THE MODERN ENGINEERING COMPANIONS: A SYSTEMS APPROACH COLLECTION

Christopher H. Jenkins, *Editor*

An Engineering Companion to the Mechanics of Materials

A Systems Approach

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ENGINEERING

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Dedication

Author Jenkins would like to thank his wife Mo for her support and guidance throughout writing this book. Many thanks also to Joel Stein at Momentum Press for continuing to believe in our projects.

Author Khanna would like to express his gratitude to his wife Vinita for her support and their children for their patience during the writing of this book. He also thanks his past students for their inquisitiveness.

Abstract

An Engineering Companion to Mechanics of Materials is the first volume in the Momentum Press collection *The Modern Engineering Companions: A Systems Approach to the Study of Engineering*. In *Mechanics of Materials*, we apply the intuitive "systems approach" to learning, the advantages of which are several. The student first gets a broad overview of the entire subject rather than the narrow piecemeal vision afforded by the traditional "component approach" common to most engineering texts. *Mechanics of Materials* comes with additional features to improve student learning, including Common Confusing Concepts (C^3) noted and clarified, indication of key concepts, side bar discussions, worked examples, and exercises for developing engineering intuition. The *Companions* are intended as a supplementary resource to help both undergraduate, graduate, and post-graduate students better learn and understand engineering concepts.

Keywords

mechanics of materials, solid mechanics, structures, strength of materials, supplemental learning materials, study guide

CONTENTS

Preface

An Engineering Companion to Mechanics of Materials is the first volume in the Momentum Press collection *The Modern Engineering Companions: A Systems Approach to the Study of Engineering.* Mechanics of materials is a fundamental engineering topic concerned with the response of structures to loads. It is a typical course of study in most engineering curriculums, often taken in the second year along with associated courses in statics and dynamics.

In *Mechanics of Materials*, we apply the intuitive "systems approach" to learning to the study of mechanics of materials. The advantages of the systems approach are several. The student first gets a broad overview of the entire subject rather than the narrow piecemeal vision afforded by the traditional "component approach" common to most engineering texts. Crucial, core topics can be reached early to provide motivation for student learning. Rather than studying a component and then leaving it behind, never to be considered again, the systems approach continually passes through components, reviewing and refreshing, then adding layers of increasing complexity.

Mechanics of Materials comes with additional features to improve student learning, including indication of key concepts, side bar discussions, worked examples, and exercises for developing engineering intuition. It is suitable as a text for a first undergraduate course, or as a companion for the advanced undergraduate, beginning graduate student, and the practicing professional in engineering.

The sole purpose of the *Engineering Companions* collection is to dramatically improve the learning of STEM (science, technology, engineering, and mathematics) topics for students of engineering. The *Companions* are intended as a supplementary resource to help both undergraduate and graduate students better learn and understand engineering concepts. They will also be found useful to the engineering post graduate looking to brush up on a topic long since forgotten. In all cases, it is assumed that *Companion* readers have a textbook, homework assignments, problem solutions, etc., either from a current course or a course previously taken. Unfortunately, what they also have is confusion and limited understanding of one or more crucial concepts in the subject.

Mechanics of Materials is of modest length with the intent of giving the reader an effective and efficient enhancement of important underlying concepts and applications, rather than trying to provide an expansive or comprehensive treatment of the foundational fields. *Mechanics of Materials* is an Engineering Fundamentals volume, which covers basic engineering topics most common among the various disciplines.

Books in this collection will appeal to college instructors as texts in fundamental and advanced topics courses, to students as a learning resource, and to practitioners in the field at all levels. The books will also be appealing to the technically savvy reader who wants a quick, but effective, orientation to collection topics though their background might be in a different technical field.

Navigating the *Mechanics of Materials Companion*

Nearly every mechanics of materials text has the chapters arranged by components (topics) of the subject. After an introductory chapter, there appear chapters on stress and strain, axial structures, torsion structures, beams, combined loading, etc. This is a perfectly reasonable approach for a reference handbook, typically consulted after one has learnt the subject. But it is non-optimal to support primary learning for the reasons already discussed here and the *Engineering Companion Preface*.

This *Companion* approaches achieving the learning objectives by *expanding circles of increasing complexity*. After an introductory chapter on fundamentals (Level 0 or Chapter 1), the next “circle” -- Level 1 (Chapter 2) considers configuration, equilibrium, deformation, and constitution for all three “strength design” structures (rod, shaft, beam). Level 2 adds complexity to what has come before, e.g., introducing the simpler combined flexure-torsion loading. Finally, Level 3 adds still more complexity, such as full tension-torsion-flexure combined loading.

A word about figure numbering is in order. There are four uses of figures in this *Companion*: introduction/motivation (I), representative problems (RP), examples (E), and general text. Thus a figure labeled Figure I1.2 is the second introductory/motivational figure in Chapter 1, Figure E3.4b is the second figure in the fourth worked example in Chapter 3, and so on. A figure label without an I, RP, or E is a figure in the general text.

Other Features

The *Engineering Companions* incorporate many additional features to aid in learning:

- Consistent nomenclature from volume to volume
- Chapter learning objectives and their practical importance
- Sidebars highlighting key concepts
- Common Confusing Concepts (C^3) noted and clarified
- Summary of key concepts
- Worked examples that demonstrate concepts
- Exercises to build engineering intuition

How to Use a *Companion*

The best way to use the *Companion* for your subject is start at the beginning and work your way through to the end. The *Companion* will lead you through the subject in an ever expanding pattern of increasing knowledge and complexity, while not losing the big picture of where you're going. However, should you want, say, as a refresher to follow the traditional path along a component, a traditional Component Index is there to guide you.

The ultimate success of any *Engineering Companion* will be measured by the extent to which confusing concepts are made clear (or at least clearer). That is our mission: to make clear the unclear. We wish the reader the best in their search for clarity.

—Chris Jenkins Bozeman, MT
Sanjeev Khanna Columbia, MO
March 2015

CHAPTER 1

Fundamentals

Learning Objectives: This chapter will provide the student with a systematic introduction to fundamental concepts in the study of mechanics of materials. Along the way, we will clear up several common confusing concepts (C^3s).

Clarifications: Studying this chapter will help clarify several C^3s, including definitions, physical concepts, methods of analysis, and where to go for help.

Importance: The design of structures crosses many disciplines and has a long history. Some of the earliest structures were civil structures, mostly habitats like tents and huts. Simple mechanical structures also were in early use, such as levers, and supports for processing and cooking game. Later, marine structures such as rafts and canoes came into existence. Other structures followed, including carts, construction equipment, and weapons. Roads, aquaducts, and bridges were built. Much later, early aerospace structures, such as balloons and parachutes, appeared.

Today, the fields of aerospace, civil, marine, and mechanical engineering are all involved in structural design. On a cursory level, we can define a *structure* as any physical body that must carry loads (other than its own weight), and hence develops stresses and strains. Often times these stresses and strains are trivial, and the body can be considered a *secondary structure*. However, in many cases, inadequate design for carrying loads can lead to significant, even catastrophic, failure in *primary structures*, those structures that have a primary function to carry loads. Unfortunately, examples of such failure are readily found, including the Tacoma Narrows Bridge disaster, the Kansas City Hyatt

> **C^3**
>
> 1.1 The word "structure" is used differently here than in my chemistry class. Why?
> *(Answer: Section 1.6)*

Regency skywalk collapse, and the space shuttle Challenger explosion. Adequate structural design is critically important!

1.0 Representative Problems

The following is an example problem that exhibits some fundamental concepts from mechanic

RP1.1. Equilibrium

Consider an equilateral truss loaded as shown in Figure RP1.1a. Determine the internal forces in the truss.

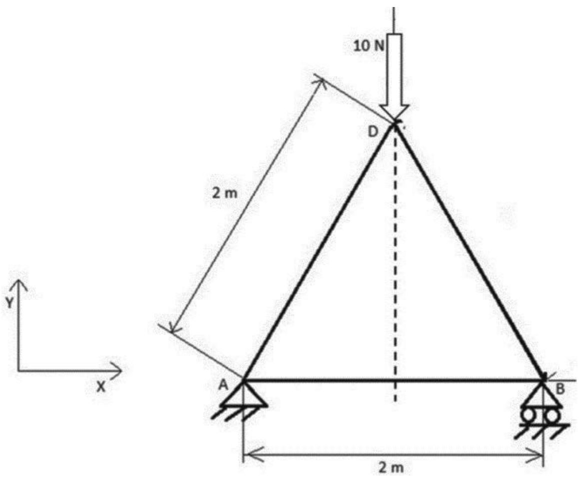


Figure RP1.1.

1.1 Mechanics and materials

C^3
1.2 My mechanics of materials course and text have very little in it about "materials". Does it matter? *(Answer: Section 1.6)*

Applied mechanics and materials science are broad fields of human endeavor with long histories. *Mechanics* deals with the theoretical and experimental analysis of forces on material bodies, and the resultant motions and deformations that follow. *Materials science* is concerned with the atomistic structure of material and the properties resulting therefrom. For our purposes here, a working, albeit over-simplified, definition is taken to be: mechanics is *physics*, materials science is *chemistry*.

Today, examples abound which show the need for engineers and scientists who have an integrated, interdisciplinary background that bridges mechanics and materials science. Consider, for example, the important and active area of high-performance composite materials.

Here, an intimate knowledge of structure-property relations is demanded for technological advancement. Bulk response can be predicted in an averaged sense using a mechanics approach, which is necessary to design a real composite structure; but only knowledge of the fine-scale (micro- to nano-scale) structure-property relations and interactions among the constituents can lead to an optimal "engineering" of these materials for an intended application.

Other topics of current interest include: computational modeling of materials with evolving microstructures; advanced manufacturing and processing challenges to mechanics and materials; mechanics and statistical physics of particulate materials; mechanics and materials science of contact; and processing and mechanics of nanoscale, multilayered materials. We will show in what follows that *every* structural design should be an integration of mechanics and materials technology.

1.2 Loads and Structures

1.2.1 Structural Loads

In structural design, we use the term "loads" to mean forces and moments applied to the structure, either externally on the surface (*surface loads*), or developed within the structure (*body loads*). (Recall that moments are forces acting through a "moment arm" so as to produce torsion or flexure.)

Loads are further considered to be either static or dynamic. Static loads are loads that do not depend on time, i.e., they are of constant magnitude, direction, and location. Although it might seem that certain structures are static, for example, on a civil structure such as a building, this is rarely the case. "Live loads" from occupant activity, wind loads, seismic (earthquake) loads, thermal cycling, etc., all may give rise to a dynamic load environment. However, if the loads vary slowly with time, they are often considered *quasi-static*, and taken as static loads. (*Slowly* usually means that structure inertia force due to accelerations may be neglected with respect to the difference of the externally applied forces and the internal resistance forces.)

Dynamic loads are divided into two main categories:

1. *Steady-state* loads are loads that maintain the same character (frequency, amplitude, etc.) over the long-term.
2. *Transient* loads are loads that change their character (for example, they may *decay*) with time.

Common structural loads are summarized in Table 1.1.

Table 1.1. Summary of common structural loads.

Surface Loads (Common name)	**Units SI (US)**	**Body Loads**	**Units SI (US)**
Concentrated force ("Point load")	N (lb)	Gravitational force ("Gravity load" or "weight")	N (lb)
Distributed force ("Line load") ("Pressure")	N/m (lb/in) N/m^2 (psi)	Thermal stress ("Thermal load")	N/m^2 (psi)
Moment or couple ("Bending moment" "Torsion moment")	N-m (lb-in)		

The *orientation* of the load on a structural member is also important. Although this issue will be discussed in more detail in latter chapters, a brief summary is given in Figure 1.1.

> **C³**
>
> 1.3 Can a "line" load or a "point" load really exist?
>
> *(Answer: Section 1.6)*

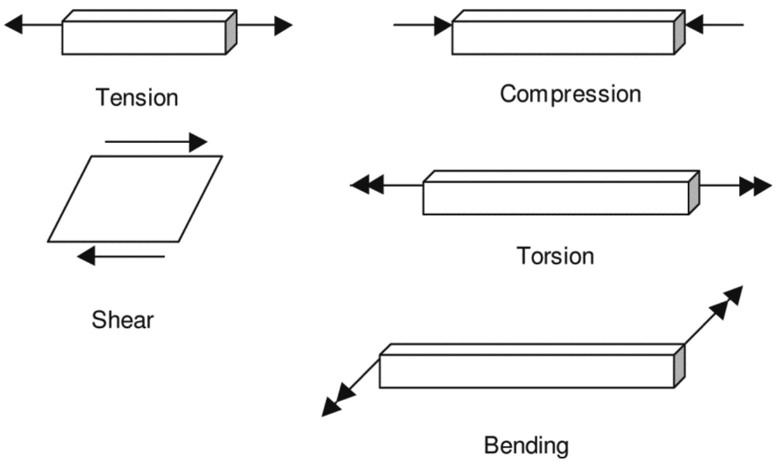


Figure 1.1. Simple load orientations. Single headed arrows are forces, double headed arrows are moments (and follow the right hand rule).

1.2.2 A Taxonomy of Structures

Humans have always tried to understand complex systems by "decomposing" them into a number of simpler, more manageable parts. The hope is that when this compartmentalized knowledge is "summed up" (*synthesized*), an accurate representation of the whole system results. While this approach has worked well in countless human enterprises, it is based on the *linear* assumption of *superposition*, which fails as systems become *nonlinear* and more complex.

Forewarned by this knowledge, we will attempt here a *taxonomy* (classification) of structures (complex systems) by decomposing them into their structural elements (simpler parts). Most real structures are comprised of a number of different types of structural elements, any one of which may assume a variety of different roles. The primary characteristic used to classify a structural element is: <u>how does it carry loads?</u>

Carrying loads is the primary *function* of a structure, and it is this characteristic that largely determines the *form* of the structural elements. Loads carried include: tensile and compressive axial loads; shear loads; torsion moment loads; bending moment loads; distributed loads; gravity loads; and thermal loads.

Structural forms fall into two major categories: *line*-forming and *surface-* forming; surface-forming elements may be further subdivided into *area*-forming and *volume*-forming elements. A taxonomy of structural elements is given in Table 1.2.

A review of Table 1.2 reveals that beams, plates, and shells are structural elements fully capable of carrying all types of loads. Specialty elements such as cables, rods, and membranes carry limited types of loads. Schematics of the elements are shown in Figure 1.2.

Line-forming elements (LFEs) are slender structures having one spatial dimension (length) significantly greater than any other dimension (width, height, thickness, etc.). Schematically, for analysis purposes, LFEs may be represented as lines, either straight or curved as required. LFEs may carry loads in tension, compression, torsion, bending, or some combination of these, depending on the nature of the applied loads, structural geometry, material properties, and boundary conditions. LFEs may form *axial, torsional*, or *bending* structures such as *rods, cables*, and *beams*.

Table 1.2. A taxonomy of structural elements.

FUNCTION	**FORM**						
⇓	Line-forming			Surface-forming			
				Area-forming		Volume-forming	
Loads carried:	Cable	Rod	Beam	Plane membrane	Plate	Curved membrane	Shell
Tensile axial	✓	✓	✓	✓	✓	✓	✓
Compressive axial	-	✓	✓	-	✓	-	✓
Direct shear	-	-	✓	-	✓	-	✓
Torsion moment	-	✓	✓	-	✓	-	✓
Bending moment	-	-	✓	-	✓	-	✓
Distributed force	✓		✓	✓	✓	✓	✓
Thermal	✓	✓	✓	✓	✓	✓	✓

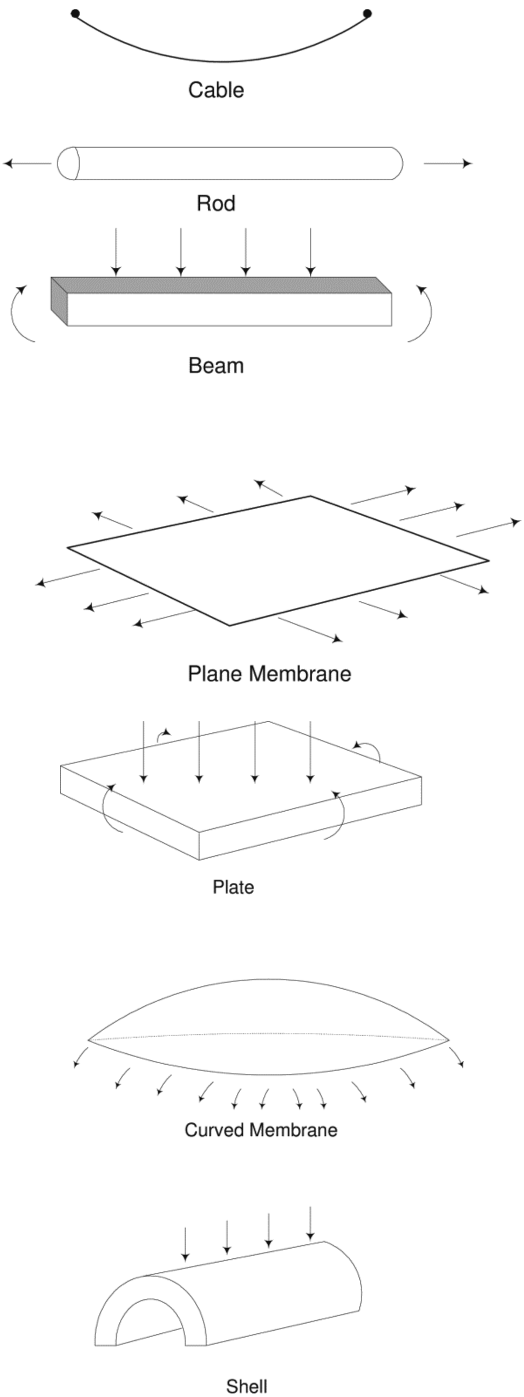


Figure 1.2. Schematic of basic structural elements.

1. *Rods* carry concentrated axial forces in either tension or compression (no moments or transverse forces). Thus they are *axial structures*. Only *pinned* ("simply supported") boundary conditions are required. Cross-sectional geometries may take any shape, but simple shapes are most common, such as circular and rectangular. The cross-section may be solid or hollow.
2. *Cables* carry concentrated axial tensile forces (but not compressive axial forces – "you cannot push a rope"!), as well as concentrated or distributed transverse forces (but not moments!). In the latter case, cables deform under the action of these transverse loads in such a way that they remain *axial* structures (specifically "no-compression axial structures"). (Of course, in the former case of axial loads, cables are by definition axial structures.) As with rods, only pinned boundary conditions are required. Cross-sectional geometries may be of any shape, but solid circular cross-sections are common.
3. *Beams* are the most complete LFEs since they can carry axial compressive or tensile forces as in rods or cables, as well as transverse concentrated or distributed forces as in cables; moreover, they can carry torsion and bending moments. In cases where moments are applied, at least one boundary condition must be able to support moment reactions (for example, a "fixed" or "built-in" condition). Cross-sectional shapes may be of any geometry, solid or hollow.

Surface-forming elements (SFEs) are thin structures having two spatial dimensions (length and width) significantly greater than the third dimension (thickness). Schematically, for analysis purposes, SFEs may be represented as surfaces, either straight or curved as required. SFEs may carry loads in tension, compression, torsion, bending, or some combination of these, depending on the nature of the applied loads, structural geometry, material properties, and boundary conditions. SFEs may form axial, torsional, or bending structures such as *plates* and *shells*. Being more complicated structures, SFEs are outside the scope of this book and will not be described further.

1.2.3 Boundary Conditions and Determinacy

It should be obvious that if a structure is not tied down somewhere, it will not be able to carry loads in many situations. Imagine trying to hang a weight from a hook not connected to the ceiling! Thus we see that the conditions at the structure boundaries, i.e., the *boundary conditions*, are critically important in structural design and analysis.

A number of "idealized" boundary conditions can be defined mathematically. (We say "idealized" since some boundary conditions may not be physically realized exactly.) Mathematically, boundary conditions provide additional *equations of constraint*. If there are fewer constraints than the minimum required, the structure is *under-constrained* and *unstable*. If there are more constraints than the minimum required, the structure is *over-constrained*. The structure is *statically indeterminate* if there are fewer independent equations available than unknowns to solve for, respectively. (See Examples 1.2 and 1.3 for further discussion on determinacy.)

> **C³**
>
> 1.4 Is a statically indeterminate structure a bad structure?
> *(Answer: Section 1.6)*

For a truss, the degree of indeterminacy i is given by:

$$i = m + r - 2(j) \tag{1.1}$$

where
m = total number of members
r = total number of unknown boundary reactions
j = total number of joints
If $i = 0$, the truss is statically determinant; if $i \geq 0$, the truss is statically indeterminant.

A number of common boundary conditions are provided in Table 1.3 and Figure 1.3.

Table 1.3. Common boundary conditions. The prime or ′symbol indicates differentiation with respect to a spatial coordinate, i.e., a gradient or "slope".

Name (other names)	Translation Constraint	Rotation Constraint
Free ~ 2D or 3D	None	None
Fixed (clamped) ~ 2D or 3D	No translation ($u = v = w = 0$)	No rotation ($u' = v' = w' = 0$)
Ball and socket (swivel) ~ 3D	No translation ($u = v = w = 0$)	None
Simple (pinned) ~ 2D	No translation ($u = v = 0$)	None
Simple (roller) ~ 2D	No translation ⊥ to roller surface (e.g., v = 0)	None

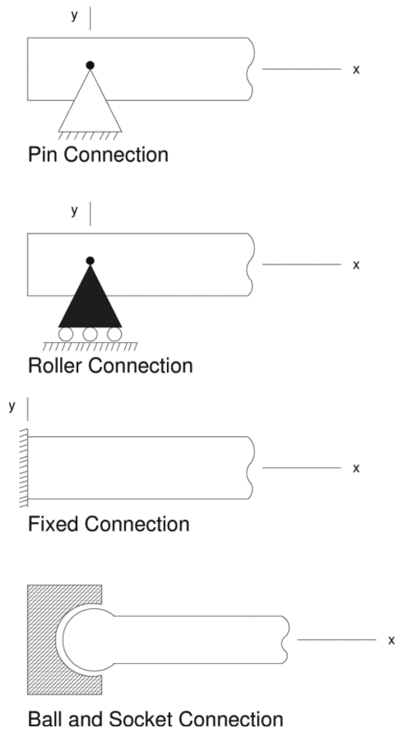


Figure 1.3. Some examples of boundary conditions listed in Table 1.3.

An example of a real boundary condition can be seen in the pin joint in the truss shown in Figure 1.4.

Figure 1.4. (a) Pin joints in a truss. (b) Pin joint detail.

Example 1.1. Statically equivalent forces and moments

Determine the reactions for the cantilever beam with a distributed load as shown in Figure E1.1a.

Figure E1.1a shows a cantilever beam with a distributed triangular load of peak magnitude q_0 per unit length.

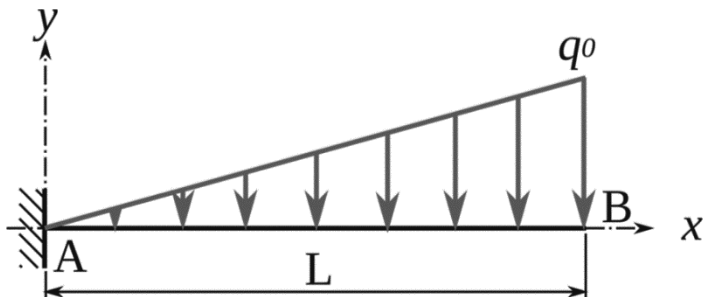


Figure E1.1a.

We know that the fixed or clamped boundary at A provides both translation and moment resistance. We have the equilibrium equations to apply globally. What we need is how to deal with the distributed load. What would be convenient is to replace the distributed load with a concentrated force F (magnitude and location to be determined) that resulted in an equivalent static response, i.e., resulted in the same reactions, internal forces and moments, etc., as the original load.

> **C³**
>
> 1.5 Why is integration needed here? *(Answer: Section 1.6)*

First, we want the equivalent concentrated force F to represent the same total load as the distributed load. If we recognize that the distributed load $q(x)$ is a linear function of the form $y = mx + b$, the slope of the function $m = q_0/L$ and the y-intercept $b = 0$. Then $q(x) = (q_0/L)x = q_0(x/L)$. Setting F equal to the total load gives:

$$F = \int_0^L q(x)\,dx = \int_0^L \frac{q_0}{L} x\; dx = \frac{q_0}{L}\frac{x^2}{2}\Big|_0^L = \frac{q_0 L}{2}$$

So now we know the magnitude of F, we need to determine its location x_F so that the beam experiences the same total moment. Thus we set the moment of F equal to the moment provided by $q(x)$:

$$Fx_F = \int_0^L xq(x)\,dx = \int_0^L \frac{q_0}{L} x^2\; dx = \frac{q_0}{L}\frac{x^3}{3}\Big|_0^L = \frac{q_0 L^2}{3}$$

Then the location x_F is:

$$x_F = \frac{1}{F}\frac{q_0 L^2}{3} = \frac{2}{q_0 L}\frac{q_0 L^2}{3} = \frac{2L}{3}$$

This is exactly the location of the *centroid* of the triangular distribution (see Figure E1.1b)!

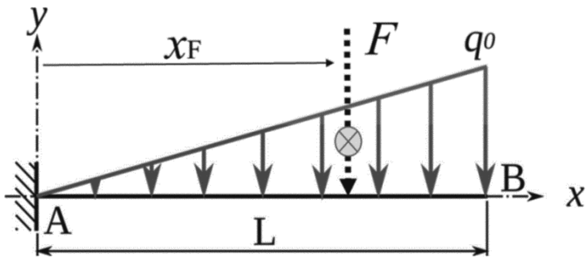


Figure E1.1b.

Example 1.2. Statically determinate or indeterminate

Let's check the truss in RP1.1 to see whether or not it is statically determinate.

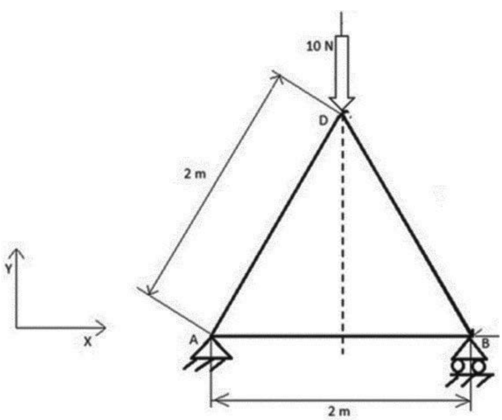


Figure E1.2.

Referring to Figure E1.2 (same as Figure RP1.1), we see there are three members, three joints, and three (possible) unknown reactions (R_{Ax}, R_{Ay}, and R_{By}). Then Equation (1.1) gives:

$$i = 3 + 3 - 2(3) = 0$$

This truss is thus statically determinate.

Example 1.3. Statically determinant or indeterminate. Determine whether the trusses in Figures E1.3a and E1.3b are statically determinate or indeterminate.

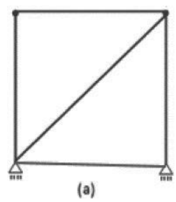


Figure E1.3a.

The above truss has five members, four joints, and three (possible) unknown reactions (two translation and one rotation). Then Equation (1.1) gives:

$$i = 5 + 3 - 2(4) = 0$$

This truss is thus statically determinate.

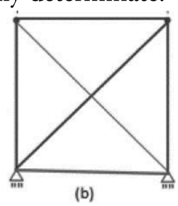


Figure E1.3b.

The above truss has six members, four joints, and three (possible) unknown reactions (two translation and one rotation). Then Equation (1.1) gives:

$$i = 6 + 3 - 2(4) = 1$$

This truss is thus statically indeterminate.

1.2.4 Materials

Every structure is a material. The response of a structure is intrinsically interrelated among geometry, boundary conditions, loads, and material properties. One of the most basic of material properties is mass, which we take a look at next.

Mass, density, and weight

> **C³**
>
> 1.6 How do I know whether the density value is mass density or weight density?
> *(Answer: Section 1.6)*

Students often find the concepts of *mass*, *density* and *weight* confusing. This is exacerbated by the confusion in *systems of units* (see the review section in the appendix), and by the fact that two types of density are often reported: *mass* density and *weight* density.

By definition we take *density* to mean *mass density* ρ, specifically the *mass* per unit *volume* of a given material. *Mass* relates to the *amount of material*; unit volume could be a cube, say, 1 mm on a side (but it could also be a spherical unit volume, etc.). The mass of a given material is the same *anywhere* in the universe, and hence is a very fundamental property. The mass density then quantifies how tightly packed is that material. Units of mass are, for example, kilogram or slug; units of mass density are, for example, kg/m^3 or slug/ft^3.

Weight, on the other hand, is a *force*, specifically the gravitational force. It depends both on the material (its mass) and the intensity (i.e, acceleration *g*) of the local gravitational field:

$$w = mg \tag{1.2}$$

Units of weight are force units, like newton or pound. *Weight* density is also commonly used, i.e., the *weight* per unit volume; unfortunately, the same symbol is often used for weight density as for mass density, and sometimes simply *density* is reported without definition, and one has to take special note of the units used.

We recognize that the weight of an object on the moon is about 1/6 of its weight on earth. Same mass, different weight! Thus we take mass or mass density as fundamental, not weight.

It is common to talk about wanting to *minimize the weight* of a structure. That's a bit sloppy, because what we really want to do is *minimize the mass* of the structure. This can be done either by reducing the density (again, we take density in this text to always mean *mass* density), or by reducing the volume of material—either way, we reduce the mass, and for a given gravitational field, reduce the weight.

1.3 Strength and Stiffness

When we think of a structure carrying loads, two primary questions must be asked:

- Is it *strong* enough?
- Is it *stiff* enough?

In other words, we wish to know how well the primary functional requirement of a given structure—carrying loads—is performed; this is the primary role of *structural analysis*. (Ultimately, a number of other questions must be asked as well, such as: Is it light enough? Cheap enough? Repairable enough? …)

Strength and stiffness are independent quantities that depend on the constituents of the material in different ways. We'll come to understand these concepts more in later sections as well. At this point, we just mention that strength considerations are usually of greater importance in structures than stiffness. If you carry a load with a large rubber band, you are usually not too worried if it stretches quite a bit, <u>as long as it doesn't break</u> (i.e., as long as it is strong enough). However, if the band stretches too much, your arms may not be long enough to keep the load from interfering with the ground, and now stiffness (or lack of it!) does become of concern.

An important issue we cannot avoid is: "How much is <u>enough</u>?" Unfortunately, the answer is a somewhat unsatisfying—"<u>It depends</u>." It depends on the application, the failure mode, how catastrophic would be the failure (for example, loss or injury of human life), the past history of similar designs, how well known are the loads and material properties, etc. In any case, we must always compare <u>what we have</u> (stress, strain, deflection—the *load effects*) to what we can allow (strength, stiffness—*the resistance*). This comparison is done generally in one of two ways (a third being a combination of these two): *resistance factor design* and *load factor design* (the third being *load and resistance factor design).*

By way of illustration, one common, elementary approach to determining "enough" is to define a *factor of safety* FS (some might say *factor of ignorance*!) that tries to account for the effect of many of the issues described above into one quantity. The factor of safety (FS ≥ 1) is applied to the *resistance* side of the comparison, and as such creates a *knockdown factor* resulting in an *allowable resistance*. For example, if the resistance considered is *strength*, then we have:

$$\frac{S_{failure}}{FS} = \sigma_{\max} \tag{1.3}$$

where

$S_{failure}$ = designated failure strength (tensile yield, tensile ultimate, compressive yield, etc.)

σ_{max} = maximum stress (at the critical section) associated with the appropriate failure mode (shear, von Mises, etc.)

The most general form of the comparison is through *load and resistance factor design* (LRFD):

$$\alpha\, R = \sum_{i=1}^{I} \beta_i Q_i \tag{1.4}$$

where

α = resistance factor (≤ 1)

R = nominal resistance

β_i = ith load factor (usually > 1)

Q_i = ith nominal load effect, associated with the ith nominal applied load, I total loads being applied.

We will not pursue this later approach in this text. Organizations such as the American Institute for Steel Construction (AISC) provide tables of load and resistance factors for various applications.

1.4 Modeling and Analysis

1.4.1 Characteristic Tasks in Structural Analysis

Structural *analysis* supports structural *design*. Structural analysis can include a variety of tasks aimed at understanding the complex response of structures, specifically the resultant stress, strain, and displacement to applied loads, which answer the questions about strength and stiffness. A sample of such tasks is given below:

- Loads analysis
- Strength analysis
- Stiffness analysis
- Natural frequency analysis
- Dynamic response analysis
- Damping analysis
- Thermal (heat transfer) analysis
- Thermoelastic (thermal deformation) analysis
- Structural life (fracture, fatigue) analysis
- Mass property (weight, center of gravity) analysis
- Precision (shape and position accuracy) analysis
- Sensitivity (of the design to perturbations or small changes) analysis

1.4.2 Methods of Structural Analysis

Structural analysis consists of three fundamental parts:

i. *Equilibrium*. Here we consider forces, moments, and the application of Newtonian mechanics (or the potential and kinetic energies and the application of Lagrangian mechanics), and stress. If the body is in *global* equilibrium, then every *local* particle of the body is also in equilibrium. Equilibrium implies negligible accelerations (inertia forces), and hence implies *static analysis*. (*Dynamic analysis* would consider the motion of the body, requiring inertia forces in the full *equations of motion*.)

ii. *Deformation*. Here we consider the geometry of material particle displacement and the concept of strain. We assume that any material is *continuous* and fully populated with *material particles* (the *continuum hypothesis)*. We further assume in this text that deformations are small enough that only a linear analysis is required.

iii. *Constitution*. Stress and strain are *dual* quantities that are intimately related within a given material/structural system. These are *stress-strain relations.*

The methods of structural analysis are *mathematical* analysis and *experimental* analysis. Mathematical analysis may result in *closed-form* solutions, *series* solutions, *asymptotic* solutions, *numerical* solutions, etc. There are two major approaches for performing mathematical analysis: Newtonian or vector mechanics and Lagrangian or scalar mechanics. Except for relatively simple structural problems, numerical solutions are usually required. The *direct stiffness method*, and its descendant the *finite element method*, are the most ubiquitous and powerful numerical methods for structural analysis today.

Newtonian or vector mechanics is the most common for structural analysis for at least two reasons. First, it is the most intuitive due to the arrow-like representation of forces, displacements, etc. Second, vector mechanics solves for the critical quantities like internal forces and moments directly. The central tool used in Newtonian analysis is the Free Body Diagram (FBD). The FBD is used to account for all forces and moments when applying Newton's laws. (See Appendix 2 for further details.)

Experimental analysis involves the testing of real or prototypical structures, and uses various techniques to assess strain and/or displacement. Some of these experimental techniques are:

- Strain gages
- Optical interferometers
- Extensometers
- Videography
- Thermography

Quite often, combinations of mathematical and experimental methods are used. Many standard results are available in reference sources (see Section 1.8).

In every case of analyzing real structures, certain *assumptions* about their behavior or character must be made. This leads to an *idealized structure model* that will be divorced from the real structure to some greater or lesser degree. **The engineer must always be aware of this discrepancy**.

1.5 Structural Design

In this section we review some concepts from statics that will be useful later in the analysis and design of structures.

RP1.1. Equilibrium

Consider an equilateral truss loaded as shown in Figure RP1.1. Determine the internal forces in the truss.

The symmetric truss problem (symmetric geometry about the line of action of the load) that follows is a good place to start because intuition tells us that the reaction loads should also be symmetrical about the line of action (the dashed line in the figure). In this case, since the total global reaction equals the load, or $R_A + R_B = 10$ N, the individual global reactions must be equal, or $R_A = R_B = 5$N.

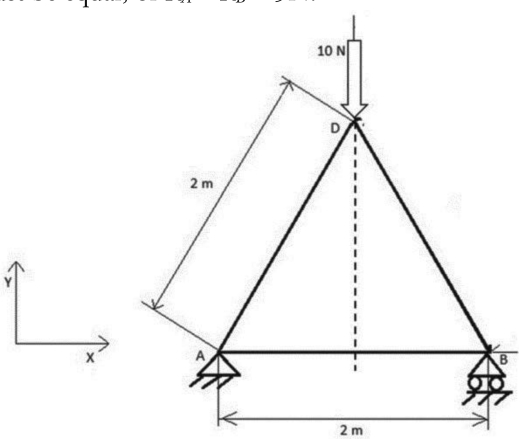


Figure RP1.1a.

Also note that the action of the load will try to increase the x-distance between joints A and B. But the truss member AB resists that spreading so the boundary conditions don't have to! In fact, for this problem both boundary conditions could be rollers without changing a thing. It's just good structural design to restrain the structure from rigid body motion in case the load isn't perfectly vertical.

Let's verify our intuition by checking global equilibrium. In Example 1.2 we've already verified that this truss is statically determinate and thus should be solvable. So next we need a FBD of the entire truss (Figure RP1.1b):

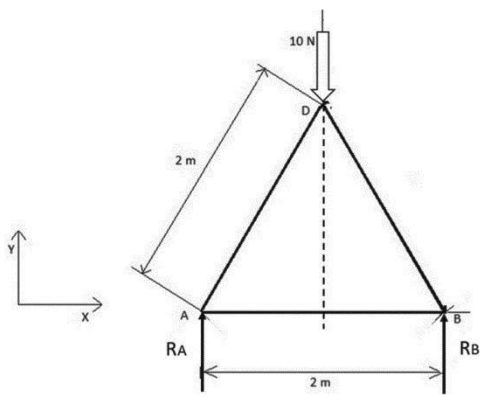


Figure RP1.1b.

Note that the global reaction forces R_A and R_B are assumed to be acting in the +y-direction, opposing the applied load. This is the only choice that makes physical sense. But for static problems, the assumed direction doesn't matter since the sign of the solution will tell you whether you got the direction correct (positive solution) or wrong (negative solution).

Applying the equilibrium equations globally now:

$$+\uparrow \sum F_y = 0 : R_A + R_B - 10N = 0 \Rightarrow R_A + R_B = 10N$$

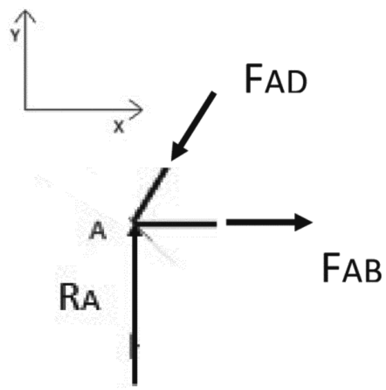
$$+\circlearrowleft \sum M_A = 0 : -10N * 1m + R_B * 2m = 0 \Rightarrow R_B = 5N$$

Note that the global reaction forces R_A and R_B are assumed to be acting in the *+y*-direction, opposing the applied load. This is the only choice that makes physical sense. But for static problems, the assumed direction doesn't matter since the sign of the solution will tell you whether you got the direction correct (positive solution) or wrong (negative solution).

This is the answer we expected. (Note that since there were no reaction forces in the *x*-direction, $\sum \mathbf{F}_x = 0$ was automatically satisfied.)

Now let's apply the equilibrium equations locally to determine the internal reaction forces in the truss members. We need a *local* FBD of the truss, a FBD that contains at least one known force, such as joint A (Figure RP1.1c).

Note that the internal reaction forces F_{AD} and F_{AB} are assumed to be acting as shown. This is the only choice that makes physical sense. But for static problems, the assumed direction doesn't matter since the sign of the solution will tell you whether you got the direction correct (positive solution) or wrong (negative solution).



Note that the internal reaction forces F_{AD} and F_{AB} are assumed to be acting as shown. This is the only choice that makes physical sense. But for static problems, the assumed direction doesn't matter since the sign of the solution will tell you whether you got the direction correct (positive solution) or wrong (negative solution).

Figure RP1.1c.

Now we apply the equilibrium equations locally. In order to do that, we need to resolve F_A into *x* and *y* components. Then, with θ = 60 degrees:

$$+\rightarrow \sum F_x = 0 : -F_{AD} cos\theta + F_{AB} = 0 \Rightarrow F_{AB} = F_{AD}\, cos\theta$$

$$+\uparrow \sum F_y = 0 : R_A - F_{AD}\, sin\theta = 0 \Rightarrow F_{AD} = \frac{R_A}{sin\theta} = \frac{5.00\,N}{0.866} = 5.77\ \boldsymbol{N}$$

Knowing F_{AD} allows us to solve for F_{AB} from the first equation above, or F_{AB} = 5.77 N * 0.5 = 2.89 N. (Note that since the line of action of all forces pass through joint A, $\sum M_A = 0$ was automatically satisfied. Also note that we've assumed we know the load, and hence R_A, to at least three significant figures, i.e., 10.0 N and 5.00 N, respectively.

The last activity left is to ask: Does our result make sense? Let's take one more look (Figure RP1.1d):

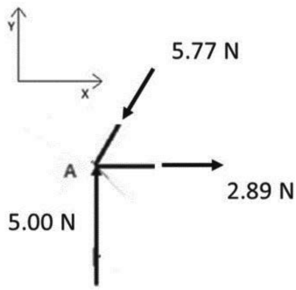


Figure RP1.1d.

Note two things:

1. The vector addition of forces is correct, i.e., $5.00^2 + 2.89^2 = 5.77^2$.
2. Intuition suggests that as theta decreases, AB and AD approach parallelism and would carry equivalent loads. As theta increases, the load carried by AD increases and that by AB decreases. (You may need to sketch these for it to make sense.) If we plot our solutions for FA and FAB Figure RP1.1e), this exactly what we have:

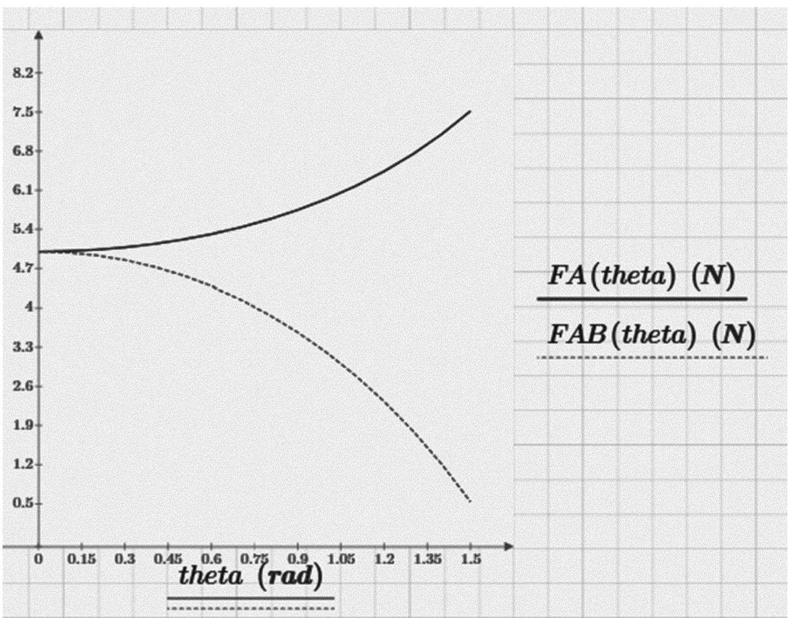


Figure RP1.1e.

1.6 C^3 Clarified

C^3 1.1 The word "structure" is used differently here than in my chemistry class. Why?

Answer. Unfortunately, our language contains many words with dual meanings and this makes learning even more challenging than it is already! We cannot escape and the best we can do is be internally consistent (for example, see the discussion on moment of inertia). Another meaning of the word "structure" is "organization" and a chemist might talk about the structure of bonds in a certain polymer, that is, the organization of the bonds.

C^3 1.2 My mechanics of materials course and text have very little in it about materials. Does it matter?

Answer: Yes it does! Every structure is a material and structural performance depends greatly on material performance. For example, consider axial stiffness EA/L—it is easy to see the material contribution through the elastic modulus E. Failure to recognize the significance of the material in structural design limits optimal solutions at the least and risks structural collapse at the worst.

C^3 1.3 Can a "line" load or a "point" load really exist?

Answer: In principal, "point" and "lines" loads cannot exist, since any load will act over a finite spatial "area", no matter how small; however, if that area is small relative to the structure size, then for practical purposes the load may be considered a "point" load or a "line" load.

C^3 1.4 Is a statically indeterminate structure a bad structure?

Answer: No. In fact, most real structures are statically indeterminate in that they have more constraints (boundary conditions) than needed. This redundancy can actually be a good thing in case a load path fails (see Section 1.7.1).

C^3 1.5 Why is integration needed here?

Answer: Note that to find the total of the distributed load, we must "add up" all of its "parts". But there are an infinite number of "parts", one for every x between 0 and L. So we must do an infinite sum or an integration.

C^3 1.6 How do I know whether the density value is mass density or weight density?

Answer: If you mistake weight for mass, you are off right away by at least 10 in SI (g = $9.81 m/s^2$) or 32 or even 386 (g = 386 in/s^2) in the US Customary system. These are serious errors. You have got to check the units for the data given, for example, mass density would be in kg/m^3 or $slugs/ft^3$ or lb s^2/in^4. On the other hand, weight density would go like N/m^3 or lb/ft^3 or lb/in^3. If in doubt, ask the person who supplied the data.

1.7 Developing Engineering Intuition

1.7.1 Load Paths

Since structures carry loads, a useful tool in structural analysis is the concept of *load path*, i.e., the path by which the load is carried. It helps if one can imagine a "flow of stress or load" along the path. In many cases, this is trivial and obvious, particularly when there is only a single, unique path. For example, the load from a sign is carried (literally *to ground* in this case) by the cantilevered arm and the beam-column (post) as in Figure 1.5 below.

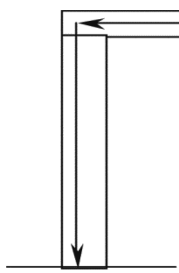


Figure 1.5. Load path for a simple sign, arm, and post.

A sign supported by a truss presents a more complicated load paths in Figure 1.6.

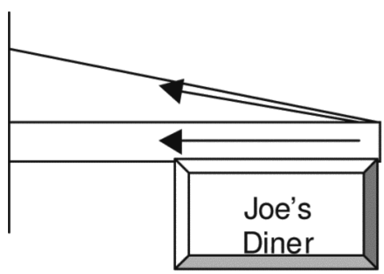


Figure 1.6. Load paths in truss-supported sign.

To continue the discussion, imagine that in Figure 1.6, the vertical beam-column is attached to ground through a bolted flange connection (Figure 1.7 is a detailed section of the lower portion of Figure 1.6). What is the load path to ground now?

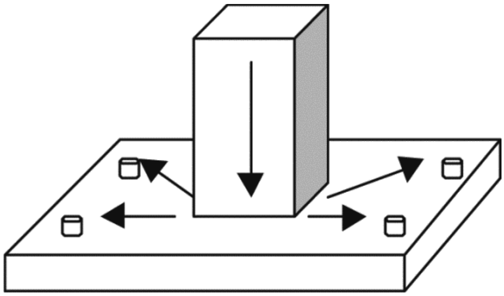
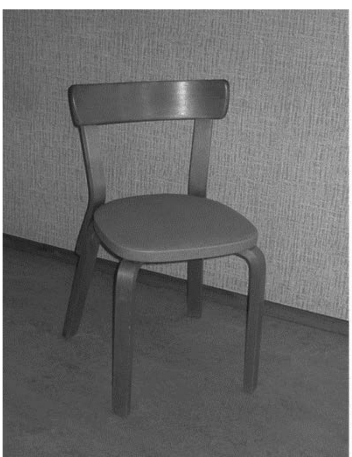
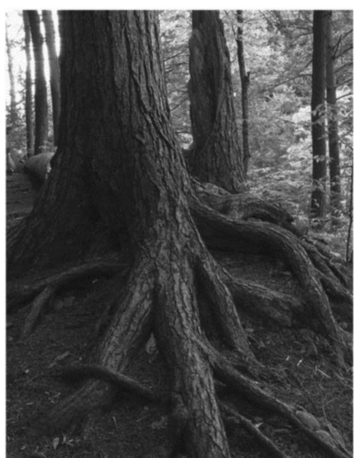


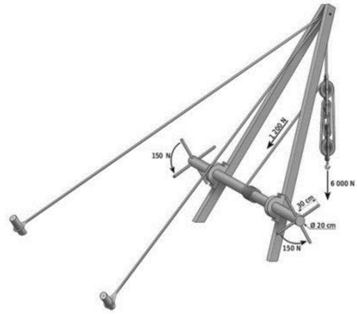
Figure 1.7. Load paths at the base of the beam-column in Figure 1.11.

We will find as we proceed that the load sharing among multiple paths apportions itself in large part according to the *path stiffness*. As a simple example, consider a load carried by two parallel springs, one considerably stiffer than the other. It is not hard to imagine that the stiffer spring carries a greater portion of the load, and we will prove this result later. Developing an intuition for load paths will prove to be a very useful asset for the structural engineer.

Find the load paths for the following structures:







c)

d)

1.8 Resources

Crandall, S. H., Dahl, N. C. and Lardner, T. J. (1999). *An Introduction to the Mechanics of Solids: Second Edition with SI Units.* New York, NY: McGraw-Hill.

Jenkins, C.H. and Khanna, S. (2005). *Mechanics of Materials.* Amsterdam: Elsevier.

Roylance, D. (1996). *Mechanics of Materials.* New York: John Wiley & Sons.

Schodek, D.L. and Bechtold, M. (2001). *Structures* (7th ed.). Upper Saddle River, NJ: Prentice Hall.

Young, W., Budynas, R., and Sadegh, A. (2011). *Roark's Formulas for Stress and Strain* (8th ed.). New York: McGraw-Hill.

As they become available, various other volumes in the *Engineering Companions* collection, such as statics and mathematics, would be excellent resources as well.

CHAPTER 2

Basic Structures Level I

Learning Objectives: This Chapter will provide the student with a systematic introduction to the three basic structure types: axial, torsional, and flexural. Strength design will be the focus of this Chapter. (Stiffness design is covered next in Chapter 3.) Along the way, we will clear up several common confusing concepts (C^3s).

Clarifications: Studying this Chapter will help clarify several C^3s, including:

- Definition and configuration of the three basic structures
- Basic structures loading and response
- Uniaxial material response
- Solving strength design problems
- Moment of inertia
- Stress vector
- Global vs local equilibrium
- Shear and bending sign conventions

Importance: Structures fall down! Most structures fail because they are not strong enough. A classic example is the Hyatt Regency walkway collapse in 1981 (Figure I2.1).

Figure I2.1. Hyatt Regency Walkway Collapse

We'll explore this example later.

2.0 Representative Problems

The following are examples of basic strength design problems we want to be able to solve as engineers. Studying this Chapter will help us solve them, some of which we will do along the way, before finishing up in Section 2.5.

RP2.1

The stepped rod shown in Figure RP2.1 consists of two co-axial circular cylindrical sections carrying concentrated forces: 800 lb at location B (x = 5 inch) and 1000 lb at location D (x = 22 inch). Cylinder ABC is 0.375 inch in diameter and CD is 0.25 inch in diameter. The material is high-carbon steel (E = 30 x 10^6 psi). Determine the overall extension of the rod and check for strength design.

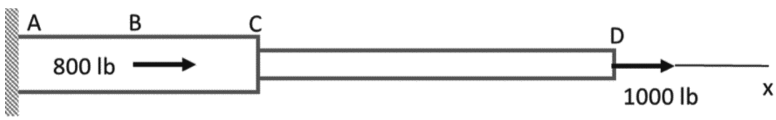


Figure RP2.1.

RP2.2

A 2024-0 aluminum shaft 30 inches long is built in at one end as shown in Figure RP2.2. A torque is applied at the free end of magnitude 6000 in lb.

a. For a 2 inch diameter solid circular cross-section, determine the maximum shear stress and the shaft weight.
b. Now consider an annular cross-section of outer radius R_o and inner radius R_i. If the wall thickness $h = R_o - R_i$ and $R_o/R_i = 1.1$, determine R_o, R_i, and the shaft weight if the annular shaft has the same maximum shear stress as the solid circular shaft.

RP2.3

We wish to minimize the amount of material in a beam such that, under a specific loading condition, each cross-section will be at the maximum allowable stress (at the outer fiber). Applications are leaf springs, gear teeth, and bridge girders. Consider a cantilever beam of rectangular cross-section b by $h(x)$, with tip load P as shown in Figure RP2.3. Find $h(x)$.

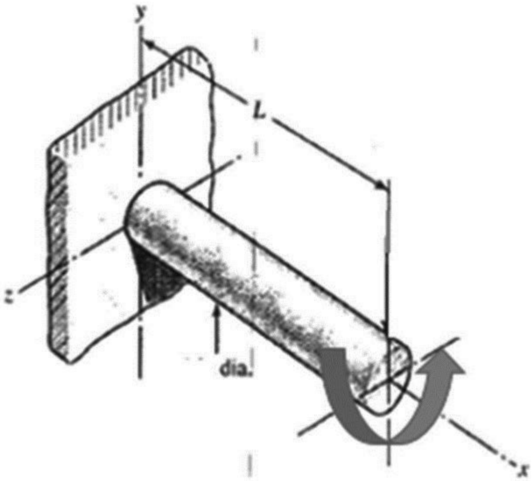


Figure RP2.2.

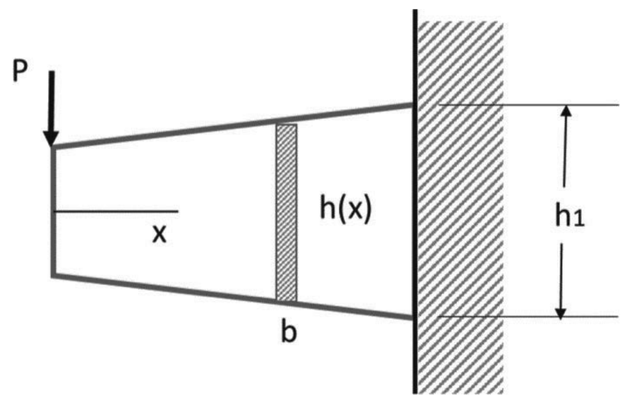


Figure RP2.3.

RP2.4

Compare the maximum stress due to a central concentrated load P between two beams that are identical except for the boundary conditions: one beam is fixed-fixed and the other is pinned-pinned.

RP2.5

For the rod in RP2.1, calculate the total strain energy stored due to the loads applied.

2.1 Definitions

2.1.1 Axial Structures

Axial structures carry loads primarily in tension or compression. They are typically long, straight, slender structures, which could be categorized as "one-dimensional (1-D)" structures. Axial structures are often called by the names *bar, link, rod*, or *strut.* Whether or not a given structure responds, or can be represented to respond, as an axial structure depends on the nature of the loading, and on the structure's boundary conditions. For example, a *truss* will be an axial structure if its ends (joints) are pinned, and if the loading occurs only through these pin joints, as shown in Figure 2.1.

In thinking of these structures as one-dimensional, we are ignoring the fact that all real structures deform simultaneously in three dimensions under the action of any load. We can get away with this neglect, for now, because we will only consider the structure's response to a single load type, and that its multi-axial response may not be important for the given design. Later, when we consider combined loading (Chapter 4), we will be required to look at multi-dimensional response.

Figure 2.1. While the various members that make up the truss assembly may or may not be axial members, the rods supporting the hanging lights clearly are axial.

Tensile axial loads act to extend the axial structure, while compressive axial loads act to compress the structure. If an axial member is rather *compact*, that is "short and stout" (Figure 2.2), its response is not unlike that of tensile axial structures. Failure is governed largely by strength, since a compact member should be overly stiff.

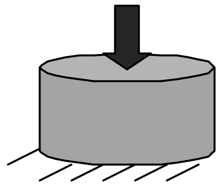


Figure 2.2. Compact compressive axial structure.

On the other hand, if the member is relatively slender (e.g., a *column*), compressive failure may be due to lack of stiffness, and a completely different response occurs (this becomes a *stability* problem which we do not treat here). The axial member, at a given *critical load*, will *buckle* or warp (Figure 2.3). Excessive buckling can lead to complete collapse upon continued loading.

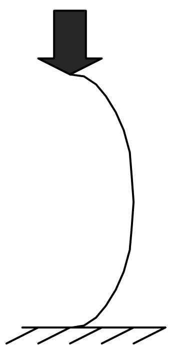


Figure 2.3. Slender compressive axial structure undergoing buckling.

An example of slender compression members can be seen in Figure 2.1. Note the columns that carry roof loads down to the truss. We will defer to a later volume the study of buckling behavior and the design of slender compressive axial structures (we need to first explore flexural structures).

2.1.2 Torsion Structures

Thus far we have introduced axial structures, capable of carrying axial loads either in tension or compression. We have seen that axial structures can form elements of more complicated structural systems, e.g., trusses, capable of carrying limited transverse loads.

But what of *twisting loads*? If you put a wrench on a nut rusted tight to a bolt and twist it so as to tighten or loosen it, how is the load carried (Figure 2.4)? What is the internal response of the bolt?

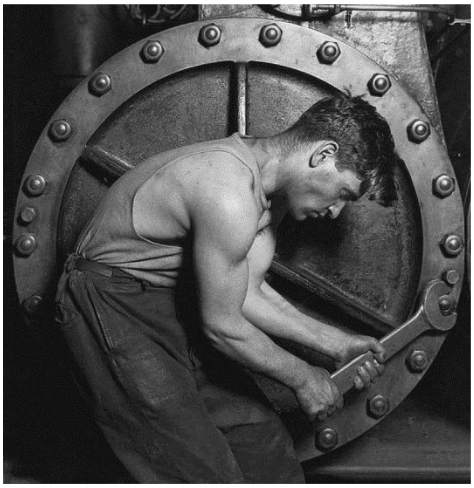


Figure 2.4. Wrench being used to tighten a nut.

Many other examples of such torsion structures are common, such as the shafts on an automobile (Figure 2.5).

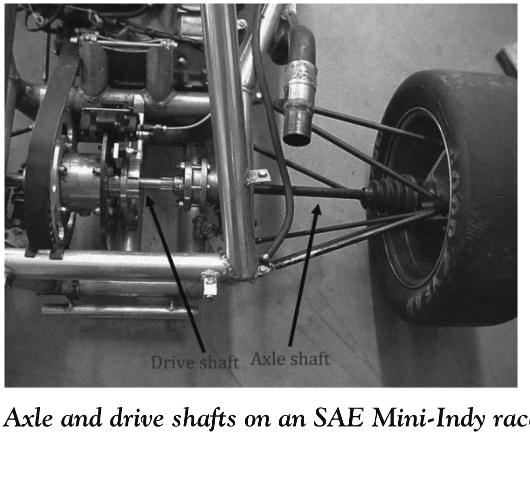


Figure 2.5. Axle and drive shafts on an SAE Mini-Indy racecar.

An infamous example of a torsion loading is the failure of the Tacoma Narrows Bridge in the state of Washington. On November 7, 1940, just several months after it opened, the bridge experienced severe torsion oscillations that ultimately lead to its collapse. The peak amplitudes of the oscillations reached around 20 ft (Figure 2.6), and fortunately there was no loss of human life.

Figure 2.6. Torsion oscillations of the Tacoma Narrows Bridge.

In practice, it is difficult to load any structural member purely in torsion. All of the examples above have transverse loads applied in addition to torques (*combined loading*). In this Chapter, however, we assume the loads to be purely torsional (we'll consider cases of combined loading later in Chapter 4).

2.1.3 Flexural Structures

Flexural structures carry *transverse* loads primarily through *bending* (flexing), which is accompanied by a stress state of combined tension and compression. The tension and compression state creates an internal *bending moment*. Common flexural structures, called *beams*, are typically straight, narrow members that are longer than they are deep (but short and stubby, as well as curved, beam configurations are also used). A plate or shell-like structure can also be used as a flexural structure. Whether or not a given structure responds, or can be represented to respond, as a flexural structure depends on the nature of the loading, and on the structure's boundary conditions. A structural element will be a flexural structure if its boundaries and loading are such that an internal bending moment can be developed.

Simplifying assumptions are often made in analyzing flexural structures. For example, we will ignore the fact that all real structures deform simultaneously in three dimensions under the action of any load. We can get away with this neglect, for now, because we will only consider the structure's response to transverse loads. Later, when we consider *combined loading*, we will be required to look at multi-dimensional response. We will in this text also disregard the effects of a small amount of shear that may develop in flexural deformation.

Beams are one of the most common of structural types. In nature, tree limbs are a most familiar example (Figure 2.7):

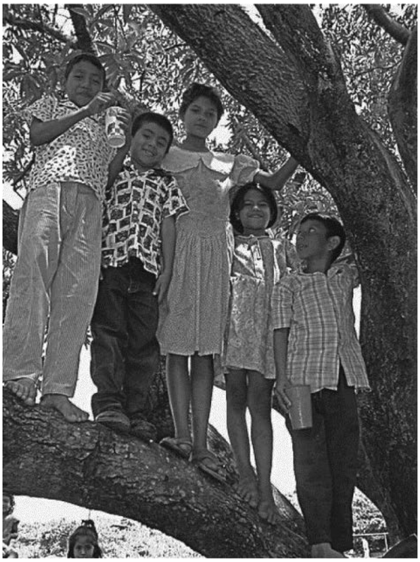


Figure 2.7. Occasionally, tree limbs must carry loads in addition to their own weight. Limbs are nonprismatic cantilever beams. Note that the limbs (and trunks) are always thicker near their attached ends. Why?

Beams resting on columns are fundamental building blocks of the built environment, as seen in Figure 2.8.

Figure 2.8. A beam (foreground top) supported by a column. Trusses are seen in the upper right.

The wings of a modern airplane are highly evolved and complex beams (Figure 2.9).

Figure 2.9. The Museum of Flight. The wings of planes are easily recognized as cantilever beams.

2.2 Geometry, Boundary Conditions, and Loads

2.2.1 Axial Structure Geometry and Boundary Conditions

In Figure 2.10, an axial structure called a *bar, link, rod*, or *strut* of round cross-sectional shape is shown. A force P is applied to one end of the rod, resulting in a displacement *u*. The other end of the rod has a boundary condition described as either fixed or built-in. The rod has geometric properties of length L and constant cross-sectional area A. (Material properties will be discussed later in this Chapter.)

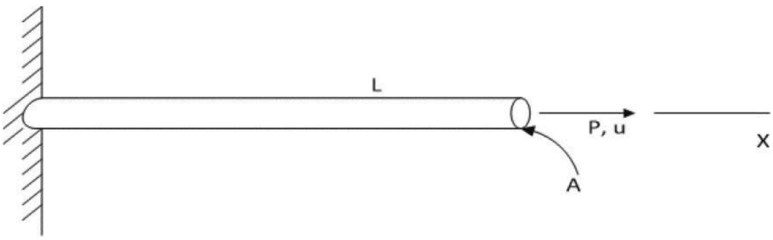


Figure 2.10. Definition sketch of an axial structure.

Comments:

i. Keep in mind that although we have shown a rod with constant, circular cross-section, primarily for convenience and simplicity, a rod may have a non-constant and/or, non-circular cross-section.
ii. However, to remain an axial structure, the axis of every cross-section, which is the line passing perpendicularly through the centroid of that cross-section, must remain co-linear.
iii. Structures with constant cross-sectional area, or more generally constant second moment of area, are called *prismatic* structures.
iv. Axial structures have one *coordinate degree of freedom* (CDOF): *x*-displacement (designated as *u*).

We need to be cautious here about boundary conditions. In the case of tensile axial structures, we can readily allow the case of pinned-free boundary conditions. Such a configuration (of co-axial tensile loads and boundary conditions combined) is *unconditionally stable*. That is, any slight *perturbation* (disturbance) of the rod from its equilibrium position (Figure 2.11a) would simply be returned to the equilibrium position (Figure 2.11b):

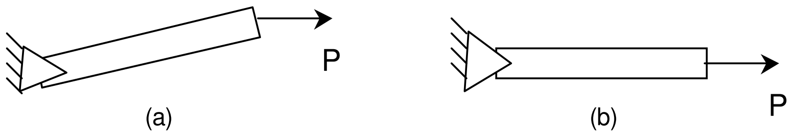


Figure 2.11. Tensile axial structure slightly perturbed from equilibrium returns to the equilibrium position.

Now in the analogous compression case, if we perturb the rod slightly from its equilibrium position (Figure 2.12a), a radically different response occurs:

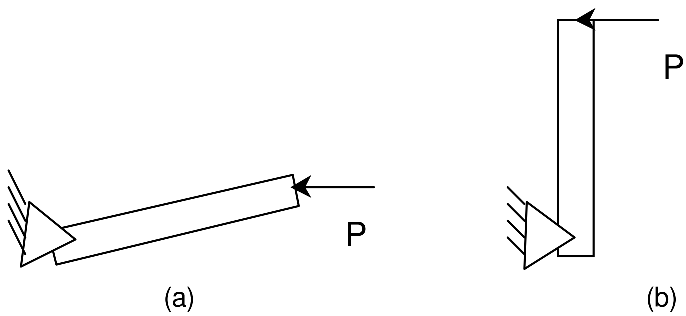


Figure 2.12. Compressive axial structure slightly perturbed from equilibrium does not return to the equilibrium position, but seeks a new equilibrium position.

In this case, the rod undergoes large excursions away from the original equilibrium position (until a new equilibrium position is found, as seen in Figure 2.12b.). The original equilibrium configuration then must have only been *conditionally stable* (also called *unstable equilibrium*).

For our present discussions, we assume that the configurations (tensile or compressive co-axial load plus boundary conditions) are unconditionally stable. In the compressive case, pinned-pinned boundary conditions are a common example of such configurations.

2.2.2 Torsion Structure Geometry and Boundary Conditions

We begin by considering that the torsion structure forms a structural member that is straight and relatively narrow, i.e., the dimension of any cross-section is small compared to the member length L. The cross-section is characterized by its area $A(x)$ and *polar second moment of area (polar moment of inertia)* $J(x)$ (this later quantity will fall naturally out of the analysis later).

C³

2.1. What is the difference between the moment of inertia and the second moment of area?
(Answer: Section 2.6)

We first consider only cross-sections that are symmetric with respect to rotations about the long axis (x-axis in the figure), i.e., *axi-symmetric* cross-sections, and that are at the same time *prismatic*, i.e., with $A(x)$ = constant. The simplest embodiment of an axi-symmetric prismatic torsion structure is the *solid circular shaft*, whose cross-section is a circle of radius R (Figure 2.13). In this case, $A = \pi R^2$ and

$$J = \int_A r^2 dA \tag{2.1}$$

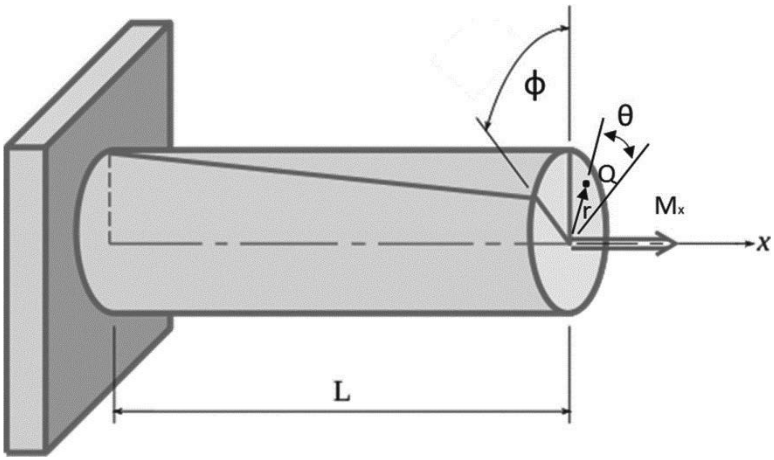


Figure 2.13. Solid circular shaft of radius R. Note that the cylindrical-polar coordinates of a point Q are Q(r, θ, x). θ is a rotational coordinate and ϕ is the rotational displacement.

For now, we will consider our torsion structural members to be right circular cylinders (solid or hollow). Any point on the circular cross section x = constant can be identified by the cylindrical-polar coordinates (r, θ, x).

Torsion structures as defined here have one CDOF: a θ-displacement or ϕ. Boundary conditions for the torsion structure must include at least one fixed boundary. (Why?) Hence either fixed-free or fixed-fixed are the required torsional boundary conditions.

Example 2.1

a. Determine the general expression for the polar second moment of area for a solid circular shaft of radius *R*. If $R = 1.0$ inch as in RP 2.2a, what is the value of *J*?

The incremental area $dA = 2\pi r dr$ (see Figure E2.1a). Then Equation (2.1) gives

$$J = \int_A r^2 dA = \int_0^R r^2 \left(2\pi r dr\right) = \frac{\pi R^4}{2}$$

If $R = 1.0$ inch, $J = 1.57$ in^4.

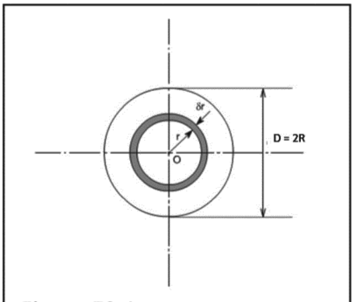


Figure E2.1a

b. Determine the general expression for the polar second moment of area for a hollow circular shaft of inner and outer radius R_i and R_o, respectively.

The incremental area is still $dA = 2\pi r dr$, only the integration limits change (see Figure E2.1b). Then Equation (2.1) gives

$$= \int^A r^2 dA = \int_{R_o}^{R_i} r^2 (2\pi r dr) = \frac{\pi R^4}{2} \Big|_{R_i}^{R_o} = \frac{\pi}{2}(R_o^4 - R_i^4)$$

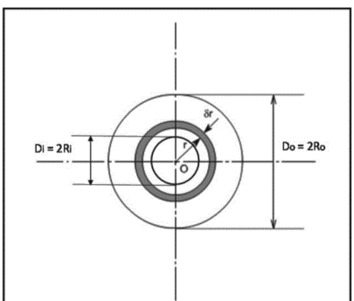


Figure E2.1b

2.2.3. Flexural Structure Geometry and Boundary Conditions

The beams we consider here have cross-sections that are relatively narrow and compact. We will for now consider that the cross-sections are symmetrical with respect to the plane of bending (say the *x*-*y* plane in Figure 2.14). (This makes for simplification in the analysis of flexural structures. The added complications from non-symmetrical cross-sections are outside the scope of this volume.)

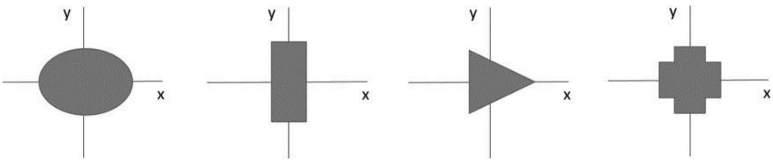


Figure 2.14. Simple beam cross-sections symmetrical with the bending plane (arbitrarily selected as the x-z plane).

The cross-section is characterized by its area $A(x)$ and *second moment of area* (*area moment of inertia*) $I(x)$, where in general A and I can be functions of position x along the beam. The second moment of area is given by:

$$I(x) = \int_A y^2(x)dA \qquad (2.2)$$

For now, however, we will only consider *prismatic* beams, i.e., beams with constant cross-sectional properties, where A and I are constants along the length of the beam. In this case, we can represent the beam as a <u>one-dimensional</u> structural element of length L, with cross-sectional properties A and I (Figure 2.15).

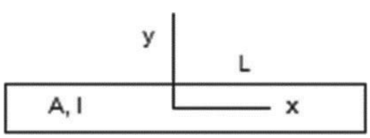


Figure 2.15. Prismatic beam as a one-dimensional structural element.

It is convenient to think about the beam as being comprised of longitudinal "fibers", running the length of the beam and parallel to the long axis (*x*-axis in the above figure). (The "fiber" imagined here is just a collection or locus of material points along a given y = constant line and not a real fiber, although real fibers can exist, e.g., in fiber-reinforced composite materials.) In a rectangular cross-section h high by b wide, the *outer fibers* would be located on the surfaces $\pm h/2$. Another important fiber is co-located with the *neutral axis* of the beam. The neutral axis (really, neutral *plane*) is a locus of material points that have zero stress during flexure and will be described in more detail in Section 4. (Unless otherwise specified, we will assume the x-axis to be coincident with the neutral axis of the beam.)

Unlike the axial and torsion structures, there is considerable variety of boundary conditions available for beam support, since the loading and subsequent reactions can be more complex. The supports can occur at any x location along the length of the beam, although the location may limit the type of boundary conditions applicable.

A beam has three *coordinate degrees of freedom* (CDOF) associated with its response: axial translation u, transverse translation v, and rotation in the plane of bending θ (Figure 2.16). One or more of these CDOF may be constrained at the boundary, defining the boundary condition, depending on the nature of response desired in the beam. (Because the beam is modeled as one-dimensional, we automatically assume that all other translations and rotations are constrained.)

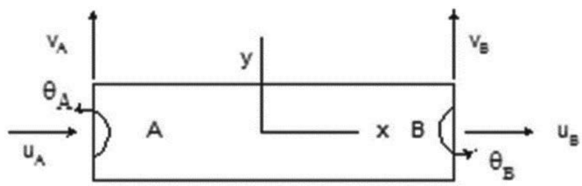


Figure 2.16. Coordinate degrees of freedom for a beam.

The principal beam boundary conditions are:

- Pin or simple support ($u = 0$, $v = 0$, $\theta \neq 0$)
- Roller support ($u = 0$, $v \neq 0$, $\theta \neq 0$) or ($u \neq 0$, $v = 0$, $\theta \neq 0$)
- Guide support ($u = 0$, $v \neq 0$, $\theta = 0$) or ($u \neq 0$, $v = 0$, $\theta = 0$)
- Fixed or clamped support ($u = 0$, $v = 0$, $\theta = 0$)

Example 2.2

a. Determine the general expression for the second moment of area of a solid rectangular beam of width b and depth h.

The incremental area dA = $2x$ dy = b dy (see Figure E2.2a). Then Equation (2.2) gives

$$I_{xx} = 2\int_A y^2 dA = 2\int_0^{h/2} y^2 b dy = 2b\frac{y^3}{3}\Big|_0^{h/2} = \frac{bh^3}{12}$$

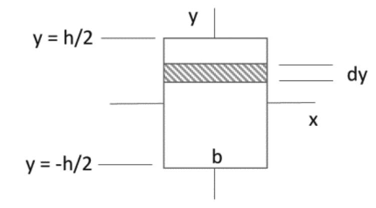


Figure E2.2a.

b. Determine the general expression for the second moment of area of a solid circular beam of radius R. Show that $J = I_{xx} + I_{yy}$.

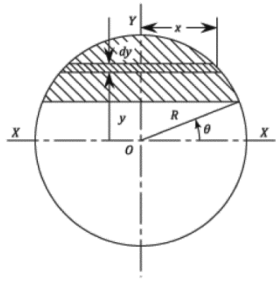


Figure E2.2b.

The incremental area dA = $2x$ dy = $2(R^2 - y^2)^{1/2}$dy (see Figure E2.2b). Then Equation (2.2) gives

$$I_{xx} = 2\int_A y^2 dA = 2\int_0^R 2y^2 \sqrt[2]{R^2 - y^2}\, dy = \ldots = \frac{\pi R^4}{4}$$

(The integral was computed by referring to a table of intergrals.)

Due to symmetry, $I_{xx} = I_{yy}$. $J = I_{xx} + I_{yy} = \pi R^4/2$ as before (see Appendix 3).

Note: As an engineer, you will almost never calculate a second moment of area directly using Equation (2.1) or (2.2). It is important to understand Equation (2.1) or (2.2), but the engineer will usually look up the second moment of area or perhaps determine it from a CAD program.

2.2.4. Loads

Recall that the definition of a *structure* is a physical artifact that carries *loads* other than its own weight. *Loads* then are the forces and moments carried by structural members in accordance with the structure type (as discussed in Section 1).

A structure that undergoes *simple loading* carries only a single type of load. For example, that could be forces or moments but not both. Simple loading results in the simplest structural response, which is a good place for us to start!

The loads on an axial structure must be *co-axial* loads (we'll see this shortly), i.e., loads acting co-axially with the significant structural axis (x-axis here), such as the *concentrated force* applied to the ends of the rod in Figure 2.10. The force shown in Figure 2.10 is tensile, since it acts to create tension in the bar, i.e., acts to displace material particles further away from one another. In Figure 2.2 the force is compressive, so as to create a compressive action on material particles, i.e., acting to displace material particles closer together than in their unloaded configuration. No moments are applied to axial structures.

On the other hand, the torsion structure is loaded by a *twisting moment* or *torque*. In practice, this could arise from a *couple* of magnitude Pd applied to the shaft of diameter d (Figure 2.17). The couple is *self-equilibrating* since no net force results. (Why?) Application of *global equilibrium* would conclude that a moment of magnitude $M = Pd$ (but oppositely directed) must be acting as a reaction for equilibrium to occur.

As we saw in Section 1, beams can carry any of the load types: concentrated forces, distributed forces, or moments.

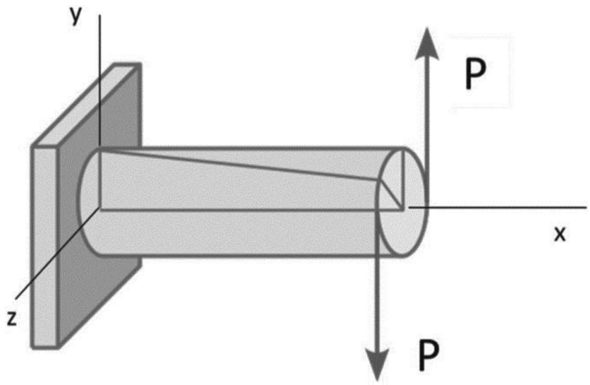


Figure 2.17. Cylindrical shaft of diameter d under the action of a couple of magnitude M = Pd. The end x = 0 is imagined clamped in the figure.

2.3 Equilibrium under Simple Loading

Now as we describe the response of structures to simple loads, what we are really describing are models of response. Our models are based on the laws of mechanics and our understanding of material response.

There are several key concepts to keep in mind as we discuss equilibrium and deformation of any structure.

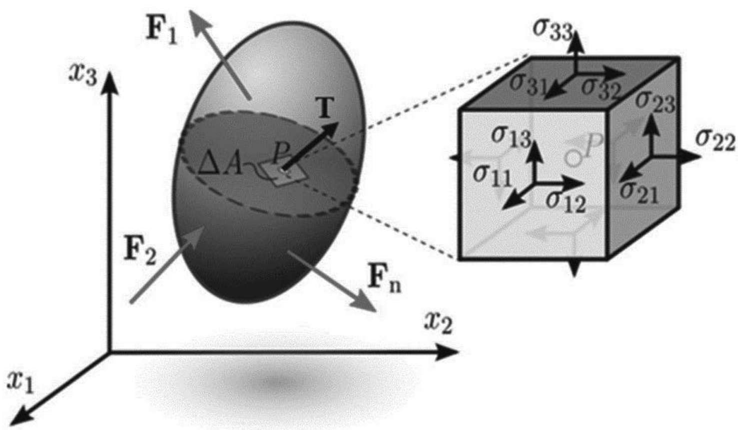
1. The quantities *stress* and *strain* are, by definition, measures of the body's response at a point. That is, in general stress and strain vary from point to point within the structure.
2. Stress and stain also vary from direction to direction at a given point.
3. Think of it this way: stress and strain vary depending on where you are and where you look when you're there. This makes stress and strain vector-like quantities called *tensors* (see Appendix 5), specifically *2nd order tensors*.
4. Three coordinate directions, each requiring three components (think vectors) to describe the stress state on that face. Thus nine components are required to fully describe the state of stress (or state of strain) at a point. The components can be arranged in a 3 by 3 matrix.

5. Any continuous body that is in global equilibrium must have all of its “parts” in equilibrium, that is, it must be in local equilibrium as well.

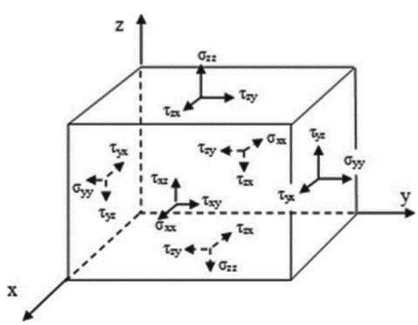
We take advantage of these concepts to go from the externally applied loads to the internal stress and strain response. The process goes as follows:

External load → internal traction vector → stress components

Imagine a body under the action of *n* concentrated loads F_1, F_2, …F_n as seen in Figure 2.18a.



(a)



(b)

Figure 2.18. (a) The body is sectioned so we can look inside and observe the internal reaction on an infinitesimal volume element. T is the traction vector. (b) Stress components in the x-y-z coordinate system.

We section the body so we can look inside and observe the internal response to external loads, starting with the *traction vector* **T**. The three components of **T** on each of three orthogonal surfaces are the nine stress components, σ_{11}, σ_{12}, etc., as seen in Figure 2.18a. The nine *stress tensor* components can be arranged in a 3 by 3 array or matrix as follows:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{2.3}$$

Replacing x_1 with x, x_2 with y, and x_3 with z (Figure 2.18b), Equation (2.3) becomes:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \tag{2.4}$$

We will study this in much more detail in Chapter 4, but for now we consider simple loading to further develop the concept of stress.

2.3.1. Axial Structure Equilibrium

We now consider a "global" *free-body diagram* (FBD) of a rod (see Appendix 2). The right end of the rod is "freed" from the body at some position x from the origin and the corresponding FBD looks as shown in Figure 2.19:

> **C³**
>
> 2.2 What is the difference between global and local equilibrium?
>
> *(Answer: Section 2.6)*

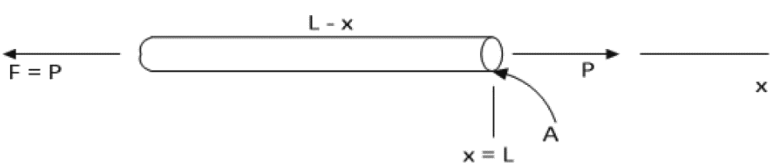


Figure 2.19. Global free body diagram of an axial structure.

Shown on the FBD is the internal reaction force **F**, which from global static equilibrium analysis

$$\Sigma \mathbf{F} = 0 \tag{2.5}$$

can be shown to be

$$\mathbf{P} - \mathbf{F} = 0 \Rightarrow \mathbf{F} = \mathrm{P} \tag{2.6}$$

Furthermore, global moment equilibrium

$$\Sigma \mathbf{M} = 0 \tag{2.7}$$

shows that **F** and **P** are *co-linear* (*co-axial*), since they must present no net *couple* (moment) on the rod.

We now look more closely at the stress in an axial structure and the corresponding local equilibrium. Consider the local FBD or *stress element* carved out of the cross-section A, shown in Figure 2.20:

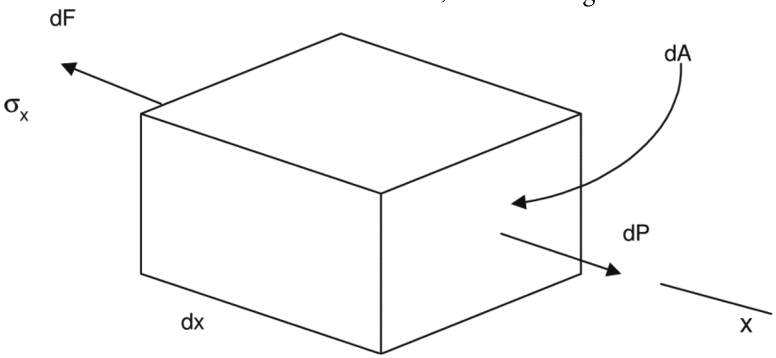


Figure 2.20. Local free body diagram of an axial structural element.

Now an increment of internal force, d**F**, can be seen to act over an increment of area, d*A*, which balances the increment of applied load d**P** (i.e., d**F** = d**P**). We take as a fundamental postulate (known formally as *Cauchy's Stress Principal*) that the externally applied load is resisted internally (to enforce local equilibrium) by a *traction vector* **T** such that

$$\mathbf{T} = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{F}}{\Delta A} = \frac{d\mathbf{F}}{dA} \tag{2.8}$$

C³

2.3 Are the traction vector and stress vector the same thing?
(Answer: Section 2.6)

Cauchy's Stress Principal also provides that the components of **T** normal and parallel to the surface d*A* are the *normal stress* σ_n and *shear stress* τ components on that face (see Figure 2.21).

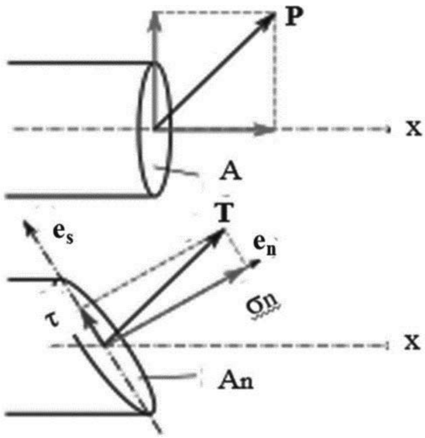


Figure 2.21. Internal stress components.

The *normal stress* component is found by taking the *projection* (see Appendix 5) of **T** on to the surface normal e_n:

$$\sigma_n = T \cdot e_n \tag{2.9}$$

Similarly, the *shear stress* component is found from

$$\tau = T \cdot e_s \tag{2.10}$$

Note:

i. Comparing Figures 2.18 and 2.21, Equation (2.4) can be written

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{2.11}$$

ii. The *stress state* and *strain state* in an axial structure are considered to be relatively simple, in that it is assumed that only *normal* stresses and strains exist on cross-sections normal to the rod axis. In that case, **T** and e_n would be collinear with the x-axis and we can neglect the shear stress component $\tau = \boldsymbol{T} \cdot \boldsymbol{e}_s$ (for now). Then for an axial structure, the only non-zero stress component is the normal stress in the axial direction, or

$$\begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.12}$$

iii. At the point of load application (the bar end in this case), the stress distribution is in fact not simple. But it also turns out that the stress develops to the simple state in a fairly short length of the bar, so assuming uniformity over the entire length is reasonable.
iv. Keep in mind that what we really calculate by Equation (2.8) is an "average" stress acting <u>everywhere</u> over the cross-section of the bar, even though stress is fundamentally a measure of the load response by a structure <u>at a point</u>.
v. We are representing the cross-section surface (area) as a vector quantity, with both magnitude A_n and direction e_n (see Appendix 5).
vi. The stress σ_{xx} is assumed to be constant as long as no intermediate axial loads are applied (see RP1.1).
vii. For one-dimensional stress states, we will often change from the double subscript notation to a single subscript notation. For example, $\sigma_{xx} = \sigma_x$.
viii. From this point forward, we will suspend the use of bold font for vector and tensor quantities except where that emphasis is essential.

Local and global equilibrium can be related through the following equation:

$$F = \int dF = \int_A \sigma_{xx} dA = P \tag{2.13}$$

Example 2.3.

Two rods are identical except one has a square cross-section while the other has an equilateral triangle cross-sectional shape. The identical cross-sectional areas of 100 mm^2 each carries an identical load P = 1 kN. Determine the axial stress for each rod and discuss the comparison.

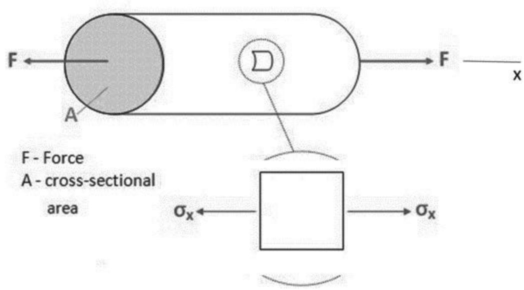


Figure E2.3.

Since the area is constant, Equation (2.13) gives $\sigma_{xx} = P/A$ and σ_{xx} = 1000 N/100 x 10^{-6} m^2 = 10 MPa. This is true for both rods since only the area matters in the tensile axial structure; shape does not play a role in axial tension or compact axial compression.

2.3.2. Torsion Structure Equilibrium

The applied couple in Figure 2.17 is resisted internally by a shear force of magnitude V. Figure 2.22 shows a cylindrical differential element of the shaft of length Δx and cross-section ΔA (radius $r < R$), which is coaxial with the original shaft of radius R. The associated traction vector T on the element face is given by:

$$T = \lim_{\Delta A \to 0} \frac{\Delta V}{\Delta A} = \frac{dV}{dA} \qquad (2.14)$$

where ΔV is the incremental shear force. Since V lies entirely along the surface (parallel with e_s), then so does T and there is only one stress component, namely $\tau_{x\theta}$. Due to symmetry, V or $\tau_{x\theta}$ can only be functions of r and x, but not θ.

> **C³**
>
> 2.4 How can V be used for both "volume" and "shear force"?
> *(Answer: Section 2.6)*

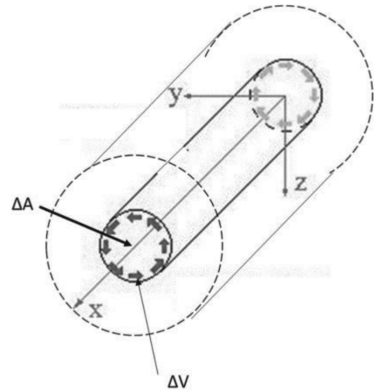


Figure 2.22. Internal force system on a differential shaft element of area ΔA, consisting of a shear force ΔV. There is a shear force acting over every material particle in ΔA

Note that there is no net shear force in the *y-z* plane, since each ΔV forms a self-equilibrating system of couples. That is, each ΔV has an equal but opposite counterpart canceling its shearing action (but not its moment).

Now the magnitude of the total internal moment M resisting the applied load C is given by the equilibrium equation $M = C$, where:

$$M = \int_A r dV = \int_A r\tau_{x\theta} dA = C \tag{2.15}$$

Let's multiply both sides of Equation (2.15) by r again, and since the shear stress is constant over dA, we have

$$rM = \tau_{x\theta} \int r^2 dA$$

or

$$\boxed{\tau_{x\theta} = \frac{Mr}{J}} \tag{2.16}$$

where

$$J = \int r^2 dA \tag{2.17}$$

Equation (2.16) is an important result that tells us the shear stress in our circular cylindrical shaft varies linearly from zero at the shaft center to a maximum at the outer fiber ($r = R$), as shown in Figure 2.23. J is the *polar second moment of area* (polar moment of inertia) (see Appendix 3).

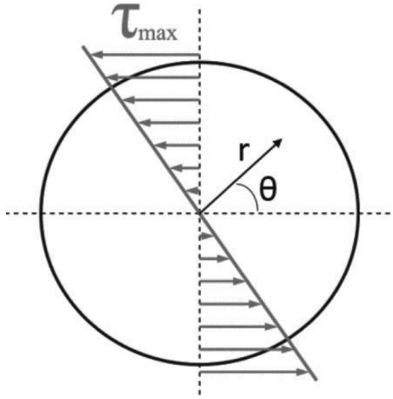


Figure 2.23. Shear stress in a circular cylindrical shaft varies linearly from zero at the shaft center to a maximum at the outer fiber (r = R),

In matrix component form, the torsion stress state as we've defined it is

$$\begin{bmatrix} 0 & \tau_{x\theta} & 0 \\ \tau_{\theta x} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.18}$$

Example 2.4

The shaft in Figure E2.4 is subject to a torque $M = PD$. Determine the maximum shear stress $\tau_{x\theta,max}$ if $D = 2R$ is the shaft diameter.

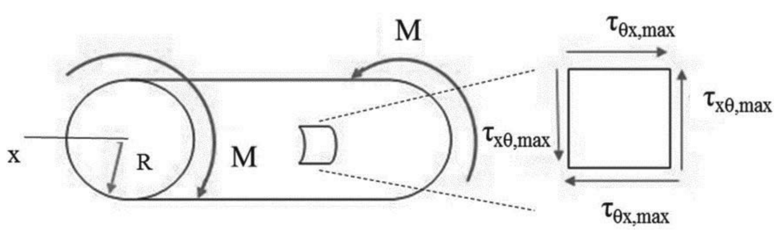


Figure E2.4.

$J = \pi R^4/2$ in this case and the maximum stress occurs at $r = R$. Then Equation (2.16) looks like

$$\tau_{x\theta,max} = \frac{2(PD)R}{\pi R^4} = \frac{2PD}{\pi R^3} = \frac{4P}{\pi R^2}$$

2.3.3. Flexural Structure Equilibrium

Figure 2.24 shows the free body diagram of a beam subjected to a transverse concentrated force P, in keeping with our current limitation of simple loading only. (We'll consider a distributed force $p(x)$ [force/length] and concentrated moments M [force*length] in the next chapter.)

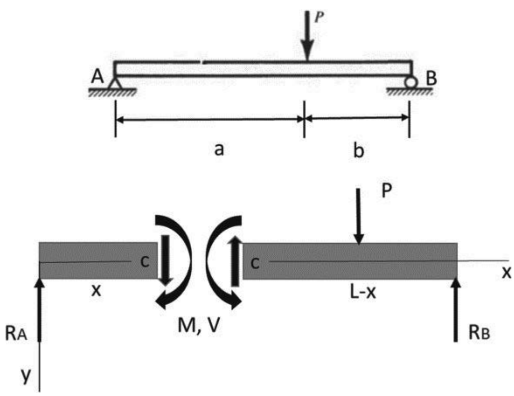


Figure 2.24. Free body diagrams of a beam under simple concentrated loading. All internal reactions are shown positive according to the coordinate and stress sign conventions.

The internal reaction system required to enforce equilibrium under an applied concentrated load includes a shear force V and moment M. In order to relate the internal forces and moments to the externally applied loads, we sum forces and moments in the usual way from the right-side FBD for convenience (the same results arise from a summation over the left-side FBD and this is left as an exercise for the student):

$$\begin{aligned} &+\downarrow \sum F_y = 0: \quad -V(x) + P = 0 \\ &+cw\sum M_z = 0: \ -M(x) + P(a-x) = 0 \end{aligned} \qquad (2.19\text{a, b})$$

Note that:

1. The signs and magnitudes of the boundary reaction forces and moments would be determined from global statics in the usual way.
2. We have summed the internal moments about the cut at x, in order not to have to include the moment due to V(x), which at the present is unknown.
3. No assumptions about the sign convention need be made other than the usual right hand rule.
4. Equations (2.19a, b) can now be solved for V(x) and M(x) in terms of the applied loads.
5. It is reasonable to assume that the stress is maximum wherever the moment is maximum (we will verify this later).

C³

2.5 Why do most mechanics of materials books discuss an additional sign convention for shear force and moment in beam bending?
(Answer: Section 2.6)

We will also show later that the expression for the magnitude of maximum flexural stress is

$$\boxed{\sigma_{max} = M_{max}c/I} \qquad (2.20)$$

where $I = \int y^2 dA$ as before and c is the distance from the neutral axis to the outer fiber.

Example 2.5

Compare the maximum stress in a "2 × 4" placed first with height "2 inches up", then with "4 inches up".

$$I_{2up} = bh^3/12 = (4 \text{ in})(2 \text{ in})^3/12 = 2.67 \text{ in}^4$$

$$I_{4up} = bh^3/12 = (2 \text{ in})(4 \text{ in})^3/12 = 10.67 \text{ in}^4$$

$$\sigma_{max,2up}/\sigma_{max,4up} = (c/I)_{2up}/(c/I)_{4up} = (2/4)(10.67/2.67) = 2.0$$

The "weak way" maximum stress is twice the "strong way" maximum stress!

2.4 Uniaxial Material Response

So far we have studied the definition, configuration, and equilibrium of the three basic structures. These considerations apply to all three structures. So why would three rods, identical in configuration, except one is made of steel, on of wood, and one of rubber, behave different under identical loads. We will see that the nature of the material, its *constitution*, plays an important role in the structural response. For now, we will only consider uniaxial material response.

2.4.1 Uniaxial Deformation

Under the action of the tensile force F, a material particle located at position x in a rod will displace an amount $u(x)$ in the $+x$ direction. The corresponding strain is a strictly geometrical quantity with a number of possible definitions. The simplest definition we can take is that strain (at a point!) is the comparison of a local change in length between two material particles relative to some "gage length", which is taken as the length between the same particles in the undeformed state, namely dx. If we take the deformed length to be dx^*, then the strain in the x-direction is defined as

$$\varepsilon_x \equiv \frac{dx^* - dx}{dx} \tag{2.21a}$$

$$\boxed{\varepsilon_x = \frac{du}{dx}} \tag{2.21b}$$

(There is no single, unique definition for strain. We use here the common definition of one-dimensional *engineering strain*.)

Equation (2.21b) is a very fundamental and general requirement of deformation that relates strain to displacement, and is called the *strain-displacement relation* (although it is more correctly the strain—displacement gradient relation).

We see that the strain ε_x is a normal (here tensile or extensional) strain in the axial direction. The axial displacement anywhere along the rod (at any cross-section) can then be found from

$$u(x) = \int_0^x \varepsilon_x dx \tag{2.22}$$

with the total increase in the rod length given by:

$$u(L) = \int_0^L \varepsilon_x dx \tag{2.23}$$

We emphasize that for a rod modeled as a 1-D structure, all fibers (which are axial fibers) deform identically.

2.4.2 Axial Structure Constitution

In 1678, the English mechanician Robert Hooke published a paper titled "De Potentia Restitutiva" ("The Spring"). In it, he detailed the results of experiments he had conducted on materials, such as metallic wire. He showed that the deflection u in the wire was proportional to the applied tensile load P:

$$P = ku \tag{2.24}$$

where k is the proportionality constant (called the *stiffness*) to be defined later. It is critical at this point to realize that every elastic element can be considered as a "spring" of one sort or another, just as Hooke did. We will come back to this concept time and time again.

Dividing both sides of Equation (2.24) by AL (the product of the initial area and length of the wire)

$$P/AL = ku/AL$$

and rearranging terms,

$$P/A = (kL/A)(u/L)$$

this relation can be re-written in terms of stress and strain as:

$$\sigma = E\varepsilon \tag{2.25}$$

> **C^3**
>
> 2.6 Can stress be measured? Strain?
>
> *(Answer: Section 2.6)*

where σ is the *engineering stress* P/A, ε is the *engineering strain* u/L, and E is the *elastic modulus* of the material. The stiffness k is easily seen to be $k = EA/L$. We will take a more detailed look at strain in the next section but we can see it is a non-dimensional ratio of deformed displacement (change in length here) to original length.

Equation (2.25) shows the proportionality now between stress and strain, and is formally called the *1-D Hooke's Law*. We have just discovered something very fundamental: the *mechanics* quantities stress and strain are related physically in the material through *material* quantities (such as the elastic modulus in this case).

Example 2.6

Three round rods, each having a diameter D and length L, are fixed at one end ($u(x = 0) = 0$) and carry a load P at the free end ($x = L$). One rod is steel (E = 210 GPa), one is aluminum (E = 70 GPa), and one hard rubber (7 GPa). For L = 1 m, D = 25.4 mm, and P = 10 kN, determine the stress, strain, and displacement in each rod.

Solution using spread sheet:
Area = $\pi D^2/4$
Stress = P/Area
Strain = Stress/E
Displacement = Strain*L

Rod length L (m)	1			
Rod dia. D (mm)	25.4	area=	5.07E-04	m^2
Load P (N)	10,000			
Material	E(Pa)	Stress (Pa)	Strain (m/m)	Displ (m)
Steel	2.10E+11	1.97E+07	9.39E-05	9.39E-05
Aluminum	7.00E+10	1.97E+07	2.82E-04	2.82E-04
Rubber	7.00E+09	1.97E+07	2.82E-03	2.82E-03

2.4.3. Axial Structure Strength

Why do we care about the internal response of a structure to loads called "stress"? It is because we can account for structural failure by claiming the internal response reached its "failure stress". The failure stress is called the material "strength" and an over-arching design objective is to always keep the stress less than the strength.

Just as there are different kinds of loads and stress, there are also different kinds of failure modes and different kinds of strength. Here, we only consider axial loading and uni-axial failure. Figure 2.25 shows the stress response of an axial structure ("ductile tension specimen") under increasing tensile axial loading.

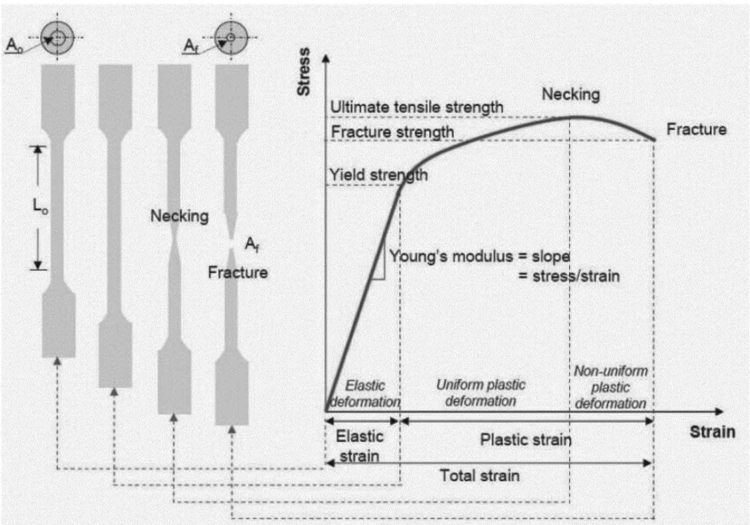


Figure 2.25. Uni-axial stress response.

There's much to discuss in Figure 2.25 but for now we only need to be concerned with two features:

1. The *elastic region.* An *elastic deformation* is one wherein the structure returns identically to its unloaded configuration upon removal of the load. That is, there is no permanent deformation (that occurs in the *plastic region*). *Linear elastic* response indicates a straight line stress-strain curve in the elastic region. If the structure exhibits a linear elastic response, the slope of the stress-strain curve in the elastic region is the elastic modulus E in Equation (2.25).

2. *Failure*. In Figure 2.25, you see various modes of failure: yield, ultimate tensile, and fracture. There are others, too, but for now it is enough to recognize that the stress at a failure mode is the *strength* in that mode and our goal is to keep the stress < strength.

2.4.4 Uniaxial Energy

In this book, we make considerable use of *Newtonian mechanics*, which is a vector mechanics of forces, displacements, and displacement derivatives. For one, Newtonian mechanics is a very intuitive approach—it is easy to visualize forces and displacements as arrows! For another, the vector approach leads directly to finding forces and moments, which are needed to determine the internal stress response.

However, there is also considerable reason to take a *Lagrangian mechanics* approach, which is a scalar mechanics of work and energy, and their derivatives. Energy concepts are important because they provide fundamental principles for formulating the governing structural equations, and as such they are the gateway to more advanced analysis techniques. Also, some failure theories are based on the amount of stored energy in the structure.

We know from earlier studies that mechanical energy can be divided into *potential energy* and *kinetic energy*. Since we only deal with static problems here, we can disregard kinetic energy considerations. The *total potential energy* Π of a structure includes both the energy of deformation stored within the structure (i.e., the *strain energy*) and the potential energy associated with the *work* of the forces and moments responsible for the deformation. The *incremental work* dU of a force is defined as the scalar product of the vector force F with the resulting incremental displacement vector ds or F$\cdot$ds = Fcosθds, where θ is the angle between F and ds The *incremental work* dU of a force is defined as the scalar product of the vector force F with the resulting incremental displacement vector ds or F$\cdot$ds = Fcosθds, where θ is the angle between F and ds. The strain energy U is equal to the *recoverable work* done on the body.

Example 2.7

Consider a linear elastic translational spring of stiffness k under the action of a net (axial) force P. From the definition of work, we have for the spring that $dU = P\,du$ (du being the increment of displacement in the x-direction). Integrating over the total displacement u we obtain the strain energy in the spring as:

$$U = \int_0^u P d\xi = \int_0^u k\xi d\xi = \frac{1}{2}ku^2$$

(Here ξ is just a dummy variable of integration.)

The potential energy E of the applied loads relevant to the structure is the negative of the work done by the applied loads during the deformation:

$$E = -Pu$$

The total potential energy $\Pi = U + E$ is then

$$\Pi = \frac{1}{2}ku^2 - Pu \ (= \frac{1}{2}ku^2 - ku^2 = -\frac{1}{2}ku^2)$$

Note that Π is a *quadratic function* (*quadratic form*) of u. A plot of Π vs. u looks like (Figure E2.7):

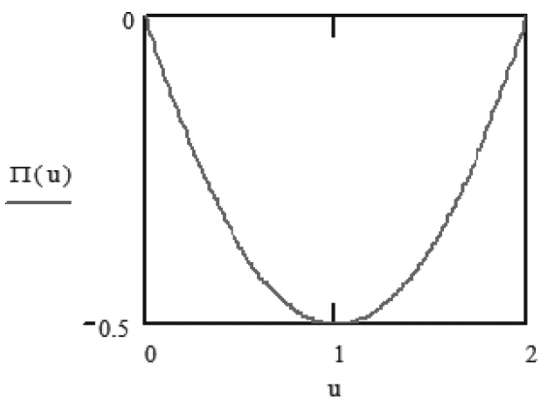


Figure E2.7.

Apparently Π has a minimum, which in this case can be analyzed by elementary calculus (more powerful methods of the *calculus of variations* will be needed for more complex energy functions):

$$d\Pi/du = ku - P = 0 \Rightarrow \underline{ku = P} \tag{2.26}$$

$$d^2\Pi/du^2 = \underline{k > 0}$$

These two simple results are very important:

1. We have derived the equilibrium equation $ku = P$ from the minimization of the total potential energy. That means that the equilibrium configuration achieved under a set of applied loads is the unique one that minimizes the total potential energy. This is the *principle of minimum potential energy*. It is a cornerstone of structural mechanics!
2. The stiffness k is guaranteed to be positive.

One final point can be made. The stored energy of a rod can be calculated in terms of stress and strain through use of the constitutive relations. For the one-dimensional Hooke's Law model, we get a strain energy per unit volume of the rod as follows:

$$U = ½\ ku^2 = ½\ Pu = ½\ \sigma_x A\ \varepsilon_x L$$

or

$$U = ½\ \sigma_x \varepsilon_x (AL)$$

Hence the strain energy per unit volume is given by $½\ \sigma_x\varepsilon_x = ½\ E\varepsilon_x^2 = ½\ \sigma_x^2/E$. In general, for the one-dimensional axial case, the strain energy is given by Equation (2.27) as:

$$U = \frac{1}{2}\int_{volume} \sigma_x \varepsilon_x dV \tag{2.27}$$

Similarly in the case of uni-axial torsion, the torsional strain energy is given by Equation (2.28)

$$U = \frac{1}{2}\int_{volume} \tau_x \gamma_x dV = \frac{1}{2}\int_{volume} \frac{\tau_x^{\ 2}}{G} dV \qquad (2.28)$$

Two and three-dimensional forms of the strain energy will be discussed later.

2.5 Structural Design

In the case of *strength design*, our solution strategy goes something like the following:

1. Global Free Body Diagram
2. Apply global equilibrium to find external reactions
3. Local Free Body Diagram
4. Apply local equilibrium to determine internal reactions
5. Determine internal stress, compare to strength criteria

The ratio of strength to stress is often called the *Factor of Safety* (see Chapter 1). Since we want the stress to be less than the strength, the Factor of Safety will be greater than one.

Let's apply this to the problems introduced at the beginning of this Chapter.

RP2.1

The stepped rod shown in Figure RP2.1 consists of two co-axial circular cylindrical sections carrying concentrated forces: 800 lb at location B (x = 5 inch) and 1000 lb at location D (x = 22 inch). Cylinder ABC is 0.375 diameter and CD is 0.25 in diameter. The material is high-carbon steel (E = 30 x 10^6 psi). Determine the overall extension of the rod and check for strength design.

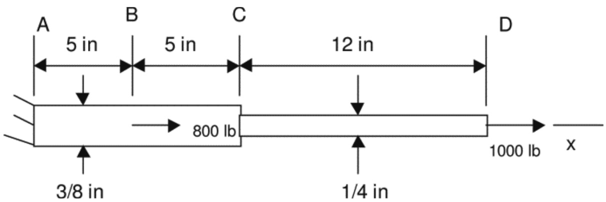


Figure RP2.1a.

First we complete three FBDs associated with each of the discontinuities (Figure RP2.1b):

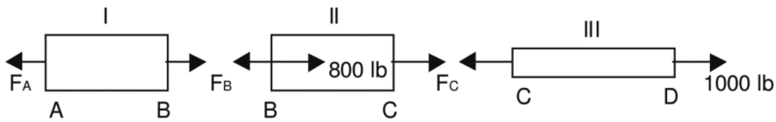


Figure RP2.1b.

From the strain definition (Section 2.4), we can find the total elongation:

$$u = \sum_i u_i, \qquad i = I, II, III$$

or

$$u = u_I + u_{II} + u_{III}$$

In the present example, the displacement is constant over each section I, II, and III.

Next we determine the loading over each section by applying the equilibrium relations:

- Section III: $\Sigma \mathbf{F}_x = 0$: $-\mathbf{F}_c + 1000\text{ lb} = 0 \Rightarrow \underline{\mathbf{F}_C = 1000\text{ lb}}$
- Section II: $\Sigma \mathbf{F}_x = 0$: $-\mathbf{F}_B + 800\text{ lb} + \mathbf{F}_c = 0 \Rightarrow \underline{\mathbf{F}_B = 1800\text{ lb}}$
- Section I: $\Sigma \mathbf{F}_x = 0$: $-\mathbf{F}_A + \mathbf{F}_B = 0 \Rightarrow \underline{\mathbf{F}_A = 1800\text{ lb}}$

Now we can integrate separately over each section of the rod:

$$u = \int_0^{L_1} \frac{P_1(x)}{EA_1(x)}dx + \int_{L_1}^{L_2} \frac{P_2(x)}{EA_2(x)}dx + \int_{L_2}^{L} \frac{P_3(x)}{EA_3(x)}dx$$

$$= \frac{F_A}{EA_1}\int_0^{5"} dx + \frac{F_C}{EA_1}\int_{5"}^{10"} dx + \frac{F_C}{EA_3}\int_{10"}^{22"} dx$$

$$= \frac{1}{30x10^6\, psi}\left[\begin{array}{l} \dfrac{1800lb}{\frac{\pi}{4}(0.375in)^2}(5in) + \dfrac{1000lb}{\frac{\pi}{4}(0.375in)^2}(5in) \\ + \dfrac{1000lb}{\frac{\pi}{4}(0.25in)^2}(12in) \end{array}\right]$$

$$= \underline{0.124in}$$

As a check on strength design, we can compute the stress in each segment of the rod:

$$\sigma_1 = \frac{P_1}{A_1} = \frac{1800lb}{\frac{\pi}{4}(0.375in)^2} = 16,300\,psi$$

$$\sigma_2 = \frac{P_2}{A_2} = \frac{1000lb}{\frac{\pi}{4}(0.375in)^2} = 9,050\,psi$$

$$\sigma_3 = \frac{P_3}{A_3} = \frac{1000lb}{\frac{\pi}{4}(0.25in)^2} = 20,400\,psi$$

Factor of safety = 70,000/16,300 = 4.3

Factor of safety = 70,000/9050 = 17.3

Factor of safety = 70,000/20,400 = 3.4

The stress is seen to be less than the strength (S_y = 70 x 10^3 psi) in every case. However, we have not taken into account the effects of stress concentrations near boundaries and discontinuities—the stresses there could be considerably higher!

RP2.2

A 2024-0 aluminum shaft 30 inches long is built in at one end as shown in Figure RP2.2. A torque is applied at the free end of magnitude 6000 in lb.

a. For a 2 inch diameter solid circular cross-section, determine the maximum shear stress and the shaft weight.
b. Now consider an annular cross-section of outer radius R_o and inner radius R_i. If the wall thickness $h = R_o - R_i$ and $R_o/R_i = 1.1$, determine R_o, R_i, and the shaft weight if the annular shaft has the same maximum shear stress as the solid circular shaft.

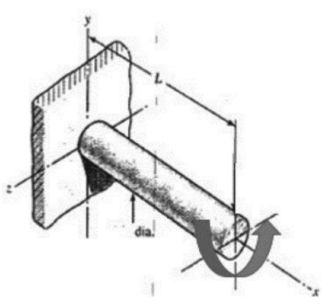


Figure RP2.2.

a. Equation (2.11) provides for the maximum shear stress at the outer fiber $R = 1.0$ inch. Then, with $J = \pi R^4/2 = 1.571$ in^4 from Example 2.1,

$$\tau_{x\theta,max} = \frac{MR}{J} = \frac{(6000\, in \cdot lb)(1.0\, in)}{1.571\, in^4} = 3819\, psi$$

The weight density ρ_w of 2024 aluminum is about 0.1 lb/in^3. Then this shaft weighs

$$w = \rho_w V = (0.1 \text{ lb/in}^3)\ \pi(1.0 \text{ in})^2(30 \text{ in}) = 9.4 \text{ lb}$$

b. For the annular shaft, $J = \pi(R_o^4 - R_i^4)/2$ from Example 2.1. Substituting $R_i = R_o/1.1$, $J = 0.498\ R_o^4$. Now solving Equation (2.11) for R_o/J (since $\tau_{x\theta,max}$ and M are known),

$$R_o/J = R_o/0.498\ R_o^4 = 2.01/R_o^3 = 3819 \text{ psi}/6000 \text{ in lb} = 0.637\ 1/\text{in}^3.$$

or

$$R_o = 1.47 \text{ inch}$$

Then

$$R_i = 1.33 \text{ inch}$$

with

$$w = \rho_w V = (0.1 \text{ lb/in}^3)\ \pi[(1.47 \text{ in})^2 - (1.33 \text{ in})^2](30 \text{ in}) = 3.6 \text{ lb}$$

The annular shaft weighs one-third that of the solid shaft but carries the same maximum stress!

The yield strength and ultimate tensile strength of this material are 11 ksi and 27 ksi, respectively. Compared to the 3819 psi stress, the FS ranges between 2.9 and 7.1.

RP 2.3

We wish to minimize the amount of material in a beam such that, under a specific loading condition, each cross-section will be at the maximum allowable stress (at the outer fiber). Applications are leaf springs, gear teeth, and bridge girders. Consider a cantilever beam of rectangular cross-section b by $h(x)$, with tip load F as shown in Figure RP2.3a:

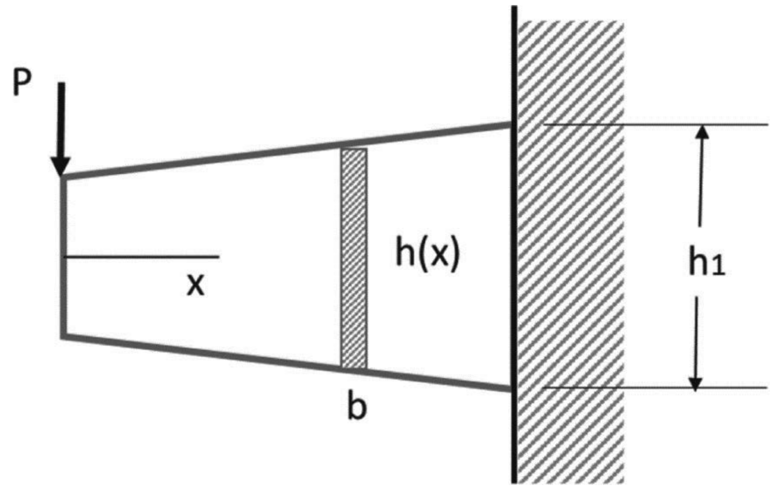


Figure RP2.3a.

Note that for this beam, $I = bh^3(x)/12$ from Example 2.2. At every section, we vary $h(x)$ to maintain:

$$\sigma_{allow} = \frac{6M}{bh^2(x)} = \frac{6Px}{bh^2(x)}$$

At the fixed end:

$$\sigma_{allow} = \frac{6PL}{bh_1^2} \Rightarrow h_1 = \sqrt{\frac{6PL}{b\sigma_{allow}}}$$

Anywhere else:

$$h(x) = \sqrt{\frac{6Px}{b\sigma_{allow}}}\left(\sqrt{\frac{L}{L}}\right) = h_1\sqrt{\frac{x}{L}}$$

or

$$\mathrm{x} = \frac{\mathrm{L}}{\mathrm{h}_1^2}\mathrm{h}^2$$

This is the equation of a parabola, and the beam is called a "constant strength parabolic beam." Let's plot $h(x)$ and see what it looks like (Figure RP2.3b):

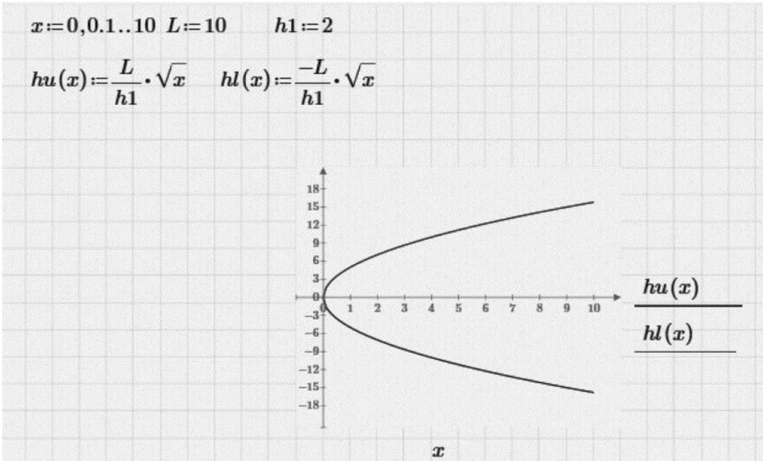


Figure RP2.3b.

One application of a non-prismatic beam geometry is the teeth of gears, such as the spur gear seen in Figure RP2.3c:

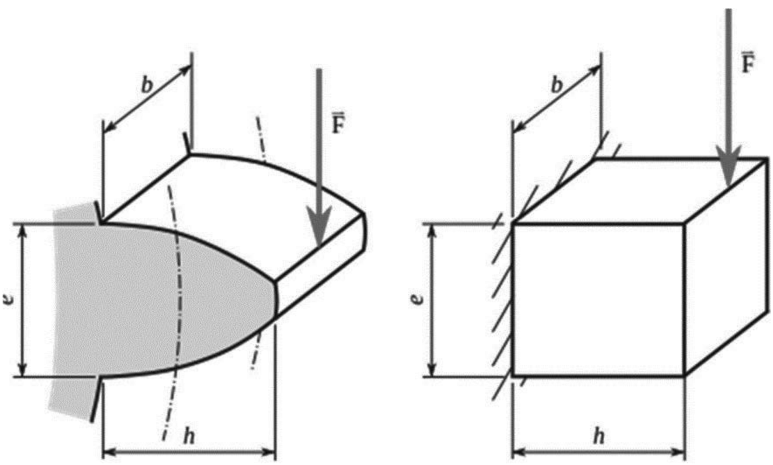


Figure RP2.3c.

RP2.4

Compare the maximum stress due to a central concentrated load P between two beams that are identical except for the boundary conditions: one beam is fixed-fixed and the other is pinned-pinned.

Let's solve this problem by using an engineering reference. For structural analysis, one of the most famous is "Roark" (see Chapter 1).

Roark organizes beam response first by load type and then by boundary conditions. Figure RP2.4 provides an excerpt from the appropriate table for our problem.

TABLE 8.1 Shear, moment, slope, and deflection formulas for elastic straight beams

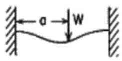
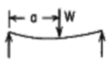
End restraints, reference no.	Selected maximum values of moments and deformations
1d. Left end fixed, right end fixed 	Max $+M = \frac{2Wa^2}{l^3}(l-a)^2$ at $x = a$; max possible value $= \frac{Wl}{8}$ when $a = \frac{l}{2}$ Max $-M = M_A$ if $a < \frac{l}{2}$; max possible value $= -0.1481Wl$ when $a = \frac{l}{3}$ Max $y = \frac{-2W(l-a)^2a^3}{3EI(l+2a)^2}$ at $x = \frac{2al}{l+2a}$ if $a > \frac{l}{2}$; max possible value $= \frac{-Wl^3}{192EI}$ when $x = a = \frac{l}{2}$
1e. Left end simply supported, right end simply supported 	Max $M = R_A a$ at $x = a$; max possible value $= \frac{Wl}{4}$ when $a = \frac{l}{2}$ Max $y = \frac{-Wa}{3EIl}\left(\frac{l^2-a^2}{3}\right)^{3/2}$ at $x = l - \left(\frac{l^2-a^2}{3}\right)^{1/2}$ when $a < \frac{l}{2}$; max possible value $= \frac{-Wl^3}{48EI}$ at $x = \frac{l}{2}$ when $a = \frac{l}{2}$ Max $\theta = \theta_A$ when $a < \frac{l}{2}$; max possible value $= -0.0642\frac{Wl^2}{EI}$ when $a = 0.423l$

Figure RP2.4.

Our problem is contained in Cases 1d and 1e, where $a = L/2$ and $W = P$. To determine the maximum stress we need only the maximum moment. Since we're asked to compare, we can form a ratio where all like terms will cancel:

$\sigma_{max,f\text{-}f}/\sigma_{max,p\text{-}p} = (M_{max}c/I)_{f\text{-}f}/(M_{max}c/I)_{p\text{-}p} = (M_{max})_{f\text{-}f}/(M_{max})_{p\text{-}p} = (PL/8)/(PL/4) = 1/2$

Thus the max stress in the fixed-fixed support case is one-half that of the pinned-pinned case. (Why?)

RP2.5

For the rod in RP2.1, calculate the total strain energy stored due to the loads applied.

In general, the strain energy is given by:

$$U = \frac{1}{2}\int_{vol} \sigma_x \varepsilon_x dV = \frac{1}{2}\int_{vol} \frac{\sigma_x^2}{E} dV, \qquad dV = dxdydz$$

Again, due to the discontinuities in stress and volume, the integral must be broken as before:

$$U = \frac{1}{2E}\left(\int_{vol.I} \sigma_x^2 dV + \int_{vol.II} \sigma_x^2 dV + \int_{vol.III} \sigma_x^2 dV\right)$$

Note that for constant stress over any volume element, the volume integral becomes

$$\int_{vol} dV = \iiint dxdydz = \iint_{area} dAdx = A\int dx$$

for constant area *A*.

Now computing the total strain energy is straightforward:

$$\begin{aligned} U &= \frac{1}{2E}\left(A_1\sigma_1^2\int_0^{5"} dx + A_2\sigma_2^2\int_{5"}^{10"} dx + A_3\sigma_3^2\int_{10"}^{22"} dx\right) \\ &= \frac{\pi/4}{2(30x10^6\ psi)}\left[\begin{matrix}(0.375in)^2(5in)(16{,}300\,psi)^2 + (0.375in)^2 \\ (5in)(9{,}050\,psi)^2 + (0.25in)^2(12in)(20{,}400\,psi)^2\end{matrix}\right] \\ &= \underline{\underline{10.5\ \ lb\cdot in}} \end{aligned}$$

IP2.1

Finally, let's take a look at the Hyatt Regency walkway collapse mentioned at the beginning of this Chapter. The collapse occurred at the Hyatt Regency Kansas City in Kansas City, Missouri, on July 17, 1981. Two vertically contiguous walkways collapsed onto a dance competition being held in the hotel's atrium. The falling walkways killed 114 and injured another 216. According to the NIST report[1]:

> "Three suspended walkways spanned the atrium at the second, third, and fourth floor levels. The second floor walkway was suspended from the fourth floor walkway which was directly above it. In turn, this fourth floor walkway was suspended from the atrium roof framing

[1] Marshall, R.D., et al. (1982). *Investigation of the Kansas City Hyatt Regency Walkways Collapse*. National Bureau of Standards Building Science Series 143, Washington, DC: U.S. Department of Commerce, National Bureau of Standards.

by a set of six hanger rods. The third floor walkway was offset from the other two and was independently suspended from the roof framing by another set of hanger rods. In the collapse, the second and fourth floor walkways fell to the atrium floor, with the fourth floor walkway coming to rest on top of the lower walkway. Based on the results of this investigation, it is concluded that the most probable cause of failure was insufficient load capacity of the box beam-hanger rod connections. Observed distortions of structural components strongly suggest that the failure of the walkway system initiated in the box beam-hanger rod connection on the east end of the fourth floor walkway's middle box beam (Figure IP2.1a).

Two factors contributed to the collapse: inadequacy of the original design for the box beam-hanger rod connection, which was identical for all three walkways, and a change in hanger rod arrangement during construction that essentially doubled the load on the box beam-hanger rod connections at the fourth floor walkway (Figure IP2.1b). As originally approved for construction, the contract drawings called for a set of continuous hanger rods which would attach to the roof framing and pass through the fourth floor box beams and on through the second floor box beams. As actually constructed, two sets of hanger rods were used, one set extending from the fourth floor box beams to the roof framing and another set from the second floor box beams to the fourth floor box beams."

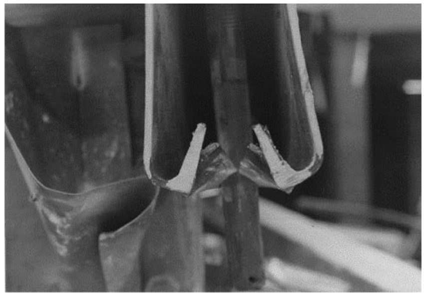


Figure IP2.1a. Distorted box beam (cross beam) hanger.

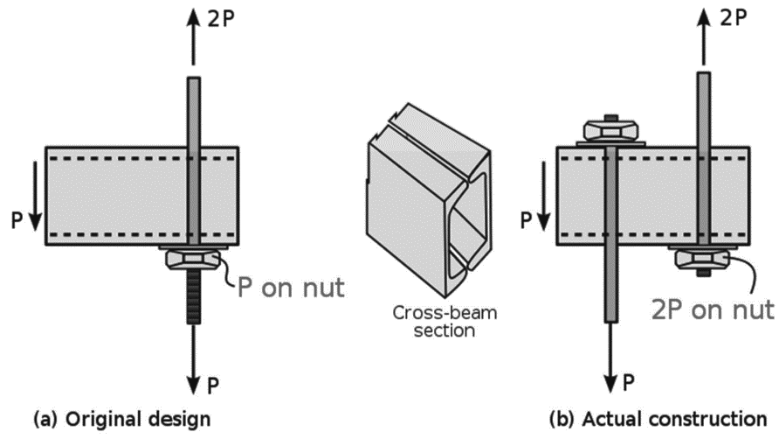


Figure IP2.1b. Original hanger design vs as built.

2.6 C³ Clarified

C³ 2.1: What is the difference between the moment of inertia (MOI) and the second moment of area (SMOA)?

Answer: Both refer to the same thing but MOI is an inappropriate choice. "Inertia" means "resistance to acceleration" (a). It is the meaning of "mass" (m) in Newton's 2nd Law, i.e., $m = F/a$. Inertia has no role in the static equilibria that we deal with here in mechanics of materials. SMOA is the correct description of the quantity $\int r^2 dA$. See Appendix 3 for further discussion.

C³ 2.2: What is the difference between global and local equilibrium?

Answer: Global equilibrium refers to the structure as a whole and not the internal reaction to loads. Reactions at boundaries to structure loads are found using local equilibrium considers equilibrium at any point in the structure, e.g., where stress or strain is of interest. If a structure is in global equilibrium, it is also in local equilibrium, i.e., every point is in equilibrium.

C³ 2.3: Are the traction vector and stress vector the same thing?

Answer: Yes, both refer to the same quantity but traction vector is the preferred name. Stress is not a vector quantity but a second order tensor and we should not confuse the two.

C³ 2.4: How can V be used for both "volume" and "shear force"?

Answer: Unfortunately, there are more engineering variables than there are letters and symbols, so there will be overlap. The best we can do is be consistent and make clear in what context we are using the variable.

C³ 2.5 Why do most mechanics of materials books discuss an additional sign convention for shear forces and moment in beam bending?

Answer: This is indeed unfortunate as it is not necessary. For a complete discussion, see Appendix 4.

C³ 2.6: Can stress be measured? Strain?

Answer: Interestingly enough, stress cannot be directly measured. This is a little troubling, since engineers like to directly measure important results if necessary. Strain can be directly measured and then stress inferred from strain through material models (Table 2.1).

Table 2.1. Comparison of some key relationships between axial, torsional, and flexural structures. Keep in mind that stress and strain are point functions, i.e., they are defined at a point in a structure and (in general) vary from point to point within a structure.

	Axial	Torsional	Flexural
Stress	$\sigma_{xx} = \frac{P}{A}$	$\tau_{x\theta} = \frac{Mr}{J}$	$\sigma_{xx} = \frac{My}{I}$
Deformation	$\varepsilon_{xx} = du/dx$	Chap. 4	Chap. 3
Constitution	$\sigma_{xx} = E\varepsilon_{xx}$	Chap. 4	Chap. 3

2.7 Developing Engineering Intuition

1. AXIAL: You and a partner grab the opposite ends of a rope or wire and pull as hard as you can. Now reverse the loading from tensile to compressive. Explain what you observe.
2. TORSION: Take a standard wire paper clip, partially open it, then try to twist it into two pieces. Repeat with four more clips (five total) and think about what you observe.
3. FLEXURE: Place a "2 × 4" lumber about 8 ft long across two shorter 2 × 4s, one at each end of the long lumber. Do this first the "short" way (2 inch sides vertical), stand on the center and observe the deformation (e.g., central sag). Now repeat with the 4 inch sides vertical. Explain your observations.
4. MATERIAL RESPONSE: Take an ordinary metal paperclip, open it up into approximately two halves, then bend it back and forth in the middle until it fractures (which should take less than 20 or so cycles). Record the number of cycles. Repeat for another four clips (five total). Discuss the results, particularly in comparison to the twisted paper clips. Comment on the clip's temperature change.

CHAPTER 3

Basic Structures Level II

Learning Objectives: This chapter will introduce the student to additional complexity from Level I, such as combined loading and beam deflection. Stiffness design will be considered in this chapter. (Strength design was covered in Chapter 2.) Along the way we will clarify several common confusing concepts (C^{3}s).
Clarifications: Studying this chapter will help clarify:

- Body loads
- General loading of flexural structures
- Combined axial and flexural loads
- Beam deflections
- Material response in flexure
- Solving stiffness design problems

Importance: Catastrophic failure of many modern structures can lead to extensive property damage as well as the unfortunate loss of life. A classic example is the crane (Figure I3.1a).

The crane, in existence at least from the early Greek and Roman periods, is a machine used for moving materials vertically and/or horizontally. Used extensively in a variety of applications today, crane failures are not uncommon, some ending badly (see Figure I3.1b). For example. Between 2003 and 2006, the National Bureau of Labor Statistics reported over 70 crane-related fatalities per year.

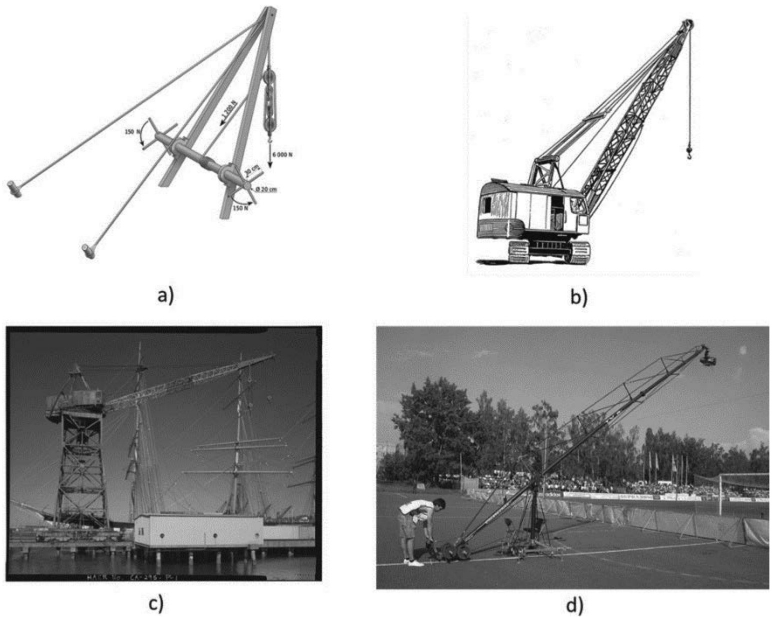


Figure I3.1a. (a) Roman crane; (b) mobile crane; (c) shipyard crane; (d) portable crane.

Figure I3.1b. Crane failure in Bellevue, WA, November 2006.

3.0 Representative Problems

The following are examples of basic structural design problems we want to be able to solve as engineers. Studying this chapter will help us solve them, which we will do in Section 3.6.

RP3.1

A cantilever beam of length L, modulus E, and second moment of area I, is tip-loaded by a transverse concentrated force P as shown in Figure RP3.1a. Determine the following quantities:

$v(x) = ?$	$v'(x) = ?$	$\sigma_x(x) = ?$
$v_{max} = ?$	$v'_{max} = ?$	$\sigma_{x,max} = ?$

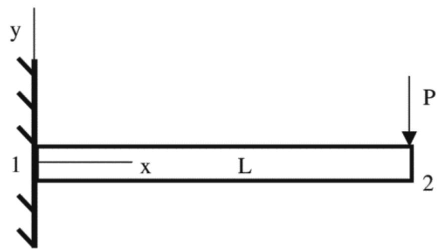


Figure RP3.1a. Here v is the beam y-displacement and v′ is the beam slope.

RP3.2

A jib crane consists of a wide flange steel beam attached by a pin (momentless) connection to a vertical support (Figure RP3.2a). The beam has an intermediate tie rod support. The wide flange beam has the following section properties: h = 153 mm, A = 22.9 cm^2, I = 916 cm^4.

If a 500 kg load is suspended at the free end of the beam, compute the maximum stress in the beam.

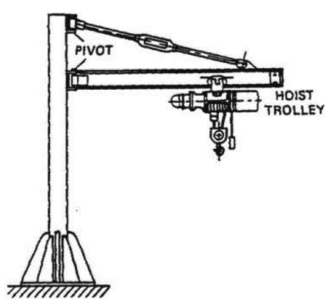


Figure RP3.2a. Jib crane.

RP3.3

A rod of solid circular cross-section is used as a cantilever beam by welding one end to a metal wall and leaving the other end free. Of the diameter *D*, modulus *E*, and length *L*, which has the greatest impact on the beam deflection?

3.1 Body Loads and Axial Response

As discussed in Chapter 1, *body loads* are those loads that act within and throughout the body by forces "acting at a distance". This is in contrast to *surface loads"*, which result from direct contact of one body with another. The most common body loads are gravity loads and thermal loads. Other body loads include magnetic and electro-magnetic.

3.1.1 Gravity Loads

Loads due to the local gravitational attraction (acceleration) are called gravity loads. Simply, this is the structure's self-weight, which is often ignorable compared to the applied load being carried.

The only real challenge arising with gravity loads is with structures that have a significant vertical component to them. In that case, we have an internal response *gradient* (spatially varying response). This is simply the fact that (for example, in compression) each successively lower material particle feels not only the tug of gravity on itself but additionally all of the weight of the material particles above it.

Consider a uniform rod of total mass m_0 in a vertical configuration as shown in Figure 3.1:

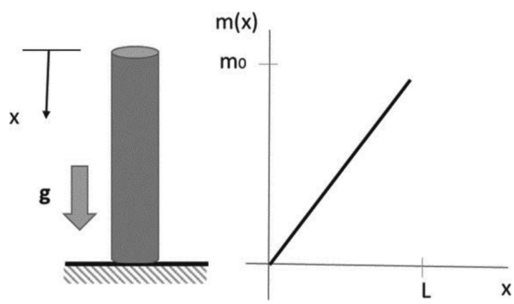


Figure 3.1. Uniform rod under gravity loading.

The rod mass must increase linearly from top to bottom such that $m(x) = \rho Ax$, with $m(x = L) = \rho AL = m_0$. Hence the self-weight loading follows the same linear trend

$$w(x) = m(x)g = \rho Agx \tag{3.1}$$

Example 3.1

Three rods are hung vertically from a ceiling, each suspended from their top (Figure E3.1). Each rod is 10 m in length, with one made from steel, another from aluminum, and a third from wood. Compute the total elongation of each rod due to its own weight, and compare one to another. Why are the results independent of the cross-sectional area?

Material	Density ρ (kg/m^3)	Modulus E (GPa)
Steel	7850	200
Aluminum	2700	70
Wood	500	12

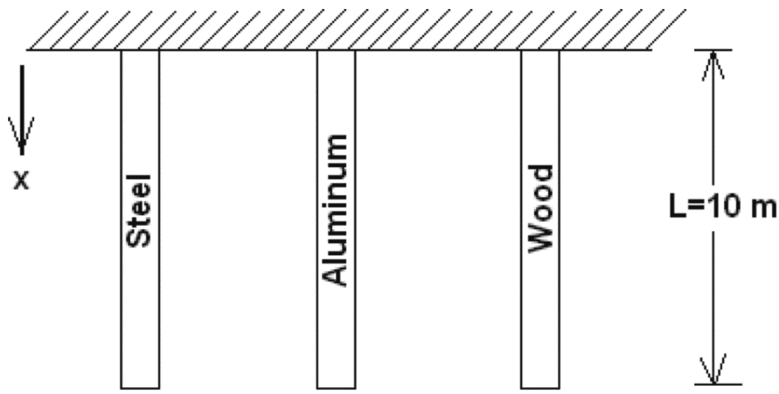


Figure E3.1.

The total elongation is

$$u_{TOT} = \int_0^L \varepsilon(x)\,dx$$

where for the 1-D case $\sigma = E\varepsilon$. Then

$$u_{TOT} = \int \frac{\sigma(x)}{E} dx$$

But

$$\sigma(x) = \frac{P(x)}{A} = \frac{m(x)\,g}{A}$$

Upon substituting,

$$u_{TOT} = \int_0^L \frac{g}{EA} m(x)\,dx$$

Since

$$m = \rho v = \rho A l \quad \Rightarrow m(x) = \rho A x$$

Then

$$\begin{aligned} u_{TOT} &= \int \frac{g}{EA}(\rho A x)\,dx \\ &= \frac{g\rho}{E}\int_0^L x\,dx \\ &= \frac{g\rho}{E}\frac{x^2}{2}\bigg|_0^L \end{aligned}$$

Finally,

$$u_{TOT} = \frac{g\rho L^2}{2E}$$

Let's check units:

$$u_{TOT} = \frac{g\left[\frac{m}{s^2}\right]\rho\left[\frac{kg}{m^3}\right]L^2\left[m^2\right]}{2E\left[\frac{kg\cdot m/s^2}{m^2}\right]} = [m] \checkmark$$

Knowing we get the correct units (length) gives us confidence the equation is correct.

Computing u_{TOT} for each rod gives the final result:

$$u_{stl} = 1.925\times10^{-5}\,m$$
$$u_{Al} = 1.391\times10^{-5}\,m$$
$$u_{wood} = 2.043\times10^{-5}\,m \qquad \Leftarrow \qquad \text{greatest elongation!}$$

Note that the resultants are independent of the cross-sectional area. The force causing the elongation is due only to gravity acting on the total mass.

> **C³**
>
> 3.1 Why should we take the time to check units?
>
> *(Answer: Section 3.7)*

Note also that for the same g and L, the variation in u_{TOT} depends only on the variation in ρ/E.

3.2 Equilibrium under General Loading I

3.2.1 General Flexural Structure Equilibrium

Consider a beam subjected to general flexural loading: transverse concentrated forces **P**, distributed forces $p(x)$ [force/length], and concentrated moments **M** [force*length] as shown in Figure 3.2:

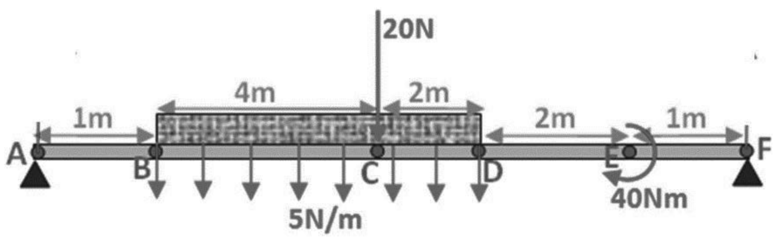


Figure 3.2. A beam under general loading.

The internal reaction system required to enforce equilibrium under any general system of applied loads includes an axial force **N**, shear force **V**, and moment **M**. (We will shortly consider loads that generate a net axial force

> **C³**
>
> 3.2 In many textbooks, the shear stresses are shown oppositely directed, that is, one up and one done. In Figure 3.3 they are shown in the same direction. Which is correct?
>
> *(Answer: Section 3.7)*

N in the beam. Formally, such a structure would be called a *beam-column*.) The signs and magnitudes of the reaction forces and moments would be determined from global statics in the usual way.

To relate the internal forces and moments to the applied loads, let's first consider an incremental element (Δx long) of a beam under a distributed load $p(x)$ only (Figure 3.3):

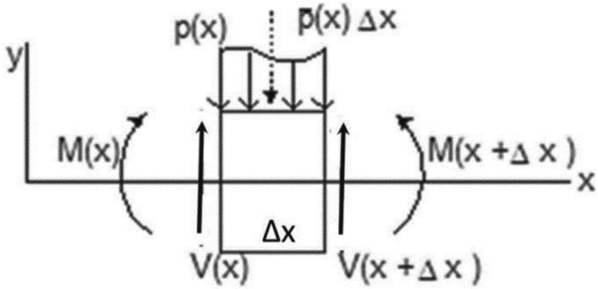


Figure 3.3. Incremental element of a beam subjected to transverse distributed load p(x).

In Figure 3.3, $\overline{p}(x)\Delta x$ represents an average force over the element. Satisfying force equilibrium in the y-direction gives:

$$\Sigma F_y = 0 : -V(x) - \overline{p}(x)\Delta x + V(x+\Delta x) = 0 \Rightarrow \overline{p}(x) = \frac{V(x+\Delta x) - V(x)}{\Delta x} \tag{3.2}$$

In the limit, as $\Delta x \to 0$ (remember stress and strain are "point" quantities), $\overline{p}(x) \to p(x)$ and

$$\boxed{p(x) = \lim_{\Delta x \to 0} \frac{V(x+\Delta x) - V(x)}{\Delta x} = \frac{dV(x)}{dx}} \tag{3.3}$$

Satisfying moment equilibrium at x gives:

$$\Sigma M_z = 0 : \quad -M(x) - \overline{p}(x)\Delta x \frac{\Delta x}{2} + V(x+\Delta x)\Delta x + M(x+\Delta x) = 0 \tag{3.4}$$

or

$$\frac{M(x+\Delta x)-M(x)}{\Delta x}=\bar{p}(x)\frac{\Delta x}{2}-V(x+\Delta x) \tag{3.5}$$

Again, in the limit as $\Delta x \to 0$ and using Equation (3.2)

$$\boxed{\frac{dM(x)}{dx}=-V(x)} \tag{3.6}$$

(Note that in the limit as $\Delta x \to 0$, the difference between Δx and $\Delta x/2$ becomes insignificant.)

> **C³**
>
> 3.3 I understand the need to know the magnitude, direction, and location of the maximum moment, but why not also for the maximum shear force?
>
> *(Answer: Section 3.7)*

Equations (3.3) and (3.6) are fundamental differential equations relating external beam loads and internal beam reactions. (The sign in Equation 3.6 has no immediate practical significance.) These two equations are useful in beam design since they allow us to locate the position along the beam length of the maximum internal shear force and bending moment, the later particularly driving the design. Note that by Equation (3.6), <u>the location of the maximum internal moment is found by setting the internal shear function $V(x) = 0$ and solving for the roots (one of which gives the location of the global maxima).</u> This is a critical concept as it allows us to find the maximum bending stress when doing a strength design, and we will come back to this point later.

From Equation (3.3) we see that the internal shear force can be found by an integration of the distributed load:

$$\boxed{\int V(x)dx=\int p(x)dx} \tag{3.7}$$

Similarly, from Equation (3.6) we see that the internal moment can be found from an integration of the shear force:

$$\boxed{\int M(x)dx=-\int V(x)dx} \tag{3.8}$$

Equations (3.7) and (3.8) are the integral equation versions of Equations (3.3) and (3.6). They can be used to construct what are called *shear* and *moment diagrams*, respectively. For simple loading $p(x)$, the shear and moment diagrams can be manually constructed by simple graphical methods. For more complicated loadings, however, the modern engineer usually relies directly on Equations (3.7) and (3.8).

Remember, what we are primarily interested in is the magnitude, sign, and location of the maximum moment(s). We'll illustrate this in the next couple of examples.

Example 3.2 Maximum moment by statics alone

A simply supported beam (pinned—roller) of length L has a distributed load as shown in Figure E3.2a:

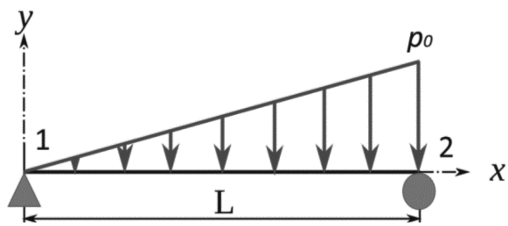


Figure E3.2a.

Here, the loading function $p(x) = p_0x/L$, where p_0 is the intensity of the load at the right end of the beam in dimensions of [force/length]. Let's find the internal shear force and moment expressions and the maximum moment simply using statics.

We start as always with the equations of statics applied globally and solve for the reaction forces and moments (which, among others things, satisfies us as to whether the problem is statically determinate or not). The first thing we do is a global FBD (Figure E3.2b):

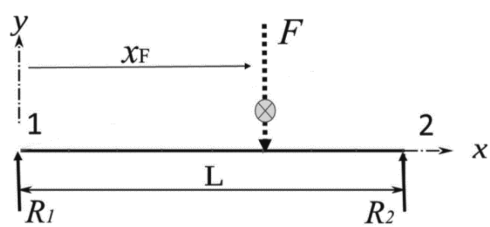


Figure E3.2b.

Notice that we've replaced the distributed load with its statically equivalent concentrated force F = $p_0L/2$ acting at x_F = $2L/3$, as we reviewed in Chapter 1. Now the global analysis gives:

$$\Sigma F_y = 0 : R_1 - \frac{p_0 L}{2} + R_2 = 0$$
$$\Sigma M_1 = 0 : -\frac{p_0 L}{2}\frac{2L}{3} + R_2 L = 0$$

Solving these two equations for the unknown reactions gives R_2 = $p_0L/3$ and R_1 = $p_0L/6$.

In order to find the internal response V(x) and M(x), we now need to apply statics locally, following the same procedure as above but this time using a local FBD (Figures E3.2c and E3.2d):

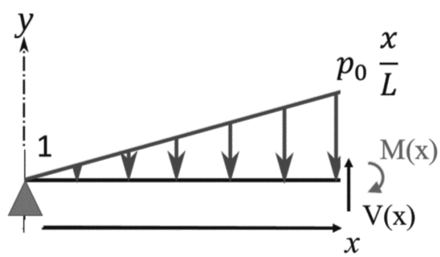


Figure E3.2c. Local FBD showing distributed load.

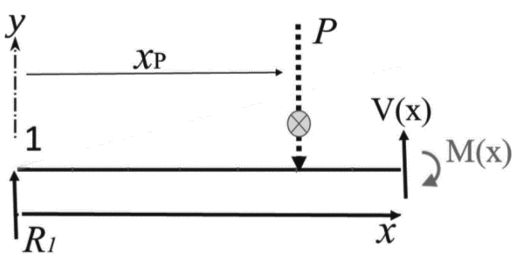


Figure E3.2d. Local FBD showing statically equivalent concentrated load P.

Following the same procedures as above and in Chapter 1, $P(x) = \frac{1}{2} p_0 \frac{x}{L} x = \frac{p_0}{2L} x^2$ and x_P = $2x/3$. A local analysis then gives

$$+\uparrow \sum F_y = 0: \quad -P + R_1 + V(x) = 0$$
$$+ccw \sum M_z = 0: \quad -M(x) - R_1 x + P(x - 2x/3) = 0$$

The first of these equations gives $V(x) = -R_1 + P$, while the second gives $M(x) = R_1x - Px/3$. Finally then our internal shear force and moment distributions are:

$$\underline{\underline{V(x) = -\frac{p_0 L}{6} + \frac{p_0}{2L}x^2}}$$

$$\underline{\underline{M(x) = \frac{p_0 L}{6}x - \frac{p_0}{6L}x^3}}$$

We can check that $M(0) = 0$ from the condition at boundary 1 and from the other moment boundary condition $M(L) = 0$. Also we find $V_1 = V(0) = -p_0L/6 = -R_1$, and $V(L) = V_2 = p_0L/3 = R_2$.

Setting $V(x_{max}) = 0$ gives

$$V(x_{\max}) = 0 = -\frac{p_0 L}{6} + \frac{p_0}{2L}x_{\max}^2 \Rightarrow \underline{x_{\max} = \frac{L}{\sqrt{3}}}$$

Then M_{max} is given by:

$$M_{\max}(x_{\max}) = \frac{p_0 L}{6}\left(\frac{L}{\sqrt{3}}\right) - \frac{p_0}{6L}\left(\frac{L}{\sqrt{3}}\right)^3 = \cdots = \underline{\underline{\frac{p_0 L^2}{9\sqrt{3}}}}$$

Let's plot the internal shear force and moment distributions in Figure E3.2e and see what they look like:

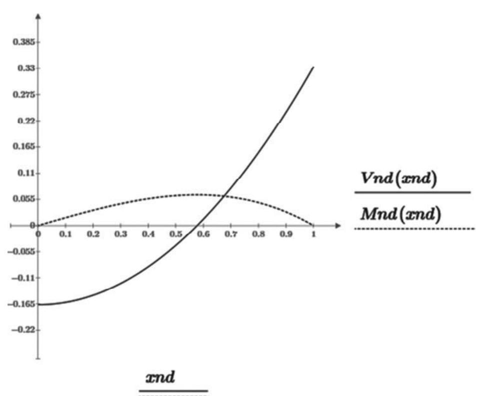


Figure E3.2e. Internal shear force and moment reactions along the beam. The variables have been non-dimensionalized such that $V_{nd}(x_{nd}) = V(x_{nd})/p_0L$ and $M(x_{nd}) = M(x_{nd})/p_0L^2$, where $x_{nd} = x/L$. (Note that xnd in the plot is xnd in the text.)

Note that the shear force is maximum at boundary 2, and that the moment is maximum when V(x) = 0 at $x = \underline{x_{max}}$ = 0.577L as predicted.

Example 3.3 Maximum moment from beam theory

Let's now solve the same problem posed in Example 3.2, namely, finding the shear and moment distributions along a simply supported beam carrying a triangular distributed load. Except this time, let's apply the differential equilibrium relations (Equations (3.3) and (3.6)) in integral form (Equations (3.7) and (3.8)). That is,

$$\int_0^x V(\zeta)d\zeta = \int_0^x \frac{p_0}{L}\zeta d\zeta$$

$$V(x) - V_1 = \frac{p_0}{2L}x^2$$

where we have used $V_1 = V(0)$. From Equation (3.8),

$$\int_0^x M(\zeta)d\zeta = -\int_0^x \left[V_1 + \frac{p_0}{2L}\zeta^2 \right] d\zeta$$

$$M(x) = -V_1 x - \frac{p_0}{6L}x^3$$

where we have taken that M(0) = 0 from the condition at boundary 1. Using the other moment boundary condition M(L) = 0, we solve for $V_1 = p_0L/6 = R_1$, which should come as no surprise and makes a good check. Finally then our internal shear force and moment distributions are:

$$\underline{\underline{V(x) = -\frac{p_0 L}{6} + \frac{p_0}{2L}x^2}}$$

$$\underline{\underline{M(x) = \frac{p_0 L}{6}x - \frac{p_0}{6L}x^3}}$$

These are the same as we found in Example 3.2.

Example 3.4. Two concentrated loads

Consider now a simply supported beam with two concentrated loads, each of magnitude F, spaced a distance "a" from each end of the beam, i.e., at $x = a$ and $x = L - a$ (Figure E3.4a). Find V(x), M(x), V_{max}, and M_{max}.

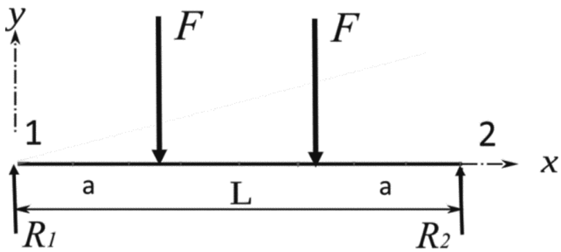


Figure E3.4a.

First, find the external reactions by global statics. Here's a good opportunity for you to practice, since by intuition we know the answer. We've already got the global FBD in Figure E3.4a. Sum forces in the y-direction and sum moments about end 1 or end 2. You should be able to show what you would intuit, that $R_1 = R_2 = F$.

Now to find the shear and moment distributions, we need three local FBDs, since the discontinuous loading breaks the beam into three distinct regions:

$$0 \le x < a$$
$$a \le x < L - a$$
$$L - a \le x \le L$$

These regions are captured in Figure E3.4b:

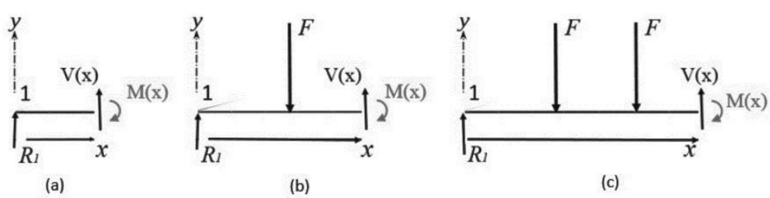


Figure E3.4b.

Now, summing forces region by region:

$0 \le x < a$: $\sum F_y = 0 : R_1 + V(x) = 0 \Rightarrow V(x) = -R_1 = -F$

$a \le x < L - a$: $\sum F_y = 0 : R_1 - F + V(x) = 0 \Rightarrow V(x) = 0$
$(since\ R_1 = F)$

$L - a \le x \le L$: $\sum F_y = 0 : R_1 - F - F + V(x) = 0 \Rightarrow V(x) = F$
$(since\ R_1 = F)$

Let's see what the shear distribution looks like (Figure E3.4c):

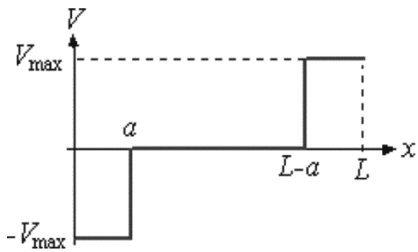


Figure E3.4c.

Now, summing moments (about the "cut") region by region:

$0 \le x < a$:
$$\sum M_c = 0 : -R_1 x - M(x) = 0 \Rightarrow M(x) = -R_1 x = -Fx$$

$a \le x < L - a$:
$$\sum M_c = 0 : -R_1 x + F(x - a) - M(x) = 0 \Rightarrow M(x) = -Fa$$

$L - a \le x \le L$:
$$\sum M_c = 0 : -R_1 x + F(x - a) + F(x - (L - a)) - M(x) = 0 \Rightarrow M(x) = -F(L - x)$$

Let's see what the moment distribution looks like (Figure E3.4d):

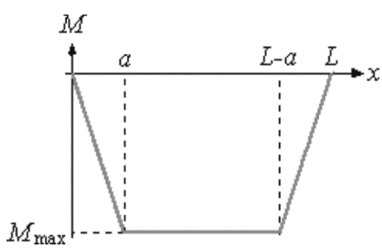


Figure E3.4d.

The maximum moment exists between the two concentrated forces and with magnitude $M_{max} = -Fa$.

3.2.2. *Combined Axial and Flexural Loading*

Thus far we have considered structural loading that acts to cause extension (axial) only, twist (torsional) only, or bending (flexural) only (Table 3.1). However, most structures will experience simultaneous combinations of two or all three loading conditions. In Chapter 4 we will consider the most general case of combined loading and see the critical issue that arises: the determination of the maximum stress.

Figure 3.4 shows the combination of all three loading conditions. Each loading condition results in a corresponding stress state (internal response); these combine *tensorially* to give the total stress state. Moving from left to right in rows (b) or (c) shows flexure, extension, and twist.

By way of introduction, here we consider only the simplest case of combined loading: axial and flexural. Our scope in this section is thus confined to the box outlined in Figure 3.4.

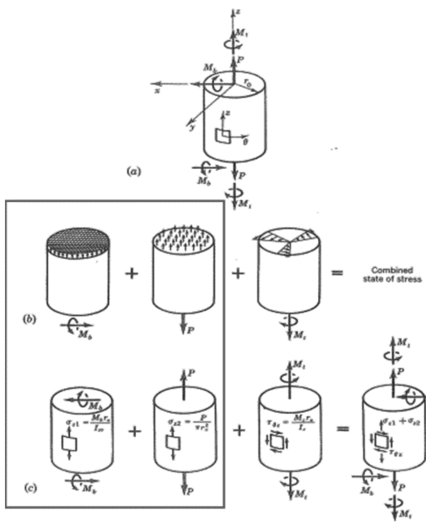


Figure 3.4. Graphic Representation of Combined Loading.

The matrix stress component representation would have the following form:

$$\begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{axial} + \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{flexural} = \begin{bmatrix} P/A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{axial} + \begin{bmatrix} My/I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{flexural} \tag{3.9}$$

Example 3.5

A 60 inch long beam of rectangular cross-section (2 × 3 inches) carries both transverse load P and axial load N (Figure E3.5a). Determine the maximum tensile and compressive stresses.

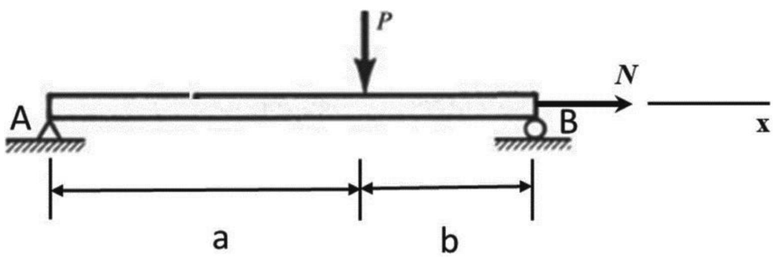


Figure E3.5a.

Let a = 45 in, b = 15 in, P = 0.8 kip, N = 6 kip. Also area A = 2 in * 3 in = 6 in^2 and second moment of area $I = (2^*(3)^3)/12$ in^4 = 4.5 in^4.

Student should verify: R_A = 0.2 kip, R_B = 0.6 kip, M_{max} = 9,000 in lb at x = 45 inches

Axial stress (N only): $\sigma_{xx} = N/A$ = 6,000 lb/6 in^2 = 1,000 psi

Flexural stress (P only): $\sigma_{xx} = My/I$

$$\sigma_{xx,\,max} = M_{max}c/I = (9{,}000 \text{ in lb})(3/2 \text{ in})/4.5 \text{ in}^4 = 3{,}000 \text{ psi}$$

In matrix stress component form, for a point located along the outer compressive fiber ($y = +h/2$) at the section of maximum moment, Equation (3.9) becomes

$$\begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{axial} + \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{flexural} = \begin{bmatrix} 1000\,psi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{axial}$$

$$+ \begin{bmatrix} -3000\,psi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{flexural} = \begin{bmatrix} -2000\,psi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A graphical representation is given in Figure E3.5b:

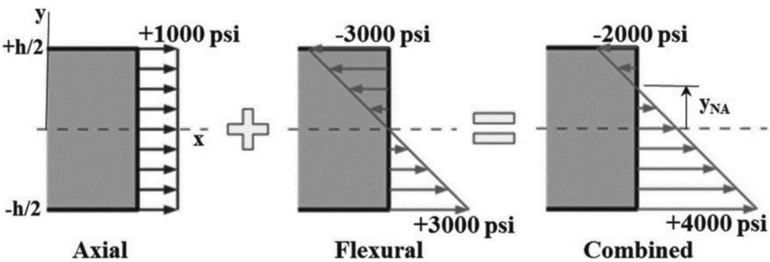


Figure E3.5b.

Note that the neutral axis has shifted from $y = 0$ to y_{NA}, found by similar triangles as:

$4000/(h/2) = 1000/y_{NA} \Rightarrow y_{NA} = (1000/4000)\ 1.5\text{in} = 0.375$ inch.

3.3 Flexural Structure Deformation I

For a beam undergoing small deflection, it is convenient and reasonable to make the following simplifying assumptions:

- The beam bends symmetrically about a so-called *neutral axis*.
- The neutral axis is co-located with a unique fiber, the *neutral fiber*, which remains <u>inextensional</u> during deformation, i.e., the axial strain along the fiber is zero. Due to symmetry and zero neutral fiber strain, it is convenient to associate the origin of coordinates along the neutral axis.
- Cross-sectional planes initially perpendicular to the neutral axis remain plane and perpendicular to the neutral axis during deformation.

(These assumptions are known as the *Euler-Bernoulli assumptions*.)

Consider a beam under the action of transverse forces and/or moments applied to the surface (Figure 3.5):

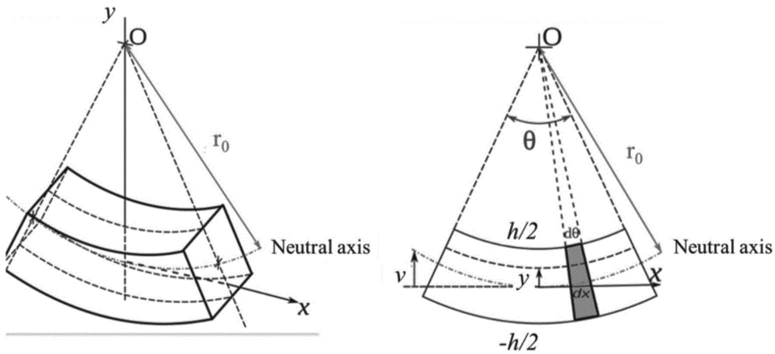


Figure 3.5. Deformation of a beam. The beam deflections shown are greatly exaggerated given the Euler-Bernoulli assumptions. (Why?) The radius of curvature of an axial fiber is r. Note that the fiber co-located at y = -h/2 is extended, while the one at y = +h/2 is contracted.

Let's now examine a slice of the beam of width *dx*, taken arbitrarily along the beam at *x*, before and after the deformation. We consider an arbitrary fiber of initial length *ds* = *dx* located at a position *y* in the slice (Figure 3.6):

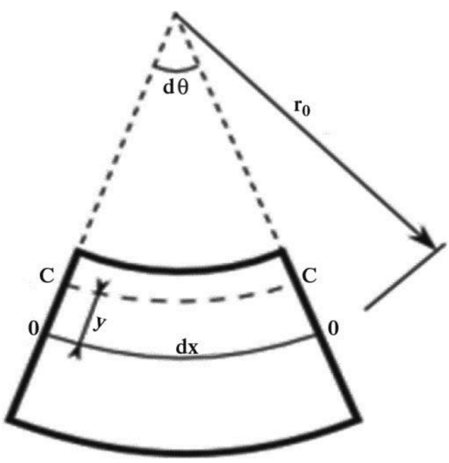


Figure 3.6. Elemental slice through the beam, before and after deformation. Line c-c has an undeformed length ds = dx and a deformed length ds*.

Note that the outer fibers are shown extended and contracted as previously discussed (e.g., $ds \rightarrow ds^*$), and the neutral fiber does not change length (equal to dx before and after deformation). The cutting planes perpendicular to the neutral axis remain plane and perpendicular after deformation. The initially straight neutral fiber is bent into a curved line, with radius of curvature $r_0(x)$, which is a large number since the deflections are assumed very, very small.

In the deformed slice, the neutral axis has length $dx^* = r_0 d\theta = dx$, which is equal to dx due to the inextensional condition. The new length of the arbitrary fiber segment ds is ds^*, with length:

$$ds^* = (r_0 - y)d\theta \tag{3.10}$$

From the definition of extensional strain:

$$\varepsilon_x(y) = \frac{ds^* - ds}{ds} = \frac{(r_0 - y)d\theta - r_0 d\theta}{r_0 d\theta} = -\frac{y}{r_0} \tag{3.11}$$

(We have assumed here that the y-location of ds is the same before and after deformation. For the small deflections assumed here, this is quite reasonable.)

We see that the extensional strain of a fiber ε_x is a linear function of its distance y from the neutral axis. The strain distribution must look like Figure 3.7:

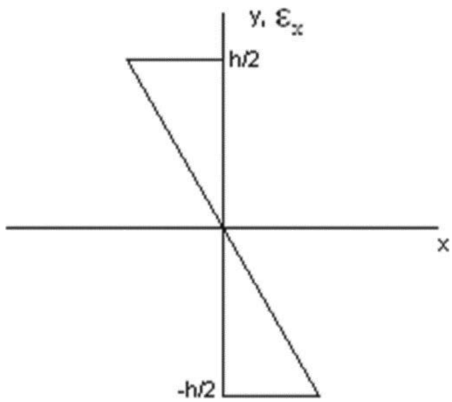


Figure 3.7. Extensional fiber strain distribution at a cross-section.

Along any given y = constant fiber, the strain increases with increasing curvature, as would be expected. From elementary calculus, the curvature $1/r_0$ (say) of a line is related to the first and second derivatives of the function describing the line by:

$$\frac{1}{r_0} = \frac{d^2u/dx^2}{\sqrt[3]{1+(du/dx)^2}} \tag{3.12}$$

where u is the displacement variable in the x-direction.

For small deflections, $(dv/dx)^2 << 1$, and $1/r_0 \approx d^2v/dx^2$. Then the strain in Equation (3.11) is given by

$$\boxed{\varepsilon_x = -\frac{y}{r_0} \approx -y\frac{d^2u}{dx^2} = -yu''(x)} \tag{3.13}$$

This is the fundamental *strain-curvature relationship* for a beam.

Example 3.6

A cantilever beam of length L and depth h, such that $L/(h/2) = 30$, is loaded by a pure moment of magnitude M_0 as shown in Figure E3.6.

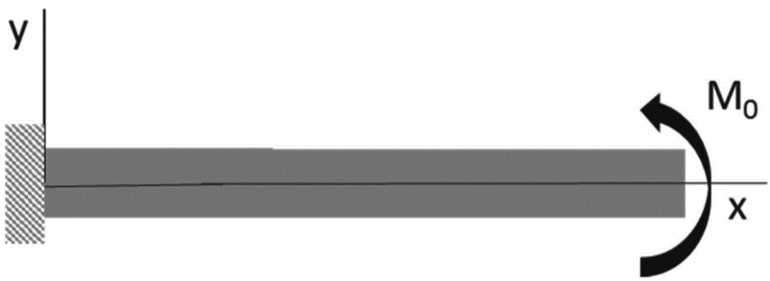


Figure E3.6.

a. Determine an expression for the normalized radius of curvature $r_0/(h/2)$ if the bottom fiber (at $y = -h/2$) is at the tensile yield strain of the material, $\varepsilon_{Y,T}$.

$$\varepsilon_x = -\frac{y}{r_0} = -\frac{(-h/2)}{r_0} = \varepsilon_{Y,T}$$

$$\underline{\frac{r_0}{h/2} = \frac{1}{\varepsilon_{Y,T}}}$$

b. Determine the normalized tip deflection $\delta/(h/2)$ for the given loading.

For small deflection, a pure moment loading results in a circular arc for a deflection curve. (A discussion of *circular bending* is given at the end of Example 3.8.) Then $L = r_0\alpha$, and $\delta_{max} = r_0 - r_0\cos\alpha = r_0(1 - \cos\alpha)$.

$$\alpha = \frac{L}{r_0} = \frac{L}{h/2}\frac{h/2}{r_0} = 30\varepsilon_{Y,T}$$

$$\underline{\frac{\delta_{\max}}{h/2} = \frac{1}{\varepsilon_{Y,T}}\left[1-\cos(30\varepsilon_{Y,T})\right]}$$

Determine r_0 and δ_{max} if $L = 15$ ft and $\varepsilon_{Y,T} = 0.001$.

$$r_0 = \frac{L}{30\varepsilon_{Y,T}} = \frac{15ft}{30(0.001)} = \underline{\underline{500ft}}$$

$$\delta_{\max} = r_0[1-\cos(30\varepsilon_{Y,T})] = 500ft[1-\cos(0.03rad)] = \underline{\underline{0.225ft}} = \underline{\underline{2.70in}}$$

3.4 Material Response in Flexure

3.4.1 Flexural Structure Constitution

For a linear elastic isotropic homogeneous beam, Hooke's law provides an adequate model for the constitutive relations. As discussed in Chapter 4, in general there are nine components of strain to relate to nine components of stress. (Physical and mathematical requirements reduce the number of relationships.) In the Hooke's Law model, normal stress and strain components are unrelated to shear stress and strain components, and vice versa. Equations (3.14) shows the Hookean relations for normal strains and stresses:

C³

3.4 Where does Hooke's Law come from?

(Answer: Section 3.7)

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] \tag{3.14}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$$

Now for the beams we study here, there will only be transverse loads in the *x-y* plane (i.e., transverse to the long axis of the beam in the bending plane). Thus it is reasonable to assume that $\sigma_y = \sigma_z = 0$ (especially since the surfaces $x = 0$, L and $z = \pm h/2$ are traction free, i.e., unloaded). Then the constitutive relations simplify to

$$\begin{aligned} \varepsilon_x &= \frac{\sigma_x}{E} \\ \varepsilon_y &= \varepsilon_z = -\nu\frac{\sigma_x}{E} \end{aligned} \tag{3.15}$$

Note that the negative sign in the latter equations implies that the extensional fibers ($\sigma_x > 0$) contract due to Poisson's effect, while the contracting fibers ($\sigma_x < 0$) expand! In most practical cases, however, we need only concern ourselves with the first of Equations (3.15).

3.4.2 Flexural Energetics

If we maintain that the stress—strain state in a flexural structure is one-dimensional, then the stored energy (internal or strain energy) under flexural loading is the same as it was for the axial structures we considered in Chapter 2. Recalling Equation (2.27):

$$U = \frac{1}{2}\int_{vol}\sigma_x\varepsilon_x dV = \frac{1}{2}\int_{vol}\frac{\sigma_x^2}{E}dV, \qquad dV = dxdydz \tag{3.16}$$

If, on the other hand, we were to consider the contribution of shear to the stress—strain state, then the strain energy relation would include shear terms. Such multi-axial strain energy is discussed in Chapter 4.

In this chapter, we will use the one-dimensional strain energy relation in the context of the flexural test.

There are several advantages of *flexural testing* over tensile testing. Flexural specimens are simpler in design and to manufacture than tensile specimens. For materials where the clamping force of the grips may cause problems, such as very brittle materials like ceramics, flexural testing obviates those concerns. Even for small strains, the displacements under flexure can be considerably more than those for tensile loading, making measurements easier.

The standard configurations for flexural testing are either *three-point bending* or *four-point bending*. The three-point bend configuration is shown in Figure 3.8:

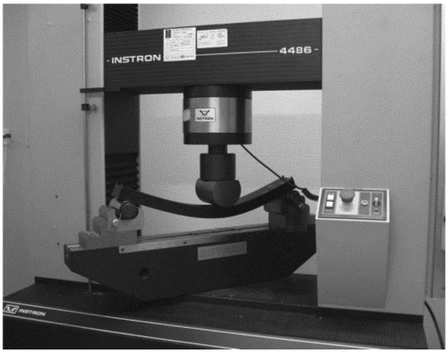


Figure 3.8. Three point flexural test on a composite beam at speed=10 mm/min. Universal testing machine (Instron brand) equipped with a 300 kN dynamometer.

If the Young's Modulus of a material is found from flexural testing, it should be very close to that found from tensile testing if the compression modulus is the same as the tension modulus (which it is for many materials). Flexural strengths may or may not correlate as well with tensile strengths.

If the beam is relatively slender (standard test methods recommend that $L/h > 15$), then the contribution of shear to the deflection Δ is more than 10 times smaller than the normal deflection, and are thus usually ignored in the modulus calculation.

3.5 Flexural Structure Deformation II

Now we substitute the constitutive relation (Equation 3.15) into the strain-curvature relation (Equation 3.13), which gives a *stress-curvature* relationship:

$$\sigma_x = -yEv''(x) \tag{3.17}$$

Computing the *stress resultant* N by integrating the force $\sigma_x dA$ through the symmetric cross-section gives, using (3.17):

$$N = \int_A \sigma_x dA = \int_{-b/2}^{b/2}\int_{-h/2}^{h/2} -yEv''(x)dydz = -bEv''(x)\left.\frac{y^2}{2}\right|_{-h/2}^{h/2} = 0 \tag{3.18}$$

This is an important result that verifies the symmetry of the bending stress distribution about the neutral plane, i.e., there is no net axial force on the beam as a result of the transverse loads.

A *stress-couple* or moment can be calculated in a similar manner:

$$\begin{aligned} M(x) = -\int_A \sigma_x y dA = \int_{-b/2}^{b/2}\int_{-h/2}^{h/2} \\ y^2 Ev''(x)dydz = Ev''(x)\int_{-b/2}^{b/2}\int_{-h/2}^{h/2} y^2 dydz = EIv''(x) \end{aligned} \tag{3.19}$$

where $\sigma_x dA$ is again the force, y is the moment arm, and we have again used (3.17). Equation (3.19) is the important *moment-curvature* relation, where we have used the definition of moment of inertia I given earlier. Note that if the moment produced by the load is a known function of x (meaning the beam is statically determinate), two integrations of the moment-curvature relation Equation (3.19) can be performed to find the resulting deflection $v(x)$. The first integration gives $v'(x)$, i.e., the slope of the beam at x or $v'(x) = \theta(x)$:

$$v'(x) = \frac{1}{EI}\int_0^x M(x)dx + C_1 \tag{3.20}$$

while the second gives the deflected shape or *elastic curve*:

$$v(x) = \frac{1}{EI}\int_0^x [\int_0^x M(x)dx]dx + C_1 x + C_2 \tag{3.21}$$

The constants C_1 and C_2 are determined from the two boundary conditions at the supported end of the free body.

Returning to the beam boundary conditions, we now see that $v = 0$ implies $V \neq 0$ at the boundary and vice versa. Similarly, $\theta = 0$ at the boundary implies $M \neq 0$ at the boundary and vice versa. Hence the simple and roller supports provide a momentless boundary with a shear force, while the fixed and guided supports each provide a moment, but no shear force in the latter case.

Substituting the moment-curvature relation (Equation 3.19) into the stress-curvature relation (Equation 3.17) gives

$$\boxed{\sigma_x(x) = -\frac{M(x)y}{I}} \tag{3.22}$$

which is useful in calculating the stress in a beam cross-section from a given moment-producing load.

Example 3.8

A cantilever beam has a moment M_0 applied at the free end (Figure E3.8a). Determine the maximum transverse displacement of the beam and its location.

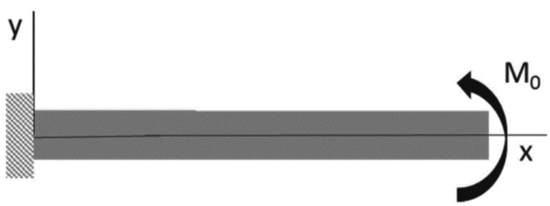


Figure E3.8a.

Let's look at a FBD (Figure E3.8b):

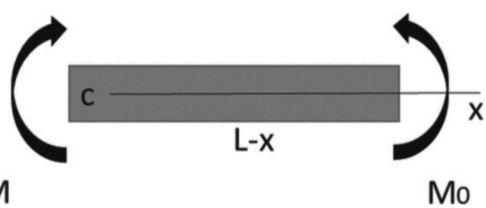


Figure E3.8b.

The internal moment must equal the external moment everywhere along the beam, i.e., M(x) = M_0. The x = 0 boundary is fixed (clamped); the boundary conditions are $v(0) = v'(0) = 0$ accordingly.

Applying Equation (3.20), with $\int_0^x M(x)\,dx = \int_0^x M_0 dx = M_0 x$:

$$v'(0) = 0 = \frac{1}{EI} M_0(0) + C_1 \Rightarrow C_1 = 0$$

Applying Equation (3.21), with $\int_0^x \left[\int_0^x M(x)\,dx \right] dx$ $= \int_0^x (M_0 x)dx = M_0 \frac{x^2}{2}$:

$$v(0) = 0 = \frac{1}{2EI} M_0 (0)^2 + C_2 \Rightarrow C_2 = 0$$

Then

$$v(x) = \frac{M_0}{2EI} x^2$$

which is maximum at $x = L$ (plot the function $v(x)$ if you don't see this):

$$v_{max} = v(L) = \frac{M_0}{2EI} L^2$$

Finally, imagine the beam in Figure E3.8a as one-half (the "right half") of a beam of length 2L held at mid-span. With moment loading applied as before, the right-side elastic curve would still look like $v(x) \sim x^2$ and the left-side would be symmetrical:

$$v(-x) \sim (-x)^2$$

For small deflection, the curve $v(x)$ is a quadradic function that represents the arc of a circle. Thus pure moment loading as we have examined it is referred to a *circular bending*.

Example 3.9

Let's determine the Young's modulus from a three-point bend test, using a beam of length L with rectangular cross-section beam of depth h and width b (Figure E3.9a).

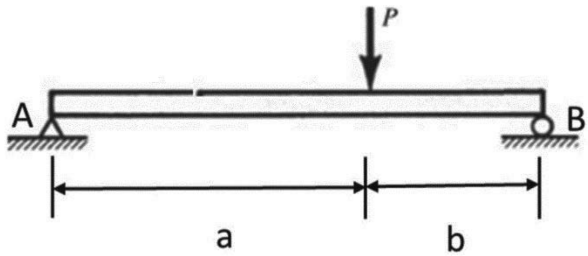


Figure E3.9a. For three-point bending, a = b.

We start with Equation (3.16), and upon substituting in Equation (3.22):

$$U = \frac{1}{2E}\int\left(\int \frac{M^2 y^2}{I^2} dA\right)dx$$

The moment is easily found by the method of sections. First we find the reactions $R_1 = R_2 = P/2$ by global statics. Then a FBD shown in Figure E3.9b is used for finding the moment M.

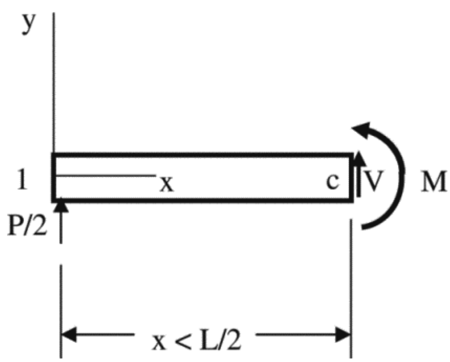


Figure E3.9b.

Summing moments around c we find that $M = (P/2)x$. Noting that

$$\int y^2 dA = I$$

and that M is only a function of x, the strain energy equation above becomes:

$$U = \frac{1}{2EI}\int M^2 dx$$

We note that the moment is discontinuous at $x = L/2$ (M changes sign but not magnitude at $x = L/2$), but since M^2 is symmetrical about $x = L/2$, we complete the integration by integrating from $x = 0$ to $x = L/2$ and multiply by 2:

$$U = \frac{2}{2EI}\int_0^{L/2} \frac{P^2 x^2}{4} dx = \frac{P^2 L^3}{96EI}$$

For the rectangular cross-section, $I = bh^3/12$. Recasting the strain energy in terms of the work done on the beam, $U = ½ P\Delta$, where Δ is the deflection incurred under the load P (see Section 3.6):

$$U = \frac{P^2 L^3}{8Ebh^3} = \frac{P}{2\Delta} \Rightarrow \qquad E = \underline{\underline{\frac{PL^3}{4bh^3\Delta}}}$$

One last issue remains to be discussed here, and that is the *shear deformation* in flexure. Looking back on Figure 3.5 (which, recall, is a greatly exaggerated view) and Figure 3.6, it is clear that there must be a relative "slip" or "sliding" between fibers. That is, the outermost tensile fiber extends just infinitesimally more than its contiguous neighbor, which extends just infinitesimally more than its contiguous neighbor, and so on and so forth. Thus there is a shearing that takes place through the cross-section of the beam. Flexing a deck of cards will give you a very quick intuition of this effect. The minimum relative slip exists at the outer fiber (the shear is zero on the free surface of the beam) and the maximum relative slip (maximum shear) takes place at the neutral axis (the difference between the maximum strain of the outer fiber and the zero strain of the neutral axis).

For us, the question is: "Is this important?" The answer is: "It depends." For a large class of flexural structures, those that are homogenous and respond within the assumptions of the linear theory (think of metallic beams used in commercial buildings, for example), the shear contribution is likely an order of magnitude or more smaller than the normal contribution. In such cases, the shear effects can likely be disregarded.

On the other hand, there are many flexural structures where shear deformation is important, even within the linear theory, such as short study beams. In particular are inhomogeneous beams, such as *laminated* beams, those that are built up by adhering several thin layers or “lamina” together (sometimes called “glue-lam beams”). For these beams, the interfacial shear strength, i.e., the strength in shear of the adhesive between the lamina, may be the controlling design feature. Also, whenever the beam deformations become large, the shear contribution becomes increasing large as well, and may need to be investigated by the engineer. However, such topics are outside the scope of this text, and we refer the interested reader elsewhere for further information.

3.6 Structural design

In the case of *stiffness design*, our solution strategy goes something like the following:

1. Global Free Body Diagram
2. Apply global equilibrium to find external reactions
3. Local Free Body Diagram
4. Apply local equilibrium to determine internal reactions
5. Determine deformation, compare to stiffness criteria

> **C³**
>
> 3.5 Can a structure fail in stiffness but still be strong enough?
> *(Answer: Section 3.7)*

Let’s apply this to problem RP3-1 introduced at the beginning of this chapter.

RP3.1

A cantilever beam of length L, modulus E, and area moment of inertia I, is tip-loaded by a transverse concentrated force P as shown in Figure RP3.1a. Determine the following quantities:

$v(x)$ = ?	$v'(x)$ = ?	$\sigma_x(x)$ = ?
v_{max} = ?	v'_{max} = ?	$\sigma_{x,max}$ = ?

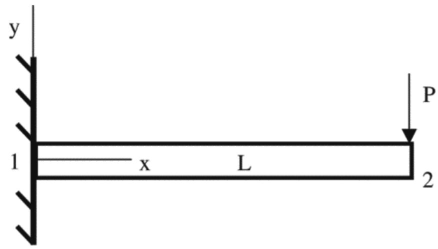


Figure RP3.1a.

$$v(x) = \frac{1}{EI}\int_0^x \left[\int_0^x M(x)dx\right]dx + C_1 x + C_2$$

The moment can easily be found from a section analysis (the FBD is shown in Figure RP3.1b):

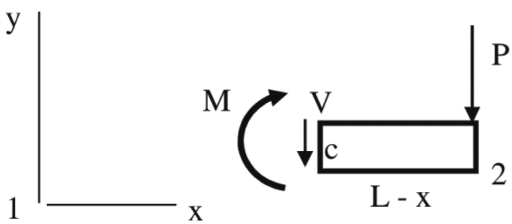


Figure RP3.1b.

$$+\ \text{ccw}\ \Sigma M_c = 0:\ -M(x) - P(L-x) = 0$$

$$M(x) = -P(L-x)$$

$$\int_0^x M(x)dx = \int_0^x -P(L-x)dx = -P(Lx - \frac{1}{2}x^2)$$

$$v(x) = -\frac{P}{EI}\int_0^x \left(Lx - \frac{x^2}{2} \right) dx + C_1 x + C_2$$
$$= -\frac{P}{EI}\left(\frac{L}{2}x^2 - \frac{1}{6}x^3 \right) + C_1 x + C_2$$

$$v'(x) = -\frac{P}{EI}\left(Lx - \frac{x^2}{2} \right) + C_1$$

$$v(0) = 0 \Rightarrow C_2 = 0$$

$$v'(0) = 0 \Rightarrow C_1 = 0$$

$$\underline{\underline{v(x) = -\frac{P}{EI}\left(\frac{L}{2}x^2 - \frac{x^3}{6} \right)}}$$

$$\underline{\underline{v'(x) = -\frac{P}{EI}\left(Lx - \frac{x^2}{2} \right)}}$$

$$v_{\max} = v(L) = -\frac{P}{EI}\left(\frac{L^3}{2} - \frac{L^3}{6} \right) = \underline{\underline{-\frac{PL^3}{3EI}}}$$

$$\left.\frac{dv'}{dx}\right|_{x=x_{\max}} = -\frac{P}{EI}(L - x_{\max}) = 0 \Rightarrow x_{\max} = L$$

$$v'_{\max} = v'(L) = -\frac{P}{EI}\left(L^2 - \frac{L^2}{2} \right) = \underline{\underline{-\frac{PL^2}{2EI}}}$$

$$\sigma(x) = -\frac{M(x)y}{I} = \underline{\underline{\frac{P(L-x)y}{I}}}$$

$$\sigma_{x,\max} = \sigma_x(0) = \underline{\underline{\frac{PLc}{I}}}$$

RP3.2

A jib crane consists of a wide flange steel beam attached by a pin (momentless) connection to a vertical support (Figure RP3.2a). The beam has an intermediate tie rod support. The wide flange beam has the following section properties: h = 153 mm, A = 22.9 cm^2, I = 916 cm^4.

If a 500 kg load is suspended at point D, compute the maximum stress in the beam

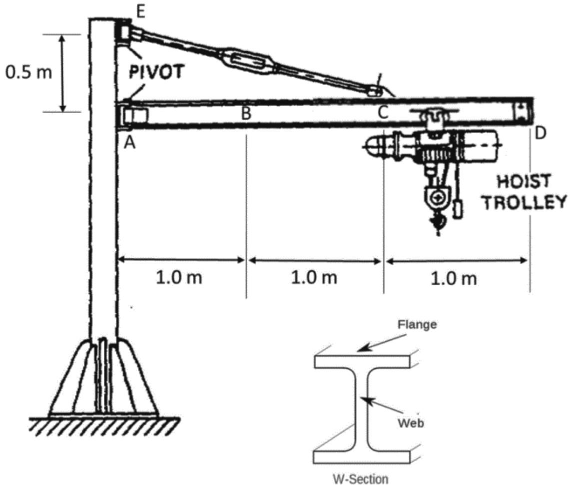


Figure RP3.2a.

First, a global FBD to determine boundary reactions (Figure RP3.2b).

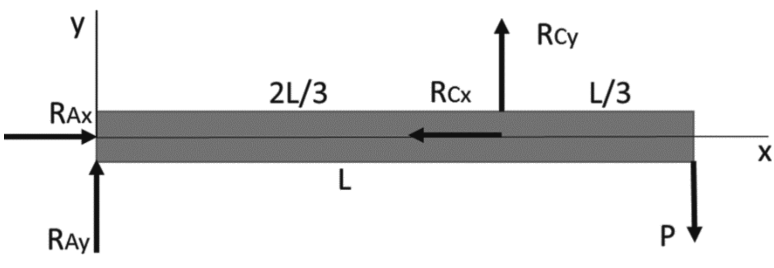


Figure RP3.2b.

P = 500 kg (9.81 m/s^2) = 4905 N

R_{Cy}/R_{Cx} = 0.5 m/ 2.0 m = ¼

$$\sum M_A = 0 : R_{Cy}(2m) - P(3m) = 0 \Rightarrow R_{Cy} = 1.5P = 7358N$$

$$R_{Cx} = 4R_{Cy} = 29{,}430 \text{ N}$$

$$\sum F_y = 0 : R_{Ay} + R_{Cy} - P = 0 \Rightarrow R_{Ay} = 4905N - 7358N = -2453N$$

$$\sum F_x = 0 : R_{Ax} - R_{Cx} = 0 \Rightarrow R_{Ax} = 29430\,N$$

Then local FBDs to determine internal reactions (Figures RP3.2c and RP3.2d):

Figure RP3.2c.

2 m ≤ x ≤ 3 m: $\sum F_x = 0 : 0 = 0$

2 m ≤ x ≤ 3 m: $\sum M_c = 0 : -M(x) - P(L-x) = 0 \Rightarrow M(x) = P(L-x)$

M(2 m) = 4905 N (3 m – 2 m) = 4905 Nm

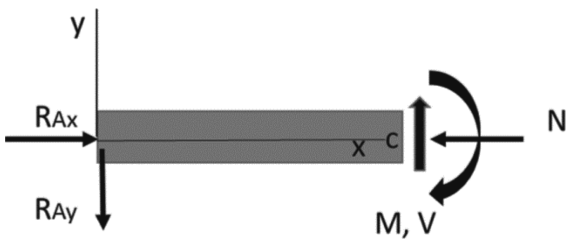


Figure RP3.2d.

0 ≤ x ≤ 2 m: $\sum F_x = 0 : -N + R_{Ax} = 0 \Rightarrow N = R_{Ax} = 29430$ N

0 ≤ x ≤ 2m: $\sum M_c = 0 : -M(x) + R_{Ay}x = 0 \Rightarrow M(x) = -R_{Ay}x$

M(2 m) = –(2453 N)(2 m) = –4905 Nm

Plotting the maximum moment (Figure RP3.2e)

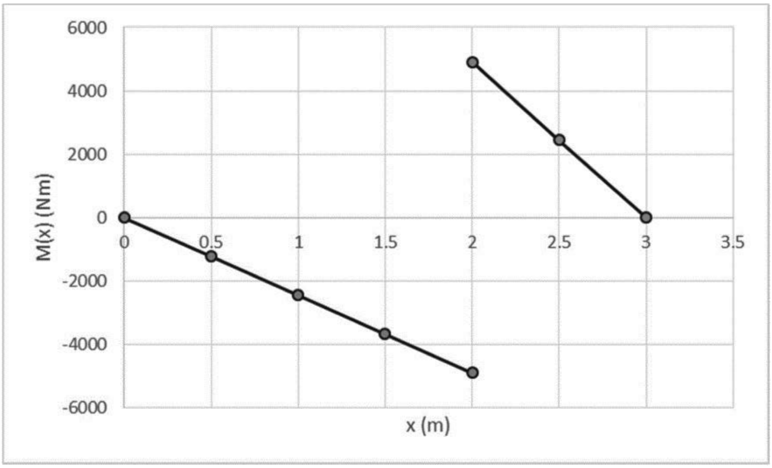


Figure RP3.2e.

M_{max} = 4905 Nm

$$\sigma_{axial,max} = N/A = \frac{29430\,N}{22.9\,cm^2\,(1m/100\,cm)^2} = 128.5\,kPa\ (compression)$$

$$\sigma_{bending,max} = \frac{M_{max}\,h/2}{I} = \frac{(4905\,Nm)(0.765\,m)}{916\,cm^4\,(1m/100\,cm)^4} = 40.96\,kPa$$

$$\sigma_{max} = 128.5\,kPa + 40.96\,kPa = 169.5\,kPa\ (compression)$$

RP3.3

A rod of solid circular cross-section is used as a cantilever beam by welding one end to a metal wall and leaving the other end free. Of the diameter D, modulus E, and length L, which has the greatest impact on the beam deflection?

We showed in RP3.1 that the maximum deflection of a tip-loaded cantilever occurred at the tip as well:

$$v_{\max} = v(L) = -\frac{P}{EI}\left(\frac{L^3}{2} - \frac{L^3}{6}\right) = \underline{\underline{-\frac{PL^3}{3EI}}}$$

Since stiffness is force per unit displacement, the beam stiffness is

$$k = \frac{P}{v} = \frac{CEI}{L^3}$$

where C is a constant that depends on the boundary conditions (e.g., $C = 3$ for fixed-free boundary conditions).

Since $I \sim D^4$, $k \sim D^4/L^3$ and a change to D will have more impact than the same change to L.

3.7 C³ Clarified

C³ 3.1 Why should we take the time to check units?

Answer: Checking units is a quick way of assessing if an equation returns the expected *dimensions*. *Dimensions* as used here refers to the primitive variables of length, mass, time, etc. (See Appendix 1 for further discussion on primitive variables.) For example, if we are expecting an equation to return an answer in force/length (or stiffness k), will the combination of variables in the equation correctly do that?

$$k = \frac{CEI}{L^3} = \frac{C[constant]E\left[{}^{N}\!/\!{}_{m^2}\right]I[m^4]}{L^3[m^3]} = \left[{}^{N}\!/\!{}_{m}\right] \checkmark$$

The combination of variables in the equation correctly return an answer in force/length.

C³ 3.2 In many textbooks, the shear stresses are shown oppositely directed, that is, one up and one done. In Figure 3.3 they are shown in the same direction. Which is correct?

Answer: This is one of the most confusing concepts in mechanics of structures. Let's start to clarify the confusion by appealing to our intuition. If an external force P is pushing down on the beam in some region (say Δx), then it stands to reason that every point on the beam in that region is pushing back (upward). So, intuition tells us that the shear force directions in Figure 3.3 are physically correct.

Unfortunately, many authors adopt a separate (and unnecessary) sign convention for shear and moment in beams. Often in these conventions, a positive shear force is defined as one that creates a (clockwise or counter-clockwise) rotation—hence the oppositely directed shear forces. But this leaves a net rotation in the beam, which, if the beam is in equilibrium, cannot exist. See Appendix 4 for further discussion on sign conventions.

Bottom line: use physical intuition where possible to draw forces and moments, and only use the rational sign conventions of mechanics.

C^3 3.3 I understand the need to know the magnitude, direction, and location of the maximum moment, but why not also for the maximum shear force?

Answer: It depends. See the discussion at the end of Example 3.9.

C^3 3.4 Where does Hooke's Law come from?

Answer: Robert Hooke was a 17th century scientist/engineer who, among other pursuits investigated the force—displacement response of materials such as metal wires. In 1660 he first published his findings (*ut tensio, sic vis* or "as the extension, so the force") that force and displacement are proportional in some materials.

Keep in mind that Hooke's Law is a phenomenological model, i.e., it relates empirical (experimental) observations but does not explain "why".

C^3 3.5 Can a structure fail in stiffness but still be strong enough?

Answer: Sure, this happens frequently. Consider the unfortunate bungy jumper who stretches the cord a little farther than planned but never snaps it in two. Or the floor board that sags when you walk on it but that never fractures.

Table 3.1. Comparison of some key relationships between axial, torsional, and flexural structures

	Axial	Torsional	Flexural
Stress	$\sigma_{xx} = \frac{P}{A}$	$\tau_{x\theta} = \frac{Mr}{J}$	$\sigma_{xx} = \frac{My}{I}$
Deformation	$\varepsilon_{xx} = du/dx$	Chap. 4	$\varepsilon_x = -\frac{y}{r_0} \approx -y\frac{d^2u}{dx^2} = -\frac{d}{dx}\left(y\frac{du}{dx}\right)$
Constitution	$\sigma_{xx} = E_{\varepsilon xx}$	Chap. 4	$\sigma_{xx} = E\,\varepsilon_{xx}$

3.8 Developing Engineering Intuition

As the structural loading and response becomes more complicated, developing intuition about them becomes more challenging. That is certainly true of topics in this chapter. However, there are some simple things we can do to develop intuition:

- Be aware of the self-weight gravity body loads that exist on all structures. Start with your own body as a structure!
- Observe structures from the point of view of being designed based on strength, stiffness, or both.
- Think about the combination of loads in common structures:
  - Flag pole: bending and axial compression
  - Bolted joint: torsion, axial tension, and axial compression
  - Bicycle fork: bending and axial compression

CHAPTER 4

Basic Structures Level III

Learning Objectives: This chapter will introduce the student to added complexity from Level II, such as more complicated combined loading and stress transformation. The strength design of structures under combined loading is considered.
Clarifications: Studying this chapter will help clarify:

- How to analyze structural elements under torsion loading
- How to analyze structures that are subjected to combined loading
- The general three-dimensional stress and strain response to combined loading
- The three-dimensional constitutive behavior of materials
- Design of structures for combined loading

Importance: Up to this point, we have considered structures and the loads they carry in a very constrained way. That is, we have either constrained the loading to be one-dimensional: axial, torsion, and flexural loads applied in isolation from one another or we have considered only the combination of axial and flexural loads. In the case of flexural structures, where we applied both forces and moments, we made sure that the loads resulted in flexure in a single plane only. In each case, the resulting stress/strain state could reasonably be considered uni-axial. But you would be correct to ask: there must be situations where we would need a structure to carry various kinds of loads, or even a single type of load in various directions?

Yes, of course, that happens all of the time. Consider a drill bit: first and foremost the bit carries torsion loads as it twists into the work piece. Yet at the same time, axial force is applied to advance the bit through the piece. Next time on the road look at the large sign boards supported at the top of a pole (Figure I4.1), which carries both the compressive axial loading of its own weight as well as the sign board weight, but also transverse wind loads, and possible seismic base loads.

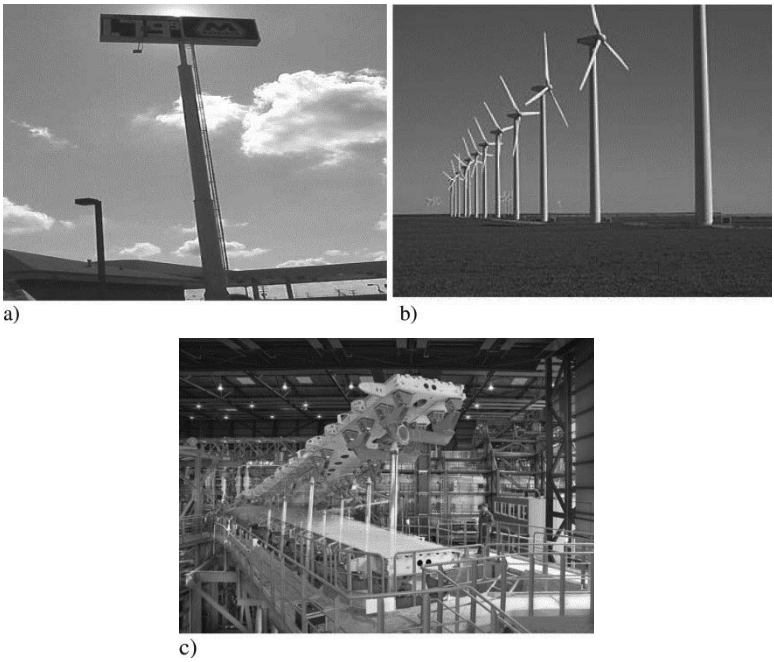


Figure I4.1. Structures subjected to three dimensional loading (a) The tall column supporting the bill board, (b) The blades and the supporting structure of a windmill, and (c) structural loading test on the wing box of an Airbus A380, at the IABG company in Dresden.

So what's the problem? Can't we just analyze each of these loads and their effects separately? Unfortunately, the answer is no. As we'll study further in this chapter, the stresses resulting from the loading are *tensor* quantities, and can't simply be added or decoupled in a algebraic fashion. We have to look at the stress resulting from the combined loading.

Two other problems arise as well. First, the convenient coordinate system chosen for analysis, for example that we are led to by the structure geometry, will in general not lead us to the critical or maximum stress we need for a strength design. We need some way to hunt for the *principal stress*. Secondly, materials fail differently under multi-axial stress than in the uni-axial case, so again simply trying to analyze the structure separately for each kind of loading won't suffice.

4.0 Representative Problems

RP4.1: Shown in Figure RP4.1 is a beam in which the shear stress τ_{xy} on the top surface is zero. If the beam is cut along a plane x = constant, what is the shear stress on the new surface?

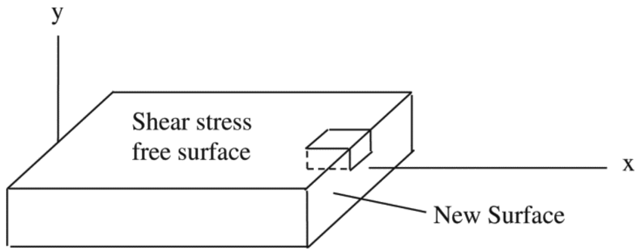


Figure RP4.1.

RP4.2 A thin rectangular rubber sheet is enclosed between two thick steel plates and the rubber sheet is subjected to a compressive stress of σ_{xx} and σ_{yy} in the x- and y-directions, respectively. Determine the strains in the x- and y-directions and the stress along the z-direction (thickness direction) of the rubber sheet (Figure RP4.2).

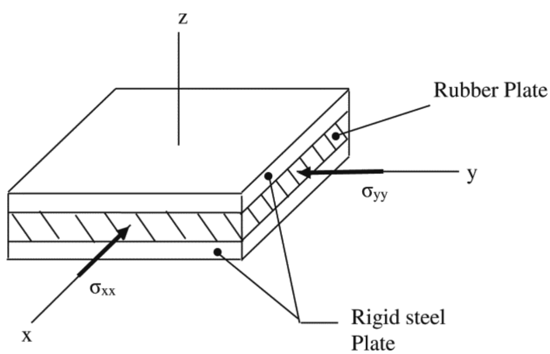


Figure RP4.2.

RP4.3 A closed cylindrical pressure vessel is fabricated from steel sheets that are welded along a helix that forms an angle of 60° with the transverse plane. The outer diameter is 1 m and the wall thickness is 0.02 m. For an internal pressure of 1.25 *MPa*, determine the stress in directions perpendicular and parallel to the helical weld (Figure RP4.3).

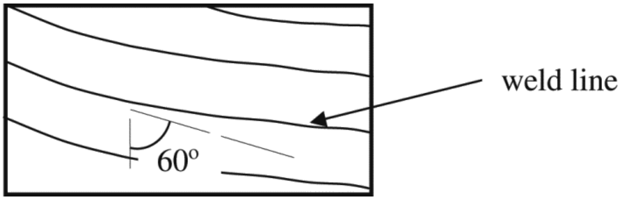


Figure RP4.3.

RP4.4 A cylindrical tube made of 2024-T4 aluminum has a diameter of 50 mm and wall thickness of 3 mm. An axial tensile load of 60 kN and a torque of 0.7 kN-m are applied. Will the tube yield? If not, how much can the tensile load or the torque can be increased before yielding occurs?

RP4.5 A thin-walled circular tube is made of AISI1020 steel. It is subjected to a torque of 6 kN-m and a pure bending moment of 4.5 kN-m. If the diameter of the tube is 50 mm, what must be the thickness so that the factor of safety against yielding is 1.25?

RP4.6 A cylindrical pressure vessel has an inner diameter of 240 mm and a wall thickness of 10 mm. The end caps are spherical and of thickness 10 mm. If the internal pressure is 2.4 *MPa*, find (a) the normal stress and the maximum shear stress in the cylindrical wall, (b) the normal stress and the maximum shear stress in the wall of the spherical end cap.

4.1 General State of Stress and Strain

4.1.1. General State of Stress at a Point

When forces are applied to a structural member, the forces are transmitted through the member as internal forces. Qualitatively, the intensity of the internal force per unit area at any point is called the *stress* at that point (see Chapter 2). Stresses in loaded members result from two basic types of forces, namely, surface forces and body forces (see Chapter 1). *Surface forces* are those that act on the surface of a body or member, for example when one body or member comes in contact with another body. *Body forces* act throughout the volume of the member; examples are gravitational, centrifugal, and electro-magnetic forces. Gravitational body forces are the most common body forces in static or quasi-static structural members (see Chapter 3), and are generally much smaller than surface forces. For this reason, body forces are often neglected in comparison to surface forces without introducing a significant error. However, there are many applications where this is not true. Consider, for example, the mirror in a large space telescope. With diameters on the order of a few meters or a few tens of meters, the self-weight of the telescope can cause enough "gravity sag" to distort the mirror considerably from its intended shape.

To determine the nature of internal forces, we divide the member into two parts by passing a cutting plane through the point of interest. Each of the two resulting parts may be considered a free body. The internal forces acting on the exposed cross-sectional area may have a distribution such that the internal force **F** (which is a vector represented by a bold quantity) varies in both magnitude and direction from point to point.

Consider a small area ΔA, around a point P of interest, on the surface generated by an arbitrary cutting plane (Figure 4.1). A system of internal forces acts on this small area, the resultant of which is $\Delta\mathbf{F}$, as is shown in Figure 4.2.

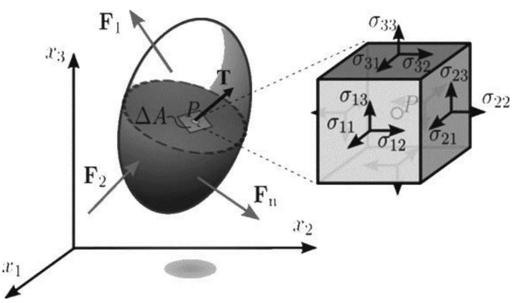


Figure 4.1. Cutting plane passing through point P.

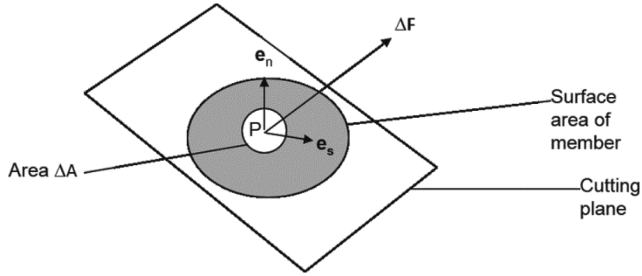


Figure 4.2. Resultant of all forces acting over an area ΔA on the arbitrary surface of a member. The vectors e_n and e_s are unit vectors perpendicular and parallel to ΔA, respectively.

It should be noted that the resultant force vector $\Delta\mathbf{F}$ does not in general coincide with the outward normal $\boldsymbol{e}_n$ associated with the element of area ΔA (see Appendix 2). The resultant traction vector $\mathbf{T}$ at point P is obtained by dividing $\Delta\mathbf{F}$ by ΔA, taking the limit as ΔA approaches zero, and is given by (see also equation 2.6)

$$\mathbf{T} = \lim_{\Delta A \to 0} \frac{\Delta\mathbf{F}}{\Delta A} = \frac{d\mathbf{F}}{dA} \qquad (4.1)$$

The line of action of $\mathbf{T}$ coincides with the line of action of resultant force $\Delta\mathbf{F}$, as shown in Figure 4.3.

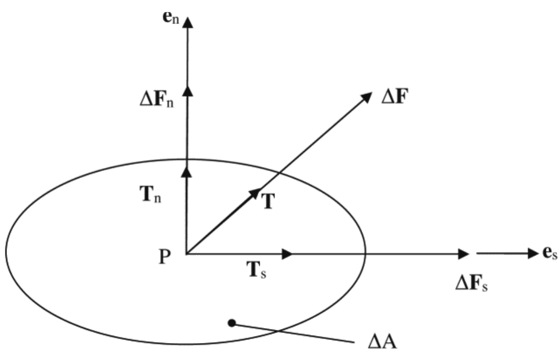


Figure 4.3. Resolution of the forces and stresses transmitted through the area ΔA.

It should be observed that the resultant stress $\boldsymbol{\sigma}_F$ is a function of the position of the point P in the member, the orientation of the cutting plane passing through the point P as identified by the outer normal $\mathbf{e_n}$, and the magnitude and direction of the resultant force on an infinitesimal area in the cutting plane around the point P. Thus to completely define the stress at a point we not only need the magnitude and direction of the force but also an additional direction of the outward normal $\mathbf{e_n}$ associated with the plane. Such quantities, for example stress (or strain), which require additional quantities besides a magnitude and a direction

C³

4.1 Stress is often computed from the expression stress=force/area and is treated as a scalar. Is that appropriate?
(Answer: Section 4.7)

(as with a *vector* quantity) for complete definition are called *'tensors'*. Thus a complete definition of stress or strain requires a magnitude and two directions. The magnitude of all vector or tensor quantities is influenced by the orientation of the coordinate system. However, the physical phenomena taking place at a point should not be dependent on the choice of a coordinate system. This implies that all operations performed with physical quantities should be independent of the orientation of the coordinate system, and the components must be obtainable from the components of the original coordinate system by appropriate transformation equations.

(A rigorous mathematical definition of a tensor is not being provided here for the sake of simplicity. However, it must be emphasized that any quantity (physical or mathematical) that transforms according to certain specific transformation laws when the original coordinate system changes its orientation, is known as a tensor. And vector quantities (such as force and displacement) are first-order tensors, while stress and strain are second order tensors. Higher order tensors can also be defined mathematically but they are generally difficult to comprehend in physical terms.)

The force $\Delta\mathbf{F}$ may be resolved into two components $\Delta\mathbf{F}_n$ and $\Delta\mathbf{F}_s$, along the normal to the small area and perpendicular to the normal $\boldsymbol{e}_n$, respectively. The force $\Delta\mathbf{F}_n$ is called the *normal force* on area ΔA and $\Delta\mathbf{F}_s$ is called the *shearing force* on ΔA. The normal and the shearing stress components at point P are obtained by letting ΔA approach zero (or become infinitesimal) and dividing the magnitude of the respective force by the magnitude of the area. Thus the normal stress σ_n and shear stress τ_s are given by the equations:

$$\sigma_n = \boldsymbol{T} \cdot \boldsymbol{e_n} \tag{4.2a}$$

$$\tau_s = \boldsymbol{T} \cdot \boldsymbol{e_s} \tag{4.2b}$$

The unit vectors associated with σ_n and τ_s are perpendicular and tangent, respectively, to the cutting plane. (From this point forward, we will drop the *n* and *s* subscripts on stress.)

C³

4.2 Does the choice of the coordinate system affect the state of stress at a point in a body under a given external load?
(Answer: Section 4.7)

Cartesian components of stress for any orientation of a rectangular-Cartesian coordinate system (x, y, z) at P can also be obtained from the resultant stress. Consider the small or incremental area ΔA, whose outer normal now coincides with the positive z-direction as shown in Figure 4.4. If the resultant traction factor **T** is resolved into components along the x, y and z-axes, the Cartesian components σ_{zz}, τ_{zx}, and τ_{zy} are obtained. (From this point forward, the bold font will be dropped on the stress notation for convenience; however, we wish to remind the student to keep in mind that stress is a tensor quantity.)

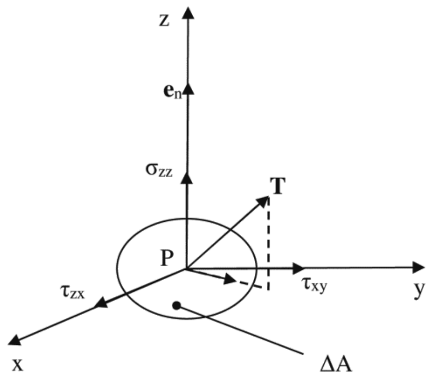


Figure 4.4. Resolution of the resultant stress on an incremental area into three rectangular-Cartesian stress components.

The first subscript refers to the outer normal and defines the plane upon which the stress component acts. The second subscript gives the direction in which the stress acts. In σ_{xx}, the first subscript x means that the outer normal of the infinitesimal area on which the force acts points along the x-axis. The second subscript x means that the force or the stress on the infinitesimal area acts in the x-direction. Thus the outward normal of the area and the force are in the same direction for normal stresses. In the term τ_{xy}, the outward normal of the area is along the x-axis, while the force is along the y-axis. Hence for shear stresses, the outward normal and force are perpendicular to each other.

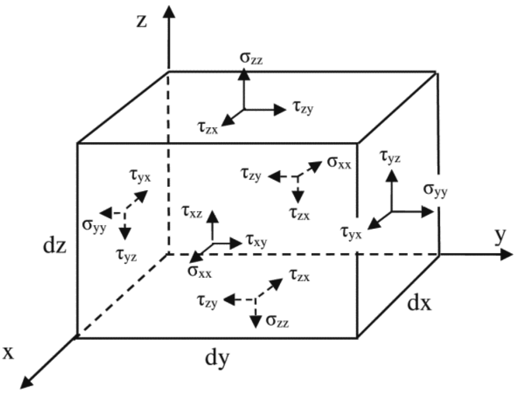


Figure 4.5. Rectangular-Cartesian stress components acting along the faces of a small cubic element around a point in a loaded member.

Normal stresses will be positive (+ sign) when they produce tension and negative (- sign) when they produce compression. Alternatively, normal stresses are positive when they act in the same direction as the outward normal to the area and negative when they act oppositely to the outward normal. As an example, a positive σ_{yy} acts on the positive y-face in the positive y-coordinate direction.

Shear stresses are termed positive (+ sign) if they point in a positive coordinate direction on a positive face or in a negative coordinate direction on a negative face, otherwise they are negative (- sign). As an example, a positive τ_{yx} acts on a negative y-face in the negative x-coordinate direction.

If the same procedure is followed using infinitesimal areas whose outer normals are in the positive x and y directions, two more sets of Cartesian components, (σ_{xx}, τ_{xy}, τ_{xz}) and (σ_{yy}, τ_{yx}, τ_{yz}), respectively, can be obtained. Hence at any point P in a member under an arbitrary load, nine Cartesian stress components can be identified. These nine components of stress constitute the *general state of stress* and are tabulated in the form of an array as follows:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{array}{l} \longrightarrow \text{Outer normal parallel to } x\text{-axis} \\ \longrightarrow \text{Outer normal parallel to } y\text{-axis} \\ \longrightarrow \text{Outer normal parallel to } z\text{-axis} \end{array}$$

By arranging the stress components in the above array and as will be shown later that the shear components are related by three relations, $\tau_{yz} = \tau_{zy}$, $\tau_{xy} = \tau_{yx}$, and $\tau_{xz} = \tau_{zx}$, the stress matrix becomes symmetric about the diagonal.

It is conventional to show these nine stress components on the faces of a small cubic element around a point in the loaded member, as shown in Figure 4.5.

Thus the three dimensional state of stress in a body loaded by surface and body forces and couples is defined by three normal stress components and six shear stress components.

4.1.2. General State of Strain at a Point

In the previous section the state of stress at any point in a body was determined. The relationships obtained were based on conditions of equilibrium and no assumptions were imposed regarding the deformation in the object or the physical properties of the material that constituted the object. Hence those results are valid for any material and any amount of deformation in the object. In this section, the state of deformation and the associated strains will be analyzed. Since strain is a pure geometric quantity, no restrictions on the object material will be required.

When a body is subjected to a system of forces, if individual points in the body change their relative positions the body is said to be in a state of *deformation*. The movement of any point is a vector quantity known as *displacement*. Translation or rotation of a body as a whole, with no change in the relative positions of points in the body, is known as *rigid-body motion* and does not result in strain within the body.

In introductory mechanics of materials two types of strains are generally used: (i) Extensional or *normal strain*, and (ii) *shear strain*. Figure 4.6 illustrates these two types of strains for a two dimensional case.

Normal strain is defined as the change in length of a line segment between two points divided by the original length of the line segment. Thus in Figure 4.6a the normal strain along the x-direction can be written as

$$\varepsilon_{xx} = \frac{\Delta u}{\Delta x}$$

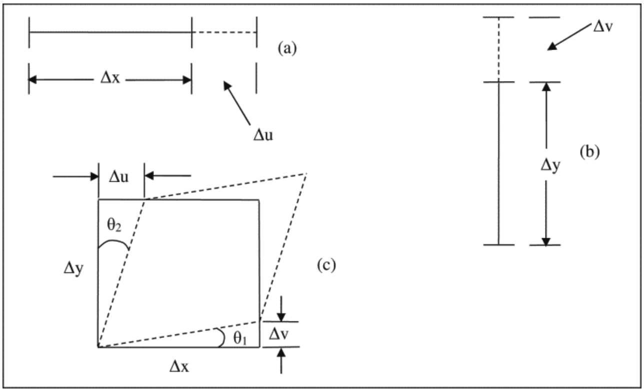


Figure 4.6. (a) Normal strain in the x-direction, (b) Normal strain in the y-direction, (c) Shear strain in the xy-plane. The dotted segments indicate the deformed position.

In the limit $\Delta x \to 0$, (that is, using an infinitesimal line segment) the normal strain equation can be written as

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \tag{4.3a}$$

It may be noted that in general the displacement u along the *x*-axis (or the *x*-component of the displacement of a point from one state to another) would be a function of both *x*- and *y*-coordinates such as $u = u(x,y)$. A partial derivative has been used in Equation (4.3a) as u is the displacement only in the *x*-direction. A partial derivative of a function of several variables is its derivative with respect to one of those variables, with the others held constant.

Similarly from Figure 4.6b, we can write

$$\varepsilon_{xx} = \frac{\partial v}{\partial y} \tag{4.3b}$$

Also, along the *z*-direction the normal strain can be expressed as

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \tag{4.3c}$$

The *shear strain* is defined as the angular change between the two line segments which were originally perpendicular and parallel to the

x- and y-axes. In Figure 4.6c the total change in the angle is $\theta_1 + \theta_2$. In the limiting case if θ_1 and θ_2 are very small, the total angle change can be expressed as

$$\theta_1(\text{radians}) + \theta_2(\text{radians}) = \tan\theta_1 + \tan\theta_2$$

where

$$\tan(\theta_1) = \frac{\partial v}{\partial x} \text{ and } \tan(\theta_2) = \frac{\partial u}{\partial y}$$

Thus the shearing strain is

$$\gamma_{xy} = \theta_1 + \theta_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{4.3d}$$

When shear deformation involves a reduction of the right angle between two line segments oriented respectively along the positive x- and y-axes, the shearing strain γ_{xy} is said to be positive, otherwise it is negative.

Similarly if two line segments are considered parallel to the y- and z-axes, and parallel to the x- and z-axes, two more expressions for the corresponding shear strains can be written as

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{4.3e}$$

and

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{4.3f}$$

Equations (4.3a) to (4.3f) constitute the *strain-displacement equations*, and are valid only for small strains of the order of 0.2% or less. For larger strains higher order terms have to be added to equations (4.3). The more general equations are not provided here as we will only consider problems with small strains in this book. Analogous to the general state of stress the *general state of strain* at a point can be written in matrix notation as

$$\begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_{zz} \end{bmatrix} \tag{4.4}$$

The strain components shown in the strain matrix in Equation (4.4) are called the engineering strains and the matrix is called the strain tensor.

4.1.3 Equilibrium and Deformation

In Sections 4.1.1 and 4.4.2 we have seen that a general state of stress and strain are completely defined by nine respective components in an isotropic material. However, if body and surface couples are zero, we can show for this case of stresses that:

$$\begin{aligned} \tau_{yz} &= \tau_{zy} \\ \tau_{xy} &= \tau_{yx} \\ \tau_{xz} &= \tau_{zx} \end{aligned} \tag{4.5}$$

Equation (4.5) can be proven as follows. Consider the *x-y* plane as shown in Figure 4.7, writing the moment equilibrium equation about the *z*-axis (by arbitrarily choosing counterclockwise direction as positive), one obtains the equation:

$$\sum M_o = 0: \ \tau_{xy}(dydz)dx - \tau_{yx}(dxdz)dy = 0$$

$$\Rightarrow \tau_{xy} = \tau_{yx}$$

Note that the equal and opposite normal force components cancel out and hence have not been shown in the equation above.

Hence the stress tensor for an isotropic, homogeneous material can be written as shown in equation (4.6) below

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \tag{4.6}$$

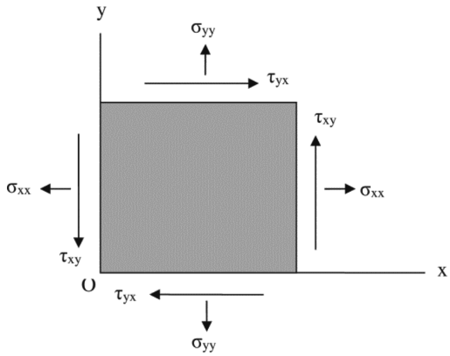


Figure 4.7. Stresses acting on the element in x-y plane to demonstrate the equality $\tau_{xy} = \tau_{yx}$

Similarly, we can write the strain components in the form

$$\begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_{zz} \end{bmatrix} \tag{4.7}$$

Note that analogous to the stress tensor in Equation (4.6), the normal strains are placed along the diagonal and the shear strains are symmetric about the diagonal of the strain matrix.

4.2 Analysis of Thin-Walled Torsion Structures

We have so far considered the torsion of shafts of solid circular cross-section, be they prismatic or non-prismatic. We found that the twist angle varies linearly along the shaft length and varies inversely with the polar moment of inertia. We also found that the shear stress varies linearly from the center of the shaft, so that only the outer fibers are fully stressed. It makes sense then that high-performance structures, those like aerospace vehicles that demand high torsion resistance at minimum weight, would rely on thin-walled shafts.

In this section we will examine the analysis of thin-walled torsion elements with both open and closed cross-sections

C³

4.3 Torque, moment and couple -- what is the difference?

(Answer: Section 4.7)

4.2.1 Shear Flow

Consider a thin-walled prismatic tube of arbitrary cross-section under pure torsion loading as shown in Figure 4.8. Figure 4.9 shows details of the elements in Figure 4.8.

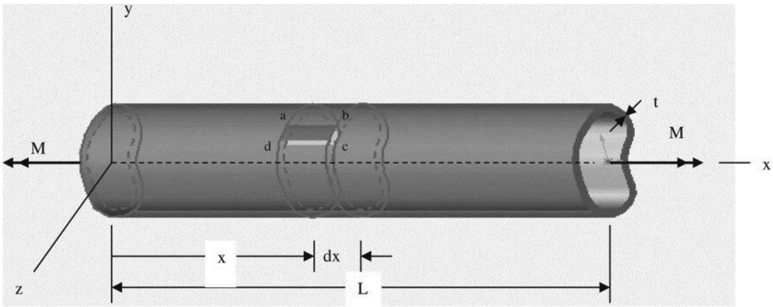


Figure 4.8. Thin-walled prismatic tube of arbitrary cross-section under pure torsion loading.

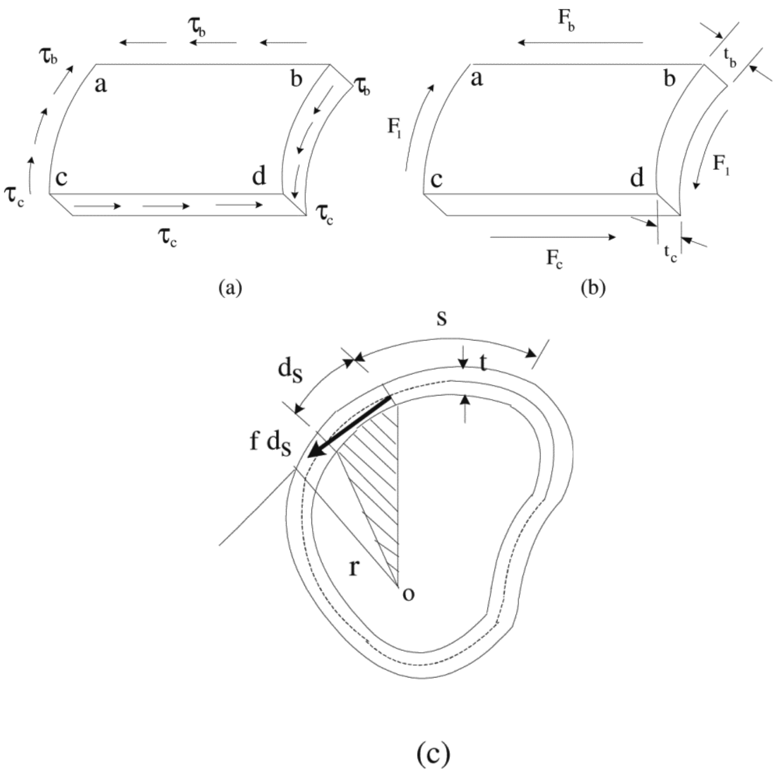


Figure 4.9. Section details of Figure 4.8.

We assume that, since the thickness t is so small, the shear stress is approximately constant across the wall thickness. In Figure 4.9, for the arbitrary element abcd to be in equilibrium, the force $\mathbf{F}_b$ (= $\tau_b t_b dx$) must be equal in magnitude to $\mathbf{F}_c$ (= $\tau_b t_b dx$). Then

$$\tau_b t_b = \tau_c t_c \tag{4.8}$$

or

$$\tau t = \text{constant} \tag{4.9}$$

The quantity τt is called the *shear flow*. Note that this implies that the largest shear stress occurs where the thickness is smallest. (Hereafter we drop the subscript on τ and ϕ for convenience, since no confusion should arise.)

We now need to relate the shear flow to the applied torque **M**. Looking at Figure 4.9, we see that an increment of torque dM is applied by the shear flow as:

$$dM = r\tau t ds \tag{4.10}$$

where ds is an increment of arc length along the mean circumference l_m. Integrating along the entire mean length l_m gives:

$$M = \tau t \int_0^{l_m} r ds = 2\tau t A_m \tag{4.11}$$

where A_m is the mean area enclosed by the mean circumference. Note that the integral

$$\int_0^{l_m} r ds = 2A_m \tag{4.12}$$

since from Figure 4.9, $r ds$ represents twice the area of the shaded triangle. Finally, we then have a useful relationship for the shear stress on a thin-walled shaft of arbitrary cross-section:

$$\tau = \frac{M}{2tA_m} \tag{4.13}$$

4.2.2 Thin-Walled Shafts of Closed Cross-Section

Consider a shaft of thin circular closed cross-section as shown in Figure 4.10:

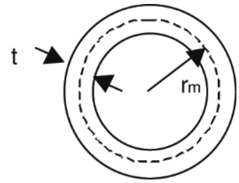


Figure 4.10. Shaft of thin circular closed cross-section. The mean radius is given by r_m and t is the wall thickness.

The polar moment of inertia J is calculated in the usual way as before:

$$J = \frac{\pi}{2}\left[\left(r_m + \frac{t}{2}\right)^4 - \left(r_m + \frac{t}{2}\right)^4\right] \tag{4.14}$$

After expansion and some simplification this becomes

$$J = \frac{\pi r_m t}{2}(4r_m^2 + t^2) = 2\pi r_m^3 t\left[1 + \left(\frac{t}{2r_m}\right)\right] \tag{4.15}$$

Now for a thin wall, where $t/r_\mathrm{m} << 1$ and $(t/2r_\mathrm{m})^2 << 1$, Equation (4.15) simplifies to:

$$J \approx 2\pi r_\mathrm{m}^3 t = 2A_\mathrm{m} r_\mathrm{m} t \tag{4.16}$$

where A_m is the area enclosed by the mean radius.

The shear stress is then

$$\tau = \frac{Mr_m}{J} \approx \frac{Mr_m}{2\pi r_m^3 t} = \frac{M}{2\pi r_m^2} = \frac{M}{2A_m t} \tag{4.17}$$

For values of t/r_m = 0.2, 0.1, and 0.05, the <u>approximate</u> shear stress is 92%, 95%, and 98% of the <u>exact</u> shear stress, respectively. Note also that since τ is assumed not to vary across the thickness, the stress found above is the maximum shear stress.

The total twist angle is given as before:

$$\varphi_{total} = \frac{ML}{GJ} = \frac{ML}{2GA_m r_m t} \tag{4.18}$$

Of course, for the thin-walled closed circular cross-section, the approximation to *J* provides us little advantage. However, we can apply this same approximation technique to more complex thin-walled shapes, of either open or closed section, where calculating the exact polar moments of inertia would be difficult. (In these cases of non-circular cross-section, the factor *J* is no longer the polar moment of inertia but is called more generally the *torsion constant*, and is in general less than the polar moment of inertia.)

A formal method for finding the torsion constant can be developed from strain energy considerations. Recall that the strain energy expression developed previously was

$$U = \frac{1}{2}\int_{\text{volume}} \frac{\tau^2}{G} dv \tag{4.19a}$$

For the thin-walled section, dV = dAdx = tdsdx, where ds is an increment of wall length measured along the mean circumference. Upon substitution and some manipulation we get:

$$U = \frac{\tau^2 t^2}{2G}\int_0^{l_m}\left(\int_0^{L} dx\right)\frac{ds}{t} = \frac{(\tau t)^2 L}{2G}\int_0^{l_m}\frac{ds}{t} = \frac{M^2 L}{8GA_m^2}\int_0^{l_m}\frac{ds}{t} = \frac{M^2 L}{2GJ} \tag{4.19b}$$

where the final term has the same form as the strain energy for a circular shaft, except that here J is the torsion constant found from

$$J = \frac{4A_m^2}{\int_0^{l_m} \frac{\mathrm{ds}}{\mathrm{t}}} \tag{4.20}$$

Example 4.1

Consider for example the thin square closed cross-section shown in Figure E4.1:

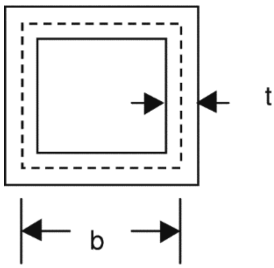


Figure E4.1. Shaft of thin square closed cross-section. A side has mean length given by b and a wall thickness t.

The above formula for J can be applied here. The value of the integral in the denominator is simply $4b/t$, A_m is b^2, giving $J = b^3t$. Hence the average shear stress (maximum shear stress must take into account the stress concentrations in the corners) is

$$\tau = \frac{M}{2tA_m} = \frac{M}{2tb^2}$$

and the total twist angle is

$$\varphi_{total} = \frac{ML}{Gb^3t}$$

Example 4.2

Compare the shear stress and total twist angle in two thin-walled shafts having the same length L and net cross-sectional area, which are subjected to the same torque T, except that one shaft is of circular cross-section while the other has a square cross-section.
For the circular tube:

$A_c = 2\pi r_m t$ $\qquad$ $A_{mc} = \pi r_m^2$ $\qquad$ $J_c = 2\pi r_m^3 t$

For the square tube:

$A_s = 4bt = A_c = 2\pi r_m t \Rightarrow b = \pi r_m/2 \qquad A_{ms} = b^2 \; J_s = b^3 t = \pi^3 r_m^3 t/8$

Now

$$\frac{\tau_s}{\tau_c} = \frac{M/2tA_{ms}}{M/2tA_{mc}} = \frac{A_{mc}}{A_{ms}}\frac{\pi r_m^2}{\pi^2 r_m^2/4} = \frac{4}{\pi} = 1.27$$

$$\frac{\phi_s}{\phi_c} = \frac{J_c}{J_s} = \frac{2\pi r_m^3 t}{\pi^3 r_m^3 t/4} = \frac{16}{\pi^2} = 1.62$$

Hence the shear stress and total twist angle for the square tube are 27% and 62% greater than the circular tube. The square shape is clearly less efficient in torsion.

4.2.3 Torsional Shape Factor

We can now begin to see how we might characterize the shape efficiency of a given cross-sectional configuration. We define a *shape factor* β that compares the relative efficiency of a given cross-section to a reference cross-section. Following Ashby, we arbitrarily take a solid circular shaft as the reference. This reference shaft has length L_0, cross-sectional area $A_0 = \pi R_0^2$, shear modulus G_0, and torsion constant

$$J_0 = \frac{\pi}{2}R_0^4 = \frac{1}{2}A_0 R_0^2 \tag{4.22}$$

Under the action of a torque **M**, the reference shaft has a maximum shear stress and total twist angle given as before:

$$\tau_0 = \frac{MR_0}{J_0} = \frac{2M}{A_0 R_0} \tag{4.23}$$

$$\varphi_0 = \frac{ML_0}{G_0 J_0} = \frac{2ML_0}{G_0 A_0 R_0^2} \tag{4.24}$$

Then we use β to compare the efficiency of other shaft configurations having shear modulus G, but <u>equivalent cross-sectional area</u> A_0 <u>and length</u> $L = L_0$, under the action of the same torque **T**. That is:

$$\beta_\tau = \frac{\tau}{\tau_0} = \frac{\tau}{2M / A_0 R_0} \tag{4.25}$$

$$\beta_\varphi = \frac{\varphi}{\varphi_0} = \frac{ML / GJ}{ML_0 / G_0 J_0} = \frac{G_0 J_0}{GJ} = \frac{G_0}{G} \frac{A_0 R_0^2}{2J} \tag{4.26}$$

A value of β less than one means that the shaft under consideration is more efficient than the reference solid circular shaft, that is, for the same load the stress and/or stiffness are less than for the reference shaft.

Example 4.3

Determine β for the thin-walled circular shaft of Example 4.2.

$$A_c = z\pi r_m t = A_o = \pi R_o^2 \Rightarrow R_o = \sqrt{r_m t}$$

$$A_{mc} = \pi r_m^2$$

$$\beta_\tau = \frac{\tau_c}{\tau_0} = \frac{M/2tA_{mc}}{2M/A_0 R_0} \frac{A_0 R_0}{4tA_{mc}} = \frac{(2\pi r_m t)\sqrt{2r_m t}}{4t\pi r_m^2} = \ldots = \sqrt{\frac{t}{2r_m}} = 0.707\sqrt{\frac{t}{r_m}}$$

$$\beta_\tau = \frac{J_o}{J_c} = \frac{\frac{1}{2} A_o R_o^2}{2\pi r_m^3 t} = \frac{1}{4\pi} \frac{(2\pi r_m t)(2r_m t)}{r_m^3 t} = \frac{t}{r_m}$$

Table E4.1 below shows a comparison of both the strength and stiffness efficiency of the thin-walled circular shaft for various wall thicknesses (compared to the solid circular shaft of same cross-sectional area). As shown, as the wall thickness becomes thinner, the shaft becomes more efficient. (Note that this trend cannot go on indefinitely, since the shaft is getting larger to keep the area constant and equal to A_0. As the wall thickness gets thinner, other failure modes will start to take place, such as local wall buckling.)

Table E4.1.

t/r_m	β_τ	β_Φ
1/10	0.224	0.1
1/100	0.0707	0.01
1/1000	0.0224	0.001

4.2.4 Thin-Walled Shafts of Open Cross-Section

Slicing a thin-walled shaft of closed circular cross-section longitudinally results in an *open cross-section* as shown in Figure 4.11.

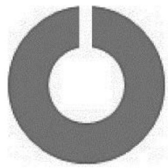


Figure 4.11. A longitudinal cut transforms a closed cross-section into an open cross-section.

However, this cut "releases" the shear stress depicted in Figure 4.9(c). This significantly increases flexibility of the structure in torsion, and hence thin-walled open sections are considerably less efficient in torsion than comparable closed cross-sections.

The exact form of the torsion constant for a thin solid rectangular section h by t ($h > t$) (Figure 4.12) can be shown to be

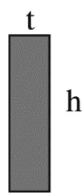
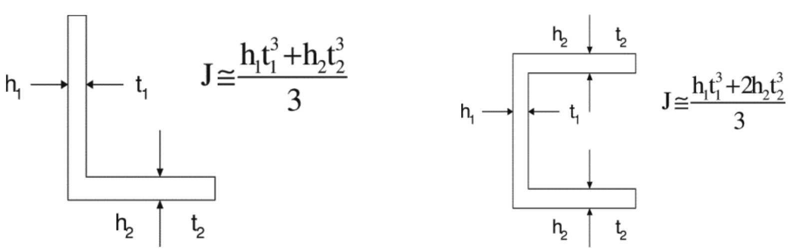


Figure 4.12. Cross-section of a thin solid rectangular shaft.

$$J = \frac{ht^3}{16}\left[\frac{16}{3} - 3.36\frac{t}{h}\left(1 - \frac{t^4}{12h^4}\right)\right] \qquad (4.27)$$

For $t/h << 1$, $J \approx ht^3/3$. Thus the torsion constant for open sections can be approximated as a sum of thin rectangular sections. Figure 4.13 provides several examples:



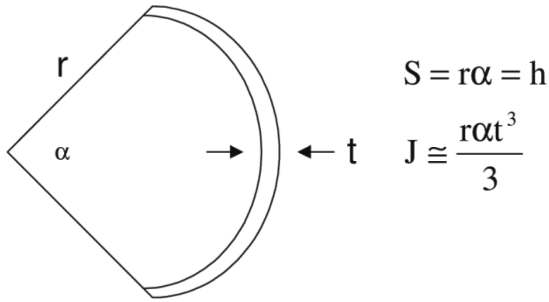


Figure 4.13. Approximate torsion constants for some thin-walled open sections.

The (average) shear stress (away from sharp corners) and the total twist angle are found as:

$$\tau_{avg} = \frac{Mt}{J}, \quad \varphi = \frac{ML}{GJ} \qquad (4.28a, b)$$

Example 4.4

Compare the torsion efficiency under the load **T** of 2 thin-walled tubes of circular section and equal cross-sectional area, one being closed and the other formed by a thin longitudinal cut in the wall.

$$\frac{\tau_{open}}{\tau_{closed}} = \frac{Mt/J_{open}}{Mr_m/J_{closed}} = \frac{t}{r_m}\,\frac{2\pi r_m^2 t}{2\pi \frac{r_m}{3} t^2} = 3\frac{r_m}{t}$$

$$\frac{\phi_{open}}{\phi_{closed}} = \frac{J_{\text{closed}}}{J_{\text{open}}} = \frac{2\pi r_m^3 t}{2\pi \frac{r_m}{3} t^3} = \underline{\underline{3\left(\frac{r_m}{t}\right)^2}}$$

As can be seen, the open cross-section carries 3 times the shear stress and rotates more than 3 times than that of the equivalent closed cross-section.

4.3 Three-Dimensional Stress-Strain Relationship

4.3.1 Three-Dimensional Elastic Material.

The general state of stress at a point is shown in Figure 4.5, with three normal stresses in three coordinate directions and three shear stresses on three orthogonal planes. In the elastic region, that is if the strains are small or less than about 0.2%, a linear relationship exists between stress and strain, called *Hooke's Law* (see Section 3.4). With the small strain assumption the strains caused by each normal stress can be added.

A normal stress in the *x*-direction causes a strain in the *x*-direction of σ_{xx}/E, and a strain of $-\upsilon\sigma_{xx}/E$ in both the *y*- and *z*-directions. Similarly stresses in the *y*- and *z*-directions cause strains in all three directions. The stress—strain relationships can be summarized in a tabular form as shown in Table 4.1 below:

Table 4.1. Stress-strain relationships.

Normal Stress	Normal Strains		
	x-direction	*y*-direction	*z*-direction
σ_{xx}	σ_{xx}/E	$-\upsilon\sigma_{xx}/E$	$-\upsilon\sigma_{xx}/E$
σ_{yy}	$-\upsilon\sigma_{yy}/E$	σ_{yy}/E	$-\upsilon\sigma_{yy}/E$
σ_{zz}	$-\upsilon\sigma_{zz}/E$	$-\upsilon\sigma_{zz}/E$	σ_{zz}/E

Adding the columns in the table above results in the tensile stress-strain equations shown in Equation (4.29), where E is the elastic modulus and υ the Poisson's ratio of the material and are also known as the elastic constants of the isotropic material. For a three-dimensional stress-strain state, the tensile stress—strain relations given in equation (4.29) are expressions for obtaining the strains in terms of stress components and elastic constants.

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] \\
\varepsilon_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})] \qquad (4.29) \\
\varepsilon_{zz} &= \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]
\end{aligned}$$

The three shear strains that occur on three orthogonal planes are related to the corresponding shear stresses by an elastic material constant called the 'shear modulus', G (note that this is the third elastic constant for isotropic materials besides E and υ), and the shear stress-strain expressions are shown below

$$\begin{aligned}\gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy}\\ \gamma_{yz} &= \frac{2(1+\nu)}{E}\tau_{yz}\\ \gamma_{zx} &= \frac{2(1+\nu)}{E}\tau_{zx}\end{aligned} \tag{4.30}$$

We can re-write Equations (4.29) and (4.30) to express stresses in terms of strain as shown below

$$\begin{aligned}\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{xx} + \nu(\varepsilon_{yy} + \varepsilon_{zz})]\\ \sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{yy} + \nu(\varepsilon_{zz} + \varepsilon_{xx})]\\ \sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{zz} + \nu(\varepsilon_{xx} + \varepsilon_{yy})]\\ \tau_{xy} &= \frac{E}{2(1+\nu)}\gamma_{xy}\end{aligned}$$

$$\begin{aligned}\tau_{yz} &= \frac{E}{2(1+\nu)}\gamma_{yz}\\ \tau_{zx} &= \frac{E}{2(1+\nu)}\gamma_{zx}\end{aligned} \tag{4.31}$$

If pure shear stresses are applied, the elastic response of the material is characterized by the last three equations in equations (4.31). For isotropic materials, in general three normal stresses and six shear stresses exist. However, three relations between the shear stresses can be obtained as given in equation (4.32)

$$\tau_{xy} = \tau_{yx}, \quad \tau_{yz} = \tau_{zy}, \quad \tau_{zx} = \tau_{xz} \tag{4.32}$$

and thus only three shear stresses need to be obtained independently.

For an isotropic material, only two of the material properties G, E, and ν, are independent, the third can be calculated from the relation shown in equation (4.33)

$$G = \frac{E}{2(1+\nu)} \tag{4.33}$$

Example 4.5

The strain components are measured at a point in a steel component in a machine and are listed below. Determine the stress at this point. For steel E=207 GPa and ν=0.3.

$$\varepsilon_{xx} = 300\mu\varepsilon, \varepsilon_{yy} = 300\mu\varepsilon, \varepsilon_{zz} = 300\mu\varepsilon$$
$$\gamma_{xy} = 200\mu\varepsilon, \gamma_{yz} = 200\mu\varepsilon, \gamma_{zx} = 200\mu\varepsilon$$

Answer: From the first expression in equation (4.31)

$$\begin{aligned}\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{xx} + \nu(\varepsilon_{yy} + \varepsilon_{zz})] \\ &= \frac{207 * 10^9}{(1=0.3)(1-2*0.3)}[(1-0.3)(300*10^{-6}) + 0.3(200+100)10^{-6}] \\ &= 119.4 * 10^6 \mathrm{N/m^2} \\ &= 119.4\ MPa\end{aligned}$$

Similarly using the second and third expressions from equation (4.31), we obtain the two remaining normal stress components as:

$$\begin{aligned}\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{yy} + \nu(\varepsilon_{zz} + \varepsilon_{xx})] \\ &= \frac{207 * 10^9}{(1=0.3)(1-2*0.3)}[(1-0.3)(200*10^{-6}) + 0.3(100+300)10^{-6}] \\ &= 103.5 * 10^6 \mathrm{N/m^2} \\ &= 103.5 MPa\end{aligned}$$

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{zz} + \nu(\varepsilon_{xx} + \varepsilon_{yy})]$$

$$\frac{207 * 10^9}{(1 = 0.3)(1 - 2 * 0.3)}[(1-0.3)(100 * 10^{-6}) + 0.3(300 + 200)10^{-6}]$$

$$= 87.6 * 10^6 \text{ N / m}^2$$

$$= 87.6\ MPa$$

The shear stress components are obtained by using the last three expressions in equation (4.31) as:

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

$$= \frac{207 * 10^9}{2(1+0.3)} 200 * 10^{-6}$$

$$= 15.9 * 10^6 \text{ N / m}^2$$

$$= 15.9 MPa$$

$$\tau_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz}$$

$$= \frac{207 * 10^9}{2(1+0.3)} 100 * 10^{-6}$$

$$= 7.9 * 10^6 \text{ N / m}^2$$

$$= 7.9 MPa$$

$$\tau_{zx} = \frac{E}{2(1+\nu)}\gamma_{zx}$$

$$= \frac{207 * 10^9}{2(1+0.3)} 150 * 10^{-6}$$

$$= 11.9 * 10^6 \text{ N / m}^2$$

$$= 11.9 MPa$$

Example 4.6

A rubber cube is inserted into a cavity of the same size and shape in a thick steel block, as shown in Figure E4.6. The rubber cube is pressed by a steel block with a pressure of *p*. Considering the thick steel cavity to be rigid and there is no friction between the cube and the cavity walls, find the pressure exerted by the rubber against the cavity walls.

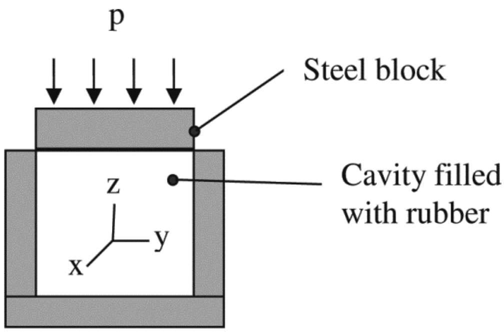


Figure E4.6.

Answer: Since the cube is constrained in the *x*- and *y*-directions, the strain components along these directions are zero,

$$\Rightarrow \varepsilon_{xx} = 0 \text{ and } \varepsilon_{yy} = 0 \qquad \text{(E4.6.1)}$$

In the *z*-direction the stress in the rubber cube must balance the pressure applied to maintain equilibrium, thus

$$\sigma_{zz} = -P \qquad \text{(E4.6.2)}$$

Using equations (E4.6.1) and (E4.6.2) and the first two expressions of equation (4.29) we can write

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} - P)] = 0 \\ \varepsilon_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu(-P + \sigma_{xx})] = 0 \end{aligned} \qquad \text{(E4.6.3)}$$

simultaneously solving equations (Equation E4.6.3), we obtain the pressure exerted by the cube against the cavity walls as:

$$\sigma_{xx} = \sigma_{yy} = -\frac{\nu}{1-\nu}P$$

4.3.2 Energetics

Strain energy is the potential energy stored in a body by virtue of an elastic deformation, equal to the work done by the applied forces, to provide both normal and shear strains. In a body loaded within the elastic limit, the work done during loading is stored as recoverable strain energy in the material of the body. If the body is unloaded, the body does work (on the loading frame or the human or other interfacing objects) and releases all its energy. The work done to deform the body depends only on the state of strain at the end of the test; it is independent of the history of loading.

Strain energy stored in a material is a very useful metric for evaluating the suitability of a material for structural applications. Higher the strain energy stored or higher the strain energy absorbed under a given loading condition the better, else if the energy is not dissipated it could likely cause failure or fracture in the material. In this regard a quantity called *strain energy density*, *U*, is defined as the work done per unit volume to deform the material from a stress free state to a loaded state. Defining this quantity for a unit volume of material eliminates the effect of size of the body. If the strain energy is divided by the density, we obtain a quantity called *specific strain energy*. Strain energy density has the dimensions of J/m^3 in the SI metric units, or in-lb/in^3 in the US system of units. The strain energy density is equal to the area under the stress-strain curve measured from zero strain to a given strain value and the mathematical form is given in equation (4.34), which is graphically represented by the shaded area in Figure 4.14.

$$U = \int_{volume} \sigma \ d\varepsilon \tag{4.34}$$

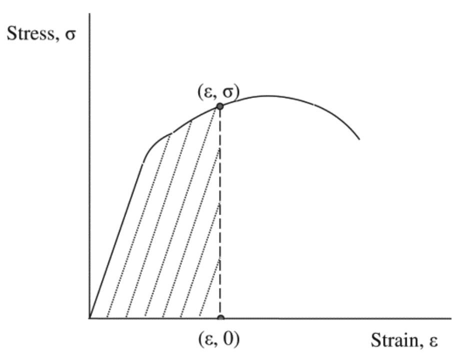


Figure 4.14. Shaded region represents the strain energy per unit volume.

If the specimen is loaded beyond its elastic limit and plastic deformation occurs and then the specimen is unloaded, only the energy represented by the shaded region in Figure 4.15 is recovered, the remainder of the energy is spent in deforming the material and is dissipated as heat energy. Note that line $P\varepsilon_p$ is the unloading path and is parallel to the linear or elastic region of the stress-strain curve. The area under the stress-strain curve up to fracture is called *modulus of toughness* of the material.

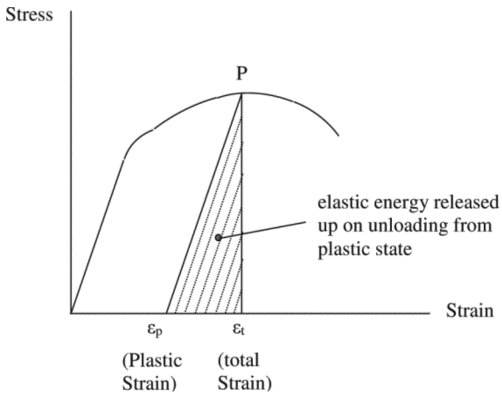


Figure 4.15.

The strain energy density, U, for a member under a uniaxial normal stress state is given by Equation (2.27), for a member under shear stress only by equation (2.28). If a body is subjected to a general state of stress, that is any element will have at least two or more non-zero stress components from among the six stress components, σ_{xx}, σ_{yy}, σ_{zz}, τ_{xy}, τ_{yz}, τ_{xz}, the strain energy density can be obtained by adding the expressions given in equations (2.27) and (2.28), as well as the four other expressions obtained through a permutation of the subscripts of the stress and strain components. The six expressions can be added to obtain the total strain energy density according to the *principle of superposition*, which states that the effect of a given combined loading on a structure can be obtained by determining separately the effects of various loads and combining the results, provided the loads produce small deformations and each effect is linearly related to the load that produces it (which is true if the material remains within the elastic limit). Thus, assuming elastic deformations in the body, we can write the strain energy density as

$$U = \frac{1}{2}\left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz}\right) \quad (4.35)$$

Substituting for the strain components from equations (4.29) and (4.30), we obtain an expression for the strain energy density in terms of stress components as

$$\begin{aligned} U = {} & \frac{1}{2E}\left[\sigma_{xx}{}^2 + \sigma_{yy}{}^2 + \sigma_{zz}{}^2 - 2\upsilon\left(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}\right)\right] \\ & + \frac{1}{2G}\left(\tau_{xy}{}^2 + \tau_{yz}{}^2 + \tau_{xz}{}^2\right) \end{aligned} \quad (4.36)$$

4.4 Combined Axial, Torsional and Flexural Loading

4.4.1 Stress Analysis for Combined Loading

Figure 4.16 shows the combination of all three loading conditions. Each loading condition results in a corresponding stress state (internal response) that combine in some fashion to give the actual stress state. The last figure in row (c) shows three loads: flexure, axial and torsion, applied simultaneously and the stress state on a small element on the surface of the object.

We now develop a process to determine the maxima's in the values of stresses created at a point in a body under combined loading. If the coordinate system $x'y'z'$ shown in Figure 4.17 is rotated through an arbitrary angle Θ about the same origin to a new axes system $x'y'z'$, with the assumed requirement that the z'-axis and the z'-axis remain coincident, the values of the normal and shear stresses will change in the new axes system. However, the new stress components do not represent a new state of stress but rather an equivalent representation of the original state of stress. In a general three-dimensional case it can be shown that for a certain specific orientation of the coordinate axes system the normal stresses achieve a maxima and a minima, while for another specific set of axes system the maxima and minima in the shear stresses are obtained. To illustrate this phenomena a three-dimensional stress state will be approximated as a two-dimensional (2-D) state of stress without loss of generality, and this has been achieved in

Figure 4.17 by effectively rotating the coordinate system only in the *xy*-plane. Moreover a 2-D stress state is only a special case of a 3-D case. The special 2-D stress state is called 'Plane Stress', which occurs when the three components of stress acting on any one face of the cubic element (Fig. 4.17) are all zero.

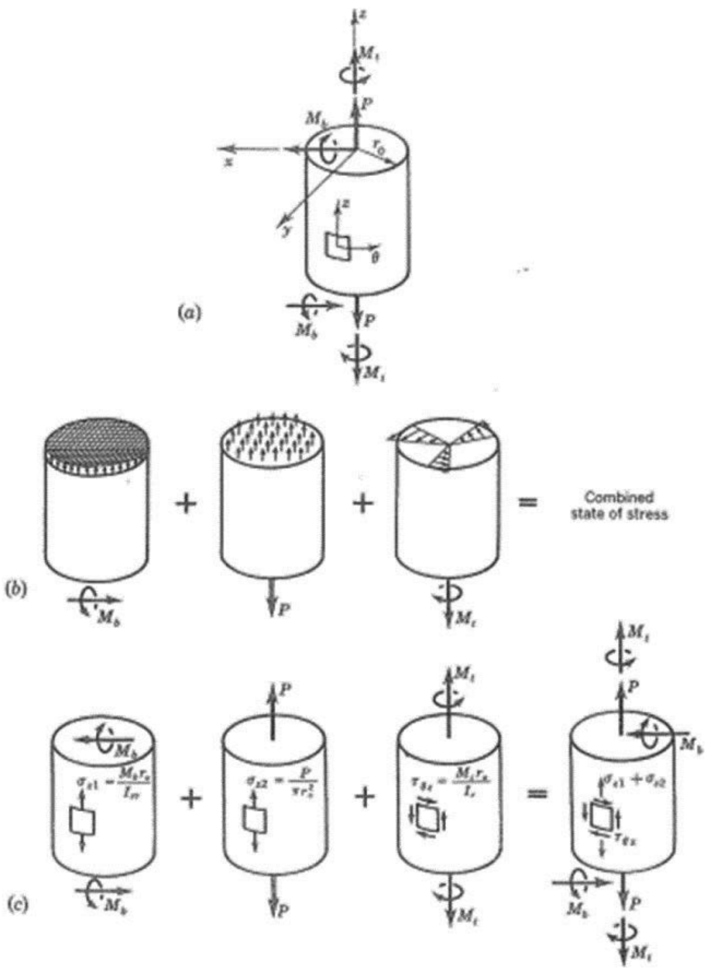


Figure 4.16. Various loading conditions and the corresponding stress states created in the simple object. (note that this figure was originally introduced in Chapter 3).

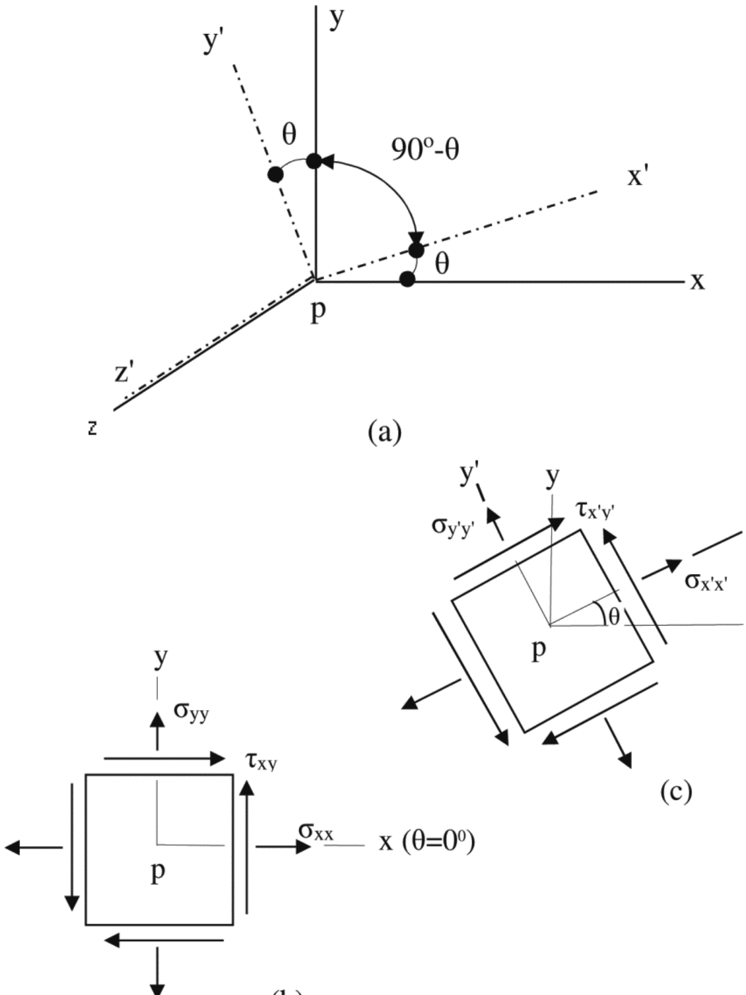


Fig. 4.17 Stress transformation in Plane Stress.

Thus in Figure 4.17 if the planes parallel to the *X-Y* plane are considered stress free, then

$$\sigma_{zz} = \tau_{xy} = \tau_{yz} = 0$$

Equilibrium of forces on the element in Figure 4.17 requires that the moments must sum to zero about both the *x*- and *y*-axis, requiring that the shear components τ_{xz} and τ_{yz} acting on the other two planes must also be zero. Hence the remaining non-zero stress components are σ_{xx},

σ_{yy}, and τ_{xy}, as illustrated in Figure 4.7 or see Figure 4.17(b). Figure 4.17(c) shows the stress state $\sigma_{x'x'}$, $\sigma_{y'y'}$ and $\tau_{x'y'}$ if the coordinate axes are rotated by an angle Θ.

The stress components in the rotated axes system can be obtained by cutting the 2-D stress element in Figure 4.18 by an oblique plane (shown as a dashed line) at an arbitrary angle Θ and then drawing the free body diagram of the portion of element ABC. The free body diagram is shown in Figure 4.18b.

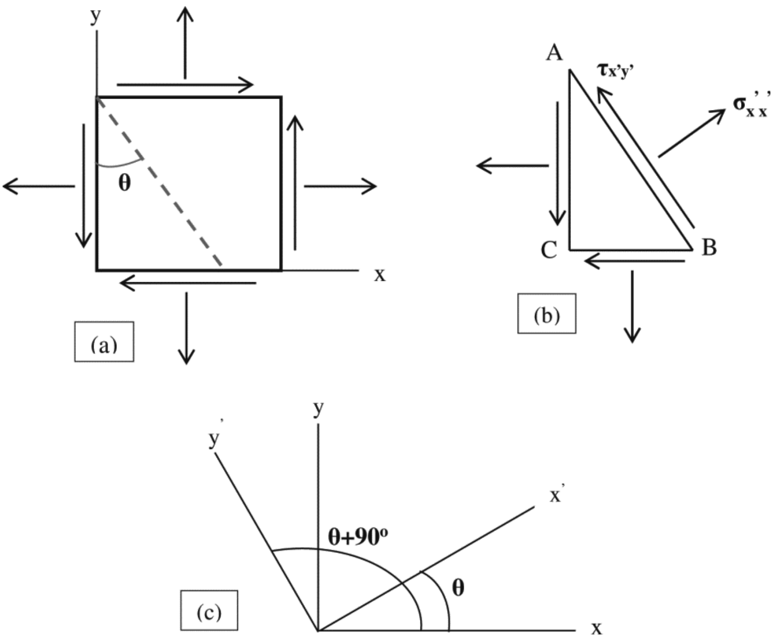


Figure 4.18. Stresses on an oblique plane. (Note that all stress components are not listed for better clarity).

Equilibrium of forces in the *x*- and *y*-directions requires that the sum of the forces is equal to zero. Thus the two equations can be solved to obtain the unknown normal and shear stress components $\sigma_{x'x'}$ and $\tau_{x'y'}$ on the arbitrary inclined plane, which are shown below

$$\sigma_{x'x'} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \tau_{xy}\sin 2\theta \quad (4.37a)$$

$$\tau_{x'y'} = -\frac{1}{2}(\sigma_{xx} - \sigma_{yy})\sin 2\theta + \tau_{xy}\cos 2\theta \quad (4.37b)$$

The Equations (4.37a,b) directly give $\sigma_{x'x'}$ and $\tau_{x'y'}$ in the new coordinate system, while substituting Θ with Θ + 90° in Equation (4.37a) results in $\sigma_{y'y'}$ shown below as Equation (4.37c):

$$\sigma_{y'y'} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) - \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta - \tau_{xy}\sin 2\theta \quad (4.37c)$$

Now let us determine the specific orientation of the coordinate axes that provides the maximum normal stress, as was mentioned earlier in this section. Using the established mathematical procedure we take the derivative $d\sigma/d\Theta$ of Equation 4.37a and equate the results to zero. Then solving for Θ we obtain the rotation required of the coordinate axes for the maximum and minimum values of the normal stress σ,

$$\tan 2\theta_n = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \quad (4.38)$$

Equation 4.38 defines two particular values of angle $2\Theta_n$ (subscript n has been introduced to indicate that the orientation of axes system provides the extreme values of normal stresses), one of which provides the maximum normal stress termed as σ_1 and the other, the minimum normal stress σ_2. These stresses have a special name and are called 'Principal Normal Stresses', given by

$$\sigma_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sqrt{\frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \sqrt{\frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \tau_{xy}^2} \quad (4.39\text{ a, b})$$

It can be shown that for the axis orientation of Θ_n the shear stress is zero on the planes where the principal normal stresses occur. This can be proven by rearranging equation 4.38 and substituting in Equation 4.37a, and is left as an exercise for the student. It should be noted that the converse is also true: if the shear stress on a plane is zero, then the normal stresses on this plane are principal normal stresses.

In a similar manner, by solving the derivative $d\tau/d\Theta = 0$ of Equation 4.37b gives the coordinate axes rotation for the maximum and minimum shear stresses.

$$\tan 2\theta_s = -\frac{\sigma_{xx} - \sigma_{yy}}{2\tau_{xy}} \tag{4.40}$$

Substituting for Θ_s in Equation (4.40) provides the maximum value of the shear stress in the *x-y* plane and is called the 'principal shear stress'.

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \tag{4.41}$$

It can be shown that the two orthogonal planes where the principal shear stress occurs has the same normal stress of

$$\sigma_s = (\sigma_{xx} + \sigma_{yy})/2 \tag{4.42}$$

Figure 4.19 shows the stress element orientation and the schematic stress state consisting of the principal normal stresses and the principal shear stresses.

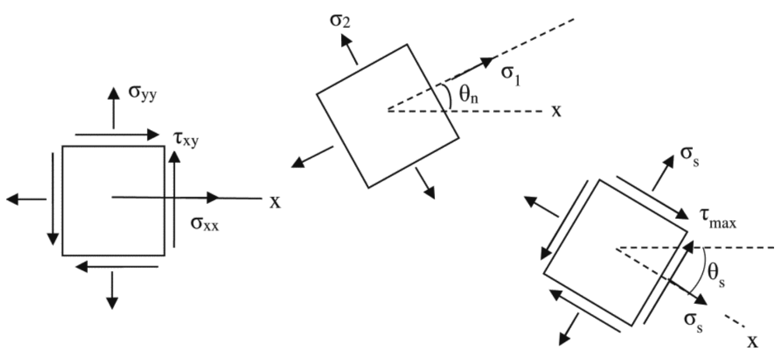


Figure 4.19. Stress element orientation and the schematic stress state consisting of the principal normal stresses and the principal shear stresses.

By substituting Equations (4.41) and (4.42) into Equations (4.39 a,b) and solving for the maximum shear stress we obtain

$$\tau_{\max} = \frac{|\sigma_1 - \sigma_2|}{2} \tag{4.43}$$

The absolute value of maximum shear stress is used due to the two roots of Equation (4.41).

4.4.2 Mohr's Circle for Plane Stress Problems

Let us again consider the transformation employed in Section 4.4.1 wherein the (x, y, z) coordinate axes were rotated through an arbitrary angle to (x', y', z') coordinate axes with the requirement that the z-axis and the z'-axis remain coincident. Let the angle between the x & x' axes and the y & y' axes be θ after the transformation, as shown in Figure 4.17. Thus effectively Figure 4.17 represents a transformation in two-dimensions or in a plane, namely, the x-y plane.

The equations (4.37 a,b) can written as:

$$\left\{\sigma_{x'x'} - \frac{1}{2}(\sigma_{xx} + \sigma_{yy})\right\}^2 = \left\{\frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \tau_{xy}\sin 2\theta\right\}^2 \tag{a}$$

$$\tau_{x'y'}^{\ 2} = \left\{-\frac{1}{2}(\sigma_{xx} - \sigma_{yy})\sin 2\theta + \tau_{xy}\cos 2\theta\right\}^2 \tag{b}$$

adding equations (a) and (b) results in the expression:

$$\left\{\sigma_{x'x'} - \frac{1}{2}(\sigma_{xx} + \sigma_{yy})\right\}^2 + \left\{\tau_{x'y'}\right\}^2$$

$$= \frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2(\cos^2 2\theta + \sin^2 2\theta) + \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\tau_{xy}\sin 2\theta\cos 2\theta$$

$$-\frac{1}{2}(\sigma_{xx} - \sigma_{yy})\tau_{xy}\sin 2\theta\cos 2\theta + \tau_{xy}^{\ 2}(\sin^2 2\theta + \cos^2 2\theta)$$

$$\Rightarrow\left\{\sigma_{x'x'}-\frac{1}{2}(\sigma_{xx}+\sigma_{yy})\right\}^2+\left\{\tau_{x'y'}-0\right\}^2=\frac{1}{4}(\sigma_{xx}-\sigma_{yy})^2+\tau_{xy}^{\ 2} \quad (4.44)$$

Equation (4.44) represents the equation of a circle in the ($\sigma_{x'x'}$, $\tau_{x'y'}$) plane, whose center C has the coordinates $\left[\frac{1}{2}\left(\sigma_{xx}+\sigma_{yy}\right)\ ,\ 0\right]$ and has a radius of $r=\left\{\frac{1}{4}\left(\sigma_{xx}+\sigma_{yy}\right)^2+\tau_{xy}^{\ 2}\right\}^{1/2}$. A graphical representation of this circle, popularly knows as *Mohr's circle* in honor of the German engineer Otto Mohr, who first employed it to study plane stress problems, is shown in Figure 4.20.

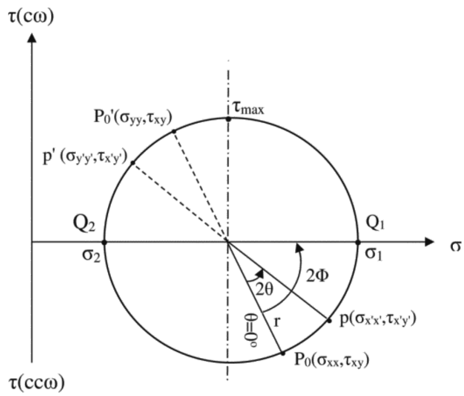


Figure 4.20. Mohr's circle for plane stress.

In the Mohr's circle diagram the normal stress components σ are plotted on the horizontal axis while the shear stress components τ are plotted on the vertical axis. Tensile normal stresses are plotted along the positive σ-axis, that is to the right of the τ-axis, while compressive normal stresses are plotted to the left of the τ-axis. To determine the shear stress the general convention is as follows. First define the positive face of an element as one for which the outward normal is in the positive direction of the coordinate axis. For example, in Figure 4.21a the outward normal along the positive z-axis is the positive face, and the outward normal on the bottom face is along the negative z-direction and is thus the negative face. Then if the shear stress acts in the positive direction of the coordinate axis on the positive face it is termed as positive (+ve) shear stress (or it is also termed positive if the shear stress acts in the negative direction on a negative face). The shear stress is termed as negative (–ve) if the shear stress acts in the negative direction of the coordinate axis on the positive face (or if a positive shear stress acts on a negative face). As can be seen from Figure 4.21 the positive shear on the *z*-faces produces a clockwise rotation of the element, while a negative shear produces a counterclockwise rotation of the element at point P. A shear stress couple that produce a clockwise rotation (cw) of an infinitesimal element around the point P under consideration, as shown in Figure 4.21a, is plotted above the σ-axis. Shear stress couple that produces a counter-clockwise rotation (ccw) (Figure 4.21b) is plotted below the σ-axis.

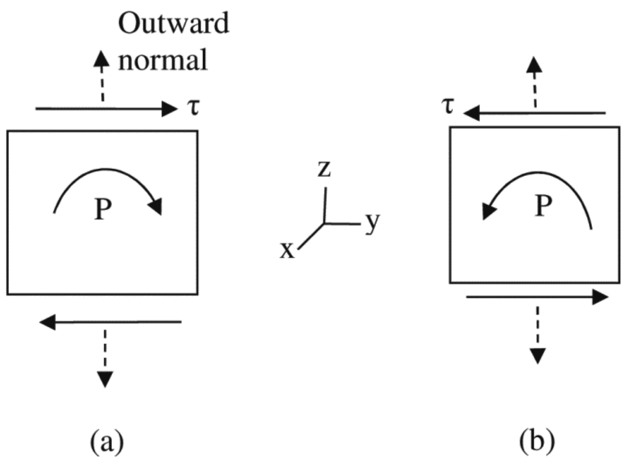


Figure 4.21. Shear stress states: (a) positive shear, and (b) negative shear. The positive shear on the z-faces produces a clockwise rotation of the element as in (a), while a negative shear produces a counterclockwise rotation of the element at point P as in (b). (Note that the complementary shear stress to maintain moment equilibrium not shown here).

From Figure 4.17, for the case θ = 0°, the equations (4.37 a,b) give

$$\sigma_{x'x'} = \sigma_{xx} \text{ and } \tau_{x'y'} = \tau_{xy}$$

which are the coordinates of point P_0 in Figure 4.20 (Note: for θ = 0° the face with outer normal along *x*-axis has a tensile stress σ_{xx} and a counterclockwise shear τ_{xy}). The coordinates of the center C are

$$\frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{\sigma_{x'x'} + \sigma_{y'y'}}{2}$$

which can be shown by adding the equations (4.37 a,c). Thus using C as the center and length CP_0 as the radius, the Mohr's circle can be plotted. Another point P_0' $(\sigma_{yy}, \sigma_{xy})$ can be located though it is not needed for drawing the Mohr's circle. This point represents the stress on the faces with outer normal along the y-axis or $\theta = \pi/2$. The points P and P' represent the stresses on the element with faces along x'-axis and y'-axis as shown in Figure 4.17. In other words, stress components associated with each plane through a point are represented by a point on the Mohr's circle.

C³

4.4 Do you need to know the shear stresses on both the x- and y-faces of the element to draw the Mohr's circle?

(Answer: Section 4.7)

The principal stresses are located at points Q_1 and Q_2. By definition, in the principal planes the shear stress is zero. Thus from the equation (4.37b) when $\tau_{x'y'} = 0$, we obtain:

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \tag{4.45}$$

Solution of Equation (4.45) will yield two values of θ, say $\theta = \Phi$ and $\Phi + \pi/2$, which are shown as 2Φ and $2\Phi + \pi$ on the Mohr's circle. The magnitude of the principal stresses can be obtained from the Mohr's circle as

$$\begin{aligned} \sigma_1 &= \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sqrt{\frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \tau_{xy}^2} \\ \sigma_2 &= \frac{\sigma_{xx} + \sigma_{yy}}{2} - \sqrt{\frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \tau_{xy}^2} \end{aligned} \tag{4.46}$$

Note that $\sigma_3 = 0$ as a plane stress case is being considered.

Example 4.7

For the state of plane stress shown in Figure E4.7a, determine (a) the principal planes, (b) the principal stresses, and (c) the maximum shearing stress and the corresponding normal stress using (i) the Mohr's circle method, and (ii) numerically without using the Mohr's circle diagram.

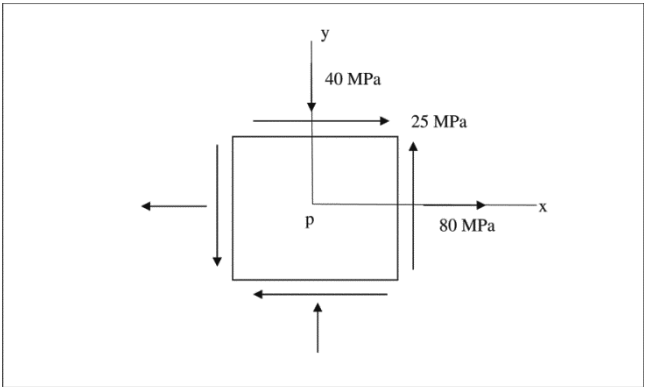


Figure E4.7a.

(i) Mohr's Circle Method:

The stress state on the face with outer normal along the *x*-axis consists of a tensile normal stress (+ normal stress) and a shear stress producing a counter-clockwise rotation of the element. Thus this stress state is plotted to the right of the τ-axis (as tensile normal stress) and below the σ-axis (as *ccw* rotation of element due to shear) at point P_0. Similarly point P_0' is plotted to represent the stress state on the face with outer normal along the *y*-axis in Figure E4.7b. (Note: One could use a reverse sign convention too, that is plot the clockwise rotation below the σ-axis, and it would make no difference in the construction of the Mohr's circle and the analysis of the stress states. However, you must be consistent in the use of the chosen sign convention).

C³

4.5 What is the effect on principal stress determination if one shows the shear stress on an element as clockwise instead of the correct direction of anti-clockwise?
(Answer: Section 4.7)

On drawing the line $P_0 P_0'$, it intersects the σ-axis at C, which is the center of the Mohr's circle. The abscissa of the center C is

$$\frac{\sigma_{xx}+\sigma_{yy}}{2}=\frac{80+(-40)}{2}=20MPa$$

The radius of the circle is CP_0 and is given by

$$r=CP_0=\sqrt{(CA)^2+(P_0A)^2}=\sqrt{(80-20)^2+(25)^2}=65MPa$$

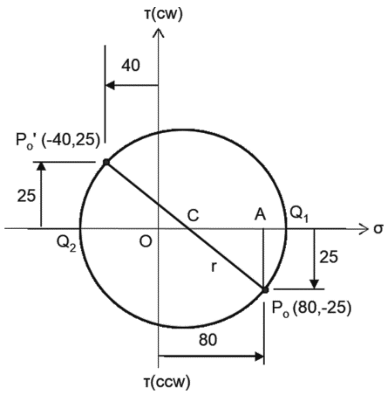


Figure E4.7b.

The principal stresses are represented by the points Q_1 and Q_2, where Q_1 represents the maximum principal stress σ_1 and Q_2 represents the minimum principal stress σ_2. Their magnitudes are given by

$$|\sigma_1| = OQ_1 = OC + CQ_1 = OC + CP_0$$
$$= 20 + 65 = 85\ MPa$$

$$|\sigma_2| = OQ_2 = OC - CQ_1 = OC + CP_0'$$
$$= 20 - 65 = -45\ MPa$$

The angle Q_1CP_0 represents 2θ [see Figures E4.7b and E4.7c], and is obtained as

$$\tan 2\theta = \frac{AP_0}{CA} = \frac{25}{60}$$
$$\Rightarrow 2\theta = 22.6^0, \Rightarrow \theta = 11.3^0$$

Thus in the Mohr's circle line CP_0 must be rotated counterclockwise through an angle 22.6^0 to bring CP_0 into CQ_1. In the actual material the element should be rotated counterclockwise through half the angle $\theta = 11.3°$ to obtain the principal stress state, as shown in Figure E4.7c.

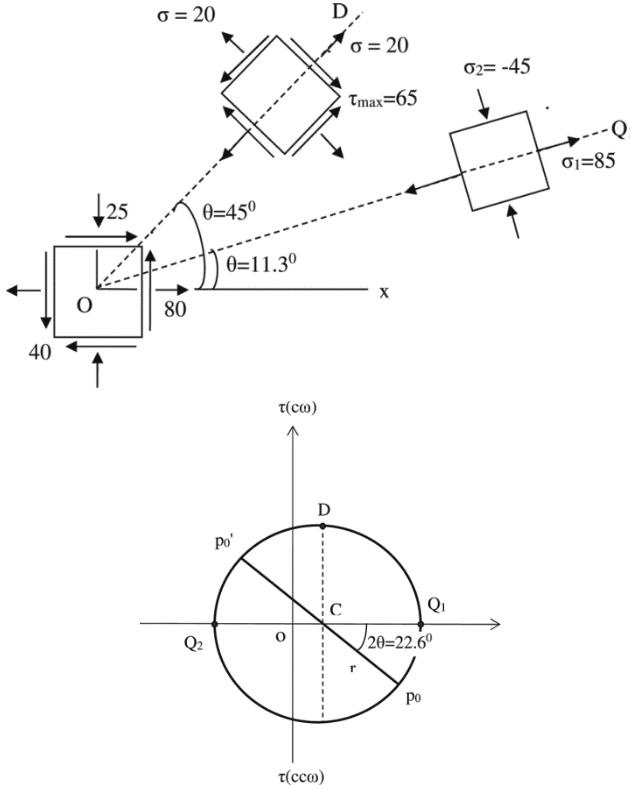


Figure E4.7c. Orientation of principal normal stress and shear stress elements, and magnitude of principal normal stresses and maximum shear stress. All stress in MPa.

Point D in Figure E4.7c represents the maximum shear stress state. CQ_1 can be rotated counterclockwise through 90° to bring CQ_1 into CD. The magnitude of the maximum shear stress is equal to the radius r of the Mohr's circle, that is $\tau_{max} = 65$ *MPa*. In the real material the stress element is rotated counterclockwise through an angle of $\theta + 90°/2 = 11.3 + 45° = 56.3°$, to bring the axis Ox into the axis OD and the orientation of the element and the associated stress represent point D on the Mohr's circle. Since point D is located above the σ-axis (that is the τ(cw) axis), the shearing stress exerted on the faces of the element perpendicular to OD in Figure E4.7c must be directed so that they will tend to rotate the element clockwise. The normal stress is the same as that at C, which is 20 *MPa*.

(ii) Numerical Solution

(a) Equation (4.45) can be used to obtain the orientation of the principal plane as

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} = \frac{2(+25)}{(+80)-(-40)} = \frac{50}{120}$$
$$\Rightarrow 2\theta = 22.6^0 \; and \; 180° + 22.6° = 202.6°$$
$$\Rightarrow \theta = 11.3^0 \; and \; 90° + 11.3° = 101.3°$$

Thus the element is oriented at an angle of 11.3° (measured ccw) to the element containing the applied plane stress state. The second value of θ = 101.3° can also be used to define the element's σ orientation (Figure E4.7d). A plane that contains the face on which σ_1 acts perpendicularly is the principal plane, and similarly a plane that contains the face on which σ_2 acts is the other principal plane.

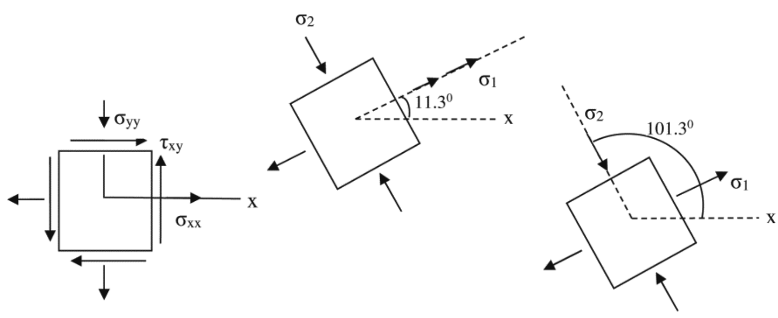


Figure E4.7d.

(b) Equations (4.39 a, b) reproduced below can be used to obtain the magnitude of the principal stresses as

$$\sigma_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sqrt{(\frac{\sigma_{xx} - \sigma_{yy}}{2})^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \sqrt{(\frac{\sigma_{xx} - \sigma_{yy}}{2})^2 + \tau_{xy}^2}$$

$$\Rightarrow \sigma_1 = \frac{80-40}{2} + \sqrt{\left(\frac{80-(-40)}{2}\right)^2 + 25^2} = 85 \ MPa$$

$$\sigma_2 = \frac{80-40}{2} - \sqrt{(\frac{80-(-40)}{2})^2 + 25^2} = -45 \ MPa$$

(c) The maximum shear stress can be obtained analytically as follows:

Differentiating equation (4.37b) with respect to $\theta = \theta_S$ and setting the results equal to zero, we obtain

$$\tan 2\theta_s = \frac{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)}{\tau_{xy}}$$

This value of θ gives the orientation of the element corresponding to the maximum shear stress.

Solving the above equation for $\sin 2\theta_s$ and $\cos 2\theta_s$ and substituting into the equation (4.37b) we obtain

$$\tau_{xy} = \tau_{\max} = (\frac{\sigma_{yy} - \sigma_{xx}}{2}) \frac{(\frac{\sigma_{yy} - \sigma_{xx}}{2})}{\sqrt{(\frac{\sigma_{yy} - \sigma_{xx}}{2})^2 + \tau_{xy}^2}} + \tau_{xy} \frac{\tau_{xy}}{\sqrt{(\frac{\sigma_{yy} - \sigma_{xx}}{2})^2 + \tau_{xy}^2}}$$

$$= \frac{(\frac{\sigma_{yy} - \sigma_{xx}}{2})^2 + \tau_{xy}^2}{\sqrt{(\frac{\sigma_{yy} - \sigma_{xx}}{2})^2 + \tau_{xy}^2}}$$

$$= \sqrt{(\frac{\sigma_{yy} - \sigma_{xx}}{2})^2 + \tau_{xy}^2}$$

which is customarily written as

$$\tau_{\max} = \sqrt{(\frac{\sigma_{xx} - \sigma_{yy}}{2})^2 + \tau_{xy}^2}$$

Note that the value of τ_{max} is also the radius of the Mohr's circle.

$$\therefore \ \tau_{\max} = \sqrt{(\frac{80-(-40)}{2})^2 + 25^2} = 65 \ MPa$$

To obtain the value of the normal stress acting on the element with the maximum shear stress, we substitute the value of $\sin 2\theta_s$ and $\cos 2\theta_s$ into equations (4.37 a,c) to obtain

$$\sigma_{xx}^{\ s} = \sigma_{yy}^{\ s} = \frac{\sigma_{xx} + \sigma_{yy}}{2}$$

Thus the normal stress on each of the four faces of the maximum shear stress element is the same.

Therefore, $\sigma_{xx}^{\ s} = \sigma_{yy}^{\ s} = \dfrac{80+(-40)}{2} = 20\,MPa$

Example 4.8

Show that the following relationship exists between the three elastic constants

$$G = \frac{E}{2(1+\nu)}$$

A circular cross-section solid rod under torsion is in a state of pure shear stress. On a plane rotated 45° with respect to the directions of pure shear the normal stresses are the principal stress and thus $\sigma_1 = \tau$, $\sigma_2 = 0$, and $\sigma_3 = -\tau$. Similarly, $\varepsilon_1 = \gamma/2$, $\varepsilon_2 = 0$, and $\varepsilon_3 = -\gamma/2$. If we let the *x*-, *y*-, and *z*-directions coincide with the principal 1-2-3 directions then substituting for $\sigma_{xx} = \tau$, $\sigma_{zz} = 0$, and $\sigma_{yy} = -\tau$ and $\varepsilon_{xx} = \gamma/2$ into the first expression in Equation (4.29) yields

$$G = \frac{\tau}{\gamma} = \frac{E}{2(1+\nu)}$$

4.4.3 Principal Stresses in a Three-Dimensional Stress State

> **C^3**
>
> 4.6. Is Mohr's circle useful for 3D stress analysis?
> *(Answer: Section 4.7)*

It was mentioned in Section 4.4.1 that the resultant stress on a plane depends on the orientation of the cutting plane Q on which the stress acts. If the plane Q is such that the outer normal to the plane coincides with the resultant stress the shear stress vanishes on the plane Q. In such a case the resultant stress is the normal stress on the plane. It can be shown that for any point P in a member there exist three mutually perpendicular planes at the point P on which the shear stress vanishes. (For two-dimensional stress states, as is evident from Mohr's Circle analysis in Section 4.4.2 that there are two mutually perpendicular planes.)

In a general 3-D state of stress a cubic characteristic equation shown below is solved to obtain the three principal normal stresses:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \tag{4.47}$$

Where the first, second and third *stress invariants*, I_1, I_2 and I_3, respectively, are

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \text{constant} \tag{4.48a}$$

$$\begin{aligned} I_2 &= \begin{vmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \tau_{xz} \\ \tau_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \tau_{yz} \\ \tau_{yz} & \sigma_{zz} \end{vmatrix} = \\ &= \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}{}^2 - \tau_{yz}{}^2 - \tau_{zx}{}^2 \\ &= \text{constant} \end{aligned} \tag{4.48b}$$

$$\begin{aligned} I_3 &= \begin{vmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{vmatrix} \\ &= \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_{xx}\tau_{yz}{}^2 - \sigma_{yy}\tau_{zx}{}^2 - \sigma_{zz}\tau_{xy}{}^2 \\ &= \text{constant} \end{aligned} \tag{4.48c}$$

The three roots of the cubic equation are the three "principal stresses" at point P where all the Cartesian stress components are known. Generally, the three principal stresses are represented by σ_1, σ_2 and σ_3. Typically, these three principal stresses are algebraically ordered as $\sigma_1 \geq \sigma_2 \geq \sigma_3$, and implies that σ_1, has the largest algebraic value and σ_3 has the smallest algebraic value. Remember that in this ordering process, tensile stress are considered as positive and compression stresses are considered negative.

The magnitude and directions of σ_1, σ_2 and σ_3, for any given equilibrium system of forces applied to a body, are uniquely determined and are independent of the orientation of the Cartesian coordinate axes. Thus the coefficients in Equation (4.47) are called "*stress invariants*" (or constant) and must have the same magnitude for all orientations of the coordinate axes. Thus σ_1 the largest principal stress, is the maximum normal stress that can occur at any place passing through the point.

After determining the magnitudes of the principal stresses finding the directions of the principal stresses or the direction of planes containing the three stresses is more involved and not elaborated here. Typically, in a material strength based design the directions are not important.

It may be noted that the maximum shear stress is equal to ½($\sigma_1 - \sigma_3$) and the associated normal stress is ½($\sigma_1 + \sigma_3$). The other two extremum values for the shear stresses are ½($\sigma_2 - \sigma_3$) and ½($\sigma_1 - \sigma_2$) with the associated normal stresses of ½($\sigma_2 + \sigma_3$) and ½($\sigma_1 + \sigma_2$), respectively.

Example 4.9

The 1 in. diameter, L-shaped steel lever is loaded as shown in Figure E4.9. Determine the critical point where the stress is expected to be highest. Determine the Cartesian stress components and the principal stresses at that point.

> **C³**
>
> 4.7. In shear force and bending moment analysis where should be the origin of the coordinate axis?
> *(Answer: Section 4.7)*

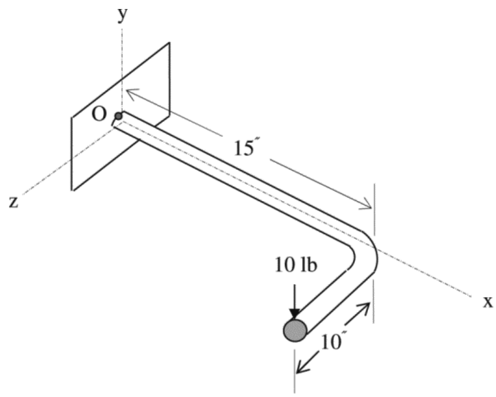


Figure E4.9.

Answer: The point marked O is the critical point as the bending stresses are highest here. This point is also subjected to torsion-induced shear stresses.

$$\sigma_{xx} = \frac{Mc}{I} = \frac{(10lb \times 14in)\frac{1.0}{2}in}{\frac{\pi(1.0in)^4}{64}} = 1426 \text{ lb/in}^2$$

$$\tau_{xz} = \frac{Tr}{J} = \frac{(10lb \times 15in)\frac{1.0}{2}in}{\pi\frac{1.0^4}{32}} = 764 \text{ lb/in}^2$$

All other stress components on a cubic element at point O are zero, that is

$\sigma_{yy} = \sigma_{zz} = \tau_{xy} = \tau_{yz} = 0$ (Note that $\tau_{xy} = \tau_{yx} = 0$ and $\tau_{yz} = \tau_{zy} = 0$)

The principal stresses are determined from the cubic equation (4.47)

$$\sigma^3 - \sigma^2(1426) + \sigma(-764)^2 = 0$$
$$\Rightarrow \sigma(\sigma^2 - 1426\sigma - 583696) = 0$$
$$\Rightarrow \sigma_1 = 1758 \text{ lb/in}^2$$
$$\sigma_2 = 0$$
$$\sigma_3 = -332 \text{ lb/in}^2$$

Note: σ_1, σ_2, σ_3 are ordered such that $\sigma_1 > \sigma_2 > \sigma_3$. Note also that the original bending stress found was <u>not</u> the maximum stress.

The maximum shear stress is given by:

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \frac{1}{2}(1758 + 332) = 1045 \ \ lb / in^2$$

Example 4.10

At a point in a machine component the stress state is given below:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & -3 \\ 1 & -3 & 4 \end{bmatrix}$$

All stresses are in *MPa*. Find the principal stresses.

Answer: From the given stress tensor, it is implied that $\sigma_{xx} = 1$, $\sigma_{yy} = -2$, $\sigma_{zz} = 4$, $\tau_{xy} = \tau_{yx} = 2$, $\tau_{zy} = \tau_{yz} = -3$, $\tau_{xz} = \tau_{zx} = 1$.

The three stress invariants are determined using equation (4.48), as

$I_1 = 1 - 2 + 4 = 3\ MPa$
$I_2 = (1 * -2) + (-2 * 4) + (4 * 1) - 2^2 - (-3)^2 - (1)^2 = -20\ MPa^2$
$I_3 = (1 * -2 * 4) + 2\ (2 * -3 * 1) -1\ (-3)^2 - (-2)(1)^2 - 4(2)^2 = -43\ MPa^3$

Using equation (4.47) we obtain the cubic equation

$$\sigma^3 - 3\sigma^2 - 20\sigma + 43 = 0$$

the three roots of the cubic equation are the three principal stresses, which when algebraically ordered ($\sigma_1 > \sigma_2 > \sigma_3$) are $\sigma_1 = 5.25\ MPa$, $\sigma_2 = 1.95\ MPa$, $\sigma_3 = -4.2\ MPa$.

4.5 Structural Design under Combined Loading

4.5.1. Yielding under Multi-Axial Stresses

Most engineering structures function in the elastic region for the majority of their life. However, the presence of stress concentrations and development of defects during service that can act as stress risers, or general degradation in the material for example due to corrosion, can cause the localized stress state to exceed the elastic limit of the material.

This state represents the beginning of *inelastic deformation* or *plastic deformation* or the initiation of yielding at that point(s). In a very severe case yielding may occur over the whole cross-section of a component. Generally, engineering designs are such that initiation of yielding will cause minimal loss of function or change in the steady state deformation of a statically loaded structure. However, initiation of yield can cause development of residual stresses in the structure (generally residual stresses are harmful if tensile), provide locations for crack initiation, provide locations for increased chemical attack, etc. Hence, if possible the structure should function in the elastic region over its whole life span.

4.5.2. Yield Criteria

The simplest state of stress is the uniaxial stress state. Such a stress condition is produced in a simple tension test experiment to obtain the uniaxial stress-strain curve of a material. This stress-strain curve provides information about the yield point of the material. Thus if an actual component, such as a tie-rod or a linkage in a four bar mechanism, is in pure tension then its failure is predictable at the yield point determined from a simple tension test.

If a structural component is subjected to biaxial or tri-axial state of stress, the prediction of failure (or yielding) is no longer as easy as the uniaxial loading case. In the multiaxial stress condition, we can no longer say that the material will yield when the largest normal stress reaches the yield point obtained from a uniaxial tension test, as the other normal stress components also influence yielding. Furthermore it is practically impossible to conduct experiments to obtain the yield condition for a whole range of stress combinations in all three possible orthogonal directions, and also consider factors such as stress concentration, temperature and environmental effects. To overcome these difficulties, designers have relied on developing theories that relate failure behavior in the

> **C³**
>
> 4.8. Are there standard tests for materials properties under combined loading?
> *(Answer: Section 4.7)*

multiaxial stress situation to the failure behavior in a simple tension test in the same mode through a selected quantity such as stress, strain, or energy. Thus the failure theories will predict failure to occur when the maximum value of the selected mechanical quantity in the multiaxial stress state becomes equal to or exceeds the value of the same quantity that produces failure in a uniaxial tension test using the same material.

In general the elastic limit or the *yield stress* is a function of the state of stress represented by six stress components for an isotropic material. The yield condition can generally be written as

$$f\left(\sigma_{xx},\sigma_{yy},\sigma_{zz},\tau_{xy},\tau_{yz},\tau_{zx},M_1,M_2,....\right)=0 \tag{4.49}$$

where M_1 and M_2 are materials constants. For isotropic materials the orientation of the principal stresses is immaterial, and the values of the three principal stresses are sufficient to describe the state of stress uniquely. The yield criteria therefore can be written as

$$f\left(\sigma_1,\sigma_2,\sigma_3,M_1,M_2,....\right)=0 \tag{4.50}$$

Based on this philosophy, several theories have been developed to predict the yield point when a component is subjected to multiaxial stresses, namely the maximum normal stress theory, the maximum shear stress theory and the distortion energy theory. In this chapter, we only discuss the maximum shear stress theory, which provides fairly accurate results for yielding in ductile materials and is relatively simple to use. As described earlier, a general multiaxial state of stress at any point can be fully described by three principle normal stresses and their directions. Hence all failure theories express the yielding criteria in terms of the principle normal stresses. It may be noted that there is only one non-zero principal stress in a uniaxial tension test situation.

4.5.3. Maximum Shear Stress Theory

This theory proposes that yielding will occur under a multiaxial stress state when the maximum shear stress becomes equal to or exceeds the maximum shear stress at the yield point in a uniaxial tension test using a specimen of the same material. The principal shearing stresses are

$$\begin{aligned}\tau_1 &= \pm\frac{1}{2}(\sigma_1 - \sigma_2)\\ \tau_2 &= \pm\frac{1}{2}(\sigma_1 - \sigma_3)\\ \tau_3 &= \pm\frac{1}{2}(\sigma_1 - \sigma_2)\end{aligned} \tag{4.51}$$

Also, for a uniaxial tension test, the only non-zero principal stress at the yield point is $\sigma_1 = S_y$ = the yield strength, and hence the principal shearing stress at the yield point is

$$\tau_{tension} = \frac{S_y}{2} \tag{4.52}$$

Thus according to the maximum shear stress yielding theory the yield conditions can be written as

$$\begin{aligned}|\sigma_2 - \sigma_3| &\geq S_y\\ |\sigma_2 - \sigma_3| &\geq S_y\\ |\sigma_1 - \sigma_2| &\geq S_y\end{aligned} \tag{4.53}$$

Failure by yielding occurs if any one of the above expressions is satisfied. These yield conditions are represented graphically for a three-dimensional stress field in Figure 4.22. The yield surface is a hexagonal cylinder whose axis makes equal angles with the three principal stress axes. And as with all yield theories, stress states that lie within the hexagonal cylinder (or the yield surface) does not result in yielding, while stress states lying outside the cylinder result in yielding. A stress state lying exactly on the yield surface signifies that the material is ready to yield.

C³

Stress or strength, what is the difference?

(Answer: Section 4.7)

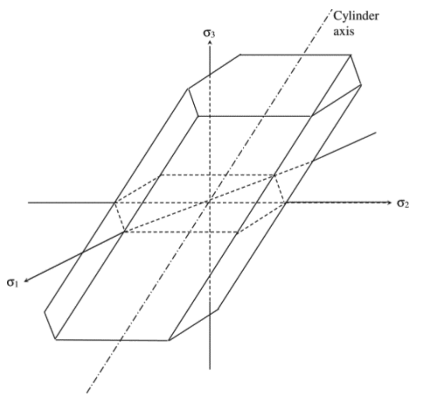


Figure 4.22. Graphical representation of the maximum shear stress theory for a general three-dimensional stress state

For a *biaxial stress field,* i.e., any one principal stress equals zero, say $\sigma_1 \neq 0$, $\sigma_2 \neq 0$, $\sigma_3 = 0$, the graphical representation is shown in Figure 4.23 assuming that the yield strength in tension is equal to the yield strength in compression, which is approximately true for most structural metals.

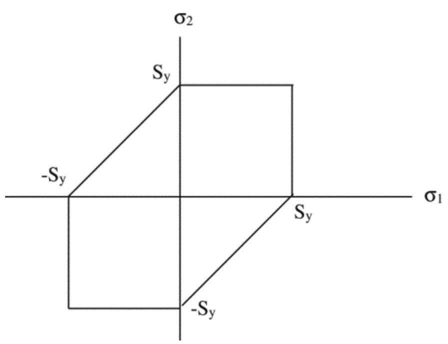


Figure 4.23. Graphical representation of the maximum shear stress theory for a biaxial stress field.

Experimental results have shown that the maximum shear stress theory predicts yielding in ductile materials with reasonable accuracy. This theory also predicts the experimentally observed behavior of ductile materials under hydrostatic stress state. If all principal stresses are equal, the shear stresses τ_1, τ_2 and τ_3 are all equal to zero and hence yielding will never initiate regardless of the magnitude of the hydrostatic stress state. In the graphical representation, the hydrostatic stress state always lies on the axis of the hexagonal cylinder and hence within the yield surface, which implies yielding will never occur.

Example 4.11

A point in a structural component has a stress state given by σ_{xx} = 50, σ_{yy} = 70 and τ_{xy} = 200 *MPa*. The material of the component is ductile and has yield strength of 300 *MPa*. Will the material yield at the point under consideration according to the maximum shear stress theory?

Answer: The given stress state is a biaxial state of stress. Hence the two in-plane principal normal stresses can be determined using equations (4.37 a,b) as:

$$\sigma_1, \sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

$$\Rightarrow \sigma_1 = 260MPa, \quad \sigma_2 = -140MPa$$

The third principal stress σ_3 = 0. (Note that according to convention, the principal stress σ_1, σ_2, σ_3 should be ordered as $\sigma_1 > \sigma_2 > \sigma_3$. However, we have *not* followed the convention *only* for this example. You may follow the convention and re-number the principal stresses, and draw the yield plot axes accordingly.)

The principal stress state (σ_1, σ_2) ≡ (260, –140) can be plotted on the biaxial yield plot to determine the yield condition. The stress state at the point lies outside the yield locus as shown in Figure E4.11. Hence yielding occurs.

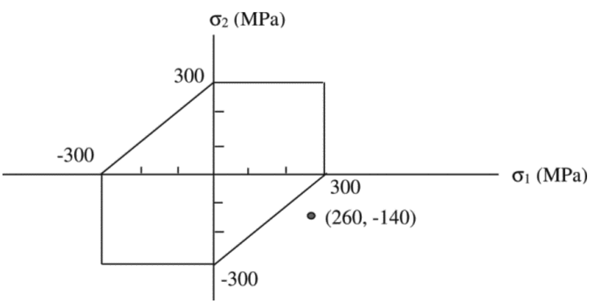


Figure E4.11.

4.6 Design III

RP4.1: Shown in Figure RP4.1 is a beam in which the shear stress τ_{xy} on the top surface is zero. If the beam is cut along a plane x = constant, what is the shear stress on the new surface?

Consider a small element as shown below

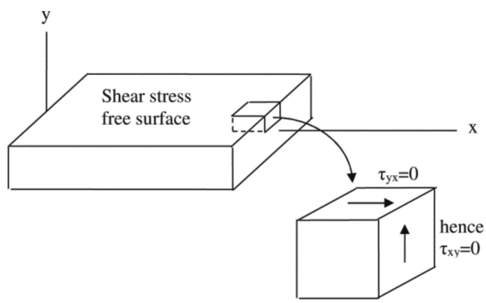


Figure RP4.1.

The top surface is given to be shear stress free, hence $\tau_{yx} = 0$. Thus to maintain equilibrium the cross shear τ_{xy} must be zero. Similarly if a section is taken at plane A, the cross shear on the newly created surface will be zero.

RP4.2

A thin rectangular rubber sheet is enclosed between two thick steel plates and the rubber sheet is subjected to a compressive stress of σ_{xx} and σ_{yy} in the x- and y-directions, respectively. Determine the strains in the x- and y-directions and the stress along the z-direction (thickness direction) of the rubber sheet (Figure RP4.2).

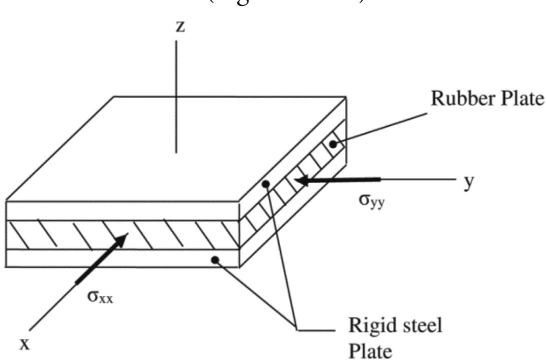


Figure RP4.2.

From Equations (4.29)

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(-\sigma_{xx} - \sigma_{yy})\right] = 0$$

$$\Rightarrow \quad \sigma_{zz} = -\nu(\sigma_{xx} + \sigma_{yy})$$

and

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(-\sigma_{yy} + \sigma_{zz})\right]$$
$$= \frac{1}{E}\left[\sigma_{xx}(1+\nu^2) + \nu(1+\nu)\sigma_{yy}\right]$$

(which is obtained by substituting for σ_{zz}.)
Similarly,

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(-\sigma_{xx} + \sigma_{zz})\right]$$
$$= \frac{1}{E}\left[\sigma_{yy}(1+\nu^2) + \nu(1+\nu)\sigma_{xx}\right]$$

RP4.3

A closed cylindrical pressure vessel is fabricated from steel sheets that are welded along a helix that forms an angle of 60° with the transverse plane. The outer diameter is 1 m and the wall thickness is 0.02 m. For an internal pressure of 1.25 *MPa*, determine the stress in directions perpendicular and parallel to the helical weld (Figure RP4.3).

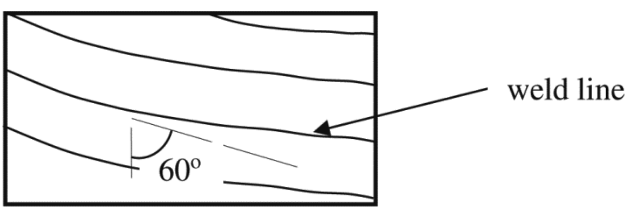


Figure RP4.3.

Ratio of diameter to wall thickness: 0.5/0.02 = 25 > 10 $\Rightarrow$ thin-walled vessel

$$\sigma_1 = \frac{pr}{t} = 1.25(0.5)/(0.02) = 31.25\ MPa,\ \sigma_2 = \frac{\sigma_1}{2} = 15.62\ MPa$$

$$\sigma_{ave} = \frac{1}{2}(\sigma_1 + \sigma_2) = 23.44 MPa$$

$$R = \frac{1}{2}(\sigma_1 - \sigma_2) = 7.82 MPa \text{ (Mohr's circle radius)}$$

The normal stress to the weld is $\sigma_w = \sigma_{ave} - R\cos 60° = 19.53\ MPa$

The shear stress on the weld is $\tau_w = R\sin 60° = 6.77\ MPa$

RP4-4

A cylindrical tube made of 2024-T4 aluminum has a diameter of 50 mm and wall thickness of 3 mm. An axial tensile load of 60 kN and a torque of 0.7 kN-m are applied. Will the tube yield? If not, how much can the tensile load or the torque can be increased before yielding occurs?

2024-T4 aluminum yield strength= 330 *MPa*

Torque: T = 0.7 kN-m

Axial tensile load: 60 kN

Polar second moment: $J = \pi(25^4 - 22^4)/2 = 0.25(10^6)\ mm^4 = 0.25(10^{-6})\ m^4$

The shearing stress due to torsion T is:

$$\tau = \frac{Td}{2J} = \frac{0.7(50)}{2(0.25)(10^{-6})} = 70(10^6) N/m^2$$

Normal stress due to tension is:

$$\sigma = 60(10^3)/\pi(25^2 - 22^2)(10^{-6}) = 0.14(10^9)\ N/m^2 = 140(10^6)\ N/m^2$$

Therefore:

$$\sigma_{1,3} = \frac{1}{2}\sigma \pm \frac{1}{2}\sqrt{\left(\sigma^2 + 4\tau^2\right)} = 70 MPa \pm 99 MPa\ ;\ \sigma_2 = 0$$

According to the maximum shear stress theory yielding occurs if

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) \geq \frac{S_y}{2}$$

$\Rightarrow$ *LHS* = 99 MPa, which is less than *RHS* = 165 MPa. Hence no yielding occurs.

RP4.5

A thin-walled circular tube is made of AISI1020 steel. It is subjected to a torque of 6 kN-m and a pure bending moment of 4.5 kN-m. If the diameter of the tube is 50 mm, what must be the thickness so that the factor of safety against yielding is 1.25?

AISI 1020 steel yield strength= 225 MPa

Factor of safety 1.25

Torque: T = 6 kN-m

Bending moment: 4.5 kN-m

For thin-walled tubes, the shearing stress due to torsion T is:

$$\tau = \frac{T}{2At} = \frac{6(10^3)}{2\pi(25)^2(10^{-6})t} = \frac{1.5(10^6)}{t} N/m^2 = \frac{1.5}{t} MPa$$

Normal stress due to bending is:

$$\sigma = \frac{Mc}{I} = \frac{M}{2\pi d^2 t} = \frac{(4.5)(10^3)}{2\pi(50^2)(10^{-6})t} = \frac{0.3}{t} MPa$$

Therefore:

$$\sigma_{1,3} = \frac{1}{2}\sigma \pm \frac{1}{2}\sqrt{\left(\sigma^2 + 4\tau^2\right)} = \frac{0.15}{t} \pm \frac{1}{2}\sqrt{\left(\frac{0.09}{t^2} + \frac{9}{t^2}\right)} MPa \text{ ; } \sigma_2 = 0$$

From the maximum shear stress theory

$$\tau_{\max} = \frac{1}{2}(1.25)(\sigma_1 - \sigma_3) \geq \frac{S_y}{2}$$

$\Rightarrow t$ = 16.6 mm

RP4.6

A cylindrical pressure vessel has an inner diameter of 240 mm and a wall thickness of 10 mm. The end caps are spherical and of thickness 10 mm. If the internal pressure is 2.4 *MPa*, find (a) the normal stress and the maximum shear stress in the wall of the spherical end cap, (b) the normal stress and the maximum shear stress in the cylindrical wall.

Check diameter to thickness ratio, 120/10 = 12 > 10 $\Rightarrow$ thin-walled vessel

(a) Spherical cap:

$$\sigma_1 = \sigma_2 = \frac{pr}{2t} = 2.4(120)/2(10) = 14.4\ MPa$$

Note that $\sigma_3 = 0$.

$$\tau_{\max} = \frac{1}{2}\sigma_1 = 7.2 MPa$$

(b) Cylindrical body of the vessel:

$$\sigma_1 = \frac{pr}{t} = 2.4(120)/(10) = 28.8\ MPa,\ \sigma_2 = \frac{\sigma_1}{2} = 14.4\ MPa$$

$$\tau_{\max} = \frac{1}{2}\sigma_1 = 14.4 MPa$$

4.7 C³ Clarified

C³ 4.1. Stress is often computed from the expression stress = force/area and is treated as a scalar. Is that appropriate?

Answer: It is appropriate only if the force is aligned (or parallel) with the direction of the outward normal to the area on which the force acts.

C³ 4.2. Does the choice of the coordinate system affect the state of stress at a point in a body under a given external load?

Answer: The Cartesian stress components change with the orientation of the coordinate system, however, the principal stresses are independent of the choice and orientation of the coordinate system.

C³ 4.3. Torque, moment and couple -- what is the difference?

Answer: These terms are often used synonymously albeit inaccurately, but the context of the problem may help arrive at the correct result. The magnitude of both torque and moment are obtained by multiplying the force with a distance between the force and the point (or an axis) about which the quantity is desired. However, strictly the moment and torque are vector product of distance vector $\boldsymbol{r}$ and force vector **F**. Thus for example in the case of a rod, if the resultant of the vector cross product $r \times \mathbf{F}$ is in a plane perpendicular to the axis of the rod it is a moment (that causes bending), and if the resultant is in the plane containing the axis of the rod it is a torque (that causing twisting). A couple is a pair of self-equilibrating forces separated by a distance, which results in force equilibrium but not moment equilibrium.

C^3 4.4. Do you need to know the shear stresses on both the *x*- and *y*-faces of the element to draw the Mohr's circle?

Answer: The Mohr's circle can be drawn by knowing the center given by the σ-axis coordinate $\dfrac{\sigma_{xx}+\sigma_{yy}}{2}$ and only one point on the circle, which determines the radius as the distance between the center and the point representing the stress on one face of the element.

C^3 4.5. What is the effect on principal stress determination if one shows the shear stress on an element as clockwise instead of the correct direction of anti-clockwise?

Answer: The principal stress magnitudes will not be affected, however, the directions would be in error by 45°.

C^3 4.6. Is Mohr's circle useful for 3D stress analysis?

Answer: Mohr's circle is primarily used for two-Dimensional or plane stress analysis, though it can also be used for 3-D stress analysis.

C^3 4.7. In shear force and bending moment analysis where should be the origin of the coordinate axis?

Answer: The best option is to use an origin that minimizes the number of unknown reactions in the two static equilibrium equations, namely the force equilibrium and moment equilibrium equations.

C^3 4.8. Are there standard tests for materials properties under combined loading?

Answer: Material properties under combined loading are not generally tested. However, tests are conducted under very specific combined load conditions for a particular application or to validate a certain analytical/numerical solution.

C^3 4.9. Stress or strength, what is the difference?

Answer: Strength is a property of the material. Stress depends on the magnitude and direction of force on an area, and is also dependent on stress concentrations.

Key Points to Remember:

- The general state of stress at any point in a stressed homogeneous, isotropic material is given by three normal stress components and six shear stress components. Three relationships exist between the six shear stress components and hence only three shear stress components need to be independently known.
- The general state of strain at any point in a deformed homogeneous, isotropic material due to applied loads is given by three normal strain components and six shear strain components. Three relationships exist between the six shear strain components and hence only three shear strain components need to be independently known (Table 4.2).
- The Cartesian components of stress and strain depend up on the choice of the coordinate system, though this choice of the coordinate system does not affect the physical phenomenon occurring at the point.
- The three principal stresses and three principal strains at a point are independent of the choice of the coordinate system, or in other words do not vary with the orientation of the coordinate system.
- The Mohr's circle is an excellent graphical method for visualizing stresses (and strains) primarily in two dimensional loading conditions and stress states.

Table 4.2. Comparison of some key relationships between axial, torsional, and flexural structures.

	Axial	Torsional	Flexural
Stress	$\sigma_{xx} = \frac{P}{A}$	$\tau_{x\theta} = \frac{Mr}{J}$	$\sigma_{xx} = \frac{My}{I}$
Deformation	$\varepsilon_{xx} = du/dx$	$\gamma_{x\theta} = r(d\phi/dx)$	$\varepsilon_x = -\frac{y}{r_0} \approx -y\frac{d^2u}{dx^2} = -\frac{d}{dx}\left(y\frac{du}{dx}\right)$
Constitution	$\sigma_{xx} = E\,\varepsilon_{xx}$	$T_{x\theta} = G\gamma_{x\theta}$	$\sigma_{xx} = E\varepsilon_{xx}$

APPENDIX 1

System of Units

Table A1.1. A taxonomy of systems of units.

	"The Good"	*"The Bad"*	*"The Ugly"*	*"The Dead"*
System	SI	U.S. Customary or British Gravitational	English Engineering	cgs (dyne, cm, gm, s) fps (poundal, ft, lb_m, s)
Basis	Absolute	Gravitational	?	
Base (primitive, fundamental) mechanics dimensions and units	mass kg length m time s	force lb length ft time s	force lb_f mass lb_m length ft time s	
Derived dimension and unit (mechanics only)	force **N**	mass slug	----	
Newton's 2nd law $\mathbf{F} \sim d/dt\,(mv)$	$\mathbf{F} = (1)\ ma$ Proportionality constant = 1	$\mathbf{F} = (1)\ ma$ Actually: $m = (1)\ \mathbf{F}/a$	$\mathbf{F} = (1/g_c)\ ma$ Since force and mass are defined, prop. const. $\neq 1$	
	$1\ \mathbf{N} = (1\ \text{kg})(1\ \text{m/s}^2)$	$1\ \text{lb} = (1\ \text{slug})(1\ \text{ft/s}^2)$ Actually: $1\ \text{slug} = (1\ \text{lb})/(1\ \text{ft/s}^2)$	$1\ lb_f = \dfrac{(1\ lb_m)(32.2\ \text{ft/s}^2)}{32.2\ lb_m \text{ft}/lb_f \text{s}^2}$	$1\ \text{dyne} = (1\ \text{gm})(1\ \text{cm/s}^2)$ $1\ \text{poundal} = (1\ lb_m)(1\ \text{ft/s}^2)$
Magnitudes	Decades	--	--	

Notes:

- The SI system is the system of units in use by most of the rest of the world outside of the USA, and is the standard for nearly all current engineering and scientific publications. The SI system is an "absolute" system since mass is taken as a primitive or fundamental dimension (as compared to force or weight); force is a derived dimension. Force and mass have separate and distinct units. The constant of proportionality in Newton's law is nondimensional and of unit value. SI also uses order of magnitude descriptors (in decades).
- All other systems of units are less desirable in practice, but a working knowledge of them is required in the USA.
- The U.S. Customary system is a "gravitational" system ("relative" system) since it takes force (weight) as a primitive dimension (but weight obviously depends on the local gravitational acceleration). It does however have separate and distinct units for force and mass, and a constant of proportionality in Newton's law that is nondimensional and of unit value. Limited order of magnitude descriptors are used (e.g., ksi).
- The English Engineering system is the least desirable of commonly used systems of units. Both force and mass are taken as primitives. Hence, the constant of proportionality in Newton's law is neither nondimensional nor of unit value ($1/g_c$). Force and mass do not have separate and distinct units. Limited order of magnitude descriptors are used (e.g., ksi).
- The cgs and fps are rarely used at the present time, and that is for the best.

APPENDIX 2

Free Body Diagrams

The *free body diagram* (FBD) is a tool for formulating mathematical models via Newton-Euler mechanics. It provides for bookkeeping of all of the externally applied forces and moments on the *system of interest* (SOI), as well as internal forces and moments at the free body boundaries.

There are three simple steps for developing FBDs:

1. Define the SOI (i.e., a particle, body, portion of a body, etc.).
2. The definition must include a coordinate system. The SOI must be considered in its *displaced* configuration. For convenience in writing the equations of motion, displace the SOI in the positive coordinate direction(s).
3. Completely isolate the free body.
   Isolate the free body from the SOI, and show on the FBD all forces and moments:
   - applied by external agents (e.g., gravity, concentrated forces, etc.
   - occuring internally as reactions at the boundaries when the SOI is isolated
3. Show on the FBD the relevant geometry and any other information required to do a complete *accounting*.

An example follows in Figure A2.1:

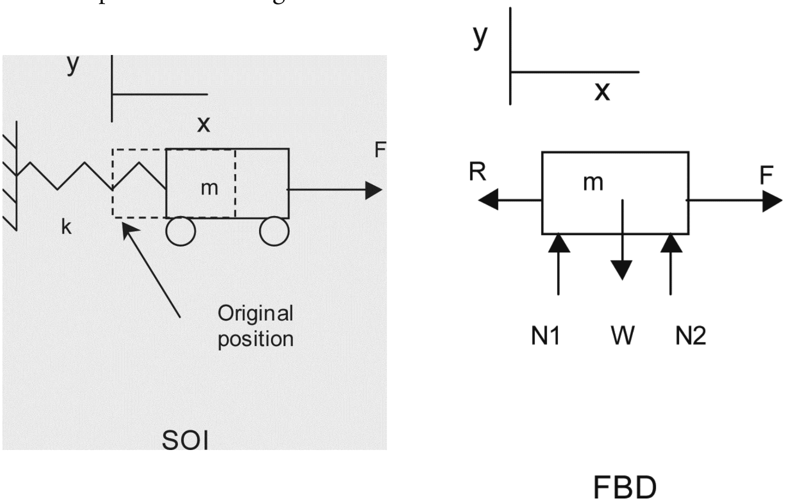


Figure A2.1. Simple free body diagram.

APPENDIX 3

Centroid and Second Moment of Area

Mechanics of materials texts do an adequate job of illustrating how to calculate the centroid and second moment of area of plane sections. Needing not to cover that ground, this brief appendix seeks to clarify common confusion over these very important topics.

Centroid

- The centroid is the "geometric center" of a plane section. It is specified as a coordinate pair, e.g., $(\overline{x}, \overline{y})$.
- The centroid uses the *first moment of area* to calculate a "weighted average of areas" that represents the geometric center. Each increment of area dA is (linearly) weighted by its distance from the reference axis—the further away, the greater the weight.
- If the plane section is smooth and continuous, then the centroidal coordinates relative to rectangular Cartesian coordinates x, y are given by

$$\overline{x} = \frac{\int x \, \mathrm{d}A}{\int \mathrm{d}A}, \; \overline{y} = \frac{\int y \, \mathrm{d}A}{\int \mathrm{d}A}$$

- If the plane section is not smooth and continuous, but consists of N discontinuous elements, then the centroidal coordinates relative to rectangular Cartesian coordinates x, y are given by

$$\overline{x} = \frac{\sum_{i=1}^{N} \overline{x}_i A_i}{\sum_{i=1}^{N} A_i}, \; \overline{y} = \frac{\sum_{i=1}^{N} \overline{y}_i A_i}{\sum_{i=1}^{N} A_i}$$

where $(\overline{x}_i, \overline{y}_i)$ and A_i are the centroidal coordinates and area of the i^{th} element, respectively. For example, an angle section would have $N = 2$ discontinuous elements.

- If the material is homogeneous and the mass of the material is uniformly distributed, then the centroid and *center of mass* coincide (in other cases they may not).

Second Moment of Area

- The *second moment of area* (SMoA) is commonly, but incorrectly, called the "moment of inertia" or the "area moment of inertia". *Inertia* is a <u>mass</u> property describing the "resistance" to <u>acceleration</u>, as correctly used in the dynamical *mass moment of inertia*. However, there is no mass nor dynamics in our mechanics of materials "resistance" (to torsion or flexure) and any reference to inertia should be avoided.
- If the plane section is smooth and continuous, then the flexural SMoA I_{xx} and I_{yy} relative to rectangular Cartesian coordinates x, y are given by

$$I_{xx} = \int y^2 \mathrm{d}A,\ I_{yy} = \int x^2 \mathrm{d}A$$

- The torsional SMoA I_{zz} (also J_{zz}), sometimes called the polar SMoA, is found as $I_{zz} = I_{xx} + I_{yy}$ or

$$I_{zz} = \int (x^2 + y^2)\mathrm{d}A$$

- The SMoA referenced to a centroidal coordinate axis $(\overline{I})$ can be referenced to another parallel coordinate axis by following the so-called "Parallel Axis Theorem". For example, consider the SMoA referenced to an x'-axis that is parallel to the x-axis passing through the section centroid:

$$I_{x'x'} = \overline{I}_{xx} + A * \mathrm{d}^2$$

where d is the perpendicular distance between the x'-axis and the x-axis.

- If the plane section is not smooth and continuous, but consists of *N* discontinuous elements, then the SMoA of each element can be summed to give the composite SMoA if each element SMoA is referenced to the composite coordinate axis, accomplished by using the Parallel Axis Theorem. For example,

$$I_{x'x'} = \sum_{i=1}^{N}\left[\overline{I}_{i,xx} + A_i\left(\overline{y}_i - \overline{y}\right)^2\right]$$

where $\overline{I}_{i,xx}$, $\overline{y}_i$, and A_i refer to the i^{th} element and $\overline{y}$ refers to the composite section.

APPENDIX 4

Sign Conventions in Mechanics of Materials

One of the leading sources of confusion in Mechanics of Materials is that of sign convention. Perhaps every MoM textbook covers this subject, unfortunately often in confusing and arbitrary ways. For example, one popular MoM text, now in its 8th edition, states the following:

> "Let us now consider the sign conventions for shear forces and bending moments. It is customary to assume that shear forces and bending moments are positive when they act in the directions shown in the figure. Note that the shear force tends to rotate the material clockwise and the bending moment tends to compress the upper part of the beam and elongate the lower part. Also, in this instance, the shear force acts downward and the bending moment acts counterclockwise."

Definitions like "rotate clockwise" and "compress the upper part of the beam" are arbitrary definitions that have no rational or physical basis, are confusing and an unnecessary burden to learning.

First of all, let's be clear on what is important and what is the goal. The sign of the shear force is not important. The sign of the shear force will not play a significant role in structural design. The sign of the moment is important in so far as it informs us where the beam is in tension and where it is in compression. Failure in tension is much more likely than it is in compression.

Fortunately, there is no need for any "additional" sign conventions. We have everything we need with the two fundamental mechanics sign conventions: coordinate/right-hand rule and stress.

Coordinate Sign Convention

A vector parallel to a coordinate axis is considered positively directed ("positive") if it points in the direction of increasing coordinate values. A vector parallel to a coordinate axis is considered negatively directed ("negative") if it points in the direction of decreasing coordinate values.

Right-Hand Rule

Coordinate systems are assumed to be *right-handed*. In the case of the rectangular Cartesian coordinate system (Figure A4.1), this means that the unit vectors **i**, **j**, and **k** are related by

$$\mathbf{k} = \mathbf{i} \times \mathbf{j}$$

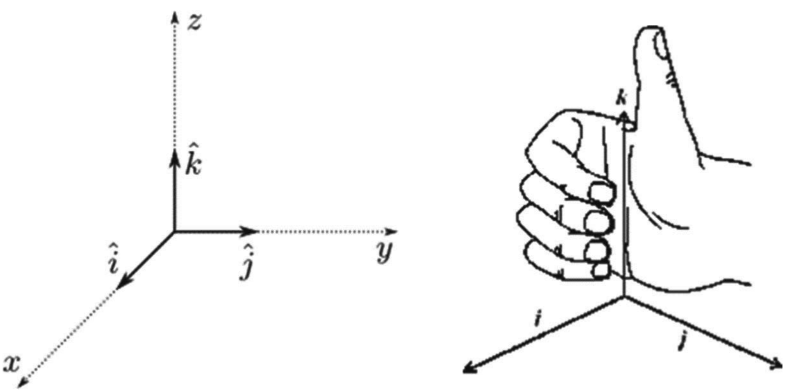


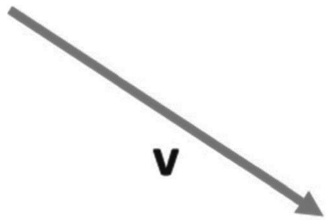
Figure A4.1. (a) unit vectors defined; (b) right-hand rule.

Forces and Moments

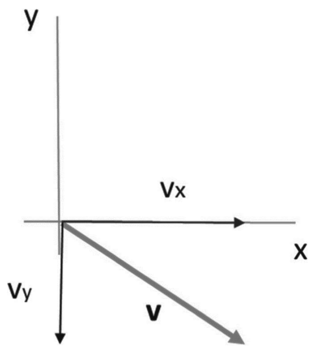
It is important to keep the following in mind: forces and moments are <u>vectors.</u> We treat them as scalars where allowable for convenience, but they are nonetheless vectors! Vectors in general do not have a "sign" associated with them. However, the *components* of a vector, which exist only in reference to a coordinate system, by definition have a sign associated with them. Only in the case where the vector can be represented by a single component, say $\mathbf{v} = 10\ \mathbf{j}$, can the vector have an associated sign (in this case **v** is in the positive y-direction). Consider the following: is the vector **v** in Figure A4.2a positive or negative?

The vector **v** can now be represented by components that are positive or negative, i.e., $\mathbf{v} = \mathbf{v}_x\,\mathbf{i} - \mathbf{v}_y\,j$, by way of the basic coordinate sign convention. The *x*-component of **v** is positively directed in the *x-y* coordinate system and the *y*-component negatively directed. However, use of the terms "positive" or "negative" to describe the vector *v* itself has no meaning (Figure A4.2b).

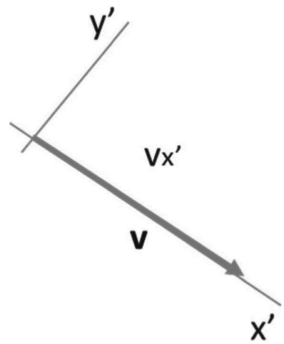
Now we could also reference the vector components to another coordinate system, say the *x′-y′* system (Figure A4.2c):



a. General vector.



b. Vector components tied to a coordinate system.



c. Vector v referenced to a different coordinate system.

Figure A4.2.

Here $\mathbf{v} = \mathbf{v}_{x'}\,\mathbf{i}$ can be described as positive.

Areas

Areas can also be treated as vector quantities. Areas have both magnitude and direction. The *direction* of an area (or surface) is given by the direction of the *outward seeking normal n* to the surface forming the area. (see Figure A4.3).

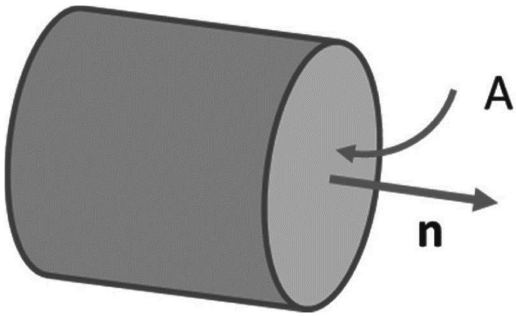


Figure A4.3. Outward seeking normal vector n.

We can now express A as $\boldsymbol{A} = A\boldsymbol{n}$, where A is the magnitude of the surface area and $\mathbf{n}$ is the outwardly-directed unit vector orthogonal to the surface.

If $\mathbf{n}$ aligns with the positive direction of a coordinate axis, the surface A is called a "positive surface" or "positive face". If $\mathbf{n}$ aligns with the negative direction of a coordinate axis, the surface A is called a "negative surface" or "negative face".

Stress Sign Convention

In addition to the usual sign convention for coordinates, we also have a sign convention for stress. The stress sign convention arises due to the critical dependence of the tensile or compressive nature of stress on the failure of a material. Tensile stresses are defined as positive stresses and compressive stresses are defined as negative stresses.

Recall our definition of normal stress

$$\sigma = \boldsymbol{T} \cdot \boldsymbol{n}$$

It is important to note that $\boldsymbol{\sigma}$ can be positive in two ways -- when $\mathbf{T}$ and $\mathbf{n}$ are both positive or are both negative. Same for the shear stress:

$$\tau = \boldsymbol{T} \cdot \boldsymbol{t}$$

Recall also the "stress cube", a 2-D view of which is shown in Figure A4.4:

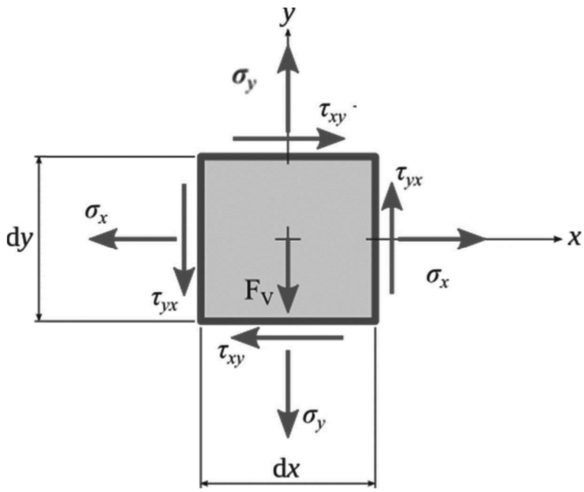


Figure A4.4. 2-D view of the stress cube. All forces, moments, and stresses are shown positive.

The stresses shown are all positive. Why? (Remember: positive * positive or negative * negative!)

Forces and Moments in Beam Bending

So what does this tell us about a sign convention for forces and moments in beam bending? Do we need a third sign convention, like positive moment if "holding water" or "sagging" (see Figure A4.5)?

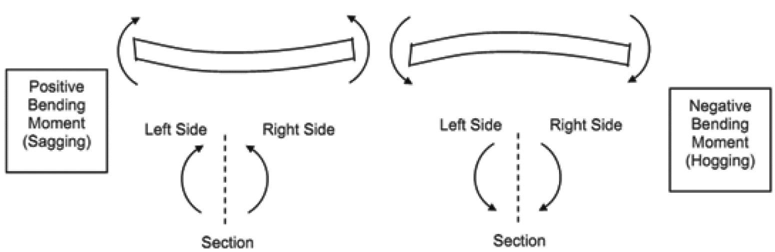


Figure A4.5.

What's wrong with this picture? A coordinate system—the most fundamental thing—is missing! (See Appendix 2 for FBD requirements.) Let's add one (Figure A4.6):

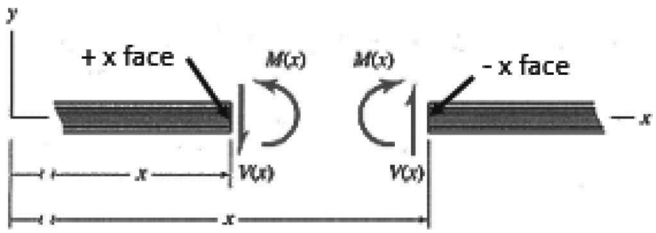


Figure A4.6.

Looking at the left-hand cut, we see that on the $+x$ face the shear force is negative and the moment is positive. Similarly, on the $-x$ face the shear is negative and the moment positive, which is internally consistent. (This is, by the way, just Newton's 3rd Law. That is, for every action there is an equal and opposite reaction.)

So we use the coordinate sign convention, along with the stress sign convention at the structure scale, to determine the sign of any internal shear force and any moment, internally or externally derived. We don't need any other convention. Given our y-coordinate is "up" and our moment—stress relationship is

$$\sigma_{xx} = -\frac{My}{I}$$

then a positive moment gives a negative (compressive) stress above ($y > 0$) the neutral axis and positive (tensile) stress below ($y < 0$) the neutral axis.

One arbitrary choice for a beam bending sign convention is the following (Figure A4.7):

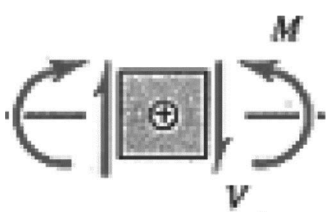


Figure A4.7. Arbitrary beam bending sign convention.

This is problematic for several reasons:

(i) The element seems to not be in rotational equilibrium since the shear forces provide a net moment on the slice;

(ii) Herein lies confusion: the internal moment **M** has been clearly defined as a positive moment if it is oriented as shown above, yet **M** can show up as a negative moment in the moment equilibrium equation! The same confusion applies to the shear force **V**.

Example A4.1. Consider a simply supported beam of length L that carries a concentrated force **P** as shown in Figure EA4.1a.

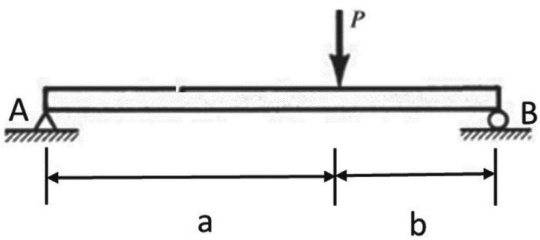


Figure EA4.1a.

We first determine expressions for the reaction forces R_A and R_B using $\sum \mathbf{F} = \mathbf{0}$ and $\sum \mathbf{M} = \mathbf{0}$ globally (the student should verify):

$$R_A = Pb/L \quad R_B = Pa/L$$

Then we investigate the internal (local) response by sectioning the beam (at $x = c$), creating two Free Bodies, a left FBD and a right FBD.

I. By additional beam bending sign convention (Figure A4.7)

We now include on our sectioned beam the internal shear and moment reactions, shown in their positive sense according to the above definition (Figure EA4.1b):

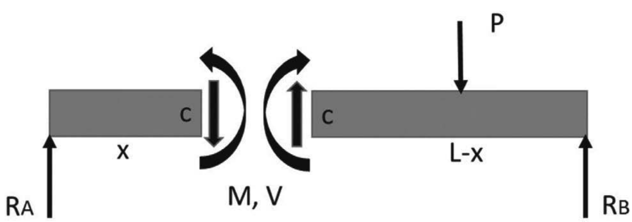


Figure EA4.1b.

We choose arbitrarily (since no coordinate system definition is given) summation directions that are consistent with the positive definition of the right section FBD shear and moment, and correctly expect these to apply to the left section FBD as well:

$$+\uparrow \sum F = 0 : R_A - V = 0 \;\; => \underline{V = Pb/L}$$

$$+\circlearrowright \sum M = 0 : R_A x - M = 0 \;\; => \underline{M = Pbx/L}$$

Herein lies the confusion: the internal moment **M** has been clearly defined as a <u>positive</u> moment if it is oriented as shown on the FBD above, yet **M** shows up as <u>negative</u> moment in the moment equilibrium equation! The same confusion applies to the shear force **V**.

II. By fundamental sign conventions only (*y*-axis up)

We again include on our sectioned beam (now correct FBDs) the internal shear and moment reactions, shown in their positive sense according to the fundamental coordinate and stress sign conventions (Figure EA4.1c):

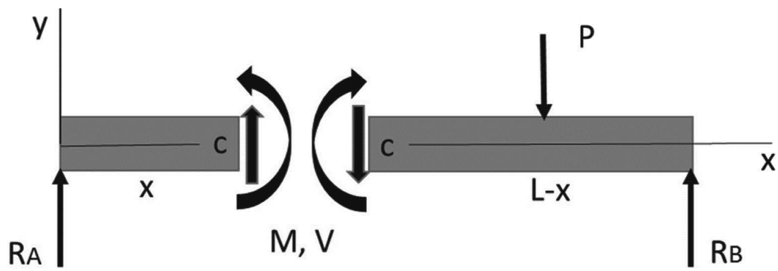


Figure EA4.1c.

$$+\uparrow \sum F = 0 : R_A + V = 0 \;\; => \underline{V = -Pb/L}$$

$$+\circlearrowleft \sum M = 0 : -R_A x + M = 0 \;\; => \underline{M = Pbx/L}$$

The minus sign on V indicates only that our original choice of direction was wrong, obvious now when we look at it.

III. By fundamental sign conventions only (y-axis down in Figure EA4.1d)

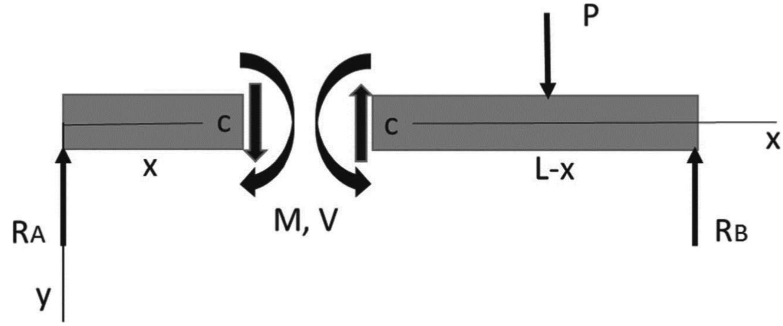


Figure EA4.1d.

$$\sum F = 0 : -R_A + V = 0 \Rightarrow \underline{V = Pb/L}$$

$$\sum M = 0 : R_A x + M = 0 \Rightarrow M = -\underline{Pbx/L}$$

The minus sign on M indicates only that our original choice of direction was wrong, obvious now when we look at it.

The correct solution is obtained in II and III without introducing an arbitrary and unnecessary sign convention.

APPENDIX 5

Vectors and Tensors

A vector is a mathematical object that can be used to represent physical objects that have both magnitude and direction. Representing physical objects as vectors can greatly simplify analysis.

For example, in mechanics we use *position vectors* to describe where something is in the space under consideration, or how its position is changing at that moment in time. The position of an object in space is its distance from a reference point. Distance must necessarily include both magnitude ("three meters") and direction ("northwest").

Any vector may be referred to a *basis*. A common basis is a rectangular Cartesian coordinate system. A vector is referred to a basis by being equivalent to a sum of components that lie along the coordinate directions. The coordinate directions are represented by *basis vectors*. Basis vectors are *unit vectors* (i.e., vectors that have magnitude equal to one) that point in the coordinate directions; a set of basis vectors for a rectangular Cartesian coordinate system could be written as $\left(\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}\right)$ or $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$. For example, the general position vector is written as $\boldsymbol{r} = x\hat{\boldsymbol{e}}_1 + y\hat{\boldsymbol{e}}_2 + z\hat{\boldsymbol{e}}_3$ (see Figure A5.1).

(Note: in text, vector quantities are typically shown in bold font; unit vectors are indicated by ^. However, for simplicity in what follows, we will drop the ^ symbol on the basis vectors.)

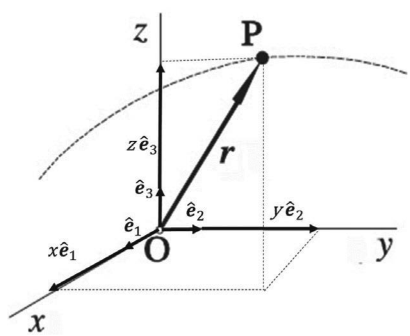


Figure A5.1.

Vector Algebra Operations

Addition and Subtraction

If **a** and **b** are vectors, then the sum **c= a +b** is also a vector (see Figure A5.2a). The two vectors can also be subtracted from one another to give another vector **d= a – b**.

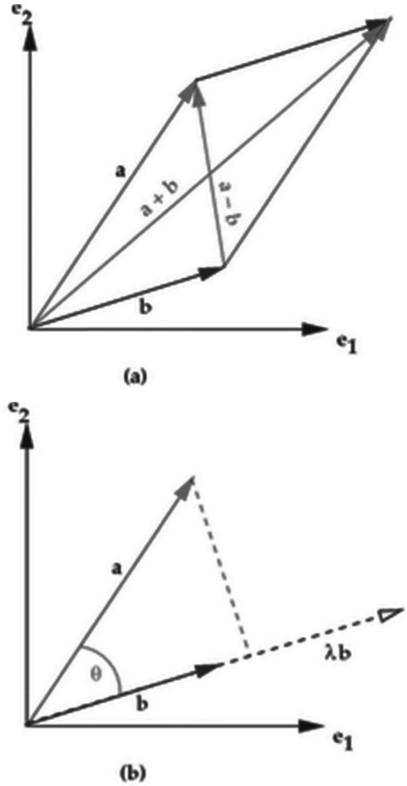


Figure A5.2. Vector operations.

Multiplication by a Scalar

Multiplication of a vector **b** by a scalar λ has the effect of stretching or shrinking the vector (see Figure A5.2b).

You can form a unit vector $\hat{b}$ that is parallel to **b** by dividing by the length of the vector $|\mathbf{b}|$. Thus,

$$\hat{b} = \frac{b}{|b|}$$

Scalar Product of Two Vectors

The *scalar* product or *inner* product or *dot* product of two vectors is defined as

$$a \cdot b = |a||b|\cos(\theta)$$

where θ is the angle between the two vectors (see Figure A5.2b). If **a** and **b** are perpendicular to each other, $\theta = \pi / 2$ and $\cos(\theta) = 0$. Therefore, $a \cdot b = 0$. The dot product therefore has the geometric interpretation as the length of the projection of **a** onto the unit vector $\hat{b}$ when the two vectors are placed so that they start from the same point.

The *direction cosines* of a vector are the cosines of the angles between the vector and the three coordinate axes. The direction cosines for the vector **r** shown in Figure A5.1 are given by:

$$\cos\theta_i = \frac{r}{|r|} \cdot e_i$$

The scalar product leads to a scalar quantity and can also be written in component form (with respect to a given basis) as

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1..3} a_i b_i$$

Using the Einstein summation convention, we can also write the scalar product as $a \cdot b = a_i b_i$.

Notice that the following also hold for the scalar product:

$$a \cdot b = b \cdot a \text{ (commutative law).}$$

$$a \cdot (b + c) = a \cdot b + a \cdot c \text{ (distributive law).}$$

Vector Product of Two Vectors

The *vector* product (or *cross* product) of two vectors **a** and **b** is another vector **c** defined as

$$c = a \times b = |a||b|\sin(\theta)\,\hat{c}$$

where θ is the angle between **a** and **b**, and $\hat{c}$ is a unit vector perpendicular to the plane containing **a** and **b** in the right-handed sense (see Figure A5.3 for a geometric interpretation).

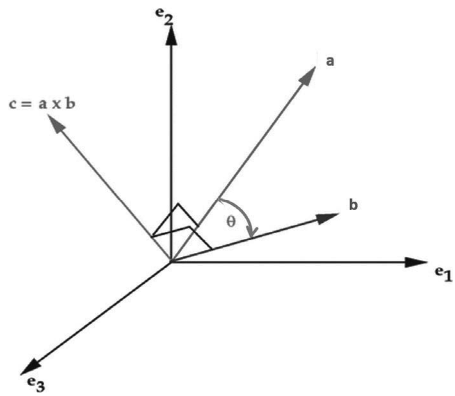


Figure A5.3. Vector product of two vectors.

In terms of the orthonormal basis (e_1, e_2, e_3), the cross product can be written in the form of a determinant

$$a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Tensors

Tensors are mathematical generalizations of vectors. In this way, vectors are first order tensors and scalars are zero order tensors. A vector has three components and a second order tensor has nine components. It is convenient to write a second order tensor, say **A**, as a 3 x 3 matrix:

$$\mathrm{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Tensors can be thought of as linear transformations. For example, second order tensors transform vectors into vectors, such as the tensor [**A**] transforms the vector {**a**} into a vector {**b**}.

$$\{\mathbf{b}\} = [\mathbf{A}]\{\mathbf{a}\}$$

Index

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A Systems Approach

Christopher H. Jenkins • Sanjeev K. Khanna

An Engineering Companion to Mechanics of Materials is the first volume in the Momentum Press collection *The Modern Engineering Companions: A Systems Approach to the Study of Engineering*. In *Mechanics of Materials*, we apply the intuitive "systems approach" to learning, the advantages of which are several. The student first gets a broad overview of the entire subject rather than the narrow piecemeal vision afforded by the traditional "component approach" common to most engineering texts. *Mechanics of Materials* comes with additional features to improve student learning, including Common Confusing Concepts (C^3) noted and clarified, indication of key concepts, side bar discussions, worked examples, and exercises for developing engineering intuition. The *Companions* are intended as a supplementary resource to help both undergraduate, graduate, and post-graduate students better learn and understand engineering concepts.

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