Principles of Electromagnetics 5— Wave Applications

Arlon T. Adams
Jay K. Lee

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Preface

Electromagnetics is not an easy subject for students. The subject presents a number of challenges, such as: new math, new physics, new geometry, new insights and difficult problems. As a result, every aspect needs to be presented to students carefully, with thorough mathematics and strong physical insights and even alternative ways of viewing and formulating the subject. The theoretician James Clerk Maxwell and the experimentalist Michael Faraday, both shown on the cover, had high respect for physical insights.

This book is written primarily as a text for an undergraduate course in electromagnetics, taken by junior and senior engineering and physics students. The book can also serve as a text for beginning graduate courses by including advanced subjects and problems. The book has been thoroughly class-tested for many years for a two-semester Electromagnetics course at Syracuse University for electrical engineering and physics students. It could also be used for a one-semester course, covering up through Chapter 8 and perhaps skipping Chapter 4 and some other parts. For a one-semester course with more emphasis on waves, the instructor could briefly cover basic materials from statics (mainly Chapters 2 and 6) and then cover Chapters 8 through 12.

The authors have attempted to explain the difficult concepts of electromagnetic theory in a way that students can readily understand and follow, without omitting the important details critical to a solid understanding of a subject. We have included a large number of examples, summary tables, alternative formulations, whenever possible, and homework problems. The examples explain the basic approach, leading the students step by step, slowly at first, to the conclusion. Then special cases and limiting cases are examined to draw out analogies, physical insights and their interpretation. Finally, a very extensive set of problems enables the instructor to teach the course for several years without repeating problem assignments. Answers to selected problems at the end allow students to check if their answers are correct.

During our years of teaching electromagnetics, we became interested in its historical aspects and found it useful and instructive to introduce stories of the basic discoveries into the classroom. We have included short biographical sketches of some of the leading figures of electromagnetics, including Josiah Willard Gibbs, Charles Augustin Coulomb, Benjamin Franklin, Pierre Simon de Laplace, Georg Simon Ohm, Andre Marie Ampère, Joseph Henry, Michael Faraday, and James Clerk Maxwell.

The text incorporates some unique features that include:

- Coordinate transformations in 2D (Figures 1-11, 1-12).
- Summary tables, such as Table 2-1, 4-1, 6-1, 10-1.
- Repeated use of equivalent forms with R (conceptual) and $|r-r'|$ (mathematical) for the distance between the source point and the field point as in Eqs. (2-27), (2-46), (6-18), (6-19), (12-21).
- Intuitive derivation of equivalent bound charges from polarization sources, including piecewise approximation to non-uniform polarization (Section 3.3).
- Self-field (Section 3.8).
- Concept of the equivalent problem in the method of images (Section 4.3).
- Intuitive derivation of equivalent bound currents from magnetization sources, including piecewise approximation to non-uniform magnetization (Section 7.3).
- Thorough treatment of Faraday's law and experiments (Sections 8.3, 8.4).
- Uniform plane waves propagating in arbitrary direction (Section 9.4.1).
- Treatment of total internal reflection (Section 10.4).
- Transmission line equations from field theory (Section 11.7.2).
- Presentation of the retarded potential formulation in Chapter 12.
- Interpretation of the Hertzian dipole fields (Section 12.3).

Finally, we would like to acknowledge all those who contributed to the textbook. First of all, we would like to thank all of the undergraduate

and graduate students, too numerous to mention, whose comments and suggestions have proven invaluable. As well, one million thanks go to Ms. Brenda Flowers for typing the entire manuscript and making corrections numerous times. We also wish to express our gratitude to Dr. Eunseok Park, Professor Tae Hoon Yoo, Dr. Gokhan Aydin, and Mr. Walid M. G. Dyab for drawing figures and plotting curves, and to Professor Mahmoud El Sabbagh for reviewing the manuscript. Thanks go to the University of Poitiers, France and Seoul National University, Korea where an office and academic facilities were provided to Professor Adams and Professor Lee, respectively, during their sabbatical years. Thanks especially to Syracuse University where we taught for a total of over 50 years. Comments and suggestions from readers would be most welcome.

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CHAPTER 1

Introduction to Reflection and Transmission of Electromagnetic Waves

1.1 Introduction

In Volume 4 we considered a wave propagating in a single unbounded medium which can be lossless or lossy, along with its propagation characteristics, dispersion, and polarization. We will now consider what happens when an electromagnetic wave hits the boundary of another medium. We expect that a part of the wave will be reflected from the surface and a part of it will transmit through the second medium. Thus we study the reflection and transmission of the wave at a boundary separating two different media. In each medium the fields satisfy Maxwell's equations with appropriate medium parameters or constitutive relations. At the interface, the fields on both sides must satisfy the boundary conditions that are given by Eq. volume 4 (1-27). Depending on the types of the media, the boundary conditions may take different forms. Here we solve the boundary value problems for time-varying fields in bounded media. First we consider the problem of a wave incident normally upon a dielectric interface, which is simpler to solve. Then we solve the problem of oblique incidence with two different distinct polarizations—perpendicular and parallel. The interpretation of the results on the solutions for the fields is given. Interesting phenomena such as total internal reflection and Brewster's angle effect are discussed. Lastly, the reflection (and no transmission) of waves from a perfectly conducting surface is studied along with the concept of standing waves.

1.2 Normal Incidence at a Dielectric Boundary

Let's consider a uniform plane wave, propagating in the +x direction in medium 1 having material constants ε_1, μ_1. The wave is incident normally upon medium 2 having material constants ε_2, μ_2. The interface is at x = 0 (yz plane) as shown in Figure 1-1. We assume now that both media are lossless dielectrics ($\sigma_1 = \sigma_2 = 0$), but each of them can be lossy in our formulation, in which case ε_1 and/or ε_2 can be simply replaced by complex effective permittivities, making our results still valid for lossy media.

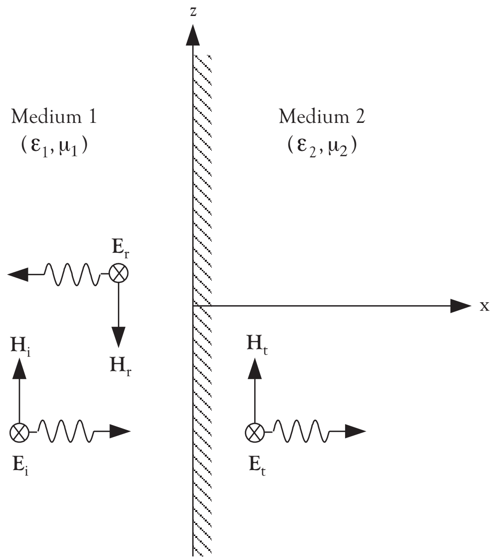


Figure 1-1. ***Reflection and transmission of waves at normal incidence***

Assuming the incident wave is linearly polarized in the y direction (we can assume z-polarization, which will give the same result), since the wave propagates in the +x direction, its electric and magnetic fields can be written as

$$\mathbf{E}_i = \mathbf{a}_y E_0 e^{-jk_1 x} \tag{1-1a}$$

$$\mathbf{H}_i = \frac{1}{\eta_1}\mathbf{a}_x \times \mathbf{E}_i = \mathbf{a}_z \frac{E_0}{\eta_1} e^{-jk_1 x} \tag{1-1b}$$

where

$$k_1 = \omega\sqrt{\mu_1 \varepsilon_1} \tag{1-2a}$$

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} \tag{1-2b}$$

k_1 is the propagation constant of the incident wave and η_1 the intrinsic impedance of medium 1. We have used the property of the uniform plane wave in deriving Eq. (1-1b), instead of using Ampère's law. The presence of the boundary gives rise to a reflected wave in medium 1 and a transmitted wave in medium 2. Since the reflected wave travels in the –x direction in medium 1, its electric and magnetic fields can be written in the following form:

$$\mathbf{E}_r = \mathbf{a}_y E_r e^{+jk_1x} \equiv \mathbf{a}_y R E_0 e^{jk_1x} \tag{1-3a}$$

$$\mathbf{H}_r = \frac{1}{\eta_1}(-\mathbf{a}_x) \times \mathbf{E}_r = -\mathbf{a}_z \frac{RE_0}{\eta_1} e^{jk_1x} \tag{1-3b}$$

where

$$R \equiv \frac{E_r}{E_0} \tag{1-4}$$

is the ratio of **E** field amplitudes of the reflected vs. incident waves, called the **reflection coefficient.** Note that the reflected wave has the same k and η as the incident wave but its spatial dependence is changed to $\exp(+jk_1x)$. The transmitted wave propagates in +x direction in medium 2 and its fields are written as

$$\mathbf{E}_t = \mathbf{a}_y E_t e^{-jk_2x} \equiv \mathbf{a}_y T E_0 e^{-jk_2x} \tag{1-5a}$$

$$\mathbf{H}_t = \frac{1}{\eta_2}\mathbf{a}_x \times \mathbf{E}_t = -\mathbf{a}_z \frac{TE_0}{\eta_2} e^{-jk_2x} \tag{1-5b}$$

where

$$k_2 = \omega\sqrt{\mu_2\varepsilon_2} \tag{1-6a}$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} \tag{1-6b}$$

k_2 is the propagation constant of the transmitted wave and η_2 is the intrinsic impedance of medium 2.

$$T \equiv \frac{E_t}{E_0} \tag{1-7}$$

is the ratio of **E** field amplitudes of the transmitted vs. incident waves, called the **transmission coefficient.**

All the parameters are known except the two coefficients, R and T. They are determined by applying the boundary conditions for the fields at the boundary (x = 0). Although we have four boundary conditions in Eq. volume 4 (1-27), we only need to use two of them on tangential components, i.e., Eqs. volume 4 (1-27a) and (1-27b). $\mathbf{E}_1$, $\mathbf{H}_1$ consist of the incident and reflected fields and $\mathbf{E}_2$, $\mathbf{H}_2$ have only transmitted fields. Since medium 2 is not a perfect conductor for the given problem, there exists no surface current at the boundary, i.e., $\mathbf{J}_s = 0$. Thus we have the following two boundary conditions at the dielectric interface:

$$\mathbf{E}_{1t} = \mathbf{E}_{2t} \tag{1-8a}$$

$$\mathbf{H}_{1t} = \mathbf{H}_{2t} \tag{1-8b}$$

where the subscript "t" means the tangential component.

For this problem, we have

$$\mathbf{E}_i + \mathbf{E}_r = \mathbf{E}_t \text{ at } x = 0 \tag{1-9a}$$

$$\mathbf{H}_i + \mathbf{H}_r = \mathbf{H}_t \text{ at } x = 0 \tag{1-9b}$$

Substituting x = 0 in Eqs. (1-1), (1-3), (1-5) and applying Eq. (1-9), we obtain

$$1 + R = T \tag{1-10a}$$

$$\frac{1}{\eta_1}(1 - R) = \frac{1}{\eta_2} T \tag{1-10b}$$

Solving Eq. (1-10) for R and T, we obtain

$$R = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \text{ (Reflection coefficient for normal incidence)} \tag{1-11a}$$

$$T = \frac{2\eta_2}{\eta_2 + \eta_1} \text{ (Transmission coefficient for normal incidence)} \tag{1-11b}$$

Note that $|R| \leq 1$, although the magnitude of T may exceed 1. R and T are real when both media are lossless, and they become complex when either one of the two is lossy. As will be seen later in Section 2.7, the expression of R is very similar to that for the reflection coefficient of the voltage wave along the transmission line.

Let's calculate the Poynting vectors of the incident, reflected and transmitted waves and find the relationships regarding the power conservation. The time-average Poynting vectors are given by (assuming lossless media)

$$\mathbf{S}_{av,i} = \frac{1}{2}\mathrm{Re}\left(\mathbf{E}_i \times \mathbf{H}_i^*\right) = \mathbf{a}_x \frac{|E_0|^2}{2\eta_1} \tag{1-12a}$$

$$\mathbf{S}_{av,r} = \frac{1}{2}\mathrm{Re}\left(\mathbf{E}_r \times \mathbf{H}_r^*\right) = -\mathbf{a}_x |R|^2 \frac{|E_0|^2}{2\eta_1} \tag{1-12b}$$

$$\mathbf{S}_{av,t} = \frac{1}{2}\mathrm{Re}\left(\mathbf{E}_t \times \mathbf{H}_t^*\right) = \mathbf{a}_x |T|^2 \frac{|E_0|^2}{2\eta_2} \tag{1-12c}$$

According to the conservation of power, it should hold that

$$P_i = P_r + P_t \tag{1-13a}$$

or

$$\left|\mathbf{S}_{av,i}\right| = \left|\mathbf{S}_{av,r}\right| = \left|\mathbf{S}_{av,t}\right| \tag{1-13b}$$

where P_i, P_r, P_t are the powers of the incident, reflected and transmitted waves, respectively. Eq. (1-13b) reduces to

$$1 = |R|^2 + \frac{\eta_1}{\eta_2}|T|^2 \tag{1-14}$$

Eq. (1-14) can be easily verified by substituting Eq. (1-11). The fraction or percentage of power reflected from the boundary is given by

$$\frac{P_r}{P_i} = \frac{\left|\mathbf{S}_{av,r}\right|}{\left|\mathbf{S}_{av,i}\right|} = |R|^2 \tag{1-15}$$

and the fraction of power transmitted through medium 2 is given by

$$\frac{P_t}{P_i} = \frac{\left|\mathbf{S}_{av,t}\right|}{\left|\mathbf{S}_{av,i}\right|} = |T|^2 \frac{\eta_1}{\eta_2} = 1 - |R|^2 \tag{1-16}$$

Note that the fraction of power transmitted is not equal to $|T|^2$.

Example 1-1.

An electromagnetic plane wave of 1 MHz is incident normally on Onondaga Lake. Assume that the lake water has $\varepsilon = 81\,\varepsilon_0$, $\mu = \mu_0$, and $\sigma = 10^{-4}$ [S/m].

(a) Calculate the reflection and transmission coefficients of the incident wave, ignoring the loss ($\sigma = 0$).

(b) Calculate the reflection and transmission coefficients, including the loss ($\sigma = 10^{-4}$). Write down the complete expression for the electric field of the transmitted wave.

Solutions:

(a) When the wave is incident from air upon the lake, η_1 and η_2 are given by

$$\eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \eta_0, \ \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \sqrt{\frac{\mu_2}{81\varepsilon_2}} = \frac{1}{9}\eta_0$$

R and T are given by Eq. (1-11):

$$R = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\frac{1}{9}\eta_0 - \eta_0}{\frac{1}{9}\eta_0 + \eta_0} = \frac{\frac{1}{9} - 1}{\frac{1}{9} + 1} = -0.8$$

$$T = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2 \cdot \frac{1}{9}\eta_0}{\frac{1}{9}\eta_0 + \eta_0} = \frac{\frac{2}{9}}{\frac{1}{9} + 1} = 0.2$$

Note that R and T satisfy the relationship 1 + R = T.

(b) When the lake is lossy, η_2 becomes complex and

$$\breve{\eta}_2 = \sqrt{\frac{\mu_2}{\varepsilon_{2,\text{eff}}}} = \sqrt{\frac{\mu_0}{81\varepsilon_0 - j\frac{\sigma}{\omega}}}$$

At

$$f = 1 \text{ MHz, L.T.} = \frac{\sigma}{\omega\varepsilon} = \frac{10^{-4}}{2\pi \cdot 10^6 \times 81 \times \frac{36\pi}{10^{-9}}} = 0.022 \ll 1$$

The lake is *slightly lossy* at f = 1 MHz, so $\frac{\sigma}{\omega} \ll \varepsilon = 81\,\varepsilon_0$ and $\breve{\eta}_2 \approx \sqrt{\frac{\mu_2}{81\varepsilon_0}} = \frac{1}{9}\eta_0$, which is approximately the same as η_2 when the lake is lossless. Thus,

$$R = \frac{\breve{\eta}_2 - \eta_1}{\breve{\eta}_2 + \eta_1} \approx -0.8, \;\; T = \frac{2\breve{\eta}_2}{\breve{\eta}_2 + \eta_1} \approx 0.2$$

You can calculate $\breve{\eta}_2$ exactly in complex numbers and find more accurate results for R and T. Since the lake is lossy, the transmitted wave will propagate with some attenuation and the electric field can be written as

$$\mathbf{E}_t = \mathbf{a}_y T E_0 e^{-j(\beta - j\alpha)x} = \mathbf{a}_y 0.2 E_0 e^{-j\beta x} e^{-\alpha x}$$

where E_0 is the amplitude of the incident electric field and β and α can be calculated from Eq. volume 4 (2-89) for slightly lossy media:

$$\beta \approx \omega\sqrt{\mu\varepsilon} = 2\, f\sqrt{\mu_0 \cdot 81\varepsilon_0} = 2\pi \cdot 10^6 \cdot 9 \cdot \frac{1}{3\times 10^8} = 1.88\times 10^3 \left[\frac{\text{rad}}{\text{m}}\right]$$

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = \frac{10^{-4}}{2}\sqrt{\frac{\mu_0}{81\varepsilon_0}} = \frac{10^{-4}}{2}\cdot\frac{377}{9} = 2.09\times 10^{-3}\left[\frac{\text{Np}}{\text{m}}\right]$$

1.3 Oblique Incidence at a Dielectric Boundary

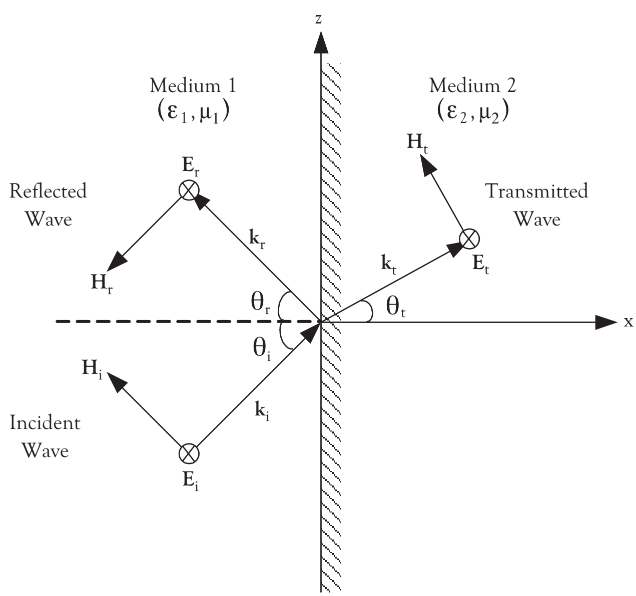


Figure 1-2. ***Reflection and transmission of waves with perpendicular polarization at oblique incidence***

We now consider the case when a uniform plane wave is incident from medium 1 at an oblique angle upon medium 2 as shown in Figure 1-2. The interface is at x = 0. The angle between the incident wave vector $\mathbf{k}_i$

and the normal of the boundary (x axis) is θ_i, called the *angle of incidence.* Let the angle between the reflected wave vector $\mathbf{k}_r$ and the normal be θ_r, called the *angle of reflection* and the angle between the transmitted wave vector $\mathbf{k}_t$ and the normal be θ_t, called the *angle of transmission.* The objective of the problem is to determine the fields of the reflected and transmitted waves along with θ_r, θ_t, given the fields of the incident wave with angular frequency ω at an angle θ_i. In the case of oblique incidence, the polarization of the incident wave affects how much the wave is reflected and transmitted. Thus we decompose the fields of the incident wave into two *linearly polarized* waves, orthogonal to each other. The decomposition is done with respect to a plane, known as the *plane of incidence,* which is formed by the incident wave vector ($\mathbf{k}_i$) and the normal of the boundary (x axis). In Figure 1-2, the xz-plane is the plane of incidence. We decompose the electric field into a component perpendicular to the plane of incidence and one parallel to the plane of incidence, and solve each problem separately. The former is said to have **perpendicular polarization,** *horizontal* polarization, *s*-polarization (*s* for "senkrecht", the German word for perpendicular), or *TE* (transverse electric) polarization. The latter is said to have **parallel polarization,** *vertical* polarization, *p*-polarization (*p* for "parallel"), or *TM* (transverse magnetic) polarization.

1.3.1 Perpendicular Polarization

First we assume the incident wave is *perpendicularly polarized.* Since the electric field $\mathbf{E}_i$ is perpendicular to the plane of incidence, it has only the y component (see Figure 1-2). Since the incident wave propagates in the positive x and positive z directions, the wave vector can be written as

$$\mathbf{k}_i = \mathbf{a}_x k_x + \mathbf{a}_z k_z \tag{1-17}$$

The electric and magnetic fields of the incident wave are written as

$$\mathbf{E}_i = \mathbf{a}_y E_0 e^{-j\mathbf{k}_i \cdot \mathbf{r}} = \mathbf{a}_y E_0 e^{-jk_x x - jk_z z} \tag{1-18a}$$

$$\mathbf{H}_i = \frac{\nabla \times \mathbf{E}_i}{-j\omega\mu_1} = (-\mathbf{a}_x k_z + \mathbf{a}_z k_x)\frac{E_0}{\omega\mu_1} e^{-jk_x x - jk_z z} \tag{1-18b}$$

Note that the magnetic field is parallel to the plane of incidence for a perpendicularly polarized wave and Eq. (1-18b) can also be obtained by $\mathbf{H}_i = \dfrac{\mathbf{k}_i \times \mathbf{E}_i}{\omega\mu_1}$ from Eq. (1-65). The components of the wave vector, k_x

and k_z, can be expressed in terms of medium constants, the frequency of the wave, and the angle of incidence as follows:

$$k_x = |\mathbf{k}_i| \cos\theta_i = k_1 \cos\theta_i \tag{1-19a}$$

$$k_z = |\mathbf{k}_i| \sin\theta_i = k_1 \sin\theta_i \tag{1-19b}$$

$$k_x^2 + k_z^2 = k_1^2 = \omega^2 \mu_1 \varepsilon_1 \tag{1-19c}$$

Since the reflected wave propagates in the negative x and positive z directions, its wave vector and fields can be written as

$$\mathbf{k}_r = -\mathbf{a}_x k_{rx} + \mathbf{a}_z k_{rz} \tag{1-20}$$

$$\mathbf{E}_r = \mathbf{a}_y E_r e^{-j\mathbf{k}_r \cdot \mathbf{r}} = \mathbf{a}_y R_\perp E_o e^{+jk_{rx}x - jk_{rz}z} \tag{1-21a}$$

$$\mathbf{H}_r = \frac{\nabla \times \mathbf{E}_r}{-j\omega\mu_1} = (-\mathbf{a}_x k_{rz} + \mathbf{a}_z k_{rx}) \frac{R_\perp E_0}{\omega\mu_1} e^{jk_{rx}x - jk_{rz}z} \tag{1-21b}$$

where $R_\perp$ is the reflection coefficient for the perpendicularly polarized wave. Because of the choice of signs in Eq. (1-20), k_{rx} and k_{rz} are considered to be positive and they can be expressed in terms of the angle of reflection θ_r.

$$k_{rx} = |\mathbf{k}_r| \cos\theta_r = k_1 \cos\theta_r \tag{1-22a}$$

$$k_{rz} = |\mathbf{k}_r| \sin\theta_r = k_1 \sin\theta_r \tag{1-22b}$$

$$k_{rx}^2 + k_{rz}^2 = k_1^2 = \omega^2 \mu_1 \varepsilon_1 \tag{1-22c}$$

Note that $|\mathbf{k}_r| = k_1 = |\mathbf{k}_i|$ because the reflected wave propagates in the same medium as the incident wave. For the transmitted wave, we can write

$$\mathbf{k}_t = -\mathbf{a}_x k_{tx} + \mathbf{a}_z k_{tz} \tag{1-23}$$

$$\mathbf{E}_t = \mathbf{a}_y E_t e^{-j\mathbf{k}_t \cdot \mathbf{r}} = \mathbf{a}_y T_\perp E_0 e^{-jk_{tx}x - jk_{tz}z} \tag{1-24a}$$

$$\mathbf{H}_t = \frac{\nabla \times \mathbf{E}_t}{-j\omega\mu_1} = (-\mathbf{a}_x k_{tz} + \mathbf{a}_z k_{tx}) \frac{T_\perp E_0}{\omega\mu_2} e^{-jk_{tx}x - jk_{tz}z} \tag{1-24b}$$

where $T_\perp$ is the transmission coefficient for the perpendicularly polarized wave. Similarly, k_{tx} and k_{tz} satisfy

$$k_{tx} = |\mathbf{k}_t| \cos\theta_t = k_2 \cos\theta_t \tag{1-25a}$$

$$k_{tz} = \left|\mathbf{k}_t\right| \sin\theta_t = k_1 \sin\theta_t \qquad (1\text{-}25b)$$

$$k_{tx}^2 + k_{tz}^2 = k_2^2 = \omega^2 \mu_2 \varepsilon_2 \qquad (1\text{-}25c)$$

Now the problem reduces to determining θ_r, θ_t, $R_\perp$, $T_\perp$, given θ_i, ω, (ε_1, μ_1), (ε_2, μ_2). Once we obtain θ_r, θ_t, $R_\perp$, $T_\perp$, we have a full knowledge on the reflected and transmitted waves. As we have done in Section 1.2, we apply the boundary conditions, Eq. (1-8), to determine these unknowns. For this problem, the tangential component of **H** is only its z component. Substituting x = 0 in Eqs. (1-18), (1-21), (1-24) and applying Eq. (1-8), we obtain

$$E_y := e^{-jk_z z} + R_\perp e^{-jk_{rz} z} = T_\perp e^{-jk_{tz} z} \qquad (1\text{-}26a)$$

$$H_z := \frac{k_x}{\omega\mu_1} e^{-jk_z z} - \frac{k_{rx}}{\omega\mu_1} R_\perp e^{-jk_{rz} z} = \frac{k_{tx}}{\omega\mu_2} T_\perp e^{-jk_{tz} z} \qquad (1\text{-}26b)$$

In order that Eq. (1-26) be satisfied at all points on the boundary (x = 0), i.e., for *all values of z,* the following condition must be met first:

$$k_z = k_{rz} = k_{rz} \qquad (1\text{-}27)$$

so that the exponential functions match on both sides. Eq. (1-27) implies that the tangential components of three wave vectors $\mathbf{k}_i$, $\mathbf{k}_r$, $\mathbf{k}_t$ are equal. The condition is known as the *phase matching condition.* Substituting Eqs. (1-19b), (1-22b), (1-25b) into Eq. (1-27) gives rise to

$$k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t \qquad (1\text{-}28)$$

The first equality gives

$$\theta_i = \theta_r \ \text{(law of reflection)} \qquad (1\text{-}29)$$

It says that the angle of reflection is equal to the angle of incidence. Making use of Eqs. (1-19c) and (1-25c), the second equality gives

$$\sqrt{\mu_1 \varepsilon_1} \sin\theta_i = \sqrt{\mu_2 \varepsilon_2} \sin\theta_t$$

or

$$n_1 \sin\theta_i = n_2 \sin\theta_t \ \text{(Snell's law)} \qquad (1\text{-}30)$$

where n_1, n_2 are the indices of refraction of media 1, 2, respectively. Eq. (1-30) is known as **Snell's law** or *law of refraction,* from which θ_t can be expressed in terms of θ_i as

$$\theta_t = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_i\right) \tag{1-31}$$

Eqs. (1-29) and (1-30) are well-known fundamental laws of geometrical optics.

Making use of Eq. (1-27), Eq. (1-26) becomes

$$1 + R_\perp = T_\perp \tag{1-32a}$$

$$\frac{k_x}{\mu_1}\left(1 - R_\perp\right) = \frac{k_{tx}}{\mu_1} T_\perp \tag{1-32b}$$

where we have used

$$k_{rx} = k_x = k_1 \cos\theta_i \tag{1-33}$$

because of Eq. (1-29). Now solving Eq. (1-32) for $R_\wedge$ and $T_\wedge$, we obtain

$$R_\perp = \frac{\mu_2 k_x - \mu_1 k_{tx}}{\mu_2 k_x - \mu_1 k_{tx}} \begin{pmatrix}\text{Reflection coefficient for} \\ \text{perpendicular polarization}\end{pmatrix} \tag{1-34a}$$

$$T_\wedge = \frac{2\mu_2 k_x}{\mu_2 k_x - \mu_1 k_{tx}} \begin{pmatrix}\text{Transmission coefficient} \\ \text{for perpendicular polarization}\end{pmatrix} \tag{1-34b}$$

Making use of Eqs. (1-19), (1-25), (1-2), (1-6), Eq. (1-34) can be written as

$$R_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t} \tag{1-35a}$$

$$T_\perp = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t} \tag{1-35b}$$

where η_1, η_2 are intrinsic impedances of the two media. These are known as the **Fresnel equations.** We note that at normal incidence ($\theta_i = \theta_t = 0$), Eq. (1-35) reduces to Eq. (1-11). When both media are non-magnetic, i.e., $\mu_1 = \mu_2 = \mu_0$, Eq. (1-35) reduces to

$$\left.\begin{aligned} R_\perp &= \frac{\eta_1 \cos\theta_i - \eta_2 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} \\ T_\perp &= \frac{2\eta_1 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} \end{aligned}\right. \begin{pmatrix}\text{Fresnel coefficients for} \\ \text{non-magnetic media} \\ \text{for } \perp \text{ polarization}\end{pmatrix} \tag{1-36}$$

where n_1, n_2 are refractive indices of the two media.

Example 1-2. Reflection and transmission of perpendicularly polarized wave

A 1 GHz uniform plane wave with 2 V/m is incident obliquely at $\theta_i = 60°$ from air upon a lossless dielectric (glass) of $\varepsilon = 1.96\,\varepsilon_0$ and $\mu = \mu_0$, whose surface is at x = 0 (see Figure 1-2). The incident wave has *perpendicular* polarization.

(a) Find the complete expression for the electric field $\mathbf{E}_i$ of the incident wave.

(b) Find the electric field $\mathbf{E}_r$ of the reflected wave.

(c) Find the magnetic field $\mathbf{H}_t$ of the transmitted wave.

Solutions:

(a) Assuming that the plane of incidence is the xz plane, the electric field of the perpendicularly polarized wave is given by

$$\mathbf{E}_i = \mathbf{a}_y E_0 e^{-j(k_x x + k_z z)}$$

where

$$E_0 = 2[\text{V/m}],\ k_1 = \frac{\omega}{c}(\text{in air}) = \frac{2\pi\times 10^9}{3\times 10^9} = \frac{20\pi}{3}[\text{rad/m}]$$

$$k_x = k_1\cos\theta_i = \frac{20\pi}{3}\cos 60^\circ = \frac{20\pi}{3}\cdot\frac{1}{2} = \frac{10\pi}{3} = 10.47$$

$$k_z = k_1\sin\theta_i = \frac{20\pi}{3}\sin 60^\circ = \frac{20\pi}{3}\cdot\frac{\sqrt{3}}{2} = \frac{10\pi}{3}\sqrt{3} = 18.14$$

$$\therefore \mathbf{E}_i = \mathbf{a}_y 2e^{-j\frac{10\pi}{3}(x+\sqrt{3}z)} = \mathbf{a}_y 2e^{-j(10.5x+18.1z)}[\text{V/m}]$$

(b) First, we need to find θ_r and θ_t to calculate R.

By the law of reflection, $\theta_r = \theta_i = 60°$.

By Snell›s law,

where

$$\theta_t = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_i\right) = \sin^{-1}\left(\frac{1}{\sqrt{1.96}}\sin 60^\circ\right) = 38.2^\circ$$

$$n_2 = \sqrt{\frac{\mu_2 \varepsilon_2}{\mu_0 \varepsilon_0}} = \sqrt{\frac{\mu_0 \cdot 1.96\varepsilon_0}{\mu_0 \varepsilon_0}} = \sqrt{1.96} = 1.4.$$

The electric field of the reflected wave is given by

$$\mathbf{E}_r = \mathbf{a}_y R_\perp E_0 e^{+jk_x x - jk_z z} = \mathbf{a}_y (-0.75) e^{+j(10.5x - 18.1z)} [\text{V/m}]$$

where

$$R_\perp = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i - n_2 \cos\theta_t} = \frac{1\cos 60^\circ - 1.4\cos 38.2^\circ}{1\cos 60^\circ + 1.4\cos 38.2^\circ} = -0.375[\text{V/m}]$$

Note that the sign of the normal (x–) component of the **k** vector is changed.

(c) The magnetic field of the transmitted wave is given by

$$\mathbf{H}_t = (-\mathbf{a}_x k_{tz} + \mathbf{a}_z k_{tx}) \frac{T_\perp E_0}{\omega \mu_2} e^{-jk_{tx} x - jk_{tz} z}$$

where

$$T_\perp = \frac{2\eta_1 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} = 1 + R_\perp = 0.625$$

$$k_2 = \omega\sqrt{\mu_2 \varepsilon_2} = \omega\sqrt{\mu_0 \cdot 1.96\varepsilon_0} = 1.4k_1 = 1.4 \times \frac{20\pi}{3} = 29.32$$

$$k_{tz} = k_z = \frac{10\pi}{3}\sqrt{3} = 18.14,\ k_{tx} = k_2 \cos\theta_t = 1.4 \times \frac{20\pi}{3}\cos 38.2^\circ = 23.04$$

$$\omega\mu_2 = 2\pi \times 10^9 \times 4\pi \times 10^{-7} = 7.90 \times 10^3$$

$$\therefore \mathbf{H}_i = (-\mathbf{a}_x 18.1 + \mathbf{a}_z 23.0) 1.58 \times 10^{-4} e^{-j(23.0x + 18.1z)} [\text{A/m}]$$

Note that the tangential (z-) component of the **k** vector is unchanged (due to the *phase matching* condition) and the normal (x-) component is changed.

1.3.2 Parallel Polarization

In the second case, we assume the incident wave is *parallel polarized.* Since the electric field is parallel with the plane of incidence, the magnetic field **H** is perpendicular to the plane of incidence as shown in Figure 1-3. In this case it is advantageous to formulate the magnetic field first.

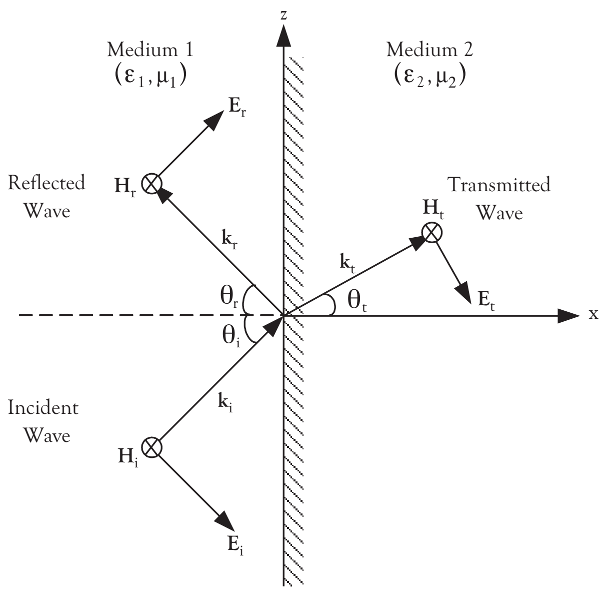


Figure 1-3. ***Reflection and transmission of waves with parallel polarization at oblique incidence***

Considering the direction of the incident wave vector as shown, the magnetic and electric fields of the incident wave are written as follows:

$$\mathbf{H}_i = \mathbf{a}_y H_t e^{-j\mathbf{k}_i \cdot \mathbf{r}} = \mathbf{a}_y H_0 e^{-jk_x x - jk_z z} \tag{1-37a}$$

$$\mathbf{E}_i = \frac{\nabla \times \mathbf{H}_i}{j\omega\varepsilon_1} = (\mathbf{a}_x k_z + \mathbf{a}_z k_x) \frac{H_0}{\omega\varepsilon_1} e^{-jk_x x - jk_z z} \tag{1-37b}$$

where H_0 is the amplitude of the incident magnetic field and k_x, k_z are given in Eq. (1-19). $\mathbf{E}_i$ can also be obtained by $\mathbf{E}_i = \dfrac{\mathbf{k}_i \times \mathbf{H}_i}{-\omega\varepsilon_1}$ from Eq. volume 4 (2-66). The fields of the reflected and transmitted waves can be formulated in the similar way as done in Section 1.3.1.

$$\mathbf{H}_r = \mathbf{a}_y H_r e^{-j\mathbf{k}_r \cdot \mathbf{r}} = \mathbf{a}_y R_{\|} H_0 e^{+jk_{rx} x - jk_{rz} z} \tag{1-38a}$$

$$\mathbf{E}_r = \frac{\nabla \times \mathbf{H}_r}{j\omega\varepsilon_1} = (\mathbf{a}_x k_{rz} + \mathbf{a}_z k_{rx}) \frac{R_{\|} H_0}{\omega\varepsilon_1} e^{jk_{rx} x - jk_{rz} z} \tag{1-38b}$$

$$\mathbf{H}_t = \mathbf{a}_y H_t e^{-j\mathbf{k}_t \cdot \mathbf{r}} = \mathbf{a}_y T_{|} H_0 e^{-jk_{tx} x - jk_{tz} z} \tag{1-39a}$$

$$\mathbf{E}_t = \frac{\nabla \times \mathbf{H}_t}{j\omega\varepsilon_1} = (\mathbf{a}_x k_{tz} - \mathbf{a}_z k_{tx}) \frac{T_{\|} H_0}{\omega\varepsilon_1} e^{-jk_{tx}x - jk_{tz}z} \tag{1-39b}$$

where $R_{\|}$ and $T_{\|}$ are, respectively, the reflection and transmission coefficients for the magnetic field for the parallel polarized wave. k_{rx}, k_{rz}, k_{tx}, k_{tz} are given in Eqs. (1-22) and (1-25). Now the problem again reduces to determining θ_r, θ_t, $R_{\|}$, $T_{\|}$, given θ_i, ω, (ε_1, μ_1), (ε_2, μ_2). Applying the boundary conditions – continuity of tangential components of **H** (y component) and **E** (z component) – at x = 0, we obtain

$$H_y := e^{-jk_z z} + R_{|} e^{-jk_{rz}z} = T_{\|} e^{-jk_{tz}z} \tag{1-40a}$$

$$E_z := -\frac{k_x}{\omega\varepsilon_1} e^{-jk_z z} + \frac{k_{rx}}{\omega\varepsilon_1} R_{\|}\, e^{-jk_{rz}z} = \frac{k_{tx}}{\omega\varepsilon_2} T_{\|}\, e^{-jk_{tz}z} \tag{1-40b}$$

Following the same procedures done in Section 1.3.1 in solving Eq. (1-40), we find the *same law of reflection and Snell's law* given by Eqs. (1-29) and (1-30) and $R_{\|}$, $T_{\|}$ are obtained as follows.

$$R_{\|} = \frac{\varepsilon_2 k_x - \varepsilon_1 k_{tx}}{\varepsilon_2 k_x - \varepsilon_1 k_{tx}} \begin{pmatrix} \text{Reflection coefficient for } \mathbf{H} \\ \text{for parallel polarization} \end{pmatrix} \tag{1-41a}$$

$$T_{\|} = \frac{2\varepsilon_2 k_x}{\varepsilon_2 k_x - \varepsilon_1 k_{tx}} \begin{pmatrix} \text{Transmission coefficient for } \mathbf{H} \\ \text{for parallel polarization} \end{pmatrix} \tag{1-41b}$$

The results in Eq. (1-41) can also be obtained simply by replacing μ by ε in Eq. (1-34) for perpendicular polarization, because the mathematical expressions for **E** and **H** for parallel polarization, given by Eqs. (1-37), (1-38), (1-39), are identical to those for perpendicular polarization, given by Eqs. (1-18), (1-21), (1-24), if we replace **E** by **H, H** by **–E,** ε by μ, and μ by ε. This is known as *duality* in electromagnetic theory. Making use of Eqs. (1-19), (1-25), (1-21), (1-6), Eq. (1-41) can be rewritten in terms of angles of incidence and transmission.

$$R_{\|} = \frac{\eta_1 \cos\theta_i - \eta_2 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} \tag{1-42a}$$

$$T_{\|} = \frac{2\eta_1 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} \tag{1-42b}$$

We note that at normal incidence ($\theta_i = \theta_t = 0$), Eq. (1-42) doesn't reduce to Eq. (1-11) because Eqs. (1-41) and (1-42) are the ratio of magnetic fields. If we take the ratios of the z components of the electric fields, we find the reflection and transmission coefficients for **E** for parallel polarization as follows.

$$R_{\|}^{E} = \frac{E_{rz}}{E_{iz}} = -R_{\|} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \left(\begin{array}{l}\text{Reflection coefficient for } \mathbf{E} \\ \text{for parallel polarization}\end{array}\right) \tag{1-43a}$$

$$T_{\|}^{E} = \frac{E_{tz}}{E_{iz}} = T_{\|} = \frac{k_{tx}\varepsilon_1}{k_x \varepsilon_2} = \frac{2\eta_2 \cos\theta_t}{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i} \left(\begin{array}{l}\text{Transmission coefficient for } \mathbf{E} \\ \text{for parallel polarization}\end{array}\right) \tag{1-43b}$$

Now at normal incidence, Eq. (1-43) reduces to Eq. (1-11) as expected. When both media are non-magnetic, i.e., $\mu_1 = \mu_2 = \mu_o$, Eqs. (1-42) and (1-43) reduce to

$$\left.\begin{array}{l} R_{\|} = \dfrac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i - n_1 \cos\theta_t} \\ T_{\|} = \dfrac{2n_2 \cos\theta_i}{n_2 \cos\theta_i - n_1 \cos\theta_t} \end{array}\right. \left(\begin{array}{l}\text{Fresnel coefficients for} \\ \text{non-magnetic media} \\ \text{for } \| \text{ polarization}\end{array}\right) \tag{1-44a}$$

$$R_{\|}^{E} = \frac{n_1 \cos\theta_t - n_2 \cos\theta_i}{n_1 \cos\theta_t + n_2 \cos\theta_i} = -R_P \tag{1-44b}$$

$$T_{\|}^{E} = \frac{2n1 \cos\theta_i}{n_1 \cos\theta_t + n_2 \cos\theta_i} \tag{1-45}$$

Power Conservation

In order to observe power conservation, the time-average Poynting vectors of the incident, reflected and transmitted waves for the perpendicular polarization are calculated from the fields in Eqs. (1-18), (1-21), (1-24).

$$\mathbf{S}_{av}^{i} = \frac{1}{2}\text{Re}\left(\mathbf{E}_i \times \mathbf{H}_i^*\right) = \frac{|E_0|^2}{2\omega\mu_1}\left\{\mathbf{a}_x k_x + \mathbf{a}_z k_z\right\} \tag{1-46a}$$

$$\mathbf{S}_{av}^{r} = \frac{1}{2}\text{Re}\left(\mathbf{E}_r \times \mathbf{H}_r^*\right) = \frac{|E_0|^2}{2\omega\mu_1}|R_\perp|^2\left\{-\mathbf{a}_x k_x + \mathbf{a}_z k_z\right\} \tag{1-46b}$$

$$\mathbf{S}^t_{av} = \frac{1}{2}\mathrm{Re}\left(\mathbf{E}_t \times \mathbf{H}^*_t\right) = \frac{|E_0|^2}{2\omega\mu_2}|T_\perp|^2 \,\mathrm{Re}\left\{\mathbf{a}_x k^*_{tx} + \mathbf{a}_z k_z\right\} \quad (1\text{-}46c)$$

where Eqs. (1-27) and (1-33) have been used. Note that k_{tx} can become complex in some instances as will be discussed later. Power conservation holds for the x components of the power. The fraction of power reflected is given by

$$\frac{P_r}{P_i} = \frac{\mathbf{S}^r_{av,x}}{\mathbf{S}^t_{av,x}} = |R_\perp|^2 \quad (1\text{-}47a)$$

The fraction of power transmitted is given by

$$\frac{P_t}{P_i} = \frac{\mathbf{S}^t_{av,x}}{\mathbf{S}^t_{av,x}} = |T_\perp|^2 \frac{\mu_1}{\mu_2}\frac{\mathrm{Re}(k^*_{tx})}{k_x} \quad (1\text{-}47b)$$

Substituting Eq. (1-34) into Eq. (1-47), one can show that

$$\frac{P_r}{P_i} + \frac{P_t}{P_i} = 1 \text{ or } P_i = P_r + P_t \quad (1\text{-}48)$$

We can derive the same power conservation relationship (1-48) for parallel polarization (see Problem 1-16). Eq. (1-47b) should be adjusted for parallel polarization. The reflection and transmission coefficients for perpendicular and parallel polarized waves are summarized in Table 1-1.

Table 1-1. Reflection and Transmission Coefficients

1. Perpendicular (TE) Polarization
$R_\perp = \frac{E_r}{E_i} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} \quad n_1\sin\theta_i = n_2\sin\theta_t$
$T_\perp = \frac{E_t}{E_i}\frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} \quad \eta_i = \sqrt{\frac{\mu_i}{\varepsilon_i}},\ i = 1,2$

Example 1-3. Reflection and transmission of parallel polarized wave

A uniform plane wave whose electric field is given by

$$\mathbf{E}_i = (\mathbf{a}_x\sqrt{3} - \mathbf{a}_z)e^{-j(2x+2\sqrt{3}z)}[\mathrm{V/m}]$$

is incident from air upon a lossless dielectric (glass) of $\varepsilon = 1.96\,\varepsilon_0$ and $\mu = \mu_0$, whose surface is at x = 0 (see Figure 1-3).

(a) Determine the angles of incidence, reflection and transmission.
(b) Find the electric field of the reflected wave.
(c) Find the magnetic field of the transmitted wave.
(d) Calculate the fraction of power reflected from the dielectric surface.

Solution:

(a) Noting that $k_x = 2$, $k_z = 2\sqrt{3}$, we find

$$k_1 = \sqrt{k_x^2 + k_z^2} = \sqrt{(2)^2 + (2\sqrt{3})^2} = 4$$

$$k_z = k_1 \sin\theta_i \rightarrow \theta_i = \sin^{-1}\left(\frac{k_z}{k_1}\right) = \sin^{-1}\left(\frac{2\sqrt{3}}{4}\right) = 60^\circ$$

By the law of reflection, $\theta_r = \theta_i = 60°$

By Snell's law,

$$\theta_t = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_i\right) = \sin^{-1}\left(\frac{1}{\sqrt{1.96}}\sin 60^\circ\right) = 38.2^\circ$$

(b) Since the incident electric field **E** is parallel to the plane of incidence (xz plane), the incident wave is *parallel polarized.* Because the reflection (R_p) and transmission (T_p) coefficients for parallel polarized are defined in terms of ratio of magnetic fields, we calculate the magnetic field $\mathbf{H}_i$ of the incident wave

$$\mathbf{H}_i = \frac{\nabla \times \mathbf{E}_i}{-j\omega\mu_1} = \frac{\mathbf{a}_{k_1} \times \mathbf{E}_i}{\eta_1} = \frac{1}{377}(\mathbf{a}_x \frac{1}{2} + \mathbf{a}_z \frac{\sqrt{3}}{2}) \times (\mathbf{a}_x \sqrt{3} - \mathbf{a}_z) e^{-j(2x - 2\sqrt{3}z)}$$
$$= \mathbf{a}_y \frac{2}{377} e^{-j(2x+2\sqrt{3}z)} [\text{A/m}]$$

where

$$\mathbf{a}_{ki} = \frac{\mathbf{k}_i}{k_1} = \frac{\mathbf{a}_x 2 + \mathbf{a}_z 2\sqrt{3}}{4} = \mathbf{a}_x \frac{1}{2} + \mathbf{a}_z \frac{\sqrt{3}}{2}$$

As expected, $\mathbf{H}_i$ is perpendicular to the plane of incidence. Now we calculate R_p and T_p:

$$R_p = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t} = \frac{1.4\cos 60^\circ - 1\cos 38.2^\circ}{1.4\cos 60^\circ + 1\cos 38.2^\circ} = -0.058$$

$$T_p = 1 + R_p = 0.942$$

The fields of the reflected wave are given by

$$\mathbf{H}_r = \mathbf{a}_y R_p \frac{2}{377} e^{+j(2x-2\sqrt{3}\,z)} = -\mathbf{a}_y 3.08\times 10^{-4} e^{+j(2x-2\sqrt{3}\,z)} [\text{A/m}]$$

$$\mathbf{E}_r = \eta_1 \mathbf{H}_r \times \mathbf{a}_{k_r} = R_p 2\mathbf{a}_y \times \left(-\mathbf{a}_x \frac{1}{2} + \mathbf{a}_z \frac{\sqrt{3}}{2} \right) e^{+j(2x-2\sqrt{3}\,z)}$$

$$= R_p \left(\mathbf{a}_x \sqrt{3} + \mathbf{a}_z \right) e^{+j(2x-2\sqrt{3}\,z)} = -0.058 \left(\mathbf{a}_x \sqrt{3} + \mathbf{a}_z \right) e^{+j(2x-2\sqrt{3}\,z)} [\text{V/m}]$$

Note that one component of the reflected electric field is changed in sign because the sign of the x component of $\mathbf{k}_r$ is changed.

(c) The magnetic field of the transmitted wave is given by

$$\mathbf{H}_r = \mathbf{a}_y T_p \frac{2}{377} e^{-j(2x+2\sqrt{3}\,z)} = \mathbf{a}_y 5.0\times 10^{-3} e^{-j(4.40x-2\sqrt{3}\,z)} [\text{A/m}]$$

where

$$k_{tx} = k_2 \cos\theta_t = \omega\sqrt{\mu_o 1.96\varepsilon_o}\cos\theta_t = 1.4k_1 \cos 38.2^\circ = 4.40.$$

(d) The fraction (or percentage) of the power reflected is given by

$$\frac{P_r}{P_i} = \left| R_p \right|^2 = \left| -0.058 \right|^2 = 0.0034$$

Thus, 0.34% of the incident power is reflected at $\theta_i = 60°$.

1.4 Total Internal Reflection

Let us examine Snell's law in Eq. (1-30) or equivalently, the phase matching condition in Eq. (1-27) or (1-28). We first note:

If $n_1 < n_2$, then $\theta_i > \theta_t$

If $n_1 > n_2$, then $\theta_i < \theta_t$

When the wave is incident from a less dense medium to a denser medium, for example, from air to glass, the transmitted wave is bent toward the normal. When the wave is incident from a denser medium to a less dense medium, the transmitted wave is bent toward the surface. This can be illustrated graphically by the k_x–k_z diagram shown in Figure 1-4, (a) and (b). The radius of the semicircle represents the wave number (k) for each of the incident, reflected and transmitted waves (k_1 and k_2). Since $k_i = \omega\sqrt{\mu_i \varepsilon_i} = n_i \omega \sqrt{\mu_o \varepsilon_o}$, the less dense medium has smaller semicircle. The arrows are the wave vectors, $\mathbf{k}_i$, $\mathbf{k}_r$ and $\mathbf{k}_t$. The phase matching condition in Eq. (1-27) or (1-28) requires that the z component of **k** or the component parallel to the boundary should match. In (a), since $k_1 < k_2$, we have $\theta_t < \theta_i$. In (b), since $k_1 > k_2$, we have $\theta_t > \theta_i$.

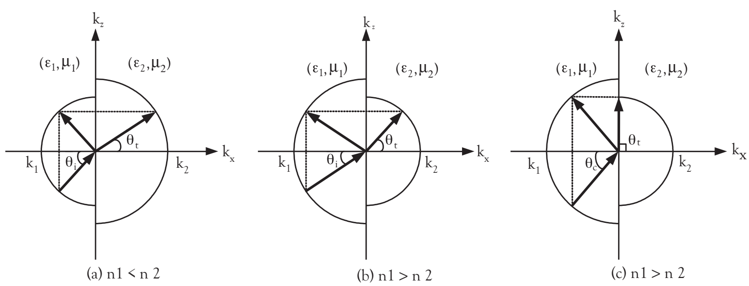


Figure 1-4. ***Snell's law and total internal reflection***

Now in the case of $n_1 > n_2$, there exists an angle of incidence ($\theta_i = \theta_c$) such that the angle of transmission becomes 90°, as shown in Figure 1-4(c).

$$n_1 \sin\theta_c = n_2 \sin 90^\circ = n_2$$

or

$$\theta_c = \sin^{-1}\frac{n_2}{n_1} \text{ (critical angle)}$$

$$= \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \text{ (for non-magnetic media)} \qquad (1\text{-}49)$$

θ_c is called the **critical angle.** It is interesting to see what happens when the angle of incidence exceeds the critical angle ($\theta_i > \theta_c$). For discussion, let us repeat the dispersion relations and Snell's law here.

$$k_x^2 + k_z^2 = k_1^2 = \omega^2 \mu_1 \varepsilon_1 = \left(\frac{\omega}{c} n_1\right)^2 \qquad (1\text{-}19c)$$

$$k_{tx}^2 + k_{tz}^2 = k_1^2 = \omega^2 \mu_2 \varepsilon_2 = \left(\frac{\omega}{c} n_2 \right)^2 \qquad (1\text{-}25c)$$

$$k_z + k_{tz} \text{ or } k_1 \sin\theta_i = k_2 \sin\theta_t \qquad (1\text{-}28)$$

When $\theta_i > \theta_c$,

$$k_z = k_1 \sin\theta_i > k_1 \sin\theta_c = k_2 \sin 90^\circ = k_2$$

Then, from Eq. (1-25c) and Eq. (1-27),

$$k_{tx}^2 = k_2^2 - k_{tz}^2 = k_{tz}^2 = k_2^2 - k_z^2 < 0$$

Thus k_{tx} has to become *purely imaginary,* so we let

$$k_{tx} = -j\alpha_2 \qquad (1\text{-}50)$$

where

$$\alpha_2 = \sqrt{k_z^2 - k_2^2} = \sqrt{k_1^2 \sin\theta_i - k_2^2} \qquad (1\text{-}51)$$

is real. Note that we choose $-j\alpha_2$ instead of $j\alpha_2$ so that $\mathbf{E}_t$ does not grow, i.e., $j\alpha_2$ gives a physically unacceptable solution. The transmitted electric field in Eq. (1-24) for perpendicular polarization becomes

$$\mathbf{E}_t = \mathbf{a}_y T_\perp E_0 e^{-\alpha_2 x} e^{-jk_{tz} z} \qquad (1\text{-}52)$$

The transmitted wave *propagates in the z direction* (parallel to the surface) with the phase constant $k_{tz} = k_z = k_1 \sin\theta_i$ and *attenuates exponentially in the x direction.* It is a *nonuniform* plane wave propagating along the boundary because $\left|\mathbf{E}_t\right| = \left|T_\perp E_0\right| e^{-\alpha_2 x}$ is not constant over a plane perpendicular to the direction of propagation (xy plane) but is attenuated away from the boundary. Such a wave is called the **surface wave.** The wave can be tightly bound to the surface of the interface if α_2 is large. More importantly, this wave carries no real power into medium 2. We show this by examining the reflection coefficient for perpendicular polarization in Eq. (1-34a).

$$R_\perp = \frac{\mu_2 k_x - \mu_1 k_{tx}}{\mu_2 k_x + \mu_1 k_{tx}} = \frac{\mu_2 k_x + j\mu_1 \alpha_2}{\mu_2 k_x - j\mu_1 \alpha_2} \frac{1 + j\dfrac{\mu_1 \alpha_x}{\mu_2 k_x}}{1 - j\dfrac{\mu_1 \alpha_x}{\mu_2 k_x}} \qquad (1\text{-}53)$$

The fraction of power reflected is given by

$$\frac{P_r}{P_i} = |R_\perp|^2 = \frac{1+\left(\dfrac{\mu_1\alpha_x}{\mu_2 k_x}\right)^2}{1+\left(\dfrac{\mu_1\alpha_x}{\mu_2 k_x}\right)^2} = 1 \qquad \text{(1-54a)}$$

or

$$P_r = P_i,\ P_t = P_i - P_r = 0 \qquad \text{(1-54b)}$$

The wave is *totally reflected* and there is *no time-average (or real) power transmitted* in the x direction or into medium 2. Hence we call this phenomenon **total internal reflection.** This can also be shown by substituting Eq. (1-50) into Eq. (1-46c). The total internal reflection is the major principle of the dielectric waveguide such as optical fiber which guides the light beam with very little loss because the metal is not used.

In summary, there are two conditions for total internal reflection (TIR) to occur:

(i) $n_1 > n_2$
(ii) $\theta_i > \theta_c = \sin^{-1}\dfrac{n_2}{n_1}$

The consequences of TIR are:

(i) $k_{tx} = -j\alpha_2 \rightarrow \mathbf{E}_t$ becomes a surface wave.
(ii) $|R_\perp| = 1$. The wave is totally reflected.

As a final note, although we illustrated TIR for perpendicular polarization, it also applies to parallel polarized wave

Example 1-4. Guidance of a light wave along optical fiber
An optical fiber is a dielectric pipe made of two concentric glass cylinders as shown in Figure 1-5. The inner cylinder, called the *core,* has the refractive index n_1 and the outer cylinder, called the *cladding,* has the index n_2, which is slightly smaller than n_1, i.e., $n_2 < n_1$. Light waves traveling in the core are totally reflected from the cladding and thus guided if their angle of incidence is greater than the critical angle.

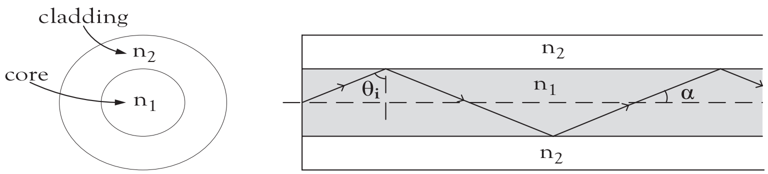


Figure 1-5. ***Optical fiber***

(a) A silica glass fiber has n_1 = 1.475 and n_2 = 1.460. What should the angle between the fiber axis and a light ray be in order that the light will propagate along the fiber?

(b) When the cladding is removed, i.e., when the core is surrounded by air, what should that angle be?

Solution:

(a) In order for the TIR to occur at the interface between the core and the cladding, θ_i must be greater than the critical angle (θ_c).

$$\theta_i > \theta_c = \sin^{-1}\frac{n_2}{n_1} = \sin^{-1}\frac{1.460}{1.475} = 81.8^\circ$$

Then the angle between the fiber axis and the ray (α) should be

$$\alpha = 90^\circ - \theta_i < 90^\circ - \theta_c = 8.2^\circ$$

(b) When the core is surrounded by air, n_2 = 1.

$$\theta_i > \theta_c = \sin^{-1}\frac{1}{1.475} = 42.7^\circ$$

Then $\alpha < 90^\circ - \theta_c = 47.3^\circ$

1.5 Brewster Angle Effect

Let us now investigate how the reflection coefficients vary as a function of incidence angle θ_i. The expressions for $R_\wedge$ and $R_\parallel$ for electric fields are given in Eqs. (1-36) and (1-45) in the case of non-magnetic media. Consider the case when the wave is incident from air (n_1 = 1) upon glass with $\varepsilon_2 = 4\varepsilon_o$. The glass has the index of refraction $n_2 = \sqrt{\frac{\mu_o\varepsilon_2}{\mu_o\varepsilon_o}} = 2$.

Figure 1-6(a) shows the plot of $|R_\perp|$ and $|R_\parallel|$ versus θ_i when n_1 = 1, n_2 = 2. It is observed that as θ_i varies from 0° (normal incidence) to 90°

(grazing), $|R_\perp|$ increases monotonically, reaching a maximum value of 1 at 90°, while $|R_\parallel|$ decreases initially, reaches a minimum value of 0 at some angle $\theta_i = \theta_B$ and then increases up to 1 at 90°.

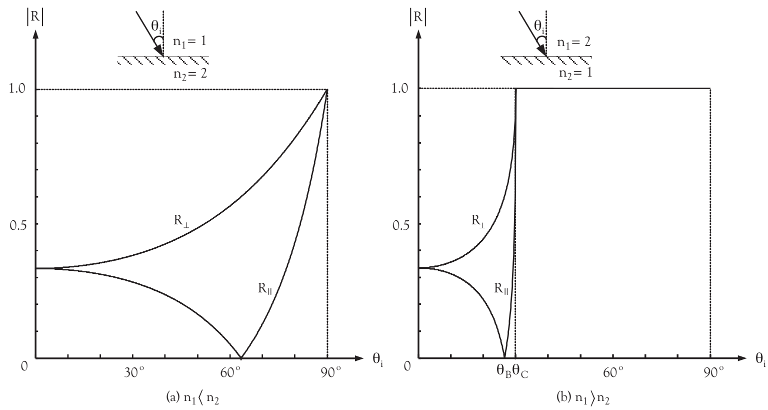


Figure 1-6. ***Reflection coefficients vs. incidence angle***

Thus we have

(i) $|R_\perp| \geq |R_\parallel|$ for all angles θ_i

(ii) $R_\parallel = 0$ at $\theta_i = \theta_B$

The perpendicularly polarized wave is reflected more than is the parallel polarized wav e. There exists an angle of incidence $\theta_i = \theta_B$ at which $R_\parallel = 0$, i.e., there is *no reflection,* thus *total transmission* for parallel polarized wave. Such an angle θ_B is called **Brewster's angle.** θ_B can be calculated by letting $R_\parallel = 0$ in Eq. (1-45) for non-magnetic media, with the use of Snell's law. Letting $R_\parallel = 0$ leads to

$$n_1 \cos\theta_t = n_2 \cos\theta_B \tag{1-55}$$

From Snell's law, Eq. (1-30), we have

$$n_2 \sin\theta_t = n_1 \sin\theta_B \tag{1-56}$$

Eliminating θ_t from Eqs. (1-55) and (1-56), we derive Brewster's angle as follows (see Problem 1-28):

$$\theta_B = \tan^{-1}\frac{n_2}{n_1} \begin{pmatrix} \text{Brewster's angle} \\ \text{for non-magnetic media} \end{pmatrix} \tag{1-57}$$

When $n_1 = 1$, $n_2 = 2$, Brewster's angle occurs or $\theta_B = 63.4°$. When the wave with elliptical polarization or the unpolarized wave is incident on the dielectric surface exactly at Brewster's angle, the reflected wave will contain only the perpendicularly polarized component because the parallel polarized component of the incident wave is totally transmitted into the dielectric. The reflected wave becomes *linearly polarized.* **Polaroid sunglasses** reduce glares based on this principle. When the unpolarized sunlight is reflected from the ground such as the asphalt and snow surface, the reflected light, that is, the glare becomes *partially polarized,* having mostly the perpendicularly polarized component for which the electric field is horizontal with respect to the reflecting surface. The material used in sunglasses is *anisotropic* such that it passes the vertical component of the electric field but blocks (or absorbs) the horizontal component (perpendicularly polarized component).

In Figure 1-6(b), $|R_\perp|$ and $|R_\parallel|$ versus θ_i are plotted when the wave is incident from glass (denser, $n_1 = 2$) upon air (less dense, $n_2 = 1$). We find similar characteristics except that $|R_\perp| = |R_\parallel| = 1$ for $\theta_i \geq \theta_c = \sin^{-1} n_2/n_1 = 30°$ because of the total internal reflection, as discussed in Section 1.4. Brewster's angle occurs at $\theta_B = \tan^{-1} n_2/n_1 = 26.6°$. Finally, we note that Brewster's angle exists only for parallel polarized waves when both media are non-magnetic ($\mu_1 = \mu_2 = \mu_o$). There can exist Brewster's angle for perpendicularly polarized wave when two media have different magnetic properties ($\mu_1 \neq \mu_2$) (see Problem 1-30).

1.6 Reflection from Perfect Conductor – Standing Waves

In this section we consider the situation when the wave is incident upon a *perfectly conducting* medium which has an infinite conductivity. Referring to Figure 1-2, we consider the oblique incidence of the *perpendicularly polarized* wave whose fields are given by Eq. (1-18). Now the medium 2 has μ_2, ε_2 and $\sigma_2 = \infty$. First we note that inside a perfect conductor the fields are zero, therefore, we have

$$\mathbf{E}_t = \mathbf{H}_t = 0 \text{ in medium 2} \tag{1-58}$$

and there will be no transmitted wave, i.e., $T_\perp = 0$. The electric and magnetic fields of the reflected wave can be written as in Eq. (1-21). The boundary conditions at the surface of the perfect conductor (x = 0) are given from Eqs. Volume 4 (1-27 a b) by letting $\mathbf{E}_2 = \mathbf{H}_2 = 0$ or

$$\mathbf{E}_{1t} = 0 \tag{1-59a}$$

$$\mathbf{a}_{n2} \times \mathbf{H}_1 = \mathbf{J}_s \tag{1-59b}$$

where $\mathbf{J}_s$ is the (free) surface current density. Substituting x = 0 in Eqs. (1-18a), (1-21a) and applying the boundary condition, Eq. (1-59a), we obtain

$$e^{-jk_z z} + R_\perp e^{-jk_{rz} z} = 0 \tag{1-60}$$

Following a similar procedure shown in Section 1.3, we have

$$k_z = k_{rz} \text{ or } \theta_r = \theta_i$$

and

$$1 + R_\perp = 0$$

Thus we have

$$R_\perp = -1,\ T_\perp = 0 \tag{1-61}$$

At the surface of the perfectly conducting medium, the wave is *totally reflected,* i.e., the perfect reflection occurs, and there will be no transmission. This fact becomes a major principle of metallic waveguides for which the wave bounces off the conducting walls by perfect reflection and is guided in a certain direction. We can also obtain the results of Eq. (1-61) from the results of a dielectric boundary, Eq. (1-35), by treating the medium 2 as a lossy dielectric with infinite conductivity,

$$\varepsilon_{2,eff} = \varepsilon_2 - j\frac{\sigma_2}{\omega},\ \sigma_2 \to \infty$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2 - j\frac{\sigma_2}{\omega}}} \to 0 \text{ as } \sigma_2 \to \infty.$$

Letting $\eta_2 = 0$ in Eq. (1-35) leads to Eq. (1-61).

In the case of a parallel polarized wave, substituting x = 0 in Eqs. (1-37b), (1-38b) and applying Eq. (1-59a), we obtain

$$R_{\parallel} = 1, \; T_{\parallel} = 0 \tag{1-62a}$$

$$R_{\parallel}^{E} = -1, \; T_{\parallel}^{E} = 0 \tag{1-62b}$$

We note that the reflection coefficient is +1 for the magnetic field and –1 for the electric field. For both polarizations, we find that

$$P_r = P_i, \; P_t = 0 \tag{1-63}$$

Standing Waves

Let us now look at the total fields in medium 1 for the perpendicular polarization and observe the field pattern as a function of distance from the surface of the perfect conductor (or medium 2).

$$\begin{aligned}\mathbf{E}_1 &= \mathbf{E}_i + \mathbf{E}_r = \mathbf{a}_y E_0 \; e^{-jk_x x - jk_z z} - \mathbf{a}_y E_0 \; e^{jk_x x - jk_z z} \\ &= \mathbf{a}_y E_0 \; e^{-jk_z z}\left\{e^{-jk_x x} - e^{jk_x x}\right\} = \mathbf{a}_y E_m \; e^{-jk_z z} \sin(k_x x)\end{aligned} \tag{1-64a}$$

where

$$E_m = -j2E_0.$$

Similarly, we can obtain

$$\mathbf{H}_1 = \mathbf{H}_i + \mathbf{H}_r = \frac{2E_0}{\omega\mu_1} e^{-jk_z z}\left\{\mathbf{a}_y jk_z \sin(k_x x) + \mathbf{a}_z k_x \cos(k_x x)\right\} \tag{1-64b}$$

Assuming $E_m = Ae^{j\phi}$, the electric field in real time is given by

$$\mathbf{E}_1(\mathbf{r},t) = \mathrm{Re}\left\{\mathbf{E}_1 e^{j\omega t}\right\} = \mathbf{a}_y A \sin(k_x x) \sin(\omega t - k_z z + \phi) \tag{1-65}$$

The wave represented by Eqs. (1-64) and (1-65) propagates in the +z direction but does not propagate or travel in the x direction. Such a wave is called a **standing wave** in the x direction while $e^{-jk_x x}$ and $e^{jk_x x}$ represent traveling waves. The amplitude of the electric field is given by

$$|E_1| = |E_m||\sin(k_x x)| \tag{1-66}$$

and plotted as a function of x in Figure 1-7. It gives a periodic pattern for which the minima (with zero value) occur at $x = -n\pi/k_x$ $(n = 0, 1, 2, \ldots.)$ and the maxima (with the value of $2|E_o|$) at $x = -\left(n + \frac{1}{2}\right)/k_x\,(n = 0, 1, 2, \ldots.)$. The period of the standing wave pattern is

$$\frac{\pi}{k_x} = \frac{\pi}{k_1 \cos\theta_i} = \frac{\lambda_1}{2}\frac{1}{\cos\theta_i} \tag{1-67}$$

Note that at normal incidence ($\theta_i = 0$), the period is half the wavelength ($\lambda/2$).

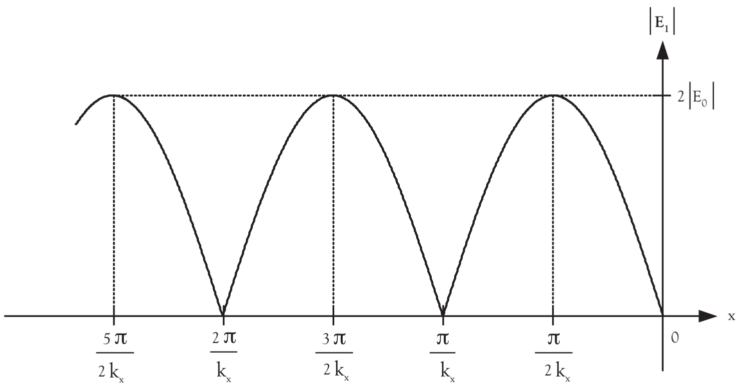


Figure 1-7. ***Standing wave pattern***

Noting that

$$E_1 = 0 \text{ at } x = -n\frac{\pi}{k_x} = -n\frac{\lambda}{2}\frac{1}{\cos\theta_i},\; n = 0, 1, 2, \ldots.,$$

perfectly conducting plates can be placed at these points without affecting the solution between the original and added conductors. This interpretation will be quite useful in understanding the field solutions of the metallic waveguide. In fact, the total fields we just discussed are proved to be the TE (transverse electric) mode solutions for the parallel-plate waveguide as we shall see in Chapter 2.

The time-average power density or Poynting vector of the total wave is given by

$$\mathbf{S}_{av} = \frac{1}{2}\text{Re}\left(\mathbf{E}_1 \times \mathbf{H}_1^*\right) = \mathbf{a}_z \frac{2|E_0|^2}{\omega\mu_1} k_z \sin^2(k_x x) = \mathbf{a}_z \frac{2|E_0|^2}{\eta_1} \sin\theta_i \sin^2(k_x x) \quad (1\text{-}68)$$

Thus, the time-average power flow is in the +z direction. Note that the x component of $\mathbf{S}_{av}$ is zero because E_y and H_z are 90° out of phase in time. The phase velocity in the z direction is given by

$$v_p = \frac{\omega}{k_z} = \frac{\omega}{k_1 \sin\theta_i} = \frac{v_1}{\sin\theta_i}$$

where v_1 is the phase velocity of a wave in an unbounded medium 1.

Now, if we use the second boundary condition given in Eq. (1-59b), we can obtain the surface current $\mathbf{J}_s$ that flows on the surface of the perfect conductor from the magnetic field as follows.

$$\mathbf{J}_s = (-\mathbf{a}_x) \times \mathbf{H}_1 \Big|_{x=0} = \mathbf{a}_y H_z \Big|_{x=0} = \mathbf{a}_y \frac{2E_0}{\omega\mu_1} k_x e^{-jk_z z} \quad (1\text{-}69)$$

If the conducting plate is replaced by a grating of parallel conducting wires arranged in the direction of the surface current flow (y direction in this case), these wires can serve as a good reflector as effectively as a solid conducting plate. The gratings are used often as reflecting antennas because they reduce weight, save material cost, and reduce wind loading.

Normal Incidence

As a special case, when the wave is incident normal to the boundary, letting $\theta_i = 0$, we obtain the following results:

$$k_x = k_1 \cos\theta_i = k_1, \; k_z = k_1 \sin\theta_i = 0$$

$$\mathbf{E}_1 = \mathbf{a}_y E_0 \left\{e^{-jk_1 x} - e^{jk_1 x}\right\} = -\mathbf{a}_y(-j2E_0)\sin(k_x x)$$

$$\mathbf{H}_1 = \mathbf{a}_z \frac{E_0}{\eta_1}\left\{e^{-jk_1 x} + e^{jk_1 x}\right\} = \mathbf{a}_z \frac{2E_0}{\eta_1}\cos(k_1 x)$$

$$\mathbf{E}_1 = 0 \text{ at } x = -n\frac{\pi}{k_1} = -n\frac{\lambda}{2}, \; n = 0, 1, 2, \ldots$$

$$\mathbf{S}_{av} = 0$$

$$\mathbf{J}_s = \mathbf{a}_y \frac{2E_0}{\eta_1}$$

We can also obtain similar results and interpretations when the parallel polarized wave is incident upon a perfectly conducting medium.

Example 1-5. Standing wave in front of a perfect conductor
A uniform plane wave of 1 GHz is incident normally on a large copper plate at x = 0 (see Figure 1-1). The electric field is z-polarized and has an amplitude of 10 V/m.

(a) Find the total electric field and the locations of its first maximum and minimum in front of the plate.
(b) Find the total magnetic field and the locations of its first maximum and minimum.

Solutions:

(a) The electric fields of the incident and reflected waves can be written as

$$\mathbf{E}_i = \mathbf{a}_z 10e^{-jkx}, \; k = \frac{\omega}{c} = \frac{2\pi \times 10^9}{3 \times 10^8} = \frac{20\pi}{3}, \; \lambda = \frac{2}{k} = 30[\text{m}]$$

$$\mathbf{E}_r = \mathbf{a}_z 10e^{+jkx} \;\; \text{Note that } R = -1 \text{ for } \mathbf{E}.$$

The total electric field is given by

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r = \mathbf{a}_z(-j20)\sin(kx)$$

The first maximum occurs at

$$kx = \frac{2\pi}{\lambda}x = -\frac{\pi}{2}, \; x_{max} = -\frac{\lambda}{4} = -7.5[\text{m}]$$

The first minimum (excluding x = 0) occurs at

$$kx = \frac{2\pi}{\lambda}x = -\pi, \; x_{min} = -\frac{\lambda}{2} = -15[\text{m}]$$

(b) The magnetic fields are given by

$$\mathbf{H}_i = \frac{1}{\eta_0}\mathbf{a}_x \times \mathbf{E}_i = -\mathbf{a}_y \frac{10}{377}e^{-jkx}$$

$$\mathbf{H}_r = \frac{1}{\eta_0}(-\mathbf{a}_x) \times \mathbf{E}_r = -\mathbf{a}_y \frac{10}{377}e^{+jkx} \;\; \text{Note that } R = +1 \text{ for } \mathbf{H}.$$

$$\mathbf{H}_1 = \mathbf{H}_i + \mathbf{H}_r = -\mathbf{a}_y \frac{20}{377}\cos(kx)$$

The first maximum (excluding x = 0) occurs at

$$kx = -\pi,\ x_{max} = -\frac{\lambda}{2} = -15[m]$$

The first minimum occurs at $kx = -\frac{\pi}{2},\ x_{min} = -\frac{\lambda}{4} = -7.5[m]$

Note that the electric and magnetic fields alternate their maxima and minima. This behavior is similar to that of the transmission line with short-circuited load for which the voltage and current alternate their maxima and minima, as will be shown in Section 2-7.

CHAPTER 2

Basics of Waveguides, Resonators and Transmission Lines

2-1 Introduction

In the previous chapters we studied propagation, polarization, reflection, and transmission of electromagnetic waves. In this chapter, we study how the electromagnetic wave is "guided" from one point to another, using certain transmission structures called *waveguides.* There are two ways to guide the wave in a bounded system, i.e., use either highly conducting materials or dielectric materials. When very good (so-called perfect) conductors such as metals are used, the wave is guided due to total reflection of waves on a perfectly conducting surface and the system is called the *metallic waveguide.* When dielectric material is used, the wave is guided based on the principle of total internal reflection and the system is called *dielectric waveguide.*

There are various forms of metallic waveguides. The simplest structure would be to use a pair of good conductors where the waves bounce off these conducting walls and are guided along the axis of these conducting structures. The examples are the parallel plate waveguide, the two-wire transmission line and the coaxial cable as shown in Figure 2-1(a)-(c). All of these waveguides support the transverse electromagnetic (TEM) mode which has no cutoff frequency. They are also called transmission lines. The TEM mode can propagate along these waveguides from dc up to some high frequency where the cross-section of the waveguide becomes electrically large. Hence they are useful in guiding signals with relatively low frequencies.

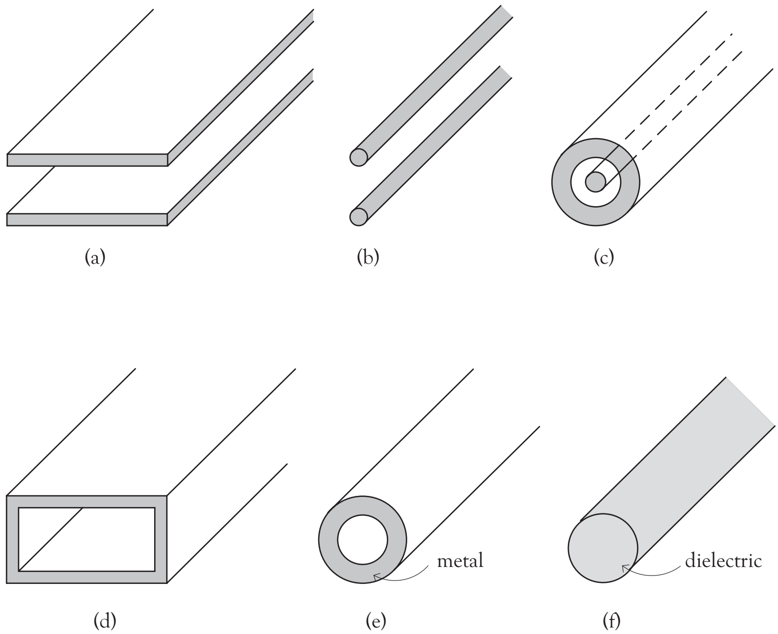


Figure 2-1. ***Various waveguides. (a) Parallel-plate waveguide, (b) two-wire transmission line, (c) coaxial cable, (d) rectangular waveguide, (e) circular waveguide, (f) optical fiber***

At high frequencies such as microwaves, the use of a two-conductor guiding system becomes impractical because the attenuation of TEM waves along the waveguide due to conductor loss (because the conductor is imperfect) increases with frequency. So at microwave frequencies, other forms of transmission system are used. The single-conductor hollow waveguides [see Figure 2-1(d), (e)] are able to transmit or guide waves in the GHz range over long distances without incurring large losses. Since metal pipes are single conductors with large surface areas, we expect that the attenuation due to conductor loss would be smaller. As will be shown, the hollow waveguides do not support the TEM mode but higher order modes – the TM (transverse magnetic) and TE (transverse electric) modes.

At optical frequencies higher than 1 THz, the metallic waveguides become impractical again due to increased conductor loss at higher frequencies, but the dielectric waveguides such as optical fibers shown in Figure 2-1(f) can provide guidance of optical signals with very small loss.

In this chapter we first discuss the general solution methods for waveguides with uniform cross section and introduce the concepts of TE, TM and TEM modes. We consider the following waveguides and resonators and find the solutions for the electric and magnetic fields and their applications.

1.Parallel plate waveguide
2.Rectangular waveguide
3.Coaxial cable
4.Rectangular cavity resonator

In the last section, we study characteristics of the transmission lines and their TEM mode analysis using the concepts of the voltage, current, and impedance.

2.2 Solution Methods for Uniform Waveguides

For the waveguiding structure whose cross section is uniform (or invariant) along certain direction (say, z-axis), the following method of analysis is generally applicable. We assume that the metallic walls are perfect conductors ($\sigma = \infty$) and the medium inside the waveguide is lossless ($\sigma = 0$) and has the real material constants μ, ε, as shown in Figure 2-2. The medium could be air.

In the medium within the guiding structure, **E** and **H** satisfy the Helmholtz wave equation

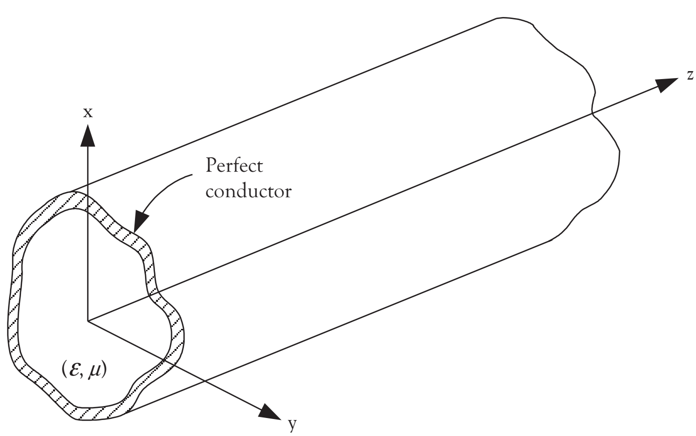


Figure 2-2. ***Uniform Waveguide***

$$(\nabla^2 + k^2)\mathbf{E} = 0 \tag{2-1a}$$

$$(\nabla^2 + k^2)\mathbf{H} = 0 \tag{2-1b}$$

where

$$k^2 = \omega^2\mu\varepsilon \tag{2-2}$$

Let us assume that we wish to *guide* the wave in the +*z direction.* Then the field solutions are assumed to take the following form:

$$\mathbf{E}(x, y, z) = \breve{\mathbf{E}}(x, y)e^{-jk_z z} \tag{2-3a}$$

$$\mathbf{H}(x, y, z) = \breve{\mathbf{H}}(x, y)e^{-jk_z z} \tag{2-3b}$$

because $e^{-jk_z z}$ represents a wave traveling in the +z direction.

Noting that

$$\frac{\partial}{\partial z}\begin{Bmatrix}\mathbf{E}\\ \mathbf{H}\end{Bmatrix} = -jk_z\begin{Bmatrix}\mathbf{E}\\ \mathbf{H}\end{Bmatrix}, \tag{2-4}$$

substitution of Eq. (2-3) into Eq. (2-1) yields

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k_z^2 + k^2\right)\begin{Bmatrix}\mathbf{E}\\ \mathbf{H}\end{Bmatrix} = 0 \tag{2-5}$$

Making use of the identity, Eq. (2-4), from the two Maxwell's vector curl equations

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{2-6a}$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} \tag{2-6b}$$

we obtain the following six scalar equations:

$$\frac{\partial E_z}{\partial y} + jk_z E_y = -j\omega\mu H_x \tag{2-7a}$$

$$-jk_z E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \tag{2-7b}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \tag{2-7c}$$

and

$$\frac{\partial H_z}{\partial y} + jk_z H_y = j\omega\varepsilon E_x \tag{2-8a}$$

$$-jk_z H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \tag{2-8b}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -j\omega\varepsilon E_z \tag{2-8c}$$

Because of the assumed form of the z dependence, from Eqs. (2-7a,b) and (2-8a,b) we can express E_x, E_y, H_x, H_y in terms of E_z and H_z. For example, from Eq. (2-7b), we find

$$H_y = \frac{k_z}{\omega\mu} E_x + \frac{1}{j\omega\mu}\frac{\partial E_z}{\partial x} \tag{2-9}$$

Substituting Eq. (2-9) into Eq. (2-8a) and solving for E_x, we obtain

$$E_x = \frac{-1}{k^2 - k_z^2}\left\{ jk_z \frac{\partial E_z}{\partial x} + j\omega\mu \frac{\partial H_z}{\partial y} \right\} \tag{2-10a}$$

Similarly, we can derive (see Problem 2-1)

$$E_y = \frac{-1}{k^2 - k_z^2}\left\{ -jk_z \frac{\partial E_z}{\partial y} + j\omega\mu \frac{\partial H_z}{\partial x} \right\} \tag{2-10b}$$

$$H_x = \frac{1}{k^2 - k_z^2}\left\{ j\omega\varepsilon \frac{\partial E_z}{\partial y} - jk_z \frac{\partial H_z}{\partial x} \right\} \tag{2-10c}$$

$$H_y = \frac{-1}{k^2 - k_z^2}\left\{ j\omega\varepsilon \frac{\partial E_z}{\partial x} + jk_z \frac{\partial H_z}{\partial y} \right\} \tag{2-10d}$$

Note that E_x and H_y are derived from Eqs. (2-7b) and (2-8a), while E_y and H_x are derived from Eqs. (2-7a) and (2-8b). Here we have expressed the transverse components (E_x, E_y, H_x, H_y) of the fields in terms of the longitudinal components (E_z, H_z) of the fields. We mean *"transverse"* with respect to the direction of guidance (z axis in this case). Thus we only need to solve for the z components (E_z, H_z) of the fields, which satisfy the following wave equation from Eq. (2-5).

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + (k^2 - k_z^2)E_z = 0 \tag{2-11a}$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + (k^2 - k_z^2)H_z = 0 \tag{2-11b}$$

Eqs. (2-11) are the second-order partial differential equations which are similar to Laplace's equation we treated in Volume 6 except for the existence of the third term. Thus the general solutions can be obtained by using the same technique, that is, the *method of separation of variables.* As will be shown in Section 2.4, the general solutions of Eq. (2-11) for E_z (or H_z) take the following form.

$$E_z(x, y, z) = \left\{A\sin k_x x + B\cos k_x x\right\} \left\{C\sin k_y y + D\cos k_y y\right\} e^{-jk_z z} \tag{2-12}$$

for which k_x, k_y, k_z should satisfy the dispersion relation:

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2\mu\varepsilon \tag{2-13}$$

The allowed or acceptable values for k_x, k_y and some of the coefficients (A, B, C, D) are determined by applying the appropriate boundary conditions at the surface of the conducting walls, i.e., $\mathbf{E}_{tan} = 0$, in the case of metallic waveguide.

It is convenient to classify the propagating waves in a waveguiding system into three types according to whether E_z or H_z exists.

(i) $E_z = 0$, $H_z \neq 0$: Transverse electric (TE) mode
(ii) $H_z = 0$, $E_z \neq 0$: Transverse magnetic (TM) mode
(iii) $E_z = H_z = 0$: Transverse electromagnetic (TEM) mode

It will be easier to find solutions for each type, separately. This also reduces the complexity of finding general waveguide solutions. For TE modes, the electric field has only transverse (x- and y-) components while the magnetic field may have both transverse and longitudinal (z-) components. Thus in order to obtain the TE mode solutions we only need to determine H_z ; then all the other components of **E** and **H** can be obtained through Eq. (2-10). For TM modes, the magnetic field has only transverse components while the electric field may have both components. Thus for the

TM mode solutions we seek to determine E_z first, from which all the other components can be calculated. In the case of the TEM mode, one has to find the transverse component directly. It is known that there exists a TEM mode for a two-conductor transmission system such as a two-wire transmission line, a parallel-plate waveguide and a coaxial cable. The TEM wave does not exist for a one-conductor system such as rectangular waveguide as will be shown later.

In summary, to obtain the field solutions for metallic waveguides whose cross section is uniform in the z direction, we take the following steps:

Step 1. Solve for E_z (for TM modes) or H_z (for TE modes) that satisfy Eq. (2-11), subject to the boundary conditions on the perfectly conducting surface [$E_{tan} = 0$].

Step 2. Determine the transverse components (E_x, E_y, H_x, H_y) from E_z or H_z, using Eq. (2-10).

We will start with the simplest waveguide, the parallel-plate waveguide, and then consider the rectangular waveguide and also the rectangular cavity resonator, whose solutions are a simple extension of the rectangular waveguide solutions.

2.3 Parallel-Plate Waveguide

A parallel-plate waveguide consists of two perfectly conducting plates separated by a distance *a* and filled with a medium having material constants μ, ε as shown in Figure 2-3. The width w of the conducting plate is assumed to be much greater than the separation a. for the purpose of finding simple and analytical solutions we assume that the cross section is infinite in extent in the y direction, i.e., $w \to \infty$. As mentioned in the previous section, we look for waves propagating in the +z direction.

First of all, because the plates are infinite in extent in the y direction and edge effects are neglected, we assume that all fields do not vary in the y direction. They are independent of y and all derivatives with respect to y are zero, i.e., $\frac{\partial}{\partial y} = 0$.

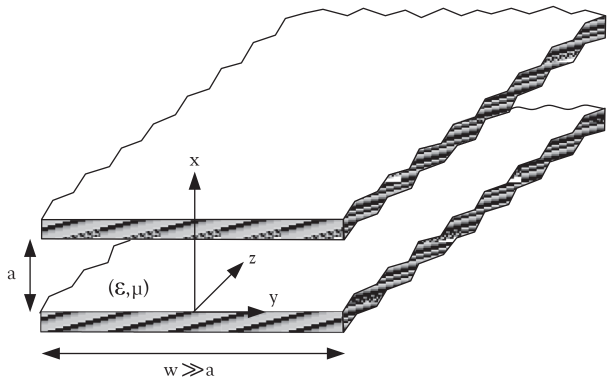


Figure 2-3. ***A parallel-plate waveguide***

Then Eqs. (2-10) reduce to

$$E_x = -\frac{jk_z}{k_x^2}\frac{\partial E_z}{\partial x},\ E_y = \frac{j\omega\mu}{k_x^2}\frac{\partial H_z}{\partial x}$$

$$H_x = -\frac{jk_z}{k_x^2}\frac{\partial H_z}{\partial x},\ E_y = -\frac{j\omega\mu}{k_x^2}\frac{\partial E_z}{\partial x} \qquad (2\text{-}14)$$

where

$$k_x^2 = k^2 - k_z^2 \qquad (2\text{-}15)$$

Eqs. (2-11) reduce to

$$\frac{\partial^2 E_z}{\partial x^2} + k_x^2 E_z = 0, \frac{\partial^2 H_z}{\partial x^2} + k_x^2 H_z = 0 \qquad (2\text{-}16)$$

Eq. (2-16) has two independent solutions, sin $k_x x$ and cos $k_x x$. We now consider two types of solutions separately.

2.3.1 TM Mode Solutions

For TM modes, $H_z = 0$ and we seek the solutions for $E_z(x,z)$ which satisfy Eq. (2-16). The solutions for E_z can be written as

$$E_z(x,z) = \left\{A \sin k_x x + B \cos k_x x\right\} e^{-jk_z z} \qquad (2\text{-}17)$$

where A and B are arbitrary constants. We now apply the boundary conditions (BC's) on the surface of the perfect conductor:

$$\mathbf{E}_{tan} = 0 \text{ or } E_y = E_z = 0 \text{ at } x = 0 \text{ and } x = a$$

Since we have formulated E_z, we can apply the BC's on E_z.

$$\text{BC (i) } E_z = 0 \text{ at } x = 0$$

$$\text{BC (ii) } E_z = 0 \text{ at } x = a$$

BC (i) leads to B = 0, thus

$$E_z = A\sin(k_x x)e^{-jk_z z} \tag{2-18}$$

BC (ii) gives the following condition and determines the allowed values of k_x.

$$\sin(k_x a) = 0 \rightarrow k_x a = mp, \ m = \text{integer}$$

$$k_x = \frac{m\pi}{a}, \ m = 1,2,3,... \tag{2-19}$$

The values obtained for k_x are known as the *eigenvalues* or the *characteristic values.* Making use of Eq. (2-15) and Eq. (2-2), we obtain the following dispersion relation:

$$k_x^2 + k_z^2 = k^2 \text{ or } \left(\frac{m\pi}{a}\right) + k_z^2 = \omega^2\mu\varepsilon \tag{2-20}$$

Therefore, the final solutions for E_z for TM modes are

$$E_z = E_0 \sin\left(\frac{m\pi}{a}x\right)e^{-jk_z z}, \ TM_m \text{ mode} \tag{2-21}$$

where

$$k_z = \sqrt{\omega^2\mu\varepsilon - (\frac{m\pi}{a})^2}, \ m = 1, 2, 3, ... \tag{2-22}$$

We have let A = E_o in Eq. (2-18). Eq. (2-22) is obtained from Eq. (2-20). It is to be noted that the field distribution and the propagation constant k_z depend on the value of m; thus given m we call it the *TM_m mode.*

The transverse components of the fields are obtained from Eq. (2-14):

$$E_x = -j\frac{k_z}{m\pi/a}E_0\cos\left(\frac{m\pi}{a}x\right)e^{-jk_z z}$$
$$H_y = -j\frac{\omega\varepsilon}{m\pi/a}E_0\cos\left(\frac{m\pi}{a}x\right)e^{-jk_z z} \qquad (2\text{-}23)$$
$$E_y = H_x = 0$$

Propagating Mode vs. Cutoff

First of all, as seen in Eq. (2-21), we note that in order for the wave to be guided in the z direction without attenuation the propagation constant k_z must be real. From Eq. (2-22), we observe that

(i) When $\omega\sqrt{\mu\varepsilon} > \frac{m\pi}{a}$, k_z becomes real.

$$k_z \equiv \beta = \sqrt{\omega^2\mu\varepsilon - (\frac{m\pi}{a})^2} \qquad (2\text{-}24)$$

then the wave propagates.

(ii) When $\omega\sqrt{\mu\varepsilon} < \frac{m\pi}{a}$, k_z becomes purely imaginary.

$$k_z = -j\alpha = -j\sqrt{(\frac{m\pi}{a})^2 - \omega^2\mu\varepsilon} \rightarrow e^{-jk_z z} = e^{-\alpha z} \qquad (2\text{-}25)$$

then the wave attenuates rapidly with z and is called an *evanescent* wave. We say that the wave is in the *cutoff* region. The frequency at which cutoff occurs, i.e., where the propagation constant k_z changes from β to −jα, is called the **cutoff frequency** and it is obtained by letting $k_z = 0$ in Eq. (2-24).

$$(2\pi f_c)\sqrt{\mu\varepsilon} = \frac{m\pi}{a} \rightarrow f_c = \frac{1}{\sqrt{\mu\varepsilon}}\frac{m}{2a} = \frac{v}{2a}m \qquad (2\text{-}26)$$

The cutoff frequency depends on the mode number m. Eq. (2-26) gives the cutoff frequency for the TM_m mode. Then the propagating mode vs. cutoff are determined by whether the operating frequency f is greater than the cutoff frequency f_c.

(i)If $f > f_{c,m} = \frac{v}{2a}m$, then the TM_m mode propagates.

(ii) If $f < f_{c,m} = \frac{v}{2a} m$, then the TM_m mode is cutoff.

Dominant Mode

The mode that has the lowest cutoff frequency is called the **dominant mode.** From Eq. (2-26), when m = 0, $f_c = 0$, thus the lowest possible value for the cutoff frequency. Therefore, the **TM_0 mode** is the dominant mode among the TM modes of the parallel-plate waveguide. When m = 0, it is seen from Eqs. (2-21) and (2-23) that

$$E_z = 0,\ E_x = E_0' e^{-jkz},\ H_y = \frac{E_0'}{\eta} e^{-jkz} \text{ for } TM_0 \text{ mode} \quad (2\text{-}27)$$

where we have redefined $E_0' = -j\frac{k_z}{m\pi/a}E_0$ and used $k_z = k$ and $\frac{k_z}{\omega\varepsilon} = \eta$ when m = 0.

In this case, $E_z = 0$ as well as $H_z = 0$. Both electric and magnetic fields are transverse to the direction of guidance (z). Thus the TM_0 mode is a **TEM** (transverse electromagnetic) mode. The mode that has the next higher cutoff frequency is the TM_1 mode for which

$$f_{c,1} = \frac{v}{2a} \quad (2\text{-}28)$$

and the corresponding cutoff wavelength is

$$\lambda_{c,1} = \frac{v}{f_{c,1}} = 2a \quad (2\text{-}29)$$

If we choose the operating frequency such that $f_{c,o} < f < f_{c,1}$ or $0 < f < \frac{v}{2a}$ or $\lambda > 2a$, only the dominant TEM mode will propagate and all the higher-order modes will be cutoff, because $f < f_{c,m} = \frac{v}{2a} m \ (m = 1, 2, \ldots.)$. Such choice is known as the *single mode operation.* In most practical applications, the waveguide operates this way because the presence of multiple modes creates a dispersion problem where each mode propagates with different velocity, leading to the distortion of the signal.

Guide Wavelength, Phase Velocity and Group Velocity

When the wave is in the propagating mode, the phase constant is given by

$$k_z \equiv \beta = \omega\sqrt{\mu\varepsilon} = \sqrt{1-(\frac{v}{f}\frac{m}{2a})^2} = k\sqrt{1-(\frac{f_c}{f})^2} \qquad (2\text{-}30)$$

The corresponding wavelength in the waveguide, called the **guide wavelength,** is given by

$$\lambda_g \equiv \frac{2\pi}{k_z} = \frac{2\pi}{k\sqrt{1-(f_c/f)^2}} = \frac{\lambda}{\sqrt{1-(\frac{f_c}{f})^2}},\ f > f_c \qquad (2\text{-}31)$$

where λ is the wavelength of the wave in an unbounded medium. We note that $\lambda_g > \lambda$.

The phase velocity of the wave in the waveguide is given by

$$V_p \equiv \frac{\omega}{k_z} = \frac{\omega}{k\sqrt{1-(\frac{f_c}{f})^2}} = \frac{v}{\sqrt{1-(\frac{f_c}{f})^2}},\ f > f_c \qquad (2\text{-}32)$$

where v is the phase velocity of the wave in an unbounded medium. We note that $v_p > v$ and v_p depends on the frequency and the mode number. Thus *the waveguide is a dispersive transmission system,* similar to the dispersion in a lossy medium. The group velocity will be different from the phase velocity and is given by

$$V_g = \frac{\partial\omega}{\partial k_z} = \frac{1}{(\frac{\partial k_z}{\partial\omega})} = v\sqrt{1-(\frac{f_c}{f})^2} \qquad (2\text{-}33)$$

We note that $v_g < v$, i.e., the group velocity cannot exceed v because the velocity of the energy transport in a lossless waveguide is equal to the group velocity.

Wave Impedance

The wave impedance defined as the ratio of the electric and magnetic fields in their transverse components is given by

$$Z_{TM} \equiv \frac{E_x}{H_y} = \frac{k_z}{\omega\varepsilon} = \eta\sqrt{1-(\frac{f_c}{f})^2},\ f > f_c \qquad (2\text{-}34)$$

It is to be noted that while the wave impedance is real in the propagating mode (when $f > f_c$), it becomes purely imaginary or reactive when the wave is cutoff ($f < f_c$) because $k_z = -j\alpha$.

Time-Average Power Flow

The time-average Poynting vector for the TM_m mode is given by
·D. K. Cheng, *Field and Wave Electromagnetics,* pp. 541-543, Addison-Wesley Publishing Co., 1989, 2[nd] ed.

$$\begin{aligned}\mathbf{S}_{av} &= \frac{1}{2}\mathrm{Re}\{\mathbf{E}\times\mathbf{H}^*\} = \frac{1}{2}\mathrm{Re}\left\{\left(\mathbf{a}_x E_x + \mathbf{a}_z E_z\right)\mathbf{a}_r H_y^*\right\} \\ &= \frac{1}{2}\mathrm{Re}\left\{-\mathbf{a}_x E_z H_y^* + \mathbf{a}_z E_x H_y^*\right\} \\ &= \mathbf{a}_z\frac{1}{2}E_x H_y^* = \mathbf{a}_z\frac{1}{2}k_z\omega\varepsilon(\frac{a}{m\pi})^2\left|E_0\right|^2\cos^2(\frac{m\pi}{a}x),\ f > f_c\end{aligned}$$

(2-35)

when the wave is in the propagating mode. $E_z H_y^*$ is purely imaginary. When the wave is cutoff, k_z becomes purely imaginary and $\mathbf{S}_{av} = 0$, i.e., the wave will not propagate as expected. The time-average power flowing through the waveguide can be easily calculated by integrating $\mathbf{S}_{av}$ over the cross section of the waveguide.

Finally we interpret the propagation of the wave within the waveguide as represented by the field solutions. The solutions given in Eqs. (2-21) and (2-23) represent a standing wave in the x direction and a traveling wave in the z direction. The standing wave can be written as a superposition of two oppositely traveling waves:

$$\sin(\frac{m\pi}{a}x) = \frac{1}{2j}\left(e^{j\frac{m\pi}{a}x} - e^{-j\frac{m\pi}{a}x}\right)$$

Thus

$$E_z = \frac{(-E_0)}{2j}\left(e^{-j\frac{m\pi}{a}x} e^{-jk_z z} - e^{j\frac{m\pi}{a}x} e^{-jk_z z} \right) \qquad (2\text{-}36)$$

The first term represents a plane wave traveling in the $+_x$ and $+_z$ directions and the second term represents a wave traveling in the $-_x$ and $+_z$ directions. These two waves can be illustrated in Figure 2-4 where the upward traveling wave corresponds to the first term and the downward traveling wave corresponds to the second term, which is due to the perfect reflection from the upper conducting boundary. The ratio of the amplitudes of the two waves should be a reflection coefficient for an electric field at the perfectly conducting surface, R = −1. Eq. (2-36) is consistent with this interpretation. When m ≠ 0, the wave doesn't travel straight down within the waveguide but bounces between the two conducting walls as it is guided.

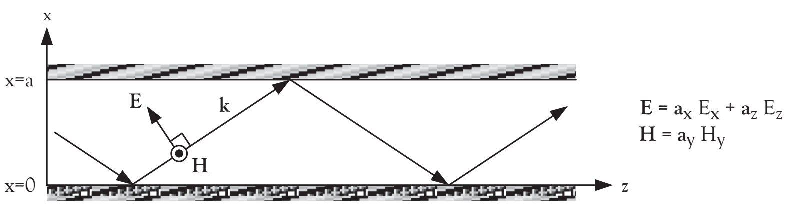


Figure 2-4. ***Guidance of the TM wave in a parallel-plate waveguide***

2.3.2 TE Mode Solutions

For the second type of solutions, the TE modes, $E_z = 0$ and we seek the solutions for $H_z(x,z)$ which satisfy Eq. (2-16). The solutions for H_z can be similarly written as

$$H_z(x,z) = \left\{ C \sin k_x x + D \cos k_x x \right\} e^{-jk_z z} \qquad (2\text{-}37)$$

where C and D are arbitrary constants. The boundary conditions (BC's) on the conducting plates are

$$E_y = \frac{j\omega\mu}{k_x^2}\frac{\partial H_z}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = a$$

The first BC, $E_y \propto \frac{\partial H_z}{\partial x} = 0$ at x = 0, leads to C = 0, thus

$$H_z = D\cos(k_x x)e^{-jk_z z} \tag{2-38}$$

The second BC, $E_y \propto \frac{\partial H_z}{\partial x} = 0$ at x = a, determines the allowed values of k_x.

$$\sin(k_x a) = 0 \rightarrow k_x = \frac{m\pi}{a},\ m = 1, 2, 3, \ldots \tag{2-19}$$

which is the same result as that of the TM modes, Eq. (2-19). k_x and k_z will also satisfy the same dispersion relation, Eq. (2-20).

The final solutions for H_z for the TE modes are

$$H_z = H_0 \cos\left(\frac{m\pi}{a}x\right)e^{-jk_z z},\ TE_m \text{ mode} \tag{2-39}$$

where

$$k_z = \sqrt{\omega^2\mu\varepsilon - (\frac{m\pi}{a})^2},\ m = 1, 2, 3, \ldots. \tag{2-22}$$

We have let D = H_o in Eq. (2-38). We again notice that the field distribution and the propagation constant k_z depend on the mode number m. The transverse components of the fields are obtained from Eq. (2-14) as follows.

$$\begin{aligned} E_y &= -j\frac{\omega\varepsilon}{m\pi/a}H_0\sin\left(\frac{m\pi}{a}x\right)e^{-jk_z z} \\ H_x &= j\frac{k_z}{m\pi/a}H_0\sin\left(\frac{m\pi}{a}x\right)e^{-jk_z z} \\ E_x &= H_y = 0 \end{aligned} \tag{2-40}$$

The TM and TE solutions for the parallel-plate waveguide are summarized in Table 2-1.

Table 2-1 Field solutions for parallel-plate waveguide

TM_m mode	TE_m mode
$\mathbf{E} = \mathbf{a}_x E_x + \mathbf{a}_z E_z$ $\mathbf{H} = \mathbf{a}_y H_Y$ $E_z = E_o \sin\left(\frac{m\pi}{a}x\right)e^{-jk_z z}$ $E_x = -\frac{jk_z}{m\pi/a}E_o \cos\left(\frac{m\pi}{a}x\right)e^{-jk_z z}$ $H_y = -\frac{j\omega\varepsilon}{m\pi/a}E_o \cos\left(\frac{m\pi}{a}x\right)e^{-jk_z z}$	$\mathbf{E} = \mathbf{a}_y E_y$ $\mathbf{H} = \mathbf{a}_x H_x + \mathbf{a}_z H_z$ $H_z = H_o \cos\left(\frac{m\pi}{a}x\right)e^{-jk_z z}$ $H_x = -\frac{jk_z}{m\pi/a}H_o \sin\left(\frac{m\pi}{a}x\right)e^{-jk_z z}$ $E_y = -\frac{j\omega\varepsilon}{m\pi/a}H_o \sin\left(\frac{m\pi}{a}x\right)e^{-jk_z z}$
Phase Constant	$k_z = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2}$
Cutoff Frequency	$f_{c,m} = \frac{\upsilon}{2a}m = \frac{1}{\sqrt{\mu\varepsilon}}\frac{m}{2a}$
Guide Wavelength	$\lambda_g = \frac{2\pi}{k_z} = \frac{\lambda}{\sqrt{1-(f_c/f)^2}}$
Phase Velocity	$\upsilon_p = \frac{\omega}{k_z} = \frac{\upsilon}{\sqrt{1-(f_c/f)^2}}$
Wave Impedance	$Z_{TM} = \sqrt{1-\left(\frac{f_c}{f}\right)^2}, \; Z_{TE} = \frac{\eta}{\sqrt{1-(f_c/f)^2}}$
Dominant Mode: TM_o or TEM mode $\mathbf{E} = \mathbf{a}_x E_o \; e^{-jkz}, \; \mathbf{H} = \mathbf{a}_y \frac{E_o}{\eta} \; e^{-jkz}$ $k = \omega\sqrt{\mu\varepsilon}, \; f_{c,0} = 0$ (no cutoff frequency)	

All the discussions about the cutoff frequency, guide wavelength, phase velocity, group velocity, wave impedance and the standing wave interpretation, presented in the previous section, are valid also for the TE modes except the following differences.

(i) The dominant TE mode which has the lowest cutoff frequency among the TE modes is not the m = 0 mode but the *TE_1 mode* for which $f_{c,1} = v/2a$.
When m = 0, $E_y = H_x = 0$; such mode does not exist. Overall, the TM_o or TEM mode is *the* dominant mode for parallel plate waveguide. The field lines for the TE_1 mode and the TM_1 mode are shown in Figure 2-5.

(ii) The wave impedance for the TE mode is given by

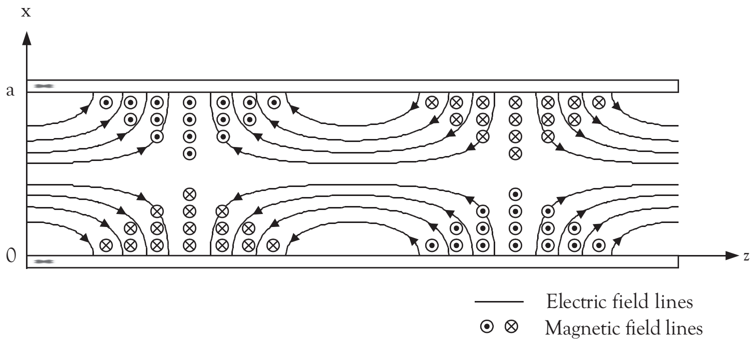
$$Z_{TE} = -\frac{E_y}{H_x} = \frac{\omega\mu}{k_z}\frac{\eta}{\sqrt{1-(\frac{f_c}{f})^2}}, \; f > f_c \tag{2-41}$$

We note that $Z_{TM} < \eta$ and $Z_{TE} > \eta$ for the propagating modes and both Z_{TM} and Z_{TE} approach η (the intrinsic impedance of the medium) as $f \to \infty$.

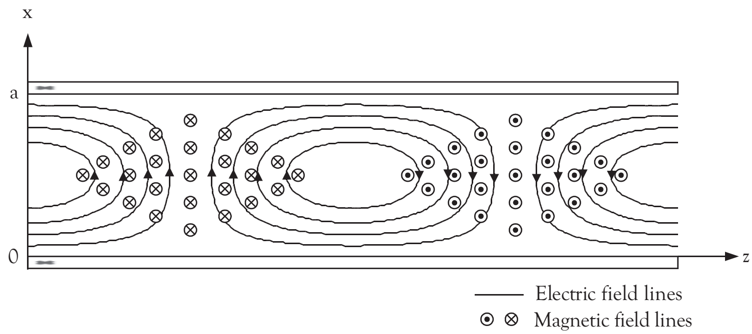
(iii) The time-average Poynting vector for the TE_m mode is given by

$$\begin{aligned}\mathbf{S}_{av} &= \frac{1}{2}\mathrm{Re}\{\mathbf{E}\times\mathbf{H}^*\} = \frac{1}{2}\mathrm{Re}\left\{\mathbf{a}_y E_y \times \left(\mathbf{a}_x H_x + \mathbf{a}_z H_z\right)^*\right\} \\ &= \frac{1}{2}\mathrm{Re}\left\{\mathbf{a}_x E_y H_z^* + \mathbf{a}_z E_y H_x^*\right\} \\ &= \mathbf{a}_z\left\{-\frac{1}{2}E_y H_x^*\right\} = \mathbf{a}_z\frac{1}{2}k_z\omega\mu(\frac{a}{m\pi})^2\left|H_0\right|^2\sin^2(\frac{m\pi}{a}x), \; f > f_c\end{aligned}$$

Note that when the wave is in the propagating mode, $E_y H_z^*$ is purely imaginary.



(a) TM_1 mode



(b) TE_1 mode

Figure 2-5. ***Field lines for the TM_1 and TE_1 modes in a parallel-plate waveguide***

Example 2-1. Microstrip Line

Mircrostrip line is one of the most commonly used planar transmission line structures in microwave integrated circuits because it can be easily fabricated by printed- circuit techniques. It consists of a metallic strip of width w and a large conducting ground plane separated by a thin layer of dielectric substrate of thickness d as shown in Figure 2-6.

The complete analysis of the electromagnetic fields that propagate in microstrip line is very complicated. The fields exist inside the dielectric layer as well as in the upper air region. However, if the width of the strip is much larger than the thickness of the layer, then the microstrip line can be modeled as a parallel-plate waveguide and the fields of the dominant mode can be approximated by the TEM mode (or the TM_o mode) discussed earlier.

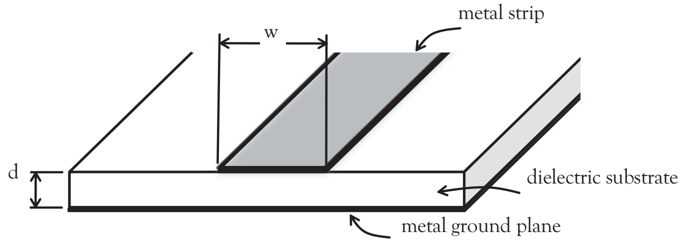


Figure 2-6: ***Microstrip transmission line***

Suppose the microstrip line has w = 0.7 cm and d = 1.4 mm, and the substrate has $\varepsilon = 1.96\,\varepsilon_o$, $\mu = \mu_o$, $\sigma = 0$.

(c) What is the range of frequencies for which only the dominant mode propagates?

(d) Calculate the time-average power for the dominant mode that is transmitted by the line when the amplitude of the electric field is 7 kV/m.

Solutions:

(c) If we model the microstrip line as parallel-plate waveguide, the dominant mode is the TEM mode which has no cutoff frequency (or zero cutoff frequency $f_{c,0} = 0$) and the next higher mode is either TM_1 or TE_1 mode whose cutoff frequency is given by

$$f_{c,1} = \frac{v}{2d} = \frac{1}{\sqrt{\mu\varepsilon}}\frac{1}{2d} = \frac{1}{\sqrt{\mu_o 1.96\varepsilon_o}}\frac{1}{2\times1.4\times10^{-3}} = \frac{3\times10^8}{1.4\times2.8\times10^{-3}} = 7.65\times10^{10}\,\text{Hz}$$

Thus the range of frequencies for which only the dominant mode propagates is

$$0 < f < 76.5\ \text{GHz}$$

(d) For the TEM mode, the fields are given by

$$\mathbf{E} = \mathbf{a}_x E_o\ e^{-jkz},\ \mathbf{H} = \mathbf{a}_y \frac{E_o}{\eta}\ e^{-jkz}$$

The time-average Poynting vector is given by

$$\mathbf{S}_{av} = \frac{1}{2}\text{Re}\left(\mathbf{E}\times\mathbf{H}^*\right) = a_z \frac{1}{2}\text{Re}\left\{E_o\left(\frac{E_o}{\eta}\right)^*\right\} = \mathbf{a}_z \frac{|E_o|^2}{2\eta}$$

The time-average power for the TEM mode transmitted by the line is approximately obtained by integrating over the dielectric cross section of width w and thickness d:

$$P_{av} = \iint S_{av} \cdot ds = \int_o^w \int_o^d \frac{|E_o|}{2\eta} dxdy = \frac{|E_{o^2}| wd}{2\sqrt{\frac{\mu_o}{1.96\varepsilon_o}}} = \frac{\left(7\times10^3\right)^2 \cdot 7\times10^{-3} \cdot 1.4\times10^{-3}}{2\times\frac{377}{1.4}} = 0.89[w]$$

Note that we have ignored the field that exists in the dielectric beyond the width w.

Example 2-2. Dielectric Breakdown

A dielectric substrate in a waveguide breaks down when a strong electric field is applied. The maximum electric field magnitude that a dielectric material can withstand without breakdown is called the **dielectric strength** of the material. Suppose you design a parallel-plate waveguide that has w = 20 d, filled with the dielectric (polystyrene) whose dielectric constant is 2.6 and dielectric strength is 20×10^6 V/m. What is the smallest thickness of the dielectric that can propagate 1 kW of power?

Solutions:

Assuming that the dominant (TEM) mode propagates in the waveguide, the time-average power transmitted (see Example 2-1) is given by

$$P_{av} = \frac{|E_o|^2}{2\eta} wd = \frac{|E_{o^2}|^2}{2\sqrt{\frac{\mu_o}{2.6\varepsilon_o}}} 20d^2 = \frac{|E_{o^2}|^2}{2\times 233.8} 20d^2 = 10^3 [w]$$

Since the dielectric strength is 20×10^6 V/m, $|E_o| \leq 2\times10^7$ [V/m].

The thickness of the dielectric should be

$$d = \sqrt{\frac{10^3 \times 2 \times 233.8}{20|E_o|^2}} \geq \sqrt{\frac{10^2 \times 233.8}{\left(2\times10^7\right)^2}} = 7.65\times10^{-6}[m]$$

Attenuation in the Waveguide

In this analysis, we have assumed that the conducting plates are perfect conductors ($\sigma = \infty$). In practice, the metals that are used in the waveguide are *not perfect* and they have high, but *finite,* conductivities. Thus the

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* L. C. Shen and J. A. Kong, *Applied Electromagnetism,* PWS Publishing Company, 1995, 3rd Ed.

wave can penetrate into the conducting walls as it propagates along the waveguide. Some of the power leaks into the conducting surfaces and is dissipated as heat. Then the signal in the waveguide is attenuated as it propagates. The attenuation constant (α) can be obtained by calculating the power flowing along the waveguide and the power dissipated on the conducting walls

Additional Notes on Solution Method

In our approach we have formulated the longitudinal components first, namely, E_z for TM modes and H_z for TE modes and then we obtained the transverse components, E_x, H_y for the TM modes and E_y, H_x for TE modes. There is another approach taken by others [Shen and Kong (1995)*] for the parallel-plate waveguide. They formulate the transverse components first, namely, H_y for TM modes and E_y for TE modes and then calculate the remaining components of the fields using Maxwell's curl equations. This approach works for the parallel-plate waveguide but doesn't work for the rectangular waveguide because the fields depend on both x and y. For the parallel-plate waveguide for which $\frac{\partial}{\partial y} = 0$, the fields for the TE and TM modes are obtained as follows.

TE (Transverse electric) modes:

$$\mathbf{E} = \mathbf{a}_y E_y, \ \mathbf{H} = \mathbf{a}_x H_x + \mathbf{a}_z H_z$$

$$H_x = \frac{1}{j\omega\mu}\frac{\partial E_y}{\partial z}, \ H_z = -\frac{1}{j\omega\mu}\frac{\partial E_y}{\partial x} \tag{2-42}$$

TM (Transverse magnetic) modes:

$$\mathbf{H} = \mathbf{a}_y H_y, \ \mathbf{E} = \mathbf{a}_x E_x + \mathbf{a}_z E_z$$

$$E_x = -\frac{1}{j\omega\mu}\frac{\partial H_y}{\partial z}, \ E_z = \frac{1}{j\omega\mu}\frac{\partial H_y}{\partial x} \tag{2-43}$$

11.4 Rectangular Waveguide

In practice, we cannot construct an *"ideal"* parallel-plate waveguide that is infinite in extent in one (y) direction. Since the width w in the y direction is finite, we will have some fringing fields at both ends. This leads to some loss of energy and imperfect guidance. Such a problem can be avoided by placing another two parallel conducting walls at both ends in the y direction. Then it becomes a rectangular waveguide. Now the waves inside the waveguide bounce off the side walls as well as the top and bottom conductors as they are guided within the closed structure. Since all four conducting walls are connected, they form a *one-conductor* transmission system. The geometry of a rectangular waveguide is shown in Figure 2-7. We again assume that the metallic walls are perfect conductors ($\sigma = \infty$) and the medium inside the waveguide is lossless ($\sigma = 0$) and has the real material constants μ, ε. The x and y dimensions of the cross section are *a* and *b,* respectively.

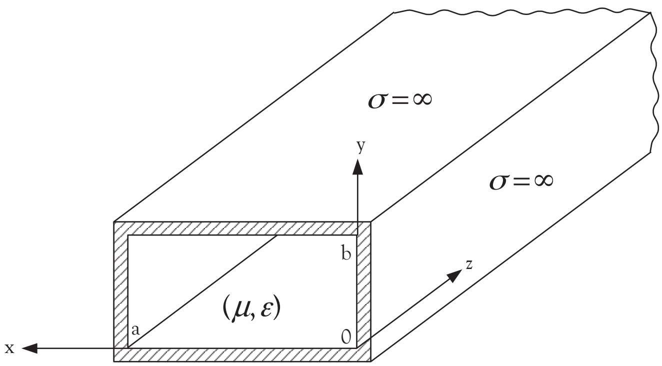


Figure 2-7. ***A rectangular waveguide***

2.4.1 General Field Solutions

Since the cross section of the waveguiding structure is uniform (it is a rectangle), the method of analysis described in Section 2.2 is applicable. We will first find the general form of field solutions for the longitudinal components E_z, H_z by solving Eq. (2-11) with the method of separation of variables. The unknown coefficients that appear in Eq. (2-12) are determined separately for the TM and TE modes, by applying the boundary conditions. Finally, the transverse components are obtained by Eq. (2-10).

First we rewrite the wave equation for uniform waveguides that E_z and H_z should satisfy.

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + (k^2 - k_z^2)E_z = 0 \qquad \text{(2-11a)}$$

In order to separate the two variables, we assume the solution of Eq. (2-11a) for $E_z(x,y,z)$ in the following form:

$$E_z(x,y,z) = X(x)\ Y(y)e^{-jk_z z} \quad \text{(2-44)}$$

where X(x) and Y(y) are functions of only x and y, respectively. Substitution of Eq. (2-44) into Eq. (2-11a) yields

$$Y(y)\frac{\partial^2 X(x)}{\partial x^2} + X(x)\frac{\partial^2 Y(y)}{\partial x^2} + (k^2 - k_z^2)X(x)\ Y(y) = 0 \quad \text{(2-45)}$$

Dividing Eq. (2-45) by X(x) Y(y), we obtain

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2} + (k^2 - k_z^2) = 0 \quad \text{(2-46)}$$

Note that the partial derivatives are replaced by the ordinary derivatives because X(x) and Y(y) are functions of one variable. In Eq. (2-46), we observe that the first term is a function of x only, the second term is a function of y only, the third term is a constant, and the sum has to be zero. In order for Eq. (2-46) to be satisfied for *all values of x and y,* each term must be a constant which is independent of x and y.

Thus we let

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = C_1 \equiv -k_z^2 \to \frac{d^2 X(x)}{dx^2} + k_z^2 X(x) = 0 \quad \text{(2-47a)}$$

$$\frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2} = C_2 \equiv -k_z^2 \to \frac{d^2 Y(y)}{dy^2} + k_z^2 Y(y) = 0 \quad \text{(2-47b)}$$

A choice of separation constants in a particular form allows the solutions of Eq. (2-47) to be written in a physically meaningful way. As discussed in Volume 6 the general solutions of Eqs. (2-47a) and (2-47b) are, when $k_x \neq 0$, $k_y \neq 0$, respectively,

$$X(x) = A\ \sin(k_x x) + B\ \cos(k_x x) \quad \text{(2-48a)}$$

$$Y(y) = C\ \sin(k_y y) + D\ \cos(k_y y) \quad \text{(2-48b)}$$

Therefore, the general solution for $E_z(x,y,z)$ or $H_z(x,y,z)$ can be written as

$$F(x,y,z) = \left\{A\sin k_x x + B\cos k_x x\right\}\left\{C\sin k_y y + D\cos k_y y\right\}e^{-jk_z z}$$
$$F(x,y,z) = E_z(x,y,z) \text{ or } H_z(x,y,z) \tag{2-49}$$

We recognize that the fields in Eq. (2-49) represent a standing wave in both the x and y directions and a traveling wave in the z direction, as expected. The values of k_x and k_y are not arbitrary but related to the propagation constant k_z. The relationship, known as the *dispersion relation,* is obtained by substituting Eq. (2-47) into Eq. (2-46).

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2\mu\varepsilon \tag{2-50}$$

As was done in Section 2.3, we will seek field solutions for the TM and TE modes, separately.

2.4.2 TM Mode Solutions

For TM modes, $H_z = 0$ and we seek the solutions for $E_z(x,y,z)$ in the form of Eq. (2-49). The boundary conditions are

$$\mathbf{E}_{tan} = 0 \text{ on four counducting walls}$$

Thus we have

$$\text{(i) } E_z = 0,\ E_y = 0 \text{ at } x = 0 \tag{2-51a}$$

$$\text{(ii) } E_z = 0,\ E_y = 0 \text{ at } x = a \tag{2-51b}$$

$$\text{(iii) } E_z = 0,\ E_x = 0 \text{ at } y = 0 \tag{2-51c}$$

$$\text{(iv) } E_z = 0,\ E_x = 0 \text{ at } y = b \tag{2-51d}$$

Since we have formulated E_z, we can use the boundary condition (BC), $E_z = 0$ directly. We follow the similar procedures which were used in the parallel-plate waveguide.

$$\text{BC (i): } E_z = 0 \rightarrow X(x) = 0 \text{ at } x = 0 \rightarrow X(x) = A\sin(k_x x)$$

$$\text{BC (iii): } E_z = 0 \rightarrow Y(y) = 0 \text{ at } y = 0 \rightarrow Y(y) = C\sin(k_y y)$$

Then $E_z(x,y,z)$ can be written as

$$E_z(x,y,z) = E_z \sin(k_x x)\sin(k_y y)e^{-jk_z z} \quad (2\text{-}52)$$

where E_o = AC. The separation constants k_x, k_y are determined by applying the BC's, (ii) and (iv).

$$\text{BC (ii): } E_z = 0 \rightarrow X(x) = 0 \text{ at } x = a \rightarrow \sin k_x a = 0$$

$$k_x a = m\pi \text{ or } k_x = \frac{m\pi}{a},\ m = 1,2,3,... \quad (2\text{-}53a)$$

$$\text{BC (iv): } E_z = 0 \rightarrow Y(y) = 0 \text{ at } y = b \rightarrow \sin k_y b = 0$$

$$k_y b = n\pi \text{ or } k_y = \frac{n\pi}{b},\ n = 1,2,3,... \quad (2\text{-}53b)$$

Note that when m or n = 0, the field vanishes. Each choice of the integers m and n yields the TM_{mn} mode. The final solution for E_z for the TM modes is then given by

$$E_z(x,y,z) = E_0 \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-jk_z z},\ TM_{mn}\text{mode} \quad (2\text{-}54)$$

The propagation constant (or phase constant) k_z is obtained from Eq. (2-50).

$$k_z = \sqrt{k^2 - (k_x^2 + k_y^2)} = \sqrt{\omega^2\mu\varepsilon - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]} \quad (2\text{-}55)$$

The transverse components of the fields can be calculated by using Eq. (2-10) with H_z = 0. They are listed in Table 2-2. Unlike the parallel-plate waveguide, we have all four components of the fields, E_x, E_y, H_x, H_y.

Propagating Mode vs. Cutoff

As explained in Section 2.3, in order for the wave to be guided in the z direction without attenuation the propagation constant k_z must be real. From Eq. (2-55), we observe that

(i) When $\omega\sqrt{\mu\varepsilon} > \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$, k_z becomes real.

Table 2-2 Field Solutions for Rectangular Waveguide

	TM_{mn} mode	TM_{mn} mode
E_z	$E_o \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$	0
H_z	0	$H_o \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$
E_x	$-\frac{jk_z}{k_\rho^2}\left(\frac{m\pi}{a}\right)E_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$	$\frac{j\omega\mu}{k_\rho^2}\left(\frac{n\pi}{b}\right)H_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$
E_y	$-\frac{jk_z}{k_\rho^2}\left(\frac{n\pi}{b}\right)E_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$	$-\frac{j\omega\mu}{k_\rho^2}\left(\frac{m\pi}{a}\right)H_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$
H_x	$\frac{j\omega\varepsilon}{k_\rho^2}\left(\frac{n\pi}{b}\right)E_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$	$\frac{jk_z}{k_\rho^2}\left(\frac{m\pi}{a}\right)H_o \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$
H_y	$-\frac{j\omega\varepsilon}{k_\rho^2}\left(\frac{m\pi}{a}\right)E_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$	$\frac{jk_z}{k_\rho^2}\left(\frac{n\pi}{b}\right)H_o \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$
$k_\rho^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = k_z^2$		

Phase Constant	$k_z = \sqrt{\omega^2 \mu\varepsilon - \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]}$
Cutoff Frequency	$f_{c,mn} = \frac{\upsilon}{2} \sqrt{\left(\frac{m}{a} \right)^2 + \left(\frac{n}{b} \right)^2}$
Dominant Mode: TE_{10} mode $(a > b)$	
$\mathbf{E} = \mathbf{a}_y E_y$, $\mathbf{H} = \mathbf{a}_x H_x + \mathbf{a}_z H_z$	
$H_z = H_o \cos\left(\frac{\pi}{a} x \right) e^{-jk_z z}$ (same as the TE_1 mode of the parallel plate waveguide)	
$f_{c,10} = \frac{\upsilon}{2} \frac{1}{\sqrt{\mu\varepsilon}} \frac{1}{2a}$, $\lambda_{c,10} = 2a$	

$$k_z \equiv \beta = \sqrt{\omega^2\mu\varepsilon - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}, \tag{2-56}$$

then the wave propagates.

(ii) When , k_z becomes purely imaginary.

$$k_z = -j\alpha = -j\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon} \rightarrow e^{-jk_z z} e^{-\alpha z} \tag{2-57}$$

then the wave attenuates rapidly with z and becomes an *evanescent* wave. The wave is in the *cutoff* region. The frequency at which cutoff occurs, the *cutoff frequency,* is obtained by letting $k_z = 0$ in Eq. (2-56).

$$(2\pi f_c)\sqrt{\mu\varepsilon} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}, \rightarrow f_c = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \tag{2-58}$$

The cutoff frequency depends on the mode numbers m and n. Eq. (2-58) gives the cutoff frequency for the TM_{mn} mode. Then the propagating mode vs. cutoff are determined by whether the operating frequency f is greater than the cutoff frequency f_c.

(i)if $f > f_{c,mn} = \frac{v}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$ then the TM_{mn} mode is propagates.

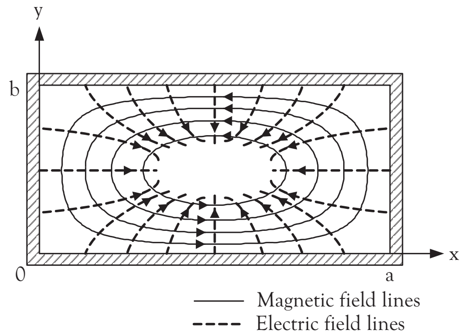
(ii)If $f < f_{c,mn} = \frac{v}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$, then the TM_{mn} mode is cutoff.

Dominant Mode

For TM modes, m and n cannot be zero because m = 0 or n = 0 leads to no fields. The mode that has the lowest cutoff frequency occurs when m = 1, n = 1. Thus the TM_{11} mode is the dominant mode among TM modes of the rectangular waveguide. Its cutoff frequency is given by

$$f_{c,11} = \frac{v}{2}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2} \tag{2-59}$$

The field lines of the TM_{11} mode are plotted in Figure 2-8.



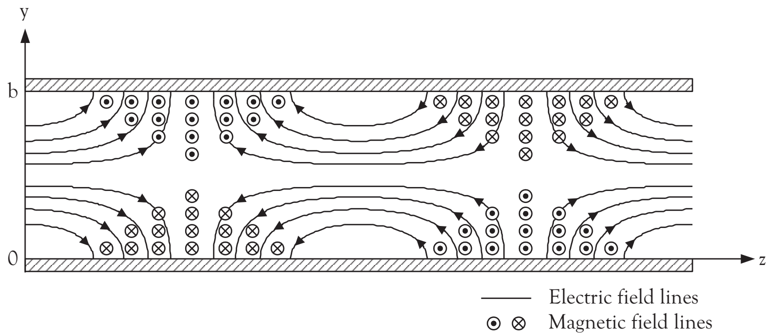


Figure 2-8. ***Field lines of the TM_{11} mode in a rectangular waveguide***

When the wave is in the propagating mode, the phase constant ($k_z = \beta$), the guide wavelength (λ_g), the phase velocity (v_p) and the group velocity (v_g) are given by the same expressions obtained for the parallel-plate waveguide, namely, Eqs. (2-30), (2-31), (2-32), and (2-33), respectively, except that f_c is given by Eq. (2-58). If we take the ratios of the transverse components of the electric and magnetic fields, we obtain the wave impedance as follows.

$$Z_{TM} = \frac{E_x}{H_y} = -\frac{E_y}{H_z} = \frac{k_z}{\omega\varepsilon} = \eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}, \ f > f_c \qquad (2\text{-}60)$$

Again the wave impedance is real in the propagating mode and it becomes purely imaginary or reactive when the wave is cutoff ($f < f_c$). The minus sign in the second ratio is due to the wave traveling in the +z direction.

The time-average Poynting vector for the TM_{mn} mode of the rectangular waveguide consists of two terms:

$$\mathbf{S}_{av} = \mathbf{a}_z \frac{1}{2}\left\{E_x H_y^* - E_y H_z^*\right\} = \mathbf{a}_z \frac{1}{2Z_{TM}}\{|E_x|^2 + |E_y|^2\} \quad (2\text{-}61)$$

where we have used the following relationships:

$$E_x = Z_{TM} H_y, \; E_y = -Z_{TM} H_x \quad (2\text{-}62)$$

The time-average power flowing through the waveguide can be calculated by integrating $\mathbf{S}_{av}$ over the cross section of the waveguide ($0 < x < a$, $0 < y < b$).

We can interpret the propagation of the wave within the rectangular waveguide as represented by the field solution in Eq. (2-54). sin $\sin\left(\frac{m\pi}{a}x\right)$ represents a standing wave in the x direction which consists of two oppositely traveling (in ±x) waves as described in Section 2.3.1. The wave bounces off two conducting walls at x = 0 and x = a as it is guided. sin $\sin\left(\frac{n\pi}{b}x\right)$ similarly represents a standing wave in the y direction, which consists of two oppositely traveling (in ±y) waves. The wave bounces off two conducting walls at y = 0 and y = b. Overall the wave doesn't travel straight down within the waveguide but bounces between the four conducting walls and travels zigzag as it is guided.

2.4.3 TE Mode Solutions

For transverse electric (TE) modes, we let $E_z = 0$ and seek the solutions for $H_z(x,y,z)$ in the form of Eq. (11-49).

$$H_z(x,y,z) = \left\{A \sin k_x x + B \cos k_x x\right\}\left\{C \sin k_y y + D \cos k_y y\right\} e^{-jk_z z} \quad (2\text{-}49)$$

The boundary conditions are given by Eqs. (2-51). From Eq. (2-10) with $E_z = 0$, we have

$$E_x = \frac{-j\omega\mu}{k^2 - k_z^2}\frac{\partial H_z}{\partial y}, \; E_x = \frac{j\omega\mu}{k^2 - k_z^2}\frac{\partial H_z}{\partial x} \quad (2\text{-}63)$$

We now apply the four boundary conditions (BC's).

$$\text{BC (i): } E_y = 0 \rightarrow \frac{\partial H_z}{\partial x} = 0 \text{ at } x = 0 \rightarrow X(x) = B\cos(k_x x)$$

$$\text{BC (iii): } E_x = 0 \rightarrow \frac{\partial H_z}{\partial x} = 0 \text{ at } y = 0 \rightarrow Y(y) = D\cos(k_y y)$$

Then $H_z(x,y,z)$ can be written as

$$H_z(x,y,z) = H_0 \cos(k_x x)\cos(k_y y)e^{-jk_z z} \qquad (2\text{-}64)$$

where $H_0 = BD$. Note that H_z for TE modes has the cosine dependence in x, y directions whereas E_z for TM modes has the sine dependence in x, y directions. The separation constants k_x, k_y are again determined by the remaining BC's

$$\text{BC (ii): } E_y = 0 \rightarrow \frac{\partial H_z}{\partial x} = 0 \text{ at } x = a \rightarrow \boxed{k_x = \frac{m\pi}{a}},\ m = 0,1,2, \qquad (2\text{-}65a)$$

$$\text{BC (iv): } E_x = 0 \rightarrow \frac{\partial H_z}{\partial y} = 0 \text{ at } y = b \rightarrow \boxed{k_y = \frac{n\pi}{a}},\ n = 0,1,2,\ldots \qquad (2\text{-}65b)$$

Note that the results in Eqs. (2-65) are identical to Eqs. (2-53) for TM modes except that *m or n can be zero* for TE modes. Each choice of the integers m and n defines the TE_{mn} mode. The final solution for H_z for the TE modes is then given by (2-66)

$$\boxed{H_z(x,y,z) = H_0 \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-jk_z z},\ TM_{mn}\text{mode}}$$

The propagation constant (or phase constant) k_z is again given by Eq. (2-55). The transverse components of the fields can be calculated by using Eq. (2-10) and $E_z = 0$. They are listed in Table 2-2.

All the discussions about the cutoff frequency, guide wavelength, phase velocity, wave impedance, time-average Poynting vector and the standing wave interpretation, presented in the previous section, are valid also for the TE modes except the following differences.

(i)The wave impedance for the TE mode is given by

$$Z_{TE} = \frac{E_x}{H_y} = -\frac{E_y}{H_z} = \frac{\omega\varepsilon}{k_z} = \frac{\eta}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}},\ f > f_c \qquad (2\text{-}67)$$

From Eqs. (2-60) and (2-67), we find that $Z_{TM} < \eta$ and $Z_{TE} > \eta$ for the propagating modes and both Z_{TM} and Z_{TE} approach the intrinsic impedance of the medium as $f \rightarrow \infty$. We also note that Eqs. (2-61) and (2-62) for the time-average Poynting vector are valid for TE modes when Z_{TM} is replaced by Z_{TE}.

(ii) Dominant Modes

One of the most important issues in any waveguide system is the question of what the dominant mode is because waveguides are often designed so as to support the lowest-order mode and suppress all the higher-order modes. Recall that the lowest-order (dominant) mode for TM modes is the TM_{11} mode because when m = 0 or n = 0 fields vanish. However, for TE modes in rectangular waveguides, *either m or n can be zero, but not both.* If m = n = 0, then all the transverse components of the fields vanish and H_z alone would not satisfy Maxwell's equations. Thus the lowest-order mode for TE modes can be either the TE_{01} mode (m = 1, n = 0) or the TE_{01} mode (m = 0, n = 1). Which mode gives the lowest cutoff frequency depends on the sizes a,b. If a > b (we will assume this throughout the book), the TE_{10} mode is dominant. If b > a (recall that a is the x dimension and b is the y dimension), the TE_{01} mode is dominant. Assuming a > b, the **TE_{10} mode** is the *dominant mode overall,* considering all TM and TE modes in rectangular waveguide, because it gives the lowest cutoff frequency which is obtained from Eq. (2-58) by letting m = 1, n = 0.

$$f_{c,10} = \frac{1}{2\sqrt{\mu\varepsilon}}\frac{1}{a} = \frac{v}{2a} \tag{2-68}$$

The corresponding cutoff wavelength is given by

$$\lambda_{c,10} = \frac{v}{f_{c,10}} = 2a \tag{2-69}$$

Eq. (2-69) is easy to remember. The cutoff wavelength is twice the longer (x) dimension of the rectangular waveguide. The TE_{10} mode is of particular importance because it also gives the lowest attenuation among all modes of the rectangular waveguide. It is useful to know the field distributions of the dominant TE_{10} mode, which are given by

$$\begin{aligned} H_z &= H_0 \cos\left(\frac{\pi}{a}x\right)e^{-jk_z z} \\ H_x &= j\frac{k_z}{\pi/a}H_0 \cos\left(\frac{\pi}{a}x\right)e^{-jk_z z} \\ E_y &= -j\frac{\omega\varepsilon}{\pi/a}H_0 \cos\left(\frac{\pi}{a}x\right)e^{-jk_z z} \end{aligned} \tag{2-70}$$

The electric field of the TE_{10} mode is polarized only in one (y) direction everywhere. The field distributions are identical to those of the TE_1 mode of the parallel-plate waveguide. The field lines of the TE_{10} mode are plotted in Figure 2-9. It is to be emphasized that the TEM mode does not exist in the rectangular waveguide. In general, *a single conductor waveguiding system does not support a TEM mode.*

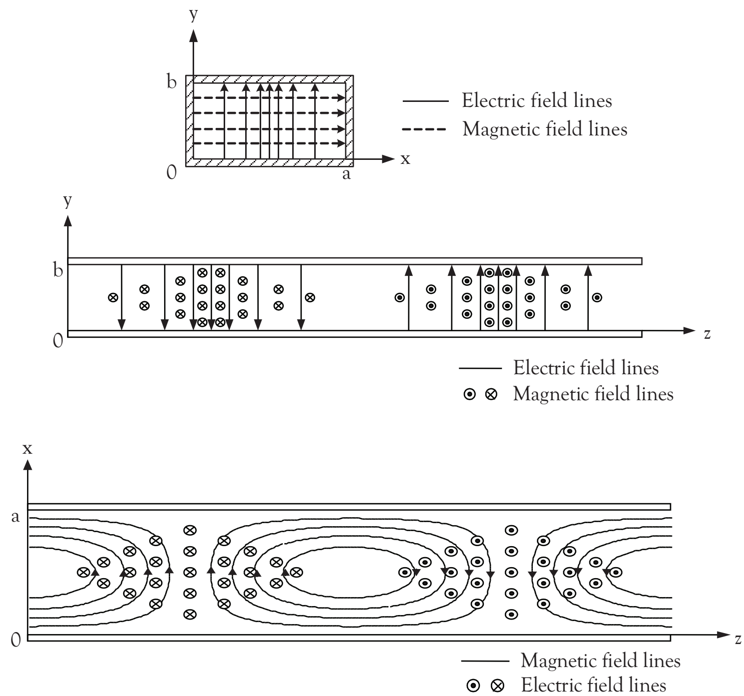


Figure 2-9. ***Field lines of the TE_{10} mode in a rectangular waveguide***

Example 2-3. X-band Waveguide

Consider an air-filled rectangular waveguide having dimensions of a = 0.9 in. (2.286 cm) and b = 0.4 in (1.016 cm).

(a) Suppose only one mode (i.e., the dominant mode) should be transmitted. What is the range of frequencies that can be used?

(b) When the operating frequency is 18 GHz, which TE and TM modes can propagate in the waveguide?

Solutions:

(a) The dominant mode of the rectangular waveguide is TE_{10} mode and its cutoff frequency is given by

$$f_{c,10} = \frac{v}{2a} = \frac{3\times10^{8}}{2\times2.286\times10^{-2}} = 6.56\times10^{9}[\mathrm{Hz}]$$

The next possible higher-order mode is either TE_{01} or TE_{20} mode:

$$f_{c,10} = \frac{v}{2b} = \frac{3\times10^{8}}{2\times1.016\times10^{-2}} = 13.56\times10^{9}[\mathrm{Hz}]$$

$$f_{c,20} = \frac{v}{2b}\cdot2 = \frac{3\times10^{8}}{2.286\times10^{-2}} = 13.12\times10^{9}[\mathrm{Hz}] < f_{c,01}$$

Thus, the range of frequencies for single-mode operation is

$$6.56\ \mathrm{GHz} < f < 13.12\ \mathrm{GHz}$$

Note that this waveguide is used for radar applications at X-band frequencies (8 – 12.4 GHz).

(b) Since f = 18 GHz > $f_{c,10}$, $f_{c,01}$, $f_{c,20}$, all the above three modes will propagate. The next higher-order modes to be considered are 11, 21 and 30 modes:

$$f_{c,11} = \frac{v}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}} = 16.16\ \mathrm{GHz} < f < 18\ \mathrm{GHz}$$

$$f_{c,21} = \frac{v}{2}\sqrt{\left(\frac{2}{a}\right)^2+\frac{1}{b^2}} = 19.74\ \mathrm{GHz} > f$$

$$f_{c,30} = \frac{v}{2}\cdot3 = 19.68\ \mathrm{GHz} > f$$

Therefore, TE_{10}, TE_{20}, TE_{01}, TE_{11} and TM_{11} modes can propagate.

Example 2-4

A rectangular waveguide having dimensions of a = 3.484 cm and b = 1.58 cm (WR-137 waveguide) is filled with polyethylene ($\varepsilon = 2.25\ \varepsilon_o$, $\mu = \mu_o$). The operating frequency is 4 GHz.

(a) Find the phase constant (β), the guide wavelength (λ_g), the phase velocity (υ_p), the group velocity (υ_g), and the wave impedance (Z) of the dominant mode.

(b) Compute the propagation constant when the operating frequency is 2 GHz. How much does the wave attenuate over the distance of 2 cm?

Solutions:

(a) The cutoff frequency of the dominant (TE_{10}) mode is

$$f_{c,10} = \frac{\upsilon}{2a} = \frac{c}{\sqrt{2.25}}\frac{1}{2a} = \frac{2\times10^8}{2\times3.484\times10^{-2}} = 2.87\times10^9[\text{Hz}]$$

Since most of the above quantities can be written in terms of $\sqrt{1-\left(\frac{f_c}{f}\right)^2}$, we calculate this factor first:

$$F \equiv \sqrt{1-\left(\frac{f_c}{f}\right)^2} = \sqrt{1-\left(\frac{2.87\times10^9}{4\times10^9}\right)} = 0.697$$

Using Eqs. (2-30) – (2-33) and (2-41), we obtain

$$k_z = \beta = \omega\sqrt{\mu\varepsilon}F = 2\pi\times4\times10^9\times\frac{1}{2\times10^8}\times0.697 = 87.5\left[\frac{1}{m}\right]$$

$$\lambda_g = \frac{2\pi}{k_z} = \frac{2\pi}{87.5} = 7.17\ [\text{cm}]$$

$$\upsilon_p = \frac{\upsilon}{F} = \frac{2\times10^8}{0.697} = 2.87\times10^8\left[\frac{m}{s}\right]$$

$$\upsilon_g = \upsilon F = 2\times10^8\times0.697 = 1.39\times10^8\left[\frac{m}{s}\right]$$

$$Z_{TE} = \frac{\eta}{F} = \sqrt{\frac{\mu_o}{2.25\varepsilon_o}}\frac{1}{F} = \frac{377}{1.5}\cdot\frac{1}{0.697} = 360.6\ [\Omega]$$

(b) When f = 2 GHz, since $f < f_{c,10}$ the wave is in the cutoff region (evanescent wave) and k_z becomes purely imaginary:

$$k_z = -j\alpha = -j\sqrt{\left(\frac{\pi}{a}\right)^2 - \omega^2\mu\varepsilon} = -j\omega^2\sqrt{\mu\varepsilon}\sqrt{\left(\frac{f_c}{f}\right)^2 - 1}$$

$$= -j\frac{2\pi\times2\times10^9}{2\times10^9}\sqrt{\left(\frac{2.87}{2.0}\right)^2 - 1} = -j64.7\left(\frac{1}{m}\right)$$

The field attenuates as $e^{-\alpha z}$ and over 2 cm propagation the field amplitude is reduced by a factor of $e^{-64.7_x 0.02} = 0.274$, which is a quite rapid attenuation. This demonstrates that the signal attenuates very rapidly with distance when its frequency is below the cutoff frequency.

2.5 Rectangular Cavity Resonator

In the rectangular waveguide, if we place two perfectly conducting walls normal to the z axis (direction of guidance), then the fields will form a standing wave in the z direction, too. The structure becomes a **resonator.** The electromagnetic fields exist in a source-free region enclosed by a perfect conductor, i.e., in a *cavity* and they exist only at specific frequencies, called *resonant frequencies,* which will depend on the geometry of the resonator. This cavity resonator behaves like an LC resonator of low frequencies. At high frequencies (microwaves), ordinary lumped circuit elements such as capacitors and inductors become impractical due to their sizes comparable to operating wavelength and their large losses at high frequencies. The losses come from radiation and high resistance. The cavity resonator having an enclosure with a large area of conducting surface can eliminate radiation and resistive losses, yielding a very high quality factor (Q).

Figure 2-10 shows the geometry of a rectangular cavity resonator. It consists of six perfectly conducting ($\sigma = \infty$) walls at x = 0, x = a, y = 0, y = b, z = 0, and z = d. The medium inside the resonator is assumed to be lossless ($\sigma = 0$) and has permittivity ε and permeability μ.

Since the resonator is constructed of the rectangular waveguide with two additional perfect conductors at z = 0 and z = d, the field solutions for the resonator can be obtained by taking the solutions of the rectangular waveguide and applying two additional boundary conditions at the added two walls. Thus we will again seek the field solutions for the TM and TE modes, separately. It is to be noted that here the TM and TE modes are with respect to the z direction, although they can also be formulated with respect to the x or y direction.

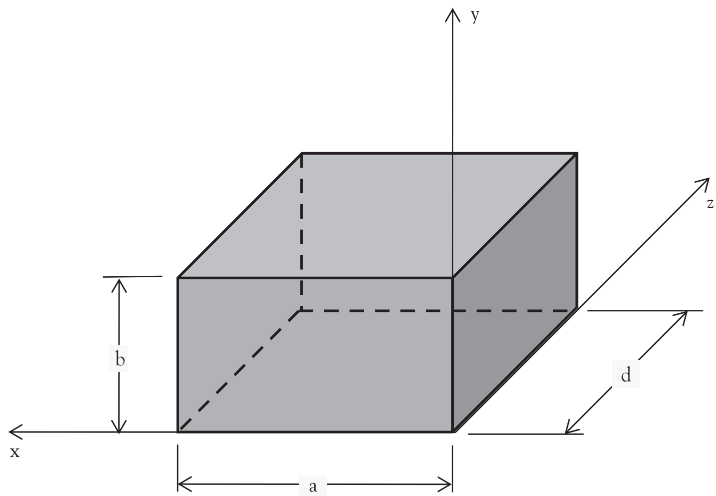


Figure 2-10. ***A rectangular cavity resonator***

2.5.1 TM Mode Solutions

For TM modes, $H_z = 0$ and we seek solutions for $E_z(x,y,z)$. In addition to four boundary conditions at x = 0, x = a, y = 0, and y = b, given by Eq. (11-51), the fields must also satisfy

$$\text{(v) } E_x = 0,\ E_y = 0 \text{ at } z = 0 \tag{2-71a}$$

$$\text{(vi) } E_x = 0,\ E_y = 0 \text{ at } z = d \tag{2-71b}$$

The TM_{mn} mode of the rectangular waveguide, shown in Eq. (2-54), represents a guided wave traveling in +z direction. In the rectangular cavity resonator, there will be a combination of a wave traveling in +z direction ($e^{-jk_z z}$) and a wave traveling in –z direction ($e^{jk_z z}$) due to reflections from the conducting walls at z = 0 and z = d, forming a *standing wave in the z direction.* Therefore, the z dependence of the field can be written as a linear combination of $e^{-jk_z z}$ and $e^{jk_z z}$ or a linear combination of $\sin(k_z z)$ and $\cos(k_z z)$. Hence the solution for E_z for the TM (to z) modes can be written as

$$E_z(x,y,z) = \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\left\{Fe^{-jk_z z} + Ge^{+jk_z z}\right\} \tag{2-72}$$

From Eq. (11-10) with $H_z = 0$, we have

$$E_x = \frac{-jk_z}{k^2 - k_z^2}\frac{\partial E_z}{\partial x},\ E_y = \frac{-jk_z}{k^2 - k_z^2}\frac{\partial E_z}{\partial y} \qquad (2\text{-}73)$$

for the wave traveling in +z direction ($e^{-jk_z z}$). Similarly, for the wave traveling in –z direction ($e^{jk_z z}$), we will have

$$E_x = \frac{jk_z}{k^2 - k_z^2}\frac{\partial E_z}{\partial x},\ E_y = \frac{jk_z}{k^2 - k_z^2}\frac{\partial E_z}{\partial x} \qquad (2\text{-}74)$$

Now we apply the two boundary conditions (BC's):

$$\begin{aligned} &\text{BC (v): } E_x = 0 \rightarrow -F\,e^{-jk_z z} + G\,e^{jk_z z} = 0 \text{ at } z = 0 \rightarrow F = \\ &\text{BC (vi): } E_x = 0 \rightarrow -F\,e^{-jk_z z} + G\,e^{jk_z z} = 0 \text{ at } z = d \\ &F\left(e^{jk_z d} - e^{-jk_z d}\right) = F(2j)\sin(k_z d) = 0 \\ &k_z d = p\pi \text{ or } \boxed{k_z = \frac{p\pi}{d}},\ p = 0,1,2,\ldots \end{aligned} \qquad (2\text{-}75)$$

We can also rewrite

$$F\,e^{-jk_z z} + G\,e^{jk_z z} = F\left\{e^{-jk_z z} + e^{jk_z z}\right\} = 2F\cos(k_z z)$$

The final solution for E_z for the TM mode is then given by

$$\boxed{E_z(x,y,z) = E_o \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right),\ TM_{mnp} \text{ mode}} \qquad (2\text{-}76)$$

$$m = 1,2,3,\ldots;\ n = 1,2,3,\ldots;\ p = 0,1,2,\ldots.$$

Each choice of the integers m, n, p defines a particular mode (TM_{mnp}). $k_x = \frac{m\pi}{a}$, $k_y = \frac{n\pi}{b}$ and $k_z = \frac{p\pi}{d}$ should satisfy the following dispersion relation (see Eq. (2-50)):

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2 = k^2 = \omega^2\mu\varepsilon \qquad (2\text{-}77)$$

It means that each mode exists only at a certain single frequency that satisfies Eq. (2-77):

$$\omega = 2\pi f = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2} \qquad (2\text{-}78)$$

$$f = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2} \equiv f_{r,mnp}$$

This frequency is called the **resonant frequency** of the TM_{mnp} mode. The mode that has the lowest resonant frequency among TM modes is the TM_{110} mode whose f_r is given by

$$f_{r,110} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2} = \frac{v}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} \qquad (2\text{-}79)$$

The remaining field components (E_x, E_y, H_x, H_y) can be obtained by using Eqs. (2-10) with substitution of $-jk_z$ by $\frac{\partial}{\partial z}$ and $k^2 - k_z^2$ by $k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$. The fields of the dominant TM_{110} mode are given by

$$\mathbf{E} = \mathbf{a}_z E_o \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right), \quad TM_{110} \text{ mode}$$

$$\mathbf{E} = \frac{jE_o}{\omega\mu}\left\{\mathbf{a}_x \frac{\pi}{b}\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right) - \mathbf{a}_y \frac{\pi}{a}\cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right)\right\} \qquad (2\text{-}80)$$

2.5.2. TE Mode Solutions

For TE modes, we let $E_z = 0$ and seek the solutions for $H_z(x,y,z)$. The boundary conditions are still given by Eq. (2-71). We again take the TE_{mn} mode solutions of the rectangular waveguide, shown in Eq. (2-66), and consider the standing wave (in z direction) nature of the resonator as discussed earier. The solution for H_z for the TE (to z) modes can be written as

$$H_z(x,y,z) = \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\left\{Fe^{-jk_z z} + Ge^{jk_z z}\right\} \qquad (2\text{-}81)$$

From Eq. (2-10) with $E_z = 0$, we have

$$E_x = \frac{-j\omega\mu}{k^2 - k_z^2}\frac{\partial H_z}{\partial y},\ E_y = \frac{j\omega\mu}{k^2 - k_z^2}\frac{\partial H_z}{\partial x} \qquad (2\text{-}82)$$

This time Eq. (2-82) is valid for the wave traveling in either +z or –z direction. Now we apply two BC's at z = 0 and z = d:

$$\begin{aligned}
&\text{BC (v): } E_x = 0 \text{ or } \frac{\partial H_z}{\partial y} = 0 \rightarrow F\,e^{-jk_z z} + G\,e^{jk_z z} = 0 \text{ at } z = 0 \rightarrow F = -G \\
&\text{BC (vi): } E_x = 0 \text{ or } \frac{\partial H_z}{\partial y} = 0 \rightarrow F\,e^{-jk_z z} + G\,e^{jk_z z} = 0 \text{ at } z = d \\
&F\left(e^{-jk_z d} - e^{jk_z d}\right) = F(-j2)\sin(k_z d) = 0 \\
&k_z d = p\pi \text{ or } k_z = \frac{p\pi}{d},\ p = 0,1,2,3,\ldots
\end{aligned} \qquad (2\text{-}83)$$

We can also rewrite,

$$Fe^{-jk_z z} + Ge^{jk_z z} = F\left(e^{-jk_z z} + e^{jk_z z}\right) = (-j2)F\sin(k_z z)$$

The final solution for H_z for the TE modes is then given by

$$H_z(x,y,z) = H_o \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right),\ TE_{mnp}\ \text{mode} \qquad (2\text{-}84)$$

m = 0, 1, 2, …; n = 0, 1, 2, …; p = 1, 2, 3, …
m = n = 0 is excluded

Note that p cannot be zero because it leads to no field. Each choice of the integers m,n,p defines a particular mode (TE_{mnp}). Again $\frac{m\pi}{a}, \frac{n\pi}{b}, \frac{p\pi}{d}$ satisfies the dispersion relation, Eq. (2-77), which gives the same expression, Eq. (2-78), for the resonant frequency of the TE_{mnp} mode. The mode that has the lowest resonant frequency among TE modes is the TE_{101} mode (when a > b) or the TE_{011} mode (when b > a). Assuming a >b, the resonant frequency of the dominant TE mode is given by

$$f_{r,101} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{d}\right)^2} = \frac{v}{2}\sqrt{\frac{1}{a^2} + \frac{1}{d^2}} \qquad (2\text{-}85)$$

Comparing Eq. (2-85) with Eq. (2-79), the overall dominant mode (including both TE and TM modes) depends upon d > b or b > d. The dominant mode of a rectangular cavity resonator is

$$TE_{101} \text{ mode when } a \geq d > b \text{ or } d > a > b$$
$$TM_{110} \text{ mode when } a > b > d$$

The remaining field components (E_x, E_y, H_x, H_y) can again be obtained by using Eqs. (2-10) with appropriate substitutions. The fields of the dominant TE_{101} mode are given by

$$\mathbf{E} = \mathbf{a}_y(-j\omega\mu)\frac{E_o}{\pi/a}\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{d}z\right), \ TE_{101} \text{ mode} \qquad (2\text{-}86)$$
$$\mathbf{H} = H_o\left\{\mathbf{a}_z\cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{d}z\right) - \mathbf{a}_x\frac{a}{d}\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{d}z\right)\right\}$$

2.5.3. Quality Factor of the Cavity Resonator

In practice, the conducting enclosure is not perfect (finite σ) and there will be power dissipated in the walls. The quality factor (or Q) of the resonator measures the bandwidth of the resonator and the degree of power loss. As in the case of lumped resonant circuits, the quality factor is defined by

$$Q = \frac{\omega W}{P_d} = 2\pi f_r \frac{(\text{time-average energy stored})}{(\text{time-average power dissipated})} \qquad (2\text{-}87)$$

An alternative definition of Q is

$$Q = \frac{\omega}{\Delta\omega} = \frac{f_r}{B} \qquad (2\text{-}88)$$

where f_r is the resonant frequency and B is the bandwidth or the difference in frequencies between two half-power points where the oscillation amplitude is reduced from its maximum value at f_r by 3 dB. Q or B indicates the sharpness of resonance; a high Q means narrower bandwidth and smaller loss.

Example 2-5 Quality factor for TE_{101} mode
Find the quality factor for the TE_{101} mode of the rectangular cavity resonator with dimensions a × b × d.
Solution:

Letting $(-j\omega\mu)H_o\dfrac{a}{\pi} = E_o$ in Eq. (2-86), the *time-average* electric energy stored in the resonator volume is given by

$$W_e = \frac{1}{2}\mathrm{Re}\left\{\iiint_V \frac{1}{2}\mathbf{E}\cdot\mathbf{D}^* \, dv\right\} = \frac{1}{4}\iiint_V \varepsilon|\mathbf{E}|^2 \, dv \quad (11\text{-}89)$$

where a factor of ½ is due to the time average.

$$W_e = \frac{\varepsilon}{4}\int_0^d\int_0^b\int_0^a |E_o|^2 \sin^2\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{d}z\right) dxdydz = \frac{\varepsilon}{4}|E_o|^2 \frac{abd}{4}$$

The time-average magnetic energy stored in the resonator is

$$W_m = \frac{1}{2}\mathrm{Re}\left\{\iiint_V \frac{1}{2}\mathbf{H}^* \cdot \mathbf{B} \, dv\right\} = \frac{1}{4}\iiint_V \mu|\mathbf{H}|^2 \, dv$$

$$= \frac{\mu}{4}\int_0^d\int_0^b\int_0^a |H_o|^2 \left\{\cos^2\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{d}z\right)\right. \quad (2\text{-}90)$$

$$\left. + \left(\frac{a}{d}\right)^2 \sin^2\left(\frac{\pi}{a}x\right)\cos^2\left(\frac{\pi}{d}z\right)\right\} dxdydz = \frac{\mu}{4}|H_o|^2 \frac{abd}{4}\left\{1 + \left(\frac{a}{d}\right)^2\right\}$$

It can be shown that at the resonant frequencies ($f = f_r$ or $\omega = 2\pi f_r$), $W_e = W_m$ Thus, the total time-average stored energy is

$$W = W_e + W_m = 2W_m = \frac{1}{8}\mu|H_o|^2 abd\left\{1 + \left(\frac{a}{d}\right)^2\right\} \quad (2\text{-}91)$$

The time-average power dissipated or power loss *per unit area* of a conducting wall is given by

$$P_d = \frac{1}{2}|H_{tan}|^2 R_s \left[W/m^2\right] \quad (2\text{-}92)$$

where H_{tan} is the tangential component of the magnetic field on the conducting surface and R_s is the surface resistance of the conducting wall. Recognizing that the power losses on two opposite sides of the walls are equal to each other, the time-average power dissipated on six walls is calculated by

$$P_d = \frac{1}{2}R_s\left\{2\int_0^b\int_0^a |H_x|^2_{z=0} dxdy + 2\int_0^d\int_0^b |H_z|^2_{x=0} dydz\right.$$

$$\left. +2\int_0^d\int_0^a \left(|H_x|^2 + |H_z|^2\right)_{y=0} dxdz\right\} \quad (2\text{-}93)$$

$$= R_s|H_x|^2\left\{\left(\frac{a}{d}\right)^2\frac{ab}{2} + \frac{bd}{2} + \frac{ad}{2}\left[\left(\frac{a}{d}\right)^2 + 1\right]\right\}$$

Substituting Eqs. (2-91) and (2-93) into Eq. (2-87), we obtain Q for the TE_{101} mode:

$$Q = \frac{\pi f_{r,101} \mu abd\left(a^2 + d^2\right)}{2a^3b + 2bd^3 + a^3d + ad^3} \tag{2-94}$$

For an air-filled cavity with a = d = 2b = 2 cm, the resonant frequency of the dominant mode is 10.6 GHz and the quality factor (Q) is approximately 10^4 when the cavity is made of copper walls Q is considerately larger than that of lumped-circuit resonator at much lower frequencies.

2.6 Coaxial Cable

The coaxial line, commonly known as the coaxial cable, is probably one of the most commonly used two-conductor waveguiding structures. The coaxial line can be viewed as a circular cylindrical version of a parallel plate waveguide. It consists of an inner conductor of radius a and an outer conductor of radius b, as shown in Figure 2-11(a). The space in between is normally filled with the dielectric of permittivity ε and permeability μ. Both conductors are usually made with many strands of thin copper wire in order to provide flexibility. Polyethylene or Teflon is commonly used for dielectric filling in coaxial cable.

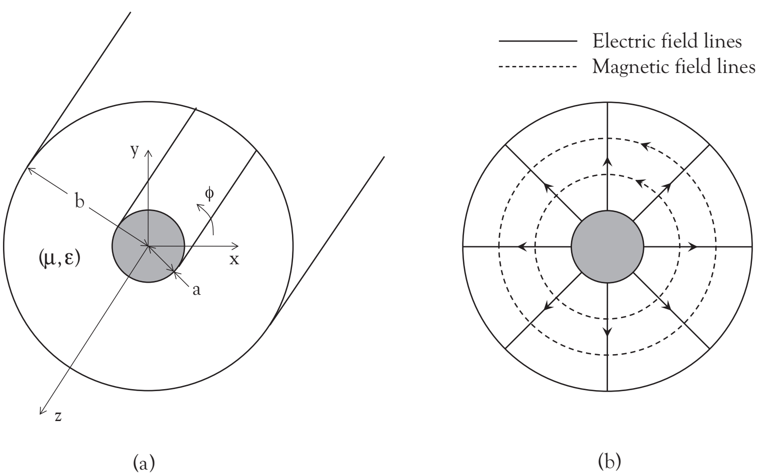


Figure 2-11. ***Coaxial cable: (a) geometry (b) field lines of a TEM mode***

One major advantage of the coaxial line is that the fields are completely confined within the dielectric region while in parallel plate waveguide there will be fringing or leakage fields at the edges of parallel plates.

The field solutions for the coaxial line include the TEM mode as the dominant mode (because it has two conductors) as well as higher-order TE and TM modes, just as in the case of the parallel plate waveguide. The general solution for the fields can be obtained by solving the Helmholtz wave equation, Eq. (2-1), in cylindrical coordinates, which involves the Bessel functions. This is beyond the scope of this book, so we will not consider general higher-order TE and TM solutions. However, the dominant TEM mode solution can be obtained without advanced mathematics.

2.6.1. TEM Mode Solution

First of all, for the TEM mode, $E_z = H_z = 0$ since both electric and magnetic fields are transverse to the direction of guidance (z). Since the coaxial line can be viewed as a cylindrical version of the parallel plate waveguide (PPW), we expect that the TEM mode fields of the coax (short word for coaxial line) will behave much like those of the PPW. From Eq. (2-27), we re-write here the fields of the TEM mode of the PPW of Figure 2-3.

$$\mathbf{E} = \mathbf{a}_x E_o e^{-jkz}, \ \mathbf{H} = \mathbf{a}_y \frac{E_o}{\eta} e^{-jkz} \tag{2-95}$$

where

$$k(=k_z) = \omega\sqrt{\mu\varepsilon}, \ \eta = \sqrt{\frac{\mu}{\varepsilon}} \tag{2-96}$$

Noting that for the PPW, **E** has only a component normal to the surface of the conductors and **H** has only a component tangential to the conductor surface, we assume that for the coaxial line the **E** and **H** fields take the following form:

$$\mathbf{E} = \mathbf{a}_\rho \tilde{E}_\rho(\rho) e^{-jkz}, \ \mathbf{H} = \mathbf{a}_\phi \tilde{H}_\phi(\rho) e^{-jkz} \tag{2-97}$$

whose field lines are sketched in Figure 2-11(b). Note that **E** is normal to the conductor surface and **H** is tangential to the conductor surface. Note also that no φ dependence is assumed due to azimuthal symmetry of the geometry.

Substituting Eq. (2-97) into $\nabla \cdot \mathbf{E} = 0$ and $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$ (see Eqs. (2-19)), we obtain

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\tilde{E}_\rho) = 0 \rightarrow \tilde{E}_\rho = \frac{K_o}{\rho}$$

$$\frac{\partial E_\rho}{\partial z} = -jk\tilde{E}_\rho e^{-jkz} = -j\omega\mu H_\phi = -j\omega\mu\tilde{H}_\phi e^{-jkz}$$

$$\rightarrow \tilde{H}_\phi = \tilde{E}_\rho \frac{k}{\omega\mu} = \frac{\tilde{E}_\rho}{\eta}\frac{K_o}{\eta\rho}$$

where K_o is an arbitrary constant.

Therefore, the field solutions for the TEM mode in a coaxial line are

$$\mathbf{E} = \mathbf{a}_\rho \frac{K_o}{\rho}e^{-jkz}, \; \mathbf{H} = \mathbf{a}_\phi \frac{K_o}{\eta\rho}e^{-jkz} \tag{2-98}$$

where k and η are given by Eq. (2-96). It can be shown that these fields satisfy also $\nabla \cdot \mathbf{H} = 0$ and $\nabla \times \mathbf{H} = j\omega\varepsilon\,\mathbf{E}$ (see Problem 2-38). We observe that $\tilde{E}_\rho = \frac{K_o}{\rho}$ behaves much like a *static electric field* of an infinitely long line charge and of a coaxial cable with opposite charges $\tilde{H}_\phi = \frac{K_o}{\eta\rho}$ behaves much like a *static magnetic field* of an infinitely long current-carrying wire and of two concentric cylinders carrying opposite currents. Thus, the TEM mode solutions can be obtained by multiplying the static field solutions by e^{-jkz} (wave characteristic). The same argument can also be applied to the TEM mode solutions of the PP waveguide. It is to be emphasized that the TEM mode of the coax has no cutoff frequency, i.e., its cutoff frequency is zero just like the TEM mode of the PP line. The coaxial cable is typically used at lower frequencies because its loss due to imperfect conductors increases dramatically at higher frequencies.

For example, the RG6 75Ω coaxial cable, made by CommScope, has the attenuation 0.70 dB/100 ft. (or 2.30 dB/100 m), 2.01 dB/100 ft, and 7.45 dB/100 ft at 10 MHz, 100 MHz, and 1 GHz, respectively. Today coaxial cables are widely used in the broadband cable TV industry. The subscribers are served by an HFC (Hybrid Fiber Coax) architecture which consists of optical fiber cables, transmitters and receivers' amplifiers in the transport segment and coaxial cables, amplifiers and splitters in the distribution segment that directly serves homes.

Example 2-6

Find the total time-average power transmitted along the coaxial line for the TEM mode, assuming that the voltage between the two conductors is V_o in its amplitude.

Solution:

For the TEM mode fields, given by Eq. (11-98), the voltage or potential difference between the two conductors is obtained by

$$V = -\int \mathbf{E} \cdot d\ell = -\int_b^a \frac{K_o}{\rho} e^{-jkz} \mathbf{a}_\rho \cdot \mathbf{a}_\rho d\rho = K_o \ln\frac{b}{a} e^{-jkz}$$

$$= V_o e^{-jkz} \rightarrow K_o = \frac{V_o}{\ln(b/a)} \qquad (2\text{-}99)$$

Thus

$$\mathbf{E} = \mathbf{a}_\rho \frac{V_o}{\ln(b/a)} \frac{1}{\rho} e^{-jkz} \qquad (2\text{-}100)$$

The time-average Poyinting vector is given by

$$\mathbf{S}_{av} = \mathbf{a}_z \frac{1}{2} \mathrm{Re}\left(\mathbf{E} \times \mathbf{H}^*\right) = \frac{1}{2} \mathbf{a}_\rho \times \mathbf{a}_\phi \frac{|K_o|}{\eta} \frac{1}{\rho^2} = \mathbf{a}_z \frac{1}{2\eta} \frac{|V_o|^2}{[\ln(b/a)]^2} \frac{1}{\rho^2}$$

The time-average power transmitted is

$$P = \iint \mathbf{S}_{av} \cdot \mathbf{ds} = \int_0^{2\pi} \int_a^b \frac{1}{2\eta} \frac{|V_o|^2}{[\ln(b/a)]^2} \frac{1}{\rho^2} \rho d\rho d\phi = \frac{\pi |V_o|^2}{\eta \ln(b/a)}$$

When you calculate the current flowing through the conductors from the magnetic field and define its amplitude to be I_o, it can be shown that the time-average power can be obtained from ½ Re $(V_o I_o^*)$

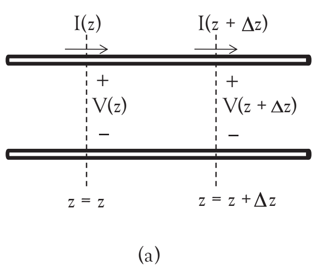
2.7 Transmission Lines

In previous sections we learned that a two-conductor waveguiding system such as parallel-plate waveguide and coaxial cable can support a *TEM wave* for which both electric and magnetic fields are *transverse* to the direction of propagation or guidance. Such a two-conductor transmission system is often called a **transmission line.** Examples, as shown in Figure 2-1(a)-(c), include the parallel-plate waveguide (also called parallel-plate transmission line), the two-wire transmission line, and the coaxial line. The parallel-plate (PP) line is fabricated in microstrip circuits at microwave frequencies, the two-wire transmission line is seen in overhead power and telephone lines, and the coaxial line is used in telephone, TV and internet cables.

When we deal with only the TEM wave in transmission lines, it is very convenient to introduce the concepts of the voltage, current, and impedance in the analysis and application of the transmission lines, since you are familiar with these concepts from circuit theory. We can use the voltage V(z) and current I(z), instead of the electric and magnetic fields, to describe the propagation and reflection of the wave. They satisfy a pair of coupled differential equations, known as the transmission-line equations, which are equivalent to the two Maxwell's curl equations. These equations will include the circuit parameters such as inductance, capacitance, resistance and conductance. Many characteristics of a TEM wave guided by transmission lines are very similar to the characteristics of a uniform plane wave in an unbounded lossless or lossy medium that you learned in Previous chapters.

2.7.1 The Transmission-Line Equations – Lumped-Circuit Model

Let's consider a section of a two-conductor transmission line shown in Figure 2-12(a). We assume that V(z) is the voltage across the conductors at z = z and I(z) is the current flowing through one conductor section at z = z, implying that the same current flows through the other conductor section in the opposite direction. Assuming that the section length (Δz) is much smaller than the operating wavelength, the equivalent circuit can be drawn in Figure 2-12(b).



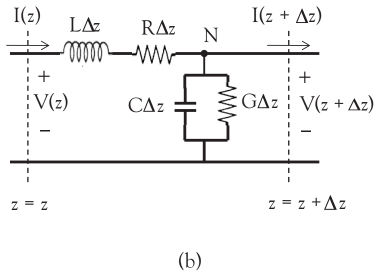


Figure 2-12. ***A transmission line and its lumped-circuit model***

A differential section (Δz) of a transmission line (T.L.) is described by four circuit parameters:

L = inductance per unit length in H/m

R = resistance per unit length in Ω/m

C = capacitance per unit length in F/m

G = conductance per unit length in S/m

When the T.L. is lossless, R and G are zero. R comes from the loss due to the imperfect conductors and G comes from the imperfect dielectric filling the T.L., i.e., the dielectric loss. Note that L and R are series elements and C and G are shunt elements.

First, we apply the Kirchhoff voltage law (KVL) around the loop and obtain:

$$V(z) = (j\omega L + R)\Delta z\, I(z) + V(z + \Delta z)$$

$$\rightarrow \frac{V(z+\Delta z) - V(z)}{\Delta z} = -(j\omega L + R)I(z)$$

Thus we have, as $\Delta z \to 0$ (incrementally small),

$$\frac{dV}{dz} = -(j\omega L + R)I = -ZI \qquad (2\text{-}101)$$

where $Z = j\omega L + R$ is the series impedance per unit length in Ω/m of the T.L.

Secondly, we apply the Kirchhoff current law (KCL) at the node N and obtain:

$$I(z) = (j\omega C + G)\Delta z\, V(z + \Delta z) + I(z + \Delta z)$$

$$\rightarrow \frac{I(z+\Delta z) - I(z)}{\Delta z} = -(j\omega C + G)V(z+\Delta z)$$

Thus we have, as $\Delta z \to 0$ (incrementally small),

$$\frac{dV}{dz} = -(j\omega C + G)V = -YV \qquad (2\text{-}102)$$

where $Y = j\omega C + G$ is the shunt admittance per unit length in S/m of the T.L. Eqs. (2-101) and (2-102) are a pair of coupled, first-order differential equations for V(z) and I(z), which are called the **transmission-line equations.**

Differentiating Eq. (2-101) with respect to z and substituting Eq. (2-102), we obtain

$$\frac{d^2V}{dz^2} = ZYV(z) \text{ or } \frac{d^2V}{dz^2} + k^2V(z) = 0 \quad (2\text{-}103)$$

where

$$k^2 = -ZY = -\left(j\omega L + R\right)\left(j\omega C + G\right) \quad (2\text{-}104)$$

Eq. (2-103) is the *wave equation* for the voltage V(z). Similarly, we can show that I(z) satisfies the same wave equation:

$$\frac{d^2I}{dz^2} = ZY\,I(z) \text{ or } \frac{d^2I}{dz^2} + k^2I(z) = 0 \quad (2\text{-}105)$$

V(z) and I(z) can now be seen as the *voltage wave* and the *current wave,* respectively.

Lossless Transmission Line

In the absence of loss, R = G = 0 and

$$k^2 = -\left(j\omega L\right)\left(j\omega C\right) = \omega^2 LC \rightarrow k = \omega\sqrt{LC} \quad (2\text{-}106)$$

The general solution to Eq. (2-103) can be written as

$$V(z) = V_o^+ e^{-jkz} + V_o^- e^{-jkz} \quad (2\text{-}107)$$

just like the field solutions in Ep. volume 4 (2-40). The first and second terms represent the voltage waves traveling in +z and –z directions, respectively. k plays the same pole as k of the uniform plane wave, so k is called the *propagation constant.* Substituting Eq. (2-107) into Eq. (2-101), we obtain

$$I(z) = \frac{1}{Z_o}(V_o^+ e^{-jkz} - V_o^- e^{-jkz}) \quad (2\text{-}108)$$

where

$$Z_o = \frac{\omega L}{k} = \sqrt{\frac{L}{C}}\,[\Omega] \quad (2\text{-}109)$$

Z_o, known as the **characteristic impedance** of the T.L., is the ratio of the voltage amplitude vs. the current amplitude and plays the same role as the wave impedance of a uniform plane wave – ratio of electric and magnetic field amplitudes – which is equal to the intrinsic impedance of the medium $\left(\eta = \sqrt{\mu/\varepsilon}\right)$.

In the presence of conductor loss (R) and/or dielectric loss (G), k in Eq. (2-104) becomes complex (β – jα) and the voltage and current waves will experience attenuation as they propagate along the T.L.

2.7.2. The Transmission-Line Equations from Field Theory

The T.L. equations shown in Eqs. (2-101) and Eq. (2-102) can also be derived from Maxwell's equations by relating the voltage V(z) and the current I(z) to the electric field **E** and the magnetic field **H** of a TEM wave in a two-conductor T.L. For illustration, we will consider the lossless parallel-plate waveguide. The following procedure gives us the physical insight as to how field concepts are converted to circuit concepts.

As shown in Section 2.3.1, the dominant TEM (or TM_o) mode fields of a parallel-plate transmission line of Figure 2-3 are described by

$$\begin{aligned} \mathbf{E} &= \mathbf{a}_x E_x(z) \\ \mathbf{H} &= \mathbf{a}_y H_y(z) \end{aligned} \tag{2-110}$$

Substituting Eq. (2-110) into $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$ and $\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$, we obtain

$$\frac{\partial E_x}{\partial z} = -j\omega\mu H_y \tag{2-111a}$$

$$\frac{\partial H_y}{\partial z} = -j\omega\varepsilon E_x \tag{2-111b}$$

Now express the voltage V(z) and the current I(z) in terms of the fields using the electrostatic and magnetostatic theory. The voltage across the two conductors is related to **E** by

$$V(z) = -\int_{C_1} \mathbf{E} \cdot d\ell \tag{2-112a}$$

where C_1 is the integration path from one conductor to the other. The current flowing through each conductor is related to **H** by Ampère's law:

$$I(z) = \oint_{C_2} \mathbf{H} \cdot d\ell \tag{2-112b}$$

where C_2 is the closed loop that encloses one conductor. The current can also be related to **H** by using the boundary condition at the conductor surface, $\mathbf{a}_n \times \mathbf{H} = \mathbf{J}_s$. The integration paths are shown in Figure 2-13,

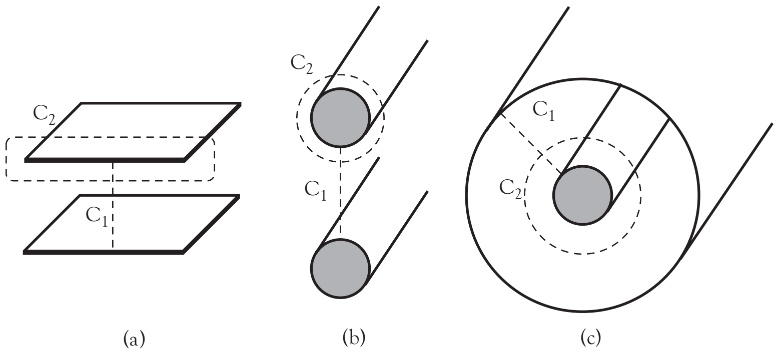


Figure 2-13. ***Integration paths defining the voltage and the current***

In the case of the PP line, Eqs. (2-112) gives

$$V(z) = E_x(z)a \tag{2-113a}$$

$$I(z) = H_y(z)w \tag{2-113b}$$

Making use of V–E_x and I–H_y relationships above, Eqs. (2-111) reduce to

$$\frac{dV}{dz} = -j\omega\mu\frac{a}{w}I(z) = -j\omega LI \tag{2-114a}$$

$$\frac{dI}{dz} = -j\omega\mu\frac{a}{w}V(z) = -j\omega CV \tag{2-114b}$$

where we define

$$L = \mu\frac{a}{w} = \text{inductance per unit length of a PP line} \tag{2-115a}$$

$$C = \varepsilon\frac{a}{w} = \text{capacitance per unit length of a PP line} \tag{2-115b}$$

Eqs. (2-114) are, in fact, the transmission-line equations for a lossless (R = G = 0) T.L., as shown earlier in Eqs. (2-101) and (2-102).

Eliminating H_y in Eqs. (2-111), E_x satisfies

$$\frac{\partial^2 E_x}{\partial z^2} + k^2\ E_x = 0,\ k^2 = \omega^2\mu\varepsilon \tag{2-116}$$

Eliminating I in Eqs. (2-114), V satisfies

$$\frac{\partial^2 V}{\partial z^2} + k^2 + V = 0,\ k^2 = \omega^2 LC \tag{2-117}$$

From Eq. (2-115), it is seen that the following relationship holds:

$$LC = \mu\varepsilon \tag{2-118}$$

For the wave traveling in +z direction, we have

$$\mathbf{E} = \mathbf{a}_x E_o^+ e^{-jkz},\ \mathbf{H} = \mathbf{a}_y \frac{E_o^+}{\eta} e^{-jkz} \tag{2-119}$$

$$V = V_o^+ e^{-jkz},\ I = \frac{V_o^+}{Z_o} e^{-jkz} \tag{2-120}$$

The time-average power along the PP transmission line can be obtained by

$$P_{av} = \iint \frac{1}{2} \mathrm{Re}\left(\mathbf{E} \times \mathbf{H}^*\right) \cdot \mathbf{a}_z dx\, dy = \frac{\left|E_o^+\right|^2}{2\eta} wa \tag{2-121}$$

or

$$P_{av} = \frac{1}{2} \mathrm{Re}\left\{V(z)I(z)\right\} \cdot = \frac{\left|V_o^+\right|^2}{2Z_o} \tag{2-122}$$

It can be easily shown that Eq. (2-121) and Eq. (2-122) are equal, using

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\varepsilon}}\left(\frac{a}{w}\right) \text{ for a PP line} \tag{2-123}$$

The analogy between the characteristics of the voltage and the current on the T.L. and those of the electric and magnetic fields of a uniform plane wave or a TEM mode on the T.L. is summarized in Table 2-3.

Table 2-3. Analogy between Transmission-Line Waves and Uniform Plane Waves (or TEM Waves)

Transmission Line (Lossless)	Uniform Plane Wave or TEM Wave
$V(z) = -\int_{C_1} \mathbf{E} \cdot d\ell$ $I(z) = \oint_{C_2} \mathbf{H} \cdot d\ell$	$\mathbf{E} = \mathbf{a}_x E_x(z)$ $\mathbf{H} = \mathbf{a}_y H_y(z)$
$\frac{dV}{dz} = -(j\omega L)I$ $\frac{dI}{dz} = -(j\omega C)V$	$\frac{\partial E_x}{\partial z} = -j\omega\mu H_y$ $\frac{\partial H_y}{\partial z} = -j\omega\mu E_x$
$\frac{d^2V}{dz^2} + k^2V = 0$ $\frac{d^2I}{dz^2} + k^2I = 0$ $k^2 = \omega^2 LC$	$\frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0$ $\frac{\partial^2 H_y}{\partial z^2} + k^2 H_y = 0$ $k^2 = \omega^2 \mu\varepsilon$
$V(z) = V_o^+ e^{-jkz} + V_o^- e^{jkz}$ $I(z) = \frac{1}{Z_o}(V_o^+ e^{-jkz} + V_o^- e^{jkz})$ $Z_o = \frac{V_o^+}{I_o^+} = \sqrt{\frac{L}{C}} \left(\begin{matrix}\text{characteristic} \\ \text{impedance}\end{matrix}\right)$	$E_x(z) = E_o^+ e^{-jkz} + E_o^- e^{jkz}$ $H_y(z) = \frac{1}{\eta}(E_o^+ e^{-jkz} - E_o^- e^{jkz})$ $\eta = \frac{E_o^+}{H_o^+} = \sqrt{\frac{\mu}{\varepsilon}} \left(\begin{matrix}\text{intrinsic} \\ \text{impedance}\end{matrix}\right)$
$P_{av} = \frac{1}{2}\text{Re}(VI^*)$	$\mathbf{S}_{av} = \frac{1}{2}\text{Re}\left(\mathbf{E} \times \mathbf{H}^*\right)$

2.7.3. Transmission-Line Circuit Parameters

As shown in the transmission-line equations, the properties of a T.L. are characterized by its four circuit parameters L, C, R and G. We would like to learn how to calculate these parameters for a given T.L. from the electromagnetic analysis. These parameters depend on the geometry and the medium parameters of the T.L.

The capacitance (C) and inductance (L) per unit length are calculated from the electrostatic and magnetostatic analysis as they are defined by

$$C = \frac{Q/l}{V} = \frac{(\text{Charge per unit length})}{(\text{Potential difference})} \tag{2-26}$$

$$L = \frac{\Phi/l}{I} = \frac{(\text{Magnetic flux per unit length})}{(\text{Current of closed loop})} \tag{2-20}$$

Since L and C are related to $\mu\varepsilon$ through Eq. (2-118), if L is known for a T.L., C can be easily obtained, and vice versa.

The shunt conductance (G) per unit length is due to dielectric loss when the dielectric medium (of permittivity ε) in the T.L. has a small conductivity σ. In Volume 2, we have shown a relationship between capacitance and conductance when the dielectric medium surrounding two conductors has ε and σ as follows:

$$\frac{C}{G} = \frac{\varepsilon}{\sigma} \tag{2-19}$$

Hence once C of a T.L. is known, G can be calculated easily by

$$G = C\frac{\sigma}{\varepsilon} \tag{2-124}$$

Finally, the series resistance (R) per unit length is due to conductor loss when the conductors of the line are not perfectly conducting, i.e., σ_c (conductivity) is finite. For good conductors at high frequencies, the current mainly flows or penetrates over the skin depth as discussed in Volume 4. Thus R can be approximately calculated by assuming that the total currents in the conductors are uniformly distributed over the skin depth d_p. The resistance of a homogeneous resistor of length ℓ and uniform cross sections S is obtained in Section 5.4 as follows.

$$R = \frac{l}{\sigma S}$$

For the transmission line of unit length that has two conductors, the series resistance per unit length (R) is given by

$$R = R_1 + R_2 = \frac{1}{\sigma_c}\left(\frac{1}{S_1} + \frac{1}{S2}\right) = \frac{1}{\sigma_c d_p}\left(\frac{1}{C_1} + \frac{1}{C2}\right) \quad (2\text{-}125)$$

where C_1 and C_2 are the widths or the circumferences of the two conductors. The skin depth d_p of the good conductor is given by Eq. (1-90):

$$d_p = \sqrt{\frac{2}{\varepsilon\mu_c\sigma_c}} = \sqrt{\frac{1}{\pi f\mu_c\sigma_c}} \quad (2\text{-}126)$$

where μ_c is the permeability of the conductor. Note that $\frac{1}{\sigma_c d_p} = \sqrt{\frac{\omega\mu_c}{2\sigma_c}}$ is known as the *surface resistance* (R_s) of the conductor. For a parallel plate line,

$$R = 2R_1 = \frac{2}{\sigma_c d_p w} = R_s\frac{2}{w} \quad (2\text{-}127)$$

Example 2-7 Transmission line parameters of a coaxial line
Find L, C, R, G and Z_o of a coaxial transmission line of Figure 2-11.

Solution:
The capacitance per unit length of a coaxial line is obtained from the capacitance of a concentric cylindrical capacitor

$$C = \frac{2\pi\varepsilon}{\ln\left(\frac{b}{a}\right)} \quad (2\text{-}128)$$

Making use of the relationship, LC = $\mu\varepsilon$, we obtain the inductance per unit length:

$$L = \frac{\mu\varepsilon}{C} = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) \quad (2\text{-}129)$$

which is also treated in Problem 7-34. Applying the C-G relationship in Eq. (2-124), we obtain

$$G = \frac{2\pi\sigma}{\ln\left(\frac{b}{a}\right)} \quad (2\text{-}130)$$

when σ is the conductivity of the dielectric filling. The series resistance per unit length is obtained from Eq. (2-125) by substituting $C_1 = 2\pi a$, $C_2 = 2\pi b$:

$$R = \frac{1}{\sigma_2 d_p}\left(\frac{1}{2\pi a} + \frac{1}{2\pi b}\right) = \frac{R_s}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right) \tag{2-131}$$

The characteristic impedance is obtained by

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\varepsilon}}\frac{1}{2\pi}\ln\left(\frac{b}{a}\right) = \frac{\eta}{2\pi}\ln\frac{b}{a} \tag{2-132}$$

The transmission line parameters (L, C, R, G and Z_o) are summarized in Table 2-4 for three types of transmission lines – parallel-plate line, coaxial line and two-wire line.

Table 2-4. Transmission Line Parameters (L, C, R, G, Z_o)

Parameters	Parallel-Plate Line	Coaxial Line	Two-Wire Line
L	$\mu\frac{a}{w}$	$\frac{\mu}{2\pi}\ln\frac{b}{a}$	$\frac{\mu}{\pi}\cosh^{-1}\left(\frac{D}{2a}\right)$, D = separation
C	$\varepsilon\frac{w}{a}$	$\frac{2\pi\varepsilon}{\ln(b/a)}$	$\frac{\pi\varepsilon}{\cosh^{-1}(D/2a)}$
R	$R_s\frac{2}{w}$	$\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{R_s}{\pi a}$, $R_s = \frac{1}{\sigma_c d_p} = \sqrt{\frac{\omega\mu_c}{2\sigma_c}}$
G	$\sigma\frac{w}{a}$	$\frac{2\pi\sigma}{\ln(b/a)}$	$\frac{\pi\sigma}{\cosh^{-1}(D/2a)}$
ZO	$\eta\frac{a}{w}$	$\frac{\eta}{2\pi}\ln\frac{b}{a}$	$\frac{\eta}{\pi}\cosh^{-1}\left(\frac{D}{2a}\right)$

2.7.4 Finite Transmission Line with Load

Let's consider a section of transmission line of characteristic impedance Z_o terminated with a load impedance Z_L as shown in Figure 2-14. We investigate the wave characteristics, in particular, how the wave reflected from the load is related to the wave traveling to the right at any point on the T.L.

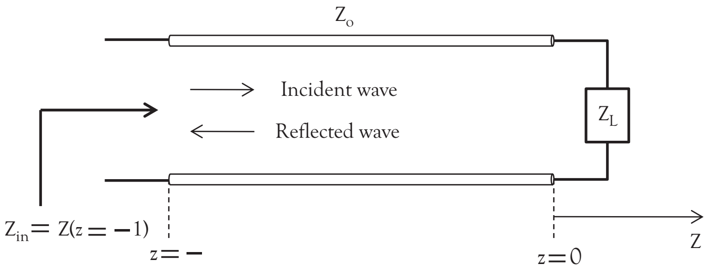


Figure 2-14. ***A finite transmission line terminated with load***

The general solution for the voltage wave on the line can be written as

$$V(z)=\underbrace{V_o^{+}e^{-jkz}}_{\substack{\text{forward travelling}\\ \text{incident wave}}} + \underbrace{V_o^{-}e^{jkz}}_{\substack{\text{backward travelling}\\ \text{reflected wave}}} \qquad (2\text{-}107)$$

The current wave on the line is similarly given by

$$I(z)=\frac{V_o^{+}}{Z_o}e^{-jkz}+\frac{V_o^{-}}{Z_o}e^{jkz} \qquad (2\text{-}108)$$

At the location of the load (z = 0),

$$V(z=0)=V_L=V_o^{+}+V_o^{-}$$
$$I(z=0)=I_L=\frac{1}{Z_o}\left(V_o^{+}+V_o^{-}\right) \qquad (2\text{-}133)$$

If we define the ratio of V_o^- (reflected amplitude) and V_o^+ (incident amplitude) to be the reflection coefficient Γ_L (at the load)

$$\Gamma_L \equiv \frac{V_o^{-}}{V_o^{+}} \qquad (2\text{-}134)$$

We can express Γ_L in terms of Z_o and Z_L using the following relationship:

$$\frac{V_L}{I_L} = Z_L \frac{V_o^+ + V_o^-}{\frac{1}{Z_o}\left(V_o^+ + V_o^-\right)} = Z_o \frac{V_o^+(1+\Gamma_L)}{V_o^+(1-\Gamma_L)} = Z_o \frac{1+\Gamma_L}{1-\Gamma_L}$$

which leads to

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{2-135}$$

Note that the reflection coefficient in Eq. (2-135) is very similar to the reflection coefficient of a uniform plane wave normally incident from the medium of η_1 upon the medium of η_2, as shown in Section 2.2:

$$R = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \tag{2-116a}$$

We observe the following three special cases:

(i)Short circuit $(Z_L = 0):\ \Gamma_L = \frac{V_o^-}{V_o^+} = -1$. This is equivalent to having a perfect conductor in the second medium for the uniform plane wave problem.

(ii)Open circuit $(Z_L = \infty):\ \Gamma_L = \frac{V_o^-}{V_o^+} = 1$. In this case, $I(z = 0) = 0$.

(iii)Matched load $(Z_L = Z_o):\ \Gamma_L = 0$ or $V_o^- = 0$. There is no reflected wave.

The impedance $Z(z)$ at any point on the line is defined as

$$Z(z) = \frac{V(z)}{I(z)} = Z_o \frac{V_o^+\left(e^{-jkz} + \Gamma_L e^{jkz}\right)}{V_o^-\left(e^{-jkz} - \Gamma_L e^{jkz}\right)} \tag{2-136}$$

At the source end $(z = -\ell)$ of the line, looking into the line, we see an input impedance $Z_{in} = Z(z = -\ell)$:

$$\begin{aligned} Z(z=-l) &= Z_o \frac{e^{jkz} + \Gamma_L e^{-jkz}}{e^{jkz} - \Gamma_L e^{-jkz}} = Z_o \frac{(Z_L + Z_o)e^{jkz} + (Z_L - Z_o)e^{-jkz}}{(Z_L + Z_o)e^{jkz} - (Z_L - Z_o)e^{-jkz}} \\ &= Z_o \frac{Z_L 2\cos(kl) + Z_o 2j\sin(kl)}{Z_L 2j\sin(kl) + Z_o 2\cos(kl)} = Z_o \frac{Z_L + jZ_o \tan(kl)}{Z_o + jZ_L \tan(kl)} \end{aligned} \tag{2-137}$$

Thus, we can evaluate the input impedance as a function of line length ℓ, given the characteristic impedance of the line and load impedance. $Z(-\ell)$ will vary periodically with period $k\ell = (2\pi/\lambda)\ell = \pi$ or $\ell = \lambda/2$. There are a few special cases for discussion:

(i) Short circuit ($Z_L = 0$): $Z(-\ell) = jZ_o \tan(k\ell)$. (2-138)
Z_{in} is reactive. For a short section ($k\ell \ll 1$), $Z(-\ell) \approx jZ_o k\ell$. The short-circuited line behaves like an inductor.

(ii) Open circuit $(Z_L = \infty):\ Z(-l) = \dfrac{Z_o}{j\tan(kl)} = -jZ_o \cot(kl)$ (2-139)
Z_{in} is also reactive. For a short section $(k\ell \ll 1),\ Z(-l) \approx -j\dfrac{Z_o}{kl}$.
The open-circuited line behaves like a capacitor.

(iii) Matched load ($Z_L = Z_o$): $Z(-\ell) = Z_o$. The input impedance is equal to the characteristic impedance at all points because there is no reflected wave.

(iv) Quarter-wave section ($\ell = \lambda/4$): $k\ell = \pi/2$, $\tan k\ell \to \infty$

$$\frac{Z_{in}}{Z_o} = \left(\frac{Z_L}{Z_o}\right)^{-1}$$

$$Z_{in} = \frac{Z_o^2}{Z_L} \quad \text{or}$$

(2-140)

A quarter-wave lossless transmission line transforms the normalized load impedance (normalized by Z_o) to its inverse. It is called the **quarter wave transformer.** For example, an open-circuited, quarter wave line appears as a short circuit.

(v) Half-wave section ($\ell = \lambda/2$): $k\ell = \pi$, $\tan k\ell = 0$, $Z_{in} = Z_L$, as expected. When the length of a line is an integer multiplier of $\lambda/2$, Z_{in} does not change.

Analysis of generalized reflection coefficient and the impedance of the transmission line involves tedious manipulations of complex numbers. A graphical chart, known as the **Smith Chart,** greatly simplifies the analysis. See D.K. Cheng (1993) or Shen and Kong (1995) for detailed analysis using the Smith Chart.

Standing Wave Pattern

Whenever the load is mismatched ($Z_L \neq Z_o$), $\Gamma_L \neq 0$ and $V_o^- \neq 0$. Then both forward and backward waves propagate on the line. The total voltages and currents are the superposition of these two and form a **standing wave** just like the standing wave formed by a uniform plane wave incident upon a perfect conductor as discussed in Section 2.6. Voltage maxima occur where the two (incident and reflected) waves are in phase; minima where they are 180° out of phase. Similarly, there are maxima and minima for current along the line. There are local accumulations of electric energy at voltage maxima and of magnetic energy at current maxima, which may reduce the power handling capacity of the line.

Defining the generalized reflection coefficient (at z)

$$\Gamma(z) = \frac{V_o^- e^{jkz}}{V_o^+ e^{-jkz}} = \Gamma_L e^{j2kz}, \tag{2-141}$$

the voltage wave from Eq. (2-107) can be re-written as

$$V(z) = V_o^+ e^{-jkz} \{1 + \Gamma(z)\} \tag{2-142}$$

The current wave is similarly given by

$$I(z) = \frac{V_o^+}{Z_o} e^{-jkz} \{1 - \Gamma(z)\} \tag{2-143}$$

It is observed from Eq. (2-139) that for a lossless line

$$|\Gamma(z)| = |\Gamma_L|$$

Since $|\Gamma(z)|$ is constant while its phase varies with z, the amplitude of V(z), i.e., $|V(z)|$, will be a maximum when $\Gamma(z) = |\Gamma_L|$, and it will be a minimum when $\Gamma(z) = -|\Gamma_L|$. But the current amplitude $|\Gamma(z)|$ is a minimum at the maximum of the voltage and vice versa. The ratio of the voltage maximum to the voltage minimum is defined as **(voltage) standing wave ratio** or VSWR:

$$\text{VSWR} = \frac{|V(z)|_{max}}{|V(z)|_{min}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \tag{2-144}$$

$|\Gamma_L|$ can also be written in terms of VSWR:

$$\left|\Gamma_L\right| = \frac{VSWR - 1}{VSWR + 1} \tag{2-145}$$

Note that since the distance between $\Gamma(z) = |\Gamma_L|$ and $\Gamma(z) = -|\Gamma_L|$ is $\lambda/4$ ($2kd = \pi$), the distance between two maxima is $\lambda/2$. Thus, although V(z) and I(z) are periodic with λ, $|V(z)|$ and $|I(z)|$ are periodic with $\lambda/2$. Two special cases are for discussion.

(i)Matched load ($Z_L = Z_o$): $\Gamma_L = 0$, VSWR = 1, $V(z) = V_o^+ e^{-jkz}$. There is no standing wave.

(ii)Short circuit ($Z_L = 0$): $\Gamma_L = -1$, VSWR $\to \infty$. $|V(z)|$ will be minimum (zero) and $|I(z)|$ will be maximum at the load. They will switch at $z = -\lambda/4$.

The standing wave patterns for $|V(z)|$ and $|I(z)|$ of the above two cases are plotted in Figure 2-15.

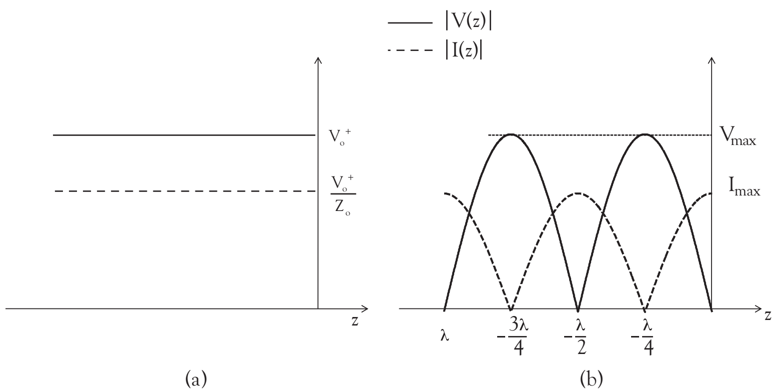


Figure 2-15. ***Standing wave patterns on a transmission line with (a) matched load and (b) short-circuited load***

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Principles of Electromagnetics 5—Wave Applications

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