Principles of Electromagnetics 3—Magneto Statics

Arlon T. Adams
Jay K. Lee

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Brief Contents

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Preface

Electromagnetics is not an easy subject for students. The subject presents a number of challenges, such as: new math, new physics, new geometry, new insights and difficult problems. As a result, every aspect needs to be presented to students carefully, with thorough mathematics and strong physical insights and even alternative ways of viewing and formulating the subject. The theoretician James Clerk Maxwell and the experimentalist Michael Faraday, both shown on the cover, had high respect for physical insights.

This book is written primarily as a text for an undergraduate course in electromagnetics, taken by junior and senior engineering and physics students. The book can also serve as a text for beginning graduate courses by including advanced subjects and problems. The book has been thoroughly class-tested for many years for a two-semester Electromagnetics course at Syracuse University for electrical engineering and physics students. It could also be used for a one-semester course, covering up through Chapter 8 and perhaps skipping Chapter 4 and some other parts. For a one-semester course with more emphasis on waves, the instructor could briefly cover basic materials from statics (mainly Chapters 2 and 6) and then cover Chapters 8 through 12.

The authors have attempted to explain the difficult concepts of electromagnetic theory in a way that students can readily understand and follow, without omitting the important details critical to a solid understanding of a subject. We have included a large number of examples, summary tables, alternative formulations, whenever possible, and homework problems. The examples explain the basic approach, leading the students step by step, slowly at first, to the conclusion. Then special cases and limiting cases are examined to draw out analogies, physical insights and their interpretation. Finally, a very extensive set of problems enables the instructor to teach the course for several years without repeating problem assignments. Answers to selected problems at the end allow students to check if their answers are correct.

During our years of teaching electromagnetics, we became interested in its historical aspects and found it useful and instructive to introduce stories of the basic discoveries into the classroom. We have included short biographical sketches of some of the leading figures of electromagnetics, including Josiah Willard Gibbs, Charles Augustin Coulomb, Benjamin Franklin, Pierre Simon de Laplace, Georg Simon Ohm, Andre Marie Ampère, Joseph Henry, Michael Faraday, and James Clerk Maxwell.

The text incorporates some unique features that include:

- Coordinate transformations in 2D (Figures 1-11, 1-12).
- Summary tables, such as Table 2-1, 4-1, 6-1, 10-1.
- Repeated use of equivalent forms with R (conceptual) and |r−r′| (mathematical) for the distance between the source point and the field point as in Eqs. (2-27), (2-46), (6-18), (6-19), (12-21).
- Intuitive derivation of equivalent bound charges from polarization sources, including piecewise approximation to non-uniform polarization (Section 3.3).
- Self-field (Section 3.8).
- Concept of the equivalent problem in the method of images (Section 4.3).
- Intuitive derivation of equivalent bound currents from magnetization sources, including piecewise approximation to non-uniform magnetization (Section 7.3).
- Thorough treatment of Faraday's law and experiments (Sections 8.3, 8.4).
- Uniform plane waves propagating in arbitrary direction (Section 9.4.1).
- Treatment of total internal reflection (Section 10.4).
- Transmission line equations from field theory (Section 11.7.2).
- Presentation of the retarded potential formulation in Chapter 12.
- Interpretation of the Hertzian dipole fields (Section 12.3).

Finally, we would like to acknowledge all those who contributed to the textbook. First of all, we would like to thank all of the undergraduate and graduate students, too numerous to mention, whose comments and suggestions have proven invaluable. As well, one million thanks go to Ms. Brenda Flowers for typing the entire manuscript and making corrections numerous times. We also wish to express our gratitude to Dr. Eunseok Park, Professor Tae Hoon Yoo, Dr. Gokhan Aydin, and Mr. Walid M. G. Dyab for drawing figures and plotting curves, and to Professor Mahmoud El Sabbagh for reviewing the manuscript. Thanks go to the University of Poitiers, France and Seoul National University, Korea where an office and academic facilities were provided to Professor Adams and Professor Lee, respectively, during their sabbatical years. Thanks especially to Syracuse University where we taught for a total of over 50 years. Comments and suggestions from readers would be most welcome.

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CHAPTER 1

Introduction to Magnetic Fields

1.1 Introduction

In the year 1820, a startling discovery was announced. A Danish scientist Hans Christian Oersted reported that an electric current flowing in a wire produced a magnetic field. The magnetic field was perpendicular to the wire and consisted of circles about the axis of the wire. His original paper, written in Latin, was published in July 1820 and was immediately translated into several languages. Before Oersted's discovery, most scientists had assumed that electricity and magnetism were unconnected. The two fields had been studied separately for centuries and no one had theorized any relationship between them, although a few researchers had begun to speculate that the two sciences might be connected in some way. Oersted proved that a steady or dc electric current produced a steady or *static* magnetic field. His finding had apparently come about during a classroom demonstration designed to show that electric current in a wire caused the wire to glow.

When the wire was connected to a battery, a compass needle nearby suddenly moved to a position nearly perpendicular to the wire. Eureka! Electricity and magnetism could no longer be viewed as separate phenomena. They were connected in some as yet unknown manner.

The response to Oersted's discovery was striking. Oersted's work was reported to the French Academy in September 1820. New results were then reported by Ampere and Biot almost every week. Ampère published ten papers in 1820; Biot published four papers that year, including the beginnings of the Biot-Savart law. By the end of 1820, they had begun to delineate the relation between electricity and magnetism.

In this chapter, we consider steady (dc) electric currents in free space. The resultant magnetic fields are constant, i.e., they do not vary with time. Thus we are concerned with *static magnetic* or *magnetostatic* fields. In practical situations, we often deal with ac current sources. The magnetostatic analysis is very useful for situations in which fields are varying slowly with time. The study of magnetostatics yields insight into fundamental magnetic phenomena.

1.2 Magnetic Field

We define the **magnetic induction** or **magnetic flux density B** by considering the forces on a small test charge q. Suppose the charge q is placed in the region where both the electric field **E** and the magnetic field **B** exist. If the charge is at rest, it experiences only the electric force $\mathbf{F}_e$ where

$$F_e = q\mathbf{E} \tag{1-1}$$

If the charge is in motion with velocity **v,** then it experiences an additional force $\mathbf{F}_m$, the *magnetic force,* where

$$F = q\,\mathbf{v} \times \mathbf{B} \tag{1-2}$$

The total force **F** on the moving test charge is

$$F = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \text{ Lorentz Force} \tag{1-3}$$

Eq. (1-3) is called the **Lorentz force equation.** It can serve as a definition of both **E** and **B** which can be determined by measurement. For a stationary test charge, **v** = 0, and **E** can be determined by a single measurement of force. Once **E** is known, the components of **B** may be determined by two different measurements of force **F** for two orthogonal velocities **v** (see Example 1-1). Although **B** is formally named *magnetic flux density* or *magnetic induction,* people often call it simply **magnetic field.** We will also use the latter term. The unit of **B** is the tesla [T]. From Eq. (1-3) we see that the unit of **B** can be represented as force divided by charge times velocity:

$$1\text{ T} = \frac{\text{N}}{\text{C}\frac{\text{m}}{\text{sec}}} = \frac{\text{Nsec}}{\text{Cm}}$$

Example 1-1. Determination of Magnetic Flux Density **B** from Measurements

Determine **B** from the Lorentz force equation and measurements.

Solution:

Assume that **E** has already been determined from a rest velocity measurement (**v** = 0) of the force and therefore components E_x, E_y, E_z are known.

Consider a velocity **v** in the x direction only. We move the charge at velocity $\mathbf{v} = \mathbf{a}_x v_x$ and measure the force **F.**

$$\mathbf{v} = \mathbf{a}_x v_x ;\ \text{Let } \mathbf{B} = \mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z$$

$$\mathbf{F} = q\,\mathbf{E} + q\,\mathbf{v} \times \mathbf{B} = \mathbf{a}_x F_x + \mathbf{a}_y F_y + \mathbf{a}_z F_z$$

$$\mathbf{F} = q\,\mathbf{E} + q(\mathbf{a}_x \mathbf{v}_x) \times [\mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z]$$

$$= \mathbf{a}_x (qE_x) + \mathbf{a}_y (qE_y)\mathbf{a}_z (qE_z) + \mathbf{a}_z (qv_x B_y) - \mathbf{a}_y (qv_x B_z)$$

Taking z components, we obtain $B_y = \frac{F_z}{q\,v_x} - \frac{E_z}{v_x}$

Taking y components, we obtain $B_z = \frac{-F_y}{q\,v_x} - \frac{E_y}{v_x}$

You can complete the determination of **B** by considering a velocity in either the y or z direction. Therefore **E, B** can be determined by the Lorentz force equation and three measurements of vector force **F.**

1.3 Basic Laws of Magnetostatics

The basic laws of magnetostatics specify the divergence and curl of the static magnetic field **B** in free space just as the basic laws of electrostatics specify the divergence and curl of **E.** The first law, in point form (or differential form) and integral form, is

$$\nabla \cdot \mathbf{B} = 0 \quad \mathbf{B} \text{ is a solenoidal field} \tag{1-4}$$

and

$$\oiint_S \mathbf{B} \cdot d\mathbf{s} = 0 \tag{1-5}$$

where S is any *closed* surface. The integral form $\oiint_S \mathbf{B} \cdot d\mathbf{s} = 0$ may be derived from the point form by integrating both sides over an arbitrary volume V and applying the divergence theorem: $\iiint_V \nabla \cdot \mathbf{B} \cdot dv = \oiint_S \mathbf{B} \cdot d\mathbf{s} = 0$ where S bounds V. It is also possible to derive the point form from the integral form; the point and integral forms are thus identical statements. It can also be shown from Eq. (6-5) that the magnetic flux $\iint_S \mathbf{B} \cdot d\mathbf{s}$ through an open surface S is *independent of surface* as long as the surfaces are bounded by the same contour C. This is a property of a *solenoidal* field just as the line integral is independent of path for conservative fields.

The second law, in point and integral form, is

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad (1\text{-}6)$$

and

$$\oint_C \mathbf{B} \cdot d\ell = \mu_0 I_{enc} \quad \text{Ampère's law} \qquad (1\text{-}7)$$

where I_{enc} is the total current enclosed by C and $\mu_0 = 4\pi \times 10^{-7}$ [T · m/A] or [H/m]. μ_0 is a universal constant, called the *permeability of free space.* **J** is the volume current density measured in A/m^2. The direction of I_{enc} is related to C by a right-hand rule.

The integral form may be derived from the point form. Integrate both sides of the point form over an arbitrary surface S and apply Stokes' theorem:

$$\iint_S \nabla \times \mathbf{B} \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\ell = \mu_0 \iint_S \mathbf{J} \cdot d\mathbf{s} = \mu_0 I_{enc}$$

Stokes' theorem specifies a right-hand rule relationship between C and $\iint_S \mathbf{J} \cdot d\mathbf{s}$ is the current I_{enc} in the direction ds. Therefore *I_{enc} is the total current passing through S (enclosed by C) in a direction ds related by a right-hand rule to path C.* The second law is called **Ampère's law.** The point form of the second law may also be derived from the integral form and thus the point and integral forms are identical statements. The first and second laws may also be derived from the Biot-Savart law in Section 1.5.

1.4 Ampère's Law

Ampère's law,

$$\oint_C \mathbf{B} \cdot d\ell = \mu_0 \iint_S \mathbf{J} \cdot d\mathbf{s} = \mu_0 I_{enc}$$

allows us to treat certain magnetostatic problems involving symmetry to obtain the magnetic field **B,** given the current distributions (**J, J**$_s$, **I**). In the expression above, the closed path C bounds the surface S and the directions of C and d**s** are related by a right-hand rule, as in Stokes' theorem. I_{enc} is the total current passing through the surface S, in the direction d**s,** enclosed by the closed path C.

We assume that current **J** is specified and thus the right-hand side of Eq. (1-7) is known or can be determined. **B** is unknown. If **B** were parallel to path C and constant in magnitude over the path chosen, then we could take the unknown outside the integral and solve for **B.** The key to an Ampère's law problem is to choose a closed path or contour C so that it is parallel to the **B** field lines and the magnitude of **B** is constant along the path C. The examples which follow will give us some idea of the applicability of Ampère's law. They will also introduce us to some typical field configurations in magnetostatics. The steps of applying Ampère's law are as follows:

1. Recognize the symmetry.
2. Sketch the magnetic field lines.
3. Choose C parallel to the field lines. Form a closed contour.
4. Solve for B.

Example 1-2. Current-Carrying Filaments, Wires and Cylinders.

(a) *A Current-Carrying Filament*

Figure 1-1 shows a straight current-carrying filament of infinite length which lies along the z axis.

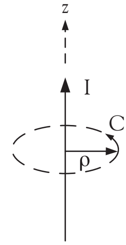


Figure 1-1. A *current-carrying filament*

The filament is of zero cross-sectional area and carries a current I. We first note that, by the curl relationship between **B** and **J,** the magnetic field **B** will curl or circulate around the z-directed current source (I); we expect that **B** is φ-directed (circumferential), i.e., $\mathbf{B} = \mathbf{a}_\varphi B_\varphi$. We have also seen a φ-directed magnetic field in Oersted's 1820 experiment. Figure 1-1 also shows a circle C of radius ρ, centered on the z axis and lying in a plane z = constant. $\oint_C \mathbf{B} \cdot d\ell$ becomes $\int B_\phi \rho d\phi$ and B_φ is constant over C by rotational symmetry ($\partial/\partial\varphi = 0$). The current I_{enc} enclosed by C, in a direction related to that of C by a right-hand rule, is I. Then application of Ampère's law, Eq. (1-7), leads to

$$\oint_C \mathbf{B} \cdot d\ell = \int_0^{2\pi} B\phi\rho d\phi = B_\phi 2\pi\rho = \mu_0 I$$

and

$$B_\phi = \frac{\mu_0 I}{2\pi\rho} \tag{1-8}$$

Note that Eq. (1-8) satisfies both basic laws, i.e., Eqs. (1-4) and (1-6).

(b) *A Current-Carrying Wire of Radius a*

Figure 1-2 shows a current-carrying wire of radius a with current I. The current is assumed to be *uniformly* distributed throughout the cross section and thus the volume current density is

$$\mathbf{J} = \mathbf{a}_z \left(\frac{I}{\pi a^2} \right) \; [\text{A/m}^2]$$

This is a two-region problem. **B** is circumferential, i.e., $\mathbf{B} = \mathbf{a}_\varphi B_\varphi$. Figure 1-2 shows circle C of radius ρ about the axis for the two cases $\rho \leq a$ and $\rho \geq a$. Circles C lie in constant z planes. B_φ will again be constant along C.

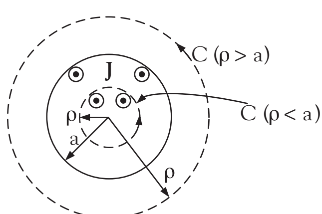


Figure 1-2. A *current-carrying wire*

Consider first the field *inside* the wire ($\rho \le a$):

$$\oint_C \mathbf{B} \cdot d\ell = B_\phi 2\pi\rho = \mu_0 \iint_S \mathbf{J} \cdot d\mathbf{s} = \mu_0 \int_0^{2\pi}\int_0^{\rho} \left(\frac{I}{\pi a^2}\right) \rho d\rho d\phi$$

$$= \mu_0 \left(\frac{I}{\pi a^2}\right) \pi\rho^2 = \frac{\mu_0 I \rho^2}{a^2} (\rho \le a)$$

Then

$$B_\phi = \frac{\mu_0 I \rho}{2\pi a^2} \ (\rho \le a)$$

Next consider the field *outside* the wire ($\rho \ge a$):

$\oint_C \mathbf{B} \cdot d\ell = B_\phi 2\pi Á = \mu_0 I \ (\rho \le a)$ (since the entire current I is enclosed),

and

$$B_\phi = \frac{\mu_0 \ I}{2\pi\rho} \ (\rho \le a)$$

Collecting expressions for B_φ:

$$B_\phi = \frac{\mu_0 I \rho}{2\pi a^2} \ (Á \le a) \quad (1\text{-}9a)$$

$$= \frac{\mu_0 I}{2\pi Á} \ (\rho \ge a) \quad (1\text{-}9b)$$

Notice that the field B_φ outside ($\rho \ge a$) is identical to the field of a filament.

(c) *A Current-Carrying Hollow Cylinder*

Figure 1-3 shows a hollow current-carrying cylinder of radius a and infinite length. The cylinder carries a longitudinal (z-directed) current I. The current is *uniformly* distributed over the surface and thus the surface current density is

$$\mathbf{J} = \mathbf{a}_z \left(\frac{I}{2\pi a}\right) \ [A/m] \quad (1\text{-}10)$$

This is also a two-region problem. **B** will be circumferential. Now consider circles C of radius ρ for the two cases ($\rho < a$) and ($\rho > a$), as shown in Figure 1-3:

$$\oint_C \mathbf{B} \cdot d\ell = B_\phi 2\pi\rho = 0 \ (\rho < a) \text{ (since no current I is enclosed)}$$

$$= \mu_0 I \ (\rho > a) \text{ (since the total current I is enclosed)}$$

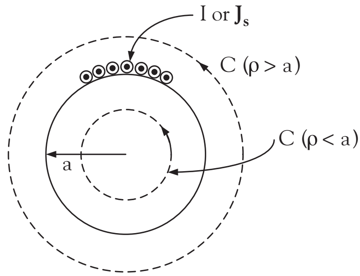


Figure 1-3. A ***current-carrying hollow cylinder***

Thus

$$B_\phi = 0 \ (\rho < a) \tag{1-11a}$$

$$= \frac{\mu_0 I}{2\pi\rho} \ (\rho > a) \tag{1-11b}$$

Note that the magnetic field inside a hollow cylinder with uniform surface current is zero.

(d) *Concentric Current-Carrying Cylinders (Coaxial Line)*

Figure 1-4 shows concentric hollow cylinders of infinite length and radii a, b carrying total currents I, –I, respectively, in the z direction. This is the configuration of a coaxial line.

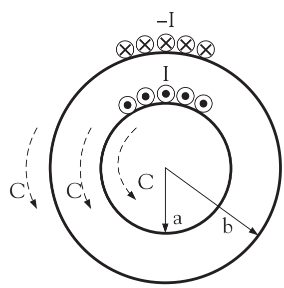


Figure 1-4. ***Concentric current-carrying cylinders***

The surface current densities are

$$\mathbf{J}_s = \mathbf{a}_x \left(\frac{I}{2\pi a} \right) \ (\rho = a)$$

$$= \mathbf{a}_z \left(\frac{-I}{2\pi b} \right) \ (\rho = b)$$

This is a three-region problem.

Consider circles C of radius ρ for the three cases (ρ < a), (a < ρ < b) and (ρ > b):

$$\oint_C \mathbf{B}\cdot d\ell = B_\phi 2\pi\rho = 0 \ (\rho < a)$$
$$= \mu_0 I \ (a < \rho < b)$$
$$= \mu_0 I + (-\mu_0 I) \ (\rho > b)$$

Thus

$$B_\phi = 0 \ (\rho < a) \quad (1\text{-}12a)$$
$$= \frac{\mu_0 I}{2\pi\rho} \ (a < \rho < a) \quad (1\text{-}12b)$$
$$= 0 \ (\rho > a) \quad (1\text{-}12c)$$

Note that, in each of the examples (a)-(d), there is a region with fields identical to that of a current-carrying filament along the z axis.

Example 1-3. Current-Carrying Toroids and Infinite Solenoid of Arbitrary Cross-Sections.

Figure 1-5(a) shows a toroid of doughnut shape and rectangular cross section, which is closely wound with N turns of wire carrying current I. The toroid has an inner radius a and an outer radius b. Find the magnetic field **B** everywhere.

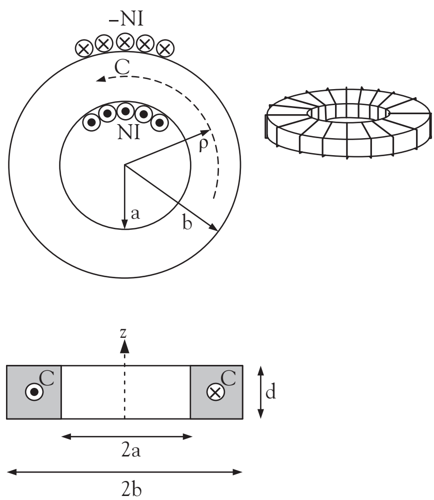


Figure 1-5(a). ***A toroid of rectangular cross-section***

Solution:

The problem has azimuthal symmetry ($\partial/\partial\varphi = 0$). This is a good approximation for a large number of turns, assuming that the winding is uniform and tight so that each turn can be considered as a closed loop of current. Then the magnetic fields **B** point perpendicular to the loop cross section and thus are circumferential about the z axis at all points, both inside and outside the toroidal coil. $\mathbf{B} = \mathbf{a}_\varphi B_\varphi$. Consider circles C of radius ρ for three cases ($\rho < a$), ($a < \rho < b$), ($\rho > b$). We further limit ourselves to the case $0 < z < d$.

Then

$$\oint_C \mathbf{B}\cdot d\ell = B_\phi 2\pi\rho = \mu_0 I_{enc} = 0 \ (\rho < a)$$

$$= \mu_0 NI \ (a < \rho < b)$$

$$= 0 \ (\rho > b)$$

Notice that the geometry is very similar to that of Example 1-2(d). For the cases ($z < 0$) and ($z > d$) the current enclosed is zero.

Thus

$$B_\phi = \frac{\mu_0 NI}{2\pi\rho} \ \text{(inside the toroid)} \qquad 1\text{-}13a)$$

$$= 0 \ \text{(outside the toroid)} \qquad (1\text{-}13b)$$

Note that we have not limited our results in any way to the *rectangular* cross-section. In fact, Eq. (1-13) is valid for a toroid of any arbitrary cross-section including the circular cross-section (doughnut).

Now consider what happens in a toroid of very large radius. Let a, b become very large as (b–a) remains constant. Then the percentage variation of ($1/\rho$) over the toroid ($a < \rho < b$) is negligible. The flux density B, is constant over the toroid. Let N_ℓ denote the number of turns per unit length of the toroid. Then

$$N = N_\ell(2\pi\rho)$$

Substituting this in Eq. (1-13a) yields

$$B_\phi = \frac{\mu_0 NI}{2\pi\rho} = \frac{\mu_0 N_l 2\pi\rho I}{2\pi\rho} = \mu_0 N_l I \qquad (1\text{-}14a)$$

Thus the flux density is uniform throughout the toroid. For other toroid cross-section shapes let a, b be minimum and maximum radii, respectively, and the above discussion still holds.

An *infinite solenoid* with its axis along the z axis (Figure 1-5(b)) and with circumferential windings may be regarded as a toroid of infinite radius.

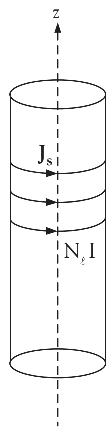


Figure 1-5(b). ***A solenoid of infinite length and arbitrary cross-section***

The flux density is uniform throughout the solenoid and z-directed:

$$B_z = \mu_0 N_l I \tag{1-14b}$$

The conclusions (Eqs. (1-14a,b)) apply to toroids and infinite solenoids of *any cross-sectional shape* such as circular, elliptical cross sections. $N_\ell I$ may also be interpreted as surface current density (J_s).

Another class of problems that can be handled by Ampère's law is the uniform current distributions on infinite sheet or slab.

Table 1-1 summarizes the magnetostatic fields and potentials for certain simple current distributions. The flux density **B** is obtained by Ampère's law. Magnetic vector potentials **A** are obtained by integrating the differential equation $\nabla \times \mathbf{A} = \mathbf{B}$, as shown in the next section (see Example 1-8). Note the simple dependence on distance for the particular cases shown:

For cylindrical symmetry $\left(\frac{\partial}{\partial \phi} = \frac{\partial}{\partial z} = 0\right)$: $\mathbf{B} \sim \frac{1}{\rho}$; $\mathbf{A} \sim \ln \rho$

For planar symmetry $\left(\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0\right)$: **B** is constant; $\mathbf{A} \sim z$

Table 1-1 Basic Magnetostatic Fields and Potentials

Current Distribution	Magnetic Flux Density	Magnetic Vector Potential
Filament $\mathbf{I} = \mathbf{a}_z I$	$B_\phi = \dfrac{\mu_0 I}{2\pi\rho}$	$A_z = \dfrac{\mu_0 I}{2\pi} \ln \dfrac{\rho}{\rho_0}$ Reference point: $\rho = {}_\rho 0$
Cylindrical Shell $J_z = \mathbf{a}_z \dfrac{I}{2\pi a} (\rho = a)$	$B_\phi = \dfrac{\mu_0 I}{2\pi\rho}$ $(p > a)$ $= 0$ $(\rho < a)$	$A_z = \dfrac{\mu_0 I}{2\pi} \ln \left(\dfrac{\rho}{\rho_0}\right)$ $(\rho \geq a)$ Reference point: $\rho = a$
Coaxial Shell $J_z = \mathbf{a}_z \dfrac{I}{2\pi a} (\rho = a)$ $= -\mathbf{a}_z \dfrac{I}{2\pi b} (\rho = b)$	$B_\phi = \dfrac{\mu_0 I}{2\pi\rho}$ $(a < p < b)$ $= 0$ $(\rho < a, \rho > b)$	$A_z = \dfrac{\mu_0 I}{2\pi} \ln \left(\dfrac{\rho}{a}\right)$ $(a \leq \rho \leq b)$ Reference point: $\rho = a$
Toroid (any cross section) N turns Current I	$B_\phi = \dfrac{\mu_0 NI}{2\pi\rho}$ (inside) $= 0$ (outside)	$A_z = \dfrac{\mu_0 NI}{2\pi} \ln \left(\dfrac{\rho}{\rho_0}\right)$ (inside) = constant (outside) Reference point: $\rho = {}_\rho 0$
Solenoid (any cross section) N_t turns per unit length Current I	$B_z = \mu_0 N_1 I = \mu_0 J_s$ (inside) = 0 (outside)	Circular Cross section (radius a) $A_z = \dfrac{\mu_0 N_1 I}{2} \rho$ $(\rho \leq a)$ $= \dfrac{\mu_0 N_1 I}{2} \dfrac{a^2}{\rho}$ $(\rho \geq a)$
Current Sheet $\mathbf{J}_s = \mathbf{a}_y J_{sy}$ $(z = 0)$	$B_x = \pm \dfrac{\mu_0 J_{sy}}{2}$ $+(z > 0$ $-(z < 0$	$A_y = m \dfrac{\mu_0 J_{sy}}{2} z$ Reference point: $z = 0$
Anti-parallel Current Sheets $\mathbf{J}_s = \mathbf{a}_y J_{sy}$ $(z = 0) = -\mathbf{a}_y J_{sy}$ $(z = d)$	$B_x = \mu_0 J_{sy}$ $(0 < z < d) = 0$ $(z < 0, z > d)$	$A_y = -\mu_0 J_{sy} z$ $(0 < z < d)$ Reference point: $z = 0$

1.5 The Magnetic Vector Potential And The Biot-Savart Law (The Magnetic Fields of Arbitrary Current Distributions)

In this section we consider the magnetic field **B** due to an arbitrary steady current distribution **J.** The methods developed are completely general but are more difficult than the Ampere's law methods of the previous section. We develop two forms for the fields of arbitrary steady currents, one which utilizes a magnetic vector potential **A** and another (the Biot-Savart law) which determines **B** directly. First let us consider the vector potential **A.**

The magnetic field **B** is divergenceless by the first law of Section 1.3; therefore it can be expressed as the curl of a vector **A** which we will call the **magnetic vector potential.**

$$B = \nabla \times \mathbf{A} \tag{1-15}$$

Recall now that a vector is completely defined by its curl and divergence. Therefore Eq. (1-15) only partially defines **A;** we are free to choose $\nabla \cdot \mathbf{A}$ for convenience. Choose $\nabla \cdot \mathbf{A} = 0$; this will turn out to be convenient mathematically. It also corresponds to the choice made in Helmholtz's theorem and is called the *Coulomb condition* or *Coulomb gauge.*

$$\nabla \cdot \mathbf{A} = 0 \tag{1-16}$$

Now take the curl of Eq. (1-15) and utilize the second law (Ampère's law) of Section 1.3 (Eq. (1-6)) and the vector identity $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$.

$$\nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

Making use of Eq. (1-16), we obtain

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{1-17}$$

or, in rectangular coordinates,

$$\nabla^2 A_x = \mu_0 J_x;\ \nabla^2 A_y = -\mu_0 J_y;\ \nabla^2 A_z = \mu_0 J_z$$

Eq. (1-17) is similar to Poisson's equation, (4-3), in Chapter 4 and thus it is called *vector Poisson's equation.* The scalar equations encountered here have already been solved in electrostatics:

$$\nabla^2 V = -\rho_v / \varepsilon_0 \Rightarrow V = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho_v dv'}{R} \qquad (1\text{-}46a)$$

Hence

$$\nabla^2 A_x = -\mu_0 J_x \Rightarrow V = A_x \frac{\mu_0}{4\pi} \iiint \frac{J_x dv'}{R}$$

Collecting the solutions for A_x, A_y, A_z:

$$\begin{aligned} A(x, y, z) &= \frac{\mu_0}{4\pi} \iiint \frac{J(x', y', z')dv'}{R} \\ &= \frac{\mu_0}{4\pi} \iiint \frac{J(x', y', z')dv'}{|\mathbf{r} - \mathbf{r}'|} \end{aligned} \qquad (1\text{-}18a)$$

where **r, r**′ are the position vectors of the field and source points, respectively.

For the cases of surface and filamentary currents, **J** dv′ is replaced with $\mathbf{J}_s$ ds′ and **I** dℓ′, respectively, to obtain

$$\mathbf{A} = \frac{\mu_0}{4\pi} \iiint \frac{J_s ds'}{R} \quad \text{for surface current density } \mathbf{J}_s \quad (1\text{-}18b)$$

$$= \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}dl'}{R} \quad \text{for filamentary current } \mathbf{I} \ (1\text{-}18c)$$

Eq. (1-18) can also be obtained directly from Helmholtz's theorem (Eq. (1-49)) since $(\nabla\times \mathbf{F})' = (\nabla \times \mathbf{B})' = \mu_0 \mathbf{J}(x', y', z')$. Note that **A** has 1/R dependence just as V has 1/R dependence in Eq. (2-46).

Eq. (1-18) allows us to obtain the vector potential **A** for an arbitrary current distribution, then **B** is obtained by taking the curl of **A.** Alternatively, we may obtain a direct expression for **B** by taking the curl of Eq. (1-18):

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \iiint \nabla \times \left(\frac{J}{R}\right) dv'$$

and, using the vector identity for the curl of a product of a vector and a scalar

$$\nabla \times \left(\frac{J}{R}\right) = \left(\frac{1}{R}\right) \nabla \times J(x', y', z') + \nabla\left(\frac{1}{R}\right) + \mathbf{J} = \mathbf{J} \times \frac{\mathbf{R}}{R^3}$$

Note that $\nabla \times \mathbf{J}(x',y',z') = 0$ since $\mathbf{J}$ is a function of primed coordinates. $\nabla(1/R)$ is equal to $-\mathbf{a}_R\left(\frac{1}{R^2}\right) = -\mathbf{R}/R^3$; it may be found by calculating the gradient in spherical coordinates.

Thus

$$\begin{aligned} \mathbf{B}(x, y, z) &= \frac{\mu_0}{4\pi}\iiint \frac{J(x', y', z')\mathbf{a}_R \, dv'}{R^2} \\ &= \frac{\mu_0}{4\pi}\iiint \frac{J(x', y', z')\times \mathbf{R} \, dv'}{R^3} \\ &= \frac{\mu_0}{4\pi}\iiint \frac{J(x', y', z')\times (\mathbf{r}-\mathbf{r}') \, dv'}{|\mathbf{r}-\mathbf{r}'|^3} \end{aligned} \tag{1-19a}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, $R = |\,\mathbf{r} - \mathbf{r}'\,|$.

For the surface and filamentary currents, $\mathbf{J}\, dv'$ is replaced with $\mathbf{J}_s\, ds'$ and $\mathbf{I}\, d\ell'$, respectively, to obtain

$$\mathbf{B} = \frac{\mu_0}{4\pi}\iiint \frac{\mathbf{J}_s \times \mathbf{R}\, ds'}{R^3} \quad \text{for surface current density } \mathbf{J}_s \tag{1-19b}$$

$$= \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\mathbf{R}\, dl'}{R^3} = \frac{\mu_0 I}{4\pi}\int \frac{d\ell'\times\mathbf{R}}{R^3} \quad \text{for filamentary current } \mathbf{I} \tag{1-19c}$$

($d\ell'$ has the direction of **I**)

Eq. (1-19) is known as the **Biot-Savart law.** Note that **B** has $1/R^2$ dependence just as **E** has $1/R^2$ dependence in Eq. (2-27).

Eqs. (1-18) and (1-19) provide two methods for the determination of **B** from current distributions:

1. Find **A** directly from (1-18), then $\mathbf{B} = \nabla \times \mathbf{A}$.
2. Find **B** directly from (1-19) (Biot-Savart law).

Each method has certain advantages. Method 1 has a simpler integral but involves two steps. Method 2 involves a more complex integral but has only one step. One method may have distinct advantages in a particular problem. However, for most magnetostatic problems, we use method 2 (Biot-Savart law).

To find the vector potential **A** from the current distribution **J,** we have a choice of two methods:

1. Find **A** directly from (1-18).
2. Find **B** first either from (1-19) or by Ampere's law, then integrate the equation $\mathbf{B} = \nabla \times \mathbf{A}$
   to find **A** (Example 1-8).

Example 1-4. On-Axis Field of a Current-Carrying Loop

(a) *Complete Loop*

Figure 1-6 shows a circular current-carrying loop of radius *a* with current I. Find the magnetic field **B** on the loop axis.

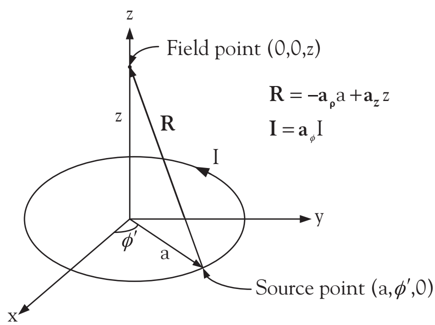


Figure 1-6. ***A current-carrying loop***

Solution:

We use Eq. (1-19c) appropriate to a filamentary current:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\mathbf{R}\ dl}{R^3}$$

We then identify the fixed field point (0, 0, z) and the variable source point $(\rho', \varphi', z') = (a, \varphi', 0)$ in cylindrical coordinates.

Then

$$\mathbf{r} = \mathbf{a}_z z;\ \mathbf{r}' = \mathbf{a}_\rho \rho' = \mathbf{a}_\rho a$$

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = -\mathbf{a}_\rho a + \mathbf{a}_z z\ (\mathbf{a}_\rho \text{ is a function of } \phi')$$

$$\mathbf{I} = \mathbf{a}_\phi I\ (\mathbf{a}_\phi \text{ is a function of } \phi')$$

$$R = \sqrt{a^2+z^2};\ dl' = a\ d\phi'$$

$$\mathbf{I}\times\mathbf{R} = (\mathbf{a}_\phi I)\times(-\mathbf{a}_\rho a + \mathbf{a}_z z) = \mathbf{a}_z(Ia) + \mathbf{a}_\rho(Iz)$$

$$= \mathbf{a}_z(Ia) + \mathbf{a}_x(Iz\cos\phi') + \mathbf{a}_y(Iz\sin\phi')$$

Take the z component of **B:**

$$B_z = \frac{\mu_0}{4\pi}\int_0^{2\pi} \frac{Ia^2 d\phi'}{[a^2+z^2]^{3/2}} = \frac{\mu_0 Ia^2}{2[a^2+z^2]^{3/2}} \qquad (1\text{-}20)$$

It can be easily shown that $B_x = B_y = 0$. Note that a φ-directed current produces a z-directed magnetic field on the axis.

For a distant point $(z \gg a)$, the loop appears small and the field approaches that of the magnetic dipole which is discussed in the next section:

$$B_z \Rightarrow \frac{\mu_0(I\pi a^2)}{2\pi z^3} = \frac{\mu_0 m}{2\pi z^3}$$

Compare this result with Eq. (1-31).

(b) *Partial Loop*

Consider a partial loop which is part of a closed circuit:

$$\mathbf{I} = \mathbf{a}_\phi \; I \; (\rho = a, \phi_1 \le \phi \le \phi_2, \; z = 0)$$

Find the magnetic field **B** along the z axis.

Solution:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int_{\phi_1}^{\phi_2} \frac{\mathbf{a}_z(Ia) + \mathbf{a}_x(Iz\cos\phi') + \mathbf{a}_y(Iz\sin\phi')}{[a^2+z^2]^{3/2}} \; a\,d\phi'$$

$$B_z = \frac{\mu_0 Ia^2(\phi_2 - \phi_1)}{4\pi[a^2+z^2]^{3/2}} \qquad (1\text{-}21a)$$

$$B_x = \frac{\mu_0 Ia^2 z}{4\pi[a^2+z^2]^{3/2}}(\sin\phi_2 - \sin\phi_1) \qquad (1\text{-}21b)$$

$$B_y = \frac{\mu_0 Ia^2 z}{4\pi[a^2+z^2]^{3/2}}(\cos\phi_1 - \cos\phi_2) \qquad (1\text{-}21c)$$

Example 1-5. The On-Axis Field of a Solenoid of Circular Cross Section Figure 1-7 shows a tightly wound solenoid of length ℓ and radius a. There are N turns of wire carrying a current I. This corresponds to a surface current as follows:

$$\mathbf{J}_s = \mathbf{a}_\phi \frac{NI}{l}\left[\frac{A}{m}\right]$$

Find the magnetic field **B** along the z axis.

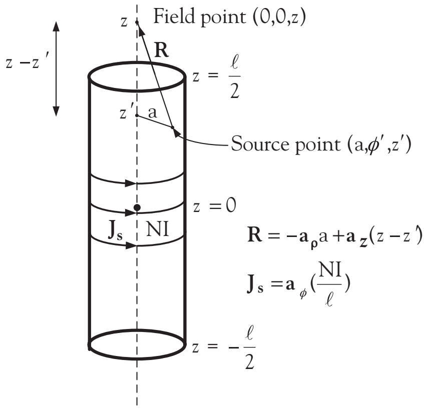


Figure 1-7. ***A solenoid of circular cross-section***

Solution:

We use Eq. (1-19b) appropriate to surface current:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\iint \frac{\mathbf{J}_s \times \mathbf{R} ds'}{R^3}$$

The fixed field point is (0, 0, z) and the variable source point is (ρ', φ', z') = (a, φ', z').

Then

$$\mathbf{r} = \mathbf{a}_z z; \; \mathbf{r}' = \mathbf{a}_\rho a = \mathbf{a}_\rho z'$$

$$\mathbf{R} = --\mathbf{a}_\rho a + \mathbf{a}_z(z - z')$$

$$\mathbf{J}_s \times \mathbf{R} = (\mathbf{a}_\phi \frac{NI}{l}) \times \left(-\mathbf{a}_\rho a + \mathbf{a}_z(z - z')\right)$$

$$= \mathbf{a}_z\left(\frac{NIa}{l}\right) + \mathbf{a}_\rho\left(\frac{NI(z - z')}{l}\right)$$

$$R = \sqrt{a^2 + (z - z')^2}; \; ds' = a d\phi' dz'$$

Taking the z component of **B:**

$$B_z = \frac{\mu_0}{4\pi}\int_{-\frac{l}{2}}^{\frac{l}{2}}\int_0^{2\pi}\frac{NIa^2 d\phi'\, dz'}{l[a^2+(z-z')^2]^{3/2}} \qquad \begin{array}{l}\text{Let } z'' = z - z' \\ dz'' = -dz'\end{array}$$

$$= \frac{\mu_0}{4\pi}\int_{z+\frac{l}{2}}^{z-\frac{l}{2}}\frac{NIa^2 dz'}{l[a^2+(z'')^2]^{3/2}} = -\frac{\mu_0 NIa^2}{2l}\left[\frac{z''}{a^2\sqrt{a^2+(z'')^2}}\right]_z^z \quad \text{1-22a)}$$

$$= \frac{\mu_0 NI}{2l}\left[\frac{z+\frac{l}{2}}{\sqrt{a^2+(z+\frac{l}{2})^2}} - \frac{z-\frac{l}{2}}{\sqrt{a^2+(z-\frac{l}{2})^2}}\right]$$

and $B_x = B_y = 0$ as shown in Example 1-4(a).

At the center $(z = 0)$:

$$B_z = \frac{\mu_0 NI}{2\sqrt{a^2+\left(\frac{l}{2}\right)^2}} \Rightarrow \frac{\mu_0 NI}{l} = \mu_0 N_l I \text{ (when } l \text{ ? } a) \quad \text{(1-22b)}$$

where $N_\ell = N/\ell$ (turns per unit length).

At the end $(z = \ell/2)$:

$$B_z = \frac{\mu_0 NI}{2\sqrt{a^2+l^2}} \Rightarrow \frac{\mu_0 NI}{2l} = \frac{\mu_0 N_l I}{2} \text{ (when } l \text{ ? } a) \quad \text{(1-22c)}$$

Thus for long, thin solenoids $(\ell \gg a)$, the on-axis field B_z at the ends $(z = \pm\, \ell/2)$ is half that at the center. Also, compare Eq. (1-22b) with Eq. (1-14b) and note that the on-axis field at the center approaches that of the infinite solenoid.

Example 1-6. The Magnetic Vector Potential and Field of a Straight Current-Carrying Wire.

Figure 1-8 shows a straight section of current-carrying wire *which is part of a closed circuit.* The filament carries current I between $z = z_1$ and $z = z_2$. Find the magnetic field **B** and the vector potential **A** due to this section of a closed circuit.

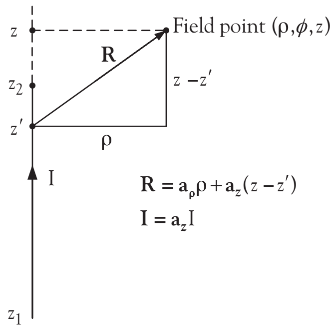


Figure 1-8. ***A straight current-carrying wire***

Solution:

(a) *Magnetic Flux Density* ***B***

We again use

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}_s \times \mathbf{R} dl'}{R^3}$$

We identify the field point (ρ, φ, z) and the source point (0, 0, z′). Then

$$\mathbf{r} = \mathbf{a}_\rho \rho + \mathbf{a}_z z = \mathbf{r}' = \mathbf{a}_z z'$$
$$\mathbf{R} = \mathbf{a}_\rho \rho + \mathbf{a}_z (z - z')$$
$$\mathbf{I} = \mathbf{a}_\phi I$$
$$\mathbf{I} \times \mathbf{R} = (\mathbf{a}_z I) \times \left(\mathbf{a}_\rho \rho + \mathbf{a}_z (z - z')\right) = \mathbf{a}_\phi (I\rho)$$

(A z-directed current produces a φ-directed magnetic field)

$$R = \sqrt{\rho^2 + (z - z')};\ dl' = dz'$$

Since $\mathbf{a}_\varphi$ is independent of z′, $\mathbf{B} = \mathbf{a}_\varphi B_\varphi$:

$$B_\phi = \frac{\mu_0}{4\pi}\int_{z_1}^{z_2} \frac{I\rho\, dz'}{l[\rho^2 + (z - z')^2]^{3/2}} \quad (\text{Let } z'' = z - z')$$

$$= \frac{-\mu_0}{4\pi}\int_{z-z_1}^{z-z_2} \frac{I\rho^2 dz'}{[\rho^2 + (z'')^2]^{3/2}} = -\frac{\mu_0\; I\rho}{4\pi}\left[\frac{z''}{\rho^2\sqrt{\rho^2 + (z'')^2}}\right]_{z-z_1}^{z-z_2} \qquad (1\text{-}23a)$$

$$= \frac{\mu_0 I}{4\pi\rho}\left[\frac{z + z_1}{\sqrt{\rho^2 + (z + z_1)^2}} - \frac{z - z_2}{\sqrt{\rho^2 + (z - z_2)^2}}\right]$$

Several special cases are of interest here:

For a semi-infinite wire ($z_1 = -\infty$, $z_2 = 0$):

$$B_\phi = \frac{\mu_0 I}{4\pi\rho}\left[1 - \frac{z}{\sqrt{\rho^2 + z^2}}\right] \qquad (1\text{-}23b)$$

$$\Rightarrow \frac{\mu_0 I}{4\pi\rho} \quad (z = 0) \qquad (1\text{-}23c)$$

Thus in the plane ($z = 0$) perpendicular to an end, the field of a semi-infinite wire is half that of the infinite wire. For an infinite wire ($z_1 = -\infty$, $z_2 = +\infty$):

$$B_\phi = \frac{\mu_0 I}{4\pi\rho}$$

which agrees with Eq. (1-8).

(b) *Magnetic Vector Potential **A***

To find the magnetic vector potential **A** we use the filamentary form of **A:**

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \mathbf{I}\frac{dl'}{R}$$

Since $\mathbf{I} = \mathbf{a}_z I$, $\mathbf{A} = \mathbf{a}_z A_z$:

$$A_E = \frac{\mu_0}{4\pi}\int_{z_1}^{z_2} \frac{I\, dz'}{\sqrt{\rho^2 + (z - z')^2}} \quad (\text{Let } z'' = z - z')$$

$$= -\frac{\mu_0}{4\pi}\int_{z-z_1}^{z-z_2} \frac{I\, dz''}{\sqrt{\rho^2 + (z'')^2}} = -\frac{\mu_0 I}{4\pi}\left[\ln\left(z'' + \sqrt{(z'')^2 + \rho^2}\right)\right]_{z-z_1}^{z-z_2} \qquad (1\text{-}23d)$$

$$= \frac{\mu_0 I}{4\pi}\ln\left[\frac{(z + z_1) + \sqrt{\rho^2 + (z + z_1)^2}}{(z + z_2) + \sqrt{\rho^2 + (z - z_2)^2}}\right]$$

Now the magnetic field **B** can also be calculated by $\mathbf{B} = \nabla \times \mathbf{A} = -\mathbf{a}_\varphi\, \partial A_z/\partial\rho$. You can obtain the result of Eq. (1-23a). *The results of this example must be used with care. They are valid only when used as a portion of a closed circuit.*

Example 1-7. On-Axis Field of a Square Loop

Figure 1-9 shows a square loop of side a and current I. Find the on-axis field **B.** The loop is in the xy plane, with sides parallel to x, y axes, and is centered at the origin.

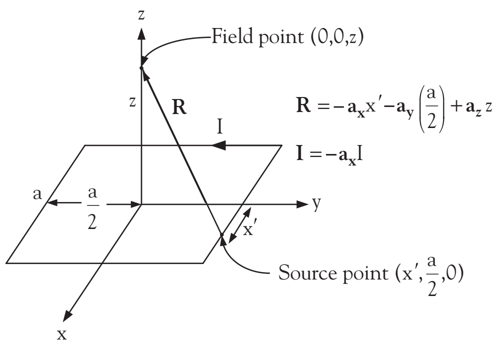


Figure 1-9. A *square loop*

Solution:

We conclude from a sketch of the field lines that only B_z exists along the z axis. We will calculate only the z component of the contribution of one side of the loop and then multiply by four since all sides are symmetrically disposed with respect to the axis.

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{(\mathbf{I}\times\mathbf{R})}{R^3}\,dl'$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int_{\text{Loop}} \frac{(\mathbf{I}\times\mathbf{R})_z}{R^3}\,dl'$$

$$= \frac{4\mu_0}{4\pi}\int_{\text{Side}} \frac{(\mathbf{I}\times\mathbf{R})_z}{R^3}\,dl'$$

Choosing one side on the right, we have the source point $(x', \frac{a}{2}, 0)$ and the field point (0, 0, z) as shown in Figure 1-9.

Then

$$\mathbf{r} = \mathbf{a}_z z;\ \mathbf{r}' = \mathbf{a}_x x' + \mathbf{a}_y \left(\frac{a}{2}\right)$$

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = \mathbf{a}_x x' + \mathbf{a}_y \left(\frac{a}{2}\right) + \mathbf{a}_z z$$

$$\mathbf{I} = \mathbf{a}_x I;\ dl' = dx'$$

$$\mathbf{I} \times \mathbf{R} = \mathbf{a}_z \left(\frac{Ia}{2}\right) + \mathbf{a}_y (Iz)$$

$$B_z = \frac{\mu_0}{2\pi} \int_{-\frac{a}{z}}^{\frac{a}{z}} \frac{Ia}{2} \frac{dx'}{[(x') + (\frac{a}{2})^2 + z^2]^{3/2}} \left(\text{let } \left(\frac{a}{2}\right)^2 + z^2 = u^2 \right)$$

$$= \frac{\mu_0 Ia}{\pi} \left[\frac{x'}{\sqrt{(x') + (\frac{a}{2})^2 + z^2}} \frac{1}{(\frac{a}{2})^2 + z^2} \right]_{-\frac{a}{2}}^{\frac{a}{2}} \qquad (1\text{-}24)$$

$$= \frac{\mu_0 Ia}{2\pi} \frac{a}{\sqrt{\frac{a^2}{2} + z^2}} \frac{1}{(\frac{a}{2})^2 + z^2}$$

Note that as $z \to \infty$:

$$B_z \Rightarrow \frac{\mu_0 Ia^2}{2\pi z^3} = \frac{\mu_0 (Ia^2)}{2\pi z^3} = \frac{\mu_0 m}{2\pi z^3}$$

in agreement with Eq. (1-31). At distant points (z ? a), the loop appears small and the field approaches that of a magnetic dipole. The component B_y contributed by this side is cancelled by the side opposite.

Example 1-8 Magnetic Vector Potential of a Long Filament

In this example, we show how the vector potential **A** can be obtained from the magnetic field **B** for an infinitely long current-carrying filament (see Table 1-1). In Example 1-2(a), we obtained **B** using Ampère's law:

$$\mathbf{B} = \mathbf{a}_\phi \frac{\mu_0 I}{2\pi\rho}$$

Note that $B_\rho = B_z = 0$ and B_φ depends only on ρ. The vector potential **A** satisfies

$$\nabla \times \mathbf{A} = \mathbf{B}$$

whose scalar equation for the φ-component is

$$\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} = B_\phi$$

Noting the rotational symmetry ($\partial/\partial_\varphi = 0$) and z-independence ($\partial/\partial_z = 0$), we conclude that

$$\mathbf{A} = \mathbf{a}_z A_z(\rho)$$

where A_z satisfies

$$-\frac{\partial A_z}{\partial \rho} = B_\phi = \frac{\mu_0 I}{2\pi\rho}$$

Integrating with respect to ρ and taking $\rho = \rho_0$ as a reference point, we obtain

$$A_z = -\int_{\rho_0}^{\rho} \frac{\mu_0 I}{2\pi\rho} d\rho = -\frac{\mu_0 I}{2\pi}\left[\ln \rho\right]_{\rho_0}^{\rho} = -\frac{\mu_0 I}{2\pi} \ln \frac{\rho}{\rho_0}$$

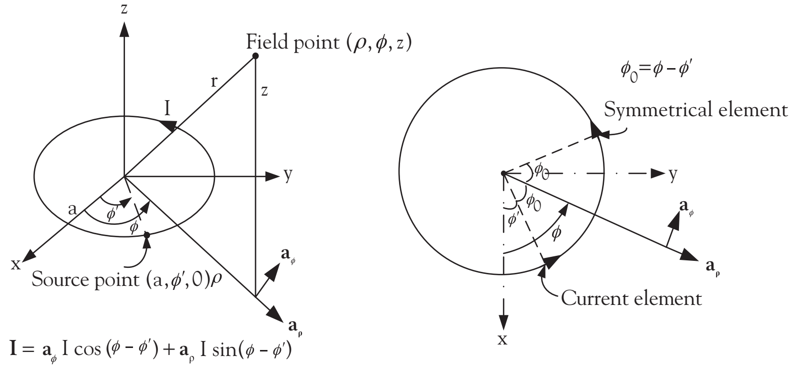
The vector potential of other current distributions in Table 1-1 can be obtained in a similar way.

1.6 The Magnetic Dipole*

In this section we consider the magnetic field **B** at an arbitrary point (not necessarily along the z-axis) of a *small* current-carrying loop. We will find that the field **B** is identical in form to the electric field of an electric dipole. We will also find that many of the characteristics of the loop can be simplified by defining a magnetic dipole moment for the loop.

*The mathematical detail of this section may be omitted.

Figure 1-10(a) shows a circular loop of radius *a* and current I.



(a) A small circular loop (r >> a). (b) A projection on the *xy* plane.

Figure 1-10. ***Magnetic Dipole***

The loop lies in the xy plane and is centered at the origin. The source point is (a, φ′, 0) and the field point is (ρ, φ, z). It can be shown that the current vector **I** can be written in terms of φ′, φ, $\mathbf{a}_\rho$, $\mathbf{a}_\varphi$ as follows:

$$\mathbf{I} = \mathbf{a}_\phi I \cos(\phi - \phi') + \mathbf{a}_\rho I \sin(\phi - \phi') \qquad (1\text{-}25)$$

where $\mathbf{a}_\rho$, $\mathbf{a}_\varphi$ are unit vectors with respect to the field point. See Problem 1-31.

The vector potential **A** is

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \mathbf{I}\frac{dl'}{R} = \frac{\mu_0}{4\pi}\int_0^{2\pi} \frac{\mathbf{a}_\phi I\cos(\phi - \phi') + \mathbf{a}_\rho I\sin(\phi - \phi')}{R}\, \mathbf{a} d\phi'$$

First consider the φ component of **A:**

$$\mathbf{A} = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi} \frac{\cos(\phi - \phi')d\phi'}{R}$$

$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \quad x = \rho\cos\phi,\ x' = a\cos\phi',$$

$$= \sqrt{\rho^2 + z^2 - 2a\rho\cos(\phi - \phi') + a^2} \quad y = \rho\sin\phi,\ y' = a\sin\phi',$$

$$= \sqrt{r^2 - 2a\rho\cos(\phi - \phi') + a^2} \quad z' = 0$$

$$r^2 = \rho^2 + z^2; \quad \frac{\rho}{r} = \sin\theta$$

$$\sqrt{1 \pm x} \approx 1 \pm x/2(|x| \ll 1)$$

$$\frac{1}{1 - x} \approx 1 + x(|x| \ll 1)$$

At far points $(r \gg a)$,

$$R = r\sqrt{1 - 2\frac{a}{r}\frac{\rho}{r}\cos(\phi - \phi') + \frac{a^2}{r^2}} \approx r\left[1 - \frac{a}{r}\sin\theta\,\cos(\phi - \phi')\right](r \gg a)$$

$$\frac{1}{R} \approx \frac{1}{r}\left[1 + \frac{a}{r}\sin\theta\,\cos(\phi - \phi')\right](r \gg a)$$

$$\mathbf{A}_\phi = \frac{\mu_0 Ia}{4\pi r}\left[\int_0^{2\pi}\cos(\phi - \phi')d\phi' + \frac{a\sin\theta}{r}\int_0^{2\pi}\cos^2(\phi - \phi')d\phi'\right]$$

$$= \frac{\mu_0 Ia}{4\pi r}\left[0 + \frac{a\sin\theta}{r}\frac{2\pi}{2}\right] = \frac{\mu_0(I\pi a)}{4\pi r}\sin\theta$$

Similarly, we can show that

$$\mathbf{A} = \frac{\mu_0 Ia}{4\pi}\int_0^{2\pi}\frac{\sin(\phi - \phi')d\phi'}{R} = 0$$

Substituting our expression for (1/R) in the integral for A_ρ, we obtain integrals of $\sin(\varphi-\varphi')$ and $\sin(\varphi-\varphi')\cos(\varphi-\varphi')$ (which is equal to 1/2 $\sin 2(\varphi-\varphi')$), each of which yields zero when integrated over a range of 2π. It can also be seen by symmetry that $A_\rho = 0$ since for each current element there is a symmetrically disposed element whose contribution to A_ρ is opposite (see Fig. 1-10(b) which shows a projection on the xy plane.) Thus

$$\mathbf{A} = \mathbf{a}_\phi\frac{\mu_0(I\pi a^2)}{4\pi r^2}\sin\theta \tag{1-26}$$

and

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0(I\pi a^2)}{4\pi r^2}(\mathbf{a}_r 2\cos\theta + \mathbf{a}_\phi\sin\theta) \tag{1-27}$$

The fields above are identical in form to the electric fields of an electric dipole. Recognizing this identity, we call a small loop of current a **magnetic dipole** and define a *magnetic dipole moment* **m** as follows:

$$\mathbf{m} = a_z m = a_z(I\pi a^2) = a_n(IS) \tag{1-28}$$

where S is the surface area of the loop and $\mathbf{a}_n$ is the normal to the plane of the loop with a right-hand rule relationship to the current **I.**

Then the vector potential **A** and field **B** of the loop of Figure 1-10 may be expressed as follows:

$$\mathbf{A} = \frac{\mu_0 \mathbf{m} \times \mathbf{a}_r}{4\pi r^2} = \mathbf{a}_\phi \frac{\mu_o m}{4\pi r^2} \sin\theta \tag{1-29}$$

$$\mathbf{B} = \frac{\mu_o m}{4\pi r^3}(\mathbf{a}_r 2\cos\theta + \mathbf{a}_\theta \sin\theta) \tag{1-30}$$

Along the loop axis (z-axis) where $\theta = 0°$, $r = z$ and $\mathbf{a}_r = \mathbf{a}_z$:

$$B_z = \frac{\mu_0 m}{2\pi r^3} \tag{1-31}$$

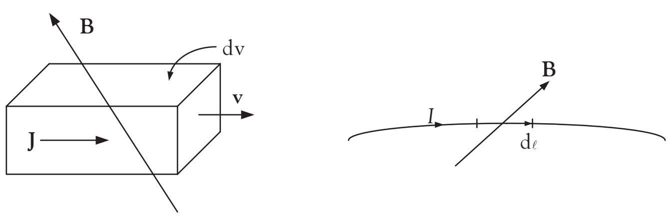
which was shown earlier in Example 1-4. In the plane of the loop (xy plane) where $\theta = 90°$:

$$B_\theta = \frac{\mu_0 m}{4\pi r^3} \tag{1-32}$$

We note that Eqs. (1-29) and (1-30) are valid for any shape of the loop, as long as it is small $(a \ll r)$. We can also view Eq. (1-30) as the field at far points $(r \gg a)$ of a current-carrying loop. The magnetic dipole plays an important role in explaining the behavior of magnetic field in matter in Chapter 2.

1.7 Forces and Torques on Currents in Magnetic Fields

In this section we consider the forces on currents in the presence of magnetic fields. Since the charge in motion experiences a force in the region of **B,** the currents, which consist of charges in motion, experience a force in the presence of magnetic fields. First let us consider the elementary force on a portion of a volume current distribution **J.** Figure 1-11(a) shows a volume element of current in the presence of the field **B.**



(a) An element of volume current **J** in field **B.**

(b) An element of filamentary current in field **B**.

Figure 1-11. ***A volume current and an element of filamentary current***

The results that follow can be used to find the force acting on a current-carrying wire, a conducting surface or a semiconductor.

The magnetic force $\mathbf{F}_m$ on a charge q moving at velocity $\mathbf{v}$ is

$$F_m = q\,\mathbf{v} \times \mathbf{B}$$

so that if current $\mathbf{J}$ consists of charges moving at drift velocity $\mathbf{v}$ then the elementary force $d\mathbf{F}_m$ on the volume element dv is

$$dF_m = dq\,\mathbf{v} \times \mathbf{B} = \rho_v dv\,\mathbf{v} \times \mathbf{B}$$

$$\frac{dF_m}{dv} = (\rho_v \mathbf{v}) \times \mathbf{B} = \mathbf{J} \times \mathbf{B} = \text{force per unit volume}$$

where Eq. (5-4) has been used. For surface current $\mathbf{J}_s$ the force per unit area is $\frac{dF_m}{ds} = \mathbf{J}_s \times \mathbf{B}$ and, finally, for filamentary current $\mathbf{I}$ (see Figure 1-11(b)), the force per unit length is $\frac{dF_m}{dl} = \mathbf{I} \times \mathbf{B}$.

This latter expression may also be written as $d\mathbf{F}_m = \mathbf{I}\,(d\boldsymbol{\ell} \times \mathbf{B})$ if $d\boldsymbol{\ell}$ is chosen as a vector in the direction of **I.** Table 1-2 summarizes the force relationships.

Table 1-2 Magnetic Forces on Currents

$\frac{dF_m}{dv} = \mathbf{J} \times \mathbf{B}$	(1-33a)
$\frac{dF_m}{ds} = \mathbf{J}_s \times \mathbf{B}$	(1-33b)
$\frac{dF_m}{dl} = \mathbf{I} \times \mathbf{B}$ or $dF_m = I\,d\ell \times \mathbf{B}$	**(1-33c)**

To obtain the magnetic force on an entire current distribution we merely integrate over the current distribution

$$F_m = \iiint \mathbf{J} \times \mathbf{B}\,dv \text{ for volume current} \quad (1\text{-}34a)$$

$$= \iint J_s \times \mathbf{B}\,ds \text{ for surface current} \quad (1\text{-}34b)$$

$$= \int \mathbf{I} \times \mathbf{B}\ dl = I \oint_C d\ell \times \mathbf{B} \text{ for filamentary current} \quad (1\text{-}34c)$$

Note that there is no net force $\mathbf{F}_m$ on a filamentary loop in a *uniform* field **B** since

$$\mathbf{F}_m = I \oint_C d\ell \times \mathbf{B} = I \left[\oint_C d\ell \right] \times \mathbf{B} = 0 \ \left(\text{since} \oint_C d\ell = 0\right)$$

Now let's consider the torque on current-carrying loops. The torque is important in addition to the force since the torque may exist even though the net force is zero. Recall the definition of torque. Figure 1-12 shows a force **F** applied at point **r.**

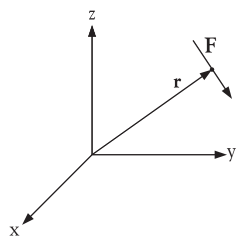


Figure 1-12. ***The definition of torque*** **T**

The torque **T** about a reference point at the origin is defined as

$$\mathbf{T} = \mathbf{r} \times \mathbf{F} \quad (1\text{-}35)$$

Note that the torque about any other reference point will, in general, be different. However, if the total force on an object is zero, then the torque is independent of reference.

To calculate the torque on a current-carrying loop we first consider the torque dT on an elementary portion of the loop:

$$d\mathbf{T} = \mathbf{r} \times d\mathbf{F}_m$$

For a filamentary loop, we integrate over the loop C to obtain

$$\mathbf{T} = \oint_C \mathbf{r} \times (d\ell \times \mathbf{B}) \quad (1\text{-}36)$$

Eq. (1-36) is the general expression for the torque on a filamentary loop. If **B** is uniform over the loop, the net force $\mathbf{F}_m$ is zero and the torque (independent of reference) is given by

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \tag{1-37}$$

where **m** is the magnetic dipole moment of the loop. For derivation of Eq. (1-37), see Reitz, Milford and Christy (1993).*

Example 1-9. The Force Between Parallel Current-Carrying Wires

Figure 1-13 shows two parallel current-carrying filaments of infinite length, separated by a distance d. Find the force per unit length between wires.

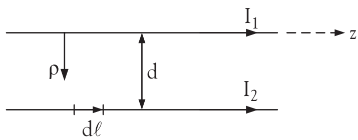


Figure 1-13. ***Two parallel current-carrying wires***

Solution:

We will find the force per unit length on wire 2 with current I_2. This will of course be the negative of the force on the other wire. We treat I_1 as the source current, which produces the field **B,** and I_2 as the test current, which experiences a force. We consider the force on an elementary length $d\ell$ of wire 2:

$$d\mathbf{F}_m = I_2 d\ell \times \mathbf{B}$$

We need to consider only the field due to wire 1. The field due to wire 2 is non-zero, but it cannot exert any net force on itself. Otherwise it could propel itself through space. Therefore we consider only the field at wire 2 due to wire 1, which is $\mathbf{a}_\varphi(\mu_0 I_1/2\pi\rho)$.

$$d\mathbf{F}_m = I_2(\mathbf{a}_z d\ell) \times \mathbf{a}_\phi \mu_0 I_1 / 2\pi d$$

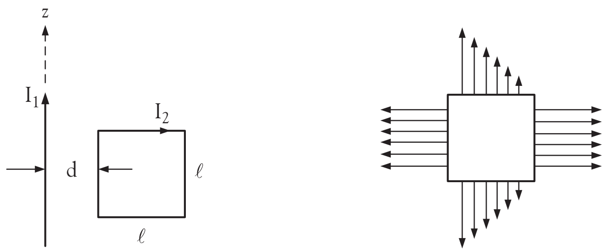
$$= \mathbf{a}_\rho \frac{\mu_0 I_1 I_2}{2\pi d} dl$$

$$\frac{d\mathbf{F}_m}{dl} = -\mathbf{a}_\rho \frac{\mu_0 I_1 I_2}{2\pi d} \tag{1-38}$$

The force is attractive if currents are parallel (I_1, I_2 both positive or negative) and is repulsive if currents are anti-parallel (one positive, one negative).

*J. R. Reitz, F. J. Milford, and R. W. Christy, *Foundations of Electromagnetic Theory,* Addison-Wesley, 1993, 4th Ed., Section 8-2.

Example 1-10. The Force Between a Filament and a Square Loop
Figure 1-14(a) shows a filament of infinite length with current I_1. Nearby is a square loop of side ℓ with two sides parallel to the filament. The square loop carries a current I_2. clockwise. The distance between the loop and the filament is d. Find the force between the loop and the filament.



(a) A filament and a square loop. (b) Forces on the square loop.

Figure 1-14. ***A filament and a square loop***

Solution:

We first choose the filament as the source current and the loop as the test current because it is easy to calculate the field due to a filament. To find the force on the loop, we consider the field **B** at the loop (due to filamentary current I_1) which is $\mathbf{a}_\varphi$ $(\mu_0 I_1/2\pi\rho)$. **B** points into the paper. We ignore the self field as before. Figure 1-14(b) shows a sketch of the forces $I_2\, d\boldsymbol{\ell} \times \mathbf{B}$ exerted on different portions of the loop. The z-directed forces cancel but the ρ-directed forces do not because the magnitude of **B** decreases as $1/\rho$.

$$
\begin{aligned}
\mathbf{F}_m &= \int_{loop} \mathbf{I} \times \mathbf{B}\, dl \\
&= \int_0^1 (\mathbf{a}_z I_2) \times \left(\mathbf{a}_\phi \frac{\mu_0 I_1}{2\pi d} \right) dz + \int_0^1 (-\mathbf{a}_z I_2) \times \left(\mathbf{a}_\phi \frac{\mu_0 I_1}{2\pi (d+1)} \right) dz \\
&= -\mathbf{a}_\rho \frac{\mu_0 I_1 I_2 l}{2\pi} \left[\frac{1}{d} - \frac{1}{d+1} \right] \\
&= -\mathbf{a}_\rho \frac{\mu_0 I_1 (I_2 l^2)}{2\pi d (d+l)}
\end{aligned}
\qquad (1\text{-}39)
$$

The force is attractive if I_1, I_2 have the same sign. Note that for this problem it will be very difficult to calculate the force acting on the filament due to the field of the loop because of difficulty in calculating the field of a loop.

Consider the torque on the loop. There is no torque about the center but there is a torque about one of the corners of the loop. Can you calculate it?

1.8 Ampère's Force Law

A particular application of the theory of Section 1.7 is the calculation of the force on a current-carrying loop due to the field **B** produced by another loop. Figure 1-15 shows two current-carrying loops with currents I_1, I_2.

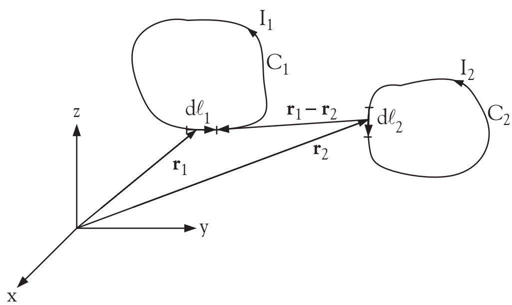


Figure 1-15. ***Two current-carrying loops***

Typical points on loops (1), (2) are (x_1, y_1, z_1), (x_2, y_2, z_2), respectively, specified by vectors $\mathbf{r}_1$, $\mathbf{r}_2$.

$$F_1 = \text{Force on loop (1)} = I_1 \oint_C d\ell_1 \times \mathbf{B} = I_1 \oint_C d\ell \times (\mathbf{B}_{11} + \mathbf{B}_{12})$$

(where $\mathbf{B}_{11}$, $\mathbf{B}_{12}$ are the fields at loop (1) due to currents I_1, I_2, respectively)

$$F_1 = I_1 \oint_C d\ell_1 \times \mathbf{B}_{12} \text{(since loop (1) exerts no net froce on itself)}$$

$$\mathbf{B}_{12} = \frac{\mu_0 I_2}{4\pi} \oint_C d\ell_2 \times \frac{\mathbf{R}_{12}}{R_{12}^3} \qquad \text{(see Eq. (1-19c))}$$

Thus

$$F_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\ell_1 \times (d\ell_1 \times \mathbf{R}_{12})}{R_{12}^3} \quad \text{Ampłre's froce law} \qquad (1\text{-}40)$$

where $\mathbf{R}_{12} = \mathbf{r}_1 - \mathbf{r}_2$.

1.9 A. M. Ampère (1775-1836) and the Magnetic Fields of Steady Currents

Andre Marie Ampère was a French scientist who made contributions to an astonishing number of different fields. He was a mathematician, chemist, physicist, engineer, and philosopher. He possessed encyclopedic knowledge and moved easily from one field to another. He devised a universal language. He wrote a drama about Columbus called *L'Americide.* He invented a table of elements before Mendeleev. His most *lasting* contribution, however, was his complete exposition of the magnetic fields and forces due to steady electric currents. The chapter which you are reading now is essentially Ampère's chapter. He combined experiment and theory, and amazing insight, in a comprehensive, thorough study unlike any of his other works. Maxwell said of his work: "The whole, theory and experiment, seems as if it had leaped, full grown and full armed, from the brain of the 'Newton of electricity'. It is perfect in form, and unassailable in accuracy, and it is summed up in a formula from which all the phenomena may be deduced, and which must always remain the cardinal formula of electrodynamics." Ampère coined the word *electrodynamics* to describe his area of study. It is no longer used since his work is limited to steady currents.

A. M. Ampère was born in Lyon in southeastern France into a prosperous middle class family. When Ampère was seven, his father, Jean Jacques Ampère, retired and moved his family to Poleymieux, a small village of several hundred inhabitants perched on a mountainside about 10 kilometers from Lyon. There was no school in Poleymieux and, as a result, Ampère had no formal schooling. He was largely self-educated under the guidance of his father. Ampère described the process as follows: "He never required me to study anything, but he knew how to inspire in me a desire to know." He became interested in mathematics at an early age and eventually was reading the encyclopedia and carrying out the calculations of Laplace's *Celestial Mechanics.*

The French Revolution had erupted in 1789 and it came to Lyon in 1792. Ampère's father found himself caught up in the struggle between two rival revolutionary groups, the moderate republican Girondins and the radical republican Jacobins. Jean Jacques was a justice of the peace when the Girondins took over Lyons. In this position, he certified the arrest of a Jacobin leader who was later guillotined. When the Jacobins took over a few months later after a siege of Lyon, Jean Jacques was arrested, tried, and guillotined. In his final letter home, he pardoned his executioners and wrote, "As for my son, there is nothing that I do not expect of him." Ampère had had a very close relationship with his father and was devastated by the loss. He went into a deep depression for over a year. After he recovered, he resumed his studies, eventually tutored mathematics in Lyon, and taught physics in Bourg. In 1801 the young Ampère presented a lecture in which he rejected action-at-a-distance in electricity: "Bodies which do not touch each other cannot interact." Therefore he rejected Coulomb's law. He retained an aversion to action-at-a-distance throughout much of his life.

He began a short but very happy marriage in 1799. His son, named Jean Jacques, was born in 1800. His wife Julie died in 1803. In 1804 Ampère went to Paris to teach at the Polytechnic School. Over the next ten years he made numerous contributions to mathematics and chemistry, leading to his election to the French Academy of Sciences in 1814.

In the fall of 1820, Ampère first learned of Oersted's discovery, which was reported to the Academy on Sept. 11 by François Arago, who had verified some of the results. This discovery was a shock to Academy members, as it was to scientists all over the world, especially because their highly respected colleague Coulomb had posited many years earlier that there could not be any connection between electricity and magnetism. It would be interesting to know the thoughts of the Academy members as they left that meeting of 11 Sept. 1820. At that time the French Academy included many of the leading scientists in the world. Certainly many of them recognized the historic importance of the discovery and some were determined to play a role in the development of this new theory. All over the world, scientists began to study the interaction of electricity and magnetism. Within a few years the scientific literature on the subject grew to enormous proportions. Ampère and Biot would lead the way in the development of the theory.

Ampère immediately set to work. He repeated Oersted's experiment, but with the terrestrial field cancelled out. In the absence of the earth's field, the compass needle rotated fully to end up perpendicular to the

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* James R. Hofmann, *André-Marie Ampère,* Blackwell Science Biographies, Cambridge, MA., p. 341, 1995.

long-wire current and perpendicular to a radial line from wire to field point (compass location). In this final position, the compass needle was either attracted to, or repelled by, the current-carrying wire. If attracted, then it would be repelled if the compass direction were reversed. Ampère also concluded that the earth's magnetic field is caused by electric currents in the earth. These discoveries were made during his first week's work, which was reported on 18 Sept. 1820. Thereafter, he reported to the Academy almost every week for several months. It's not possible to describe here all of his accomplishments but we'll describe a few of the more significant ones. Ampère showed in late September that parallel/antiparallel current-carrying wires attract/repel as shown in Example 1-9 and Eq. (1-38). He showed that helical current-carrying wires (solenoids) act like bar magnets. See Example 1-5 for an approximation to a very tightly-wound solenoid. This work led him to conclude that long bar magnets carry electric currents circumferentially around their periphery as in Figure 1-7. Later he concluded that every small portion of the cross section carried circumferential "Ampèrian" currents as in Figure 2-1(b). These Ampèrian currents then would add up to produce circumferential currents around the entire cross section as shown in Figure 2-1(c). These currents then correspond to those of the air-filled solenoid of Figure 1-7.

Since he was able to explain everything in terms of electric currents, Ampère had no need for the various theories of vortices, magnetic fluids, and magnetic charges with which the theory of magnetism was so burdened. He came to the conclusion that magnetic charges did not exist and insisted emphatically on the primacy of *electric currents* in explaining magnetic phenomena. He showed, however, that the magnetic charge model was equivalent mathematically to the electric current model. Once Ampère had become convinced that electric currents were the source of magnetism, he then recognized the primary importance of the force between two current-carrying elements. He set about in a very determined way to find this basic force law. This is a very complex relationship, as we can see by observing the double cross-product of Eq. (1-40), which is obtained by combining the single cross products of Eqs. (1-19) and (1-33). Ampère started with a special case, namely two short z-directed current elements, both located in the x-y plane. He verified that the force between elements was attractive, inverse square, and of the form $F = K\, I_1\, d\ell_1\, I_2\, d\ell_2 \,/\, R_{12}^{\,2}$, where K is a constant and R_{12} is the distance between elements. From this simple result to Eq. (1-40) is a long journey but somehow Ampère was able to get there. Actually his form was equivalent to Eq. (1-40) for the current loops of steady currents. Ampère was handicapped by his belief that the basic force must be radial, along the line connecting the elements. We can see by Eq. (1-40) that this is not necessarily so. The basic force law embedded in Eq. (1-40), or its equivalent, was the fundamental result which was praised so highly by Maxwell in the beginning of this section.

Ampère also showed that an inverse square force law ($1/r^2$) between elements leads to an inverse force law (1/p) between an element and a long current-carrying wire. In other words, if the field of a current element/point charge is of the form ($1/r^2$), then the field of a long current-carrying wire/long line charge is of the form (1/p). This is shown in Examples 1-2(a), 1-6, Laplace and Biot also showed this principle at about the same time (Laplace was apparently the first). Ampère is also given credit for the force law $d\mathbf{F} = I\, d\boldsymbol{\ell} \times \mathbf{B}$, Eq. (1-33), which gives the force exerted on a current element in a magnetic field **B.** Ampère was a person of tremendous insight. He proposed the electron shell. He postulated Ampèrian currents. He guessed that the electromagnetic wave was transverse and he invented a telegraphic arrangement, which was first put to use a year after his death.

1.10 The Hall Effect

If a current-carrying conductor is placed in a magnetic field it experiences a separation of charge due to the magnetic force $\mathbf{F}_m$ and a resultant *Hall voltage* V_H appears across the resistor. This effect can be used to measure magnetic fields. Consider a conductor of thickness d, width w, and volume current density **J.** The current is y-directed (electrons flow in the negative y direction). The magnetic force on an electron is

$$\mathbf{F}_m = q(\mathbf{v} \times \mathbf{B}) = -e(-\mathbf{a}_y v_0 \times \mathbf{a}_x B_0) = \mathbf{a}_z(-ev_0 B_0)$$

Thus there is a magnetic force deflecting electrons downwards, leading to a charge separation as shown in Figure 1-16. The charge separation creates an electric field $\mathbf{E}_H$ (the Hall field) and a voltage V_H (the Hall voltage). Associated with $\mathbf{E}_H$ is an electric force $q\mathbf{E}_H$ which opposes the magnetic force.

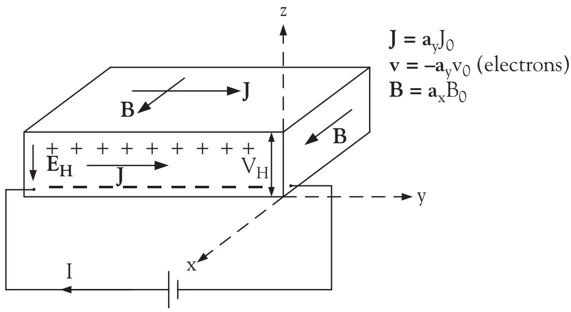


Figure 1-16. ***The Hall effect***

How long does this charge buildup continue and what is the magnitude of the electric field $\mathbf{E}_H$? Clearly the charge buildup continues until electric and magnetic forces cancel and equilibrium is reached. Equilibrium is reached in about 10^{14} seconds for copper (relaxation time). At equilibrium the total force on a charge is zero:

$$\mathbf{F} = q(\mathbf{E}_H + \mathbf{v} \times \mathbf{B}) = 0$$

$$\mathbf{E}_H = -\mathbf{v} \times \mathbf{B} \qquad (1\text{-}41)$$

For the example of Figure 1-16:

$$\mathbf{E}_H = \mathbf{a}_z(-v_0 \times B_0),\ V_H = v_0 \times B_0 d$$

we have assumed a parallel-plate relationship between $\mathbf{E}_H$ and V_H).

If the sample of Figure 1-16 were a semiconductor with both positive and negative charge carriers, the positive charge carriers would produce a charge separation opposite to that of Figure 1-16 with an opposing contribution to $\mathbf{E}_H$, V_H. If the principal charge carriers are positive, then the direction of the Hall voltage in Figure 1-16 is reversed (assuming current direction unchanged). The Hall effect may thus be used to determine whether semiconductors are p-type or n-type. The Hall effect was discovered by Edwin Hall in 1879.

CHAPTER 2

Magnetic Fields in Matter

2.1 Introduction

In this chapter, we study some of the characteristics of magnetic materials. We note the close relationships to dielectric materials, but the magnetic effects are much stronger and much more important in everyday life. Whenever we ring a doorbell, start a car, or use a motor or generator, we see firsthand the effects of magnetism.

Shortly after the astounding discoveries of Oersted, Ampère produced a relatively complete formulation for magnetostatic fields, as we have seen. One is amazed at the rapidity of the scientific response and the swift communication between countries. We will see further evidence of this breakthrough in scientific communication and the rapidity of scientific advancement. Within a few years Michael Faraday will discover the effects of time-varying magnetic fields and magnetic induction and will invent the first electric generator (the disc generator). Joseph Henry will precede Faraday in the first discoveries by a few months but will lay aside his work and will not publish immediately. Voilà!! Faraday's law instead of Henry's law. We should not feel unduly sorry for Henry because there will be ample credit for both.

But to return to the years immediately following Oersted's discovery, it is in the 1820's that Ampère has a most amazing insight into magnetic effects in material objects. He concludes that there are tiny circulating currents within magnetic materials and that they are the source of the magnetic fields of a magnet. Recall that this is about 70 years before any knowledge of the atom was available. This is certainly an amazing insight and we call the microscopic currents that do indeed exist *Ampèrian* currents in recognition of his insight. Ampère also concludes that, if the currents are oriented with parallel magnetic (dipole) moments, a surface current will result. Recall that a small loop of current is a magnetic dipole as discussed in the previous chapter.

Figure 2-1 shows an Ampèrian model of a cylindrical magnet. Here we assume that the Ampèrian currents are identical throughout the body of the material and that their magnetic moments are all z-directed. It is clear from Figure 2-1(b) that adjacent currents cancel each other and that the remaining current is a surface current. This is called a **bound surface current** denoted $\mathbf{J}_{ms}$ and is to be distinguished from *free* surface current $\mathbf{J}_s$. There are no net currents within the volume; the bound volume current density $\mathbf{J}_m$ is zero within (Figure 2-1(c)). Therefore, the magnetic material may be represented in terms of the equivalent surface currents $\mathbf{J}_{ms}$. We can ignore the currents within the volume because they cancel.

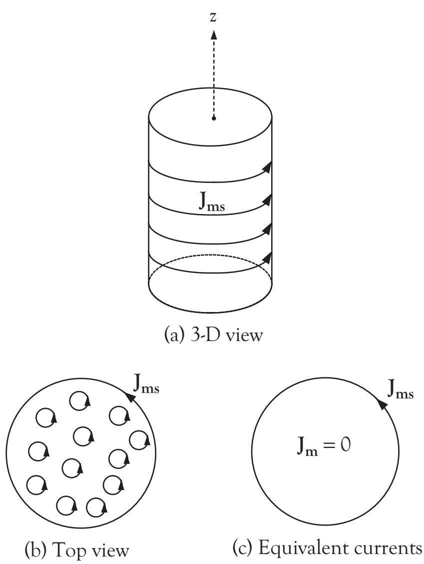


Figure 2-1. ***A magnetized cylinder***

The model of Figure 2-1 will be useful in describing the magnetic effects that occur in the presence of material objects. In the absence of an applied field **B** some materials are unmagnetized either because of an orderly arrangement of magnetic moments which cancel or because of a random arrangement with macroscopic cancellation. In addition, some materials are permanently magnetized. In the presence of an applied field **B,** most materials show an increase in magnetic moments parallel to the applied field, thus producing a net orderly arrangement of magnetic dipoles such as that of Figure 2-1.

2.2 Magnetization

We define a *magnetization* **M** (x,y,z) to characterize the magnetized state of the material. To calculate **M** at a point, we construct a small volume Δv about the point and add vectorially the individual magnetic dipole moments $\mathbf{m}_i$ within the volume Δv. Then

$$\mathbf{M}(x,y,z) \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_i \mathbf{m}_i \tag{2-1}$$

M is called the **magnetization** or the *magnetic dipole moment per unit volume.* By definition it is automatically zero in vacuum (free space). Magnetization **M** has the units of magnetic moment per unit volume or $\left(\frac{IS}{\text{volume}}\right)$ which is equal to (amps/m).

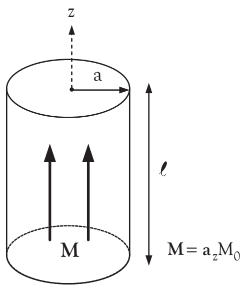
Referring to Figure 2-1, we see that our assumption there is that the magnetization is uniform:

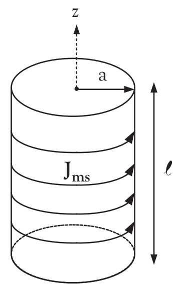
$$\mathbf{M} = \mathbf{a}_z M_0$$

2.3 The Magnetic Field B of a Magnetized Material

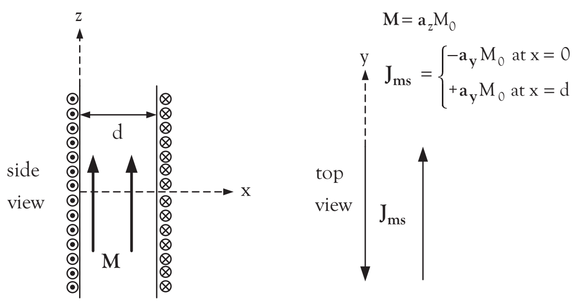
We mentioned earlier that a magnetic material becomes magnetized in the presence of an applied field **B.** An orderly arrangement of net Ampèrian currents and their net dipole moments is created such as that of Figure 2-1. For a uniform magnetization **M,** there is a uniform surface current $\mathbf{J}_{ms}$. What is the relationship between **M** and $\mathbf{J}_{ms}$? If **M** is known, what is $\mathbf{J}_{ms}$? To obtain the relationship we require that the magnetized material, and the equivalent currents, have the same distant fields. This is certainly *necessary* and it will be *sufficient* to determine the result.

First we consider a uniformly magnetized piece of material. Figure 2-2(a) shows a cylinder of magnetized matter. It is uniformly magnetized with $\mathbf{M} = \mathbf{a}_z M_0$. Figure 2-2(b) shows its equivalent bound currents in free space. Currents cancel macroscopically within the volume and there is a uniform surface current $\mathbf{J}_{ms}$ on the cylindrical surface.





(a) A cylinder - magnetic moments. (b) A cylinder - current model.



(c) An infinite slab of thickness *d*.

Figure 2-2. ***Uniformly magnetized material***

Now we require that the fields produced by the magnetized material **M** and by the equivalent currents $\mathbf{J}_{ms}$ be the same at distant points. This requires that the magnetic dipole moments of **M** and $\mathbf{J}_{ms}$ are equal. The dipole moment of the magnetized cylinder is m = M_0(volume) = $M_0(\pi a^2\ell)$ (because the magnetization M_0 is the dipole moment per unit volume) and that of the cylindrical surface current is m = IS = $(J_{ms}\ell)\,(\pi a^2)$. Both are z-directed. Thus

$$(J_{ms}\ell)\pi a^2 = M_0(\pi a^2\ell)$$

Therefore,

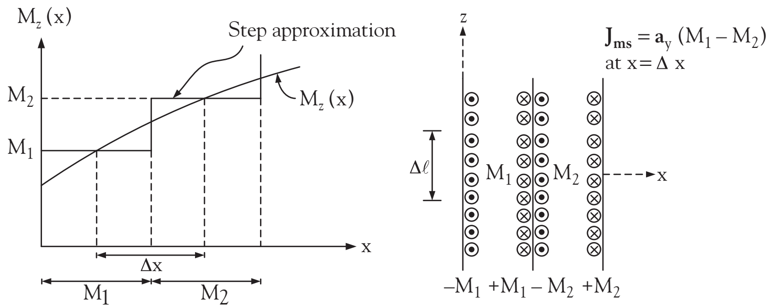
$$J_{ms} = M_0$$

In general, the result is independent of cross-sectional shape and $J_{ms} = M_0$ in all cases. By taking into account the directions of $\mathbf{J}_{ms}$ and **M,** $J_{ms} = M_0$ can be rewritten, in vector form, as

$$\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{a}_n \qquad (2\text{-}2)$$

where $\mathbf{a}_n$ is the outward unit vector normal to the surface of the magnetic material. As an example, it can be easily shown from Eq. (2-2) that an infinite slab of uniformly magnetized material ($\mathbf{M} = \mathbf{a}_z M_0$) has two sheets of oppositely directed uniform surface currents $\mathbf{J}_{ms} = \mathbf{a}_y M_0$ as shown in Figure 2-2(c).

Now what happens if the magnetization is not uniform but is a function of position? First we consider a one-dimensional variation $M_z(x)$ as shown in Figure 2-3(a). We can approximate the function $M_z(x)$ by step functions. The first two regions can thus be approximated by two slabs of uniformly magnetized material side by side (Figure 2-3(b)).



(a) Step approximation $M_z(x)$. (b) Two– slab model.

Figure 2-3. ***Non-uniform magnetization $M_z(x)$***

Surface currents on each side of the slabs are proportional to magnetization as in Figure 2-2(c). Since the magnetization of the two slabs differ, there is a net surface current density at $x = \Delta x$. The magnitude of the surface current density at $x = \Delta x$ is equal to the difference $M_1 - M_2$ which is the magnitude of the step discontinuity at $x = \Delta x$. The total current $\Delta \mathbf{I}$ in a length $\Delta \ell$ is

$$\Delta \mathbf{I} = \mathbf{J}_{ms}(\Delta l) = \mathbf{a}_y (M_1 - M_2)(\Delta l) = -\mathbf{a}_y \left(\frac{\partial M_z}{\partial x} \Delta x \right) \Delta l$$

Now let's spread the excess surface current $\Delta \mathbf{I}$ over the region (one half subsection ($\Delta x/2$) to the right and one half subsection to the left) to form the bound volume current $\mathbf{J}_m$. This is equivalent to smaller steps and greater accuracy in the approximation.

$$\Delta\mathbf{I} = \mathbf{J}_{ms}(\Delta x\Delta l) = -\mathbf{a}_y \frac{\partial M_z}{\partial x}(\Delta x\Delta l)$$

Thus the z-directed **M** nonuniform in x-direction leads to the y-directed bound current:

$$J_{ms} = -\frac{\partial M_z}{\partial x}$$

We add an additional magnetization $M_x(z)$ which can be treated in exactly the same way. Then

$$J_{ms} = \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}$$

Finally, consider all components and variations (these can be treated by coordinate rotations of Figure 2-3(b)) to obtain

$$\mathbf{J}_m = \nabla\times\mathbf{M} \tag{2-3}$$

Table 2-1 gives the equivalent bound current densities for magnetized material. These are *real* bound currents that exist in magnetized material and they are *equivalent* in the sense that we can obtain the magnetic flux density **B** by assuming those currents in free space (so that the formulations of Chapter 6 apply). The bound currents are also called the *magnetization currents.*

Table 2-1 Equivalent Bound Current Densities

Surface Current [A/m]	Volume Current [A/m^2]
$\mathbf{J}_{ms} = \mathbf{M} \times {}_a n$	$\mathbf{J}_m = \nabla \times M$

2.4 The Magnetic Intensity H

Free and bound currents are identical in their effect upon magnetic flux density **B,** as we have seen. The fundamental reason for this is that **B** is defined in terms of the Lorentz force law **F** = q(**E + v × B**) and bound currents exert forces in the same way as free currents. Therefore, in the presence of magnetized materials we can replace **J** with (**J** + $\mathbf{J}_m$) in Ampère's law 1-6) to obtain

$$\nabla\times\mathbf{B} = \mu_0(\mathbf{J}+\mathbf{J}_m) \tag{2-4a}$$

Rewriting Eq. (2-4a),

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} \right) = \mathbf{J} + \mathbf{J}_m \tag{2-4b}$$

Substituting $\mathbf{J}_m = \nabla \times \mathbf{M}$,

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) = \mathbf{J}$$

The quantity $\mathbf{B}/\mu_o - \mathbf{M}$ is defined as the *magnetic intensity* **H:**

$$\frac{\mathbf{B}}{\mu_0} - \mathbf{M} = \mathbf{H}$$

One often calls it also the *magnetic field* **H.** The magnetic field **B** can be written as

$$\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M} \tag{2-5}$$

Then Eq. (2-4b) becomes

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{2-6}$$

H has the units of amps/m. It is a vector which curls around the *free current only.* Eq. (2-6) can be recognized as a new **Ampère's law** in point form. We can obtain an integral form of Eq. (2-6) by integrating both sides over surface S and applying Stokes' theorem:

$$\iint_S \nabla \times \mathbf{H} \cdot ds = \oint_C \mathbf{H} \cdot d\ell = \iint_S \mathbf{J} \cdot ds = I_f$$

$$\oint_C \mathbf{H} \cdot d\ell = I_f \quad \text{(Ampère's law in matter)} \tag{2-7}$$

where I_f is the free current passing through the surface S in direction ds which is related by a right-hand rule to C. It is thus the current enclosed by C, with a right-hand relationship between the directions of I and C. Eq. (2-7) is a new integral form of Ampère's law. In some cases it is more useful since the free current, in contrast to the bound current, can easily be measured. Ampère's law for **B** in the presence of magnetic materials can be obtained by integrating Eq. (2-4a) over the surface S and applying Stokes' theorem:

$$\oint_C \mathbf{B} \cdot d\ell = \mu_0 (I_f + I_m) \tag{2-4b}$$

where I_f is the free current and I_m is the bound current passing through S bounded by the closed loop C. Table 2-2 summarizes the two forms of Ampère's law for **H** and **B.**

Table 2-2 Ampère's Law for Magnetic Fields in Matter

Point Form	Integral Form
$\nabla \times \mathbf{H} = \mathrm{J}$	$\oint_C \mathbf{H} \cdot d\ell = I_f$
$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \mathbf{J}_m)$	$\oint_C \mathbf{B} \cdot d\ell = \mu_0 (I_f + I_m)$

2.4.1 Linear Magnetic Materials

For *linear* magnetic materials, the magnetization **M** is proportional to magnetic intensity **H:**

$$\mathbf{M} = \chi_m \mathbf{H}$$

Then

$$\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M} = \mu_0 (1 + \chi_m) \mathbf{H} = \mu_0 \mu_r \mathbf{H} = \mu \mathbf{H}$$

χ_m, μ_r, μ are called *magnetic susceptibility, relative permeability,* and *permeability,* respectively. They represent three alternative ways of characterizing the linear relationship. We will usually specify the permeability μ. The three parameters (χ_m, μ_r, μ) may be functions of position. Thus $\mu(x, y, z)$ represents an *inhomogeneous,* linear medium and μ = constant represents a *homogeneous,* linear medium.

For a linear medium, we have

$$\mathbf{B} = \mu_0 (1 + \chi_m) \mathbf{H} = \mu \mathbf{H} \quad \text{for a linear magnetic medium}$$

$$\mathbf{M} = \left(\frac{\mu - \mu_0}{\mu_0} \right) \mathbf{H} \tag{2-8}$$

Values of μ for a variety of magnetic materials are given in Section 2.10.

2.4.2 Linear, Homogeneous Magnetic Materials

If a material is both linear and homogeneous, then

$$\mathbf{B} = \mu \mathbf{H}; \ \mu = \text{constant}$$

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \mathbf{J}_m) = \nabla \times (\mu \mathbf{H}) = \mu \nabla \times \mathbf{H} = \mu \mathbf{J}$$

$$\mathbf{J}_m = \left(\frac{\mu - \mu_0}{\mu_0} \right) \mathbf{J}$$

For a non-conducting magnetic material ($\sigma = 0$, $\mathbf{J} = 0$),
$\mathbf{J}_m = 0$ for a linear, homogeneous, non-conducting material.

Example 2-1. On-Axis Fields of a Magnetized Cylinder

Consider a uniformly magnetized cylinder of length ℓ, radius a as shown in Figure 2-2(a):

$$\mathbf{M} = \mathbf{a}_z \ M_0 \ \ (0 \le \rho \le a, \ -\frac{1}{2} \le z \le \frac{1}{2})$$

Find the on-axis fields **B, H** both inside and outside the cylinder.

Solution:

First we find the equivalent bound currents $\mathbf{J}_m$ and $\mathbf{J}_{ms}$:

$$\mathbf{J}_m = \nabla \times \mathbf{M} = 0 \ \text{(because } \mathbf{M} \text{ uniform)}$$

$$\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{a}_n$$

where $\mathbf{a}_n$ is the outward unit vector normal to the cylinder. It is equal to $\pm \ \mathbf{a}_z$ on top and bottom surfaces, respectively, and is equal to $\mathbf{a}_\rho$ on the curved surface $\rho = a$.

$$\mathbf{J}_{ms} = (\mathbf{a}_z M_0) \times (\pm \mathbf{a}_z) = 0 \text{ on top and bottom surface}$$

$$\mathbf{J}_{ms} = (\mathbf{a}_z M_0) \times \mathbf{a}_\rho = \mathbf{a}_\phi M_0 \ \text{ on the curved surface } \rho = a.$$

Now the equivalent currents $\mathbf{J}_{ms}$ may be placed in free space (ε_0, μ_0) and the field **B** can then be determined by the Biot-Savart law. To determine the magnetic intensity **H,** we must invoke the general relationship (Eq. (2-5)) between **B, M, H.**

The equivalent currents $\mathbf{J}_{ms}$ are identical in form to those of the solenoid of Example 1-5 and Figure 1-7; NI/ℓ is replaced everywhere by M_0.

The on-axis field B_z is thus obtained by replacing NI/ℓ in Eq. (1-22a) by M_0
:

$$B_z = \frac{\mu_0 M_0}{2}\left[\frac{z+1/2}{\sqrt{a^2+(z+1/2)^2}} - \frac{z-1/2}{\sqrt{a^2+(z-1/2)^2}}\right] \quad (2\text{-}9)$$

Eq. (2-9) is valid on the z axis both inside and outside the cylinder. To obtain **H** along the z axis, we use Eq. (2-5):

$$\mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{M}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

which gives

$$H_z = \frac{M_0}{2}\left[\frac{z+1/2}{\sqrt{a^2+(z+1/2)^2}} - \frac{z-1/2}{\sqrt{a^2+(z-1/2)^2}}\right] - M_0 \left(|z| < \frac{1}{2}\right) \quad (2\text{-}10a)$$

$$= \frac{M_0}{2}\left[\frac{z+1/2}{\sqrt{a^2+(z+1/2)^2}} - \frac{z-1/2}{\sqrt{a^2+(z-1/2)^2}}\right] \left(|z| > \frac{1}{2}\right) \quad (2\text{-}10b)$$

Example 2-2. Fields at the Center of a Magnetized Sphere
Figure 2-4 shows a magnet sphere of radius a which is uniformly magnetized:

$$\mathbf{M} = \mathbf{a}_z M_0 \;\; (r \le a)$$

Find the fields **B, H** at the center of the sphere.

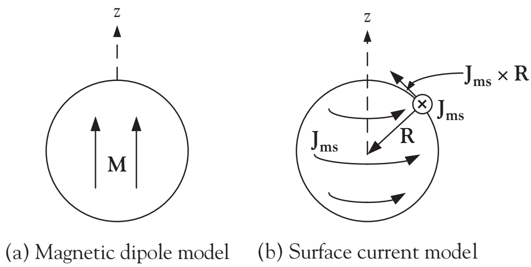


Figure 2-4. ***A uniformly magnetized sphere***

Solution:

First, the equivalent bound currents are determined as follows:

$$\mathbf{J}_m = \nabla \times \mathbf{M} = 0 \text{ (because } \mathbf{M} \text{ uniform)}$$
$$\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{a}_n = (\mathbf{a}_z M_0) \times \mathbf{a}_r = M_0(\mathbf{a}_r \cos\theta - \mathbf{a}_\theta \sin\theta) \times \mathbf{a}_r$$
$$= \mathbf{a}_\phi M_0 \sin\theta \quad (\text{at } r = 0)$$

Next we apply the Biot-Savart law:

$$\mathbf{B} = \frac{\mu_0}{4\pi} \iint_S \frac{\mathbf{J}_{ms} \times \mathbf{R} ds'}{R^3}$$

The typical source point is a point $(r', \theta', \varphi') = (a, \theta', \varphi')$ on the surface of the sphere. The field point is at the center of the sphere.

Then

$$\mathbf{R} = -\mathbf{a}_r \ a; R = a \text{ (since } \mathbf{r} = 0, \ \mathbf{r}' = \mathbf{a}_r a)$$

$$\mathbf{J}_{ms} \times \mathbf{R} = (\mathbf{a}_\phi M_0 \sin\theta') \times (-\mathbf{a}_r a)$$
$$= \mathbf{a}_\phi(-M_0 \sin\theta') = (\mathbf{a}_\rho \cos\theta - \mathbf{a}_z \sin\theta')(-M_0 a \sin\theta')$$

$$(\mathbf{J}_{ms} \times \mathbf{R})_z = M_0 \sin^2\theta'$$

By symmetry, the ρ component of **B** is cancelled out, giving rise to the z-directed **B** field:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\iint \frac{\mathbf{J}_{ms} \times \mathbf{R} ds'}{R^3}$$

$$= \mathbf{a}_z \frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{M_0 a \sin^2\theta' a^2 \sin\theta' d\theta' d\phi'}{a^3} \tag{2-11}$$

$$B_z = \frac{\mu_0}{2} M_0 \int_0 \sin^3\theta \, d\theta = \frac{\mu_0 M_0}{2}\left(\frac{4}{3}\right) = \frac{2\mu_0 M_0}{3}$$

and

$$H_z = \frac{B_z}{\mu_0} - M_z = -\frac{M_0}{3} \tag{2-12}$$

Note that B_z and H_z are oppositely directed, a *demagnetizing* effect. It is interesting to note that the magnetic field is given by Eq. (2-11) *at all points* inside the sphere. The fields at arbitrary points both inside and outside the sphere for this problem can be found using the *magnetic scalar potential* approach and using the solutions of Laplace's equation in.

Example 2-3. Current-Carrying Filaments, Wires, Cylinders and Toroids with Permeable Materials

The problems treated by Ampère's law in Examples 1-2 and 1-3 can readily be modified in the presence of permeable magnetic materials. For instance, consider Example 1-2(a) with a filament of infinite length and current I in an unbounded medium of meability μ. We apply Ampère's law for **H** to the geometry of Figure 1-1, which is modified by replacing free space (μ_0) with material of permeability μ:

$$\oint_C \mathbf{H} \cdot d\ell = \int_0^{2\pi} H_\phi \rho d\phi = H_\phi 2\pi\rho = I_f = I$$

Then

$$H_\phi = \frac{I}{2\pi\rho} \tag{2-13a}$$

and

$$B_\phi = \mu H_\phi = \frac{\mu I}{2\pi\rho} \tag{2-13b}$$

Note that in the presence of highly permeable ($\mu \gg \mu_0$) material, the flux density **B** is greatly increased as compared to that of free space.

All of the problems of Examples 1-2 and 1-3 can be treated in this manner. Let the field regions (the regions in which fields are non-zero) be completely filled with material of permeability μ. Equations (1-8), (1-11b), (1-12b), (1-13a), (1-14a), and (1-14b) are modified by replacing μ_0 with μ.

For the wire of finite radius (Example 1-2(b)), let the wire of radius a be of permeability μ_1, and the region outside ($\rho > a$) be of permeability μ_2. Then μ_0 is replaced with μ_1 in Eq. (1-9a) and μ_0 is replaced with μ_2 in Eq. (1-9b).

· See, for example, D. K. Cheng, *Field and Wave Electromagnetics,* Addison-Wesley, 1989, pp. 242-249.

2.5 Boundary Conditions

Boundary conditions are fundamental to magnetostatics, as to electrostatics. We need to understand the behavior of the fields **B** and **H** at an interface, i.e., a boundary, between two different media.

First we consider the normal component of **B** using the divergence property that Ñ · **B** = 0. For this component we could carry out a pillbox analysis. However, let's use the mathematical analogy between electrostatics and magnetostatics and merely substitute **B** for **D.** Consider the divergence law and boundary condition concerning **D:**

$$\nabla \cdot \mathbf{D} = \rho_v, \; {}'\iint_S \mathbf{D} \cdot d\mathbf{s} = Q_f \text{ and } D_{1n} - D_{2n} = \rho_s$$

Let $\rho_v = 0$ in regions 1, 2:

$$\nabla \cdot \mathbf{D} = 0, {}'\iint_S \mathbf{D} \cdot ds = 0 \text{ and } D_{1n} - D_{2n}$$

Substituting **B** for **D:**

$$\nabla \cdot \mathbf{B} = 0, {}'\iint_S \mathbf{B} \cdot ds = 0 \text{ and } \mathbf{B}_{1n} - \mathbf{B}_{2n}$$

Hence, *normal **B** is continuous.*

$$B_{1n} - B_{2n} \tag{2-14}$$

Next we consider the tangential component of **H** using the curl law ∇ × **H** = **J.** Figure 2-5 shows an interface between two arbitrary media.

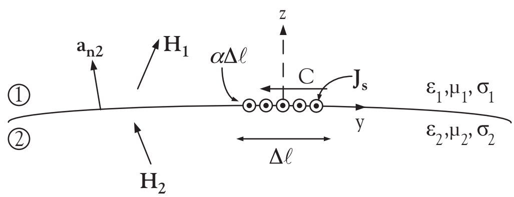


Figure 2-5. ***Tangential*** **H** ***at a point on a boundary***

We are interested in the behavior of **H** at a point on the boundary. At that point we construct a coordinate system (x, y, z) so that the normal is z-directed. The x, y directions lie in the tangent plane. Contour C lies in a plane normal to the interface. $\Delta\ell$ is made small so that $\mathbf{H}_1$, $\mathbf{H}_2$ do not vary over C. α is made arbitrarily small compared to unity so that the end sections of C do not contribute to $\oint_C \mathbf{H} \cdot d\ell$.

Then, orienting length $\Delta\ell$ in the y direction:

$$\nabla \times \mathbf{H} = \mathbf{J};\ \oint_C \mathbf{H} \cdot d\ell = (H_{2y} - H_{1y})\nabla\ell = I_f = J_{sx}\nabla\ell$$

if there exists an x-directed free surface current (J_{sx}). Thus,

$$H_{2y} - H_{1y} = -J_{sx}$$

If we orient length $\Delta\ell$ in the x direction, we have

$$H_{2x} - H_{1x} = -J_{sy}$$

Combining the above results:

$$\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) - \mathbf{J}_s \tag{2-15}$$

where $\mathbf{a}_{n2}$ is a unit vector normal outward from region 2, i.e, it points into region 1. Equation (2-15) indicates that a *free surface current on the boundary, in the tangent plane, causes a jump in the tangential component of **H**, also lying in the tangent plane, orthogonal to $\mathbf{J}_s$.*

If there is no $\mathbf{J}_s$ on the surface, then tangential **H** is continuous, i.e,

$$\mathbf{H}_{1t} - \mathbf{H}_{2t} \;\; \text{if } \mathbf{J_s} = 0 \tag{2-16}$$

Returning to the general boundary condition (Eq. (2-15)) for tangential **H,** we note that **H** determines $\mathbf{J}_s$ but not vice versa. $\mathbf{J}_s$ can be determined by a specification of $\mathbf{H}_1$, $\mathbf{H}_2$. $\mathbf{J}_s$ determines the odd portion of **H** but not the even portion.

We might at this point raise the question, "when can (free) surface current $\mathbf{J}_s$ exist at a boundary?" The answer is quite simple. In some cases, we wish to place an ideal source $\mathbf{J}_s$ at the interface to represent, for instance, thin current-carrying wires wound around a magnetic material. In no other cases will $\mathbf{J}_s$ arise except at the surface of a perfect conductor ($\sigma = \infty$). If conductivity σ is finite, volume currents may arise but no surface currents.

2.5.1 Linear Magnetic Materials

For linear magnetic materials, the following relationships are obtained

$$\mathbf{B} = \mu\mathbf{H};\;\; \mathbf{M} = \left(\frac{\mu - \mu_0}{\mu_0}\right)\mathbf{H}$$

$$B_{ln} = B_{2n} \tag{2-17a}$$

$$H_{ln} = \frac{\mu_2}{\mu_1} H_{2n} \tag{2-17b}$$

$$M_{ln} = \frac{1 - 1/\mu_{1r}}{1 - 1/\mu_{2r}} M_{2n} \tag{2-17c}$$

And if, in addition, $\mathbf{J}_s = 0$, then

$$H_{1t} = H_{2t} \tag{2-18a}$$

$$B_{1t} = B_{2t}\left(\frac{\mu_1}{\mu_2}\right) \tag{2-18b}$$

$$M_{1t} = \left(\frac{\mu_{1r} - 1}{\mu_{2r} - 1}\right) M_{2t} \tag{2-18c}$$

Behavior of **B, H** Lines at an Interface

Figure 2-6 shows a vector **B** at an interface between two linear isotropic media ($\mathbf{B} = \mu_1 \mathbf{H}_1$, $\mathbf{B}_2 = \mu_2 \mathbf{H}_2$).

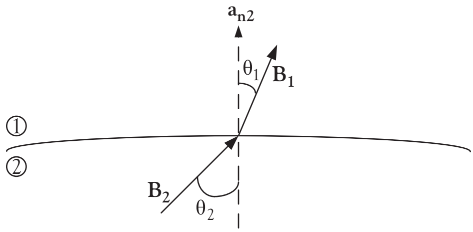


Figure 2-6. B, H *lines at an interface*

We are interested in the relationship of angles θ_1, θ_2. In other words, how does the direction of the vector change in passing from one medium to another? Let B_1, B_{1n}, B_{1t} represent the magnitudes of total **B,** normal **B,** and tangential **B,** respectively, in region 1 at the interface. Then

$$B_{1n} = B_1 \cos\theta_1$$
$$B_{1t} = B_1 \sin\theta_1$$

and $\tan\theta_1 \dfrac{B_{1t}}{B_{1n}}$; similarly, $\tan\theta_2 \dfrac{B_{2t}}{B_{2n}}$. Using $H_{1t} = H_{2t}$ and $B_{1n} = B_{2n}$, we obtain

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2} \qquad \text{(2-19)}$$

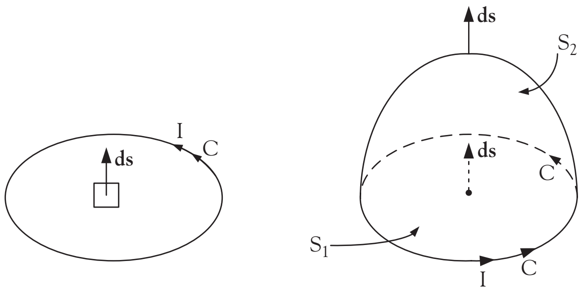
Note that Eq. (2-19) is also valid for the **H** vector since **H** is parallel to **B.** The directions of **B** and **H** are thus identical but the magnitude changes across the interface differ since

$$\frac{B_2}{B_1} = \frac{\mu_2}{\mu_1}\frac{H_2}{H_1}$$

In some cases the ratio μ_2/μ_1 may be very large, for instance, $\mu_r \approx 5000$ for medium silicon steel and $\mu_r \approx 1$ for air. In this case the field lines for **B, H** will be, for the most part, nearly normal to the interface in air and nearly parallel to the interface in the steel. The **B, H** lines passing from steel to air are bent very sharply towards the normal. Note that Eq. (2-19) is also valid for vectors **D, E** at the interface between two isotopic linear dielectric media if we replace μ_1/μ_2 with $\varepsilon_1/\varepsilon_2$.

2.6 Inductance

In this section, we study the inductance of the current circuit. Consider a current-carrying loop with current I (Figure 2-7a).



(a) Planar surface (b) Non-planar surface

Figure 2-7. ***Current-carrying loops***

Path C is chosen in the direction of I. The magnetic flux density **B** may be obtained by the Biot-Savart law (Eq. 1-19c)):

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\ell \cdot \mathbf{R}}{R^3}$$

The magnetic flux Φ through the loop (through any surface bounded by the loop*) is defined as

$$\boldsymbol{\Phi} = \iint_S \mathbf{B} \cdot \mathbf{ds}$$

where S is bounded by C and ds is related to C by a right-hand rule relationship. Then the **inductance** L (or **self-inductance**) is defined by

$$L = \frac{\Phi}{I} \tag{2-20}$$

* Ñ · **B** = 0, therefore ' $\iint_S \mathbf{B} \cdot \mathbf{ds} = 0$, and the flux through C is independent of the open surface bounded by C.

Note that we are free to choose any surface S bounded by C. For instance, with a circular loop (Figure 2-7b) we may choose the planar surface S_1, or the hat-shaped surface S_2. The unit of inductance is the henry [H] (named after Joseph Henry). 1 H (henry) is equal to 1 $(T \cdot m^2/A)$ or (Webers/A).

There are two steps in the process of finding L. Assuming the current I, we first find **B.** Then we integrate to find the flux Φ and thus Φ/I. The current I will cancel in the process. L is independent of excitation and depends only on geometry and the surrounding medium. L is always positive. It increases as the wire cross section is reduced. *For a coil of N turns, the flux* Φ *linking all turns should be counted*:

$$\Phi = N\Phi_s \tag{2-21}$$

where Φ_s is the flux through a surface bounded by a single turn. The above result (Eq. (2-21)) assumes that the same flux links all turns. When would this be a reasonable approximation?

Network Model for Inductance

Figure 2-8 shows the network model for an inductance L.

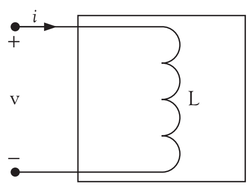


Figure 2-8. ***Inductance (network model)***

The network relationship between v and i across the inductor is:

$$v = \frac{d\Phi}{dt} = \frac{d}{dt}(Li) = L\frac{di}{dt} + i\frac{dL}{dt} \Rightarrow L\frac{di}{dt} \quad (\text{if } \frac{dL}{dt} = 0)$$

$$v = L\frac{di}{dt} \tag{2-22}$$

Note that $v = \frac{d\Phi}{dt}$ comes from Faraday's law that will be discussed later in Volume 4.

Mutual Inductance

Consider a set of N current loops with currents $I_1, \ldots, I_N$ as shown in Figure 2-9.

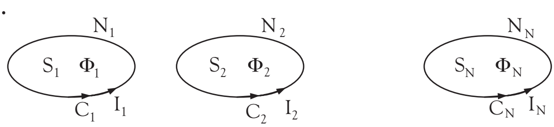


Figure 2-9. ***Mutual inductance***

Consider the total flux Φ_j through loop j:

$$\Phi_j = \Phi_{j1} + \Phi_{j2} + \cdots + \Phi_{jN} = \sum_{k=1}^{N} \Phi_{jk}$$

where Φ_{jk} is the flux through loop j due to current I_k (all other currents set equal to zero):

$$\Phi_{jk} = \iint_{S_j} \mathbf{B}_k \cdot d\mathbf{s}$$

Then the **mutual inductance** L_{jk} is defined as follows:

$$L_{jk} = \frac{\Phi_{jk}}{I_k} \tag{2-23}$$

If N_j is the number of turns in loop j, then L_{jk} is proportional to $(N_j\, N_k)$. The inductance L_{kk} is a self inductance, of course. It can be shown that

$$L_{jk} = L_{kj} \tag{2-24}$$

for wires in free space and in general unless a non-reciprocal material such as ferrite is present. Since L_{ij} is usually equal to L_{ji} we can choose to calculate either Φ_{ij} or Φ_{ji}. The calculations may differ considerably.

Note also that $L_{ii} \geq L_{ji}$ since *all* of the flux generated by loop i passes through loop i but only a *portion* passes through (or links) loop j. Mutual inductance L_{ij} may be positive or negative. Switching the direction of current I_j will switch the sign of L_{ij}. Similarly, reversing one set of terminals in the network model switches the sign.

Network Model

Figure 2-10 shows the network (two-port) model for a pair of coupled coils.

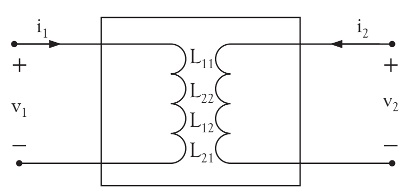


Figure 2-10. ***Coupled coils (network model)***

The network v-i relationships are:

$$v_1 = L_{11}\frac{di_1}{dt} + L_{12}\frac{di_2}{dt} \tag{2-25a}$$

$$v_2 = L_{21}\frac{di_1}{dt} + L_{22}\frac{di_2}{dt} \tag{2-25b}$$

(if L_{11}, L_{12}, L_{21}, L_{22} are all constant in time). Then, for a set of N coupled coils,

$$v_j(t) = \sum_{k=1}^{N} L_{jk}\frac{di_k}{dt} \tag{2-25c}$$

Just as capacitance was increased by the addition of dielectrics, inductance is greatly increased by the addition of magnetic materials.

Example 2-4. The Inductance of a Toroid
Consider the toroid of Figure 2-5(a), of rectangular cross section, with N turns and current I. The toroid is completely filled with material of permeability μ. Find the inductance of the toroid.

Solution:
As discussed in Example 2-3, the flux density in the toroid with magnetic core is obtained from Eq. (2-13a) by replacing μ_0 by μ:

$$\mathbf{B} = \frac{\mu_0 IN}{2\pi\rho} \quad (a < \rho < b,\ 0 < z < d) \tag{2-26}$$

Next we find Φ_s, the flux through the surface bounded by a single turn, i.e, the cross-section surface of the toroid:

$$\mathbf{\Phi}_s = \int_0^d\int_a^b \frac{\mu_0 IN\, d\rho dz}{2\pi\rho} = \frac{I\mu Nd}{2\pi}\ln\frac{b}{a}$$

Then the total flux is

$$\mathbf{\Phi} = N\mathbf{\Phi}_s = \frac{I\mu N^2 d\ \ln(b/a)}{2\pi}$$

Therefore,

$$L = \frac{\Phi}{I} = \frac{\mu N^2 d \ln(b/a)}{2\pi} \tag{2-27}$$

Note that the inductance is *proportional* to the permeability (μ) and the *square of turns (N^2).* Addition of highly permeable materials greatly increases the inductance.

Example 2-5 The Inductance per Unit Length of an Infinite Solenoid

The inductance per unit length of an infinite solenoid may be obtained readily. Consider the field **B** of an infinite solenoid of arbitrary cross-section as given in Eq. (1-14b):

$$B_z = \mu_0 N_\ell I$$

Now find the flux Φ linked by a unit length of the solenoid with N_ℓ turns:

$$\mathbf{\Phi} = N_\ell (\mu_0 N_\ell I)\ A,$$

where A is the cross-section area of the solenoid. Then the inductance per unit length (L/ℓ) is given by

$$\frac{L}{l} = \frac{\Phi}{I} = \mu_0 N_1^2 A \tag{2-28}$$

The above result is valid for arbitrary cross-section.

Example 2-6. The Mutual Inductance of a Filament and a Square Loop.

Consider the filament and square loop of Figure 1-14(a). Find the mutual inductance.

Solution:

It is simpler to find the flux Φ_{21} through the square loop due to the filament of current I_1 than finding Φ_{12} through the filament due to the loop current:

$$\mathbf{\Phi}_{21} = \iint_{S_2} \mathbf{B}_1 \cdot d\mathbf{s} = \int_0^l \int_d^{d+l} \frac{\mu_0 I_1 d\rho dz}{2\pi\rho} = \frac{\mu_1 I_1 l}{2\pi} \ln\left(\frac{d+l}{d}\right)$$

where we have used Eq. (1-8) for $\mathbf{B}_1$. Then

$$L_{21} = L_{12} = \frac{\Phi_{21}}{I_1} = \frac{\mu_0 l \ln\left(\frac{d+l}{d}\right)}{2\pi} \tag{2-29}$$

Example 2-7. The Mutual Inductance of a Filament and a Toroid.

Figure 2-11 shows a filament of infinite length and current I_1 along the axis of a toroid of rectangular cross-section, with current I_2 and N_2 turns. Find the mutual inductance.

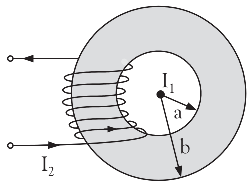


Figure 2-11. ***A filament and a toroid***

Solution:

Once again it is simpler to find the flux through the toroid due to the filament. The **B** field is given by Eq. (1-8). The total flux Φ_{21} is N_2 times the flux Φ_{21s} through the surface bounded by a single turn of the toroid, i.e., the cross-section of the toroid (air core):

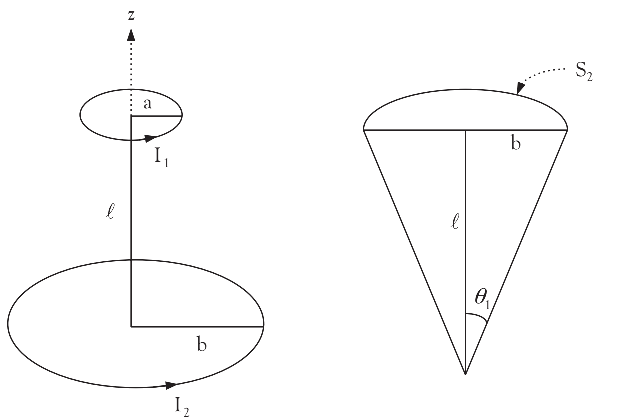
$$\Phi_{21s} = \int_0^d \int_a^b \frac{\mu_0 I_1 d\rho dz}{2\pi\rho} = \frac{\mu_1 I_1 d}{2\pi} \ln(b/a)$$

$$\Phi_{21} = N_2 \Phi_{21s} = \frac{\mu_0 I_1 d N_2 \ln(b/a)}{2\pi}$$

$$L_{21} = L_{12} = \frac{\Phi_{21}}{I_1} = \frac{\mu_0 d N_2 \ln(b/a)}{2\pi} \tag{2-30}$$

Example 2-8. The Mutual Inductance of Two Loops.

Figure 2-12(a) shows two loops, each centered on the z axis, with radii a, b and currents I_1, I_2, respectively. The loop axis of each is the z axis. The loops are separated by a distance ℓ where $\ell \gg$ a, b $\gg$ a. Find the mutual inductance.



(a) Loop geometry (b) Surface S_2 for calculation of L_{21}

Figure 2-12. ***Two coaxial loops***

Solution:

Method 1 - Using the Field ***B****$_2$ due to I_2:*

Since $\ell \gg a$ and $b \gg a$, we may assume that the field at loop 1 due to current I_2 is approximately equal to the on-axis field of loop 2 (see Example 1-4). The on-axis field B_{2z} of loop 2 at $z = \ell$ is

$$\mathbf{B}_{2z} = \frac{\mu_0 I_2 b^2}{2[b^2 + l^2]^{3/2}}$$

$$\Phi_{12} = \iint_{S_1} \mathbf{B}_2 \cdot d\mathbf{s} = \frac{\mu_0 I_2 b^2}{2[b^2 + l^2]^{3/2}}(\pi a^2) \tag{2-31}$$

$$L_{12} = \frac{\Phi_{12}}{I_2} = \frac{\mu_0 \pi a^2 b^2}{2[b^2 + l^2]^{3/2}}$$

Method 2 – Using The Field ***B****$_1$ due to I_1:*

Since $\ell \gg a$, we may use the magnetic dipole field developed in Section 1.6 for the field at loop 2 due to loop 1. To find the flux Φ_{21}, we need to integrate over a surface bounded by loop 2. It is more convenient to use a *spherical* surface of radius $\sqrt{\ell^2 + b^2}$ as shown in Figure 2-12(b) rather than a planar surface because of the form of the magnetic dipole fields. The radial component B_{1r} of loop 1 at $r = \sqrt{l^2 + b^2}$ is

$$B_{1r} = \frac{\mu_0 I_1 (\pi a^2)}{4\pi[l^2 + b^2]^{3/2}}(2\cos\theta)$$

and

$$L_{21} = \frac{\Phi_{21}}{I_1} = \frac{\mu_0 \pi a^2 b^2}{2[l^2 + b^2]^{3/2}}$$

which agrees with Eq. (2-31). This confirms that Eq. (2-24), the reciprocal property, is satisfied.

2.7 Joseph Henry (1797-1878) and the Discovery of Magnetic Induction

In the summer of 1831, Joseph Henry, an instructor at the Albany Academy in Albany, New York, knew that he was on the verge of an important discovery and ready to make a name for himself in the new science of electromagnetism. Henry was already beginning to be known throughout the United States for his practical work with powerful new magnets. He had started by insulating wires from each other by carefully wrapping them in silk (from his wife's petticoat). Thus he was able to use a very large number of turns. He was able to produce electromagnets of great lifting power by (1) insulating wires as described (2) using powerful batteries in series to produce high voltage and (3) using many coils in parallel. These were simple techniques, but clearly the young Henry knew what he was doing. Prof. Silliman of Yale requested that Henry construct an electromagnet for his laboratory. Built in 1831, this magnet was able to lift a ton and was the most powerful electromagnet of its day.

Henry, however, had his sights set on bigger fish. For several years he had been experimenting on magnetic induction (which is explained in Volume 4). His powerful magnets were very useful because they created large magnetic fields which magnified the small effects of induction. By August of 1831, he had completed his work. He understood that inductive effects occurred only when magnetic fields were changing. He was ready to publish but he had not heard anything at all from Europe on the subject and assumed that he was several years ahead of everyone else. He could afford to set aside his work until next summer and publish later with more complete results. Unfortunately, Michael Faraday, an English chemist and the greatest experimental scientist of his time, was hot on his trail. In ten famous days of experiment in the fall of 1831, he wrapped up the problem very effectively. Faraday read his results on November 1831 and published them in 1832. Henry received the paper in May 1832 and learned that he had been thoroughly beaten.

At first, he was in despair, as he was no longer a young man and had missed his great opportunity. But his friend Professor Silliman encouraged him to publish anyway. Thereby he secured the primacy of his work in self-induction, which would lead eventually to the naming of the unit of inductance as the Henry.

There are several similarities between Henry and Faraday. They were both raised in somewhat impoverished circumstances. Both had minimal schooling, could barely read and, by happenstance almost, both became avid readers. Both became outstanding lecturers who prepared their talks very thoroughly, with dramatic experimental display. Both were interested primarily in scientific principles and would not develop or patent their discoveries. Both worked primarily on their own.

Henry was born in Albany, NY. His father was a day laborer who died young. Joseph Henry had very little schooling and worked as a farm hand and store clerk. He was apprenticed at 14 to a watchmaker and silversmith. He was a member of an amateur theatrical company. At age 16 he became very interested in scientific questions while reading a book on general science. Later in life, he said, "This book, although by no means a profound work, has under providence exerted a remarkable influence on my life. It opened to me a new world of thought and enjoyment, fixed my mind on the study of nature, and caused me to resolve that I would immediately commence to devote my life to the acquisition of knowledge." Attending the Albany Academy, he soon learned enough to become a teacher in a country school. He began his college education at the Albany Academy in 1819 and finished in 1822, studying and teaching simultaneously with a 16 hour schedule.

It was said that the young Henry easily made major decisions but had difficulty with some minor ones. The story is told of a pair of shoes which he ordered. He changed the order from square to round toes and back so many times that the cobbler, out of frustration, finally delivered his shoes with one round and one square toe.

Henry made many additional contributions before retiring from research in 1846. He produced a motor (in 1831), which he regarded as a "scientific toy" and did not patent. It had the basic elements of a D.C.

motor including a commutator. He also invented the electric relay and the electric telegraph. He built a telegraph line in 1835 but did not patent it. He also built a transformer capable of stepping the voltage up or down.

Joseph Henry possessed significant talents as an administrator. In 1846, he left Princeton, where he had been a professor since 1832, and became head of the new Smithsonian Institution, where he served ably until his death. He was a key scientific advisor to Abraham Lincoln during the Civil War, and served on many government advisory boards. He played a role in the founding of the National Academy of Sciences, the American Association for the Advancement of Science, the Philosophical Society of Washington, the Lick Observatory, and the U.S. Weather Bureau. In 1893 the International Electrical Congress honored him by naming the unit of inductance the Henry.

2.8 Magnetic Energy

In Volume 2 we obtained the electric energy required to assemble a set of charges in the presence of dielectrics. In this section we derive the magnetic energy of a system of current-carrying coils in the presence of magnetic materials. There are many similarities with the analysis of Volume 2 and it may be helpful to make comparisons. Except for the very first steps, the analysis is limited to the *linear* case (Φ = Li). It is also assumed that all changes are made slowly enough so that the maximum system dimension is small compared to wavelength. This is called the *quasistatic* condition. It is necessary to ensure that radiation does not occur.

First we consider the magnetic energy W_m of a single current-carrying loop (Figure 2-13).

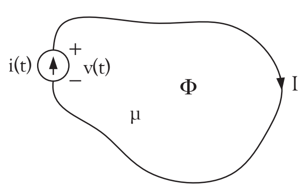


Figure 2-13. A *current-carrying loop in the presence of magnetic materials*

Magnetic materials may be present. The current of an ideal current source i(t) is increased slowly from zero to a final value I. Final flux is Φ. You may recall that the current in an inductor does not change readily; work is required. The work W_m done in establishing the current and flux is

$$W_m = \int P(t)\,dt = \int v(t)\,i(t)\,dt = \int \frac{d\Phi}{dt}\,i\,dt = \int i\,d\Phi$$
$$W_m = \int i\,d\Phi \qquad (2\text{-}32)$$

Eq. (2-32) is the general relationship for magnetic energy. It is valid for nonlinear problems, too. Note that $v = d\Phi/dt$ comes from Faraday's law. For linear problems where $\Phi = Li$,

$$W_m = \int i\frac{d\Phi}{dt}\,dt = \int i\left(\frac{di}{dt} + \int i\,\frac{dL}{dt}\right)dt \Rightarrow \int_0^I L\,i\,di\ \left(\text{if } \frac{dL}{dt} = 0\right)$$
$$= \frac{1}{2}L\,I^2 = \frac{1}{2}I\,\Phi = \frac{1}{2}\frac{\Phi^2}{L}\ \left(\text{if } \frac{dL}{dt} = 0\right)$$

$$W_m = \frac{1}{2}L\,I^2 = \frac{1}{2}I\,\Phi = \frac{1}{2}\frac{\Phi^2}{L}\quad \begin{array}{l}\text{Linear case}\\ \text{L constant in time}\end{array} \qquad (2\text{-}33)$$

In the process described above, current and voltage change with time but inductance is constant since there is no distortion of the loop or motion with respect to magnetic materials present. Now consider N coupled coils in the presence of magnetic materials (Figure 2-14).

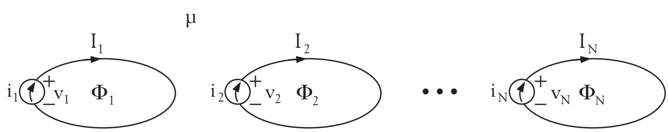


Figure 2-14. *N coupled coils in the presence of magnetic materials*

The currents are increased slowly from zero to their final values $I_1, \ldots, I_N$. For simplicity, bring $i_1(t)$ to its final value I_1 with $i_2, \ldots, i_N$ zero. Then bring $i_2(t)$ to its final value I_2 with $i_1 = I_1$ and $i_3, \ldots, i_N$ zero, etc. From our network model (Eq. 2-25) we know that $v_j(t) = \sum_{k=1}^{N} L_{jk}\,di_k/dt$ and that the total work done is

$$W_m = \sum_{j=1}^{N}\int i_j(t)\,v_j(t)\,dt = \sum_{j=1}^{N}\sum_{k=1}^{N} L_{jk}\int i_j(t)\,di_k$$

Consider the contribution of the pair of numbers (1, 2) which is

$$L_{11}\int_0^{I_1} i_1 \; di_1 + L_{21}\int_2^{I_1} i_2 \; di_1 + L_{22}\int_2^{I_2} i_2 \; di_2 + L_{12}\int_2^{I_2} i_1 \; di_2$$

Now note that $i_2 = 0$ in the second integral and $i_1 = I_1$ in the fourth integral. The contribution of (1, 2), i.e., the total work for N = 2, is

$$\begin{aligned} W_m &= \frac{1}{2}L_{11}I_1^2 + \frac{1}{2}L_{22}I_2^2 + L_{12}I_1I_2 \\ &= \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} L_{jk}I_jI_k \;\; \text{(assuming reciprocity, i.e., } L_{12} = L_{21}) \end{aligned} \tag{2-34}$$

For each pair of numbers (j, k) we have four integrals which yield contributions $\frac{1}{2}L_{jj}I_j^2 + \frac{1}{2}L_{kk}I_k^2 + L_{jk}I_jI_k$ as before. One of the integrals is zero because one of the currents (say, i_j) is raised to its final value when the other (say, i_k) is zero. Every pair of numbers has a similar contribution and the final result, assuming reciprocity, is

$$W_m = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} L_{jk} \; I_j \; I_k \tag{2-35}$$

Note also that

$$W_m = \frac{1}{2}\sum_{j=1}^{N} I_j \Phi_j \tag{2-36}$$

To see this, substitute $\Phi_j = \sum_{k=1}^{N} L_{jk} I_j$ in Eq. (2-36) to obtain Eq. (2-35).

Magnetic Energy of Volume Current Distributions

Next let's consider a current-carrying loop of finite cross section with volume current density **J.** Divide the loop into a large number N of loops in parallel. The typical loop has cross section ΔS_k about path C_k, current ΔI_k and flux Φ_k. Then

$$\Phi_k = \iint_{S_k} \mathbf{B}\cdot d\mathbf{s} = \iint_{S_k} (\nabla \times \mathbf{A})\cdot d\mathbf{s} = \oint_{C_k} \mathbf{A}\cdot d\ell \;\; \text{(by Stokes' theorem)}$$

where C_k bounds S_k. Using Eq. (2-36),

$$W_m = \frac{1}{2}\sum_{k=1}^{N}(\Delta I_k)\,\Phi_k = \frac{1}{2}\sum_{k=1}^{N}\Delta I_k \oint_{C_k} \mathbf{A}\cdot d\ell$$

Since

$$\Delta I_k d\ell = J\Delta IS_k d\ell = \mathbf{J}\Delta S_k d\ell$$

we have

$$W_m = \frac{1}{2}\sum_{k=1}^{N}\oint_{C_k}\mathbf{A}\cdot\mathbf{J}\,\Delta S_k dl \Rightarrow \iiint_V \mathbf{A}\cdot\mathbf{J}dv \text{ as } N\to\infty$$
$$W_m = \frac{1}{2}\iiint_V \mathbf{A}\cdot\mathbf{J}dv \qquad (2\text{-}37)$$

Note that Eq. (2-37) involves an integration over that volume V of space which includes currents since $\mathbf{J} = 0$ elsewhere. The magnetic energy is expressed in terms of the current density and the vector potential distribution just as the electric energy was in terms of the charge density and the scalar potential distribution.

Magnetic Energy in Terms of Fields

An alternative form of W_m may be obtained by expanding the range of integration to cover a larger volume V′ which includes V.

$$W_m = \frac{1}{2}\iiint_V \mathbf{A}\cdot\mathbf{J}dv = \frac{1}{2}\iiint_V \mathbf{A}\cdot(\nabla\times\mathbf{H})dv$$

Using the vector identity,

$$\nabla\cdot(\mathbf{A}\times\mathbf{H}) = \mathbf{H}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{H}) = \mathbf{H}\cdot\mathbf{B} - \mathbf{A}\cdot(\nabla\times\mathbf{H})$$

Then

$$W_m = \frac{1}{2}\iiint_V \mathbf{B}\cdot\mathbf{H}dv = \frac{1}{2}\iiint_V \nabla\cdot(\mathbf{A}\times\mathbf{H})dv$$
$$= \frac{1}{2}\iiint_V \mathbf{B}\cdot\mathbf{H}dv = \frac{1}{2}{}'\iint_S (\mathbf{A}\times\mathbf{H})d\mathbf{s}$$

The above result is valid for any volume V′ which includes V. As V′ approaches the entire region of space (say, it's a sphere of radius r) then the surface integral vanishes as $r \to \infty$ because $|\mathbf{A}| \sim 1/r$ and $|\mathbf{H}| \sim 1/r^2$ for a current distribution $\mathbf{J}$ of finite extent, and the surface area of the sphere is proportional to r^2. Thus

$$W_m = \frac{1}{2} \iiint_{\text{all space}} \mathbf{B} \cdot \mathbf{H} dv \qquad (2\text{-}38)$$

In Eq. (2-37), the energy is viewed as being stored with the current source, whereas Eq. (2-38) shows the energy being stored with the fields. Both equations yield the same results for a magnetostatic system, but for time-varying sources and fields, Eq. (2-38) makes more sense because the fields move in space, carrying the energy, as will be shown in Volume 4.

Finally, we note that Eqs. (2-37) and (2-38) can be used to determine the inductance of a circuit, using the relationship $W_m = 1/2\ LI^2$.

$$L = \frac{2W_m}{I^2} \frac{1}{I^2} \iiint \mathbf{B} \cdot \mathbf{H}\ dv = \frac{1}{I^2} \iiint \mathbf{A} \cdot \mathbf{J}\ dv \qquad (2\text{-}39)$$

We can drop the volume specifications at this point. The integrals exist over the regions of (**B, H**) and (**A, J**), respectively.

2.9 Magnetic Forces and Torques

Just as the electric force can be calculated from the electric energy for some system as shown in Volume 2, the magnetic force can also be calculated from the magnetic energy for some systems. Consider a solenoid into which is inserted a bar of magnetic material (Figure 2-15).

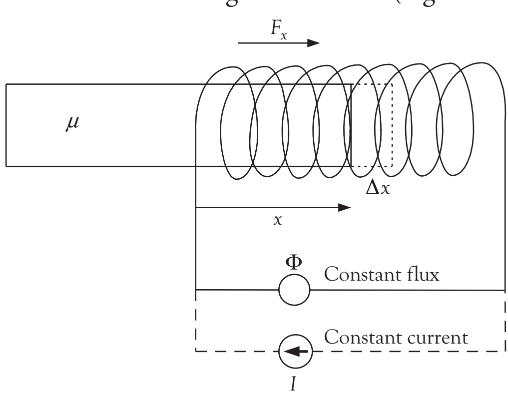


Figure 2-15. ***The force on a bar of magnetic material in a solenoid***

Its location is specified by the variable x. The source may be either an ideal current source or an ideal flux source. We may think of an ideal flux source as a device which senses flux and varies current to maintain flux constant. Thus it could be merely a variable current or voltage source. With the ideal flux source connected, I varies with x. With the ideal current source connected, Φ varies with x. Magnetic energy may be expressed as $W_m(\Phi, x)$ or $W_m(I, x)$.

Let's consider the two cases shown in Figure 2-15, the constant flux case (shown solid) and the constant current case (shown dotted) separately.

2.9.1 Constant Flux Case

Consider a small movement dx of the bar. Flux does not change and therefore the flux generator does no work since $\int i(t)\, v(t)\, dt = \int i\, d\Phi = 0$ (upper and lower limits are identical). In general there is a force F_x on the bar. It is either attracted towards or repelled from the solenoid. Mechanical work done is F_x dx. The mechanical work done may lead to mechanical energy stored in a spring or to kinetic energy. When x varies, the fields **B, H** may change, leading to a change dW_m in magnetic energy storage.

Conservation of energy thus requires that

$$F_x dx + dW_m = 0$$

$$F_x = -\frac{dW_m}{dx}\Big|_{\Phi\ \text{const.}} = -\frac{\partial W_m(\Phi, x)}{\partial x}$$

Similarly, $F_y = -\dfrac{\partial W_m(\Phi, y)}{\partial y}$ and $F_z = -\dfrac{\partial W_m(\Phi, z)}{\partial z}$. In general,

$$\mathbf{F} = -\nabla W_m(\Phi, x,y,z) \text{ (constant flux)} \qquad (2\text{-}40a)$$

If the bar is free to rotate in the θ direction, then work done is (Torque) $d\theta$ and

$$\text{Torque} = -\frac{\partial W_m(\Phi, \theta)}{\partial \theta} \text{ (constant flux)} \qquad (2\text{-}40b)$$

2.9.2 Constant Current Case

First we disconnect the constant flux source and connect the constant current source. Then consider a small movement dx. Flux changes from Φ to (Φ + dΦ). W_m changes from 1/2 IΦ to 1/2 I(Φ + dΦ); dW_m is 1/2 I dΦ. Work done by the current source is ∫ i(t) v(t) dt = ∫ i dΦ = I dΦ = 2 dW_m. Therefore energy supplied by the current source is equally divided, half to W_m and half to mechanical energy. Conservation of energy requires that

$$F_x dx + dW_m = 2dW_m$$

$$F_x = -\frac{dW_m}{dx}\bigg|_{\text{I const.}} \frac{\partial W_m(\Phi, x)}{\partial x}$$

In general,

$$\mathbf{F} = \nabla W_m(I, x, y, z) \text{ (constant current)} \qquad \text{(2-41a)}$$

If the bar is free to rotate in the θ direction,

$$\text{Torque} = -\frac{\partial W_m(I, \theta)}{\partial \theta} \text{ (constant current)} \qquad \text{(2-41b)}$$

Example 2-9. The Force Between Coupled Coils

Consider the force between two coupled coils (see Figure 1-15). Let coil 1 be fixed and let coil 2 be free to move.

Then

$$W_m = \frac{1}{2} L_{11} I_1^2 + \frac{1}{2} L_{22} I_2^2 + L_{12} I_1 I_2$$

The force between the two coils is

$$F_x = \frac{\partial W_m}{\partial x} \text{ (constant current)} = I_1 I_2 \frac{\partial L_{12}}{\partial x}$$

$$F_y = I_1 I_2 \frac{\partial L_{12}}{\partial y}$$

and the total force on coil 2 is

$$\mathbf{F} = I_1 I_2 \nabla L_{12} \qquad \text{(2-42)}$$

2.10 Magnetic Materials

Before we close this chapter, we discuss physical characteristics of magnetic materials. There are several types of magnetic materials with distinctly different characteristics. Three of the most common are diamagnetic, paramagnetic, and ferromagnetic materials. We call a material

$$\text{diamagnetic if } |\chi_m| \ll 1,\ \chi_m \text{ negative.}$$

$$\text{paramagnetic if } |\chi_m| \ll 1,\ \chi_m \text{ positive.}$$

$$\text{ferromagnetic if } |\chi_m| \gg 1.$$

Table 2-3 shows typical values of μ_r, χ_m for some magnetic materials.

Table 2-3 Magnetic Susceptibilities of Magnetic Materials

Diamagnetic Materials	
Material	χ_m
Aluminum Oxide	-0.5×10^{-5}
Copper	-0.94×10^{-5}
Gold	-3.6×10^{-5}
Lead	-1.7×10^{-5}
Silver	-2.6×10^{-5}
Bismuth	-1.7×10^{-4}
Sodium Chloride	-1.2×10^{-5}
Paramagnetic Materials	
	χ_m
Aluminum	2.1×10^{-5}
Cr_2O_3	1.7×10^{-3}
Platinum	2.9×10^{-4}
Palladium	8.2×10^{-4}
Oxygen	1.8×10^{-6}
Ferromagnetic Materials	
	$\mu_r \approx \chi_m$
Nickel	250
Cobalt	600
Iron	5×10^{3}
Permalloy	10^{5}
Super Malloy	10^{6}

Since $\mu_r = \chi_m + 1$, the relative permeability of diamagnetic and paramagnetic materials is very close to unity. For ferromagnetic materials μ_r and χ_m are practically equal. Note that $|\chi_m| \ll 1$ for both diamagnetic and paramagnetic materials. Typical values of χ_m are $\pm 10^{-5}$. Thus the magnetization **M** is much smaller in magnitude than the applied field **H** since

$$\mathbf{M} = \chi_m \mathbf{H}$$

For instance, consider a long thin solenoid with a current I and N turns per unit length. In the air-filled case, the magnetic flux density is

$$B_1 = \mu_0 H = \mu_0 NI.$$

If the solenoid is completely filled with a diamagnetic or paramagnetic material, then

$$B_2 = \mu_0 H + \mu_0 M = \mu_0 NI(1 + \chi_m) = B_1(1 + \chi_m)$$

The fractional change in flux density is of the order of 10^5. Thus the effect of the insertion of the cylindrical bar of diamagnetic or paramagnetic material is practically negligible.

Diamagnetic and paramagnetic materials are often called nonmagnetic materials because of the small magnitude of the magnetic effects. For instance, silver and aluminum are often described as non-magnetic although one is diamagnetic and the other paramagnetic. Michael Faraday discovered diamagnetism and paramagnetism in 1846. He noted that certain materials are attracted towards regions of higher fields and certain materials are repelled. Diamagnetic materials are repelled and all other materials are attracted (Figure 2-16).

Another example is shown in Figure 2-15. We recall Volume 2 that the electric force attracting a dielectric block into a parallel plate capacitor is proportional to ($\varepsilon - \varepsilon_0$ or $\chi_e\, \varepsilon_0$). By analogy, the magnetic force attracting a bar into a solenoid is proportional to ($\mu - \mu_0$ or $\chi_m\, \mu_0$). Thus the force in Figure 2-15 is repulsive if χ_m is negative (diamagnetic). There is no dielectric effect corresponding to diamagnetism because there are no materials with negative χ_e.

Diamagnetic and paramagnetic effects can be explained in terms of the magnetic moments of the molecules involved. There are two principle contributions to the magnetic moment, namely, (1) the orbital motion of electrons about the nucleus and (2) electron spin. (There is also a very small nuclear spin moment).

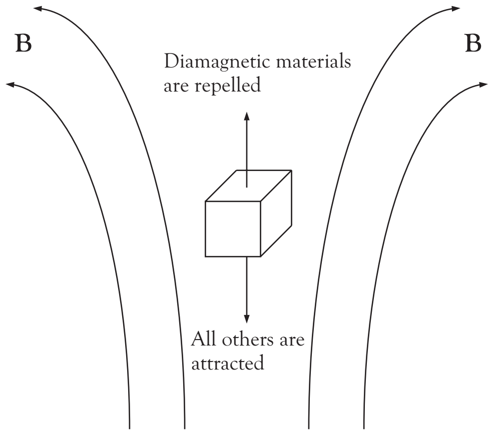


Figure 2-16. ***Diamagnetic material***

A diamagnetic material has molecules with zero net magnetic moment due to (1) and (2) in the absence of an applied magnetic field. When a magnetic field is applied, the forces on the moving electron change the orbital velocity to create a net magnetic moment for the magnetic dipole (an electron in an orbital motion constitutes a tiny loop of current). The net magnetic moment **m** and the magnetization **M** per unit volume are such as to oppose the applied field. Thus

$$\mathrm{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}$$

where **H** is the applied field and **M** opposes **H.** When the applied field **H** is removed, the magnetization **M** returns to zero, i.e., the effects are reversible.

In diamagnetic materials, there are other effects which occur but the principal one is the effect of the applied field upon the electron orbital motion. Even though the diamagnetic effect is very small, it is always present in all materials. Its effect may be hidden by other, more powerful, effects. The diamagnetic effect is linear as we might expect because of its small, perturbational effect. μ_r and χ_m are independent of temperature for diamagnetic materials. χ_m is of the order of 10^{-5} for most diamagnetic materials. The largest negative magnetic susceptibility χ_m is that of Bismuth, the material used by Faraday in his discovery of diamagnetism.

In paramagnetic materials, the molecules each have a net magnetic moment **m,** but magnetization **M** = 0, because of random orientation, in the absence of an applied field. If a magnetic field **H** is applied, there are several effects which occur. The principal one is the effect of the applied field upon the electron spin moments, which tend to align parallel to the direction of the applied field. However, random thermal agitation reduces significantly the magnetization **M** obtainable by this alignment of spins. Temperature would have to be lowered to a few degrees kelvin to approach the maximum effect. At room temperatures the random thermal motion almost completely overwhelms the tendency to align leaving only a very small magnetic moment in the direction of the applied field. This moment is, however, strong enough to overcome the diamagnetism which is always present. As we might expect, temperature has a significant effect on χ_m, which decreases rapidly as temperature increases from absolute zero.

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Principles of Electromagnetics 3—Magneto Statics

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