Principles of Electromagnetics 6—Radiation and Antennas

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Preface

Electromagnetics is not an easy subject for students. The subject presents a number of challenges, such as: new math, new physics, new geometry, new insights and difficult problems. As a result, every aspect needs to be presented to students carefully, with thorough mathematics and strong physical insights and even alternative ways of viewing and formulating the subject. The theoretician James Clerk Maxwell and the experimentalist Michael Faraday, both shown on the cover, had high respect for physical insights.

This book is written primarily as a text for an undergraduate course in electromagnetics, taken by junior and senior engineering and physics students. The book can also serve as a text for beginning graduate courses by including advanced subjects and problems. The book has been thoroughly class-tested for many years for a two-semester Electromagnetics course at Syracuse University for electrical engineering and physics students. It could also be used for a one-semester course, covering up through Chapter 8 and perhaps skipping Chapter 4 and some other parts. For a one-semester course with more emphasis on waves, the instructor could briefly cover basic materials from statics (mainly Chapters 2 and 6) and then cover Chapters 8 through 12.

The authors have attempted to explain the difficult concepts of electromagnetic theory in a way that students can readily understand and follow, without omitting the important details critical to a solid understanding of a subject. We have included a large number of examples, summary tables, alternative formulations, whenever possible, and homework problems. The examples explain the basic approach, leading the students step by step, slowly at first, to the conclusion. Then special cases and limiting cases are examined to draw out analogies, physical insights and their interpretation. Finally, a very extensive set of problems enables the instructor to teach the course for several years without repeating problem assignments. Answers to selected problems at the end allow students to check if their answers are correct.

During our years of teaching electromagnetics, we became interested in its historical aspects and found it useful and instructive to introduce stories of the basic discoveries into the classroom. We have included short biographical sketches of some of the leading figures of electromagnetics, including Josiah Willard Gibbs, Charles Augustin Coulomb, Benjamin Franklin, Pierre Simon de Laplace, Georg Simon Ohm, Andre Marie Ampère, Joseph Henry, Michael Faraday, and James Clerk Maxwell.

The text incorporates some unique features that include:

- Coordinate transformations in 2D (Figures 1-11, 1-12).
- Summary tables, such as Table 2-1, 4-1, 6-1, 10-1.
- Repeated use of equivalent forms with R (conceptual) and |r−r′| (mathematical) for the distance between the source point and the field point as in Eqs. (2-27), (2-46), (6-18), (6-19), (12-21).
- Intuitive derivation of equivalent bound charges from polarization sources, including piecewise approximation to non-uniform polarization (Section 3.3).
- Self-field (Section 3.8).
- Concept of the equivalent problem in the method of images (Section 4.3).
- Intuitive derivation of equivalent bound currents from magnetization sources, including piecewise approximation to non-uniform magnetization (Section 7.3).
- Thorough treatment of Faraday's law and experiments (Sections 8.3, 8.4).
- Uniform plane waves propagating in arbitrary direction (Section 9.4.1).
- Treatment of total internal reflection (Section 10.4).
- Transmission line equations from field theory (Section 11.7.2).
- Presentation of the retarded potential formulation in Chapter 12.
- Interpretation of the Hertzian dipole fields (Section 12.3).

Finally, we would like to acknowledge all those who contributed to the textbook. First of all, we would like to thank all of the undergraduate and graduate students, too numerous to mention, whose comments and suggestions have proven invaluable. As well, one million thanks go to Ms. Brenda Flowers for typing the entire manuscript and making corrections numerous times. We also wish to express our gratitude to Dr. Eunseok Park, Professor Tae Hoon Yoo, Dr. Gokhan Aydin, and Mr. Walid M. G. Dyab for drawing figures and plotting curves, and to Professor Mahmoud El Sabbagh for reviewing the manuscript. Thanks go to the University of Poitiers, France and Seoul National University, Korea where an office and academic facilities were provided to Professor Adams and Professor Lee, respectively, during their sabbatical years. Thanks especially to Syracuse University where we taught for a total of over 50 years. Comments and suggestions from readers would be most welcome.

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CHAPTER 1

Introduction to Radiation and Antennas

1.1 Introduction

We have so far studied propagation, reflection, transmission, guidance, and resonance of the electromagnetic plane waves in the source-free region, that is, at points away from the source, without knowing how the waves were generated by the sources. In this chapter, we consider how the time-varying current and charge sources generate the waves and radiate the electromagnetic energy. It is to be noted that a charge at rest does not generate electromagnetic waves, nor does a steady (dc) electric current. It takes accelerating charges or time-varying currents to produce electromagnetic waves. An antenna is a device or structure that is designed to launch or radiate electromagnetic waves efficiently into space and to focus or direct these waves in a certain direction. The antenna is also used to receive an electromagnetic signal.

We first present the potential formulation where the electric and magnetic fields are expressed in terms of the scalar and vector potentials. The retarded potentials are found for both arbitrary and time-harmonic charge and current sources. Then we consider the elementary (Hertzian) dipole antenna and the linear antenna with sinusoidal current distribution. Finally we study a linear array of identical, parallel antennas with different phases and amplitudes. Several special cases are considered.

1.2 Potential Formulation – Method of Solutions for Radiation Problems

The antenna (or radiation) problem consists of solving for the fields that are generated by an impressed current distribution – **J,** $\mathbf{J}_s$ or **I,** depending on the type of the antenna. In many practical problems the current distribution is not known in advance, but it is obtained during the solution process. How to obtain the current distribution is beyond the scope of this book. For the moment, assuming that the current distribution **J** is known, we wish to determine the electric field **E** and the magnetic field **H.** The typical approach is the use of the scalar electric potential V and the (magnetic) vector potential **A,** known as the *potential formulation.* Since we deal with time-varying fields, we expect that some of the results we obtained in the static problems will not be valid here.

We again start with Maxwell's equations that are re-written here.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1-20a}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{1-20b}$$

$$\nabla \cdot \mathbf{D} = \rho_v \tag{1-20c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1-20d}$$

First, since the magnetic field **B** is divergenceless from Eq. (8-20d), it can be expressed as the curl of another vector function **A,** as this was done in magnetostatics (Section 6.5).

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (1\text{-}1)$$

A is called the **magnetic vector potential** or simply **vector potential.** The magnetic field **H** is obtained as $\mathbf{H} = \frac{1}{\mu}\mathbf{B} = \frac{1}{\mu}\nabla \times \mathbf{A}$. Substituting Eq. (1-1) into Faraday's law, we obtain

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A})$$

or

$$\nabla \times \left\{ \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right\} = 0 \tag{1-2}$$

Since the sum of two quantities inside the bracket is curl-free, it can be expressed as the gradient of a scalar function V (see Section 1.5), as this was done in electrostatics

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V \tag{1-3}$$

from which we obtain

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{1-4}$$

V is called the **scalar electric potential** or simply **scalar potential.** Note that in the static case, $\frac{\partial \mathbf{A}}{\partial t} = 0$, thus Eq. (1-4) reduces to $\mathbf{E} = -\nabla V$, Eq. (2-41). For time-varying fields, the electric field depends on both V and **A.** Now the **E** and **H** fields are expressed in terms of the potential functions, V and **A.** If we find V and **A,** we can obtain easily the fields.

In order to find the solutions for the potentials, we derive the partial differential equations that will be satisfied by V and **A** as follows. Let's substitute Eqs. (1-1) and (1-4) into Eq. (8-20b) and make use of the constitutive relations:

$$\mathbf{D} = \varepsilon \mathbf{E}, \ \mathbf{B} = \mu \mathbf{H} \tag{1-21}$$

We have

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) = \mathbf{J} + \varepsilon \frac{\partial}{\partial t} \left\{ -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \right\} \tag{1-5}$$

Using the vector identity, $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, we obtain

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} - \nabla \left(\mu\varepsilon \frac{\partial V}{\partial t} \right) - \mu\varepsilon - \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

or

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \nabla \left\{ \nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial V}{\partial t} \right\} \tag{1-6}$$

Now, according to Helmholtz's theorem discussed in Volume 1, a vector field is completely determined if its divergence and curl are specified everywhere. For the vector potential **A,** its curl is given as **B** in Eq. (1-1) but its divergence is not specified. Thus we are free to choose $\nabla \cdot \mathbf{A}$ for convenience. We let

$$\nabla \cdot \mathbf{A} = -\mu\varepsilon \frac{\partial V}{\partial t} \tag{1-7}$$

which makes the second term in Eq. (1-6) disappear and reduces Eq. (1-6) to a simple partial differential equation (PDE). The relation between **A** and V in Eq. (1-7) is called the **Lorentz condition** or **Lorentz gauge** for potentials. For static fields, it reduces to $\nabla \cdot \mathbf{A} = 0$, which is known as the Coulomb gauge, for magnetostatic fields, as discussed in Section 6.5. Applying Eq. (1-7), we have the following PDE for **A**:

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} \tag{1-8}$$

Eq. (1-8) is called the **inhomogeneous wave equation for the vector potential A** in the presence of the current source **J.** In the source-free region, Eq. (1-8) reduces to the wave equation we have seen earlier for **E** and **H** in Eqs. (9-4) and (9-5). Similarly, we can also derive the PDE for the scalar potential V, by substituting Eq. (1-4) into Eq. (8-20c) and making use of Eqs. (8-21) and (1-7), as follows.

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon} \tag{1-9}$$

which is called the **inhomogeneous wave equation for the scalar potential** V in the presence of the charge source ρ_v. The Lorentz condition made it possible to decouple the wave equations for **A** and V. We find striking similarities between the two wave equations. They are identical when the following substitutions are made:

$$\mathbf{A} \rightarrow V, \ \mu \mathbf{J} \rightarrow \frac{\rho_v}{\varepsilon}$$

Thus we expect that the solutions for **A** and V, given **J** and ρ_v, respectively, will take similar forms. However, we don't need to find both **A** and V in order to determine the **E** and **H** fields because V is related to **A** through the Lorentz condition, Eq. (1-7). This is also expected through the relationship between the current and charge densities (**J** and ρ_v), given by the equation of continuity:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \qquad (5\text{-}7)$$

Therefore, for radiation problems, we try to determine only the vector potential **A** by solving Eq. (1-8) and calculate the **E** and **H** fields from **A** using Eqs. (1-1) and (1-4) along with Eq. (1-7).

1.2.1 Retarded Potentials

Now our task is to find the solutions of Eqs. (1-8) and (1-9) for **A**(**r,** t) and V(**r,** t), given the sources **J**(**r,** t) and ρ_v(**r,** t). The solutions to these equations for time-varying sources can be obtained by making use of the potential solutions for electrostatic and magnetostatic problems. In the static limit where $\frac{\partial}{\partial t} = 0$, Eqs. (1-8) and (1-9) reduce to the vector and scalar Poisson's equations, (6-17) and (2-3), which are repeated here:

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \qquad (6\text{-}17) \text{ or } (1\text{-}10a)$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \qquad (2\text{-}3) \text{ or } (1\text{-}10b)$$

The solutions to Poisson's equations for **A** and V were obtained in Sections 6.5 and 2.8 as follows:

$$\mathbf{A}(\mathbf{r}) = \iiint \frac{\mu \mathbf{J}(\mathbf{r}')dv'}{4\pi R} \qquad (6\text{-}18a) \text{ or } (1\text{-}11a)$$

$$V(\mathbf{r}) = \iiint \frac{\rho_v(\mathbf{r}')dv'}{4\pi\varepsilon R} \qquad (2\text{-}46a) \text{ or } (1\text{-}11b)$$

where $R = |\mathbf{r} - \mathbf{r}'|$ is the distance from the source point $\mathbf{r}'$ to the field point **r.** The integrals in Eqs. (1-11) show how the static sources, **J** and ρ_v at the source point $\mathbf{r}'$, generate the potentials, **A** and V at the field point **r.** Knowing these relationships, we expect that the time-varying sources, **J** and ρ_v at the source point $\mathbf{r}'$, would generate the time-varying potentials,

A(**r,** t) and V(**r,** t) at the field point **r,** in a similar fashion except for some time delay because it takes time for temporal variation of the sources to reach the field point. Using these arguments, without mathematical proof we present the solutions to Eqs. (1-8) and (1-9) as follows.

$$\mathbf{A}(\mathbf{r},t) = \iiint \frac{\mu \mathbf{J}(\mathbf{r}',t')}{4\pi R} dv' \tag{1-12}$$

$$V(\mathbf{r},t) = \iiint \frac{\rho_v(\mathbf{r}',t')}{4\pi\varepsilon R} dv' \tag{1-13}$$

where

$$\begin{aligned} R &= |\mathbf{r} - \mathbf{r}'| \\ t' &= t - \frac{R}{v} = t - \sqrt{\mu\varepsilon}\,|\mathbf{r} - \mathbf{r}'| \end{aligned} \tag{1-14}$$

A(**r,** t) and V(**r,** t) given in Eqs. (1-12) and (1-13) are called the **retarded vector potential** and the **retarded scalar potential,** respectively. t' in Eq. (1-14), called the **retarded time,** is a time *earlier* than t by $\Delta t = \frac{R}{v}$, which is how long it takes the electromagnetic wave to travel a distance R. The physical interpretation of the retarded potentials is given as follows. As shown in Eqs. (1-12) and (1-13), at any given point **r** and given time t, the potentials (**A** and V) are determined by the current and charge distributions that existed at other points $\mathbf{r}'$ (source point) at earlier time t'. The time difference between t and t' is the time required for the electromagnetic "news," due to temporal change of **J** and ρ_v, to travel from the source point $\mathbf{r}'$ to the field point **r.** In other words, any temporal change that occurs at the sources will be felt *later* to the observer at some distance away.

The mathematical proof that the retarded potentials in Eqs. (1-12) and (1-13) satisfy the wave equations, Eqs. (1-8) and (1-9), can be done by evaluating the Laplacian $\nabla^2 V = \nabla \cdot (\nabla V)$ or $\nabla^2 A$ (each component), recognizing that t' has a spatial dependence in it, and making use of the static solutions of Poisson's equation. See Griffiths (1999)* for proof of the retarded scalar potential. The integral expressions for the retarded potentials can also be derived by considering the point sources at the origin,

* D. J. Griffiths, *Introduction to Electrodynamics,* Prentice-Hall, 3rd Ed., 1999.

solving the wave equation in spherical coordinates with the use of spherical symmetry, and lastly applying the superposition principle for the distributed sources. See, for example, Reitz, Milford and Christy (1993)† for this alternative approach.

One final note is that the potential integrals in Eqs. (1-12) and (1-13) with $t' = t + \frac{R}{v}$ (advanced time) also satisfy Eqs. (1-8) and (1-9). However, we discard these solutions because they represent potentials that depend on the sources at future time. These are physically unacceptable solutions and they violate the principle of *causality.*

1.2.2 Retarded Vector Potential for Time-Harmonic Fields

When the current sources are sinusoidal with single frequency, i.e., time-harmonic, the expressions for the fields and potentials can be simplified. As discussed in Volume 4, for time-harmonic problems the time derivative $\frac{\partial}{\partial t}$ can be replaced by the algebraic product of $j\omega$. Thus the electric field in Eq. (1-4) can be written as

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla V \tag{1-15}$$

and the scalar potential is related to the vector potential from Eq. (1-7) (the Lorentz gauge) as follows:

$$\nabla \cdot \mathbf{A} = -j\omega\mu\varepsilon V \tag{1-16}$$

or

$$V = \frac{-\nabla \cdot \mathbf{A}}{j\omega\mu\varepsilon} \tag{1-17}$$

Substituting Eq. (1-17) into Eq. (1-15) leads to

$$\mathbf{E} = -j\omega\mathbf{A} + \frac{\nabla(\nabla \cdot \mathbf{A})}{j\omega\mu\varepsilon} \tag{1-18}$$

Therefore, the electric field can be completely determined once the vector potential **A** is obtained. The vector potential integral in Eq. (1-12)

† J. R. Reitz, F. J. Milford and R. W. Christy, *Foundations of Electromagnetic Theory,* Addison-Wesley, 4th Ed., 1993.

for time-harmonic sources can be obtained as follows. Assume the time-harmonic current density to be of the form:

$$\mathbf{J}(\mathbf{r},t) = \mathrm{Re}\left\{\mathbf{J}(\mathbf{r})e^{j\omega t}\right\} \tag{1-19}$$

in which $\mathbf{J}(\mathbf{r})$ is interpreted as the complex phasor representation of $\mathbf{J}(\mathbf{r},t)$. The current density that appears in the retarded vector potential [see Eq. (1-12)] is then given by

$$\begin{aligned}\mathbf{J}(\mathbf{r}',t') &= \mathbf{J}\left(\mathbf{r}',t-\frac{R}{v}\right) = \mathrm{Re}\left\{\mathbf{J}(\mathbf{r}')e^{j\omega(t-\frac{R}{v})}\right\} \\ &= \mathrm{Re}\left\{\mathbf{J}(\mathbf{r}')e^{-j\frac{\omega}{v}R}e^{j\omega t}\right\} = \mathrm{Re}\left\{\mathbf{J}(\mathbf{r}')e^{-jkR}e^{j\omega t}\right\}\end{aligned} \tag{1-20}$$

where $\frac{\omega}{v} = \omega\sqrt{\mu\varepsilon}$ has been used. In Eq. (1-20), we observe that $\mathbf{J}(\mathbf{r}')\,e^{-jkR}$ is the complex phasor representation of $\mathbf{J}(\mathbf{r}', t')$. Therefore, the retarded vector potential for time-harmonic fields is given by

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \iiint_V \frac{\mu \mathbf{J}(\mathbf{r}')e^{-jkE}}{4\pi R}dv' \\ &= \iiint_V \frac{\mu \mathbf{J}(\mathbf{r}')e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}dv'\end{aligned} \tag{1-21}$$

where $\mathbf{r}$ indicates the position of the field point where $\mathbf{A}$ is to be evaluated, $\mathbf{r}'$ indicates the position of the source point where the current source $\mathbf{J}$ is located and $\mathbf{R} = |\mathbf{r} - \mathbf{r}'|$ is the distance between the source point $\mathbf{r}'$ and the field point $\mathbf{r}$, as shown in Figure 1-1.

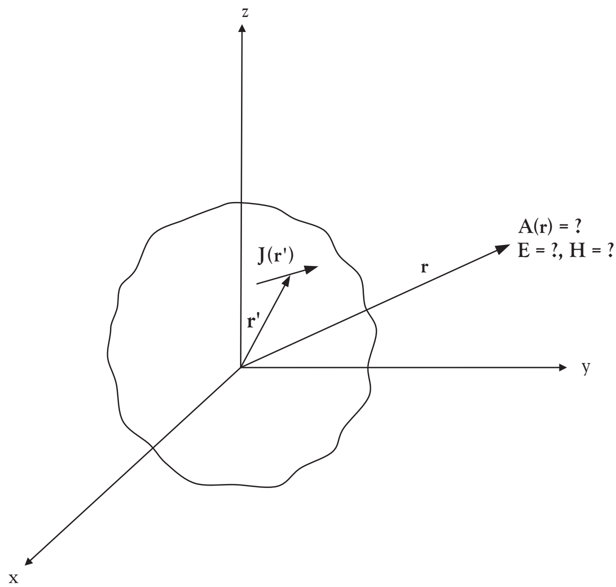


Figure 1-1. The vector potential **A** at **r** due to the current source **J** at **r**′

1.2.3 Finding Antenna Fields

Given the time-harmonic current density **J** distributed over the antenna source, the procedures of finding the electric and magnetic fields produced by this antenna can be summarized in Table 1-1.

Table 1-1. Road Map for Finding Antenna Fields

Step 1. Calculate the retarded vector potential **A**: $$\mathbf{A}(\mathbf{r}) = \iiint_V \frac{\mu \mathbf{J}(\mathbf{r}')e^{-jk\lvert\mathbf{r}-\mathbf{r}'\rvert}}{4\pi\lvert\mathbf{r}-\mathbf{r}'\rvert} dv'$$
Step 2. Find the magnetic field **H:** $$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A}$$
Step 3. Find the electric field **E:** $$\mathbf{E} = \frac{\nabla\times\mathbf{H}}{j\omega\varepsilon}$$

Note that in Step 3 the electric field can also be calculated directly from the vector potential **A,** using Eq. (1-18). Normally this requires more computation, so Ampere's law is more often used to calculate **E** from **H.** As a final note, although Eq. (1-21) gives the vector potential for the volume current distribution **J,** we can easily write the potential integral for the surface current distribution $\mathbf{J}_s$ (for example, dish antenna), simply by replacing $\mathbf{J}(\mathbf{r}')\ dv'$ by $\mathbf{J}_s(\mathbf{r}')\ ds'$ and changing the volume integral to the surface integral. Similarly, the vector potential integral for the line current distribution **I** (for example, wire antenna) is obtained by replacing $\mathbf{J}(\mathbf{r}')$ dv' by $\mathbf{I}(\mathbf{r}')\ d\ell'$ as follows:

$$\mathbf{A}(\mathbf{r}) = \int_L \frac{\mu \mathbf{I}(\mathbf{r}')e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} dl' \tag{1-22}$$

(Potential intergral for wire antenna)

1.3 Elementary Dipole Antenna – Hertzian Dipole

The first antenna configuration to consider is the so-called **Hertzian dipole,** the simplest and most fundamental antenna. It is a thin wire piece of *uniform amplitude* current of *infinitesimal length* or of very small finite length much shorter than wavelength, i.e., $\nabla\ell \ll \lambda$. It does not exist by itself in practice, but it may be considered to be a piece of an actual wire antenna. Understanding fully the radiation characteristics of the Hertzian dipole antenna is very important because the radiation properties of the actual wire antenna result from the superposition of many infinitesimal (Hertzian) dipoles. The Hertzian dipole also gives a good approximation of an electrically small ($\Delta\ell \ll \lambda$) dipole antenna.

We assume that the antenna has an infinitesimal length $\delta\ell$, and the current (phasor) flowing through the antenna has amplitude I, pointing in the z direction, i.e., the antenna is placed along the z axis (Figure 1-2). Then the line current source is given by

$$\mathbf{I} = \mathbf{a}_z I, -\frac{\Delta l}{2} \le z \le \frac{\Delta l}{2} \tag{1-23}$$

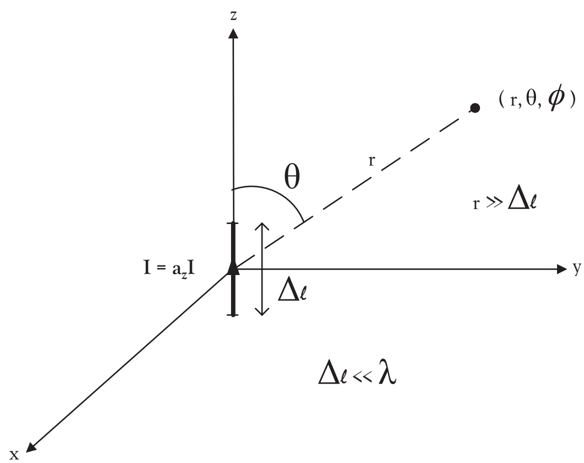


Figure 1-2. ***Hertzian dipole antenna.***

We will use Eq. (1-22) to calculate the vector potential **A.** As mentioned above, there are three conditions that define a Hertzian dipole:

(i) *I is constant,* independent of the position, i.e., the wire has a uniform current distribution.

(ii) $\Delta l \ll r = |\mathbf{r}|$ (r is the distance to the field point), i.e., we observe the field at points far from the source. As $\Delta\ell \to 0$, fields we will obtain are valid *everywhere.*

(iii) $\Delta\ell \ll \lambda$, λ is the free-space wavelength of the time-harmonic current source and $\lambda = \frac{v}{f} = \frac{2\pi}{k}$.

Note that the second condition (ii) applies to all antenna problems. In other words, we don't observe the fields at points close to the antenna because the main purpose of the antenna is to radiate the electromagnetic energy to far distances. Since we observe the field at *far* points, it is more convenient to work the problem in spherical coordinates (r, θ, φ) than rectangular or cylindrical coordinates.

Now in order to find the **E** and **H** fields we follow the procedures outlined in Table 1-1.

Step 1. Calculate the retarded vector potential **A** first.

Substituting Eq. (1-23) into Eq. (1-22), we obtain

$$\mathbf{A}(\mathbf{r}) = \int_{-\frac{\Delta l}{2}}^{\frac{\Delta l}{2}} \frac{\mathbf{a}_z \mu I e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} dz' = \mathbf{a}_z \frac{\mu I}{4\pi} \int_{-\frac{\Delta l}{2}}^{\frac{\Delta l}{2}} \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} dz' \quad (1\text{-}24)$$

where $\mathbf{r} = \mathbf{a}_r r$ and $\mathbf{r}' = \mathbf{a}_z z'$. Noting that $|\mathbf{r}'|_{max} = (z')_{max} = \frac{\Delta l}{2}$ and using the condition (ii), $r \gg \Delta\ell$, we can approximate

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{1}{r} \quad (1\text{-}25a)$$

Now consider the effect of $\mathbf{r}'$ on $e^{-jk|r-r'|}$. We write

$$\begin{aligned} |\mathbf{r}-\mathbf{r}'| &= \sqrt{|\mathbf{r}-\mathbf{r}'|^2} = \sqrt{(\mathbf{r}-\mathbf{r}')\cdot \mathbf{r}-\mathbf{r}'} \\ &= [\mathbf{r}\cdot\mathbf{r} - 2\mathbf{r}\cdot\mathbf{r}' + \mathbf{r}'\cdot\mathbf{r}']^{1/2} = \left[|\mathbf{r}|^2 - 2r\mathbf{a}_r\cdot\mathbf{r}' + |\mathbf{r}'|^2\right]^{1/2} \quad (1\text{-}25b) \\ &\approx \left[r^2 - 2r\mathbf{a}_r\cdot\mathbf{r}'\right]^{1/2} \ \text{ when } r \gg |\mathbf{r}'| = z' \\ &\approx r - z'\cos\theta \end{aligned}$$

where θ is the angle between two vectors $\mathbf{r}$ and $\mathbf{r}'$ (z-axis).

The complex exponential term in Eq. (1-24) is given by

$$e^{-jk|\mathbf{r}-\mathbf{r}'|} \approx e^{-jk(r-z'\cos\theta)} = e^{-jkr}e^{-jkz'\cos\theta} \quad (1\text{-}26)$$

Under the third condition (iii), $\Delta\ell \ll \lambda$,

$$kz' = \frac{2\pi}{\lambda}z' \ll 1 \text{ becomes very small)}$$

Therefore, the correction term in Eq. (1-26) is small and we can approximate the complex exponential, for the Hertzian dipole, as follows:

$$e^{-jk|\mathbf{r}-\mathbf{r}'|} \approx^{-jkr} \quad (1\text{-}27)$$

Making use of the two approximations, Eqs., (1-25) and (1-27), we obtain finally

$$\mathbf{A}(\mathbf{r}) = \mathbf{a}_z \frac{\mu I}{4\pi} \int_{-\frac{\Delta l}{2}}^{\frac{\Delta l}{2}} \frac{e^{-jkr}}{r} dz' = \mathbf{a}_z \frac{\mu I \Delta l\, e^{-jkr}}{4\pi r} \equiv \mathbf{a}_z A_z \quad (1\text{-}28)$$

Step 2. Find the magnetic field **H** by taking the curl of **A.**

We note that **A** in Eq. (1-28) is written in mixed coordinates – $\mathbf{a}_z$ in rectangular or cylindrical coordinate and the component A_z in terms of spherical coordinate (r). In order to calculate $\nabla\times\mathbf{A}$, we need to write **A** in one coordinate system. Using

$$\mathbf{a}_z = \mathbf{a}_r \cos\theta - \mathbf{a}_\theta \sin\theta$$

from Table 1-3, **A** can be rewritten as

$$\begin{aligned} A &= (\mathbf{a}_r \cos\theta - \mathbf{a}_\theta \sin\theta)A_z \\ &= \mathbf{a}_r(A_z \cos\theta) + \mathbf{a}_\theta(-A_z \sin\theta) \equiv \mathbf{a}_r\ A_z + \mathbf{a}_\theta\ A_\theta \end{aligned} \tag{1-29}$$

Noting that $A_\varphi = 0$ and $\dfrac{\partial}{\partial\phi} = 0$ (A_r and A_θ have no φ dependence), and referring to Table 1-4, we calculate **H** as follows.

$$\begin{aligned} \mathbf{H} &= \frac{1}{\mu}\nabla\times\mathbf{A} = \frac{1}{\mu}\mathbf{a}_\phi \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right] \\ &= \mathbf{a}_\phi \frac{1}{\mu}\frac{1}{r}\left\{\frac{\partial}{\partial r}\left[\frac{\mu}{4\pi}I\Delta l\ e^{-jkr}(-\sin\theta)\right] - \frac{\partial}{\partial r}\left[\frac{\mu}{4\pi}I\Delta l\ \frac{e^{-jkr}}{r}\cos\theta\right]\right\} \\ &= \mathbf{a}_\phi \frac{I\Delta l}{4\pi} jk\frac{e^{-jkr}}{r}\left[1+\frac{1}{jkr}\right]\sin\theta \equiv \mathbf{a}_\phi H_\phi \end{aligned} \tag{1-30}$$

Step 3. Find the electric field **E** by taking the curl of **H.**

Noting that $H_r = H_\theta = 0$ and $\partial/\partial\varphi = 0$, we calculate **E** (see Table 1-4):

$$\begin{aligned} \mathbf{E} &= \frac{1}{j\omega\varepsilon}\nabla\times\mathbf{H} = \frac{1}{j\omega\varepsilon}\left[\mathbf{a}_r \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(H_\phi \sin\theta) - \mathbf{a}_\theta \frac{1}{r}\frac{\partial}{\partial\theta}(rH_\phi)\right] \\ &= \frac{I\Delta l}{4\pi} j\omega\mu\frac{e^{-jkr}}{r}\left\{\mathbf{a}_r\left[\frac{1}{jkr}+\frac{1}{(jkr)^2}\right]2\cos + \mathbf{a}_\theta\left[1+\frac{1}{jkr}+\frac{1}{(jkr)^2}\right]\sin\theta\right\} \\ &\equiv \mathbf{a}_r E_r + \mathbf{a}_\theta E_\theta \end{aligned} \tag{1-31}$$

where $k^2 = \omega^2\mu\varepsilon$ has been used. Thus, Eqs. (1-30) and (1-31) provide a complete expression of **E** and **H** for the Hertzian dipole antenna. We observe that the magnetic field has only $\mathbf{a}_\varphi$ component, circulating around the axis of the dipole, and the electric field has both $\mathbf{a}_r$ and $\mathbf{a}_\theta$ components that are perpendicular to **H.** In the following we will discuss the properties of these fields in two different ranges – (i) radiation fields in the far-field region and (ii) the near field behavior. We define the *far-field region* where the observation distance is much greater than the wavelength ($r \gg \lambda$) and the *near-field* region where the field point is closer to the antenna relative to the wavelength ($r \ll \lambda$).

1.3.1 Radiation Fields of a Hertzian Dipole

In the far-field region where the field point is at great distances from the antenna as compared to wavelength of the signal ($r \gg \lambda$), the parameter jkr that appears in Eqs. (1-30) and (1-31) becomes very large, i.e.,

$$kr = \frac{2\pi}{\lambda} r \gg 1$$

Among the terms of three different orders of magnitude

$$1 \gg \frac{1}{jkr} \gg \frac{1}{(jkr)^2}$$

the first term is dominant. The fields in the far-field region, called the **radiation fields,** are given by

$$\begin{aligned} \mathbf{E} &= \mathbf{a}_\theta \frac{I\Delta l}{4\pi} j\omega\mu \frac{e^{-jkr}}{r} \sin\theta \\ \mathbf{H} &= \mathbf{a}_\phi \frac{I\Delta l}{4\pi} jk \frac{e^{-jkr}}{r} \sin\theta \end{aligned} \tag{1-32}$$

The interpretation of these results is as follows.

(i) *The dependence on the distance r*:

The radiation fields are proportional to $\frac{e^{-jkr}}{r}$, which is known to be true for all types of antennas. First and foremost, $\frac{1}{r}$ dependence guarantees the fact that the electromagnetic signal can be sent and detected at far distances without wire ("wireless") because the fields decrease much slower as r increases, as compared to the static fields (electric or magnetic) that behave like $\frac{1}{r^2}$ or $\frac{1}{r^3}$ at best, Secondly, the e^{-jkr} term implies that the constant phase front, given by kr = constant, is a spherical surface. Thus, we have a *spherical wave.* $\frac{e^{-jkr}}{r}$ typically represents a spherical wave.

(ii) *The plane wave characteristics:*

At far distances, a spherical surface with large radius looks more like a plane, so we expect to see some aspects of the plane wave characteristics. First, we note that

$$\mathbf{E} = \mathbf{a}_\theta E_\theta,\ \mathbf{H} = \mathbf{a}_\phi E_\phi,\ \mathbf{k} = \mathbf{a}_r k$$

E, H and **k** are perpendicular to each other and they make a right-handed orthogonal coordinate system as they do for a uniform plane wave. The directions of these vectors are depicted in Figure 1-3.

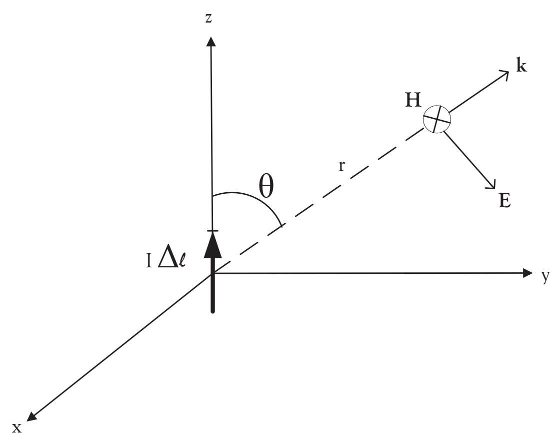


Figure 1-3. ***Radiation fields of a Hertzian dipole.***

Secondly, the ratio of the electric and magnetic field amplitudes is given by

$$\frac{E_\theta}{H_\phi} = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$$

which is the intrinsic impedance of the medium. This is also one of the properties of a uniform plane wave. Due to this property, we don't have to calculate both **E** and **H** if we are interested only in radiation fields. You just need to calculate one of the two and obtain one from the other.

It makes sense to see that the electromagnetic wave leaves the antenna as a spherical wave and eventually becomes at very far distances something that looks like a plane wave locally while retaining the 1/r dependence of a spherical wave.

(iii) *The dependence on observation angle (θ) – Radiation pattern*:

From Eq. (1-32), we observe that both E_θ and H_φ have the sinθ dependence:

$$\left|E_\theta\right| \text{ or } \left|H_\phi\right| \propto \sin\theta$$

Thus, the field amplitude is maximum at θ = 90°, which corresponds to observation points in the xy plane, perpendicular to the axis of the dipole. $|E_\theta|$ is zero at θ = 0° and θ = 180°, which correspond to the direction along the axis of the dipole. The Hertzian dipole antenna gives maximum radiation in the horizontal plane and no radiation along its axis. The field amplitude versus the observation angle θ can be plotted in Figure 1-4, which is called the **radiation pattern** of the antenna. In general, the radiation pattern of the antenna depends on both θ and φ, one needs to draw it in three-dimensional space or draw two separate plots in a plane – one vs. θ on the constant-φ plane and one vs. φ on the constant-θ plane. The radiation pattern of a Hertzian dipole is independent of φ.

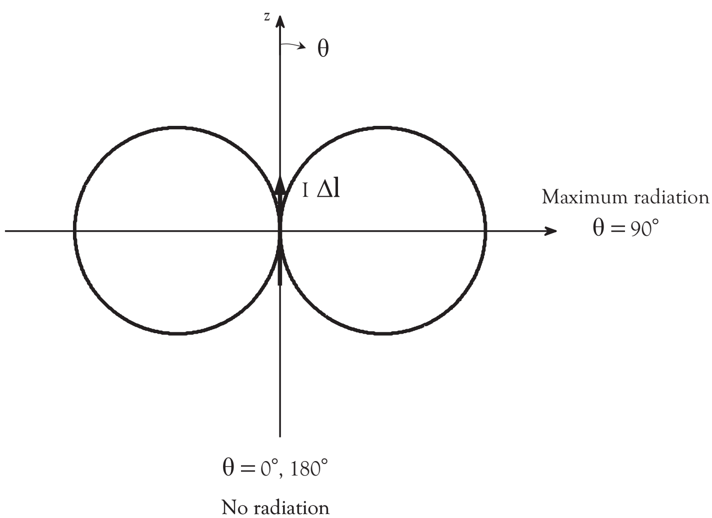


Figure 1-4. ***Radiation pattern of a Hertzian dipole.***

1.3.2 Near Fields of a Hertzian Dipole

In the near-field region where the field point is at close distances from the antenna as compared to wavelength ($r \ll \lambda$, or equivalently, $kr \ll 1$, but still $r \gg \Delta\ell$) we have,

$$1 \ll \frac{1}{jkr} \ll \frac{1}{(jkr)^2}$$

and the third term becomes dominant. From Eqs. (1-30) and (1-31), the dominant fields in the near-field region are given by

$$\begin{aligned} \mathbf{E} &\approx \frac{I\Delta l}{4\pi} j\omega\mu \frac{e^{-jkr}}{r} \frac{1}{(jkr)^2} \left\{ \mathbf{a}_r 2\cos\theta + \mathbf{a}_\theta \sin\theta \right\} \\ &= \frac{I\Delta l\omega\mu}{j4\pi k^2} \frac{1}{r^3} \left\{ \mathbf{a}_r 2\cos\theta + \mathbf{a}_\theta \sin\theta \right\} \\ \mathbf{H} &\approx \mathbf{a}_\phi \frac{I\Delta l}{4\pi} \frac{1}{r^2} \sin\theta \end{aligned} \qquad (1\text{-}33)$$

It is interesting to note that the electric field resembles the *static* electric field of a (static) electric dipole as shown in Eq. (2-68). If we calculate $\frac{1}{2}\mathbf{E} \times \mathbf{H}^*$ from near fields in Eq. (1-32), it becomes purely imaginary, indicating that no real power is carried by the near field terms ($1/r^3$ and $1/r^2$) in Eq. (1-33); instead it is carried by the far field ($1/r$) terms in Eq. (1-32). In the near fields of the Hertzian dipole there are both electric and magnetic stored energy but the electric stored energy is much greater than the magnetic stored energy (see Problem 1-11).

Example 1-1. Radiated Power of a Hertzian Dipole
Calculate the total power radiated by a Hertzian dipole antenna of $I\,\delta\ell$ at the frequency f [Hz] in open air.

Solutions:
The total radiated power can be calculated by integrating the time-average Poynting vector over the entire spherical surface with radius r, using the radiation fields we obtained in Eq. (1-32). It is expected that the total power should be independent of the distance r since the air medium is lossless.

The time-average Poynting vector is given by

$$\mathbf{S}_{av} = \frac{1}{2}\mathrm{Re}\{\mathbf{E}\times\mathbf{H}^*\} = \mathbf{a}_\theta \times \mathbf{a}_\phi \frac{1}{2}\mathrm{Re}\left\{E_\theta H_\phi^*\right\} = \mathbf{a}_r \frac{1}{2}\mathrm{Re}\left\{E_\theta\left(\frac{E_\theta}{\eta}\right)^*\right\}$$
$$= \mathbf{a}_r \frac{|E_\theta|^2}{2\eta} = \mathbf{a}_r \frac{|I\Delta\ell|^2}{32\pi^2}\frac{(\omega\mu)^2}{\eta}\frac{\sin^2\theta}{r^2}\left[\frac{W}{m^2}\right] \tag{1-34}$$

The total radiated power is given by

$$P_r = \iint_{sphere} \mathbf{S}_{av}\cdot\mathbf{a}_r\ ds = \int_0^{2\pi}\int_0^{\pi}\frac{|I|^2}{32\pi^2}(\Delta\ell)^2\frac{(k\eta)^2}{\eta}\frac{\sin^2\theta}{r^2}r^2\sin\theta\ d\theta d\phi$$
$$= \frac{|I|^2}{32\pi^2}(\Delta\ell)^2\left(\frac{2\pi}{\lambda_o}\right)^2\eta(2\pi)\int_o \sin^3\theta d\theta = \frac{\pi\eta_o}{3}\left(\frac{\Delta\ell}{\lambda_o}\right)^2|I|^2\,[W] \tag{1-35}$$

where $\eta = \eta_o = 377\ [\Omega]$ in air and $\lambda_o = c/f$ is the wavelength in air.

Radiation Resistance

A useful measure of the amount of power radiated by an antenna is the quantity called the **radiation resistance.** The radiation resistance R_r of an antenna is defined by the following equation:

$$P_r = \frac{1}{2}I_A^2 R_r \ \text{ or } R_r = \frac{2P_r}{I_A^2} \tag{1-36}$$

where I_A is the current at the input terminals of an antenna or the maximum current along the antenna. Thus the radiation resistance is the value of a resistance that would dissipate an amount of power equal to the radiated power P_r when the current in the resistance is equal to the input or maximum current of the antenna.

The radiation resistance of a Hertzian dipole is obtained from Eq. (1-35):

$$R_r = \frac{2P_r}{|I|^2} = \frac{2\pi}{3}\eta_o\left(\frac{\Delta\ell}{\lambda_o}\right)^2 = 80\pi^2\left(\frac{\Delta\ell}{\lambda_o}\right)^2[\Omega] \tag{1-37}$$

For example, an elementary dipole of $\Delta\ell = 0.1\lambda_o$ has $R_r = 7.9\ [\Omega]$, which is small for an antenna. As will be shown later, a more practical straight wire antenna that is half-wavelength long has $R_r = 73\ [\Omega]$.

Note the rapid change of R_r with *electrical length* ($\Delta\ell/\lambda_0$). If we reduce the length $\Delta\ell$ or the frequency by a factor of 10, then R_r is reduced by a factor of 100. This is our first indication of the basic limitations of "electrically small antennas" of any shape. As the electrical length is decreased, the efficiency and bandwidth is lowered very rapidly. Half wavelength antennas can be very efficient and of reasonable bandwidth.

Example 1-2. Directive Gain of an Antenna
One very important characteristic of an antenna is how much the antenna sends energy in a certain direction in preference to radiation in other directions. A parameter that measures such characteristic is called the **directive gain,** $D(\theta, \varphi)$, which is defined by

$$D(\theta,\phi) = \frac{S_{av}(\theta,\phi)}{P_r / 4\pi r^2} \quad (1\text{-}38)$$

where S_{av} is the time-average Poynting power density in a given direction denoted by the observation angles (θ, φ) and P_r is the total power radiated over the entire space. Thus, the directive gain measures a ratio of the power density in a certain direction at given r (distance) to the average power density at that range. Find the directive gain and its maximum value for a Hertzian dipole antenna.

Solutions:
In Example 1-1, we have calculated S_{av} and P_r of the Hertzian dipole:

$$S_{av} = \frac{|I\Delta l|^2}{32\pi^2}\frac{(\omega\mu)^2}{\eta}\frac{\sin\theta}{r^2} \equiv K\frac{\sin^2\theta}{r^2}$$

$$P_r = \iint S_{av} r^2 \sin\theta \; d\theta d\phi = \int_0^{2\pi}\int_0^{\pi} \sin^3\theta \; d\theta d\phi = K(2\pi)\frac{4}{3}$$

The directive gain of the Hertzian dipole is

$$D(\theta,\phi) = \frac{S_{av}}{P_r / 4\pi r^2} = \frac{K\sin^2\theta}{K\frac{8\pi}{3} / 4\pi} = \frac{3}{2}\sin^2\theta \quad (1\text{-}39)$$

The plot of D(θ, φ) vs. observation angles (θ, φ) is basically the radiation pattern of the antenna shown in Figure 1-4. In a direction θ = 90°, the gain is maximum with the value of 1.5 whereas at θ = 0° and 180°, the gain is zero. The value of the directive gain in the direction of its maximum value is called the **directivity** of an antenna. Thus the directivity of a Hertzian dipole is

$$D_{max} = 1.5 \text{ for Hertizian dipole} \tag{1-40}$$

As will be shown in the next section, the directivity of a half-wavelength dipole antenna is 1.67.

1.4 Linear Antenna – Long Dipole

In the previous section we considered an elementary dipole antenna, which is electrically short ($\Delta\ell \ll \lambda$) and is assumed to have uniform current distribution. However, such a short antenna has a very small radiation resistance and is not a good radiator. A more practical and efficient antenna would be a straight wire whose length is comparable to a wavelength, known as the **linear dipole antenna.** If the current distribution along the wire is known we can easily incorporate it in the vector potential integral and find the radiation fields. However, the exact current distribution on the thin wire antenna is not known and its approximate distribution can be obtained through the use of advanced numerical techniques such as the method of moments. We will not deal with such sophisticated techniques and assume that the current distribution on the wire is *sinusoidal in space* along the wire, with zero at the ends. This functional dependence on z is known to be a good approximation to actual current distribution.

Figure 1-5 shows a straight thin wire antenna of length ℓ excited by a source in the middle, whose current distribution is assumed to take the following form:

$$I(z) = I_o \sin\left[k\left(\frac{\ell}{2} - |z|\right)\right], \quad |z| < \frac{\ell}{2} \tag{1-41}$$

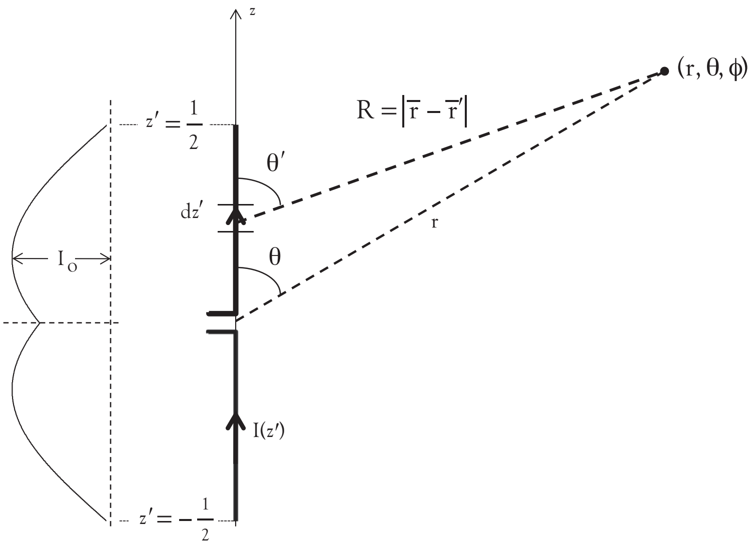


Figure 1-5. ***Long dipole antenna with sinusoidal current distribution.***

where k = 2π/λ is a wavenumber corresponding to an excitation frequency. The current is symmetric about z = 0 (center-driven) and goes to zero smoothly at the ends (z = ±ℓ/2). I_o is the maximum value of the current on the wire and its position depends on the length of the antenna.

In order to obtain the fields radiated by the antenna we would normally calculate the vector potential first. However, since we are primarily interested in the radiation (far) fields and we know the radiation fields of an elementary dipole, we can make use of the result of Eq. (1-32) and apply superposition to calculate directly the radiation fields of the long dipole.

Consider a differential element I(z′) dz′ located at z = z′. Its radiation electric field can be written from Eq. (1-32) as follows:

$$dE = \mathbf{a}_\theta \frac{I(z')dz'}{4\pi} j\omega\mu \frac{e^{-jrK}}{R} \sin\theta' \tag{1-42}$$

where R = |**r** − **r**′| is the distance from the source point (z = z′) to the field point (r, θ, φ) and θ′ is the angle between the wire axis (z-axis) and the vector **R = r − r.** At far points (r ≫ z′), the following approximations can be made (see Eq. (1-25)).

$$R = |\mathbf{r} - \mathbf{r}'| \approx r - z'\cos\theta$$
$$\frac{1}{R} \approx \frac{1}{r},\ \sin\theta' \approx \sin\theta \qquad \text{when } r \gg |\mathbf{r}'| = z'$$

Then Eq. (1-42) reduces to

$$d\mathbf{E} \approx \mathbf{a}_\theta \frac{I(z')dz'}{4\pi} j\omega\mu \frac{e^{-jkr}}{r} e^{jkz'\cos\theta} \sin\theta' \tag{1-43}$$

The radiation electric field due to the entire wire can be obtained by integrating d**E** over the source distribution.

$$\mathbf{E} = \int d\mathbf{E} = \mathbf{a}_\theta \frac{j\omega\mu}{4\pi} \frac{e^{-jkr}}{r} \sin\theta \int_{-1/2}^{1/2} I(z')\, e^{jkz'\cos\theta} dz' \tag{1-44}$$

$$\begin{aligned} E_\theta &= \frac{j\omega\mu}{4\pi} \frac{e^{-jkr}}{r} \sin\theta \left\{ \int_0^{1/2} I_o \sin\left[k\left(\frac{1}{2} - z\right)\right] e^{jkz'\cos\theta} dz' \right. \\ &\quad \left. + \int_{-1/2}^{0} I_o \sin\left[k\left(\frac{1}{2} + z\right)\right] e^{jkz'\cos\theta} dz' \right\} \\ &= j\eta 2 I_o \frac{e^{-jkr}}{4\pi r} \frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta} \end{aligned} \tag{1-45}$$

The radiation magnetic field is simply given by

$$\mathbf{H} = \mathbf{a}_\phi H_\phi = \mathbf{a}_\phi \frac{E_\theta}{\eta} \tag{1-46}$$

due to its plane wave characteristic at far distances. The radiation pattern of the long dipole antenna is given by

$$F(\theta) = \frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta} \tag{1-47}$$

Note that it is independent of φ (because the current is z-directed) and depends on the length of the dipole. Making use of $k = \frac{2\pi}{\lambda}$, we have $\frac{k\ell}{2} = \pi\frac{\ell}{\lambda}$. For example,

$$\ell = \frac{\lambda}{2} \rightarrow F(\theta) = \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \tag{1-48a}$$

$$l = \lambda \rightarrow F(\theta) = \frac{\cos(\pi\cos\theta) + 1}{\sin\theta} \tag{1-48b}$$

$$\ell = \frac{3\lambda}{2} \rightarrow F(\theta) = \frac{\cos\left(\frac{3}{2}\pi\cos\theta\right)}{\sin\theta} \tag{1-48c}$$

$$l = 2\lambda \rightarrow F(\theta) = \frac{\cos(2\pi\cos\theta) - 1}{\sin\theta} \tag{1-48d}$$

The radiation patterns of the long dipole for four different lengths, $\ell = \frac{\lambda}{2}$, λ, $\frac{3}{2}\lambda$ and 2λ, are plotted in Figure 1-6. It is observed that a full-wavelength dipole ($\ell = \lambda$) has a more directive beam along $\theta = 90°$ than a half-wave dipole $\left(\ell = \frac{\lambda}{2}\right)$. As the length of the dipole increases further, the main beam (or maximum radiation) shifts away from $\theta = 90°$ and nulls (zero radiation) occur in certain directions. For example, $\ell = \frac{3}{2}\lambda$ dipole has a null at $\theta = 70.5°$ and $\ell = 2\lambda$ dipole has a null at $\theta = 90°$.

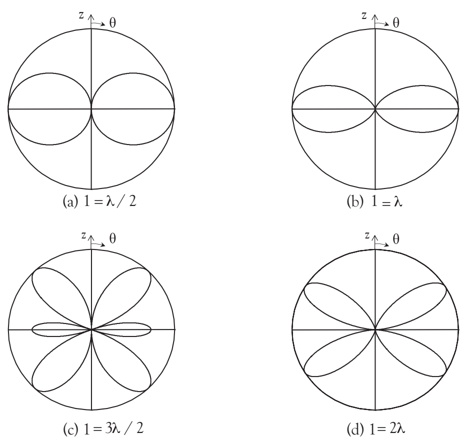


Figure 1-6. *Radiation patterns of long dipole antennas*. (a) $\ell = \lambda/2$, (b) $\ell = \lambda$, (c) $\ell = 3\lambda/2$, (d) $\ell = 2\lambda$

1.4.1 Half-Wave Dipole

A linear antenna having a length $\ell = \lambda/2$ (half wavelength), known as the **half-wave dipole,** is very widely used and is of practical importance because of its impedance characteristic. The half-wave dipole has a current distribution that is one-half of a sine wave with maximum at the center.

$$I(z) = I_o \sin\left[k\left(\frac{\lambda}{4} - |z|\right)\right] = I_o \sin\left[\frac{\pi}{2} - \frac{2\pi}{\lambda}|z|\right], \ |z| < \frac{\lambda}{4} \quad (1\text{-}49)$$

A unique advantage of a half-wave dipole is that it can be made to *resonate* and to have zero reactance components, making it easy to tune. To make the dipole resonant its physical length must be a little bit shorter than a free-space half wavelength.

The radiation fields of the half-wave dipole are given by

$$E_\theta = j\eta 2I_o \frac{e^{-jkr}}{4\pi r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}, \ H_\phi = \frac{E_\theta}{\eta} \quad (1\text{-}50)$$

whose pattern is shown in Figure 1-6(a). The pattern is very similar to that of a Hertzian dipole, i.e., it has a maximum radiation along $\theta = 90°$ and no radiation along its axis ($\theta = 0°$). However, the beam width is a little bit sharper.

Example 1-3. Radiation Resistance of a Half-Wave Dipole.
Calculate the radiation resistance and directivity of a half-wave dipole antenna.

Solutions:
The total radiated power of a half-wave dipole is given by

$$P_r = \iint S_{av} \cdot \mathbf{a}_r \, ds = \iint \frac{|E_\theta|^2}{2\eta} r^2 \sin\theta \, d\theta d\phi$$
$$= \int_0^{2\pi}\int_0^{\pi} \frac{\eta}{2}\left(\frac{2I_o}{4\pi}\right)^2 \frac{\left[\cos\left(\frac{\pi}{2}\cos\theta\right)\right]^2}{\sin\theta} d\theta d\varphi = \frac{\eta}{4\pi} I_o^2 \int_0^{\pi} \frac{\left[\cos\left(\frac{\pi}{2}\cos\theta\right)\right]^2}{\sin\theta} d\theta$$
$$(1\text{-}51)$$

The integral can be written in terms of the cosine integral function or can be evaluated numerically to give a value of 1.22.
Then

*W.L. Stutzman and G.A. Thiele, *Antenna Theory and Design,* John Wiley & Sons, 1998, 2^{nd} Ed., Section 5.1.

$$P_r = \frac{\eta}{4\pi} I_o^2 (1.22) = 36.6\ I_o^2\ [W] \tag{1-52}$$

The radiation resistance is given by (see Eq. (1-36)).

$$R_r = \frac{2P_r}{I_o^2} = 73\ [\Omega] \tag{1-53}$$

The directive gain of the dipole is given by

$$D(\theta,\phi) = \frac{S_{av}}{P_r / 4\pi r^2} = \frac{|E_\theta|^2 / 2\eta}{P_r / 4\pi r^2} = \frac{\frac{\eta}{2}\left(\frac{2I_o}{4\pi}\right)^2}{\eta\left(\frac{I_o}{4\pi}\right)^2 (1.22)} \left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]^2$$
$$= 1.46\ [F(\theta)]^2 \tag{1-54}$$

As shown in Figure 1-6(a), $F(\theta)$ has a maximum value 1 when $\theta = 90°$. Therefore, the directivity of a half-wave dipole is

$$D_{max} = 1.64 \text{ for half-wave dipole} \tag{1-55}$$

This is a little bit higher than D_{max} (= 1.5) of a Hertzian dipole, which results in a slightly sharper main beam along $\theta = 90°$. A full-wave dipole ($\ell = \lambda$) gives a more directive pattern, as shown in Figure 1-6(b), and it has the directivity of 2.41.

1.5 Antenna Arrays

In the previous sections we considered the radiation of a single-element antenna – the Hertzian dipole, the small loop antenna the long dipole and the monopole antenna. We observe that the radiation patterns of all these antennas are omnidirectional (independent of φ) in the plane perpendicular to the antenna and they have low directivity due to a broad main beam. Then how do we send energy in certain directions or areas and avoid radiation in other directions? We can achieve this by using an **array** of several antennas in different configurations – on a line, circle or plane, etc. By changing the spacing between antennas and the phase differences between excitation currents of antenna elements, we can create a more directional pattern of the array. For example, many broadcast antenna stations use an array of multiple vertical antennas above ground (monopoles). An array of many small antennas can also be used to obtain a performance similar to that of a single large antenna. We will first treat the simplest array with two-element antennas and then extend to an array of multiple antennas.

1.5.1 Two-Element Array

Let's consider two identical antennas separated by a distance d on the x axis as shown in Figure 1-7. For example, they could be a pair of vertical half-wave dipoles or quarter-wave monopoles above ground. The current excitations are I_1 and I_2; they have the same amplitude and the phase difference of α. For simplicity, let

$$I_1 = I_o,\ I_2 = I_o\ e^{j\alpha} \qquad (1\text{-}56)$$

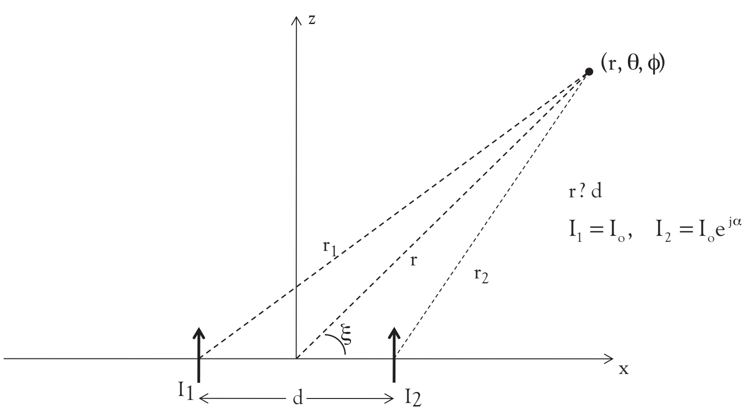


Figure 1-7. ***A two-element array.***

The radiation field at far point P(r, θ, φ) ($r \gg d$) is a superposition of radiation fields due to two antennas. Since two antennas are identical, they will have the same radiation pattern but with different current excitations. Hence we can write

$$E_1 = I_1 K F(\theta,\phi)\frac{e^{-jkr_1}}{r_1} \qquad (1\text{-}57)$$

$$E_2 = I_2 K F(\theta,\phi)\frac{e^{-jkr_2}}{r_2} \tag{1-58}$$

where $F(\theta, \varphi)$ is the radiation pattern function of each individual antenna and K is a multiplying constant. Substituting Eq. (1-56) into E_1 and E_2, the total radiation field is given by

$$E = E_1 + E_2 = I_0 \; K \; F\,(\theta,\phi) \left\{\frac{e^{-jkr_1}}{r_1} + \frac{e^{-jkr_2}e^{j\alpha}}{r_2}\right\} \tag{1-59}$$

When the field (or observation) point P is in the far distances ($r \gg d$),

$$r_1 \approx r + \frac{d}{2}\cos\xi \tag{1-60a}$$

$$r_2 \approx r - \frac{d}{2}\cos\xi \tag{1-60b}$$

where ξ is the angle between the direction of the observation ($\mathbf{a}_r$) and the axis of the array ($\mathbf{a}_x$) and its cosine is given by

$$\cos\xi = \mathbf{a}_r \cdot \mathbf{a}_x = \sin\theta\cos\theta$$

The radiation field can be approximated by

$$\begin{aligned} E &\approx I_0 K F(\theta,\phi)\left\{\frac{e^{-jk\left(r+\frac{d}{2}\cos\xi\right)}}{r_1} + \frac{e^{-jk\left(r-\frac{d}{2}\cos\xi\right)}e^{j\alpha}}{r_2}\right\} \\ &\approx I_0 K F(\theta,\phi)\frac{e^{-jkr}}{r}e^{j\frac{\alpha}{2}}\left\{e^{-j\left(\frac{1}{2}kd\cos\xi+\frac{\alpha}{2}\right)} + e^{j\left(\frac{1}{2}kd\cos\xi+\frac{\alpha}{2}\right)}\right\} \\ &= I_0 K e^{j\frac{\alpha}{2}}\frac{e^{-jkr}}{r}F(\theta,\phi)\cdot 2\cos\left[\frac{1}{2}\left(kd\cos\xi+\alpha\right)\right] \end{aligned} \tag{1-61}$$

The magnitude of the electric field of the array is

$$|E| = \frac{|2I_0K|}{r}|F(\theta,\phi)|\left|\cos\frac{\psi}{2}\right| \tag{1-62}$$

where

$$\psi = kd\cos\xi + \alpha = \frac{2\pi d}{\lambda}\sin\theta\,\cos\phi + \alpha \tag{1-63}$$

Note that $|F(\theta,\varphi)|$ represents the radiation pattern of an individual antenna, called the **element pattern,** and $\left|\cos\frac{\psi}{2}\right|$ is the additional pattern factor due to the array configuration, called the **array factor** (AF). The array factor depends on the array geometry and the phase difference between current excitations of the antenna elements. From Eq. (1-62), we observe that *the radiation pattern of an array of identical antennas can be obtained by a product of the element pattern and the array factor.* Such property is known as the **principle of pattern multiplication.** For a more detailed discussion, see, for example, Stutzman and Thiele (1998).

Example 1-4. Radiation Pattern of a Two-Element Array
Sketch the electric field pattern of an array of two vertical (z-directed) half-wave dipoles in the xy plane $\left(\theta=\frac{\pi}{2}\right)$ for the following four cases: (i) $d = \lambda/2$, $\alpha = 0$, (ii) $d = \lambda/2$, $\alpha = \pi$, (iii) $d = \lambda/4$, $\alpha = -\pi/2$, (iv) $d = \lambda$, $\alpha = 0$.

Solution:
Since we observe the pattern in the xy plane ($\theta = 90°$), the element pattern of a half-wave dipole is $F(\theta = \pi/2, \varphi) = 1$. The radiation pattern of the array is simply given by the array factor:

$$AF = \left|\cos\frac{\psi}{2}\right| = \left|\cos\left[\frac{\pi d}{\lambda}\cos\phi + \frac{\alpha}{2}\right]\right| \tag{1-64}$$

Case 1: $d = \lambda/2$, $\alpha = 0$

$$AF = \left|\cos\left(\frac{\pi}{2}\cos\phi\right)\right| \tag{1-65}$$

The pattern is sketched in Figure 1-8(a). It has the maximum radiation along $\varphi = 90°$ (perpendicular to the array axis) and no radiation along the axis of the array ($\varphi = 0°, 180°$). Because two antennas are identical and in phase, in the direction along $\varphi = 90°$ the radiation fields from the two antennas add, whereas in the direction along $\varphi = 0°$ or $180°$ the radiation fields from the two antennas are out of phase by $kd = (2\pi/\lambda) \cdot \lambda/2 = \pi$ and they cancel each other. This type of array is known as the *broadside array.*

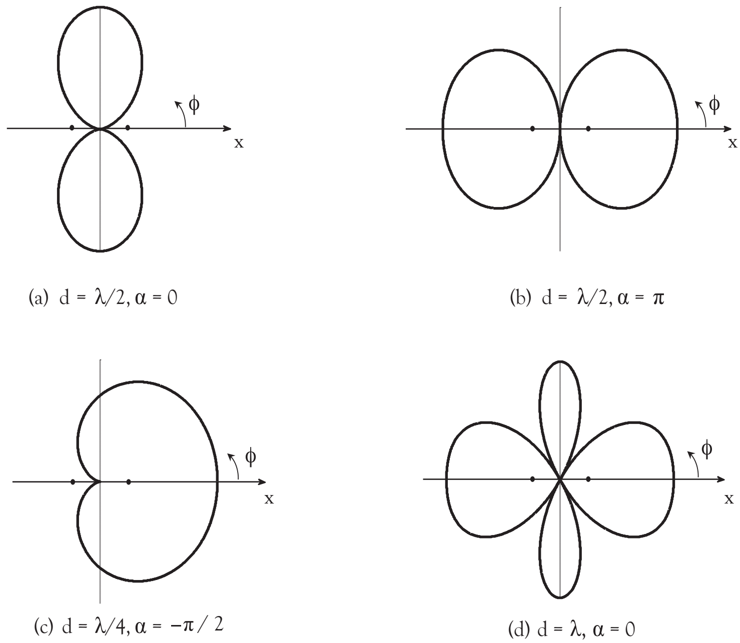


Figure 1-8. ***Radiation pattern of a two-element array***

(a) d = λ/2, α = 0, (b) d = λ/2, α = π, (c) d = λ/4, α = −π/2,

(d) d = λ, α = 0.

Case 2: d = λ/2, α = π

$$AF = \left|\cos\left(\frac{\pi}{2}\cos\phi + \frac{\pi}{2}\right)\right| = \left|\sin\left(\frac{\pi}{2}\cos\phi\right)\right| \qquad (1\text{-}66)$$

The pattern, shown in Figure 1-8(b) is completely opposite to that of Case 1. It has the maximum radiation along the array axis (φ = 0°, 180°) and no radiation (nulls) along φ = 90°. This time two antennas are 180° out of phase, thus the radiation fields from the two cancel each other along φ = 90° and add along φ = 0° because the total phase difference is kd + α = π + π = 2π in the direction along the array axis. An array which has the maximum in the direction along the array axis is called an *end fire array.*

Case 3: d = λ/4, α = −π/2

$$AF = \left|\cos\left(\frac{\pi}{4}\cos\phi + \frac{\pi}{4}\right)\right| \tag{1-67}$$

The radiation pattern is sketched in Figure 1-8(c). It has a maximum along φ = 0° and null at φ = 180°, known as the *cardioid pattern.* This array can be used when two tall antennas need to be placed near the sea. You want to send signal in the residential area but no signal in the direction of the seaside. This array is also frequently used in acoustics for microphone patterns.

Case 4: d = λ, α = 0

$$AF = \left|\cos\pi\cos\phi\right| \tag{1-68}$$

The pattern, shown in Figure 1-8(d), has maximum radiation in four directions (φ = 0°, 90°, 180°, 270°) and nulls in between (φ = ±60°, ±120°). Nulls (perfect cancellation) occur when the path length difference from two antennas is one-half wavelength, which is equivalent to a phase difference of π, i.e., d cosφ = λ cosφ = λ/2 or cosφ = 1/2.

1.5.2 Uniform Linear Array

An array of multiple (more than two) elements can be analyzed in a similar manner. Among various types of an array, we consider an array of *equally spaced* identical antennas that have current excitations with the same amplitude and *uniform progressive phase shift* because its analysis is straightforward. Such an array is called a **uniform linear array.** Suppose N identical antennas are placed along the x axis. They are separated by d and the phase difference between two neighboring elements is α, i.e., the current excitations can be written as

$$I_1 = I_o,\ I_2 = I_o e^{j\alpha},\ I_3 = I_o e^{j2\alpha},\ldots, I_N = I_o e^{j(N-1)\alpha},$$

The configuration of the array is shown in Figure 1-9. At far points (r ≫ d),

$$r_1 = r,\ r_2 \approx r - d\cos\xi,\ r_3 = r_2 - 2d\cos\xi,\ldots,\ r_N = r - (N-1)d\cos\xi$$

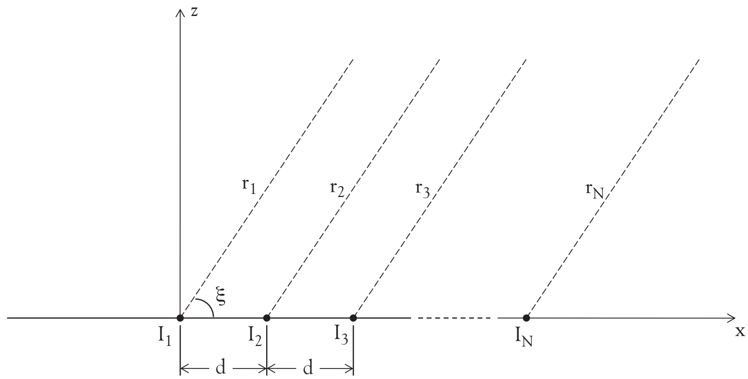


Figure 1-9. A *uniform linear array*

Following a similar analysis of the two-element array (see Eq. (1-61)), the array factor of the uniform linear array is given by

$$\begin{aligned} AF &= 1 + e^{j(kd\cos\xi)}e^{j\alpha} + e^{j(2kd\cos\xi)}e^{j2\alpha} + \cdots + e^{j(N-1)kd\cos\xi}e^{j(N-1)\alpha} \\ &= 1 + e^{j\psi} + e^{j2\psi} + \cdots + e^{j(N-1)\psi} = \sum_{n=0}^{N-1} e^{jn\psi} \end{aligned} \tag{1-69}$$

where

$$\psi = kd\cos\xi + \alpha \tag{1-63}$$

Eq.(1-69) is a geometric series and can be summed in closed form:

$$AF = \frac{1-e^{jN\psi}}{1-e^{j\psi}} = \frac{e^{j\frac{N\psi}{2}}}{e^{j\frac{\psi}{2}}}\frac{e^{-j\frac{N\psi}{2}} - e^{j\frac{N\psi}{2}}}{e^{-j\frac{\psi}{2}} - e^{-j\frac{\psi}{2}}} = e^{j\frac{(N-1)\psi}{2}}\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{\psi}{2}\right)} \tag{1-70}$$

The array factor in Eq. (1-70) is maximum when $\psi = 0$ and its value is given by

$$\lim_{\psi\to 0}\frac{\sin(N\psi/2)}{\sin(\psi/2)} = \lim_{\psi\to 0}\frac{\frac{N}{2}\cos(N\psi/2)}{\frac{1}{2}\cos(\psi/2)} = N$$

where L'Hopital's rule is used. Then the normalized array factor is given by

$$|AF|_n = \frac{1}{N}\left|\frac{\sin(N\psi/2)}{\sin(\psi/2)}\right| \tag{1-71}$$

As we plot $|AF|_n$ as a function of ψ for different values of N, we observe the following:

1. The function is periodic in ψ, with period 2π.
2. As N increases, the main lobe becomes narrower (more directivity).
3. As N increases, the number of side lobes increases. The total number of lobes in one period is N−1 (one main lobe and N−2 side lobes).
4. The level of side lobes decreases as N increases.
5. The main beam occurs when

$$\psi = kd\cos\xi + \alpha = 0 \text{ or } \cos\xi = -\frac{\alpha}{kd} \qquad (1\text{-}72)$$

When α = 0 (all elements are in phase), the main beam occurs at ξ = 90°; it is a *broadside array.*

When α = −kd, the main beam occurs at ξ = 0°, which gives an *end fire array.*

6. The side lobes (minor maxima) occur approximately when sin (Nψ/2) is at maximum, i.e., when |sin(Nψ/2)| = 1.

$$\frac{N\psi}{2} = (2n+1)\frac{\pi}{2}, n = 1,2,3,.... \qquad (1\text{-}73)$$

The first side lobe occurs when Nψ/2 = 3π/2 (n=1).

7. The nulls occur when

$$\frac{N\psi}{2} = n\pi = 1,2,3,.... \qquad (1\text{-}74)$$

Example 1-5. Four-Element Linear Array

Plot the normalized array factor, as a function ψ (0≤ψ≤2π), of a four-element array, uniformly excited and equally spaced. Sketch the radiation pattern of this array (assuming the element pattern is omnidirectional) when d = λ/2 and α = π.

Solution:

$$|AF|_n = \left|\frac{\sin(2\psi)}{4\sin(\psi/2)}\right| (N = 4)$$

The side lobes occur approximately at $2\psi = 3\pi/2, 5\pi/2$, or $\psi = 3\pi/4, 5\pi/4$. The nulls occur at $2\psi = \pi, 2\pi, 3\pi$, or $\psi = \pi/2, \pi, 3\pi/2$. $|AF|_n$ is plotted in Figure 1-10(a).

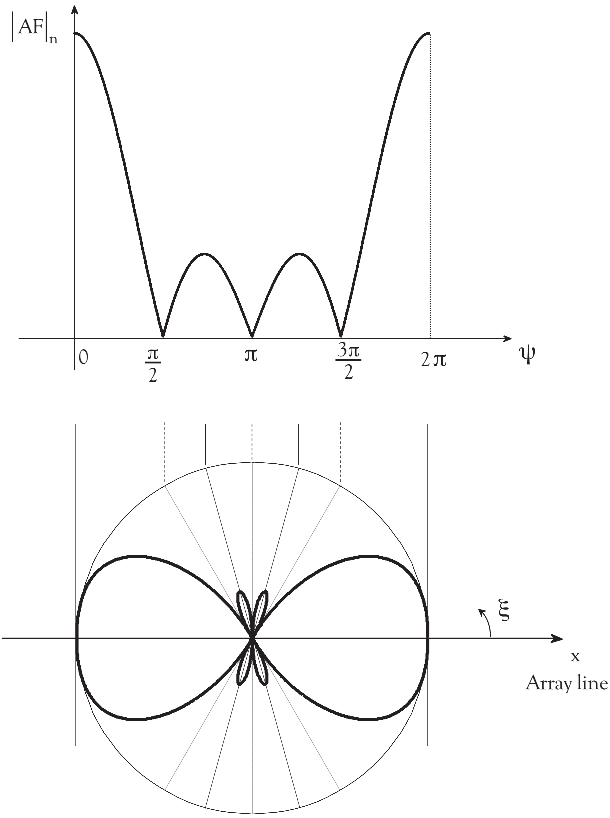


Figure 1-10. ***Normalized array factor (a) and radiation pattern (b) of a four element linear array with d = λ/2, α = π***

When $d = \lambda/2$ and $\alpha = \pi$, $kd = (2\pi/\lambda)d = \pi$.

$$\psi = kd\cos\xi + \pi = \pi(\cos\xi + 1)$$

The main beam occurs when $\cos\xi + 1 = 0, 2 \rightarrow \xi = 180°, 0°$.

The nulls occur when $\cos\xi + 1 = 1/2, 1, 3/2 \rightarrow \xi = 60°, 90°, 120°$.

The side lobes occur when $\cos\xi + 1 \approx 3/4, 5/4 \rightarrow \xi \approx 75.5°, 104.5°$.

The radiation pattern is sketched in Figure 1-10(b). For a more systematic way of sketching the array pattern, see Stutzman and Thiele (1998), pp. 101-106.

We previously mentioned some basic limitations of electrically small antennas. The trend continues. All of the electrical characteristics of an antenna or antenna system such as an array depend on electrical size, not on its physical length in meters but on its electrical size in wavelengths or in square wavelengths.

Furthermore, if we list all of the desirable characteristics of an antenna, such as (1) high efficiency, (2) large bandwidth, and (3) high directivity, then the more characteristics (1, 2, 3) that we ask for and the higher the levels that we demand, the larger the required electrical size of our antenna or array.

CHAPTER 2

Special Solutions—Laplace's and Poisson's Equations

2.1 Introduction

Thus far we have learned how one could find the electric field, given the stationary charge distribution or the polarization. The latter gives bound charge distribution. If the known charge distributions (either free or bound) have certain symmetries, Gauss's law, is useful in finding the electric field.

However, in many practical electrostatic problems involving conductors, the exact charge distribution is not known in advance, although we can control the total charge or the potential of each conductor. In such cases, the above methods are not useful in finding the fields; thus we need to seek other methods. In this chapter, we discuss two special techniques that have proven to be very useful in finding the electric potential. They are the method of images and the method of separation of variables. With both methods, one basically solves Laplace's equation, which is the second-order partial differential equation that the potential V must satisfy. These methods can also be used in solving certain magnetostatic problems.

2.2 Laplace's and Poisson's Equations

In this section we derive Poisson's equation and Laplace's equation. To derive these equations, we go back and visit the two fundamental laws for the electric field.

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{D} = \rho_v$$

In a linear homogeneous dielectric medium of permittivity ε, Eq. (3-11) reduces to

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon} \tag{2-1}$$

As discussed

$$\mathbf{E} = -\nabla V \tag{2-2}$$

Substituting Eq. (2-2) into Eq. (2-1), we obtain

$$\nabla \cdot (-\nabla V) = -\nabla \cdot (\nabla V) = -\nabla^2 V = \frac{\rho_v}{\varepsilon}$$

thus we have

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \tag{2-3}$$

where $\nabla^2 V$ is the Laplacian of V Equation (2-3) is known as **Poisson's equation.** It is a second-order partial differential equation that the electric potential should satisfy when the charge distribution ρ_v is given in a linear dielectric. One may note that it takes only *one* differential equation (Poisson's equation) to determine V, because V is a scalar. On the other hand, for **E** we need *two,* involving the divergence and the curl.

Very often, we are interested in finding V in a region where there is no charge, $\rho_v = 0$, a so-called *source-free* region. In such a region, Eq. (2-3) reduces to

$$\nabla^2 V = 0 \text{ in a source-free region} \tag{2-4}$$

which is called **Laplace's equation.** You may ask how one could have V or **E** if there is no charge, $\rho_v = 0$. Of course, if $\rho_v = 0$ *everywhere,* then V = 0. We explain that charges may exist *outside* the source-free region. Laplace's equation, (2-4), is the theme of this chapter. It is fundamental to the subject of electrostatics and also appears in many different branches of physics, such as the gravitational field, magnetostatics, and heat conduction. In Cartesian coordinates, Laplace's equation becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{2-5}$$

2.2.1 Laplace's Equation in One Dimension

In this subsection, we consider the solutions of Laplace's equation in one dimension through examples.

Example 2-1. A Parallel-Plate Capacitor
A voltage V_0 is applied between two parallel conducting plates separated by a distance d as shown in Figure 2-1.

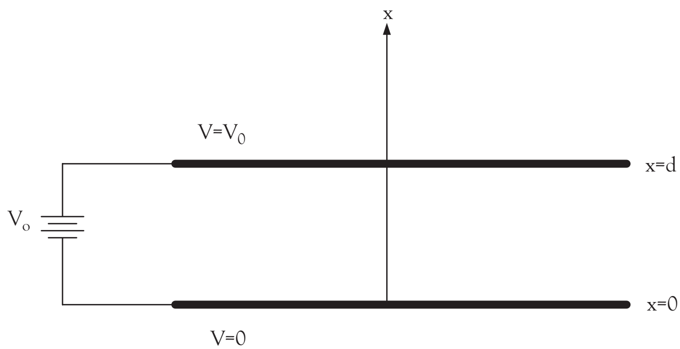


Figure 2-1. ***A parallel-plate capacitor for Example 2-1***

Find the potential distribution and the electric field in the region ($0 \le x \le d$) between the two plates. Assume both plates are infinitely large.

Solution:

Since the region $0 \le x \le d$ is free of charge, V will satisfy Laplace's equation. Because both plates are *infinitely large* planar surfaces and the potential is uniform at fixed x, we expect that V will not depend on y and $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$. In this case, Laplace's equation becomes one-dimensional:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} = 0 \tag{2-6}$$

Integrating Eq. (2-6) with respect to x twice, we obtain

$$V(x) = Ax + B \tag{2-7}$$

To determine the unknown coefficients A and B, we use the boundary conditions:

$$\begin{aligned} V(x = d) &= V_0 \\ V(x = 0) &= 0 \end{aligned} \tag{2-8}$$

which result in

$$A = \frac{V_0}{d}, \; B = 0 \tag{2-9}$$

Substituting Eq. (2-9) into Eq. (2-7), we obtain

$$V(x) = V_0 \frac{x}{d}, \; 0 \le x \le d \tag{2-10}$$

The electric field is given by

$$\mathbf{E} = -\nabla V = -\mathbf{a}_x \frac{\partial V}{\partial x} = -\mathbf{a}_x \frac{V_0}{d} \tag{2-11}$$

The field is uniform. The surface charge density at the upper plate can be obtained by

$$\rho_S = \varepsilon_0 E_n = -\varepsilon_0 E_n \Big|_{x=d} = \varepsilon_0 \frac{V_0}{d} \left[\frac{C}{m^2} \right] \tag{2-12}$$

Example 2-2. A Coaxial Cable

A coaxial cable consists of two coaxial conducting cylinders with radii a and b (a < b). It is filled with a linear dielectric of permittivity ε. A voltage V_0 is applied between inner and outer conductors, positive at the inner conductor (Figure 2-2). Find the potential distribution and the electric field in the region a ≤ ρ ≤ b. Assume that the cylinders are infinitely long.

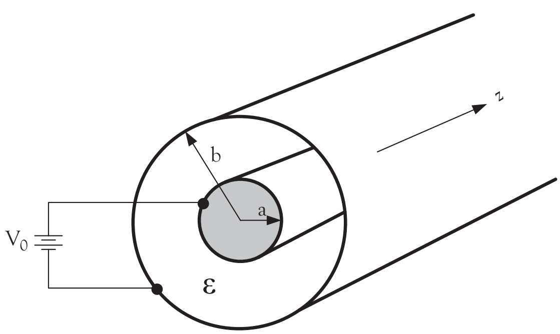


Figure 2-2. ***A coaxial cable for Example 2-2***

Solution:

Since the region $a \le \rho \le b$ is free of charge and the boundary surfaces coincide with the cylindrical coordinate surfaces, V satisfies Laplace's equation in cylindrical coordinates:

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{(2-13)}$$

Because the cylinders are *infinitely long* in the z direction and the potential is *uniform* (or constant) on the cylindrical surfaces ($\rho = a$ and $\rho = b$), we expect that V will not depend on z and φ, i.e., $\frac{\partial}{\partial \phi} = \frac{\partial}{\partial z} = 0$. In this case, Eq. (2-13) becomes

$$\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0 \qquad \text{(2-14)}$$

Integrating Eq. (2-14) with respect to ρ, we obtain

$$\rho \frac{\partial V}{\partial \rho} = C_1 \text{ or } \frac{\partial V}{\partial \rho} = \frac{C_1}{\rho} \qquad \text{(2-15)}$$

Integrating Eq. (2-15) once more with respect to ρ, we have

$$V(\rho) = C_1 \ln \rho + C_2 \qquad \text{(2-16)}$$

We apply the two boundary conditions:

$$\begin{aligned} V(\rho = a) &= V_0 \\ V(\rho = b) &= 0 \end{aligned} \qquad \text{(2-17)}$$

which gives

$$C_1 \ln a + C_2 = V_0 \qquad \text{(2-18a)}$$

$$C_1 \ln b + C_2 = 0 \qquad \text{(2-18b)}$$

Solving Eqs. (2-18), we obtain

$$C_1 = \frac{V_0}{\ln \frac{a}{b}},\ C_2 = -\frac{V_0 \ln b}{\ln \frac{a}{b}} \qquad \text{(2-19)}$$

Substituting Eq. (2-19) into Eq. (2-16), we obtain

$$V(\rho) = \frac{V_0}{\ln\frac{a}{b}}\{\ln\rho - \ln b\}$$
$$= V_0 \frac{\ln\frac{\rho}{b}}{\ln\frac{a}{b}}, a \le \rho \le b \qquad (2\text{-}20)$$

The electric field is given by

$$\mathbf{E} = -\nabla V = -\mathbf{a}_\rho \frac{\partial V}{\partial \rho}$$
$$= -\mathbf{a}_\rho \frac{V_0}{\ln\frac{a}{b}}\frac{1}{\rho} = \mathbf{a}_\rho \frac{V_0}{\ln\frac{b}{a}}\frac{1}{\rho} \qquad (2\text{-}21)$$

The radial dependence of the electric field agrees with the result of an infinitely long cylinder of uniform charge density. The surface charge density on the surface of the inner conductor can be obtained by

$$\rho_S = \varepsilon E_n = \varepsilon E_\rho \Big|_{\rho=a} = \varepsilon \frac{V_0}{\ln\frac{b}{a}}\frac{1}{a} \qquad (2\text{-}22)$$

The capacitance of a cylindrical capacitor of length ℓ $(\ell \gg b)$ can also be calculated from Eq. (2-22) as follows:

$$C = \frac{Q}{V_0} = \frac{\rho_S \cdot (2\pi a l)}{V_0} = \frac{2\pi\varepsilon l}{\ln\frac{b}{a}} \qquad (2\text{-}23)$$

2.2.2 Uniqueness theorem

Laplace's equation itself alone does not determine V uniquely. In addition, a suitable set of boundary conditions must be supplied as shown in Examples 2-1 and 2-2. The general statement about the uniqueness of the solutions of Laplace's equation (or Poisson's equation) is as follows:

> The solution to Laplace's equation in some region is uniquely determined if the value of V or $\frac{\partial V}{\partial n}$ (normal derivative) is specified on all boundaries of the region (Figure 2-3).

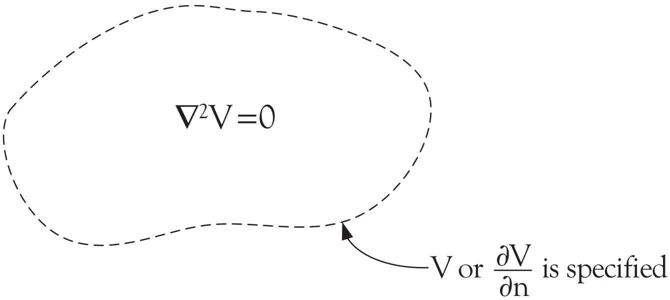


Figure 2-3. *Uniqueness theorem*

Due to this uniqueness theorem, a solution of an electrostatic problem satisfying the set of boundary conditions is *the only* possible solution. It does not matter how you find a solution. As long as the solution satisfies (i) Laplace's equation and (ii) the correct boundary conditions, it is the only correct solution. The uniqueness theorem is so powerful that even intelligent guessing may give a correct solution easily. The formal proof of this theorem can be found in, for example, Cheng (1989).*

We note from the statement of the uniqueness theorem that there is a choice of specifying boundary values: V or $\frac{\partial V}{\partial n}$. The problem for which the values of V are specified everywhere on the boundaries is called the *Dirichlet* boundary-value problem (BVP). The BVP for which the values of $\frac{\partial V}{\partial n}$ are specified everywhere on the boundaries is called the *Neumann* problem. There is also a mixed BVP for which V is specified on some boundaries and $\frac{\partial V}{\partial n}$ is specified on other remaining boundaries. In the following two sections, we will consider only the Dirichlet BVP's.

2.2.3 Marquis Pierre-Simon de Laplace (1749-1827)

Laplace was a French astronomer and mathematician. He spent most of his working life focusing on astronomy, particularly on the solar system, and the mathematical problems that arose from these studies. He was born Pierre-Simon Laplace into a middle class family in Normandy. The "Marquis" and the "de" were added by King Louis XVIII, who came to power in 1814 when the monarchy was restored after the French Revolution and the Napoleonic Empire.

* D. K. Cheng, *Field and Wave Electromagnetics*, Addison-Wesley, 1989, 2nd Ed., Chapter 4.

Laplace's father expected him to establish a career in the Church. He studied theology at the University of Caen for two years, and then, interrupting his plans, went off to Paris. His mathematical talents were quickly recognized in Parisian academic circles, and he obtained a professorship in mathematics at the Military school of Paris at the young age of 20.

Pierre was a young man in a hurry. He proceeded to flood the Academy of Science with four or five papers a year on planetary orbits, satellites, probability, difference equations, differential equations, etc. In this way, he quickly established himself as one of the brightest astronomers of his day and was elected to the Academy at the age of 24. Some of his early work was concerned with the stability of the solar system. Newton had felt that the solar system was unstable over time and that it would not be maintained without some (divine?) intervention. Laplace showed that, within some substantial restrictions, the solar system is, in fact,stable. He also refined the nebular hypothesis, which attributed the formation of the planets to gaseous nebulae from the equatorial rim of the sun that had collapsed and coalesced into these bodies. This was in stark contrast to the Comte de Buffon's hypothesis, which assumed that a comet striking the sun was the cause of planetary formation.

Laplace discovered his eponymous equation, $\nabla^2 V = 0$, while he was studying the gravitational potential and gravitational forces. This equation is also satisfied by the electrostatic potential, the magnetostatic potential, the temperature, the velocity potential of hydrodynamics, etc.

Laplace is known as the French Newton because of his many contributions to our understanding of the solar system. *Celestial Mechanics* is his best-known astronomical work. It contains sixteen books, the last published just two years before his death. In addition to his work on astronomy, he wrote a treatise on probability and is regarded as one of the founders of the discipline. Of course, he also laid the groundwork for the Laplace transform.

Laplace was a very compelling writer. Eventually he was elected to the French (literary) Academy. But he did have certain quirks and foibles. We are all familiar with the expression "It is easy to see …" which then requires several hours of work to fill in the missing proof. Certainly Laplace did not invent this trick, but he was an inveterate user, to the dismay of his many readers. He also failed often to give proper credit to other scientific contributors, thus clouding his own contributions. But, in contrast, he was very generous to neophytes. The story is told by Biot of how he once gave a talk at the Academy. Laplace had actually performed the same work many years before, but rather than embarrassing the young man in public, he showed his notebook to Biot in private and urged him to publish.

In his latter years, Laplace was elected to the French (literary) Academy as noted and became its president in 1817. He died in his seventy-eighth year in 1827. His last words were "we know so little; what we do not know is immense."

2.3 Method of Images

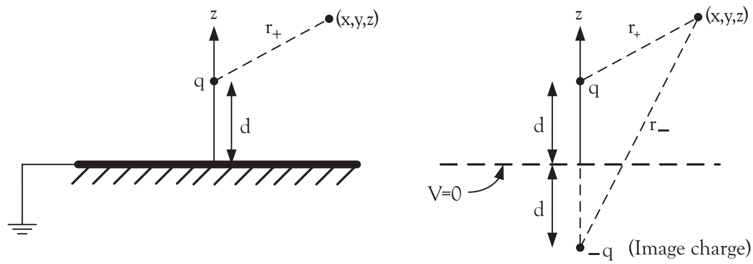
The method of images is very useful in handling the problems of charges in the vicinity of conductors. The idea is to replace the conductors by the so-called image charges to take into account the effect of conductors. For example, a power transmission line hanging above the conducting earth can be treated by the method of images.

2.3.1 A Point Charge Near A Grounded Conducting Plane

Let's consider the problem of a point charge q located a distance d above an infinite grounded conducting plane as shown in Figure 2-4(a). You are asked to find the electric potential everywhere above the plane. Before we solve this problem, think first what happens with respect to the conductor. Since the conductor has many free charges inside, the point charge q, (if positive), will attract negative free charges inside the conductor; thus negative charges will appear on the surface of the conducting plane. Then the total electric potential or the field will be due to the original point charge q and the induced surface charges $\rho_s(x,y)$, which will be a function of position (x,y) on the plane. So we can write the potential as follows:

$$V(x,y,z) = \frac{q}{4\pi\varepsilon_0\sqrt{x^2 + y^2 + (z-d)^2}} + \iint \frac{\rho_S(x',y')\,dx'dy'}{4\pi\varepsilon_0\sqrt{(x-x')^2 + (y-y')^2 + z^2}}$$

The trouble is that (i) ρ_s must be obtained first and (ii) the surface integral is very difficult to evaluate. In fact, ρ_s can be found only *after* the potential or the field is obtained. Therefore, we will seek other methods.



(a) Original problem. (b) Equivalent problem.

Figure 2-4. ***A point charge near a grounded conducting plane***

The problem of finding the electric potential V(x,y,z) for $z > 0$ (note that $V = 0$ everywhere for $z < 0$, i.e., for the region inside the conductor) can be formulated as follows:

V satisfies $\nabla^2 V = 0$ everywhere for $z > 0$ except at the point charge, subject to the boundary conditions:

(a) $V = 0$ when $z = 0$ (since the conducting plane is grounded)

(b) $V \to 0$ as $r = \sqrt{x^2 + y^2 + z^2} \to \infty$ (far from the charge)

(c) $V \to \dfrac{q}{4\pi\varepsilon_0 r_+}$ as $r_+ = \sqrt{x^2 + y^2 + (z-d)^2} \to 0$ (very close to the charge)

It appears very difficult to construct a solution for V that satisfies all these conditions. The method of images greatly simplifies the difficulty. Suppose we remove the conductor and replace it by another point charge − q at $z = -d$ as shown in Figure 2-4(b). This is a completely different problem, but it will give the correct solution that we seek. The solution of this new problem (or the *equivalent* problem) is very easy: the potential due to this charge configuration, i.e., a pair of point charges, is given by

$$V(x,y,z) = \frac{q}{4\pi\varepsilon_0 r_+} + \frac{(-q)}{4\pi\varepsilon_0 r_-}$$

$$= \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}}\right\} \qquad (2\text{-}24)$$

Now one can show that V given by Eq. (2-24) satisfies $\nabla^2 V = 0$ everywhere for z > 0 except at the point (0,0,d) and satisfies all the boundary conditions (a), (b) and (c) described above. By the uniqueness theorem, this is the only solution to the original problem for z > 0. The complete solution for the original problem is

$$V(x,y,z) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{q}{r_+} - \dfrac{q}{r_-}\right), & z \geq 0 \\ 0\,, & z \leq 0 \end{cases} \qquad (2\text{-}25)$$

We call the charge – q at (0,0,–d) the **image charge.** In summary, we find the solution of the original problem by solving the completely different *equivalent* problem for which

(i)the conductor (z < 0) is replaced by air (the same medium as above)
(ii)an image charge (– q) is placed at (0,0,–d).

The difference in solutions of the two problems—the original and equivalent problems—lies in the region z < 0, as shown in Eqs. (2-25) and (2-24). The latter is a complete solution for the equivalent problem everywhere including z < 0. We call the region including the point charge (z < 0) the *region of interest* and the conducting region (z < 0) the *region of no interest.* We note that the image charge must be placed outside the region of interest (i.e., inside the conductor) so as not to disturb the requirement of $\nabla^2 V = 0$ in that region.

Since the potential is obtained, we can now calculate the actual induced surface charge density at z = 0 by using Eq. (2-63).

$$\rho_S(x,y) = \varepsilon_0 E_n = -\varepsilon_0 \frac{\partial V}{\partial z}\Big|_{z=0} \qquad (2\text{-}26)$$

From Eq. (2-24)

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left\{\frac{-q(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{q(z+d)}{[x^2+y^2+(z-d)^2]^{3/2}}\right\}$$

So

$$\rho_S(x,y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}} \qquad (2\text{-}27)$$

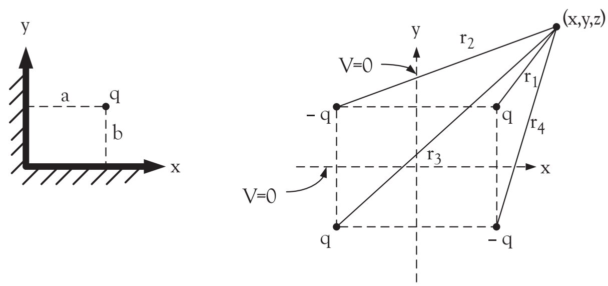
The induced charge is negative (assuming q is positive) as expected. The total induced charge

$$Q = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho_S(x,y)\, dx\, dy$$

is equal to the magnitude of the image charge, – q<ill/>

Example 2-3

A point charge q is in the vicinity of a corner where two grounded conducting plates meet at right angle (Figure 2-5(a)). Find the potential in the region $x > 0$, $y > 0$.



(a) Original problem. (b) Equivalent problem.

Figure 2-5. ***A point charge in a right-angle conducting corner***

Solution:

This problem can be solved by the method of images. The boundary conditions to be satisfied are:

$$V = 0 \text{ at } x = 0 \text{ (vertical plane)} \tag{2-28a}$$

$$V = 0 \text{ at } y = 0 \text{ (horizontal plane)} \tag{2-28b}$$

In order to satisfy Eq. (2-28a), we need an image charge $-q$ at the point $(-a, b)$. In addition, the boundary condition, Eq. (2-28b), requires two more image charges, $-q$ at $(a, -b)$ and $+q$ at $(-a, -b)$ so that each pair of $+q$ and $-q$ guarantee the zero potential on the horizontal plane. Therefore, the equivalent problem is to find the potential due to a group of four point charges as shown in Figure 2-5(b). Then the potential is given by

$$V(x,y) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{r_1} + \frac{(-q)}{r_2} + \frac{q}{r_3} + \frac{(-q)}{r_4}\right\} \tag{2-29}$$

2.3.2 A Line Charge and a Parallel Conducting Cylinder

Example 2-4

Consider an infinitely long line charge ρ_ℓ [C/m] located a distance d away from the axis of an infinitely long, parallel, conducting circular cylinder of radius a as shown in Figure 2-6(a). Find the electric potential everywhere.

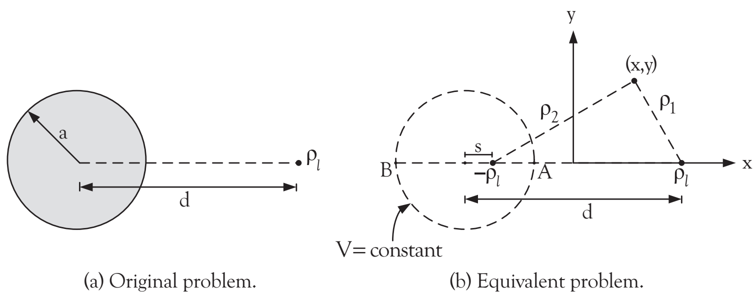


Figure 2-6. *Cross section of a line charge and a parallel conducting cylinder*

Solution:

We again solve this problem by the method of images. The boundary condition is V = constant on the surface of the conducting cylinder. The main task is to replace the conducting cylinder by an image charge ρ_ℓ'.

Let's first consider the potential of a pair of infinitely long, opposite line charges, ρ_ℓ and – ρ_ℓ (Figure 2-6(b)).

$$\begin{aligned} V &= \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{\rho_o}{\rho_1}\right) + \frac{(-\rho_l)}{2\pi\varepsilon_0}\ln\left(\frac{\rho_o}{\rho_2}\right) \\ &= \frac{\rho l}{2\pi\varepsilon_0}\left\{\ln\left(\frac{\rho_o}{\rho_1}\right) - \ln\left(\frac{\rho_o}{\rho_2}\right)\right\} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{\rho_2}{\rho_1}\right) \end{aligned} \tag{2-30}$$

where ρ_o is the reference point. Letting the (x, y) coordinates of ρ_ℓ and $-\rho_\ell$ be $(x_0,0)$ and $(-x_0,0)$, the equipotential surfaces – lines in the cross section – are given by

$$\frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{\rho_2}{\rho_1}\right) = \text{constant}, \tag{2-31a}$$

or

$$\frac{\rho_2}{\rho_1} = \frac{\sqrt{(x+x_o)^2+y^2}}{\sqrt{(x-x_o)^2+y^2}} = \text{constant} \equiv \sqrt{C} \tag{2-31b}$$

where $C \geq 0$. Squaring both sides of Eq. (2-31b), we obtain

$$(x + x_o)^2 + y^2 = C(x + x_o)^2 + Cy^2 \tag{2-32a}$$

$$(1 - C)x^2 + (1 - C)y^2 + 2(1 + C)x_o x + (1 - C)x_0^2 = 0 \tag{2-32b}$$

When $C \neq 1$, after algebraic manipulation, we have

$$\left\{x + \frac{1 + C}{1 - C} x_o\right\}^2 + y^2 = \left(\frac{2\sqrt{C}\; x_o}{1 - C}\right)^2 \tag{2-33a}$$

which is the equation of a circle of radius $\frac{2\sqrt{C}x_o}{1 - C}$, centered at $\left(-\frac{1 + C}{1 - C} x_o, 0\right)$. When $C = 1$, Eq. (2-32b) reduces to

$$4x_0 x = 0 \text{ or } x = 0 \tag{2-33b}$$

which is the equation of the plane that bisects the line connecting the two line charges. We conclude from Eqs. (2-33) that the equipotential lines (surfaces) are a family of circles (cylinders) including a bisecting plane.

Consider the original problem now. If the surface of the conducting cylinder coincides with one of these equipotential surfaces, then it will satisfy the boundary condition that V = constant. Therefore, we place the image line charge

$$\rho_1' = -\rho_1 \tag{2-34}$$

at a distance s away from the axis of the cylinder, on the line connecting ρ_l and ρ_l' as shown in Figure 2-6(b). Note that the image charge is on the right side of the axis of the cylinder because $\left(\frac{1 + C}{1 - C}\right) x_o > x_o$ or a family of circles on the left half plane ($0 < C < 1$). To determine the value of s, we consider the potential of two points A and B on the surface of the conductor for which it is simple to write the potential or the ratio of ρ_2 and ρ_1 as follows:

$$\left.\frac{\rho_2}{\rho_1}\right|_A = \frac{a - s}{d - a} = \left.\frac{\rho_2}{\rho_1}\right|_B = \frac{a + s}{d + a} = \text{constant} \tag{2-35a}$$

$$(a - s)(d + a) = (a + s)(d - a) \tag{2-35b}$$

$$ad + a^2 - sd - sa = ad - a^2 + sd - sa$$

Thus

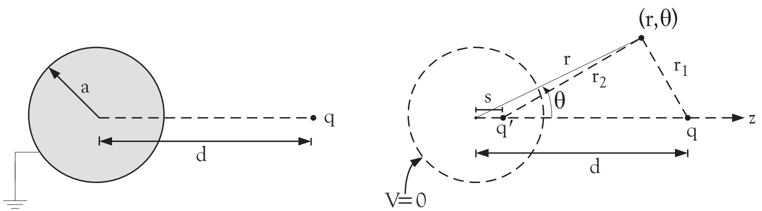
$$S = \frac{a^2}{d} \tag{2-36}$$

Knowing ρ_1' and s from Eqs. (2-34) and (2-36) and expressing ρ_1 and ρ_2 in terms of the coordinates (x,y), Eq. (2-30) gives the electric potential everywhere in the region outside the conducting cylinder (i.e. in the region of interest).

2.3.3 A Point Charge Near a Grounded Conducting Sphere

Example 2-5

Consider a point charge q located a distance d away from the center of a grounded conducting sphere of radius a as shown in Figure 2-7 (a). Find the electric potential everywhere.



(a) Original problem. (b) Equivalent problem.

Figure 2-7. ***A point charge near a grounded conducting sphere***

Solution:

We can solve this problem by the method of images. We first notice that the boundary condition is V = 0 on the surface of the conductor because it is grounded. As mentioned earlier, we replace the conducting sphere by an image charge q′ at an appropriate point inside the conductor region (region of no interest) such that the potential at the surface of the conductor is zero, i.e., V = 0 at r = a (Figure 2-7(b)). Because of symmetry, we place the image charge at a point on the line connecting the center of the sphere and the location of the original charge q. Let the line be the z axis and the distance of q′ from the center of the sphere be s. Now the question becomes, "Can we find the values of q′ and s that will satisfy the boundary condition, V = 0 at r = a ?" The potential at (r,θ) (spherical coordinates) due to q and q′ is given by

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1} + \frac{q'}{r_2}\right)$$
$$= \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{\sqrt{r^2 + d^2 - 2rd\cos\theta}} + \frac{q'}{\sqrt{r^2 + s^2 - 2rs\cos\theta_2}}\right\}$$
(2-37)

where the law of cosines is used.

It can be shown that if

$$q' = -\frac{a}{d},\ s = \frac{a^2}{d} \tag{2-38}$$

then $V(r,\theta)$ vanishes when $r = a$ for any arbitrary value of θ, which corresponds to all points on the surface of the conducting sphere (see Problem 2-18). Thus the boundary conditions for the original problem are satisfied. Then the potential for $r > a$ (in the region of interest) is given by Eq. (2-37) with q' and s as given in Eq. (2-38).

2.4 Method of Separation of Variables

The separation of variables is a typical technique for solving partial differential equations (PDE's). Laplace's equation is a PDE involving three variables. The key idea is to look for solutions that are products of functions each of which depends on only one variable. In the process of separation of variables, a PDE involving n variables is transformed into a set of n ordinary differential equations, each involving one variable; thus the PDE can be solved more readily. The method is applicable to boundary value problems for which the potential (V) or its normal derivative $\frac{\partial V}{\partial n}$ is specified on the boundaries of some region and the potential in the interior is to be found. We will deal with only two-dimensional problems for which the potential depends on only two coordinates: (x,y), (ρ,φ), and (r,θ). The method of separation of variables will also be used in solving the wave equation in Chapter 11.

2.4.1 Boundary Value Problems in Cartesian Coordinates

For an electric potential V(x,y) that depends only on x and y and is independent of z, i.e., $\left(\frac{\partial}{\partial z}\right) = 0$, Laplace's equation (2-5) becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad (2\text{-}39)$$

To achieve separation of variables, we assume the solution of Eq. (2-39) for V(x,y) in the following form:

$$V(x,y) = X(x)Y(y) \qquad (2\text{-}40)$$

where X(x) and Y(y) are functions of only x and y, respectively. Substituting Eq. (1-40) in Eq. (1-39), we have

$$Y(y)\frac{\partial^2 X(x)}{\partial x^2} + X(x)\frac{\partial^2 Y(y)}{\partial y^2} = 0 \qquad (2\text{-}41)$$

Dividing Eq. (2-41) by X(x) Y(y), we obtain

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = 0 \qquad (2\text{-}42a)$$

or

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2} \qquad (2\text{-}42b)$$

Note that the partial derivatives are replaced by the ordinary derivatives because X(x) and Y(y) are functions of one variable. In order for Eq. (2-42) to be satisfied for all values of x and y, the functions on both sides of Eq. (2-42b) must be a constant which is independent of x and y. Setting the constant equal to k^2, we have

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2} = k^2 \qquad (2\text{-}43)$$

which leads to two ordinary differential equations:

$$\frac{d^2 X(x)}{dx^2} - k^2 X(x) = 0 \qquad (2\text{-}44a)$$

$$\frac{d^2Y(y)}{dy^2} + k^2Y(y) = 0 \qquad \text{(2-44b)}$$

k is called the separation constant. We can choose k^2 or $-k^2$ in Eq. (2-43). Note that we have converted a PDE into two ordinary differential equations. Solutions to Eqs. (2-44) are well known as they are second-order ordinary differential equations with constant coefficients.

When $k \neq 0$, the general solutions of Eqs. (2-44a) and (2-44b) are, respectively,

$$X(x) = A_k \sinh kx + B_k \cosh kx \qquad \text{(2-45a)}$$

or

$$X(x) = A'_k e^{kx} + B'_k e^{-kx} \qquad \text{(2-45b)}$$

$$Y(y) = C_k \sin ky + D_k \cos ky \qquad \text{(2-46)}$$

The solutions for X(x) can be written as either Eq. (2-45a) or Eq. (2-45b) because the hyperbolic functions and the exponential functions are related as follows:

$$\sinh u = \frac{1}{2}(e^u - e^{-u}) \qquad \text{(2-47a)}$$

$$\cosh u = \frac{1}{2}(e^u + e^{-u}) \qquad \text{(2-47a)}$$

These functions are plotted in Figure 2-8. Finally, the general solutions for V(x,y) are given by

$$V(x,y) = (A_k \sinh kx + B_k \cosh kx)(C_k \sin ky + D_k \cos ky) \qquad \text{(2-48)}$$

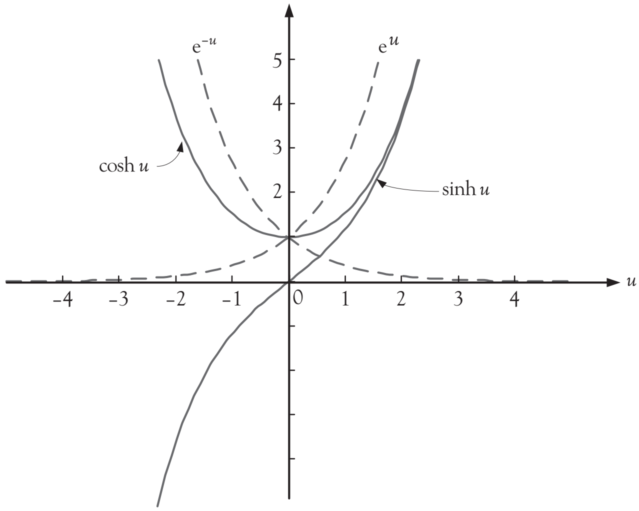


Figure 2-8. ***Hyperbolic and exponential functions***

There are another set of possible solutions when the separation constant $-k^2$ is chosen in Eq. (2-43). Then k^2 is replaced by $-k^2$ in Eqs. (2-44). It is clear that the new solutions are

$$V(x,y) = (A_k \sin kx + B_k \cos kx)(C_k \sinh ky + D_k \cosh ky) \tag{2-49}$$

The choice of appropriate separable solutions and the determination of coefficients and values of k depend on the specified boundary conditions of the problems. As seen in Eq. (2-48) and Eq. (2-49), one of the characteristics of the solutions to Laplace's equation is that *they are oscillatory or harmonic (sine and cosine) in one direction and are non-oscillatory in the other direction.*

When k = 0, solutions in Eqs. (2-48) and (2-49) are not valid. Letting k = 0 and solving Eqs. (2-44), we find

$$X(x) = A_o x + B_o \tag{2-50a}$$

$$Y(y) = C_o x + D_o \tag{2-50b}$$

Then

$$V(x,y) = (A_o x + B_o)(C_o x + D_o) \tag{2-51}$$

Solutions in the form of Eq. (2-50) were shown in Example 2-1. In summary, Eqs. (2-48), (2-49), and (2-51) constitute a complete set of separable solutions to two-dimensional Laplace's equation and they are listed in Table 2-1.

Example 2-6

Consider an infinitely long rectangular pipe as shown in Figure 2-9. Three metal strips at x = 0, x = a and y = 0 are grounded and the metal strip at y = b is insulated from other strips and maintained at a constant potential V_0. Find the potential distribution inside the pipe.

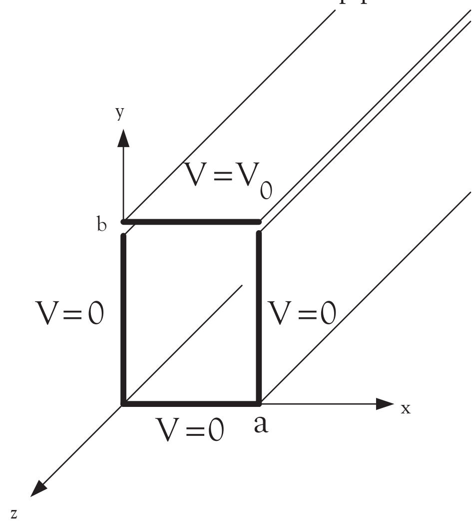


Figure 2-9. ***Boundary value problem for Example 2-6.***

Table 2-1 Solutions of Laplace's Equation in Two-Dimensions

I. Cartesian Coordinates, $\frac{\partial}{\partial z} = 0$ (independent of z):

(1) $k \neq 0$,

$V(x,y) = (A_k \sinh kx + B_k \cosh k_x)\,(C_k \sin ky + D_k \cos ky)$
or Replace sinh kx and cosh kx by e^{kx} and e^{-kx}
or Switch the x-dependence and the y-dependence ($x \leftrightarrow y$).

(2) $k = 0$,

$$V(x,y) = (A_o x + B_o)\,(C_o y + D_o)$$

II. Cylindrical Coordinates, $\frac{\partial}{\partial z} = 0$ (independent of z):

(1) $n \neq 0$,

$$V(\rho,\phi) = (A_n \rho^n + B_n \frac{1}{\rho^n})(C_n \sin n\phi + D_n \cos n\theta)$$

[Note] $\begin{cases} \rho\cos\phi \text{ or } \rho\sin\phi \ (n=1) \text{ gives a uniform field.} \\ \frac{1}{\rho}\cos\phi \text{ or } \frac{1}{\rho}\sin\phi (n=1) \text{ gives a line-dipole field.} \end{cases}$

(2) n = 0,

$V(\rho, \rho) = (A_o \ln \rho + B_o)(C_o \varphi + D_o)0$

[Note] In $\frac{\rho_o}{\rho}$ is a potential of an infinite line charge.

III. Spherical Coordinates, $\frac{\partial}{\partial\phi} = 0$ (independent of φ):

$$V(r, \theta) = (A_n r^n P_n(\cos\theta) + B_n \frac{1}{r^{n+1}} P_n(\cos\theta)$$

where

$P_o(\cos\theta) = 1,\ P_1(\cos\theta) = \cos\theta,\ P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$, etc. are the Legendre polynomials.

[Note]

(1) $\frac{1}{r}(n=0)$ is a potential of a point charge.

(2) $\frac{1}{r^2}\cos\theta\ (n=1)$ gives a dipole field.

(3) $r\cos\theta\ (n=1)$ gives a uniform field.

Solution:
First of all, because the pipe is infinitely long in the z-direction with uniform rectangular cross section, we expect that the potential will not depend on z, i.e. $\frac{\partial}{\partial z} = 0$. Since there is no charge inside the pipe, V(x,y) satisfies Laplace's equation subject to the following boundary conditions:

$$\text{(i)}\ V(x=0, y) = 0,\ 0 \le y \le b \quad (2\text{-}52)$$

$$\text{(ii)}\ V(x=a, y) = 0,\ 0 \le y \le b \quad (2\text{-}53)$$

$$\text{(iii)}\ \ V(x = y, 0) = 0,\ 0 \le x \le a \tag{2-54}$$

$$\text{(iv)}\ \ V(x = y, b) = V_o,\ 0 < x < a \tag{2-55}$$

We will solve this problem step by step.

Step 1. The first task is to choose between Eq. (2-48) and Eq. (2-49). To answer this question, we look at the boundary conditions and see in which direction V vanishes (or takes the same value) at two points (or more than once). In this problem, V vanishes at x =0 and x = a. So the potential is oscillatory in the x direction. Therefore, V(x,y) for this problem can be written as Eq. (2-49).

Step 2. Considering the combination of two solutions for X(x), the boundary condition (i) indicates that sine function is appropriate because it vanishes at the origin (x = 0):

$$X(x) = A\sin kx \tag{2-56}$$

Step 3. Considering the combination of two hyperbolic solutions for Y(y), the boundary condition (iii) indicates the sinh function is appropriate because sinh vanishes at the origin (y = 0):

$$Y(y) = C\sinh ky \tag{2-57}$$

Then we can write

$$V(x, y) = AC\sin kx \sinh ky \tag{2-58}$$

Step 4. Application of the boundary condition (ii) determines the acceptable values of k, the separation constant, as follows:

$$\sin ka\ \sinh ky = 0 \text{ for } 0 \le y \le b \tag{2-59}$$

i.e.,

$$\sin ka = 0$$

which is satisfied only if

$$ka = n\pi \text{ or } k = \frac{n\pi}{a},\ n = 0, 1, 2, 3, \ldots \tag{2-60}$$

For this particular set of values of k, Eq. (2-58) is a solution to $Ñ^2V = 0$ which satisfies the first three homogeneous boundary conditions. Then the general solution that satisfies (i), (ii) and (iii) can be written as a linear combination of solutions with all possible values of k:

$$V(x, y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}y \tag{2-61}$$

where the n = 0 term is not included because X(x) = 0 for n = 0.

Step 5. The final task is to determine the unknown arbitrary coefficients A_n, n = 1, 2, 3, … by using the last inhomogeneous boundary condition (iv). Application of (iv) in Eq. (2-61) yields

$$V(x, y = b) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}b = V_o,\ 0 < x < a \tag{2-62}$$

Rearranging coefficients,

$$\sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{a}x = V_o \equiv f(x),\ 0 < x < a \tag{2-63}$$

where

$$C_n = A_n \sinh\frac{n\pi}{a}b,\ n = 1, 2, 3, \ldots \tag{2-64}$$

We have an infinite number of unknowns (A_n or C_n) and one equation: How can we determine all these unknowns? The answer is the powerful technique known as *Fourier's method.* What we see in Eq. (2-63) is basically an expansion of the function f(x) = V_o in terms of a Fourier sine series in the interval 0 < x < a. The formal procedure is:

Multiply both sides of Eq. (2-63) by another harmonic function $\sin\frac{m\pi}{a}x$ and integrate over x from 0 to a.

We then obtain

$$\int_0^a \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{a}x \sin\frac{m\pi}{a}x\, dx = \int_0^a V_o \sin\frac{m\pi}{a}x\, dx \tag{2-65}$$

The integral on the right hand side (RHS) is easily evaluated:

$$\text{RHS} = \left[V_o \frac{(-a)}{m\pi} \cos\frac{m\pi}{a}x\right]_0^a = \begin{cases} \dfrac{2V_o a}{m\pi}, & \text{if m is odd} \\ 0, & \text{if m is even} \end{cases} \tag{2-66}$$

To evaluate the sum of integrals on the left hand side (LHS) of Eq. (2-65), we first evaluate the following integral:

$$\int_0^a \sin\frac{n\pi}{a}x \sin\frac{m\pi}{a}x\, dx = \frac{1}{2}\int_0^a \left\{\cos\frac{(n-m)\pi}{a}x - \cos\frac{(n+m)\pi}{a}x\right\}dx$$

$$= \begin{cases} \frac{1}{2}\left[\frac{a}{(n-m)\pi}\sin\frac{(n-m)\pi}{a}x - \frac{a}{(n-m)\pi}\sin\frac{(n+m)\pi}{a}x\right]_0^a = 0 \text{ if } n \neq m \\ \frac{1}{2}\left[x - \frac{a}{(2m)\pi}\sin\frac{(2m)\pi}{a}x\right]_0^a = \frac{a}{2} \text{ if } n = m \end{cases} \tag{2-67}$$

This particular integral is zero for all values of n except when n is equal to m. Such a property is known as the *orthogonality* of the Fourier sine and cosine functions. Using this result,

$$\begin{aligned} \text{LHS} &= \sum_{n=1}^{\infty} C_n \int_0^a \sin\frac{n\pi}{a}x \sin\frac{m\pi}{a}x\, dx \\ &= C_1 \cdot 0 + C_2 \cdot 0 + \cdots + C_m \cdot \frac{a}{2} + C_{m+1} \cdot 0 + \cdots \\ &= C_m \frac{a}{2} \end{aligned} \tag{2-68}$$

From Eqs. (2-66) and (2-68), we obtain

$$C_m \frac{a}{2} = \begin{cases} \frac{2V_o a}{m\pi}, \text{ if m is odd } (m = 1,3,5,...) \\ 0, \text{ if m is even } (m = 2,4,6,...) \end{cases} \tag{2-69}$$

Thus

$$C_n = A_n \sinh\frac{n\pi}{a}b = \begin{cases} \frac{4V_o}{n\pi}, & \text{if n is odd} \\ 0, & \text{if n is even} \end{cases}$$

i.e.,

$$A_n = \begin{cases} \frac{4V_o}{n\pi \sinh\frac{n\pi}{a}b}, & \text{if } n = 1,3,5... \\ 0, & \text{if } n = 2,4,6... \end{cases} \tag{2-70}$$

We replaced m by n since it is a dummy integer now. Substituting Eq. (2-70) into Eq. (2-61), we obtain the final answer as follows:

$$V(x, y) = \sum_{n=1,3,\ldots}^{\infty} \frac{4V_o}{n\pi \sinh \frac{n\pi}{a} b} \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} y \qquad (2\text{-}71)$$

Although the final solution is given in the form of an infinite series, we can find a good approximate potential by retaining the first few terms because the coefficient decreases rapidly as n increases. The equipotential lines and the **E** field lines are sketched in Figure 2-10.

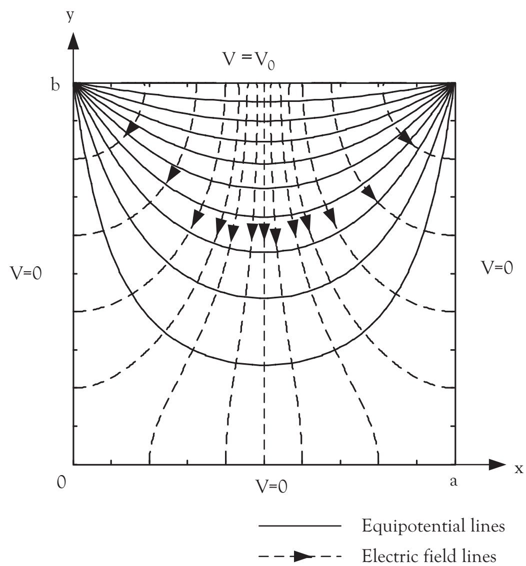


Figure 2-10. ***The equipotential lines and the electric field lines for the problem of Figure 2-9***

Before we close this example, it is interesting to visit Eq. (2-63) with the substitution of C_n:

$$f(x) = V_o = \sum_{\substack{n=1 \\ n=odd}}^{\infty} \frac{4V_o}{n\pi} \sin \frac{n\pi}{a} x, \quad 0 < x < a \qquad (2\text{-}72)$$

and see how the sum of the first few terms in this Fourier expansion constructs the constant function. Figure 2-11 shows the plot of the summation of n terms for different values of n. As n increases, the sum forms a better approximation to the function f(x).

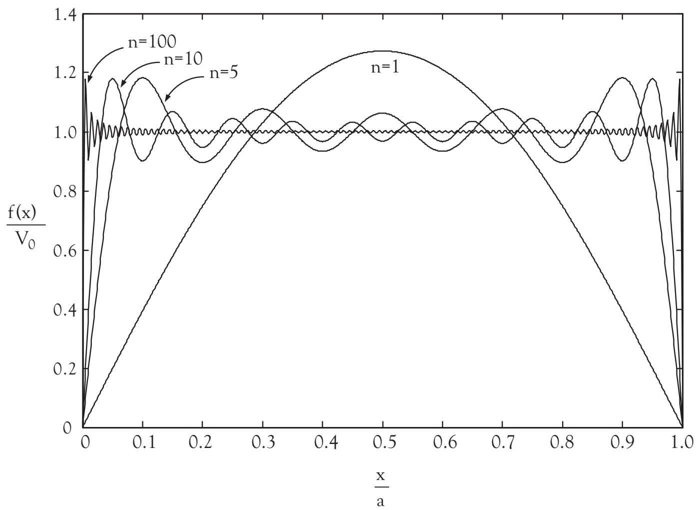


Figure 2-11. ***Plot of sum of n terms in the Fourier series of Eq. (2-72), (a) n = 1, (b) n = 5, (c) n = 10, (d) n = 100.***

Example 2-7

Consider two half-infinite parallel conducting plates separated by a distance b as shown in Figure 2-12. A very long conducting strip is placed between the plates at the left end and is insulated from the plates. The two plates are connected together and grounded, and a constant potential V_o is applied between the strip and the plates. Find the potential distribution in the region between the plates to the right of the strip.

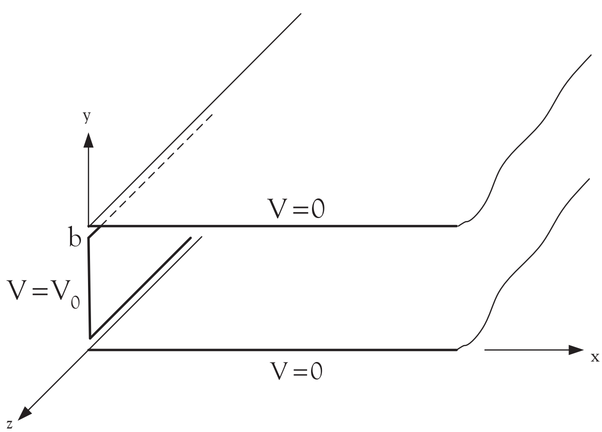


Figure 2-12. ***Boundary value problem for Example 2-7.***

Solution:

Setting the rectangular coordinate system as in Figure 2-12, the potential will be independent of z. Since there is no charge inside the region ($x > 0$, $0 \le y \le b$), $V(x,y)$ satisfies Laplace's equation subject to the following boundary conditions:

$$\text{(i)}\quad V(x, y = 0) = 0,\ x > 0 \tag{2-73a}$$

$$\text{(ii)}\quad V(x, y = b) = 0,\ x > 0 \tag{2-73b}$$

$$\text{(iii)}\quad V(x = 0, y) = V_o,\ 0 < y < b \tag{2-73c}$$

$$\text{(iv)}\quad V(x, y) \to 0,\ \text{as } x \to \infty \tag{2-73d}$$

Note that the last condition is included to guarantee physically acceptable solutions. First, since $V = 0$ at $y = 0$ and $y = b$, the potential will be oscillatory in the y direction. Thus $V(x,y)$ for this problem can be written as Eq. (2-48). Secondly, the boundary condition (i) implies

$$Y(y) = C\sin ky \tag{2-74}$$

Thirdly, in order to satisfy the boundary condition (iv), the choice of exponential function is more appropriate than that of the hyperbolic functions. From (2-45b) and (2-73d), we have

$$X(x) = B'e^{-kx} \tag{2-75}$$

Then we can write

$$V(x, y) = B'\,Ce^{-kx}\sin ky \tag{2-76}$$

Apply the boundary condition (ii),

$$e^{-kx}\sin kb = 0 \text{ or } \sin kb = 0 \tag{2-77}$$

which is satisfied only if

$$kb = n\pi \text{ or } k = \frac{n\pi}{b},\ n = 0,1,2,3,\ldots \tag{2-78}$$

Now the general solution that satisfies (i), (ii), and (iv) can be written as

$$V(x, y) = \sum_{n=1}^{\infty} A_n e^{-\frac{n\pi}{a}x}\sin\frac{n\pi}{a}y \tag{2-79}$$

Applying the remaining inhomogeneous boundary condition (iii),

$$V(x=0,\,y)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{b}y=V_o,\;0<y<b \qquad (2\text{-}80)$$

which is the same Fourier series as Eq. (2-63). Following the same procedures used in Example 2-6, we obtain, from Eq. (2-69),

$$A_n=\begin{cases}\dfrac{4V_o}{n\pi},\ \text{if n is odd}\\ 0,\ \text{if n is even}\end{cases} \qquad (2\text{-}81)$$

Therefore,

$$V(x,\,y)=\sum_{n=1,3,\ldots}^{\infty}\frac{4V_o}{n\pi}e^{-\frac{n\pi}{b}x}\sin\frac{n\pi}{b}y \qquad (2\text{-}82)$$

2.4.2 Boundary Value Problems in Cylindrical Coordinates

Laplace's equation in cylindrical coordinates is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial V}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial^2 V}{\partial z^2}=0 \qquad (2\text{-}13)$$

For the boundary value problems where the geometry is infinitely long in the z direction, V is independent of z, i.e., $\frac{\partial}{\partial z}$. In such cases, Laplace's equation becomes

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial V}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2}=0 \qquad (2\text{-}83)$$

To find the general solutions, we again use the method of separation of variables. We let

$$V(\rho,\phi)=R(\rho)\Phi(\phi) \qquad (2\text{-}84)$$

and substitute into Eq. (2-83):

$$\Phi(\phi)\frac{1}{\rho}\frac{\partial}{\partial\rho}\left\{\rho\frac{\partial R(\rho)}{\partial\rho}\right\}+\frac{R(\rho)}{\rho^2}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=0 \qquad (2\text{-}85)$$

We divide this equation by $\dfrac{R(\rho)\Phi\phi}{\rho^2}$ to obtain

$$\frac{\rho}{R\rho}\frac{\partial}{\partial\rho}\left\{\rho\frac{\partial R(\rho)}{\partial\rho}\right\}+\frac{1}{\Phi\phi}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=0 \tag{2-86a}$$

or

$$\frac{1}{R(\rho)}\rho\frac{d}{d\rho}\left\{\rho\frac{dR(\rho)}{d\rho}\right\}=-\frac{1}{\Phi\phi}\frac{d^2\Phi(\phi)}{d\phi^2} \tag{2-86b}$$

In order for Eq. (2-86) to be satisfied for all values of ρ and φ, the functions on both sides must be a constant. Letting the constant be n^2, we have

$$\frac{1}{R(\rho)}\rho\frac{d}{d\rho}\left\{\rho\frac{dR(\rho)}{d\rho}\right\}=-\frac{1}{\Phi\phi}\frac{d^2\Phi(\phi)}{d\phi^2}=n^2 \tag{2-87}$$

which leads to two ordinary differential equations:

$$\rho\frac{d}{d\rho}\left\{\rho\frac{dR(\rho)}{d\rho}\right\}-n^2R(\rho)=0 \tag{2-88a}$$

or

$$\rho^2\frac{d^2R(\rho)}{d\rho}+\rho\frac{dR(\rho)}{d\rho}-n^2R(\rho)=0 \tag{2-88b}$$

and

$$\frac{d^2\Phi(\phi)}{d\phi^2}+n^2\Phi(\phi)=0 \tag{2-89}$$

The solutions for Φ(φ) are simple: when n ≠ 0,

$$\Phi(\phi)=C_n\sin n\phi+D_n\cos n\phi(n\neq 0) \tag{2-90}$$

where n is an integer because Φ(φ) must be a single-valued function for 0 ≤ φ ≤ 2π, i.e., it should satisfy

$$\Phi(\phi+2\pi)=\Phi(\phi) \tag{2-91}$$

When φ goes through the full range from 0 to 2π, the function must join smoothly at φ = 2π. When n = 0, the solutions are

$$\Phi(\phi) = C_o\phi + D_o \;\; (n = 0) \tag{2-92}$$

where C_o and D_o are arbitrary constants.

Eq. (2-88) is known as the Euler-Cauchy equation.* When n ≠ 0, the two independent solutions are

$$R(\rho) = A_n\rho^n + B_n \frac{1}{\rho^n} \;\; (n \neq 0) \tag{2-93}$$

When n = 0, the solutions are

$$R(\rho) = A_o \ln \rho + B_o \;\; (n = 0) \tag{2-94}$$

Note that In $\frac{\rho_o}{\rho}$ is a potential of an infinite line charge as shown in Example 2-14.

Combining Eqs. (2-90), (2-92), (2-93) and (2-94), the most general solutions for V(ρ,φ) are given by

$$V(\rho,\phi) = (A_o \ln \rho + B_o)\,(C_o\phi + D_o) + \sum_{n=1}^{\infty}\left\{A_n\rho^n + \frac{B_n}{\rho^n}\right\}\left\{C_n \sin n\phi + D_n \cos n\phi\right\}$$

A complete set of separable solutions to two-dimensional Laplace's equation in (ρ,φ) are summarized in Table 2-1.

Example 2-8 A Conducting Cylinder in a Uniform Electric Field

An infinitely long uncharged cylindrical conductor of radius a is placed in an initially uniform electric field $\mathbf{E}_o = \mathbf{a}_x\mathbf{E}_o$. The axis of the cylinder is the z axis (Figure 2-13(a)). Find the electric field at points (ρ, φ) exterior to the cylinder.

Solution:

Since the conductor is a circular cylinder, it is natural to use cylindrical coordinates. Because the cylinder is infinitely long in the z direction, V will not depend on z, i.e., $\frac{\partial}{\partial z} = 0$. Thus Eq. (2-95) represents a solution of this problem. However, since the conductor is uncharged, i.e., the net charge is zero; we don't expect to have the term In ρ. Since V has to be

*See E. Kreyszig, *Advanced Engineering Mathematics,* John Wiley & Sons, 1993, 7th Ed., Section 2.6.

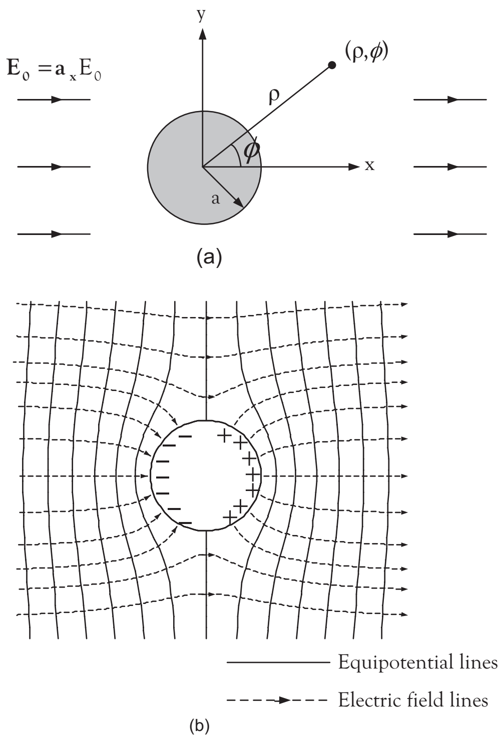


Figure 2-13. ***Cross section of a conducting cylinder in a uniform electric field.***

single-valued for $0 \le \varphi \le 2\pi$, $C_o\,\varphi + D_o$ is not an appropriate solution. Then the general solution for $V(\rho,\varphi)$ can be written as

$$V(\rho,\phi) = \sum_{n=1}^{\infty}\left\{A_n\rho^n + \frac{B_n}{\rho^n}\right\}\left\{C_n \sin n\phi + D_n \cos n\phi\right\} \quad (2\text{-}96)$$

Now the boundary conditions (B.C.'s) are:

(i) $$V(\rho = a,\ \phi) = \text{constant} = 0,\ 0 \le \phi \le 2\pi \quad (2\text{-}97)$$

(ii) $$\mathbf{E} = -\nabla V \rightarrow \mathbf{E}_o = a_x E_o = \nabla\{E_o x\},\ \text{as } \rho \rightarrow \infty \quad (2\text{-}98a)$$

$$V \rightarrow E_o x = -E_o \rho \cos\phi \ \text{as } \rho \rightarrow \infty \quad (2\text{-}98b)$$

One could set the potential of the conductor to be any other constant; only the reference potential will change. So we assume it to be zero. The boundary condition (ii) indicates that the solution in the form of sin nφ is not appropriate (it can not match the boundary condition). Thus we can rewrite the potential as follows:

$$V(\rho,\phi)=\sum_{n=1}^{\infty}\left\{A_n\rho^n\cos n\phi+\frac{B_n}{\rho^n}\cos n\phi\right\} \qquad (2\text{-}99)$$

where A_n and B_n are redefined. Applying the B.C. (ii),

$$V(\rho=\infty,\phi)=\sum_{n=1}^{\infty}A_n\rho^n\cos n\phi=-E_o\rho\cos n\phi,\ \ 0\le\phi\le 2\pi \qquad (2\text{-}100)$$

In order for this equation to be satisfied for all values of ρ and φ, it is necessary that

$$A_1=-E_o,\ A_2=A_3=\cdots=0 \qquad (2\text{-}101)$$

Then

$$V(\rho,\phi)=-E_o\rho\cos\phi+\sum_{n=1}^{\infty}\frac{B_n}{\rho^n}\cos n\phi \qquad (2\text{-}102)$$

Application of the B.C. (i) gives

$$V(\rho=a,\phi)=-E_o a\cos\phi+\sum_{n=1}^{\infty}\frac{B_n}{a^n}\cos n\phi=0 \qquad (2\text{-}103a)$$

or

$$\sum_{n=1}^{\infty}C_n\cos n\phi=E_o a\cos\phi\equiv g(\phi),\ 0\le\phi\le 2\pi \qquad (2\text{-}103b)$$

where

$$C_n=\frac{B_n}{a^n},\ \ n=1,2,3,\ldots \qquad (2\text{-}104)$$

Eq. (2-103b) is again the expansion of the function g(φ) = E_o a cos φ in terms of the Fourier cosine series. Multiplying both sides of Eq. (2-103b) by another harmonic function cos mφ and integrating over φ from 0 to 2π, we can show that

$$C_1=E_o a,\ C_2=C_3=\cdots=0 \qquad (2\text{-}105a)$$

or

$$B_1=C_1 a=E_o a^2,\ B_2=B_3=\cdots=0 \qquad (2\text{-}105b)$$

Substituting Eq. (2-105b) into Eq. (2-102), we finally obtain

$$V(\rho,\phi) = -E_o \rho \cos\phi + E_o \frac{a^2}{\rho}\cos\phi \qquad (2\text{-}106)$$

The electric field is given by

$$\begin{aligned}\mathbf{E} = -\nabla V &= -\mathbf{a}_\rho \frac{\partial V}{\partial \rho} - \mathbf{a}_\phi \frac{1}{\rho}\frac{\partial V}{\partial \rho} \\ &= E_o(\mathbf{a}_\rho \cos\phi - \mathbf{a}_\phi \sin\phi) + E_o \frac{a^2}{\rho^2}(\mathbf{a}_\rho \cos\phi + \mathbf{a}_\phi \sin\phi) \qquad (2\text{-}107) \\ &= \mathbf{a}_x E_o + E_o(\frac{a}{\rho})^2(\mathbf{a}_\rho \cos\phi + \mathbf{a}_\phi \sin\phi)\end{aligned}$$

We notice that the first term is the initially applied uniform field and the second term is the additional field due to the *induced charge* on the surface of the conductor. The $\frac{1}{\rho^2}$ dependence of the latter indicates that it comes from the *dipole* configuration of the line charges. This can be easily confirmed by calculating the surface charge density at $\rho = a$:

$$\rho_s = \varepsilon_o E_n = \varepsilon_o E_\rho \Big|_{\rho=a} = 2\varepsilon_o E_o \cos\phi,\ 0 \le \phi \le 2\pi \qquad (2\text{-}108)$$

Note that $\rho_s > 0$ for the right-half portion of the cylinder surface and $\rho_s < 0$ for the left-half portion. It is a *line dipole.* The field lines and the equipotential lines are plotted in Figure 2-13(b).

2.4.3 Boundary Value Problems in Spherical Coordinates

Laplace's equation in spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial V}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial V}{\partial \theta}) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

(2-109)

For the boundary value problems where the potential is independent of φ (*azimuthally symmetric*), i.e., $\frac{\partial}{\partial \phi} = 0$, Laplace's equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial V}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial V}{\partial \theta}) = 0 \qquad (2\text{-}110)$$

Applying the method of separation of variables, we assume

$$V(r,\theta) = R(r)\Theta(\theta) \qquad (2\text{-}111)$$

Following the procedures similar to those in the previous two sections, we obtain

$$\frac{1}{R(r)}\frac{d}{dr}\left\{r^2\frac{dR(r)}{dr}\right\} = -\frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{d\Theta(\theta)}{d\theta}\right\} \qquad (2\text{-}112)$$

If Eq. (2-112) is to be satisfied for all values of r and θ, the functions on both sides must be a constant. Letting the constant be k, we obtain

$$\frac{d}{dr}\left\{r^2\frac{dR(r)}{dr}\right\} - k\,R(r) = 0 \qquad (2\text{-}113)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{d\Theta(\theta)}{d\theta}\right\} + k\Theta(\theta) = 0 \qquad (2\text{-}114)$$

There exist solutions for $\Theta(\theta)$ that are physically acceptable for $0 \le \theta \le \pi$, when

$$k = n(n+1),\ n = 0,\ 1,\ 2,\ .... \qquad (2\text{-}115)$$

The solutions are called the **Legendre polynomials***

$$\Theta(\theta) = P_n(\cos\theta) \qquad (2\text{-}116)$$

where $P_n(x)$ is defined by

$$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2 - 1)^n \qquad (2\text{-}117)$$

The first few Legendre polynomials are listed in Table 2-2.

* R. V. Churchill and J. W. Brown, *Fourier Series and Boundary Value Problems,* Mc-Graw-Hill, 1987, 4^{th} Ed., Chapter 9.

Table 2-2 Legendre Polynomials

$$P_0(\cos\theta) = 1$$
$$P_1(\cos\theta) = \cos\theta$$
$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$
$$P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

Substituting Eq. (2-115) into Eq. (2-113), we find the following two independent solutions for R(r):

$$R(r) = A_n r^n + \frac{B_n}{r^{n+1}} \quad (2\text{-}118)$$

Combining Eqs. (2-116) and (2-118), the general solutions for V(r,θ) can be written as

$$V(r,\theta) = \sum_{n=0}^{\infty}\left\{A_n r^n + \frac{B_n}{r^{n+1}}\right\}P_n(\cos\theta) \quad (2\text{-}119)$$

Note that we have seen two of the second solutions with n = 0 and n = 1 before:

$$\frac{B_o}{r}(n = 0), \ \frac{B_1}{r^2}\cos\theta(n = 1)$$

They are the potentials of a single point charge and an electric dipole, respectively. The set of separable solutions to two-dimensional Laplace's equation in (r,θ) are summarized in Table 2-1. The following example illustrates the application of these solutions.

Example 2-9 A Dielectric Sphere in a Uniform Electric Field
An uncharged dielectric sphere of radius a and permittivity ε is placed in an initially uniform electric field $\mathbf{E}_o = \mathbf{a}_z E_o$ as shown in Fig. 2-14(a). Find the electric field everywhere.

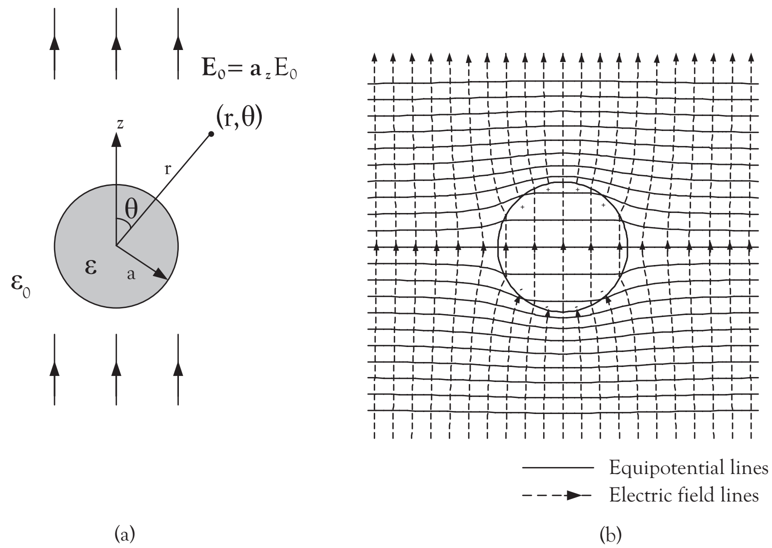


Figure 2-14. *A dielectric sphere in a uniform electric field.*

Solution:

This problem is very similar to that of Example 2-8 except that the object is a dielectric and a sphere. Thus the electric field is not zero inside the sphere and the potential is not a constant on the surface of the sphere. Since the applied electric field is uniform in the z direction, the problem has azimuthal symmetry; the potential will be independent of φ. Thus Eq. (2-119) represents a general solution of this problem. We write the potential inside and outside the sphere as follows:

$$V_1(r,\theta) = V_{app} + V_{ind,\,1},\ r \le a \tag{2-120a}$$

$$V_2(r,\theta) = V_{app} + V_{ind,\,2},\ r \ge a \tag{2-120b}$$

where V_{app} is the potential due to an initially applied uniform field and V_{ind} is the *induced* potential due to the presence of dielectric. V_{app} can be easily obtained from

$$\mathbf{E}_o = \mathbf{a}_z E_o = -\nabla V_{app} = -\nabla\{-E_o z\} \quad (2\text{-}121)$$

Thus

$$V_{app} = -E_o z = -E_o r\cos\theta = -E_o\ r\,P_1(\cos\theta) \quad (2\text{-}122)$$

Since V_{ind} must be solutions of $\nabla^2 V = 0$ and it should be finite at the origin (r = 0) and at infinity (r→∞), V_{ind} can be formulated as

$$V_{ind,1} = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta),\ r \le a \tag{2-123a}$$

$$V_{ind,2} = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta),\ r \ge a \tag{2-123b}$$

Note that the solutions $\frac{1}{r^{n+1}} P_1 = (\cos\theta)$ are not appropriate in the region r ≤ a because they blow up at the origin (r = 0). The solutions $r^n\ P_n(\cos\theta)$ are not appropriate in the region r ≥ a because they blow up at infinity ($r \rightarrow \infty$). The unknown coefficients A_n, B_n can be determined by applying the boundary conditions at the dielectric interface (r = a) (see Table 3-1):

$$E_{1t} = E_{2t}\ \text{ or }\ V_1 = V_2 \tag{2-124a}$$

$$D_{1n} = D_{2n}\ \text{ or }\ \varepsilon \frac{\partial V_1}{\partial r} = \varepsilon_0 \frac{\partial V_2}{\partial r} \tag{2-124b}$$

for all values of θ. The θ-dependence of V_{app} in Eq. (2-122) and the boundary conditions suggest that we need only the term with n = 1. Therefore, we rewrite V(r,θ) as follows:

$$V_1(r,\theta) = -E_o r\ \cos\theta + A_1 r\ \cos\theta \equiv A\ r\ \cos\theta,\ r \le a \tag{2-125a}$$

$$V_2(r,\theta) = -E_o r\ \cos\theta + \frac{B_1}{r^2}\ \cos\theta,\ r \ge a \tag{2-125b}$$

Applying the two boundary conditions, Eqs. (2-124), at r = a,

$$A\, a\cos\theta = -E_o a\cos\theta\ + \frac{B_1}{a^2}\cos\theta \tag{2-126a}$$

$$\varepsilon A\ \cos\theta = \varepsilon_0 \left\{ -E_o - \frac{2B_1}{a^3} \right\} \cos\theta \tag{2-126b}$$

Solving these equations for A and B_1, we obtain

$$A = -\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_o,\ B_1 = -\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} E_o a^3 \tag{2-127}$$

Substituting Eq. (2-127) into Eq. (2-125),

$$V_1(r,\theta) = -\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_o r\ \cos\theta = -\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_o\ z,\ r \le a \tag{2-128}$$

$$V_2(r, \theta) = -E_o r\cos\theta = +\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} E_o \frac{a^3}{r^2}\cos\theta, \ r \le a \qquad (2\text{-}129)$$

The electric fields are given by

$$\begin{aligned} \mathbf{E}_1 &= -\nabla V_1 = -\mathbf{a}_z \frac{\partial V_1}{\partial z} = \mathbf{a}_z \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_o, \ r \le a \\ &= \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_o (\mathbf{a}_r \cos\theta - \mathbf{a}_\theta \sin\theta) \end{aligned} \qquad (2\text{-}130)$$

$$\begin{aligned} \mathbf{E}_2 &= -\nabla V_2 = -\mathbf{a}_r \frac{\partial V_2}{\partial r} - \mathbf{a}_\theta \frac{1}{r}\frac{\partial V_2}{\partial r} \\ &= \mathbf{a}_z E_o + \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} E_o \left(\frac{a}{r}\right)^3 (\mathbf{a}_r \cos\theta - \mathbf{a}_\theta \sin\theta) \end{aligned} \qquad (2\text{-}131a)$$

$$\begin{aligned} &= \mathbf{a}_\theta E_o \left\{1 + \frac{2(\varepsilon - \varepsilon_0)}{\varepsilon + 2\varepsilon_0}\left(\frac{a}{r}\right)^3\right\}\cos\theta \\ &\quad -\mathbf{a}_\theta E_o \left\{1 - \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\left(\frac{a}{r}\right)^3\right\}\sin\theta \ , \ r \ge a \end{aligned} \qquad (2\text{-}131b)$$

It is interesting to note that the field inside the dielectric sphere is *uniform* at all points as seen in Eq. (2-130). The field outside the dielectric sphere shown in Eq. (2-131a) consists of two parts: the first term is the initially applied uniform field and the second term is the additional field due to the *induced bound charge* on the surface of the dielectric sphere. The latter is a *dipole* field. In fact, the induced surface bound charge ρ_{ps} at r = a has a dipole nature because from Eq.

$$\begin{aligned} \rho_{\rho s} &= \mathbf{P}\cdot\mathbf{a}_n = (\mathbf{D}_1 - \varepsilon_0 \mathbf{E}_1)\cdot\mathbf{a}_r = (\varepsilon - \varepsilon_0)E_{1r}\Big|_{r=a} \\ &= \frac{3\,\varepsilon_0(\varepsilon - \varepsilon_0)}{\varepsilon + 2\varepsilon_0} E_o \cos\theta \ , \ 0 \le \theta \le \pi \end{aligned} \qquad (2\text{-}132)$$

Note that $\rho_{ps} > 0$ for $0 < \theta < \pi/2$ and $\rho_{ps} < 0$ for $\pi/2 < \theta < \pi$. The field lines and the equipotential lines are plotted in Figure 2-14(b).

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Principles of Electromagnetics 6—Radiation and Antennas

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