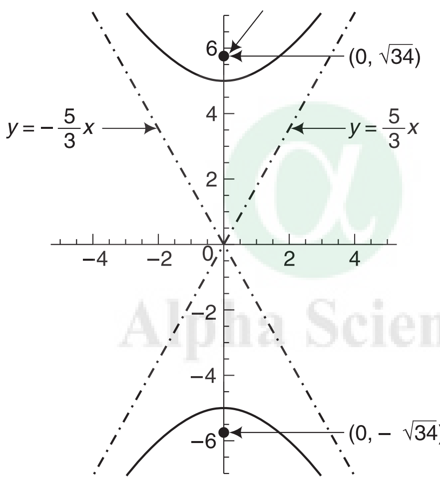
3. (i)


$y = -\frac{1}{2}x$
$y = \frac{1}{2}x$
(– √5, 0)
(√5, 0)
(2, 0)

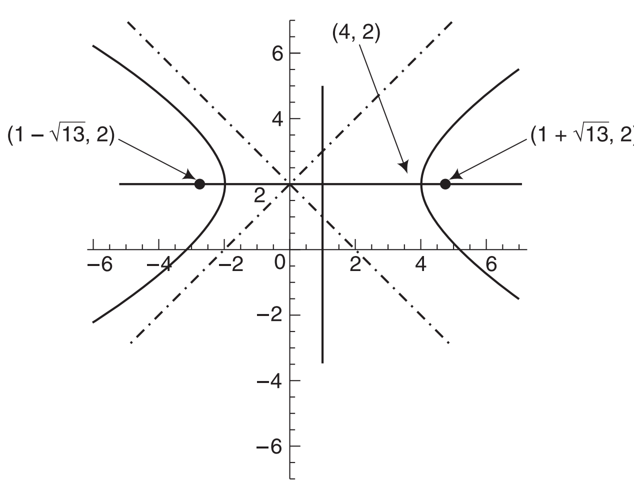
(ii)



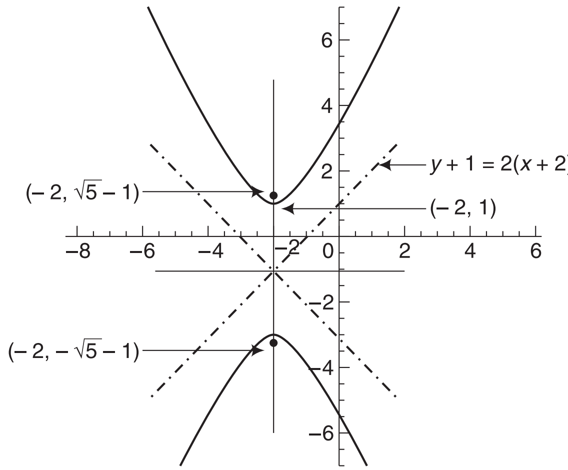
(iii)



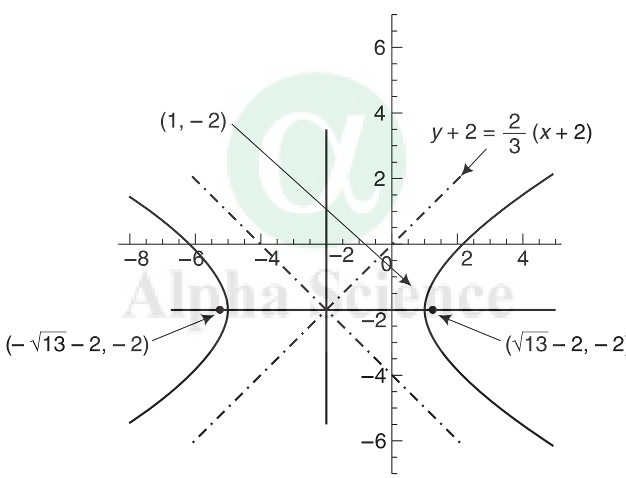
(iv)



(v)



(vi)



4. (i) $y^2 = 8x$, (ii) $y^2 = -12x$,

(iii) $x^2 = -8y$, (iv) $y^2 = 20x$,

(v) $(y - 4)^2 = -16(x - 3)$, (vi) $y^2 = -5\left(x - \frac{19}{5}\right)$

5. (i) $\frac{x^2}{16} + \frac{y^2}{9} = 1$, (ii) $\frac{x^2}{100} + \frac{y^2}{91} = 1$,

(iii) $\frac{x^2}{7} + \frac{y^2}{3} = 1$, (iv) $\frac{x^2}{81} + \frac{y^2}{77} = 1$,

(v) $\frac{x^2}{4} + \frac{y^2}{5} = 1$. (vi) $\frac{(x - 1)^2}{12} + \frac{(y + 1)^2}{16} = 1$.

6. (i) $\dfrac{x^2}{1} - \dfrac{y^2}{3} = 1$, (ii) $\dfrac{x^2}{4} - \dfrac{y^2}{36} = 1$,

(iii) $\dfrac{y^2}{1} - \dfrac{x^2}{8} = 1$, (iv) $\dfrac{x^2}{16} - \dfrac{y^2}{36} = 1$, and $\dfrac{y^2}{81} - \dfrac{x^2}{36} = 1$,

(v) $\dfrac{(x-6)^2}{16} - \dfrac{(y-2)^2}{9} = 1$, (vi) $\dfrac{(y-4)^2}{7} - \dfrac{9(x-1)^2}{7} = 1$,

(vii) $\dfrac{(x-1)^2}{16} - \dfrac{(y+2)^2}{16} = 1$.

6.5 SECOND-DEGREE EQUATION AND ROTATION OF AXIS

The curves we have discussed in section 6.5 had their axes parallel to the coordinate axes, but this is not always true i.e., Fig. 6.107.

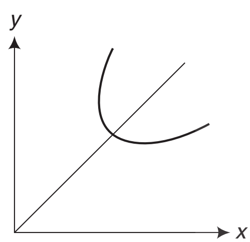


Fig. 6.107(a)

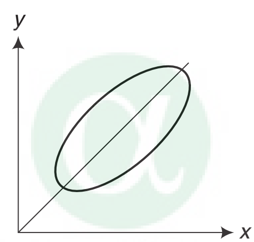


Fig. 6.107(b)

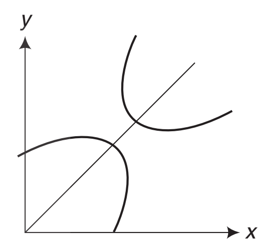


Fig. 6.107(c)

To obtain curves like those sketched in Fig. 6.107, we must rotate the coordinate axes through an appropriate angle. How do we do so. Let us suppose that the x-axes and y-axes are rotated an angle of θ with respect to the origin and let $P(x, y)$ represent a point in the coordinate x and y Fig. 6.108.

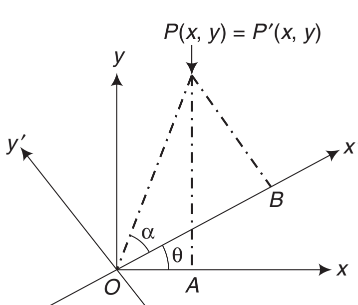


Fig. 6.108(a)

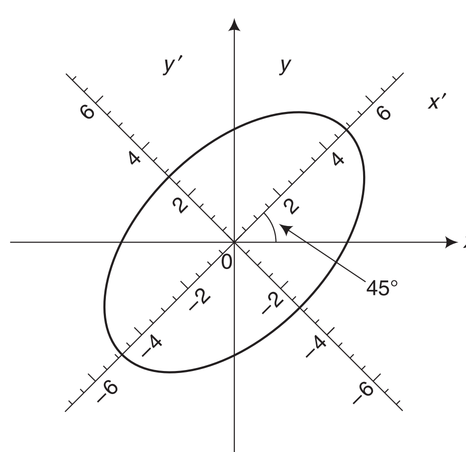


Fig. 6.108(b)

Now seek a represent of the point $P(x, y)$ in the new coordinates x' and y'. From Fig. 6.108(a)

In ΔOAP, we have

$$x = OA = OP\cos(\theta + \alpha) = OP(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \tag{6.41}$$

And

$$y = AP = OP\sin(\theta + \alpha) = OP(\sin\theta\cos\alpha + \cos\theta\sin\alpha) \tag{6.42}$$

Similarly In ΔOBP, we have

$$x' = OP\cos\alpha \text{ and } y' = OP\sin\alpha \tag{6.43}$$

From equations (6.41), (6.42) and (6.43), we get

$$\boldsymbol{x = x'\cos\theta - y'\sin\theta} \tag{6.44}$$

$$\boldsymbol{y = x'\sin\theta + y'\cos\theta} \tag{6.45}$$

These equations (6.44) and (6.45) are called the **rotation equations**

From the equations (6.44) and (6.45), we also get

$$\boldsymbol{x' = x\cos\theta + y\sin\theta} \tag{6.46}$$

$$\boldsymbol{y' = -x\sin\theta + y\cos\theta} \tag{6.47}$$

Example 22 Find the equation of the curve obtained from the graph of $x^2 - xy + y^2 = 12$ by rotating the axes through an angle of $\theta = 45°$.

Solution Since $\cos 45° = \sin 45° = \dfrac{1}{\sqrt{2}}$ and from equations (6.42) and (6.43), we have

$$x = \frac{x' - y'}{\sqrt{2}} \text{ and } y = \frac{x' + y'}{\sqrt{2}}$$

Substitution of these into the given equation $x^2 - xy + y^2 = 12$, yields

$$\left(\frac{x' - y'}{\sqrt{2}}\right)^2 - \left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) + \left(\frac{x' + y'}{\sqrt{2}}\right)^2 = 12$$

After simplification, we get

$$\frac{(x')^2 + 3(y')^2}{2} = 12$$

Or

$$\frac{(x')^2}{24} + \frac{(y')^2}{8} = 1$$

Hence in new coordinate system x' and y' this is the equation of an ellipse with $a = \sqrt{24}$, $b = \sqrt{8}$ and $c = \sqrt{24 - 8} = \pm 4$, Fig. 108(b).

Example 23 Find the new coordinates of the point (1, 2) if the coordinate axes are rotated through an angle of $\theta = 60°$

Solution From equations (6.46) and (6.47), we get

$$x' = 1 \cdot \frac{1}{2} + 2\frac{\sqrt{3}}{2}$$

$$y' = -\frac{\sqrt{3}}{2} + 2 \cdot \frac{1}{2}$$

Thus, the new coordinates are $\left(\frac{1 + 2\sqrt{3}}{2}, \frac{2 - \sqrt{3}}{2}\right)$.

Quadratic equation

Consider the quadratic equation

$$\boldsymbol{Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0} \tag{6.48}$$

$B = 0$ and $A \neq 0$ and $C \neq 0$ then

(i) If $AC > 0$, then the equation represents an ellipse, a circle, a point, or has no graph.

(ii) If $AC < 0$, then the equation represents a hyperbola or a pair of intersecting lines.

(iii) If $AC = 0$, then the equation represents a parabola or a pair of parallel lines or has no graph. i.e. the equation $2x^2 - y^2 - 4x + 4y = 10$ has $A = 2$ and $C = -1 \Rightarrow AC = -2, \Rightarrow AC < 0$ then the equation represents

a hyperbola or a pair of intersecting lines. Now this equation can be written as $\frac{(x - 1)^2}{4} - \frac{(y - 2)^2}{8} = 1$, which is a hyperbola while the equation $x^2 - 4y^2 - 8y = 4$ also has $AC < 0$ and this equation can be written as $x^2 - 4(y + 1)^2 = 0 \Rightarrow x = \pm 2(y + 1)$ which represents the pair of intersecting lines.

$B \neq 0$ and $A \neq 0$ and $C \neq 0$ then

(i) If $B^2 - 4AC < 0$, then the equation represents an ellipse, a circle, a point, or has no graph.

(ii) If $B^2 - 4AC > 0$, then the equation represents a hyperbola or a pair of intersecting lines.

(iii) If $B^2 - 4AC = 0$, then the equation represents a parabola or a pair of parallel lines or a line or has no graph.

The quantity $\mathbf{B^2 - 4AC}$ is called **discriminant** of the equation (6.48). i.e. the equation $x^2 + y^2 - xy = 4$ has $A = 1$, $B = -1$ and $C = 1 \Rightarrow B^2 - 4AC = 1 - 4.1.1 < 0$. So the given equation represents an ellipse, a circle, a point, or has no graph. Further we observed that the points $(0, \pm 2)$ lies on the curve, thus the curve is an ellipse. Now the equation $x^2 + y^2 + 2xy + 2 = 0$ satisfies $B^2 - 4AC = 0$; but there is no real values of x and y that satisfy the given equation $(x + y)^2 = -2$. Hence the equation has no graph.

If the equation (6.48) is such that $B \neq 0$, and if an $x'y'$-coordinate system is obtained by rotating the x y-axes through an angle θ satisfying $\cot 2\theta = \frac{A - C}{B}$ then, in $x'y'$-coordinate system the equation (6.48) will have the form

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0 \tag{6.49}$$

To prove this substitute the value of x and y from equations (6.44) and (6.45) into (6.48) and after simplifying we have $A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$ where

$$\left.\begin{aligned}
A' &= A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta \\
B' &= B(\cos^2\theta - \sin^2\theta) + 2(C - A)\sin\theta\cos\theta = B\cos 2\theta + (C - A)\sin 2\theta \\
C' &= C\cos^2\theta - B\sin\theta\cos\theta + A\sin^2\theta \\
D' &= D\cos\theta + E\sin\theta \\
E' &= E\cos\theta - D\sin\theta \\
F' &= F
\end{aligned}\right\} \tag{6.49a}$$

We know that in equation (6.49) there is no $B' \Rightarrow B' = 0$ $\cot 2\theta = \frac{A - C}{B}\left(0 < \theta < \frac{\pi}{2}\right)$.

Rotating a curve through an angle θ has the effect of rotating the axes through an angle $(-\theta)$.

Example 24 Identify and sketch the curve $xy = 1$

Solution Comparing the equation $xy = 1$ from equation (6.48), we have $A = 0 = C$, $B = 1$ and $\cot 2\theta = \frac{A - C}{B} = \frac{0 - 0}{1} = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$.

Hence from (6.44) and (6.45), we have

$$x = \frac{x' - y'}{\sqrt{2}} \text{ and } y = \frac{x' + y'}{\sqrt{2}}$$

Substituting these in the given equation $xy = 1$ yields

$$\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) = 1 \text{ or } \frac{(x')^2}{2} - \frac{(y')^2}{2} = 1$$

This is the equation of the hyperbola in $x'y'$-coordinate system, Fig. 109.

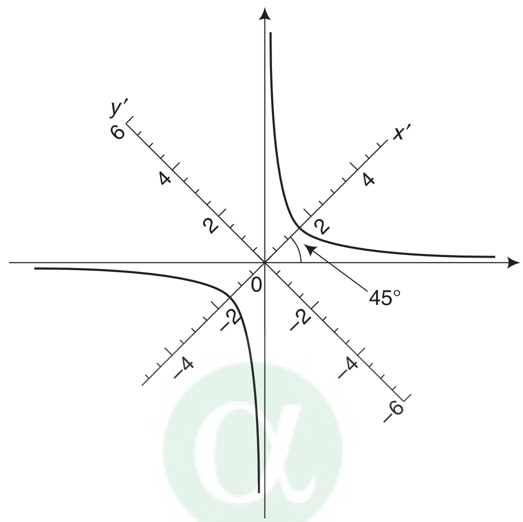


Fig. 6.109

Example 25 Identify and sketch the curve $x^2 + 3xy + 5y^2 + 4 = 0$

Solution We have $A = 1$, $B = 3$, $C = 5$.

Then $\cot 2\theta = \frac{A - C}{B} = \frac{1 - 5}{3} = -\frac{4}{3}$. We know that here is inconvenient to solve the value of θ. In this case we can solve the θ as.

The θ lies between 0 and $\frac{\pi}{2}$, this relationship is represented by the Fig. 6.110.

From Fig. 6.110, we obtained

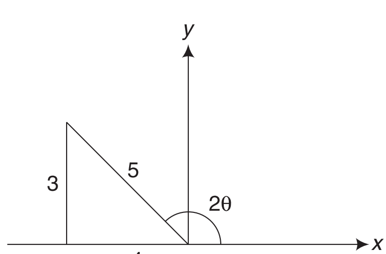


Fig. 6.110

$$\cos 2\theta = -\frac{4}{5} \Rightarrow \cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\frac{1-\frac{4}{5}}{2}} = \frac{1}{\sqrt{10}} \text{ and}$$

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1+\frac{4}{5}}{2}} = \frac{3}{\sqrt{10}}$$

Hence from (6.44) and (6.45), we have

$$x = \frac{x' - 3y'}{\sqrt{10}} \text{ and } y = \frac{3x' + y'}{\sqrt{10}}$$

Substituting these in the given equation $x^2 + 3xy + 5y^2 + 4 = 0$ yields

$$\left(\frac{x' - 3y'}{\sqrt{10}}\right) + 3\left(\frac{x' - 3y'}{\sqrt{10}}\right)\left(\frac{3x' + y'}{\sqrt{10}}\right) + 5\left(\frac{3x' + y'}{\sqrt{10}}\right) + 4 = 0$$

After some simplification we have

$$\frac{11(x')^2}{8} + \frac{(y')^2}{8} = 1$$

This is the equation of the ellipse in $x'\,y'$-coordinate system with $\theta = \sin^{-1}\left(\frac{3}{\sqrt{10}}\right) = .9486 \approx 71°$, Fig. 6.111.

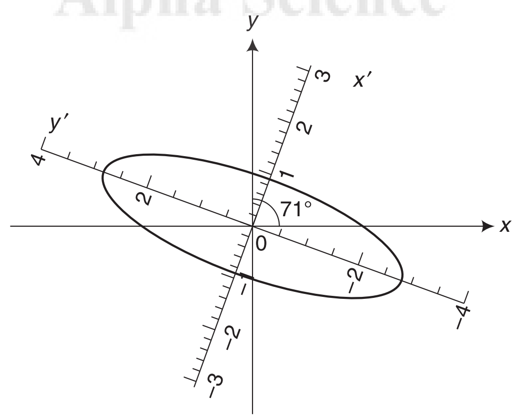


Fig. 6.111

Example 26 State about the graph of the equation $\sqrt{x} + \sqrt{y} = 2$.

Solution $\sqrt{x} + \sqrt{y} = 2 \Rightarrow x = (2 - \sqrt{y})^2 \Rightarrow x^2 - 2xy + y^2 - 8x - 8y + 16 = 0$

Now $\cot 2\theta = \frac{A - C}{B} = \frac{1-1}{-2} = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$

Hence from Equation (6.44) and (6.45), we have

$$x = \frac{x' - y'}{\sqrt{2}} \text{ and } y = \frac{x' + y'}{\sqrt{2}}$$

Substituting these in the equation $x^2 - 2xy + y^2 - 8x - 8y + 16 = 0$ yields

$$\left(\frac{x' - y'}{\sqrt{2}}\right)^2 - 2\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) + \left(\frac{x' + y'}{\sqrt{2}}\right)^2 - 8\left(\frac{x' - y'}{\sqrt{2}}\right) - 8\left(\frac{x' + y'}{\sqrt{2}}\right) + 16 = 0$$

After some simplification we have
$(y')^2 - 2(x' - 4) = 0$ Which is a parabola in x' y'-coordinate system and from $\sqrt{x} + \sqrt{y} = 2$

We know that $0 \le x \le 1$ and $0 \le y \le 1$, so the graph is just a portion of a parabola.

Exercises

1. By rotating x y-coordinate system through an angle of $\theta = 30°$ we obtained x' y'-coordinate system.
   (i) Find the x' y'-coordinates when $x = -1$ and $y = 3$.
   (ii) Find the x' y'-coordinates when $x = 2$ and $y = -5$
   (iii) Find an equation of the curve $2xy + x^2 = 3$ in x' y'- coordinates when $x = -1$ and $y = 3$.
2. Identify and sketch the curve
   (i) $2xy = -6$,
   (ii) $x^2 - xy + y^2 = 16$,
   (iii) $x^2 - 3xy - 3y^2 + 2 = 0$,
   (iv) $16x^2 - 24xy + 9y^2 + 100x - 200y + 200 = 0$,
   (v) $153x^2 - 192xy + 97y^2 - 30x - 40y - 100 = 0$,
   (vi) $4x^2 + 4xy + y^2 + 20x - 10y = 0$,
   (vii) $x^2 + 4xy - 2y^2 = 10$,
   (viii) $x^2 + xy + y^2 = 1$,
   (ix) $3x^2 + 4\sqrt{3}xy - y^2 = 7$.
3. By equation (ixa) prove that $B^2 - 4AC = B'^2 - 4A'C'$ and $A + C = A' + C'$
4. Prove that $x^2 + y^2 = r = x'^2 + y'^2$.

5. Use the discriminant to identify the graph of the following equations

   (i) $2x^2 + 3xy + y^2 + 3 = 0$,

   (ii) $x^2 + 2xy + y^2 - 7 = 0$,

   (iii) $x^2 + 3\sqrt{3}xy + y^2 + \sqrt{3}x - y = 0$,

   (iv) $5x^2 + 4xy + y^2 - 12x + 20y = 11$.

6. The following equations represents the degenerate conic section. Where possible, sketch the graph

   (i) $x^2 - 3y^2 = 0$, (ii) $x^2 + 5y^2 + 3 = 0$,

   (iii) $7x^2 + 5y^2 = 0$, (iv) $x^2 + 4xy + 4y^2 = 0$,

   (v) $4x^2 + 72xy + 9y^2 - 36 = 0$.

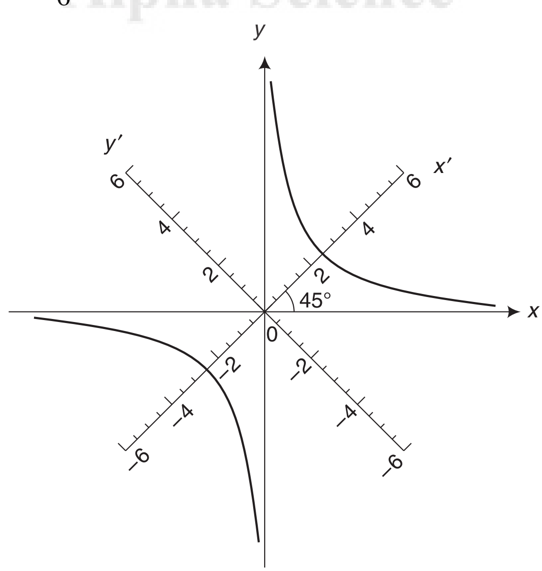
Answers

1. (i) $x' = -\dfrac{\sqrt{3}}{2} + \dfrac{3}{2}$ and $y' = \dfrac{1}{2} + \dfrac{3\sqrt{3}}{2}$,

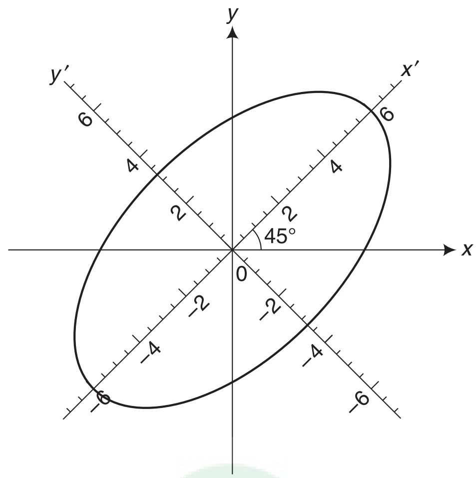
   (ii) $x' = \sqrt{3} - \dfrac{5}{2}$ and $y' = -1 - \dfrac{5\sqrt{3}}{2}$,

   (iii) $\dfrac{(x')^2}{2} + \dfrac{(y')^2}{3} = 1$.

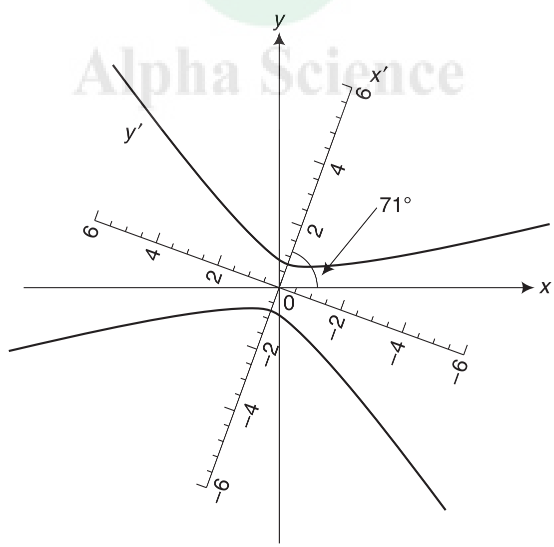
2. (i) $\dfrac{(x')^2}{6} - \dfrac{(y')^2}{6} = 1$, Hyperbola;



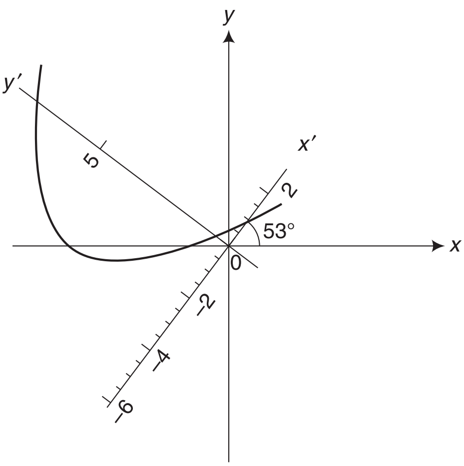
(ii) $\dfrac{(x')^2}{32} + \dfrac{3(y')^2}{32} = 1$, Ellipse;



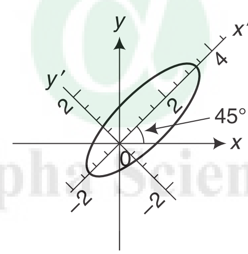
(iii) $\dfrac{7(x')^2}{4} - \dfrac{3(y')^2}{4} = 1$, Hyperbola;



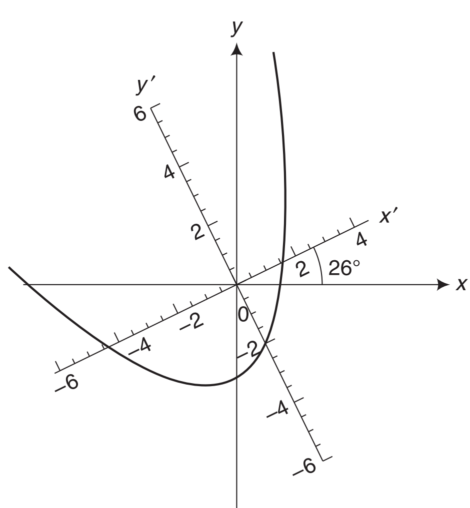
(iv) $(y' - 4)^2 = 4(x' + 3)$, Parabola;



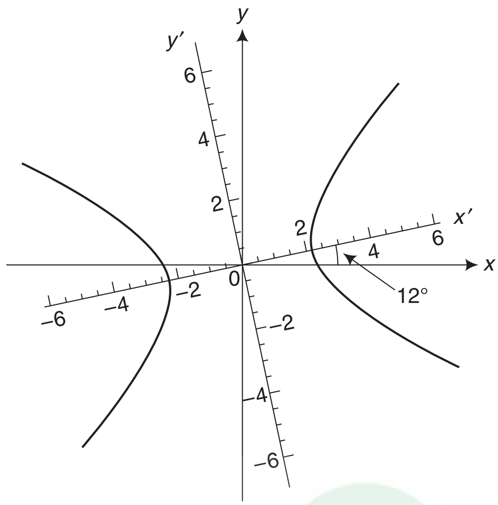
(v) $\dfrac{(x' - 1)^2}{5} + \dfrac{9(y')^2}{5} = 1$, Ellipse;

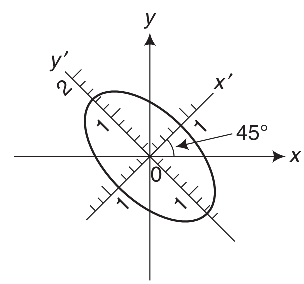


(vi) $\left(x' + \dfrac{3}{\sqrt{5}}\right)^2 = \dfrac{8}{\sqrt{5}}\left(y' + \dfrac{9\sqrt{5}}{8}\right)$, Parabola;

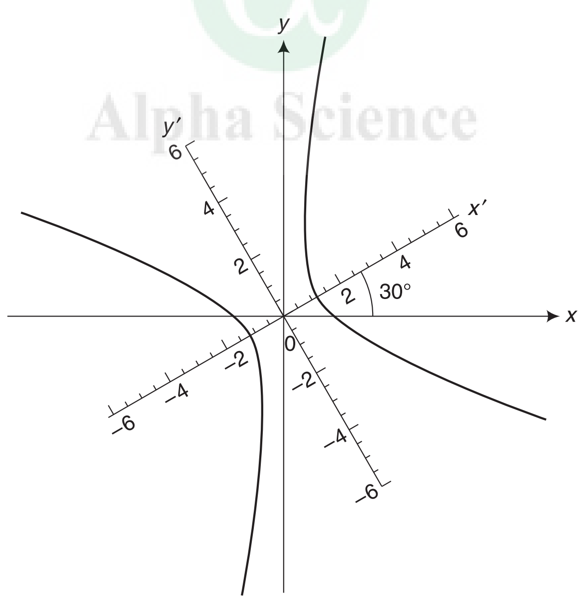


(vii) $\dfrac{(x')^2}{5} - \dfrac{3(y')^2}{10} = 1$, Hyperbola; (viii) $\dfrac{3(x')^2}{2} + \dfrac{(y')^2}{2} = 1$, Ellipse;



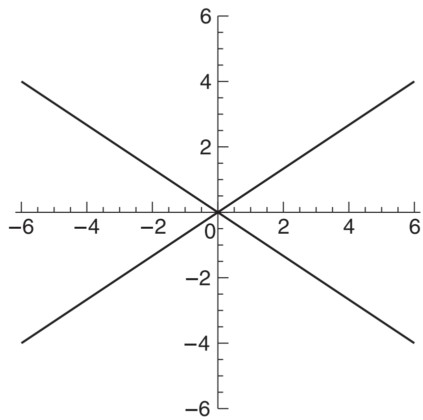


(ix) $\dfrac{5(x')^2}{7} - \dfrac{3(y')^2}{7} = 1$, Hyperbola;

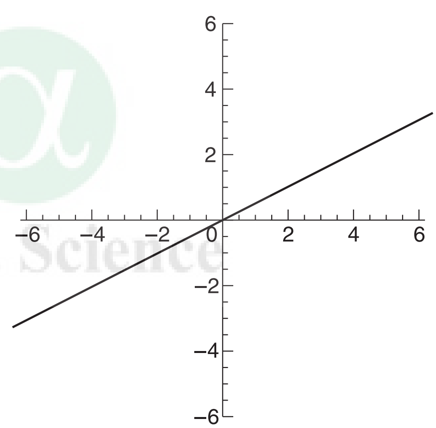


5. (i) Hyperbola or a pair of intersecting lines,
   (ii) Parabola or a pair of parallel lines or a line or has no graph,
   (iii) Hyperbola or a pair of intersecting lines,
   (iv) Ellipse, a circle, a point, or has no graph.

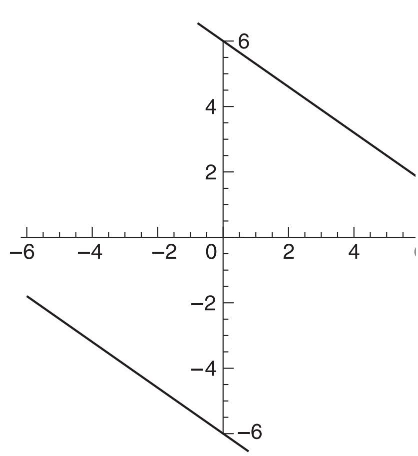
6. (i) Pair of straight lines, (ii) No graph,



(iii) Point $x = 0$, $y = 0$, (iv) A line,



(v) Parallel lines



6.6 CONIC SECTION IN POLAR COORDINATE

Let a fixed point O and a fixed line AB at a distance D from O, Fig. 6.112. Suppose that a point P lies in the plane of O and AB moves so that the ratio of its distance from the point O to its distance from line AB is always equal to the positive constant e i.e. $\frac{r}{d} = e$.

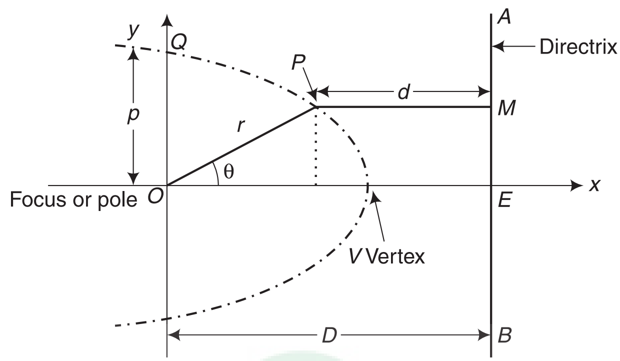


Fig. 6.112

The point O is called the **focus** or **pole** the line AB is called the **directrix** and the ratio e is called the **eccentricity**. The curve described by P is given in polar coordinate (r, θ) as

$$\boldsymbol{r = \frac{p}{1 + e\cos\theta}} \tag{6.50}$$

or

$$\boldsymbol{r = \frac{eD}{1 + e\cos\theta}}$$

And this curve is often called a **conic section**. We know that there are three types of conic section. These three conic sections depending on the value of the eccentricity. To derive the equation (6.50)

$$\frac{r}{d} = e \quad \text{or} \quad \frac{r}{e} = d \tag{6.51}$$

At a particular point Q, we have

$$\frac{p}{D} = e \quad \text{or} \quad p = eD \tag{6.52}$$

But $D = d + r\cos\theta = \frac{r}{e} + r\cos\theta = \frac{r}{e}(1 + e\cos\theta)$ (6.53)

From (6.52) and (6.53), we have

$$p = r(1 + e\cos\theta) \text{ or } r = \frac{p}{1 + e\cos\theta} = \frac{eD}{1 + e\cos\theta}.$$

The equation (6.50) represent the

Parabola if $e = 1$, Fig. 6.113, **Ellipse if $0 < e < 1$,** Fig. 6.114, **Hyperbola if $e > 1$** Fig. 6.115,

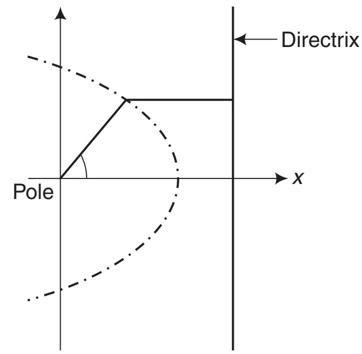


Fig. 6.113

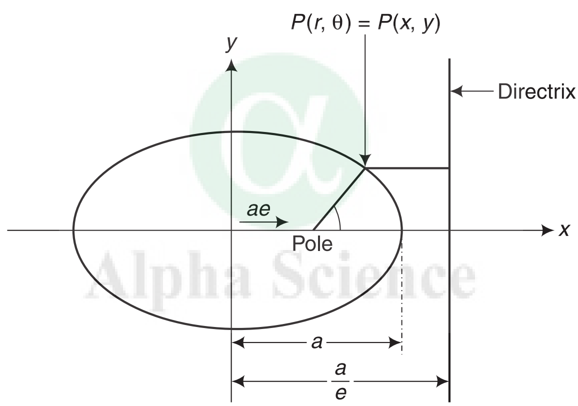


Fig. 6.114

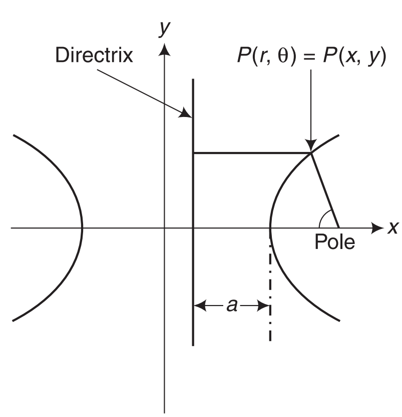


Fig. 6.115

The equation of the parabola is $\boldsymbol{r} = \dfrac{\boldsymbol{D}}{\mathbf{1 + \cos\theta}} = \dfrac{\boldsymbol{p}}{\mathbf{1 + \cos\theta}}$, **if** $\boldsymbol{e} = \mathbf{1}$ (6.54)

The equation of the ellipse is

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}, \text{ if } 0 < e < 1 \left(D = \frac{a}{e} - ae = \frac{a(1 - e^2)}{e}\right) \quad (6.55)$$

The equation of the hyperbola is

$$\boldsymbol{r} = \frac{\boldsymbol{a}(\boldsymbol{e}^2 - \mathbf{1})}{\mathbf{1} + \boldsymbol{e}\cos\theta}, \text{ if } e > 1 \left(D = ae - \frac{a}{e} = \frac{a(e^2 - 1)}{e}\right) \quad (6.56)$$

Theorem If a conic section defined in polar coordinate system such that the focus at the pole(origin) and the corresponding directrix is at a distance *D* from the pole and is either parallel or perpendicular to the polar axis with eccentricity *e*, then the equation of the conic has one of four possible forms, depending on its orientation:

$$r = \frac{eD}{1 + e\cos\theta} \text{ directrix right of the pole} \quad (6.57)$$

$$r = \frac{eD}{1 - e\cos\theta} \text{ directrix left of the pole} \quad (6.58)$$

$$r = \frac{eD}{1 + e\sin\theta} \text{ directrix above of the pole} \quad (6.59)$$

$$r = \frac{eD}{1 - e\sin\theta} \text{ directrix below of the pole} \quad (6.60)$$

Some useful steps for sketching the conic section in polar coordinates:

(i) Identifies the conic

For example we want to sketch the graph of $r = \dfrac{8}{2 + \cos\theta}$

This equation can be written in standard form as $r = \dfrac{4}{1 + \frac{1}{2}\cos\theta}$, now this equation is an exact match to equation (6.57) with $e = \dfrac{1}{2} \Rightarrow e < 1$, So given curve is ellipse where $eD = 4 \Rightarrow D = 8$ which implies that the directrix of the ellipse is 8 units to the right of the pole.

(ii) Obtain r_0 and r_1

r_0 is the distance from the focus to the closet vertex (called **perigee or perihelion**) and r_1 is the distance from the focus to the farthest vertex (called **apogee or aphelion**).

(a) If the given equation is an exact match to (6.57) then r_0 and r_1 can be obtained by setting $\theta = 0$ and $\theta = \pi$ in the given equation respectively.

(b) If the given equation is an exact match to (6.58) then r_0 and r_1 can be obtained by setting $\theta = \pi$ and $\theta = 0$ in the given equation respectively.

(c) If the given equation is an exact match to (6.59) then r_0 and r_1 can be obtained by setting $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$ in the given equation respectively.

(d) If the given equation is an exact match to (6.60) then r_0 and r_1 can be obtained by setting $\theta = \frac{3\pi}{2}$ and $\theta = \frac{\pi}{2}$ in the given equation respectively. for the ellipse $a = \frac{1}{2}(r_0 + r_1)$, $b = \sqrt{r_0 r_1}$, $c = \frac{1}{2}(r_1 - r_0)$. For the hyperbola $a = \frac{1}{2}(r_1 - r_0)$, $b = \sqrt{r_0 r_1}$, $c = \frac{1}{2}(r_1 + r_0)$. for the ellipse $e = \frac{c}{a} = \frac{\frac{1}{2}(r_1 - r_0)}{\frac{1}{2}(r_0 + r_1)} = \frac{(r_1 - r_0)}{(r_0 + r_1)}$, and hyperbola $e = \frac{(r_1 + r_0)}{(r_1 - r_0)}$.

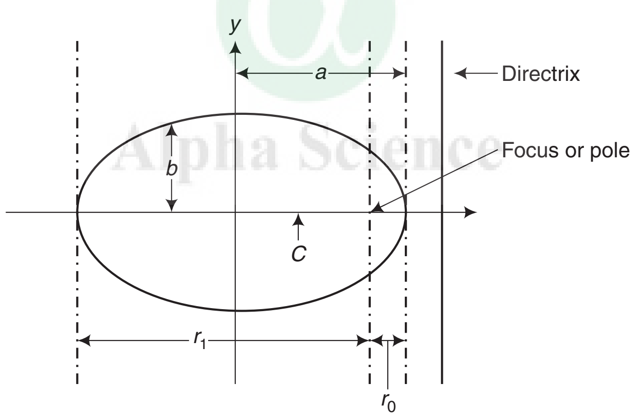


Fig. 6.115(a)

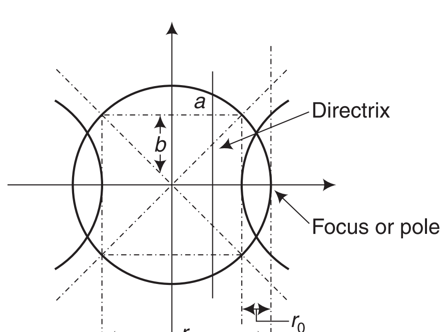


Fig. 6.115(b)

In given example the equation match exact to equation (6.57), hence $r_0 = \dfrac{4}{1 + \frac{1}{2}\cos 0} = \dfrac{8}{3}$ and $r_1 = \dfrac{4}{1 + \frac{1}{2}\cos \pi} = 8$. Distance from the pole to the vertices are $\dfrac{8}{3}$ and 8.

According to the definition of the ellipse $a = \dfrac{1}{2}(r_0 + r_1) = \dfrac{16}{3}$, $b = \sqrt{r_0 r_1} = \dfrac{8}{\sqrt{3}}$, $c = \dfrac{1}{2}(r_1 - r_0) = \dfrac{8}{3}$. Center $\left(-\dfrac{8}{3}, 0\right)$. So the equation of the ellipse in rectangular coordinates is $\dfrac{9\left(x + \frac{8}{3}\right)^2}{16^2} + \dfrac{3y^2}{64} = 1$

Now we are able to sketch the graph, Fig. 6.116

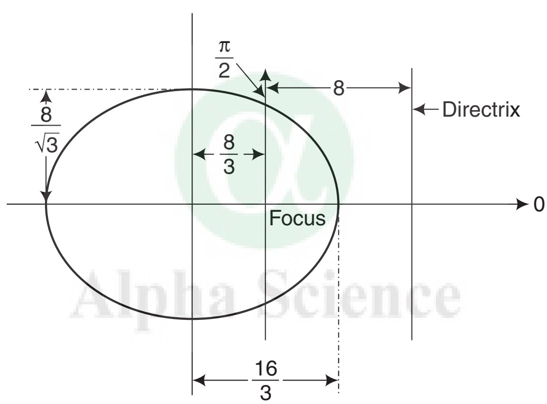


Fig. 6.116

Example 27 Find the eccentricity the distance from the pole to the directrix, the distance from the pole to the vertices, find the equation in rectangular coordinates and sketch the graph of $r = \dfrac{3}{1 + 3\sin\theta}$

Solution Given equation is an exact match to equation (6.59) with $e = 3 \Rightarrow e > 1$, So given curve is hyperbola where $eD = 3 \Rightarrow D = 1$ which implies that the directrix of the hyperbola is 1 units above the pole. Hence $r_0 = \dfrac{3}{1 + \sin\frac{\pi}{2}} = \dfrac{3}{4}$, $r_1 = \left|\dfrac{3}{1 + 3\sin\frac{3\pi}{2}}\right| = \left|\dfrac{3}{-2}\right| = \dfrac{3}{2}$, Distance from the pole to the vertices are $\dfrac{3}{4}$ and $\dfrac{3}{2}$. $a = \dfrac{1}{2}(r_1 - r_0) = \dfrac{3}{8}$, $b = \sqrt{r_0 r_1} = \dfrac{3}{\sqrt{8}}$, $c = \dfrac{1}{2}(r_1 + r_0)$

$= \frac{9}{8}$. Centre $\left(0, \frac{9}{8}\right)$ So the equation of the hyperbola in rectangular coordinates is $\frac{-8x^2}{9} + \frac{64\left(y - \frac{9}{8}\right)^2}{9} = 1$, Fig. 6.117.

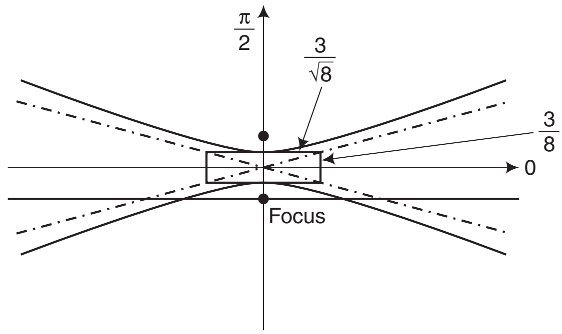


Fig. 6.117

Example 28 Sketch the graph of $r = \frac{4}{1 - \sin\theta}$

Solution Given equation is an exact match to equation (6.60) with $e = 1 \Rightarrow e > 1$, So given curve is parabola where $eD = 4 \Rightarrow D = 4$ which implies that the directrix of the parabola is 4 units below the pole. So the equation of the parabola in rectangular coordinates is $x^2 = 8y$. This tell us that the parabola opens to the up side with $p = 2$, Fig. 6.118.

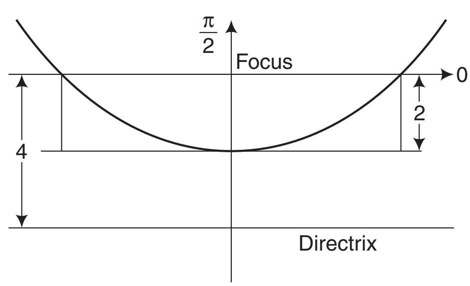


Fig. 6.118

Exercises

1. Find the eccentricity the distance from the pole to the directrix D, the distance from the pole to the vertices, find the equation in rectangular coordinates and sketch the graph of the following:

(i) $r = \dfrac{6}{2 + 3\cos\theta}$, (ii) $r = \dfrac{1}{2 - \cos\theta}$,

(iii) $r = \dfrac{4}{1 - \cos\theta}$, (iv) $r = \dfrac{6}{2 + \sin\theta}$,

(v) $r = \dfrac{1}{2 + 2\sin\theta}$, (vi) $r = \dfrac{1}{2 - 3\sin\theta}$.

2. Find a polar equation for the conic which has the focus at the pole and satisfies the following conditions:

(i) Directrix to the right of the pole; $a = 6$; $e = \dfrac{1}{2}$.

(ii) Directrix to the left of the pole; $b = \sqrt{2}$; $e = \dfrac{1}{3}$

(iii) Directrix to the above of the pole; $c = 4$; $e = \dfrac{2}{3}$.

(iv) Directrix to the below of the pole; $a = 5$; $e = \dfrac{1}{5}$.

(v) Directrix $x = 2$; $e = \dfrac{1}{5}$

(vi) Directrix $x = -2$

(vii) Directrix $y = 1$; $e = \dfrac{5}{3}$

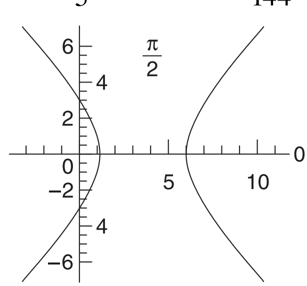
(viii) Vertices $(8, 0)$ and $(6, \pi)$, Ellipse.

(ix) Vertices $\left(4, \dfrac{\pi}{2}\right)$ and $\left(-6, \dfrac{3\pi}{2}\right)$, hyperbola.

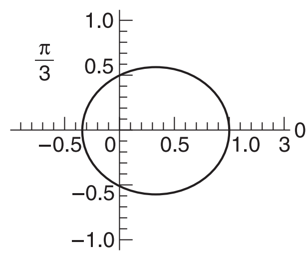
(x) Vertex $\left(1, \dfrac{3\pi}{2}\right)$, Parabola.

Answers

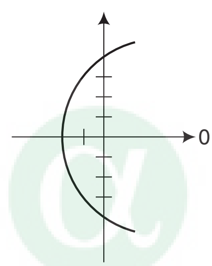
1. (i) $e = \dfrac{3}{2}$, $D = 2$, $r_0 = \dfrac{6}{5}$, $r_1 = 6$, $\dfrac{25\left(x - \dfrac{18}{5}\right)^2}{144} - \dfrac{5y^2}{36} = 1$



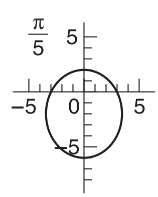
(ii) $e = \frac{1}{2}$, $D = 1$, $r_0 = \frac{1}{3}$, $r_1 = 1$, $\frac{9\left(x - \frac{1}{3}\right)^2}{4} + 3y^2 = 1$



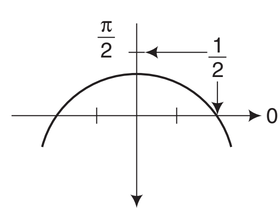
(iii) $e = 1$, $D = 4$, $a = 2$.



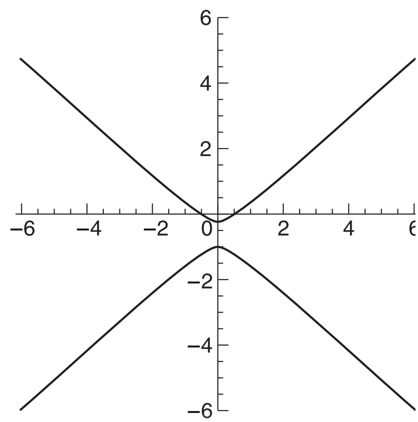
(iv) $e = \frac{1}{2}$, $D = 6$, $r_0 = 2$, $r_1 = 6$, $\frac{x^2}{12} + \frac{(y + 2)^2}{16} = 1$



(v) $e = 1$, $D = \frac{1}{2}$, $a = \frac{1}{4}$.



(vi) $e = \frac{3}{2}$, $D = \frac{1}{3}$, $r_0 = \frac{1}{5}$, $r_1 = 1$, $\frac{25\left(x + \frac{3}{5}\right)^2}{4} - 5x^2 = 1$



2. (i) $r = \frac{9}{2 + \cos\theta}$, (ii) $r = \frac{4}{3 - \cos\theta}$,

(iii) $r = \frac{10}{3 + 2\sin\theta}$, (iv) $r = \frac{24}{5 - \sin\theta}$,

(v) $r = \frac{2}{5 + \cos\theta}$, (vi) $r = \frac{2}{1 - \cos\theta}$,

(vii) $r = \frac{5}{3 + 5\sin\theta}$, (viii) $r = \frac{48}{7 - \cos\theta}$,

(ix) $r = \frac{24}{1 + 5\sin\theta}$, (x) $r = \frac{2}{1 - \sin\theta}$.

7

CHAPTER

Integration

7.1 INTRODUCTION

As we have discussed in chapter 2 that the invention of the calculus was based on the four major problems in which the fourth problem was to find the area enclosed by a given curve. The solution of this fourth problem led to what is now termed **integral calculus.** Before discussing the method for computing areas, we will define a new function, called the **antiderivative**. An antidarivatve of a function f is a new function F having the property that $\frac{dF}{dx} = f$

Antiderivative: Suppose a function f defined on $[a, b]$. If there exists a function $y = F(x)$ such that F is continuous on $[a, b]$ differentiable on open interval$]a, b[$, and the derivative of F is f for every x in $]a, b[$; that is, if

$$F'(x) = \frac{dF}{dx} = f(x), \quad \forall x \in [a, b]$$

Then F is called an **antiderivative** of f on $[a, b]$ and

$$F = \int f(x)dx = \int f. \tag{7.1}$$

Above notation is read "$F(x)$ is the integral of $f(x)$ with respect to x", or "F is an antiderivative of f". If the function F exists, then f is said to be **integrable,** the process of calculating an integral is called **integration.** The variable x is called the **variable of integration,** and the function f is called the **integrand**.

We know that $\frac{d(x^2)}{dx} = 2x$, hence $\int 2x \; dx = x^2$. But the derivative of any constant term is zero, so that $x^2 + 1$, $x^2 - 2$, $x^2 + 2\pi$ are also the antiderivative of $2x$. To see this, we have $\dfrac{d(x^2 + c)}{dx} = 2x$. So $2x$ have an infinite number of antiderivatives. If F is an antiderivative of f, then so is $F + c$ for every constant c, since $\dfrac{d(F + c)}{dx} = \dfrac{d(F)}{dx} + \dfrac{d(c)}{dx} = f$.

Theorem If there are two differentiable functions F and G have the same derivative, then they differ by a constant. That is,

If $\quad F'(x) = G'(x)$, then $F(x) - G(x) = c$.

The above theorem allows us to define the general antiderivative or **indefinite integral** of f as $F(x) + c$, where F some antiderivative of f and c is an arbitrary constant.

The above theorem allows us to define the general antiderivative or **indefinite integral** of f as $F(x) + c$, where F some antiderivative of f and c is an arbitrary constant.

In the earlier classes we have find the indefinite integral of so many functions, for example

$$\int k\, dx = kx + c,\ \int x^r,\, dx = \frac{x^r + 1}{r + 1} + c,\ \int \sin x\, dx = -\cos x + c,\ \int \cos x\, dx = \sin x + c \ldots$$

Some other basic indefinite integration formulas are:

$$\int \frac{1}{x}\, dx = \log |x| + c,\ \int a^x\, dx = \frac{1}{\log a}\, a^x + c,\ \int e^x\, dx = e^x + c,\ \int \sec^2 x\, dx = \tan x + c,$$

$$\int \sec x \tan x\, dx = \sec x + c,\ \int csc^2 x\, dx = -\cot x + c = -\csc x + c,$$

$$\int \tan x\, dx = \log |\sec x| + c,\ \int \cot x\, dx = \log |\sin x| + c,\ \int \sec x\, dx = \log |\sec x + \tan x| + c,$$

$$\int \csc x\, dx = -\log |\csc x + \cot x| + c,\ \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c,$$

$$\left\{\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1}\left|\frac{x}{a}\right| + c,\ \int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left(x + \sqrt{x^2 - a^2}\right) + c,\ (0 < a < |x|)\right\},\ \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left|\frac{x}{a}\right| + c,\ |x| < a,$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \log\left(x + \sqrt{a^2 + x^2}\right) + c, \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left|\frac{a + x}{a - x}\right| + c,$$

$$\int \frac{dx}{x\sqrt{x^2 + a^2}} = \frac{-1}{a} \log \left|\frac{a + \sqrt{a^2 + x^2}}{x}\right| + c. \int \log x \, dx = x \log x - x + c.$$

7.2 APPROXIMATION AREA UNDER THE CURVE

Suppose we want to find out the area of the region which is bounded above by a curve $y = f(x)$, below by the x-axis and the line $x = a$ and $x = b$, Fig. 7.1. Let the interval $[a, b]$ is divided into n subintervals $[x_{i-1}, x_i]$ of equal length. Choose an arbitrarily point x_i^* in each interval $[x_{i-1}, x_i]$. The number $f(x_1^*)$ give us height of the our n rectangles and base of each rectangle is of length $x_i - x_{i-1} = \Delta x$. Then the area A_i of the *ith* rectangle is A_i = length × breadth = $f(x_1^*)\ \Delta x$, and the total area is

$$A_a^b \approx \sum_{i=1}^{n} f(x_i^*)\ \Delta\ x\ (i) \tag{7.2}$$

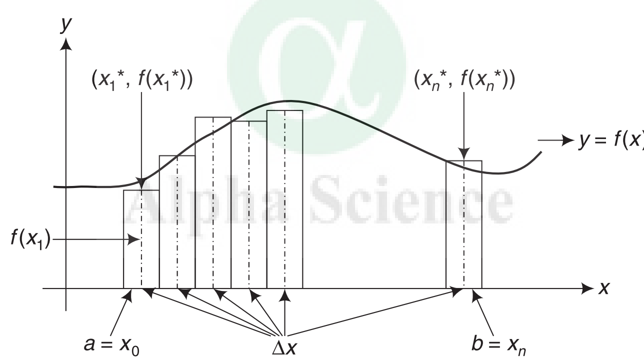


Fig. 7.1

The Definite integral: Let the function f defined on $[a, b]$ with $a < b$. Then the **definite integral** of the function f over the interval$[a, b]$ written $\int_a^b f(x)dx$, is given by

$$\int_a^b f(x)dx = \lim_{\max\ \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\ \Delta x_i \tag{7.3}$$

(Each rectangle becomes "thinner and thinner", the area of the region enclosed by the rectangles seems to get closer to the area of the regions under the curve).

The area of the region bounded above by a curve $y = f(x)$, below by the x-axis and the line $x = a$ and $x = b$ for $a < b$, define by the formula $\int_a^b |f(x)|dx$.

Note: If $a < b$ and if $f(x)$ is nonnegative for $a \le x \le b$, then $|f(x)| = f(x)$, and in this case area $= \int_a^b f(x)dx$, if $f(x)$ is negative for some values of x in $[a, b]$, then the area is sometimes called the net area of f over $[a, b]$

7.3 SOME RESULTS RELATED TO THE DEFINITE INTEGRAL

(i) If c be a constant, then $\int_a^b c\ dx = c(b - a)$ particularly $c = 1$ $\int_a^b 1\ dx = (b - a)$.

(ii) If $a = b$, $\int_a^a f(x)\ dx = 0$.

(iii) If $a < b$, and $\int_a^b f(x)\ dx$ exists, then $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$

i.e. $\int_1^0 x^4 dx = -\int_0^1 x^4\ dx = -\frac{1}{5}$.

(iv) If $f(x)$ and $g(x)$ are define and continuous in $[a, b]$, then

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)\ dx.$$

Proof: Suppose $F(x)$ and $G(x)$ are the antiderivatives of $f(x)$ and $g(x)$ respectively

Then $\int_a^b f(x)dx + \int_a^b g(x)dx = [F(x)]_a^b + [G(x)]_a^b = [F(b) - F(a)] + [G(b) - G(a)] = [F(b) + G(b)] - [F(a) + G(a)] = [F(x) + G(x)]_a^b = \int_a^b [f(x) + g(x)]\ dx$.

i.e. $\int_0^{\frac{\pi}{2}} [x + \sin x]dx = \int_0^{\frac{\pi}{2}} x\ dx + \int_0^{\frac{\pi}{2}} \sin x\ dx = \left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} + [-\cos x]_0^{\frac{\pi}{2}}$

$= \frac{\pi^2}{8} - 0 + 0 - (-1) = \frac{\pi^2}{8} + 1.$

(v) If the function f is integrable on $[a, b]$ and if $a < c < b$, then f is integrable on $[a, c]$

And $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Proof: $\int_a^c f(x)dx + \int_c^b f(x)dx = [F(x)]_a^c + [F(x)]_c^b = [F(c) - F(a)] + [F(b) - F(c)] = [F(b) - F(a)] = [F(x)]_a^b = \int_a^b f(x)dx.$

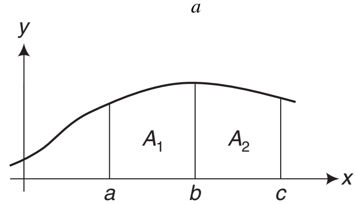


Fig. 7.2

Figure 7.2 shoes that the area $\int_a^b f(x)dx$ = area A_1 + area $A_2 = \int_a^c f(x)dx + \int_c^b f(x)dx$.

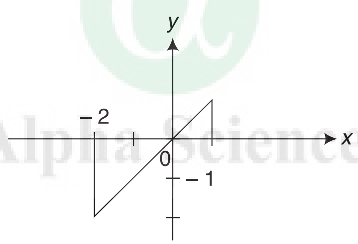


Fig. 7.3

i.e. To find the area of the region enclosed by the line $y = x$, and $x = -2$ and $x = 1$. Figure 7.3 we can use the above formula as

$$\int_{-2}^{1} |x|dx = \int_{-2}^{0} |x|dx + \int_{0}^{1} |x|dx = \frac{5}{2}.$$

(vi) If f be integrable, then

1. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. and

2. $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx.$

Proof 1: Let $x = a - t$, so $dx = -dt$, and when $x = 0$ then $t = a$, and when $x = a$ then $t = 0$

$$\therefore \quad \int_0^a f(x)\,dx = -\int_a^0 f(a-t)dt = \int_0^a f(a-x)dx \text{ or } \int_0^a f(a-x)dx$$

Proof 2: Now $0 < a < 2a$, therefore

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$$

In the second integral on the right

Let $x = 2a - t$, so $dx = -dt$, and when $x = a$ then $t = a$, and when $x = 2a$ then $t = 0$

$$\therefore \quad \int_a^{2a} f(x)dx = -\int_a^0 f(2a-t)dt = \int_0^a f(2a-t)dt \text{ or } \int_0^a f(2a-x)dx \text{ Hence}$$

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

Corollary If $f(2a - x) = f(x)$, then above formula gives

$$\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$$

And if $f(2a - x) = -f(x)$, then 2. gives $\int_0^{2a} f(x)dx = 0$.

(vii) If f is integrable, then

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(x) \text{ is even and}$$

$$\int_{-a}^a f(x)dx = 0, \text{ if } f(x) \text{ is odd}$$

Proof: We know that $\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$ (7.4)

In the first integral on the right

Let $-x = t$, so $dx = -dt$, and when $x = -a$ then $t = a$, and when $x = 0$ then $t = 0$

$\therefore \int_{-a}^0 f(x)dx = -\int_a^0 f(-t)dt = \int_0^a f(-t)dt$ or $\int_0^a f(-x)dx$. Put this value in (7.4), we have

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(-x)dx + \int_{0}^{a} f(x)dx \tag{7.5}$$

If $f(x)$ is even, then $f(-x) = f(x)$, from (7.5), we have

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

And if $f(x)$ is odd, then $f(-x) = -f(x)$, from (7.5), we have

$$\int_{-a}^{a} f(x)dx = 0.$$

7.4 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

First fundamental theorem of integral calculus: Let f be continuous on $[a, b]$ if F is any antiderivative of f on $[a, b]$ then

$$\int_{a}^{b} f(x)dx = F(b) - F(a).$$

Proof: Let F be the antiderivative of a function f which is continuous and integrable on $[a, b]$ then by definition of the antiderivative, F is continuous on $[a, b]$ and differentiable on$]a, b[$.

Let $a = x_0 < x_1 < x_2 < x_3 \ldots \ldots \ldots \ldots < x_n = b$ be a regular subinterval of $[a, b]$ use the mean value theorem in the subinterval $[x_{i-1}, x_i]$ we have

$$F(x_i) - F(x_{i-1}) = F'(x_i^*)(x_i - x_{i-1}) = F'(x_i^*)\,\Delta x_i \tag{7.6}$$

Where $x_{i-1} < x_i^* < x_i$. Thus

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} F'(x_i^*)\,\Delta x_i \tag{7.7}$$

But we know that

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})] = [F(x_1) - F(x_0)] + [F(x_2) - F(x_1)] + \ldots [F(x_n) - F(x_{n-1})]$$
$$= F(x_n) - F(x_0) = F(b) - F(a)$$

Hence from (7.7), we have

$$\sum_{i=1}^{n} F'(x_i^*)\,\Delta x_i = F(b) - F(a)$$

Taking limit as $\Delta x_i \to 0$ of both side, we obtain

$$\lim_{\Delta x \to 0}[F(b) - F(a)] = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} F'(x_i^*)\,\Delta x_i \tag{7.8}$$

Since F is antiderivative of f, we have

$$F'(x_i^*) = f(x_i^*) \tag{7.9}$$

From (7.8) and (7.9), we obtain

$$[F(b) - F(a)] = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)dx$$

(use equation (7.3) of section (7.2)

Example 1 Calculate $\int_0^1 x dx$

Solution We know that $\frac{x^2}{2}$ is an antiderivative for x. Thus

$$\int_0^1 x dx = \frac{1}{2} - 0 = \frac{1}{2}.$$

Example 2 Calculate $\int_0^{\pi} \cos x \, dx$

Solution We know that $\sin x$ is an antiderivative for $\cos x$. Thus

$$\int_0^{\pi} \cos x \, dx = \sin \pi - \sin 0 = 0.$$

Second fundamental theorem of integral calculus: If a function f is continuous on $[a, b]$, then the function $G(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ differentiable on $]a, b[$ and for every x in $]a, b[$

$$\boldsymbol{G'(x) = f(x)}$$

That is, G is an antiderivative of f on the interval $[a, b]$

Proof: We know that

$$G'(x) = \lim_{\Delta x \to 0} \frac{G(x + \Delta x) - G(x)}{\Delta x} \tag{7.10}$$

Since $G(x) = \int_a^x f(t)dt$ = area under the curve above $y = f(x)$, below x-axis and between x and a, Fig. 7.4 then assuming that $f > 0$ on $[a, b]$.

If Δx is small, then Fig. 7.4 shows that $G(x + \Delta x) - G(x) \approx f(x)\Delta x$, or

$$\frac{G(x + \Delta x) - G(x)}{\Delta x} \approx \frac{f(x)\ \Delta x}{\Delta x} = f(x) \tag{7.11}$$

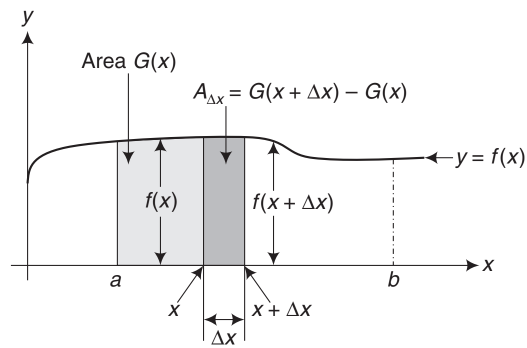


Fig. 7.4

As for as $\Delta x \to 0$, then from equations (7.10) and (7.11), we have

$$G'(x) = \lim_{\Delta x \to 0} \frac{G(x + \Delta x) - G(x)}{\Delta x} = f(x),$$

Hence proved.

7.5 INTEGRATION BY SUBSTITUTION

We know that the derivative of $F(g(x))$ is

$$\frac{d[F(g(x))]}{dx} = F'(g(x))g'(x) \quad \textit{(by chain rule)}$$

In integral form it can be written as

$$\int F'(g(x))g'(x)dx = F(g(x)) + c$$

If F is antiderivative of f, then

$$\int f\,(g(x))\; g'(x)\; dx = F(g(x)) + c \tag{7.12}$$

Now let $u = g(x)$, then $\frac{du}{dx} = g'(x)\; dx$ and in differential form it can be written as $du = g'(x)dx$, with this notation above equation can be expressed as

$$\int \boldsymbol{f(u)\; du = F(u) + c} \tag{7.13}$$

If we evaluate an integral by converting the equation (7.12) into (7.13) with the substitution $u = g(x)$ and $du = g'(x)\; dx$ then this process is called the **method of *u*-substitution**.

Example 3 Calculate $\int \sqrt{2 + x}\; dx$

Solution Let $u = g(x) = 2 + x$ Then $du = dx$ and

$$\int\sqrt{2+x}\,dx = \int\sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}(2+x)^{\frac{3}{2}} + c$$

Example 4 Calculate $\int x\sqrt[3]{2+x^2}\,dx$

Solution Let $u = g(x) = 2 + x^2$. Then $du = 2x\,dx$ and

$$\int x\sqrt[3]{2+x^2}\,dx = \frac{1}{2}\int u^{\frac{1}{3}}\,du = \frac{1}{2}.\frac{3}{4}u^{\frac{4}{3}} + c = \frac{3}{8}(2+x^2)^{\frac{4}{3}} + c$$

Warring: This method of u-substitution will do not work if the chosen u and the computed du cannot be used to produce an integrand in which no expressions involving x remain, or if its not possible to evaluate the resulting integral. For example, the substitution $u = 2 + x^2$, $du = 2xdx$ and $u = x^2$, $du = 2xdx$ will not work respectively of the following integrals

(i) $\int\sqrt{2+x^2}\,dx = \frac{1}{2x}\int u^{\frac{1}{2}}\,du$ cannot be evaluate, (ii) $\int 2x\sin x^4\,dx$ $= \int\sin u^2\,dx$ also cannot be evaluate.

Example 5 Calculate $\int x^2\sqrt{x-3}\,dx$

Solution Let $u = g(x) = x - 3$. Then $du = dx$ and we have $x^2 = (u+3)^2 = u^2 + 9 + 6u$

$$\int x^2\sqrt{x-3}\,dx = \int(u^2+9+6u)\sqrt{u}\,du = \frac{2}{7}u^{\frac{7}{2}} + 6u^{\frac{3}{2}} + \frac{12}{5}u^{\frac{5}{2}} + c$$

$$= \frac{2}{7}(x-3)^{\frac{7}{2}} + 6(x-3)^{\frac{3}{2}} + \frac{12}{5}(x-3)^{\frac{5}{2}} + c$$

Example 6 Calculate $\int\sin^3 x\,dx$

Solution $\int\sin^3 x\,dx = \int\sin^2 x\sin x\,dx = \int(1-\cos^2 x)\sin x\,dx$

Now let $u = g(x) = \cos x$. Then $du = -\sin x\,dx$ and

$$\int\sin^3 x\,dx = \int(1-\cos^2 x)\sin x\,dx = -\int(1-u^2)du$$

$$= \frac{1}{3}u^3 - u + c = \frac{1}{3}\cos^3 x - \cos x + c.$$

Some other useful trigonometric substitutions are

For $\sqrt{a^2-x^2}$ $\quad x = a\sin\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

For $\sqrt{a^2+x^2}$ $\quad x = a\tan\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

For $\sqrt{x^2 - a^2}$ $\quad x = a \sec \theta \quad \begin{cases} 0 \le \theta < \dfrac{\pi}{2} \text{ if } x \ge a \\ \dfrac{\pi}{2} < \theta \le \pi \text{ if } \le -a \end{cases}$

7.6 INTEGRATION BY PARTS

There is an another important method of integration which is called the **integration by parts,** and this method is derived from the product rule of differentiation.

$$d(uv) = u\, dv + v\, du \tag{7.14}$$

Integrating both sides, we have

$$uv = \int u\, dv + \int v\, du \text{ or}$$

$$\int \boldsymbol{u\, dv = uv - \int v\, du} \tag{7.15}$$

Example 7 Calculate $\int xe^x\, dx$

Solution This cannot be integrate directly because the x term gets in the way. However, if we set $u = x$ and $dv = e^x$, then $du = dx$, $v = \int e^x\, dx = e^x$ and

$$\int xe^x\, dx = \int u\, dv = uv - \int v\, du = xe^x - \int e^x dx = xe^x - e^x + c$$

Example 8 Calculate $\int \log x\, dx$

Solution There are two terms $\log x$ and dx, now let $u = \log x$ and $dv = dx$, then $du = \dfrac{dx}{x}$, $v = x$ and

$$\int \log x\, dx = \int u\, dv = uv - \int v\, du = x \log x - \int x.\ \frac{dx}{x} = x \log x - x + c.$$

The success of this method is depend on the choice of $u = \log x$ and dv. For example if we choose $u = \sin x$, and $dv = x\,dx$, then $du = \cos x\, dx$ and $v = \dfrac{x^2}{2}$, to calculate the integration

$\int x \sin x\, dx$, then we have $\int x \sin dx = \dfrac{x^2}{2}.\ \sin x - \int \dfrac{x^2}{2}.\cos x\, dx$,

Which is more complicated than the original. In general there are no any hard and fast rules for choosing u and dv.

According to Herbert Kasube when the integrand of an integration by parts problem consists of the product of two different types of functions, we should let u designate the function that appears first in **LIATE** (logarithmic, inverse trigonometric, algebraic, trigonometric, and exponential) and let dv denote the rest. i.e. in above example according to **LIATE** we should be take $u = x$

(algebraic) and $dv = \sin x$ (trigonometric) then the example will be solve easily, but this choice does not always produce the correct choice of u and dv.

7.7 REDUCTION FORMULAS

A *reduction formula*, by which we can solve an integral problem by *reducing* it to a problem of solving an easier integral problem, and then reducing that to the problem of solving an easier problem, and so on, and the integration by parts can be used to derive reduction formulas for integral. For example, if n is a positive integer such that $n \geq 2$ then the integration by parts can be used to obtain the reduction formulas

$$\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x\, dx \qquad (7.16)$$

$$\int \cos^n dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x\, dx \qquad (7.17)$$

To obtain (7.16), write

$$\int \sin^n x dx = \int \sin^{n-1} x \sin x\, dx$$

Let $u = \sin^{n-1} x$, $dv = \sin x\, dx$

$$du = (n-1) \sin^{n-2} x \cos x\, dx, \quad v = -\cos x$$

$$\int \sin^n x\, dx = \int \sin^{n-1} x \sin x\, dx = \int u\, dv = uv - \int v\, du$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos x. \cos x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$$

$$n \int \sin^n dx = -\sin^{n-1} x \cos x + (n-1) \sin^{n-2} x\, dx$$

$$\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \sin^{n-2} x dx.$$

Example 9 Use reduction formula to evaluate

(i) $\int \cos^6 x\, dx$, (ii) $\int_0^{\frac{\pi}{4}} \sin^4 x\, dx$.

Solution (i) $\int \cos^6 x\, dx$

By equation (7.17), we have

$$\int \cos^6 x\, dx = \frac{1}{6} \cos^5 x \sin x + \frac{5}{6} \int \cos^4 x\, dx$$

Use the reduction formula for $n = 4$, we have

$$= \frac{1}{6}\cos^5 x \sin x + \frac{5}{6}\left\{\frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x\, dx\right\}$$

Now again use the reduction formula for $n = 2$, we have

$$= \frac{1}{6}\cos^5 x \sin x + \frac{5}{6}\left\{\frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left(\frac{1}{2}\cos x \sin x + \frac{1}{2}\int dx\right)\right\}$$

$$= \frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{5}{16}\cos x \sin x + \frac{5}{16}x + c$$

(ii) By equation (7.16), we have

$$\int \sin^4 x dx = -\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}\int \sin^2 x\, dx$$

Use the reduction formula for $n = 2$, we have

$$= -\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}\left(-\frac{1}{2}\cos x \sin x + \frac{1}{2}\int dx\right)$$

$$\int_0^{\frac{\pi}{4}} \sin^4 x dx = \left\{-\frac{1}{4}\sin^3 x \cos x - \frac{3}{8}\cos x \sin x + \frac{3}{8}x\right\}_0^{\frac{\pi}{4}}$$

$$= -\frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^3 \frac{1}{\sqrt{2}} - \frac{3}{8}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + \frac{3\pi}{32}.$$

Example 10 Use the reduction formula show that

$$\int \sin^3 x dx = -\frac{1}{3}\cos^3 x - \cos x + c$$

Solution By reduction formula (7.16), we have

$$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x + \frac{2}{3}\int \sin x + c$$

$$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + c$$

$$= -\frac{1}{3}(1 - \cos^2 x)\cos x - \frac{2}{3}\cos x + c$$

$$\int \sin^3 x dx = \frac{1}{3}\cos^3 x - \cos x + c.$$

Integrating product of trigonometric functions: Suppose m and n are positive integers, then the integral

$$\int \sin^m x \cos^n x\, dx \tag{7.18}$$

Can be evaluated as

When m is odd then substitute $u = \cos x$ and break $\sin^2 x = 1 - \cos^2 x$

When n is odd then substitute $u = \sin x$ and break $\cos^2 x = 1 - \sin^2 x$

When both m and n are even then break $\sin^2 x = \dfrac{1 - \cos 2x}{2}$ and $\cos^2 x = \dfrac{1 + \cos 2x}{2}$ or $\sin x \cos x = \dfrac{1}{2} \sin 2x$.

Example 11 Calculate

(i) $\int \sin^4 x \cos^5 x \, dx$, (ii) $\int \sin^6 x \cos^6 x \, dx$,

(iii) $\int_0^{\frac{\pi}{4}} \sin^2 x \cos^4 x \, dx$, (iv) $\int \sin^3 x \cos^{5/3} x dx$,

(v) $\int_0^{\frac{\pi}{3}} \sin^4 3x \cos^3 3x \, dx$.

Solution (i) $\int \sin^4 x \cos^5 x \, dx = \int \sin^4 x \cos^4 x \cos x dx$

here n is odd break $\cos^2 x = 1 - \sin^2 x$, we have

$$\int \sin^4 x \cos^4 x \cos x \, dx = \int \sin^4 x \, (1 - \sin^2 x)^2 \cos x \, dx$$

substitute $u = \sin x$ $du = \cos x \, dx$

$$= \int u^4 (1 - u^2)^2 du = \int (u^4 - 2u^6 + u^8) du$$

$$= \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + c$$

$$= \frac{1}{5} \sin^5 x - \frac{2}{7} \sin^7 x + \frac{1}{9} \sin^9 x + c$$

(ii) $\int \sin^6 x \cos^6 x \, dx$, here both m and n are even so let $\sin x \cos = \dfrac{1}{2} \sin 2x$, we have

$$\int (\sin x \cos x)^6 \, dx = \int \left(\frac{1}{2} \sin 2x\right)^6 dx \text{ let } u = 2x \; du = 2 \, dx$$

$\displaystyle\int \frac{1}{128} \sin^6 u \, du = \frac{1}{128} \int \sin^6 u \, du$. Now use the formula (7.16), we have

$$\frac{1}{128} \int \sin^6 u \, du = \frac{1}{128} \left(- \frac{1}{6} \sin^5 u \cos u + \frac{5}{6} \int \sin^4 u du\right)$$

Use the reduction formula for $n = 4$, we have

$$= \frac{1}{128} \left[- \frac{1}{6} \sin^5 u \cos u + \frac{5}{6} \left\{-\frac{1}{4} \sin^3 u \cos u + \frac{3}{4} \int \sin^2 u \, du\right\}\right]$$

Now again use the reduction formula for $n = 2$, after some simplification we have

$$= \frac{1}{128}\left[-\frac{1}{6}\sin^5 u\cos u + \frac{5}{6}\left\{-\frac{1}{4}\sin^3 u\cos u + \frac{3}{4}\left(-\frac{1}{4}\sin 2u + \frac{u}{2}\right)\right\}\right]$$

$$= -\frac{1}{768}\sin^5 2x\cos x - \frac{5}{3072}\sin^3 2x\cos 2x - \frac{15}{12288}\sin 4x + \frac{15x}{3072} + c$$

(iii) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 x\cos^4 x\,dx = \int_0^{\frac{\pi}{4}} (\sin x\cos x)^2\cos^2 x\,dx$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2}\sin 2x\right)^2 \frac{1}{2}(1 + \cos 2x)dx$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{4}}\sin^2 2x\,dx + \frac{1}{8}\int_0^{\frac{\pi}{4}}\sin^2 2x\cos 2x\,dx$$

Let $u = 2x,\ du = 2dx,$

$$= \frac{1}{16}\left(\int_0^{\frac{\pi}{4}} \sin^2 u\,du + \int_0^{\frac{\pi}{4}}\sin^2 u\cos u\,du\right)$$

In first integral we use the formula (7.16) and in second integral formula (7.18), we get

$$= \frac{1}{16}\left\{-\frac{1}{4}\sin 2u + \frac{u}{2}\right\}_0^{\frac{\pi}{4}} + \frac{1}{16}\left\{\frac{\sin^3 u}{3}\right\}_0^{\frac{\pi}{4}}$$

$$= \frac{1}{16}\left\{-\frac{1}{4}\sin 4x + x\right\}_0^{\frac{\pi}{4}} + \frac{1}{16}\left\{\frac{\sin^3 2x}{3}\right\}_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{64} + \frac{1}{48}.$$

(iv) $\displaystyle\int \sin^3 x\cos^{\frac{5}{3}} x\,dx = \int \sin^2 x\sin x\cos^{\frac{5}{3}} x\,dx = \int (1 - \cos^2 x)\sin x\cos^{\frac{5}{3}} x\,dx$

substitute $u = \cos x\ du = -\sin x\,dx$

$$= \int -(1 - u^2)u^{\frac{5}{3}}du = -\int u^{\frac{5}{3}} + \int u^{\frac{11}{3}}$$

$$= -\frac{3}{8}u^{\frac{8}{3}} + \frac{3}{14}u^{\frac{14}{3}} + c$$

(v) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^4 3x\cos^3 3x\,dx$

Let $t = 3x \quad dt = 3\ dx$

$$\frac{1}{3}\int_0^{\frac{\pi}{3}} \sin^4 t \cos^3 t\, dt$$

Now use the formula (7.18), we have

$$\frac{1}{3}\int_0^{\frac{\pi}{3}} \sin^4 t \cos^3 t\, dt = \frac{1}{3}\int_0^{\frac{\pi}{3}} \sin^4 t \cos^2 t \cos t\, dt = \frac{1}{3}\int_0^{\frac{\pi}{3}} \sin^4 t\,(1 - \sin^2 t)\cos t\, dt$$

Let $u = \sin t \quad du = \cos t\, dt$

$$= \frac{1}{3}\int_0^{\frac{\pi}{3}} \sin^4 t\,(1 - \sin^2 t)\cos t\,dt = \frac{1}{3}\int_0^{\frac{\pi}{3}} u^4 (1 - u^2)du = \frac{1}{3}\left[\frac{u^5}{5} - \frac{u^7}{7}\right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{3}\left[\frac{\sin^5 t}{5} - \frac{\sin^7 t}{7}\right]_0^{\frac{\pi}{3}} = \frac{1}{3}\left[\frac{\sin^5 3x}{5} - \frac{\sin^7 3x}{7}\right]_0^{\frac{\pi}{3}} = 0.$$

Some Other Reduction Formulas

$$\int \sec^n x\, dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx \tag{7.19}$$

$$\int \tan^n x\, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\, dx \tag{7.20}$$

$$\int x^n e^x dx = x^n e^x - n\int x^{n-1} e^x\, dx \tag{7.21}$$

$$\int (\log x)^n dx = x(\log x)^n - n\int (\log x)^{n-1}\, dx \tag{7.22}$$

Proof: $\int \sec^n x\, dx = \int \sec^{n-2} x \sec^2 x\, dx$

Let $u = \sec^{n-2} x \quad dv = \sec^2 x\,dx \quad du = (n-2)\sec^{n-2} x \tan x\, dx$

$$\int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx = \int u\, dv = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x\,(\sec^2 x - 1)\, dx$$

$$= \sec^{n-2} x \tan x + (n-2)\int (\sec^{n-2} x - \sec^n x)\, dx$$

$$= \sec^{n-2} x \tan x + (n-2)\int (\sec^{n-2} x\, dx - (n-2)\int \sec^n x\, dx$$

$$\int \sec^n x\, dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx$$

Proof: $\int \tan^n x\,dx = \int \tan^{n-2} \tan^2 x\,dx = \int \tan^{n-2} (\sec^2 x - 1)\,dx$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx \tag{7.23}$$

In first integral of (7.23)

Let $u = \tan^{n-2} x \quad dv = \sec^2 x\,dx \quad du = (n-2)\tan^{n-3} x \sec^2 x\,dx$ then

$$\int \tan^{n-2} x \sec^2 x\,dx = \int u\,dv = \tan^{n-2} x \tan x - (n-2)\int \tan^{n-3} x \sec^2 x \tan x\,dx$$

$$= \frac{\tan^{n-1} x}{(n-1)} \tag{7.24}$$

From (7.23) and (7.24), we have

$$\int \tan^n x\,dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x\,dx$$

Proof: To prove formula (7.21)

Let $u = x^n \quad dv = e^x dx \quad du = n\,x^{n-1}\,dx$ then

$$\int x^n e^x dx = \int u\,dv = x^n e^x - n \int x^{n-1} e^x dx$$

Proof: To prove formula (7.22)

Let $u = (\log x)^n \quad dv = dx \quad du = n(\log x)^{n-1} \frac{1}{x} dx$ then

$$\int (\log x)^n\,dx = \int u\,dv = x(\log x)^n - n\int (\log x)^{n-1} dx.$$

Similarly,

$$\boldsymbol{\int \cot^n x\,dx = -\frac{1}{n-1} \cot^{n-1} x - \int \cot^{n-2} x\,dx}$$

$$\boldsymbol{\int \mathrm{cosec}^n x\,dx = -\frac{1}{n-1} \mathrm{cosec}^{n-2} x \cot x + \frac{n-2}{n-1} \int \mathrm{cosec}^{n-2} x\,dx} \tag{7.25}$$

Now the integral

$\boldsymbol{\int \tan^m x \sec^n x\,dx}$ can be evaluated as

When m is odd break the factor of $\sec x \tan x$ then substitute $u = \sec x$ and make $\tan^2 x = \sec^2 x - 1$

When n is even break the factor of $\sec^2 x$ then substitute $u = \tan x$ and make $\sec^2 x = 1 + \tan^2 x$

When m even and n odd then use $\sec^2 x = 1 + \tan^2 x$ to write everything in terms of $\sec x$ and after that use the formula for power of $\sec x$.

Example 12 Calculate

(i) $\int \sec^4 x\,dx$, (ii) $\int \tan^3 x\,dx$,

(iii) $\int \tan^5 x \sec^2 x \, dx$, (iv) $\int \tan^2 x \sec^4 x \, dx$,

(v) $\int \tan^2 x \sec x \, dx$.

Solution

(i) $\int \sec^4 x \, dx$

from formula (7.19), we have

$\int \sec^4 x \, dx = \frac{1}{3} \sec^2 x \tan x + \frac{2}{3} \int \sec^2 x \, dx = \frac{1}{3} \sec^2 x \tan x + \frac{2}{3} \tan x + c$

(ii) $\int \tan^3 x \, dx$

from formula (7.20), we have

$\int \tan^3 x \, dx = \frac{1}{2} \tan^2 x - \int \tan x \, dx = \frac{1}{2} \tan^2 x - \log |\sec x| + c$

(iii) $\int \tan^5 x \sec^2 x \, dx$

Here m is odd; we will use the formula (7.25) for odd m

$\int \tan^5 x \sec^2 x \, dx = \int \tan^4 x \sec x \sec x \tan x \, dx = \int (\sec^2 - 1)^2 \sec x \sec x \tan x \, dx$

Now let $u = \sec x$ $du = \sec x \tan x \, dx$

$\int \tan^5 x \sec^2 x \, dx = \int (\sec^2 - 1)^2 \sec x \sec x \tan x \, dx = \int (u^2 - 1)^2 u \, du = \int (u^5 - 2u^3 + u) \, du$

$$= \frac{u^6}{6} - \frac{u^4}{2} + \frac{u^2}{2} + c = \frac{\sec^6 x}{6} - \frac{\sec^4 x}{2} + \frac{\sec^2 x}{2} + c$$

(iv) $\int \tan^2 x \sec^4 x \, dx$

Here n is even, we will use the formula (7.25) for even n

$\int \tan^2 x \sec^4 x \, dx = \tan^2 x (1 + \tan^2 x) \sec^2 x \, dx$

Now let $u = \tan x$ $du = \sec^2 x \, dx$

$\int \tan^2 x \sec^4 x \, dx = \int \tan^2 x (1 + \tan^2 x) \sec^2 x \, dx = \int u^2 (u^2 + 1) \, du$

$$= \frac{u^5}{5} + \frac{u^3}{3} + c = \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + c$$

(v) $\int \tan^2 x \sec x \, dx$.

Here m is odd and n is even we will use again the formula (7.25)

$$\int \tan^2 x \sec x \, dx = \int (\sec^2 - 1) \sec x \, dx$$

$$\int \sec^3 x \, dx - \int \sec x \, dx$$

$$\frac{1}{2} \sec x \tan x + \frac{1}{2} \log |\sec x + \tan x| - \log |\sec x + \tan x| + c$$

$$\frac{1}{2} \sec x \tan x - \frac{1}{2} \log |\sec x + \tan x| + c.$$

$$\int \cos^m x \sin nx \, dx = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} \int \cos^{m-1} x \sin (n-1) x \, dx \quad (7.26)$$

Proof: Let $u = \cos^m x \quad dv = \sin nx\, dx \quad du = -m \cos^{m-1} x \sin dx$ then

$$\int \cos^m x \sin nx \, dx = \int u \, dv = -\frac{\cos^m x \cos nx}{n} - m \int \frac{\cos^{m-1} x \sin x \cos nx \, dx}{n} \quad (7.27)$$

Now we know that

$$\sin (n-1)x = \sin nx \cos x - \cos nx \sin x$$

$$\therefore \int \cos^{m-1} x \cos nx \sin x dx = \int \cos^m x \sin nx \, dx - \int \cos^{m-1} x \sin (n-1) x dx \quad (7.28)$$

From (7.27) and (7.28), we have

$$\int \cos^m x \sin nx \, dx = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} \int \cos^{m-1} x \sin (n-1) x dx$$

Similarly, we can establish the following:

$$\int \cos^m x \cos nx \, dx = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} \int \cos^{m-1} x \cos (n-1) x \, dx \quad (7.29)$$

$$\int \sin^m x \cos nx \, dx = -\frac{n \sin^m x \sin nx + m \sin^{m-1} x \cos x \cos nx}{m^2 - n^2} + \frac{m(m-1)}{m^2 - n^2} \int \sin^{m-2} x \cos nx \, dx \quad (7.30)$$

$$\int \sin^m x \sin nx \, dx = \frac{n \sin^m x \cos nx - m \sin^{m-1} x \cos x \sin nx}{m^2 - n^2} + \frac{m(m-1)}{m^2 - n^2} \int \sin^{m-2} x \sin n x \, dx \quad (7.31)$$

$$\boldsymbol{\int x^n \sin mx\,dx = \frac{x^{n-1}}{m^2}(n\sin mx - mx\cos mx) - \frac{n(n-1)}{m^2}\int x^{n-2}\sin mx\,dx} \quad (7.32)$$

Proof: Let $u = x^n$ $dv = \sin mx\ dx$ $du = nx^{n-1}\ dx$ then

$$\int x^n \sin mx\,dx = \int u\ dv = -\frac{x^n \cos mx}{m} + n\int \frac{x^{n-1}\cos mx\,dx}{m} \quad (7.33)$$

Now for right integral of (7.33)

$$\int x^{n-1}\cos mx\,dx = \int u\,dv = \frac{x^{n-1}\sin mx}{m} - (n-1)\int \frac{x^{n-2}\sin mx\,dx}{m} \quad (7.34)$$

Where $u = x^{n-1}$ $dv = \cos mx\,dx$ $du = (n-1)x^{n-2}dx$

From (7.33) and (7.34), we have

$$\int x^n \sin mx\,dx = \frac{x^{n-1}}{m^2}(n\sin mx - mx\cos mx) - \frac{n(n-1)}{m^2}\int x^{n-2}\sin mx\ dx$$

Similarly,

$$\boldsymbol{\int x^n\cos mx\,dx = \frac{x^{n-1}}{m^2}(n\cos mx + mx\sin mx) - \frac{n(n-1)}{m^2}\int x^{n-2}\cos mx\,dx} \quad (7.35)$$

$$\boldsymbol{\int x\sin^n x\ dx = \int \frac{(\sin x - nx\cos x)\sin^{n-1}x}{n^2} + \frac{n(n-1)}{n}\int x\sin^{n-2}x\ dx} \quad (7.36)$$

Where $n \geq 2$

Proof: Since $n \geq 2$ $\int x\sin^n x\ dx = \int x\sin^{n-1}x\sin x\ dx$

Let $u = x\sin^{n-1}x$ $dv = \sin x\ dx$ $du = \{(n-1)x\sin^{n-2}x\cos x + \sin^{n-1}x\}\ dx$ then

$$\int x\sin^{n-1}x\sin x\ dx = \int u\ dv$$

$$= -x\sin^{n-1}x\cos x + \int x\ \{(n-1)x\sin^{n-2}x\cos x + \sin^{n-1}x\}\cos x\,dx$$

$$= -x\sin^{n-1}x\cos x + \frac{\sin^n x}{n} + (n-1)\int x\sin^{n-2}x\ dx - (n-1)\int x\sin^n x\ dx$$

$$\therefore \int x\sin^n x\ dx = -x\sin^{n-1}x\cos x + \frac{\sin^n x}{n} + (n-1)\int x\sin^{n-2}x\ dx - (n-1)\int x\sin^n x\ dx$$

Hence

$$\int x \sin^n x\, dx = \int \frac{(\sin x - nx \cos x) \sin^{n-1} x}{n^2} + \frac{n(n-1)}{n} \int x \sin^{n-2} x\, dx$$

Similarly,

$$\int x \cos^n x\, dx = \int \frac{(\cos x + nx \sin x) \cos^{n-1} x}{n^2} + \frac{n(n-1)}{n} \int x \cos^{n-2} x\, dx \quad (7.37)$$

Exercises

1. Use the reduction formulas evaluate the following

(i) $\int \sin^5 x\, dx$, (ii) $\int \sin^7 x\, dx$,

(iii) $\int \sin^6 x\, dx$, (iv) $\int \cos^4 x\, dx$,

(v) $\int_0^{\frac{\pi}{2}} \sin^7 x\, dx$ (vi) $\int_0^{\frac{\pi}{2}} \sin^3\, dx$,

(vii) $\int_0^{\frac{\pi}{6}} \cos^3 3x\, dx$, (viii) $\int_0^{\frac{\pi}{8}} \sin^4 4x\, dx$,

(ix) $\int_0^{\frac{\pi}{4}} \sin^4 x\, dx$, (x) $\int \sin^4 x \cos^4 x\, dx$,

(xi) $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^2 x\, dx$, (xii) $\int_0^{\frac{\pi}{2}} \sin^6 x \cos^4 x\, dx$,

(xiii) $\int_0^{\frac{\pi}{3}} \sin^4 5x \cos^3 5x\, dx$, (xiv) $\int \sin^3 3x \cos^2 3x\, dx$,

(xv) $\int \sin^2 4x \cos^3 4x\, dx$, (xvi) $\int_0^{\frac{\pi}{4}} \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}\, dx$,

(xvii) $\int \cos^{\frac{1}{3}} x \sin x\, dx$ (xviii) $\int \cos^{\frac{5}{3}} x \sin^3 x\, dx$,

(xix) $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^3 x\, dx$, (xx) $\int \sec^3 x\, dx$,

(xxi) $\int \tan^4 x \, dx$, (xxii) $\int \tan^3 5x \, dx$,

(xxiii) $\int \sec^5 x \, dx$, (xxiv) $\int_0^{\frac{\pi}{4}} \tan^7 x \sec^4 x \, dx$,

(xxv) $\int_0^{\frac{\pi}{3}} \tan^5 x \sec^3 x \, dx$, (xxvi) $\int \tan^4 4x \sec 4x \, dx$,

(xxvii) $\int \tan^2 x \sec^3 x \, dx$, (xxviii) $\int \tan^3 3x \sec^4 3x \, dx$,

(xxix) $\int_0^{\frac{\pi}{4}} \tan^4 x \sec^4 x \, dx$, (xxx) $\int \tan x \sec^{\frac{5}{2}} x \, dx$,

(xxxi) $\int \tan^{\frac{3}{2}} x \sec^4 x \, dx$, (xxxii) $\int_0^{\frac{\pi}{2}} \cos^6 x \cos 6x \, dx$,

(xxxiii) $\int_0^{\frac{\pi}{2}} x \sin^5 x \, dx$, (xxxiv) $\int \cot^5 3x \, dx$.

2. If $l_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$, show that $l_n + l_{n-2} = \frac{1}{n-1}$.

3. Show that

   (i) $\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \frac{\pi}{2} \cdot \frac{1.3.5 \ldots (n-1)}{2.4.6 \ldots n}$ when n is even and $n \geq 2$

   (ii) $\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \frac{2.4.6 \ldots (n-1)}{3.5.7 \ldots n}$ when n is odd and $n \geq 3$

4. If $l_n = \int_0^{\frac{\pi}{4}} x \sin^n x \, dx$, $(n > 1)$, then show that $nl_n = (n-1) \, l_{n-2} + \frac{1}{n}$.

5. Show that $\int (\log x)^5 \, dx = x(\log x)^5 - 5x(\log x)^4 + 20x(\log x)^3 - 60x(\log x)^2 + 120 \, (x \log x - x) + c$.

Answers

1. (i) $-\cos x + \frac{2}{3}\cos^2 x - \frac{1}{5}\cos^5 x + c,$

(ii) $-\frac{1}{7}\sin^6 x\cos x - \frac{6}{5}\sin^4 x\cos x - \frac{8}{35}\sin^2 x\cos x - \frac{16}{35}\cos x + c$

(iii) $-\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{16}\sin x\cos x + \frac{5}{16}x + c,$

(iv) $\frac{1}{4}\cos^3 x\sin x + \frac{3}{8}(\sin x\cos x + x) + c,$

(v) $\frac{16}{35},$ (vi) $\frac{2}{3},$

(vii) $\frac{2}{9},$ (viii) $\frac{3\pi}{64},$

(ix) $\frac{3\pi}{32} - \frac{1}{4},$

(x) $\frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + c,$

(xi) $\frac{\pi}{32},$ (xii) $\frac{3\pi}{512},$

(xiii) 0, (xiv) $-\frac{1}{9}\cos^3 3x + \frac{1}{15}\cos^5 3x + c,$

(xv) $\frac{1}{12}\sin^3 4x - \frac{1}{20}, \sin^5 4x + c,$

(xvi) $\frac{\pi}{32} - \frac{1}{16},$

(xvii) $-\frac{3}{4}\cos^{\frac{4}{3}} + c,$ (xviii) $-\frac{3}{8}\cos^{\frac{8}{3}} + \frac{3}{14}\cos^{\frac{14}{3}} + c,$

(xix) $\frac{1}{12},$

(xx) $\frac{1}{2}\sec x\tan x + \frac{1}{2}\log|\sec x + \tan x|$

(xxi) $\frac{1}{3}\tan^3 x - \tan x + x + c,$ (xxii) $\frac{1}{10}\tan^2 x - \frac{1}{5}\log|\sec x| + c,$

(xxiii) $\frac{1}{4}\sec^3 x\tan x + \frac{3}{8}\sec x\tan x + \frac{3}{8}\log|\sec x + \tan x| + c,$

(xxiv) $\frac{9}{40},$ (xxv) $\frac{418}{35},$

(xxvi) $\frac{1}{16}\sec^3 x\tan x - \frac{5}{32}\sec x\tan x + \frac{3}{32}\log|\sec x + \tan x| + c$,

(xxvii) $\sec^3 x\tan x - \frac{1}{2}\sec x\tan x - \frac{1}{2}\log|\sec x + \tan x| + c$,

(xxviii) $\frac{1}{18}\tan^6 x + \frac{1}{12}\tan^4 x + c$, (xxix) $\frac{12}{35}$,

(xxx) $\frac{2}{5}\sec^{\frac{5}{2}} x + c$, (xxxi) $\frac{2}{5}\tan^{\frac{5}{2}} x + \frac{2}{9}\tan^{\frac{7}{2}} x + c$,

(xxxii) $\frac{\pi}{128}$, (xxxiii) $\frac{149}{225}$,

(xxxiv) $-\frac{1}{12}\cot^4 x + \frac{1}{6}\cot^2 x + \frac{1}{3}\log\sin x + c$,

7.8 AREA BETWEEN TWO CURVES

As we have seen in section 7.2 that the area between a curve $y = f(x)$ and the x-axis can be defined as

$$\text{Area} = \int_a^b |f(x)|\,dx$$

Similarly the area between the curve $x = f(y)$ and the y-axis can be defined as

$$\text{Area} = \int_c^d |f(y)|\,dy.$$

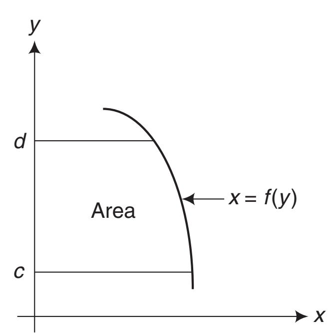


Fig. 7.5

We will now extend the above area problem as:

Suppose we want to find the area of a region which is bounded above by $y = f(x)$ below by $y = g(x)$ left by $x = a$ and right by $x = b$, Fig. 7.7. To find the area we divide this region into n rectangles, Fig. 7.6. Now let a typical rectangle with breath Δx_i and length $\{f(x_i^*) - g(x_i^*)\}$, then the

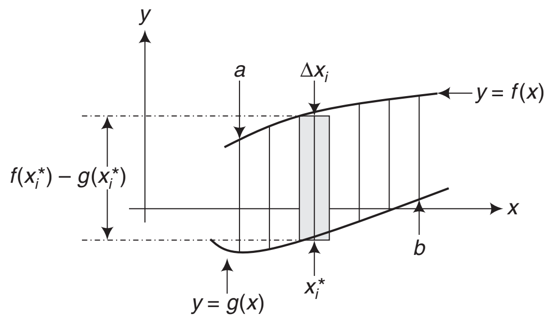


Fig. 7.6

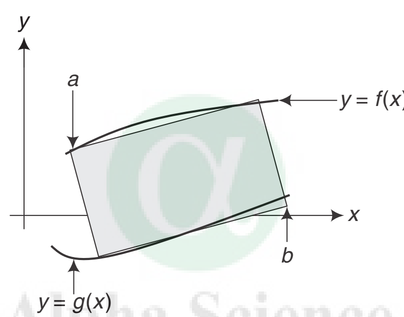


Fig. 7.7

area of this rectangle is

$$\{f(x_i^*) - g(x_i^*)\}\,\Delta x_i$$

Where $x_i^* \in [x_{i-1} - x_i]$ and $\Delta x_i = x_i - x_{i-1}$.

Now taking the limit as n increases and the breath of the rectangle approach zero we get the area between the curves:

Area $\lim_{\Delta xi \to 0} \sum_{i=1}^{n} \{f(x_i^*) - g(x_i^*)\}\,\Delta x_i = \int_a^b |f(x) - g(x)|\, dx$

Hence the area between two curves are

$$\textbf{Area} = \int_a^b |\, f(x) - g(x)|\, dx \tag{7.38}$$

If the function f and g are positive on the interval $[a, b]$ then the area is

$$\textbf{Area} = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \tag{7.39}$$

Example 13 Find the area of the region bounded by $y = 2 - x$ and $y = x^2$

Solution (To find the area first we sketch the curves it not necessary to sketch the accurate curve because the purpose of the sketch is to determine which curve is the upper boundary and which is the lower).

Therefore the Fig. 7.8 shows the upper boundary is $y = 2 - x$ and lower boundary is $y = x^2$. The limit of the integration will be the point of intersection of these two curves, to obtain these points we equate $y = x^2$ and $y = 2 - x$ and this gives

$$x^2 = 2 - x \Rightarrow x^2 + x - 2 = 0 \Rightarrow x^2 + 2x - x - 2 = 0 \text{ or } (x + 2)(x - 1) = 0$$

Hence, we get

$$x = -2 \text{ and } x = 1 \Rightarrow a = -2 \text{ and } b = 1. \; f(x) = 2 - x, \; g(x) = x^2$$

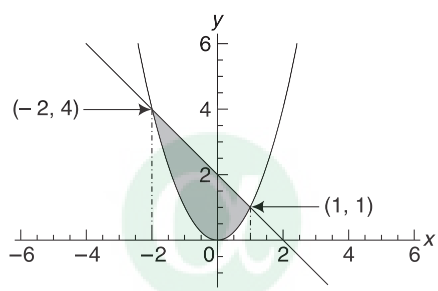


Fig. 7.8

We obtain the area

$$\text{Area} = \int_{-2}^{1} |[(2 - x) - x^2]| \, dx = \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1} = \frac{7}{6} - \left(-\frac{10}{3}\right) = \frac{27}{6}.$$

Example 14 Find the area of the region bounded by $y = \sin 2x$, $y = 0$, $x = \frac{\pi}{2}$ and $x = \pi$.

Solution The Fig. 7.9 shows the upper boundary is $y = 0$ and lower boundary is $y = \sin 2x$.

Therefore $f(x) = 0$, $g(x) = \sin 2x$, $a = \frac{\pi}{2}$ and $b = \pi$

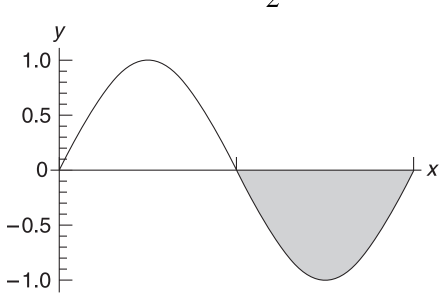


Fig 7.9

We obtain the area

$$\text{Area} = \int_{\frac{\pi}{2}}^{\pi} |[0 - \sin 2x]|\, dx = \left[\frac{\cos 2x}{2}\right]_{\frac{\pi}{2}}^{\pi} = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1.$$

Example 15 Find the area of the region bounded by $y = \frac{x}{4}$ $y = 4x$, and $y = 4 - x$.

Solution Here the region divide in two parts. Fig. 7.10

In first part the upper boundary is $y = 4x$ and lower boundary is $y = \frac{x}{4}$

In second part the upper boundary is $y = 4 - x$ and lower boundary is $y = \frac{x}{4}$

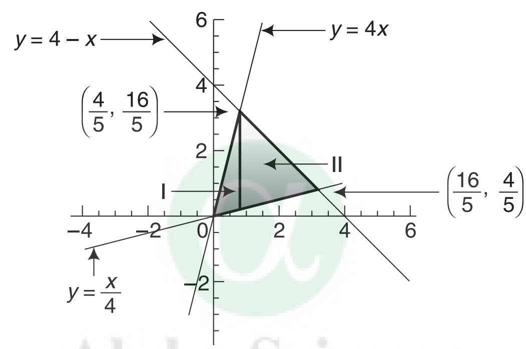


Fig. 7.10

For first part

$$f(x) = 4x,\ g(x) = \frac{x}{4},\ a = 0 \text{ and } b = \frac{4}{5}$$

And for second part

$$f(x) = 4 - x,\ g(x) = \frac{x}{4},\ a = \frac{4}{5} \text{ and } b = \frac{16}{5}$$

$$\text{Total area} = \int_0^{\frac{4}{5}} \left|\left[(4x) - \frac{x}{4}\right]\right| dx + \int_{\frac{4}{5}}^{\frac{16}{5}} \left|\left[(4 - x) - \frac{x}{4}\right]\right| dx$$

$$= \left[2x^2 - \frac{x^2}{8}\right]_0^{\frac{4}{5}} + \left[4x - \frac{x^2}{2} - \frac{x^2}{8}\right]_{\frac{4}{5}}^{\frac{16}{5}}$$

$$= \frac{6}{5} + \frac{154}{25} = \frac{184}{25}.$$

Reversing the roles of x and y

We are in the habit of writing every relation between x and y in terms of x as a function of x. However, there are so many problems which are more convenient to treat x as a function of y. For example to find the area of the region between the curves $x = y^2 - 4$ and $x + 2y = 4$ is more convenient to treat x as a function of y. We will now show how this can be done. Suppose we want to find the area of a region which is bounded above by $y = d$ below by $y = c$ left by $x = v(y)$ and right by $x = w(y)$, (Fig. 7.11). In this case the formula of the area will be

$$\textbf{Area} = \int_c^d |w(y) - v(y)|\,dy \qquad (7.40)$$

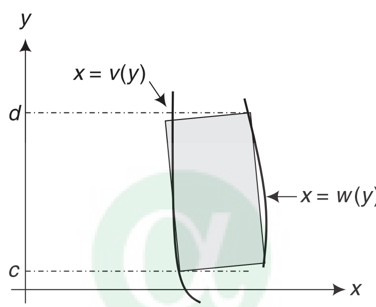


Fig. 7.11

Example 16 Find the area of the region bounded by $y^2 = x + 4$ and $x + 2y = 4$.

Solution The Fig. 7.12 shows the right boundary is $x + 2y = 4$ and left boundary is $y^2 = x + 4$. The limit of the integration will be the point of intersection of these two curves, and this gives

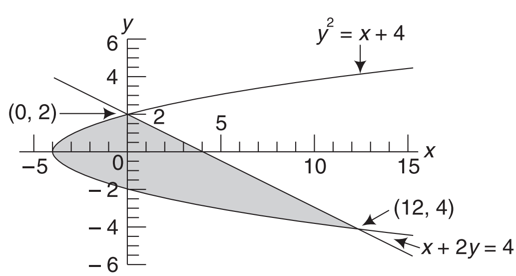


Fig. 7.12

$$y^2 = 4 - 2y + 4 \Rightarrow y^2 + 2y - 8 = 0 \text{ or } (y - 2)(y + 4) = 0$$

Hence, we get

$y = -4$ and $y = 2 \Rightarrow c = -4$ and $d = 2$. $w(y) = 4 - 2y$, $v(y) = y^2 - 4$

Now from formula (7.40), we have

$$\text{Area} = \int_{-4}^{2} |(4 - 2y) - (y^2 - 4)|\, dy = \int_{-4}^{2} |8 - 2y - y^2|$$

$$dy = \left[8y - y^2 - \frac{y^3}{3}\right]_{-4}^{2} = 36.$$

Example 17 Find the area of the region bounded by $y^3 - 4y^2 + 3y = x$ and $x = y^2 - y$.

Solution The Fig. 7.13 shows that the region divide in two parts. In first part the right boundary is $y^3 - 4y^2 + 3y = x$ and left boundary is $x = y^2 - y$, and in second part the right boundary is $x = y^2 - y$ and left boundary is $y^3 - 4y^2 + 3y = x$.

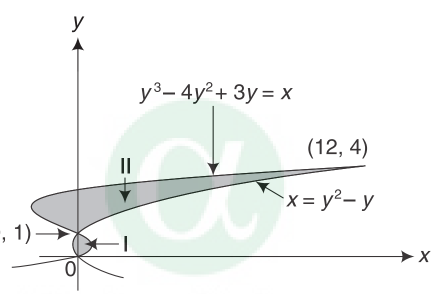


Fig. 7.13

The limit of the integration will be the point of intersection of these two curves, and this gives

$$y^3 - 4y^2 + 3y = y^2 - y$$

$$\Rightarrow \quad y\,(y^2 - 5y + 4) = 0 \text{ or } y(y - 1)(y - 4) = 0$$

Hence, we get

Now from formula (7.40), we have

$$\text{Total area} = \int_{0}^{1} |[y^3 - 4y^2 + 3y) - (y^2 - y)]|\, dy + \int_{1}^{4} |[(y^2 - y)$$

$$- (y^3 - 4y^2 + 3y)]|\, dy = \frac{7}{12} + \frac{45}{4} = \frac{71}{6}.$$

Note: There is no any general rule for choose the formula (7.38) or (7.40). In most cases an inspection of the graph of the area being considered will indicate which is suitable.

Exercises

1. Find the area bounded by the curves

   (i) $y = x^3$ and $y = \sqrt{x}$, (ii) $y = x^2$ and $y = 2x$,

   (iii) $y = x^2$ and $y = 3x + 4$, (iv) $y = e^{3x}$, $y = e^x$ $x = 0$, $x = \log 3$,

   (v) $y = \cos 2x$, $y = 0$, $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$.

   (vi) $y = \cos x$, $y = \sin x$, $x = 0$, $x = 2\pi$

   (vii) $y = x^3 - 4x$, $y = 0$, $x = -2$, $x = 2$,

   (viii) $y = \left|\frac{x}{2}\right|$, $y = \frac{1}{1+x^2}$,

   (ix) $y = 2 + |x - 1|$, $y = -\frac{1}{6}x + 8$,

   (x) $x = 2 \sin y$, $x = 0$, $y = \frac{\pi}{4}$, $y = \frac{3\pi}{4}$.

2. Find the area bounded by the curves $y = x^2 + 3x + 5$, $y = -x^2 + 5x + 9$, $x = -1$ and $x = 3$.

3. Find the area of the region bounded by the curves $y = x^2$ and $y = 3x$ by integrating

   (i) With respect to x (ii) With respect to y.

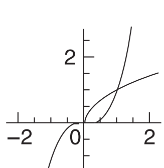
4. Find the area of the region bounded by the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the coordinate axes.

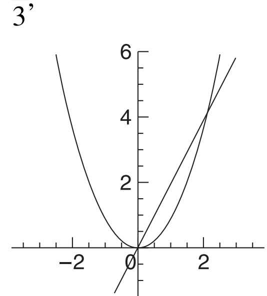
5. Find the area of the region bounded by the parabola $y = 2x - x^2$ and the x-axes.

6. Find the area of the region bounded by the parabolas $y^2 = 4a\,(x + a)$ and $y^2 = 4b\,(b - x)$.

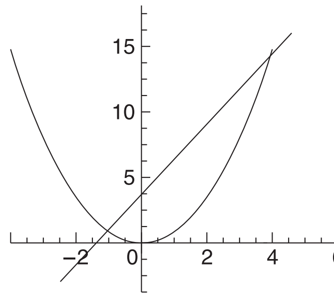
Answers

1. (i) $\frac{5}{12}$, (ii) $\frac{4}{3}$,

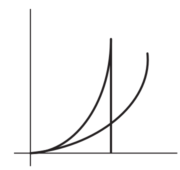




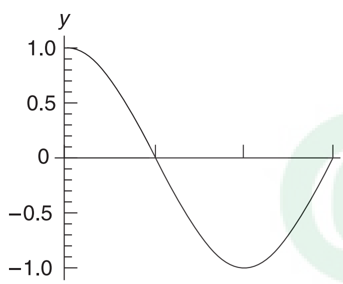
(iii) $\dfrac{125}{6}$,



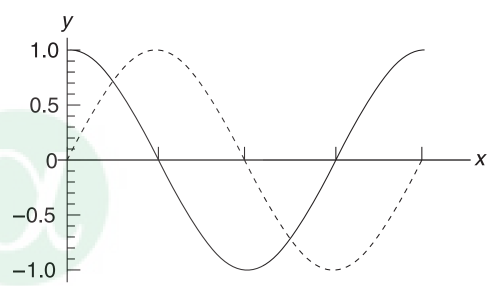
(iv) $\dfrac{2}{3}$,



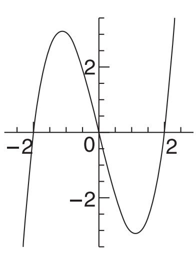
(v) 1,



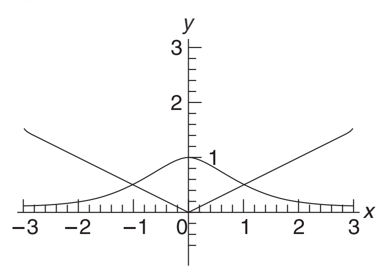
(vi) $4\sqrt{2}$,



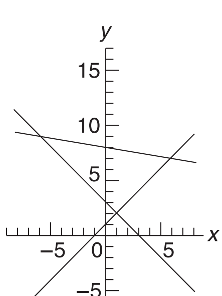
(vii) 8,



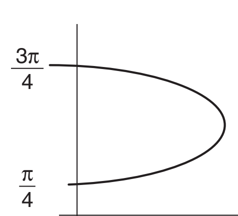
(viii) $\dfrac{\pi}{2} - \dfrac{1}{2}$,



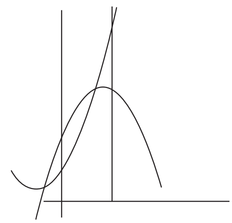
(ix) 35,



(x) $2\sqrt{2}$,



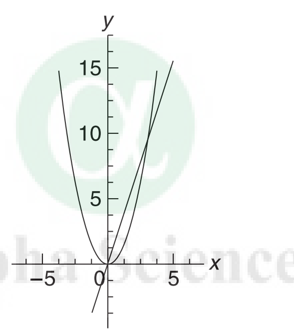
2. $\frac{38}{3}$,



3. (i) $\frac{9}{2}$, (ii) $\frac{9}{2}$,

4. $\frac{a}{6}$, **5.** $\frac{4}{3}$,

6. $\frac{8}{3}\sqrt{ab}\ (a + b)$.



7.9 AREAS OF SURFACE OF REVOLUTION

A surface of revolution is formed when a curve rotated about a line, Fig 7.14.

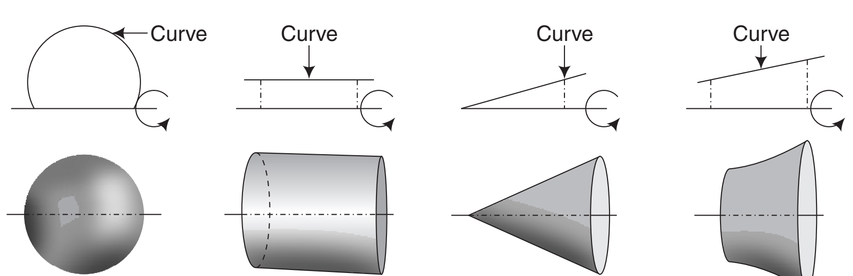


Fig. 7.14 Some surfaces

Above figures shows that if we revolve a semi circle bout a line then we get a surface of the sphere and we know that the surface area of the sphere with radius r is $4\pi r^2$, if we revolve a parallel line about a line then we get a surface of a cylinder and the lateral surface area of the cylinder with radius r and height h is $2\pi rh$, and if we revolve a line which makes a fixed angle with a line then we get a surface of a right circular cone and we also know that the lateral surface area of this right circular cone with base radius r and slant height l is πrl.

Now suppose we want to find the area of the surface generated by revolving the graph of a positive function $y = f(x)$, about the x-axis, where $a \le x \le b$, Fig. 7.15. Now divide $[a, b]$ into n partition, in which one partition is PQ, Fig. 7.16. As we revolve this arc PQ about x-axis, the line segment sweeps out a part of a cone called **frustum** of the cone whose axis lies along the x-axis, Fig. 7.17.

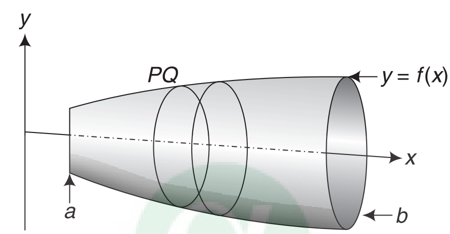


Fig. 7.15

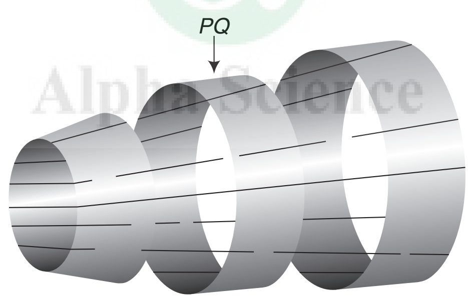


Fig 7.16

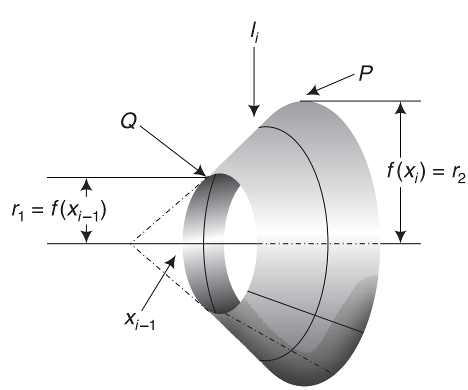


Fig 7.17

The surface area of the frustum approximates the surface area of the band swept out by the arc *PQ*.

The surface area A_i of the *ith* frustum is

$$A_i = \pi\,[r_2\,(L + l_i) - r_1 L] = \pi\,[(r_2 - r_1)L + r_2 l_i] \quad (7.41)$$

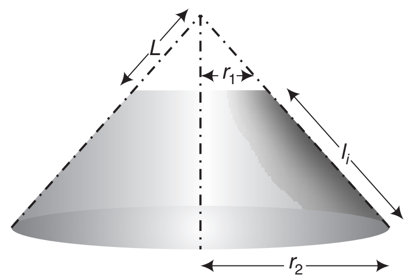


Fig. 7.18

By similar triangles we have $\dfrac{L}{r_1} = \dfrac{L + l_i}{r_2}$ or $(r_2 - r_1)L = r_1 l_i$ (7.42)

From (7.41) and (7.42), we have

$$A_i = \pi\,(r_2 + r_1)\,l_i$$

$$A_i = \pi\,\{f(x_i) + f(x_{i-1})\}\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \quad (7.43)$$

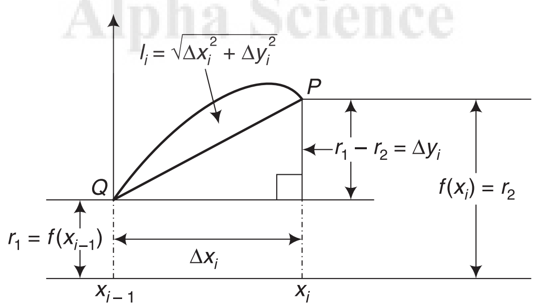


Fig. 7.19

$$A_i = \pi\{f(x_i) + f(x_{i-1})\}\sqrt{(\Delta x_i)^2 + \{f(x_i) - f(x_{i-1})\}^2} \quad (7.44)$$

The area *A* of the original surface will be the some of the areas of the bands generated by arcs like arc *PQ*, is approximated by the frustum area sum

$$\therefore \qquad A \approx \sum_{i=1}^{n} \pi\,\{f(x_i) + f(x_{i-1})\}\sqrt{(\Delta x_i)^2 + \{f(x_i) - f(x_{i-1})\}^2} \quad (7.45)$$

Now by Mean - Value theorem, we have

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(x_i^*) \text{ where } x_{i-1} < x_i^* < x_i$$

Or

$$f(x_i) - f(x_{i-1}) = (x_i - x_{i-1})\, f'(x_i^*) = \Delta x_i f'(x_i^*)$$

From (7.45), we get

$$A \approx \sum_{i=1}^{n} \pi \{f(x_i) + f(x_{i-1})\} \sqrt{1 + \{f'(x_i^*)\}^2}\ \Delta x_i \qquad (7.46)$$

By Intermediate-Value theorem, we have

$$\frac{1}{2}\{f(x_i) + f(x_{i-1})\} = f(x_i^{**}) \text{ where } x_{i-1} < x_i^{**} < x_i$$

We know the function f is continuous on $[a, b]$. If we assume that $x_i^{**} = x_i^*$, then

$$A = \lim_{max\Delta xi \to 0} \sum_{i=1}^{n} 2\pi\ f(x_i^{**})\sqrt{1 + \{f'(x_i^*)\}^2}\ \Delta x_i$$

$$= \int_a^b 2\pi\, f(x) \sqrt{1 + \{f'(x)\}^2}\, dx$$

Hence

If f is a smooth (function with a continuous first derivative) nonnegative function on $|a, b|$ then the surface area A of the surface of revolution that is generated by revolving the portion of the curve $y = f(x)$ about the x-axis is

$$\boldsymbol{A = \int_a^b 2\pi\, f(x)\sqrt{1 + \{f'(x)\}^2}\ dx = \int_a^b 2\pi\, y\sqrt{1 + \left\{\frac{dy}{dx}\right\}^2} dx} \qquad (7.47)$$

For revolution about the y-axis, we interchange x and y in equation (7.47) and obtained the formula as

$$\boldsymbol{A = \int_c^d 2\pi g(y)\sqrt{1 + \{g'(y)\}^2}\, dy = \int_c^d 2\pi x}\sqrt{1 + \left\{\frac{dx}{dy}\right\}^2} dy \qquad (7.48)$$

Where the function $x = g(y)$ is smooth nonnegative on $[c, d]$.

If we have the equation of the curve in parametric form as

$$x = x(t),\ y = y(t),\ (a \le t \le b)$$

Then the surface area A of the surface of revolution that is generated by revolving the portion of the curve about the x-axis is define as

$$\boldsymbol{A = \int_a^b 2\pi\, y(t)\sqrt{\left\{\frac{dx}{dt}\right\}^2 + \left\{\frac{dy}{dt}\right\}^2}\ dt}$$

And about the y-axis is define as

$$A = \int_a^b 2\pi\, x(t) \sqrt{\left\{\frac{dx}{dt}\right\}^2 + \left\{\frac{dy}{dt}\right\}^2}\, dt$$

Where $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous functions in $[a, b]$.

Example 18 Find the area of the surface generated by revolving the curve $y = \sqrt{x}$, $1 \le x \le 2$, about the x-axis, Fig. 7.20.

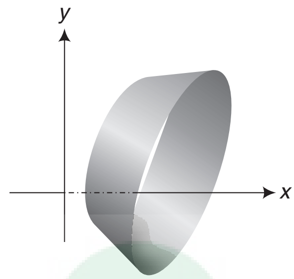


Fig. 7.20

Solution Here $y = \sqrt{x},\ \frac{dy}{dx} = \frac{1}{2\sqrt{x}},$

$$A = \int_a^b 2\pi y \sqrt{1 + \left\{\frac{dy}{dx}\right\}^2}\, dx = \int_1^2 2\pi\sqrt{x} \sqrt{1 + \left\{\frac{1}{2\sqrt{x}}\right\}^2}\, dx = \int_1^2 2\pi\sqrt{x}\ \frac{\sqrt{4x+1}}{2\sqrt{x}}\, dx$$

$$A = \pi \int_1^2 \sqrt{4x+1}\, dx = \frac{\pi}{6}\left[\{4x+1\}^{\frac{3}{2}}\right]_1^2 = \frac{\pi}{6}\left(27 - 5\sqrt{5}\right)$$

$$A = \frac{\pi}{6}(27 - 5\sqrt{5}).$$

Example 19 Find the area of the surface generated by revolving the curve $y = \sqrt{16 - x^2}\ -1 \le x \le 2$, about the x–axis.

Solution Here $y = \sqrt{16 - x^2},\ \frac{dy}{dx} = \frac{-x}{\sqrt{16 - x^2}},\ 1 + \left\{\frac{dy}{dx}\right\}^2$

$$= 1 + \frac{x^2}{16 - x^2} = \frac{16}{16 - x^2}$$

$$A = \int_a^b 2\pi\, y \sqrt{1 + \left\{\frac{dy}{dx}\right\}^2}\, dx = \int_{-1}^2 2\pi\sqrt{16 - x^2}\sqrt{\frac{16}{16 - x^2}}\, dx = 8\pi \int_{-1}^2 dx$$

$$A = 8\pi\, [x]_{-1}^2 = 24\pi.$$

Example 20 Find the area of the surface generated by revolving the curve $xy^2 = \frac{1}{4}y^6 + \frac{1}{8}, -1 \leq y \leq 2$, about the y-axis.

Solution Here $x = \frac{1}{4}y^4 + \frac{1}{8y^2}, \frac{dx}{dy} = y^3 - \frac{1}{4y^3}, 1 + \left\{\frac{dx}{dy}\right\}^2 = 1$

$$+ \left\{y^3 - \frac{1}{4y^3}\right\}^2 = \left\{y^3 + \frac{1}{4y^3}\right\}^2$$

$$A = \int_{-1}^{2} 2\pi x \left\{y^3 + \frac{1}{4y^3}\right\} dy = 2\pi \int_{-1}^{2} \left(\frac{1}{4} y^4 + \frac{1}{8y^2}\right)$$

$$\left\{y^3 + \frac{1}{4y^3}\right\} dy$$

$$= \frac{\pi}{16} \int_{-1}^{2} \left(8y^7 + 6y + \frac{1}{y^5}\right) dy = \frac{\pi}{16} \left[y^8 + 3y^2 - \frac{1}{4y^4}\right]_{-1}^{2}$$

$$A = \frac{10511}{1024} \pi.$$

7.10 ARC LENGTH

Suppose we want to find the length of a curve $y = f(x)$ from to $x = a$ to $x = b$, Fig. 7.21.

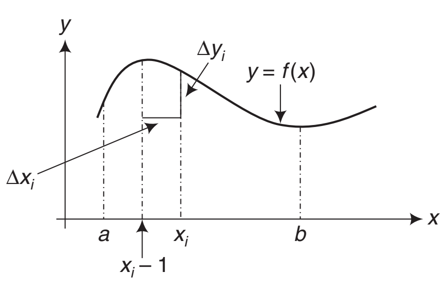


Fig. 7.21

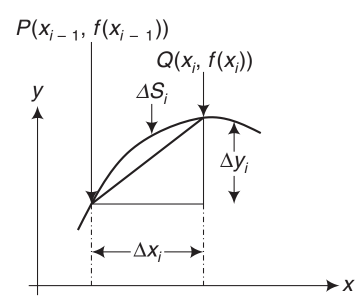


Fig. 7.22

To find the length of the curve divide it into n subintervals of equal length, then find ΔS_i the approximate length of the arc in these each subinterval and add up. In Fig. 7.22 we have taken a typical subinterval $|x_{i-1}, x_i|$ then the length ΔS_i of the curve between x_{i-1} and x_i can be approximated by the line segment PQ.

$$\therefore \qquad PQ = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \approx \Delta S_i \qquad (7.49)$$

If the function f is continuous on $[a, b]$ and differentiable on (a, b) then f is also continuous on $[x_{i-1}, x_i|$ and differentiable on (x_{i-1}, x_i) By the Men-Value theorem we have

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(x_i^*) \text{ where } x_{i-1} < x_i^* < x_i$$

Or

$$\frac{\Delta y_i}{\Delta x_i} = f'(x_i^*),\ \Delta y_i = \Delta x_i\ f'(x_i^*) \tag{7.50}$$

From (7.49) and (7.50), we have

$$\Delta S_i \approx \sqrt{(\Delta x_i)^2 + (\Delta x_i f'(x_i^*))^2}$$

So that

$$S = \sum_{i=1}^{n} \Delta S_i \approx \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\ \Delta x_i$$

If f' is also continuous on $[a, b] \Rightarrow (f')^2$ is continuous on $[a, b] \Rightarrow$ $\sqrt{1 + (f'(x_i^*)^2}$ is continuous on $[a, b] \Rightarrow \sqrt{1 + (f'(x_i^*)^2}$ is integrable in $[a, b]$ and

$$\int_a^b \sqrt{1 + \{f'(x)\}^2}\ dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \sqrt{1 + \{f'(x_i^*)\}^2}\ \Delta x_i$$

Hence

If f and f' are continuous in $[a, b]$ then the length of the curve $f(x)$ in $[a, b]$ is

$$\boldsymbol{S = \int_a^b \sqrt{1 + \{f'(x)\}^2}dx = \int_a^b \sqrt{1 + \left\{\frac{dy}{dx}\right\}^2}\ dx} \tag{7.51}$$

Moreover, if the function given of the form $x = g(y)$, where g′ is continuous on $[c, d]$ then the arc length can be define as

$$S = \int_c^d \sqrt{\mathbf{1 + \{g'(y)\}^2}}\ \boldsymbol{dy} = \int_c^d \sqrt{\mathbf{1} + \left\{\frac{dx}{dy}\right\}^2}\ \boldsymbol{dy} \tag{7.52}$$

If we have the equation of the curve in parametric form as

$$x = x(t),\ y = y(t),\ (a \le t \le b)$$

Then the length of the curve define as

$$S = \int_a^b \sqrt{\left\{\frac{\boldsymbol{dx}}{\boldsymbol{dt}}\right\}^2 + \left\{\frac{\boldsymbol{dy}}{\boldsymbol{dt}}\right\}^2}\ \boldsymbol{dt}$$

Where $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous functions in $[a, b]$.

Example 21 Find the length of the curve $y = x^{\frac{3}{2}} + 4,\ 1 \le x \le 2.$

Solution $y = x^{\frac{3}{2}} + 4,\ \dfrac{dy}{dx} = \dfrac{3}{2}\, x^{\frac{1}{2}},\ \left\{\dfrac{dy}{dx}\right\}^2 = \dfrac{9}{4}x,$

$$S = \int_1^2 \sqrt{1+ \frac{9}{4}\, x}\ dx = \frac{1}{2}\int_1^2 \sqrt{4 + 9x}\ dx = \frac{1}{27}\left[(4 + 9x)^{\frac{3}{2}}\right]_1^2.$$

$$S = \frac{1}{27}\left\{(22)^{\frac{3}{2}} - (13)^{\frac{3}{2}}\right\}$$

Example 22 Find the length of the curve $x = (2 + t)^2$ and $y = (1 + \mathrm{t})^2$, $0 \le x \le 1$.

Solution $x = (2 + t)^2,\quad y = (2 + t)^3,\quad \dfrac{dx}{dt} = 2\ \ (2 + t)$, and $\dfrac{dy}{dt} = 3(2 + t)^2$

$$S = \int_0^1 \sqrt{4\ (2 + t)^2 + 9\ (2 + t)^4}\ dt$$

$$= S = \int_0^1 (2 + t)\ \sqrt{4 + 9\ (2 + t)^2}\ dt$$

$$S = \frac{1}{27}\left[(4 + 9\ (2 + t)^2)^{\frac{3}{2}}\right]_0^1 = \frac{1}{27}\left\{(85)^{\frac{3}{2}} + (40)^{\frac{3}{2}}\right\}.$$

Exercises

1. Find the area of the surface generated by revolving the curve about the stated axis

   (i) $y = x^3,\ 0 \le x \le \dfrac{1}{2}$, about x-axis,

   (ii) $y = 8x,\ 0 \le x \le 2$, about x-axis,

   (iii) $y = \sqrt{5 - x^2},\ 0 \le x \le 1$, about x-axis,

   (iv) $x = \sqrt[3]{y},\ 0 \le y \le 27$, about x-axis,

   (v) $y = \dfrac{1}{3}\, x^3 + \dfrac{1}{4}\, x^{-1},\ 1 \le x \le 3$, about x-axis

   (vi) $x = 5y + 1,\ 0 \le y \le 1$, about y-axis,

   (vii) $y = x^2,\ 1 \le x \le 2$, about y-axis,

   (viii) $x = 2\sqrt{2 - y},\ 0 \le x \le 1$, about y-axis.

2. Find the area of the surface generated by revolving the curve $y = \sqrt{r^2 - x^2}$, $-r \le x \le r$, about the x-axis.
3. Find the area of the surface generated by revolving parametric curve $x = t^2$, $y = 2t$, $0 \le t \le 4$, about the x-axis.
4. Find the area of the surface generated by revolving parametric curve $x = t^2$, $y = 2t$, $0 \le t \le 2$, about the y-axis.
5. Find the area of the surface generated by revolving parametric curve $x = 4 \cos t$, $y = \sin^2 t$, $0 \le t \le \frac{\pi}{2}$, about the y-axis.
6. Find the area of the surface generated by revolving cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $0 \le t \le 2\pi$, about the x-axis.
7. Show that the area of the surface generated by revolving cardioide $r = a(1 - \cos t)$, about the initial line is $\frac{32\pi a^2}{5}$.
8. Show that the area of the surface generated by revolving asteroid $x = a \cos^3 t$, $y = a \sin^3 t$ about the x-axis is $\frac{12\pi a^2}{5}$.
9. Find the length of the following curves
   (i) $y = \frac{4\sqrt{2}}{3} x^{\frac{3}{2}} - 1$, $0 \le x \le 1$, (ii) $y = \frac{x^4}{96} + 3x^{-2}$, $1 \le x \le 2$,
   (iii) $x = \frac{1}{3}(y^2 + 2)^{\frac{3}{2}}$, $0 \le y \le 2$, (iv) $x = \frac{y^3}{6} + \frac{1}{2y}$, $0 \le y \le 2$,
   (v) $y = \frac{x^4}{96} + 3x^{-2}$, $1 \le x \le 2$,
10. Find the length of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.
11. Find the length of the curve $x = \cos 4t$, $y = \sin 4t$, $0 \le t \le \frac{\pi}{2}$.
12. Show that the arc of the cardioide $r = a(1 - \cos\theta)$ lying above the initial line is bisected at $\theta = \frac{2\pi}{3}$.
13. Find the length of the curve $x = a \cos^3 t$, $y = a \sin^3 t$.
14 Show that the arc length of the cardioide $r = a(1 + \cos\theta)$ measured from the point $\theta = 0$ upto the point $\theta = \frac{2\pi}{3}$ is $2\sqrt{3}a$.

Answers

1. (i) $\frac{61\pi}{1728}$, (ii) $32\pi\sqrt{65}$,

(iii) 4π, (iv) $\frac{\pi}{27}\left\{(730)^{\frac{3}{2}} - (10)^{\frac{3}{2}}\right\}$,

(v) $\frac{1505}{36}$, (vi) $7\pi\sqrt{26}$,

(vii) $\frac{\pi}{6}\left\{(17)^{\frac{3}{2}} - (5)^{\frac{3}{2}}\right\}$, (viii) $\frac{8\pi}{36}\left\{(2)^{\frac{3}{2}} - (3)^{\frac{3}{2}}\right\}$,

2. $4\pi r^2$, **3.** $\frac{8\pi}{3}\left\{(17)^{\frac{3}{2}} - 1\right\}$,

4. $\frac{\pi}{24}\left\{(65)^{\frac{3}{2}} - 1\right\}$, **5.** $\frac{16\pi}{3}\left\{(5)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\right\}$,

6. $\frac{64\pi a^2}{3}$,

9. (i) $\frac{13}{6}$, (ii) $\frac{77}{32}$, (iii) $\frac{14}{3}$, (iv) $\frac{9}{8}$,

10. 6, **11.** 2π, **13.** $6a$,

7.11 VOLUMES BY SLICING METHOD

The slice means a part, a portio, a share or a cross section of an object Fig. 7.23. In this section we see how can we find the volume of an object by use the slice.

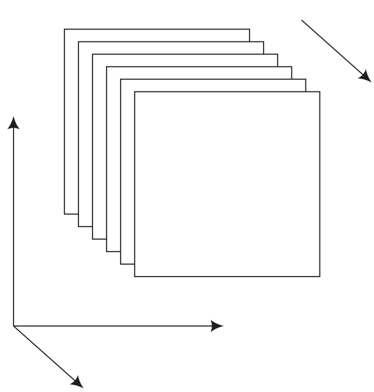


Fig. 7.23(a)

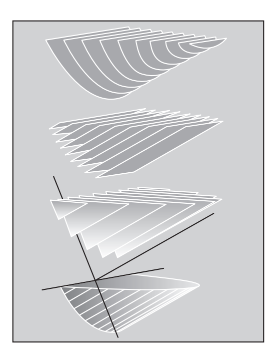


Fig. 7.23(b)

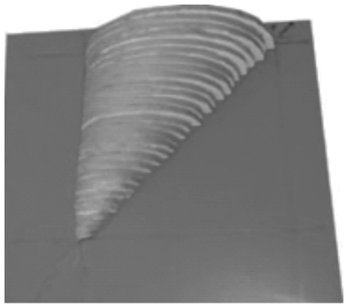


Fig. 7.23(c)

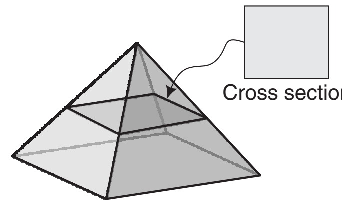


Fig. 7.23(d)

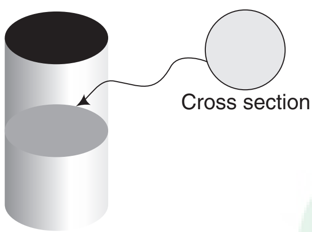


Fig. 7.23(e)

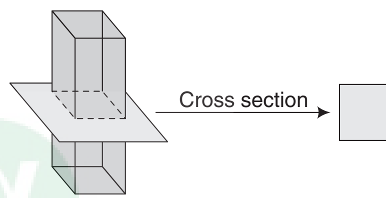


Fig. 7.23(f)

Suppose we want to find the volume of a solid Fig. 7.24

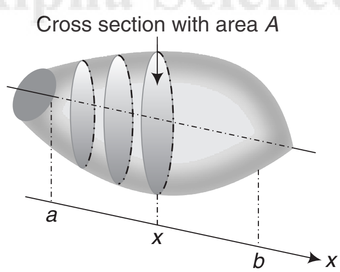


Fig. 7.24 Enlarged view of the slice

Let $A(x)$ be the area of the cross section at each point x in $[a, b]$ is a continuous real valued function. Partition the interval $[a, b]$ along the x-axis and slice the solid by the planes perpendicular to the x-axis at the partition points. The volume of the *ith* slice between the planes at x_{i-1} and x_i will be the approximately same as the cylinder between these planes. Figure 7.25. The volume of this slice or cylinder is

$$V_i = \text{base area} \times \text{height}$$

$$= A(x_i)\,\Delta x_i$$

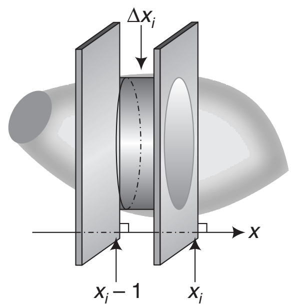


Fig. 7.25

Therefore the volume V of the solid is approximated by the sum of the cylinder volume

$$V \approx \sum_{i=1}^{n} A(x_i)\ \Delta x_i$$

This is Riemann sum for the function $A(x)$ on the interval $[a, b]$. Taking the limit as the number of partitions increases and the widths of the partitions approach zero yields the definite integral

$$V \approx \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} A(x_i)\ \Delta x_i = \int_a^b A(x)\ dx$$

Let a solid bounded by the two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. Let $A(x)$ be area of the cross section of the solid perpendicular to the x-axis for $x \in [a, b]$ the volume of the solid is

$$\boldsymbol{V = \int_a^b A(x)\ dx} \qquad (7.53)$$

Similarly if the cross section is perpendicular to the y-axis the volume of the solid is

$$\boldsymbol{V = \int_a^d A(y)\ dy} \qquad (7.54)$$

Where solid bounded by the two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. And $A(y)$ be area of the cross section of the solid perpendicular to the y-axis for $x \in [c, d]$.

Example 23 Find the volume of a solid such that the solid lies between the planes perpendicular to the x-axis at $x = -2$ and $x = 2$, the cross section perpendicular to the x-axis between these planes run from the semicircle $y = -\sqrt{2 - x^2}$ and $y = \sqrt{2 - x^2}$.

(i) The cross sections are circular disks with diameters in the xy-plane.

(ii) The cross sections are squares with bases in the xy-plane.

Solution (i) To find the volume by slicing methods, first we sketch the solid and a typical cross section, Fig. 7.26

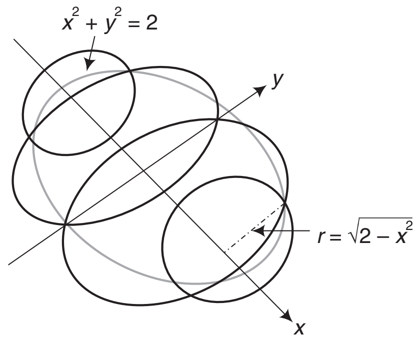


Fig. 7.26

Now we find the area of cross section and here the cross sections are circular disk with diameters in the xy-plane. Therefore the area $A(x)$ of cross section is

$$A(x) = \pi r^2 = \pi(2 - x^2)$$

Hence the volume is

$$V = \int_a^b A(x)\,dx = \pi \int_{-2}^{2} (2 - x^2)\,dx = \pi\left[2x \frac{-x^3}{3}\right]_{-2}^{2} = \frac{8\pi}{3}$$

.

(ii) From Fig 7.27 the cross sections are the squares and the area $A(x)$ of the cross section is

$$A(x) = (\text{side})^2 = 4(2 - x^2)$$

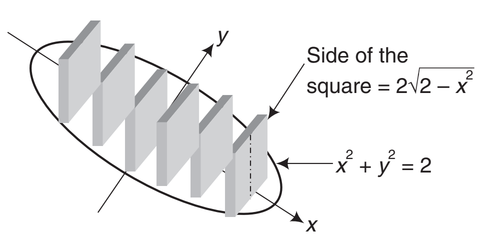


Fig. 7.27

Hence the volume is

$$V = \int_a^b A(x)\,dx = 4\int_{-2}^{2} (2 - x^2)\,dx = 4\left[2x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}$$

.

Example 24 Find the volume of right pyramid whose altitude is h and whose base is a square with sides of length a.

Solution The cross section of the right pyramid whose base is the square is a square Fig. 7.28 the area $A(x)$ of this cross section of side s from the height y from the base is.

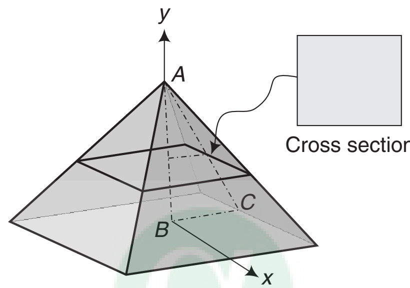


Fig. 7.28

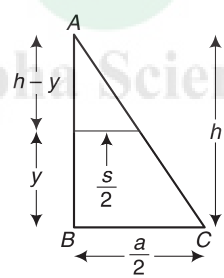


Fig. 7.29

$$A(x) = (\text{side})^2 = (s)^2$$

To find the relation between s and y from triangle ABC, we have

$$\frac{\frac{s}{2}}{\frac{a}{2}} = \frac{h - y}{h} \text{ or } s = \frac{a(h-y)}{h}$$

Hence the volume is

$$V = \int_c^d A(y)\,dy = \int_0^h \left(\frac{a(h-y)}{h}\right)^2 dy = \frac{a^2}{h^2}\left[-\frac{(h-y)^3}{3}\right]_0^h = \frac{a^2 h}{3}$$

.

7.12 VOLUMES BY DISKS METHOD

Figure 7.14 shows that if we revolve the curve about a line we get some familiar surfaces similarly if we revolve a plane region about a line that lies in the same plane as the region then we get some solids.

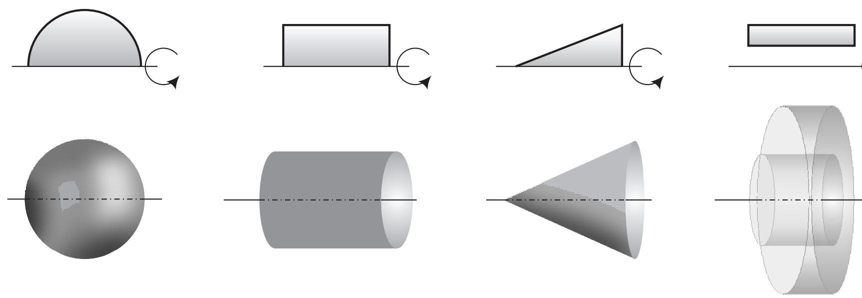


Fig. 7.30

We can find the volume of these solids as:

Let f be a continuous and nonnegative function define on the interval $[a, b]$ let R be a region that is bounded above by $y = f(x)$ below by the x-axis left by $x = a$ and right by $x = b$. Now we want to find the volume of the solid generated by the region R revolving about the x-axis, Fig. 7.31. To solve this problem take a cross section of the solid which will be a circular disk of radius $f(x)$ perpendicular to the x-axis at the point x, Fig. 7.32.

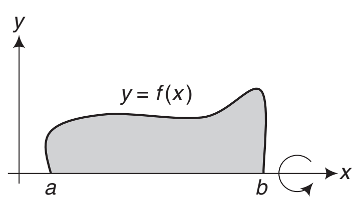


Fig. 7.31

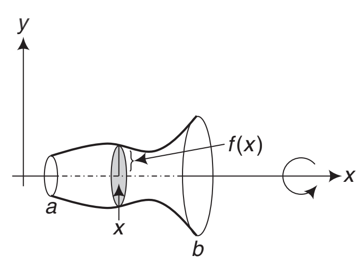


Fig. 7.32

Now the area $A(x)$ of this cross section is

$$A(x) = \pi\,[f(x)]^2$$

Hence, from (7.53) the volume V of the solid is

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\,[f(x)]^2 dx \tag{7.55}$$

Similarly,

Let g be a continuous and nonnegative function define on the interval $[c, d]$ let R be a region that is bounded right by $x = g(y)$, left by the x-axis above by $y = d$ and below by $y = c$. Now we want to find the volume of the solid generated by the region R revolving about the y-axis, Fig. 7.33. To solve this problem take a cross section of the solid which will be a circular disk of radius $g(y)$ perpendicular to the y-axis at the point y, Fig. 7.34.

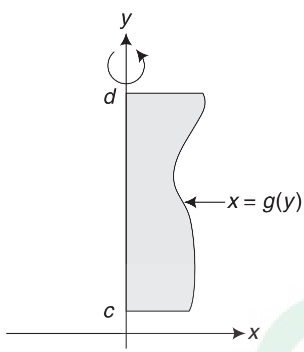


Fig. 7.33

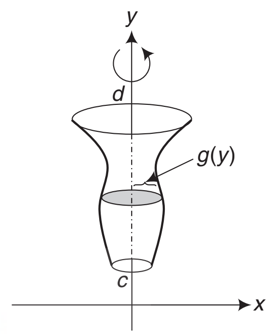


Fig. 7.34

Now the area $A(y)$ of this cross section is

$$A(y) = \pi[g(y)]^2$$

Hence, from (7.54) the volume V of the solid is

$$\boldsymbol{V = \int_c^d A(y)\, dy = \int_c^d \pi\, [g(y)]^2\, dy} \qquad (7.56)$$

For example a sphere of radius r can be generated by revolving the upper semicircular disk enclosed between the x-axis and $x^2 + y^2 = r^2$, Fig. 7.35.

$$y^2 = [f(x)]^2 = r^2 - x^2$$

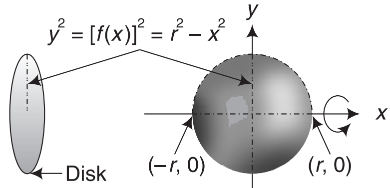


Fig. 7.35

And the formula of the volume V of this sphere can be derive as

$$V = \int_a^b A(x)\,dx = \int_{-r}^{r} \pi[f(x)]^2\,dx = \pi = \int_{-r}^{r} (r^2 - x^2)\,dx = \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} = \frac{4}{3}\pi r^3.$$

Or

From Fig. 7.36

$$V = \int_c^d A(y)\,dy = \int_{-r}^{r} \pi[g(y)]^2\,dy = \pi\int_{-r}^{r} (r^2 - y^2)\,dy = \pi\left[r^2y - \frac{y^3}{3}\right]_{-r}^{r} = \frac{4}{3}\pi r^3.$$

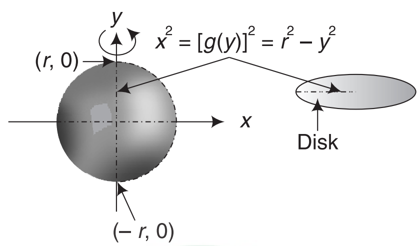


Fig. 7.36

Example 25 Find the volume of solid that is generated by the region under the curve $y = 2\sqrt{x}$ which is revolved about the x-axis over the interval [0, 4].

Solution From (7.53), and with the help of Fig. 7.37 volume is

$$V = \int_a^b A(x)\,dx = \int_0^4 \pi[f(x)]^2\,dx = 4\pi\int_0^4 x\,dx = 4\pi\left[\frac{x^2}{2}\right]_0^4 = 32\pi.$$

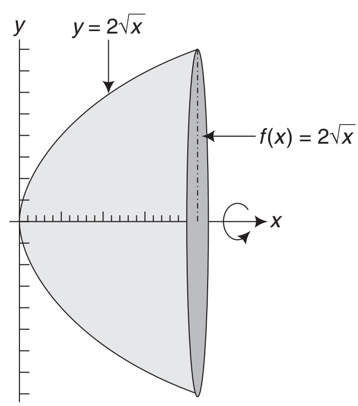


Fig. 7.37

Example 26 Find the volume of solid that is generated by the region under the curve $y = \sqrt{x}$ and the lines $y = 1$, $x = 5$ which is revolved about the line $y = 1$.

Solution The Fig. 7.38 shows that the radius of the required region is $\sqrt{x} - 1$ and the solid generated in Fig. 7.39.

The line $y = 1$ intersect the curve $y = \sqrt{x}$ at the point (1, 1).

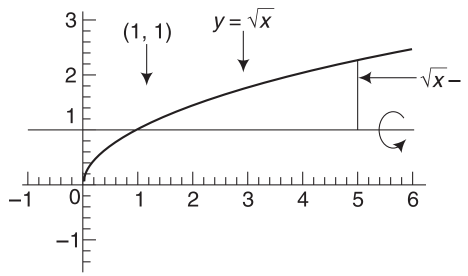


Fig. 7.38

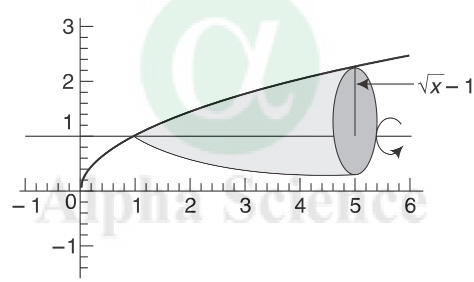


Fig. 7.39

Therefore the volume is

$$V = \int_a^b A(x)\,dx = \int_1^5 \pi\,[f(x)]^2\,dx = \pi \int_1^5 (\sqrt{x} - 1)^2\,dx = \pi \int_1^5 (x - 2\sqrt{x} + 1)\,dx$$

$$= \pi \left[\frac{x^2}{2} - 2.\frac{2}{3}x^{3/2} + x\right]_1^5 = \pi \left[13 - \frac{4}{3}(5)^{3/2}\right].$$

Example 27 Find the volume of solid that is generated by the region enclosed the curve $y = \sqrt{x}$ and the lines $y = 3$, $x = 0$ which is revolved about the y-axis.

Solution The Fig. 7.40 shows that the required region and the solid generated in Fig. 7.41.

The line $y = 3$ intersect the curve $y = \sqrt{x}$ at the point (9, 3).

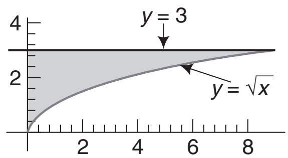


Fig. 7.40

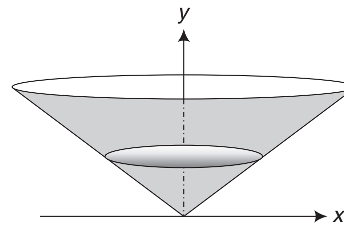


Fig. 7.41

Therefore the volume is

$$V = \int_c^d A(y)\,dy = \int_c^d \pi[g(y)]^2\,dy = \pi\int_0^3 y^4\,dx = \pi\left[\frac{y^5}{5}\right]_0^3 = \pi\,\frac{243}{5}.$$

7.13 VOLUMES BY WASHERS METHOD

The last part of Fig. 7.30 shows that if we revolve the region between two curves about a line then we get a solid which have the hole or channels. The volume of this solid can be obtained as:

Let f and g be two nonnegative continuous functions define in the interval $[a, b]$ and $f(x) \geq g(x)\ \forall x \in [a, b]$ and let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, left by $x = a$ and right by $x = b$, Fig. 7.42.

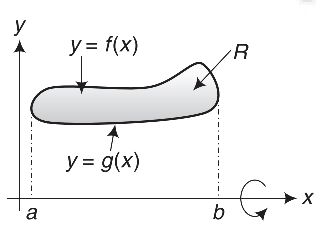


Fig. 7.42

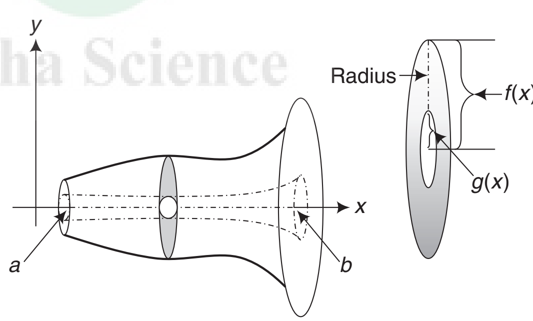


Fig. 7.43

Then the volume V of the solid of revolution that is generated by revolving the region R about the x-axis is

$$\boldsymbol{V = \int_a^b \pi\,([f(x)]^2 - [g(x)]^2)\,dx} \tag{7.57}$$

Similarly,

Let u and v be two nonnegative continuous functions define in the interval $[c, d]$ and $u(y) \geq v(y)\ \forall y \in [c, d]$ and let R be the region that is bounded above by $y = d$ below by $y = c$, left by $x = v(y)$ and right by $x = u(y)$, Fig. 7.44.

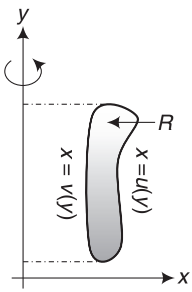


Fig. 7.44

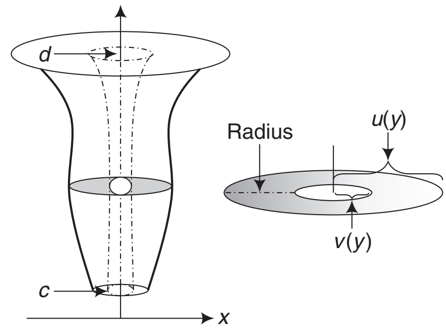


Fig. 7.45

Then the volume V of the solid of revolution that is generated by revolving the region about the y-axis is

$$\boldsymbol{V = \int_c^d \pi([u(y)]^2 - [v(y)]^2)\, dy} \tag{7.58}$$

Example 28 Find the volume of solid that is generated by the region enclosed the curve $y = x^2 + 1$ and the lines $y = x$, $x = 0$ over the interval [0, 2] which is revolved about the x-axis.

Solution The Fig. 7.46 shows that the required region and the solid generated in Fig. 7.47.

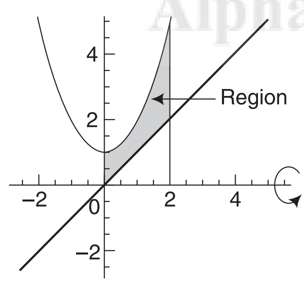


Fig. 7.46

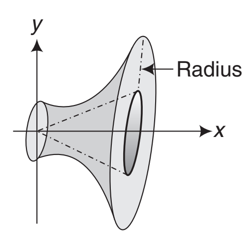


Fig. 7.47

$$V = \int_a^b \pi(|(x)]^2 - [g(x)]^2)\, dx = \int_0^2 \pi([x^2 + 1]^2 - [x]^2)\, dx = \int_0^2 \pi\,(x^4 + x^2 + 1)\, dx$$

$$= \pi \left[\frac{x^5}{5} + \frac{x^3}{3} + x\right]_0^2 = \pi\,\frac{166}{15}.$$

Example 29 Find the volume of solid that is generated by the region enclosed the curve $x = y^2$ and the line $y = x$ which is revolved about the line (i) $y = -1$. (ii) $x = -1$.

Solution (i) The Fig. 7.48 shows the required region.

$$f(x) = \sqrt{x} + 1$$

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\, dx = \int_0^1 \pi\,([\sqrt{x} + 1]^2 - [x + 1]^2)\, dx$$

$$= \int_0^1 \pi\,(2\sqrt{x} - x^2 - x)\, dx = \pi\left[4\,\frac{x^{\frac{3}{2}}}{3} - \frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 = \frac{\pi}{2}.$$

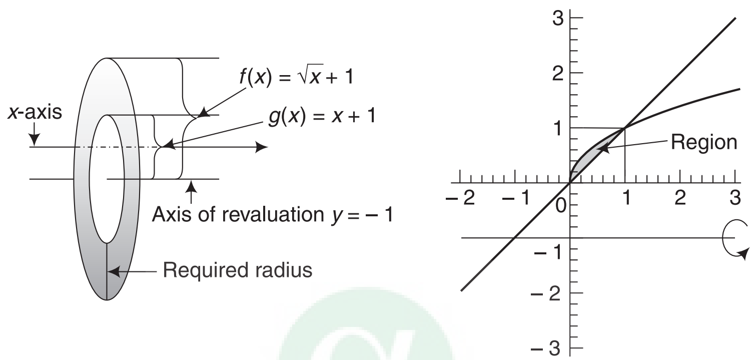


Fig. 7.48

(ii) The Fig. 7.49 shows the required region.

$$V = \int_c^d \pi([u(y)]^2 - [v(y)]^2)\, dy = \int_0^1 \pi\,([y + 1]^2 - ([y^2 + 1]^2)\, dy$$

$$= \int_0^1 \pi\,(2y - y^2 - y^4)\, dy = \pi\left[y^2 - \frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 = \frac{7\pi}{15}.$$

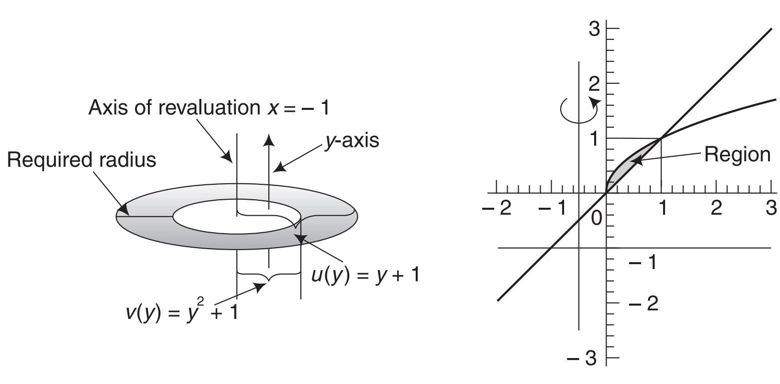


Fig. 7.49

7.14 VOLUMES BY SHELL METHOD

In last two sections we have discussed the methods for computing the volumes of the solid by compute the area of the cross section. There are so many problems in which we cannot find the cross section or the integration is too difficult. To solve this type of problems we will develop an another method called the Shell Method.

Let f be nonnegative continuous functions define in the interval $[a, b]$, $\forall x \in [a, b]$ and let R be the region that is bounded above by $y = f(x)$, below by x-axis, left by $x = a$ and right by $x = b$, Fig. 7.50. Suppose we want to find the volume of a solid generated by this region revolving about the y-axis. Divide the interval $[a, b]$ into n subintervals, thereby subdividing the region R into n small rectangles $R_1, R_2 \ldots R_n$ Fig. 7.52. When we revolved the region R about the y-axis then these rectangles generate **Cylindrical shells** $S_1, S_2, \ldots S_n$ Fig. 7.53.

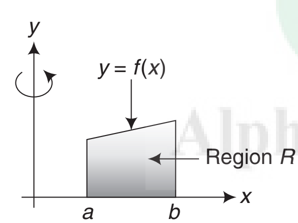


Fig. 7.50

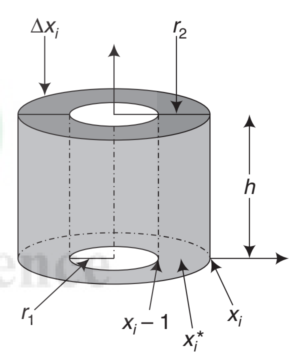


Fig. 7.51

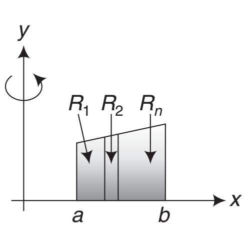


Fig. 7.52

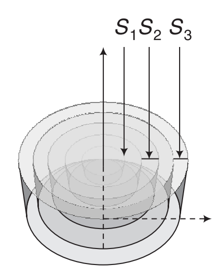


Fig. 7.53

Cylindrical shell is a solid enclosed by two concentric right circular cylinders, Fig. 7.51.

We can obtain the volume of a shell with outer radius r_2 and inner radius r_1 as

$$V = (\pi r_2^2 - \pi r_1^2)\ h = 2\pi\ (r_2 - r_1).\ \frac{1}{2}\ (r_2 + r_1).h$$

$$= 2\pi.\ \text{Thickness. Average, Height of the radius.}$$

Therefore the volume of the *ith* shell is

$$V_i = 2\pi\ x_i^*\ f(x_i^*)\ \Delta x_i$$

Where $\frac{1}{2}\ (r_2 + r_1) = x_i^*$ = average radius, $(r_2 - r_1) = \Delta x_i$ difference of the radius or width of the shell, $h = f(x_i^*)$ height of the shell Fig. 7.51.

Add up the volumes of the n cylindrical shells to obtain an approximation to the volume of the solid.

$$V \approx \sum_{i=1}^{n} 2\pi\ x_i^*\ f(x_i^*)\ \Delta x_i$$

Taking the limit as n increases and $\Delta x_i \to 0$

$$V = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} 2\pi\ x_1^*\ f(x_i^*)\ \Delta x_i = \int_a^b 2\pi x f(x)\ dx.$$

Hence,

$$\boldsymbol{V = \int_a^b 2\pi x f(x)\ dx} \tag{7.59}$$

Similarly,

Let f be nonnegative continuous functions define in the interval $[c, d]$, $\forall x \in [c, d]$ and let R be the region that is bounded above by $y = d$, below by $y = c$, left by y-axis and right by $x = f(y)$, Fig. 7.54. Then the volume of a solid generated by this region revolving about the x-axis is

$$\boldsymbol{V = \int_c^d 2\pi y f(y)\ dy} \tag{7.60}$$

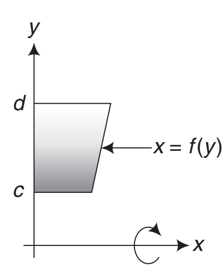


Fig. 7.54

Example 30 Find the volume of solid that is generated by the region enclosed the curve $y = x^2$ and the line $y = x$, which is revolved

(i) about the x-axis, by Washer Method,

(ii) about the y-axis, by Washer Method

(iii) about the x-axis, by Shell Method,

(iv) about the y-axis, by Shell Method

(i) The Fig. 7.55 shows the required region

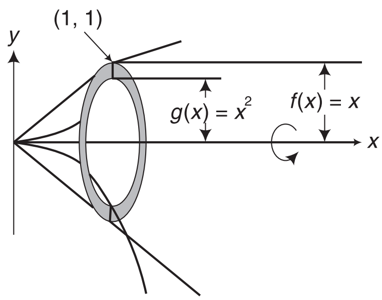


Fig. 7.55

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\, dx = \int_0^1 \pi([x]^2 - [x^2]^2)\, dx = \int_0^1 \pi\, (x^2 - x^4)\, dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2\pi}{15}.$$

(ii) The Fig. 7.56 shows the required region

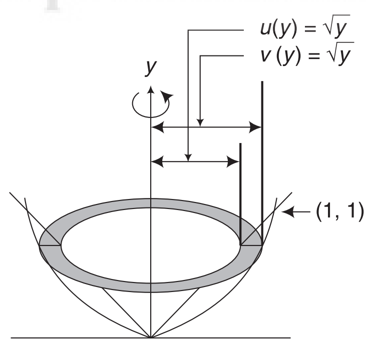


Fig. 7.56

$$V = \int_c^d \pi([u(y)]^2 - [v(y)]^2)\, dy = \int_0^1 \pi([\sqrt{y}]^2 - [y]^2)\, dy = \int_0^1 \pi(y - y^2)\, dy$$

$$= \pi \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{\pi}{6}.$$

(iii) The Fig. 7.57 shows the required region

$$V = \int_c^d 2\pi y\, f(y)\, dy = \int_0^1 2\pi y\, (\sqrt{y} - y)\, dy = 2\pi \int_0^1 \left(y^{\frac{3}{2}} - y^2\right) dy$$

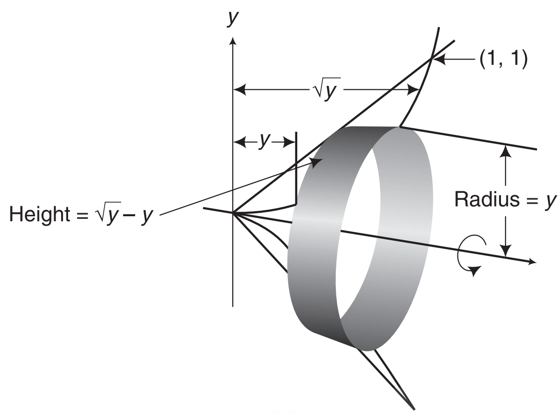


Fig. 7.57

(iv) The Fig. 7.58 shows the required region

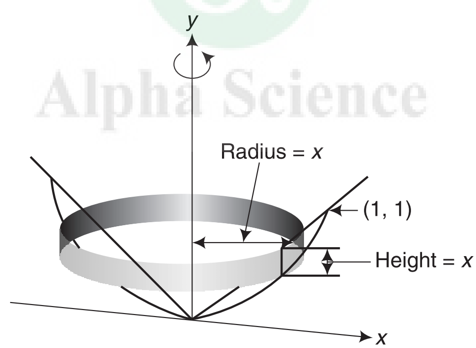


Fig. 7.58

$$V = \int_a^b 2\pi x\, f(x)\, dx = \int_0^1 2\pi x\, (x - x^2)\, dx = 2\pi \int_0^1 (x^2 - x^3)\, dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}.$$

Example 31 Use cylindrical shell to find the volume of solid that is generated by the region enclosed the curve $y = x^2$ over the interval [0, 1] which is revolved about the x-axis

Solution The Fig. 7.59 shows the required region

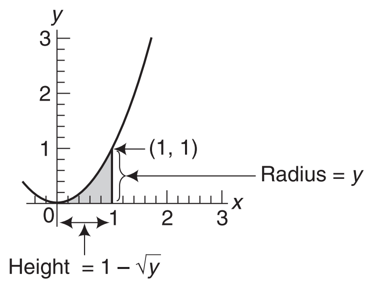


Fig. 7.59

Exercises

1. Use the slicing method to find the volume of a solid such that the solid lies between the planes perpendicular to the x-axis at $x = -2$ and $x = 2$, the cross section perpendicular to the x-axis between these planes run from the semicircle $y = -\sqrt{2 - x^2}$ and $\sqrt{2 - x^2}$.
   (i) The cross sections are equilateral triangle with base in the xy-plane.
   (ii) The cross sections are squares with diagonal in the xy-plane.
2. Use the slicing method to find the volume of a solid such that the base of the solid is the region between the curve $y = 4\sqrt{\sin x}$ and the interval $\left[0, \frac{\pi}{2}\right]$ on the x-axis. The cross section perpendicular to the x-axis are
   (i) equilateral triangles with bases running from the x-axis to the curve.
   (ii) squares with bases running from the x-axis to the curve.
3. (i) Use the slicing method to find the volume of a solid such that the solid lies between the planes perpendicular to the y-axis at $y = 0$ and $y = 4$, the cross section perpendicular to the y-axis are circular disks with diameters running from the y-axis to the curve $x = \sqrt{3}y^2$.
   (ii) Find the volume of the solid whose base is the region bounded between the curves $y = 2x$ and $y = x^2$ and whose cross sections perpendicular to the x-axis are squares.
4. Find the volumes of the solids generated by revolving the regions bounded by the lines and the curves about the y-axis.

(i) $y = \sqrt{3x},\ y = 0,\ 0 \le x \le 2,$ (ii) $y = \sqrt{4 - x^2},\ y = 0$

(iii) $y = \sqrt{\cos 2x},\ 0 \le x \le \dfrac{\pi}{4},$ (iv) $y = x^2,\ x = 0,\ x = 2,\ y = 0,$

(v) $y = x^2,\ y = x^3$ (vi) $y = \sqrt{29 - x^2},\ y = 2$

(vii) $y = 4 - x^2,\ y = 0$ (viii) $x = \sqrt{y},\ x = \dfrac{y}{2}$

5. Find the volumes of the solids generated by revolving the regions bounded by the lines and the curves about the y-axis.

(i) $x = \dfrac{3}{4},\ x = 0,\ 1 \le y \le 2,$ (ii) $y = x^3,\ x = 0,\ y = 1$

(iii) $x = 4 - y^2,\ x = 0$ (iv) $x = y^2\ x = y + 6,$

(v) $x = 1 - y^2,\ x = 2 + y^2,\ y = -1,\ y = 1$

(vi) $y = x^2,\ x = y^2$ (vii) $x = \sqrt{1 + y},\ x = 0,\ y = 4,$

(viii) $y = x^2 - 2,\ x = 3,\ y = 0.$

6. Find the volume of the solid that generated by the region between the curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and x-axis.

7. Find the volume of the solid that generated by the region enclosed by the curve $y = \sqrt{x + 2}\ y = \sqrt{2x}$ and x-axis is revolved about the x-axis.

8. Find the volume of the solid that generated by the region enclosed by the curve $y = \sqrt{2x}\ y = 4 - x$ and x-axis is revolved about the x-axis.

9. Find the volume of the solid that generated by the region enclosed by the curve $y = \sqrt{x}$ x-axis and $x = 4$ is revolved about the line (i) $x = 4$, (ii) $y = 2$,

10. Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the said axis.

(i) $y = \sqrt{2x},\ y = 0,\ x = 4$ about the y-axis,

(ii) $y = x^3,\ y = 0,\ x = 2$ about the y-axis,

(iii) $y = \sqrt{x},\ x = 1,\ x = 4,\ y = 0$ about the y-axis,

(iv) $y = \dfrac{4}{x}\ y = 0,\ x = 1,\ x = 4$ about the y-axis,

(v) $y = 2x - 1,\ y = -2x + 3,\ x = 3$ about the x-axis,

(vi) $y^2 = 2x$, $y = 2$, $x = 0$ about the x-axis.

(vii) $x = 3y$, $y = 3$, $y = 4$, $x = 0$ about the x-axis.

(viii) $xy = 5$, $x + y = 6$ about the x-axis.

11. Use cylindrical shells to find the volume of the solid generated when the region under the curve $y = x^3 - 5x^2 + 7x$ over the interval [0, 1] is revolved about the y-axis.
12. Use cylindrical shells to find the volume of the solid generated when the region under the curve $y = \frac{1}{x^5}$ $x = 1$, $x = 2$, $y = 0$ is revolved about the line $x = -1$.

Answers

1. (i) $\frac{8\sqrt{3}}{3}$, (ii) $\frac{16}{3}$

2. (i) $4\sqrt{3}$, (ii) 32.

3. (i) $\frac{768\pi}{5}$, (ii) $\frac{16}{15}$,

4. (i) 6π, (ii) $\frac{32\pi}{3}$, (iii) $\frac{\pi}{2}$, (iv) $\frac{32\pi}{3}$,

(v) $\frac{2\pi}{35}$, (vi) $\frac{500\pi}{3}$, (vii) $\frac{512\pi}{15}$, (viii) $\frac{64\pi}{15}$.

5. (i) $\frac{9\pi}{2}$, (ii) $\frac{3\pi}{5}$, (iii) $\frac{512\pi}{15}$, (iv) $\frac{500\pi}{3}$,

(v) 10π, (vi) $\frac{3\pi}{10}$, (vii) $\frac{25\pi}{2}$, (viii) $\frac{49\pi}{2}$.

6. $\frac{4\pi ab^2}{3}$. **7.** 4π, **8.** $20\frac{\pi}{3}$,

9. (i) $\frac{256\pi}{15}$, (ii) $8\pi\left(-1 + \frac{\sqrt[3]{2}}{3}\right)$.

10. (i) $\frac{128\sqrt{2}\pi}{5}$ (ii) $\frac{32\pi}{5}$, (iii) 12π, (iv) 24π,

(v) $\frac{112\pi}{3}$, (vi) 4π, (vii) 74π, (viii) $\frac{64\pi}{3}$.

11. $\frac{154\pi}{60}$. **12.** $\frac{202\pi}{192}$.

8

CHAPTER

Hyperbolic Functions

8.1 INTRODUCTION

In many physical applications, functions arise that are combinations of exponential function e^x and e^{-x} for example

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{Even part}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{Odd part}}$$

The even part of e^x is called the **hyperbolic cosine** function and odd part of e^x is called the **hyperbolic sine** function of x so that the hyperbolic cosine function define as

$$\cos hx = \frac{e^x + e^{-x}}{2} \quad \text{for all } x$$

The domain of cos hx is $(-\infty, \infty)$.

The hyperbolic sine function define as

$$\sin hx = \frac{e^x - e^{-x}}{2} \quad \text{for all } x$$

The domain of sin hx is $(-\infty, \infty)$.

Other Hyperbolic functions are

$$\tan hx = \frac{\sin hx}{\cos hx} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \text{domain is } (-\infty, \infty).$$

$$\cot hx = \frac{\cos hx}{\sin hx} = \frac{1}{\tan hx} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \quad \text{domain is all } x \text{ except } x = 0.$$

$$\sec hx = \frac{1}{\cos hx} = \frac{2}{e^x + e^{-x}} \quad \text{domain is } (-\infty, \infty).$$

$$\csc hx = \frac{1}{\sin hx} = \frac{2}{e^x - e^{-x}} \quad \text{domain is all } x \text{ except } x = 0.$$

8.2 DERIVATIVE OF HYPERBOLIC FUNCTIONS

The derivative of Hyperbolic functions define as

$$\frac{d}{dx}\cos hx = \sin hx$$

Proof: $\frac{d}{dx}\cos hx = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \left(\frac{e^x - e^{-x}}{2}\right) = \sin hx$

Similarly, $\frac{d}{dx}\sin hx = \cos hx.$

Proof: $\frac{d}{dx}\sin hx = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \left(\frac{e^x + e^{-x}}{2}\right) = \cos hx$

$$\frac{d}{dx}\tan hx = \sec h^2x$$

Proof: $\frac{d}{dx}\tan hx = \frac{d}{dx}\left(\frac{\sin hx}{\cos hx}\right) = \left(\frac{\cos h^2x - \sin h^2x}{\cos h^2x}\right) = \frac{1}{\cos h^2x} = \sec h^2x$

Because

$$\cos h^2x - \sin h^2x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = 1.$$

Similarly we can proof

$$\frac{d}{dx}\cot hx = -\csc h^2x, \frac{d}{dx}\sec hx = -\sec hx \tan hx,$$

$$\frac{d}{dx}\csc h = -\cos hx \cot hx.$$

If u be a function of x. Then

$$\frac{d}{dx}\sin hu = \cos hu\frac{du}{dx}, \frac{d}{dx}\cos hu = \sin hu\frac{du}{dx}, \frac{d}{dx}\tan hu = \sec h^2u\frac{du}{dx}, \frac{d}{dx}\cot hu$$

$$= -\cos h^2u\frac{du}{dx}, \frac{d}{dx}\sec hu = -\sec hu \tan hu\frac{du}{dx}, \frac{d}{dx}\csc hu = -\csc hu \cot hu\frac{du}{dx}.$$

For example $\frac{d}{dx}\tan h(x^2 + 2) = \sec h^2(x^2 + 2)\frac{d(x^2 + 2)}{dx} = 2x \sec h^2(x^2 + 2).$

To draw the graphs of the hyperbolic functions we know that $\sin hx > 0$ if $x > 0$, $\sin hx < 0$ if $x < 0$. If $x = 0$ then $\sin hx = 0$ and $\cos hx = 1$. Therefore, $\cos hx$ is decreasing if $x < 0$, and increasing if $x > 0$, and $x = 0$ is the critical point. Further $\frac{d^2}{dx^2}(\cos hx) = \cos hx > 0$, so that $\cos hx$ is concave up and the point (0, 1) is a minimum, Fig. 8.1. Since $\frac{d}{dx}\sin hx = \cos hx > 0$, therefore $\sin hx$ always increasing.

$\frac{d^2}{dx^2}(\sin hx) = \sin hx$, so that $\sin hx$ is concave down for $x < 0$ and concave up for $x > 0$, and (0, 0) is a point of inflection, Fig. 8.2.

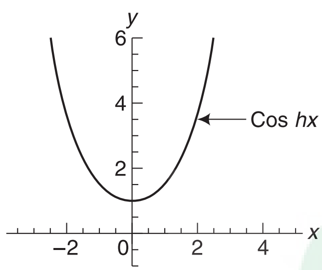


Fig. 8.1(a)

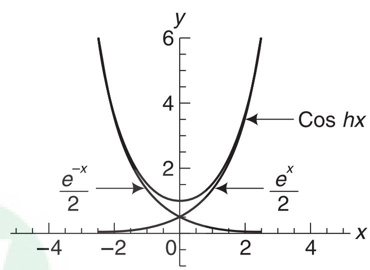


Fig. 8.1(b)

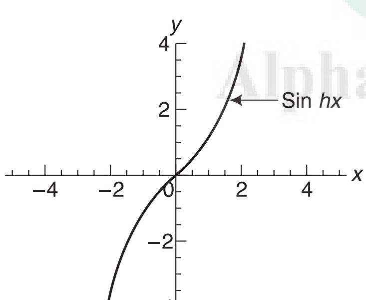


Fig. 8.2(a)

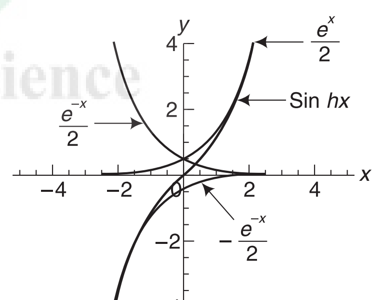


Fig. 8.2(b)

Similarly

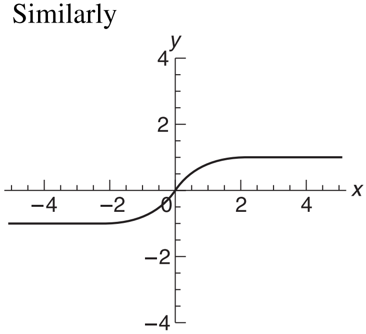


Fig. 8.3(a) $y = \tan hx$

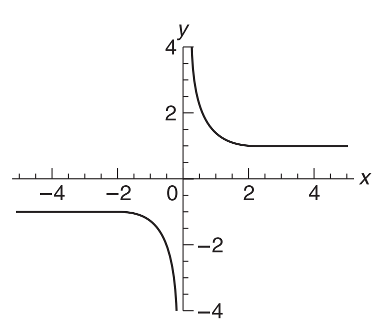


Fig. 8.3(b) $y = \cot hx$

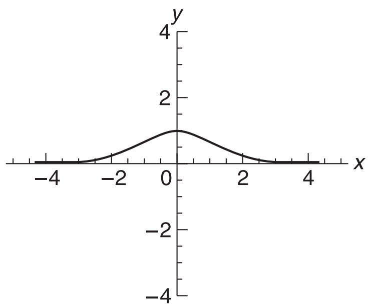


Fig. 8.3(c) $y = \sec hx$

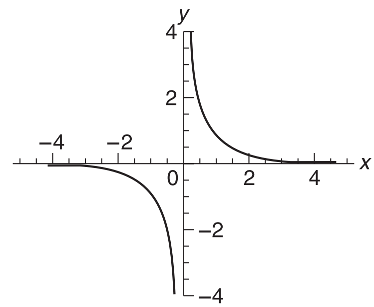


Fig. 8.3(d) $y = \cos hx$

8.3 INTEGRATION FORMULAS OF HYPERBOLIC FUNCTIONS

$\int \sin hx\, dx = \cos hx + c$, $\int \cos hx\, dx = \sin hx + c$, $\int \sec h^2 x\, dx = \tan hx + c$, $\int \csc h^2 x\, dx = -\cot hx + c$, $\int \sec hx \tan hx\, dx = -\sec hx + c$, $\int \cos hx \cot hx\, dx = -\csc hx + c$.

Examples

(i) $\int \sin h^4 x \cos hx\, dx$, (ii) $\int x \sec h^2(x^2)$, (iii) $\int \tan hx\, dx$.

Solutions

(i) $\int \sin h^4 x \cos hx\, dx$

Let $u = \sin hx$ $du = \cos hx\, dx$

$$\int u^4\, du = \frac{u^5}{5} + c = \frac{\sin h^5 x}{5} + c.$$

(ii) $\int 2x \sec h^2(x^2)\, dx$

Let $u = x^2$ $du = 2x\, dx$

$$\int 2x \sec h^2(x^2)\, dx = \int \sec h^2(u)\, du = \tan hu + c = \tan hx^2 + c$$

(iii) $\int \tan hx\, dx = \int \frac{\sin hx}{\cos hx}$ Let $u = \cos hx$ $du = \sin hx\, dx$

$$\int \frac{\sin hx}{\cos hx}\, dx = \int \frac{du}{u} = \log u + c = \log(\cos hx) + c.$$

8.4 PROPERTIES OF THE HYPERBOLIC FUNCTIONS

$\cos h^2 x - \sin h^2 x = 1$

$$\sin h(-x) = \frac{e^{-x} - e^{x}}{2} = -\frac{e^{x} - e^{-x}}{2} = -\sin hx$$

Similarly,

$$\cos h(-x) = \cos hx$$

$$\tan h(-x) = -\tan hx$$

$$\sin h(x+y) = \sin hx \cos hy + \cos hx \sin hy,\ \sin h(x-y)$$
$$= \sin hx \cos hy - \cos hx \sin hy$$

$$\cos h(x+y) = \cos hy \cos hx + \sin hy \sin hx,\ \cos h(x-y)$$
$$= \cos hy \cos hx - \sin hy \sin hx$$

$$\sec h^2x = 1 - \tan h^2x$$

$$\csc h^2x = \cot h^2x - 1,\ \cos h^2x = \frac{\cos h\ 2x + 1}{2},$$

$$\sin h^2x = \frac{\cos h\ 2x - 1}{2},\ \sin h(0) = \frac{\frac{1}{e^0} - e^0}{2} = 0$$

$$\cos h(0) = 1$$

$$\sin hx + \cos hx = e^x$$

$$\cos hx - \sin hx = e^{-x}$$

$$\sin h\ 2x = 2 \sin hx \cos hx,\ \cos h\ 2x = \cos h^2x + \sin h^2x.$$

9 CHAPTER

Vectors

9.1 INTRODUCTION

Various quantities of physics, such as length, mass and time, require for their specification of a single real number is called magnitude of the quantity. For example the length of a book is 6 inch so the magnitude of the length is 6 inch, and such quantities are called the scalars quantities.

Other quantities of physics, such as displacement, velocity, require for their specification a magnitude as well as direction. Such quantities are called **vectors.**

In Fig. 9.1 cars have the same (scalar) speed but different (vector) velocities because they move in different directions. The bow in Fig. 9.2 converts potential (stored) (scalar) energy into kinetic (motive) (vector) energy. In Fig. 9.4 a ball has some mass (scalar) and in Fig. 9.5 a kick give the displacement (vector) quantity.

Fig. 9.1

Fig. 9.2

Fig. 9.3

Fig. 9.4

Fig. 9.5

A vector is represented analytically by a bold faced letter such as **A.** Geometrically speaking; a vector is a **directed line segment.** If a vector extend from a point *P* (called initial point) to a point *Q* (called terminal point), then we denote the vector as ***PQ*** **= A.** The magnitude of a vector **A** is denoted by |**A**|.

Suppose a car moves from a point *P* to a point *Q* and the distance between them is 5 meter which is shown in Fig. 9.6. The point *P* is called the initial point and the point *Q* is called the terminal point and, is |***A***| = 5.

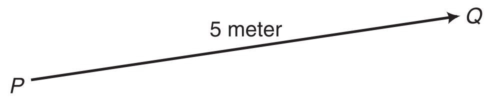


Fig. 9.6

Vectors having unit length are called unit vectors, or a vector whose magnitude is unity is called a unit vector, and this unit vector obtained as $\frac{\boldsymbol{A}}{|\boldsymbol{A}|}$. Two vectors **A** and **B** are equal if they have the same magnitude and direction regardless of their initial points.

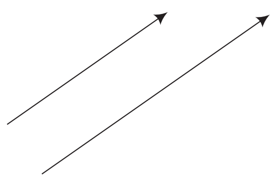


Fig. 9.7

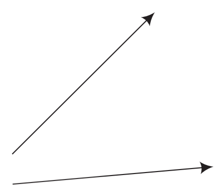


Fig. 9.8

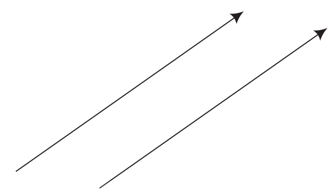


Fig. 9.9

The negative of a vector **A**, denoted as **– A**, Fig. 9.11 which has the same magnitude as **A** but where direction is opposite to that of **A**.

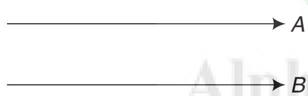


Fig. 9.10 $A = B$

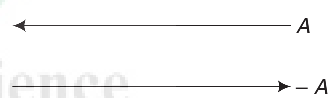


Fig. 9.11 $A = -A$

The rectangular unit vectors **i, j** and **k** are mutually perpendicular unit vectors having direction of the positive *x*-axis, *y*-axis and *z*-axis respectively of a rectangular coordinate system, Fig. 9.12.

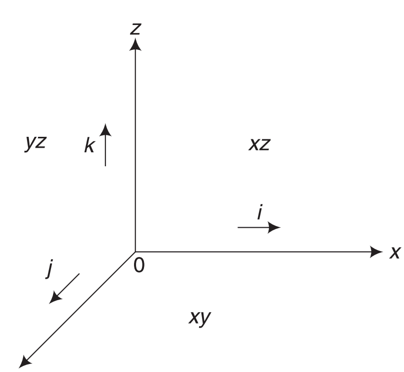


Fig. 9.12

Let (A_1, A_2, A_3) be the rectangular coordinates of the terminal point of vector **A** with initial point at origin, then A_1, A_2, A_3 are called the rectangular components or simply components of **A** in the x, y and z direction respectively in $\mathbb{R}^3$.

Hence $$\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}.$$

The magnitude of $\boldsymbol{A}$ is

$$|\boldsymbol{A}| = \sqrt{(A_1 - 0)^2 + (A_2 - 0)^2 + (A_3 - 0)^2} = \sqrt{A_1^2 + A_2^2 + A_3^2}, \text{ Fig. 9.13.}$$

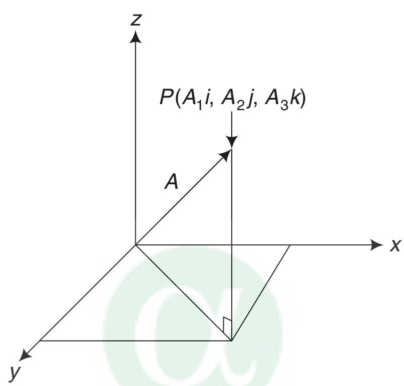


Fig. 9.13

If (A_1, A_2, A_3) are the coordinates of the terminal point and (A_{11}, A_{21}, A_{31}) are the coordinates of the initial point of the vector $\boldsymbol{A}$, then

$$\boldsymbol{A} = (A_1 - A_{11})\boldsymbol{i} + (A_2 - A_{21})\boldsymbol{j} + (A_3 - A_{31})\boldsymbol{k}.$$

And the magnitude is

$$|\boldsymbol{A}| = \sqrt{(A_1 - A_{11})^2 + (A_2 - A_{21})^2 + (A_3 - A_{31})^2}.$$

For example,

If $(5, 6, -3)$ are the coordinates of the terminal point and $(2, -1, 4)$ are the coordinates of the initial point of the vector $\boldsymbol{A}$, then

$$\boldsymbol{A} = (5 - 2)\boldsymbol{i} + (6 - (-1))\boldsymbol{j} + (-3 - 4)\boldsymbol{k} = 3\boldsymbol{i} + 7\boldsymbol{j} - 7\boldsymbol{k}$$

The magnitude is

$$|\boldsymbol{A}| = \sqrt{(3)^2 + (7)^2 + (-7)^2} = \sqrt{107}.$$

And the unit vector is

$$\frac{\mathbf{A}}{|\boldsymbol{A}|} = \frac{3\boldsymbol{i} + 7\boldsymbol{j} - 7\boldsymbol{k}}{\sqrt{107}} = \frac{3}{\sqrt{107}}\boldsymbol{i} + \frac{7}{\sqrt{107}}\boldsymbol{j} - \frac{7}{\sqrt{107}}\boldsymbol{k}.$$

If $\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}$. And $\boldsymbol{B} = B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k}$, then

$$\boldsymbol{A} \pm \boldsymbol{B} = (A_1 \pm B_1)\boldsymbol{i} + (A_2 \pm B_2)\boldsymbol{j} + (A_3 \pm B_3)\boldsymbol{k}.$$

If $\boldsymbol{A} = \boldsymbol{r}$, $A_1 = x$, $A_2 = y$ and $A_3 = z$

Then the vector $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ is called the position vector of a point P.

9.2 THE DOT PRODUCT

If $\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}$, and $\boldsymbol{B} = B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k}$ are two vectors then the dot product or scalar product, written $\boldsymbol{A}\cdot\boldsymbol{B}$ is

$$\boldsymbol{A}\cdot\boldsymbol{B} = A_1B_1 + A_2B_2 + A_3B_3,$$

Properties of dot product:

$$\boldsymbol{A}\cdot\boldsymbol{A} = |A|^2$$

$$(\boldsymbol{i}\cdot\boldsymbol{i} = \boldsymbol{j}\cdot\boldsymbol{j} = \boldsymbol{k}\cdot\boldsymbol{k} = 1),\ (\boldsymbol{i}\cdot\boldsymbol{j} = \boldsymbol{j}\cdot\boldsymbol{k} = \boldsymbol{k}\cdot\boldsymbol{i} = 0).$$

$$\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{A},\ c\,(\boldsymbol{A}\cdot\boldsymbol{B}) = (c\cdot\boldsymbol{A})\cdot\boldsymbol{B} = (c\cdot\boldsymbol{B})\cdot\boldsymbol{A},\ \boldsymbol{0}\cdot\boldsymbol{A} = 0.$$

If $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are three vectors then

$$\boldsymbol{A}\cdot(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A}\cdot\boldsymbol{C}$$

If θ be the angle between two non-zero vectors $\boldsymbol{A}$ and $\boldsymbol{B}$,

then $$\boldsymbol{A}\cdot\boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}|\cos\theta.$$

Proof: Suppose $\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}$, and $B = B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k}$, are the sides of a triangle whose one vertex at the origin, Fig. 9.14.

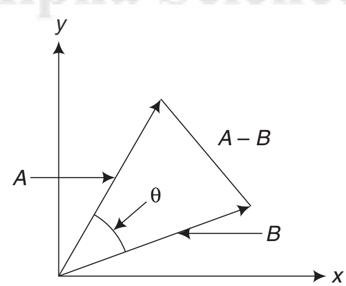


Fig. 9.14

The lengths of the sides are

$$|\boldsymbol{A}| = \sqrt{(A_1 - 0)^2 + (A_2 - 0)^2 + (A_3 - 0)^2} = \sqrt{A_1^2 + A_2^2 + A_3^2},$$

$$|\boldsymbol{B}| = \sqrt{(B_1 - 0)^2 + (B_2 - 0)^2 + (B_3 - 0)^2} = \sqrt{B_1^2 + B_2^2 + B_3^2}$$

and

$$|\boldsymbol{A} - \boldsymbol{B}| = \sqrt{(A_1 - B_1)^2 + (A_2 - B_2)^2 + (A_3 - B_3)^2}$$

By the law of cosines, we have

$$|\boldsymbol{A} - \boldsymbol{B}|^2 = |\boldsymbol{A}|^2 + |\boldsymbol{B}|^2 - 2|\boldsymbol{A}|\,|\boldsymbol{B}| \cos \theta$$

Or

$$\cos \theta = \frac{|\boldsymbol{A}|^2 + |\boldsymbol{B}|^2 - |\boldsymbol{A} - \boldsymbol{B}|^2}{2|\boldsymbol{A}|\,|\boldsymbol{B}|}$$

$$= \frac{A_1^2 + A_2^2 + A_3^2 + B_1^2 + B_2^2 + B_3^2 - \{(A_1 - B_1)^2 + (A_2 - B_2)^2 + (A_3 - B_3)^2\}}{2|\boldsymbol{A}|\,|\boldsymbol{B}|}$$

$$= \frac{2A_1B_1 + 2A_2B_2 + 2A_3B_3}{2|\boldsymbol{A}|\,|\boldsymbol{B}|} = \frac{\boldsymbol{A} \cdot \boldsymbol{B}}{|\boldsymbol{A}|\,|\boldsymbol{B}|}$$

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}| \cos \theta.$$

If non-zero vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal then $\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}| \cos \left(\frac{\pi}{2}\right) = 0$.

Projections

Suppose we have two none-zero vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ with common initial point, shown in Fig. 9.15. The **vector projection** of $\boldsymbol{A}$ onto $\boldsymbol{B}$ is the vector $\boldsymbol{OP}$ determined by dropping a perpendicular from Q to the line OS, and notation for this vector is $projection_B\boldsymbol{A}$ (vector projection of **A** onto **B**)

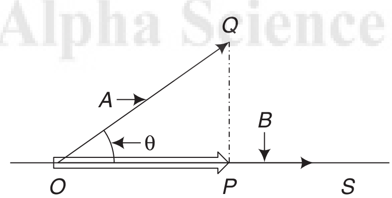


Fig. 9.15

If we pull an object with a force **A**, then the effective force moving the object forward in the direction **B** is the projection of **A** onto **B** shown in Fig. 9.16.

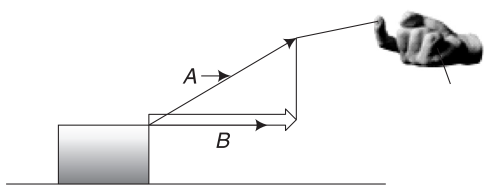


Fig. 9.16

Let θ be the acute angle between the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, then the length of projection ${}_B\boldsymbol{A}$ is $|\boldsymbol{A}| \cos\theta$ and direction $\frac{B}{|B|}$. If θ be the obtuse, and $\cos\theta < 0$ then the length of projection ${}_B\boldsymbol{A}$ is $-|A| \cos\theta$ and direction $-\frac{B}{|B|}$, Fig. 9.17.

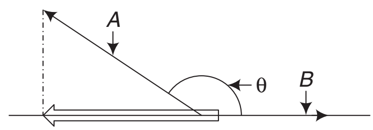


Fig. 9.17

Now in both cases

$$\text{projection } {}_B\boldsymbol{A} = (|\boldsymbol{A}| \cos\theta)\, \frac{\boldsymbol{B}}{|\boldsymbol{B}|}$$

Length . Unit vector

$$= \left(\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{|\boldsymbol{B}|}\right) \frac{\boldsymbol{B}}{|\boldsymbol{B}|}$$

$$= \left(\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{|\boldsymbol{B}|^2}\right)\boldsymbol{B}$$

Hence, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are two non-zero vectors then the vector projection of $\boldsymbol{A}$ onto $\boldsymbol{B}$ is the vector

$$\text{projection } {}_B\boldsymbol{A} = \left(\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{|\boldsymbol{B}|^2}\right)\boldsymbol{B}.$$

The scalar component of $\boldsymbol{A}$ in the direction of $\boldsymbol{B}$

$$|\boldsymbol{A}| \cos\theta = \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{|\boldsymbol{B}|} = \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{|\boldsymbol{B}|}.$$

Work as a dot product

Let W be work done by a force $\boldsymbol{F}$ on an object moving along the line from a point P to a point Q, then

$$\text{Work done} = W = \boldsymbol{F} \cdot \boldsymbol{PQ}$$

Where $\boldsymbol{PQ}$ is the displacement vector of the object's motion from P to Q.

9.3 THE CROSS PRODUCT

If $\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}$ and $\boldsymbol{B} = B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k}$ are two vectors then the cross product or vector product, written as $\boldsymbol{A} \times \boldsymbol{B}$, is a vector

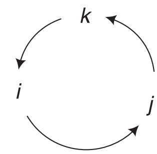
$$\boldsymbol{A} \times \boldsymbol{B} = (A_2B_3 - A_3B_2)\boldsymbol{i} + (A_3B_1 - A_1B_3)\boldsymbol{j} + (A_1B_2 - A_2B_1)\boldsymbol{k}.$$

We can calculate the cross product by determinant

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Properties of Cross Product

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = -\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = -(\boldsymbol{B} \times \boldsymbol{A})$$



$$(\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = 0), (\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i} \cdot \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j})$$

$$(c\boldsymbol{A}) \times (d\boldsymbol{B}) = cd(\boldsymbol{B} \times \boldsymbol{A}),\ \boldsymbol{0} \times \boldsymbol{A} = \boldsymbol{A} \times \boldsymbol{0} = 0.$$

If $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are three vectors, then

$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C}$$

$$(\boldsymbol{A} + \boldsymbol{B}) \times \boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{C} + \boldsymbol{B} \times \boldsymbol{C}$$

$$|\boldsymbol{A} \times \boldsymbol{B}|^2 = |\boldsymbol{A}|^2\ |\boldsymbol{B}|^2 - (\boldsymbol{A} \cdot \boldsymbol{B})^2.$$

If θ be the angle between two non-zero vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ then

$$|\boldsymbol{A} \times \boldsymbol{B}| = (|\boldsymbol{A}|\ |\boldsymbol{B}| \sin \theta)\ \mathbf{n}$$

The cross product of two vectors is also a vector. For this reason it is also called the **vector product.** The vector $\boldsymbol{A} \times \boldsymbol{B}$ is orthogonal to both $\boldsymbol{A}$ and $\boldsymbol{B}$ because it is a scalar multiple of $\boldsymbol{n}$, Fig. 9.18.

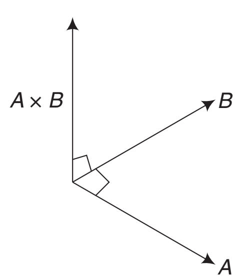


Fig. 9.18

Proof: We know that $|\boldsymbol{A} \times \boldsymbol{B}|^2 = |\boldsymbol{A}|^2\ |\boldsymbol{B}|^2 - (\boldsymbol{A}.\boldsymbol{B})^2 = |\boldsymbol{A}|^2\ |\boldsymbol{B}|^2 - |\boldsymbol{A}|^2\ |\boldsymbol{B}|^2 \cos^2\theta = |\boldsymbol{A}|^2\ |\boldsymbol{B}|^2\ (1 - \cos^2\theta) = |\boldsymbol{A}|^2\ |\boldsymbol{B}|^2 \sin^2\theta.$

Non-zero vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are parallel, if and only if $\boldsymbol{A} \times \boldsymbol{B} = 0$.

Cross product as area

The parallelogram with adjacent sides AB and AC has

$$\text{Area} = |\boldsymbol{A}|\ |\boldsymbol{C}| \sin\theta = |\boldsymbol{AB} \times \boldsymbol{AC}|$$

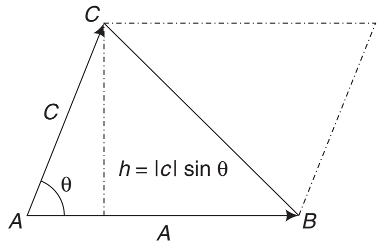


Fig. 9.19

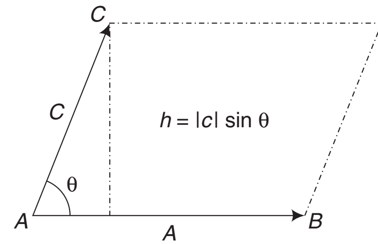


Fig. 9.20

Let A, B and C are the vertices of a triangle in $\mathbb{R}^3$, then the area of the triangle is

$$\text{Area} = \frac{1}{2}\,[\boldsymbol{AB} \times \boldsymbol{AC}].$$

9.4 TRIPLE PRODUCT

In this section we will discuss two types of triple product, **scalar triple product** and **vector triple product**.

Scalar Triple Product

Let $\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}$, $\boldsymbol{B} = B_1\boldsymbol{i} + B_2\boldsymbol{j} + B_3\boldsymbol{k}$ and $\boldsymbol{C} = C_1\boldsymbol{i} + C_2\boldsymbol{j} + C_3\boldsymbol{k}$ are three vectors then the scalar triple product define as

$$\boldsymbol{A}\cdot(\boldsymbol{B} \times \boldsymbol{C}) = A_1\,(B_2C_3 - B_3C_2)\boldsymbol{i} + A_2\,(B_3C_1 - B_1C_3)\boldsymbol{j} + A_3\,(B_1C_2 - B_2C_1)\boldsymbol{k}.$$

We can Calculate the scalar triple product by determinant

$$\boldsymbol{A}\cdot(\boldsymbol{B} \times \boldsymbol{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Similarly,

$$(\boldsymbol{A} \times \boldsymbol{B})\cdot\boldsymbol{C} = \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Since both the determinants are equal, hence

$$\boldsymbol{A}\cdot(\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B})\cdot\boldsymbol{C} = (\boldsymbol{C} \times \boldsymbol{A})\cdot\boldsymbol{B}$$

Example 1 Find the volume of a parallelepiped in $\mathbb{R}^3$ with side $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$.

Solution The volume of a parallelepiped = height of parallelepiped × area of the base of parallelepiped

From Fig. 9.22, height $h = |\boldsymbol{C}| \cos\theta$ and area of the base is equal to the area of a parallelogram with side $\boldsymbol{A}$ and $\boldsymbol{B}$.

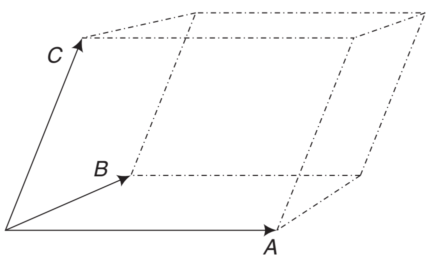


Fig. 9.21

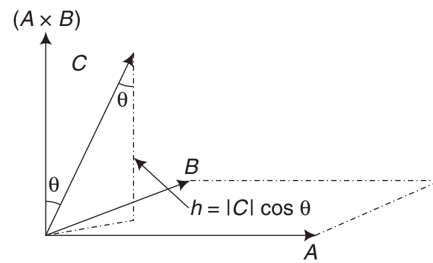


Fig. 9.22

$\therefore$ Volume $V = |\boldsymbol{C}| \cos\theta \times |\boldsymbol{A} \times \boldsymbol{B}|$

$$\text{Volume } V = \left|\frac{(\boldsymbol{A} \times \boldsymbol{B})\cdot \boldsymbol{C}}{|\boldsymbol{A} \times \boldsymbol{B}|}\right| \times |\boldsymbol{A} \times \boldsymbol{B}| \quad \{(\boldsymbol{A} \times \boldsymbol{B})\cdot\boldsymbol{C} = |\boldsymbol{A} \times \boldsymbol{B}|\cdot|\boldsymbol{C}| \cos\theta\}$$

Hence,

Volume of a parallelepiped = $|(\boldsymbol{A} \times \boldsymbol{B})\cdot \boldsymbol{C}|$.

Example 2 Find the volume of a parallelepiped determined by the vectors $\boldsymbol{A} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{B} = 2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{C} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$.

Solution
$$(\boldsymbol{A} \times \boldsymbol{B})\cdot\boldsymbol{C} = \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \begin{vmatrix} 3 & -2 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & -3 \end{vmatrix} = -20$$

Hence the volume is $(\boldsymbol{A} \times \boldsymbol{B})\cdot\boldsymbol{C} = |-20| = 20$.

Example 3 Following Fig. 9.23 shows that the volume of a tetrahedron is $\frac{1}{3}$ (area of ΔABC) (height of tetrahedron)

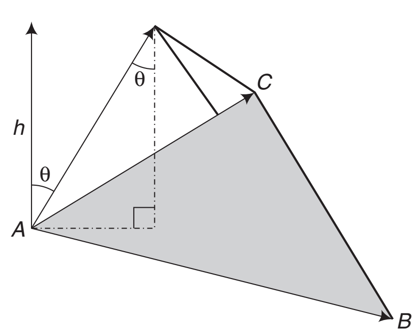


Fig. 9.23

Use the above result show that the volume V of the tetrahedron is

$$V = \frac{1}{6}\,|(\boldsymbol{AB} \times \boldsymbol{AC})\cdot \boldsymbol{AD}|$$

Solution We know that the area of the triangle is $\frac{1}{2}\,|\boldsymbol{AB} \times \boldsymbol{AC}|$ and

$$\text{Height} = |\boldsymbol{AD}| \cos\theta = \left|\frac{(\boldsymbol{AB} \times \boldsymbol{AC})\cdot \boldsymbol{AD}}{|\boldsymbol{AB} \times \boldsymbol{AC}|}\right|$$

$$\therefore \qquad V = \frac{1}{3}\left\{\frac{1}{2}\,|\boldsymbol{AB} \times \boldsymbol{AC}|\right\}\cdot\left|\frac{(\boldsymbol{AB} \times \boldsymbol{AC})\cdot \boldsymbol{AD}}{|\boldsymbol{AB} \times \boldsymbol{AC}|}\right| = \frac{1}{6}\,|(\boldsymbol{AB} \times \boldsymbol{AC})\cdot \boldsymbol{AD}|.$$

Example 4 Show that if $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are vectors in $\mathbb{R}^3$ with $\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C} = 0$. Show that

$$\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{B} \times \boldsymbol{C} = \boldsymbol{C} \times \boldsymbol{A}$$

Solution $\boldsymbol{A} \times (\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \times \boldsymbol{A} + \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C} = 0$

$$\Rightarrow \qquad \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C} = 0$$

$$\Rightarrow \qquad \boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{C} \times \boldsymbol{A}$$

Similarly,

$$\boldsymbol{B} \times (\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{B} \times \boldsymbol{A} + \boldsymbol{B} \times \boldsymbol{B} + \boldsymbol{B} \times \boldsymbol{C} = 0$$

$$\Rightarrow \qquad \boldsymbol{B} \times \boldsymbol{A} + \boldsymbol{B} \times \boldsymbol{C} = 0$$

$$\Rightarrow \qquad \boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{B} \times \boldsymbol{C}$$

Hence

$$\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{B} \times \boldsymbol{C} = \boldsymbol{C} \times \boldsymbol{A}.$$

Vector Triple Product

For any three vectors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ the vector triple product define as

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}.$$

Proof: Let $\boldsymbol{A} = A_1\boldsymbol{i}$, $\boldsymbol{B} = B_1\boldsymbol{i} + B_2\boldsymbol{j}$ and $\boldsymbol{C} = C_1\boldsymbol{i} + C_2\boldsymbol{j} + C_3\boldsymbol{k}$.

We have $\boldsymbol{A}\cdot\boldsymbol{C} = A_1C_1$, $\boldsymbol{A}\cdot\boldsymbol{B} = A_1B_1$,

$$\boldsymbol{B} \times \boldsymbol{C} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ B_1 & B_2 & 0 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= B_2C_3\boldsymbol{i} - B_1C_3\boldsymbol{j} + (B_1C_2 - B_2C_1)\boldsymbol{k}.$$

$$\therefore \qquad \boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_1 & 0 & 0 \\ B_2C_3 & -B_1C_3 & B_1C_2 - B_2C_1 \end{vmatrix}$$

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = -A_1(B_1C_2 - B_2C_1)\boldsymbol{j} - A_1B_1C_3\boldsymbol{k} \tag{9.1}$$

Now

$$(\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C} = A_1C_1(B_1\boldsymbol{i} + B_2\boldsymbol{j}) - A_1B_1(C_1\boldsymbol{i} + C_2\boldsymbol{j} + C_3\boldsymbol{k})$$

$$(\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C} = -(A_1B_1C_2 - A_1B_2C_1)\boldsymbol{j} - A_1B_1C_3\boldsymbol{k} \tag{9.2}$$

From (9.1) and (9.2), we have

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}.$$

9.5 PARAMETRIC REPRESENTATION OF CURVES

When we draw a curve by an equation $y = f(x)$ we must always restrict ourselves to a single-valued branch. Hence it is often more convenient to introduce other analytical methods of representation. The most general and the most useful representation of a curve is parametric representation. Instead of considering one of the rectangular coordinates as a function of the other, we think the coordinates x and y as functions of a third independent variable t, the so called **parameter**; the point with the coordinates x and y then describes the curve as t transverses a closed interval.

Therefore, a curve in the plane is said to be parameterised if the coordinates on the curve, (x, y), are represented as functions of a variable t. Namely,

$$x = f(t), \; y = g(t)$$

Where f and g are continuous functions of real number t define on an interval I. The variable t is called a parameter and the relations between x, y and t are called parametric equations. The set I is called the domain of f and g and it is the set of values t takes.

Conversely, given a pair of parametric equations with parameter t, the set of points $\{f(t), g(t)\}$ form a curve in the plane.

For example, the graph of any function $y = f(x)$ can be parameterised as:

let $\qquad t = x$ so that

$$x = t, \; y = f(t).$$

is a pair of parametric equations with parameter t whose graph is identical to that of the function. The domain of the parametric equations is the same as the domain of f

For example the curve $y = ax^2$ can be parameterized by the parametric equation $x = t$ and $y = at^2$, the equation of the circle $x^2 + y^2 = a^2$, can be parameterised by the parametric equation $x = a \sin t$ and $y = a \cos t$.

Similarly, the polar equation $r = a \cos^5\theta$ can be parameterised as:

Let $x = r \cos\theta$ and $y = r \sin\theta$. Then $x = (a \cos^5\theta) \cos\theta = a \cos^6\theta$, and $y = (a \cos^5\theta) \sin\theta$.

Example 5 Sketch the path of the following curves.

(i) $x = t^2 - 4,\ y = \frac{1}{2}t,\ -2 \le t \le 2$

(ii) $x = 3 \tan 2\theta,\ y = 2 \sec 2\theta,\ 0 \le \theta \le \pi$.

Solution (i) In the following table we have given the values of x, y and t.

Table 9.1

t	x	y
−2	0	−1
−1	−3	−1/2
0	−4	0
1	−3	1/2
2	0	1

We have $y = \frac{1}{2}t$ then $t = 2y$ and substituting these values in $x = t^2 - 4$, we obtain $x = 4y^2 - 4$, which is a parabola, Fig. 9.24 (dot denote the position of t).

Parameterization are not unique. For example the curve $x = 4(4t^2 - 1)$, $y = 2t$ for $-\frac{1}{2} \le t \le \frac{1}{2}$, represent the same curve.

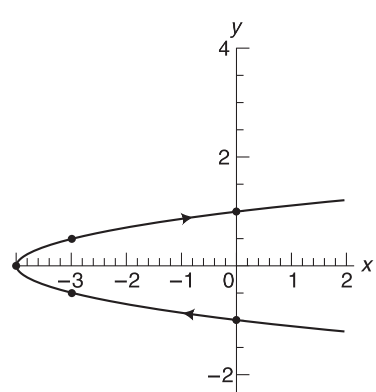


Fig. 9.24

(ii) We have, $x = 3 \tan 2\theta$, $y = 2 \sec 2\theta$.

We can write given equation as $\frac{x}{3} = \tan 2\theta$, $\frac{y}{2} = \sec 2\theta$ or $\sec^2 2\theta - \tan^2 2\theta = \frac{y^2}{4} - \frac{x^2}{9} = 1$.

Therefore the path is an hyperbola shown in Fig. 9.25.

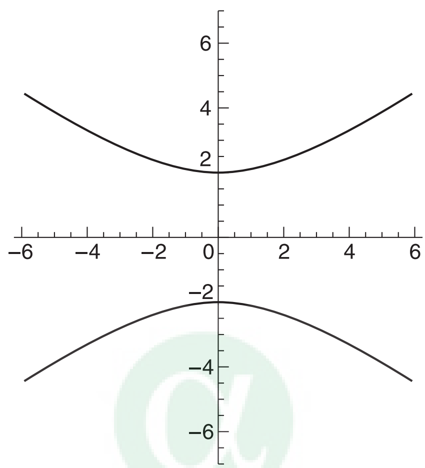


Fig. 9.25

9.6 VECTOR VALUED FUNCTIONS

In section 9.1, we defined the vectors as $\boldsymbol{A} = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}$. In this section, we see what happens when the numbers A_1, A_2 and A_3 are replaced by functions $f_1(t)$, $f_2(t)$ and $f_3(t)$.

Definition Let f_1, f_2 and f_3 be functions of the real variable t. Then for all values of t for which $f_1(t)$, $f_2(t)$ and $f_3(t)$ are defined, we define the **vector-valued function $\boldsymbol{f}$** in $\mathbb{R}^3$ as

$$\boldsymbol{f}(t) = (f_1(t), f_2(t), f_3(t)) = f_1(t)\boldsymbol{i} + f_2(t)\boldsymbol{j} + f_3(t)\boldsymbol{k} \qquad (9.3)$$

The **domain D** of $\boldsymbol{f}$ is the intersection of the domains of f_1, f_2 and f_3.

For example $\boldsymbol{f}(t) = t\boldsymbol{i} + \frac{4}{t}\boldsymbol{j} + \sqrt{t+1}\,\boldsymbol{k}$ is a **vector-valued function.** The domain of $\boldsymbol{f}$ is the set of all t for which $f_1(t) = t$, $f_2(t) = \frac{4}{t}$ and $f_3(t) = \sqrt{t+1}$ are defined. We know that $f_1(t)$ defined for all real values of t, $f_2(t)$ defined for all real values of t except $t = 0$ and $f_3(t)$ defined for all real values of $t \geq -1$. Therefore the domain of $\boldsymbol{f}$ is the set $\{t: t \geq -1 \text{ and } t \neq 0\}$.

Graph of vector-valued function: The graph a vector valued function we can just graph the parametrically defined function or let $\boldsymbol{f}$ be a vector function such that the initial point of the vector $\boldsymbol{f}(t)$ is at the origin then the graph of the function $\boldsymbol{f}$ is the curve which traced by terminal point of the vector $\boldsymbol{f}(t)$ as t varies over the domain $\boldsymbol{D}$.

Example 6 Describe the graph of the following vector-valued functions

(i) $\boldsymbol{f}(t) = (2 - t)\boldsymbol{i} + t\boldsymbol{j} + (3t - 2)\boldsymbol{k}$, (ii) $\boldsymbol{f}(t) = 5 \sin t\boldsymbol{i} - 3 \cos t\boldsymbol{j} + 3t\boldsymbol{k}$,

Solution (i) $\boldsymbol{f}(t) = (2 - t)\boldsymbol{i} + t\boldsymbol{j} + (3t - 2)\boldsymbol{k}$,

$x = f_1(t) = (2 - t)$, $y = f_2(t) = t$ and $z = f_3(t) = (3t - 2)$, the graph of $\boldsymbol{f}(t)$ is the collection of the all points (x, y, z) for different values of t which is a line, Fig. 9.26.

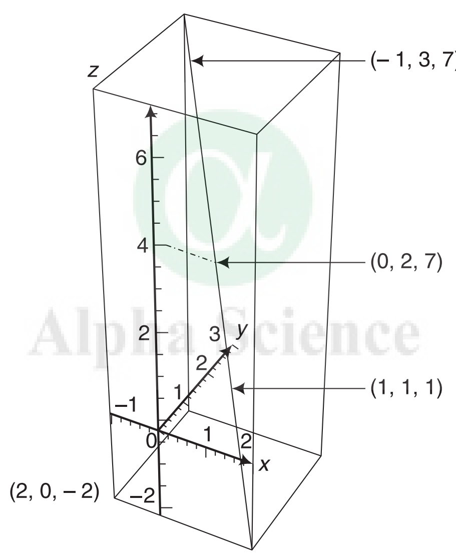


Fig. 9.26

(ii) $\boldsymbol{f}(t) = 5 \sin t\,\boldsymbol{i} - 3 \cos t\,\boldsymbol{j} + 3t\boldsymbol{k}$,

$x = f_1(t) = 5 \sin t$, $y = f_2(t) = -3 \cos t$ and $z = f_3(t) = 3t$, we obtain $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$, which is a equation of an ellipse in the xy-plane and $z = 3t$ increases as t increases, therefore the curve is a spiral that climbs up the side of an elliptical cylinder. This type of curve is called an **elliptical helix**. the graph of $\boldsymbol{f}(t)$ is the collection of the all points (x, y, z) for different values of t, Fig. 9.27.

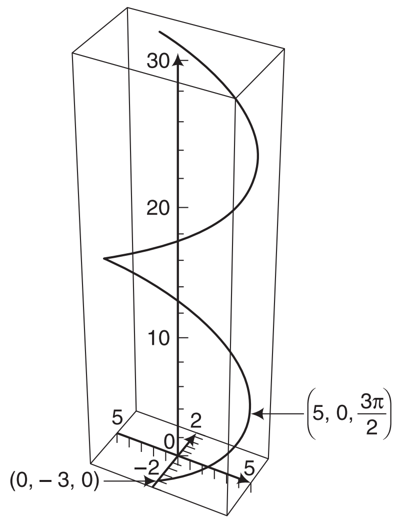


Fig. 9.27

Vector function operations: Suppose $\boldsymbol{f}(t)$ and $\boldsymbol{g}(t)$ are two vector valued functions of the real variable t, and let $f(t)$ be a scalar function. Then

$$\left.\begin{aligned}(\boldsymbol{f}+\boldsymbol{g})t &= \boldsymbol{f}(t)+\boldsymbol{g}(t)\\ (\boldsymbol{f}-\boldsymbol{g})t &= \boldsymbol{f}(t)-\boldsymbol{g}(t)\\ (f\boldsymbol{f})(t) &= f(t)\,\boldsymbol{f}(t)\\ (\boldsymbol{f}\times\boldsymbol{g})t &= \boldsymbol{f}(t)\times\boldsymbol{g}(t)\end{aligned}\right\}\rightarrow \text{Vector functions}$$

And

$$(\boldsymbol{f}\cdot\boldsymbol{g})(t) = \boldsymbol{f}(t)\cdot\boldsymbol{g}(t) \text{ is a scalae function.}$$

Example 7 If $\boldsymbol{f}(t)= 2t\,\boldsymbol{i} - 3t^2\boldsymbol{j} + \cos t\boldsymbol{k}$ and $\boldsymbol{g}(t) = t\boldsymbol{i} + e^t\boldsymbol{j} + 5\boldsymbol{k}$.

Then find

(i) $(\boldsymbol{f}+\boldsymbol{g})t$, (ii) $(e^{2t}\boldsymbol{f})(t)$, (iii) $(\boldsymbol{f}\,.\,\boldsymbol{g})(t)$, (iv) $(\boldsymbol{f}\times\boldsymbol{g})(t)$.

Solution (i) $(\boldsymbol{f}+\boldsymbol{g})t = \boldsymbol{f}(t)+\boldsymbol{g}(t)$

$$\{2t\boldsymbol{i} - 3t^2\boldsymbol{j} + \cos t\boldsymbol{k}\} + \{t\boldsymbol{i} + e^t\boldsymbol{j} + 5\boldsymbol{k}\}$$

$$= 3t\boldsymbol{i} + (-3t^2 + e^t)\boldsymbol{j} + (\cos + 5)\boldsymbol{k}.$$

(ii) $(e^{2t}\boldsymbol{f})(t) = e^{2t}\boldsymbol{f}(t)$

$$e^{2t}\{2t\boldsymbol{i} - 3t^2\boldsymbol{j} + \cos t\boldsymbol{k}\} = 2te^{2t}\boldsymbol{i} - 3t^2e^{2t}\boldsymbol{j} + e^{2t}\cos t\boldsymbol{k}.$$

(iii) $(\boldsymbol{f}\cdot\boldsymbol{g})(t) = \boldsymbol{f}(t)\cdot\boldsymbol{g}(t)$

$$\{2t\boldsymbol{i} - 3t^2\boldsymbol{j} + \cos t\boldsymbol{k}\}.\{t\boldsymbol{i} + e^t\boldsymbol{j} + 5\boldsymbol{k}\} = 2t^2 - 3t^2e^t + 5\cos t.$$

(iv) $(\boldsymbol{f} \times \boldsymbol{g})\,(t) = \boldsymbol{f}(t) \times \boldsymbol{g}\,(t)$

$$\boldsymbol{f}(t) \times \boldsymbol{g}\,(t) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2t & -3t^2 & \cos\ t \\ t & e^t & 5 \end{vmatrix}$$

$$= (-15t^2 - e^t \cos \text{t})i + (t \cos t - 10t)\boldsymbol{j} + (2te^t + 3t^3)\boldsymbol{k}.$$

Limit of vector function: Let f_1, f_2 and f_3 all have the finite limit as $t \to t_0$, where t_0 is any number or $\pm\ \infty$. Then the limit of the function $\boldsymbol{f}(t)$ as $t \to t_0$ is a vector

$$\lim_{t\to t_0} \boldsymbol{f}(t) = \left\{\lim_{t\to t_0} f_1(t)\right\}\boldsymbol{i} + \left\{\lim_{t\to t_0} f_2(t)\right\}\boldsymbol{j} + \left\{\lim_{t\to t_0} f_3(t)\right\}\boldsymbol{k}.$$

Or

Let $\boldsymbol{f}(t)$ be a vector-valued function define for all real value t in some open interval I containing the number t_0, except that the function $\boldsymbol{f}(t)$ need not be defined at the t_0. Then

$$\lim_{t\to t_0} f(t) = l$$

If any given number $\varepsilon > 0$ there exist a number $\delta > 0$ such that

$$|\boldsymbol{f}(t) - \boldsymbol{l}| < \varepsilon \quad \text{if} \quad |t - t_0| < \delta.$$

Geometric interpretation of limits: If $\boldsymbol{f}(t)$ be a vector-valued function, then

$$\lim_{t \to t_0} \boldsymbol{f}(t) = \boldsymbol{l}$$

If and only if the vector $f(t)$ approaches l in both length and direction as $t \to t_0$, Fig. 9.28.

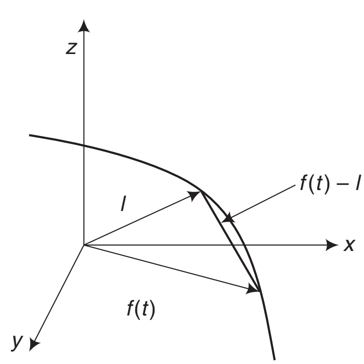


Fig. 9.28

Example 8 Find the limit.

(i) $\lim_{t\to 1} \boldsymbol{f}(t)$, Where $\boldsymbol{f}(t) = t\boldsymbol{i} - 3e^t\boldsymbol{j} + \cos 2t\boldsymbol{k}$,

(ii) $\lim_{t\to 0} \boldsymbol{f}(t)$, Where $\boldsymbol{f}(t) = \dfrac{\sin}{t}\boldsymbol{i} - 3e^t\boldsymbol{j} + \sin 2t\boldsymbol{k}$,

Solution

(i)
$$\lim_{t\to 1} \boldsymbol{f}(t) = \left\{\lim_{t\to 1} f_1(t)\right\}\boldsymbol{i} + \left\{\lim_{t\to 1} f_2(t)\right\}\boldsymbol{j} + \left\{\lim_{t\to 1} f_3(t)\right\}\boldsymbol{k}.$$

$$\lim_{t\to 1} \boldsymbol{f}(t) = \left\{\lim_{t\to 1} t\right\}\boldsymbol{i} + \left\{\lim_{t\to 1} (-3e^t)\right\}\boldsymbol{j} + \left\{\lim_{t\to 1} \cos 2t\right\}\boldsymbol{k}$$

$$= \boldsymbol{i} - 3e\boldsymbol{j} + \cos 2\boldsymbol{k}.$$

(ii)
$$\lim_{t\to 0} \boldsymbol{f}(t) = \left\{\lim_{t\to 0} \frac{\sin t}{t}\right\}\boldsymbol{i} + \left\{\lim_{t\to 0} (-3e^t)\right\}\boldsymbol{j} + \left\{\lim_{t\to 0} (\sin 2t)\right\}\boldsymbol{k}$$
$$= \boldsymbol{i} - 3\boldsymbol{j}.$$

Rules for vector limit: Suppose $\boldsymbol{f}(t)$ and $\boldsymbol{g}(t)$ are two vector valued functions of the real variable t, and let $f(t)$ be a scalar function such that all three functions have finite limits as $t \to t_0$, then

(i) $\lim\limits_{t\to t_0} \{\boldsymbol{f}(t) + \boldsymbol{g}(t)\} = \lim\limits_{t\to t_0} \boldsymbol{f}(t) + \lim\limits_{t\to t_0} \boldsymbol{g}(t)$

(ii) $\lim\limits_{t\to t_0} \{\boldsymbol{f}(t) - \boldsymbol{g}(t)\} = \lim\limits_{t\to t_0} \boldsymbol{f}(t) - \lim\limits_{t\to t_0} \boldsymbol{g}(t)$

(iii) $\lim\limits_{t\to t_0} \{\boldsymbol{f}(t)\cdot\boldsymbol{g}(t)\} = \left\{\lim\limits_{t\to t_0} \boldsymbol{f}(t)\right\}\cdot\left\{\lim\limits_{t\to t_0} \boldsymbol{g}(t)\right\}$

(iv) $\lim\limits_{t\to t_0} \{\boldsymbol{f}(t) \times \boldsymbol{g}(t)\} = \left\{\lim\limits_{t\to t_0} \boldsymbol{f}(t)\right\} \times \left\{\lim\limits_{t\to t_0} \boldsymbol{g}(t)\right\}$

(v) $\lim\limits_{t\to t_0} \{f(t)\,\boldsymbol{f}(t)\} = \left\{\lim\limits_{t\to t_0} f(t)\right\}\left\{\lim\limits_{t\to t_0} \boldsymbol{f}(t)\right\}.$

Example 9 For the vectors $\boldsymbol{f}(t) = 2\boldsymbol{i} + t^2\boldsymbol{j} + t\boldsymbol{k}$ and $\boldsymbol{g}(t) = (1 - t)\boldsymbol{i} + e^{2t}\boldsymbol{j} + 2\boldsymbol{k}$. Show that

$$\lim_{t\to 0} \{\boldsymbol{f}(t) \times \boldsymbol{g}(t)\} = \left\{\lim_{t\to 0} \boldsymbol{f}(t)\right\} \times \left\{\lim_{t\to 0} \boldsymbol{f}(t)\right\}$$

Solution
$$\boldsymbol{f}(t) \times \boldsymbol{g}(t) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & t^2 & t \\ (1 - t) & e^{2t} & 2 \end{vmatrix}$$

$$= (2t^2 - te^{2t})\boldsymbol{i} + (t(1 - t) - 4)\boldsymbol{j} + (2e^{2t} + (1 - t)t^2)\boldsymbol{k}.$$

Hence, the limit of the cross product is

$$\lim_{t\to 0} \{\boldsymbol{f}(t) \times \boldsymbol{g}(t)\} = \left\{\lim_{t\to 0} (2t^2 - te^{2t})\right\}\boldsymbol{i} + \left\{\lim_{t\to 0} (t(1-t) - 4)\right\}\boldsymbol{j}$$
$$+ \left\{\lim_{t\to 0} (2e^{2t} + (1-t)t^2)\right\}\boldsymbol{k} = -4\boldsymbol{j} + 2\boldsymbol{k}.$$

Now $\quad \lim_{t\to 0} \boldsymbol{f}(t) = \lim_{t\to 0} (2\boldsymbol{i} + t^2\boldsymbol{j} + t\boldsymbol{k}) = 2\boldsymbol{i}$

and $\quad \lim_{t\to 0} \boldsymbol{g}(t) = \lim_{t\to 0} ((1-t)\boldsymbol{i} + e^{2t}\boldsymbol{j} + 2\boldsymbol{k}) = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$

$$\left\{\lim_{t\to 0} \boldsymbol{f}(t)\right\} \times \left\{\lim_{t\to 0} \boldsymbol{f}(t)\right\} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 0 & 0 \\ 1 & 1 & 2 \end{vmatrix} = -4\boldsymbol{j} + 2\boldsymbol{k}.$$

Therefore,

$$\lim_{t\to 0}\{\boldsymbol{f}(t) \times \boldsymbol{g}(t)\} = \left\{\lim_{t\to 0} \boldsymbol{f}(t)\right\} \times \left\{\lim_{t\to 0} \boldsymbol{f}(t)\right\}.$$

Continuity of vector function: A vector-valued function $\boldsymbol{f}(t)$ is said to be continuous at a point t_0 if t_0 is in the domain of $\boldsymbol{f}$ and

$$\lim_{t\to t_0} \boldsymbol{f}(t) = \boldsymbol{f}(t_0)$$

$$\Rightarrow \quad \lim_{t\to t_0} f_1(t) = f_1(t_0),\ \lim_{t\to t_0} f_2(t) = f_2(t_0),\ \lim_{t\to t_0} f_3(t) = f_3(t_0).$$

Example 10 Find the value of t for which the function

$\boldsymbol{f}(t) = 2\sin t\boldsymbol{i} + (2-t)^{-1}\boldsymbol{j} + 2\log t\boldsymbol{k}$ is continuous.

Solution Vector-valued function $\boldsymbol{f}$ is continuous where the functions

$f_1(t) = 2\sin t \qquad f_2(t) = (2-t)^{-1} \qquad f_3(t) = 2\log t$

are continuous. The function f_1 is continuous for all t; f_2 is continuous when $2 - t \neq 0$ and f_3 is continuous when $t > 0$. Hence $\boldsymbol{f}$ is continuous when $t > 0$, $t \neq 2$.

Example 11 Determine whether $\boldsymbol{f}(t)$ is continuous at given points.

(i) $\boldsymbol{f}(t) = t\boldsymbol{i} + e^{2t}\boldsymbol{j} + 2\sqrt{1+t}\,\boldsymbol{k}$. at $t = 1$,

(ii) $\boldsymbol{f}(t) = \dfrac{2}{t}\boldsymbol{i} + t^2\boldsymbol{j} + t\boldsymbol{k}$, at $t = 0$

Solution (i) Vector-valued function $\boldsymbol{f}$ is continuous where the functions

$f_1(t) = t \qquad f_2(t) = e^{2t} \qquad f_3(t) = 2\sqrt{1+t}$

are continuous and here all functions are continuous at $t = 1$

$\lim_{t\to 1} f_1(t) = f_1(1) \Rightarrow 1 = 1$, $\lim_{t\to 1} f_2(t) = f_2(1) \Rightarrow e^1 = e^1$, $\lim_{t\to 1}$ $f_3(t) = f_3(1) \Rightarrow 2\sqrt{2} = 2\sqrt{2}$. Hence the function $\boldsymbol{f}(t)$ is continuous.

(i) $\boldsymbol{f}(t) = t\boldsymbol{i} + e^{2t}\boldsymbol{j} + 2\sqrt{1+t}\,\boldsymbol{k}$. at $t = 1$,

(ii) $\boldsymbol{f}(t) = \frac{2}{t}\boldsymbol{i} + t^2\boldsymbol{j} + t\boldsymbol{k}$, at $t = 0$

Solution (ii) Vector-valued function $\boldsymbol{f}$ is continuous where the functions

$f_1(t) = \frac{2}{t}$ $\qquad f_2(t) = t^2$ $\qquad f_3(t) = t$

are continuous and here $f_1(t)$ are not continuous at $t = 0$

$\Rightarrow$ the function $\boldsymbol{f}(t)$ is not continuous at $t = 0$.

9.7 DIFFERENTIATION AND INTEGRATION OF VECTOR VALUED FUNCTIONS

In section (3.4) we defined how the derivative $\frac{dy}{dx}$ could be calculated when x and y were given parametrically in terms of t. In this section, we will show how to calculate the derivative of a vector function.

Derivative of a Vector Function

Let f be defined at a point t. Then $\boldsymbol{f}$ is differentiable at t if

$$\boldsymbol{f}'(t) = \lim_{\Delta t\to 0} \frac{\Delta \boldsymbol{f}}{\Delta t} = \lim_{\Delta t\to 0} \frac{\boldsymbol{f}(t+\Delta t) - \boldsymbol{f}(t)}{\Delta t} = \frac{d\boldsymbol{f}}{dt}$$

exist and is finite.

Or

The vector function $\boldsymbol{f}(t) = f_1(t)\boldsymbol{i} + f_2(t)\boldsymbol{j} + f_3(t)\boldsymbol{k}$ is differentiable if the functions $f_1(t)$, $f_2(t)$ and $f_3(t)$ are differentiable, therefore

$$\boldsymbol{f}'(t) = f_1'(t)\boldsymbol{i} + f_2'(t)\boldsymbol{j} + f_3'(t)\boldsymbol{k}$$

Proof: We know that the function $\boldsymbol{f}$ differentiable if the limit exist.

$$\boldsymbol{f}'(t) = \lim_{\Delta t\to 0} \frac{\boldsymbol{f}(t+\Delta t) - \boldsymbol{f}(t)}{\Delta t}$$

$$\lim_{\Delta t\to 0} \frac{\boldsymbol{f}(t+\Delta t) - \boldsymbol{f}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\{f_1(t+\Delta t)\boldsymbol{i} + f_2(t+\Delta t)\boldsymbol{j} + f_3(t+\Delta t)\boldsymbol{k}\} - \{f_1(t)\boldsymbol{i} + f_2(t)\boldsymbol{j} + f_3(t)\boldsymbol{k}\}}{\Delta t}$$

$$= \left\{\lim_{\Delta t \to 0} \frac{f_1(t+\Delta t) - f_1(t)}{\Delta t}\right\}\boldsymbol{i} + \left\{\lim_{\Delta t \to 0} \frac{f_2(t+\Delta t) - f_2(t)}{\Delta t}\right\}\boldsymbol{j}$$

$$+ \left\{\lim_{\Delta t \to 0} \frac{f_3(t+\Delta t) - f_3(t)}{\Delta t}\right\}\boldsymbol{k}$$

$$\boldsymbol{f}'(t) = f_1'(t)\,\boldsymbol{i} + f_2'(t)\,\boldsymbol{j} + f_3'(t)\,\boldsymbol{k}.$$

Example 12 Find the derivative of the function $\boldsymbol{f}(t) = 2e^{2t}\boldsymbol{i} + \cos t\boldsymbol{j} + (t^2 + 5)\boldsymbol{k}$.

Solution $\boldsymbol{f}'(t) = (2e^{2t})'\boldsymbol{i} + (\cos t)'\boldsymbol{j} + (t^2 + 5)'\boldsymbol{k} = 4e^{2t}\,\boldsymbol{i} - \sin t\boldsymbol{j} + 2t\boldsymbol{k}$.

Example 13 Find the value of t for which the function $\boldsymbol{f}(t) = 2|t - 1|\boldsymbol{i} + \cos t\boldsymbol{j} + 5t\boldsymbol{k}$ is differentiable.

Solution The function $f_2(t) = \cos t$ and $f_3(t) = 5t$ are differentiable for all values of t but $f_1(t) = 2|t - 1|$ is not differentiable at $t = 1$, therefore the function $\boldsymbol{f}(t)$ is differentiable for all t except $t = 1$.

Geometric interpretation of derivative f': Figure 9.29 shows that the vector

$$\frac{\boldsymbol{f}(t+\Delta t) - \boldsymbol{f}(t)}{\Delta t}$$

has the direction of a secant vector whose direction approaches that of the tangent vector as $\Delta t \to 0$.

Hence, $\boldsymbol{f}'(t)$ is tangent to the graph of $\boldsymbol{f}$ at the point P.

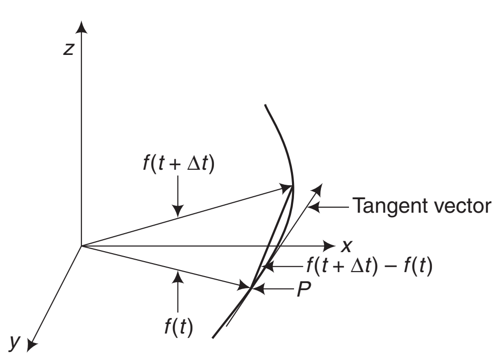


Fig. 9.29

Unit tangent vector: The unit tangent vector denoted by $\boldsymbol{T}$ of a curve is a tangent vector whose magnitude is 1. Hence,

$$\boldsymbol{T} = \frac{\boldsymbol{f}'(t)}{|\boldsymbol{f}'(t)|}$$

For any number t, as long as $\boldsymbol{f}'(t) \neq 0$.

Smooth curve: The graph of a vector function $\boldsymbol{f}(t)$ is smooth on any interval I

(i) If f' is continuous in I

(ii) $\boldsymbol{f}'(t) \neq 0$ in I.

***Example* 14** Determine whether the graph of the vector function

(i) $\boldsymbol{f}(t) = 2e^{2t}\,\boldsymbol{i} + 2t\,\boldsymbol{j} + 2\boldsymbol{k}$,

(ii) $\boldsymbol{f}(t) = 2\boldsymbol{i} + \cos t\,\boldsymbol{j} + t^2\boldsymbol{k}$ is smooth.

Solution

(i) The derivative

$$\boldsymbol{f}'(t) = 4e^{2t}\boldsymbol{i} + 2\boldsymbol{j}$$

is continuous and not zero for all t, hence the function is smooth for all t.

(ii) The derivative

$$\boldsymbol{f}'(t) = -\sin t\boldsymbol{j} + 2t\boldsymbol{k}$$

is continuous for all t but $\boldsymbol{f}'(0) = 0$, Hence the function is smooth for all t except $t = 0$.

Rules for differentiable vector functions: Suppose two vector functions $\boldsymbol{f}(t)$ and $\boldsymbol{g}(t)$ are differentiable at t, and let $f(t)$ be a scalar differentiable function at t and if a and b are constant then $a\boldsymbol{f} + b\boldsymbol{g}$, $f\,\boldsymbol{f}$, $\boldsymbol{f}\cdot\boldsymbol{g}$ and $\boldsymbol{f} \times \boldsymbol{g}$ are also differentiable at t and

(i) $\{a\boldsymbol{f} + b\boldsymbol{g}\}'(t) = a\boldsymbol{f}'(t) + b\boldsymbol{g}'(t)$

(ii) $\{f\,\boldsymbol{f}\}'(t) = f'(t)\,\boldsymbol{f}(t) + f(t)\,\boldsymbol{f}'(t)$

(iii) $\{\boldsymbol{f}\cdot\boldsymbol{g}\}'(t) = \boldsymbol{f}'(t)\cdot\boldsymbol{g}(t) + \boldsymbol{f}(t)\cdot\boldsymbol{g}'(t)$

(iv) $\{\boldsymbol{f} \times \boldsymbol{g}\}'(t) = \boldsymbol{f}'(t) \times \boldsymbol{g}(t) + \boldsymbol{f}(t) \times \boldsymbol{g}'(t)$

(v) $\{\boldsymbol{f}(f(t))\}' = f'(t)\,\boldsymbol{f}'(f(t))$.

Proof: (i) We know that

$$\{a\boldsymbol{f} + b\boldsymbol{g}\}'(t) = \lim_{\Delta t \to 0} \left\{\frac{\Delta(a\boldsymbol{f} + b\boldsymbol{g})}{\Delta t}\right\}$$

$$= \lim_{\Delta t \to 0} \left\{\frac{a\Delta \boldsymbol{f}}{\Delta t} + \frac{b\Delta \boldsymbol{g}}{\Delta t}\right\} = a \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{f}}{\Delta t} + b \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{g}}{\Delta t}$$

$$(a\boldsymbol{f} + b\boldsymbol{g})'(t) = a\boldsymbol{f}'(t) + b\boldsymbol{g}'(t).$$

Othet rules leave for exercises.

Example 15 If $\boldsymbol{f}(t) = 2\boldsymbol{i} + 3t\boldsymbol{j} + e^t\boldsymbol{k}$, and $\boldsymbol{g}(t) = t\boldsymbol{i} + 5\boldsymbol{j} + t^2\boldsymbol{k}$. Show that

(i) $\{\boldsymbol{f} \times \boldsymbol{g}\}'(t) = \boldsymbol{f}'(t) \times \boldsymbol{g}(t) + \boldsymbol{f}(t) \times \boldsymbol{g}'(t)$.

(ii) $\dfrac{d}{dt}\{2\boldsymbol{f}(t) + t^2\boldsymbol{g}(t)\} = \{(3t^2)\boldsymbol{i} + (6 + 10t)\boldsymbol{j} + (2e^t + 4t^3)\boldsymbol{k})\}$

Solution

(i) $\{\boldsymbol{f} \times \boldsymbol{g}\}(t) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 3t & e^t \\ t & 5 & t^2 \end{vmatrix} = (3t^3 - 5e^t)\boldsymbol{i} + (te^t - 2t^2)\boldsymbol{j} + (10 - 3t^2)\boldsymbol{k}.$

Therefore,

$$\{\boldsymbol{f} \times \boldsymbol{g}\}'(t) = (9t^2 - 5e^t)\boldsymbol{i} + (e^t + te^t - 4t)\boldsymbol{j} + (-6t)\boldsymbol{k}.$$

$$\boldsymbol{f}'(t) = 3\boldsymbol{j} + e^t\boldsymbol{k},\ \boldsymbol{g}'(t) = \boldsymbol{i} + 2t\boldsymbol{k},$$

$$\{\boldsymbol{f}'(t) \times \boldsymbol{g}(t)\} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 3 & e^t \\ t & 5 & t^2 \end{vmatrix} = (3t^2 - 5e^t)\boldsymbol{i} + (te^t)\boldsymbol{j} + (-3t)\boldsymbol{k}.$$

And

$$\{\boldsymbol{f}(t) \times \boldsymbol{g}'(t)\} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 3t & e^t \\ 1 & 0 & 2t \end{vmatrix} = (6t^2)\boldsymbol{i} + (e^t - 4t)\boldsymbol{j} + (-3t)\boldsymbol{k}$$

Add these two vector, we have

$\{\boldsymbol{f}'(t) \times \boldsymbol{g}(t)\} + \{\boldsymbol{f}(t) \times \boldsymbol{g}'(t)\} = (9t^2 - 5e^t)\boldsymbol{i} + (e^t + te^t - 4t)\boldsymbol{j} + (-6t)\boldsymbol{k}.$

Hence $\{\boldsymbol{f} \times \boldsymbol{g}\}'(t) = \boldsymbol{f}'(t) \times \boldsymbol{g}(t) + \boldsymbol{f}(t) \times \boldsymbol{g}'(t)$.

(ii) $\dfrac{d}{dt}\{2\boldsymbol{f}(t) + t^2\boldsymbol{g}(t)\} = \dfrac{d}{dt}\{2(2\boldsymbol{i} + 3t\boldsymbol{j} + e^t\boldsymbol{k}) + t^2(t\boldsymbol{i} + 5\boldsymbol{j} + t^2\boldsymbol{k})\}$

$$= \frac{d}{dt}\{(4\boldsymbol{i} + 6t\boldsymbol{j} + 2e^t\boldsymbol{k}) + (t^3\boldsymbol{i} + 5t^2\boldsymbol{j} + t^4\boldsymbol{k})\}$$

$$= \frac{d}{dt}\{(4 + t^3)\boldsymbol{i} + (6t + 5t^2)\boldsymbol{j} + (2e^t + t^4)\boldsymbol{k})\}$$

$$= \{(3t^2)\boldsymbol{i} + (6 + 10t)\boldsymbol{j} + (2e^t + 4t^3)\boldsymbol{k})\}.$$

Vector in motion: If a partical moves in such a way that the position vector at any time t is $\boldsymbol{r}$. Then

(i) The velocity vector is $\boldsymbol{v} = \dfrac{dr}{dt}$,

(ii) Speed $|\boldsymbol{v}|$ is magnitude of the velocity $\boldsymbol{v}$

(iii) Direction of motion is the unit vector $\dfrac{v}{|v|}$

(iv) Acceleration vector is the derivative of the velocity vector

$$A = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2}.$$

Theorem If $\boldsymbol{r}(t)$ is a vector function and $|\boldsymbol{r}(t)|$ (length) is constant for all t, then

$$\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t) = 0, \text{ Fig. 9.30}$$

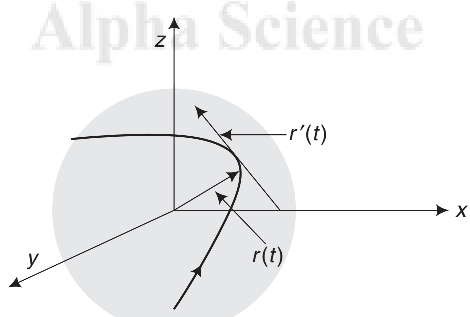


Fig. 9.30

That is, $\boldsymbol{r}(t)$ and $\boldsymbol{r}'(t)$ are orthogonal vectors for all t.

Proof: It is given that $|\boldsymbol{r}(t)|^2$ = constant.

$$\frac{d}{dt}|\boldsymbol{r}(t)|^2 = 2\boldsymbol{r}(t)\cdot\frac{dr}{dt} = 0 \Rightarrow \boldsymbol{r}(t)\cdot\boldsymbol{r}'(t) = 0.$$

Example 16 If the position vector of a particle at a time t is $\boldsymbol{r}(t) = t^2\boldsymbol{i} + 3\sin t\boldsymbol{j} + e^t\boldsymbol{k}$ then find the particle's velocity, speed, acceleration and direction of motion at a time $t = 1$.

Solution $\boldsymbol{v} = \dfrac{d\boldsymbol{r}}{dt} = 2t\boldsymbol{i} + 3\cos t\boldsymbol{j} + e^t\boldsymbol{k}$

At $t = 1$ $\boldsymbol{v} = 2\boldsymbol{i} + 3\cos 1\boldsymbol{j} + e\boldsymbol{k}$

$$A = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2} = 2\boldsymbol{i} - 3\sin t\boldsymbol{j} + e^t\boldsymbol{k}$$

At $t = 1$

$$A = 2\boldsymbol{i} - 3\sin 1\boldsymbol{j} + e\boldsymbol{k}$$

The speed is

$$|\boldsymbol{v}| = \sqrt{4 + 9\cos^2 t + e^{2t}}$$

At $t = 1$

$$|\boldsymbol{v}| = \sqrt{4 + 9\cos^2 1 + e^2}$$

Direction derivative is the unit vector

$$\frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{2t\boldsymbol{i} + 3\cos t\boldsymbol{j} + e^t\boldsymbol{k}}{\sqrt{4 + 9\cos^2 t + e^{2t}}}$$

At $t = 1$

$$\frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \frac{2\boldsymbol{i} + 3\cos 1\boldsymbol{j} + e\boldsymbol{k}}{\sqrt{4 + 9\cos^2 1 + e^2}}.$$

Vector integrals: Let f_1, f_2 and f_3 all are continuous on the closed interval $[a, b]$ such that $t \in [a, b]$ then the **indefinite integral** of $\boldsymbol{f}(t) = f_1(t)\boldsymbol{i} + f_2(t)\boldsymbol{j} + f_3(t)\boldsymbol{k}$ is the vector function

$$\int \boldsymbol{f}(t)dt = \left\{\int f_1(t)dt\right\}\boldsymbol{i} + \left\{\int f_2(t)dt\right\}\boldsymbol{j} + \left\{\int f_3(t)dt\right\}\boldsymbol{k} + \boldsymbol{c}$$

Where $\boldsymbol{c} = c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k}$ is an arbitrary constant vector.

The **definite integral** of $\boldsymbol{f}(t) = f_1(t)\boldsymbol{i} + f_2(t)\boldsymbol{j} + f_3(t)\boldsymbol{k}$ is the vector

$$\int_a^b \boldsymbol{f}(t)dt = \left\{\int_a^b f_1(t)dt\right\}\boldsymbol{i} + \left\{\int_a^b f_2(t)dt\right\}\boldsymbol{j} + \left\{\int_a^b f_3(t)dt\right\}\boldsymbol{k}$$

Where $t \in [a, b]$.

Example 17 If the velocity of a particle at a time t is $\boldsymbol{v}(t) = t^2\boldsymbol{i} + 2\cos t\boldsymbol{j} + e^t\boldsymbol{k}$ then find the particle's position as a function of t if $t = 0$ is $\boldsymbol{r}(t) = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$.

Solution To find the position of the particle integrate the velocity vector both side with respect to t:

$$\int \boldsymbol{v}(t)dt = \left\{\int t^2\, dt\right\}\boldsymbol{i} + \left\{\int 2\cos t\, dt\right\}\boldsymbol{j} + \left\{\int e^t\, dt\right\}\boldsymbol{k}$$

$$\boldsymbol{r}(t) = \left(\frac{t^3}{3} + c_1\right)\boldsymbol{i} + (2\sin t + c_2)\boldsymbol{j} + (e^t + c_3)\boldsymbol{k}$$

Use the initial condition to find c_1, c_2 and c_3

$$3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k} = \left(\frac{0^3}{3}\right)\boldsymbol{i} + (2\sin 0)\boldsymbol{j} + (e^0)\boldsymbol{k} + c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k}$$

$$3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k} = \boldsymbol{k} + c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{k}$$

Compare the coefficients of $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$, we have

$$3 = c_1,\ -1 = c_2 \text{ and } 2 = 1 + c_3$$

So $$3 = c_1,\ -1 = c_2 \text{ and } 1 = c_3.$$

Hence, the particle position at any time t is

$$\boldsymbol{r}(t) = \left(\frac{t^3}{3} + 3\right)\boldsymbol{i} + (2\sin t - 1)\boldsymbol{j} + (e^t + 1)\boldsymbol{k}.$$

Example 18 If $\int_0^{3a} \{(\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} + (\sin 2t)\boldsymbol{k}\}dt = \boldsymbol{i} + \boldsymbol{j} + \frac{1}{2}\boldsymbol{k}$

Then find the value of a.

Solution $$\left[(\sin t)\boldsymbol{i} + (-\cos t)\boldsymbol{j} + \left(\frac{-\cos 2t}{2}\right)\boldsymbol{k}\right]_0^{3a} = \boldsymbol{i} + \boldsymbol{j} + \frac{1}{2}\boldsymbol{k}$$

Or

$$\left\{(\sin 3a)\boldsymbol{i} + (1 - \cos 3a)\boldsymbol{j} + \left(\frac{1}{2} - \frac{\cos 6a}{2}\right)\boldsymbol{k}\right\} = \boldsymbol{i} + \boldsymbol{j} + \frac{1}{2}\boldsymbol{k}$$

From above equation, we obtain

$$\sin 3a = 1 \Rightarrow a = \frac{\pi}{6}.$$

Exercises

1. Find two vectors that are orthogonal to both of $\boldsymbol{A} = -3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{B} = 2\boldsymbol{i} + \boldsymbol{k}$,
2. Let $\boldsymbol{A} = \boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$, $\boldsymbol{B} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{C} = \boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$,

   find (i) $\boldsymbol{A} \times \boldsymbol{C}$, (ii) $|\boldsymbol{A} \times \boldsymbol{B}|$,

   (iii) $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$, (iv) $|(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C}|$,

   (v) $(\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{B} \times \boldsymbol{C})$.

   (vi) $\mathbf{A} \times (\boldsymbol{B} + \boldsymbol{C}) + \boldsymbol{B} \times (\boldsymbol{C} + \boldsymbol{A}) + \boldsymbol{C} \times (\boldsymbol{A} + \boldsymbol{B})$.

3. Find the projection of the vector $\boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}$ on the vector $\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$.
4. Find the area of the triangle PQR where $P(-1, 2, 3)$, $Q(0, 3, 2)$, $R(1, 2, 3)$ If θ be the angle between PQ and PR then show that $\theta = \sqrt{2}$.
5. Consider the parallelepiped with adjacent sides $\boldsymbol{A} = 2\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$, $\boldsymbol{B} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{C} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$.
   (i) Find the volume,
   (ii) Find the area of the face determined by $\boldsymbol{A}$ and $\boldsymbol{C}$.
6. Find the volume of the parallelepiped with adjacent sides $\boldsymbol{A} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{B} = -4\boldsymbol{i} + 7\boldsymbol{j} - 11\boldsymbol{k}$, $\boldsymbol{C} = 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.
7. Show that the vectors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are coplanar if $\boldsymbol{A}\cdot(\mathbf{B} \times \mathbf{C}) = 0$ or $(\boldsymbol{A} \times \boldsymbol{B})\cdot\boldsymbol{C} = 0$
8. Find the constant a such that the following vectors are coplanar
   (i) $\boldsymbol{A} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{B} = \boldsymbol{i} + a\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{C} = \boldsymbol{i} + 2\boldsymbol{k}$,
   (ii) $\boldsymbol{A} = 2\boldsymbol{i}$, $\boldsymbol{B} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{C} = \boldsymbol{i} + \boldsymbol{j} + a\boldsymbol{k}$.
9. Find an explicit relationship between x and y of the following parametric equations and sketch the path.
   (i) $x = t + 2$, $y = t - 2$, $-2 \le t \le 4$,
   (ii) $x = 2t$, $y = 3 + 16t^2$, $-2 \le t \le 2$,
   (iii) $x = 2 + \sin t$, $y = -3 + \cos t$, $0 \le t \le 2\pi$,
   (iv) $x = 3 \tan 3t$, $y = 2 \sec 3t$, $0 \le t \le \pi$,
   (v) $x = t^5$ $y = 2 + 5 \log x$, $t > 0$.
10. Find the parametric equations for each of the following curves.
    (i) A circle of radius 2, centered at (1, 2), oriented counter-clockwise,
    (ii) The ellipse $\dfrac{(x-1)^2}{2} + \dfrac{(y-2)^2}{3} = 1$, counter-clockwise,
    (iii) The hyperbola $\dfrac{(x-1)^2}{2} - \dfrac{(y-2)^2}{3} = 1$, counter-clockwise.
11. Find the domain of the following vector functions
    (i) $\boldsymbol{f}(t) = t\boldsymbol{i} + \dfrac{3}{2t}\boldsymbol{j} + 5\boldsymbol{k}$,
    (ii) $\boldsymbol{f}(t) = t\boldsymbol{i} + 5\sqrt{t-2}\,\boldsymbol{j} + \dfrac{3}{t-3}\boldsymbol{k}$,
    (iii) $\boldsymbol{f}(t) = \tan t\boldsymbol{i} + \cot t\boldsymbol{j} + \cos t\boldsymbol{k}$,
    (iv) $\boldsymbol{f}(t) + \boldsymbol{g}(t)$, where $\boldsymbol{f}(t) = t\boldsymbol{i} + 2 \log t\boldsymbol{j} + 5\boldsymbol{k}$, $\boldsymbol{g}(t) = 4t\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$.

12. Find the limits of the following vector functions

(i) $\lim_{t\to 1}\left(t\boldsymbol{i}+\frac{3}{2t}\,\boldsymbol{j}+5\boldsymbol{k}\right)$,

(ii) $\lim_{t\to 0}\,(t\boldsymbol{i}+\cos t\boldsymbol{j}+5t\boldsymbol{k})$,

(iii) $\lim_{t\to 0}\left(\frac{\sin t\boldsymbol{i}+e^{2t}\boldsymbol{j}+5t\boldsymbol{k}}{4t^2-t+1}\right)$,

(iv) $\lim_{t\to\infty}\left(\frac{t^2+2}{2t^2+3}\,\boldsymbol{i}+\frac{3}{2t}\,\boldsymbol{j}\right)$,

(v) $\lim_{t\to 1}\left(\frac{1}{2t^3}\boldsymbol{i}+\log t\boldsymbol{j}+\sin 3t\boldsymbol{k}\right)$,

(vi) $\lim_{t\to 0}\left(\frac{\sin 2\,t}{\sin 3\,t}\boldsymbol{i}+\log(\cos t)\,\boldsymbol{j}+e^t\boldsymbol{k}\right)$.

13. Determine all values of t for which the following vector functions are continuous

(i) $\boldsymbol{f}(t)=3\boldsymbol{i}+\frac{3}{2t}\boldsymbol{j}+4t\boldsymbol{k}$,

(ii) $\boldsymbol{f}(t)=\frac{3\boldsymbol{i}+4\boldsymbol{k}}{t^2+t}$,

(iii) $\boldsymbol{f}(t)=3\log t\boldsymbol{i}+\frac{1}{2t}\,\boldsymbol{j}+4t\boldsymbol{k}$,

(iv) $\boldsymbol{f}(t)=3t\boldsymbol{i}+\sqrt{2t+1}\;\boldsymbol{j}+4t\boldsymbol{k}$.

14. If $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ are differentiable vector function of t, then prove that

(i) $(\boldsymbol{f}\cdot\boldsymbol{g})'(t)=(\boldsymbol{f}\cdot\boldsymbol{g}')(t)+(\boldsymbol{f}'\cdot\boldsymbol{g})(t)$,

(ii) $(\boldsymbol{f}\times\boldsymbol{g})'(t)=(\boldsymbol{f}\times\boldsymbol{g}')(t)+(\boldsymbol{f}'\times\boldsymbol{g})(t)$,

(iii) $\frac{d}{dt}\{\boldsymbol{f}\cdot(\boldsymbol{g}\times\boldsymbol{h})\}=\boldsymbol{f}'\cdot(\boldsymbol{g}\times\boldsymbol{h})+\boldsymbol{f}\cdot(\boldsymbol{g}'\times\boldsymbol{h})+\boldsymbol{f}\cdot(\boldsymbol{g}\times\boldsymbol{h}')$,

(iv) $\frac{d}{dt}\{\boldsymbol{f}\times(\boldsymbol{g}\times\boldsymbol{h})\}=\{(\boldsymbol{h}\cdot\boldsymbol{f})\boldsymbol{g}\}'-\{(\boldsymbol{g}\cdot\boldsymbol{f})\boldsymbol{h}\}'$.

15. If $\boldsymbol{f}$ is differentiable vector function of t, and h is a scalar function of t, then show that

$$(h(t)\cdot\boldsymbol{f}(t))'=h(t)\cdot\boldsymbol{f}'(t)+h'(t)\cdot\boldsymbol{f}(t).$$

16. Find the value of $\boldsymbol{f}'$ and $\boldsymbol{f}''$ of the following vector functions and also find the tangent vector to the graph of the vector function at the indicate points.

(i) $\boldsymbol{f}(t) = 3t^2\boldsymbol{i} + \frac{3}{t}\boldsymbol{j} + 4t\boldsymbol{k},\ t = 1,\ t = 2,$

(ii) $\boldsymbol{f}(\theta) = \tan^2\theta\boldsymbol{i} + \sin 2\theta\boldsymbol{j} + 4\boldsymbol{k},\ \theta = 0,$

(iii) $\boldsymbol{f}(\theta) = \sin\theta\boldsymbol{i} + \sin 2\theta\boldsymbol{j} + \theta^2\boldsymbol{k},\ \theta = 0,\ \theta = \frac{\pi}{2}.$

17. If $\boldsymbol{r} = a\cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j} + b\theta\boldsymbol{k}$, prove that $\frac{d\boldsymbol{r}}{d\theta}\cdot\left(\frac{d^2\boldsymbol{r}}{d\theta^2} \times \frac{d^3\boldsymbol{r}}{d\theta^3}\right) = a^2 b.$

18. If the position vector of a particle at a time t is $\boldsymbol{r}(t) = 2t^2\boldsymbol{i} + (t^2 - 4t)\boldsymbol{j} + (3t - 5)\boldsymbol{k}$. Find the components of its velocity and the acceleration at $t = 1$ in the direction $\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$.

19. A particle moves along a curve whose parametric equation are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time.
    (i) Determine its velocity and the acceleration at any time
    (ii) Find the magnitude of the velocity and acceleration at $t = 0$

20. Find the position vector $\boldsymbol{r}(t)$, given the velocity $\boldsymbol{v}(t)$ and the initial position $\boldsymbol{r}(0)$ as:
    (i) $\boldsymbol{v}(t) = e^t\boldsymbol{i} + t^2\boldsymbol{j} + (\cos 2t)\boldsymbol{k},\ \boldsymbol{r}(0) = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k},$
    (ii) $\boldsymbol{v}(t) = t^3\boldsymbol{i} - e^{3t}\boldsymbol{j} + \sqrt{t}\,\boldsymbol{k},\ \boldsymbol{r}(0) = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}.$

21. Find the position vector $\boldsymbol{r}(t)$ and velocity vector $\boldsymbol{v}(t)$, given the acceleration $\boldsymbol{a}(t)$ and initial position and initial velocity vectors $\boldsymbol{r}(0)$ and $\boldsymbol{v}(0)$, respectively as:

    $\boldsymbol{a}(t) = (\sin 2t)\boldsymbol{i} + (t\cos 2t)\boldsymbol{k},\ \boldsymbol{r}(0) = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k},\ \boldsymbol{v}(0) = 2\boldsymbol{i} - \boldsymbol{k}.$

Answers

1. $\pm\left\{\frac{-3\boldsymbol{i}}{\sqrt{45}} + \frac{5\boldsymbol{j}}{\sqrt{45}} + \frac{4\boldsymbol{k}}{\sqrt{45}}\right\},$

2. (i) $13\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k},$ (ii) $5\sqrt{3},$ (iii) $-20,$
   (iv) $5\sqrt{26},$ (v) $-40\boldsymbol{i} - 20\boldsymbol{j} + 20\boldsymbol{k},$ (vi) $0,$

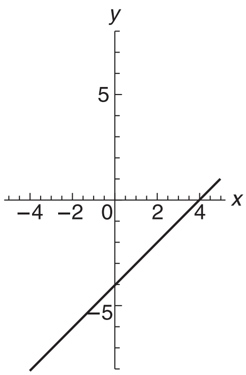
3. $\frac{5}{3},$ 4. $\sqrt{2}$

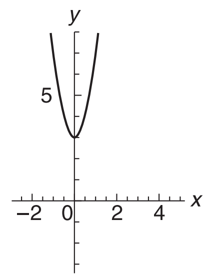
5. (i) $7,$ (ii) $\sqrt{117},$

6. $3,$

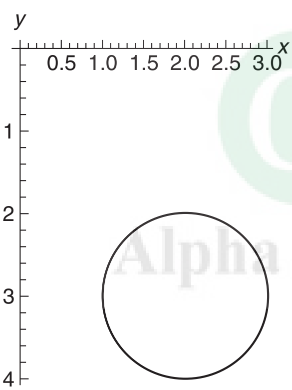
8. (i) $a,$ (ii) $-1.$

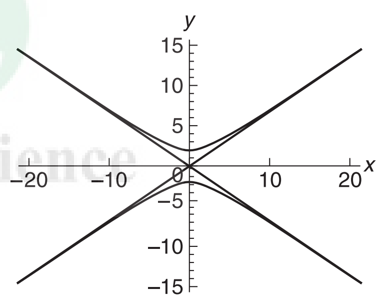
9. (i) $x = y + 4$, (ii) $y = 3 + 4x^2$,



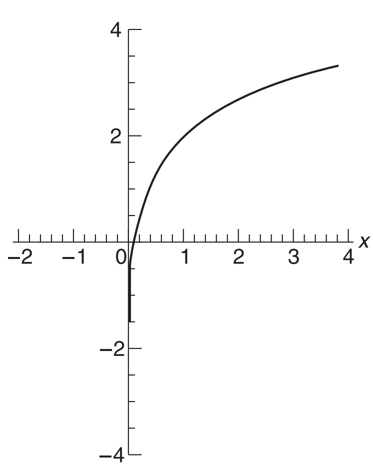


(iii) $(x - 2)^2 + (y + 3)^2 = 1$. (iv) $\dfrac{y^2}{4} - \dfrac{x^2}{9} = 1$,





(v) $y = 2 + \log x$,



10. (i) $x = \sqrt{2}\cos t - 1$, $y = \sqrt{2}\sin t - 2$,
(ii) $x = 1 + \sqrt{2}\cos t$, $y = 2 + \sqrt{3}\sin t$,
(iii) $x = 1 + \sqrt{2}\sec t$, $y = 2 + \sqrt{3}\tan t$.

11. (i) $t \neq 0$ (ii) $t \geq 2$, $t \neq 3$,
(iii) $t \neq \frac{n\pi}{2}$ n integer, (iv) $t > 0.12$.

12. (i) $\left(\boldsymbol{i} + \frac{3}{2}\boldsymbol{j} + 5\boldsymbol{k}\right)$, (ii) $\frac{\pi}{2}\boldsymbol{j}$, (iii) $\boldsymbol{j}$,
(iv) $\frac{1}{2}\boldsymbol{i}$, (v) $\frac{1}{2}\boldsymbol{i} + \sin 3\boldsymbol{k}$, (vi) $\frac{2}{3}\boldsymbol{i} + \boldsymbol{k}$.

13. (i) all t except $t = 0$, (ii) all t except $t = 0$ and $t = -1$,
(iii) For all $t > 0$, (iv) For all $t > -\frac{1}{2}$.

16. (i) $\boldsymbol{f}'(t) = 6t\boldsymbol{i} - \frac{3}{t^2}\boldsymbol{j} + 4\boldsymbol{k}$, $\boldsymbol{f}''(t) = 6\boldsymbol{i} + \frac{3}{t^3}\boldsymbol{j}$, $\boldsymbol{f}'(1) = 6\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$,
$\boldsymbol{f}'(2) = 12\boldsymbol{i} - \frac{3}{4}\boldsymbol{j} + 4\boldsymbol{k}$,
(ii) $\boldsymbol{f}'(\theta) = 2\tan\theta\sec^2\theta\boldsymbol{i} + 2\cos 2\theta\boldsymbol{j}$, $\boldsymbol{f}''(\theta) = (2\sec^4\theta + 4\tan^2\theta\sec^2\theta)\boldsymbol{i} - 4\sin 2\theta\boldsymbol{j}$, $\boldsymbol{f}'(0) = 2\boldsymbol{j}$,
(iii) $\boldsymbol{f}'(\theta) = \cos\theta\boldsymbol{i} + 2\cos 2\theta\boldsymbol{j} + 2\theta\boldsymbol{k}$, $\boldsymbol{f}''(\theta) = -\sin\theta\boldsymbol{i} - 4\sin 2\theta\boldsymbol{j} + 2\boldsymbol{k}$. $\boldsymbol{f}'(0) = \boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{f}'\left(\frac{\pi}{2}\right) = -2\boldsymbol{j} + \pi\boldsymbol{k}$.

18. $\frac{8\sqrt{14}}{7}, \frac{-\sqrt{14}}{7}$.

19. (i) $\boldsymbol{v} = -e^{-t}\boldsymbol{i} - 6\sin 3t\boldsymbol{j} + 6\cos 3t\boldsymbol{k}$, $\frac{d^2\boldsymbol{r}}{dt^2} = e^{-t}\boldsymbol{i} - 18\cos 3t\boldsymbol{j} - 18\sin 3t\boldsymbol{k}$,
(ii) $\sqrt{37}, \sqrt{325}$.

20. (i) $\boldsymbol{r}(t) = (e^t + 1)\boldsymbol{i} + \left(1 + \frac{t^3}{3}\right)\boldsymbol{j} + \left(\frac{1}{2}\sin 2t - 1\right)\boldsymbol{k}$,
(ii) $\boldsymbol{r}(t) = \left(\frac{t^4}{4} + 1\right)\boldsymbol{i} + \left(\frac{4}{3} - \frac{e^{3t}}{3}\right)\boldsymbol{j} + \left(\frac{3}{2}t^{\frac{3}{2}} - 1\right)\boldsymbol{k}$.

21. $\boldsymbol{v} = \left(\frac{5}{2} - \frac{\cos 2t}{2}\right)\boldsymbol{i} + \left(\frac{t\sin 2t}{2} + \frac{\cos 2t}{4} - \frac{5}{4}\right)\boldsymbol{j}$, $\boldsymbol{r} = \left(\frac{5}{2}t - \frac{\sin 2t}{4} + 2\right)\boldsymbol{i} + \left(\frac{\sin 2t}{4} - \frac{t\cos 2t}{4} - \frac{5}{4}t - 1\right)\boldsymbol{j}$.

9.8 MODELING BALLISTICS AND PLANETARY MOTION

Dictionary meaning of ballistics is the scientific study of the movement of objects that are through the air such as bullets, shot from a gun or an object through by a muzzle. An object fired from a gun or dropped from a moving aircraft is often called projectile. If air resistance is negligible then the path of a projectile is a parabola, Fig. 9.31.

Fig. 9.31(a)

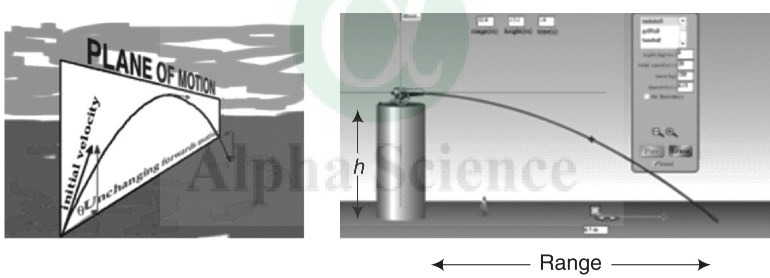


Fig. 9.31(b)

9.9 MOTION OF PROJECTILES

Suppose ***r*** is the position vector and ***v*** is the velocity vector at any time t of a projectile of mass m, Fig. 9.32.

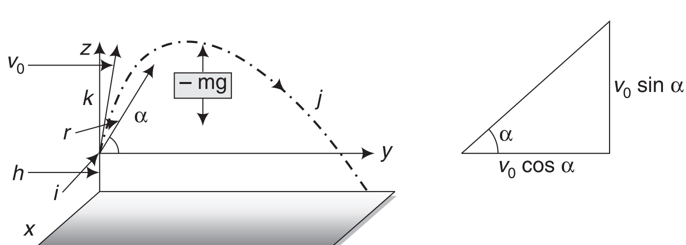


Fig. 9.32

By Newton's second law, we have

Mass × acceleration = Force

$$m\frac{d^2\boldsymbol{r}}{dt^2} = -mg\,\boldsymbol{k} \tag{9.4}$$

(g is the free fall acceleration due to the gravity = 9.8 m/s^2 approximately)

Or $$\frac{d\boldsymbol{v}}{dt} = -g\boldsymbol{k} \qquad \left(\frac{d^2\boldsymbol{r}}{dt^2} = \frac{d\boldsymbol{v}}{dt}\right)$$

Integrate both sides

$$\int \frac{d\boldsymbol{v}}{dt} = \int -g\boldsymbol{k}$$

$$\Rightarrow \qquad \boldsymbol{v} = -gt\boldsymbol{k} + \boldsymbol{c} \tag{9.5}$$

($\boldsymbol{c}$ is constant of integration)

Suppose that the initial velocity of the projectile is in the yz-plane so when $t = 0$ velocity is

$$\boldsymbol{v}_0 = v_0 \cos\alpha\boldsymbol{j} + v_0 \sin\alpha\boldsymbol{k} \tag{9.6}$$

At $t = 0$ from (9.5) and (9.6), we have

$$\boldsymbol{v}(t) = v_0 \cos\alpha\boldsymbol{j} + (v_0 \sin\alpha - gt)\boldsymbol{k} \tag{9.7}$$

Now $$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v} = v_0 \cos\alpha\boldsymbol{j} + (v_0 \sin\alpha - gt)\boldsymbol{k}$$

Again integrate both sides, we have

$$\boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = v_0 \cos\alpha t\boldsymbol{j} + \left\{v_0 \sin\alpha t - g\frac{t^2}{2}\right\}\boldsymbol{k} + \boldsymbol{c}_1$$

When $t = 0$, then $\boldsymbol{r}(0) = 0$, but if the object start from height h (Fig. (9.30), last part), then

$$\boldsymbol{c}_1 = h\boldsymbol{k}$$

And we get,

$$x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k} = v_0 \cos\alpha t\boldsymbol{j} + \left\{v_0 \sin\alpha\, t - g\frac{t^2}{2} + h\right\}\boldsymbol{k} \tag{9.8}$$

From (9.8), we obtain

$$x(t) = 0,\ y(t) = v_0 \cos\alpha t \text{ and } z(t) = v_0 \sin\alpha t - g\frac{t^2}{2} + h \tag{9.9}$$

Suppose that an object moves in a projectile motion in a coordinate plane such that xy-plane along a level ground, Fig. 9.31. If a object fired in a vacuum from a height of h with initial speed v_0 and angle of elevation α, then at time t ($t \geq 0$), it will be at the point

$$y(t) = v_0 \cos \alpha\, t \quad \text{and} \quad z(t) = v_0 \sin \alpha\, t - g\,\frac{t^2}{2} + h.$$

At the highest point of the path the component of velocity $\boldsymbol{v}$ in the direction of $\boldsymbol{k}$ is zero, hence from (9.7), we have

$$v_0 \sin \alpha - gt = 0$$

Or

$$t = \frac{v_0 \sin \alpha}{g}$$

Substitute the value of t in last part of (6), we have

$$= v_0 \sin \alpha \left(\frac{v_0 \sin \alpha}{g}\right) - g\,\frac{\left(\dfrac{v_0 \sin \alpha}{g}\right)^2}{2} + h$$

Or

$$= \frac{v_0^2 \sin^2 \alpha}{2g} + h$$

Therefore the object have **maximum height** $\dfrac{v_0^2 \sin^2 \alpha}{2g} + h$, at the ground level at a **time** $\dfrac{v_0 \sin \alpha}{g}$.

The time of flight of a projectile is the time between launch and impact, and range is the total distance which the projectile travel horizontally, therefore when the object fired from ground level then the **time of flight** can be obtain when

$$z(t) = v_0 \sin \alpha t - g\frac{t^2}{2} + 0 = 0$$

$$\Rightarrow \qquad t = \frac{2v_0 \sin \alpha}{g}.$$

And the **range** R given by the equation

$$R = v_0 \cos \alpha \left(\frac{2v_0 \sin \alpha}{g}\right) = \frac{v_0^2 \sin 2\alpha}{g}.$$

And this is maximum when $\sin 2\alpha = 1 \Rightarrow 2\alpha = \dfrac{\pi}{2} \Rightarrow \alpha = \dfrac{\pi}{4}$.

Hence the maximum range is $\dfrac{v_0^2}{g}$ and it occurs when $\alpha = \dfrac{\pi}{4}$.

The parametric equation for the motion of the projectile can be obtain by substituting the value of t from the second part of the equation (9.9) into last part of same equation (9.9) as

$$y(t) = v_0 \cos \alpha t \text{ and } z(t) = v_0 \sin \alpha t - g\frac{t^2}{2} + h$$

$$t = \left(\frac{y}{v_0 \cos \alpha}\right)$$

By substitute this in $z(t) = v_0 \sin \alpha t - g\frac{t^2}{2} + h$, we have

$$z = \left\{\frac{-g}{2(v_0 \cos \alpha)^2}\right\} y^2 + y(\tan \alpha) + h$$

This is a Cartesian equation for the trajectory of the projectile has the general form

$$z = ay^2 + by + h$$

With $a < 0$, and this is an equation of a parabola opening downward in yz-pale.

When the projectile fired from the height h from the ground then the time of flight T satisfies the equation

$$v_0 \sin \alpha T - g\frac{T^2}{2} + h = 0 \tag{9.10}$$

And range R given by the equation

$$R = (v_0 \cos \alpha)T. \tag{9.11}$$

With the help of (9.10) and (9.11) We can show that the range R satisfy the equation

$$g(\sec^2 \alpha)R^2 - 2v_0^2(\tan \alpha)R - 2v_0^2\, h = 0 \tag{9.12}$$

Maximum range occurs when $\frac{dR}{d\alpha} = 0$, therefore from above equation we have

$$2g(\sec^2 \alpha) \tan \alpha R^2 - 2v_0^2 (\sec^2 \alpha)R + 2g(\sec^2 \alpha)R \frac{dR}{d\alpha} - 2v_0^2 \tan \alpha \frac{dR}{d\alpha} = 0$$

So maximum range occurs when $R \tan \alpha = \frac{v_0^2}{g}$.

Now we know that $\tan \alpha = \frac{v_0^2}{Rg}$ and $\sec \alpha = \sqrt{1 + \frac{v_0^4}{(Rg)^2}}$

Put these values of $\tan \alpha$ and $\sec \alpha$ in (9.12), we have

$$R = \frac{v_0}{g}\sqrt{v_0^2 + 2gh}.$$

Example 19 A ball is thrown upward from the ground level at an angle 45° hits the ground 1000 m away. Find the initial speed and time of the flight of the ball.

Solution We know that

$$R = \frac{v_0^2 \sin 2\alpha}{g} = 1000 = \frac{v_0^2 \sin 90^\circ}{9.8}$$

$$v_0^2 = 1000 \times 9.8$$

Or $$v_0 = 98.99.$$

Time of the flight is

$$t = \frac{v_0^2 \sin \alpha}{g} = \frac{2 \times 98.99 \times \frac{1}{\sqrt{2}}}{9.8} \approx 14 \text{ sec.}$$

Example 20 A ball is thrown upward from the edge of cliff at a 30° angle with initial speed 68 ft/sec. Suppose the height of the cliff from the ground is 50 then

(i) Find the time of flight of the ball and its range.

(ii) Find the velocity and the speed of the ball at the time of impact.

(iii) Find the highest point where, the ball reached during the flight.

Solution (i) We have given $\alpha = 30^\circ$, $v_0 = 68$ ft/sec, and $h = 50$, and we know that $g = 32$ ft/sec.

From (9.10), we have

$$v_0 \sin \alpha T - g\frac{T^2}{2} + h = 68 \sin 30^\circ T - 32\frac{T^2}{2} + h = 0$$

Or $$-16T^2 + 34T + 50 = 0$$

$\Rightarrow$ $$(8T + 25)(2 - 2T) = 0$$

Hence the ball hit the ground when $T = 1$, because $T \geq 0$,

From (9.11) we can find the range as

$$R = 68 \cos 30^\circ \times 1 = \frac{68.\sqrt{3}}{2} = 34\sqrt{3} \approx 58.889.$$

(ii) The velocity $\boldsymbol{v}$ can be obtain from (9.7) as

$$\boldsymbol{v}(t) = v_0 \cos \alpha \boldsymbol{j} + (v_0 \sin \alpha - gt)\boldsymbol{k} = 34\sqrt{3}\boldsymbol{j} + (34 - 32.1)\boldsymbol{k} = 34\sqrt{3}\boldsymbol{j} + 2\boldsymbol{k}$$

The speed is $$|\boldsymbol{v}| = \sqrt{(34\sqrt{3})^2 + 4} \approx 58.9237.$$

(iii) we know that the time when the ball reached at the highest point is $\dfrac{v_0 \sin \alpha}{g}$ and maximum height is $\dfrac{v_0^2 \sin^2 \alpha}{2g} + h$

Hence time $= \dfrac{v_0 \sin \alpha}{g} = \dfrac{34}{32} \approx 1$ and maximum height is $\dfrac{v_0^2 \sin^2 \alpha}{2g} + h$

$= \dfrac{68^2 \sin^2 30}{64} + 50 \approx 68.06$

So the ball attend the maximum height at (58.889, 68.06 ft.).

9.10 KEPLER'S LAWS

In the seventeenth century the German astronomer Johennes Kepler (1571-1630) formulated three useful laws for describing the planetary motion (motion of the planets) which are

(i) Every planet moves in an elliptical orbit with sun at on focus.

(ii) The radius vector drawn from the sun to any planet sweeps out equal area in equal time.

(iii) The squares of the periods of revolution of the planets are proportional to the cubs of the semi-major axis of their orbits (If T is the period and a is semi-major axis then ($T^2 \approx a^3$), Fig. 9.33.

Fig. 9.33 Sweeps out equal area in equal time

Velocity and Acceleration in polar form when an object moves in a curve:

Suppose $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$ are the unit vectors along the radial axis and perpendicular to the radial axis respectively, Fig. 9.33. Now in terms of the unit Cartesian vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, we have

$$\boldsymbol{u}_r = \cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j} \text{ and } \boldsymbol{u}_\theta = -\sin\theta \boldsymbol{i} + \cos\theta \boldsymbol{j}$$

Now $$\frac{d\boldsymbol{u}_r}{d\theta} = -\sin\theta \boldsymbol{i} + \cos\theta \boldsymbol{j} = \boldsymbol{u}_\theta$$

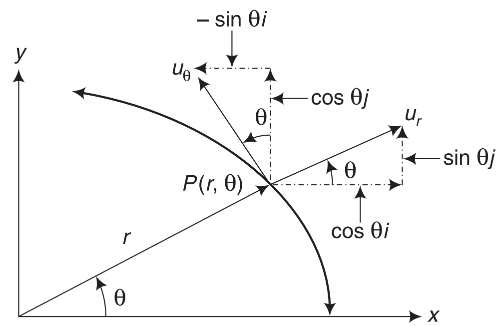


Fig. 9.33(a)

And $$\frac{d\boldsymbol{u}_\theta}{d\theta} = -\cos\theta\boldsymbol{i} - \sin\theta\boldsymbol{j} = -\boldsymbol{u}_r$$

Suppose the sun S is at the origin and an object O moves about the sun, Fig. 9.32. Then in polar coordinate system the radial vector $\boldsymbol{r} = \boldsymbol{SO}$ can be written as

$$\boldsymbol{r} = r\boldsymbol{u}_r = r\cos\theta\boldsymbol{i} + r\sin\theta\boldsymbol{j}$$

Now, we know

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{dr}{dt}\boldsymbol{u}_r + r\frac{d\boldsymbol{u}_r}{dt}$$

$$= \frac{dr}{dt}\boldsymbol{u}_r + r\frac{d\boldsymbol{u}_r}{d\theta}\frac{d\theta}{dt}$$

$$= \frac{dr}{dt}\boldsymbol{u}_r + r\frac{d\theta}{dt}\boldsymbol{u}_\theta \quad (9.13)$$

Hence, the component of the velocity along the radial axis is $\frac{dr}{dt}$ and perpendicular to the radial axis is $r\frac{d\theta}{dt}$.

Differentiate (9.13), with respect to t, we have

Acceleration $$\boldsymbol{A} = \frac{d\boldsymbol{v}}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\right)\boldsymbol{u}_r + \frac{dr}{dt}\frac{d\boldsymbol{u}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\boldsymbol{u}_\theta + r\frac{d}{dt}\left(\frac{d\theta}{dt}\right)\boldsymbol{u}_\theta + r\frac{d\theta}{dt}\frac{d\boldsymbol{u}_\theta}{dt}$$

$$= \frac{d^2r}{dt^2}\boldsymbol{u}_r + \frac{dr}{dt}\frac{d\boldsymbol{u}_r}{d\theta}\frac{d\theta}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\boldsymbol{u}_\theta + r\frac{d}{dt}\left(\frac{d\theta}{dt}\right)\boldsymbol{u}_\theta + r\frac{d\theta}{dt}\frac{d\boldsymbol{u}_\theta}{d\theta}\frac{d\theta}{dt}$$

$$= \frac{d^2r}{dt^2}\boldsymbol{u}_r + \frac{dr}{dt}\boldsymbol{u}_\theta\frac{d\theta}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\boldsymbol{u}_\theta + r\frac{d}{dt}\left(\frac{d\theta}{dt}\right)\boldsymbol{u}_\theta - r\frac{d\theta}{dt}\boldsymbol{u}_r\frac{d\theta}{dt}$$

$$= \left\{ \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \right\} \boldsymbol{u}_r + \left\{ r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \right\} \boldsymbol{u}_\theta.$$

Hence, the component of the acceleration along the radial axis is $\left\{ \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \right\}$ and perpendicular to the radial axis is $\left\{ r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \right\}$.

Kepler's second Law: According to the Kepler's second law, the radius vector drawn from the sun to any planet sweeps out equal area in equal time, which is described in Fig. 9.34. We will assume that the only force acting on a planet is the gravitational attraction of the sun and by the universal law of gravitation and force of attraction is given by

$$\boldsymbol{F} = -G\,\frac{mM}{r^2}\boldsymbol{u}_r$$

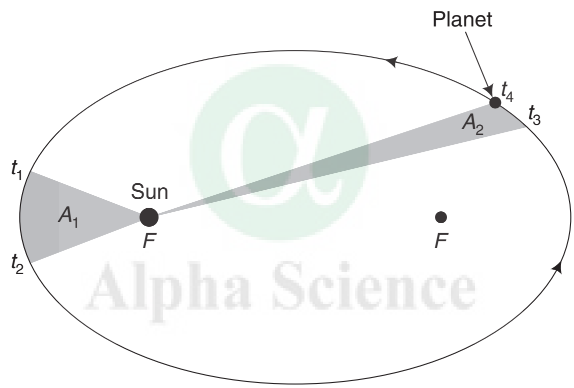


Fig. 9.34

Where G is a physical constant, M is the mass of the sun and m is the mass of the planet. By Newton's second law of motion, we have

$$\boldsymbol{F} = m\boldsymbol{A} = -G\,\frac{mM}{r^2}\boldsymbol{u}_r$$

$$\boldsymbol{A} = -G\,\frac{M}{r^2}\boldsymbol{u}_r \qquad (9.14)$$

Where $\boldsymbol{A}$ is the acceleration of the planet.

Equation (9.14) says that the acceleration of a planet has only a radial component, this means that the $\boldsymbol{u}_\theta$ component of the planet's acceleration is zero. Hence

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0$$

which is the derivative of a certain expression namely,

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$$

Integrate w.r.t. t giving us

$$r^2\frac{d\theta}{dt} = C$$

for some constant C. Now let two time intervals $[t_1, t_2]$ and $[t_3, t_4]$ of equal length, Fig. 9.34. Now area A_1 of the region SPQ, Fig. 9.35 is

$$A_1 = \int_{t_1}^{t_2} \frac{1}{2}\, r\cdot rd\theta = \int_{t_1}^{t_2} \frac{1}{2}\, r^2 d\theta = \int_{t_1}^{t_2} \frac{1}{2}\, r^2\left(\frac{d\theta}{dt}\right) dt = \int_{t_1}^{t_2} \frac{1}{2}\, Cdt$$

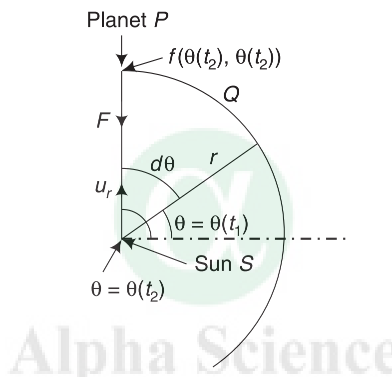


Fig. 9.35

Hence, $A_1 = \frac{1}{2}\, C(t_2 - t_1)$. Similarly we can show that the area swept out in $[t_3, t_4]$ is $A_2 = \frac{1}{2}\, C(t_4 - t_3)$. Therefore,

$$A_1 = \frac{1}{2}\, C(t_2 - t_1) = A_2 = \frac{1}{2}\, C(t_4 - t_3).$$

So equal area is swept out in equal time.

Example 21 Suppose the position vector $\boldsymbol{r}$ of a moving body is $\boldsymbol{r}(t) = t\boldsymbol{i} + 2t^2\boldsymbol{j}$ for $t \geq 0$. Express the position vector $\boldsymbol{r}$ and velocity vector $\boldsymbol{v}(t)$ in terms of $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$.

Solution

$$r = |\boldsymbol{r}| = \sqrt{(t)^2 + (2t^2)^2} = t\sqrt{1 + 4t^2}$$

$$\boldsymbol{r} = r\boldsymbol{u}_r = t\sqrt{1 + 4t^2}\,\boldsymbol{u}_r$$

$$\frac{dr}{dt} = \sqrt{1 + 4t^2} + \frac{t}{2}(1 + 4t^2)^{-\frac{1}{2}}\, 8t = \frac{1 + 8t^2}{\sqrt{1 + 4t^2}}$$

We know that $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2t^2}{t}\right)$

$$\frac{d\theta}{dt} = \frac{1}{1 + 4t^2}$$

Hence, $$\frac{1 + 8t^2}{\sqrt{1 + 4t^2}}\boldsymbol{u}_r + t\sqrt{1 + 4t^2}\;\frac{1}{1 + 4t^2}\boldsymbol{u}_\theta.$$

9.11 TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

Suppose you are riding in a car which is accelerates forward then you are pressed to the back of your seat. If the car turns sharply to one side, then you will be thrown out other side. In this case both motions are due to acceleration. It is of interest to know how much of the acceleration acts in the direction of motion, as indicated by the unit tangent vector **T,** and how much in the direction of the principal unit normal **N**. The answers of these questions can be found by following theorem.

Theorem When an object moving along a smooth curve then the velocity $\boldsymbol{v}$ and the acceleration **A** of the object define as

$$\boldsymbol{v} = \left(\frac{ds}{dt}\right)\boldsymbol{T} \text{ and } \boldsymbol{A} = \left(\frac{d^2s}{dt^2}\right)\boldsymbol{T} + k\left(\frac{ds}{dt}\right)^2\boldsymbol{N}$$

Where s is the length of the arc along the curve and k is the curvature of the curve. (The curvature is a measure of how fast the curve turns as we move along with the curve).

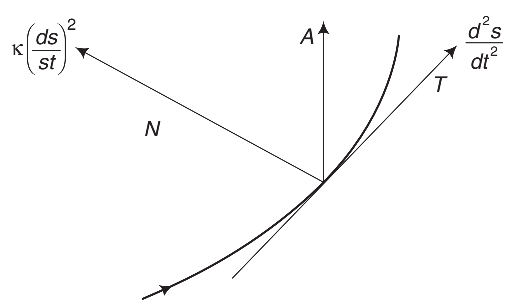


Fig. 9.36

Proof: $$\boldsymbol{A} = \frac{d\boldsymbol{v}}{dt} = \frac{d}{dt}\left\{\left(\frac{ds}{dt}\right)\boldsymbol{T}\right\} = \left(\frac{d^2s}{dt^2}\right)\boldsymbol{T} + \frac{ds}{dt}\frac{d\boldsymbol{T}}{dt}$$

$$= \left(\frac{d^2s}{dt^2}\right)\boldsymbol{T} + \frac{ds}{dt}\frac{d\boldsymbol{T}}{ds}\frac{ds}{dt}$$

$$= \left(\frac{d^2s}{dt^2}\right)\boldsymbol{T} + \left(\frac{ds}{dt}\right)^2 \frac{d\boldsymbol{T}}{ds}$$

$$= \left(\frac{d^2s}{dt^2}\right)\boldsymbol{T} + \left(\frac{ds}{dt}\right)^2 \kappa\boldsymbol{N} \qquad \left[N = \frac{1}{\kappa}\frac{d\boldsymbol{T}}{ds}\right]$$

Hence,

$A_T = \left(\frac{d^2s}{dt^2}\right)$ is the tangential component of the acceleration

$A_N = \left(\frac{ds}{dt}\right)^2 \kappa$ is the normal component of the acceleration.

If an object moves along a curve then the velocity $\boldsymbol{v}$ and the acceleration $\boldsymbol{A}$ of the object at each point are related to A_T, A_N and κ by the formulas

$$A_T = \frac{\boldsymbol{v}\cdot\boldsymbol{A}}{|\boldsymbol{v}|},\ A_N = \frac{|\boldsymbol{v}\times\boldsymbol{A}|}{|\boldsymbol{v}|},\ \kappa = \frac{|\boldsymbol{v}\times\boldsymbol{A}|}{(|\boldsymbol{v}|)^3}.$$

Proof: From Fig. 9.37, we have s

$$A_T = |\boldsymbol{A}|\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{A}}{|\boldsymbol{v}|}$$

$$A_N = |\boldsymbol{A}|\sin\theta = \frac{|\boldsymbol{v}\times\boldsymbol{A}|}{|\boldsymbol{v}|}$$

$$k = \frac{A_N}{\left(\frac{ds}{dt}\right)^2} = \frac{A_N}{(|\boldsymbol{v}|)^2} = \frac{1}{(|\boldsymbol{v}|)^2}\frac{|\boldsymbol{v}\times\boldsymbol{A}|}{|\boldsymbol{v}|} = \frac{|\boldsymbol{v}\times\boldsymbol{A}|}{(|\boldsymbol{v}|)^3}.$$

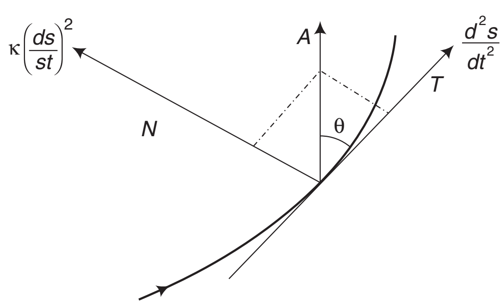


Fig. 9.37

Example 22 Suppose the position vector $\boldsymbol{r}$ of a moving object are

(i) $\boldsymbol{r}(t) = t\boldsymbol{i} + 2t^2\boldsymbol{j} + t^3\boldsymbol{k}$ for $t \geq 0$.

(ii) $\boldsymbol{r}(t) = \sin t\boldsymbol{i} + \cos t\boldsymbol{j} + 2t\boldsymbol{k}$ for $t \geq 0$.

Find the tangential and normal components of the acceleration.

Solution (i) $\boldsymbol{r}(t) = t\boldsymbol{i} + 2t^2\boldsymbol{j} + t^3\boldsymbol{k}$, $\boldsymbol{v} = \dfrac{dr}{dt} = \boldsymbol{i} + 4t\boldsymbol{j} + 3t^2\boldsymbol{k}$ and

$$\boldsymbol{A} = \frac{d\boldsymbol{v}}{dt} = 4\boldsymbol{j} + 6t\boldsymbol{k}$$

$$\boldsymbol{v}\cdot\boldsymbol{A} = (\boldsymbol{i} + 4t\boldsymbol{j} + 3t^2\boldsymbol{k})\cdot(4\boldsymbol{j} + 6t\boldsymbol{k}) = 16t + 18t^3$$

$$\boldsymbol{v} \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 4t & 3t^2 \\ 0 & 4 & 6t \end{vmatrix}$$

$$\boldsymbol{v} \times \boldsymbol{A} = (12t^2\boldsymbol{i} - 6t\boldsymbol{j} + 4\boldsymbol{k})$$

$$|\boldsymbol{v}| = \sqrt{1 + 16t^2 + 9t^4}$$

Hence, the components are

$$A_T = \frac{\boldsymbol{v}\cdot\boldsymbol{A}}{|\boldsymbol{v}|} = \frac{16t + 18t^3}{\sqrt{1 + 16t^2 + 9t^4}},\ A_N = \frac{|\boldsymbol{v} \times \boldsymbol{A}|}{|\boldsymbol{v}|} = \frac{\sqrt{144t^4 + 36t^2 + 16}}{\sqrt{1 + 16t^2 + 9t^4}}.$$

Solution (ii) $\boldsymbol{r}(t) = \sin t\boldsymbol{i} + \cos t\boldsymbol{j} + 2t\boldsymbol{k}$, $\boldsymbol{v} = \dfrac{dr}{dt} = \cos t\boldsymbol{i} - \sin t\boldsymbol{j} + 2\boldsymbol{k}$ and

$$\boldsymbol{A} = \frac{d\boldsymbol{v}}{dt} = -\sin t\boldsymbol{i} - \cos t\boldsymbol{j}$$

$$\boldsymbol{v}\cdot\boldsymbol{A} = (\cos t\boldsymbol{i} - \sin t\boldsymbol{j} + 2\boldsymbol{k})\cdot(-\sin t\boldsymbol{i} - \cos t\boldsymbol{j})$$

$$= -\cos t \sin t + \cos t \sin t = 0$$

$$\boldsymbol{v} \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos t & -\sin t & 2 \\ -\sin t & -\cos t & 0 \end{vmatrix} = 2\cos t\boldsymbol{i} - 2\sin t\boldsymbol{j} - \boldsymbol{k}$$

$$|\boldsymbol{v}| = \sqrt{5}$$

Hence, the components are

$$A_T = \frac{\boldsymbol{v}\cdot\boldsymbol{A}}{|\boldsymbol{v}|} = \frac{0}{\sqrt{5}} = 0, \qquad A_N = \frac{|\boldsymbol{v} \times \boldsymbol{A}|}{|\boldsymbol{v}|} = \frac{\sqrt{5}}{\sqrt{5}} = 1.$$

Note: If an object of mass m move such that the total force acting on the object is $\boldsymbol{F}$ then by Newton's second law.

$$\boldsymbol{F} = m\boldsymbol{A} = m(A_T)\boldsymbol{T} + m(A_N)\boldsymbol{N} = F_T\boldsymbol{T} + F_N\boldsymbol{N}$$

Therefore $$F_T = m(A_T) = m\left(\frac{d^2s}{dt^2}\right) \text{ and } F_N = m\,k\left(\frac{ds}{dt}\right)^2.$$

Exercises

1. A shell fired from ground level at an angle of 45° hits the ground 3000 m away. Find the initial speed and time of the flight of the ball.
2. Find two angles of elevation so that a shell fired from ground level if its muzzle speed is 80 ft/sec and the desired range is 100 ft.
3. A boy standing at the edge of a cliff throws a ball upward at an angle of 30° with the horizontal axis and an initial speed of 64 ft/sec. Suppose that when the ball leaves the boy's hand, it is 48 ft above the ground at the base of the cliff. What are the time of the flight of the ball and its range.
4. A golf ball is hit from the tee to a target with an initial speed 145 ft/sec at an angle of elevation 45°. How long will it take for the ball to hit the target.
5. Find the maximum height, flight time, and range of a particle fired from the origin over horizontal ground at an initial speed of 600 m/sec and a launch angle of 60°.
6. Express $\boldsymbol{r}$ and the velocity vector $\boldsymbol{v}(t)$ in terms of $\boldsymbol{u}_r$ and $\boldsymbol{u}_{q*}$, when

   (i) $\boldsymbol{r}(t) = 2t\boldsymbol{i} - t^2\boldsymbol{j}$, for ≥ 0, (ii) $x = 3t^2$, $y = t$,

   (iii) $r = \sin\theta$, $\theta = 3t$.
7. Express the velocity vector $\boldsymbol{v}(t)$ and acceleration $\boldsymbol{A}$ in terms of $\boldsymbol{u}_r$ and $\boldsymbol{u}_\theta$, when $r = (1 + \cos\theta)$,

   $\theta = 2t$.
8. Suppose the position vector $\boldsymbol{r}$ of a moving body are

   (i) $\boldsymbol{r}(t) = 2t\boldsymbol{i} + t^2\boldsymbol{j}$

   (ii) $\boldsymbol{r}(t) = \sin t\boldsymbol{i} + 2t\boldsymbol{k}$.

   (iii) $\boldsymbol{r}(t) = \cos t\boldsymbol{i} + 2t\boldsymbol{j} + 7\boldsymbol{k}$

   Find the tangential and normal components of the acceleration.
9. The speed $|\boldsymbol{v}|$ of an object at an arbitrary time t is given. Find the tangential component of acceleration at the indicated time.

   (i) $|\boldsymbol{v}| = \sqrt{t^2 + 2}$; $t = 1$, (ii) $|\boldsymbol{v}| = \sqrt{t + 2e^{2t}}$; $t = 2$.

Answers

1. $v_0 = 171.464$, t ≈ 25 sec.
2. $\alpha = 15°, 75°$.
3. $t \approx 3$ sec, $R \approx 166$.
4. $t \approx 6.4$ sec.
5. Maximum height 13775.5, $t \approx 106$ sec, $R \approx 31813$.

6. (i) $\boldsymbol{r}(t) = t\sqrt{4 + t^2}\,\boldsymbol{u}_r$, $\boldsymbol{v}(t) = \dfrac{4 + 2t^2}{\sqrt{4 + t^2}}\,\boldsymbol{u}_r + t\sqrt{4 + t^2}\left(\dfrac{-2}{4 + t^2}\right)\boldsymbol{u}_\theta$,

   (ii) $\boldsymbol{r}(t) = t\sqrt{1 + 9t^2}\,\boldsymbol{u}_r$, $\boldsymbol{v}(t) = \dfrac{1 + 18t^2}{\sqrt{1 + 9t^2}}\,\boldsymbol{u}_r + t\sqrt{1 + 9t^2}\left(\dfrac{9t^2}{1 + 9t^2}\right)\boldsymbol{u}_\theta$,

   (iii) $\boldsymbol{r}(t) = \sin\theta\,\boldsymbol{u}_r$, $\boldsymbol{v}(t) = 3\cos 3t\,\boldsymbol{u}_r + 3\sin 3t\,\boldsymbol{u}_\theta$.

7. $\boldsymbol{v}(t) = -2\sin 2t\,\boldsymbol{u}_r + 2(1 + \cos 2t)\,\boldsymbol{u}_\theta$ $\boldsymbol{A}(t) = (-8\cos 2t - 4)\boldsymbol{u}_r - 8\sin 2t\,\boldsymbol{u}_\theta$.

8. (i) $A_t = \dfrac{2t}{\sqrt{1 + t^2}}, \dfrac{4}{\sqrt{1 + t^2}}$.

   (ii) $A_t = \dfrac{-\sin t\cos t}{\sqrt{4 + \cos^2 t}}$, $A_N = \dfrac{-2\sin t}{\sqrt{4 + \cos^2 t}}$.

   (iii) $A_t = \dfrac{\sin t\cos t}{\sqrt{4 + \sin^2 t}}$, $A_N = \dfrac{-2\cos t}{\sqrt{4 + \sin^2 t}}$.

9. (i) $\dfrac{1}{\sqrt{3}}$, (ii) $\dfrac{1 + 4e^{4t}}{2\sqrt{2 + 2e^{4t}}}$.

APPENDIX

Some Important Results

$$\int \frac{dx}{x\,(ax+b)} = \frac{1}{b} \log \left|\frac{x}{ax+b}\right| + c,$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + c = -\frac{1}{a} \cot h^{-1} \frac{x}{a} + c,\ x^2 > a^2,$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left|\frac{a+x}{a-x}\right| + c = \frac{1}{a} \tan h^{-1} \frac{x}{a} + c,\ a^2 > x^2,$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{|a|} + c,$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left|x + \sqrt{x^2 - a^2}\right| + c,$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|} \sec^{-1} \left|\frac{x}{a}\right| + c,$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left|x + \sqrt{x^2 + a^2}\right| + c,$$

$$\int \frac{dx}{x\sqrt{x^2 + a^2}} = -\frac{1}{a} \log \frac{a + \sqrt{x^2 + a^2}}{x} + c,$$

$$\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a} \log \left|\frac{a + \sqrt{a^2 - x^2}}{x}\right| + c,$$

$$\int \frac{xdx}{(ax+b)} = \frac{x}{a} - \frac{b}{a^2} \log |ax+b| + c,$$

$$\int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2} \log \left|\frac{ax+b}{x}\right| + c,$$

$$\int \frac{dx}{(ax+b)^2} = \frac{-1}{a(ax+b)} + c,$$

$$\int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} + \frac{1}{b^2} \log \left|\frac{x}{ax+b}\right| + c,$$

$$\int \frac{xdx}{(ax+b)^2} = \frac{b}{a^2(ax+b)} + \frac{1}{a^2} \log |ax+b| + c,$$

$$\int \frac{dx}{\sqrt{ax+b}} = \frac{2\sqrt{ax+b}}{a} + c,$$

$$\int \frac{xdx}{\sqrt{ax+b}} = \frac{2(ax-2b)\sqrt{ax+b}}{3a^2} + c,$$

$$\int \frac{dx}{x\sqrt{ax+b}} = \begin{cases} \frac{1}{\sqrt{b}} \log \left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + c, & b > 0 \\ \frac{2}{\sqrt{-b}} \tan^{-1} \sqrt{\frac{ax+b}{-b}} + c, & b < 0 \end{cases}$$

$$\int \sqrt{ax+b}\, dx = \frac{2\sqrt{(ax+b)^3}}{3a} + c,$$

$$\int \frac{x^2 dx}{x^2+a^2} = x - a \tan^{-1} \frac{x}{a} + c,$$

$$\int \frac{xdx}{x^2+a^2} = \frac{1}{2} \log(x^2+a^2) + c,$$

$$\int \frac{dx}{x(x^2+a^2)} = \frac{1}{2a^2} \log \left(\frac{x^2}{x^2+a^2}\right) + c,$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + c, \; x^2 > a^2,$$

$$\int \frac{xdx}{x^2-a^2} = \frac{1}{2} \log(x^2-a^2) + c, \; x^2 > a^2,$$

$$\int \frac{x^2dx}{x^2 - a^2} = x + \frac{a}{2} \log \left|\frac{x-a}{x+a}\right| + c,\ x^2 > a^2$$

$$\int \frac{dx}{x^2(x^2 - a^2)} = \frac{1}{xa^2} + \frac{1}{2a^3} \log \left|\frac{x-a}{x+a}\right| + c,\ x^2 > a^2,$$

$$\int \frac{dx}{x(x^2 - a^2)} = \frac{1}{2a^2} \log \left|\frac{x^2 - a^2}{x^2}\right| + c,\ x^2 > a^2$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left|\frac{a+x}{a-x}\right| + c = \frac{1}{a} \tan h^{-1} \frac{x}{a} + c,\ x^2 < a^2,$$

$$\int \frac{x^2dx}{a^2 - x^2} = -x + \frac{a}{2} \log \left|\frac{a+x}{a-x}\right| + c,\ x^2 < a^2,$$

$$\int \frac{xdx}{a^2 - x^2} = -\frac{1}{2} \log |a^2 - x^2| + c,\ x^2 < a^2,$$

$$\int \frac{xdx}{(a^2 - x^2)^2} = \frac{1}{2} \frac{1}{(a^2 - x^2)} + c,\ x^2 < a^2,$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left(x + \sqrt{x^2 + a^2}\right) + c, = \sin h^{-1} \frac{x}{|a|} + c,$$

$$\int \frac{xdx}{\sqrt{x^2 + a^2}} = \sqrt{x^2 + a^2} + c,$$

$$\int \frac{x^2dx}{\sqrt{x^2 + a^2}} = \frac{x\sqrt{x^2 + a^2}}{2} - \frac{a^2}{2} \log\left(x + \sqrt{x^2 + a^2}\right) + c,$$

$$\int \frac{dx}{x\sqrt{x^2 + a^2}} = -\frac{1}{a} \log \left|\frac{a + \sqrt{x^2 + a^2}}{x}\right| + c,$$

$$\int \sqrt{x^2 + a^2}\ dx = \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \log\left(x + \sqrt{x^2 + a^2}\right) + c,$$

$$\int x \sqrt{x^2 + a^2}\ dx = \frac{(x^2 + a^2)^{\frac{3}{2}}}{3} + c,$$

$$\int x^2\sqrt{x^2 + a^2}\ dx = \frac{x(x^2 + a^2)^{\frac{3}{2}}}{4} - \frac{a^2x\sqrt{x^2 + a^2}}{8} - \frac{a^4}{8} \log\left(x + \sqrt{x^2 + a^2}\right) + c,$$

$$\int \frac{\sqrt{x^2 + a^2}}{x}\, dx = \sqrt{x^2 + a^2} - a \log \left|\frac{a + \sqrt{x^2 + a^2}}{x}\right| + c,$$

$$\int \frac{\sqrt{x^2 + a^2}}{x^2}\, dx = -\frac{\sqrt{x^2 + a^2}}{x} + \log\left(x + \sqrt{x^2 + a^2}\right) + c,$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left(x + \sqrt{x^2 - a^2}\right) + c,$$

$$\int \frac{xdx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2} + c,$$

$$\int \frac{x^2 dx}{\sqrt{x^2 - a^2}} = \frac{x\sqrt{x^2 - a^2}}{2} + \frac{a^2}{2} \log\left(x + \sqrt{x^2 - a^2}\right) + c,$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left|\frac{x}{a}\right| + c,$$

$$\int \sqrt{x^2 - a^2}\, dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \log\left(x + \sqrt{x^2 - a^2}\right) + c,$$

$$\int x\sqrt{x^2 - a^2}\, dx = \frac{(x^2 - a^2)^{\frac{3}{2}}}{3} + c,$$

$$\int x^2\sqrt{x^2 - a^2}\, dx = \frac{x(x^2 - a^2)^{\frac{3}{2}}}{4} + \frac{a^2 x\sqrt{x^2 - a^2}}{8} - \frac{a^4}{8} \log\left(x + \sqrt{x^2 - a^2}\right) + c,$$

$$\int \frac{\sqrt{x^2 - a^2}}{x}\, dx = \sqrt{x^2 - a^2} - a \sec^{-1} \left|\frac{x}{a}\right| + c,$$

$$\int \frac{\sqrt{x^2 + a^2}}{x^2}\, dx = -\frac{\sqrt{x^2 - a^2}}{x} + \log\left(x + \sqrt{x^2 - a^2}\right) + c,$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{|a|} + c,$$

$$\int \frac{xdx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2} + c,$$

$$\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = \frac{-x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{|a|} + c,$$

$$\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a} \log \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right| + c,$$

$$\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = \frac{-\sqrt{a^2 - x^2}}{a^2 x} + c,$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{|a|} + c$$

$$\int x\sqrt{a^2 - x^2}\, dx = -\frac{(a^2 - x^2)^{\frac{3}{2}}}{3} + c,$$

$$\int \frac{\sqrt{a^2 - x^2}}{x}\, dx = \sqrt{a^2 - x^2} - a \log \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right| + c,$$

$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$,

$\csc(-\theta) = -\csc\theta$, $\sec(-\theta) = \sec\theta$, $\cot(-\theta) = -\cot\theta$.

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$, $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$,

$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$, $\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$, $\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$,

$\sin(\pi - \theta) = \sin\theta$, $\cos(\pi - \theta) = -\cos\theta$, $\tan(\pi - \theta) = -\tan\theta$,

$\csc(\pi - \theta) = \csc\theta$, $\sec(\pi - \theta) = -\sec\theta$, $\cot(\pi - \theta) = -\cot\theta$,

$\sin(\pi + \theta) = -\sin\theta$, $\cos(\pi + \theta) = -\cos\theta$, $\tan(\pi + \theta) = \tan\theta$,

$\csc(\pi + \theta) = -\csc\theta$, $\sec(\pi + \theta) = -\sec\theta$, $\cot(\pi + \theta) = \cot\theta$,

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$,

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$,

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$,

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$,

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}, \qquad \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\theta}$$

$\sin 2\alpha + \cos^2\alpha = 1$, $\tan^2\alpha + 1 = \sec^2\alpha$, $1 + \cot^2\alpha = \csc^2\alpha$.

$\sin 2\alpha = 2\sin\alpha\cos\alpha$,

$$\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha = \cos^2\alpha - \sin^2\alpha,$$

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}, \qquad \cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2},$$

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}, \qquad \tan\alpha = \frac{2\tan\alpha/2}{1-\tan^2\alpha/2},$$

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha,$$

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha, \quad \tan\alpha/2 = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}},$$

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta),$$

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta),$$

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta),$$

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta),$$

$$\sin\alpha + \sin\beta = 2\sin\frac{(\alpha+\beta)}{2}\cos\frac{(\alpha-\beta)}{2},$$

$$\sin\alpha - \sin\beta = 2\cos\frac{(\alpha+\beta)}{2}\sin\frac{(\alpha-\beta)}{2},$$

$$\cos\alpha + \cos\beta = 2\cos\frac{(\alpha+\beta)}{2}\cos\frac{(\alpha-\beta)}{2},$$

$$\cos\alpha - \cos\beta = 2\sin\frac{(\alpha+\beta)}{2}\sin\frac{(\beta-\alpha)}{2},$$

	$\alpha = 0°$	$\frac{\pi}{6}$ 30°	$\frac{\pi}{4}$ 45°	$\frac{\pi}{3}$ 60°	$\frac{\pi}{2}$ 90°	$\frac{2\pi}{3}$ 120°	$\frac{3\pi}{4}$ 135°	$\frac{5\pi}{6}$ 150°	π 180°	$\frac{3\pi}{2}$ 270°	2π 360°
sin α	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0	–1	0
cos α	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0	$-1/2$	$-1/\sqrt{2}$	$-\sqrt{3}/2$	–1	0	1
tan α	0	$1/\sqrt{3}$	1	$\sqrt{3}$	–	$-\sqrt{3}$	–1	$-1/\sqrt{3}$	0	–	0
csc α	–	2	$\sqrt{2}$	$2/\sqrt{3}$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	–	–1	–
sec α	1	$2/\sqrt{3}$	$\sqrt{2}$	2	–	–2	$-\sqrt{2}$	$-2/\sqrt{3}$	–1	–	1
cot α	–	$\sqrt{3}$	1	$1/\sqrt{3}$	0	$-1/\sqrt{3}$	–1	$-\sqrt{3}$	–	0	–

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