Principles of Electromagnetics 2—Dielectric and Conductive Materials

Arlon T. Adams
Jay K. Lee

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Preface

Electromagnetics is not an easy subject for students. The subject presents a number of challenges, such as: new math, new physics, new geometry, new insights and difficult problems. As a result, every aspect needs to be presented to students carefully, with thorough mathematics and strong physical insights and even alternative ways of viewing and formulating the subject. The theoretician James Clerk Maxwell and the experimentalist Michael Faraday, both shown on the cover, had high respect for physical insights.

This book is written primarily as a text for an undergraduate course in electromagnetics, taken by junior and senior engineering and physics students. The book can also serve as a text for beginning graduate courses by including advanced subjects and problems. The book has been thoroughly class-tested for many years for a two-semester Electromagnetics course at Syracuse University for electrical engineering and physics students. It could also be used for a one-semester course, covering up through Chapter 8 and perhaps skipping Chapter 4 and some other parts. For a one-semester course with more emphasis on waves, the instructor could briefly cover basic materials from statics (mainly Chapters 2 and 6) and then cover Chapters 8 through 12.

The authors have attempted to explain the difficult concepts of electromagnetic theory in a way that students can readily understand and follow, without omitting the important details critical to a solid understanding of a subject. We have included a large number of examples, summary tables, alternative formulations, whenever possible, and homework problems. The examples explain the basic approach, leading the students step by step, slowly at first, to the conclusion. Then special cases and limiting cases are examined to draw out analogies, physical insights and their interpretation. Finally, a very extensive set of problems enables the instructor to teach the course for several years without repeating problem assignments. Answers to selected problems at the end allow students to check if their answers are correct.

During our years of teaching electromagnetics, we became interested in its historical aspects and found it useful and instructive to introduce stories of the basic discoveries into the classroom. We have included short biographical sketches of some of the leading figures of electromagnetics, including Josiah Willard Gibbs, Charles Augustin Coulomb, Benjamin Franklin, Pierre Simon de Laplace, Georg Simon Ohm, Andre Marie Ampère, Joseph Henry, Michael Faraday, and James Clerk Maxwell.

The text incorporates some unique features that include:

- Coordinate transformations in 2D (Figures 1-11, 1-12).
- Summary tables, such as Table 2-1, 4-1, 6-1, 10-1.
- Repeated use of equivalent forms with R (conceptual) and $|\mathbf{r}-\mathbf{r}'|$ (mathematical) for the distance between the source point and the field point as in Eqs. (2-27), (2-46), (6-18), (6-19), (12-21).
- Intuitive derivation of equivalent bound charges from polarization sources, including piecewise approximation to non-uniform polarization (Section 3.3).
- Self-field (Section 3.8).
- Concept of the equivalent problem in the method of images (Section 4.3).
- Intuitive derivation of equivalent bound currents from magnetization sources, including piecewise approximation to non-uniform magnetization (Section 7.3).
- Thorough treatment of Faraday's law and experiments (Sections 8.3, 8.4).
- Uniform plane waves propagating in arbitrary direction (Section 9.4.1).
- Treatment of total internal reflection (Section 10.4).
- Transmission line equations from field theory (Section 11.7.2).
- Presentation of the retarded potential formulation in Chapter 12.
- Interpretation of the Hertzian dipole fields (Section 12.3).

Finally, we would like to acknowledge all those who contributed to the textbook. First of all, we would like to thank all of the undergraduate

and graduate students, too numerous to mention, whose comments and suggestions have proven invaluable. As well, one million thanks go to Ms. Brenda Flowers for typing the entire manuscript and making corrections numerous times. We also wish to express our gratitude to Dr. Eunseok Park, Professor Tae Hoon Yoo, Dr. Gokhan Aydin, and Mr. Walid M. G. Dyab for drawing figures and plotting curves, and to Professor Mahmoud El Sabbagh for reviewing the manuscript. Thanks go to the University of Poitiers, France and Seoul National University, Korea where an office and academic facilities were provided to Professor Adams and Professor Lee, respectively, during their sabbatical years. Thanks especially to Syracuse University where we taught for a total of over 50 years. Comments and suggestions from readers would be most welcome.

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CHAPTER 1

Introduction to Dielectrics

1.1 Introduction

We turn our attention now to ideal (perfect) dielectrics. Wood, glass, chalk, plastics, rubber, paper, quartz, and distilled water are all considered close to perfect dielectics. Ideal dielectrics do not contain *free* charges which can move from molecule to molecule. Instead they possess *bound* charges which are tightly bound to the atomic or molecular structure.

Now what happens when an electric field is applied to a dielectric as in Figure 1-1? The bound charges are not free to move from molecule to molecule but they can move over very small distances. Positive and negative charges tend to move in opposite directions. A typical result is shown in Figure 1-1 with dipoles existing in the body of the dielectric and surface charges on two surfaces. Dipoles are lined up with their dipole moments parallel to **E.** We say that the dielectric is *polarized.* We see that in Figure 1-1 bound surface charge density is formed on the right and left hand surfaces of the dielectric. Bound volume charge density may or may not exist inside the dielectric, depending on the type of polarization.

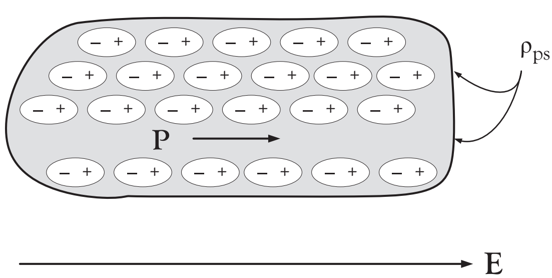


Figure 1-1. ***A polarized dielectric***

We distinguish bound charges from free charges by adding a subscript p. Thus bound or polarization charges may exist as bound volume charge density ρ_{pv} or as bound surface charge density ρ_{ps}. The net bound charge in a volume is represented as Q_p. The definitions for ρ_{pv}, ρ_{ps} are identical to Eqs. (2-2) except that ΔQ is replaced with ΔQ_p. Note that ρ_v, ρ_s, Q represent *free* charges; we add the subscript p for *bound* or *polarization* charge densities.

1.2 Polarization

We have seen an example of a polarized dielectric with dipoles existing within the body of a dielectric. Since dipoles give rise to electric fields and potentials, we need to characterize the strength of the dipoles within the dielectric. How many dipoles are there per unit volume and what are the dipole moments? We define a **polarization vector P**(x,y,z) to characterize the polarized state of the dielectric material. To calculate **P** at a particular point, we construct a small volume Δv around the point in question, and add vectorially all the dipole moments $\mathbf{p}_i$ within the volume Δv.

Then

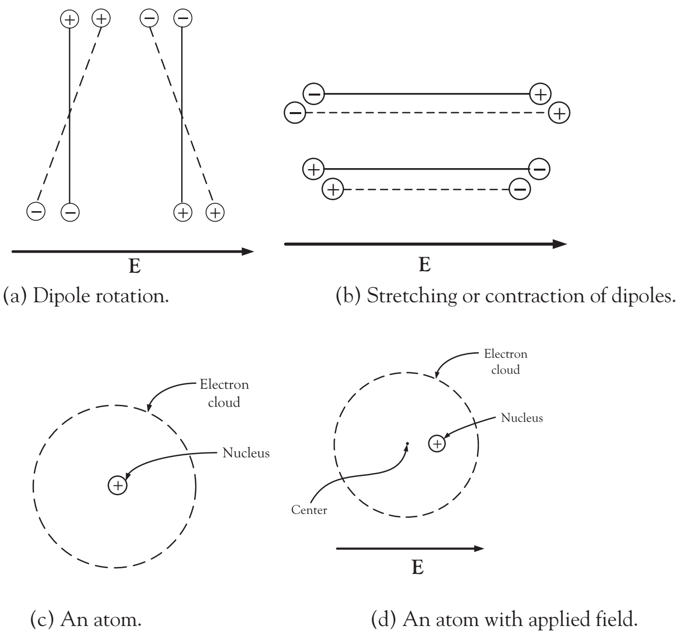
$$\mathbf{P} = \lim_{\Delta V \to 0} \frac{1}{\Delta v} \sum_i \mathbf{P}_i \tag{1-1}$$

P is called the polarization or the *dipole moment per unit volume.* By definition it is automatically zero in vacuum (free space).

In the absence of an applied electric field, most dielectrics are unpolarized (**P** = 0) either because of an orderly arrangement of dipoles whose moments cancel as in Figures 2-17(a),(b) or because of a random orientation of dipoles. The individual molecules may be polarized or not. *Nonpolar* molecules have no net dipole moment. *Polar* molecules each have a net dipole moment but a random orientation often tends to produce an unpolarized dielectric macroscopically in the absence of an electric field.

There are some dielectrics which are *permanently* polarized, i.e., they may remain polarized in the absence of an electric field. An example is barium titanate, which is one of the ferroelectrics. Such materials are called **electrets,** the electrical analogue of magnets.

If an electric field is applied, dielectrics which were previously unpolarized become polarized (**P** ≠ 0), as in Figure 1-1, with a polarization **P** which tends to be in the direction of the electric field **E.** There are several contributors to the polarization. First, consider an unpolarized dielectric with dipoles whose moments cancel in the absence of an electric field. The cancellation may be orderly or disorderly. Figure 1-2(a) shows an orderly cancellation. The application of an electric field leads to forces on the positive and negative bound charges which make up the dipoles. We assume that the dipoles can rotate slightly about their centers. The dipoles will tend to rotate to the new positions shown dotted, each rotation resulting in a dipole moment with an increased component in the direction of **E.** We also assume that the dipole can stretch or contract. Consider a dipole pair with moment parallel, antiparallel to **E** (Figure 1-2(b)). The applied field stretches one dipole and contracts another, in each case increasing the dipole moment in the direction of **E.** Figures 3-2(c), (d) show the effect of an applied field on an individual atom. The nucleus is displaced with respect to the center of the electron cloud. This produces a net dipole moment in the direction of **E.** Thus we see that given an unpolarized dielectric (**P** = 0), the addition of an applied electric field tends to polarize the dielectric with a **P** in the direction of **E.**



(a) Dipole rotation. (b) Stretching or contraction of dipoles.

(c) An atom. (d) An atom with applied field.

Figure 1-2. ***Dipoles in an electric field***

We should emphasize that the movements of the bound charges and the resultant deformation of the structure, i.e., the resultant rotations and stretching of dipoles, are very small indeed. The basic reason for this is that the applied fields produce forces on the bound charges which are very small compared with the strong internal forces holding the structure together. The movement shown in our figures is much greater than that usually obtained. The perturbation or deformation of the structure is usually minimal and for this reason the process is often linear (**P** ~ **E**). On the other hand, if extremely large electric fields are applied, the bound charges may be torn from their molecules and *breakdown* will occur.

1.3 The Electric Field of a Polarized Dielectric

As we have seen, a dielectric becomes polarized in the presence of an electric field. An orderly arrangement of dipoles is created with a concomitant bound surface charge and perhaps, as we will soon see, a bound volume charge as well. What are the bound electric charge densities associated with the state of polarization and what is the electric field due to those charges?

First, let us consider the charge distribution of a uniformly polarized dielectric. Figure 1-3(a) shows a block of dielectric with *uniform* polarization $\mathbf{P} = \mathbf{a}_z \mathbf{P}_o$.

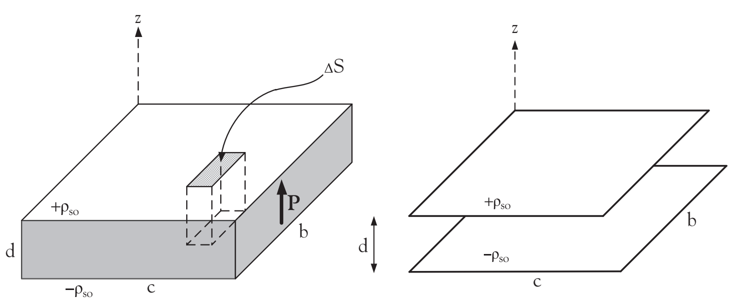


Figure 1-3 (a). ***A uniformly polarized block of dielectric*. (b) *An equivalent charge distribution***

We note that an unknown *uniform* bound surface charge density $\pm\ \rho_{so}$ is present on top and bottom surfaces, respectively. There is *no* bound volume charge density within the block since every small volume contains equal amounts of positive and negative charges, i.e., $\rho_{pv} = 0$. Since there is no volume charge density, we can represent the charges of the system as in Figure 1-3(b), which shows uniform surface charge densities $\pm\ \rho_{so}$ separated by a distance d. Now we will solve for ρ_{so} by requiring that the dipole moments of Figures 1-3(a), (b) be equal. We consider a small patch of surface Δs above and below and the intervening volume $d(\Delta s)$. The dipole moment of the intervening volume in Figure 1-3(a) is $\mathbf{a}_z\ (\mathbf{P}_o\ d\ \Delta s)$ because $\mathbf{P}_o$ is the dipole moment per unit volume. The dipole moment of the same volume in Figure 1-3(b) is $\mathbf{a}_z\ (q\ d) = \mathbf{a}_z\ (\rho_{so}\ \Delta s\ d)$. The moments must be equal and therefore

$$\rho_{so} = P_o \tag{1-2a}$$

We can also compare the total dipole moments of Figures 1-3(a), (b), i.e., $\mathbf{a}_z\ P_o\ (bcd) = \mathbf{a}_z\ \rho_{so}\ (bcd)$, with the same result. The bound surface charge density ρ_{ps} of Figure 1-3(a) may be characterized as follows:

$$\begin{aligned} \rho_{ps} &= \pm P_o = \mathbf{P}\cdot(\pm\mathbf{a}_z) \text{ on top and bottom} \\ &= 0 \text{ on vertical sides} \end{aligned} \tag{1-2b}$$

or, in general,

$$\rho_{ps} = \mathbf{P}\cdot\mathbf{a}_n \tag{1-2c}$$

where $\mathbf{a}_n$ is the outward unit normal to the dielectric surface.

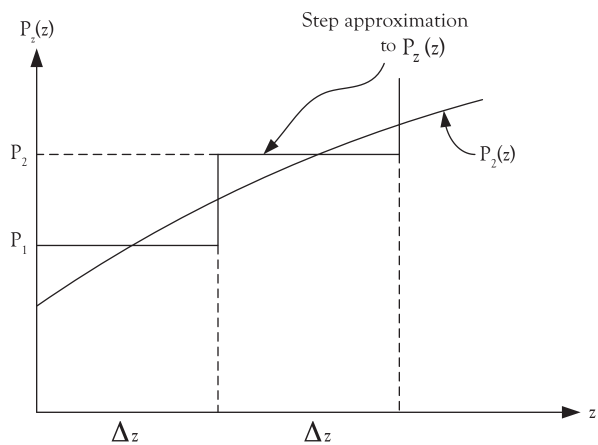


Figure 1-4. ***A non-uniform polarization $P_z(z)$***

It is perhaps surprising that we can throw away all the dipoles in the body of the dielectric since the individual dipoles do produce electric fields. This is essentially the same process as replacing two dipoles laid end-to-end with the two end charges. The dipole moment doubles and two charges in the middle cancel.

Now let us consider what happens if **P** is *not uniform* but varies in one dimension. Assume that **P** is z-directed, $\mathbf{P} = \mathbf{a}_z P_z$, and let P_z be a continuous function of z as shown in Figure 1-4.

We approximate the continuous function with a series of steps, each of length Δz. We then represent each step by a model such as that of Figure 1-3 with a uniform **P** in each step and bound surface charges at the beginning and end of each step. The resulting bound charges are shown in Figure 1-5.

The first two regions are represented by two slabs of uniformly polarized material with polarization P_1 and P_2, respectively. Since the polarization of the two slabs differ, there is a net surface charge density $P_1 - P_2$ which is equal in magnitude to the step discontinuity at z = Δz. Assume a surface of cross section area of Δs at z = Δz. Then the total bound surface charge at z = Δz is given as follows:

$$\Delta Q_P = \rho_{ps}\Delta s = (P_1 - P_2)\Delta s = -\frac{\partial P_z}{\partial z}\Delta z \Delta s$$

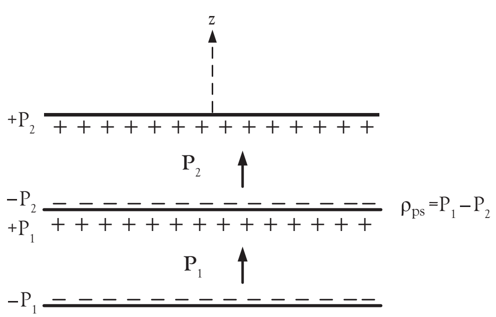


Figure 1-5. ***The charge distribution for the step approximation to the non-uniform polarization $P_z(z)$***

Now let's spread the excess bound surface charge ΔQ_P over the region (one half subsection (or $\frac{\Delta z}{2}$) to the right and one half subsection to the left) to obtain a representation of the bound volume charge density ρ_{pv}. This is equivalent to taking smaller and smaller steps in the approximation of the continuous function $P_z(Z)$. Then $\Delta Q_P = \rho_{pv} \Delta s \, \Delta z = -\frac{\partial P_z}{\partial z} \Delta s \, \Delta z$ and

$$\rho_{pv} = -\frac{\partial P_z}{\partial z}$$

If we consider additional components of **P**, then

$$\rho_{pv} = -\frac{\partial P_x}{\partial x} - \frac{\partial P_y}{\partial y} - \frac{\partial P_z}{\partial z} = -\nabla \cdot \mathbf{P} \qquad (1\text{-}3)$$

The result above corresponds to the limit $\Delta z \to 0$. As $\Delta z \to 0$, we obtain an infinite number of steps in the approximation and the surface charge becomes volume charge density.

Table 1-1 gives the equivalent bound charge densities for polarized material. These are *real* bound charges that exist in polarized material. They are *equivalent* in the sense that we can obtain the electric field **E** by assuming those charges in free space (so that the formulations of Chapter 2 apply). The bound charges are also called the *polarization charges.*

Table 1-1 Equivalent Bound Charge Densities

Surface Charge [C/m^2]	**Volume Charge [C/m^3]**
$\rho_{ps} = \mathbf{P} \cdot$ an	$\rho_{pv} = -\nabla \cdot P$

Equations (1-2) and (1-3) above specify the bound charge densities which are associated with the polarization vector **P.** Note that **P** in Eq. (1-2) should be evaluated *at the surface* and $\mathbf{a}_n$ is a unit vector normal *outward* from the dielectric surface. The negative sign in Eq. (1-3) is due to the previous choice that dipole moment **p** is directed from negative to positive charges. We note that for uniform polarization **P** there is no volume charge:

$$\rho_{pv} = 0 \ (\text{if } \mathbf{P} \text{ is constant}) \qquad (1\text{-}4)$$

Later we will see that uniform **P** is sufficient but not *necessary,* in order that $\rho_{pv} = 0$. We will see that $\rho_{pv} = 0$ if the dielectric is linear and homogeneous.

Given polarization **P**, the bound volume and surface charge densities are known. We can use this information to obtain the electric potential V and the electric field **E** merely by substituting ρ_{pv}, ρ_{ps} for ρ_v, ρ_s, respectively, in Eqs. (2-46) and (2-27), to obtain

$$V = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho_{pv}dv'}{|\mathbf{r}-\mathbf{r}'|} \text{ or } \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho_{pv}dv'}{R} \quad \text{for volume charge} \tag{1-5a}$$

$$V = \frac{1}{4\pi\varepsilon_0}\iint \frac{\rho_{ps}ds'}{|\mathbf{r}-\mathbf{r}'|} \text{ or } \frac{1}{4\pi\varepsilon_0}\iint \frac{\rho_{ps}ds'}{R} \quad \text{for surface charge} \tag{1-5b}$$

and

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho_{pv}(\mathbf{r}-\mathbf{r}')dv'}{|\mathbf{r}-\mathbf{r}'|^3} \text{ or } \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho_{pv}\mathbf{R}dv'}{R^3} \quad \text{for volume charge} \tag{1-6a}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\iint \frac{\rho_{ps}(\mathbf{r}-\mathbf{r}')ds'}{|\mathbf{r}-\mathbf{r}'|^3} \text{ or } \frac{1}{4\pi\varepsilon_0}\iint \frac{\rho_{ps}\mathbf{R}ds'}{R^3} \quad \text{for surface charge} \tag{1-6a}$$

Note that Eqs. (2-46) ad (2-27) can now be interpreted in a more general light. A charge is a charge; *bound and free charges contribute identically to V and **E**. Thus we can interpret ρ_v, ρ_s in Eqs. (2-46) and (2-27) as free charge densities, bound charge densities, or total (bound plus free) charge densities.*

Example 1-1. On-Axis Potential of a Permanently Polarized Cylinder (Electret)

Consider a uniformly polarized cylinder (Figure 1-6):

$$\mathbf{P} = \mathbf{a}_z P_o \begin{Bmatrix} 0 < \rho < a \\ z_1 \le z \le z_2 \end{Bmatrix}$$

Find the electrostatic potential along the z axis.

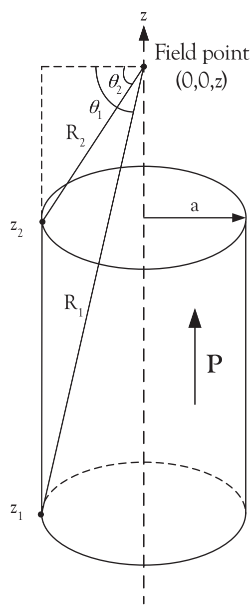


Figure 1-6. ***A permanently polarized dielectric cylinder (electret)***

Solution:

First we find the bound charge densities.

$$\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n = P_o \text{ (upper disk, } z = z_2)$$
$$= -P_o \text{ (lower disk, } z = z_2)$$

We can use directly the results from Example 2-9. Substituting $(z - z_2)$ for z everywhere in Eq. (2-49), we obtain the following contribution to the potential from the upper disk ($\rho_{ps} = P_o$):

$$\frac{P_o}{2\varepsilon_0}\left[\sqrt{(z-z_2)^2+a^2}-|z-z_2|\right]$$

Subtracting a similar contribution from the lower disk ($\rho_{ps} = -P_o$), we obtain

$$V(z)=\frac{P_o}{2\varepsilon_0}\left[\sqrt{(z-z_2)^2+a^2}-\sqrt{(z-z_1)^2+a^2}-|z-z_2|+|z-z_1|\right] \quad (1\text{-}7)$$

which can also be expressed as

$$V(z)=\frac{P_o}{2\varepsilon_0}\left[R_2-R_1-R_2\,|\sin\theta_2\,|+R_1\,|\sin\theta_1\,|\right] \quad (1\text{-}8)$$

Can you establish the following identity?

$$\begin{aligned} -|z-z_2|+|z-z_1| &= z_2-z_1(z\ge z_2) \\ &= z_1-z_2(z\ge z1) \\ &= 2z-z_2-z_1(z_1\le z\le z_2) \end{aligned}$$

Equation (1-7) can also be obtained by using Eq. (1-5b) for both upper and lower surfaces ($z = z_1, z_2$). The potential V(z) of Eq. (1-7) is continuous along the z axis. The electric field E_z jumps (is discontinuous) at $z = z_1, z_2$ because of the bound surface charges on the upper, lower surfaces of the cylinder.

The on-axis electric field E_z may be obtained in several ways: (a) by using $\mathbf{E} = -\nabla V$, (b) by direct application of Eq. (1-6b), (c) by using the results of Example 2-7 for both upper and lower discs at $z = z_1, z_2$.

1.4 The Displacement Vector D

As we have noted before, free and bound charges are identical in their effect on V, **E.** Therefore, in the presence of dielectrics, we replace ρ_v with $\rho_v + \rho_{pv}$ in Eq. (2-12) (Gauss' law) to obtain

$$\nabla \cdot \mathbf{E} = \frac{\rho_v + \rho_{pv}}{\varepsilon_0} \tag{1-9a}$$

or

$$\nabla \cdot \varepsilon_0 \mathbf{E} = \rho_v + \rho_{pv} \tag{1-9b}$$

and note that

$$\nabla \cdot \mathbf{P} = -\rho_{pv} \tag{1-3}$$

Equation (1-9) indicates that **E** lines begin and end on free or bound charges, going from positive to negative charges. Equation (1-3) indicates that **P** lines begin and end on bound charges, going from negative to positive charges. Adding Eqs. (1-9b) and (1-3),

$$\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_v \tag{1-10a}$$

The quantity ε_o **E + P** is defined as the **displacement vector D:**

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{1-11}$$

Then Eq. (1-10a) becomes

$$\nabla \cdot \mathbf{D} = \rho_v \tag{1-10b}$$

D is thus a vector whose lines begin and end on free charges. We obtain the integral form of Eq. (1-10b) by integrating both sides over volume V and applying the divergence theorem:

$$\oiint_S \mathbf{D} \cdot d\mathbf{s} = Q_f \tag{1-12}$$

where Q_f is the *free* charge enclosed within the surface S. Equation (1-12) should be recognized as a new form of Gauss' law which is valid even in the presence of the dielectric. Equation (1-12) is more useful since we often know the free charges but not the bound charges. Table 1-2 summarizes the two forms of Gauss' law for **D** and **E.** Note that

$$Q_f = \iiint_V \rho_v \; dv \text{ (total free charge enclosed)}$$

$$Q_p = \iiint_V \rho_{pv} \; dv \text{ (total bound charge enclosed)}$$

Table 1-2 Gauss' law

Point Form	**Integral Form**
$\nabla \cdot \mathbf{D} = \rho_v$	$\oiint_S \mathbf{D} \cdot d\mathbf{s} = Q_f$
$\nabla \cdot \mathbf{E} = \dfrac{1}{\varepsilon_0}(\rho_v + \rho_{pv})$	$\oiint_S \mathbf{E} \cdot d\mathbf{s} = \dfrac{Q_f + Q_p}{\varepsilon_0}$

1.4.1 Linear Dielectrics

Many dielectrics are highly *linear*; thus we can assume that **P** is proportional to **E.**

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \tag{1-13}$$

Then

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon_0 \varepsilon_r \mathbf{E} = \varepsilon \mathbf{E} \tag{1-14}$$

χ_e, ε_r, ε are called **electric susceptibility,** relative permittivity (or **dielectric constant**), **permittivity,** respectively. They represent three different ways of specifying the same linear relationship. We will usually specify the permittivity ε. The three parameters (χ_e, ε_r, ε) are not functions of **E** for linear dielectrics but may be functions of position (x,y,z). For *homogeneous* media they are constants. Thus ε(x,y,z) represents a linear, *inhomogeneous* medium, and ε = constant represents a linear, homogeneous medium.

For a linear dielectric medium,

$$\begin{aligned} \mathbf{D} &= \varepsilon \mathbf{E} \\ \mathbf{P} &= (\varepsilon - \varepsilon_0) \mathbf{E} \end{aligned} \qquad \text{Linear dielectric medium (1-15)}$$

and both **D** and **P** are known once **E** is known. Equation (1-15) is known as a **constitutive relation.** Table 1-3 shows some typical values of dielectric constant ε_r.

Table 1-3 Dielectric Constants

Material	Dielectric Constant
Air	1.0006
Balsa Wood	1.4
Paper	2-4
Teflon	2.1
Polyethylene	2.25
Polystyrene	2.55
Dry Soil	2.6
Glass	4 - 10
Quartz	4.3
Mica	6
Silicon	11.7
Alcohol	28
Water	80
Rutile	100
$BaTi_O3$	500 - 4500

1.4.2 Linear, Homogeneous Dielectrics

If a medium is both linear and homogeneous, then $\mathbf{D} = \varepsilon\,\mathbf{E}$ and ε is a constant. Thus

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\varepsilon \mathrm{E}) = \varepsilon \nabla \cdot \mathrm{E} + \underbrace{\nabla \varepsilon}_{\substack{\| \\ 0}} \cdot \mathrm{E} = \rho_v$$

(see Eq. (1-43)).

Then

$$\nabla \cdot \mathrm{E} = \frac{\rho_v}{\varepsilon} = \frac{\rho_v + \rho_{pv}}{\varepsilon_0}$$

and

$$\rho_{pv} = -\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon}\right)\rho_v \quad \text{Linar homogeneous dielectric} \quad (1\text{-}16)$$

Also, since an ideal (perfect) dielectric has no free charge ($\rho_v = 0$),

$$\rho_{\rho v} = 0 \text{ Liner, homogeneous, perfect dielectric} \qquad (1\text{-}17)$$

Example 1-2. A Point Charge and a Dielectric Shell
Figure 1-7 shows a point charge q surrounded by a dielectric shell of radii a, b and permittivity ε_1. Find the electric field in each region and the bound surface charges at r = a, b.
Solution:

Because of spherical symmetry, we expect that **E** and **D** will be radial.

$$\mathbf{E} = \mathbf{a}_r E_r, \ \mathbf{D} = \mathbf{a}_r D_r$$

Apply Gauss' law for the spherical surface of radius r.

Regardless of where r is chosen,

$$\oiint \mathbf{D} \cdot d\mathbf{s} = 4\pi r^2 D_r = Q_f = q$$

$$D_r = \frac{q}{4\pi r^2} \text{ everywhere} \qquad (1\text{-}18)$$

$$(r < a, \ a < r < b, \text{ and } r > b)$$

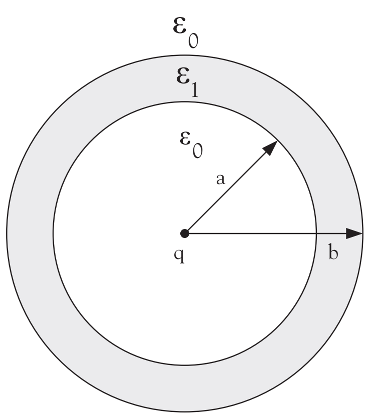


Figure 1-7. ***A point charge and a spherical dielectric shell***

$$E_r = \frac{D_r}{\varepsilon_1} = \frac{q}{4\pi\varepsilon_1 r^2} \quad (a < r < b) \text{ in the dielectric shell}$$

$$E_r = \frac{D_r}{\varepsilon_0} = \frac{q}{4\pi\varepsilon_0 r^2} \quad (r < a,\ r > b)$$

The polarization (vector) **P** in the dielectric shell is given by

$$\mathbf{P} = (\varepsilon_1 - \varepsilon_0)\mathbf{E} = \mathbf{a}_r \frac{(\varepsilon_1 - \varepsilon_0)q}{4\pi\varepsilon_1 r^2} \quad (a < r < b) \tag{1-19a}$$

Bound charges are:

$$\rho_{\rho v} = -\nabla \cdot P = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P_r\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left\{\frac{(\varepsilon_1 - \varepsilon_0)q}{4\pi\varepsilon_1}\right\} = 0$$

$$\rho_{\rho s} = \mathbf{P} \cdot \mathbf{a}_n = -\left[\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1}\right]\frac{q}{4\pi a^2} \text{ at } r = a\ [\mathbf{a}_n = -\mathbf{a}_r] \tag{1-19b}$$

$$= \left[\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1}\right]\frac{q}{4\pi b^2} \text{ at } r = b\ [\mathbf{a}_n = -\mathbf{a}_r]$$

Note that the electric charge q has drawn up a negative bound charge at r = a and left a positive bound charge at r = b. This bound charge is related to free charge (that would be present if there were a conducting shell) by the ratio indicated in Eq. (1-16). It also *weakens* the electric field within the dielectric. Note also that although **P** is not uniform, $\rho_{\rho v} = 0$, which confirms Eq. (1-17).

Example 1-3. A Dielectric-Loaded Coaxial Transmission Line

Figure 1-8 shows a coaxial line of inner radius a, outer radius b, which is loaded with a dielectric of permittivity ε_1. A voltage V_o is applied between inner and outer conductors. The inner conductor is the positive reference. Find the electrostatic fields and the bound and free charge densities.

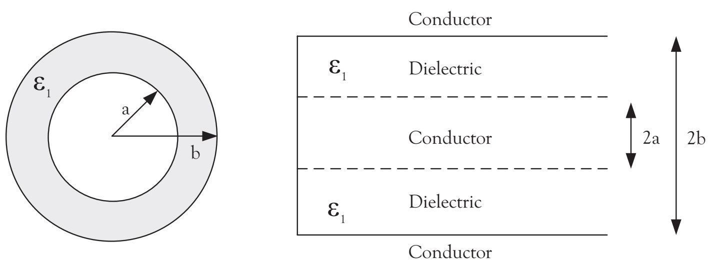


Figure 1-8. *A dielectric-loaded coaxial transmission line*

Solutions:

First we assume an unknown free charge ρℓ per unit length at $\rho = a$, which is drawn up by the application of voltage V_o. Then −ρℓ per unit length appears at $\rho = b$. Assuming that the coaxial line is infinitely long, the fields will be radial (ρ-directed) because of cylindrical symmetry.

Applying Gauss' law for the cylinder of radius ρ and length ℓ,

$$\oint\!\!\oint \mathbf{D}\cdot d\mathbf{s} = 2\pi\rho\, l\, D_p = Q_f = \rho_l l$$

$$D_\rho = \frac{\rho_l}{2\pi}\left(\frac{1}{\rho}\right),\ E_\rho = \frac{D_\rho}{\varepsilon_1} = \frac{\rho_l}{2\pi\varepsilon_1}\left(\frac{1}{\rho}\right)$$

To determine the (linear) relationship between applied voltage V_o and the resultant free charge ρℓ per unit length:

$$V_o = -\int \mathbf{E}\cdot d\ell = -\int_b^a E_\rho\, d\rho = -\frac{\rho_l}{2\pi\varepsilon_1}\int_b^a \frac{1}{\rho}\, d\rho = \frac{\rho_l}{2\pi\varepsilon_1}\ln\left(\frac{b}{a}\right)$$

and
$$\rho_l = \frac{2\pi\varepsilon_1 V_o}{\ln\left(\frac{b}{a}\right)}$$

The free surface charge density is

$$\rho_s = \frac{\rho_\ell}{2\pi a} = \frac{\varepsilon_1 V_o}{a\ln\left(\frac{b}{a}\right)} \quad \text{at } \rho = a \tag{1-20a}$$

$$= \frac{-\rho_l}{2\pi b} = \frac{\varepsilon_1 V_o}{b\ln\left(\frac{b}{a}\right)} \quad \text{at } \rho = b \tag{1-20b}$$

The electric field and the polarization are

$$\mathbf{E} = \mathbf{a}_\rho \frac{\rho_l}{2\pi\varepsilon_1}\left(\frac{1}{\rho}\right) = \mathbf{a}_\rho \frac{V_o}{\ln\left(\frac{b}{a}\right)}\left(\frac{1}{\rho}\right) \tag{1-21a}$$

$$\mathbf{P} = (\varepsilon_1 - \varepsilon_0)\mathbf{E} = \mathbf{a}_\rho \frac{(\varepsilon_1 - \varepsilon_0)\, V_o}{\ln\left(\frac{b}{a}\right)} \left(\frac{1}{\rho}\right) \quad (1\text{-}21b)$$

The bound surface charge density is

$$\rho_{ps} = \mathbf{P} = \mathbf{a}_n = -\frac{(\varepsilon_1 - \varepsilon_0)\, V_o}{a\, \ln\left(\frac{b}{a}\right)} (\rho = a) \quad (1\text{-}22a)$$

$$= \frac{(\varepsilon_1 - \varepsilon_0)\, V_o}{b \ln\left(\frac{b}{a}\right)} (\rho = b) \quad (1\text{-}22b)$$

Note that the bound and free surface charge densities at both $\rho = a$ and $\rho = b$ are related by

$$\rho_{ps} = -\left(\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1}\right) \rho_S$$

as in Eq. (3-16).

1.5 Boundary Conditions

We now turn to problems involving two or more media. To treat these problems, we need to understand how the vectors **E, D** behave at an interface. First, we analyze the tangential component of the electric field **E.** Consider a particular point P lying on the interface between two arbitrary media 1, 2 (Figure 1-9).

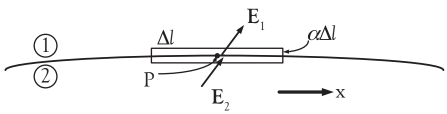


Figure 1-9. ***Tangential* E *at a boundary***

Let x, y be rectangular coordinates of a point P located in the tangent plane to the interface between media 1 and 2. We construct a closed contour C around P as shown in Figure 1-9 and consider the line integral

$$\oint_C \mathbf{E} \cdot d\ell$$

The long side $\Delta\ell$ is parallel to the x direction; $\Delta\ell$ is made small so that **E** does not vary over the length. α is made arbitrarily small compared to unity so that the short side does not contribute to the line integral. $\mathbf{E}_1$, $\mathbf{E}_2$ are the electric fields at the interface in regions 1, 2, respectively. Then from the first law, of electrostatics, Eq. (2-11),

$$\oint_C \mathbf{E} \cdot d\ell = (E_{1x} - E_{2x})\Delta\ell = 0$$

and $E_{1x} = E_{2x}$; similarly, $E_{1y} = E_{2y}$, therefore

$$\mathbf{E}_{1t} - \mathbf{E}_{2t} \tag{1-24}$$

i.e., the *tangential component of the electric field is continuous* across the boundary.

Next we analyze the normal component of the displacement **D.** Consider Figure 1-10 which shows a small closed surface S about the point P on the interface, with top and bottom (horizontal) surfaces of area Δs parallel to the tangent plane and vertical side of area $\alpha\Delta s$ normal to the tangent plane.

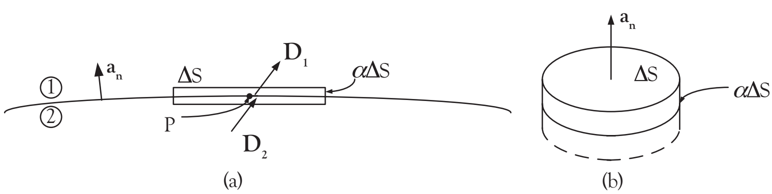


Figure 1-10. ***(a) Normal* D *at a boundary (side view).***
(b) Full view of surface S

We consider the integral $\oiint_S \mathbf{D} \cdot \mathbf{ds}$. Δs is made small so that **D** does not vary over S and α is made small compared to unity so that the flux through the vertical side does not contribute to $\oiint_S \mathbf{D} \cdot \mathbf{ds}$. A direction $\mathbf{a}_n$ normal to the interface is arbitrarily chosen pointing into region 1. Then if there exists a free surface charge ρ_s at the boundary, Gauss' law, Eq. (1-12), leads to

$$\oiint_S \mathbf{D} \cdot d\mathbf{s} = (D_{1n} - D_{2n})\Delta s = Q_f = \rho_s \Delta s$$
$$D_{1n} - D_{2n} = \rho_s \tag{1-25}$$

Normal D is discontinuous by an amount equal to the free surface charge density. The polarities can be checked by noting that a normal vector **D,** upon encountering a positive charge, increases. Note that the normal direction ($\mathbf{a}_n$) was chosen pointing into region 1; Eq. (1-25) will change if the normal is chosen pointing into region 2.

Because we have used the two basic laws $\oint_C \mathbf{E} \cdot d\ell$ and $\oiint_S \mathbf{D} \cdot d\mathbf{s} = Q_f$, which are valid for any electrostatic problem, the boundary conditions are also valid for any electrostatic problem. They hold for any media whatsoever, i.e., dielectrics (perfect and imperfect) and vacuum and any combinations thereof. There are thus many special cases for regions 1, 2. Table 1-4 outlines a few of the most important cases.

Note that for arbitrary media, the jump (discontinuity) in normal **D, E, P** is proportional to ρ_s, $\rho_s + \rho_{ps}$, $-\rho_{ps}$ respectively. *At the interface between two ideal dielectrics, normal **D** is continuous.* The boundary conditions at an air-conductor interface, Eq. (2-67), can be obtained by setting $\varepsilon_1 = \varepsilon_0$ in the last case of Table 1-4.

Table 1-4 Boundary Conditions

Arbitrary Media
$E_{1t} = E_{2t}$, $D_{1n} - D_{2n} = \rho_s$ $\varepsilon_0 (E_{1n} - E_{2n}) = \rho_s + \rho_{ps}$ $P_{1n} - P_{2n} = -\rho ps$
Linear, Ideal Dielectrics ($\rho_s = 0$)
$E_{1t} = E_{2t}$ $\quad$ $D_{1n} = D_{2n}$ $D_{1t} = D_{2t}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)$ $\quad$ $E_{1n} = E_{2n}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)$ $P_{1t} = P_{2t}\left(\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_2 - \varepsilon_0}\right)$ $\quad$ $P_{1n} = P_{2n}\left(\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_2 - \varepsilon_0}\right)\frac{\varepsilon_2}{\varepsilon_1}$

<u>Conductor, Dielectric</u>

(Region #2 is a conductor)

$$\mathbf{E}_2 = \mathbf{D}_2 = \mathbf{P}_2 = 0;\ \varepsilon_0 E_{1n} = (\rho_s + \rho_{ps}),\ E_{1t} = 0$$

$$\mathbf{D}_{1n} = \rho_s$$

$$\mathbf{P}_{1n} = -\rho ps$$

<u>Conductor, Linear Dielectric</u>

(Region #2 is a conductor and Region #1 has permittivity ε_1)

$$\mathbf{E}_2 = \mathbf{D}_2 = \mathbf{P}_2 = 0;\ E_{1n} = \frac{\rho_s}{\varepsilon_1},\ E_{1t} = 0$$

$$D_{1n} = \rho_s$$

$$P_{1n} = \rho_s \left(\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1} \right)$$

Example 1-4. Fields in a Parallel-Plate Region

Figure 1-11 shows the field lines for $\varepsilon_0\mathbf{E}$ and $\mathbf{P}$ in a parallel plate region with a homogeneous dielectric and air gaps. The fringing fields near the edges are neglected. The electric field $\mathbf{E}$ is stronger in the air gap, due to the fact that some $\mathbf{E}$ lines terminate on bound charge. Note that the sum of $\varepsilon_0\mathbf{E}$ + $\mathbf{P}$ equals $\mathbf{D}$ and that the $\mathbf{D}$ lines begin and end only on free charge. For a linear dielectric with permittivity ε_1, $\rho_{ps} = -\left(\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1} \right)$ as in Eq. (3-16).

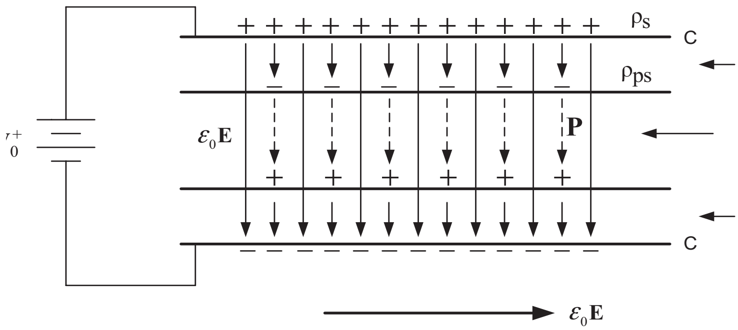


Figure 1-11. ***Field lines in a dielectric-loaded parallel-plate region with air gaps***

1.6 Capacitance

In this section we study the capacitance of a capacitor. Consider two conductors (1), (2) with charges Q, –Q, and electric field **E** and voltage V as shown in Figure 1-12(a).

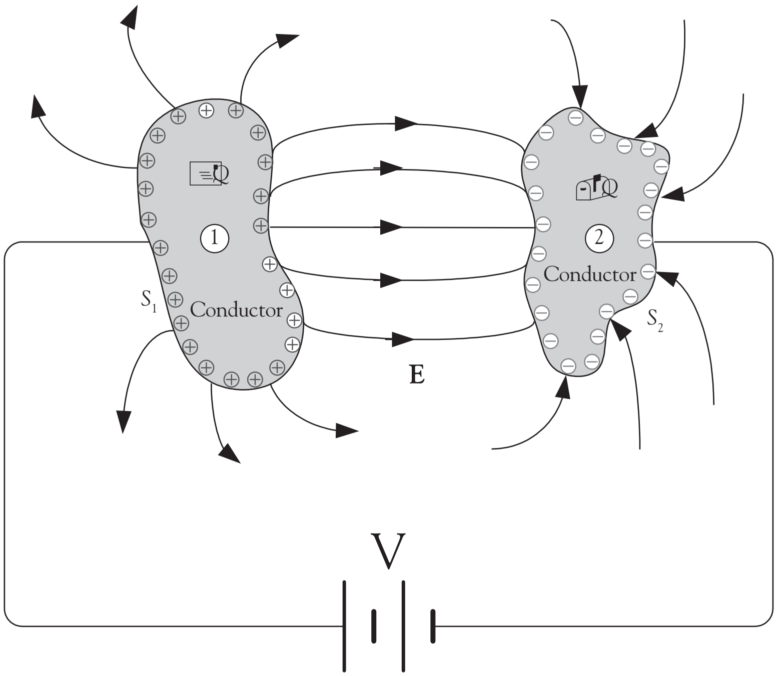


Figure 1-12(a). *Capacitance*

A linear dielectric of permittivity ε exists in the region between the conductors. The charge resides on the surfaces S_1, S_2 with a surface charge distribution ρ_s. Because of the linear relationships between ρ_s, **E,** V, there is a linear relationship between charge and voltage. Doubling one doubles the other. The ratio of charge to voltage is called the **capacitance:**

$$C = \frac{Q}{V} \tag{1-26}$$

Note that the voltage V here is the potential difference between the two conductors. The unit of the capacitance is the farad [F] (named after Michael Faraday). The capacitance may also be expressed in terms of electric field **E** and displacement **D:**

$$C = \frac{Q}{V} = \frac{\iint\limits_{S_1} \rho_s \, ds}{-\int_2^1 \mathbf{E} \cdot d\ell} = \frac{\iint\limits_{S_1} \mathbf{D} \cdot ds}{-\int_2^1 \mathbf{E} \cdot d\ell} \tag{1-27a}$$

which reduces to the following form for a linear, homogeneous dielectric

$$C = \frac{\varepsilon \iint\limits_{S_1} \mathbf{E} \, \mathbf{ds}}{-\int_2^1 \mathbf{E} \cdot d\ell} \tag{1-27b}$$

(for a linear, homonogeneous dielectric)

The capacitance of a capacitor depends on the geometry of the capacitor and the dielectric property of the medium between the conductors. The capacitance C can be calculated by the following steps:

1. Assume charges +Q, –Q on the two conductors.
2. Find the electric **E,** using one of the methods we learned so far.
3. Calculate the potential difference, $V = -\int_{(-)}^{(+)} \mathbf{E} \cdot d\ell$.
4. Find C by taking the ratio $\frac{Q}{V}$.

Example 1-5. Parallel-Plate Capacitor

Consider a parallel-plate capacitor with dielectric material of uniform permittivity ε shown in Figure 1-12(b). The area of each plate is A and plate separation is d. Neglect the fringing fields. Determine the capacitance.

Solution:

First we assume charges Q, –Q on the plates, resulting in a surface charge ρ_s as shown in Figure 1-12(b), where

$$\rho_s = \frac{Q}{A}$$

We expect that the electric field is uniform as in Example 2-6. Applying the boundary condition at x = 0,

$$D_n = D_x = \varepsilon E_x = \rho_s = \frac{Q}{A};\ E_x = \frac{Q}{\varepsilon A}$$

and the potential V across the capacitor between plates can now be determined.

$$V = -\int_d^0 \mathbf{E} \cdot d\ell = \int_0^d E_x dx = \frac{D_x d}{\varepsilon} = \frac{Qd}{\varepsilon A}$$

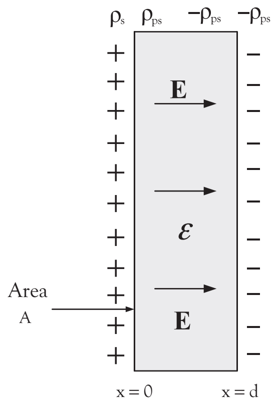


Figure 1-12(b). A *parallel-plate capacitor*

The capacitance is

$$C = \frac{Q}{V} = \frac{\varepsilon A}{d} \tag{1-28}$$

The bound charge ρ_{ps} at x = 0 is determined from boundary conditions:

$$\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n = \mathbf{P} \cdot (-\mathbf{a}_x) = -\frac{(\varepsilon_1 - \varepsilon_0)}{\varepsilon} D_x = -\left(\frac{\varepsilon_1 - \varepsilon_0}{\varepsilon}\right)\rho_S$$

as in Eq. (1-16). Note then $\mathbf{a}_n$ is normal *outward* from the dielectric.

Consider what happens when we insert the dielectric between parallel plates. Under constant charge conditions, bound charge is drawn up to weaken **E** and V, thereby increasing C. Under constant voltage conditions, the electric field must be maintained. Additional free charge must be drawn up to maintain **E** despite the weakening effect of bound charge. Q and C are therefore increased. Insertion of a dielectric increases the capacitance. C is proportional to the area of the plate and inversely proportional to the plate separation.

Example 1-6. Spherical Capacitors (see Figure 1-13)

(a) *The Isolated Sphere*

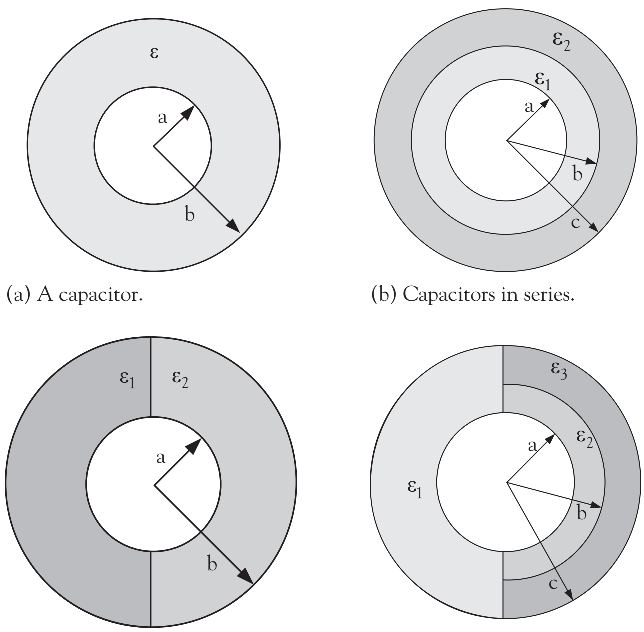
Consider an isolated conducting sphere of radius *a* in free space. To find the capacitance, we assume a charge q uniformly distributed over the surface of the sphere. Here we assume that the negative terminal is at infinity. Applying Gauss' law (see Example 2-4):

$$E_r = \frac{q}{4\pi\varepsilon_0 r^2}(r > a); V(r) = \frac{q}{4\pi\varepsilon_0 r}; V(a) = \frac{q}{4\pi\varepsilon_0 a},\ V(\infty) = 0$$

Then

$$C = \frac{q}{V(a)} = 4\pi\varepsilon_0 a \tag{1-29}$$

The capacitance of the earth is therefore less than a millifarad.



(a) A capacitor. (b) Capacitors in series.

(c) Capacitors in parallel. (d) Capacitors in a series-parallel combination.

Figure 1-13. ***Concentric spherical and cylindrical geometries.***
Inner and outer surfaces are perfect conductors

(b) *Concentric Spheres* (see Figure 1-13(a))

Consider concentric spheres (perfect conductors at r = a, b). The space between spheres is filled with a dielectric of permittivity ε. Assume a charge q at r = a and −q at r = b. Then

$$E_r = \frac{q}{4\pi\varepsilon r^2}(a < r < b); V_{ab} = -\int_b^a \mathbf{E}\cdot d\ell = \frac{q}{4\pi\varepsilon}(\frac{1}{a} - \frac{1}{b}) = \frac{q}{4\pi\varepsilon}\left(\frac{b-a}{a\,b}\right)$$

Then

$$C = \frac{q}{V_{ab}} = \frac{4\pi\varepsilon ab}{b-a} \tag{1-30}$$
$$\rightarrow 4\pi\varepsilon a \text{ as } b \rightarrow \infty$$

which agrees with Eq. (1-29).

(c) *Capacitors in Series* (Figure 1-13(b))

Figure 1-13(b) shows a spherical capacitor with two dielectrics. Two conductors are at r = a, c. We assume a charge q at r = a and −q at r = c. Applying Gauss' law:

$$D_r = \frac{q}{4\pi r^2}(a < r < c) \tag{1-31}$$

Note that normal **D** is continuous across the boundary of two dielectrics at r = b as required by the boundary conditions. The electric fields are given by

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_1} = \mathbf{a}_r\frac{q}{4\pi\varepsilon_1 r^2}(a < r < b)$$
$$= \frac{\mathbf{D}}{\varepsilon_2} = \mathbf{a}_r\frac{q}{4\pi\varepsilon_2 r^2}(b < r < c)$$

V_{ac} is then evaluated as follows

$$V_{ac} = -\int_c^a \mathbf{E}\cdot d\ell = -\int_c^b \frac{q}{4\pi\varepsilon_2 r^2}dr - \int_b^a \frac{q}{4\pi\varepsilon_1 r^2}dr = \frac{q}{4\pi}\left(\frac{1}{\varepsilon_1}\frac{(b-a)}{ab} + \frac{1}{\varepsilon_2}\frac{(c-b)}{cb}\right)$$

then

$$C = \frac{q}{V_{ac}} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} = \frac{C_1C_2}{C_1 + C_2} \tag{1-32}$$

where

$$C_1 = \frac{4\pi\varepsilon_1 ab}{b-a},\ C_2 = \frac{4\pi\varepsilon_2 bc}{c-b}$$

Note that the result corresponds to capacitances in series.

(d) *Capacitors in Parallel* (Figure 1-13(c))

Figure 1-13(c) shows a spherical capacitor with two dielectrics in parallel. We try a solution of the following form:

$$E_r = \frac{K}{r^2} (a < r < b) \tag{1-33}$$

in both dielectrics where K is an unknown constant. *Note that tangential **E** is continuous at the interface of two dielectrics as required by the boundary conditions.* The free charge densities ρ_{s1}, ρ_{s2} (at r = a) differ. Using the boundary condition (1-25), the total charge at r = a is given by

$$q = 2\pi a^2 (\rho_{s1} + \rho_{s2}) = 2\pi a^2 \left(\frac{\varepsilon_1 K}{a^2} + \frac{\varepsilon_2 K}{a^2} \right) = 2\pi K (\varepsilon_1 + \varepsilon_2)$$

$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\ell = -\int_b^a \frac{K}{r^2} dr = \frac{K(b-a)}{ab}$$

then

$$C = \frac{q}{V_{ab}} = \frac{2\pi(\varepsilon_1 + \varepsilon_2)}{(\frac{b-a}{ab})} = C_1 + C_2 \tag{1-34}$$

where

$$C_1 = \frac{2\pi\varepsilon_1 ab}{b-a}, \ C_2 = \frac{2\pi\varepsilon_2 ab}{b-a}$$

The result corresponds to capacitors in parallel.

(e)*A Series–Parallel Combination* (Figure 1-13(d))

Figure 3-13(d) shows a capacitor with three dielectrics. We are tempted to call this a series–parallel combination. *There is no simple form for **E** and normal **D** which will satisfy continuity of both tangential **E** and normal **D**.* In fact, this is a difficult problem involving fringing fields and requiring numerical computation for precise results. However, for thin spherical shells, the fringing is less significant and the capacitance may be approximated by the result given below:

$$C \approx C_1 + \frac{C_2 C_3}{C_2 + C_3} \quad (\text{if a ? c} - a) \tag{1-35}$$

where

$$C_1 = \frac{2\pi\varepsilon_1 ac}{c - a}$$

$$C_2 = \frac{2\pi\varepsilon_2 ab}{b - a}$$

$$C_3 = \frac{2\pi\varepsilon_3 bc}{c - b}$$

The result in Eq. (1-35) corresponds to a series-parallel combination.

Example 1-7. A Concentric Cylindrical Capacitor (Figure 1-13(a))

Note that Figure 1-13 applies to both *spherical* and *cylindrical* geometries.

The cross section of a concentric cylindrical capacitor of infinite length is shown in Figure 1-13(a). Surfaces ρ = a, b are perfect conductors. Find the capacitance per unit length appropriate to this cylindrical geometry.

Solution:

We assume a charge ρ_ℓ per unit length uniformly distributed over the surface ρ = a and $-\rho_\ell$ at ρ = b. Then, applying Gauss' law (see Example 1-3)

$$E_\rho = \frac{\rho_l}{2\pi\varepsilon}\left(\frac{1}{\rho}\right)(a < \rho < b)$$

$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\ell = -\int_b^a E_\rho d\rho = \frac{\rho_l}{2\pi\varepsilon} \ln\left(\frac{b}{a}\right)$$

The capacitance per unit length is given as follows:

$$\frac{C}{l} = \frac{Q/l}{V} = \frac{\rho_l}{V_{ab}} = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)}\left[\frac{F}{m}\right] \tag{1-36}$$

and

$$C = \frac{2\pi\varepsilon l}{\ln\left(\frac{b}{a}\right)} \tag{1-37}$$

For a concentric cylindrical capacitor of finite length ℓ, fringing at the ends invalidates the results given in Eq. (1-37), which will, however, be reasonably accurate if $\ell \gg a,b$.

1.7 Benjamin Franklin (1706-1790) and the Beginnings of Electrical Science

Benjamin Franklin can accurately be described as a *Renaissance man*: he did so many things and he did them to perfection. The breadth and depth of his accomplishments are largely unparalleled. History remembers him as the inventor of the lightning rod, bifocals, and the Franklin stove. We recall his role in the American Revolution as Ambassador to France. He appears in numerous famous paintings: performing his kite experiment, presenting the Declaration of Independence to John Hancock, and in his fur cap and spectacles in Paris. As memorable as these well-known images are, they commemorate only a very small number of his accomplishments. To gain some idea of the scope of his activities, we would have to read some of the biographies of Franklin; the 1938 biography by Carl Van Doren is still one of the best.

Benjamin Franklin was born in 1706, the youngest boy of a very large family. He attended school for about two years and, at age ten, began to work with his father, a soap and candle maker. At age twelve he was apprenticed to his brother James, a printer. He ran away to Philadelphia at age seventeen, worked as a printer, and eventually set up his own printing shop at age twenty-two. Within four years, he had started writing and printing "Poor Richard's Almanac", a yearly publication which would become immensely popular, eventually selling 10,000 copies a year and continuing for 27 years. By his late twenties, then, Franklin was well-established as a local printer and businessman, and was well on his way to wider recognition.

Benjamin Franklin wrote two outstanding books during his lifetime. The first was a description of his discoveries in electricity, *Experiments and Observations on Electricity,* published in 1751. This first publication was crucial in establishing Franklin's reputation as a scientist. In it, Franklin put forward the idea that lightning was electricity, proposed a specific experiment (the sentry box experiment) for verification of the theory, and recommended the use of grounded rods to protect buildings from lightning strikes. A year later, several French scientists carried out his sentry box experiment, while Franklin performed his famous kite experiment, resulting in a thorough confirmation of his basic theory. The book and these lightning experiments catapulted Franklin into the limelight and established him as one of the world's foremost scientists and a leading figure in the new science of electricity. Accolades and awards from around the world followed soon after. Two years later, in 1753, Benjamin Franklin received the Copley medal, an award that carried prestige equivalent to today's Nobel Prize.

Carl Van Doren, Benjamin Franklin, Bramhall House, 1987, Reprint of Viking Press, NY, 1938.

Benjamin Franklin wrote in a very clear and readable style. His judgment in reaching for a conclusion or drawing an hypothesis is very sure. His style is engaging and frank, and he readily acknowledges his ignorance or uncertainty: "You require the reason. I do not know it. Perhaps you may discover it and then you will be so good as to communicate it to me." His treatment of his own hypotheses was very refreshing. He readily acknowledged them as only tentative models and he was ready to abandon them if they proved inadequate: "I am still at a loss about the manner in which they (clouds) become charged with electricity; no hypothesis I have yet formed perfectly satisfying me."

Benjamin Franklin's second book is his *Autobiography,* which remains extremely popular to this day. Particularly interesting is the description of his self education and self discipline. It is the story of a brilliant and unusual man struggling to emerge from ignorance and poverty. He relates many of his mistakes and failures and does not bask in his successes and honors.

Throughout his life, Benjamin Franklin was also a prolific writer of newspaper articles. These were usually written to advance some social or political purpose. They were entertaining as well as educational, and their impact was always considerable. The undergraduate of today can read with edification and delight any of these famous articles, including "Miss Polly Baker," "Rattlesnakes for Felons," "The King of Prussia," and "Rules by Which a Great Empire May be Reduced to a Small One." In the age of Enlightenment, Franklin's articles led to personal recognition and to profound respect for the power of his arguments.

To return to Franklin's role as a scientist, what were the key scientific accomplishments which established his reputation? To begin with, one of his initial contributions was his analysis of the "Leyden jar," essentially a glass jar filled with water and held in one's hand. We may think of it as a jar which is coated separately with a conductor on the inside and outside. The Leyden jar is thus a "warped" parallel plate capacitor. Because of the large surface area and the thinness of the glass, the capacity is large and considerable electric energy can be stored in the glass. The Leyden jar was a convenient source of energy and was therefore used in a wide variety of experiments after it was discovered by the Dutch physicist Pieter Van Mussenbroek in 1746.

Franklin demonstrated that the conductors inside and outside were oppositely charged, that the total charge was zero and was therefore unchanged as the capacitor was discharged. He wrote: "These two states of electricity, the plus and the minus, are combined and balanced in this miraculous bottle in a manner that I can by no means comprehend". The Leyden jar analysis led to a general theory of positive and negative charges, the conservation of total charge, and the single fluid theory. Franklin explained his experiments with the motion of a single type of charge, namely free electrons. He also concluded that the energy was contained in the glass itself. He "unwrapped" the Leyden jar and formed the parallel plate capacitor. This was merely a flat glass plate coated with conductors on both sides. He then combined a set of plates in either series or parallel combinations. The series combination was called a "battery". The Leyden jar experiments were crucial because they led directly to Franklin's general theory of electricity. His theory swept aside the vague previous theories of "effluvia" and "affluvia" espoused by the Abbé Nollet, one of the leading figures in the early study of electricity.

Franklin's earliest writings on electricity in 1747 described some of his observations on the effects of pointed objects. His colleague Thomas Hopkinson had carried out some of the early experiments. By 1750 Franklin had proposed the lightning rod as a protection for houses and for ships and had added the very important grounding wire for both. He had also proposed his "sentry box" experiment for determining whether clouds were electrified, and whether lightning was, in fact, electricity. In 1752 the experiment was performed successfully in France with the sentry box and in America with the kite.

The drama of these experiments cannot be overrated. Lightning was proven to be electrical in origin and identical to the small-scale sparks produced in the laboratory. The clouds were shown to be electrically charged and the cloud-earth combination analogous to a giant Leyden jar. Foremost, a solution to the important practical problem of lightning hazard was proposed, namely the grounded lightning rod. The potential usefulness of lightning rods was enormous. It is estimated that in one thirty-three year period in the eighteenth century 386 church towers were struck and 120 bell ringers were killed in Germany alone.

Ultimately, Benjamin Franklin's most important contributions to electrical science were

(1) The conservation of electric charge.
(2) The single-fluid theory.
(3) The concept of positive and negative charges.
(4) The analysis and extension of the Leyden Jar.
(5) The effects of pointed objects.
(6) The grounded lightning rod.
(7) The electrification of clouds.

Now we begin to see why Franklin, who first organized in a coherent theory the basic ideas of electricity, was regarded in his time, and for some time after, as one of the leading scientists of electricity. His experiments and the book which described them gave order and structure to what had previously been a disorganized collection of theories. The importance of his work was immediately recognized by leading European scientists. He received the highest scientific awards of the day, including the prestigious Copley medal. He was also elected Fellow of the Royal Society and a member of the French Academy of Sciences. Long before the American Revolution, Franklin had established himself as the leading scientist of electrical phenomena.

So Franklin, in his time and for years afterwards, was widely recognized for his pioneering work in electricity. In recent times, however, we seem to overlook Franklin's scientific contributions. It is certainly possible that you do not think of Benjamin Franklin as a scientist, and there are several reasons for this oversight. Franklin left no equations which we recognize as fundamental parts of the theory. We have absorbed many of his contributions and use them unconsciously. As well, he was a prolific inventor, and it is easy to think of his inventions as representing his primary technical accomplishments, whereas they were actually the fruits of his scientific curiosity and scientific theories.

When Benjamin Franklin came to France as Ambassador, his scientific reputation preceded him. He was not an unknown Philadelphian, but an eminent scientist. His greatest experiment, the lightning experiment, had first been carried out by Frenchmen. His book had been translated into French and was in wide circulation. His greatest antagonist, the French Abbé Nollet, wrote two books attacking the Franklinian theory; he was particularly averse to lightning rods. Franklin chose not to reply directly to any of the Abbé's many attacks, and said, "I have an extreme aversion to public altercation on philosophic points, and have never yet disputed with anyone who thought fit to attack my opinions." He was directly defended by several Frenchmen, and, if anything, his reputation was enhanced by this controversy.

How much of his life did Benjamin Franklin devote to his key electrical experiments? We are surprised to note that it was a relatively short period of time, namely, the six years from 1747 thru 1752. Having achieved financial independence, Benjamin Franklin retired from the printing business in 1748 in order to devote himself more completely to his electrical research. His book on experiments was first published in 1751 and the lightning rod experiments were performed in 1752.

Franklin was the quintessential public relationist. He was very conscious of his image and strove at an early age to project a positive one. This is not to say that the image necessarily conflicted with reality but that Franklin fully realized its importance. Thus, "To show that I was not above my business, I sometimes brought home the paper I purchased at the stores through the streets on a wheelbarrow." In his scientific writings he projected the image of an open, receptive person. As a result, other scientists flocked to share their results with him and to perform experiments for him. When Franklin first came to France as Ambassador during the American Revolution, he was already known as an eminent scientist. What image then was appropriate for the diplomat Franklin? Could he possibly outdo the other diplomats in their courtly finery? He decided to do just the opposite of what was expected and appeared in simple dress with a fur cap, thereby projecting a unique and lasting image.

Franklin was a very highly skilled diplomat. His efforts on behalf of the fledgling republic led to diplomatic recognition by France, to massive military aid, and to huge financial loans. There are many examples of Franklin's diplomatic skill but one of the most striking occurred at the conclusion of the Peace Treaty negotiations. The Americans had decided to break their agreement with France and their instructions from Congress and carry out separate negotiations with England. Any other course of action would have made the negotiations almost impossible. An extremely favorable treaty had resulted, including extensions of United States Territory to the Mississippi and to Canada. The negotiations could not have gone better. The Americans had come to the table with their most talented statesmen (Franklin, Adams, Jefferson, Jay, Laurens) and had overwhelmed the English negotiators. Now it was necessary to tell the French about the negotiations. Franklin reported to his French counterpart, Charles Gravier, the comte de Vergennes, and sent him a copy of the treaty which had already been signed. Vergennes sent him back a note of reproach. Franklin now sent Vergennes a famous letter in which he put forth the American position in the most favorable light, made a partial apology, and said "the English, I just now learn, flatter themselves they have already divided us." Somehow this did the trick! Not only were fences mended but Vergennes arranged yet another much-needed loan to the United States. The years of careful diplomacy to establish cordial relationships with Vergennes combined with just the right touch in the letter, had preserved America's unique relationship with France.

Additionally, Franklin had recognized early the injustice of the institution of slavery and had written many articles on the subject, including those that highlighted unrecognized talents of African Americans. Two months before his death he submitted to Congress a petition for the abolition of slavery. It was turned down.

1.8 The Force on a Conductor in an Electric Field

We turn now to the calculation of the force on a conductor in an electric field. This would seem to be straightforward matter but it's not. There are subtleties involved which are concerned with the *self* field $\mathbf{E}_s$. In the course of the discussion we will learn the very useful fact that in calculating forces we can always subtract the self force. For the force on a conductor, this means that we can use $(\mathbf{E} - \mathbf{E}_s)$ in place of $\mathbf{E}$.

To find the force on a conductor we need to consider only the forces on the electric charges which exist on the surface of the conductor. Figure 3-14(a) shows a conductor with a typical patch of the surface with area ds, surface charge density ρ_s, and electric field $\mathbf{E}$. The force d$\mathbf{F}$ on the patch is equal to the charge $\rho_s$ds times the electric field vector. There is just one problem and that is, "which electric field?"

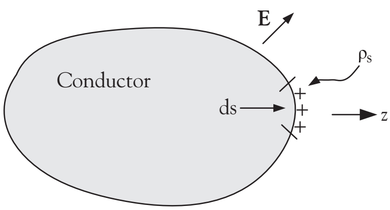


Figure 1-14(a). ***A conductor in an electric field***

Figure 1-14(b) shows a plot of the electric field near the surface of the conductor. We choose z as the direction normal to the patch. E_z is a step function which jumps from zero to $\frac{\rho_s}{\varepsilon_o}$ at the surface of the conductor. For $\mathbf{E}$ should we use the value (zero) just inside the conductor, the value $\frac{\rho_s}{\varepsilon_o}$ just outside, the average $\frac{\rho_s}{2\varepsilon_o}$, or something else?

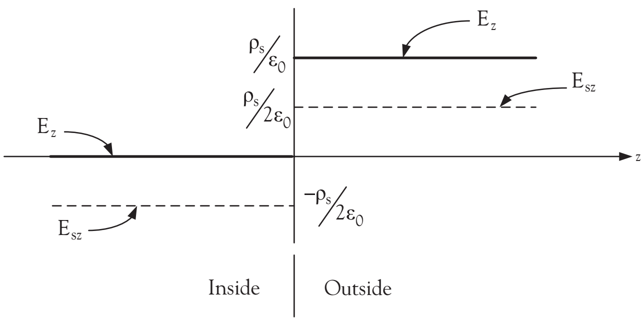


Figure 1-14(b). ***The normal electric field E_z and a self field E_{sz} at a conductor surface***

Before going any further, let's remind ourselves of the behavior of the electric field **E** and also the behavior of the self field $\mathbf{E}_s$ which is the contribution of the patch itself. We are interested in the behavior of these fields just inside and just outside the conductor surface.

$$\mathrm{E} = \mathbf{a}_n \frac{\rho_s}{\varepsilon_0} = \mathbf{a}_z \frac{\rho_s}{\varepsilon_0} \text{ (just outside)} \tag{1-38a}$$

$$= 0 \text{ (just inside)} \tag{1-38b}$$

$$\mathbf{E}_{av} = \frac{1}{2} \left\{\mathrm{E(outside)} + \mathrm{E(inside)}\right\} = \mathbf{a}_z \frac{\rho_s}{2\varepsilon_0} > \tag{1-39}$$

The self field $\mathbf{E}_s$ due to the small patch itself is the same as that of an infinite plane of charge:

$$\mathbf{E}_s = \pm\mathbf{a}_n \frac{\rho_s}{2\varepsilon_0} = \mathbf{a}_z \frac{\rho_s}{2\varepsilon_0} \text{ (just outside)} \tag{1-40a}$$

$$= -\mathbf{a}_z \frac{\rho_s}{2\varepsilon_0} \text{ (just inside)} \tag{1-40b}$$

Figure 1-14(b) shows a plot of **E** (solid) and $\mathbf{E}_s$ (dotted). We note that **E** and $\mathbf{E}_s$ have the same jump (discontinuity) at the surface and therefore we consider using their difference ($\mathbf{E} - \mathbf{E}_s$) in calculating forces. This is legitimate since the conductor exerts no net force on itself. The difference $\mathbf{E} - \mathbf{E}_s$ is continuous:

$$\mathbf{E} - \mathbf{E}_s = \mathbf{a}_z \frac{\rho_s}{2\varepsilon_0} = \mathbf{E}_{av} \tag{1-41}$$

Now we see the culprit. It is the self field $\mathbf{E}_s$ which is discontinuous at the surface. Its presence in **E** leads to an ambiguity; its removal resolves the ambiguity. Its removal is also equivalent to taking the average of fields inside and out (since $\mathbf{E}_s$ is odd). $\mathbf{E}_s$ is the field at the patch due to the patch itself; $\mathbf{E} - \mathbf{E}_s$ is the field at the patch due to the rest of the conductor, i.e., everything but the patch.

Thus the force **F** on a conductor in an electric field is given as

$$\mathrm{F} = \iint_S (\mathrm{E} - \mathrm{E}_S)\rho_S ds = \iint \mathrm{E}_{av}\ \rho_S ds = \iint_S \mathbf{a}_n \frac{\rho_S^2}{2\varepsilon_0} ds$$

$$\mathbf{F} = \iint_S \mathbf{a}_n \frac{\rho_S^2}{2\varepsilon_0} ds \qquad (1\text{-}42)$$

1.9 Energy and Forces in the Electrostatic Field

In this section, we calculate the energy of, or the work required to assemble, electrostatic systems. In addition, we are interested in the forces exerted on portions of the system. First we consider the work required to assemble a collection of point charges.

1.9.1 Energy of a Collection of Point Charges

Figure 1-15 shows a set of N point charges at arbitrary locations. The distance between q_1 and q_2 is r_{12} and the distance between q_i and q_j is r_{ij}. Let us determine the work required to bring up the charges from infinity. We start with all the charges an infinite distance from the origin and from each other. The point charges have been given to us; we do not, for the moment, inquire into the work required to construct them.

No work is required to bring up the first charge (q_1) since there are no forces exerted on it. Charge q_2 is brought up working against the potential of q_1. The potential through which q_2 is moved is $\frac{q_1}{4\pi\varepsilon_0 r_{12}}$; the work required is thus $\frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}}$. Similarly, charge q_3 works against the forces of both q_1 and q_2. The work required to bring up q_3, working against the forces of both q_1 and q_2, is

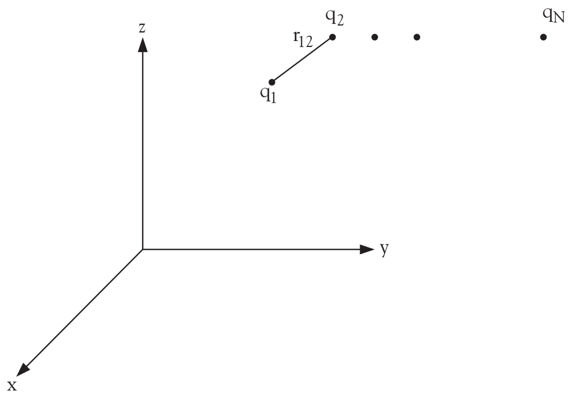


Figure 1-15. ***A collection of point charges***

$$q_3\left(\frac{q_1}{4\pi\varepsilon_0 r_{13}}+\frac{q_2}{4\pi\varepsilon_0 r_{23}}\right)$$

Finally, the work required to bring up q_N is

$$q_N\left(\frac{q_1}{4\pi\varepsilon_0 r_{1N}}+\frac{q_2}{4\pi\varepsilon_0 r_{2N}}+\cdots+\frac{q_{N-1}}{4\pi\varepsilon_0 r_{(N-1)N}}\right)$$

Consider an N × N matrix of terms $\frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}}$. We have taken into account all the terms above or below the main diagonal and may thus express the total electric energy W_e as follows:

$$W_e=\sum_{i=1}^{N}\sum_{\substack{j=1\\ j>i}}^{N}\frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}} \tag{1-43a}$$

where j > i includes terms above the main diagonal. Because of symmetry about the main diagonal,

$$W_e=\frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}} \tag{1-43b}$$

$$=\frac{1}{2}\sum_{i=1}^{N}q_i\left(\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}}\right)=\frac{1}{2}\sum_{i=1}^{N}q_i V_i \tag{1-44}$$

where V_i is the potential at the location of q_i due to all charges except q_i itself. The factor of $\frac{1}{2}$ in Eq. (1-44) is related to the linear nature of the problem. The first charge is brought up through zero potential, the last charge is brought up through the total potential V, and the average charge is brought up through $\frac{1}{2}V$.

1.9.2 Energy of Continuous Charge Distribution

For continuous charge distribution $\rho_v(x,y,z)$, we divide it into infinitesimally small volumes and consider the charge of an infinitesimally small volume element to be dQ = $\rho_v$dv. We then use the summation of Eq. (1-44) which becomes an integral. Integrating over all volume elements yields

$$W_e = \frac{1}{2}\iiint \rho_v \left(x,y,z\right)V(x,y,z\right)dv \tag{1-45}$$

Eq. (1-45) is the energy expression for volume charge distribution. For surface charge distribution it reduces to $\frac{1}{2}\iint \rho_s Vds$ if the surface is a conductor it reduces to $\frac{1}{2}$ where Q is the total charge on the conductor and V is the potential of the conductor, since the potential V is constant on the conductor. The energy stored in the capacitor, which consists of two conductors carrying charges
+Q, –Q (see Figure 1-12(a)), can be expressed as follows:

$$W_e = \frac{1}{2}QV_1 + \frac{1}{2}(-Q)V_1 = \frac{1}{2}Q(V_1 - V_2) = \frac{1}{2}QV = \frac{1}{2}CV^2$$

where V_1 and V_2 are the potentials of the two conductors and V is the potential difference or voltage.

Energy in Terms of Fields

A second general form may be obtained by extending the volume of Eq. (1-45) to infinity. This is permitted since $\rho_v = 0$ outside the charge distribution. Then

$$W_e = \frac{1}{2}\iiint_{\text{all space}} \rho_v Vdv = \frac{1}{2}\iiint_{\text{all space}} (\nabla \cdot D)Vdv$$
$$= \frac{1}{2}\iiint_{\text{all space}} \left\{\nabla \cdot (V\mathbf{D}) - \mathbf{D} \cdot \nabla V\right\}dv$$

where we have used the vector identity

$$\nabla \cdot (V\mathbf{D}) = V(\nabla \cdot \mathbf{D}) + (\nabla V) \cdot \mathbf{D}$$

We now use $\mathbf{E} = -\nabla V$ and apply the divergence theorem to the first term of the integral above to obtain

$$W_e = \frac{1}{2}\left\{ \iiint\limits_{\text{all space}} \mathbf{E}\cdot\mathbf{D}\ dv + ' \iint\limits_{\text{sphere at }\infty} V\,\mathbf{D})\cdot d\mathbf{s} \right\}$$

For a charge distribution of finite extent, $V \sim \frac{1}{r},\ |\mathbf{D}| \sim \frac{1}{r^2},\ ds \sim r^2,\ \text{as } r \to \infty$,

and the surface integral above vanishes. Thus

$$W_e = \frac{1}{2} \iiint\limits_{\text{all space}} \mathbf{E}\cdot\mathbf{D}\ dv \qquad (1\text{-}46)$$

In Eq. (1-45), the energy is viewed as being stored with the charge source, whereas Eq. (1-46) shows the energy being stored with the fields. Both equations yield the same result for an electrostatic system, but for time-varying sources and fields, Eq. (1-46) makes more sense because the fields move in space, carrying the energy,

Example 1-8. A Spherical Cloud of Charge

Consider a spherical cloud of uniform volume charge density ρ_v:

$$\rho_v = \rho_o \quad (r < a)$$

Find the work required to assemble the cloud of charge.

Solution:

Applying Gauss' Law:

$$'\iiint\limits_S \mathbf{D}\cdot ds = 4\pi r^2 D_r = \rho_o \frac{4}{3}\pi r^3 \quad (r \le a)$$

$$= \rho_o \frac{4}{3}\pi a^3 \quad (r \ge a)$$

$$D_r = \varepsilon_o E_r = \frac{\rho_o r}{3} (r \le a)$$

$$= \frac{\rho_o a^3}{3r^2} (r \ge a)$$

The electric potential is given by

$$V = -\int \mathbf{E}\cdot d\ell = -\int_{\infty}^{r} E_r\big|_{r\geq a}\, dr = -\int_{\infty}^{r} \frac{\rho_o a^3}{3\varepsilon_0 r^2}\, dr = \frac{\rho_o a^3}{3\varepsilon_0 r} \quad (r \geq a)$$

$$= -\int_{\infty}^{a} E_r\big|_{r\geq a}\, dr - \int_{a}^{r} E_r\big|_{r\leq a}\, dr = \frac{\rho_o a^2}{2\varepsilon_0} - \frac{\rho_o r^2}{6\varepsilon_0} \quad (r \leq a) \qquad (1\text{-}47)$$

$$W_e = \frac{1}{2}\iiint \rho_v V\, dv$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho_o^2 \left[\frac{a^2}{2\varepsilon_0} - \frac{r^2}{6\varepsilon_0}\right] r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{4\pi\rho_o^2 a^5}{15\varepsilon_0} = \frac{3Q^2}{20\pi\varepsilon_0 a}$$

(where $Q = \frac{4}{3}\pi a^3 \rho_o$). We can also derive the same result, using the energy in terms of fields, Eq. (1-46) [see Problem 3-26]. Note that

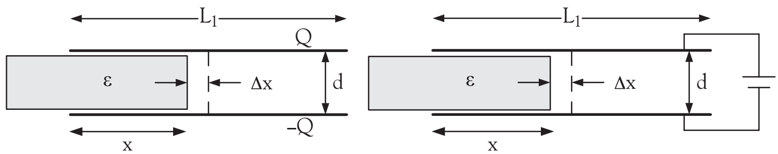
$W_e \to 0$ as $a \to 0$ (ρ_o constant)

$W_e \to \infty$ as $a \to 0$ (Q constant)

This implies that (a) a small volume of uniform finite ρ_o has negligible energy and (b) a point charge Q has infinite energy.

1.9.3 Forces and Torques in Terms of Energy

Let us consider an electrostatic system consisting of conductors and dielectrics. Allow one of the parts of the system to move under the influence of the force **F** acting upon it. We allow a small (virtual) displacement Δx of the movable part. Consider two situations: (a) constant charge for which conductors are isolated from each other and the total charge on each conductor remains constant during the displacement, (b) constant potentials for which potentials between conductors are constant during the displacement. Figure 1-16 shows parallel plates with a dielectric block partially inserted between plates. In the constant charge case the plates are unconnected and charge Q remains constant during a small displacement Δx. x is defined as the distance between a point on the fixed plates and the movable block of dielectric. In the constant voltage case an ideal voltage source is connected between plates. In this case, additional charge ΔQ is deposited on the upper plate during the displacement Δx.



(a) Constant charge case. (b) Constant voltage case.

Figure 1-16. *A dielectric slab partially inserted between parallel plates*

Constant Charge

In the constant charge case, charge is fixed but potentials and fields may vary. Therefore the electrostatic energy W_e may vary with displacement Δx. As the dielectric block moves through displacement Δx, under the influence of force F_x, mechanical work $F_x \Delta x$ is done. The mechanical energy is thereby changed. It could be stored (as in a spring) or it could show up as kinetic energy $(\frac{1}{2} mv^2)$. In any case, both the mechanical and electrostatic energy may change. Conservation of energy may be expressed as follows:

$$F_x = \Delta x + W_e = 0 .$$

$$F_x = -\frac{\Delta W_e}{\Delta x} \to -\frac{\partial W_e(Q,x)}{\partial x} (\text{as } \Delta x \to 0) \qquad (1\text{-}48a)$$

Constant Voltage

In the constant voltage case, the voltage is fixed but the charge may vary. If the charge is increased by ΔQ then the electrostatic energy, which is equal to $\frac{1}{2} VQ$, increases by $\frac{1}{2} V\Delta Q$ to $\frac{1}{2} V(Q + \Delta Q)$. The mechanical work done is $F_x \Delta x$ as before. The battery is a third contributor to energy storage. The battery does work (supplies energy) $V(\Delta Q)$ in moving charge ΔQ through voltage V. Conservation of energy may be expressed as follows:

$$\Delta W_e + F_x \Delta x = V\Delta Q = 2\Delta W_e \;\; (\text{since } \Delta W_e = \frac{1}{2} V\Delta Q).$$

$$F_x = \frac{\Delta W_e}{\Delta x} \to -\frac{\partial W_e(V,x)}{\partial x} \;\; \text{as } \Delta x \to 0 \qquad (1\text{-}48b)$$

If the dielectric block is hinged at one end and free to rotate within the parallel plates, the mechanical work done is $T_\theta \, \Delta\theta$ in each case, where

θ is an angle between the fixed plates and the rotatable block of dielectric. T_θ is the torque. The corresponding results are

$$T_\theta = -\frac{\partial W_e(Q,\theta)}{\partial \theta} \quad \text{(constant charge)} \tag{1-49a}$$

$$= \frac{\partial W_e(V,\theta)}{\partial \theta} \quad \text{(constant voltage)} \tag{1-49b}$$

Forces and displacement in directions other than x may also be considered. The results may be generalized as follows:

$$F_Q = -\nabla W_e \tag{1-50a}$$

$$F_V = +\nabla W_e \tag{1-50b}$$

where the subscripts Q and V represent constant charge and voltage, respectively.

Example 1-9. Force on a Dielectric Block between Parallel Plates

Consider the configuration of Figure 1-16. The plates are rectangular with plate area equal to L_1L_2. Assuming that L_1, $L_2 \gg d$, we neglect fringing fields. In other words, we assume that the electric field is uniform in the region between the plates and zero outside. The electrostatic energy of the system and the forces are given by

$$W_e = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C} \tag{1-51}$$

$$F_x = \frac{\partial W_e(V,x)}{\partial x} = \frac{1}{2}V^2\frac{dC(x)}{dx} \tag{1-52a}$$

$$= -\frac{\partial W_e(Q,x)}{\partial x} = -\frac{1}{2}Q^2\frac{d}{dx}\left[\frac{1}{C(x)}\right] = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC(x)}{dx} \tag{1-52b}$$

Note that the constant charge and constant voltage cases yield identical expressions, as they must, for F_x. Considering the configuration as a parallel connection of two parallel-plate capacitors and using the result of Eq. (1-28),

$$C(x) = \frac{\varepsilon L_2 x}{d} + \frac{\varepsilon_0 L_2 (L_1 - x)}{d}$$

$$\frac{dC(x)}{dx} = \frac{L_2(\varepsilon - \varepsilon_0)}{d}$$

Then

$$F_x = \frac{V^2 L_2 (\varepsilon - \varepsilon_0)}{2d} \tag{1-53}$$

Since $\varepsilon > \varepsilon_o$, F_x is always positive and the dielectric slab is *pulled in* between the plates. Note that we have assumed a uniform electric field in finding the force. However, it is not this uniform field but a *nonuniform fringing field* that pulls the dielectric into the capacitor. Without knowing the fringing field, which is difficult to calculate, we were able to find the force on the dielectric from the change in the electrostatic energy of the system.

CHAPTER 2

Electric Currents and Conductive Materials

2.1 Introduction

Now we turn to problems which involve moving charges and resultant current flow. We are particularly interested in steady currents such as those which arise when steady voltages are applied across conductive or resistive materials. In Chapter 1, perfect dielectrics with zero conductivity were studied. Now we consider dielectrics with finite conductivity.

First we consider definitions of current. Then we establish some general relationships, including the equation of continuity, Ohm's law, Joule's law, and relaxation time. These relationships may be applied to the general time-varying case, as well as to the special case, of steady currents. Finally, we consider a new boundary condition which is required for steady currents, namely, the continuity of normal volume current density.

2.2 Electric Current

Let's consider charges which are moving within a conductor. Consider any surface S within the conductor. Let Q(t) be the charge which has passed through the surface. Then the current I(t) through the surface is defined as

$$I(t) = \frac{dQ(t)}{dt} \tag{2-1}$$

The unit of current is the ampere [A], or coulomb/second, named after A. M. Ampère. The current is thus a measure of the rate at which charges are crossing the surface S.

2.2.1 Volume Current Density

Consider charges which are moving through a volume. To define the volume current density **J** at a point, construct a small area ds perpendicular to current flow. dI is the current passing through the surface. Then the magnitude of vector **J** may be represented as follows:

$$J = \frac{dI}{ds} \ [A/m^2] \tag{2-2}$$

Vector **J** has magnitude J and a direction identical to that of positive charge movement or current flow. The current through a differential surface ds not necessarily perpendicular to current flow (Figure 2-1) is given by

$$dI = \mathbf{J} \cdot d\mathbf{s}$$

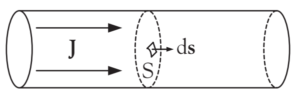


Figure 2-1. ***Volume current flow through an arbitrary surface***

and the total current I passing through surface S is

$$I = \iint_S \mathbf{J} \cdot d\mathbf{s} \tag{2-3}$$

Next consider charges which are moving at a steady drift velocity **v** (Figure 2-2). Construct a rectangular solid of volume (ds v dt) as shown with surface ds perpendicular to current flow. In time dt all charges in the box will pass through the surface ds. Let there be N charges per unit volume, each of charge q. The current passing through the surface ds is then

$$dI = \frac{dQ(t)}{dt} = \frac{Nq(ds\,vdt)}{dt} = Nq\,dsv$$

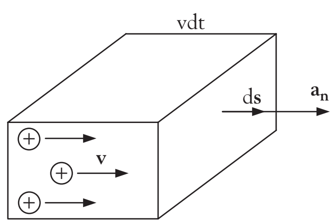


Figure 2-2. ***Drift velocity***

and

$$J = \frac{dI}{ds} = Nq\ v \text{ where } v = |\mathbf{v}|$$

$$\mathbf{J} = N\ q\ \mathbf{v} = \rho_v \mathbf{v}$$

$$\mathbf{J} = \rho_v \mathbf{v} \qquad (2\text{-}4)$$

In the most general case where charges are of different types,

$$\mathbf{J} = \sum_i N_i q_i \mathbf{v}_i = \sum_i \rho_{vi} \mathbf{v}_i$$

where N_i, q_i, $\mathbf{v}_i$, ρ_{vi} are number of charges per unit volume, charge per carrier, velocity and volume charge density, respectively, for each type of charge carrier.

2.2.2 Surface Current Density

Surface current flows in a zero-thickness layer on a surface (Figure 2-3). To define the surface current density $\mathbf{J}_s$ at a point we construct a line $d\ell_\perp$. The subscript $\perp$ denotes that the line is perpendicular to the direction of current flow. Then

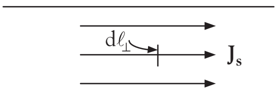


Figure 2-3. ***Surface current density*** $\mathbf{J}_s$

$$J_s = \frac{dI}{dl_\perp} [A/m] \qquad (2\text{-}5)$$

and the vector $\mathbf{J}_s$ has magnitude J_s and a direction identical to that of current flow. Note that surface current density has the unit of amps/m whereas volume current density has the unit of amps/m^2. The total current passing through the strip (Figure 2-3) is given by

$$I = \int J_s d\ell_\perp \qquad (2\text{-}6)$$

2.2.3 Line Current

For line current, that is, current which flows in a filament of zero cross section, we merely count the charges passing through a point to determine the current I there. Then

$$I = \frac{dQ}{dt}$$

I is the current vector, which has magnitude I and whose direction is that of current flow.

2.3 The Equation of Continuity

Current is produced by the motion of charges. Current and charge quantities are thus related. The fundamental relationship may be obtained by considering an arbitrary volume V bounded by surface S (Figure 2-4). Current I(t) is flowing inward through the surface and charge Q(t) is accumulating within.

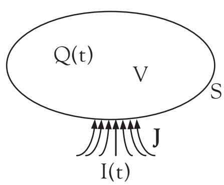


Figure 2-4. ***The equation of continuity***

The total current flowing into the volume V is

$$I(t) = {}'\iint_S \mathbf{J} \cdot d\mathbf{s} = -\iiint_V \nabla \cdot \mathbf{J} \; dv$$

(by the divergence theorem)

$$= \frac{dQ(t)}{dt} = \frac{d}{dt}\iiint_V \rho_v dv = \iiint_V \frac{\partial \rho_v}{\partial t} dv$$

(by Leibnitz's rule for the derivative of an integral)

The above relationship is equivalent to *conservation of charge* within volume V.

$$\iiint_V \nabla \cdot \mathbf{J}\ dv = \iiint_V \frac{\partial \rho_v}{\partial t}\, dv$$

The volume V of integration is arbitrary and thus the integrands must be equal:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \tag{2-7}$$

Equation (2-7) is called the **equation of continuity.** It expresses the fundamental relationship between current and charge densities. It plays an important role for time-varying fields in Chapter 8.

In the steady-state (charge and current do not vary in time), Eq. (2-7) reduces to

$$\nabla \cdot \mathbf{J} = 0 \qquad \text{for steady currents (2-8)}$$

2.4 Ohm's Law and Conductive Materials

The electromagnetic form of **Ohm's law** gives the relationship between volume current density **J** and the electric field **E** in a conducting material:

$$\mathbf{J} = \sigma \mathbf{E} \tag{2-9}$$

where σ is the **conductivity** of the material. The conductivity of various materials is given in Table 2-2. The table includes both good conductors and good insulators. Silver, copper, and gold are the three best conductors. Silver is best but it is expensive and tarnishes readily which reduces the conductivity. Often materials are silver-plated to improve their conductivity. Copper is much less expensive than silver or gold and so is the conductor of choice where large quantities are needed. Gold is often used in microelectronics. It is the most corrosion-free of the three. Finally, we note that rubber and fused quartz are two of the best insulators.

Table 2-1 Conductivities

Material	Conductivity (S/m)
Silver	6.2×10^{7}
Copper	5.8×10^{7}
Gold	4.1×10^{7}
Aluminum	3.5×10^{7}
Brass	2.6×10^{7}
Iron	10^{7}
Sea Water	4
Silicon	4.4×10^{-4}
Distilled Water	10^{-4}
Dry Earth	10^{-5}
Glass	10^{-12}
Rubber	10^{-15}
Fused Quartz	10^{-17}

To see that Eq. (2-9) is equivalent to the usual form of Ohm's law in circuit theory, consider a rectangular conductor with uniform **J, E, σ** throughout (Figure 2-5). Multiply both sides of Eq. (2-9) by S, the cross sectional area, and equate components to obtain

$$JS = \sigma ES = \frac{\sigma S}{\ell}(E\ell)$$

or

$$I = \frac{\sigma S}{l} V \text{ (since } I = JS,\ V = El).$$

or

$$V = \frac{l}{\sigma S} I$$

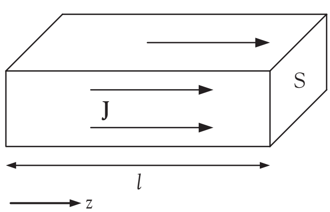


Figure 2-5. ***Uniform current flow in a resistor***

We have, then, a linear relationship between V and I. The term $\frac{\ell}{\sigma S}$ is called the **resistance** R. The familiar circuit relation

$$V = IR \tag{2-10}$$

is thus obtained. The resistance,

$$R = \frac{V}{I} = \frac{1}{\sigma S} \tag{2-11}$$

applies to any homogeneous resistor of length ℓ and uniform cross section S. $\left(\frac{1}{\surd}\right)$, the inverse of conductivity, is called *resistivity.*

"Why should **J** be proportional to **E**? Why should **J** not increase indefinitely as electrons are accelerated by the electric field **E**?" To answer this question, we need to understand the processes of current flow in a conductor, which can be explained as follows. In the absence of an applied electric field, at room temperature, the conduction electrons are moving at about 10^6 m/s through the "lattice" structure of the material. The movement is in random directions so that the average vector velocity is zero and there is no net current (**J** = 0). We then apply a voltage across the conductor and an electric field **E** is established within the conductor. The conduction electrons are accelerated by the force q**E** = – e**E** and produce a current **J** in the direction of **E.** However, the electrons are accelerated for only a very short time about (about 10^{-14} s). They travel only a very short distance (about 10^{-8}m) before colliding with the lattice structure and being scattered in arbitrary directions. Before collision, the electron motion contributes to current **J** in the direction of **E.** Immediately after collision, the direction is arbitrary and so the electron motion does not, on average, contribute to **J** in the direction of **E.** The contribution to current is "lost", so to speak. Then the process starts all over again. The electron is accelerated, contributes to **J** between collisions and loses its contribution upon collision. There is however an average *"drift"* velocity contributing to **J.** This is merely the average velocity acquired during the brief acceleration period between collisions. The drift velocity is very slow, on the order of 10^{-4} m/s.

Thus we have electrons which are moving very rapidly (10^6 m/s) in random directions. Superimposed on this very rapid motion is a very slow drift velocity (10^{-4} m/s) which contributes to **J** in the direction of **E.** If we double the voltage, the time between collisions (the acceleration period) is unchanged since it is determined primarily by the much larger random velocity. The acceleration time is constant, the acceleration force $q\mathbf{E}$ is doubled, and so the average "drift" velocity is doubled. Thus we have a linear system and $\mathbf{J}$ is proportional to $\mathbf{E}$ as in Eq. (2-9). The constant σ is called conductivity.

It should be pointed out that a conductor is usually electrically neutral. The moving negative charges travel through a stationary lattice structure which is oppositely charged. For linear homogeneous media characterized by a constant conductivity σ, the conductor or resistor is uncharged (ρ_v = 0) within the volume. This is analogous to the case for the linear homogeneous dielectric (ρ_{pv} = 0). Surface charge will in general accumulate at interfaces. Volume charge density may exist ($\rho_v \neq 0$) for inhomogeneous conductors.

2.5 Georg Simon Ohm (1789-1854) and the Discovery of Ohm's Law

1827 should have been a very good year for Georg Simon Ohm. He had just published his comprehensive book on resistive circuits. He had investigated the subject very thoroughly and had confirmed his theory with very carefully performed experiments. He was now almost 40 years old. Surely he would receive the recognition which he deserved and with it, perhaps, the university appointment which he had long sought.

In his book, *The Galvanic Circuit Mathematically Investigated,* Ohm had explored the subject as completely as was possible at the time. He had shown that in a simple loop circuit with a battery of voltage V and uniform conductor size, the current in the wire is uniform throughout the length. Thus the circuit can be characterized with a single current, I. He also demonstrated that the potential falls uniformly with distance along the wire. Both of these essential facts, which are taken for granted by modern students, were unknown at the time.

Then, he had established the linear relationship between voltage and current

$$V = IR \tag{2-10}$$

where the resistance R of the circuit is proportional to wire length ℓ and inversely proportional to wire cross section S:

$$R = \frac{kl}{S} = \frac{l}{\sqrt{S}} \tag{2-11}$$

where k is a constant, the resistivity, the inverse of conductivity, which depends on the material of the wire. Ohm also determined the relative conductivity of various metals, the best conductors being silver, copper, gold, in that order, as shown in Table 2-2. Ohm's experiments were very sophisticated and reliable. He used the Seebeck thermoelectric battery, invented in 1821, because of its predictable internal resistance. To measure the current and its magnetic field, he used the torsion balance compass needle galvanometer invented by Coulomb in his measurements of the inverse square law for magnetic poles.

There is a direct analogy between resistive circuits and heat flow. The quantity of heat flowing in a wire is proportional to the temperature difference between the ends of the wire and inversely proportional to thermal resistance, which, in turn, is proportional to the length of the wire and inversely proportional to its cross-sectional area. Ohm had studied Fourier's book, *The Analytical Study of Heat,* and had, with great insight, seen the analogy to electrical current flow. Voltage, current, and electrical resistance are analogous to temperature difference, heat flow, and thermal resistance, respectively.

Note that our familiar rules for combinations of resistances in series and parallel follow directly from Ohm's law. For instance, placing two identical sections of wire in series doubles the resistance since total length ℓ is doubled; placing them in parallel halves the resistance since cross-section area S is doubled.

Georg Ohm was born in Erlangen, Germany, in 1789, the son of a locksmith. He and his younger brother Martin were well-educated and Georg received a Ph.D. from the University of Erlangen in 1812. He then held several teaching positions in secondary schools, finally settling in Cologne, where he taught for about nine years. It was in Cologne that he carried out most of his basic research. It was a long and difficult process. He had to educate himself in the mathematics and science of the times and become an expert in the current experimental techniques of electrical science, all the while carrying the heavy teaching load of a secondary school. We are told that he was a very serious, effective and innovative teacher.

The early 19th century was a time of great advances in electrical science. In 1800 Alessandro Volta invented the voltaic cell or battery, which supplied a steady source of current. This was much preferable to the Leyden jar or capacitor as a source. The voltaic cell served as a great stimulus to research. Oersted's dramatic discovery in 1820 of the magnetic field of an electric current is described in Volume 4. Within a year, this discovery led to a method of measuring current by determining its magnetic field. This was done by noting the deflection of a compass needle. In this way, Volta and Oersted provided a steady current and the means to measure it, making it possible for Ohm to carry out his experiments.

Unfortunately, Ohm's book was not well received in Germany. In fact, his theory was widely rejected, condemned as too complicated by some and too simple by others. In general, reaction to his work was very negative and the backward educational establishment criticized him in response. Ohm felt that he must resign his post in Cologne and went to teach at a military school in Berlin.

However, scientists in other countries slowly began to recognize his work and to give it the respect which it deserved. He received the Copley medal in 1841 and finally, in 1849, twenty-two years after the publication of his book, he was appointed Professor of Physics at the University of Munich, where he continued to teach, honored and revered, until his death at 65 in 1854.

Henry Cavendish and other researchers had preceded Ohm in his discovery of Ohm's law but the thoroughness and completeness of Ohm's work guaranteed that he would receive primary credit for the discovery.

2.6 Power – Joule's Law

The power dissipated in a conductive or resistive material may be calculated from the electric field and the current density as follows:

$$P = \iiint_V \mathbf{E} \cdot \mathbf{J} \, dv \ [\mathrm{W}] \tag{2-12}$$

where V is the volume of the resistor. $\mathbf{E} \cdot \mathbf{J}$ is thus a volume density of power dissipated with units of watts/m^3. Equation (2-12) is known as **Joule's law.**

To see the correspondence with circuit theory, consider the resistor of Figure 2-5 with uniform **E, J,** σ. Making use of Eq. (2-9), we have

$$P = \iiint_V \sigma E^2 \ dv = \iiint_V \frac{1}{\sigma} J^2 \ dv = \sigma E^2 (Sl) = \frac{J^2}{\sigma}(Sl)$$
$$= (El)^2 \frac{S\sigma}{l} = \frac{V^2}{R}$$
$$= (JS)^2 \frac{l}{S\sigma} = I^2 R$$
$$= \int E \ dl \iint J \ ds = (El)(JS) = VI$$

2.7 Relaxation Time

In Volume 1, we noted that charges placed within a conductor move almost immediately by mutual repulsion toward the surface of the conductor. In other words, volume charge density disappears very rapidly giving rise to surface charge density. Now we are in a position to examine the phenomenon quantitatively and ask the question *"how fast?"*. Consider a linear homogeneous medium of conductivity σ, permittivity ε, volume V and surface S (Figure 2-6).

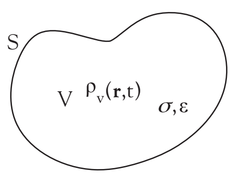


Figure 2-6. ***Relaxation time***

We start with the equation of continuity:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} = \nabla \cdot (\sigma \mathbf{E}) = \nabla \cdot \left(\frac{\sigma}{\varepsilon} \mathbf{D} \right)$$
$$= \frac{\sigma}{\varepsilon} \nabla \cdot \mathbf{D} = \frac{\sigma}{\varepsilon} \rho_v \quad \text{[Eq.(3-10b) is used.]}$$

$$\frac{\partial \rho_v}{\partial t} = \left(-\frac{\sigma}{\varepsilon} \right) \rho_v$$

The solution to the differential equation above is

$$\rho_v(x, y, z, t) = \rho_{vo}(x, y, z, t)\, e^{-\frac{\sigma}{\varepsilon}t} \tag{2-13}$$

or

$$\rho_v(x, y, z, t) = \rho_{vo}(x, y, z, t)\, e^{-t/\tau}$$

where $\tau = \frac{\varepsilon}{\sigma}$ is called the *relaxation time.* We may interpret Eq. (2-13) in the following way.

Suppose we start with an initial volume charge density ρ_{vo} (x,y,z) within volume V at time t = 0. Then the volume charge density decays exponentially at every point inside the volume. The net charge will eventually reside on the surface S since charge is conserved. So the surface charge density is increasing as the volume charge density decreases. τ is the time required for the volume charge density to decay to 1/e or about 37% of its value at any point in V. Table 2-2 shows relaxation times for some conductors and insulators. Relaxation time is so short, on the order of 10^{-19} seconds, for good conductors, that we may consider the process to be practically instantaneous for good conductors.

Table 2-2. Relaxation Times

Material	**Relaxation Time**
Copper	2.5×10^{-19} (s)
Sea Water	2×10^{-10} (s)
Distilled Water	10^{-6} (s)
Fused Quartz	10^{5} (s) or about 11 days

2.8 Boundary Conditions for Steady Currents

In the presence of steady currents it is necessary to add one boundary condition, namely, the continuity of normal **J,** to the basic boundary conditions already derived in Chapter 1. Figure 2-7 shows a boundary between two arbitrary media (1), (2) with characteristics (σ_1, ε_1) and (σ_2, ε_2), respectively.

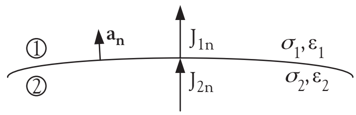


Figure 2-7. *Continuity of normal* J

Consider the normal components J_{1n}, J_{2n} of volume current density at the interface. J_{2n} specifies the charge flowing into the interface and J_{1n} determines charge flowing out from the interface. J_{1n}, J_{2n} must be equal or else charge would accumulate indefinitely at the interface. Therefore

$$J_{1n} = J_{2n} \text{ for steady currents} \tag{2-14}$$

In summary, the three basic boundary conditions at an interface are:

$E_{1t} = E_{2t}$ Tangential **E** is continuous

$D_{1n} - D_{2n} = \rho_s$ Normal **D** is discontinuous by the amount of free surface charge

$J_{1n} = J_{2n}$ Normal **J** is continuous

Example 2-2. Dissimilar Dielectrics in a Parallel Plate Capacitor

Figure 2-8 shows parallel plates of area A and separation $d_1 + d_2$. The parallel plate region is filled with two different materials of permittivity, conductivity, ε_1, σ_1, and ε_2, σ_2, respectively. The switch is closed and the steady state is eventually reached. Find the current, the total resistance, and the free surface charge at the interface between dissimilar materials.

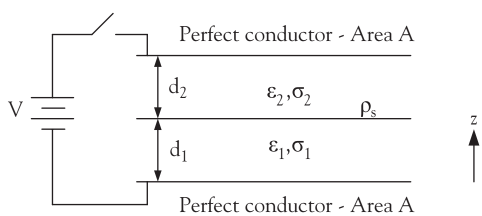


Figure 2-8. *Dissimilar dielectrics in a parallel-plate capacitor*

Solution:

In such problems, we concentrate on the current. Because of the continuity of normal **J** in the steady state, we assume a single volume current density for both regions:

$$\mathbf{J} = -\mathbf{a}_z J_o$$

We find the voltage by applying Ohm's law [Eq. (5-9)]:

$$V = -\int \mathbf{E} \cdot d\ell = E_1 d_1 + E_2 d_2 = \frac{J_o d_1}{\sigma_1} + \frac{J_o d_2}{\sigma_2} = J_o \left(\frac{d_1}{\sigma_1} + \frac{d_2}{\sigma_2} \right)$$

and

$$J_o = \frac{V}{\left(\frac{d_1}{\sigma_1} + \frac{d_2}{\sigma_2} \right)}$$

then

$$I = \iint \mathbf{J} \cdot d\mathbf{s} = J_o A = \frac{VA}{\left(\frac{d_1}{\sigma_1} + \frac{d_2}{\sigma_2} \right)}$$

and

$$R = \frac{V}{I} = \frac{d_1}{\sigma_1 A} + \frac{d_2}{\sigma_2 A} \tag{2-15}$$

Note that the potential at the interface ($z = d_1$) is

$$E_1 d_1 = J_o \frac{d_1}{\sigma_1} \text{ or } V = \frac{\frac{d_1}{\sigma_1}}{\frac{d_1}{\sigma_1} + \frac{d_2}{\sigma_2}}$$

with respect to the potential at $z = 0$. It is determined solely by the conductivities σ_1, σ_2.

To find the surface charge density at the interface we use the boundary condition for the normal component of **D,**

$$\mathbf{D}_1 = -\mathbf{a}_z \frac{J_o}{\sigma_1} \varepsilon_1, \ \mathbf{D}_1 = -\mathbf{a}_z \frac{J_o}{\sigma_2} \varepsilon_2$$

$$\rho_S = D_{2z} - D_{1z} = J_o \left(\frac{\varepsilon_1}{\sigma_1} - \frac{\varepsilon_2}{\sigma_2} \right)$$

$$\rho_S \Rightarrow 0 \text{ if } \frac{\varepsilon_1}{\sigma_1} = \frac{\varepsilon_2}{\sigma_2} \tag{2-16}$$

Where did the charge ρ_s come from? Normal **J** is continuous in the steady state and leads to no charge accumulation. We may consider that the charge ρ_s is deposited on the interface during the transient period when normal **J** is not necessarily continuous. A net charge is deposited only if relaxation times are unequal.

2.9 A Relationship Between Capacitance and Resistance

Figure 2-9 shows two conductors in an infinite homogeneous medium of permittivity ε or conductivity σ or both.

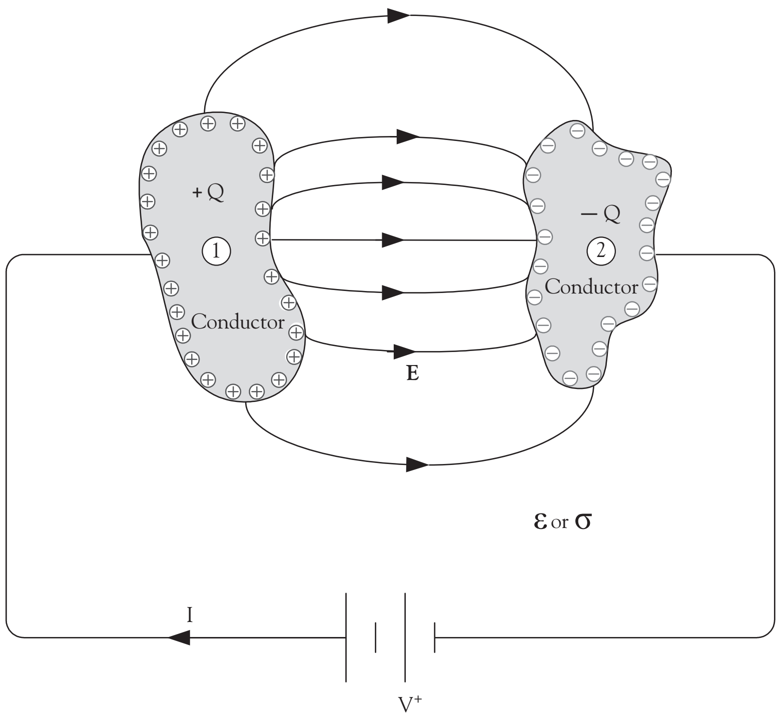


Figure 2-9. ***A relationship between capacitance and resistance***

Let's consider first the ideal dielectric with permittivity ε. A voltage V is applied between (perfect) conductors, an electric field **E** is established in the dielectric and charges Q, –Q reside on the conductors. The capacitance is

$$C = \frac{Q}{V} = \frac{\iint_S \mathbf{D} \cdot d\mathbf{s}}{V} = \frac{\varepsilon \iint_S \mathbf{E} \cdot d\mathbf{s}}{V} \tag{2-17}$$

Next consider the resistor with conductivity σ. Voltage V is applied and a steady current I flows in the circuit and through the resistor ($\mathbf{J} = \sigma\,\mathbf{E}$ in the resistor). The resistance is

$$R = \frac{1}{G} = \frac{V}{I} = \frac{V}{\sigma \iint_S \mathbf{E} \cdot d\mathbf{s}} \tag{2-18}$$

since

$$I = \iint_S \mathbf{J} \cdot d\mathbf{s} = \sigma \iint_S \mathbf{E} \cdot d\mathbf{s}$$

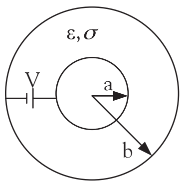
G, the inverse of resistance, is called the **conductance.** We assume that the voltage V is identical in the two cases. The electric field **E** then is also identical in the two cases because of uniqueness and, multiplying Eqs. (2-17) and (2-18), we obtain

$$RC = \frac{C}{G} = \frac{\varepsilon}{\sigma} \tag{2-19}$$

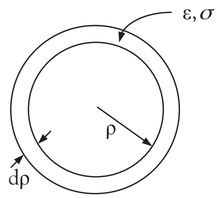
What happens as we change from a perfect to an imperfect dielectric in Figure 5-9? The only thing that changes is that current flows and increases with increasing conductivity σ. V, **E,** Q, C remain unchanged as σ increases. Only **J** and **R** vary. Thus the resistance and capacitance problems are decoupled.

Example 2-2. Shunt Resistance and Capacitance of a Coaxial Line

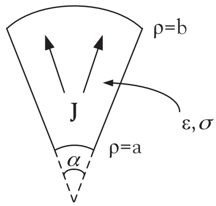
Figure 2-10(a) shows the cross section of a coaxial line with inner, outer radii a, b, respectively and length ℓ. The coaxial line is completely filled with a material of permittivity ε and conductivity σ. A voltage V is applied between inner and outer conductors and a current **I** then flows radially outward from inner to outer conductor. Find the shunt resistance and capacitance.



(a) A coaxial line.



(b) A thin cylindrical shell.



(c) A truncated wedge.

Figure 2-10. ***A coaxial line with inner, outer radii a, b, respectively and length ℓ***

Solution:

The volume current density is independent of φ and varies as $\left(\frac{1}{\rho}\right)$ so that the total current is independent of ρ:

$$\mathbf{J} = \mathbf{a}_\rho \frac{I}{2\pi\rho\ell} \tag{2-20}$$

An integration over a cylindrical surface of radius ρ and length ℓ shows that the total current is I:

$$\iint \mathbf{J} \cdot d\mathbf{s} = \int_0^1 \int_0^2 \frac{I}{2\pi\rho l} \rho \, d\phi dz = I$$

Then

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \mathbf{a}_\rho \frac{I}{2\pi\rho\sigma l}$$

$$V = -\int \mathbf{E} \cdot d\ell = -\int_b^a \frac{I}{2\pi\sigma l\rho} d\rho = \frac{I}{2\pi\sigma l} \ln\left(\frac{b}{a}\right)$$

$$I = \frac{2\pi\sigma l \, V}{\ln(b/a)}; \;\; \mathbf{E} = \mathbf{a}_\rho \frac{V}{\ln(b/a)}\left(\frac{1}{\rho}\right)$$

and

$$R = \frac{V}{I} = I = \frac{\ln(b/a)}{2\pi\sigma l} \tag{2-21}$$

$$G = \frac{2\pi\sigma\ell}{\ln(b/a)};\ \frac{G}{\ell} = \frac{2\pi\sigma}{\ln(b/a)} \tag{2-22}$$

The charge Q in length ℓ (on the inner conductor) is determined as follows:

$$Q = (\text{area})\rho_S = (\text{area})D_\rho = (2\pi\sigma l)\varepsilon\frac{V}{\ln(b/a)}\frac{l}{a} = \frac{2\pi\varepsilon l}{\ln(b/a)}V$$

Then

$$C = \frac{Q}{V} = \frac{2\pi\sigma\ell}{\ln(b/a)};\ \frac{C}{\ell} = \frac{2\pi\varepsilon}{\ln(b/a)} \tag{2-23}$$

Note that R in Eq. (2-21) and C in Eq. (2-23) satisy $RC = \frac{C}{G} = \frac{\varepsilon}{\sigma}$ as in Eq. (2-19).

The resistance R may also be determined by considering a thin cylindrical shell of radius ρ, length ℓ (Figure 2-10(b)).

$$dR = \frac{dl}{\sigma A} = \frac{d\rho}{2\pi\rho\sigma l}$$

$$R = \int_a^b dR = \int_a^b \frac{d\rho}{2\pi\sigma l\rho} = \frac{\ln(b/a)}{2\pi\sigma l}$$

which agrees with Eq. (2-21).

The conductance G may be determined by considering a truncated wedge (Figure 2-10(c)) with radial current flow:

$$\mathbf{J} = \mathbf{a}_\rho\frac{I}{\alpha\rho\ell};\iint \mathbf{J}\cdot d\mathbf{s} = I$$

$$\mathbf{E} = \mathbf{a}_\rho\frac{I}{\alpha\rho l\sigma};\ V = -\int \mathbf{E}\cdot d\ell = \frac{I\ln(b/a)}{\alpha l\sigma} \tag{2-24}$$

$$R = \frac{V}{I} = \frac{I\ln(b/a)}{\alpha l\sigma};\ G = \frac{\alpha l\sigma}{I\ln(b/a)}\ \text{(wedge)}$$

For a thin wedge of angle dφ,

$$dG = \frac{d\phi \ell \sigma}{\ln(b/a)}$$

and for the whole coaxial line,

$$G = \int dG = \int_0^{2\pi} \frac{l\sigma d\phi}{\ln(b/a)} = \frac{2\pi\sigma l}{\ln(b/a)}$$

which agrees with Eq. (2-22).

The power losses in the coaxial line are determined as follows:

$$\begin{aligned} P &= \iiint \mathbf{E} \cdot \mathbf{J}\, dv = \iiint \sigma E_\rho^2 dv \\ &= \int_0^{l}\int_0^{2\pi}\int_0^{b} \frac{\sigma V^2 \rho d\rho d\phi dz}{\ln^2(b/a)\rho^2} = \frac{V^2(2\pi l\sigma)}{\ln(b/a)} \\ &= \frac{V^2}{R} = I^2 R \end{aligned}$$

Example 2-3. A Truncated Conical Resistor

Figure 2-11 shows a truncated conical resistor bounded by surfaces r = a, r = b, and θ = 15°. The material of the resistor is inhomogeneous:

$$\sigma = \sigma_o \frac{r}{a}$$

A voltage V is applied between surfaces r = a, b. Find the electric field and the resistance between the surfaces r = a, b.

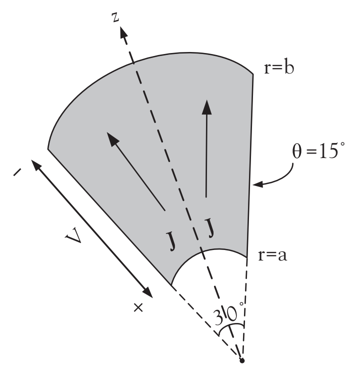


Figure 2-11. ***A truncated conical resistor***

Solution:

First we assume a radial, inverse square current density which makes the total current independent of r:

$$\mathbf{J} = \mathbf{a}_r \frac{C_1}{r^2}$$

$$I = \iint \mathbf{J} \cdot d\mathbf{s} = \int_0^{2} \int_0^{/12} \frac{C_1 r^2 \sin\theta \, d\theta d\phi}{r^2}$$

$$= 2\pi C_1 (1 - \cos 15^\circ)$$

$$\mathbf{E} = \frac{J}{\sigma} = \mathbf{a}_r \frac{C_1}{r^2} \frac{1}{\sigma_o \frac{r}{a}} = \mathbf{a}_r \frac{C_1 a}{\sigma_o r^3}$$

$$V = -\int \mathbf{E} \cdot d\ell = -\int_b^a E_r \, dr = \frac{C_1 a}{2\sigma_o}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$$

and

$$C_1 = 2\sigma_o \left(\frac{b^2 a}{b^2 - a^2}\right) V; \ \mathbf{E} = \mathbf{a}_r \frac{2b^2 a^2}{b^2 - a^2}\left(\frac{1}{r^3}\right) V$$

$$R = \frac{V}{I} = \frac{b^2 - a^2}{4\pi\sigma_o a b^2 (1 - \cos 15^\circ)} \qquad (2\text{-}25)$$

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Principles of Electromagnetics 2—Dielectric and Conductive Materials

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