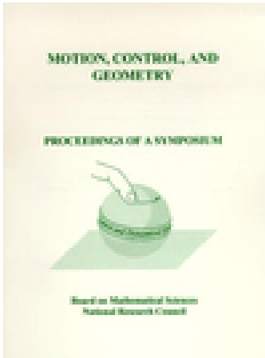


**Motion, Control, and Geometry: Proceedings of a Symposium**

Board on Mathematical Sciences, National Research Council

ISBN: 0-309-59061-2, 80 pages, 8.5 x 11, (1997)

# MOTION, CONTROL, AND GEOMETRY

**Proceedings of a Symposium**

Board on Mathematical Sciences
Commission on Physical Sciences, Mathematics, and Applications
National Research Council

National Academy Press
Washington, D.C. 1997

NOTICE: The project that is the subject of this report was approved by the Governing Board of the National Research Council, whose members are drawn from the councils of the National Academy of Sciences, the National Academy of Engineering, and the Institute of Medicine.

This report has been reviewed by a group other than the authors according to procedures approved by a Report Review Committee consisting of members of the National Academy of Sciences, the National Academy of Engineering, and the Institute of Medicine.

The National Academy of Sciences is a private, nonprofit, self-perpetuating society of distinguished scholars engaged in scientific and engineering research, dedicated to the furtherance of science and technology and to their use for the general welfare. Upon the authority of the charter granted to it by the Congress in 1863, the Academy has a mandate that requires it to advise the federal government on scientific and technical matters. Dr. Bruce Alberts is president of the National Academy of Sciences.

The National Academy of Engineering was established in 1964, under the charter of the National Academy of Sciences, as a parallel organization of outstanding engineers. It is autonomous in its administration and in the selection of its members, sharing with the National Academy of Sciences the responsibility for advising the federal government. The National Academy of Engineering also sponsors engineering programs aimed at meeting national needs, encourages education and research, and recognizes the superior achievements of engineers. Dr. William A. Wulf is interim president of the National Academy of Engineering.

The Institute of Medicine was established in 1970 by the National Academy of Sciences to secure the services of eminent members of appropriate professions in the examination of policy matters pertaining to the health of the public. The Institute acts under the responsibility given to the National Academy of Sciences by its congressional charter to be an adviser to the federal government and, upon its own initiative, to identify issues of medical care, research, and education. Dr. Kenneth I. Shine is president of the Institute of Medicine.

The National Research Council was organized by the National Academy of Sciences in 1916 to associate the broad community of science and technology with the Academy's purposes of furthering knowledge and advising the federal government. Functioning in accordance with general policies determined by the Academy, the Council has become the principal operating agency of both the National Academy of Sciences and the National Academy of Engineering in providing services to the government, the public, and the scientific and engineering communities. The Council is administered jointly by both Academies and the Institute of Medicine. Dr. Bruce Alberts and Dr. William A. Wulf are chairman and interim vice-chairman, respectively, of the National Research Council.

The National Research Council established the Board on Mathematical Sciences in 1984. The objectives of the Board are to maintain awareness and active concern for the health of the mathematical sciences and to serve as the focal point in the National Research Council for issues connected with the mathematical sciences. In addition, the Board is designed to conduct studies for federal agencies and maintain liaison with the mathematical sciences communities and academia, professional societies, and industry.

The Board on Mathematical Sciences is supported by core funding from the following federal agencies: Air Force Office of Scientific Research, Army Research Office, Department of Energy, National Science Foundation, National Security Agency, and Office of Naval Research. This activity was supported by core funding. Any opinions, findings, conclusions, or recommendations expressed in this publication are those of the authors and do not necessarily reflect the view of the organizations or agencies that provided support for the Board.


International Standard Book Number 0-309-05785-X

Additional copies of this report are available from:
Board on Mathematical Sciences
National Research Council
2101 Constitution Avenue, N.W.
Washington, D.C. 20418
Printed in the United States of America

Cover: Rolling your finger in a circular motion on a rolling sphere generates rotations. Drawing courtesy of Jerrold E. Marsden, California Institute of Technology.

# BOARD ON MATHEMATICAL SCIENCES

## COMMISSION ON PHYSICAL SCIENCES, MATHEMATICS, AND APPLICATIONS

# Preface

The symposium "Motion, Control, and Geometry" was held on April 12, 1994, at the National Academy of Sciences in Washington, D.C. This symposium focused on control theory as a fundamental aspect of motion generation in many emerging areas. Those areas include microsurgery (for example, involving microrobots or "snakes" capable of locomotion in confined spaces such as an intestinal tract), spacecraft positioning, biological and robotic movement, motor miniaturization, and motion engineering (for instance, via coupled-oscillator pattern generation). Traditional control theory methods have been supplemented by the growing body of techniques associated with dynamical systems and geometric mechanics.

This symposium addressed the exciting interdisciplinary synergy that is developing on the basis of theoretical insight and technological inventiveness. The speakers at the symposium discussed both cutting-edge research and technology developments. The symposium and proceedings will help to inform researchers, practitioners, federal and state program managers, policy experts, and decision makers, as well as the scientific, engineering, and technology communities, of important issues in the mathematical sciences and of the relation of the mathematical sciences to other areas and to national interests. The Board on Mathematical Sciences, which organized the symposium, hopes the information presented here will help foster increased awareness of how research on questions of fundamental interest often can naturally connect to practical benefits for the nation and society.

# Contents

# Introduction

Geometry has been associated with motion, either implicitly or explicitly, from very early times in human history. There are relationships between motion and geometry both in how motion is described and in how it is harnessed and directed. Geometric notions underlie such mechanical devices as the potter's wheel and the wheeled cart, the ramp (or inclined plane), the lever, the pulley, and the coil. Although formal geometrical descriptions and explicit functionality principles were not supplied until centuries after such mechanisms came into widespread use, their connections with linked linear and circular motion, horizontal and vertical or forward and sideways motion, and winding-in and-out (spiral) or winding-up and-down (helical) motion are unmistakable. The substantial interrelationships between motion and geometry have been a continuing focus of scientific study and technological development from the eras of Archimedes of Syracuse, Leonardo da Vinci and Galileo Galilei, Rene Descartes, Isaac Newton, Pierre-Louis Moreau de Maupertuis, James Clerk Maxwell, and Albert Einstein fight through to the present time. Those linkages bear heavily on how motion is modeled and ultimately controlled, be it by mechanical contrivance (for instance, in a pendulum clock) or through the discovery of how prevailing conditions influence outcomes (for example, finding the trajectory of an object that is subject to gravity and that is thrown horizontally off a cliff).

From the construction of the Great Pyramids and of Stonehenge, which both involved the transport and careful positioning of massive blocks or lintels, to the reckoning of celestial motions; from the Renaissance design or engineering of a prototype submarine, bicycle, or helicopter to latter-day satellite positioning or in vivo intestinal exploration and examination; from the movements of subatomic particles to the meanderings of computer-modeled sidewinding snakes, geometry supplies an indispensable vocabulary for the mathematical description of whatever motions are observed, achievable, or sought. As mathematics is the language of science, so geometry is the language of motion. The motivation may have changed from a desire to understand, predict, or direct motions by way of geometric modeling and analysis to a focus on designing and controlling the mechanical generation of particular motions on the basis of their geometric description, computer simulation, and robotic replication. However, the value of this geometric language is undiminished.

Some of the modem developments described in the following chapters include the geometric control of robot motion and craft orientation, how high-power precision micromotors are engineered for less invasive surgery and self-focusing lens applications, what a mobile robot on a surface has in common with one moving in three dimensions, and how the motion-control problem is simplified by a coupled oscillator's geometric grouping of degrees of freedom and motion time scales.

The four papers in these proceedings provide a view through the scientific portal of today's motion-control geometric research into tomorrow's technology. The mathematics needed to carry out this research is that of modem differential geometry, and the questions raised in the field of motion-control geometry go directly to the research frontier. Some of the mathematical tools that are useful here are Lie algebras of vector fields, differential forms and exterior algebra, and affine connections. Another tool that has proven useful is gauge theory remarkably, the same sort of geometry that is used in elementary-particle physics. It is fortunate that mathematicians have developed the mathematical tools in a general context so that they can be used for many purposes. In particular, the mathematical notion of the holonomy of a connection has been around for some time—an idea that links locomotion generation with gauge theory. Interestingly, control and locomotion generation are two of the other areas in which these ideas can be applied.

Geometry is a mathematical area too often neglected nowadays in a student's education. This publication will help adjust the control initially imposed about 2,300 years ago on one kind of "motion"— that of students entering Plato's Academy, where the following caveat was inscribed above the doorway: "Let no one ignorant of geometry enter here." Readers of these chapters will gain an appreciation of modem geometry and how it continues to play a crucial role in the context of motion control in cutting-edge science and technology.

# 1

# Geometric Foundations of Motion and Control

Jerrold E. Marsden
Department of Control and Dynamical Systems
California Institute of Technology

Some interesting aspects of motion and control, such as those found in biological and robotic locomotion and attitude control of spacecraft, involve geometric concepts. When an animal or a robot moves its joints in a periodic fashion, it can rotate or move forward. This observation leads to the general idea that when one variable in a system moves in a periodic fashion, motion of the Whole object can result. This property can be used for control purposes; the position and attitude Of a satellite, for example, are often controlled by periodic motions of parts of the satellite, such as spinning rotors. One of the geometric tools that has been used to describe this phenomenon is that of connections, a notion that is used extensively in general relativity and other parts of theoretical physics. This tool, part of the general subject Of geometric mechanics, has been helpful in the study of both the stability and instability of a system and system bifurcations, that is, changes in the nature of the system dynamics, as some parameter changes. Geometric mechanics, currently in a period of rapid evolution, has been used, for example, to design stabilizing feedback control systems in attitude dynamics. Theory is also being developed for systems with rolling constraints such as those found in a simple rolling wheel. This paper explains how some of these tools of geometric mechanics are used in the study of motion control and locomotion generation.

## INTRODUCTION

We describe below a geometric framework that leads to a better understanding of locomotion generation and motion control in mechanical systems. This introduction provides some basic examples that motivate and set the stage for this framework.

Perhaps the most popular example of the generation of rotational motion is the failing cat, which is able to execute a 180º reorientation, all the while having zero angular momentum. It achieves this by manipulating its joints to create *shape changes*. To understand this, one has to remember that the angular momentum of a rotating rigid object is its moment of inertia times its instantaneous angular velocity; this is the angular version of the familiar relation "momentum equals mass times velocity." Shape changes result in a change in the cat's moment of inertia and this, together with the constancy of the angular momentum, creates the overall orientation change. However, the exact process by which this occurs is subtle, and intuitive reasoning can lead one astray. While this problem has been long studied (e.g., by Kane and Shur, 1969), recently new and interesting insights have been discovered using geometric methods (see Enos, 1993; Montgomery, 1990, and references therein).

Astronauts who wish to reorient themselves in a free space environment can similarly do so by means of shape changes. For example, holding one of their legs straight, they can swivel it at the hip, moving their foot in a circle. When they have achieved the desired orientation, they merely stop their leg movement. Similar movements for robots and spacecraft can be controlled automatically to achieve desired objectives (see, for example, Walsh and Sastry, 1995). One often refers to the extra motion that is achieved as the *geometric phase.*

The history of this phenomenon and its applications is a long and complex story. We shall only mention a few highlights. Certainly the shift in the plane of the swing in the Foucault pendulum as the earth rotates once around its axis is one of the earliest examples of this phenomenon. Anomalous spectral shifts in rotating molecules are another. Phase formulas for special problems such as rigid body motion and polarized light in helical fibers were understood already in the early 1950s. Additional historical comments and references can be found in Berry (1990), and Marsden and Ratiu (1994). Gradually the subject became better understood, but the first paper to clarify and emphasize the ubiquity of the geometry behind all these phenomena was Berry (1985). It was also quickly realized that the phenomenon occurs in essentially the same way in both classical and quantum mechanics (Hannay, 1985), and that the phenomenon can be linked in a fundamental way with the presence of symmetry (Montgomery, 1988; Marsden et al., 1990).

The theory of geometric phases has an interesting link with noneuclidean geometry, a subject first invented for its own sake, without regard to applications. A simple way to explain this link is as follows. Hold your hand at arm's length, but allow rotation in your shoulder joint. Move your hand along three great circles, forming a triangle on the sphere, and during the motion, keep your thumb "parallel," that is, forming a *fixed* angle with the direction of motion. After completing the circuit around the triangle, your thumb will return rotated through an angle relative to its starting position (see Figure 1.1). In fact, this angle (in radians) is given by $\Theta = \Delta - \pi$ where $\Delta$ is the sum of the angles of the triangle. The fact that $\Theta \neq 0$ is of course one of the basic facts of noneuclidean geometry—in curved spaces, the sum of the angles of a triangle is not necessarily $\pi$ (i.e., 180º). This angle is also related to the *area A* enclosed by the triangle through the relation $\Theta = A/r^2$, where r is the radius of the sphere.

The examples presented so far are rather different from what one finds in many other mechanical systems of interest in one crucial aspect—the absence of constraints of rolling, sliding, or contact. For example, when one parks a car, the steering mechanism is manipulated and movement into the parking spot is generated; obviously the rolling of the wheels on the road is crucial to the maneuver. When a human or a robot manipulates an object in its fingers (imagine twirling an egg in your fingers), it can reorient the object through the rolling of its fingers on the object. This can be shown in a demonstration I learned from Roger Brockett: roll your fingers in a rotating motion on a ball resting on a table—you will find that the ball reorients itself under your finger! The amount of rotation is again related to the amount of area you capture in the rotating motion. You have *generated* rotational motion! (See Figure 1.2.)

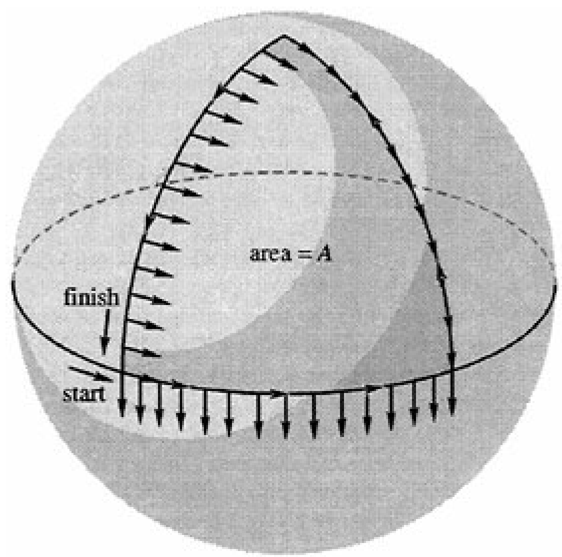


FIGURE 1.1 A parallel movement of your thumb around a spherical triangle produces a phase shift.

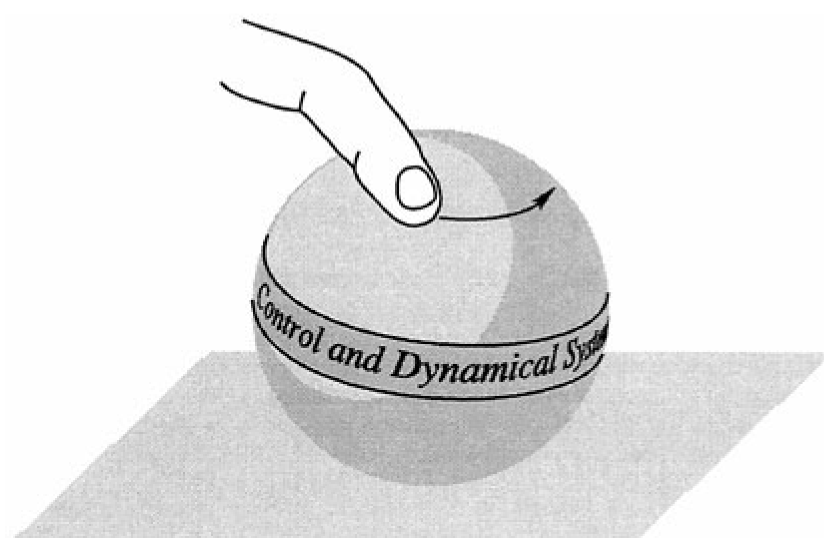


FIGURE 1.2 Rolling your finger in a circular motion on a rolling sphere generates rotations.

In all these cases, cyclic motion in one set of variables (often called the *internal* variables) produces motion in another set (often called the *group* variables). This idea is central to the basic geometric framework described in ensuing sections.

One can generate translational motion as well as rotational motion. For example, microorganisms and snakes generate translations by a very specific cyclic manipulation of their internal variables (Shapere and Wilczek, 1987). The reason for this is, in a superficial sense, that in these examples, translation is kinematically possible (translations are available as group variables) and the controls are such that these variables are activated. Often translational motion and rotational motion are coupled in interesting ways, as in the snakeboard, a modification of the familiar skateboard. This modification allows the rider to rotate the front and back wheels by rotating his feet and this, together with the rotary motion of the rider's body, allows *both* translational and rotational motion to be generated. Such motion can be controlled with the objective that desired motions be generated. We will discuss this example in a little more detail in the section entitled "The Snakeboard," below.

The generation of motion in small robotic devices is very promising for medical applications. In this context, one seeks devices that can move in confined spaces under variable conditions (flexible walls, tight comers, etc.). In fact, this general philosophy is one of the reasons one hopes that medical operations in the future will be much less intrusive than many of them are now.

There are similar links between vibratory motion and translational and rotational motion (e.g., the developments of micromotors) (Brockett, 1989), on the one hand, and, on the other hand, motion generation in animals (e.g., the generation and control of waves from coupled oscillators, as seen in the swimming of fish and in the locomotion of insects and other creatures).

A central question to address in this area is, How should one control motions of the internal variables so that the desired group (usually translational and rotational) motions are produced? To make progress on this question, one needs to combine experience with simple systems and strategies—such as steering with sinusoids, as in Murray and Sastry (1993)—with a full understanding of the mathematical structure of the mechanical systems, both analytical and geometrical. We also mention the work of Brockett (1981), which shows that for certain classes of control systems that are controllable via first level brackets, steering by sinusoids is, in fact, optimal.

## CONNECTIONS AND BUNDLES

One of the fruitful ideas from geometry that has been used in the investigation of mechanical systems is that of a connection. While the notion of a connection is quite precise, connections have many personalities. On the one hand, one thinks of them as describing how curved a space is; in fact, in the classical Riemannian setting used by Einstein in his theory of general relativity, the curvature of the space is constructed out of the connection (in that case, also called the Christoffel symbols). In other, but related, settings developed by Eli Cartan, the connection is what is responsible for a corrected measure of acceleration; for example, if one is on a rotating merry-go-round, one has to correct any measurement of acceleration to take into account the acceleration of the merry-go-round, and this correction can be described by a connection.

In the general theory, connections are associated with mappings, called *bundle mappings*, that project larger spaces onto smaller ones, as in Figure 1.3. The larger space is called the bundle and the smaller space is called the base. Directions in the larger space that project to zero are called *vertical directions*. The general definition of a connection is a specification of a set of directions, called *horizontal directions*, that complements at each point the space of vertical directions.

In the example of parallel transport of the thumb around the sphere, the larger space is the space of all tangent vectors to the sphere, and this space projects to the sphere itself by projecting a vector to its point of attachment to the sphere. The horizontal directions are the directions with zero acceleration within the intrinsic geometry of the sphere; that is, the directions determined by great circles.

In the thumb example, we saw that going around the triangle produces a change in the orientation of the thumb on return. The thumb is parallel transported, that is, it moves in horizontal directions with respect to the connection. The thumb has undergone a rotational shift from the beginning to the end of its journey.

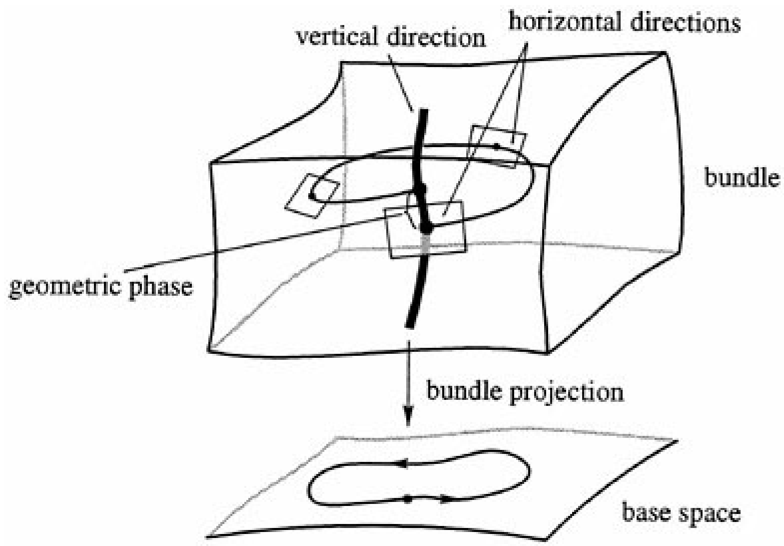


Figure 1.3 A connection divides the space into vertical and horizontal directions.

In general, we can expect that if we have a horizontal motion in the bundle and if the corresponding motion in the base is cyclic, then the horizontal motion will undergo a shift, which we will call a phase shift, between the beginning and the end of its path. The shift in the vertical direction is often given by an element of a group, such as a rotation or translation group. In many of the examples discussed so far, the base space is the control space in the sense that the path in the base can be chosen by suitable controls. The path above it in the bundle is regarded as being determined by the condition of horizontality. This condition therefore determines its phase.

This setting of connections provides a framework in which one can understand the phrase we started with: when one variable in a system moves in a periodic fashion, motion of the whole object can result. Here, the "motion of the whole object" is represented by the geometric phase. Coming along with this notion are plenty of lovely theorems and calculational tools; for example, one of these (based on Stokes' theorem) shows how to calculate the geometric phase in terms of the integral of the curvature of the connection over an area enclosed by the closed curve on the base. This is one reason that areas so commonly appear in geometric phase formulas.

Connections are ubiquitous in geometry and physics. For example, connections are one of the main ingredients in the modem theory of elementary particles, and are the primary fields in Yang-Mills theory, a generalization of Maxwell's electromagnetic theory. In fact, in electromagnetism, the equation $B = \nabla \times A$ for the magnetic field may be thought of as an expression for the curvature of the connection (or magnetic potential) $A$.

## CONSTRAINTS: ANGULAR MOMENTUM AND ROLLING

In many mechanical systems, there are conditions called "constraints." For our purposes, these are of two fundamentally different sorts. The first is typified by the constraint of zero angular momentum for the falling cat. The cat, once released, and before it reaches the ground, cannot change the fact that its angular momentum is zero, no matter how it moves its body parts. We choose the cat's base space to be its shape space, which does indeed literally mean what it says—the collection of all the shapes of its body, which can be specified by giving the angles between its body parts. The bundle in this case consists of these shapes together with a rotation and translation to specify the position and orientation in space. Since the cat is free to manipulate its shape using its muscles, it can perform maneuvers, some of them cyclic, in shape space. Meanwhile, how the cat turns in space is governed by the law of conservation of angular momentum. It turns out that this law exactly defines the horizontal space of a connection! The connection in this case is called the "mechanical connection." That this corresponds to a connection was discovered through the combined efforts of Smale (1970), Abraham and Marsden (1978), and Kummer (1981). Thus, when one puts together the theory of connections with this observation, and throws in control theory, one has the beginnings of the "gauge theory of mechanics." This theory has been extended and developed by many workers since then.

Observation of the motions of a mechanical system in its shape space shows a relation to the theory of reduction, a theory that seeks to make the configuration space of a mechanical system smaller by taking advantage of symmetries. This method has led to many interesting and unexpected discoveries about mechanics, including, for example, the explanation of the integrability of the Kowaleskaya top in terms of symmetry by Bobenko et al. (1989). (An algebraic-geometric construction with similar goals was found by Haine, Horozov, Adler, and van Moerbeke around the same time.) Observing the motion in shape space alone is similar to watching the shapes change relative to an observer riding with the object. In such a frame, one sees what seem to be extra forces, namely the Coriolis and centrifugal forces. In fact, these forces can be understood in terms of the curvature of the mechanical connection. Then the problem of finding the original complete path is one of finding a horizontal path above the given one. This is sometimes called the "reconstruction problem." We conclude that the generation of geometric phases is closely linked with the reconstruction problem.

One of the simplest systems in which one can see these phenomena is called the planar skater. This device consists of three coupled rigid bodies lying in the plane. They are free to rotate and translate in the plane, somewhat like three linked ice hockey pucks. This has been a useful model example for a number of investigations, and was studied fairly extensively in Oh et al. (1989), and Krishnaprasad (1989) and references therein. The only forces acting on the three bodies are the forces they exert on each other as they move. Because of their translational and rotational invariance, the total linear and angular momentum remains constant as the bodies move. This holds true even if one applies controls to the joints of the device; this is because the conservation of momentum depends only on externally applied forces and torques. See Figure 1.4.

The planar skater illustrates well some of the basic ideas of geometric phases. If the device starts with zero angular momentum and it moves its arms in a periodic fashion, then the whole assemblage can rotate, keeping, of course, zero angular momentum. This is analogous to our astronaut in free space who rotates his arms or legs in a coordinated fashion and finds that he rotates. One can understand this simple example directly by using conservation of angular momentum. In fact, the definition of angular momentum allows one to reconstruct the overall attitude of the device in terms of the motion of the joints using freshman calculus. Doing so, one gets a motion generated in the overall attitude, which is indeed a geometric phase. This example is sufficiently simple that one does not need the geometry of connections to understand it, but nonetheless it provides a simple situation in which one can test the ideas. For more complex examples, such as the falling cat, the geometric setting of connections has indeed proven useful.

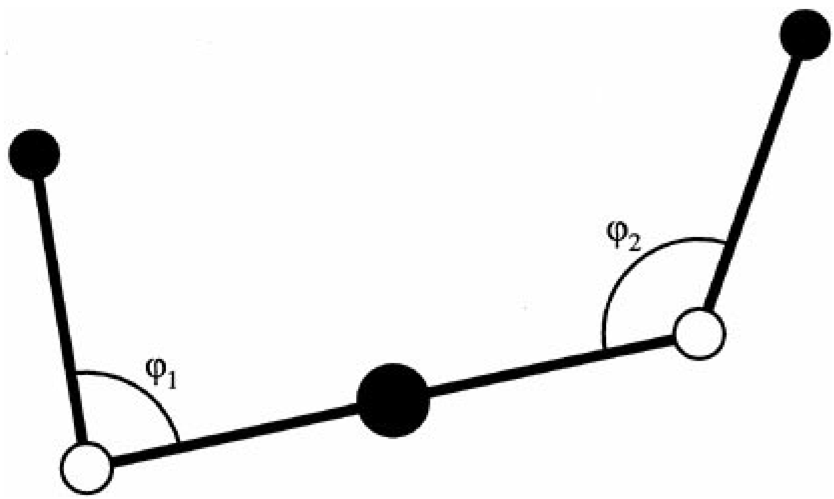


FIGURE 1.4 The planar skater consists of three interconnected bodies that are free to rotate about their joints.

To indicate some of the flavor of three-dimensional examples, we discuss the rigid body. Each position of the rigid body is specified by a Euclidean motion giving the location and orientation of the body. The motion is then governed by the equations of mechanics in this space. Assuming that no external forces act on the body, conservation of linear momentum allows us to solve for the components of the position and momentum vectors of the center of mass. Passage to the center of mass frame reduces one to the case where the center of mass is fixed, so only pure rotations remain. Each possible orientation corresponds to the specification of a proper orthogonal matrix *A*. Back in 1740, Euler parametrized the matrix *A* in terms of three (Euler) angles between axes that are either fixed in space or are attached to symmetry planes of the body's motion.

We regard the element $A \in SO(3)$ giving the configuration of the body as a mapping of a reference configuration to the current configuration. The matrix *A* takes a reference or label point *X* to a current point $x = A(X)$. For a rigid body in motion, the matrix *A* is time dependent and the velocity of a point of the body is $\dot{x} = \dot{A}X = \dot{A}A^{-1}x$. Since *A* is an orthogonal matrix, we can write $\dot{x} = \dot{A}A^{-1}x = \omega \times x$, which defines the spatial angular velocity vector ω .The corresponding body angular velocity is defined by $\Omega = A^{-1}\omega$, so that Ω is the angular velocity as seen in a body-fixed frame. The kinetic energy is given by integrating the kinetic energy expression for particles (one-half the mass density times the square of the velocity) over the reference configuration. In fact, this kinetic energy is a quadratic function of Ω. Writing $K = \frac{1}{2}\Omega^T I\Omega$ defines the (time independent) moment of inertia tensor *I*, which, if the body does not degenerate to a line, is a positive definite 3 × 3 matrix, or better, a quadratic form. Every calculus student learns how to calculate moments of inertia as illustrations of the process of multiple integration.

The equations of motion in *A* space define certain equations in Ω space that were discovered by Euler: $I\dot{\Omega} = I\Omega \times \Omega$. The body angular momentum is defined, analogous to linear momentum $p = mv$, as $\Pi = I\Omega$. In terms of Π, the Euler equations read Π = Π × Ω. This equation implies that the spatial angular momentum vector $\pi = A\Pi$ is fixed in time. One may view this fact as a conservation law that results from the rotational symmetry of the problem. These and other facts given here are proven in every mechanics textbook, such as Marsden and Ratiu (1994).

Viewing the components $(\Pi_1, \Pi_2, \Pi_3)$ of Π as coordinates in three-dimensional space, the Euler equations are evolution equations for a point in this space. A constant of motion for the system is given by the square length of the total angular momentum vector: $\|\Pi\|^2 = \Pi_1^2 + \Pi_1^2 + \Pi_1^2$. This follows from conservation of π and the fact that $\|\pi\| = \|\Pi\|$ or can be verified directly from the Euler equations by computing the time derivative of $\|\Pi\|^2$.

Because of conservation of $\|\Pi\|$, the evolution in time of any initial point Π(0) is constrained to the sphere $\|\Pi\|^2 = \|\Pi(0)\|^2 = \text{constant}.$ Thus we may view the Euler equations as describing a two-dimensional dynamical system on an invariant sphere. This sphere is called the *reduced-phase space* for the rigid body equations. Another constant of the motion is the Hamiltonian or energy: $H = \frac{1}{2}\langle \Pi, I^{-1}\Pi \rangle$. Since solutions curves are confined to the sets where *H* is constant, which are in general ellipsoids, as well as to the invariant spheres where $\|\Pi\|$ = constant, the intersection of these surfaces is precisely that of the trajectories of the rigid body, as shown in Figure 1.5.

Let us briefly indicate how geometric phases come into the rigid body example. Suppose we are given a trajectory Π(*t*) on $P_\mu$ that has period *T* and energy *E*. After time *T* the rigid body has rotated in physical 3-space about the axis μ by an angle given by

$$\Delta\theta = -\Lambda + \frac{2ET}{\|\mu\|} + 2k\pi .$$

Here is the solid angle enclosed by the curve Π($t$) on the sphere and is oriented according to the right-hand rule, and $k$ is an integer (reflecting the fact that we are really only interested in angles up to multiples of 2π).

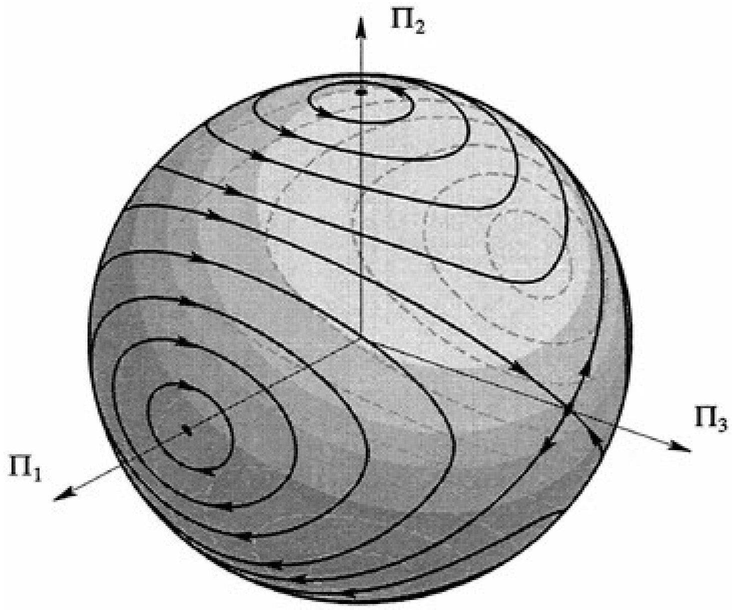


FIGURE 1.5 The solutions of Euler's equations for rigid body motion.

An interesting feature of this formula is the fact that Δθ splits into two parts. The term A is the purely geometric quantity, the geometric phase. It does not depend on the energy of the system or the period of motion, but rather on the fraction of the surface area of the sphere that is enclosed by the trajectory. The second term, the dynamic phase, depends on the system's energy and the period of the trajectory.

Geometrically we can picture the rigid body as tracing out a path in its phase space; that is, the space of rotations (playing the role of positions) and corresponding momenta with the constraint of a fixed value of the spatial angular momentum. The phase space plays the role of the bundle, and the projection map to the base, the momentum sphere, is the map we described earlier that takes the orientation $A$ and its velocity (or momentum) to the body momentum sphere. As Figure 1.5 shows, almost every trajectory on the momentum sphere is periodic, but this does not imply that the original curve of rotations was periodic, as is shown in Figure 1.6. The difference between the true trajectory and a periodic trajectory is given by the geometric plus the dynamic phase. Although this figure is given in the context of rigid body dynamics, its essential features are true for any mechanical system with symmetry.

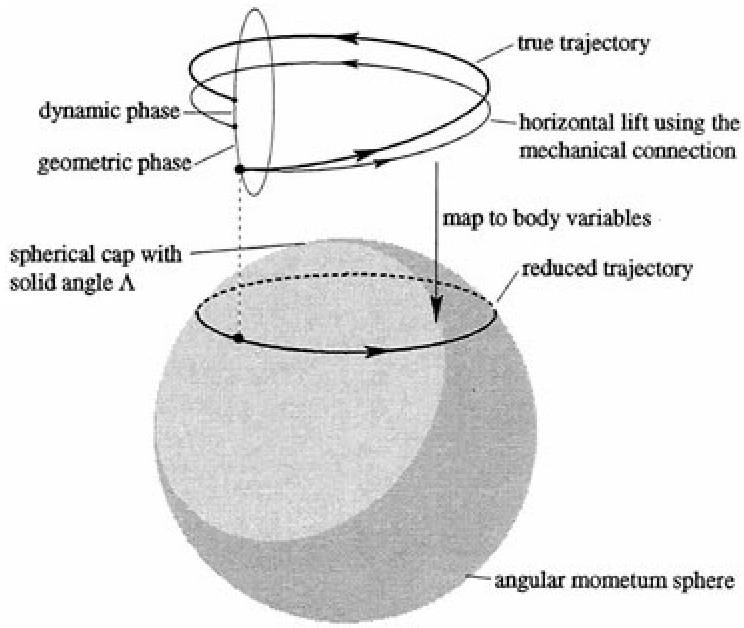


FIGURE 1.6 The geometric phase formula for rigid body motion.

This formula for the rigid body phase has a long and interesting history. It was known in classical books, such as that of Whittaker, in terms of quotients of theta functions, but not in terms of areas, as above. This aspect was discovered in the 1950s independently in work of Ishlinskii and of Goodman and Robinson. Montgomery (1991b,c) and Marsden et al. (1990) showed, following the lead of Berry and Hannay, that the formula can be interpreted in terms of holonomy of a connection. Further historical details may be found in Marsden and Ratiu (1994).

It is possible to observe some aspects of the geometric phase formula for a rigid body with a simple experiment. Put a rubber band around a book so that the cover will not open. (A tall thin book works best.) With the front cover facing up, gently toss the book in the air so that it rotates about its middle axis. Catch the book after a single rotation and you will find that it has also rotated by 180º about its long axis; that is, the front cover is now facing the floor.

In addition to its use in understanding phases, the mechanical connection has been helpful in stability theory. For example, when a rigid body such as a satellite tumbles about its long or short axis, it does so stably, but it is unstable when it rotates about the middle axis. When one takes into account small dissipative effects such as a vibrating antenna, then the rotational motion about the long axis becomes unstable as well, but this effect is more delicate. Corresponding statements for systems like rigid bodies with flexible appendages or interconnected rigid bodies are more subtle than the dynamics of a single rigid body. There is a powerful method for determining the stability of such solutions called the *energy momentum* method. This method is an outgrowth of basic work of Riemann, Poincaré, and others in the last century and more recently by Arnold; further recent developments were made by Simo et al. (1991), and Bloch et al. (1994, 1996) and references therein. Here the main problem is to split the variables properly into those that correspond to internal, or shape, changes, and to those that correspond to rotational and translational motions. Interestingly, the mechanical connection plays a key role in the solution of this problem and it makes many otherwise intractable problems soluble.

This gauge theory of mechanics has been successful for a number of important problems, such as the falling cat problem, as we shall discuss below. Nevertheless, there is another important class of problems that it does not apply to as stated, namely, mechanical systems with rolling constraints, typified by the constraint that a wheel or ball rolls without slipping on a plane. One very simple idea ties this type of problem to the zero angular momentum constraint problem that was just described. This idea is that of realizing the constraint as the horizontal space of a connection. In fact, the constraint itself defines a connection by declaring the constraint space to be the horizontal space. This, in effect, defines the connection. In the case of rolling constraints, we call this connection the kinematic connection to avoid confusion with the mechanical connection described earlier. This point of view for systems with rolling (and rolling type) constraints was developed by Koiller (1992) and by Bloch et al. (1997). For example, the equations of motion expressed on the base space again involve the curvature of the kinematic connection. This shows again that the links with geometry are strong at a very basic level.

Things get even more interesting when the system has both rolling constraints and symmetry. Then we have the kinematic connection as well as the symmetry group to deal with, but now they are interlinked. One of the things that makes systems with rolling constraints with symmetry different from free systems is that the law of conservation of angular momentum is no longer valid for them. This is already well illustrated by a toy called the rattleback, a canoe-shaped piece of wood or plastic. When the rattleback rocks on a flat surface like a table, the rocking motion induces a rotational motion, so that it can go from zero to nonzero angular momentum about the vertical axis as a result of the interaction of the rocking and rotational motion and the rolling constraint with the table. One can say that the forces of constraint that enforce the condition of rolling can upset the balance of angular momentum. This is also the case for the snakeboard discussed below, but nonetheless, this rams out to be a key point in understanding locomotion generation for this system. One of the interesting aspects of this is that, as shown by Bloch et al. (1996), there is a very nice equation satisfied by a particular combination of the linear and angular momentum, which they call the momentum equation. Because of that success, one can imagine that this understanding will be important for many other similar systems as well.

## STABILIZATION AND OPTIMAL CONTROL

Control theory is closely tied to dynamical systems theory in the following way. Dynamical systems theory deals with the time evolution of systems by writing the state of the system, say $z$ in a general space $P$, and writing an evolution equation

$$\frac{d}{dt}z = f(z, \mu)$$

for the motion, where μ includes other parameters of the system (masses, lengths of pendula, etc.). The equations themselves include things like Newton's second law, the Hodgkin-Huxley equations for the propagation of nerve impulses, and Maxwell's equations for electrodynamics, among others. Many valuable concepts have developed around this idea, such as stability, instability, and chaotic solutions.

Control theory adds to this the idea that in many instances, one can directly intervene in the dynamics rather than passively watching it. For example, while Newton's equations govern the dynamics of a satellite, we can intervene in these dynamics by controlling the onboard gyroscopes. One simple way to describe this mathematically is by making $f$ dependent on additional control variables u that can be functions of $t$, $z$, and μ. Now the equation becomes

$$\frac{d}{dt}z = f(z, \mu, u(t, z, \mu)),$$

and the objective, naively stated, is with an appropriate dependence of $f$ on $u$ to choose the function $u$ itself to achieve certain desired goals. Control engineers are frequently tempted to overwhelm the intrinsic dynamics of a system with the control. However, in many circumstances (fluid control is an example—see, for example, the discussion in Bloch and Marsden, 1989), one needs to work with the intrinsic dynamics and make use of its structure.

Two of the basic notions in control theory involve steering and stabilizability. Steering has, as its objective, the production of a control that has the effect of joining two points by means of a solution. One imagines manipulating the control, much the way one imagines driving a car so that the desired destination is attained. This type of question has been the subject of extensive study and many important and basic questions have been solved. For example, two of the main themes that have developed are, first, the Lie algebraic techniques based on brackets of vector fields (in driving a car, you can repeatedly make two alternating steering motions to produce a motion in a third direction) and, second, the application of differential systems (a subject invented by Elie Cartan in the mid-1920s whose power is only now being significantly tapped in control theory). The work of Tilbury et al. (1993) and Walsh and Bushnell (1993) typify some of the recent applications of these ideas.

The problem of stabilizability has also received much attention. Here the goal is to take a dynamic motion that might be unstable if left to itself but that can be made stable through intervention. A famous example is the F-15 fighter, which can fly in an unstable (forward wing swept) mode but which is stabilized through delicate control. Flying in this mode has the advantage that one can execute tight turns with rather little effort—just appropriately remove the controls! The situation is really not much different from what people do everyday when they ride a bicycle. One of the interesting things is that the subjects that have come before—namely, the use of connections in stability theory—an be turned around to be used to find useful stabilizing controls, for example, how to control the onboard gyroscopes in a spacecraft to stabilize the otherwise unstable motion about the middle axis of a rigid body (see Bloch et al., 1992; Kammer and Gray, 1993).

Another issue of importance in control theory is that of optimal control. Here one has a cost function (think of how much you have to pay to have a motion occur in a certain way). The question is not just if one can achieve a given motion but how to achieve it with the least cost. There are many well-developed tools to attack this question, the best known of these being what is called the Pontryagin Maximum Principle. In the context of problems like the falling cat, a remarkable consequence of the Maximum Principle is that, relative to an appropriate cost function, the optimal trajectory in the base space is a trajectory of a Yang-Mills particle. The equations for a Yang-Mills particle are a generalization of the classical Lorentz equations for a particle with charge *e* in a magnetic field *B*:

$$\frac{d}{dt}v = \frac{e}{c}v \times B,$$

where *v* is the velocity of the particle and where *c* is the velocity of light. The mechanical connection comes into play though the general formula for the curvature of a connection; this formula is a generalization of the formula $B = \nabla \times A$ expressing the magnetic field as the curl of the magnetic potential. This remarkable link between optimal control and the motion of a Yang-Mills particle is due to Montgomery (1990, 1991a).

One would like to make use of results like this for systems with rolling constraints as well. For example, one can (probably naively, but hopefully constructively) ask what is the precise connection between the techniques of steering by sinusoids mentioned earlier and the fact that a particle in a constant magnetic field also moves by sinusoids, that is, moves in circles. Of course if one can understand this, it immediately suggests generalizations by using Montgomery's work. This is just one of many interesting theoretical things that requires more investigation. One of the positive things that has already been achieved by these ideas is the beginning of a deeper understanding of the links between mechanical systems with angular momentum type constraints and those with rolling constraints. The use of connections has been one of the valuable tools in this endeavor. One of the papers that has been developing this point of view is that of Bloch et al. (1997). We shall see some further glimpses into that point of view in the next section.

## THE SNAKEBOARD

The snakeboard is an interesting example that illustrates several of the ideas we have been discussing (see Lewis et al., 1997). This device is a modification of the standard skateboard, the most important of which is that riders can use their feet to independently turn the front and back wheels—in the standard skateboard, these wheels are of course fixed to the frame of the skateboard. In addition, one can manipulate one's body using a swivelling motion and this motion is coupled to the motion of the snakeboard itself. We show the snakeboard schematically in Figure 1.7.

One of the fascinating things about the snakeboard is that one can generate locomotion without pedaling, solely by means of internal motions. When the user's feet and body are moved in the right way, rotational and translational motion of the device can be generated. The snakeboard is simple enough that one can study many parts of it analytically and numerical simulations of its motion are reasonably economical to implement. On the other hand, it seems to have all of the essential features that one would want for more complex systems, the main one for the present goals being its ability to generate rotational and translational motion. From the mathematical and mechanical point of view, it is rich in geometry and in symmetry structure, which also makes it attractive. Thus, it provides a good testing and development ground for both theoretical and numerical investigations.

From the theoretical point of view, one feature of the snakeboard that sets it truly apart from examples like the planar skater and the falling cat is that even though it has the symmetry group of rotations and translations of the plane, the linear and angular momentum is not conserved. Recall that for the planar skater, no matter what motions the arms of the device make, the values of the linear and angular momentum cannot be altered. This is not true for the snakeboard and this can be traced to the presence of the forces of constraint, just as in the rattleback mentioned earlier. Thus, one might suspect that one should abandon the ideas of linear and angular momentum for the snakeboard. However, a deeper inspection shows that this is not the case. In fact, one finds that there is a special component of the angular momentum, namely that about the point $P$ shown in Figure 1.8.

If we call this component $p$, one finds that, due to the translational and rotational invariance of the whole system, there is a "momentum equation" for $p$ of the form

$$\frac{d}{dt} p = f(x, \dot{x}, p) ,$$

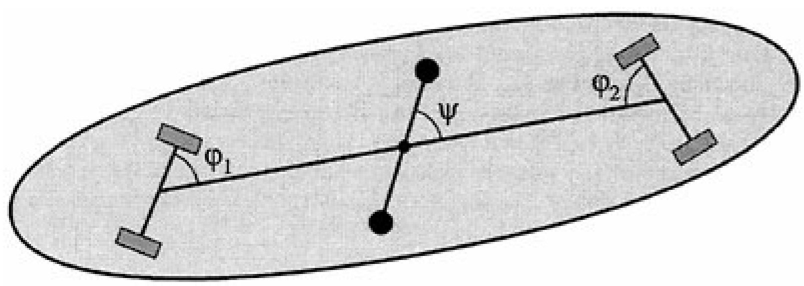


FIGURE 1.7 The snakeboard has three movable internal parts, the front and back wheels and the angle of the rider's body.

where x represents the internal variables of the system (the three angles shown in the preceding figure). The point is that this equation does not depend on the rotational and translational position of the system. Thus, if one has a given internal motion, this equation can be solved for $p$ and, from it, the attitude and position of the snakeboard calculated by means of another integration. This strategy is thus parallel to that for the falling cat and the planar skater.

With this set-up, one is now in a good position to identify the resulting geometric phase with the holonomy of a connection that is a synthesis of the kinematic and mechanical connection. Carrying this out and implementing these ideas for more complex systems is in fact the subject of current research.

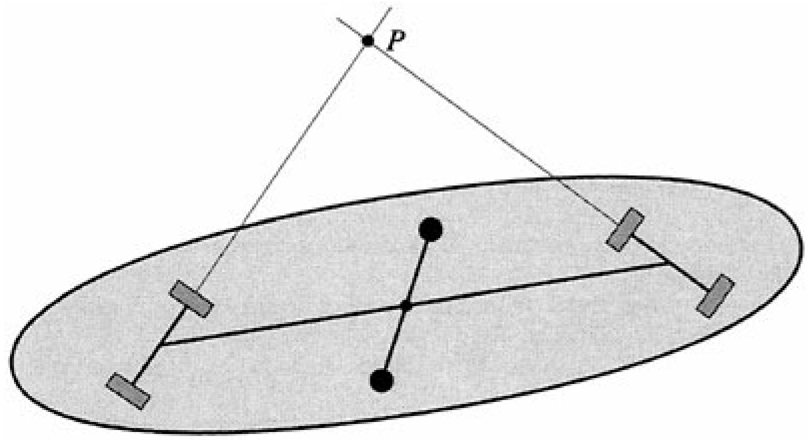


FIGURE 1.8 The angular momentum about the point P plays an important role in the analysis of the snakeboard.

## ACKNOWLEDGMENTS

Thanks are extended to John Tucker, Tony Bloch, Roger Brockett, Joel Burdick, P.S. Krishnaprasad, Andrew Lewis, Richard Montgomery, Richard Murray, Jim Ostrowski, Tudor Ratiu, Shankar Sastry, Greg Walsh, and Jeff Wentlandt for their kind advice and help.

## REFERENCES

Abraham, R. and J.E. Marsden, 1978, *Foundations of Mechanics*, Second Edition, Reading, Mass.: Addison-Wesley.
Berry, M., 1990, "Anticipations of the geometric phase," *Physics Today*, December, 34-40.
Berry, M., 1985, "Classical adiabatic angles and quantal adiabatic phase," *J. Phys. A*. **18**, 15-27.
Bloch, A.M., P.S. Krishnaprasad, J.E. Marsden, and R.M. Murray, 1997, "Nonholonomic systems with symmetry," *Arch. Rat. Mech. Anal*. (to appear).
Bloch, A.M., P.S. Krishnaprasad, J.E. Marsden, and T.S. Ratiu, 1996, "The Euler-Poincaré equations and double bracket dissipation," *Commun. Math. Phys*. **175**, 1-42.
Bloch, A.M., P.S. Krishnaprasad, J.E. Marsden, and T.S. Ratiu, 1994, "Dissipation Induced Instabilities," *Ann. Inst. H. Poincaré, Analyse Nonlinßaire* **11**, 37-90.
Bloch, A.M., P.S. Krishnaprasad, J.E. Marsden, and G. Sánchez de Alvarez, 1992, "Stabilization of rigid body dynamics by internal and external torques," *Automatica* **28**, 745-756.
Bloch, A.M. and J.E. Marsden, 1989, "Controlling homoclinic orbits," *Theor. Comput. Fluid Mech*. **1**, 179-190.

Bobenko, A.I., A.G. Reyman, and M.A. Semenov-Tian-Shansky, 1989, "The Kowalewski Top 99 years later: A Lax pair, generalizations and explicit solutions," *Commun. Math. Phys*. **122**, 321-354.

Brockett, R.W., 1989, "On the rectification of vibratory motion," *Sensors and Actuators* **20**, 91-96.

Brockett, R.W., 1981, "Control theory and singular Riemannian geometry." In: *New Directions in Applied Mathematics*, P.J. Hilton and G.S. Young (eds.), Berlin, Heidelberg, New York: Springer-Verlag.

Enos, M. (ed.), 1993, "Dynamics and Control of Mechanical Systems: The Falling Cat and Related Problems," *Fields Institute Communications* , Vol. 1, Providence, RI: American Mathematical Society.

Hannay, J., 1985, "Angle variable holonomy in adiabatic excursion of an itegrable Hamiltonian," *J. Phys. A* **18**, 221-230.

Kammer, D.C. and G.L. Gray, 1993, "A nonlinear control design for energy sink simulation in the Euler-Poinsot problem," *J. Astron. Sci*. **41**, 53-72.

Kane, T.R., and M. Shur, 1969, "A dynamical explanation of the falling cat phenomenon," *Int. J. Solid Structures* **5**, 663-670.

Koiller, J., 1992, "Reduction of some classical nonholonomic systems with symmetry," *Arch. Rat. Mech. Anal*. **118**, 113-148.

Krishnaprasad, P.S., 1989, "Eulerian many-body problems," *Cont. Math. AMS* **97**, 187-208.

Kummer, M., 1981, "On the construction of the reduced phase space of a Hamiltonian system with symmetry," *Indiana Univ. Math. J*. **30**, 281-291.

Lewis, A., R.M. Murray, J.P. Ostrowski, and J. Burdick, 1997, "Nonholonomic mechanics and locomotion: the snakeboard example," IEEE Intern. Conf. on Robotics and Automation, to appear.

Marsden, J.E. and T.S. Ratiu, 1994, "An Introduction to Mechanics and Symmetry" *Texts in Applied Mathematics*, Berlin, Heidelberg, New York: Springer-Verlag.

Marsden, J.E., R. Montgomery, and T.S. Ratiu, 1990, "Reduction, symmetry, and phases in mechanics," *Memoirs AMS* 436.

Montgomery, R., 1991a, "Optimal Control of Deformable Bodies and Its Relation to Gauge Theory." In: *The Geometry of Hamiltonian Systems* , T. Ratiu (ed.), Berlin, Heidelberg, New York: Springer-Verlag.

Montgomery, R., 1991b, "How much does a rigid body rotate? A Berry's phase from the 18$^{th}$ century," *Am. J. Phys*. **59**, 394-398.

Montgomery, R., 1991c, "Optimal control of deformable bodies and its relation to gauge theory," In *The Geometry of Hamiltonian Systems* , T. Ratiu (ed.), Berlin, Heidelberg, New York: Springer-Verlag.

Montgomery, R., 1990, "Isoholonomic problems and some applications," *Commun. Math. Phys*. **128**, 565-592.

Montgomery, R., 1988, "The connection whose holonomy is the classical adiabatic angles of Hannay and Berry and its generalization to the non-integrable case," *Commun. Math. Phys*. **120**, 269-294.

Murray, R.M. and S.S. Sastry, 1993, "Nonholonomic Motion Planning: Steering Using Sinusoids," *IEEE Trans. Autom. Control* **38**, 700-716.

Oh, Y.G., N. Sreenath, P.S. Krishnaprasad, and J.E. Marsden, 1989, "The Dynamics of Coupled Planar Rigid Bodies Part 2: Bifurcations, Periodic Solutions, and Chaos," *Dynamics and Differential Equations* **1**, 269-298.

Shapere, A. and F. Wilczek, 1987, "Self-propulsion at low Reynolds number," *Physical Review Letters* **58**, 2051-2054.

Simo, J.C., D.R. Lewis, and J.E. Marsden, 1991, "Stability of relative equilibria I: The reduced energy momentum method," *Arch. Rat. Mech. Anal*. **115**, 15-59.

Smale, S., 1970, "Topology and Mechanics," *Invent. Math*. **10**, 305-331, **11**, 45-64.

Sreenath, N., Y.G. Oh, P.S. Krishnaprasad, and J.E. Marsden, 1988, "The dynamics of coupled planar rigid bodies. Part 1: Reduction, equilibria and stability," *Dynamics and Stability of Systems* **3**, 25-49.

Tilbury, D., R. Murray, and S. Sastry, 1993, "Trajectory generation for the N-trailer problem using Goursat normal form," *Proc. IEEE Control and Decision Conf*., San Antonio, Dec. 1993.

Walsh, G. and L. Bushnell, 1993, "Stabilization of multiple input chained from control systems," *Proc. IEEE Control and Decision Conf* ., San Antonio, Dec. 1993.

Walsh, G. and S. Sastry, 1995, "On reorienting linked rigid bodies using internal motions," *IEEE Trans. on Robotics and Automation* **11** (1), 139-146.

# 2

# Cycles That Effect Change

**Roger W. Brockett**

Division of Applied Sciences
Harvard University

New generations of motors are needed for applications where, due to small dimensions, a need for high torque, a need for precise control, or all of these factors, conventional electric or internal combustion motor technology cannot be used. In some instances, engineers desire suitable analogs to the high force and precise control provided by muscle tissue. We discuss the geometrical ideas that underlie a new generation of precisely controllable high-force actuators based on suitably patterned high-frequency vibrations. Current and planned applications of such actuators include their use in automatic focusing lenses and in endoscopic surgery.

## INTRODUCTION

Prevailing modes of thought in subjects, ranging from economics to biology, show the influence of ideas originating in the study of feedback control. The linear analysis that underlies the theory of single and multiple feedback loops has played an important role in explaining both stable regulation and exponential growth. There are, however, a variety of problems involving automatic control that cannot be explained on the basis of this body of results. For example, at least since the work of Graham Brown in 1911, it has been recognized that some provision for pattern generation must be incorporated in the neural circuitry used to generate and control various animal movements such as walking, breathing, blood circulation, and peristalsis. Pattern generation also plays an important role in the design of mechanical and electrical systems. The wheels of a steam locomotive should rotate steadily, yet their motive power comes from periodic processes involving the filling and emptying of the cylinders with steam and the corresponding motion of the pistons. In fact, both conventional piston-type steam engines and internal combustion engines depend on a carefully orchestrated, repetitive motion pattern as an intermediate step in the generation of steady rotational motion; the recently introduced vibratory motors depend on an even more subtle type of pattern generation. Similar ideas play a role in certain types of electrical circuits. Technologically important examples include parametric amplifiers, switched capacitor

Note: This work was supported in part by the National Science Foundation under grant NSF D CDR-8803012 (Engineering Research Center Program), and by the U.S. Army Research Office under grant DAAL03-86-K-0171 (Center for Intelligent Control Systems).

filters, and DC to DC transformers that make use of periodic switching to transform the voltage available from some supply, such as a battery, to the voltage required by the transistor or motor that is being powered. In many cases, the explanation of the behavior of these types of examples is much more subtle than the explanation of ordinary linear regulation. In this paper we use examples from various domains to illustrate the mathematical ideas that lie at the heart of these problems.

Perhaps the most fundamental arguments as to why periodic processes are required to produce nonperiodic effects seem to be based on considerations of kinematics and force amplification. Animals and automobiles need to cover distances that far exceed the longest linear dimension in their makeup. They cannot simply reconfigure their bodies to cover the distances involved. Moreover, muscles, magnets, and expanding gases can only generate their significant force over a limited range of displacements. Having generated a force over this range, it is necessary to reconfigure before being able to generate the same level of force again. Among the possible temporal patterns of reconfiguration, some are more effective than others. Having found an effective one, it can be repeated over and over, giving a cyclic process that enables the coverage of large distances by means of repeated short distance movements.

Within this overall paradigm there is a further important distinction to be made. Certain periodic processes operate with a fixed amplitude, piston engines being a good example. Other periodic processes, such as the motion of an inchworm and the swimming motion of a fish, can operate at a variety of amplitudes. In the case of variable amplitude devices it may happen that the mechanical advantage increases as the amplitude decreases. Theories dealing with nonlinear controllability provide considerable insight into the capabilities of systems of this latter type. Understanding the dynamics of their regulatory processes requires more study, and only recently has there been an appropriate mathematical formulation of a control problem in which pattern generation plays a decisive role. We touch on this in our final section.

## ORDER SOMETIMES MATTERS

In choosing from various possible actions that one may take, it sometimes happens that a particular set of actions applied in one order has an overall effect that is different from that obtained when the same set of actions is applied in a different order. The order in which we make deposits and withdrawals in a checking account does not affect the end-of-the-month balance. Driving in a city laid out on a rectangular grid, we can go north for one block then east for one block and get to the same location as we would if we first went east for one block and then north for one block. On the other hand, there are situations in which order matters very much. The most obvious examples, such as opening a door and then walking though it versus walking through the door and then opening it, do not lead to very interesting, or general, mathematical models. However, the situation shown in Figure 2.1 does embody a rather general mathematical/physical principle. Because it is illustrative of several of the main points we will analyze it in detail.

The illustration depicts a pair of tanks. The top tank holds fluid in a vessel that is fitted to create a sealed chamber below it. The lower chamber is full of fluid. Fluid can be pumped into or out of the tanks. For the purposes of exposition, we suppose that there are individual agents responsible for the management of the flow rates *u* and ν. One agent, associated with the flow rate *u*, adds or removes fluid to the top tank. The second, associated with flow rate ν, adds or removes fluid to the lower chamber. One of the fundamental ideas in hydrostatics is that the pressure produced by a column of fluid is independent of the shape of the column and is proportional to the height of the column. The power required to pump a fluid is the flow rate times the pressure. Applying these facts to the situation at hand, we see that, when expressed in suitable units, the energy supplied by the agent responsible for the flow *u* must provide energy at the rate $u(x + y)$ per unit time whereas the agent responsible for flow ν must provide energy at rate $\nu(x + y)$ per unit time.

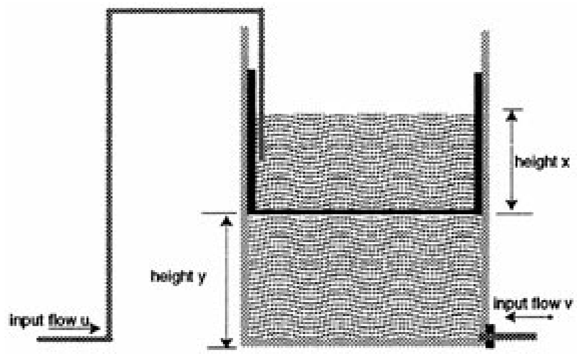


FIGURE 2.1 A simple flow-level model of a nonholonomic system.

The virtue of having a model that can be visualized this concretely is thin the differential equation description can be related to common sense in a direct way. For example, it is clear that the time rate of change of *y*, the height of the fluid in the bosom chamber measured from the base, is proportional to the flow rate ν. The time rate of change of *x*, the height of the fluid in the top tank, measured from the bosom of thin tank, is proportional to *u*. Denoting the time rate of change of a variable *x* by means of an overdot, e.g., $\dot{x}$, we may summarize the statements about flow rate and levels as

$$\dot{x}(t) = u(t)$$
$$\dot{y}(t) = v(t).$$

Now suppose that the agents are concerned with the energy they must expend to manage the levels in the ranks. The corresponding equations describing the time rate of change of the energy that must be expended by the agent controlling *u*, call it $e_u$, and the time rate of change of the energy that must be expended by the agent controlling ν, $e_\nu$ satisfy the equations

$$\dot{e}_u = (x(t) + y(t))u(t)$$
$$\dot{e}_v = (x(t) + y(t))v(t).$$

Simple as this system is, by analyzing the flow of energy one comes to conclusions that are not entirely obvious. To begin with, it is clear that if it is necessary to bring the levels in the tanks to specific values, the bottom tank to the level $y = yo$, and the top tank to $x = xo$, different courses of action on the part of individual agents will result in different expenditures of energy by the individual agents. For example, if the tanks are initially empty and if the bottom tank is filled to the level $y = 1$ while the top tank remains empty and then the top tank is filled to the level $x = 1$ while the bottom tank remains at $y = 1$, then agent *u* supplies energy 1/2 and agent ν supplies energy 3/2. If we reverse the order, i.e., if agent ν fills the top tank first and then agent *u* fills the bottom tank, it follows that the energy requirements of the agents are just reversed. A similar analysis applies to pumping fluid from the tanks.

By making use of periodic filling and emptying, the difference between the energy requirements for *u* and ν can be made as large as one wants and can favor either agent. There is no limit on the energy difference. We can find filling policies that take an arbitrary amount of energy *e* from agent *u* and take energy 2- *e* from agent ν. We can use this device as a way to transfer energy from one agent to the other. The important point is that the difference between the energy supplied by *u* and the energy supplied by ν is not just a function of the final state, but rather depends on the order, now interpreted in some infinitesimal sense, in which the tanks are filled. This is an example of what is meant by the phrase *path dependent*. The infinitesimal relationships between *x, y* and $e_u$-$e_v$ are *nonintegrable*. In a mechanical context they could be said to define a *nonholonomic* system.

Before leaving this example we want to recast, slightly, the energy equations. First of all, because the sum of the energies supplied by the two agents is just the difference between the initial and final value of $x + y$, there is no need to keep track of the sum of the energies supplied by the agents. However, the difference in the energies supplied by the agents is dependent on the path and, therefore, requires us to use a differential equation to keep track of it. This means that we can use just three equations for the levels and the energy relationships, not four. Finally, it will simplify the equations if we focus on the difference in the supplied energy plus the difference in the energy available to the individual agents, should they wish to empty their tanks. With this in mind, we define $z$ as $z = e_u$-$e_v + x^2$-$y^2 / 2$ and describe the system by

$$\begin{aligned}\dot{x}(t) &= u(t)\\ \dot{y}(t) &= v(t)\\ \dot{z} &= v(t)x(t) - u(t)y(t).\end{aligned}$$

Expressed in this form, the energy flow relationships of the situation depicted in Figure 2.1 can be explained in terms of a remarkably simple geometric picture to be developed in the next section.

## SIMPLE MACHINES AND THE AREA RULE

We emphasized, in the case just analyzed, that the difference between the energy supplied by *u* and the energy supplied by ν is strongly dependent on the path taken to accomplish the task. In fact, the way that it depends on the path is particularly simple: the difference between the amount of energy supplied by $u$ and the amount of energy supplied by $v$ is proportional to the area of the closed path defined by going around a cycle in $(u, v)$-space. It is characteristic of the simplest class of nonholonomic systems to exhibit such a proportionality. (See Figure 2.2.)

Physically speaking, the area in $(u, v)$-space has the units of flow rate squared. To equate this to energy flow one needs a constant factor having suitable units. This factor can be expressed in terms of the value of a certain component of the Lie bracket of two vector fields; as $u$ and $v$ vary in a cyclical way, $z$ increases monotonically with a rate proportional to the product of this factor and the area in $(u, v)$-space.

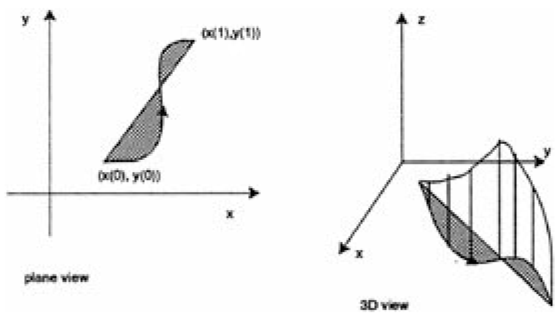


FIGURE 2.2 The area rule in x-y and x-y-z coordinates.

The tank analysis carried out above may seem far removed from mechanics and the basic mechanisms of mechanical engineering. However, there are mechanical realizations of the same mathematical model based on spheres rolling on planes. In Figure 2.3 we illustrate an elementary and useful mechanical device, consisting of a screw engaging a slider having regularly spaced gear teeth. This configuration couples the rotational motion of the screw to the horizontal motion of the slider. If $z$ represents the position of the slider and if $x$ and $y$ are the coordinates of a point on the crank handle, then the equations relating $x$, $y$, and $z$ are just a rescaled version of the ones introduced above. More precisely, if the screw has $p$ threads per unit length, then we may describe the kinematics by

$$\begin{aligned} \dot{x}(t) &= u(t) \\ \dot{y}(t) &= v(t) \\ \dot{z} &= \frac{1}{pa}(v(t)x(t) - u(t)y(t)) \end{aligned}$$

subject only to the constraint that $x^2 + y^2 = a^2$. Thus a suitably restricted version of the equations relating height and energy flow in the tanks also describes the relationship between the rotational motion and the translational motion of this mechanical system.

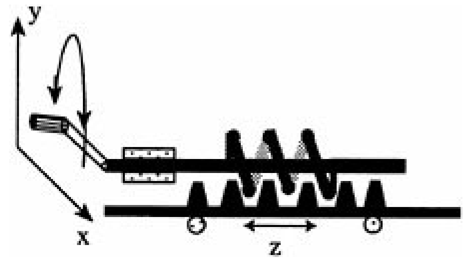


FIGURE 2.3 Converting rotational motion to rectilinear motion with a screw and slider.

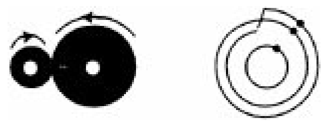

FIGURE 2.4 Rotating wheels and the corresponding covering space.

Of course the fact that this screw and slider mechanism represents only the case in which $x^2 + y^2 = a^2$ means that the overall behavior of this system is just a subset of the possible behaviors of the tank system. In this case we can say that for each point on the circle $x^2 + y^2 = a^2$ there are (in the case of an infinitely long slider extending in both directions) a countably infinite number of possible locations for the slider and that these locations are discrete, separated by the distance that the slider travels when the screw rotates by one full turn. Thus this mechanism displays a kind of discrete path dependence. In the language of topology, one would say that the space of possible $z$ coordinates forms a coveting space for the space of possible $(x, y)$ values. We illustrate this idea in a somewhat simpler setting also involving the kinematics of machines. In Figure 2.4 we show two wheels, one having twice the diameter of the other, rolling on each other without slipping. If we are told that a particular mark on the small wheel lies at 9 o'clock then we can say that a particular mark on the bigger wheel lies at one of two possible locations, separated by 180 degrees. Again, the space of possible configurations of the bigger wheel forms a covering space for the set of configurations of the smaller one.

A significant difference between the way the set of configurations of the slider covers the set of configurations of the screw versus the way that the set of configurations of the large wheel covers the set of configurations of the small wheel is illustrated on the tight side of Figure 2.4, which shows that there are only two points "sitting over" each point on the small circle, not a countably infinite number as in the screw-slider case. Of course the situation represented by the basic nonholonomic model is even more extreme. In that case there is a continuum of $z$-values corresponding to a given value of $x$ and $y$. In fact, any $z$ value can sit above a given value of $x$ and $y$. In this sense we can think of the nonholonomic system as having the capability of generating any relationship between these variables, containing any fixed-gear ratio system as a special case.

## VIBRATIONAL, ROTARY, AND LINEAR MOTION

It is a familiar story that Hero of Alexandria invented a simple steam engine in the first century AD. The drawings of it found in history books show it spinning smoothly over the fire, producing motion but not much torque. Hero's invention did not enable any industrial revolution; the greater part of two thousand years elapsed before that occurred. And when steam power did become useful, it was through the more indirect designs of Newcomen and Watt, based on the generation of reciprocating motion that proved to be effective. In fact, there are a great many examples, ranging from biped locomotion to ratcheting, in which the generation of motion involves the conversion of force developed by a periodic process into some desired steady translation or rotation.

The standard way to convert rotational motion into reciprocating motion is by means of a crank and slider mechanism illustrated in Figure 2.5. As the crank turns in a circle, steadily advancing, the connecting rod connected to the crank moves back and forth. Thought of from the opposite point of view, as the connecting rod moves back and forth, the crankshaft rotates in a steady advance. Thus the same mechanism converts reciprocating motion to circular motion. Figure 2.6 illustrates a different means for converting rotary motion into translational motion. These ideas are relevant to a discussion of nonholonomic systems because they provide a useful basis for comparison to the way in which nonholonomic systems convert oscillatory inputs into steady unidirectional motion.

Recently there has been renewed interest in a variety of rather different mechanisms for the generation of motion. Figure 2.7 provides some insight into the nature of one of these new ideas. An elastic member is shown supported at each end in such a way as to permit it to vibrate in two different modes. The first of these may be characterized as a general vertical motion, larger in the middle and smaller toward the supports. The second mode is an asymmetrical motion characterized by a shape such that when the elastic is raised on the fight it is lowered on the left and vice versa. If these modes were to vibrate at the same frequency and with the correct phase relationship, the overall motion of the tip protruding upward from the center would produce an elliptical path as shown at the bottom of the figure. Just as it takes two agents to manipulate the energy flow in the tank example, it takes two modes having the same frequency and appropriate phase relationship to generate the elliptical motion with nonzero area.

In Figure 2.8 we illustrate how this type of two-mode motion can be coupled to a rigid member in such a way as to generate translational motion. The operation of the vibratory motors referred to in the introduction can be explained in this way.

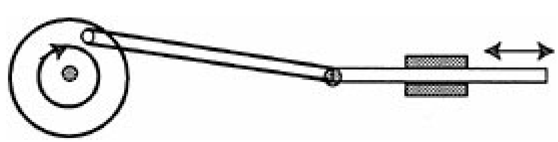

FIGURE 2.5 Converting rotary to reciprocating motion.

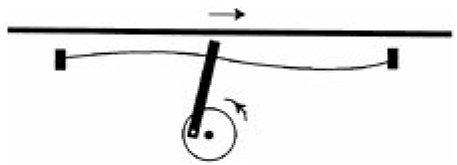

FIGURE 2.6 A mechanism for converting rotational to translational motion.

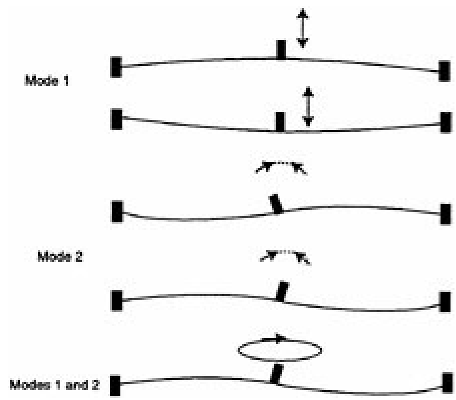


FIGURE 2.7 A multimode vibration of a beam.

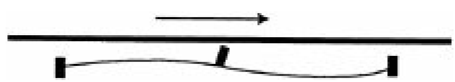

FIGURE 2.8 A mechanism for converting vibrational to translational motion.

## ELECTRICAL CIRCUITS WITH SWITCHES

The scope of the above arguments, based as they are on simple geometric ideas, is quite wide and can be applied in a variety of circumstances. One of the especially important areas is that of electrical networks whose behavior is controlled by switches. Nearly every personal computer contains one or more subsystems of this type to provide electrical power at the various voltage levels appropriate for memory chips, disk drive motors, etc. In the recent past, different, more expensive, and less efficient methods based on linear analysis were used. Although our treatment here is brief, we will try to give the reader a feeling for this type of application by recasting some of the arguments given above in terms of electrical circuits.

Electrical engineers often deal with periodic signals such as sine waves and square waves. In some cases it is natural to think of smoothly modulated vector fields, as suggested by

$$\dot{x}(t) = \sin(\omega t) g_1(x(t)) + \cos(\omega t) g_2(x(t)),$$

whereas in other cases it might be more useful to think of

$$\dot{x}(t) = ssq(\omega t) g_1(x(t)) + csq(\omega t) g_2(x(t)),$$

with the understanding that *csq* ($\omega t$) is a square wave, taking on the values =1 and -1 and having a sign that agrees with that of cos ($\omega$t). Similarly, *ssq* ($\omega t$) is a square wave whose sign always agrees with the sign of sin ($\omega t$). The above system can be thought of as one whose time evolution is governed by two vector fields. In operation, one switches periodically among the set $g_1 + g_2$, $g_1$-$g_2$,-$g_1 + g_2$, and-$g_1$-$g_2$. According to the area rule, the response to the sinusoidal system and the response of the square wave system will be nearly the same, provided that the inputs are scaled so as to make the area, in the sense described in the section "Simple Machines and the Area Rule," the same.

Electrochemical batteries generate a steady voltage, which, in the presence of suitable external circuit, will cause a current to flow through them, always in the same direction. By contrast, in almost all cases, electrical power from utility companies is distributed to customers in the form of alternating current, that is, in the form of a current that alternates between the two possible directions of flow many times per second. In some important applications, such as powering electronic equipment and charging rechargeable batteries, it is necessary to convert this alternating current into direct current before it can be used. This process of conversion, called rectification, used to be a relatively inefficient and expensive process, requiting bulky equipment and producing unwanted heat. Today there exist solid state electronic devices capable of providing elegant and efficient solutions to this problem at modest cost. A pervasive problem in electronics is that of producing the required voltages for various applications from a given source. In some cases this means converting battery voltage of 1.5 volts to 12 volts as required for typical electronics applications. To do this one now uses *switching converters* , which can be thought of as electrical analogs of the problems we have been discussing. In electrical terms, we may say that their operation is based on a sort of "inverse rectification" in which switches are used to generate alternating current. Figure 2.9 shows a circuit with two switches, two inductors, a capacitor, and a battery. The switches allow one to control the evolution of the divers variables that describe the network. The labels *u* and ν refer to the two switches and denote variables that represent the position of the switches in the diagram.

The time evolution of this network is governed by a set of equations that are similar to those describing the tanks, although the meaning of the control terms is now different. In terms of the assignments given below, they take the form

$$\dot{x}(t) = 1 - v(t) + (1 - u(t))z(t)$$
$$\dot{y}(t) = v(t) - u(t)z(t)$$
$$\dot{z}(t) = y(t)u(t) - (1 - u(t))x(t).$$

If the switch labeled *u* is closed to the left, then $u = 0$; if it is closed to the right, then $u = 1$. We use the same convention for ν. We attach no meaning to other values of *u* and ν. The variables *x, y* are the currents through the inductors. The variable *z* is the voltage across the capacitor. For simplicity, we take the voltage on the battery to be one. If the voltage across the capacitor is small compared to the voltage on the battery, then the ν*z* term in the first equation and the *uz* term in the second equation can be ignored and, after a relabeling of the controls, the equations become identical to those discussed above.

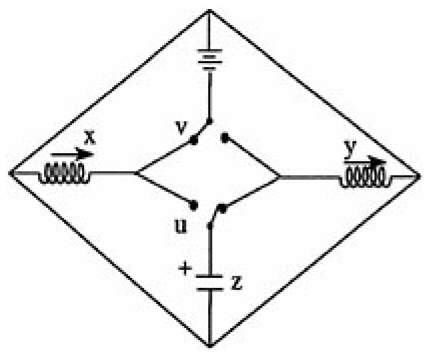


FIGURE 2.9 An electrical network controlled by switches.

## THREE THINGS FROM TWO THINGS

One important aspect of nonholonomic systems, not brought out in our discussion above, is the extraordinary possibility of using a relatively small number of inputs to force the system to follow paths in a high dimensional space with small error. We begin our discussion with an analogy from telecommunications.

A single pair of telephone wires can carry many distinct conversations at the same time. Years ago, engineers demonstrated that it is possible to superimpose multiple signals, send them to their destination, and then disentangle them for delivery to their intended recipient. This is accomplished by recognizing that a speech signal carries most of its useful information in a limited frequency range whereas copper wires can reliably carry signals whose energy occupies a much wider range of frequencies. By mapping different telephone conversations into different, nonoverlapping frequency bands at the sending end, and then reversing the process at the receiving end, one can use the transmission line more efficiently. This is an example of a more abstract circle of ideas involving a general theory of multiplexing, i.e., combining many independent signals into one, and demultiplexing, i.e., separating one signal into many independent ones.

The model nonlinear system we have been using for the purposes of this exposition displays a capability for demultiplexing. It is possible to use it to extract three signals, having a surprisingly high degree of independence, from the two inputs. In Figure 2.10 we diagram the relationship between the various causes and effects represented by the differential equations of our standard model. This type of block diagram differs from the usual physical diagrams, such as that shown in Figure 2.1, in that one does not attempt to be faithful to the physics or spatial relationships but, as happens when one represents natural phenomena in terms of equations, attempts to express the abstract relationships clearly. In this sense such diagrams are just mathematics all over again but now arranged in such a way as to make certain types of interrelationships more obvious.

This diagram depicts a system with two independent inputs and three outputs. Intuition, in many cases backed up by theorems in topology, suggests that one should not be able to control three things with two things. Indeed, if the spaces are finite dimensional and/or the relationships are linear, this idea can be made into a precise statement and proven. The present situation is different. We can get some feeling for it by examining the response to a class of inputs of a particularly simple type. Suppose that we apply the inputs

$$u(t) = \sqrt{\omega a}\cos(\omega t)$$
$$v(t) = \sqrt{\omega b}\cos(\omega t + \phi).$$

Assuming the appropriate initial conditions, the resulting response is

$$x(t) = \frac{a}{\sqrt{\omega}}\sin(\omega(t))$$
$$y(t) = \frac{b}{\sqrt{\omega}}\sin(\omega(t) + \phi)$$
$$z(t) = abt\sin\phi.$$

This shows that the effect of such an input on $x$ and $y$ can be made as small as we wish by increasing the frequency $\omega$, whereas its effect on $z$ is independent of $\omega$. In fact, one may see here the shadow of a very general fact proven by Liu and Sussmann showing that, in our language, a very wide class of nonlinear controllable systems can act as demultiplexers.

One of the possible interpretations of this idea is that systems such as the two-input, three-output system in Figure 2.10 have approximate inverses. In this case the inverse takes the form of a three-input, two-output precompensator, such that the overall precompensated system is, for low-frequency inputs, nearly the identity operator. One interesting aspect of these inverse systems is that they are necessarily time-varying even though the original system is described by time-invariant differential equations. This can be interpreted as supporting the need for pattern generation. When one approaches the difficult problem of designing stabilizing feedback control laws for nonholonomic systems from this point of view, it often becomes much easier to understand.

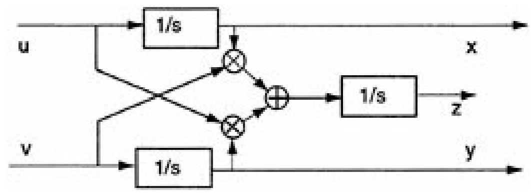


FIGURE 2.10 A block diagram representation of the basic system.

## CONCLUSIONS

New ways of thinking about nonlinear systems have lead to new ways of taking advantage of their properties. Nonholonomic systems, a particularly interesting class of nonlinear systems that were once cause for alarm because of their counterintuitive properties, are now being incorporated in practical systems. Some ideas from differential geometry, algebra, and other areas of mathematics have played an important role in this process. In the not too distant future we expect to see some of these ideas being applied to new problems in biological motion control, mechanical design, electronics, and in areas we cannot yet anticipate.

# BIBLIOGRAPHY

Brockett, R.W., 1989, "On the Rectification of Vibratory Motion," *Sensors and Actuators* **20** 91-96.

Brockett, R.W., 1983, "Asymptotic Stability and Feedback Stabilization." In: *Differential Geometric Control Theory*, R.W. Brockett, R. Millman, and H. Sussmann (eds.), Boston: Burkhauser.

Brockett, R.W., 1981, "Control Theory and Singular Riemannian Geometry." In: *New Directions in Applied Mathematics*, pp. 11-27, P. Hilton and G. Young (eds.), New York: Springer-Verlag.

Brown, G., 1911, "Intrinsic Factors in the Act of Progression of the Mammal," *Proc. R. Soc. London* **84**, 308-319.

Krishnaprasad, P.S., 1990, "Geometric Phases and Optimal Reconfiguration of Multibody Systems," *Proceedings of the 1990 Automatic Control Conference*, 2440-2444.

Liu, W. and H.J. Sussmann, 1991, "Limits of Highly Oscillatory Controls and the Approximation of General Paths by Admissible Trajectories," *Proceedings of the 30th IEEE Conference on Decision and Control*, Brighton, England, Dec., 437-442.

Murray, R.M. and S.S. Sastry, 1991, "Steering Nonholonomic Systems Using Sinusoids," *Proceedings of the 29th IEEE Conference on Decision and Control*, New York.

Severns, R. and E. Bloom, 1987, *DC to DC Converters*, New York: Van Nostrand.

Shapere, A. and F. Wilczek, 1989, *Geometric Phases in Physics*, Singapore: World Scientific.

von Euler, K., 1985, "Central Pattern Generation During Breathing." In: *The Motor System in Neurobiology*, E. Evarts, S. Wise, and D. Bousfield (eds.), Amsterdam: Elsevier Biomedical Press.

# 3

# Geometric Phases, Control Theory, and Robotics

**Richard M. Murray**
Department of Control and Dynamical Systems and
Department of Mechanical Engineering
California Institute of Technology

Differential geometry and nonlinear control theory provide essential tools for studying motion generation in robot systems. Two areas where progress is being made are motion planning for mobile robots of factory floors (or on the surface of Mars), and control of highly articulated robots—such as multifingered robot hands and robot "snakes"—for medical inspection and manipulation inside the gastrointestinal tract. A common feature of these systems is the role of constraints on the behavior of the system. Typically, these constraints force the instantaneous velocities of the system to lie in a restricted set of directions, but they do not actually restrict the reachable configurations of the system. A familiar example in which this geometric structure can be exploited is parallel parking of an automobile, where periodic motion in the driving speed and steering angle can be used to achieve a net sideways motion. By studying the geometric nature of velocity constraints in a more general setting, it is possible to synthesize gaits for snake-like robots, generate parking and docking maneuvers for automated vehicles, and study the effects of rolling contacts on multifingered robot hands. As in parallel parking, rectification of periodic motions in the control variables plays a central role in the techniques that are used to generate motion in this broad class of robot systems.

## INTRODUCTION

The earliest robots consisted of simple electromechanical devices that could be programmed to perform a limited set of tasks. They were a cross between numerically controlled milling machines and the master-slave teleoperators developed for handling radioactive material. These robots are the

Note: Research supported in part by grants from the Powell Foundation, the National Science Foundation, and the National Aeronautics and Space Administration.

precursors to the automated machines used for painting, welding, and pick-and-place operations in today's factories. For these types of automation and manufacturing tasks, the complexity of the robot can be minimized since the workspace of the robot can be carefully controlled and the robots are not required to perform particularly dextrous manipulation of objects. However, many future applications of robotics are moving toward more autonomous operation in highly uncertain environments. The robots being developed for these applications are increasingly complex and have a high degree of interaction with their environment.

Examples of the next generation of robots range from miniature robots for medical inspection and manipulation inside the human body to mobile robots for exploration in hazardous and remote environments. All these robots will require sensing, actuation, and computational capabilities that were unheard of just a few years ago. However, as we seek to design robots that can act with increasing autonomy, we move closer to endowing robots with human-like capabilities. And we begin to become limited by our own ability to understand and analyze the highly complex systems that we are trying to control.

As the complexity of robots increases, so does the importance of abstraction and theory in understanding and analyzing robot motion. One approach that has begun to yield new insights is the use of differential geometry, in the context of both geometric mechanics and nonlinear control theory. Two specific areas where progress is being made are locomotion and manipulation in robot systems.

Locomotion is defined as the act of moving from one place to another. For robots, there are several mechanisms by which this movement can occur. The use of wheels and of legs are the two traditional methods, but other possibilities, such as undulatory gaits in snake-like robots, have also been proposed and implemented. Each of these mechanisms has certain advantages over the others, but all of them fundamentally involve interaction with their environment. Locomotion is achieved by pushing, sliding, rolling, or a combination of all of these.

Robotic manipulation involves motion of an object rather than motion of the robot itself. The prototypical example is a multifingered hand manipulating a grasped object. Once again, the fundamental mechanisms that govern motion involve pushing, rolling, and sliding. The motion of a set of fingers grasping an object is constrained in much the same way as the motion of a legged robot is constrained by the contacts between its feet and the ground. Indeed, many of the tools that are used to analyze manipulation and grasping problems are easily adapted to analyze locomotion.

The most basic problem in all locomotion and manipulation systems is to devise a method for generating and controlling motion between one configuration and another. The common feature is that motion of the robot is constrained by its interactions with the environment. For example, in wheeled mobile robots the wheels must roll in the direction in which they are pointing and they must not slide sideways. In grasping, the motion of the fingers is constrained by the object being held in the grasp: motion of one finger affects the others since forces are transmitted between the fingers by the object. Even in legged robots, one usually assumes that the feet do not slip on the ground, allowing the robot to propel itself. These constraints on the motion of the system are the defining features for how locomotion and manipulation work in these systems.

Furthermore, in most locomotion and manipulation systems, the range of the actuators is small, while the desired net motion for the system may be large. A good example of this is using your fingers to screw in a light bulb: repeated grasping and twisting of the bulb is required in order to fully insert it into the socket. A large motion of the light bulb (multiple revolutions) is accomplished by repeated (i.e., periodic) small motions in your fingers.

In this paper we consider locomotion and manipulation using the notion of geometric phases as a central theme. Intuitively, geometric phases relate the motion of one parameter describing the configuration of a system to other parameters that undergo periodic motion. A simple example of geometric phase is the motion of an automobile performing a parallel parking maneuver. By moving the car backwards and forwards and turning the steering wheel in a periodic fashion, a driver is able to achieve a net sideways motion of the car even though the car cannot move sideways directly. This net sideways motion is the geometric phase associated with this choice of the car velocity and steering wheel angle.

The role of geometric phases as a means of analyzing locomotion is a relatively new perspective. One of the earliest works is that of Shapere and Wilczek (1989), who studied the motion of paramecia swimming in a highly viscous fluid. They show that periodic variations in the shape of an organism can be used to achieve net forward motion. This is very reminiscent of the type of motion present in parallel parking and this similarity can be made precise by using geometric phases.

There has also been an increased interest in the use of geometric phases for understanding motion in other biological systems, such as snakes and insects. Here again, periodic changes in one set of variables, which describe the shape of the system, are used to obtain net motion. The phasing of the inputs plays a central role, generating different gaits for achieving different types of motion. The interpretation of locomotion in terms of geometric phases is still far from complete, but it is providing a unifying view of locomotion and manipulation that has already yielded new insights and has impact on several challenging applications.

## LOCOMOTION IN MOBILE ROBOTS

Locomotion involves movement of a mechanical system by appropriate application of forces on the robot. These forces can arise in several ways, depending on the means of locomotion used. The simplest form of locomotion is to apply the forces directly, as is done in a spacecraft, where high-energy mass is ejected in the direction opposite to the desired motion. A similar technique is the use of jet engines on modem aircraft.

For ground-based systems, a much more common means of locomotion is the use of forces of constraint between a robot and its environment. For example, a wheeled mobile robot exerts forces by applying a torque to its drive wheels. These wheels are touching the ground and, in the presence of sufficient friction, are constrained so as not to slip along the ground. This constraint is enforced by the application of internal forces, which cause a net force on the robot that propels it forward. If no constraints existed between the robot and the ground, then the robot would just spin its wheels. Similarly, for legged and snake robots, the parts of the robots in contact with the environment are used to exert net forces on the robot. In fact, for a large class of robotic systems we can view constraints as the basis for locomotion.

A second common feature in robot locomotion is the notion of base (or internal) variables versus fiber (or group) variables. Base variables describe the geometry and shape of the robot, while fiber variables describe its configuration relative to its environment. For example, in a snake robot the fiber variables might be the position and orientation of a coordinate frame fixed to the robot's body, while the base variables would be the angles that describe the overall shape of the robot. These base and fiber variables are coupled by the constraints acting on the robot. Hence, by making changes in the base variables, it is possible to effect changes in the fiber variables.

In this paper we concentrate on a particular type of constraint on the configuration variables of the robot, known as a *Pfaffian* constraint. Consider a mechanical system with configuration space $Q = \mathbb{R}^n$ and configuration $q \in Q$. A Pfaffian constraint restricts the motion of the system according to the equation

$$\omega(q)\dot{q} = 0\,,$$

where $\omega(q)$ is a row vector that gives the direction in which motion is not allowed. Pfaffian constraints arise naturally in wheeled mobile robots: they model the ability of a wheel to roll along the ground and spin about its vertical axis but not slide sideways. Pfaffian constraints typically do not provide a complete model of the interaction with the environment, since frictional forces are present in both rolling and spinning, but they do capture the basic behavior of the system.

As an example, consider a simple kinematic model of an automobile, as shown in Figure 3.1. The constraints are derived by assuming that the front and rear wheels can roll and spin (about the center of the axle) but not slide. Let $q = (x, y, \theta, \phi) \in \mathbb{R}^4$ denote the configuration of the car, parameterized by the *xy* location of the center of the rear axle, the angle of the car body with respect to the horizontal, θ, and the steering angle with respect to the car body, $\phi$. To simplify the derivation, we model the front and rear pairs of wheels as single wheels at the midpoints of the axles. The constraints for the front and rear wheels are formed by setting the sideways velocity of the wheels to be zero. A simple calculation shows that the Pfaffian constraints are given by

$$\begin{aligned} \omega_1(q)\dot{q} &= \sin\theta\dot{x} - \cos\theta\dot{y} = 0 \\ \omega_2(q)\dot{q} &= \sin(\theta + \phi)\dot{x} - \cos(\theta + \phi)\dot{y} - l\cos\phi\theta = 0. \end{aligned}$$

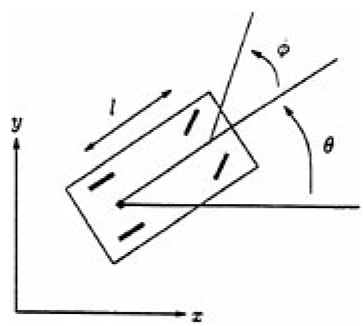


FIGURE 3.1 Kinematic model of an automobile. The configuration of the car is determined by the Cartesian location of the back wheels, the angle the car makes with the horizontal, and the steering wheel angle relative to the car body. The two inputs are the velocity of the rear wheels and the steering velocity.

For simplicity we take $l = 1$ in the sequel.

To study the motion of a system subject to a set of Pfaffian constraints $\{\omega_1,\ldots,\omega_k$, it is convenient to convert the problem to a control problem. Roughly speaking, we would like to shift our viewpoint from describing the directions in which the system cannot move to describing those in which it can. Formally, we choose a basis for the right null space of the constraints, denoted by $g_i(x) \in \mathbb{R}^n, i = 1,\ldots,n-k$. The locomotion problem can be restated as finding an input function, $u(t) \in \mathbb{R}^{n-k}$, such that the control system

$$\dot{x} = g_1(x)u_1 + \ldots + g_{n-k}(x)u_{n-k}$$

achieves a desired motion. The $g_i$'s can be regarded as vector fields on $\mathbb{R}^n$, describing the allowable motion of the system. This type of control system is called a *driftless* control system since setting the inputs to zero stops the motion of the system.

For the kinematic car, this conversion yields the control system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \theta \\ \phi \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \tan\phi \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2 .$$

For this choice of vector fields, $u_1$ corresponds to the forward velocity of the rear wheels of the car, and $u_2$ corresponds to the velocity of the steering wheel.

The first question one must consider when analyzing a control system is whether the system is *controllable*. That is, given an initial state $x_i$ and a final state $x_f$, does there exist a choice of inputs $u$ that will steer the system from one state to the other? A geometric interpretation of this question can be formulated by studying the properties of the vector fields that define the control system. A set of vector fields which is *not* controllable is shown in Figure 3.2. For these vector fields, there exists a surface whose tangent space contains the span of the vector fields: Hence, any motion of the system is necessarily restricted to this surface and it is not possible to move to an arbitrary point in the configuration space (only to other points on the same surface).

To test whether a set of vector fields are tangent to some surface, we make use of a special type of motion called a *Lie bracket motion* . Roughly, the idea is to choose two vector fields, say $g_1$ and $g_2$, and construct an infinitesimal motion by first flowing along $g_1$ for ε seconds, then flowing along $g_2$ for ε seconds, and then flowing backwards along $g_1$ for ε seconds and backwards along $g_1$ for ε seconds. This motion is illustrated in Figure 3.3. A simple Taylor series argument shows that the net motion given by this strategy is

$$q(4\varepsilon) = q_0 + \varepsilon^2\left(\frac{\partial g_2}{\partial q}g_1(q_0) - \frac{\partial g_1}{\partial q}g_2(q_0)\right) + O(\varepsilon^3) .$$

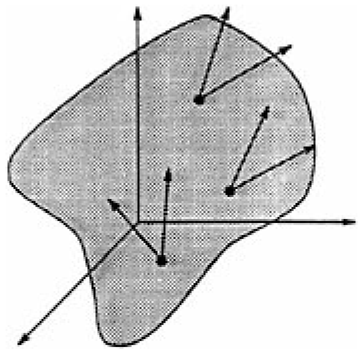

FIGURE 3.2 A set of vector fields that are tangent to a hypersurface in the configuration space. The system described by this set of vector fields is not controllable since motion is restricted to a hypersurface.

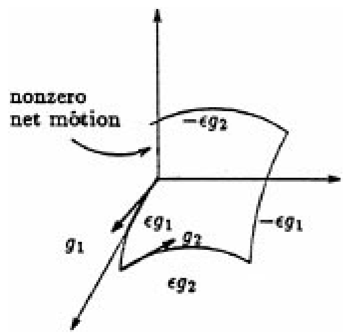


FIGURE 3.3 A Lie bracket motion.

(See Murray et al., 1994, pp. 323-324 for a derivation.) Motivated by this calculation, we define the Lie bracket of two vector fields $g_1$ and $g_2$ as

$$[g_1, g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2. \quad (3.1)$$

The Lie bracket of $g_1$ and $g_2$ describes the infinitesimal motion due to cycling between the inputs corresponding to $g_1$ and $g_2$.

The Lie bracket between two vector fields gives a potentially new direction in which we can move. In particular, if $[g_1,g_2]$ is not in the span of $g_1$ and $g_2$, then it is not possible for $g_1$ and $g_2$ to be locally tangent to a two-dimensional surface, since we could move off such a surface by executing a Lie bracket motion. Furthermore, from a controllability point of view we can treat $g_3 = [g_1,g_2]$ as a new direction in which we are free to move, and we can look at higher order bracket motions involving $g_3$. A fundamental result, proven in the 1940s by the German mathematician W.-L. Chow (1940), is the central result in controllability for control systems of this type. Let $\bar{\Delta}_q$ be the set of all directions that can be achieved by the input vector or repeated Lie brackets. That is,

$$\bar{\Delta}_q = \operatorname{span}\left\{g_i(q), \left[g_i, g_j\right](q), \left[g_i, \left[g_j, g_k\right]\right](q), \text{etc.}\right\}.$$

**Theorem 1 (Chow)** A driftless control system is controllable in a neighborhood of $q \in \mathbb{R}^n$ if $\bar{\Delta}_q = \mathbb{R}^n$.

We can verify that the kinematic car satisfies Chow's theorem by direct calculation. The input vector fields are given by

$$g_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \tan\phi \\ 0 \end{bmatrix} g_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

We call $g_1$ the *drive* vector field, corresponding to the motion commanded by the gas pedal, and $g_2$ the *steer* vector field, corresponding to the motion of the steering wheel.[1] The Lie bracket between the drive and steer vector fields turns out to be

$$g_3 = [g_1, g_2] = \begin{bmatrix} 0 \\ 0 \\ -\sec^2\phi \\ 0 \end{bmatrix}.$$

We call $g_3$ the *wriggle* vector field; it gives infinitesimal rotation of the car about the center of the rear wheels. Finally, we compute the Lie bracket between drive and wriggle, which yields

[1] These names for the vector fields are due to Nelson (1967), who considered this system as part of an example in a book on differential geometry.

$$g_4 = [g_1, [g_1, g_2]] = \begin{bmatrix} -\sin\theta \sec^2\phi \\ \cos\theta \sec^2\phi \\ 0 \\ 0 \end{bmatrix}.$$

This vector field is called the *slide* vector field since it corresponds to motion perpendicular to the direction in which the car is pointing.

The four vector fields $g_1$, $g_2$, $g_3$, and $g_4$ span $\mathbb{R}^4$ as long as $\phi \neq \pm\pi/2$ (at which point $g_1$ is not defined). By Chow's theorem this means that we can steer between any two configurations by an appropriate choice of input.

What Chow's theorem does not tell us is how to *synthesize* an input that causes the car to move to a given location. Of course, humans are very good at synthesizing trajectories for automobiles, but for more complicated situations (such as backing a truck with two or three trailers into a loading dock), the solution to the locomotion problem is not so intuitive.

One method for synthesizing trajectories is to use the Lie bracket motions described earlier. The problem is that these motions are only piecewise smooth in the inputs (a problem here since we are commanding velocities) and they only generate infinitesimal motions. A partial solution to these issues was explored in Murray and Sastry (1993), where we suggested the use of sinusoids to steer control systems of this form. The basic idea was to use sinusoids at integrally related frequencies to generate motion in the Lie bracket directions. In essence, one replaces the squares of a Lie bracket motion with circles. By varying the relative frequencies of the inputs, motion corresponding to different combinations of brackets between the inputs can be obtained. These calculations were motivated by results of Brockett (1981), who showed that under certain conditions these types of inputs are actually optimal.

An example of this type of motion is shown in Figure 3.4. The input, shown in the lower right, consists of a sequence of sinusoidal input segments with different frequencies. The first part of the path, labeled A, drives $x$ and $\phi$ to their desired values using a constant input. These are the states controlled directly by the inputs, so no periodic motion is needed.

The second portion, labeled B, uses a sine and cosine to drive $\theta$ while bringing the other two states back to their desired values. Thus, choosing $u_1 = a \sin t$ and $u_2 = b \cos t$ gives motion in the wriggle direction, $[g_1,g_1]$. By choosing $a$ and $b$ properly, we can control the net change in orientation. However, a careful examination of the motion reveals that some motion also occurs in the $y$ direction. This is due to the higher order terms that appear in the expression for a Lie bracket motion. However, the input directions, $x$ and $\phi$, return to their original values.

The last step, labeled C, uses the inputs $u_1 = a \sin t$ and $u_2 = b \sin 2t$ to steer $y$ to the desired value and returns the other states back to their correct values. This choice of inputs moves the car back and forth once while rotating the steering wheel twice in the proper phase. It generates motion in the bracket direction corresponding to *slide,* $g_4 = [g_1[g_1,g_2]]$. Notice that the Lie bracket expression contains two copies of $g_1$ and one copy of $g_2$, while the inputs move twice in $u_2$ versus once in $u_1$.

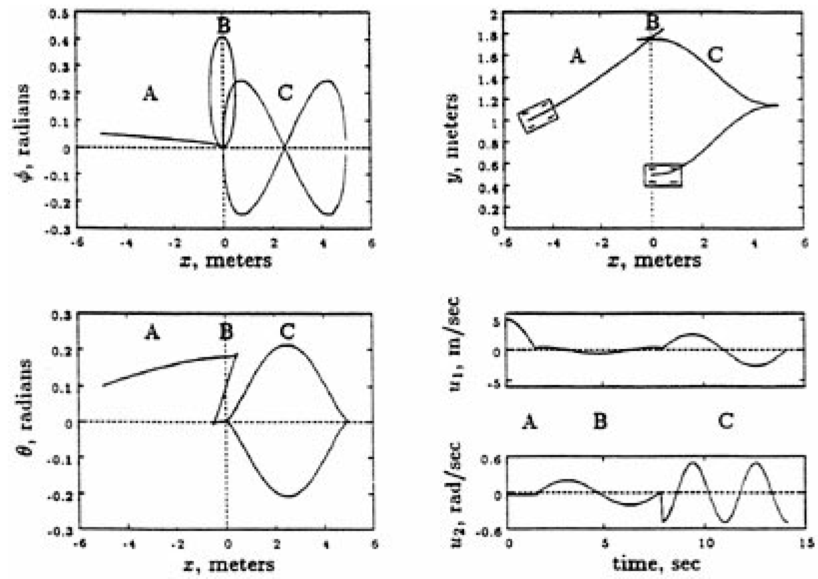


FIGURE 3.4 Motion of a kinematic car. The trajectory shown is a sample path that moves the car form (x, y,θ,ϕ) = (5,1, 0.05,1) to (0, 0.5, 0, 0). The first three figures show the states versus x, and the boom right graphs show the inputs as functions of time. Repented, by permission, form Murray and Sastry, 1993. Copyright © 1993 by Institute of Electrical and Electronics Engineer.

The Lissajous figures obtained from the phase portraits of the different variables are quite instructive. Consider the part of the curve labeled C. The upper left plot contains the Lissajous figure for *x*, ϕ (two loops); the lower left plot is the corresponding figure for *x*, θ (one loop); and the open curve in *x,y* shows the increment in the *y* variable. The interesting implication here is that the Lie bracket motions correspond to rectification of harmonic periodic motions of the driving vector fields, and the harmonic relations are determined by the order of the Lie bracket corresponding to the desired direction of motion.

It is instructive to reinterpret this example in terms of geometric phases. To do this, we rewrite the equations of motion for the system as

$$\begin{aligned} \dot{r}_1 &= u_1 \qquad & \dot{s}_1 &= \cos s_3 \dot{r}_1 \\ \dot{r}_2 &= u_2 \qquad & \dot{s}_2 &= \sin s_3 \dot{r}_1 \\ & & \dot{s}_3 &= \tan r_1 \dot{r}_1 . \end{aligned}$$

Note that the right-hand set of equations have the form of a set of Pfaffian constraints. One can directly verify that these equations describe the motion of the system by identifying ($s_1$, $s_2$, $s_3$) with (*x, y*,θ) and of $u_1$ and $u_2$ with the driving, steering, and velocity. The variable $r_1$ represents the (signed) distance traveled by the car and $r_2$ the angle of the steering wheel.

The decomposition of the problem into a set of independent variables, $r \in \mathbb{R}^2$, and dependent variables, $s \in \mathbb{R}^3$, is an example of a fiber bundle decomposition of the system. We call $r \in \mathbb{R}^2$ the base variables and we call $s \in \mathbb{R}^3$ the fiber variables. Looking back at Figure 3.4, we see that the motion of the fiber variables in segments $B$ and $C$ is obtained by using closed loops in the base variables. The amount of motion in the fiber variables, due to a trajectory in the base variables, is the geometric phase associated with the path in the base space. Parallel parking corresponds to a phase shift in the $y$ direction, while making a U-turn corresponds to a phase shift in the θ direction (sometimes combined with $y$).

The trajectories shown in Figure 3.4 show how geometric phases can be used to understand car parking, but they are not very good examples of parallel parking maneuvers. In fact, it is possible to get much better trajectories for this system by using some obvious tricks, such as using the sum of a set of sinusoids instead of applying simple periodic inputs in a piecewise fashion. One can even solve for the optimal trajectories in simple examples such as this one. Reeds and Schepp (1990) showed that the optimal trajectory between any two configurations can be obtained by following a path comprising up to five segments consisting of either straight, hard left, or hard right driving in either forward or reverse. Furthermore, they were able to show that no more than two backups are required and that only 48 different input patterns are needed to construct minimum length paths (this number has since been reduced to 46 by Sussmann and Tang, 1991).

## GRASPING AND MANIPULATION

A somewhat more complicated example of geometric phases occurs in the area of dextrous manipulation using multifingered robot hands. Here the basic issues involve the use of phases for repositioning the fingers of a hand without removing the fingers from the object. In fact, one is usually more interested in making sure that geometric phase is *not* generated, so that periodic motions of the object cause the fingers to return to their original positions. We start by discussing the repositioning problem and then make some brief comments about generating cyclic motions.

Consider the grasping control problem with rolling contacts, such as the system shown in Figure 3.5. Assuming that the fingers roll without slipping on the surface of the object (a very useful assumption to enforce, since control of sliding is very tricky), the constraints on the system can be described by a set of equations of the form

$$J_h(\theta, x)\dot{\theta} = G^T(\theta, x)\dot{x}, \qquad (3.2)$$

where θ is the vector of finger joint angles and $x$ specifies the position and orientation of the grasped object. In the robotics literature, $J_h$ is called the *hand Jacobian* and $G$ is the *grasp map* (see Murray et al., 1994, for a detailed discussion).

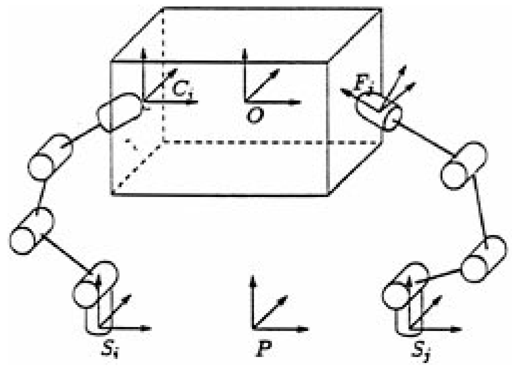


FIGURE 3.5 Multifingered hand grasping an object. Reprinted with permission from Murray et al., 1994. Copyright © 1994 by CRC Press, Boca Raton, Florida.

A detailed calculation of the hand Jacobian and grasp map is quite involved and makes use of a large amount of specialized machinery. However, the basic idea behind the grasp constraint is quite straightforward: the left-hand side of equation 3.2 is the vector of fingertip contact velocities for the robot hand, expressed in an appropriate frame of reference. The right-hand side of equation 3.2 is the vector of velocities for the contact points on the object, expressed in the same frame of reference. The condition that the fingers roll without slipping is obtained by equating these sets of velocities.

If the fingers of a grasp have sufficient dexterity, they can follow any motion of the object without slipping. A grasp of this type is called a *manipulable grasp*. For manipulable grasps, any object velocity $\dot{x}$ can be accommodated by some finger velocity vector θ. However, the vector θ may not be unique in the case that the null space of $J_h$—that is, the set of vectors that $J_h$ maps to the zero vector—is nontrivial. This situation corresponds to the existence of internal motions of the fingers that do not affect the motion of the object. If we let $u_1$ be an input that controls the velocity of the object and let $u_2$ parameterize the internal motions, then equation 3.2 can be written as

$$\begin{aligned} \dot{x} &= u_1 \\ \dot{\theta} &= J_h^{+} G^T u_1 + K u_2, \end{aligned} \tag{3.3}$$

where $J_h^{+} = J_h^T (J_h J_h^T)^{-1}$ is the right pseudo-inverse of $J_h$ and K is a matrix whose columns span the null space of $J_h$.

Equation 3.3 describes the grasp kinematics as a control system. The input $u_1$ describes the motion of the object, whose position is given by $x$. The effect of $u_1$ on θ describes how the fingers must move in order to maintain contact with the object. If the fingers have any extra degrees of freedom, $u_2$ can be used to control the internal motions that affect the shape of the fingers but leave the object position unchanged. The dynamic finger repositioning problem is to steer the system from an initial configuration ($\theta_0$,$x_0$) to a desired final configuration ($\theta_f$,$x_f$). The explicit location of the fingertip on the object at the initial and final configurations can be found by solving the forward kinematics of the system.

The general case of finding $u_1$ ($t$) and $u_2$ ($t$) such that the object and the fingers move from an initial to final position (while maintaining contact) can be very difficult. A special case is when the object position is kept fixed by setting $u_1$ at zero. If there are still sufficient degrees of freedom available for the fingers to roll on the object, then the internal motions parameterized by $u_2$ can be used to move the fingers individually around on the object. With the object position held fixed, each of the fingers can be controlled individually without regard to the motion of the others, simplifying the problem somewhat.

An example of such a path moving a spherical fingertip down the side of a planar object is shown in Figure 3.6. In this figure we consider the motion of a finger with a spherical tip on a rectangular object. The plots show trajectories that move a finger down the side of the object. The location of the contact on the finger is unchanged, as shown in the right graph, which plots the finger contact configurations ($u_f$, $v_f$), while the location of the contact on the face of the object ($u_0$,$v_0$) undergoes a displacement in the $v_0$ direction.

In addition to describing how the fingers can be repositioned on the object without releasing contact, geometric phases can also be used to understand when control laws keep the fingers from drifting, in case this is not desired. Imagine, for example, performing a complicated manipulation of an object. Depending on the geometry of the object and fingers, it is possible that the object might return to its starting configuration while the fingers would have shifted to a new location. In many cases we are interested in choosing control laws to ensure that this *does not* happen. Thus, we want to control the fingers in such a way that there is no geometric phase associated with any closed loop motions of the controls.

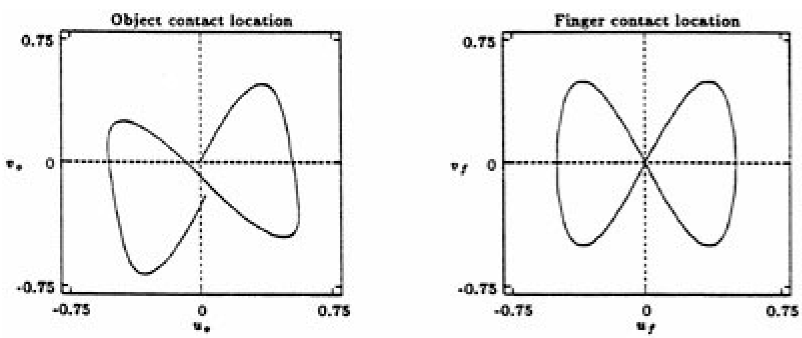


FIGURE 3.6 Steering applied to the multifingered hand shown in Figure 3.5. The left plot shows the location of the contact point on the object, and the right plot shows the corresponding contact point on the finger. The object contact moves down and slightly to the right, so the object is shifted slightly in the grasp after executing the steering maneuver.

A controller that always returns the fingers to their original configuration when the object returns to its original configuration is said to be *cyclic*. The same type of problem can occur in resolving motion in redundant robots, and has been studied a great deal in that context. In terms of the point of view described in this paper, the work of Shamir and Yomdin (1988) gives necessary and sufficient conditions in terms of Lie brackets that guarantee that a controller is cyclic. The same basic ideas can be used in the grasping case.

Using the kinematics described in equation 3.3, the input $u_1$ describes the motion of the object. To make the overall controller cyclic, we must choose $u_2$ so that any cyclic motion of $u_1$ gives a cyclic motion in $\theta$. Thus, we wish to choose a feedback law $u_2 = \alpha(x,\theta)$ such that the geometric phase associated with $u_1$ is always zero.

While dynamic finger repositioning provides a direct connection between geometric phases and manipulation, the basic ideas behind phases are present in other types of manipulation as well. Consider the problem of inserting a light bulb into a threaded hole using your fingers. One way to do this would be to grab the bulb and then walk around in circles to insert it. This obviously requires a large workspace and is completely inappropriate for manipulation inside a cluttered environment (like the inside of a refrigerator).

A much more natural way to insert the bulb is to rotate your fingers, release the bulb, and then move your fingers back to perform another rotation. If we treat "twisting" and "grasp/release" as inputs, then this type of motion corresponds exactly to a Lie bracket motion where the bracket direction corresponds to the motion of the bulb into the socket. The description of this problem does not quite fit into the differential framework described above without some modification of the underlying mathematics, but the basic notion of a Lie bracket motion is still present.

Another example along these lines is using a (miniature) multifingered hand for sewing stitches in tissue. To understand how such a hand should be designed and how such a task might be accomplished, we can use the tools from geometric phases to guide our insights and uncover the fundamental principles that govern the behavior of the system. Since these systems tend to be highly complex, it is important to understand the essential geometry of the system and its role in satisfying the overall task. The implications of some of these ideas in areas such as planetary exploration and medicine are discussed in the next section.

## APPLICATIONS

We now discuss two specific applications of the techniques outlined above to existing and future robotic systems.

### Exploration of Mars

In 1996 NASA is scheduled to launch a spacecraft to Mars containing the Microrover Flight Experiment MFEX (Pivirotto, 1993). A major part of the mission consists of landing a semiautonomous rover on the surface of Mars and using the rover to analyze soil and rock samples on the Martian surface.

Due in part to funding cuts in the program, the size of the rover has been reduced from an initial mass of 800-1000 kg to a "mini" or "micro" rover in the 5-50 kg range.

The starting point for the MFEX rover design is the Rocky IV rover pictured in Figure 3.7. Rocky IV has a mass of 7.2 kg and is approximately 60 cm in length. It uses a novel "rocker-bogie" design for the wheels, which allows it to climb over obstacles that are as much as 50 percent larger than the wheel diameter. Rocky IV is equipped with a color video camera and numerous proximity sensors. The flight version of the rover will also be equipped with an alpha proton X-ray spectrometer (AXPS), to be used to analyze the composition of Martian soil and rocks.

The three primary goals of the MFEX rover mission are to complete a set of technology experiments in at least one soil type, complete an AXPS measurement on at least one rock with a video image of that rock, and take at least one full image of the lander. When these goals are met, the rover will complete additional experiments on different soil and rock types and attempt to take two more pictures of the lander, giving a complete view from all sides.

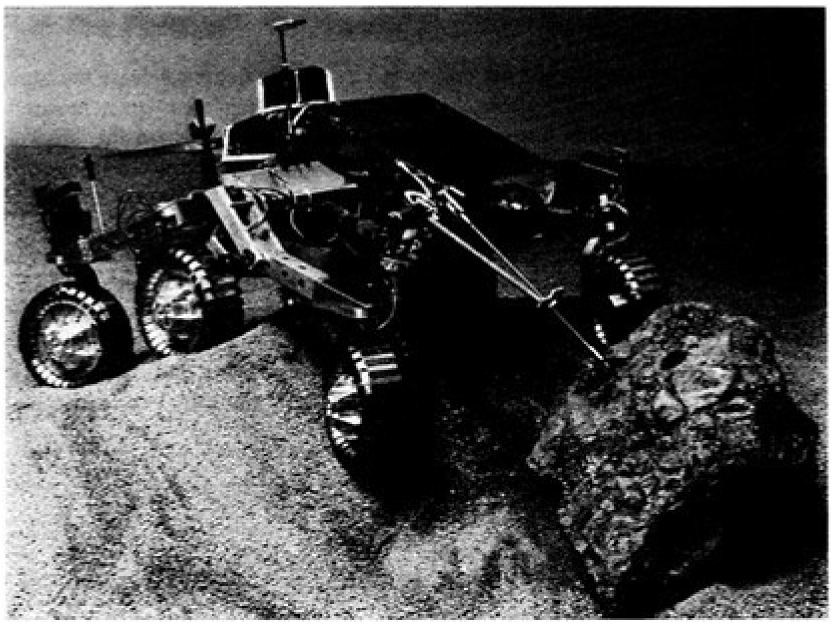

Figure 3.7 Rocky IV rover. Used by permission of the Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California.

The basic mode of operation for the Mars rover mission involves humans providing overall planning and guidance while the rover itself will be responsible for low-level navigation and control. This level of autonomy in the control of the rover is necessitated by two factors: communication delays to Mars and limited communications bandwidth. The round-trip travel time for a signal to Mars can range from approximately 10 minutes to as long as 40 minutes. This makes direct teleoperation of the rover impossible. Furthermore, the communications bandwidth of the lander is limited to approximately 4 megabits/day, or the equivalent of 10 bytes every 1.7 seconds, and can only be sustained while Earth is visible from the lander sight.

Because of these communications limitations, the Mars rover will be given commands once per day describing a sequence of actions to be carried out. Onboard computation will be used to provide low-level trajectory tracking and to ensure obstacle avoidance. Due to power constraints on the rover and the need to use flight-qualified hardware, the amount of computational power onboard the rover is quite low. Current earth-based experiments employ an 8-bit microprocessor capable of performing only 1 million operations per second and containing less than 40K of memory. Behavioral control is being explored as a means of implementing the controller and has shown good reliability in autonomous navigation and manipulation tasks in both indoor and outdoor rough-terrain environments (Gat et al., 1994).

Path planning for the Mars rover involves finding paths that satisfy the basic kinematics of the rover and also avoid obstacles. To a rough degree of approximation, the constraints on the rover can be described by a set of Pfaffian constraints and hence the geometric machinery described above can be used to understand the vehicle motion and to plan maneuvers. Examples of path planners for mobile robots in the presence of obstacles can be found in Latombe (1991) and Laumond et al. (1994).

Future Mars missions are expected to include a sample and return scenario, in which a soil sample from Mars will be returned to the lander for further analysis. Since the lander can house much larger and heavier measurement apparatus than the rover can, returning a soil sample to the lander allows much more detailed experiments to be run. In returning to the lander, the rover must accurately position itself to dock with the lander. This must be done in the presence of unknown terrain and with a reasonably high degree of precision.

One approach to this problem is the use of controllers that use real-time feedback to guide the rover to the docking bay. Stabilization of constrained systems of this type turns out to be a challenging theoretical problem that has received a large amount of attention in the controls community over the past several years (see M'Closkey and Murray, 1994, for a recent list of papers and experimental results in active stabilization). These results have relied heavily on the geometric point of view, which has evolved over the past few years and will no doubt continue to make use of these tools.

### Medical Robotics: Minimally Invasive Surgery

One of the exciting applications of robotic manipulation and locomotion is in medicine, particularly in minimally invasive surgery. Over the past ten years, the use of minimally invasive surgical techniques has increased dramatically in the U.S. and abroad. These techniques offer several advantages over conventional surgery, including reduced hospital stays after an operation and decreased risk of infection and other complications.

A typical minimally invasive surgical operation is a laparoscopic cholecystectomy (gall bladder removal). In this procedure, a doctor removes the gall bladder through several small incisions in the patient's abdomen. One of the incisions is used to insert a tube that inflates the abdominal cavity with gas, while the other incisions are used to insert medical instruments. The gall bladder is removed by cutting it with a pair of scissors and extracting it through one of the incisions.

Gall bladder removal is one of the most common surgeries performed in the United States. Ten to fifteen years ago, the percentage of such surgeries that were done using minimally invasive techniques was negligible. However, with new techniques and new medical instruments, the use of minimally invasive techniques has surged, and this is by far the most common method currently in use for gall bladder removal.

Another minimally invasive surgical technique is the use of an endoscope for inspection and removal of tissue or polyps from the gastrointestinal tract. One example is an endoscopic polypectomy. In this procedure, a flexible endoscope is maneuvered near a polyp on the interior of the colon. Using a video display connected to a camera at the end of the endoscope (via fiber optics), the surgeon is able to snare the polyp and cut it off with electrocautery while drawing the snare closed.

While the use of minimally invasive techniques has progressed rapidly, it is currently limited by several factors. Among them is the limited dexterity of the tools used by the surgeons and the large portions of the body that cannot be reached with an endoscope or laparoscope. One of the applications of the work described here is toward extending the abilities of surgeons in these directions.

Researchers at the University of California-Berkeley, Harvard, and other institutions are working to develop a *teleoperative surgical workstation*, which would allow surgeons more dexterity and ease of use than current minimally invasive technology (Cohen et al., 1994). These researchers are focusing on a number of different problems, including the design, fabrication, and control of miniature robotic hands and the use of instrumented data gloves to allow the surgeon to control the hands in a natural way. A photograph of one of the hands that has been fabricated is shown in Figure 3.8.

Figure 3.8 Three-fingered laparoscopic manipulator designed for tendon actuation (tendons omitted for clarity). The manipulator is shown in a 10 mm diameter laparoscopic trocar. Reprinted with permission from Michael Cohn, University of California-Berkeley.

The design and control of these complicated machines require a thorough understanding of the basic mechanisms that are present in manipulation tasks. A fairly simple miniature hand such as the one shown in Figure 3.8 might have up to 9 degrees of freedom, which, when combined with the dynamics of the object, can give a phase space with as many as 24 states (configurations plus velocities). The dynamics for such a system cannot be easily studied without understanding the basic structure that the dynamics inherit from the specific manipulation problem under consideration.

A second area of research in medical robotics is the development of small locomotion robots capable of navigation and inspection in the gastrointestinal tract. J. Burdick and his students at Caltech have built several prototype devices for locomotion inside an intestine, and clinical trials on pigs are in development (Burdick, J.W. and B. Slatkin, 1994, personal communication). The motion of their devices relies on the geometric phases associated with alternately inflating and deflating a set of gas-filled bags coupled with shortening and lengthening the robot along its longitudinal axis.

Other possibilities for gastrointestinal robots include the use of hyperredundant (or "snake") robots. Over the past five years, a complete theory for these robots has been developed by Chirikjian and Burdick (1994). They have explored the use of hyperredundant robots not only for locomotion tasks, but also for manipulation tasks in which the robot wraps itself around the object that it is manipulating. This provides a very stable grasp while still allowing the object to be manipulated within the grasp.

## CONCLUSIONS AND DISCUSSION

In this paper we have indicated some of the roles that geometric phases play in modern robotics, concentrating on applications in robotic locomotion and dextrous manipulation. As the robotic systems that we seek to control become increasingly complex, our ability to understand and program them is forced to rely more and more on the use of abstraction and advanced analysis. Further development of relevant theory, and applications of that theory to engineering problems, will help promote the use of advanced technology in many areas.

In addition to the specific applications discussed above, there are many other areas that overlap with the ideas presented here. For example, the use of geometric phases to understand biological motion is starting to become more clear. The use of central pattern generators (CPGs) to generate repetitive motion is common to many types of animals. One can view CPGs as the driving input to a set of kinematic constraints. Intuitively, the geometric phase associated with a particular gait pattern determines the direction and amount of motion of the system.

The explicit connection between CPGs and geometric phases remains to be established, but there are several clues that indicate that some new advances in theory might help. One such clue is the motion of the *snakeboard*, a commercial variant of a skateboard, which is discussed by Marsden in his paper and is described in detail in Lewis et al. (1994). The snakeboard relies on coupling between angular momentum and Pfaffian constraints to generate motion. Different gaits can be achieved in the snakeboard by using integrally related periodic motions in the input variables of the system. As its name indicates, the snakeboard provides an important link between wheeled mobile robots and more complicated snake-like robots. By studying the geometry of the snakeboard we are able to understand one of the mechanisms by which locomotion occurs.

In order to expand this geometric point of view to other locomotion and manipulation problems, a slightly more general framework is required. In particular, while the notion of periodic motions for locomotion is fairly ubiquitous, the generation of trajectories via Pfaffian constraints is limited. For snake-like robots, a more general framework would allow different friction models and discontinuous dependence on the velocity of the snake. For legged locomotion, a completely different approach may be required since the contacts occur in piecewise fashion. These problems are the subject of current work by researchers in the United States and around the word, and we can expect to see exciting new insights and applications in the years to come.

## ACKNOWLEDGMENTS

The author would like to thank Joel Burdick, Michael Cohn, Carl Ruoff, Shankar Sastry, and Donna Shirley for providing references and pictures used in this paper, and Roger Brockett, P.S. Krishnaprasad, and Jerry Marsden for many valuable and stimulating conversations on this material.

## REFERENCES

Brockerr, R.W., 1981, "Control Theory and singular Riemannian Geometry." In: *New Directions in Applied Mathematics*, pp. 11-27, New York: Springer-Verlag.

Chirikjian, G. and J.W. Burdick, 1995, "Kinematics of Hyperredundant Locomotion," *IEEE Transactions on Robotics and Automation* **11**(6), 781-793.

Chow, W.-L., 1940, "Über Systeme Yon Linearen Partiellen Differentialgleichungen Erster Ordnung," *Math. Annalen* **117**, 98-105.

Cohn, M., L.S. Crawford, J.M. Wendlandt, and S.S. Sastry, 1994, "Surgical Applicators of Millirobotics," *J. Robotic Sys*. **12**(6), 401-416.

Gat, E., R. Desai, R. Ivlev, J. Lock, and D. Miller, 1994, "Behavior Control for Robotic Exploration of Planetary Surfaces," *IEEE Transactions on Robotics and Automation* **10**(4), 490-503.

Latombe, J.-C., 1991, *Robot Motion Planning*, Boston: Kluwer Academic Publishers.

Laumond, J.-P., P. Jacobs, M. Taix, and R.M. Murray, 1994, "A Motion Planner for Nonholonomic Mobile Robots," *IEEE Transactions on Robotics and Automation* **10**(5), 577-593.

Lewis, A., J. Ostrowski, R.M. Murray, and J.W. Burdick, 1994, "Nonholonomic Mechanics and Locomotion: The Snakeboard Example." In: *Proceedings of the IEEE International Conference on Robotics and Automation*.

M'Closkey, R.T. and R.M. Murray, 1994, "Experiments in Exponential Stabilization of a Mobile Robot Towing a Trailer," In *Proceedings of the American Control Conference*, American Automatic Control Council, Philadelphia.

Murray, R.M., Z. Li, and S.S. Sastry, 1994, *A Mathematical Introduction to Robotic Manipulation* , Baton Rouge, LA: CRC Press.

Murray, R.M. and S.S. Sastry, 1993, "Nonholonomic Motion Planning: Steering Using Sinusoids," *IEEE Transactions on Automatic Control* **38**(5), 700-716.

Nelson, E., 1967, *Tensor Analysis*, Princeton, NJ: Princeton University Press.

Pivirotto, D.S., 1993, "Finding the Path to a Better Mars rover," *Aerospace America* **25**, 12-15.

Reeds, J.A. and L.A. Shepp, 1990, "Optimal Paths for A Car That Goes Both Forwards and Backwards," *Pacific Journal of Mathematics* **145** (2), 367-393.

Shamir, T. and Y. Yomdin, 1988, "Repeatability of Redundant Manipulators: Mathematical Solution of the Problem," *IEEE Transactions on Automatic Control* **33**(11), 1004-1009.

Shapere, A. and F. Wilczek, 1989, "Efficiencies of Self-propulsion at Low Reynolds Number," *Journal of Fluid Mechanics* **198**, 587-599.

Sussmann, H.J. and G. Tang, 1991, "Shortest Paths for the Reeds-Shepp Car: A Worked out Example of the Use of Geometric Techniques in Nonlinear Optimal Control," *Technical Report*, New Brunswick, NJ: Rutgers Center for Systems and Control.

# 4

# Motion Control and Coupled Oscillators

**P. S. Krishnaprasad**

Department of Electrical Engineering and
Institute for Systems Research
University of Maryland

It is remarkable that despite the presence of large numbers of degrees of freedom, motion control problems are effectively solved in biological systems. While feedback, regulation, and tracking have served us well in engineering as useful solution paradigms for a wide variety of control problems, including motion control, it appears that nature gives prominent roles to planning and coordination as well. There is also complex interplay between sensory feedback and motion planning to achieve effective operation in uncertain environments (in movement on uneven terrain cluttered with obstacles, for example). Recent investigations by neurophysiologists have brought to increasing prominence the idea of central pattern generators (a class of coupled oscillators) as sources of motion "scripts" as well as a means for coordinating multiple degrees of freedom. The role of coupled oscillators in motion control systems is currently under intense investigation. In this paper we examine some unifying themes relating movement in biological systems and machines. An important insight in this direction comes from the natural grouping of degrees of freedom and time scales in biological and engineering systems. Such grouping and separation can be treated from a geometric viewpoint using the formalisms and methods of differential geometry, Lie groups, and fiber bundles. Coupled oscillators provide the means to bind degrees of freedom either directly through phase locking or indirectly through geometric phases. This point of view leads to fresh ways of organizing the control structures of complex technological systems.

## INTRODUCTION

In optics, lithography applications in microelectronics, and in a variety of other contexts, the need for high-resolution motion control with high accuracy is met by specialized actuators that are quite different in their principles of operation from everyday devices such as electromagnetic motors. One such device, manufactured by Burleigh Instruments under the trademark INCHWORM™, is illustrated in Figure 4.1.

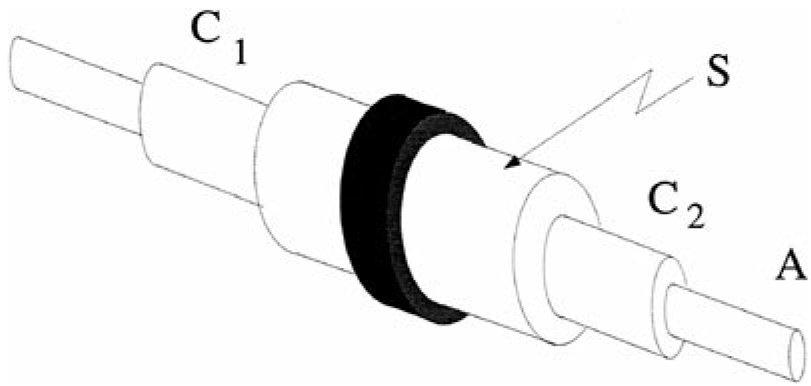


FIGURE 4.1 INCHWORM™ clamps $C_1$, $C_2$, stretcher S, and armature A.

This actuator, consisting of three sleeves/tubes made from piezoelectric material mounted on a frame and enclosing a linear armature, works on the physical principle that the piezoelectric material deforms under electrical stimulus (the outer sleeves independently clamp down, and the middle sleeve stretches in length). Running the actuator through a succession of clamp-stretch-unclamp-unstretch cycles, one generates incremental motion of the armature in a specified direction. It is possible to make linear movements as small as 4 nanometers. Other actuators based on piezoelectric effects are increasingly finding their way into consumer products, including ultrasonic motors for autofocusing in cameras based on surface wave excitation (see Ueha and Tomikawa, 1993, for detailed discussions of these devices). A common design principle in these devices is a type of *rectification* of small cyclical motions to produce gross motions.

Turning to the natural world, much attention has been devoted to the systematic understanding of how various microscopic organisms move in fluids under various conditions. Since movement is essential to reaching food particles, efficiency considerations have also been of interest (see Childress, 1981, for related discussion). Apparently, the paramecium gets around in a fluid under conditions of a very low Reynolds number through a process of cyclical change in its boundary contour (or more precisely, the envelope determined by the oscillating cilia that make up the contour). (See Figure 4.2.) In the work of Shapere and Wilczek (1989) this has been shown, under appropriate fluid mechanical assumptions, via the mathematics of gauge theory (which has played an important role elsewhere in modem physics and geometry over the last three decades). Here again a type of rectification is at work.

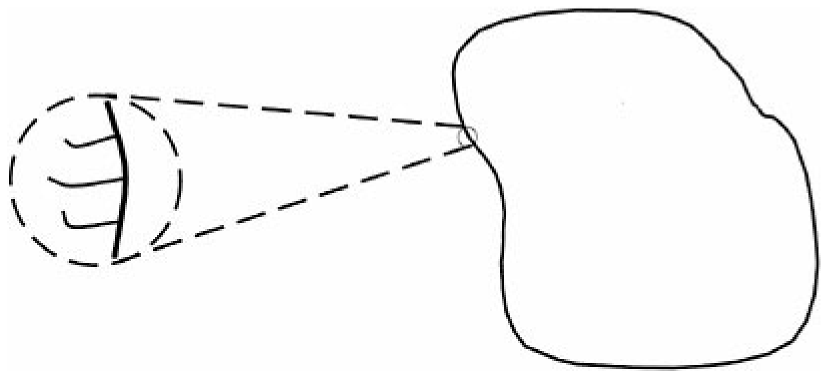

Figure 4.2 Illustration of paramecium in a fluid.

Although the low Reynolds number regime permits an essentially kinematic treatment of the paramecium, in other contexts of animal movement dynamic influences play an important part, e.g., in walking, trotting, and galloping gaits of quadrupeds (Alexander, 1968), in the swimming movement of the lamprey (Bowtell and Williams, 1991), etc. A rather striking illustration of this occurs in the *Basiliscus plumifrons*, a type of lizard found in Central America that is capable of engaging in short bursts (less than 10 meters) of walking on water, supporting itself by rapidly pushing down (at 30 Hz frequency) its hind feet on the water. The reaction forces thus generated are sufficient for support (see Lambert, 1994).

Our examples are meant to underscore the principle of movement generation by repetitive, cyclical variation in certain degrees of freedom (of a machine or an organism) while constrained by interactions with the environment (e.g., ground contact, friction). Understanding this principle has had an important influence on recent research in control theory and in robotics, as also explained by the companion papers of R.W. Brockett, J.E. Marsden, and R.M. Murray in this report. Turning this principle into a quantitative tool requires an understanding of the rectification mechanism mentioned above. It is precisely this mechanism, variously associated with geometric phases, area rules, and Lie bracket generation, that has had a crucial role as a tool for designing machines and algorithms to control them. In the context of motion generation, Brockett's paper (1989) appears to be the first to state clearly and prove a version of the rule (see Murray et al., 1994; Murray and Sastry, 1993).

Placing the rectification principle in the broader context of motion control architectures for systems with many degrees of freedom is one of the goals of this paper. To clarify matters further, an example is given in the following section, "From Shape Change to Global Movement," involving a unicycle and oscillations.

There is already a rich tradition in the biology and neuroscience of modeling movement via coupled oscillators. It is noteworthy that even in the presence of large numbers of degrees of freedom, motion control problems are effectively solved in biological systems of extraordinary variety. The work of Brown (1914) on half-centers, and the fundamental investigations, starting in the 1930s, of Bernstein (1967) on strategies for motion control, continue to have an influence in modem work (see Pearson, 1993, for a modern perspective). Bernstein clearly identified a role for planning (i.e., feedforward control) along with feedback, regulation, and tracking in motion control. In Bernstein's scheme, adaptive restructuring of motion programs on the fly, through the use of afferent feedback pathways from mechanoreceptors and other sensory modalities, had a prominent place. More recent work of neurophysiologists has focused attention on *central pattern generators* (CPGs) in the nervous system as key to understanding the control of movement and posture (Cohen, 1988; Cohen et al., 1982, 1988; Kopell and Ermentrout, 1988; Kopell, 1988). As mathematical objects, CPGs are networks of coupled oscillators and can be incorporated in the control architecture of a complex machine. Thus, if the state variables of the nodes of a CPG are in turn coupled to the degrees of freedom of the system to be controlled, it is possible to achieve coordination of the latter by prescribed phase coherence of the oscillators. The system to be controlled may be a multilegged walking machine or a multifingered, anthropomorphic mechanical hand with built-in tactile sensors on the fingers. Sensory feedback paths to correct CPG dynamics would be necessary to provide a level of robustness to changes in the environment (e.g., obstacles, failures). These elements lead us to the architecture of Figure 4.4, discussed further in "Scripts and Oscillators" below.

In the section on "Unifying Geometry," we present a unifying geometric-mechanical picture of the ideas on rectification. The language of principal bundles and connections goes hand in hand with the mechanical notions of configuration spaces, symmetries, and constraints. Complementing the perspectives presented in the companion papers in this report, we focus attention on the notion of averaging in Lie groups and its relation to rectification. In the section below on "Some Interesting Machines," we discuss novel machines that illustrate the main ideas of this paper and point the way to further extensions.

## FROM SHAPE CHANGE TO GLOBAL MOVEMENT

Our purpose here is to show how suitable notions of shape, together with cyclical shape change, can yield global movement. In the case of the INCHWORM™ actuator, the concept of shape can be identified with two pieces of information: the continuous elongation/contraction of the middle sleeve and the *discrete* state of the clamp-pair (which one is on or off?). For each such "shape," there is an associated holonomic constraint, and coordinated shape change together with switching of constraints leads to rectification and the travel of the armature. In a setting more natural for geometric arguments, piecewise holonomic constraints may be replaced by nonholonomic constraints. This is best illustrated by classical mechanical examples involving the constraint of *no sliding* of a knife edge or *rolling without slipping* of a wheel on a surface. Consider, for instance, the geometry of motion of a unicycle with rider, as shown in Figure 4.3. The kinematic equations of the unicycle are given by

$$\begin{aligned} \dot{x} &= \cos(\phi)u_2 \\ \dot{y} &= \sin(\theta)u_2 \\ \dot{\theta} &= u_1 . \end{aligned} \tag{4.1}$$

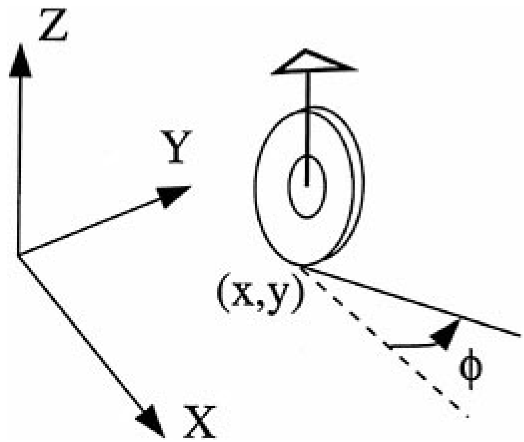


FIGURE 4.3 Geometric model of a unicycle.

Here $x$ and $y$ denote the position of a fixed reference point on the unicycle and $\phi$ denotes the orientation of the unicycle relative to a fixed laboratory/observer frame. Further, $u_1$ denotes the steering speed and $u_2$ denotes the heading speed, and these are assumed to be controllable by the unicyclist. From equation 4.1, it is clear that the constraint of no sliding

$$-\dot{x}\sin(\phi) + \dot{y}\cos(\phi) = 0 \tag{4.2}$$

is maintained at all times.

Equations 4.1 and 4.2 can also be recast in the following equivalent form:

$$\dot{g} = g \cdot (A_1 u_1 + A_2 u_2), \tag{4.3}$$

where

$$g = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & x \\ \sin(\phi) & \cos(\phi) & y \\ 0 & 0 & 1 \end{pmatrix} \tag{4.4}$$

evolves in the group of rigid motions in the plane, with

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.5)$$

Imagine a typical unicyclist implementing pedaling and steering maneuvers that give rise to $u_i(t) = \varepsilon f_i(t)$, where $f_i(.)$ are zero-mean periodic functions of time with a common period T, and $\varepsilon > 0$ is a small amplitude parameter. In this instance, the "shape variables" $\tilde{u}_i(t) = \int_0^t u_i(\sigma)d\sigma$ are also periodic/oscillatory. Where does the unicycle end up? To get a decent approximation to the exact solution, one resorts to averaging theory (Leonard and Krishnaprasad, 1994a; Leonard, 1994). In fact, *g(t)* is approximated up to quadratic terms in ε by the formula

$$g^{(2)} = g(0) \cdot \exp\left(\sum_{i=1}^{3} z_i^{(2)} A_i\right), \quad (4.6)$$

where, for $i = 1, 2$,

$$z_i^{(2)}(t) = \tilde{u}_i(t) + z_{i0}^{(2)}, \quad (4.7)$$

and

$$z_3^{(2)}(t) = \frac{t}{T}\text{Area}_{1,2}(T) + z_{30}^{(2)}. \quad (4.8)$$

Here, the $z_{i0}^{(2)}$ are initial conditions, and the matrix

$$A_3 = A_1 A_2 - A_2 A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is the Lie bracket of $A_1$ and $A_2$, and

$$\text{Area}_{1,2}(T) = \frac{1}{2}\int_0^T \left(\tilde{u}_1(\sigma)\dot{\tilde{u}}_2(\sigma) - \tilde{u}_2(\sigma)\dot{\tilde{u}}_1(\sigma)\right)d\sigma \quad (4.9)$$

is the area of the loop in shape space executed by the unicyclist in the course of the chosen oscillatory maneuver.

Equation 4.6 predicts a secular drift in the direction of the Lie bracket $A_3$ and, following 4.8 and 4.9, illustrates the rectification principle as an *area rule*. By a succession of oscillatory maneuvers, the unicyclist can get around anywhere and manage parallel parking! (For related ideas and references, see the paper by Murray in this report.) This hinges on the fact that the constraint (4.2) is nonholonomic or, equivalently, the Lie bracket $A_3$ is linearly independent of $A_1$ and $A_2$, the directions trivially controllable by the unicyclist. It is, however, important that the phase relations between the pedaling oscillations and the steering oscillations be right, or else the $\text{Area}_{1,2}$ ($T$) will vanish, killing the secular drift. This brings up the need for coupled oscillations. Appropriate shape variations may be drawn from solutions to variational problems.

The unicycle example illustrates a geometric interpretation of shape and shape change. For a six-legged insect (or machine) with legs capable of lift and swing, the shape space may be a submanifold of a 6 × 3-dimensional toms. Shape change in that case is achieved via successively lifting and swinging the legs before returning to ground contact.

## SCRIPTS AND OSCILLATORS

Area rules of the type discussed in the last section have been used in developing computer programs to synthesize feedforward control laws (motion scripts) in Leonard and Krishnaprasad (1994a) and Leonard (1994). We think of such programs as *script generators*, producing detailed streams of instructions to machines. One such program is used to control an underwater vehicle (Leonard and Krishnaprasad, 1994b; Leonard, 1994). Integrating script generators into a larger framework for *intelligent control* of movement is a major challenge (the framework has to accommodate uncertainty, limited sensing of the environment, obstacles that move about, rough terrain, etc.) and we argue that there is much insight to be gained from deeper study of biological motion control systems.

The Russian physiologist N. Bernstein, in his studies of the movement problem, proposed a variety of architectural principles. Given the large numbers of degrees of freedom involved in even elementary motor acts, binding (or synchronization) of the degrees of freedom into groups is necessary. Such binding has to be dynamic to accommodate varying stages in a movement. Bernstein viewed rhythm generators or oscillators as the means to implement binding. Bernstein also viewed as central to motor control the ability to change a motor program in the middle of a movement, possibly based on data from afferent sensory pathways. Much work since Graham Brown's proposal of half-centers has gone into understanding how neural circuitry could be organized to produce temporal patterns of neuronal firings that seem to be responsible for rhythmic movements. See, for instance, the compendium of papers in Cohen et al. (1988). The oscillations in the temporal patterns are assumed to be in correspondence with actual movements produced, for instance, a particular gait, i.e., rhythmic stepping, in a quadruped. A complex movement could be segmented into distinctive gaits and modules capable of piecing together such segments prior to initiation of a movement and altering them "on the fly" are essential to intelligent control. Further, it is plausible that in biological systems, the higher cognitive elements engaged in movement control pay attention primarily to a symbolic description of movement, ignoring detailed timing information. For instance, in the case of a six-legged insect, by identifying the legs on the left and right sides of the body with the symbols $L_i$ and $R_i$, where the index $i$ rims from 1 to 3 (3 stands for the hind legs, 2 for the middle legs, and 1 for the front legs), one can refer to a gait pattern by a string of symbols, as in

(a) $R_3, R_2, R_1, L_3, L_2, L_1$
(b) $L_2 R_3 L_1, R_2, R_1 L_3$
(c) $R_3 L_2 R_1, L_3 R_2 L_1$

These strings are to be interpreted as defining the sequence in which each leg is lifted from the ground, and symbols in a group *not* separated by a comma correspond to synchronized leg-lifts. Thus string (c) above represents the so-called alternating tripod gait, being the fastest, and string (a) above stands for the slowest. Both timing and step length information are hidden, although it is experimentally observed that swing times are independent of gait. A suitable control framework would need to be able to accommodate descriptions of movement both at the symbolic and at the detailed timing level. In fact, one can even work out an admissible language for movements by stringing together "words" as in (a) (b)(c).

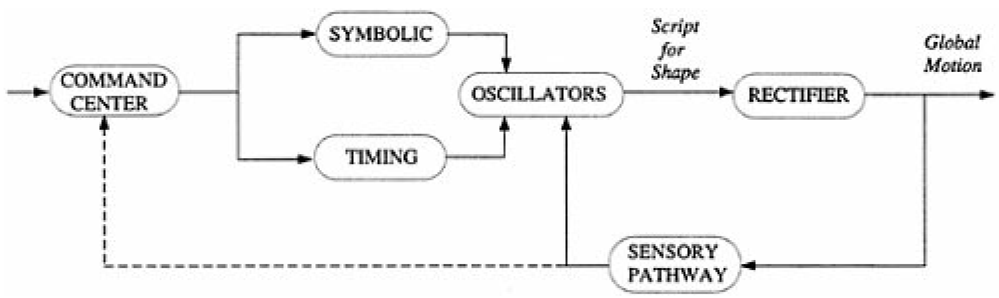


FIGURE 4.4 Architecture for motion control.

Based on these insights from biology, one is led to a possible architecture for intelligent control of movement as in Figure 4.4. Here the command center communicates a prescribed global movement command to be transformed into symbolic and timing instructions, which are then implemented by a network of coupled oscillators that produce the script for shape change. The rectification path produces the desired global movement. Sensory information is returned to the command center and possibly to the oscillators to modify/correct the motion commands and scripts. In practice, in legged animals or machines, this afferent pathway may lead to script change (gait switching). For a robust, model-independent approach to gait switching on the basis of bifurcation theory in the presence of symmetries, see Collins and Stewart (1993). In our own work, the change in control authority that accompanies the failure of actuators is one of the sources of script change (Leonard and Krishnaprasad, 1994b).

The architecture sketched out here gives prominence to what may be a missing ingredient in much of the discussion of rhythmic movements in biology, namely, the rectification module. In his paper in the present volume, Brockett takes the view that rectification is a tool for "approximate inversion" of motion specifications over time and shows how oscillators do the job. It may be possible to suggest some biological experiments to determine if indeed such approximate inverses are learned from repeated trials, thereby lessening the need to store motion scripts. Finally, it should be added that there is some software involved in the control architecture we discussed that corresponds to current thinking in reconfigurable software for robotics (Stewart et al., 1993).

We close this section by pointing out exciting new developments toward incorporating a combination of pattern-generating oscillators, elastomechanics-based models of body movement and muscle response, and models of fluid interaction with the skin to capture the complexities of lamprey movement (Bowtell and Williams, 1991). In at least one machine that we have studied (see "Some Interesting Machines"), all these ingredients prove to be necessary for complete understanding.

## UNIFYING GEOMETRY

The model (4.3) of the unicycle is not so special as it might seem at first glance. In practice, the models of mechanics governing the behavior of a wide variety of machines admit certain unifying geometric elements. The possible system configurations constitute the space $Q$. There is always the symmetry of Newtonian mechanics, namely, indifference of the dynamics to change of inertial observer. More generally, one has a Lie group $G$ of invertible transformations acting on $Q$ that leave the Lagrangian of the system invariant and possibly any applicable constraints as well. The equivalence classes defined by the orbits of $G$ can be brought to one-to-one correspondence with the space $S = Q/G$ of shapes. The triple $(Q,G,S)$ is known as a principal bundle. Most of the examples one encounters in mechanical settings can be given the structure of a *trivial* principal bundle, i.e., the configuration space looks like a product of $S \times G$, a simplification we shall assume from here on. Each configuration will then be a pair $q = (x,g)$.

If sufficiently many independent constraints (analogous to the constraint of no sliding in the unicycle example) are present then, it is possible to construct a well-defined splitting of the space of velocities (tangent bundle $TQ$ of the configuration space) at every configuration, into a set of symmetry directions along group orbits (also called vertical directions) and a set of complementary directions (called horizontal directions) *isomorphic* to the space of directions along which one can change the shape. One then says that the bundle $(Q,G,S)$ has acquired a *principal connection* (see Figure 4.5 for a sketch of the geometric set-up). The curvature of the connection has a great to deal to say about the following question. In analogy with the unicyclist's problem, where do we end up in the configuration space $Q$ when we make a cyclic movement in the space of shapes? The constraints are sufficient to determine a relation between the evolution of $g(t)$ and the shape trajectory $x(t)$, of the form

$$\dot{g} = -g \cdot \xi(x, \dot{x}), \qquad (4.10)$$

where $\xi$ represents the *connection form* and is linear in $\dot{x}$. Despite the complication arising from the connection form, equation 4.10 is a good deal like equation 4.3. The concept of holonomy in differential geometry gives a formal answer to the above question. Drift in the group variables can be built up by repetitively traversing the same loop in shape space. If the shape velocity is of the form $\dot{x} = \varepsilon \cdot f(t)$, where $\varepsilon$ is a small amplitude parameter, then, as discussed above in "From Shape Change to Global Movement," one can give an approximate solution to equation 4.10 using the theory of averaging. This is done in Leonard and Krishnaprasad (1994a,b) and in Leonard (1994), leading to area rules. Once again the area rules yield constructive procedures for generating suitable movements in shape space to achieve required transport in configuration space.

The unifying geometric point of view of this section is very useful in formulating answers to constructive controllability questions arising in the study of maneuvers of space-based robotic devices (Krishnaprasad, 1990), the problem of the paramecium at low Reynolds number studied by Shapere and Wilczek (1989), and a wide variety of nonholonomically constrained problems. There is much that needs to be done to integrate this geometric viewpoint into the control architecture presented above in "Scripts and Oscillators." In particular, the capability to adapt motion scripts in this level of generality, based on sensory inputs, probably needs new mathematical apparatus.

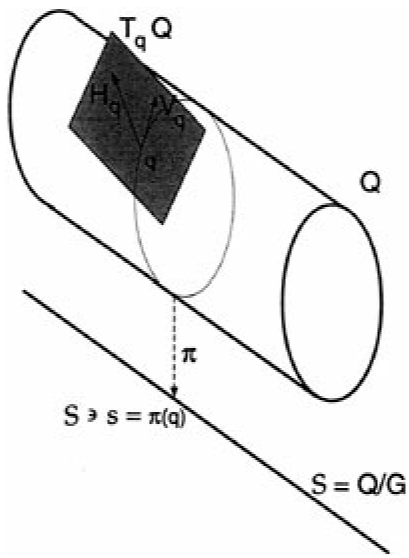


FIGURE 4.5 Principal bundle with connection.

## SOME INTERESTING MACHINES

Some of the ideas we presented here have been tested in the laboratory and in simulation. In the thesis of Manikonda (1994), motion planning for nonholonomic robots in the presence of obstacles is investigated from a perspective close to the one we discuss. Over the years, there has been growing interest in robotic machines that exploit principles of movement found in biological systems. The excellent book of Hirose (1993) contains many examples of successful designs and algorithms. Encouraged by certain designs for redundant manipulators developed by Joel Burdick and his students at

Caltech, we investigated a variety of machines that could be controlled via shape change. One such instance is the nonholonomic variable geometry truss (NVGT) in Figure 4.6. This machine consists of a pair of modules on idle wheels, rolling without slipping on a surface, with deformable bodies. The intent is to drive this machine entirely by deformations of the body using the connecting links, without any direct actuation of the wheels.

The NVGT fits nicely into the framework of this paper. The configuration space is the Cartesian product of three copies of the rigid motion group $SE(2)$, and the symmetry group is also $SE(2)$. Thus the shape space is $S = SE(2) \times SE(2)$, representing the freedom to alter the shape by changing the lengths of the connecting links in each module. Apart from certain singular configurations, determined as those for which all three axles intersect (possibly at infinity), the unifying geometry discussed previously applies and the "no sliding" constraints fix a principal connection. Cyclical shape changes produce snake-like movement of the machine (Krishnaprasad and Tsakiris, 1994; Tsakiris, 1995).

It should be clear that additional modules could be attached to the NVGT of Figure 4.6, thereby increasing the number of constraints and the number of degrees of freedom. In that case, as shown in Krishnaprasad and Tsakiris (1994b), and Tsakiris (1995), the problem becomes over constrained, thus limiting the allowable shape changes. This in itself is not a disadvantage in selecting shape change scripts.

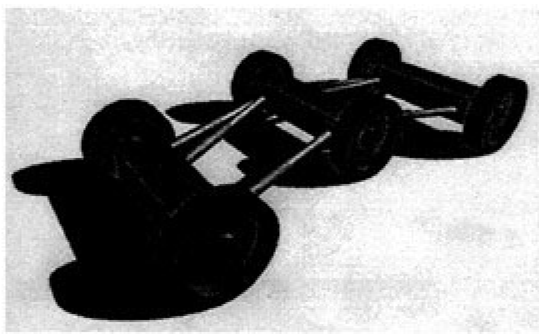

FIGURE 4.6 Two-module nonholonomic variable geometry tress. Reprinted, by permission, from Krishnaprasad and Tsakiris, 1995. Copyright © 1995 by Institute of Electrical and Electronics Engineers.

Suppose now that one of the modules in Figure 4.6 is detached and we are left with just one module. In this case the problem is under constrained, and one does not quite have the unifying geometry described above. One does not obtain a principal connection from the "no sliding" constraints alone. There is a subtler symmetry in the problem that arises from the interaction between the original Newtonian symmetry and the constraints, which yields a new momentum equation that the trajectories of the system must obey. The main ideas behind this new symmetry have only recently become clear in the work of Bloch et al. (1996). To illustrate this, a machine modeled on the patented toy Roller Racer (U.S. patent # 3663038 of May 16, 1972) was built (see Figure 4.7). This device is a special case of the single module nonholonomic variable geometry truss on wheels, with only a single degree of shape freedom.

The shape space in this case is the circle $S^1$. It is remarkable that in this case the full theory of nonholonomic momentum equation applies and using this extra equation, one formulates a principal connection on the bundle ($S^1 \times SE(2)$, $SE(2)$, $S^1$). Motion control by periodic forcing of one degree of shape freedom is accomplished. Details can be found in Tsakiris (1995).

The last-mentioned example uses dynamical information in an essential way, and in some sense there are parallels between this investigation and the work of Bowtell and Williams (1991) on the lamprey. The rich variety of global motions can be best understood by the proper synthesis of kinematic, geometric, and dynamic information, and the principle of rectification applied to cyclical shape variations is an effective guide even in this mathematically complex setting. An intelligent control architecture based on such synthesis would be of great interest. Again nature would have taught us to build better machines.

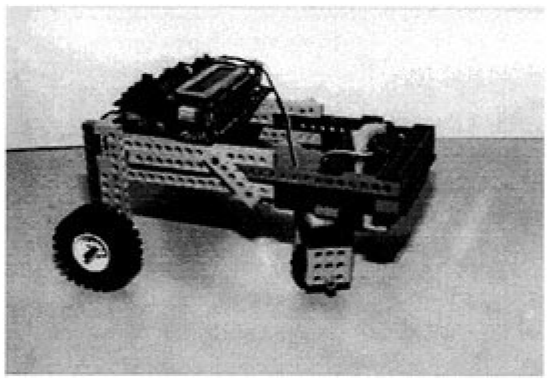

FIGURE 4.7 Computer-controlled roller racer.

## ACKNOWLEDGMENTS

It is a pleasure to acknowledge fruitful collaborations on the subject matter of this paper with Anthony Bloch, Roger Brockett, Naomi Leonard, Vikram Manikonda, Jerrold Marsden, Richard Murray, Dimitris Tsakiris, and Rui Yang. Conversations with Joel Burdick and Avis Cohen on the subject of rhythmic movement have been a source of inspiration. The theses of Mishra and Marvin were influential in the development of the ideas here. Tom Finan got us thinking about the Roller Racer. This research was supported in part by the National Science Foundation's Engineering Research Centers Program, NSFD CDR 8803012; by the AFOSR University Research Initiative Program under grant AFOSR-90-0105 and by grant AFOSR-F49620-92-J-0500; and by the Army Research Office under Smart Structures, URI Contract No. DAAL03-92-G-0121.

## REFERENCES

Alexander, R., 1968, *Animal Mechanics*, Seattle: University of Washington Press.

Bernstein, A., 1967, *The Coordination and Regulation of Movements*, Oxford: Pergamon Press.

Bloch, A.M., P.S. Krishnaprasad, J.E. Marsden, and R.M. Murray, 1996, "Nonholonomic Mechanical Systems with Symmetry," *Archive for Rational Mechanics and Analysis* (in press).

Bowtell, G. and T.L. Williams, 1991, "Anguilliform Body Dynamics: Modelling the Interaction Between Muscle Activation and Body Curvature," *Philos. Trans. R. Soc. Lond. B. Biol. Sci*. **234**, 385-390.

Brockett, R.W., 1989, "On the Rectification of Vibratory Motion," *Sensors and Actuators* **20**(1-2), 91-96.

Brown, T.G., 1914, "On the Nature of the Fundamental Activity of the Nervous Centres, Together with an Analysis of the Conditioning of Rhythmic Activity in Progression and a Theory of Evolution of Function in the Nervous System," *J. Physiol. Lond*. **48**, 18-46.

Childress, S., 1981, *Mechanics of Swimming and Flying*, Cambridge: Cambridge University Press.

Chirikjian, G. and J.W. Burdick, 1995, "Kinematics of Hyper-redundant Locomotion with Applications to Grasping," *Proceedings of the IEEE International Conference on Robotics and Automation*, New York: IEEE, 720-727.

Cohen, A.H., 1988, "Evolution oft he Vertebrate Central Pattern Generator for Locomotion." In: *Neural Control of Rhythmic Movements in Vertebrates* , A.H. Cohen, S. Rossignol, and S. Grillner (eds.), New York: John Wiley and Sons Inc.

Cohen, A.H., P.J. Holmes, and R.H. Rand, 1982, "The Nature of the Coupling Between Segmental Oscillators of the Lamprey Spinal Generator for Locomotion: A Mathematical Model," *J. Math. Biol*. **13**, 345-369.

Cohen, A.H., S. Rossignol, and S. Grillner (eds.), 1988, *Neural Control of Rhythmic Movements in Vertebrates*, New York: John Wiley and Sons Inc.

Collins, J.J. and I. Stewart, 1993, "Hexapodal Gaits and Coupled Nonlinear Oscillator Models," *Biol. Cybern*. **68**, 287-298.

Hirose, S., 1993, *Biologically Inspired Robots: Snake-Like Locomotors and Manipulators*, Oxford: Oxford University Press.

Kopell, N., 1988, "Toward a Theory of Modelling Central Pattern Generators." In: *Neutral Control of Rhythmic Movements in Vertebrates*, A.H. Cohen, S. Rossignol, and S. Grillner (eds.), New York: John Wiley and Sons Inc.

Kopell, N. and G.B. Ermentrout, 1988, "Coupled Oscillators and the Design of Central Pattern Generators," *Math. Biosci*. **90**, 87-109.

Krishnaprasad, P.S., 1990, "Geometric Phases and Optimal Reconfiguration of Multibody Systems," *Proceedings of the American Control Conference* , American Automatic Control Council, Philadelphia, 2440-2444.

Krishnaprasad, P.S. and D.P. Tsakiris, 1994a, "2-Module Nonholonomic Variable Geometry Truss Assembly: Motion Control," *Proceedings of the 4th IFAC Symposium on Robotics Control*, 263-268.

Krishnaprasad, P.S. and D.P. Tsakiris, 1994b, "G Snakes: Nonholonomic Kinematic Chains on Lie Groups," *Proceedings of the 33rd IEEE Conference on Decision and Control*, IEEE, New York, 2955-2960.

Krishnaprasad, P.S. and D.P. Tsakiris, 1995, "Oscillations, SE(2)-snakes and Motion Control," *Proceedings of the 34th IEEE Conference on Decision and Control*, IEEE, New York, 2806-2811.

Lambert, C., 1994, "To Run Atop the River," *Harvard Magazine*, May-June, 14-15.

Leonard, N.E., 1994, *Averaging and Motion Control of Systems on Lie Groups*, Ph.D. thesis (Institute for Systems Research Technical Report, Ph.D. 94-09), University of Maryland, College Park.

Leonard, N.E. and P.S. Krishnaprasad, 1994a, "Motion Control of Drift-Free Left-Invariant Systems on Lie Groups: Parts I & II," *Institute for Systems Research Technical Reports*, **94**-8 and **94**-9, University of Maryland, College Park. (Also *IEEE Transactions on Automatic Control* **40**(9), Sept. 1995, 1539-1554.)

Leonard, N.E. and P.S. Krishnaprasad, 1994b, "Motion Control of Autonomous Underwater Vehicle with an Adaptive Feature," *Proceedings of a Symposium on Autonomous Underwater Vehicle Technology*, IEEE Oceanic Engineering Society, 283-288, IEEE, New York.

Manikonda, V., 1994, *A Hybrid Control Strategy for Path Planning and Obstacle Avoidance with Nonholonomic Robots*, M.S. thesis, Institute for Systems Research Technical Report, M.S. 94-8, University of Maryland, College Park.

Murray, R.M., Z. Li, and S.S. Sastry, 1994, *A Mathematical Introduction to Robotic Manipulation*, Boca Raton, FL: CRC Press.

Murray, R.M. and S.S. Sastry, 1993, "Nonholonomic Motion Planning: Steering Using Sinusoids," *IEEE Transactions on Automatic Control* **38**(5), 700-716.

Pearson, K.G., 1993, "Common Principles of Motor Control in Vertebrates and Invertebrates," *Annu. Rev. Neurosci*. **16**, 265-297.

Shapere, A. and F. Wilczek, 1989, "Geometry of Self-Propulsion at Low Reynolds Number," *J. Fluid Mech*. **198**, 557-589.

Stewart, D.B., R.A. Volpe, and P.K. Khosla, 1993, "Design of Dynamically Reconfigurable Real-Time Software using Port-Based Objects," Carnegie Mellon University Tech. Report CMU-RI-TR-93-11, July.

Tsakiris, D.P., 1995, *Motion Control and Planning for Nonholonomic Kinematic Chains*, Ph.D. thesis, University of Maryland, College Park.

Ueha, S. and Y. Tomikawa, 1993, *Ultrasonic Motors: Theory and Applications* , Oxford: Clarendon Press.

# Appendixes

# Appendix A

# Speakers

**Roger W. Brockett** is An Wang Professor of Electrical Engineering and Computer Science in the Division of Applied Science at Harvard University. He has contributed extensively to the theory of automatic control with work on stability, nonlinear control, feedback linearization, system identification, nonlinear estimation, and design by pole placement. More recently his work has involved problems arising in the study of intelligent machines. Areas of particular interest include the problem of motion control and the investigation of new computational paradigms appropriate for control in the high-data-rate, sensory-rich environments that characterize vision-guided systems. Professor Brockett has been recognized by the American Automatic Control Council and the IEEE through their Richard Bellman Award and the Control System Science and Engineering Award, and he is a member of the National Academy of Engineering.

**P.S. Krishnaprasad** is Professor of Electrical Engineering at the University of Maryland, with a joint appointment at the Institute for Systems Research, and a member of the faculty of the Applied Mathematics Program. His research interests lie in the broad area of geometric control theory and its applications. He has contributed to the understanding of parametrization problems in linear systems, the Lie algebraic foundations of certain nonlinear filtering problems pertaining to system identification, the Lie theory and stability of interconnected mechanical systems, and symmetry principles in nonlinear control theory. In the last few years, Professor Krishnaprasad has undertaken a deeper study of the role of artificial neural networks in solving a variety of problems, and he is currently actively exploring the use of networks of coupled oscillators as a framework for perception and control in nature and in machines.

**Jerrold E. Marsden** is a professor in the department of Control and Dynamical Systems at the California Institute of Technology. He has done extensive research in the area of geometric mechanics, with applications to fluid mechanics, elasticity theory, plasma physics, and general field theory. He also works in the area of dynamical systems and control theory, focussing on how it relates to mechanical systems and systems with symmetry. He is one of the original developers (in the early 1970s) of reduction theory for mechanical systems with symmetry, which remains an active and much studied area of research today. He was the recipient in 1990 of the prestigious Norbert Wiener prize of the American Mathematical Society and the Society for Industrial and Applied Mathematics.

**Richard M. Murray** is an assistant professor of mechanical engineering at the California Institute of Technology. His major research interests are in nonlinear control of mechanical systems, with recent emphasis on the dynamics and control of mechanical systems with nonholonomic constraints. His application areas include mobile robots, nonlinear flight control, multifingered robot hands, and active control of high-performance turbomachinery. Dr. Murray was instrumental in the creation of the new Control and Dynamical Systems Department at Caltech, which emphasizes the interdisciplinary nature of dynamical systems and control, and the application of advanced theory to complex problems in engineering and science.

# Appendix B

# Symposium Participants

Gerald Andersen
George Mason University

Joseph Applebaum
U.S. Department of Labor

Louis Auslander
City University of New York

Hyman Bass
Columbia University

Daria Bielecki
Naval Research Laboratory

Yvonne Bishop
U.S. Department of Energy

Claudia Blair
National Institutes of Health

I. Edward Block
Society for Industrial and Applied Mathematics

Albert Bosse
Naval Research Laboratory

John S. Bradley
American Mathematics Society

Roger Brockett
Harvard University

Thomas Carr
Naval Research Laboratory

Jagdish Chandra
U.S. Army Research Office

Gregory Chirikjian
Johns Hopkins University

Ed Cohen
Naval Surface Warfare

Gall Corbett
Society for Industrial and Applied Mathematics

Gregory Coxson
Alexandria,Virginia

James Donaldson
Howard University

Rick Fenton
McMaster University Medical Center

Barry Fridling
Institute for Defense Analyses

Avner Friedman
University of Minnesota

Paul Gaddie
University of Louisville

Devandra P. Garg
National Science Foundation

Ionannis Georgiou
Naval Research Laboratory

Andrew Girard
Greenbelt, Maryland

James F. Glazebrook
National Science Foundation

Vmadini Greenberg
U.S. Department of Energy

Ja Gualtieri
National Aeronautics and Space Administration

Murli Gupta
George Washington University

Ralph Hartley
Naval Research Laboratory

Liam Healy
Naval Research Laboratory

John Heurtley
Federal Aviation Administration

Samuel Hollander
Naval Research Laboratory

Frederick Howes
U.S. Department of Energy

Marc Jacobs
Air Force Office of Scientific Research

Scott James
Kensington,Maryland

W. Karwowski
University of Louisville

Jon R. Kettenring
Bellcore

P. S. Krishnaprasad
University of Maryland

John Lagnese
Georgetown University

Feogyong Lee
University of Maryland

James Lightbourne
National Science Foundation

Vikram Manikonda
University of Maryland

Jerrold Marsden
California Institute of Technology

Steve Martinez
General Accounting Office

Lilly Mason
U.S. Postal Service

Peter A. McCoy
U.S. Naval Academy

Eric Mokole
Naval Research Laboratory

Paul S. Muhly
University of Iowa

Richard Murray
California Institute of Technology

Ruth O'Brien
National Research Council

Ronald F. Peierls
Brookhaven National Laboratory

Charles Perry
U.S. Department of Agriculture

Donald Richards
University of Virginia

Geoffrey Ringer
Medical College of Virginia

Robert J. Rubin
National Institutes of Health

Richard Rudolph
Bethesda, Maryland

Jerome Sacks
National Institute of Statistical Sciences

Ira Schwartz
Naval Research Laboratory

Mark Stamp
National Security Agency

Margie Stein
U.S. Postal Service

Michael Steiv
Vitro

Stephan Thiriez
George Washington University

Ana Triandaf
Naval Research Laboratory

Dimitris Tsakiris
University of Maryland

John Tucker
National Research Council

Barbara Tyson
Bethesda, Maryland

Barry Walden
Naval Research Laboratory

Shmuel Winograd
IBM Corporation