Principles of Electromagnetics 1—Understanding Vectors & Electrostatic Fields

Arlon T. Adams
Jay K. Lee

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Preface

Electromagnetics is not an easy subject for students. The subject presents a number of challenges, such as: new math, new physics, new geometry, new insights and difficult problems. As a result, every aspect needs to be presented to students carefully, with thorough mathematics and strong physical insights and even alternative ways of viewing and formulating the subject. The theoretician James Clerk Maxwell and the experimentalist Michael Faraday, both shown on the cover, had high respect for physical insights.

This book is written primarily as a text for an undergraduate course in electromagnetics, taken by junior and senior engineering and physics students. The book can also serve as a text for beginning graduate courses by including advanced subjects and problems. The book has been thoroughly class-tested for many years for a two-semester Electromagnetics course at Syracuse University for electrical engineering and physics students. It could also be used for a one-semester course, covering up through Chapter 8 and perhaps skipping Chapter 4 and some other parts. For a one-semester course with more emphasis on waves, the instructor could briefly cover basic materials from statics (mainly Chapters 2 and 6) and then cover Chapters 8 through 12.

The authors have attempted to explain the difficult concepts of electromagnetic theory in a way that students can readily understand and follow, without omitting the important details critical to a solid understanding of a subject. We have included a large number of examples, summary tables, alternative formulations, whenever possible, and homework problems. The examples explain the basic approach, leading the students step by step, slowly at first, to the conclusion. Then special cases and limiting cases are examined to draw out analogies, physical insights and their interpretation. Finally, a very extensive set of problems enables the instructor to teach the course for several years without repeating problem assignments. Answers to selected problems at the end allow students to check if their answers are correct.

During our years of teaching electromagnetics, we became interested in its historical aspects and found it useful and instructive to introduce stories of the basic discoveries into the classroom. We have included short biographical sketches of some of the leading figures of electromagnetics, including Josiah Willard Gibbs, Charles Augustin Coulomb, Benjamin Franklin, Pierre Simon de Laplace, Georg Simon Ohm, Andre Marie Ampère, Joseph Henry, Michael Faraday, and James Clerk Maxwell.

The text incorporates some unique features that include:

- Coordinate transformations in 2D (Figures 1-11, 1-12).
- Summary tables, such as Table 2-1, 4-1, 6-1, 10-1.
- Repeated use of equivalent forms with R (conceptual) and $|\mathbf{r}-\mathbf{r}'|$ (mathematical) for the distance between the source point and the field point as in Eqs. (2-27), (2-46), (6-18), (6-19), (12-21).
- Intuitive derivation of equivalent bound charges from polarization sources, including piecewise approximation to non-uniform polarization (Section 3.3).
- Self-field (Section 3.8).
- Concept of the equivalent problem in the method of images (Section 4.3).
- Intuitive derivation of equivalent bound currents from magnetization sources, including piecewise approximation to non-uniform magnetization (Section 7.3).
- Thorough treatment of Faraday's law and experiments (Sections 8.3, 8.4).
- Uniform plane waves propagating in arbitrary direction (Section 9.4.1).
- Treatment of total internal reflection (Section 10.4).
- Transmission line equations from field theory (Section 11.7.2).
- Presentation of the retarded potential formulation in Chapter 12.
- Interpretation of the Hertzian dipole fields (Section 12.3).

Finally, we would like to acknowledge all those who contributed to the textbook. First of all, we would like to thank all of the undergraduate and graduate students, too numerous to mention, whose comments and suggestions have proven invaluable. As well, one million thanks go to Ms. Brenda Flowers for typing the entire manuscript and making corrections numerous times. We also wish to express our gratitude to Dr. Eunseok Park, Professor Tae Hoon Yoo, Dr. Gokhan Aydin, and Mr. Walid M. G. Dyab for drawing figures and plotting curves, and to Professor Mahmoud El Sabbagh for reviewing the manuscript. Thanks go to the University of Poitiers, France and Seoul National University, Korea where an office and academic facilities were provided to Professor Adams and Professor Lee, respectively, during their sabbatical years. Thanks especially to Syracuse University where we taught for a total of over 50 years. Comments and suggestions from readers would be most welcome.

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June 2012
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CHAPTER 1

INTRODUCTION TO VECTORS

1.1 INTRODUCTION

In Chapter One we take up the subject of vector analysis. This subject is of fundamental importance to us in the development and the applications of electromagnetics. Vectors and coordinate systems are covered first, including differential elements of length, surface, and volume as well as vector transformations between different coordinate systems. We introduce the basic functions of divergence, gradient, and curl, along with the divergence theorem and Stokes' theorem, and we conclude with Helmholtz's theorem. In this chapter, we may need to refresh our memories on the concepts of partial derivatives and multiple integrals, both of which are fundamental to electromagnetic analysis. We also will strive to gain a deeper understanding of the three-dimensional properties of the coordinate systems.

1.1.1 Josiah Willard Gibbs (1839-1903) and the Development of Vector Analysis

Josiah Willard Gibbs has been called the greatest American scientist of the nineteenth century. He was a Professor of Mathematical Physics at Yale University from 1871 until his death in 1903. During that period he made major contributions to physical chemistry, statistical mechanics, and vector analysis. Professor Gibbs was the discoverer of the chemical potential, the phase rule, the Gibbs-Dalton Law, the absorption law of chemistry, and the Gibbs' phenomenon of Fourier series.

Willard Gibbs, as he was sometimes called, grew up in New Haven, Connecticut. His father (also named Josiah Willard Gibbs) was a Professor of Theology with an interest in comparative languages. He participated in the famous "Amistad" case of 1839-41 in which it was finally decided that a group of slaves from Africa had the right to mutiny and take over the ship (*The Amistad*) on which they were illegally held. Former President John Quincy Adams was counsel for the defense in this sensational case. After the case was over, Father Gibbs wrote biographies of the slaves involved and compiled a vocabulary of the rare language (Mendi) which they spoke. Father Gibbs died in 1861, leaving Willard and his sisters well provided for.

After completing his graduate studies, Willard Gibbs spent four years in postdoctoral study in Europe, accompanied by his two sisters. He was part of a flood of American students of the time who traveled to Europe for advanced education. He returned in 1869 and was appointed Professor of Mathematical Physics in 1871.

During the 1870's Willard Gibbs accomplished some of his most important work. He published three papers in the Transactions of the Connecticut Academy of Arts and Sciences. The third paper, which was over 300 pages in length, developed the idea of the chemical potential and outlined the phase rule. James Clerk Maxwell immediately recognized the significance of his work, was a strong supporter, and did much to help him gain some recognition. Maxwell corresponded with Gibbs, sent him a plaster cast model of one of his theories and devoted an entire chapter to Gibbs in his book *Theory of Heat.* Later, scientists in Germany, France, and Holland also recognized Gibbs' work. His third paper was translated and published as a book in Germany in 1892 and in France in 1899. Finally, he received the Copley Medal in 1901.

Gibbs became interested in vector analysis in the early 1880's. He often said that the work gave him more pleasure than any other intellectual activity. He drew up an extensive set of notes on vector analysis for his students during the 1880's but did not publish them. He himself felt that the work was not original since it derived from much earlier work by the German mathematician Grassman. Nonetheless, his work on vector analysis did attract attention. It involved him in extensive and bitter controversy with the champions of quaternions, an extremely complex competing methodology first proposed by Hamilton. Most of this controversy took place in the 1890's. Gibbs finally allowed publication of his lectures in a book *Vector Analysis* authored by E.B. Wilson, one of his students. Why did he not allow his name to appear on the book as author or even coauthor? Certainly part of the reason was that he himself did not regard it as strictly original. Perhaps also he did not want to become involved in further controversy with the fiercely combative quaternionic group.

Much has been written about Gibbs and his relative anonymity. The President of an American university went to Europe to find a physicist and was told that the best man for his purposes was Gibbs. "He can't be a very dynamic person if I haven't heard of him," was the response. It is interesting to note that he was recognized quite early by those within his field. Within ten years of his initial appointment in 1871, he was recognized and praised extensively by Maxwell, referred to by Lorentz, elected to the National Academy of Sciences and awarded the Rumford Medal.

Willard Gibbs was a quiet man, by no means an extrovert. He enjoyed an active intellectual life, the affection of a few close friends, and close relationships with his family. He lived out his professional life in the house in which he had been raised, with spinster sister Anna and the family of sister Julia.

Gibbs' writings were very abstract and unusually concise, making them accessible to only a handful of people. In addition, he published at first only in the relatively obscure *Connecticut Academy Transactions.* Moreover, Gibbs did not fully illustrate his theory with applications. That was largely left to others. Time was required for many of the applications to emerge. So, given his personality and the nature of his work, it should not be surprising that he long remained unknown in the wider sense. He was the last of the classical school, coming, as he did, just after the epochal classical contributions of Maxwell and just before the dramatic appearance of relativity and quantum theory.

· Muriel Rukeyser, *Willard Gibbs,* pp. 199-203, Oxbow Press, Woodbridge, CT, 1988, Reprint from Doubleday, Doran, Garden City, NY, 1942.

1.2 VECTOR ALGEBRA

1.2.1 Basic Operations of Vector Algebra

A **scalar** quantity is characterized by its magnitude. Temperature, charge and voltage are scalars. Ordinary type is used for scalars, e.g., V for voltage.

A **vector** has both magnitude and direction. Velocity and the electric field are vectors. Boldface type is used for vectors, e.g., **E** for electric field. A vector **A** may be represented as a *directed line segment* or arrow (Figure 1-1). The length of the arrow represents the magnitude of the vector, denoted by | **A** | or A, and the arrow head indicates its direction. Switching the head and tail of arrow changes **A** to **–A.** A vector is changed by rotation, but not by translation.

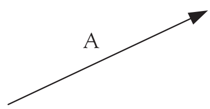


Figure 1-1. ***The vector* A *as a directed line segment.***

Vector Addition

To form the vectorial sum **A + B,** place the tail of **B** at the head of **A** as shown in Figure 1-2(a). The sum **A + B** is the arrow from the tail of **A** to the head of **B.** One may also reverse the order and place the tail of **A** at the head of **B** (Figure 1-2(b)). Note that both procedures yield the same result and one which is also identical to the parallelogram method (Figure 1-2(c)) with which you may be familiar.

Vector addition is commutative as seen in Figures 1-2(a), (b):

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1-1}$$

It is also associative

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \tag{1-2}$$

To subtract **B** from **A,** we add the negative of **B** to **A** (see Figure 1-2(d)):

$$\mathbf{A} - \mathbf{B}) = \mathbf{A} + (-\mathbf{B}) \tag{1-3}$$

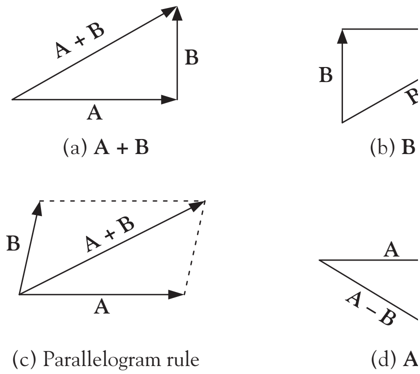


Figure 1-2. ***Addition of vectors.***

Multiplication of a Vector by a Scalar

If a vector is multiplied by a scalar *a* its magnitude is multiplied by the magnitude of *a.* The direction of the vector is unchanged if the scalar is positive and real. Scalar multiplication is distributive:

$$a(\mathbf{A}+\mathbf{B}) = a\mathbf{A} + a\mathbf{B} \tag{1-4}$$

Scalar division by *a* corresponds to multiplication by the inverse of *a*:

$$\frac{\mathbf{B}}{a} = \left(\frac{1}{a}\right)\mathbf{B} \tag{1-5}$$

If we divide a vector **A** by its magnitude A, we obtain a vector of unit length pointing in the direction of **A,** i.e., a *unit vector* **a.**

$$\mathbf{a} = \frac{\mathbf{A}}{\mathrm{A}}, \quad |\mathbf{a}| = 1 \tag{1-6}$$

All unit vectors have the same magnitude; they differ only in *direction.*

Dot or Scalar Product

The dot product of two vectors is a scalar and it is defined as

$$\mathbf{A}\cdot\mathbf{B} = \mathrm{AB}\cos\theta_{\mathrm{AB}} \quad (0 \le \theta_{\mathrm{AB}} \le \pi) \tag{1-7}$$

where A = | **A** |, B = | **B** |, θ_{AB} is the angle between **A** and **B** (Figure 1-3).

The dot product is A times the projection of **B** on **A** (B cos θ_{AB}) or B times the projection of **A** on **B** (A cos θ_{AB}). A projection is negative for $\theta_{AB} > \pi/2$.

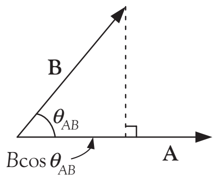


Figure 1-3. ***The dot product.***

The dot product is commutative:

$$\mathbf{B}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{B} \tag{1-8}$$

and distributive:

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C} \tag{1-9}$$

Special cases are:

$$\mathbf{A}\cdot\mathbf{B} = AB \text{ (for } \theta_{AB} = 0)$$
$$= 0 \text{ (for } \theta_{AB} = \pi/2)$$
$$\mathbf{A}\cdot\mathbf{A} = A^2$$

Cross or Vector Product

The cross product of two vectors is a vector and it is defined as follows:

$$\mathbf{A}\times\mathbf{B} = \mathbf{a}_n \; AB \sin\theta_{AB} (0 \le \theta_{AB} \le \pi) \tag{1-10}$$

Note that the magnitude of $\mathbf{A} \times \mathbf{B}$ is $AB \sin \theta_{AB}$, which is the area (base times height) of the parallelogram formed by **A** and **B** (see Figure 1-4(a)). Either A or B may be taken as the base. The direction of the cross product is that of $\mathbf{a}_n$ (Figure 1-4(a)) which is a unit vector perpendicular to the plane of the parallelogram (the plane formed by **A** and **B**). The

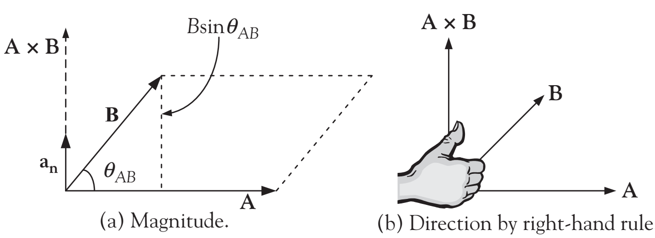


Figure 1-4. ***The cross product.***

direction of the normal $\mathbf{a}_n$ is determined by the right hand rule (Figure 1-4(b)). Note that $\mathbf{A} \times \mathbf{B}$ is perpendicular to both **A** and **B.**

The cross product is distributive:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{1-11}$$

The cross product is *not commutative*:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{1-12}$$

Special cases are:

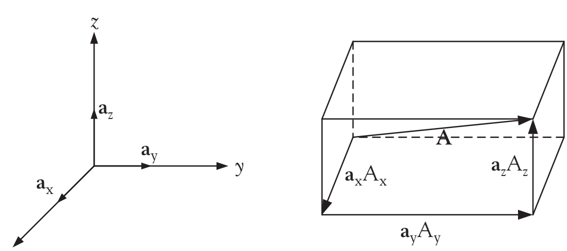
$$|\mathbf{A} \times \mathbf{B}| = AB \text{ (for } \theta_{AB} = \pi/2)$$
$$= 0 \text{ (for } \theta_{AB} = 0)$$
$$\mathbf{A} \times \mathbf{A} = 0$$

1.2.2 Vector Algebra in Rectangular Coordinates

A vector may be represented in terms of its components in any coordinate system; we choose the rectangular (Cartesian) coordinate system for this section. Figure 1-5(a) shows the Cartesian coordinate system with its unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$. Figure 1-5(b) shows a vector **A** and its projections $\mathbf{a}_x A_x$, $\mathbf{a}_y A_y$, $\mathbf{a}_z A_z$ on the x, y, z axes. By translation, we can place the projections head to tail and show that

$$\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z \quad (1\text{-}13)$$

A_x, A_y, A_z are the components of **A** or the magnitudes of its projections. The magnitudes may either precede or follow the unit vectors.



(a) Unit vectors.

(b) Components of vector **A**.

Figure 1-5. ***Representation of a vector in rectangular coordinate system.***

Vector Addition

$$\mathbf{A}+\mathbf{B} = (\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z) + (\mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z)$$
$$= \mathbf{a}_x (A_x + B_x) + \mathbf{a}_y (A_y + B_y) + \mathbf{a}_z (A_z + B_z) \qquad (1\text{-}14)$$

Multiplication by a Scalar

$$c\mathbf{A} = c(\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z)$$
$$= \mathbf{a}_x (cA_x) + \mathbf{a}_y (cA_y) + \mathbf{a}_z (cA_z) \qquad (1\text{-}15)$$

Dot Product

Unit vector relationships:

$$\mathbf{a}_x \cdot \mathbf{a}_x = \mathbf{a}_y \cdot \mathbf{a}_y = \mathbf{a}_z \cdot \mathbf{a}_z = 1$$
$$\mathbf{a}_x \cdot \mathbf{a}_y = \mathbf{a}_x \cdot \mathbf{a}_z = \mathbf{a}_y \cdot \mathbf{a}_z = 0$$

Thus

$$\mathbf{A} \cdot \mathbf{B} = (\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z) + (\mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z)$$
$$= A_x B_x + A_y B_y + A_z B_z \text{ (using the distributive property)} \qquad (1\text{-}16)$$

Special case:

$$\mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 + A_z^2 = |\mathbf{A}|^2$$

Thus

$$|\mathbf{A}| = \sqrt{A_x^2 + A_z^2 + A_z^2}$$

Cross Product

Unit vector relations:

$$\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z \qquad \mathbf{a}_y \times \mathbf{a}_x = -\mathbf{a}_z$$
$$\mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x \qquad \mathbf{a}_z \times \mathbf{a}_y = -\mathbf{a}_x$$
$$\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y \qquad \mathbf{a}_x \times \mathbf{a}_z = -\mathbf{a}_y$$

(Simply use the right-hand rule or consider the cycle xyzxyz to determine the cross product)

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= (\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z) + (\mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z) \\ &= \mathbf{a}_x (A_y B_z - A_z B_y) + \mathbf{a}_y (A_z B_x - A_x B_z) + \mathbf{a}_z (A_z B_y - A_y B_x)\end{aligned} \tag{1-17a}$$

The results above are identical to the convenient determinant form of the cross product:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{1-17b}$$

1.2.3 Triple Products

Scalar Triple Product **A · (B × C)**

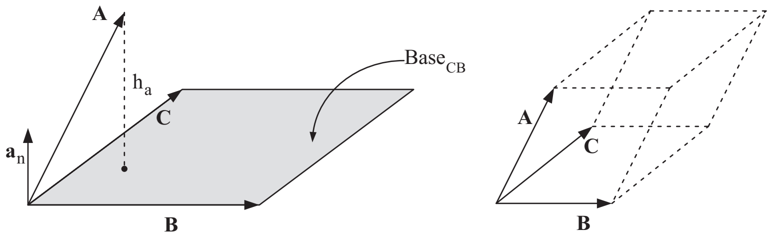
A · (B × C) is a scalar whose magnitude is equal to the volume of the parallelepiped (box) formed by the three vectors. Figure 1-6(b) shows vectors **A, B, C** and the box which they form. We note that **B** × **C** is a vector with direction $\mathbf{a}_n$ and magnitude equal to the area of the base of the parallelepiped: (see Figure 1-6(a))

$$\mathbf{B}\times\mathbf{C} = \mathbf{a}_n \ (\text{area of Base}_{CB})$$

Then

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= (\mathbf{a}_n\cdot\mathbf{A})\ (\text{area of Base}_{CB}) \\ &= \pm\, h_a (\text{area of Base}_{CB}) = \pm(\text{volume of the box})\end{aligned} \tag{1-18}$$

($\mathbf{a}_n$ · **A**) is positive in the case of Figure 1-6. If **A** were reversed, ($\mathbf{a}_n$ · **A**) would be negative.



(a) $Base_{CB}$ and height h_a

(b) The parallelepiped formed by vectors **A**, **B**, and **C**.

Figure 1-6. *The Triple Product* **A · (B × C).**

Since the volume of the box can be obtained from three different bases and heights:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$
$$= -\,\mathbf{A}\cdot(\mathbf{C}\times\mathbf{B}) = -\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}) = -\mathbf{C}\cdot(\mathbf{B}\times\mathbf{A}) \quad (1\text{-}19)$$

Thus we can change the order so long as we retain the cycle ABCABC. We may also interchange dot and cross products in Eq. (1-19) to obtain:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} \quad (1\text{-}20)$$

Note that all the parentheses of this section may be removed since there is only one possible location.

There is also a determinant form of the scalar triple product:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \quad (1\text{-}21)$$

This identity may be shown by laboriously expanding both sides in rectangular coordinates.

Vector Triple Product **A** × (**B** × **C**)

A × (**B** × **C**) is a vector. We note that the direction of **B** × **C** is that of $\mathbf{a}_n$ (Figure 1-6). Then **A** × (**B** × **C**) is perpendicular to (**B** × **C**), i.e., normal to $\mathbf{a}_n$. Thus **A** × (**B** × **C**) lies in the plane formed by **B** and **C** and has components in the **B** and **C** directions. The "BAC-CAB" rule gives these components:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \tag{1-22}$$

The identity above can be shown by expanding in rectangular components (see Problem 1-5). The parenthesis in $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ *is* necessary since it differs from $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$.

Example 1-1

Let $\mathbf{A} = \mathbf{a}_y 2 + \mathbf{a}_z 2$

$\mathbf{B} = \mathbf{a}_y 3$

$\mathbf{C} = \mathbf{a}_x (-4) + \mathbf{a}_y(4)$

Find $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.
Solution:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} 0 & 2 & 2 \\ 0 & 3 & 0 \\ -4 & 4 & 0 \end{vmatrix} = (-4)\{2\cdot 0 - 2\cdot 3\} = 24$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

$$= \mathbf{a}_y 3\cdot(8) - (-\mathbf{a}_x 4 + \mathbf{a}_y 4)\cdot 6 = \mathbf{a}_x 24$$

Sketch the vectors and the parallelepiped (box). Show by geometrical considerations that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ has only an $\mathbf{a}_x$ component.

Example 1-2

Let $\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$

Find $\mathbf{A} \cdot \mathbf{a}_z$ and $\mathbf{A} \times \mathbf{a}_z$
Solution:

$$\mathbf{A}\cdot\mathbf{a}_z = \mathbf{a}_z\cdot(\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z) = 0 + 0 + A_z = A_z$$

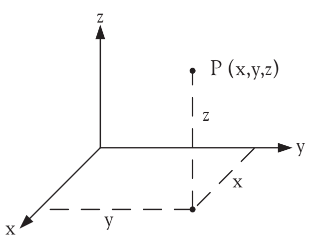
Taking the dot product of vector **A** with any unit vector yields the component of **A** in the direction of the unit vector.

$$\mathbf{A} \times \mathbf{a}_z = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{a}_x A_y - \mathbf{a}_y A_x$$

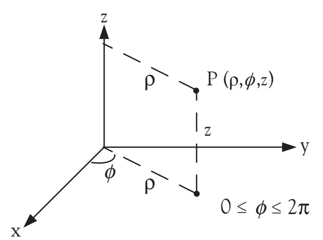
1.3 COORDINATE SYSTEMS

1.3.1 Coordinate System Geometry

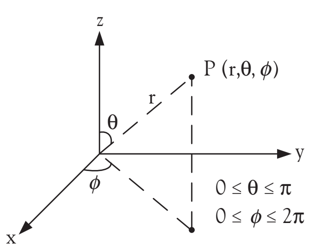
Figure 1-7 shows the three basic coordinate systems we will use in this text. An understanding of the coordinate systems and the geometry involved is most essential in the three-dimensional problems of electromagnetics. Often the geometry is at the heart of the problem.



(a) The rectangular (Cartesian) coordinate system.



(b) The cylindrical coordinate system.



(c) The spherical coordinate system.

Figure 1-7. ***The three basic coordinate systems.***

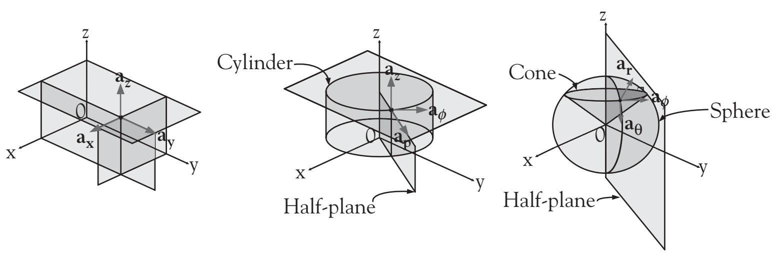
Note in Figure 1-7 the relationships among the three coordinate systems. r is the distance from the origin to a particular point P. θ (polar angle) is an angle from the z-axis to the radial line OP. It ranges from 0 to π. ρ is the radial distance from P to the z axis. ρ is also the distance from the origin to the projection of P on the xy plane and $\rho = r \sin \Theta$. z has the same definition for both rectangular and cylindrical coordinate systems and $z = r \cos \Theta$. φ (azimuth angle) is an angle from the x-axis to the radial line from the origin to the projection of P on the xy plane. It ranges from 0 to 2π. φ has the same definition for cylindrical and spherical coordinate systems.

Table 1-1 is a collection of all the equations of transformation between coordinates. Can you reconstruct some of the relationships, for instance, the first set, from the figures of the coordinate systems?

Table 1-1. Equations of Transformation between Coordinate Systems

$x = \rho \cos\varphi = r \sin\theta \cos\varphi$ $y = \rho \sin \varphi = r \sin\theta \sin\varphi$ $z = r \cos\theta$
$\rho = \sqrt{x^2 + y^2} = r \sin\theta$ $\phi = \tan^{-1} \frac{y}{x}$ $z = r \cos\theta$
$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2}$ $\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1} \frac{\rho}{z}$ $\phi = \tan^{-1} \frac{y}{x}$*
*Note that it is important to consider the signs of x and y to determine the appropriate quadrant of angle φ.

Next we consider the orthogonal surfaces in the various coordinate systems. Figure 1-8 shows the orthogonal surfaces and the unit vectors. Note that the surface θ = constant is a cone and the surface φ = constant is a half-plane. The surfaces, ρ = constant and r = constant, are cylinders and spheres, respectively. A unit vector is normal to its coordinate surface and points in the direction in which its coordinate is increasing. Thus the unit vector $\mathbf{a}_\theta$ is normal to the conical surface. It points in the direction of increasing θ and is equal, for example, to $-\mathbf{a}_z$ at $\theta = \pi/2$. Note that only the unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$ are constant in direction. Thus they can be taken outside the integral when we deal with vector integrals. However, the unit vectors $\mathbf{a}_\rho$, $\mathbf{a}_r$, $\mathbf{a}_\theta$, $\mathbf{a}_\varphi$ are not constant vectors because their direction depends on the location (see Problem 1-17). Relationships between unit vectors in different coordinates are discussed in Section 1.3.3. Finally, referring to Figure 1-8(c), can you identify the lines that correspond to latitude and longitude lines on the earth?



(a) Rectangular coordinates (b) Cylindrical coordinates (c) Spherical coordinates

Figure 1-8. ***Orthogonal surfaces and unit vectors.***

1.3.2 Differential Elements of Length, Surface and Volume

The integration of scalar and vector functions requires that we know the differential elements of length, surface and volume. You are already familiar with some of these; others will be new to you. First we consider the vector differential element of length $d\boldsymbol{\ell}$. In moving along the arrow $d\boldsymbol{\ell}$, the changes in x, y, z are dx, dy, dz, respectively. Thus:

$$d\boldsymbol{\ell} = \mathbf{a}_x dx + \mathbf{a}_y dy + \mathbf{a}_z dz \qquad \text{(1-23a)}$$

For changes in length variables ρ and r, the corresponding differential lengths are $\mathbf{a}_\rho d\rho$ and $\mathbf{a}_r dr$, respectively. The differential elements of length

$d\ell$ in cylindrical coordinates require an additional element associated with angular change ($d\varphi$), while in spherical coordinates it requires two additional elements associated with changes in angle variables ($d\theta$ and $d\varphi$). These are shown in Figure 1-9. If we consider an element of length in which only θ changes, the appropriate vector differential length is $\mathbf{a}_\theta$ r $d\theta$. For changes in φ only, the vector differential length is $\mathbf{a}_\varphi \rho d\varphi$ or $\mathbf{a}_\varphi r \sin\theta d\varphi$ as shown in Figure 1-9.

Thus in cylindrical and spherical coordinates, the vector differential elements of length are given as follows:

$$d\ell = \mathbf{a}_\rho d\rho + \mathbf{a}_\phi \rho d\phi + \mathbf{a}_z dz \quad (1\text{-}23b)$$

$$d\ell = \mathbf{a}_r dr + \mathbf{a}_\theta r d\theta + \mathbf{a}_\phi r \sin\theta d\phi \quad (1\text{-}23c)$$

The vector differential elements of length are summarized in Table 1-2(a).

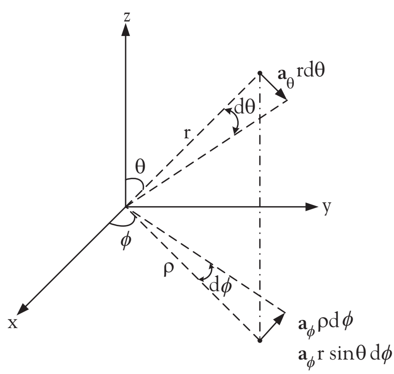
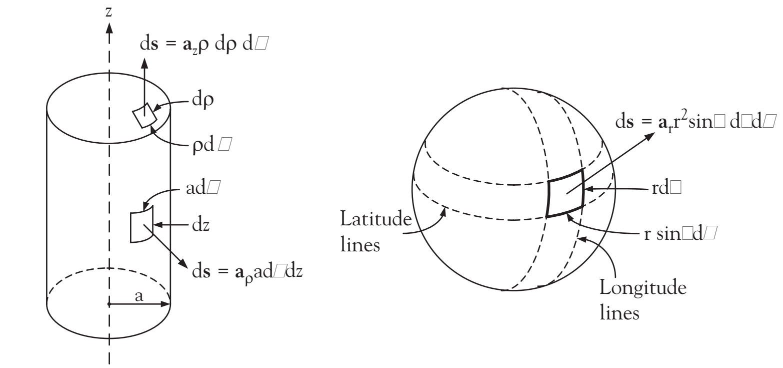


Figure 1-9. ***Basic elements of differential length in cylindrical and spherical coordinates.***

The vector surface elements are readily constructed from the elements of length. They have magnitudes equal to the elementary surface areas and directions corresponding to one of the two normals to the surface. The choice of normal direction depends on the problem. Figure 1-10 shows some typical vector surface elements, namely, a portion of a disk, a curved cylinder, and a sphere.



(a) Cylindrical coordinates. (b) Spherical coordinates.

Figure 1-10. ***Basic surface elements.***

The complete set of vector surface elements is given in Table 1-2(b). The elements of volume are given in Table 1-2(c). These may be constructed by multiplying surface elements by the length element associated with the direction normal to the surface. For instance, the disk, curved cylindrical and spherical surfaces of Figure 1-10 are multiplied by dz, dρ, dr, respectively to obtain volume elements.

Table 1-2. Differential Elements of Length, Surface, and Volume

(a) Vector Differential Length Elements

$$d\mathbf{l} = \mathbf{a}_x dx + \mathbf{a}_y dy + \mathbf{a}_z dz$$

$$= \mathbf{a}_\rho d\rho + \mathbf{a}_\varphi\, \rho\, d\varphi + az\, dz$$

$$= ar\, dr + a\theta\, r\, d\theta + \mathbf{a}_\varphi\, r \sin\theta d\varphi$$

(b) Vector Differential Surface Elements

$$d\mathbf{s} = \pm\, \mathbf{a}_x\, dy\, dz,\ \mathbf{a}_y\, dzdx,\ \mathbf{a}_z\, dx\, dy^*$$

$$= \pm\, \mathbf{a}_\rho\, \rho\, d\varphi dz,\ \mathbf{a}\varphi d\rho dz,\ az\, \rho\, d\rho d\varphi$$

$$= \pm\, ar\, r2\, \sin\theta d\theta d\varphi,\ \mathbf{a}_\theta\, r \sin\theta\, dr d\varphi,\ \mathbf{a}_\varphi\, r\, dr d\theta$$

(c) Differential Volume Elements

$$dv = dx\, dydz$$

$$= \rho\, d\rho d\varphi dz$$

$$= r2\, \sin\, \theta\, dr d\theta d\varphi$$

* The choice of sign ± depends on the problem

1.3.3 Coordinate Transformations

There are situations where it is required to change from one coordinate system to another. In order to accomplish this, it is useful to have a systematic method of transforming between different coordinate systems. There are in all six possible vector transformations (rectangular to cylindrical, rectangular to spherical, cylindrical to spherical, and their inverses) for the three basic coordinate systems. Each transformation can be expressed in terms of a matrix relationship.

Scalar transformations are relatively simple since we need only to substitute from the basic equations of transformation in Table 1-1. For example, to transform from cylindrical to rectangular coordinates, a function F(ρ, φ, z) becomes $F\left(\sqrt{x^2+y^2},\ \tan^{-1} y/x, z\right)$.

Vector transformations are also straightforward. Consider first the transformations between rectangular and cylindrical coordinates. Figure 1-11 shows the unit vectors $\mathbf{a}_\rho$, $\mathbf{a}_\varphi$ in the xy plane (or z = constant plane) because $\mathbf{a}_z$ is common in both rectangular and cylindrical coordinates. We can resolve each into x, y, z components by inspection.

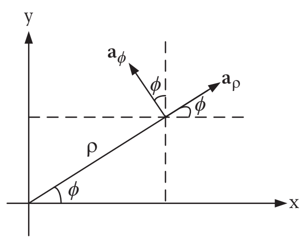


Figure 1-11. ***The transformation between rectangular and cylindrical coordinates.***

$$
\begin{aligned}
\mathbf{a}_\rho &= \mathbf{a}_x \cos\phi + \mathbf{a}_y \sin\phi + \mathbf{a}_z(0) \\
\mathbf{a}_\phi &= \mathbf{a}_x(-\sin\phi) + \mathbf{a}_y \cos\phi + \mathbf{a}_z(0) \\
\mathbf{a}_z &= \mathbf{a}_x(0) + \mathbf{a}_y(0) + \mathbf{a}_z(1)
\end{aligned}
$$

The equations above may be cast in matrix form:

$$
\begin{bmatrix} \mathbf{a}_\rho \\ \mathbf{a}_\phi \\ \mathbf{a}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \end{bmatrix} \tag{1-24a}
$$

Similarly, the inverse matrix relationship is:

$$\begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{a}_\rho \\ \mathbf{a}_\phi \\ \mathbf{a}_z \end{bmatrix} \quad \text{(1-24b)}$$

The inverse relationship may be determined by inspection of Figure 1-11 or by matrix inversion. The inversion process is particularly simple since only the 2 × 2 submatrix in the upper left hand corner needs to be considered. The determinant of each matrix is unity.

Consider next the transformations between cylindrical and spherical coordinates.

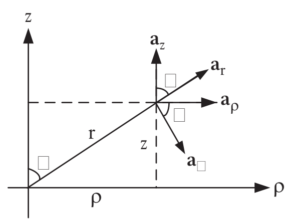


Figure 1-12. ***The transformation between cylindrical and spherical coordinates.***

Figure 1-12 shows the unit vectors $\mathbf{a}_r$, $\mathbf{a}_\theta$ and their relationships to the unit vectors $\mathbf{a}_\rho$, $\mathbf{a}_z$ in the half plane (φ = constant) because $\mathbf{a}_\varphi$ is common in both cylindrical and spherical coordinates. Resolving $\mathbf{a}_r$, $\mathbf{a}_\theta$ into cylindrical components yields.

$$\begin{aligned} \mathbf{a}_r &= \mathbf{a}_\rho \sin\phi + \mathbf{a}_z \cos\theta + \mathbf{a}_\phi(0) \\ \mathbf{a}_\theta &= \mathbf{a}_\rho \cos\theta - \mathbf{a}_z \sin\theta + \mathbf{a}_\phi(0) \\ \mathbf{a}_\phi &= \mathbf{a}_\rho(0) + \mathbf{a}_z(0) + \mathbf{a}_\phi(1) \end{aligned}$$

In matrix form:

$$\begin{bmatrix} \mathbf{a}_r \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_\rho \\ \mathbf{a}_\phi \\ \mathbf{a}_z \end{bmatrix} \quad \text{(1-25a)}$$

The inverse relationship is

$$\begin{bmatrix} \mathbf{a}_\rho \\ \mathbf{a}_\phi \\ \mathbf{a}_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_r \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix} \quad (1\text{-}25b)$$

The transformations given above (rectangular to cylindrical and cylindrical to spherical) are particularly simple because they involve only two coordinates. In each transformation, there is a common coordinate for the transformation between rectangular and cylindrical coordinates, and for the transformation between cylindrical and spherical coordinates. All other transformations can be derived by successive transformation. The six basic transformations are summarized in Table 1-3.

The transformations for unit vectors may also be used for arbitrary vector components. For example, consider a vector **A** in rectangular coordinates:

$$\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$$

We introduce the following results from Table 1-3:

$$\mathbf{a}_x = \mathbf{a}_\rho \cos\phi - \mathbf{a}_\phi \sin\phi$$
$$\mathbf{a}_y = \mathbf{a}_\rho \sin\phi - \mathbf{a}_\phi \cos\phi$$

Substitute in **A** above to obtain

$$\mathbf{A} = \mathbf{a}_\rho (A_x \cos\phi + A_y \sin\phi) + \mathbf{a}_\phi(-A_x \sin\phi + A_y \cos\phi) + \mathbf{a}_z A_z$$
$$= \mathbf{a}_\rho A_\rho + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z$$

Thus

$$A_\rho = A_x \cos\phi + A_y \sin\phi$$
$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

or, in matrix form,

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad (1\text{-}26)$$

Note that the matrix for this transformation of components is identical to the corresponding matrix for transformation of unit vectors shown in Eq. (1-24a). Thus *the six matrices of Table 1-3 may be used either for unit vectors or for arbitrary vector components.*

Table 1-3. Transformation of Unit Vectors

	RECTANGULAR			CYLINDRICAL			SPHERICAL		
	a_x	a_y	a_z	a_ρ	a_φ	a_z	a_r	a_θ	a_φ
a_x	1	0	0	$\cos\varphi$	$-\sin\varphi$	0	$\sin\theta\cos\varphi$	$\cos\theta\cos\varphi$	$-\sin\varphi$
a_y	0	1	0	$\sin\varphi$	$\cos\varphi$	0	$\sin\theta\sin\varphi$	$\cos\theta\sin\varphi$	$\cos\varphi$
a_z	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
a_ρ	$\cos\varphi$	$\sin\varphi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
a_φ	$-\sin\varphi$	$\cos\varphi$	0	0	1	0	0	0	1
a_z	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
a_r	$\sin\theta\cos\varphi$	$\sin\theta\sin\varphi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
a_θ	$\cos\theta\cos\varphi$	$\cos\theta\sin\varphi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
a_φ	$-\sin\varphi$	$\cos\varphi$	0	0	1	0	0	0	1

[Note] The unit vectors ($\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$, $\mathbf{a}_\rho$, $\mathbf{a}_\varphi$, $\mathbf{a}_r$, $\mathbf{a}_\theta$) can be replaced by the vector components (A_x, A_y, A_z, A_ρ, A_φ, A_r, A_θ). The transformation is still valid.

Example 1-3

Apply successively the rectangular to cylindrical and cylindrical to spherical transformations to obtain the rectangular to spherical transformation.

Solution:

$$\begin{bmatrix} \mathbf{a}_r \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_\rho \\ \mathbf{a}_\phi \\ \mathbf{a}_z \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \end{bmatrix}$$

which agrees with the relationship in the left bottom box of Table 1-3.

Example 1-4

Let $\mathbf{F} = \mathbf{a}_r\, 5\cos\theta - \mathbf{a}_\theta\, 5\sin\theta$. Express **F** in rectangular coordinates.

Solution:

Use the transformation from spherical to rectangular coordinates in the right top box of Table 1-3 to obtain

$$= \mathbf{a}_x[5\cos\theta\ (\sin\theta\cos\phi) - 5\sin\theta\ (\cos\theta\cos\phi)]$$
$$+\ \mathbf{a}_y[5\cos\theta\ (\sin\theta\sin\phi) - 5\sin\theta\ (\cos\theta\sin\phi)]$$
$$+\ \mathbf{a}_z[5\cos\theta\ (\cos\theta) - 5\sin\theta\ (-\sin\theta)]$$
$$= \mathbf{a}_z 5\ (\cos^2\theta + \sin^2\theta) = \mathbf{a}_z 5$$

where we used, for example,

$$F_x = F_r\sin\theta\cos\phi + F_\theta\cos\theta\cos\phi + F_\phi(-\sin\phi)$$

The following relationship is often used:

$$\mathbf{a}_z = \mathbf{a}_r\cos\theta - \mathbf{a}_\phi\sin\theta$$

1.3.4 Integrals of Vector Functions

There are many types of integrals which occur in electromagnetics. In particular, the *line integral* $\int \mathbf{F}\cdot d\ell$ over a path and the *surface integral* $\iint \mathbf{F}\cdot ds$ over a surface occur quite frequently. First, we consider the line integral.

Line Integrals

$\int_C \mathbf{F}\cdot d\ell$ is called the **line integral** of vector **F** over path C. It is a scalar quantity. C may be open or closed. Both **F**, $d\ell$ may be expressed in rectangular, cylindrical, and spherical coordinates:

$$\mathbf{F} = \mathbf{a}_x F_x + \mathbf{a}_y F_y + \mathbf{a}_z F_z \qquad d\ell = \mathbf{a}_x dx + \mathbf{a}_y dy + \mathbf{a}_z dz$$
$$= \mathbf{a}_\rho F_\rho + \mathbf{a}_\phi F_\phi + \mathbf{a}_z F_z \qquad = \mathbf{a}_\rho d\rho + \mathbf{a}_\phi \rho\ d\phi + \mathbf{a}_z dz$$
$$= \mathbf{a}_r F_r + \mathbf{a}_{,} F_{,} + \mathbf{a}_\phi F_\phi \qquad = \mathbf{a}_r dr + \mathbf{a}_\theta r\ d\theta + \mathbf{a}_\phi r\sin\theta\ d\phi$$

Taking the dot product of **F** and $d\ell$:

$$\int_C \mathbf{F}\cdot d\ell = \int_C F_x dx + F_y dy + F_z dz \qquad \text{(rectangular coordinates)} \qquad (1\text{-}27a)$$

$$= \int_C F_\rho d\rho + F_\phi d\phi + F_z dz \qquad \text{(cylindrical coordinates)} \qquad (1\text{-}27b)$$

$$= \int_C F_r dr + F_\theta r\, d\theta + F_\phi\ r \sin\theta\, d\phi \quad \text{(spherical coordinates)} \tag{1-27c}$$

In order to carry out the integration it is necessary to introduce the constraints imposed by the path C of integration.

Example 1-5

Let $\mathbf{F} = \mathbf{a}_x x^2 + \mathbf{a}_y xy + \mathbf{a}_z 2yz$. Calculate the line integral $\int_C \mathbf{F} \cdot d\ell$ where C is a straight line between the two points (0, 0, 0) and (2, 3, 4) in rectangular coordinates (Figure 1-13). Calculate also $\int_{C_a} \mathbf{F} \cdot d\ell$ where $C_a = C_x + C_y + C_z$ is the integration path shown in Figure 1-13.

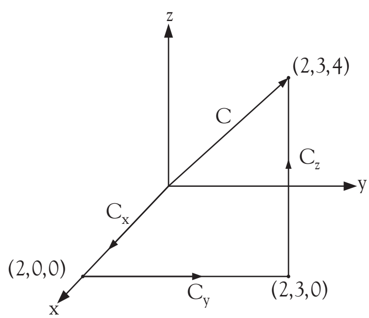


Figure 1-13. ***Line integrals.***

Solution:

First we find the equation of the straight line for path C:

$$\frac{x-0}{2-0} = \frac{y-0}{3-0} = \frac{z-0}{4-0} \quad \text{or} \quad \frac{x}{2} = \frac{y}{3} = \frac{z}{4} \quad \text{(constraints on variables)}$$

Then we differentiate to obtain

$$\frac{dx}{2} = \frac{dy}{3} = \frac{dz}{4} \quad \text{(constraints on differentials)}$$

Using the above equations of constraint we can obtain an integral in one variable. We choose x as the variable.

$$\int_C \mathbf{F}\cdot d\ell = \int_C F_x dx + F_y dy + F_z dz = \int_C x^2 dx + xydy + 2yzdz$$
$$= \int x^2 dx + x\left(\frac{3x}{2}\right)dy + 2\left(\frac{3x}{2}\right)\left(\frac{4x}{2}\right)dz \text{ (using constraints on variables)}$$
$$= \int x^2 dx + \frac{3}{2}x^2 \cdot \frac{3}{2}dx + 6x^2 \cdot 2dx \text{ (using constraints on variables)}$$
$$= \int_0^2 \frac{61}{4}x^2\ dx = \frac{122}{3} \text{ or } 40.67$$

Now calculate the integral over $C_a = C_x + C_y + C_z$ which is a different path (Figure 1-13) between the same two points (0, 0, 0) and (2, 3, 4):

$$\int_{C_a} \mathbf{F}\cdot d\ell = \int_{C_x} \mathbf{F}\Big|_{\substack{y=0\\z=0}} \cdot a_x dx + \int_{C_y} \mathbf{F}\Big|_{\substack{x=0\\z=0}} \cdot a_y dx + \int_{C_z} \mathbf{F}\Big|_{\substack{x=0\\y=0}} \cdot a_z dz$$
$$= \int_0^2 F_x(y=0,\ z=0)dx + \int_0^3 F_y(x=2,\ z=0)dy + \int_0^4 F_z(x=2,\ y=3)dz$$
$$= \int_0^2 x^2 dx + \int_0^3 2y\ dy + \int_0^4 6z\ dz = 59\tfrac{2}{3} \text{ or } 59.67$$

Note that the two results differ. In general, the line integral between two points depends on the path taken. In the integrals above note that the direction of path C is taken into account by the limits of integration. Upper and lower limits correspond to the head and tail, respectively. If the direction of C is reversed, then the upper and lower limits are interchanged.

Independence of Path

There is a special class of vector fields **F** which are **conservative,** i.e.,

$$\oint_C \mathbf{F}\cdot d\ell = 0 \text{ for all colsed paths C.} \quad (1\text{-}28a)$$

The circle on the integral sign denotes a *closed* path. The line integral on the left-hand side of Eq. (1-28a) is called the **circulation** of the vector **F** along the closed path C. For conservative fields, *the line integral is independent of path.*

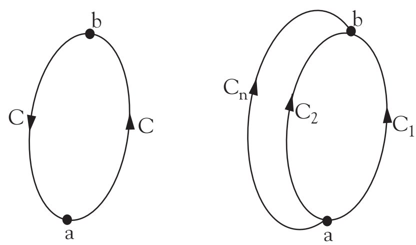
To see this, consider the line integral around path C of Figure 1-14(a) where $C = C_1 - C_2$ (Figure 1-14(b)). **F** is a conservative field and therefore

$$\oint_C \mathbf{F}\cdot d\ell = \int_{C_1} \mathbf{F}\cdot d\ell - \int_{C2} \mathbf{F}\cdot d\ell = 0$$

then

$$\int_{C_1} \mathbf{F} \cdot d\ell - \int_{C2} \mathbf{F} \cdot d\ell = \int_{C_n} \mathbf{F} \cdot d\ell \quad (1\text{-}28b)$$

We may conclude that the line integral between a and b (or between any two points) is *independent of path* for conservative fields. For this special class of vector fields, we may evaluate the line integral by choosing any convenient path. See Problem 1-27 for an example of the conservative vector field.



(a) A closed path. (b) Paths from a to b.

Figure 1-14. ***Independence of path.***

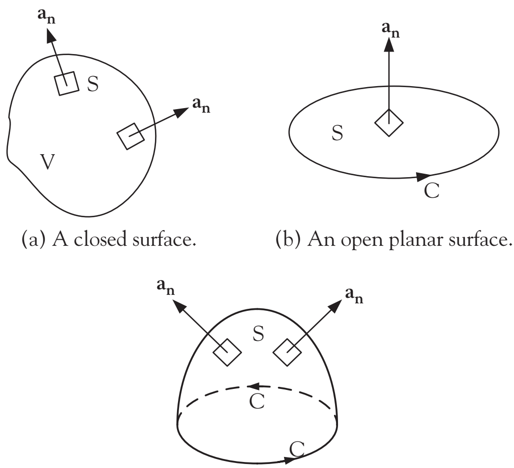
Surface Integrals: Flux

The integral $\iint_S \mathbf{F} \cdot d\mathbf{s}$ occurs often in electromagnetics. It is called the **flux** of the vector **F** through the surface S. ds is defined as $\mathbf{a}_n$ ds, where $\mathbf{a}_n$ is the unit vector normal to the surface and ds is the element of differential surface area. Thus

$$\iint_S \mathbf{F} \cdot d\mathbf{s} = \iint_S \mathbf{F} \cdot \mathbf{a}_n \ d\mathbf{s} = \iint_S F_n \ d\mathbf{s} \quad (1\text{-}29)$$

where F_n is the component of **F** normal to the surface S.

The surface S may be closed or open. For closed surfaces, such as that of Figure 1-15(a), the normal is defined as the *outward* normal. Figure 1-15(a) shows how the vector $\mathbf{a}_n$ varies over the surface. If the surface is open, then the normal direction is not unique and a direction must be chosen as part of the problem. We usually define the normal by a *right-hand rule* relationship with a path C around the periphery. Figure 1-15(b) shows an open planar surface lying in the xy plane with perimeter C. In this case the normal is constant ($\mathbf{a}_n$ = $\mathbf{a}_z$) with a right-hand rule relationship with the periphery C when C follows the counter clockwise direction. Now we perturb the surface of 1-15(b) by pushing on its central portion. The result is a hat-shaped open surface (Figure 1-15(c)). $\mathbf{a}_n$ varies over the surface and the right-hand rule relationship is maintained.



(a) A closed surface. (b) An open planar surface.

(c) An open non-planar surface.

Figure 1-15. ***Surface integrals.***

Example 1-6

Let $\mathbf{F} = \mathbf{a}_r \frac{K_1}{r^2}$ (a) Find the flux of the vector **F** through the entire surface of the sphere of radius r_0, centered at origin. (b) Find the flux of **F** through a surface of area S_0 and arbitrary shape on the surface of the sphere of radius r_0.

Solution:

(a) On the surface of the sphere of radius r_0, r = r_0. It is a closed surface and d**s** = $\mathbf{a}_n$ ds = $\mathbf{a}_r$ $r^2 \sin\theta \, d\theta d\varphi$. So

$$\oiint_S \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi}\int_0^{\pi} F_r \, r^2 \sin\theta \, d\theta \, d\phi = \int_0^{2\pi}\int_0^{\pi} K_1 \sin\theta \, d\theta \, d\phi = 4\pi K_1$$

Note that the circle on the integral signs denotes a *closed* surface.

(b) The surface S is an open surface.

$$\iint_{S_0} \mathbf{F} \cdot d\mathbf{s} = \iint_{S_0} F_r \, ds = \iint_{S_0} F_r(r = r_0) r_0^2 \sin\theta \, d\theta \, d\phi$$

$$= F_r(r = r_0) \iint_{S_0} ds \text{ [because } F_r \text{ is a constant on } S_0 (r{=}r_0)\text{, i.e., independent of } \theta \text{ ang } \phi]$$

$$= \frac{K_1}{r_0^2} S_0$$

Example 1-7

Let $\mathbf{F} = \mathbf{a}_\rho\, \rho \sin^2\varphi + \mathbf{a}_\varphi\, \rho \cos^2\varphi + \mathbf{a}_z\, z$

Find the flux of the vector **F** through the (closed) surface of a cylinder of length 2ℓ meters and radius $\rho = \rho_0$ meters, centered at the origin (see Figure 1-10(a)).

Solution:

The closed surface consists of three open surfaces: the side (cylindrical) S_1, the top S_2 and the bottom surface S_3.

On S_1, $d\mathbf{s} = \mathbf{a}_\rho\, \rho\, d\varphi dz$; $\rho = \rho_o$, $0 \le \varphi \le 2\pi$, $-\ell \le z \le \ell$.

On S_2, $d\mathbf{s} = \mathbf{a}_z\, \rho\, d\rho d\varphi$; $z = \ell$, $0 \le \rho \le \rho_o$, $0 \le \varphi \le 2\pi$.

On S_3, $d\mathbf{s} = -\mathbf{a}_z\, \rho\, d\rho d\varphi$; $z = -\ell$, $0 \le \rho \le \rho_o$, $0 \le \varphi \le 2\pi$.

Therefore,

$$\iint_{S} \mathbf{F}\cdot d\mathbf{s} = \iint_{\text{SIDE}} F_\rho\ \rho_0\ d\phi\ dz + \iint_{\text{TOP}} F_z\ \rho\ d\rho\ d\phi - \iint_{\text{BOTTOM}} F_z\ \rho\ d\rho\ d\phi$$

$$= \int_{-1}^{1}\int_{0}^{2\pi} \rho_0\ \sin^2\phi\ \rho_0\ d\phi\ dz + \int_{0}^{2\pi}\int_{0}^{0} 1\ \rho\ d\rho\ d\phi + \int_{0}^{2\pi}\int_{0}^{0} (-1)\ (-\rho\ d\rho\ d\phi$$

$$= \rho_0^2\,[z]_{-1}^{1}\left[\frac{1}{2}(\phi - \frac{1}{2}\sin 2\phi)\right]_0^{2\pi} + 2l\int_0^{2\pi} d\phi \int_0^{\rho_0} \rho\ d\rho$$

1.4 VECTOR CALCULUS

1.4.1 Definitions

In this section we define three new derivative quantities, namely, the gradient of a scalar function and the divergence and curl of a vector function. They describe how the scalar and vector fields vary as a function of position. First we define the vector differential operator Δ read **del** or **nabla:**

$$\nabla = \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z} \quad (1\text{-}30)$$

Δ operates on a vector or scalar. The partial derivatives operate on the vector or scalar operand in a manner determined by the distributive property.

$$\text{gard } V=\nabla V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z} \tag{1-31}$$

The divergence of a vector function **A** (x,y,z) is defined as

$$\text{div } \mathbf{A}=\nabla \cdot \mathbf{A} = \mathbf{a}_x \frac{\partial A_x}{\partial x} + \mathbf{a}_y \frac{\partial A_y}{\partial y} + \mathbf{a}_z \frac{\partial A_z}{\partial z} \tag{1-32}$$

The curl of a vector function **A** is defined as

$$\text{curl } \mathbf{A}=\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times \mathbf{A} = \mathbf{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \tag{1-33}$$

The partial derivatives of the second row of the determinant operate on the quantities in the third row. Note the cycle xyz in the final equation for the curl. Note also that DV, Δ × **A** are vectors and that Δ · **A** is a scalar.

The grad, div, and curl have been succinctly defined in this section in rectangular coordinates. The corresponding forms in cylindrical and spherical coordinates as well are given in Table 1-4. The operations described can readily be performed. In succeeding sections we introduce physical interpretations of grad, div, curl, and their forms in other coordinate systems.

1.4.2 Gradient

The **gradient** of a scalar function has a very simple physical description. It is a vector whose direction is that of *maximum rate of change* of the scalar with distance. Its magnitude is that of maximum rate of change. Consider the temperature T(x,y,z) in a room.

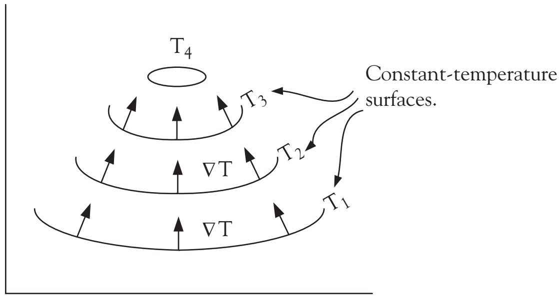


Figure 1-16. ***Temperature (T) and temperature gradient (DT) in a room.***

Figure 1-16 shows the constant temperature surfaces, called the **level surfaces.** The temperature generally increases with height; there is a "hot spot" near the ceiling caused by a light bulb. DT is normal to the constant temperature surfaces as shown. To understand this, consider a point on a constant temperature surface. Construct a tangent plane at that point. The rate of change of T with distance is zero for any direction lying in the tangent plane. It is non-zero only for directions with a normal component. But for any direction with both normal and tangential components, the rate of change is increased by dropping the tangential component; therefore the maximum rate of change occurs along the normal direction. This means that *the gradient (DT) is perpendicular to the level surfaces where T = constant.* Meanwhile the backpackers among us are rather amused by all this; they knew all along that the gradient, i.e. the steepest path up the mountain, is normal to the contour lines.

Table 1-4. Vector Derivatives

RECTANGULAR COORDINATES (x, y, z)

$$\nabla V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

CYLINDRICAL COORDINATES (θ, θ, z)

$$\nabla V = \mathbf{a}_\rho \frac{\partial V}{\partial \rho} + \mathbf{a}_\phi \frac{1}{\rho}\frac{\partial V}{\partial \phi} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{a}_\rho \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{a}_\phi \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \mathbf{a}_z \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right)$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL COORDINATES (r, θ, φ)

$$\nabla V = \mathbf{a}_r \frac{\partial V}{\partial r} + \mathbf{a}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta} + \mathbf{a}_\phi \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \mathbf{a}_r \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \phi}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right)$$

$$+ \mathbf{a}_\theta \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi)\right) + \mathbf{a}_\phi \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \theta}\right)$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta\partial\theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

It is easy to see that the above description corresponds to our previous definition of DT:

$$\nabla T = \mathbf{a}_x \frac{\partial T}{\partial x} + \mathbf{a}_y \frac{\partial T}{\partial y} + \mathbf{a}_z \frac{\partial T}{\partial z} \quad (1\text{-}34)$$

Since we are free to choose our coordinate system, choose $\mathbf{a}_z$ normal to the tangent plane to the constant temperature surfaces at (x,y,z). Then $\mathbf{a}_x$, $\mathbf{a}_y$ lie in the tangent plane. $\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}$ are zero, since $\mathbf{a}_x$, $\mathbf{a}_y$ lie in the tangent plane. DT is in the $\mathbf{a}_z$ direction normal to the tangent plane. Its magnitude is $\frac{\partial T}{\partial z}$, the rate of change in a direction normal to the tangent plane.

These concepts can be related to the so-called *directional derivative,* the rate of change in an arbitrary direction. By a chain rule relationship,

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz \tag{1-35}$$

$$= \left(\mathbf{a}_x \frac{\partial T}{\partial x} + \mathbf{a}_y \frac{\partial T}{\partial y} + \mathbf{a}_z \frac{\partial T}{\partial z} \right) \cdot (\mathbf{a}_x dx + \mathbf{a}_y dy + \mathbf{a}_z dz)$$

$$= \nabla T \cdot d\ell \tag{1-36}$$

The directional derivative is given by

$$\frac{dT}{dl} = \frac{\partial T}{\partial x}\frac{dx}{dl} + \frac{\partial T}{\partial y}\frac{dy}{dl} + \frac{\partial T}{\partial z}\frac{dz}{dl} \tag{1-37}$$

$$= \nabla T \cdot (\frac{d\ell}{dl}) = \nabla T \cdot (\mathbf{a}_1)$$

The gradient represents the maximum rate of change. The dot product with any arbitrary direction $\mathbf{a}_\ell$ yields the directional derivative or the rate of change in that particular direction. The gradient will be used in relating the electric field to the electric potential in Chapter 2.

Example 1-8

Calculate the gradient of the following scalar functions:

(a) $V = \dfrac{K}{\sqrt{x^2 + y^2 + z^2}} = \dfrac{K}{r}$. (b) $V = \dfrac{K\cos\theta \sin\phi}{r^2}$.

Solution:

(a) In rectangular coordinates,

$$\nabla V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$= \frac{-K}{\left[x^2 + y^2 + z^2\right]^{3/2}} \left(\mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z \right)$$

$$= \frac{-K}{r^3}\mathbf{r} = \mathbf{a}_r \left(\frac{-K}{r^2} \right)$$

where we have used $\mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z = \mathbf{r} = \mathbf{a}_r r$. $\mathbf{r}$ may be called the **position vector.**

In spherical coordinates, $\nabla V = \mathbf{a}_r \frac{\partial V}{\partial r} + \mathbf{a}_r\left(\frac{-K}{r^2}\right)$. The level surfaces on which V = constant or $\frac{K}{r} = C$ are given by $r = \frac{K}{C}$ constant, which is a family of spherical surfaces. It is clear from the above result that $\nabla V = -\mathbf{a}_r \frac{K}{r^2}$ is perpendicular to the level surfaces.

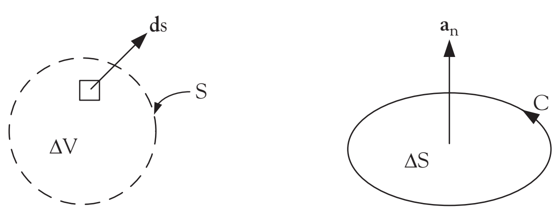
(b)
$$\nabla V = -\mathbf{a}_r \frac{\partial V}{\partial r} + \mathbf{a}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta} + \mathbf{a}_\phi \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$$
$$= \mathbf{a}_r\left(\frac{-2K\cos\theta\sin\phi}{r^3}\right) + \mathbf{a}_\theta\left(\frac{-K\sin\theta\sin\phi}{r^3}\right) + \mathbf{a}_\phi\left(\frac{K\cos\theta\sin\phi}{r^3}\right)$$

1.4.3 Divergence

The **divergence** of a vector function **A** is a measure of the net outward flux or flow at a given point. It is also a measure of how much a vector spreads out or diverges from a point. The divergence may be defined as the *net outward flux per unit volume.* To find the divergence at some given point,we construct a small volume Δv around the point and calculate the vector flux flowing out of the volume divided by Δv as Δv goes to zero:

$$\nabla \cdot \mathbf{A} = \lim_{\nabla v \to 0} \frac{1}{\nabla v} \iint_S \mathbf{A} \cdot d\mathbf{s} \qquad (1\text{-}38)$$

where the closed surface S bounds or encloses the small volume Δv (see Figure 1-17(a)). Here we take the limit Δv → 0 because the divergence is defined at a *point.* The definition above is identical to (1-32) of Section 1.4.1. The result is independent of the shape of Δv.



(a) The definition of divergence. (b) The definition of curl.

Figure 1-17. ***The definition of divergence and curl.***

Example 1-9. A Vector Field that has Divergence
Calculate the divergence of the following vector field **F** and explain why **F** has divergence using the definition (1-38).

$$\mathbf{F}(x, y, z) = \mathbf{a}_x 2x$$

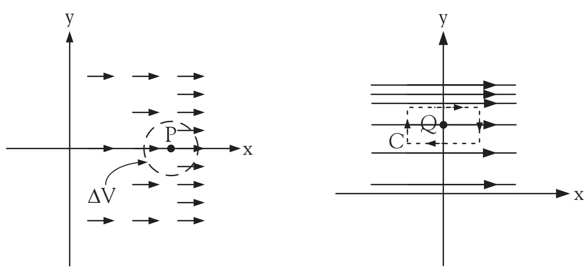
Solution:

$$F_x = 2x, F_y = 0, F_z = 0,$$

Using Eq. (1-32),

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \frac{\partial}{\partial x}(2x) = 2$$

The vector **F** has only the x component and its magnitude increases as x increases for x > 0. The field lines of **F** can be represented by a collection of arrows shown in Figure 1-18(a) (more arrows mean stronger field). When you consider a small spherical volume Δv (shown as dotted surface) around some point P and calculate the net flux flowing *out* of that volume, you will find that it is positive because the outgoing flux is greater than the incoming flux, i.e., the vector **F** has positive divergence at that point. We can also find the same result by using a small cube rather than a sphere since the divergence is independent of the shape of Δv.



(a) A vector field that has divergence. (b) A vector field that has curl.

Figure 1-18. ***A vector field that has divergence and curl.***

1.4.4 Curl

The **curl** of a vector function is a measure of the circulation about a point. It is also a measure of how much a vector curls around or rotates about a point. The curl may be defined as the *circulation per unit area.* Since the curl is a vector, we need to define both the direction and magnitude. To find the component of curl in a particular direction $\mathbf{a_n}$, we construct a surface Δs, perpendicular to $\mathbf{a_n}$ and bounded by contour C (see Figure 1-17(b)) and find the circulation per unit area as Δs goes to zero:

$$(\nabla \times \mathbf{A})_n = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \oint_C \mathbf{A} \cdot d\ell \quad (1\text{-}39)$$

where the closed contour C and the unit vector $\mathbf{a}_n$ are related by a right-hand rule. (see Figure 1-17(b)). If $\mathbf{a_n}$ is successively chosen in three orthogonal directions, we obtain the three components of the vector Δ × **A.** The definition above is identical to (1-33) of Section 1.4.1. The result is independent of the shape of Δs.

Example 1-10. A Vector Field that has Curl
Calculate the curl of the following vector field **G** and explain why **G** has curl using the definition (1-39).

$$\mathbf{G}(x, y, z) = \mathbf{a}_x 2y$$

Solution:

$$G_x = 2y,\ G_y = 0,\ G_z = 0$$

Using Eq. (1-33),

$$\nabla \times \mathbf{G} = \mathbf{a}_y \frac{\partial G_x}{\partial z} - \mathbf{a}_z \frac{\partial G_x}{\partial y} = -\mathbf{a}_z \frac{\partial}{\partial y}(2y) = \mathbf{a}_z 2$$

The vector **G** has only the x component and its magnitude increases as y increases for y > 0. The field lines are shown in Figure 1-18(b). When you consider a small closed rectangular contour C in the xy plane (shown as dotted line) around some point Q on the y axis and calculate the circulation clockwise, you will find a nonzero (positive) quantity because the upper portion of $\int \mathbf{G} \cdot d\ell$ is positive and the lower portion is negative, but of smaller magnitude. Using the right-hand rule, we see that **G** has a curl in the negative z direction (into the paper). You will find the same result with the circular loop because the curl is independent of the shape of the contour. Note that **G** has no divergence and **F** in Example 1-9 has no curl.

The basic concepts of divergence and curl are subtle and complex. Convenient analogies from our past experience are lacking. Therefore we should recognize that we have a continuing task. Our understanding of divergence and curl will grow as we proceed in our study of electromagnetics.

1.4.5 The Divergence Theorem and Stokes' Theorem – Solenoidal and Conservative Fields

The two important integral theorems — the divergence theorem and Stokes' theorem — can be obtained by applying the physical or geometrical definitions of divergence and curl, (1-38) and (1-39), to the real (not infinitesimally small) volume and surface, respectively.

The **divergence theorem** is stated as follows:

$$\iiint_V \nabla \cdot \mathbf{A}\, dv = \oiint_S \mathbf{A} \cdot d\mathbf{s} \quad (1\text{-}40)$$

where the closed surface S bounds the volume V. The theorem is valid for any vector **A,** and for any volume V and its boundary surface S. It allows us to replace a volume integral with a surface integral, or vice versa. Note that d**s** points in the direction *normal* to the surface S *outward* from the volume V.

If a vector **A** is *divergenceless,* i.e., if $\Delta \cdot \mathbf{A}$ is zero everywhere, then $\oiint \mathbf{A} \cdot d\mathbf{s} = 0$ for any closed surface S. Such a vector is also called **solenoidal.** For a solenoidal field, $\iint_S \mathbf{A} \cdot d\mathbf{s}$ is independent of the surface, as long as the surface S is bounded by the same closed contour C.

Stokes' theorem is stated as follows:

$$\iint_S \nabla \times \mathbf{A} \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\ell \quad (1\text{-}41)$$

where the closed path C bounds the open surface S and dℓ along C and d**s** = $\mathbf{a}_n$ ds are related by a *right-hand rule* as in Figure 1-15(b), (c) and Figure 1-17(b).

If **A** is *curl free,* i.e., if $\Delta \times \mathbf{A}$ is zero everywhere, then $\oint_C \mathbf{A} \cdot d\ell = 0$ for any closed path C and therefore **A** is a *conservative* field.[*] Furthermore the line integral of **A** between two points is independent of path (see Section 1.3.4). The formal proof of (1-40) and (1-41) can be found in Cheng (1989).[+]

[*] Conservative and curlfree are identical characterizations.

[+] D. K. Cheng, *Field and Wave Electromagnetics,* Addison-Wesley, 1989, 2nd Ed., Chapter 2.

Example 1-11. Illustration of the Divergence Theorem
Consider the vector field

$$\mathbf{F} = \mathbf{a}_z z^2$$

(a) Find the surface integral, $\oiint \mathbf{F} \cdot d\mathbf{s}$, i.e., the flux through the closed surfaces of the half-sphere of radius a as shown in Figure 1-19.
(b) Verify the divergence theorem, by calculating $\iiint_V \nabla \cdot \mathrm{F}\, dv$ for the volume enclosed.

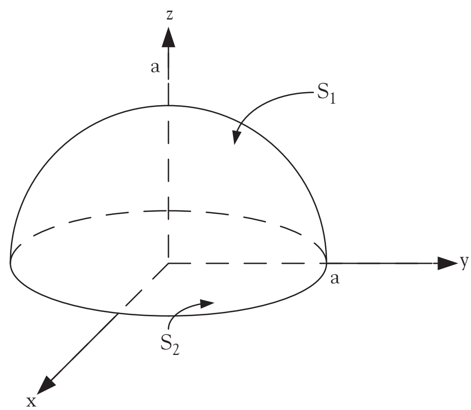


Figure 1-19. ***A hemispherical volume.***

Solution:

(a) We first recognize that there are two open surfaces: the upper half-spherical surface S_1 and the lower base surface S_2.

For the surface S_1, r = a, $0 \le \theta \le \pi/2$, $0 \le \varphi \le 2\pi$

$$d\mathbf{s} = \mathbf{a}_r r^2 \sin\theta \; d\theta \; d\phi = \mathbf{a}_r a^2 \sin\theta \; d\theta \; d\phi.$$

Since **F** is given in Cartesian coordinates, we need to transform it into spherical coordinates:

$$\begin{aligned} \mathbf{F} = a_z z^2 &= (\mathbf{a}_r \cos\theta - \mathbf{a}_\theta \sin\theta)(r\cos\theta)^2 \\ &= \mathbf{a}_r r^2 \cos^3\theta - \mathbf{a}_\theta r^2 \sin\theta \cos^2\theta = \mathbf{a}_r F_r + \mathbf{a}_\theta F_\theta \end{aligned}$$

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{s} = \iint_{S_1} (\mathbf{a}_r F_r + \mathbf{a}_\theta F_\theta) \cdot \mathbf{a}_r ds = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} F_r \Big|_{r=a} a^2 \sin\theta \; d\theta \; d\phi$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} a^2 \cos^3\theta \; a^2 \sin\theta \; d\theta \; d\phi = 2\pi \cdot a^4 \left[-\frac{1}{4} \cos^4\theta \right]_0^{\frac{\pi}{2}} = \frac{1}{2} \pi a^4$$

For the surface S_2, z = 0. Thus $\mathbf{F} = \mathbf{a}_z z^2 = 0$.

$$\iint_{S_2} \mathrm{F} \cdot d\mathbf{s} = 0$$

So,

$$\oint\!\!\!\!\iint_{S} \mathrm{F} \cdot d\mathbf{s} = \iint_{S_1} \mathrm{F} \cdot d\mathbf{s} + \iint_{S_2} \mathrm{F} \cdot d\mathbf{s} = \frac{1}{2}\pi a^4$$

(b) $\nabla \cdot \mathbf{F} = \dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z} = 0 + 0 + \dfrac{\partial}{\partial z}(z^2) = 2z$

For the hemispherical volume, $dv = r^2 \sin\theta dr d\theta d\varphi$, $0 \le r \le a$, $0 \le \theta \le \dfrac{\pi}{2}$, $0 \le \varphi \le 2\pi$.

$$\iiint_V \nabla \cdot \mathbf{F}\, dv = \iiint_V 2z\, dv = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^a 2(r\cos\theta)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= 2 \cdot 2\pi \left[-\frac{1}{2}\cos^2\theta\right]_0^{\frac{\pi}{2}} \left[\frac{1}{4} r^4\right]_0^a = 4\pi \cdot \frac{1}{2}\frac{a^4}{4} = \frac{1}{2}\pi a^4$$

Therefore, $\iiint_V \nabla \cdot \mathbf{F}\, dv = \oint\!\!\!\!\iint_S \mathbf{F} \cdot d\mathbf{s}$; the divergence theorem is verified.

Example 1-12 Illustration of Stokes' Theorem

Consider the vector field

$$\mathbf{F} = \mathbf{a}_x y - \mathbf{a}_y x$$

(a) Evaluate the circulation $\oint_C \mathbf{F} \cdot d\ell$ along a closed contour of a half-circle of radius a as shown in Figure 1-20.

(b) Verify Stokes' theorem by calculating $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{s}$ for the surface bounded by the contour.

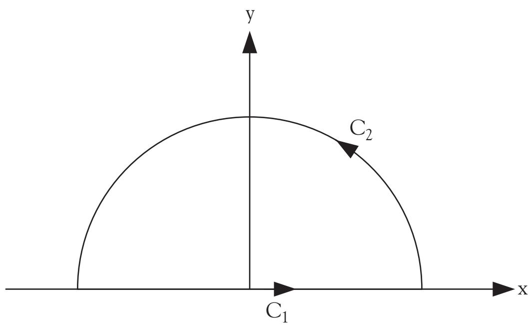


Figure 1-20. *A semi-circular contour.*

Solution:

(a) Define the straight line path C_1 and the half-circular path C_2.

Along C_1, $d\ell = \mathbf{a}_x\, dx$, $y = 0$, $-a \le x \le a$

$$\int_{C_1} \mathbf{F}\cdot d\ell = \int_{-a}^{a} \{\mathbf{a}_x 0 - \mathbf{a}_y x\}\cdot \mathbf{a}_x dx = 0$$

Along C_2, $d\ell = \mathbf{a,}\ ad\varphi$, $\rho = a$, $0 \le \varphi \le \pi$.

We also need to express **F** in cylindrical coordinates:

$$\mathbf{F} = \mathbf{a}_x y - \mathbf{a}_y x = \mathbf{a}_x \rho\sin\phi - \mathbf{a}_y \rho\cos\phi$$
$$= \rho(\mathbf{a}_x \sin\phi - \mathbf{a}_y \cos\phi) = -\mathbf{a}_\phi \rho$$

Thus

$$\int_{C_2} \mathbf{F}\cdot d\ell = \int_0^{\pi} \{-\mathbf{a}_\phi \rho\}_{\rho=a}\cdot \mathbf{a}_\phi a\ d\phi = -a^2[\phi]_0^{\pi} = -\pi a^2$$

$$\oint_C \mathbf{F}\cdot d\ell = \int_{C_1} F\cdot d\ell + \int_{C_2} F\cdot d\ell = 0 - \pi a^2 = -\pi a^2$$

(b)

$$\nabla\times\mathbf{F} = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = a_z\left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right] = -a_z 2$$

For the surface S bounded by C, $d\mathbf{s} = \mathbf{a}_z\, ds = \mathbf{a}_z\, \rho\, d\rho d\varphi$, $0 \le \rho \le a$, $0 \le \varphi \le \pi$.

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{ds} = \int_0^{\pi}\int_0^{a} \{-\mathbf{a}_z 2\} \cdot \mathbf{a}_z \rho \; d\rho d\phi = -2\pi \frac{a^2}{2} = -\pi a^2$$

Therefore, $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{ds} = \oint_C \mathbf{F} \cdot d\ell$; Stokes' theorem is verified.

Example 1-13. The Electric Field of a Point Charge.

Consider the field of Figure 1-21

$$\mathbf{E} = \mathbf{a}_r \frac{K}{r^2}$$

where K is a constant.

(a) Find $\Delta \cdot \mathbf{E}$ everywhere. What happens to $\Delta \cdot \mathbf{E}$ at the origin (r = 0)?

(b) Find $\Delta \times \mathbf{E}$ everywhere.

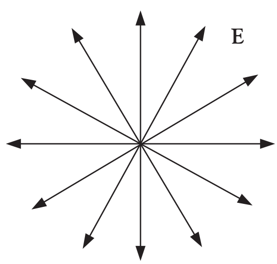


Figure 1-21. ***The electric field of a point charge.***

Solution:

(a) Consider $\Delta \cdot \mathbf{E}$ in spherical coordinates as given in Table 1-4.

$$\mathbf{E} = \mathbf{a}_r \frac{K}{r^2}, \; E_\theta = E_\phi = 0$$

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r\frac{K}{r^2}) = \frac{1}{r^2}\frac{\partial}{\partial r}(K) = 0 \; (r \neq 0)$$

However, at r = 0, $\nabla \cdot \mathbf{E} = \dfrac{\frac{\partial}{\partial r}(K)}{r^2}$ is not necessarily zero. To examine the behavior of $\Delta \cdot \mathbf{E}$ at the origin, we use the divergence theorem.

Construct a small spherical volume V of radius r about the origin and consider the following integral as r → 0:

$$\iiint_V \nabla\cdot\mathbf{E}\,dv = '\iint_{S(r\to 0)} \mathbf{E}\cdot d\mathbf{s} = \int_0^{2}\int_0 \frac{K}{r^2}\mathbf{a}_r\cdot\mathbf{a}_r\; r^2\sin\theta\; d\theta\; d\phi = \int_0^{2}\int_0 K\sin\theta\; d\theta\; d\phi = 4\pi K$$

Since there is no contribution to $\iiint_V \nabla\cdot\mathbf{E}\,dv$ for r ≠ 0, the divergence must be infinite at the origin. Thus Δ · **E** is zero when r ≠ 0 and infinite at r = 0.

(b)To find Δ × **E,** refer to Table 1-4. Since E_r is independent of θ, φ,

$$\nabla\times\mathbf{E} = \mathbf{a}_\theta\frac{1}{r\,\sin\theta}\frac{\partial E_r}{\partial\phi} - \mathbf{a}_\phi\frac{1}{r}\frac{\partial E_r}{\partial\theta} = 0 \quad (r\neq 0,\ \sin\theta\neq 0)$$

To examine the behavior at r = 0, sinθ = 0, we apply Stokes' theorem. Construct a hemisphere of radius r centered about the origin, 0 ≤ θ ≤ π/2. S is the surface of the hemisphere and C is the periphery.

$$\iint_S \nabla\times\mathbf{E}\cdot d\mathbf{s} = \oint_C \mathbf{E}\cdot d\ell = \int_0^{2} \mathbf{E}\cdot\mathbf{a}_\phi r\, d\phi = E_\phi 2\pi r = 0$$

because E_φ = 0. Thus there is no contribution to $\iint_S \nabla\times\mathbf{E}\cdot d\mathbf{s}$ from the axis, for finite or zero r, and therefore Δ × **E** may be considered to be zero everywhere.

The vector field **E** above represents the electric field of a point charge as will be shown in Chapter 2. Note that the electric field is *conservative* and has divergence.

Example 1-14. The Magnetic Field of a Filament of Current

Consider the field **H** of Figure 1-22.

$$\mathbf{H} = \mathbf{a}_\phi\frac{K}{\rho}$$

where K is a constant.

(a) Find Δ · **H** everywhere.

(b) Find Δ × **H** everywhere. What happens to Δ × **E** at points along the z-axis (ρ = 0)?

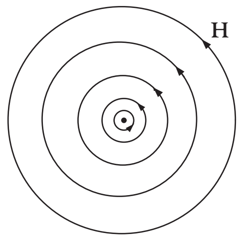


Figure 1-22. *The magnetic field of a current filament.*

Solution:

(a) Consider Δ · **H** in cylindrical coordinates as given in Table 1-4.

$$H_\rho = 0,\ H_\phi = \frac{K}{\rho},\ H_z = 0$$

$$\nabla \cdot \mathbf{H} = \frac{1}{\rho}\frac{\partial H_\phi}{\partial \phi} = \frac{1}{\rho}\frac{\partial}{\partial \phi}\left(\frac{K}{\rho}\right) = 0\ (\rho \neq 0)$$

To determine the behavior of Δ · **H** along the z axis (ρ = 0), we use the divergence theorem again. Construct a cylinder of radius ρ, length ℓ, whose axis is the z axis:

$$\iiint_V \nabla \cdot \mathbf{H}\ dv =' \iint_{S(` \to 0)} \mathbf{H} \cdot d\mathbf{s} = \iint_{SIDE} H_\rho ds + \iint_{TOP,\ BOTTOM} H_z ds = 0$$

since $H_\rho = H_z = 0$. Thus there is no contribution to $\iiint_V \nabla \cdot \mathbf{H}\ dv$ from the axis (ρ = 0) and Δ · **H** is zero everywhere.

(b) To find Δ × **H,** refer to Table 1-4

$$\cdot\ \nabla \times \mathbf{H} = \mathbf{a}_\rho \frac{\partial H_\phi}{\partial z} + \mathbf{a}_z \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi) = \mathbf{a}_z \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{K}{\rho}\right) = \mathbf{a}_z \frac{1}{\rho}\frac{\partial}{\partial \rho}(K) = 0\ (\rho \neq 0)$$

To determine the behavior of Δ × **H** along the z axis (ρ = 0), we apply Stokes' theorem. Construct a small circular disk (of radius ρ → 0) centered on and perpendicular to the z axis. S is the surface of the disk and C is the circumference. Then

$$\iint_S \nabla \times \mathbf{H} \cdot d\mathbf{s} = \oint_{C(\rho \to 0)} \mathbf{H} \cdot d\ell = \lim_{\rho \to 0} H_\phi 2\pi\rho = \frac{K}{\rho} 2\pi\rho = 2\pi K$$

On the disk, there is no contribution to $\iint_S \nabla \times \mathbf{H} \cdot \mathbf{ds}$ for $\rho \neq 0$. Therefore $\Delta \times \mathbf{H}$ must be infinite on the z axis ($\rho = 0$). Thus $\Delta \times \mathbf{H}$ is zero when $\rho \neq 0$ and infinite at $\rho = 0$. The vector field **H** above represents the magnetic field of a long current-carrying filament along the z axis as will be shown in Chapter 6. Note that the magnetic field is *solenoidal* and has curl.

1.4.6 Vector Identities

A set of vector identities is presented in Table 1-5. These identities can be verified by a process which is straightforward, although laborious in some cases. We merely expand both sides of the identity in rectangular coordinates and show that they are identical. In this section, we will go through the process for certain commonly used vector identities.

First, consider the gradient of a product of two scalar functions:

$$\nabla(VW) = \mathbf{a}_x \frac{\partial}{\partial x}(VW) + \mathbf{a}_y \frac{\partial}{\partial y}(VW) + \mathbf{a}_z \frac{\partial}{\partial z}(VW)$$

$$= \mathbf{a}_x V \frac{\partial W}{\partial x} + \mathbf{a}_y V \frac{\partial W}{\partial y} + \mathbf{a}_z V \frac{\partial W}{\partial z} + \mathbf{a}_x W \frac{\partial V}{\partial x} + \mathbf{a}_y W \frac{\partial V}{\partial y} + \mathbf{a}_z W \frac{\partial V}{\partial z}$$

(using the rule for the derivative of a product)
then

$$\nabla(VW) = V\nabla W + W\nabla V \qquad (1\text{-}42)$$

Next, consider the divergence of the product of a scalar and a vector:

$$\nabla \cdot (V\mathbf{A}) = \nabla \cdot \{\mathbf{a}_x VA_x + \mathbf{a}_y VA_y + \mathbf{a}_z VA_z\}$$

$$= \frac{\partial}{\partial x}(VA_x) + \frac{\partial}{\partial y}(VA_y) + \frac{\partial}{\partial z}(VA_z)$$

$$= V\frac{\partial A_x}{\partial x} + V\frac{\partial A_y}{\partial y} + V\frac{\partial A_z}{\partial z} + A_x \frac{\partial V}{\partial x} + A_z \frac{\partial V}{\partial y} + A_z \frac{\partial V}{\partial z}$$

Table 1-5. Vector Identities

TRIPLE PRODUCTS

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

PRODUCT RULES

(3) $\Delta(V\,W) = V(\Delta\,W) + W(\Delta\,V)$

(4) $\Delta(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\Delta \times \mathbf{B}) + \mathbf{B} \times (\Delta \times \mathbf{A}) + (\mathbf{A} \cdot \Delta)\,\mathbf{B} + (\mathbf{B} \cdot \Delta)\,\mathbf{A}$

(5) $\Delta \cdot (V\,\mathbf{A}) = V\,(\Delta \cdot \mathbf{A}) - \mathbf{A} \cdot (\Delta\,V)$

(6) $\Delta \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\Delta \times \mathbf{A}) - \mathbf{A} \cdot (\Delta \times \mathbf{B})$

(7) $\Delta \times (V\,\mathbf{A}) = V(\Delta \times \mathbf{A}) - \mathbf{A} \times (\Delta\,V)$

(8) $\Delta \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \Delta)\,\mathbf{A} - (\mathbf{A} \cdot \Delta)\,\mathbf{B} + \mathbf{A}\,(\Delta \cdot \mathbf{B}) - \mathbf{B}\,(\Delta \cdot \mathbf{A})$

SECOND DERIVATIVES

(9) $\Delta \cdot (\Delta \times \mathbf{A}) = 0$

(10) $\Delta \times (\Delta\,V) = 0$

(11) $\Delta \times (\Delta \times \mathbf{A}) = \Delta\,(\Delta \cdot \mathbf{A}) - D^2\,\mathbf{A}$

INTEGRAL THEOREMS

(1) Gradient Theorem: $\int_a^b (\nabla V) \cdot d\ell = V(b) - V(a)$

(2) Divergence Theorem: $\iiint_V (\nabla \cdot \mathbf{A})\,dv = {}'\iint_S \mathbf{A} \cdot d\mathbf{s}$

(3) Stokes' Theorem: $\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\ell$

Then

$$\nabla \cdot (V\mathbf{A}) = V(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot \nabla V \quad (1\text{-}43)$$

Finally consider the curl of a product:

$$\nabla \times (V\mathbf{A}) = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ VA_x & VA_y & VA_z \end{vmatrix}$$

Expanding the determinant, using the product rule and collecting terms yields

$$\nabla \times (V\mathbf{A}) = V\nabla \times \mathbf{A} + (\nabla V) \times \mathbf{A} \quad (1\text{-}44)$$

1.4.7 Higher Order Functions of Vector Calculus

The gradient (ΔV) is a vector so we can take its divergence to form Δ · (ΔV) which is a scalar called the **Laplacian.** There are five such quantities involving two successive del (Δ) operations:

$$\nabla \cdot (\nabla V) \quad \nabla (\nabla \cdot V) \quad \nabla \cdot (\nabla \times \mathbf{A})$$
$$\nabla \times (\nabla V) \qquad\qquad \nabla \times (\nabla \times \mathbf{A})$$

Two of these quantities, namely, Δ × (ΔV) and Δ · (Δ × **A**), are always zero, for any scalar function V and for any vector function **A.**

$$\nabla \times \nabla V = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix}$$
$$= \mathbf{a}_x (\frac{\partial^2 V}{\partial y \partial z}) - (\frac{\partial^2 V}{\partial z \partial y}) + \mathbf{a}_y (\frac{\partial^2 V}{\partial z \partial x}) - (\frac{\partial^2 V}{\partial x \partial z}) + \mathbf{a}_z (\frac{\partial^2 V}{\partial x \partial y}) - (\frac{\partial^2 V}{\partial y \partial x})$$

If the function V and its derivatives are continuous, the order of differentiation is immaterial, and thus each component of Δ × DV is zero. The divergence of the curl, Δ · (Δ × **A**), is zero for the same reason, as may be shown by expansion in rectangular coordinates.

Thus

$$\nabla \times \nabla V = \nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad (1\text{-}45)$$

$\Delta \cdot (\Delta V)$ is written as $\Delta^2 V$ and is called the *Laplacian* of the scalar V. Expanding in rectangular coordinates:

$$\nabla \cdot (\nabla V) = \nabla^2 \cdot V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \quad (1\text{-}46)$$

The Laplacian of a scalar is itself a scalar and is called the *scalar Laplacian.*

The Laplacian of a vector is written as $\Delta^2 \mathbf{A}$ and is defined as follows (see the vector identity (11) of Table 1-5):

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \quad (1\text{-}47)$$

The Laplacian of a vector is itself a vector and is called the *vector Laplacian.* It has a particularly simple form in rectangular coordinates:

$$\nabla^2 \mathbf{A} = \mathbf{a}_x \nabla^2 A_x + \mathbf{a}_y \nabla^2 A_y + \mathbf{a}_z \nabla^2 A_z \quad (1\text{-}48)$$

where $\Delta^2 A_x$ is the scalar Laplacian, etc., from whence derives the concept of the vector Laplacian. In other coordinate systems there is *no* such simple result and one must, consequently, return to Equation (1-47) to obtain the vector Laplacian.

1.5 HELMHOLTZ'S THEOREM

Helmholtz's theorem states that any vector **F** can be split into two parts, one of which is a conservative vector and the other of which is solenoidal, as follows:

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A} \quad (1\text{-}49)$$

where V and **A** are defined by

$$V = \frac{1}{4\pi} \iiint_V \frac{(\nabla \cdot \mathbf{F})' dv'}{R} \quad (1\text{-}50a)$$

$$\mathbf{A} = \frac{1}{4\pi} \iiint_V \frac{(\nabla \times \mathbf{F})' dv'}{R} \quad (1\text{-}50b)$$

The definitions of V and **A** are of interest to us primarily in that they indicate that V derives from $\Delta \cdot \mathbf{F}$ and **A** from $\Delta \times \mathbf{F}$. R is the distance between points (x, y, z) and (x′, y′, z′).

$(\Delta \cdot \mathbf{F})'$ merely indicates that we take the divergence of $\mathbf{F}$(x,y,z) and then change (x,y,z) to (x′,y′,z′). Now consider Equation (1-49)

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A}$$

$-\Delta V$ is the curlfree (conservative) portion of $\mathbf{F}$ since $\Delta \times \Delta V = 0$. $\Delta \times \mathbf{A}$ is the divergenceless (solenoidal) portion since $\Delta \cdot (\Delta \times \mathbf{A}) = 0$. Substituting the definitions into Eq. (1-49), we see that Helmholtz's theorem may be stated as a vector identity involving only the arbitrary vector **F:**

$$\mathbf{F} = -\nabla \underbrace{\left[\iiint \frac{(\nabla \cdot \mathbf{F})' \, dv'}{4\pi R} \right]}_{V} + \nabla \times \underbrace{\left[\iiint \frac{(\nabla \times \mathbf{F})' \, dv'}{4\pi R} \right]}_{V} = -\nabla V + \nabla \times \mathbf{A}$$

Now we can see that $\mathbf{F}$ is completely determined by its divergence and curl. The portion $-\Delta V$ is curlfree (conservative) and is determined solely by $\Delta \cdot \mathbf{F}$. The portion $\Delta \times \mathbf{A}$ is divergenceless (solenoidal) and is determined solely by $\Delta \times \mathbf{F}$. If both $\Delta \cdot \mathbf{F}$ and $\Delta \times \mathbf{F}$ are zero, then vector $\mathbf{F}$ is zero (or a constant vector since a constant vector has no divergence or curl). Furthermore, we note two special cases:

If $\Delta \times \mathbf{F} = 0$, the second integral vanishes and $\mathbf{F} = -\Delta V$, i.e., $\mathbf{F}$ can be expressed as the gradient of a scalar.

If $\Delta \cdot \mathbf{F} = 0$, the first integral vanishes and $\mathbf{F} = \Delta \times \mathbf{A}$, i.e., $\mathbf{F}$ can be expressed as the curl of another vector.

The electrostatic field is an example of the first type (conservative field) and the magnetostatic field is an example of the second type (solenoidal field). A proof of Helmholtz's theorem may be found in Arfken.*
In summary:

(a) A vector field **F** is completely determined if its divergence and curl are known everywhere.
(b) Every vector field **F** can be broken into two parts, one of which is conservative and the other of which is solenoidal.
(c) The conservative part is determined by $\Delta \cdot \mathbf{F}$.
(d) The solenoidal part is determined by $\Delta \times \mathbf{F}$.
(e) If $\Delta \times \mathbf{F} = 0$, then $\mathbf{F} = -\Delta V$. (1-51a)
(f)If $\Delta \times \mathbf{F} = 0$, then $\mathbf{F} = \Delta \times \mathbf{A}$. (1-51b)

* G. Arfken, *Mathematical Methods for Physicists,* Academic Press, 1985, 3rd. Ed., Section 1.15.

Example 1-15

Apply Helmholtz's theorem to the vector field of Figure 1-21 (see Example 1-13).

Solution:

$$\mathbf{E} = \mathbf{a}_r \frac{K}{r^2}$$

In Example 1-13, we have shown that $\Delta \times \mathbf{E} = 0$ everywhere. Thus, $\mathbf{E} = -\Delta V$ and

$$V = \frac{1}{4\pi} \iiint \frac{(\nabla \cdot \mathbf{F})' \, dv'}{R}$$

$\Delta \cdot \mathbf{E} = 0$ except at the origin (Example 1-13). At the origin ($x' = y' = z' = 0$), R reduces to r which can be taken outside the integral:

$$V = \frac{1}{4\pi r} \iiint (\nabla \cdot \mathbf{E})' \, dv'$$

But $\iiint (\nabla \cdot \mathbf{E})' \, dv' = \iiint (\nabla \cdot \mathbf{E}) dv = 4\pi K$ (Example 1-13) and thus

$$\mathbf{E} = -\nabla V; \; V = \frac{K}{r}$$

This result will be determined in another way in Chapter 2 (see Eq. (2-43)).

CHAPTER 2

Introduction to Electrostatic Fields and Electromagnetic Potentials

2.1 Introduction

In Chapter Two, we are concerned with stationary electric charges in free space (vacuum) and the resultant electric fields and potentials. We discuss first the concepts of electric charge and the particular forms (volume, surface, line and point charge distributions) which it takes. We consider next the electric fields, which are related to *forces* exerted by the charges. Then we take up electric potentials which are related to *work* (integrals of the electric field). General methods are described for the determination of fields and potentials due to arbitrary charge distributions. A specialized method, Gauss' law, is used to obtain the electric fields and potentials of certain symmetric charge distributions. Finally, we consider some special arrangements of charges that form dipoles and higher-order multipoles.

In this chapter, we consider only charges and conductors in free space. The charges are stationary and the resultant electric fields and potentials are constant (they do not vary with time). Thus we are concerned with *static electric* or *electrostatic* fields. Chapters Two through Four of this text are limited to stationary charges.

In the real world, nothing is constant, and so what application can electrostatics have to reality? First, there are situations in which fields are varying slowly with time (low-frequency variations). This is the so-called *quasi-static* case; the electrostatic analysis is very useful here. Second, we have situations in which transients occur. As the transients die out the electric field may approach the electrostatic solution. Finally, there are many situations in which there are only very minor random changes in electric fields with time. Thus there are many cases in which the electrostatic solution is useful. The most important result, however, of our study of electrostatic problems is, undoubtedly, the insight which we gain into electrical phenomena.

2.2 Electric Charge

Perhaps the most striking characteristic of charge is the incredible, nearly perfect balance between positive and negative charges. Most objects have only a tiny net or excess charge. For instance, if two persons each had 0.1% excess charge and were 1 meter apart, the force of repulsion between them would be about 10^{20} tons. Since no such force is observed, the excess charge must be tiny indeed.

Similarly, it takes only a tiny excess charge to exert forces which can be observed. The effects of charged objects were noted very early in history. About 600 B.C., Thales of Miletus observed and recorded the fact that the material amber, when rubbed with silk, attracted particles of straw. The Greek word for amber is *elektron* from whence come our words electron, electricity, etc.

We know now that silk removes electrons from the amber, leaving it positively charged. Similarly, rubbing a glass rod with silk also leaves it positively charged whereas rubbing a plastic comb with wool leaves it negatively charged. The names *positive* and *negative* came from the work of Benjamin Franklin. He recognized that there were two kinds of contributions to the charged state. He arbitrarily called that of the rubbed glass rod positive from which definition the charge of electron becomes negative.

The unit of charge is the coulomb (C). The charge on an electron is $-e$ where

$$e = 1.602 \times 10^{19}\,[\mathrm{C}] \tag{2-1}$$

The charge and mass of the electron are known. Little is known concerning its internal structure. From Eq. (2-1) above, we note that -1 coulomb contains $.625 \times 10^{19}$ electrons. This is a very large amount of charge. A lightning stroke carries about 20 coulombs. There are three possible types of charge distributions: volume (ρ_v), surface (ρ_s), and line (ρ_ℓ)

charge densities. When charge is distributed over a volume, we define a volume charge density ρ_v. To determine ρ_v at a point, we construct a small volume Δv around the point, measure the charge Δq within Δv, compute the ratio $\frac{\Delta q}{\Delta v}$ and then find the limit as Δv becomes infinitesimal, i.e.

$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v}\left[\frac{C}{m^3}\right] \tag{2-2a}$$

where [C/m^3] represents the unit of the volume charge density. Similarly,

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s}\left[\frac{C}{m^2}\right] \tag{2-2b}$$

$$\rho_l = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l}\left[\frac{C}{m}\right] \tag{2-2c}$$

where Δs is the area of a small surface and $\Delta \ell$ is the length of a small line segment. The total charge q in a volume V, a surface S, and a line L may then be expressed as

$$q = \iiint_V \rho_v dv \text{ (for volume charge)} \tag{2-3a}$$

$$= \iint_S \rho_s ds \text{ (for surface charge)} \tag{2-3b}$$

$$= \int_L \rho_l dl \text{ (for line charge)} \tag{2-3c}$$

Figure 2-1 illustrates some of the concepts of volume, surface, and line charge densities. Figure 2-1(a) shows a cube of continuous volume charge distribution $\rho_v(x,y,z)$.

Total charge q is defined by Eq. (2-3a). Now if we squeeze all of the charge into a zero thickness layer (Figure 2-1(b)), we have a surface charge density $\rho_s(x,y)$ with total charge q, which is unchanged, determined by Eq. (2-3b). If we compress the surface charge into a line of zero cross section (Figure 2-1(c)), we obtain a line charge distribution $\rho_\ell(x)$ with total charge q determined by Eq. (2-3c). Finally if we compress the line charge down to a point, we have a point charge of q coulombs (Fig. 2-1(d)).

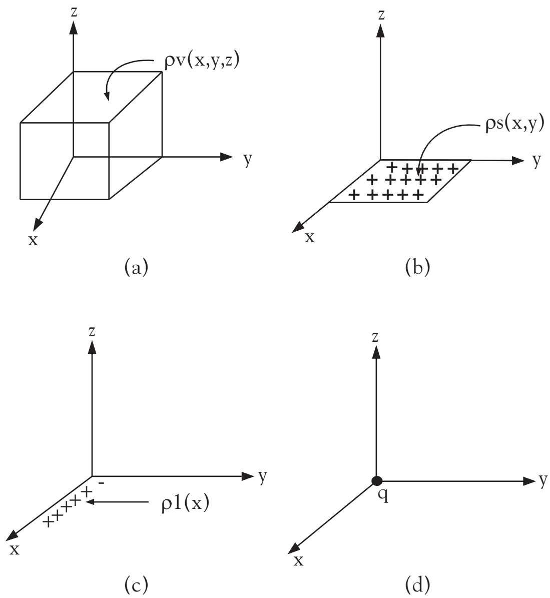


Figure 2-1. ***Electric charge distribution (a) Volume charge density*** ρ_v***. (b) Surface charge density*** ρ_s***. (c) Line charge density*** ρ_ℓ***. (d) A point charge q.***

The different types of charge densities are related to different physical situations. For instance, charge in an electron beam may be represented as volume charge density ρ_v. Charge on the surface of a good conductor may be represented by surface charge density ρ_s. Charge on a thin wire is modeled as line charge density ρ_ℓ. Finally, the electron is usually treated as a point charge.

Surface and line charge densities, and point charge may all be represented as special cases of volume charge density. Those of you familiar with delta functions can probably see how to accomplish this.

In Eqs. (2-2) and (2-3) we are neglecting the microscopic variations of charge density associated with atomic dimensions. We are interested in a smoothed-out or space-averaged distribution. This is the *macroscopic* rather than the *microscopic* point of view.

Example 2-1 Uniform Charge Distribution

Consider a spherical cloud of uniform volume charge density:

$$\rho_v = \rho_{vo} (0 < r < a) \tag{2-4a}$$

$$= 0 \text{ elsewhere} \quad \text{(2-4b)}$$

Often we will omit the latter statement (zero elsewhere). Find the total charge contained within the cloud.

Solution:

$$Q = \iiint \rho_v \, dv$$
$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho_{vo} \; r^2 \sin\theta \; dr \; d\theta \; d\phi$$
$$= \rho_{vo} \; 2\pi \left[-\cos\theta\right]_0^{\pi} \left[\frac{r^3}{3}\right]_0^a \quad \text{(2-5)}$$
$$= \rho_{vo} \left(\frac{4}{3}\pi a^3\right)$$

Note that we can obtain this result merely by multiplying the volume charge density (ρ_{vo}) by the volume $(\frac{4}{3}\pi a^3)$ because the charge density is uniformly distributed within the volume. It is useful to recognize the following integral which occurs often:

$$\int_0^{2\pi} \int_0^{\pi} \sin\theta \; d\theta \; d\phi = 4\pi \quad \text{(2-6)}$$

Example 2-2 A Non-Uniform Charge Distribution

Consider a spherical cloud of charge of *non-uniform* volume charge density:

$$\rho_v = \rho_{vo} \left(\frac{r}{a}\right)^2 \; (0 < r < a) \quad \text{(2-7)}$$

It is *assumed* that ρ_v is zero elsewhere. Find the total charge contained within the cloud.

Solution:

$$Q = \iiint \rho_v dv \qquad (2\text{-}8)$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho_{vo} \frac{r^2}{a^2} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{\rho_{vo}}{a^2} 2\pi[-\cos\theta]_0^{\pi} \left[\frac{r^5}{5}\right]_0^a$$

$$= \rho_{vo}\left(\frac{4}{5}\pi a^3\right)$$

In this case, integration is necessary since the charge density depends on position.

2.3 The Electric Field in Free Space

The **electric field intensity E** at a point (x,y,z) is defined in terms of the force **F** on a small test charge q located there:

$$\mathbf{E}(x, y, z) = \lim_{q \to 0} \frac{\mathbf{F}}{q} \left[\frac{N}{C}\right] \text{ or } \left[\frac{V}{m}\right] \qquad (2\text{-}9a)$$

The unit of the electric field is Newtons per coulomb [N/C] but is also given by volts per meter [V/m] since the voltage is related to the electric field by Eq. (2-42). In this book, *the SI units are used.* A small test charge q is used in order that the determination of **E** not be affected by q itself.

The force on a charge q in an electric field **E** is then

$$\mathbf{F} = q\mathbf{E} \ [N] \qquad (2\text{-}9b)$$

Basic Laws of Electrostatics

The basic laws of electrostatics, in point form (or differential form), specify the divergence and curl of the electric field **E.** From Helmholtz's theorem, we know that the specification of D × **E** and D · **E** completely determines the vector field intensity **E.**

The first law is

$$\nabla \times \mathbf{E} = 0 \quad \mathbf{E} \text{ is a conservative field} \tag{2-10}$$

Equation (2-10) is the point form of the law, i.e., D × **E** = 0 at every point (x,y,z). We can derive an equivalent integral form by integrating both sides of Eq. (2-10) over an arbitrary open surface S and applying Stokes' theorem:

$$\iint_S \nabla \times \mathbf{E} \cdot d\mathbf{s} = \oint_C \mathbf{E} \cdot d\ell = 0$$

for any closed path C which bounds the surface S and where d**s** and d**ℓ** are related by a right-hand rule. S is arbitrary and thus C is also arbitrary.

$$\oint_C \mathbf{E} \cdot d\ell = 0 \tag{2-11}$$

We have derived possible to derive two equations are identical statements.

The second law is

Eq. (2-11) from Eq. (2-10). It is also

Eq. (2-10) from Eq. (2-11) and thus the

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon_0} \tag{2-12}$$

where $\varepsilon_0 = 8.854 \times 10^{-12}$ [$C^2/N{\cdot}m^2$] or [F/m]. ε_0 is a universal constant, called the *permittivity of free space.* ρ_v is the volume charge density in C/m^3. We can derive an integral form by integrating both sides of Eq. (2-12) over a volume V and applying the divergence theorem:

$$\iiint_V \nabla \cdot \mathbf{E}\ dv = {}'\!\iint_S \mathbf{E} \cdot d\mathbf{s} = \frac{1}{\varepsilon_0} \iiint_V \rho_v\ dv = \frac{Q_{enc}}{\varepsilon_0}$$

S and V are both arbitrary. Q_{enc} is the *total charge enclosed within the closed surface S.*

$$'\!\iint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q_{enc}}{\varepsilon_0} \quad \text{Gauss' Law} \tag{2-13}$$

Eqs. (2-12) and (2-13) are identical statements. The second law is called **Gauss' law,** which will be explained in a later section. The basic laws above may also be derived directly from Coulomb's law.

2.4 Charles Augustin Coulomb (1736-1806) and the Discovery of Coulomb's Law

Charles Augustin Coulomb was one of the leading scientists of the latter half of the 18th century. He made significant contributions to a wide variety of fields, including civil engineering, mechanical engineering, soil mechanics, and electricity and magnetism. He established, in an extremely careful and thorough manner, the inverse square law for electricity, which is known as *Coulomb's law.* In addition to these contributions to engineering and physics, he made several important practical contributions, among which was the invention of the torsion balance, a device capable of measuring very small forces.

His contributions to engineering and physics laid the foundations for later discoveries. He discovered one of the basic laws of friction. His theory of earth pressure and the related *Coulomb's Equation* are still used as the most fundamental principles of soil mechanics. His contributions to electricity form the basis of our theory. All of the basic laws of electrostatics can be derived from Coulomb's Law.

Charles Augustin Coulomb was born in 1736 in the town of Angouleme in southern France. His father, Henry Coulomb, grew up in the family home in Montpelier where the Coulombs had lived for several generations. They were a bourgeois or middle class family and many of the Coulombs were lawyers in the Montpelier region. Henry Coulomb started out in the military and then became a minor governmental administrator, an inspector of the king's domains. He later became an administrator of the tax-farm system, a system much-hated by the French people. Many of the tax-farm administrators were guillotined during the French Revolution, some years after Henry's death.

The family moved to Paris during Coulomb's adolescence. He attended mathematical lectures at the Royal College of France, where he became very interested in mathematics and decided to become a mathematician. This decision displeased his mother, who had set her heart on his becoming a doctor. At this point, Coulomb's father Henry lost his entire fortune in speculation and returned to the family home in Montpelier. His mother, Catherine Bajet Coulomb, who had resources of her own, stayed in Paris with her two daughters and disowned her son Charles who was forced to join his father in Montpelier.

Later biographers have sometimes promoted Coulomb to the ranks of the nobility by adding the prefix "de" to his name and calling him "de Coulomb". There is, however, no true basis for this. Neither Coulomb nor any of his relatives ever signed themselves "de Coulomb," and, ironically, if he had been of the nobility, he might not have survived the revolution.

Coulomb presented papers on mathematics and astronomy in 1757, 1758 to the Royal Society of Sciences of Montpelier. He was certainly developing his mathematical and experimental skills, but as a young man of 22, he had no source of income and was forced to seek gainful employment. He decided to join the military corps of engineers and was accepted in its engineering school at Mezieres, graduating at the age of 25 in 1761.

The next eleven years Coulomb spent in active service in the engineering corps. He was posted to Martinique for eight years and was given responsibility for the complete construction of a major fortress, a project that involved overseeing several hundred workers. During the eight-year duration of the project, Coulomb faced basic problems of the design of vaults, arches, and retaining walls. He had to understand the strength of materials and to predict bending, compression, and rupture. Rather than relying on rules of thumb already available to accomplish these tasks, Coulomb went back to the most basic questions, developed his own theories, and carried out his own experiments in order to proceed on a firmer basis. All of this was done in his spare time, amid outbreaks of malaria and cases of sunstroke, in the equatorial climate of Martinique. This work, in his own words, "was meant at first only for my own use, in the different tasks for which I am responsible in my occupation". Nonetheless, the impact of his work is still felt centuries later. Upon returning to France, Coulomb presented his theory and experiment on these construction problems to the Academy of Sciences in 1773 in an historic paper on arches, strength of materials, and soil mechanics. This paper has been called the most brilliant engineering paper read at the Academy during the last half of the eighteenth century.

In 1777, the Academy offered a special prize for improvements in magnetic compasses. The basic problem is one which all Girl Scouts and Boy Scouts should recognize. The force on the compass needle is so weak that friction in the compass may give us an incorrect reading. Coulomb suspended the needle on a silk or metal thread so that it encountered virtually no friction. By these means he was able to determine magnetic declination to within a few minutes of arc. By the addition of a micrometer at the other end of the thread, one could also determine the torque on the thread and thus the magnitude of the earth's magnetic field. This device is called the torsion balance. Coulomb studied the phenomena of torsion in detail and discovered that, within the linear region of operation, the angle through which a wire is twisted is proportional to the torque exerted. The magnitude of the torque required to twist the wire through a given angle is inversely proportional to the wire length and is proportional to the fourth power of the wire radius. Thus a long, thin wire in a torsion balance is very sensitive. It is capable of measuring very small forces or torques.

Coulomb won the 1777 compass prize. However, he realized that friction theory was needed for a study of conventional compasses. The Academy was interested in friction for naval applications. A prize was proposed for a study of friction. Coulomb undertook an extremely thorough mathematical and experimental study of friction which won him a second prize in 1781. Once again, Coulomb asked himself very basic questions and reformulated the entire theory, earning himself the title of "The Father of Friction." He was elected to the Academy in 1781.

During the period 1785-1791, Coulomb conducted a series of very elegant experiments which established the inverse square force law for electric charges. He verified the law by three completely different methods. For like charges (both positive or both negative), he measured the force of repulsion directly using the torsion balance. These measurements are extremely difficult, even now, especially because of the influence of the experimenter. For unlike charges, he determined the force of attraction by timing the pendulum swing of a movable charge attracted towards a fixed charge. The third method involved a demonstration that there is no charge inside a conducting body as required by the inverse square law. Maxwell, writing much later, said of his third method, "It is impossible to overestimate the delicacy and ingenuity of his apparatus, the accuracy of his observations, and the sound scientific method of his researches."

Coulomb first reported his results in 1785. They were soon accepted by the French scientific community and the inverse square law was named *Coulomb's Law.* He also established the inverse square force law for isolated magnetic poles. This required a careful choice of magnets and distances. He chose long thin magnets which were closely spaced so that the predominant force was that between nearby poles.

For charged objects, Coulomb noted that the charge leaked off rather rapidly with time. His first experiments had to be carried out in two minutes to avoid error due to leakage. Coulomb found that the charge lost per second was proportional to the charge present, determined the differential equation governing charge decay, and deduced that the charge decay was therefore *exponential,* the exponent depending on factors such as humidity. Thus he was the first person to describe the effect which we will later call *relaxation time* (See Chapter 5).

In some of his later papers, Coulomb attacked the difficult problem of determining charge distributions over the surface of a charged metallic object. He showed that the surface charge density at a point is proportional to the electric field normal to the surface and devised a standard disk which could be used to measure the charge density at different points on the surface of a conductor.

Several scientists preceded Coulomb in the discovery of the inverse square laws of electricity and magnetism. In fact, there were over ten investigators who provided some evidence for the electric and/or magnetic laws over the forty-year period preceding Coulomb's 1785 paper. Some of these contributions were relatively minor, mere statements or assumptions of the inverse square law. In addition there were a number of claims for the inverse first and third powers. The situation was confused and a definitive, comprehensive study was called for. Henry Cavendish had carried out very careful experiments and analysis to show that the charge inside a conductor is zero, but his results were not published until a century later. John Robison had made thorough direct measurements of force between charges but he also did not publish. Thus it fell to Coulomb to carry out the definitive, comprehensive experiments and to make the arguments which convinced the French academic community.

Why was it so difficult to establish the inverse square laws? Why was Coulomb's work accepted over those who went before and why was his name attached to the law?

It is important for us to understand the time in which Coulomb worked. It was the time of the *enlightenment,* with new ideas being posited in every discipline of study. Science itself was slowly emerging from the imprecise methodology of the middle ages. The concept of action at-a-distance was still difficult for many to accept. It was believed that *objects can interact only through contact* and so the motion of planets was explained by vortices of invisible matter existing between sun and planets. The clever ideas of Descartes and Nollet involving *vortices* and *effluvia* of hidden invisible matter explained the solar system as well as magnetism and electricity. Thus one's intellect and imagination were used to create complex systems to explain the known world, but these systems were not tested. In some respects they were untestable. What can one do with material which is invisible, untouchable, unweighable? Such theories were typically accepted on faith. In the middle ages imagination and speculation, however intelligent, formed the basis of science. The discoveries made in one country did not always cross the border. The French did not fully accept Newton and the Germans did not fully accept Coulomb for many years.

Many different forms of the force law echoed through the halls of science in the latter half of the 18th century. $1/r$, $1/r^2$, $1/r^3$ and other forms were seriously pursued, so that the many statements of the inverse square law did not immediately lead to acceptance. Finally, Coulomb comes at the right time. He reports his results and confirms them by three entirely different methods. He was by then a highly respected figure, a leader of eighteenth century science within the academy. Because of the thoroughness of his methods and his position in the Academy, his work was immediately accepted in France. The scientific community did not accept the inverse square laws before Coulomb because the situation was confused and competing theories existed. Many scientists were not disposed to accept any theorem involving action at-a-distance. Some of the inverse square law results were merely statements or conjecture. Finally, the time was ripe. Vortices and effluvia were on the wane. New ideas were in the wind. The American revolution was over and the French Revolution was only four years away. Modern scientists were in the ascendancy. Coulomb himself was gaining an unmatched reputation and the confidence of his colleagues. When the new experiments were reported, they were immediately accepted and Coulomb's name was soon attached to the inverse square law.

We learn a very valuable lesson from the history of Coulomb's law. Priority is certainly important. With two equally strong contributions, the prior one is recognized. However, thoroughness and completeness, and depth of understanding may be more important than priority. In addition, the discoverer must be aware of the importance of his discovery and must be able to present his results in a convincing manner.

In the eyes of his colleagues, Coulomb was a great leader in French science. Jean Batiste Biot, of whom we will hear later, said, "It is to Borda and to Coulomb that one owes the renaissance of true physics in France, not a verbose and hypothetical physics, but that ingenious and exact physics which observes and compares all with rigor.":

Coulomb's Law

Figure 2-2 shows two point charges q_1, q_2 located at arbitrary positions separated by a distance **R.** The force on q_2 due to q_1 is

$$\mathbf{F}_2 = \text{force on } q_2 = \mathbf{a}_R \frac{q_1 q_2}{4\pi\varepsilon_0 R^2} = \mathbf{R}\frac{q_1 q_2}{4\pi\varepsilon_0 R^3} \qquad (2\text{-}14)$$

where **R** is the vector from q_1 to q_2, **R** is the distance from q_1 to q_2, and $\mathbf{a}_R = \dfrac{\mathbf{R}}{R}$.

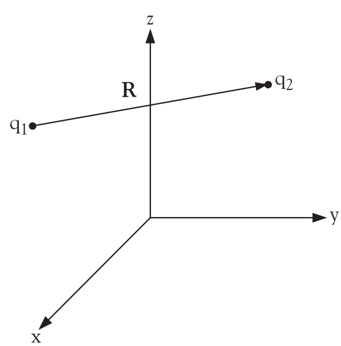


Figure 2-2. *Coulomb's law.*

The electric field **E** at q_2 due to q_1 is defined as $\mathbf{F}_2$ divided by q_2 or the *force per unit charge*:

$$\mathbf{E} = \frac{\mathbf{F}_2}{q_2} = \mathbf{a}_R \frac{q_1}{4\pi\varepsilon_0 R^3} = \mathbf{R}\frac{q_1}{4\pi\varepsilon_0 R^3} \qquad (2\text{-}15)$$

2.5 Gauss' Law

C. Stewart Gilmour, *Coulomb and the Evolution of Physics and Engineering in Eighteenth-Century France,* Princeton University Press, p. 230, 1971.

Gauss' law may be used to obtain the electric fields of certain charge distributions with a high degree of symmetry. It is therefore a specialized method, but it is very useful for that class of problems to which it can be applied. At this stage, Gauss' law will help us to gain insight into the forms of electric fields due to charge distributions. First let us look again at Gauss' law:

$$'\iint_S \mathbf{E}\cdot d\mathbf{s} = \iint_S E_n d\mathbf{s} \frac{Q_{enc}}{\varepsilon_0} \qquad (2\text{-}13)$$

where the surface S is an arbitrary closed surface, chosen for convenience, which will be called the *Gaussian surface.* Q_{enc} is the charge enclosed within the Gaussian surface S. E_n is the component of E normal to S.

We will assume that charges are specified and so the right hand side of Eq. (2-13) is known or can be determined. E_n is unknown. If E_n were constant over the surface S, then we could take it outside the integral and solve for E_n. The key to a Gauss' law problem is to choose S so that it is normal to the electric field lines. Thus we need to have some knowledge of the field lines. The steps then are as follows:

1. Recognize the symmetry.
2. Sketch the electric field lines.
3. Choose S normal to the field lines. Form a closed surface.
4. Solve for E_n.

Electric Field of a Point Charge

Consider first a point charge q located at the origin (Figure 2-3(a)). We can observe spherical symmetry ($\partial/\partial\theta = \partial/\partial\varphi = 0$). In other words, the electric field components do not vary with angles θ, φ. E_θ, E_φ are ruled out because of the first basic law $\oint \mathbf{E}\cdot d\ell = 0$ where the contour is taken around the spherical surface. Therefore, the electric field is *purely* *radial,* i.e., $\mathbf{E} = \mathbf{a}_r E_r$, and is a function only of r. We construct a Gaussian surface S normal to the electric field, i.e., a sphere of radius r centered at the origin. E_r will be constant over the surface S due to spherical symmetry.

Then

$$\oiint_S \mathbf{E}\cdot d\mathbf{s} = \oiint_S E_n d\mathbf{s} = E_r \int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta \, d\theta \, d\phi = 4\pi r^2 \; E_r = \frac{Q_{enc}}{\varepsilon_0} = \frac{q}{\varepsilon_0}$$

$$E_r = \frac{q}{4\pi\varepsilon_0 r^2}$$

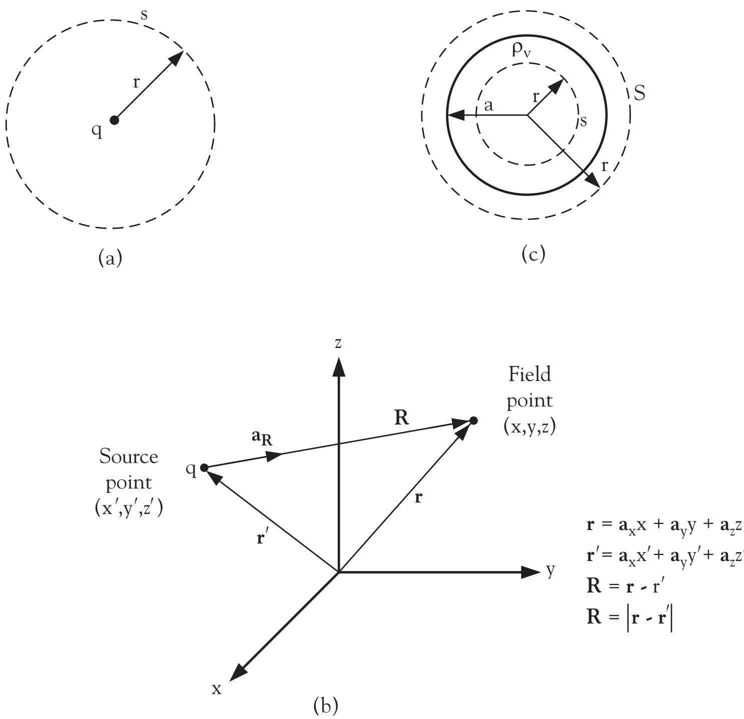


Figure 2-3. ***Spherical charge distributions.***

(a) A point charge at the origin.

(b) A point charge at source point (x′, y′, z′).

(c) A uniform volume charge density ρ_v.

For a point charge located at an arbitrary source point (x′, y′, z′) as in Figure 2-3(b), the electric field at the field point or the observation point (x,y,z) is given as follows:

$$\mathbf{E} = \mathbf{a}_R \frac{q}{4\pi\varepsilon_0 R^2} = \frac{\mathbf{R}q}{4\pi\varepsilon_0 R^3} = \frac{(\mathbf{r}-\mathbf{r}')q}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|^3} \qquad (2\text{-}17)$$

where **R** is a vector from the source point to the field point and $\mathbf{a}_R$ is a unit vector in the direction of **R.** Note that **R** is identical to $\mathbf{r} - \mathbf{r}'$ and that R is identical to $|\mathbf{r} - \mathbf{r}'|$ as indicated in Figure 2-3(b). **r** and **r**′ are the position vectors of the field point and source point, respectively. Note also that Eq. (2-17) is identical to Eq. (2-15), Coulomb's law.

Example 2-3 Volume Charge in a Spherical Shape
Consider a uniform volume charge distributed within a sphere of radius a (Fig. 2-3c):

$$\rho_v = \rho_0 \left[\frac{C}{m^3}\right] (r \le a) \qquad (2\text{-}18)$$

Find the electric field everywhere.

Solution:

The total charge Q within the sphere is $\frac{4}{3}\pi a^3 \rho_o$. Rotation of the spherical cloud of charge produces no change in charge distribution and therefore $\partial/\partial\theta = \partial/\partial\varphi = 0$. The charge distribution has *spherical symmetry.* E_θ, E_φ are ruled out because of the first law. Since the electric field diverges away from the charge distribution, we expect that $\mathbf{E} = \mathbf{a}_r E_r$. We can construct two Gaussian surfaces (a sphere of radius r), one inside ($r \le a$) and one outside ($r \ge a$) as in Figure 2-3(c). The charge enclosed within the Gaussian surface is $\frac{4}{3}\pi a^3 \rho_o$ when $r < a$ (inside the charged sphere), and is $\frac{4}{3}\pi a^3 \rho_o$ when $r > a$ (outside the charged sphere).

Then

$$\oiint_S \mathbf{E} \cdot d\mathbf{s} = 4\pi r^2 \; E_r = \frac{Q_{enc}}{\varepsilon_0} = \frac{4\pi r^3 \rho_0}{3\varepsilon_0} (r \le a) \qquad (2\text{-}19a)$$

$$= \frac{4\pi r^3 \rho_0}{3\varepsilon_0} (r \ge a)$$

$$E_r = \frac{r\,\rho_0}{3\,\varepsilon_0} = \frac{r}{a}\frac{Q}{4\pi\varepsilon_0 a^2} \quad (r \le a) \text{ inside the sphere}$$

$$E_r = \frac{\rho_0 a^3}{3\,\varepsilon_0 r^2} = \frac{Q}{4\pi\varepsilon_0 r^2} \quad (r \ge a) \text{ outside the sphere} \tag{2-19b}$$

The electric field outside is the same as if the total charge Q were concentrated at the origin. Note that the electric field E_r increases linearly with distance r inside (r < a) and decreases as the inverse square of r outside (r > a).

Example 2-4 Surface Charge on a Sphere

$$\rho_s = \rho_{so}\left[\frac{C}{m^2}\right] \;(r = a) \quad (2\text{-}20)$$

Consider a uniform surface charge density ρ_{so} distributed on a sphere of radius a: Find the electric field everywhere.

Solution:

The total charge Q on the sphere is $4\pi a^2\rho_{so}$. Note the spherical symmetry. Rotating the sphere in θ, φ does not change the charge distribution and therefore $\partial/\partial\theta = \partial/\partial\varphi = 0$. E_θ, E_φ are again ruled out because of the first law. We construct two Gaussian surfaces, one inside (r < a) and one outside (r > a). The charge enclosed is zero for r < a and is $4\pi a^2\rho_{so}$ for r > a.

Then

$$\oiint_S \mathbf{E}\cdot d\mathbf{s} = 4\pi r^2\; E_r = \frac{Q_{enc}}{\varepsilon_0} = 0 \;(r < a)$$
$$= \frac{4\pi r^2\rho_{so}}{3\varepsilon_0}(r > a) \tag{2-21a}$$

$$E_r = 0 \;(r < a) \text{ inside the sphere}$$

$$E_r = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{\rho_{so} a^2}{\varepsilon_0 r^2} \;(r > a) \text{ outside the sphere} \tag{2-21b}$$

Note that the electric field inside the hollow sphere with uniform surface charge is zero.

Example 2-5 Electric Field of a Line Charge of Infinite Length

Consider an infinitely long line charge with a uniform charge density ρ_t [C/m], located on the z axis (Figure 2-4a). Find the electric field everywhere.

Solution:

Rotation in φ and translation in z do not change the charge distribution and therefore $\partial/\partial z = \partial/\partial\varphi = 0$. We say that the charge distribution has *cylindrical symmetry.* E_φ is ruled out by the first law. E_z is also ruled out. We expect that the electric field is *purely radial* from the z-axis (E_ρ only), i.e., $\mathbf{E} = \mathbf{a}_\rho E_\rho$. Construct a cylindrical Gaussian surface S, of radius ρ and length ℓ, whose axis is the z axis. Note that the line charge is of infinite length whereas the Gaussian surface is of finite length. The electric field E_ρ is normal to the curved surface and parallel to the end surfaces of the Gaussian cylinder.

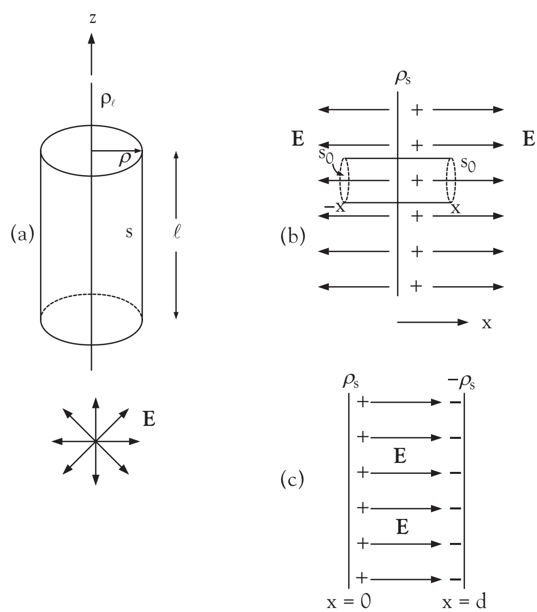


Figure 2-4. ***Cylindrical and planar charge distributions.***

(a) A uniform line charge density ρ_ℓ.

(b) A uniform planar surface charge density ρ_s.

(c) Parallel plates

Then

$$\oiint_S \mathbf{E}\cdot d\mathbf{s} = \iint_{SIDE} \mathbf{E}\cdot \mathbf{a}_\rho ds + \iint_{TOP} \mathbf{E}\cdot \mathbf{a}_z ds + \iint_{BOTTOM} \mathbf{E}\cdot(-\mathbf{a}_z)ds$$

$$= E_\rho \iint_{SIDE} ds + 0 + 0 = E_\rho 2\pi\rho l = \frac{Q_{enc}}{\varepsilon_0} = \frac{\rho_l l}{\varepsilon_0}$$

$$E_\rho = \frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{1}{\rho}\right) \tag{2-22}$$

Note that ρ, with no subscripts, is a cylindrical coordinate (a distance from the z axis), whereas ρ_ℓ is the line charge density.

Note also that the length ℓ above is arbitrary and the field doesn't depend on ℓ. The electric field decreases as the *inverse* of ρ for a very long line charge while the field of a point charge decreases as the *inverse square* of r.

Example 2-6 Planar Surface Charge Distributions
Consider a uniform surface charge density ρ_s [C/m^2] over the infinite plane x = 0 as shown in Figure 2-4(b). Find the electric field everywhere.
Solution:

Translation in y, z directions produces no change in charge distribution and thus ∂/∂y = ∂/∂z = 0. We say that the charge distribution has *planar symmetry.* We expect that E_y and E_z are zero and that E_X is odd ($E_x(x) = -E_x(-x)$) since **E** will point *away* from the charge source as shown in Figure 2-4(b). Construct a cylindrical Gaussian surface consisting of length 2x and end surfaces S_o as shown in Figure 2-4(b). The cylinder is of arbitrary cross section with area S_o. The electric field E_X is normal to the end caps and parallel to the side surface. Thus there is no contribution to the electric flux from the side surface. The charge enclosed in the cylinder is $\rho_s S_o$. Then

$$\oiint_S \mathbf{E}\cdot d\mathbf{s} = 2\iint_{END} \mathbf{E}\cdot ds = E_x 2S_o = \frac{Q_{enc}}{\varepsilon_0} = \frac{\rho_s S}{\varepsilon_0}$$

$$E_x = \frac{\rho_s}{2\varepsilon_0} \tag{2-23a}$$

$$\mathbf{E} = \mathbf{a}_x \frac{\rho_s}{2\varepsilon_o} \ (x > 0)$$
$$= -\mathbf{a}_x \frac{\rho_s}{2\varepsilon_o} \ (x < 0) \qquad (2\text{-}23b)$$

Next consider two parallel plates, separated by distance d, with uniform surface charge distributions $\pm \rho_s$ as shown in Figure 2-4(c). Now we add the electric fields due to each charged plate. The fields add inside and cancel outside, yielding

$$\mathbf{E} = \mathbf{a}_x \frac{\rho_s}{\varepsilon_0} (0 < x < d) \qquad (2\text{-}23c)$$

We see from Examples 2-3 through 2-6 that the solution for the electric field by Gauss' law is very straightforward for the class of symmetrical problems to which it can be applied. The problems treated are of three types:

1. Spherical symmetry: Spherical charge distributions which are independent of θ and φ such as

$$\rho_v = f(r) \left(\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \phi} = 0 \right) \qquad (2\text{-}24a)$$

2. Cylindrical symmetry: Infinitely long cylindrical charge distributions which are independent of z and φ such as

$$\rho_v = f(\rho) \left(\frac{\partial}{\partial z} = \frac{\partial}{\partial \phi} = 0 \right) \qquad (2\text{-}24b)$$

3. Planar symmetry: Infinite planar or slab charge distributions which are independent of y and z such as

$$\rho_v = f(x) \left(\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0 \right) \qquad (2\text{-}24c)$$

Let's look at a problem which *cannot* be treated by Gauss's law. Consider a line charge of finite length with a uniform charge distribution:

$$\rho_l = \rho_{lo} \left(-\frac{1}{2} \le z \le \frac{1}{2}, \ x = y = 0 \right) \qquad (2\text{-}25a)$$

$$= 0 \text{ elsewhere} \tag{2-25b}$$

Figure 2-5 shows the charge distribution and a sketch of the electric field lines. In this case $\partial/\partial\varphi = 0$ but $\partial/\partial z \neq 0$. If we try a cylindrical Gaussian surface, we find that two problems arise: (a) the cylindrical surface is not normal to the electric field **E** and (b) E_n is not constant over the surface. We will need more general methods to treat this problem (see Example 2-8).

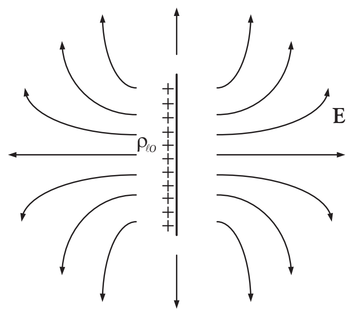


Figure 2-5. ***A uniform line charge density*** $\rho_{\ell o}$ ***of finite length.***

2.6. The Electric Fields Of Arbitrary Charge Distributions

In this section we consider the electric fields due to arbitrary charge distributions. The methods developed are completely general but more difficult to apply than are those of Gauss' law. They can handle the problems of charge distributions that do not have symmetry.

The concepts of this section are particularly difficult for many students. We introduce two independent sets of variables (x,y,z) and (x′,y′,z′) for the *field point* and the *source point,* respectively. We integrate over the primed variables (x′,y′,z′) of the source leaving our electric field in terms of the unprimed variables (x,y,z) of the field point. These concepts are new and require a careful introduction.

To begin, consider an arbitrary volume charge distribution $\rho_v(x',y',z')$ as shown in Figure 2-6. The volume charge is contained within a volume V. We want to find the electric field **E**(x,y,z) due to the entire charge distribution at a field point located at (x,y,z).

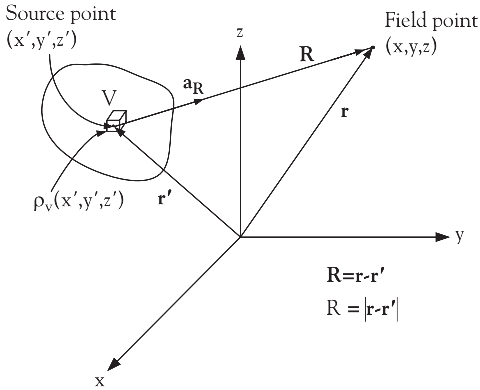


Figure 2-6. ***An arbitrary volume charge distribution*** ρ_v***(x′, y′, z′) in the basic source point-field point representation.***

First we consider the elementary contribution due to a typical volume element dv′ with volume charge density $\rho_v(x',y',z')$ located at typical source point (x′,y′,z′). Treating the volume charge element as a point charge (dQ), we can use the results already obtained for the field at (x,y,z) due to a point charge at (x′,y′,z′):

$$dE = \mathbf{a}_R \frac{dQ}{4\pi\varepsilon_0 R^2} = \mathbf{a}_R \frac{dQ}{4\pi\varepsilon_0 R^2} = \mathbf{a}_R \frac{\rho_v dv'}{4\pi\varepsilon_0 R^2} = \frac{\mathbf{R}\rho_v dv'}{4\pi\varepsilon_0 R^3} = \frac{(\mathbf{r}-\mathbf{r}')\rho_v dv'}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|^3} \tag{2-26}$$

Note that $dQ = \rho_v\, dv'$. Then, summing all such typical contributions over volume V, we obtain

$$\begin{aligned} \mathbf{E}(x,y,z) &= \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\mathbf{a}_R \rho_v(x',y',z')\, dv'}{R^2} \\ &= \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\mathbf{R}\rho_v(x',y',z')\, dv'}{R^3} \\ &= \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{(\mathbf{r}-\mathbf{r}')\rho_v(x',y',z')\, dv'}{|\mathbf{r}-\mathbf{r}'|^3} \end{aligned} \tag{2-27a}$$

(for volume charge density ρ_v)

where

$$\mathbf{R} = \mathbf{r}-\mathbf{r}' = \mathbf{a}_x(x-x') + \mathbf{a}_y(y-y') + \mathbf{a}_z(z-z')$$

$$R = |\mathbf{r}-\mathbf{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

For the special cases of surface or line charge densities, dQ is replaced with $\rho_s\, ds'$, $\rho_\ell\, d\ell'$, respectively, and integration is carried out over the surface or line to obtain:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\iint_S \frac{\mathbf{R}\rho_S ds'}{R^3} \text{ (for surface charge density } \rho_s) \quad (2\text{-}27b)$$

$$= \frac{1}{4\pi\varepsilon_0}\int_L \frac{\mathbf{R}\rho_l\, dl'}{R^3} \text{ (for line charge density } \rho_1) \quad (2\text{-}27c)$$

For a collection of point charges q_i at source points (x_i, y_i, z_i) we merely sum the contributions to obtain

$$\mathbf{E}(x,y,z) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{\mathbf{R}_i q_i}{R_i^3} \quad (2\text{-}28)$$

where

$$\mathbf{R}_i = \mathbf{a}_x(x - x_i) + \mathbf{a}_y(y - y_i) + \mathbf{a}_z(z - z_i)$$

$$R_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}$$

Note that the inverse square relationship may be represented in terms of $\frac{\mathbf{a}_R}{R^2}$ or $\frac{\mathbf{R}}{R^3}$.

Note that **R** of Eqs. (2-27a,b,c) has a very simple physical interpretation. **R** is *a vector from the variable source point to the fixed field point.* R is the distance from the source point to the field point.

Note also that in Eqs. (2-27a,b,c), there are six independent variables (x, y, z, x′, y′, z′). The integrands are functions of all six variables. The densities $\rho_v(x', y', z')$, $\rho_s(x', y', z')$, $\rho_\ell(x', y', z')$ are expressed in primed coordinates. Integration is carried out over the primed variables. The other independent variables, namely, the unprimed variables (x, y, z), may be considered as constants for purposes of the integration. Integration removes the primed variables leaving an electric field **E**(x, y, z) which is a function of unprimed variables only. To reiterate, *the field point (x, y, z) is fixed during the integration while the source point (x′, y′, z′) varies over volume* V.

Example 2-7 On-Axis Electric Field of a Charged Disk

Consider a disk of uniform surface charge density, with radius a, centered at the origin and lying in the xy plane (Figure 2-7).

$$\rho_s = \rho_{so}(\rho \le a,\ z = 0) \tag{2-29}$$

Find the electric field at points along the z-axis.

Solution:

The electric field is given by

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\iint \frac{\rho_{so}\mathbf{R}\ ds'}{R^3} \tag{2-27b}$$

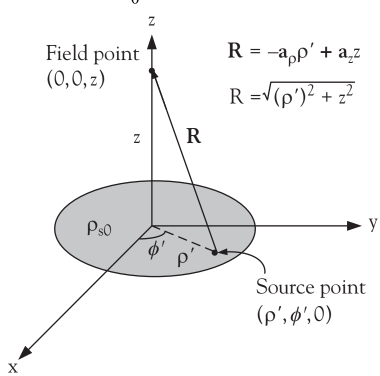


Figure 2-7. ***A uniformly charged disk.***

The electric field along the z axis is to be found. We identify the field point (0, 0, z) and the source point (ρ', φ', 0) in cylindrical coordinates.

Then

$\mathbf{r} = \mathbf{a}_z z$, $\mathbf{r}' = \mathbf{a}_\rho\ \rho'$ (Note that $\mathbf{r}'$ has no $\mathbf{a}_\varphi$ component.)

$\mathbf{R} = \mathbf{a}_z\ z - \mathbf{a}_\rho\ \rho'$ (Note that $\mathbf{a}_\rho$ is not a constant vector, i.e., it varies with φ'.)

$$= \mathbf{a}_z\ z - (\mathbf{a}_x \cos\varphi' + \mathbf{a}_y \sin\varphi')\ \rho'$$

$$= -\ \mathbf{a}_x\ \rho' \cos\varphi' - \mathbf{a}_y\ \rho' \sin\varphi' + \mathbf{a}_z\ z$$

$$ds' = \rho' d\rho' d\phi',\ R = \sqrt{(\rho')^2 + z^2}$$

First we consider the z component of **E:**

$$E_Z = \frac{1}{4\pi\varepsilon_o}\int_0^{2\pi}\int_0^a \frac{\rho_{so} z\rho' d\rho' d\phi'}{[(\rho') + z^2]^{3/2}} = \frac{\rho_{so} z}{2\varepsilon_o}\left[\frac{-1}{\sqrt{(\rho')^2 + z^2}}\right]_0^a$$

$$= \frac{\rho_{so}}{2\varepsilon_o}\left[\frac{z}{\sqrt{z^2}} + \frac{z}{\sqrt{a^2 + z^2}}\right]$$

Note that $\sqrt{z^2} = |z| = \pm z\begin{Bmatrix} +(z>0) \\ -(z<0)\end{Bmatrix}$ and thus E_z may also be represented as follows:

$$E_z = \frac{\rho_{so}}{2\varepsilon_o}\left[\pm 1 - \frac{z}{\sqrt{z^2 + a^2}}\right], \tag{2-30}$$

Note that E_z is odd ($E_z(-z) = -E_z(z)$) as it should be.

As $z \to 0$ and the field point approaches the center of the disk along the positive axis, the disk appears large and $E_z \to \frac{\rho_{so}}{2\varepsilon_o}$ which agrees, as it should, with the electric field of a charged infinite planar surface (Eq. (2-23a)).

As $z \to \infty$ along the positive z axis, the disk appears small and the electric field E_z should approach that of a point charge at the origin:

$$E_z = \frac{\rho_{so}}{2\varepsilon_o}\left[1 - \frac{1}{\sqrt{1 + a^2/z^2}}\right] \approx \frac{\rho_{so}}{2\varepsilon_o}\left(1 - (1 - \frac{a^2}{2z^2})\right)\left(\text{as } \frac{a}{z} = 1\right)$$

$$= \frac{\rho_{so}(\pi a^2)}{4\pi\varepsilon_o z^2} = \frac{Q}{4\pi\varepsilon_o r^2}$$

Thus the electric field is reduced to the appropriate form of Coulomb's law, with the entire charge concentrated at the origin. Here we have used the approximation

$$\frac{1}{\sqrt{1+x}} \approx 1 - \frac{x}{2} \quad (|x| \ll 1) \tag{2-31}$$

which represents the first two terms of the binomial series $(n = -\frac{1}{2})$:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \tag{2-32a}$$

$$\approx 1 + nx \ (|x| \ll 1) \tag{2-32b}$$

The other components, E_x and E_y, are zero along the z axis. This can be determined analytically (see Eq. (2-34) below) or by observing the symmetry of the problem.

Partial Disk

As an extension of Example 2-7, we can also calculate the field along the axis of a partial disk:

$$\rho_s = \rho_{so} (\rho_1 \le \rho \le \rho_2, \ \phi_1 \le \phi \le \phi_2) \tag{2-33}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} \frac{\rho_{so}[-\mathbf{a}_x\rho' \cos\phi' - \mathbf{a}_y\rho' \sin\phi' + \mathbf{a}_z z]\rho' d\rho' d\phi'}{[(\rho')^2 + z^2]^{3/2}} \tag{2-34}$$

First we note that integration of $\cos\varphi'$, $\sin\varphi'$ over the full range ($0 \le \varphi' \le 2\pi$) yields zero for E_x and E_y. The z component of **E** is given by

$$E_z = \frac{\rho_{so} E}{4\pi\varepsilon_o} \left[\frac{1}{\sqrt{z^2 + \rho_1^2}} - \frac{1}{\sqrt{z^2 + \rho_2^2}} \right] (\phi_2 - \phi_1) \quad (2\text{-}35$$

)

You can solve also for E_x, E_y and find the electric fields of an annular disk or a pie-shaped partial disk.

Example 2-8 Electric Fields of a Line Charge of Finite Length

Consider a uniform line charge density along the z axis (Figure 2-8a):

$$\rho_l = \rho_{lo} \left(z_1 \le z \le z_2; \ x = y = 0 \right) \tag{2-36}$$

Find the electric field at an arbitrary point (ρ, φ, z).

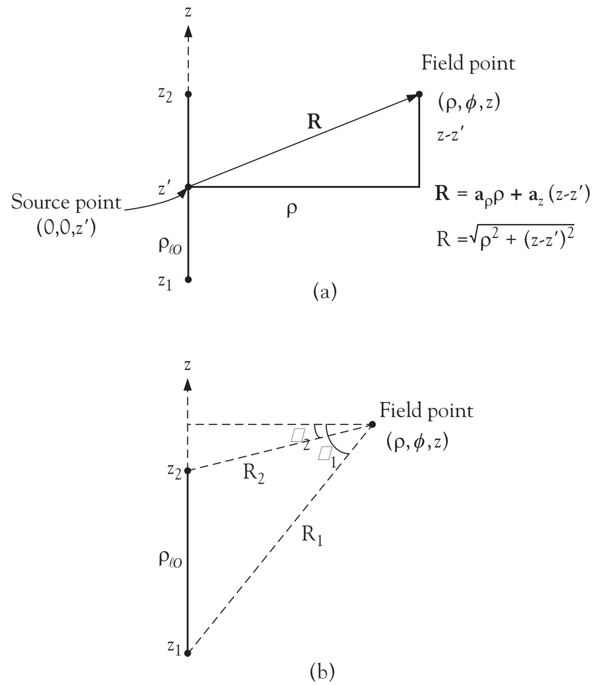


Figure 2-8. ***A uniform line charge density of finite length.***

(a) Source point-field point representation.
(b) Alternative geometry.

Solution:
The electric field is given by

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o} \int \frac{\mathbf{R}\rho_l dl'}{R^3} \qquad (2\text{-}27c)$$

where

$r = a,r + a_z z$, $r' = a_z z'$

$R = a,r + a_z (z - z')$ (a, is fixed for this problem, i.e., independent of z′)

$$R = \sqrt{\rho^2 + (z - z')^2}; \; d\ell' = dz'$$

Thus,

$$\mathbf{E} = \mathbf{a}_\rho E_\rho + \mathbf{a}_z E_z \qquad (2\text{-}37)$$

where

$$E_Z = \frac{1}{4\pi\varepsilon_o}\int_{z_2}^{z_2}\frac{\rho_{1o}(z-z')\,dz'}{[\rho^2+(z-z')^2]^{3/2}}$$

Let u = z – z′, du = – dz′ (z constant)

$$E_Z = \frac{1}{4\pi\varepsilon_o}\int_{z-z_1}^{z-z_2}\frac{\rho_{1o}\,u\,du}{[\rho^2+u^2]^{3/2}} = \frac{\rho_{1o}}{4\ \varepsilon_o}\left[\frac{1}{\sqrt{\rho^2+u^2}}\right]_{z-z_1}^{z-z_2} \qquad (2\text{-}38)$$

$$= \frac{\rho_{1o}}{4\pi\varepsilon_o}\left[\frac{1}{\sqrt{\rho^2+(z-z_2)^2}} - \frac{1}{\sqrt{\rho^2+(z-z_1)^2}}\right]$$

$$= \frac{\rho_{1o}}{4\pi\varepsilon_o}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \text{ (see Figure 2-8b)}$$

$$E_Z = \frac{1}{4\pi\varepsilon_0}\int_{z_1}^{z_2}\frac{\rho_{1o}\,\rho\,dz'}{[\rho^2+(z-z')^2]^{3/2}};\ \text{Let } u = z - z' \qquad (2\text{-}39)$$

$$= \frac{\rho_{1o}}{4\pi\varepsilon_0}\rho\left[\frac{-(z-z_2)}{\sqrt{\rho^2+(z-z')^2}}\right]_{z-z_1}^{z-z_2}$$

$$= \frac{\rho_{1o}}{4\pi\varepsilon_0}\left(\frac{1}{\rho}\right)\left[\frac{z-z_1}{\sqrt{\rho^2+(z-z_1)^2}} - \frac{z-z_2}{\sqrt{\rho^2+(z-z_2)^2}}\right]$$

which may also be represented as

$$E_\rho = \frac{\rho_{1o}}{4\pi\varepsilon_o}\left(\frac{1}{\rho}\right)(\sin\theta_1 - \sin\theta_1) \qquad \text{(see Fig. 2-8(b))} \quad (2\text{-}40)$$

Note that ρ, with no subscript, is cylindrical distance, whereas ρ_v, ρ_s, ρ_ℓ are all charge densities.

2.7. The Scalar Electric Potential V

We noted in Section 2.3 that the electric field is conservative (D × **E** = 0). Therefore it can be expressed, as noted in Section 1.5, as the gradient of a scalar potential V:

$$\mathbf{E} \text{ is the } \mathbf{E} = -\nabla \mathbf{V} \text{ gradient of a scalar} \tag{2-41}$$

As discussed in Section 1.4.2, the gradient represents the maximum rate of change; thus the electric field, which is the negative gradient of V, points in the direction of maximum rate of *decrease* of the scalar potential V. The electric field lines are perpendicular to the level surfaces where V= constant, called the **equipotential surfaces,** and they flow from high potential to low potential.

Eq. (2-41) is a point relationship. We can obtain the corresponding integral relationship by taking the line integral of both sides between two points. The lower limit is an arbitrary reference point (x_o, y_o, z_o) at which the potential is zero. The upper limit is the field point (x,y,z). Since **E** is conservative, the line integral is independent of path

$$-\int_{(x_o,y_o,z_o)}^{(x,y,z)} \mathbf{E}\cdot d\ell = \int_{\text{ref. point}}^{\text{field point}} \nabla V \cdot d\ell$$

$$= \int_{\text{Ref.}}^{(x,y,z)} \left(\mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_z \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z} \right) \cdot (\mathbf{a}_x dx + \mathbf{a}_z dy + \mathbf{a}_z dz)$$

$$= \int_{\text{Ref.}}^{(x,y,z)} \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial y} dy = \int_{\text{Ref.}}^{(x,y,z)} dV = V(x,y,z) - V(x_o,y_o,z_o) = V(x,y,z)$$

$$V(x,y,z) = \int_{\text{Ref.}}^{(x,y,z)} \mathbf{E} \cdot d\ell [V] \tag{2-42}$$

(We have proved here the gradient theorem of Table 1-5).

V(x,y,z) is thus the voltage or potential difference between two points, the field point and the reference point. Whether mentioned or not, there is always an implied reference point. Note the effect of raising all potentials by a constant. The electric field is unchanged (Eq. (2-41)) and the potential difference between any two points is unchanged. Therefore one may set the reference point potential at zero as we have done. The location of the reference point (x_o,y_o,z_o) is arbitrary. It may be chosen for convenience. It should not be changed during the course of a problem. The zero potential reference point may be chosen at the origin, at infinity, or at a finite location. It may be chosen on a ground plane. It can be identified in some cases with a location which we call *ground.* We have made two arbitrary choices, i.e., the level and the location of the reference.

Consider an automobile which is electrostatically charged so that its chassis is 20 volts above the earth potential. The negative terminal of the 12 volt battery chassis is connected to the chassis. Thus the positive end of the battery is 32 volts above earth. What do we call zero potential, the chassis or the earth? The choice is up to you.

Consider a battery of voltage V connected to parallel conducting plates as shown in Figure 2-9.

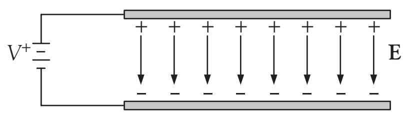


Figure 2-9. ***Parallel plates with applied voltage.***

The electric field **E** points downward as shown in agreement with the force law **F** = q**E.** The **E** field points in the direction of decreasing potential (from high to low potential) in agreement with the minus signs in Eqs. (2-41) and (2-42).

Potential of a Point Charge

Now let us apply these concepts to determine the potential of a point charge q at the origin [Figure 2-10(a)] and at an arbitrary point (x′, y′, z′) [Figure 2-10(b)].

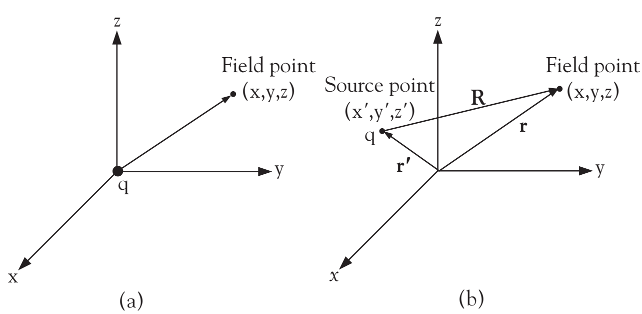


Figure 2-10. ***Electric potential of a point charge q (a) at the origin, and (b) at the source point (x′, y′, z′).***

For a point charge at the origin:

$$E_r = \frac{q}{4\pi\varepsilon_o r^2}$$

$$V(r) = \int_{r_o}^{r} \mathbf{E}\cdot d\ell = -\int_{r_o}^{r} E_r\ dr = -\int_{r_o}^{r} \frac{q\,dr}{4\pi\varepsilon_o r^2} = \left[\frac{q}{4\pi\varepsilon_o r}\right]_{r_o}^{r} = \frac{q}{4\pi\varepsilon_o}\left\{\frac{1}{r} - \frac{1}{r_o}\right\}$$

$$\Rightarrow \frac{q}{4\pi\varepsilon_o r} \text{ as } r_o \to \infty \text{ (reference point at infinity)}$$

$$V(r) = \frac{q}{4\pi\varepsilon_o r} \quad (2\text{-}43)$$

For a point charge at (x',y',z'):

The potential at the field point (x,y,z) depends only on the distance between the field point and the source point (x',y',z'), hence

$$V(x,y,z) = \frac{q}{4\pi\varepsilon_o R} \qquad (2\text{-}44)$$

where $R = |\mathbf{r} - \mathbf{r}'|$.

2.8. Potential of an Arbitrary Charge Distribution

In this section we consider the potential due to arbitrary charge distributions. The method developed here represents an alternative to that of Section 2.6. Figure 2-6 shows an arbitrary volume charge distribution $\rho_v(x',y',z')$ contained within a volume V. We wish to find the potential V(x,y,z) due to the entire charge distribution. First consider the elementary contribution due to a typical volume dv' located at (x',y',z'). We use the results obtained previously for the potential due to a point charge:

$$dV = \frac{dQ}{4\pi\varepsilon_0 R} = \frac{\rho_v dv'}{4\pi\varepsilon_0 R} = \frac{\rho_v dv'}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}'|} \quad (2\text{-}45)$$

Summing all such typical contributions dv:

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho_v(x',y',z')dv'}{R}$$
$$= \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho_v(x',y',z')dv'}{|\mathbf{r}-\mathbf{r}'|} \qquad (2\text{-}46a)$$
$$\text{(for volume charge density } \rho_v\text{)}$$

For the cases of surface or line charge densities, dQ is replaced by ρ_s ds', $\rho_\ell\, d\ell'$ and integration is carried out over the surface or line to obtain

$$V = \frac{1}{4\pi\varepsilon_0}\iint_S \frac{\rho_S ds}{R} \quad \text{(for surface charge density } \rho_S\text{)} \qquad (2\text{-}46b)$$

$$= \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho_l dl'}{R} \quad \text{(for line charge density } \rho_l\text{)} \qquad (2\text{-}46c)$$

For a collection of point charges q_i located at points (x_i, y_i, z_i) we sum contributions to obtain

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{R_i} \qquad (2\text{-}47)$$

where $R_i = \sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}$. The potential integral formulation of Eqs. (2-46a,b,c) represents an alternative to that of Eqs. (2-27a,b,c). Given the charge distribution ρ_v we have two choices for the determination of the electric field:

1. Find **E** directly from Eqs. (2-27a,b,c)
2. Find V directly from Eqs. (2-46a,b,c), then $\mathbf{E} = -\nabla V$.

Each method has advantages and may be simpler for a particular problem. Method 1 involves vector integration but has only one step. Method 2 involves scalar integration but has two steps. We should be prepared to use either method.

To find the potential from the charge we have a choice of two methods:

1. Find V directly from Eqs. (2-46a,b,c)
2. Find **E** directly from Eqs. (2-27a,b,c). Then $V = -\int \mathbf{E} \cdot d\ell$.

Example 2-9. On-Axis Potential of a Charged Disk

Consider the disk of surface charge of Example 2-7 and Figure 2-7:

$$\rho_s = \rho_{so} (\rho_s \le a,\ z = 0) \tag{2-48}$$

Find the potential at points along the z-axis.

Solution:

$$\begin{aligned} V(z) &= \frac{1}{4\pi\varepsilon_0} \iint \frac{\rho_{so}\ ds'}{R} \\ &= \frac{1}{4\pi\varepsilon_0} \int_0^2 \int_0^a \rho_{so} \frac{\rho' d\rho' d\phi'}{\sqrt{(\rho')^2 + z^2}} = \frac{\rho_{so}}{4\pi\varepsilon_0} 2\pi \left[\sqrt{(\rho')^2 + z^2} \right]_0^a \\ &= \frac{\rho_{so}}{2\varepsilon_0} \left[\sqrt{a^2 + z^2} - |z| \right] \end{aligned} \tag{2-49}$$

The electric field E_z along the positive z axis may be obtained by using the relationship $\mathbf{E} = -\nabla V$:

$$E_z = \frac{\partial V(z)}{\partial z} = \frac{\rho_{so}}{2\varepsilon_o} \left[-\frac{z}{\sqrt{a^2 + z^2}} + 1 \right],\ z > 0$$

which agrees with Eq. (2-30).

As $z \to \infty$ the potential approaches that of a point charge at the origin:

$$\begin{aligned} V(z) &= \frac{\rho_{so}}{2\varepsilon_0} \left[z\sqrt{1 + \frac{a^2}{z^2}} - z \right] \to \frac{\rho_{so}}{2\varepsilon_0} \left[z(1 + \frac{a^2}{z^2}) - z \right] \left(\text{as } \frac{a}{z} = 1 \right) \\ &= \frac{\rho_{so} (\pi a^2)}{2\pi\varepsilon_0 z} = \frac{Q}{4\pi\varepsilon_0 R} \end{aligned} \tag{2-50}$$

where we have used the approximation

$$\sqrt{1 + x} \approx 1 + \frac{x}{2} (|x| = 1) \tag{2-51}$$

which represents the first two terms of the binomial series $(n = \frac{1}{2})$:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots \tag{2-32a}$$

$$\approx 1 + nx(|x| = 1) \tag{2-32b}$$

Example 2-10. Potential of a Line Charge of Finite Length
Consider the line charge of Example 2-8 and Figure 2-8:

$$\rho_1 = \rho_{10}(z_1 \le z \le z_2;\ x = y = 0) \tag{2-52}$$

Find the electrostatic potential.
Solution:

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_l dl'}{R}$$

where $R = \sqrt{\rho^2 + (z - z')^2}$; $dl' = dz'$

$$V_{(\rho,z)} = \frac{1}{4\pi\varepsilon_0}\int_{z_1}^{z_2} \frac{\rho_{1o}\ dz'}{\sqrt{\rho^2 + (z - z')^2}} \quad \text{Let } u = z - z';\ du = -dz' \tag{2-53}$$

$$= \frac{-1}{4\pi\varepsilon_0}\int_{z-z_1}^{z-z_2} \frac{\rho_{1o}\ du}{\sqrt{\rho^2 + u^2}} = \frac{-\rho_{1o}}{4\pi\varepsilon_0}\left[\ln\left(u + \sqrt{u^2 + \rho^2}\right)\right]_{z-z_1}^{z-z_2}$$

$$= \frac{\rho_{1o}}{4\pi\varepsilon_0}\ln\left[\frac{z - z_1 + \sqrt{\rho^2 + (z - z_1)^2}}{z - z_2 + \sqrt{\rho^2 + (z - z_2)^2}}\right]$$

You can show that **E** = $-\nabla V$ will lead to the results in Eqs. (2-38) and (2-39) (see Problem 2-30). The following two examples show cases in which the potential can be easily found from the line integral of the electric field.

Example 2-11. Potential of Spherical Volume Charge

Consider the charge distribution of Example 2-3:

$$\rho_v = \rho_o(r \le a) \tag{2-54}$$

Find the electrostatic potential.

Solution:

The electric field is obtained in Example 2-3 and is given by Eq. (2-19):

$$E_r = \left(\frac{r}{a}\right)\frac{Q}{4\pi\varepsilon_0 a^2}\,(r \le a)$$
$$= \frac{Q}{4\pi\varepsilon_0 a^2}\,(r \ge a) \qquad (2\text{-}19)$$

where $Q = \frac{4}{3}\pi a^3 \rho_o$. To find the potential V(r), we integrate the electric field. Since **E** is in the radial ($\mathbf{a}_r$) direction, we choose the integration path to be also in the radial direction: $d\ell = \mathbf{a}_r dr$.

When r ≥ a,

$$V = -\int_{r_o}^{r} \mathbf{E}\cdot d\ell = -\int_{\infty}^{r} E_r\Big|_{r\ge a}\, dr$$
$$= -\int_{\infty}^{r} \frac{Q\,dr}{4\pi\varepsilon_0 r^2} = \frac{Q}{4\pi\varepsilon_0 r^2}\,(r \ge a) \qquad (2\text{-}55a)$$

When r ≤ a,

$$V = -\int_{\infty}^{r} E_r \cdot dr = -\int_{\infty}^{a} E_r\Big|_{r\ge a}\, dr - \int_{a}^{r} E_r\Big|_{r\le a}\, dr \qquad (2\text{-}55b)$$
$$= -\int_{\infty}^{a} \frac{Q\,dr}{4\pi\varepsilon_0 r^2} - \int_{a}^{r}\left(\frac{r}{a}\right)\frac{Q\,dr}{4\pi\varepsilon_0 r^2} = \frac{Q}{8\pi\varepsilon_0 a^3}(3a^2 - r^2)(r \le a)$$

Example 2-12. Potential of a Line Charge of Infinite Length
Consider the infinite-length line charge density of Example 2-5 and Figure 2-4(a). The electric field is

$$E_\rho = \frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{1}{\rho}\right) \qquad (2\text{-}22)$$

Then the potential is given by

$$V(\rho) = -\int_{\rho_o}^{\rho} \mathbf{E}\cdot d\ell = -\int_{\rho_o}^{\rho} E_\rho\, d\rho = \frac{\rho_l}{2\pi\varepsilon_o}\ln\left(\frac{\rho_o}{\rho}\right) \qquad (2\text{-}56)$$

where ρ_o is the reference point. Note that, for this charge distribution of infinite extent, the reference point must remain finite. Infinity cannot be chosen as a reference point because the potential will not be finite at infinity.

Table 2-1 summarizes the electric fields and potentials of certain simple charge distributions. Electric fields are obtained by Gauss' law. Potentials are obtained by the line integral $-\int \mathbf{E} \cdot d\ell$. Note the simple dependence on distance for the particular cases shown here:

$$\text{For spherical symmetry } \left(\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\phi} = 0\right) : |\mathbf{E}| : \frac{1}{r^2};\ V : \frac{1}{r} \qquad (2\text{-}57)$$

$$\text{For cylindrical symmetry } \left(\frac{\partial}{\partial\phi} = \frac{\partial}{\partial z} = 0\right) : |\mathbf{E}| : \frac{1}{\rho};\ V : \ln\rho \qquad (2\text{-}58)$$

$$\text{For planar symmetry } \left(\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0\right) : |\mathbf{E} \text{ is constant};\ V : x \qquad (2\text{-}59)$$

Table 2-1 Basic Electrostatic Fields and Potentials

Charge Distribution	**Electric Fields**	**Electrostatic Potential**
Point Charge at origin	$E_r = \dfrac{q}{4\pi\varepsilon_0 r^2}$	$V(r) = \dfrac{q}{4\pi\varepsilon_0 r}$
Spherical Shell Surface Charge $\rho_s = \rho_{so}$ (r=a) $q = 4\pi_{a2}\ \rho_{so}$	$E_r = \dfrac{q}{4\pi\varepsilon_0 r^2}\ (r > a)$ $= 0\ (r < a)$	$V(r) = \dfrac{q}{4\pi\varepsilon_0 r^2}\ (r \geq a)$ $= \dfrac{q}{4\pi\varepsilon_0 a}\ (r \geq a)$
Line Charge on z axis	$E_\rho = \dfrac{\rho_l}{2\pi\varepsilon_0}\left(\dfrac{1}{\rho}\right)$	$V(\rho) = \dfrac{\rho_l}{2\pi\varepsilon_0}\ln\left(\dfrac{\rho}{\rho_o}\right)$ Reference Point: $\rho = \rho_o$
Cylindrical Shell Surface Charge $\rho_s = \rho_{so}$ (ρ=a) $\rho_\ell = 2\pi a\rho so$	$E_\rho = \dfrac{\rho_l}{2\pi\varepsilon_0}\left(\dfrac{1}{\rho}\right)(\rho > a)$ $= 0\ (\rho < a)$	$V(\rho) = -\dfrac{\rho_l}{2\pi\varepsilon_0}\ln\left(\dfrac{\rho}{a}\right)(\rho \geq a)$ $= 0\ (\rho \leq a)$ Reference Point: $\rho = a$

Plane of Surface Charge Surface Charge Density ρ_s at x=0	$E_x = \pm \frac{\rho_S}{2\varepsilon_0} \begin{matrix} +(x>0) \\ -(x<0) \end{matrix}$	$V(x) = m\frac{\rho_S}{2\varepsilon_0}$ Reference Point: x = 0
Parallel Plates ρ_s at x=0 $-\rho_s$ at x=d	$E_x = \frac{\rho_S}{\varepsilon_0}(0<x<d)$ $= 0\ (x<0,\ x>d)$	$V(x) = -\frac{\rho_S}{\varepsilon_0}x(0 \le x \le d)$ Reference Point: x = 0

2.9. CONDUCTORS

Conductors are materials which possess a large number of electrons which are free to move under the influence of an electric field. The conductor has the property that, under static conditions, there is no charge or electric field within the conductor:

$$\mathbf{E} = 0$$
$$\rho_v = 0$$
inside a conductor (2-60) (2-61)

Let's see how this comes about. Consider a solid conducting sphere. Suppose that some excess charges are placed at the center of the sphere. An electric field is produced and the field exerts a force on the charges, making them move away from one another. Eventually charges will reach the surface in a symmetrical surface charge distribution (Figure 2-11(a)) so that both the charge and the field vanish inside. Gauss' law reveals that the electric field is zero inside (see Example 2-4).

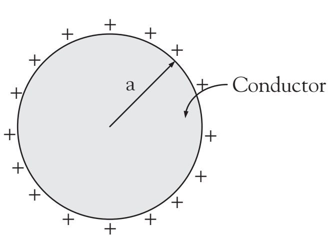


Figure 2-11(a). ***A symmetric surface charge distribution for a spherical conductor.***

For an arbitrarily-shaped conductor (Figure 2-11(b)) charges placed inside lead to a non-uniform surface charge distribution which produces zero electric field inside. If the electric field were not zero inside, charges would continue to move until equilibrium (**E** = 0) was reached.

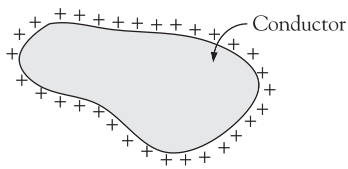


Figure 2-11(b). *A surface charge distribution for an arbitrarily-shaped conductor.*

Finally, consider a conductor in the presence of an applied electric field $\mathbf{E}_1$ (Figure 2-11(c)). Charge movement due to $\mathbf{E}_1$ sets up an opposing field. The process continues until the opposing field exactly cancels the applied field $\mathbf{E}_1$. The total electric field is then zero inside the conductor.

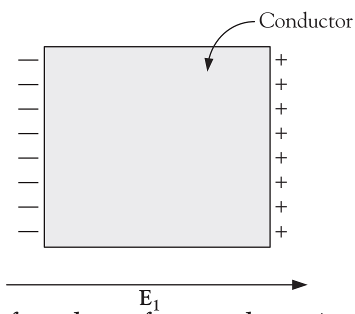


Figure 2-11(c). *A surface charge for a conductor in the presence of an applied field.*

Thus, if charges are placed within a conductor or if external electric fields are applied, charges move quickly to set both **E** and ρ_v equal to zero within the conductor. The process is nearly instantaneous as we shall see later in Chapter 5.

For an ideal conductor, we may also draw the following conclusion.

$$V = \text{constant throughout the conductor}$$

The potential is constant throughout a conductor including the surface because the potential difference, i.e., the line integral $-\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell$, between any two points P_1 and P_2 in the conductor, is zero since **E** = 0 in the conductor. Thus, the conductor surface is an *equipotential surface.*

Let's consider some of the characteristics of conductors and their charges. First we will consider a solid conductor with a net positive charge as shown in Figure 2-12(a).

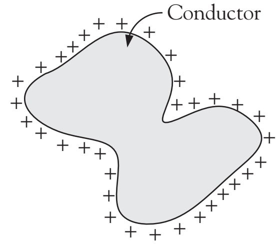


Figure 2-12(a). ***A charged conductor.***

The charge will distribute itself over the surface of the conductor in such a way as to produce zero field inside. In the absence of external sources or applied fields the surface charge density will be positive everywhere. A negative charge density is ruled out because it implies electric field lines beginning and ending on the same conductor which contradicts the assumption of a constant potential. The surface charge density may vary considerably over the conductor. We may encounter very large densities near sharp edges or corners, but the charge density cannot be negative anywhere. We can of course reverse the words positive and negative above.

If, however, there are external sources near the conductor, or external applied fields such as that of Figure 2-11(c), then the charge density may be positive in some locations and negative in others. For example, consider a point charge q located near an uncharged conductor as in Figure 2-12(b). If q is positive, negative charge is drawn to the near vicinity of q leaving a positive charge elsewhere on the conductor as shown in Figure 2-12(b).

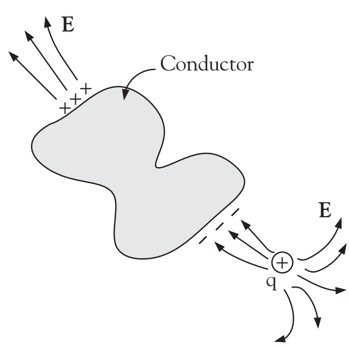


Figure 2-12(b). ***An uncharged conductor with charge source nearby.***

The conductor provides a brief interruption of some of the field lines of **E** all of which go to infinity eventually. Can you explain what will happen if the conductor of Figure 2-12(b) carries a small net charge?

Now consider a charged hollow conductor, i.e., a conductor with a cavity V_0 bounded by surface S_0 (Figure 2-13(a)).

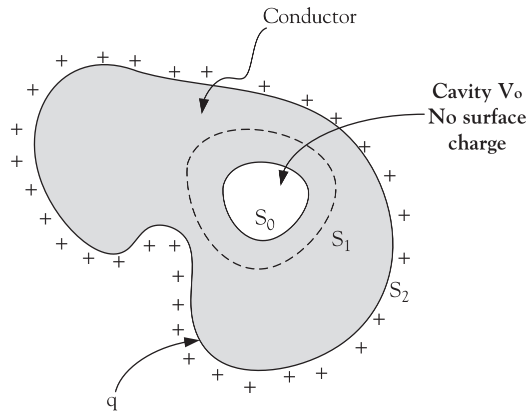


Figure 2-13(a). ***A charged hollow conductor.***

Let there be a net charge q on the conductor; there is no charge in the empty cavity V_0. Now where does the charge reside? Is there any on the inner surface So? First we apply Gauss' law over the surface S_1 to conclude that the total net charge enclosed by S_1 is zero because $\oiint \mathbf{E} \cdot d\mathbf{s} = (\mathbf{E} = 0)$ inside a conductor). Could there be both positive and negative surface charge densities on surface S_0? This would imply field lines beginning and ending on S_0 which is impossible since it is an equipotential surface. Therefore there is no charge on S_0. Now what about the electric field in V_0? Is it zero also? If there were electric field lines in V_0 they must begin and end on surface S_0, leading again to a contradiction. Therefore the electric field is zero in V_0. Finally the net charge q must reside on the outer surface S_2.

Thus *the electric field is zero inside a hollow charged conductor.* This curious fact was first noted in 1755 by Benjamin Franklin who recognized that the fact was significant and communicated it to Joseph Priestley. Priestley concluded that the electric field satisfied an inverse square law similar to that of gravity; this conclusion was reached long before Coulomb's discovery.

Now consider the same problem (the hollow conductor has a net charge q) but with a charge q_o placed inside the cavity (Figure 2-13(b)). The charge q_o will produce an electric field in the cavity, which will draw the negative charge on the inner surface S_o.

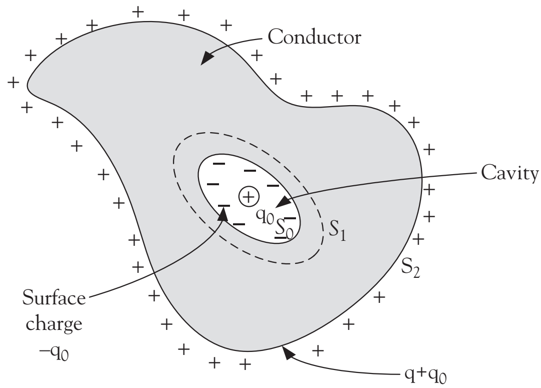


Figure 2-13(b). ***A charged hollow conductor with source inside.***

Now an application of Gauss' law over S_1 again leads to the conclusion that the net charge enclosed by S_1 is zero. Therefore a charge $-q_0$ must reside on surface S_0. Finally a net charge $q + q_0$ must reside on the outer surface S_2 so that the net total charge carried by the conductor is q. Note that in the problems shown in Figure 2-13 the electric field is zero in the conductor. Therefore no electric field lines can pass through the conductor and the interior problem is therefore isolated to some degree from the exterior problem.

Figure 2-14 shows an air-conductor interface. We can determine the behavior of the electric field at the surface. The tangential component is zero and the normal component is ρ_s/ε_0.

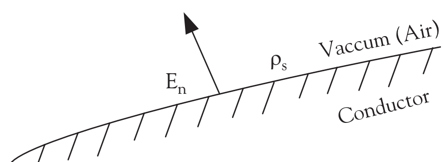


Figure 2-14. ***An air-conductor interface.***

$$E_t = 0 \tag{2-62}$$

at an air-conductor interface

$$E_n = \frac{\rho_s}{\varepsilon_o} \tag{2-63}$$

Since $E = -\nabla V$ and the conductor surface is an equipotential surface, the electric field lines are normal to the surface of the conductor. Therefore the tangential component of the electric field is zero and only the normal component of **E** exists at the conductor surface. A rigorous derivation of these results will be presented in Chapter 3.

This section has been devoted to the characteristics of ideal or perfect conductors. Metals such as copper, gold, silver, brass, aluminum are almost perfect conductors. In Chapter 5, *conductivity* is defined. The **perfect conductor** corresponds to an *infinite conductivity.* Imperfect conductors or resistors are treated in Chapter 5. In our next chapter we discuss **perfect dielectrics** or insulators which represent the other extreme of conductivity, namely, *zero conductivity.*

Example 2-13. Point Charge Within a Charged Conducting Spherical Shell

Consider a point charge Q_1 placed at the center of a conducting spherical shell of inner and outer radii a, b. *The shell carries a net charge Q_2*. Find the surface charge and the electric field everywhere.

Solution:

First we find the surface charge which is distributed as shown in Figure 2-15.

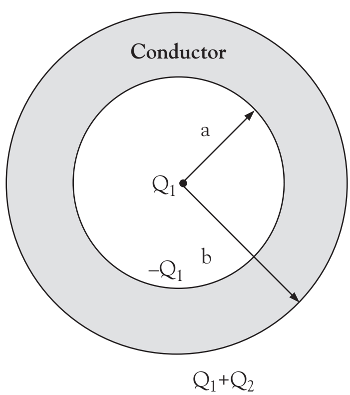


Figure 2-15. *A point charge within a conducting shell.*

A surface charge $-Q_1$ is drawn to the surface r = a to terminate the field lines emanating from the point charge Q_1. The charge $Q_1 + Q_2$ must reside on the surface r = b to make the net total charge in the conductor Q_2. The surface charge is symmetrically distributed because of a spherical shape and

$$\rho_s = -\frac{Q_1}{4\pi a^2} \text{ on the inner surface } (r = a) \quad (2\text{-}64a)$$

$$= \frac{Q_1 + Q_2}{4\pi b^2} \text{ on the outer surface } (r = b) \quad (2\text{-}64b)$$

Applying Gauss' law for a sphere of radius r, we obtain

$$E_r = \frac{Q_1}{4\pi\varepsilon_o r^2}(r < a) \quad (2\text{-}65a)$$

$$= 0(a < r < b) \text{ inside the conductor} \quad (2\text{-}65b)$$

$$= \frac{Q_1 + Q_2}{4\pi\varepsilon_o r^2}(r > b) \quad (2\text{-}65c)$$

The electrostatic potential can also be obtained from **E** (see Problem 2-51).

2.10. The Electric Dipole

Figure 2-16 shows an **electric dipole** consisting of charges ±q separated by a small distance d. The *dipole moment* **p** is defined as follows:

$$\mathbf{p} = \mathbf{a}_z qd = \mathbf{a}_z p \quad (2\text{-}66)$$

We are interested in the electric potential V and the electric field **E** at far field point (r, θ, φ). We assume that r >> d, i.e., $\frac{d}{r} << 1$ since we are interested in the potential and fields at distances r large compared to the separation d. Using superposition:

$$V(r,\theta) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{q(r_2 - r_1)}{4\pi\varepsilon_0 r_1 r_2}$$

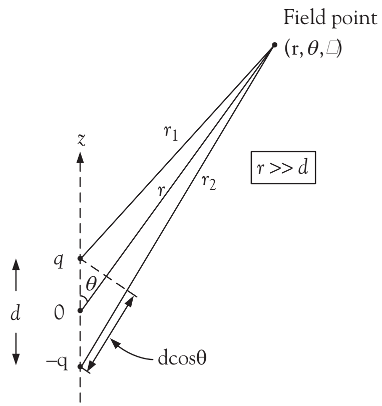


Figure 2-16. ***A dipole.***

We can see in Figure 2-16 that $r_2 - r_1 \approx d\cos\theta$. However, let's show this by an alternative method. Using the approximation $\sqrt{1 \pm x} \approx 1 \pm \frac{x}{2} (x << 1)$
,

$$r_1 = \sqrt{r^2 + \left(\frac{d}{2}\right)^2 - 2r\frac{d}{2}\cos\theta} \text{ (using the law of cosines)}$$

$$= r\sqrt{1 - \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}}$$

$$\approx r\left(1 - \frac{d}{2r}\cos\theta\right) = r - \frac{d}{2}\cos\theta \text{ (we used } \sqrt{1+x} \approx 1 + \frac{x}{2}, |x| = 1)$$

$$r_2 = \sqrt{r^2 + \left(\frac{d}{2}\right)^2 - 2r\frac{d}{2}\cos(\quad - \theta)} \approx r - \frac{d}{2}\cos(\quad - \theta) = r + \frac{d}{2}\cos\theta$$

Note that r_1 and r_2 are arbitrarily close to r (percentage wise) and that the difference $r_2 - r_1$ is equal to $d\cos\theta$ for $\frac{d}{r} << 1$.

$$V(r,\theta) = \frac{qd\cos\theta}{4\pi\varepsilon_o r^2} = \frac{p\cos\theta}{4\pi\varepsilon_o r^2} = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\varepsilon_o r^3}$$

where we have used $\mathbf{p} = \mathbf{a}_z qd$ and $\mathbf{r} = \mathbf{a}_r r$.

$$V(r,\theta) = \frac{p\cos\theta}{4\pi\varepsilon_o r^2} \quad \text{Potential of an electric dipole} \tag{2-67}$$

The electric field **E** can be found by taking the gradient of the potential in spherical coordinates:

$$\mathbf{E} = -\nabla V = -\left[\mathbf{a}_r \frac{\partial V}{\partial r} + \mathbf{a}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta} + \mathbf{a}_\phi \frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi}\right]$$

$$\mathbf{E} = \frac{qd}{4\pi\varepsilon_o r^3}(\mathbf{a}_r 2\cos\theta + \mathbf{a}_\theta \sin\theta) \ \mathbf{E} \text{ field of a dipole} \tag{2-68}$$

We note that the potential and the electric field of a dipole varies with distance r as $\frac{1}{r^2}$ and $\frac{1}{r^3}$, respectively. Thus both the potential and electric field fall off more rapidly with distance (by a factor of $\frac{1}{r}$) than do those of a point charge. The reason for this is that the dipole consists of positive and negative charges whose potentials nearly cancel, the cancellation becoming more nearly perfect as r increases. Secondly, we note the variation with angle θ. The potential is maximum, minimum at θ = 0, π, respectively. It is zero in the plane $\theta = \frac{\pi}{2}$. The electric field also depends strongly on the dipole orientation. $E_\theta = 0$ along the polar axis (θ = 0) and $E_r = 0$ in the equatorial plane $(\theta = \frac{\pi}{2})$.

Higher-order multipoles may be formed merely by combining lower-order multipoles, one reversed with respect to the other. A *quadrupole* is shown in Figure 2-17(a). It is formed by placing one dipole on the left and bringing up a reversed dipole from the right. Figure 2-17(b) shows a linear quadrupole. Can you describe how it is formed? Figure 2-18 shows the atomic structure of crystal salt which forms an *octopole.* It is constructed by forming a quadrupole on the front and then bringing up a reversed quadrupole from the back. The quadrupole and octopole have potentials which fall off as $\frac{1}{r^3}$ and $\frac{1}{r^4}$, respectively, because of higher-order cancellation of potentials. If we consider the sequence of elements: point charge, dipole, quadrupole, octopole, then each element has potentials and fields which fall off faster by the factor $\left(\frac{1}{r}\right)$ than those of the preceding element.

For dielectrics, which are treated in the next chapter, the dipole is the principal contributor to the electric potential and electric fields.

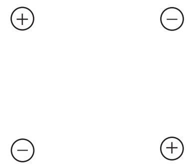


Figure 2-17 (a). ***A quadrupole.***

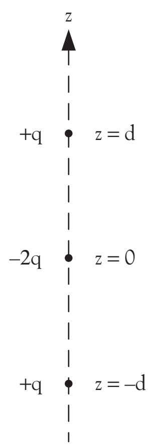


Figure 2-17(b). ***A linear quadrupole.***

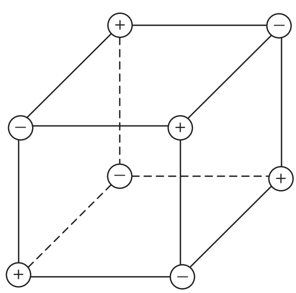


Figure 2-18. An octopole.

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Principles of Electromagnetics 1—Understanding Vectors & Electrostatic Fields

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