# Essays on Stochastic Inventory Model

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# 中文摘要

随着市场的全球化和竞争的激烈化，供应链也随之变得越来越复杂。 在如此的市场环境下，可行而又有效的供应链管理对一个公司的成败是至关重要的。 而在供应链管理中，库存问题是一个及其重要的分支。库存费用往往是巨大的。本篇论文由三篇文章组成，分别考虑了三个相关的随机库存管理问题。

第一篇文章考虑了一个动态非平稳的随机库存问题。其补货的形式是以一种批量补货， 例如用卡车，或者容器。我们考虑两种不同的情况。第一种， 价格是外在的。第二种，价格是由公司决定的。 在第一种情况，如果订货成本没有准备成本的话，我们可以证明无论是单级的，还是多级的供应链，其最优的订货策略是（r，Q）策略。如果存在准备成本的话，我们证明对于单级的供应链来讲，基于批量的（s，S）策略是最优的。在第二种情况下，在假设需求关于价格的函数是可加形式下，我们证明（r，Q，P）策略是最优的，在这种策略下，补货策略服从（r，Q）策略，而价格由库存水平决定。

第二篇文章研究了一个多期随机库存问题，其准备成本是关于订货量的一个分段函数。 具体的说，如果订货量小于 C，那么准备成本是 K1。 否则的话，准备成本是 K2。这里我们假设 K2>=K1， 这个费用结构在一些工业和生产机构有很多应用。 我们首先介绍了一种新的函数类，叫做 C-(K1，K2) 凸。在假设条件 K1<=K2<=2K1 下，我们部分的刻画了最优订货策略。而在更加松弛的条件 K1<=K2 下，我们通过另外一种新的函数叫强 K 凸，刻画了其最优订货策略的结构。通过这些分析，我们给出了一个有效的启发式算法。 数值试验表明这种启发式算法对于解决这类问题非常有效。

第三篇文章研究的是一个多期的生产库存问题，其下一期的准备成本将被这期的订货量影响。具体的说，如果这期的订货量超过一个临界值，那么下期的系统将处于“暖”状态。而此时，无论生产与否都不会产生任何准备成本。相反，如果这期的订货量小于这个临界值，那么下期的系统将处于“冷”状态。这时，任何生产都会产生一个准备成本。我们对这个系统建立了一个动态规划。在假设需求服从 Polya 或者均匀分布的前提下，我们部分地刻画了最优订货策略。 基于这些分析，我们给出了一个启发式算法。数值试验表明这种启发式算法的效率和效果都非常的不错。

Abstract of thesis entitled:

Essays on Stochastic Inventory Models

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With increased globalization and competition in the current market, supply chain has become longer and more complicated than ever before. An effective and efficient supply chain is crucial and essential to a successful firm. In a supply chain, inventories are a very important component as the investment in inventories is enormous. This dissertation consists of three essays related to stochastic inventory management.

The first essay considers a dynamic non-stationary inventory problem in which replenishment is made in fixed lot sizes (e.g., in full truckloads or full containers). We consider two separate cases: one with exogenous pricing and the other with endogenous pricing. In the first case (exogenous pricing), we show that when the ordering cost contains only a variable component, the reorder-point lot-size policy or $(r, Q)$ policy is optimal for both single-stage and multi-echelon inventory systems. In the presence of a fixed cost, we establish the optimality of batch-

based $(s, S)$ policies for the single-stage inventory system. In the second case (endogenous pricing), we show that when the demand function has the additive form and there is only a variable ordering cost, the $(r, Q)$ list-price policy is optimal for the single-stage system, where inventory replenishment follows an $(r, Q)$ policy and the optimal price in each period depends on the order-up-to level.

The second essay analyzes a periodic-review, stochastic, inventory-control system in which the fixed order-cost is a step function of the order size. In particular, if the order size is within a specified limit, $C$, then the setup cost is $K_1$; otherwise it is $K_2$, where $K_2 \geq K_1$. This cost structure is motivated from some industrial applications and transportation/production contracts used in practice. Under the condition that $K_1 \leq K_2 \leq 2K_1$, we introduce a new concept called $C-(K_1, K_2)$-convexity, which enables us to partially characterize the structure of an optimal ordering policy. For the general condition $K_1 \leq K_2$, the analysis is facilitated with a different notion called strong $K$-convexity. Based on this analysis, we provide a partial characterization of the optimal policy and construct an easy-to-implement heuristic method that has near-optimal performance in random test instances. Our study extends or redevelops (with different techniques) several existing results in the literature.

The third essay studies a firm's periodic-review production/inventory ordering decisions when the next period's setup cost depends on the quantity produced/ordered in the current period. In particular, if the current period's pro-

duction/order quantity exceeds a specified threshold value, the system starts the next period in a "warm" state and no fixed setup cost is incurred; otherwise the state is considered "cold" and a positive setup cost is required for production/ordering. We develop a dynamic programming formulation of the problem and provide a partial characterization of the optimal policy under the assumption that the demands follow a Pólya or Uniform distribution. We use the structural results to develop fairly simple heuristic policies, which perform highly effectively in our computational experiments.

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This work is dedicated to...

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# Chapter 1

# Inventory System with Batch Ordering

## 1.1 Introduction

### 1.1.1 Background and Main Contributions

Materials often flow in fixed batch sizes in supply chains. For example, packaged consumer goods typically arrive at stores in casepacks (Ketzenberg et al., 2000), finished goods may be transported in full containers from manufacturers to distributors, and work-in-process (WIP) is usually processed in convenient lot sizes between production stages. Furthermore, many companies experience *non-stationary demand because of seasonality, trends, and short life cycles.* For example, electronic products mostly have life cycles in units of months, with 6- to 12-month life cycles being common (Graves and Willems, 2008). In addition, the prices of raw materials such as metals, oil, and electricity vary significantly over time. All of these situations require a non-stationary model with batch ordering.

Facilitated by advanced information technologies, many industries beyond the airlines and hotels have been applying dynamic pricing strategies in the last decade. Dynamic pricing, where price adjusts over time to match supply with demand, has long been adopted in retail industries. For instance, MarketWatch (Cheng, 2009) recently reported:

> *"At the Banana Republic store in New York's World Financial Center, a white pleated skirt was on sale for $39.99, marked down from $69. The same skirt was discounted to $33.99 at Banana Republic's SoHo store, just two miles away.*
>
> *"The $6.00 difference wasn't a mistake. It's part of Banana Republic's parent Gap Inc.'s very deliberate move to tailor prices to fit local demand and inventory - right down to the individual store level.*
>
> *"The payoff: Gap's merchandise margins have either matched or topped year-ago levels in each of the past five months through May..."*

MarketWatch further pointed out that other retailers, including Wal-Mart Stores Inc., Home Depot, Sears, and Macy's Inc., had taken on or expanded some forms of store-based markdowns; i.e., rather than marking down by entire market, they used a markdown strategy that depends on a store inventory and its local market demand.

In response to these developments, there has emerged a growing body of academic research on integrated pricing and inventory decisions. This is because a basic yet critical success element of marketing and operations is the ability to match supply with demand. As Federgruen and Heching (1999) have shown, an optimal integrated price and inventory control policy can result in significant profit improvements by 6.5% for a specialty retailer as compared to a sequential procedure in which a price trajectory is determined first (by marketing/sales), followed by an inventory replenishment policy to optimally match the resulting sequence of demand distributions. Therefore, integrated pricing and inventory control is not only useful, but is essential. To date, however, the literature on pricing-inventory control has confined itself to models with arbitrary ordering sizes (i.e., lot-for-lot sizes), whereas in supply chains, materials often flow in fixed batch sizes. Extending some of the existing models of pricing-inventory control to take into account batch ordering is one of the primary goals of this paper.

As batch ordering is pervasive in the retail, distribution, industrial, and service environments, it is no surprise that numerous papers dealing with batch-ordering inventory systems have appeared in the literature. In a single-stage, periodic-review setting, Veinott (1965) shows the optimal inventory policy to be of the $(r, Q)$ type, or the so-called reorder-point lot-size policy: when the inventory level falls below the reorder point $r$, it is increased to the range between the reorder point and the maximum level $r + Q$ by placing an order of a single batch ($Q$) or multiple batches. His optimality result has been formally established in a stationary setting (i.e., the cost and demand parameters are identical over time, and the demand distributions over time are i.i.d.). For the non-stationary setting, Veinott has provided a set of sufficient conditions under which the same policy structure is optimal. These conditions are, however, difficult to verify (or can be verified only numerically). Moreover, Veinott's model includes only variable ordering costs. Therefore, the first two objectives of this chapter are (1) to identify easily verifiable conditions for the optimality of the reorder-point lot-size policy and (2) to analyze the structure of the optimal policy in the presence of fixed ordering costs. Both objectives will be accomplished in a non-stationary setting.

Chen (2000) further extends Veinott's analysis to multi-echelon systems with batch ordering and establishes the optimality of the echelon stock-based reorder-point lot-size ordering policy under the long-run average criterion. On the basis of the equivalence between serial and assembly systems under certain conditions (Rosling, 1989), his results also prevail for an assembly system with batch ordering. Chen's work also extends the model proposed by Clark and Scarf (1960), which assumes a base quantity of one for every stage. However, Chen (2000) points out in his concluding remarks that an extension of his results to a non-stationary setting constitutes an open problem. In this chapter, we attempt to solve this problem, which is our third objective.

To achieve the three aforementioned objectives, we introduce a new class of convex functions, which we coin as $Q$-jump-convexity, strong $Q$-jump-convexity, and $Q$-jump-$K$-convexity. These functions have appealing properties that enable us to make several contributions to the existing literature. In particular, this chapter extends both Veinott (1965) and Chen (2000). First, by virtue of our new class of convex functions, we are able to characterize the structure of the objective function and establish the optimality of the $(r, Q)$ policy for a single-stage, periodic-review, non-stationary inventory system with batch ordering. The assumptions are standard: linear variable ordering costs and convex holding-shortage costs. Our analysis is then extended to multi-echelon settings parallel to those of Chen (2000), but in a non-stationary setting. Second, we also characterize the optimal policy structure for a single-stage, batch-ordering, non-stationary inventory system with fixed ordering costs. The optimal policy structure is now of the batch-based $(s, S)$ type. Finally, to demonstrate the robustness of our approach, we further extend our analysis of the single-stage problem in which prices are exogenous to the case in which prices are endogenous. More specifically, we consider a periodic pricing-inventory control problem with batch ordering, but no fixed order costs. Demand in each period depends on the price in an additive form (see Section 4 for more details). We demonstrate that the $(r, Q)$ ordering policy remains optimal for inventory replenishment and that the list price is optimal for pricing decisions. This result also constitutes an important addition to the existing literature.

### 1.1.2 Literature Review

In this subsection, we briefly survey a few of the most related models. Four bodies of literature are related: inventory models emphasizing non-stationaries, batch ordering problems, inventory-pricing models, and generalized notions of convexity.

**Inventory models emphasizing non-stationaries.** The problem of periodic inventory replenishment with stochastic demand is pervasive in the retail, industrial, distribution, and service environments. The majority of models dealing with inventory management issues are thus discrete-time based. Moreover, underlying many of these models is a basic setting that incorporates stochastic demand in a non-stationary environment. These non-stationaries arise from a variety of different causes: for example, (i) changes in economic conditions (Karlin, 1960b; Song and Zipkin, 1993; Sethi and Cheng, 1997; Gavirneni, 2004); (ii) seasonal effects (Karlin, 1960a; Zipkin, 1989); (iii) Bayesian updates of demand information (Lovejoy, 1990); and (iv) short product life cycles (Graves and Willems, 2008; Neale and Willems, 2009).

The majority of work on stochastic, non-stationary inventory models focuses on characterizing the form of the optimal policy, for example, in addition to the above referenced papers, Scarf (1960) with setup costs and Kapuscinski and Tayur (1998) with a capacity constraint. These two papers are based on a single-stage setting. Relative to the non-stationary, single-stage literature, there is much less work for the optimality results in multi-echelon non-stationary models. Chen and Song (2001) show that an echelon base-stock policy with state-dependent order-up-to levels is optimal for a serial supply chain with Markov-modulated demand. Dong and Lee (2003) demonstrate that the structure of Clark and Scarf's (1960) optimal stocking policy holds under time-correlated demand processes using a Martingale model of forecast evolution. Because the time-varying parameters for these optimal policies are difficult to compute, a number of papers focus on heuristic algorithms to calculate them: for example, Morton and Pentico (1995) in a single-stage setting, and Graves and Willems (2008), Neale and Willems (2009), and Ettl et al. (2000) for multi-echelon systems. However, none of the work has addressed the optimality issue with batch ordering in a non-stationary environment.

**Batch-ordering models.** Batch-ordering models, which have been extensively investigated in the operations management literature, can be categorized based on a primary characteristic: single-stage or multi-echelon. Here, we survey only the most relevant papers that study the optimal policy structures. For single-stage models, as referred earlier, Veinott (1965) demonstrates the optimality of $(r, Q)$ policies for both stationary and non-stationary settings. To the best of our knowledge, there is only one paper by Alp, Huh and Tan (2009) that analyzes the optimal policies under the conditions of batch ordering and stochastic demand in a single-stage system with fixed order costs. More specially, they study the features of batch ordering, where a separate fixed cost is associated with each batch order. They partially characterize the optimal policy and propose two effective heuristic policies. Different from our model, however, they do not restrict the order quantities to integer multiples of the batch size, instead allowing the possibility of partial batches.

In addition to Chen (2000), who extends Veinott's stationary model to the multiechelon model with batch-ordering, Chao and Zhou (2009) study a serial system with batch ordering and fixed replenishment intervals (each of which may contain multiple review periods) and develop the structural properties of the system. Their paper generalizes Chen's (2000) work.

In general, however, the reorder-point, lot-size ordering policy is not optimal for many complex multi-echelon systems. Nevertheless, as it is easy to implement, many evaluation bounds and heuristics have been proposed (see De Bodt and Graves, 1985; Axsater and Rosling, 1993; Chen and Zheng, 1994, 1998; Cachon, 2001; and Shang and Zhou, 2009). For references on the computation of optimal parameters within the class of the full batch-ordering policy, the reader is referred to Zheng and Chen (1992), Federgruen and Zheng (1992), and Gallego (1998).

**Inventory-pricing problem.** Federgruen and Heching (1999) establish the optimality of the base-stock list-price policy, when the ordering cost is proportional to the order size and the batch-ordering requirement is absent. The base-stock list-price policy is characterized by the base-stock level for inventory replenishment and the respective list-price for pricing. Our inventory-pricing model is motivated in part by Federgruen and Heching. We extend their work by allowing batch ordering and demonstrate the $(r, Q)$ list-price policy to be optimal for the additive demand model (note that they consider a more general demand model). The past few years have witnessed significant progress in this area. For models with both variable and fixed costs, see Chen and Simchi-Levi (2004a&b), Huh and

Janakiraman (2008), Chen et al. (2006), and Song et al. (2009). For relatively earlier surveys, see Elmaghraby and Keskinocak (2003), Yano and Gilbert (2003), and Chan et al. (2004). To the best of our knowledge, however, no literature in this area has yet addressed the batch-ordering issue.

**Generalized notion of convexity.** The last stream of the literature we consider is generalized notions of convexity analysis. Since Scarf's (1960) K-convexity, many variations of generalized convexity have been proposed in inventory theory, based on the structure of the models, including quasi-K-convexity by Porteus (1971), symmetric-K-convexity by Chen and Simchi-Levi (2004a), $(K_1, K_2)$-convexity by Ye and Duenyas (2007) and Semple (2007), and CK-convexity by Gallego and Scheller-Wolf (1998). Our notion of $Q$-jump convexity is another generalization of convexity and $Q$-jump $K$-convexity is a variation of $K$-convexity.

In summary, this chapter strives to fill the following gaps that exists in the inventory literature: batch ordering in a non-stationary setting, batch ordering with fixed ordering costs, and batch ordering with pricing decisions. In terms of models, our work is most closely related to that of Veinott (1965) for the single-stage model, Chen (2000) for the multi-echelon setting, and Federgruen and Heching (1999) for the inventory-pricing problem. Our research is intended to complement theirs. In terms of analytical tools, we propose a set of generalized convex functions, that are new to the literature.

## 1.2 Single-stage Inventory Problem

In this subsection, we first specify our basic setting, the single-stage model, then introduce the concept of $Q$-jump-convexity, and finally apply it to structural analysis of the model.

We consider a dynamic non-stationary inventory model in which demands in different periods are independent random variables. At the beginning of each period, the system is reviewed, and then an order may then be placed for any nonnegative integral multiple of $Q$, a given positive number. Finally, demand arrives, and any unsatisfied demand is fully backordered. We assume without loss of generality that the leadtime is zero, because our analysis can easily be generalized to non-zero leadtimes by the notion of inventory position.

Three types of costs are assessed: ordering, holding, and penalty costs. Let $c_t$ be the unit ordering cost and $h_t(x)$ the holding/penalty cost incurred at the end of period $t$ with ending inventory level $x$. We assume that $h_t(x)$ is convex and $\lim_{|x|\to\infty} h_t(x) = \infty$. Let $L_t(y) = E\{h_t(y - D_t)\}$, which represents the expected holding/penalty cost in terms of the order-up-to level $y$ at the beginning of period $t$. Note that all of the costs may vary over periods, because we consider a non-stationary system. The costs in future periods are discounted with discount factor $\alpha \leq 1$.

For period $t = 1, 2, ..., T$, let

$x_t$ = the inventory level at the beginning of period $t$ before an order is placed;

$y_t$ = the inventory level (or position) after any order is placed in period $t$, but before demand is realized; and

$D_t$ = demand in period $t$.

Throughout this chapter, we define $\mathbb{Z}$ as an integer set, $\mathbb{Z}^+$ as a nonnegative integer set, and $\mathbb{R}$ as a real set.

Here, we consider the discrete version in which $D_t$, $x_t$ and $y_t$ are integer-valued, i.e., $D_t \in \mathbb{Z}^+$ and $x_t, y_t \in \mathbb{Z}$.

For any given $x_t$, the decision space can be characterized by

$$\mathcal{A}(x_t) = \{y_t | y_t = x_t + mQ, \text{ for } m \in \mathbb{Z}^+\}.$$

Let $v_t(x_t)$ be the optimal expected discounted cost from period $t$ until the end of the planning horizon $T$, when the starting inventory level in period $t$ is $x_t$, and $v_{T+1}(x) = 0$ for all $x \in \mathbb{Z}$. A dynamic program for the above problem is as follows. For each $t = 1, 2, ..., T$, we have

$$v_t(x_t) = -c_t x_t + \inf_{y_t \in \mathcal{A}(x_t)} \{\delta(y_t - x_t) \cdot K + J_t(y_t)\}, \tag{1.1}$$

where $K$ is the fixed ordering cost, $\delta(z) = 1$ if $z > 0$ and $= 0$ otherwise, and

$$J_t(y_t) = L_t(y_t) + c_t y_t + \alpha E[v_{t+1}(y_t - D_t)].$$

In what is to follow, we first provide an overview of the optimality results and then give the definition of $Q$-jump-convexity. Finally, we consider the cases without and with a fixed cost (i.e., $K = 0$ and $K > 0$), respectively.

### 1.2.1 Overview of Main Results in this Section

First, we divide $\mathbb{Z}$ into $Q$ different non-overlapping groups (or sets) as follows. Let $[j]_Q = \{z \in \mathbb{Z} | z = mQ + j,\ m = 0, \pm 1, \pm 2, \cdots\}$, for $j = 1, 2, \cdots, Q$. (Note that $[0]_Q$ can be treated as $[Q]_Q$.) $[j]_Q$ represents the set of integers with remainder $j$ when they are divided by $Q$.

Second, we define a batch-based $(s, S)$ policy, denoted by $(\mathbf{s}, \mathbf{S})_Q$ policy. Let $x_t \in [j]_Q$ be the initial inventory level (before ordering) in period $t$, $j = 1, 2, ..., Q$.

**Definition 1.1.** *(Batch-based $(s, S)$ Policy) An inventory policy is called a batch-based $(s, S)$ policy or $(\mathbf{s}, \mathbf{S})_Q$ policy, where $\mathbf{s} = (s^1, s^2, ..., s^Q)$, $\mathbf{S} = (S^1, S^2, ..., S^Q)$, and both $s^j$ and $S^j$ belong to $[j]_Q$, $j = 1, 2, \cdots, Q$, if inventory replenishment follows an $(s^j, S^j)$ policy, i.e., if $x_t < s^j$, then order up-to $S^j$, and otherwise, order nothing.*

As indicated by Figure 1.1, the batch-based $(s, S)$ policy implies that for any group $[j]_Q$, $j = 1, 2, \cdots, Q$, an ordinary $(s^j, S^j)$ policy is optimal. When $s^j = S^j$, this policy will be called batch-based base-stock policy, denoted by $(\mathbf{S})_Q$, where $\mathbf{S} = \{S^1, S^2, ..., S^Q\}$. Clearly, the $(r, Q)$ policy (Veinott 1965) is a special case of $(\mathbf{S})_Q$ policy. And, an $(\mathbf{S})_Q$ policy is the $(r, Q)$ policy *if* $\max_j\{S^j\} - \min_j\{S^j\} = Q - 1$. Then, $r = \min_j\{S^j\}$.

In the following, we will show that in the absence of fixed costs, the optimal policy is of the $(r, Q)$ type for a periodic-review, non-stationary, single-stage model. We prove this result by two steps: first prove that the batch-based base-stock or $(\mathbf{S})_Q$ policy is optimal and then show that $\max_j\{S^j\} - \min_j\{S^j\} = Q - 1$,

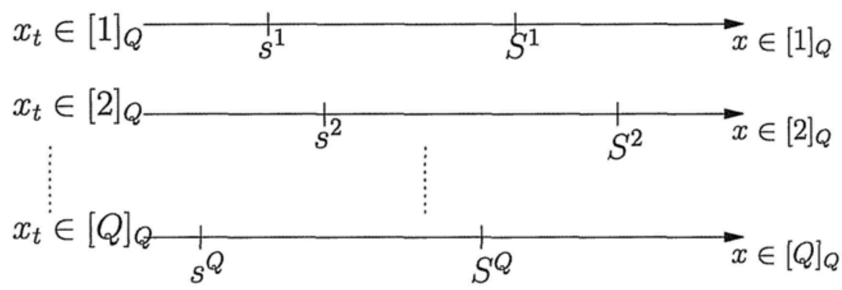


Figure 1.1: Illustration of the $(\mathbf{s}, \mathbf{S})_Q$ policy

i.e., the $(r, Q)$ policy is optimal. In the presence of fixed costs, we demonstrate that the optimal policy no longer has a simple form, however, a batch-based $(s, S)$ or $(\mathbf{s}, \mathbf{S})_Q$ policy is optimal.

All the structural analysis has been facilitated by the notion of $Q$-jump convexity and its variants. To the best of the authors' knowledge, this concept, which is formally introduced in the next subsection, has never been reported in the literature.

### 1.2.2 $Q$-Jump-Convexity

To begin with, recall the definition of ordinary convexity. As we consider discrete-valued demand and inventories, all functions are defined in a discrete manner.

**Definition 1.2.** *A function $f(x) : \mathbb{Z} \to \mathbb{R}$ is convex if for any $z_1 \in \mathbb{Z}$, $z_2 \in \mathbb{Z}$ and $z_1 \geq z_2$,*

$$f(z_1 + 1) - f(z_1) \geq f(z_2 + 1) - f(z_2).$$

This definition is equivalent to the following two statements: A function $f(x)$ :

$\mathbb{Z} \to \mathbb{R}$ is convex *if and only if* for any $y \in \mathbb{Z}$,

$$f(y+1) - f(y) \geq f(y) - f(y-1),$$

or *if and only if* for any $y \in \mathbb{Z}$, $z \in \mathbb{Z}^+$, and $b \in \mathbb{Z}^+/\{0\}$,

$$f(y+z) \geq f(y) + \frac{z}{b}\{f(y) - f(y-b)\}.$$

**Q-Jump-Convexity**

For Model (1.1) with $K = 0$ and $Q = 1$, ordinary convexity is sufficient for structural analysis. When $Q > 1$, in contrast, it is no longer sufficient.

**Definition 1.3.** *(Q-Jump-Convexity) A function $f(x) : \mathbb{Z} \to \mathbb{R}$ is $Q$-jump-convex if for any $y \in \mathbb{Z}$,*

$$f(y+Q) - f(y) \geq f(y) - f(y-Q).$$

Equivalently, a function $f(x) : \mathbb{Z} \to \mathbb{R}$ is $Q$-jump-convex *if and only if* for any $y \in \mathbb{Z}$, $z \in \mathbb{Z}^+$, and $b \in \mathbb{Z}^+/\{0\}$, $f(y+zQ) \geq f(y) + \frac{z}{b}\{f(y) - f(y-bQ)\}$.

To understand this definition in a more intuitive way, refer to Figure 1.2, which illustrates three $Q$-jump-convex functions with $Q = 3$, where for any $x \in \mathbb{Z}$,

$$g_1(x) = \begin{cases} 0.5 & if\ x \in [1]_3, \\ 1 & if\ x \in [2]_3, \\ 3 + \dfrac{x}{6} & if\ x \in [3]_3. \end{cases}$$

$$g_2(x) = x^2/6 \qquad for\ any\ \ x \in \mathbb{Z}.$$

$$g_3(x) = \begin{cases} 2.6 & if\ x \in [1]_3, \\ 3.9 & if\ x \in [2]_3, \\ 4.2 & if\ x \in [3]_3. \end{cases}$$

Clearly, $Q$-jump-convexity is an extension of convexity, and it is reduced to convexity when $Q = 1$. We summarize the properties of $Q$-jump-convex functions as follows.

**Lemma 1.1.** *(Properties of Q-Jump-Convexity)*

(a) *A convex function is also a Q-jump-convex function.*

(b) *If $f(x)$ is Q-jump-convex and $\alpha$ is a positive scalar, then so is $\alpha f(x)$.*

(c) *The sum of any two Q-jump-convex functions is also Q-jump-convex.*

(d) *If $v(x)$ is Q-jump-convex and $w$ is a random variable that takes only non-negative integer values, then $G(y) = E\{v(y - w)\}$ is also Q-jump-convex.*

(e) *$f(x)$ is Q-jump-convex if and only if for any given integer $a$, $g^a(y) = f(a + yQ)$ is convex.*

**Proof.** Parts (a), (b), and (c) follow directly from the definition of Q-jump-convexity.

Part (d). Without loss of generality, suppose that the distribution of $w$ can be characterized by $P(w = i) = \lambda_i$ for $i = 0, 1, 2, ..., W$, where $W$ is an upper bound and $\sum_{i=0}^{W} \lambda_i = 1$. Then, $G(y) = \sum_{i=0}^{W} \lambda_i v(y-i)$. Next, we prove that $G(y)$ satisfies $G(y + zQ) \geq G(y) + \frac{z}{b}\{G(y) - G(y - bQ)\}$ for any $y \in \mathbb{Z}$, $z \in \mathbb{Z}^+$, and

$b \in \mathbb{Z}^+/\{0\}$. Note that for any $i = 0, 1, 2, ..., W$, $\lambda_i v(y + zQ - i) \geq \lambda_i v(y - i) + \frac{z}{b}\{\lambda_i v(y - i) - \lambda_i v(y - bQ - i)\}$, due to the $Q$-jump-convexity of $v(x)$. Combing these $W$ inequalities, we can obtain the desired result.

Part (e). We first prove the necessity. By the $Q$-jump-convexity of $f(x)$, for any $y, a \in \mathbb{Z}$, $z \in \mathbb{Z}^+$, and $b \in \mathbb{Z}^+/\{0\}$, we have

$$f(a + (y + z)Q) \geq f(a + yQ) + \frac{z}{b}\{f(a + yQ) - f(a + (y - b)Q)\},$$

which implies that $g^a(y + z) \geq g^a(y) + \frac{z}{b}\{g^a(y) - g^a(y - b)\}$.

For the sufficiency, we need only prove that for any $x \in \mathbb{Z}$, $z \in \mathbb{Z}^+$, and $b \in \mathbb{Z}^+/\{0\}$, $f(x + zQ) \geq f(x) + \frac{z}{b}\{f(x) - f(x - bQ)\}$. By the convexity of $g^a(y)$, we have for any $y \in \mathbb{Z}$, $z \in \mathbb{Z}^+$, and $b \in \mathbb{Z}^+/\{0\}$,

$$g^a(y + z) \geq g^a(y) + \frac{z}{b}\{g^a(y) - g^a(y - b)\},$$

which implies that

$$f(a + (y + z)Q) \geq f(a + yQ) + \frac{z}{b}\{f(a + yQ) - f(a + (y - b)Q)\}.$$

Then, the result holds by letting $x = a + yQ$. □

Part (e) provides another way of interpreting $Q$-jump-convexity by connecting it with convexity. More specifically, based on a $Q$-jump-convex function $f(x)$, we can construct a new function $g^a(y)$ by picking up all those points whose distances from point $a$ are integral multiples of $Q$. Then, $f(x)$ is $Q$-jump-convex if and only if for any integer $a$, $g^a(y)$ is convex.

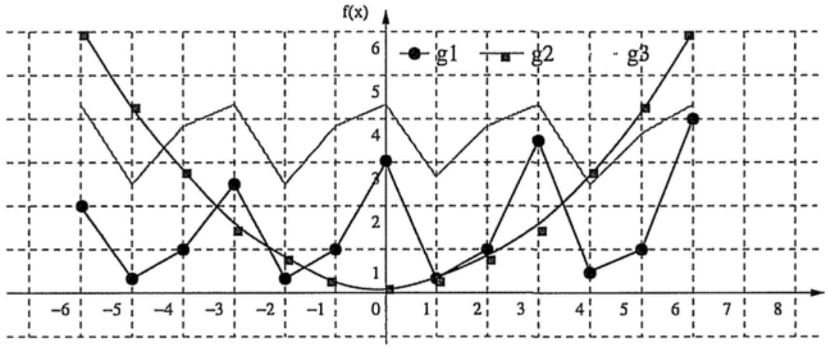


Figure 1.2: Geometric examples of $Q$-jump-convex function with $Q = 3$

**Strong Q-Jump-Convexity**

We will see that $Q$-jump-convexity can not guarantee the optimality of $(r, Q)$ policy. Hence, we need a stronger kind of convexity as follows.

**Definition 1.4.** *(Strong Q-Jump-Convexity) A function $f(x) : \mathbb{Z} \to \mathbb{R}$ is strong $Q$-jump-convex if for any $z_1 \in \mathbb{Z}$, $z_2 \in \mathbb{Z}$, and $z_1 \geq z_2$,*

$$f(z_1 + Q) - f(z_1) \geq f(z_2 + Q) - f(z_2).$$

Lemma 1.2 (a) below indicates that a convex function is also strong $Q$-jump-convex, and that a strong $Q$-jump-convex function is also $Q$-jump-convex. However, the reverse does not hold. Figure 1.2 gives an example to show that a strong $Q$-jump-convex function may not be convex and that a $Q$-jump-convex function may not be strong $Q$-jump-convex. In Figure 1.2, for any $x \in \mathbb{Z}$, $g_3(x + Q) - g_3(x) = 0$, which implies that function $g_3(x)$ is strong $Q$-jump-convex. Clearly, $g_3(x)$ is not convex. However, although $g_1(x)$ is $Q$-jump-convex, it is not strong $Q$-jump-convex, because $0.5 = g_1(-6 + Q) - g_1(-6) \geq$

$g_1(-5+Q) - g_1(-5) = 0$. In conclusion, only $g_2$ is convex, and both $g_2$ and $g_3$ are strong $Q$-jump-convex, whereas all three functions are $Q$-jump-convex.

We summarize the properties of strong $Q$-jump-convex functions as follows.

**Lemma 1.2.** *(Properties of Strong Q-Jump-Convexity)*

(a) *A convex function is also strong Q-jump-convex, and a strong Q-jump-convex function is also Q-jump-convex.*

(b) *If $f(x)$ is strong Q-jump-convex and $\alpha$ is a positive scalar, then so is $\alpha f(x)$.*

(c) *The sum of any two strong Q-jump-convex functions is also strong Q-jump-convex.*

(d) *If $v(x)$ is strong Q-jump-convex and $w$ is a random variable that takes only nonnegative integer values, then $G(y) = E\{v(y-w)\}$ is also strong Q-jump-convex.*

**Proof.** The proof is similar to that of Lemma 1.1, and is thus omitted. □

**$Q$-Jump-$K$-Convexity**

The two types of $Q$-jump-convexity thus far introduced can be used to analyze Model (1.1) with $K = 0$. To deal with the model with $K > 0$, however, we need to extend $Q$-jump-convexity to $Q$-jump-$K$-convexity.

**Definition 1.5.** *(Q-Jump-K-Convexity) A function $f(x) : \mathbb{Z} \to \mathbb{R}$ is Q-jump-K-convex if for any $x \in \mathbb{Z}$, $z \in \mathbb{Z}^+$, and $b \in \mathbb{Z}^+/\{0\}$,*

$$f(x+zQ) + K \geq f(x) + \frac{z}{b}[f(x) - f(x-bQ)].$$

Clearly, $Q$-jump-$K$-convexity is an extension of $K$-convexity (Scarf, 1960).

**Lemma 1.3.** *(Properties of Q-Jump-K-Convexity)*

(a) *A convex or $Q$-jump-convex function is also $Q$-jump-0-convex. Moreover, a $K$-convex function is $Q$-jump-$K$-convex.*

(b) *If $f(x)$ is $Q$-jump-$K$-convex and $\alpha$ is a positive scalar, then $\alpha f(x)$ is $Q$-jump-$\alpha K$-convex; moreover, if $f(x)$ is $Q$-jump-$K_1$-convex, then it is also $Q$-jump-$K_2$-convex for $K_2 > K_1 \geq 0$.*

(c) *The sum of a $Q$-jump-$K_1$-convex function and a $Q$-jump-$K_2$-convex function is $Q$-jump-$(K_1 + K_2)$-convex.*

(d) *If $v(x)$ is $Q$-jump-$K$-convex and $w$ is a discrete random variable that takes only nonnegative integer values, then $G(y) = E\{v(y - w)\}$ is also $Q$-jump-$K$-convex.*

(e) *$f(x)$ is $Q$-jump-$K$-convex if and only if for any given integer $a$, $g^a(y) = f(a + yQ)$ is $K$-convex.*

**Proof.** See the Appendix. □

### 1.2.3 Zero Fixed Cost

In this subsection, we consider the case of Model (1.1) with $K = 0$ and prove the main results, i.e., the optimality of the $(r, Q)$ policy.

We first prove that the batch-based base-stock or $(\mathbf{S})_Q$ policy is optimal.

**Lemma 1.4.** (a) *For any $t = 1, 2, ..., T$, $J_t(y_t)$ and $v_t(x_t)$ are $Q$-jump-convex.*

(b) *For any $t = 1, 2, ..., T$, there exists $S_t^j \in [j]_Q$, where $j = 1, 2, ..., Q$, such that the $(\mathbf{S})_Q$ policy is optimal.*

**Proof.** We prove (a) and (b) simultaneously by induction.

For $t = T+1$, $v_{T+1} = 0$, and the results thus hold. (Note that as long as $v_{T+1}$ is convex in $x$, the desired result remains true.) By induction, we assume that $v_{t+1}(x)$ is $Q$-jump-convex. By Lemma 1.1 parts (b) and (d), $\alpha E[v_{t+1}(y_t - D_t)]$ is also $Q$-jump-convex. Because $h_t(y)$ is assumed to be convex, $L_t(y_t)$ is also convex. Then, by Lemma 1.1 parts (a) and (c), $J_t(y_t) = L_t(y_t) + c_t y_t + \alpha E[v_{t+1}(y_t - D_t)]$ is $Q$-jump-convex.

For any $j = 1, 2, ..., Q$, define $\mu_t^j(z) = J_t(j + zQ)$. Thus, for any given $x_t \in [j]_Q$,

$$v_t(x_t) = -c_t x_t + \min_{z_t \geq \frac{x_t - j}{Q}, z_t \in \mathbb{Z}} \{\mu_t^j(z_t)\}.$$

By Lemma 1.1 part (e), $\mu_t^j(z_t)$ is convex. Note that $\frac{x_t - j}{Q}$ is an integer, as $x_t \in [j]_Q$. By the convexity of $\mu_t^j(z)$, there exists $S_t^j \in [j]_Q$ , such that $\frac{S_t^j - j}{Q}$ minimizes $\mu_t^j(z_t)$ (if not unique, then take the largest minimizer), and for any $x_t \in [j]_Q$,

$$v_t(x_t) = \begin{cases} \mu_t^j(\dfrac{S_t^j - j}{Q}) - c_t x_t = J_t(S_t^j) - c_t x_t & if\ x_t < S_t^j, \\ \mu_t^j(\dfrac{x_t - j}{Q}) - c_t x_t = J_t(x_t) - c_t x_t & if\ x_t \geq S_t^j. \end{cases} \tag{1.2}$$

(From the relationship between $\mu_t^j(z_t)$ and $J_t(y_t)$, we know that for $j = 1, 2, ..., Q$, $S_t^j$ is the largest minimum point of $J_t(y_t)$ on $[j]_Q$.)

For any given integer $a$, let $v_t^a(z) = v_t(a + zQ)$. Because $a + zQ \in [a]_Q$ for any

$z \in \mathbb{Z}$, we have

$$v_t^a(z) = \begin{cases} \mu_t^a(\frac{S_t^a - a}{Q}) - c_t(a + zQ) & if\ z < \frac{S_t^a - a}{Q}, \\ \mu_t^a(z) - c_t(a + zQ) & if\ z \geq \frac{S_t^a - a}{Q}. \end{cases} \tag{1.3}$$

By the convexity of $\mu_t^a(z)$, we know that $v_t^a(z)$ is also convex. By Lemma 1.1 part (e), $v_t(x_t)$ is $Q$-jump-convex, which completes the proof. □

Lemma 1.4 indicates that for any group $[j]_Q$, $j = 1, 2, \cdots, Q$, the base-stock policy is optimal, that is, if the inventory level $x_t \in [j]_Q$ at the beginning of the period is less than $S_t^j$, then order up to $S_t^j$. The following theorem shows the optimality of the $(r, Q)$ policy by proving $\max_j\{S_t^j\} - \min_j\{S_t^j\} = Q$.

**Theorem 1.1.** (a) *For any $t = 1, 2, ..., T$, $J_t(y_t)$ and $v_t(x_t)$ are strong $Q$-jump-convex.*

(b) *The $(r, Q)$ policy is optimal for the non-stationary system.*

**Proof.** We prove the results by induction. For $t = T + 1$, $v_{T+1} = 0$ and the results hold. Assume $v_{t+1}(x)$ to be strong $Q$-jump-convex. By Lemma 1.2 parts (b) and (d), $\omega_{t+1}(y_t) = \alpha E[v_{t+1}(y_t - D_t)]$ is also strong $Q$-jump-convex.

We first prove that the $(r, Q)$ policy is optimal for period $t$, i.e, $\max_j\{S_t^j\} - \min_j\{S_t^j\} = Q - 1$, which is clearly equivalent to $S_t^i - S_t^j \leq Q - 1$ for any $i, j = 1, 2, ..., Q$. We prove it by contradiction. Suppose there exists $i_1$ and $i_2$, such that $S_t^{i_2} - S_t^{i_1} > Q$. Because $S_t^{i_1}$ is the largest minimum point of $J_t(y_t)$ for $y_t \in [S_t^{i_1}]_Q$, we have the following inequality:

$$L_t(S_t^{i_1}) + \omega_{t+1}(S_t^{i_1}) + c_t S_t^{i_1} < L_t(S_t^{i_1} + Q) + \omega_{t+1}(S_t^{i_1} + Q) + c_t(S_t^{i_1} + Q). \tag{1.4}$$

Similarly, for $S_t^{i_2}$, we obtain:

$$L_t(S_t^{i_2}) + \omega_{t+1}(S_t^{i_2}) + c_t S_t^{i_2} \leq L_t(S_t^{i_2} - Q) + \omega_{t+1}(S_t^{i_2} - Q) + c_t(S_t^{i_2} - Q). \quad (1.5)$$

By Equations (1.4) and (1.5), the following results hold.

$$\begin{aligned}\omega_{t+1}(S_t^{i_1} + Q) - \omega_{t+1}(S_t^{i_1}) &> L_t(S_t^{i_1}) - L_t(S_t^{i_1} + Q) - c_t Q \\ &\geq L_t(S_t^{i_2} - Q) - L_t(S_t^{i_2}) - c_t Q \\ &> \omega_{t+1}(S_t^{i_2}) - \omega_{t+1}(S_t^{i_2} - Q), \quad (1.6)\end{aligned}$$

where the second inequality holds because $L_t(y_t)$ is convex and $S_t^{i_2} > S_t^{i_1} + Q$.

Thus, $\omega_{t+1}(S_t^{i_1} + Q) - \omega_{t+1}(S_t^{i_1}) > \omega_{t+1}(S_t^{i_2}) - \omega_{t+1}(S_t^{i_2} - Q)$, which contradicts the hypothesis of $\omega_{t+1}(y_t)$ being strong $Q$-jump-convex, because $S_t^{i_2} > S_t^{i_1} + Q$. Let $r_t = \min_j\{S_t^j\}$. Then, the $(r_t, Q)$ policy is optimal for period $t$.

Next, we prove that $v_t(x)$ is strong $Q$-jump-convex, i.e., for any $z_1 \in \mathbb{Z}$, $z_2 \in \mathbb{Z}$, and $z_1 \geq z_2$, $v_t(z_1 + Q) - v_t(z_1) \geq v_t(z_2 + Q) - v_t(z_2)$. Note that, by Lemma 1.2 parts (a) and (c), $J_t(y_t) = L_t(y_t) + \alpha E[v_{t+1}(y_t - D_t)]$ is strong $Q$-jump-convex. Here, for convenience, let $c_t = 0$. Suppose that $z_1 \in [j_1]_Q$ and $z_2 \in [j_2]_Q$.

Case 1: $r_t \geq z_1 \geq z_2$. It is then clear that $v_t(z_1) = J_t(S_t^{j_1})$ and $v_t(z_2) = J_t(S_t^{j_2})$. Note that $z_1 + Q \leq r_t + Q$, which implies that $v_t(z_1 + Q) = J_t(S_t^{j_1})$. Similarly, we have $v_t(z_2 + Q) = J_t(S_t^{j_2})$. Therefore, $v_t(z_1 + Q) - v_t(z_1) = v_t(z_2 + Q) - v_t(z_2) = 0$.

Case 2: $z_1 \geq r_t \geq z_2$. In this case, $v_t(z_1) = J_t(z_1)$, $v_t(z_1 + Q) = J_t(z_1 + Q)$, and $v_t(z_2 + Q) - v_t(z_2) = 0$. Because $J_t(z)$ is $Q$-jump-convex, Lemma 1.1 part (e) implies that $J_t(z)$ is non-decreasing on $[j]_Q \cap [r_t, +\infty)$, and hence, $J_t(z_1 + Q) - J_t(z_1) \geq 0$. Therefore, $v_t(z_1 + Q) - v_t(z_1) \geq 0 = v_t(z_2 + Q) - v_t(z_2)$.

Case 3: $z_1 \geq z_2 \geq r_t$. Then, $v_t(z_1) = J_t(z_1)$, $v_t(z_1+Q) = J_t(z_1+Q)$, $v_t(z_2) = J_t(z_2)$, and $v_t(z_2+Q) = J_t(z_2+Q)$. Because $J_t(y_t)$ is strong $Q$-jump-convex, $J_t(z_1+Q) - J_t(z_1) \geq J_t(z_2+Q) - J_t(z_2)$, which implies that $v_t(z_1+Q) - v_t(z_1) \geq v_t(z_2+Q) - v_t(z_2)$.

Therefore, $v_t(x)$ is also strong $Q$-jump-convex, which completes the proof. □

### 1.2.4 Positive Fixed Costs

In a supply chain, inventory replenishment typically involve a setup cost in a direct production or purchasing system. In a manufacturing setting, a setup cost may represent the cost of operations to prepare for production that can include activities such as cleaning, warming up and calibrating equipment, and readying the shop floor and workforce. In a purchasing setting, the setup cost may represent the additional administrative costs, including inspection and receiving, etc. However, no literature studies the optimal policy for the system with positive setup cost and batch ordering.

In this subsection, we consider the case with $K > 0$. It may be expected that an $(s, r, Q)$ policy is optimal, and operates as follows. If the inventory level is below the "re-order" point $s$, then it is optimal to order a minimum multiple of $Q$ to increase the inventory level to above $r$; otherwise, it is optimal to order nothing.

By the definition, the $(\mathbf{s}, \mathbf{S})_Q$ policy is an $(s, r, Q)$ policy *if* $\max_j\{s^j\} - \min_j\{s^j\} = Q - 1$ and $\max_j\{S^j\} - \min_j\{S^j\} = Q - 1$. Here, $s = \min_j\{s^j\}$ and $r = \min_j\{S^j\}$.

Unfortunately, the following example shows that an $(s, r, Q)$ policy may not be optimal (optimization was carried out in Matlab).

**Example 1.1.** *Consider a two-period problem with non-stationary parameters. Assume that the one-period cost function* $L_t(y) = h_t E[(y - D_t)^+] + p_t E[(D_t - y)^+]$, *for* $t = 1, 2$, *where* $h_t$ *and* $p_t$ *are the unit holding and shortage costs, respectively. Suppose that* $Q = 5$, $K = 20$, $c_t = 0$, *and* $\alpha = 0.9$. *For* $t = 2$, $h_2 = 3$, $p_2 = 6$, *and* $D_2$ *follows a Poisson distribution with mean* $\lambda = 20$. *For* $t = 1$, $h_1 = 1$, $p_1 = 10$, *and* $D_1$ *follows a Binomial distribution,* $b(x; n, q) = C_n^x q^x (1 - q)^{n-x}$ *with* $q = 0.75$ *and* $n = 30$. *As shown in Table 1.1, the batch-based* $(s, S)$ *policy is optimal. Moreover, the optimal policy is unique. We can see that* $S_1^1 - S_1^4 > Q = 5$. *Therefore, an* $(s, r, Q)$ *policy is not optimal.*

Table 1.1: Optimal Policy

| j | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $s_1^j$ | -14 | -13 | -17 | -16 | -15 |
| $S_1^j$ | 26 | 27 | 28 | 9 | 10 |

We now prove the optimality of $(\mathbf{s}, \mathbf{S})_Q$ policy.

**Theorem 1.2.** (a) *For any* $t = 1, 2, ..., T$, $J_t(y_t)$ *and* $v_t(x_t)$ *are* $Q$*-jump-*$K$*-convex.*

(b) *For any* $t = 1, 2, ..., T$, *there exists* $s_t^j$ *and* $S_t^j$ *with* $s_t^j \leq S_t^j$, *and* $s_t^j, S_t^j \in [j]_Q$, *where* $j = 1, 2, ..., Q$, *such that the* $(\mathbf{s}, \mathbf{S})_Q$ *policy is optimal.*

**Proof.** We prove the results by induction. For $t = T+1$, $v_{T+1} = 0$, and the results clearly hold. Assume $v_{t+1}(x)$ to be $Q$-jump-$K$-convex. By Lemma 1.3 parts (b) and (d), $\alpha E[v_{t+1}(y_t - D_t)]$ is also $Q$-jump-$K$-convex. Then, by Lemma 1.3 parts (a) and (c), $J_t(y_t) = L_t(y_t) + c_t y_t + \alpha E[v_{t+1}(y_t - D_t)]$ is $Q$-jump-$K$-convex.

For any $j = 1, 2, ..., Q$, define $\mu_t^j(z) = J_t(j+zQ)$. Thus, for any given $x_t \in [j]_Q$,

$$v_t(x_t) = -c_t x_t + \min_{z_t \geq \frac{x_t - j}{Q}, z_t \in \mathbb{Z}} \{\delta(z_t - \frac{x_t - j}{Q}) \cdot K + \mu_t^j(z_t)\}.$$

By Lemma 1.3 part (e), $\mu_t^j(z)$ is $K$-convex. Referring to Porteus (2002, page 108), there exists $s_t^j$ and $S_t^j$, with $s_t^j \leq S_t^j$, and $s_t^j, S_t^j \in [j]_Q$, such that the $(s_t^j, S_t^j)$ policy is optimal, where $\frac{S_t^j - j}{Q}$ minimizes $\mu_t^j(z_t)$ and $s_t^j$ is the smallest $\varepsilon_t$, such that $\mu_t^j(\frac{\varepsilon_t - j}{Q}) \leq \mu_t^j(\frac{S_t^j - j}{Q}) + K$. Moreover, for any $x_t \in [j]_Q$,

$$v_t(x_t) = \begin{cases} \mu_t^j(\dfrac{S_t^j - j}{Q}) - c_t x_t + K = J_t(S_t^j) - c_t x_t + K & if\ x_t < s_t^j, \\ \mu_t^j(\dfrac{x_t - j}{Q}) - c_t x_t = J_t(x_t) - c_t x_t & if\ x_t \geq s_t^j. \end{cases} \tag{1.7}$$

(From the relationship between $\mu_t^j(z_t)$ and $J_t(y_t)$, we know that for $j = 1, 2, ..., Q$, $S_t^j$ is a minimum point of $J_t(y_t)$ on $[j]_Q$ and $s_t^j$ is the smallest value less than $S_t^j$ and belonging to $[j]_Q$ such that $J_t(s_t^j) \leq J_t(S_t^j) + K$.)

For any given integer $a$, define $v_t^a(z) = v_t(a + zQ)$. Because $a + zQ \in [a]_Q$ for any $z \in \mathbb{Z}$, we have

$$v_t^a(z) = \begin{cases} \mu_t^a(\dfrac{S_t^a - a}{Q}) - c_t(a + zQ) + K & if\ z < \dfrac{s_t^a - a}{Q}, \\ \mu_t^a(z) - c_t(a + zQ) & if\ z \geq \dfrac{s_t^a - a}{Q}. \end{cases} \tag{1.8}$$

By the $K$-convexity of $\mu_t^a(z)$, we can conclude that $v_t^a(x)$ is $K$-convex. By Lemma 1.3 part (e), $v_t(x_t)$ is also $Q$-jump-$K$-convex, which completes the proof. □

Note that in the above proof, the original problem is decomposed into $Q$ subproblems based on groups $[j]_Q$, $j = 1, 2, ..., Q$. For each subproblem, an ordinary $(s, S)$ policy is optimal. We are unaware of any similar result for a batch-ordering model with fixed order costs in the literature, even in the stationary case.

## 1.3 Multi-stage Inventory Problem

In this section, we characterize the optimal policy for a periodic-review, single-item, serial inventory system with batch ordering. In a stationary setting and with the long-run average criterion, Chen (2000) shows that the $(r, Q)$ policy is optimal. Our model is different from his in that we consider a non-stationary setting and hence, a different approach is called for.

For notational simplicity, we consider the two-stage case (as Clark and Scarf, 1960, do). Customer demand arises at stage 1 only, stage 1 replenishes its inventory from stage 2, and stage 2 from an outside supplier with one-period leadtime. We assume the leadtime from stage 2 to stage 1 to be two periods. Our assumptions of two stages only and a two-period leadtime are made without loss of generality, as more stages and an arbitrary leadtime can be treated similarly.

Define the *echelon inventory level* at stage 2 to be the inventories on hand at stages 1 and 2 plus inventories in transit to stage 1, minus backorders at stage 1. Define the *echelon inventory position* at stage $i$ to be the echelon inventory level at stage $i$ plus inventories in transit to stage $i$, for $i = 1, 2$. For period

$t = 1, 2, ..., T$, let

$x_{1t}$ = the on-hand inventory at stage 1 after the receipt of order due in this period;

$y_t$ = the echelon inventory position at stage 1, after an order is placed, but before demand is realized;

$w_{1t}$ = the quantity to be delivered to stage 1 one period in the future;

$x_{2t}$ = the echelon inventory level at stage 2, before an order is placed;

$z_t$ = the quantity ordered at stage 2 from the outside supplier; and

$D_t$ = demand in period $t$.

The sequence of events that occur in each period is as follows. In-transit items due in this period arrive; decisions are made regarding how much to order from the upstream at each stage; demand is realized and costs are incurred based on the inventory levels at the end of the period. Each order placed by stage $i$ must be a nonnegative multiple of a stage-specific base batch, $Q_i$, $i = 1, 2$. As in Chen (2000), we assume that they satisfy the following integer-ratio constraint: $Q_2 = nQ_1$, where $n$ is a positive integer value.

Note that $w_{1t} \in [Q_1]_{Q_1}$, because the order from stage 1 must be a positive integer multiple of $Q_1$. Therefore, $w_{1t} + x_{1t}$ and $x_{1t}$ belong to the same group. We further assume that the on-hand inventory at stage 2 in period 1 belongs to group $[Q_1]_{Q_1}$. This assumption is reasonable, because each order from both stages 1 and 2 is a positive integer multiple of $Q_1$, and thus there is no incentive to keep an inventory at stage 2 that is a fraction of $Q_1$. Consequently, the on-hand inventory at stage 2 in any period belongs to group $[Q_1]_{Q_1}$, which implies that $w_{1t}+x_{1t}$ and $x_{2t}$ belong to the same group.

Demands in different periods are independent. When the customer demand exceeds the on-hand inventory at stage 1, the excess is backlogged. Four types of costs are assessed: ordering, shipping, holding, and penalty costs. Let $c_{2t}$ and $c_{1t}$ be the ordering cost for stage 2 and shipping cost from stage 2 to 1 in period $t$, respectively, and $L_{1t}(x_{1t})$ ($L_{2t}(x_{2t})$) be the expected holding/shortage cost (holding cost) incurred at stage 1 (2) in period $t$ in terms of their echelon inventory levels $x_{1t}$ ($x_{2t}$). We assume $L_{it}(x_{it})$, $i = 1, 2$, to be convex.

Define $\mathcal{A}_1(x) := \{\eta | \eta = x + mQ_1, \text{ for } m \in \mathbb{Z}^+\}$ and $\mathcal{A}_2 := \{\eta | \eta = mQ_2, \text{ for } m \in \mathbb{Z}^+\}$. Let $v_t(x_{1t}, w_{1t}, x_{2t})$ be the minimum expected discounted cost for the whole system from period $t$ until the end of the planning horizon $T$, and $v_{T+1}(x_{1t}, w_{1t}, x_{2t}) = 0$. For each $t = 1, 2, ..., T$, we have

$$\begin{aligned} & v_t(x_{1t}, w_{1t}, x_{2t}) \\ &= \min_{y_t \in \bar{\mathcal{A}}_1(x_{1t}+w_{1t}, x_{2t}), z_t \in \mathcal{A}_2} \{c_{2t} z_t + c_{1t}(y_t - x_{1t} - w_{1t}) + L_{1t}(x_{1t}) + L_{2t}(x_{2t}) \\ &+ \alpha E[v_{t+1}(x_{1t} + w_{1t} - D_t, y_t - x_{1t} - w_{1t}, x_{2t} + z_t - D_t)]\}, \qquad (1.9) \end{aligned}$$

where $\bar{\mathcal{A}}_1(x_{1t} + w_{1t}, x_{2t}) = \{\eta \le x_{2t} | \eta \in \mathcal{A}_1(x_{1t} + w_{1t})\}$.

Before going further, we need to investigate the following auxiliary system. Instead of treating both stages as a whole, we now consider stage 1 separately and assume that stage 2 has unlimited inventory. Let $v_t^1(x_{1t}, w_{1t})$ be the minimum expected discounted cost at stage 1, which begins with $x_{1t}$ units on hand and $w_{1t}$ units in transit. Then, for the auxiliary system, we have the following dynamic program:

$$v_t^1(x_{1t}, w_{1t}) = \min_{y_t \in \mathcal{A}_1(x_{1t}+w_{1t})} \{c_{1t}(y_t - x_{1t} - w_{1t}) + L_{1t}(x_{1t}) + \alpha E[v_{t+1}^1(x_{1t} + w_{1t} - D_t, y_t - x_{1t} - w_{1t})]\}. \quad (1.10)$$

We can see that problem (1.10) is actually a single-stage problem with a leadtime of two periods, as discussed in §1.2. Moreover, we can re-express (1.10) as

$$v_t^1(x_{1t}, w_{1t}) = L_{1t}(x_{1t}) + \alpha E[L_{1t+1}(x_{1t} + w_{1t} - D_t)] + \phi_t(x_{1t} + w_{1t}), \quad (1.11)$$

where

$$\phi_t(x) = \min_{y \in \mathcal{A}_1(x)} \{c_{1t}(y - x) + \alpha^2 E[L_{1t+2}(y - D_t - D_{t+1})] + \alpha E[\phi_{t+1}(y - D_t)]\}. \quad (1.12)$$

with $\phi_{T+1}(x) = 0$.

Clark and Scarf (1960) characterize the relationship between $v_t(x_{1t}, w_{1t}, x_{2t})$ and $v_t^1(x_{1t}, w_{1t})$ without batch ordering, which plays a central role in establishing the optimality results. The following lemma demonstrates that such a relationship remains in the batch-ordering setting.

**Lemma 1.5.** *There exists a sequence of functions $\pi_t(x_{2x})$, with $\pi_T(x_{2T}) = L_{2T}(x_{2T})$, such that*

$$v_t(x_{1t}, w_{1t}, x_{2t}) = v_t^1(x_{1t}, w_{1t}) + \pi_t(x_{2t}).$$

**Proof.** We prove this lemma by induction. For $t = T$, from (1.9) and $v_{T+1} = 0$,

$$v_T(x_{1T}, w_{1T}, x_{2T})$$
$$= \min_{y_T \in \bar{\mathcal{A}}_1(x_{1T}+w_{1T}, x_{2T}), z_T \in \mathcal{A}_2} \{c_{2t} z_T + c_{1T}(y_T - x_{1T} - w_{1T}) + L_{1T}(x_{1T}) + L_{2T}(x_{2T})\}. \tag{1.13}$$

Then, the results clearly hold, and $\pi_T(x_{2T}) = L_{2T}(x_{2T})$. Assume it to be true for period $t+1$, i.e., $v_{t+1}(x_{1t+1}, w_{1t+1}, x_{2t+1}) = v^1_{t+1}(x_{1t+1}, w_{1t+1}) + \pi_t(x_{2t+1})$. Then, Equation (1.9) can be rewritten as

$$\begin{aligned} v_t(x_{1t}, w_{1t}, x_{2t}) = & \min_{y_t \in \bar{\mathcal{A}}_1(x_{1t}+w_{1t}, x_{2t})} \{c_{1t}(y_t - x_{1t} - w_{1t}) + L_{1t}(x_{1t}) \\ & + \alpha E[v^1_{t+1}(x_{1t} + w_{1t} - D_t, y_t - x_{1t} - w_{1t})]\} \\ & + \min_{z_t \in \mathcal{A}_2} \{c_{2t} z_t + L_{2t}(x_{2t}) + \alpha E[\pi_{t+1}(x_{2t} + z_t - D_t)]\}. \end{aligned} \tag{1.14}$$

From (1.14), we can see that aside from the constraint that $y_t \leq x_{2t}$ in $\bar{\mathcal{A}}_1(x_{1t} + w_{1t}, x_{2t})$, the optimal solution of $y_t$ is the same as that for the problem of stage 1 considered separately, for which we have shown in §1.2 that there exists $\bar{r}_t$, such that $(\bar{r}_t, Q_1)$ policy is optimal. Then, if $x_{2t} \geq \bar{r}_t$, the constraint $y_t \leq x_{2t}$ is not operative, and we thus obtain

$$v_t(x_{1t}, w_{1t}, x_{2t}) = v^1_t(x_{1t}, w_{1t}) + \min_{z_t \in \mathcal{A}_2} \{c_{2t} z_t + L_{2t}(x_{2t}) + \alpha E[\pi_{t+1}(x_{2t} + z_t - D_t)]\}, \tag{1.15}$$

which implies that the result holds with $\pi_t(x_{2t}) = \min_{z_t \in \mathcal{A}_2} \{c_{2t} z_t + L_{2t}(x_{2t}) + \alpha E[\pi_{t+1}(x_{2t} + z_t - D_t)]\}$.

If $x_{2t} < \bar{r}_t$, then it is optimal to increase the stock level as high as possible at stage 1. Note that $x_{2t}$ and $x_{1t} + w_{1t}$ belong to the same group, that is, $x_{2t}$ is a feasible solution. Therefore, in this case, it is optimal to order up to $x_{2t}$, and we have

$$
\begin{aligned}
&v_t(x_{1t}, w_{1t}, x_{2t}) \\
&= c_{1t}(x_{2t} - x_{1t} - w_{1t}) + L_{1t}(x_{1t}) + \alpha E[v^1_{t+1}(x_{1t} + w_{1t} - D_t, x_{2t} - x_{1t} - w_{1t})] \\
&+ \min_{z_t \in \mathcal{A}_2} \{c_{2t} z_t + L_{2t}(x_{2t}) + \alpha E[\pi_{t+1}(x_{2t} + z_t - D_t)]\}. \qquad (1.16)
\end{aligned}
$$

Suppose that $x_{2t} \in [e_t]_Q$, for some $e_t = 1, 2, ..., Q_1$, and $S^{e_t}_{1t} = [\bar{r}_t, \bar{r}_t + Q_1) \cap [e_t]_Q$. Define

$$
\begin{aligned}
\Lambda_t(x_{2t}) = {}& c_{1t}(x_{2t} - x_{1t} - w_{1t}) + L_{1t}(x_{1t}) + \alpha E[v^1_{t+1}(x_{1t} + w_{1t} - D_t, x_{2t} - x_{1t} - w_{1t})] \\
&- v^1_t(x_{1t}, w_{1t}), \qquad (1.17)
\end{aligned}
$$

for $x_{2t} < S^{e_t}_{1t}$ and zero for $x_{2t} \geq S^{e_t}_{1t}$. Now, we need only prove that $\Lambda_t(x_{2t})$ is independent of $x_{1t}$ and $w_{1t}$ in the case of $x_{2t} < S^{e_t}_{1t}$.

Turn to the auxiliary system. Referring to (1.10) and by the definition of $S^{e_t}_{1t}$, we have

$$
v^1_t(x_{1t}, w_{1t}) = c_{1t}(S^{e_t}_{1t} - x_{1t} - w_{1t}) + L_{1t}(x_{1t}) + \alpha E[v^1_{t+1}(x_{1t} + w_{1t} - D_t, S^{e_t}_{1t} - x_{1t} - w_{1t})].
$$

Putting the above equation into (1.17), we can obtain

$$
\begin{aligned}
\Lambda_t(x_{2t}) = {}& c_{1t}(x_{2t} - S^{e_t}_{1t}) + \alpha\{E[v^1_{t+1}(x_{1t} + w_{1t} - D_t, x_{2t} - x_{1t} - w_{1t})] \\
&- E[v^1_{t+1}(x_{1t} + w_{1t} - D_t, S^{e_t}_{1t} - x_{1t} - w_{1t})]\}. \qquad (1.18)
\end{aligned}
$$

Finally, by Equation (1.11), we have

$$
\begin{aligned}
\Lambda_t(x_{2t}) = {}& c_{1t}(x_{2t} - S^{e_t}_{1t}) + \alpha^2 E[L_{1t+2}(x_{2t} - D_t - D_{t+1}) - L_{1t+2}(S^{e_t}_{1t} - D_t - D_{t+1})] \\
&+ \alpha E[\phi_{t+1}(x_{2t} - D_t) - \phi_{t+1}(S^{e_t}_{1t} - D_t)], \qquad (1.19)
\end{aligned}
$$

which is independent of $x_{1t}$ and $w_{1t}$, and hence, the proof is complete. □

From Equations (1.16) and (1.17), we have

$$v_t(x_{1t}, w_{1t}, x_{2t}) = v_t^1(x_{1t}, w_{1t}) + \Lambda_t(x_{2t}) + \min_{z_t \in \mathcal{A}_2} \{c_{2t}z_t + L_{2t}(x_{2t}) + \alpha E[\pi_{t+1}(x_{2t} + z_t - D_t)]\}.$$

Therefore, $\pi_t(x_{2t})$ satisfies the following dynamic equation.

$$\pi_t(x_{2t}) = \min_{z_t \in \mathcal{A}_2} \{cz_t + L_{2t}(x_{2t}) + \Lambda_t(x_{2t}) + \alpha E[\pi_{t+1}(x_{2t} + z_t - D_t)]\}, \qquad (1.20)$$

with $\pi_T(x_{2T}) = L_{2T}(x_{2T})$. To solve $v_t(x_{1t}, w_{1t}, x_{2t})$, it is sufficient to compute $v_t^1(x_{1t}, w_{1t})$ and $\pi_t(x_{2t})$ and consequently, to compute $\phi_t(x)$ and $\pi_t(x_{2t})$ (see (1.12)). To characterize the optimal policy, we need the following lemma.

**Lemma 1.6.** *If $f(x)$ is strong $Q_1$-jump-convex and $Q_2 = mQ_1$ for some positive integer $m$, then $f(x)$ is also strong $Q_2$-jump-convex.*

**Proof.** See the Appendix. □

The following theorem investigates the properties of $\phi_t(x)$ and $\pi_t(x_{2t})$ and characterizes the optimal policy for each stage.

**Theorem 1.3.** *For any $t = 1, 2, ..., T$, we have*

(a) *$\phi_t(x)$ and $\Lambda_t(x_{2t})$ are strong $Q_1$-jump-convex and $\pi_t(x_{2t})$ is strong $Q_2$-jump-convex.*

(b) *For stage 1, there exists $r_{1t}$, such that when $x_{2t} \geq r_{1t} + Q_1$, it is optimal to follow the $(r_{1t}, Q_1)$ policy, and when $x_{2t} < r_{1t} + Q_1$, it is optimal to order up-to $x_{2t}$, if $x_{1t} + w_{1t} < r_{1t}$, and to order nothing, if $x_{1t} + w_{1t} \geq r_{1t}$.*

(c) *For stage 2, there exists $r_{2t}$, such that the $(r_{2t}, Q_2)$ policy is optimal.*

**Proof.** We prove these results by induction.

(a). First, we prove the strong $Q$-jump-convexity of $\phi_t(x)$. As $\phi_{T+1}(x) = 0$, the result holds for $T+1$. Suppose it is also true for period $t+1$. Referring to Equation (1.12), and observing that the first two terms in (1.12) are convex in $y$, the terms within the curly brackets of (1.12) are thus strong $Q_1$-jump-convex. From the analysis of single-stage problem, we know that $\phi_t(x)$ is also strong $Q_1$-jump-convex.

Second, we prove that $\Lambda_t(x_{2t})$ is also strong $Q_1$-jump-convex. Let $\Lambda_t^1(x_{2t}) = c_{1t}x_{2t} + \alpha^2 E[L_{1t+2}(x_{2t} - D_t - D_{t+1})] + \alpha E[\phi_{t+1}(x_{2t} - D_t)]$, and $\Lambda_t^2(x_{2t}) = -c_{1t}S_{1t}^{e_t} - \alpha^2 E[L_{1t+2}(S_{1t}^{e_t} - D_t - D_{t+1})] - \alpha E[\phi_{t+1}(S_{1t}^{e_t} - D_t)]$. Then, $\Lambda_t(x_{2t}) = \Lambda_t^1(x_{2t}) + \Lambda_t^2(x_{2t})$. By Lemma 1.2 (c), we need only prove that both $\Lambda_t^1(x_{2t})$ and $\Lambda_t^2(x_{2t})$ are strong $Q_1$-jump-convex. Because $L_{1t+2}(x)$ is convex and $\phi_t(x)$ is strong $Q_1$-jump-convex, by Lemma 1.2 parts (b), (c), and (d), $\Lambda_t^1(x_{2t})$ is also strong $Q_1$-jump-convex. Note that for any $\tau \in \mathbb{Z}$, $\tau + Q_1$ and $\tau$ belong to the same group, and thus have the same $S_{1t}^{e_t}$ by the definition. Therefore, for any $\tau_1 \geq \tau_2$, $\Lambda_t^2(\tau_1 + Q) - \Lambda_t^2(\tau_1) = 0 = \Lambda_t^2(\tau_2 + Q) - \Lambda_t^2(\tau_2)$, which implies that $\Lambda_t^2(x_{2t})$ is also strong $Q_1$-jump-convex.

We now focus on $\pi_t(x_{2t})$. Suppose $\pi_{t+1}(x_{2t})$ to be $Q_2$-jump-convex. By Lemma 1.6 and the convexity of $L_{1t+2}(x)$ and $Q_1$-jump-convexity of $\Lambda_t(x_{2t})$, the terms within the curly bracket of Equation (1.20) are $Q_2$-jump-convex. By the analysis of the single-stage problem, we know that $\pi_t(x_{2t})$ is also strong $Q_2$-jump-convex.

(b). Referring to Equation (1.14), the first and second minimizations represent stage 1 and stage 2 problems, respectively. We first consider the first minimization, i.e., stage 1 problem. Substituting Equation (1.11) into the first minimization, we obtain

$$\min_{y_t \in \bar{A}_1(x_{1t}+w_{1t}, x_{2t})} \{c_{1t}y_t + \alpha^2 E[L_{1t+2}(y_t - D_t - D_{t+1})] + \alpha E[\phi_{t+1}(y_t - D_t)]\}$$

$$+ L_{1t}(x_{1t}) + \alpha E[L_{1t+1}(x_{1t} + w_{1t} - D_t)] - c_{1t}(x_{1t} + w_{1t}).$$

By the convexity of $L_{1t+2}(x)$ and the strong $Q_1$-jump-convexity of $\phi_{t+1}(x)$, the terms within the first curly bracket of Equation (1.14) are strong $Q_1$-jump-convex. Define $r_{1t}$ as the reorder point at stage 1 in period $t$. Then, if $x_{2t} \geq r_{1t} + Q_1$, the constraint $y_t \leq x_{2t}$ is not operative, and thus $(r_{1t}, Q_1)$ policy is optimal. If $x_{2t} < r_{1t} + Q_1$, then it is optimal to order as much as possible. Note that $x_{2t}$ and $x_{1t} + w_{1t}$ belong to the same group, that is, $x_{2t}$ is a feasible solution. Therefore, it is optimal to order up to $x_{2t}$.

(c). We now consider the second minimization in Equation (1.14), i.e., stage 2 problem. Because $\pi_{t+1}$ is strong $Q_2$-jump-convex, so are the terms within the second curly bracket. Similar to the single-stage problem, we know that there exists a reorder point $r_{2t}$, such that the $(r_{2t}, Q_2)$ policy is optimal. □

We conclude this section with two remarks.

**Remark 1.1.** *Our analysis can also be extended to a periodic-review assembly system with batch ordering. Under standard assumptions, Chen (2000) establishes the equivalence between an assembly system and a serial system with batch ordering. Because this result is established without assuming a stationary setting, it prevails in our model. We do not repeat his analysis here, but note that the* *analysis of the assembly system is the same as that of the serial system.*

**Remark 1.2.** *Another extension would be to allow a fixed cost $K$ at stage 2. In this case, the optimal policy for stage 1 does not change, and by the analysis in §1.2.4, we know that an $(\mathbf{s}, \mathbf{S})_Q$ policy is optimal for stage 2.*

## 1.4 Single-stage Inventory-Pricing Problem

In this section, we take pricing as an endogenous decision, that is, the firm has to make ordering and pricing decisions simultaneously in each period. We show that the $(r, Q, p)$ policy is optimal for an inventory-pricing system with zero fixed cost and additive demand. This policy can be described as follows. The inventory is replenished according to the $(r, Q)$ policy, and the price is set at $p(y_t^*)$, where $y_t^*$ is the optimal order-up-to level.

### 1.4.1 Model and Results

We retain all of the notations in Section 1.2 and introduce several additional notations. For period $t = 1, 2, ..., T$,

$p_t =$ the selling (or list) price in period $t$,

$\underline{p}_t$,$\bar{p}_t$ are the lower and upper bounds on $p_t$, respectively, and

$d_t =$ expected demand level in period $t$.

We consider the additive demand function:

$$D_t(p_t, \beta_t) = D_t(p_t) + \beta_t, \tag{1.21}$$

where $\beta_t$ is a random variable. Such a demand model is commonly seen in the OM literature e.g., Chen and Simchi-Levi (2004a), Huh and Janakiraman (2008), Chen et al. (2006), and Allon and Zeevi (2009).

Without loss of generality, we assume that $E[\beta_t] = 0$. As usual, we also assume zero replenishment leadtime.

**Assumption 1.1.** *For any $t = 1, 2, ..., T$, (1) $D_t(p_t, \beta_t)$ take nonnegative integer values, i.e., $D_t(p_t, \beta_t) \in \mathbb{Z}^+$. (2) $D_t(p_t)$ is strictly decreasing in $p_t \in \boldsymbol{P}_t$, where $\boldsymbol{P}_t = [\underline{p}_t, \bar{p}_t]$.*

Part (1) of Assumption 1.1 is in line with Section 1.2. Part (2) of Assumption 1 implies a one-to-one correspondence between the selling price and expected demand. Hence, we can replace the pricing decision with the expected demand level $d_t$ and define the feasible expected demand level set as $\mathbf{D}_t = [\underline{d}_t, \bar{d}_t]$, where $\underline{d}_t = D_t(\underline{p}_t)$ and $\bar{d}_t = D_t(\bar{p}_t)$. Moreover, Assumption 1 implies that the inverse function of $D_t(p_t)$, denoted by $D_t^{-1}$, is also strictly decreasing. We also assume that the expected revenue $R_t(d_t) := d_t D_t^{-1}(d_t)$ is a concave function of expected demand $d_t \in \mathbf{D}_t$.

Let $L_t(y, p) = E\{h_t(y - D_t(p, \epsilon_t))\}$, which represents the expected holding/penalty cost in terms of the order-up-to level at the beginning of period $t$ (after possible ordering). For technical reasons, we make the following assumptions regarding functions $h_t(y)$ and $L_t(y, p)$, which are again commonly seen in the literature (see Federgruen and Heching, 1999; Chen and Simchi-Levi, 2004a).

**Assumption 1.2.** *For any $t = 1, 2, ..., T$,*

(i) $h_t(y)$ *is a convex function in* $y$.

(ii) $\lim_{y\to\infty} L_t(y,p) = \lim_{y\to-\infty}[c_t y + L_t(y,p)] = \lim_{y\to\infty}[(c_t - c_{t+1})y + L_t(y,p)] = \infty$ *for all* $p \in \boldsymbol{P}_t$.

(iii) $0 \le L_t(y,p) = O(|y|^\rho)$ *for some integer* $\rho$.

(iv) $E[D_t(p,\epsilon_t)]^\rho < \infty$ *for all* $p \in \boldsymbol{P}_t$.

For consistency, our objective is to minimize the total expected cost instead of maximizing the expected profit. Let $v_t(x_t)$ be the optimal expected discounted cost from period $t$ until the end of planning horizon $T$, when the starting inventory level in period $t$ is $x$. Let $v_{T+1}(x) = 0$ for all $x$. Then, for any $t = 1, 2, ...T$, we have

$$v_t(x_t) = -c_t x_t + \min_{y_t \in \mathcal{A}(x_t)} \{J_t(y_t)\}, \tag{1.22}$$

where

$$J_t(y_t) = \min_{d_t \in \mathbf{D}_t} U_t(y_t, d_t), \text{ and}$$

$$U_t(y_t, d_t) = -R(d_t) + c_t y_t + E\{h_t(y_t - d_t - \beta_t) + \alpha v_{t+1}(y_t - d_t - \beta_t)\}.$$

We next present an important result, which states that strong $Q$-jump-convexity can be preserved under a minimization operation.

**Lemma 1.7.** *Suppose that* $\eta(x)$ *is a convex function,* $\gamma(x)$ *is strong* $Q$*-jump-convex and* $\mathbf{D} = [\underline{d}, \bar{d}]$, *where* $\underline{d} \le \bar{d} \in \mathbb{Z}$, *then,*

$$\Gamma(y) = \min_{d \in \mathbf{D}} \{\eta(d) + \gamma(y - d)\}$$

*is also strong* $Q$*-jump-convex.*

**Proof.** See the Appendix. □

Next, we prove the optimality of the $(r, Q, p)$ policy, i.e., the inventory strategy is a $(r, Q)$ policy and the optimal list-price depends on the order-up-to inventory level in each period.

**Theorem 1.4.** (a) *For any $t = 1, 2, ..., T$, $J_t(y_t)$ and $v_t(x_t)$ are both strong $Q$-jump-convex.*

(b) *For any $t = 1, 2, ..., T$, the $(r, Q, p)$ policy is optimal.*

**Proof.** We prove the results by induction. For $t = T + 1$, the results clearly hold. Assume $v_{t+1}(x)$ to be strong $Q$-jump-convex. By Lemma 1.2 parts (b) and (d), $\omega_{t+1}(x) = E\{h_t(x - \beta_t) + \alpha v_{t+1}(x - \beta_t)\}$ is also strong $Q$-jump-convex. Note that $-R(d_t)$ is a convex function. Then, by Lemma 1.7, $J_t(y_t)$ is also strong $Q$-jump-convex. From the analysis in Section 1.2.3, we know that the $(r, Q)$ policy is optimal for inventory replenishment, and that $v_t(x_t)$ is also strong $Q$-jump-convex, which completes the proof. □

Define $d_t(y_t)$ as the expected demand associated with the best selling price for a given inventory level $y_t$, i.e.,

$$d_t(y_t) = \arg \min_{d_t \in \mathbf{D}_t} \{U_t(y_t, d_t)\}.$$

**Corollary 1.1.** *For any $t = 1, 2, ..., T$, there exists a $d_t(y_t)$ that maximizes $J_t(y_t, d_t)$ for any given $y_t$, such that*

(a) *$y_t - d_t(y_t)$ is a nondecreasing function of $y_t$.*

(b) *$d_t(y_t)$ is nondecreasing on $[j]_Q$, $j = 1, 2, ..., Q$, i.e., $d_t(y_t + Q) \geq d_t(y_t)$.*

**Proof.** See the Appendix. □

Corollary 1.1 (a) implies that the higher the inventory level at the beginning of period $t$, the higher the expected inventory level at the end of period $t$. This result is consistent with the case without batch-ordering in Federgruen and Heching (1999).

Federgruen and Heching (1999) further show that the optimal expected demand $d_t(y_t)$ to be selected in any given period is nondecreasing in the prevailing inventory level $y_t$. This result may not hold in the batch case (see the following numerical study). However, Corollary 1.1 (b) proves that the optimal expected demand $d_t(y_t)$ increases on $[j]_Q$, for each $j = 1, 2, ..., Q$. Note that Part (a) also implies that $y_t + Q - d_t(y_t + Q) \geq y_t - d_t(y_t)$, i.e., $d_t(y_t) \geq d_t(y_t + Q) - Q$. Together with Part (b), we have the following relationship between $d_t(y_t)$ and $d_t(y_t + Q)$.

$$d_t(y_t + Q) \geq d_t(y_t) \geq d_t(y_t + Q) - Q.$$

This relationship may be helpful in computing the optimal expected demand.

Let $v_t^Q(x_t)$ be the optimal expected discounted cost from period $t$ until the end of the planning horizon $T$ for the system with batch size $Q$. The following theorem indicates the relationship between the optimal costs for systems with different batch sizes.

**Theorem 1.5.** *Suppose that* $Q_1 = nQ_2$ *for some integer* $n$*, then,* $v_t^{Q_1}(x_t) \leq v_t^{Q_2}(x_t)$*, for* $t = 1, 2..., T$.

**Proof.** We prove the result by induction. For $t = T + 1$, the result clearly holds. Suppose it is also true for $t + 1$. Then, $U_t^{Q_1}(y_t, d_t) \leq U_t^{Q_2}(y_t, d_t)$ and thus $J_t^{Q_1}(y_t) \leq J_t^{Q_2}(y_t)$. Let $\mathcal{A}^Q(x_t) = \{y_t | y_t = x_t + mQ, \text{ for some } m \in \mathbb{Z}^+\}$. Referring to Equation (1.22), we have $v_t^{Q_1}(x_t) \leq v_t^{Q_2}(x_t)$, due to the fact that $\mathcal{A}^{Q_1}(x_t) \supseteq \mathcal{A}^{Q_2}(x_t)$. □

### 1.4.2 Numerical Study

Although they do not consider batch ordering, Federgruen and Heching (1999) have explored a number of qualitative insights into the structure of optimal policies and their sensitivity with respect to several parameters. A natural question is how does the batch size affect these insights and sensitivities? To address this question, we focus in particular on the impact of $Q$. Among the major questions investigated, we plan to investigate the following insights.

*(i)* The sensitivity of the optimal list price to the initial inventory level with different batch sizes;

*(ii)* the sensitivity of the optimal reorder level to the batch size $Q$; and

*(iii)* the benefits of a dynamic pricing strategy compared to a fixed or one-off price change strategy with different batch sizes.

For the numerical experiment, we assume $D_t(p_t)$ to be stationary and linear, i.e., $D_t(p_t) = a - bp_t$. The random part $\beta_t$ is assumed to follow a binomial distribution $b(x|n,q) = C_n^x q^x (1-q)^{n-x}$. The holding and backlogging costs are proportional to the end-of-the-period inventory level or backlog, at rates of $h$ and $\pi$, respectively. The salvage value for any inventory remaining at the end of the

Table 1.2: Parameter Values for Base Scenario

| $a$ | $b$ | $n$ | $q$ | Order Cost $c$ | Holding Cost $h$ | Penalty Cost $\pi$ | $\underline{p}_t$ | $\bar{p}_t$ | Discount Rate $\alpha$ |
|---|---|---|---|---|---|---|---|---|---|
| 174 | 3 | 20 | 0.7 | 5 | 2 | 10 | 20 | 40 | 0.9 |

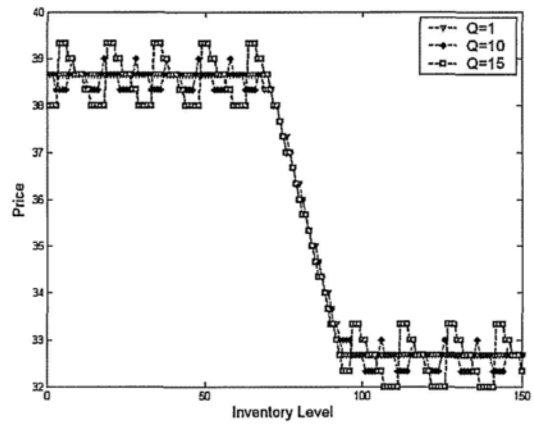


Figure 1.3: Optimal List Price as a Function of Initial Inventory Level.

horizon is assumed to be zero. Table 1.2 summarizes the values of all parameters for the base scenario. Here, we consider the infinite horizon case unless otherwise specified.

Figure 1.3 exhibits, for the base scenario, the optimal list price as a function of the starting inventory level in the first period of the horizon with different batch sizes. Consistent with Federgruen and Heching (1999), we can observe that when $Q = 1$, the optimal list price is decreasing in the inventory level. As observed, the optimal list price may not decrease in the inventory level, when $Q > 1$. However, in this case, the price is still decreasing in $[j]_Q$, for any $j = 1, 2, ..., Q$,

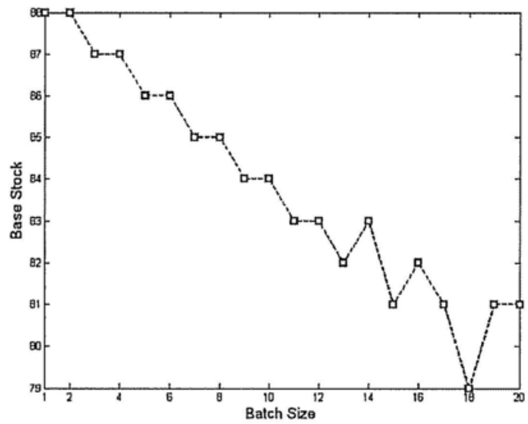


Figure 1.4: Optimal Reorder Levels as a Function of Batch Size.

as indicated by Corollary 1.1 (b). Clearly, when $Q = 1$ and the initial inventory level is low, the optimal price becomes constant because of the base-stock policy. Nevertheless, when $Q > 1$, the base-stock levels are different for the inventory levels in different groups, and the price fluctuates accordingly. Also note that when the inventory level is below the reorder points, the prices are also constant in group $[j]_Q$, $j = 1, 2, ..., Q$. We can further see that the optimal list prices in the cases of $Q = 1$ and $Q = 5$ converge at the same value: 32.7.

Figure 1.4 displays the reorder level $r$ in the first period under different batch sizes. Roughly speaking, the reorder level decreases as the batch size increases, but not monotonically. This is consistent with the $(r, Q)$ literature (e.g., Chen and Zheng, 1994).

When comparing the optimal total profits of different settings, we take the maximal relative difference over all possible states. More specifically, if we let $f^1$

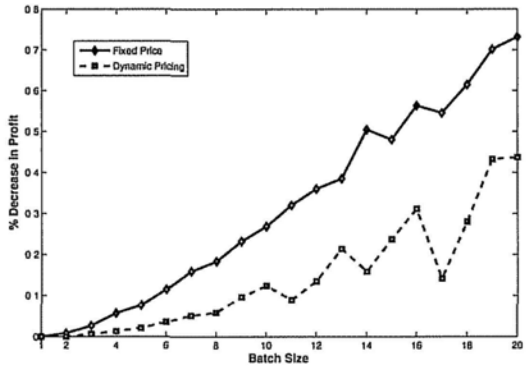


Figure 1.5: Total Optimal Profits as a Function of Batch Size.

and $f^2$ be the discounted total profits of two different settings, respectively, then the relative percentage difference is defined as

$$RD = \max_{x\in[-300,700]} \left[(f^1 - f^2)/f^1\right] \times 100\%.$$

Figure 1.5 exhibits the impact of the batch size on the optimal total profit under dynamic pricing strategy and fixed price strategy, respectively. For the fixed price strategy, we set the same price over the entire horizon. Here, we set $p = 30$. To achieve the result, we compare the total profit with different batch size to that with $Q = 1$. It can be observed that the profit may not monotonically decrease in the batch size. However, as indicated in Theorem 1.5, the profit drops as the batch size increases in integer multiples. On the other hand, the profit appears not sensitive to batch size $Q$. For example, comparing $Q = 1$ with $Q = 20$, profit declines only by 0.44% under dynamic pricing strategy, and 0.74% under fixed price strategy. Therefore, the firm has less incentive to find a supplier that can provide a smaller batch size. Furthermore, dynamic pricing can reduce

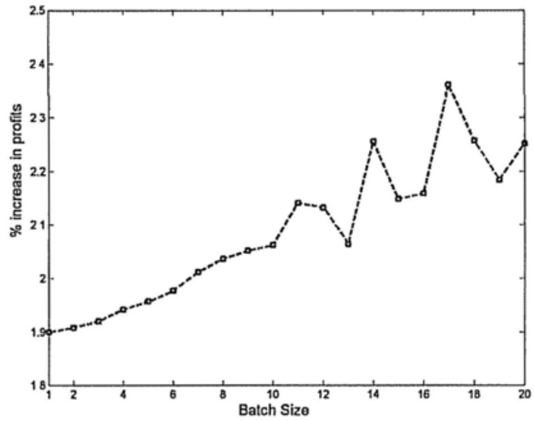


Figure 1.6: Fixed Pricing Strategy vs. Dynamic Pricing Strategy.

the impact of the batch size on the total profit.

Figure 1.6 shows that even in a stationary environment significant benefits accrue from a dynamic pricing strategy relative to a fixed price strategy. In the latter case, we choose the single best price throughout the planning horizon. For a batch size of around 17, the profit enhancement may be as much as 2.37%. As observed in the general literature of revenue management, in the retail sector, these differences may have very large impacts on bottom-line profit figures. Alternatively, the benefits of a dynamic pricing strategy increase as the batch size increases. Therefore, it is more profitable for retailers to adopt a dynamic pricing when the batch size is large.

Figure 1.7 gauges the profit improvement from a one-off change pricing strategy to a dynamic pricing strategy, i.e., there is only one price modification in the middle of the horizon. Here, we consider the case with $T = 30$, which is a

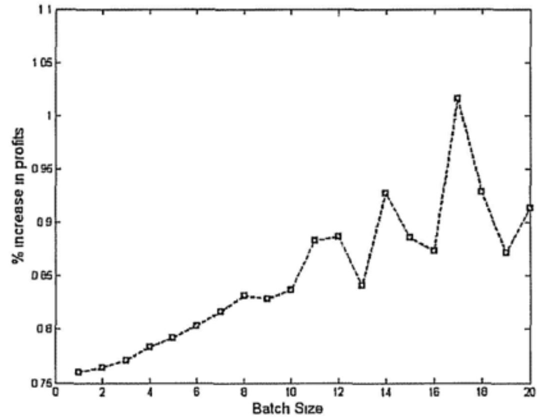


Figure 1.7: One-off Change Pricing Strategy vs. Dynamic Pricing Strategy.

very good approximation of the infinite horizon case. With the one-off change pricing strategy, we reset the prices at the middle of horizon, i.e., $t = 15$. As expected, the benefits of a dynamic pricing strategy increase as the batch size increases. Compared to Figure 1.6, we can see that although the price is modified only once in the middle of the horizon, this modification results in a significant improvement in profits compared to the fixed price strategy.

## 1.5 Concluding Remarks

In this chapter, we investigate a non-stationary periodic-review inventory system with and without fixed costs in which replenishment is made in fixed lot sizes. We introduce two new concepts, $Q$-jump-convexity and strong $Q$-jump-convexity, to show that the $(r, Q)$ policy is optimal for both single-stage and multi-echelon models without a fixed ordering cost. When a positive fixed cost is incurred, we further introduce $Q$-jump-$K$-convexity to establish the optimality of the $(\mathbf{s},\mathbf{S})_Q$ policy. We also analyze a pricing-batch system with additive demand and show that the $(r, Q, p)$ policy is optimal.

To conclude this chapter, we suggest a number of important extensions.

- **More General Cost Structure:** The analysis thus far assumes that the single-period holding/shortage cost function $L_t$ is convex. Although this assumption is commonly used in the literature, it can be extended to strong $Q$-jump-convexity in our model. The results will be guaranteed by Lemma 1.2 part (c).

- **Infinite Time Horizon:** With several technical assumptions (see Sethi and Cheng, 1997), the optimal policy for the nonstationary finite single-stage horizon case could be extended to the infinite horizon case. Another extension is to investigate the nonstationay infinite horizon multi-stage model, and essentially a generalization of Chen (2000). However, significant effort would be required for such an extension, and so we leave it as a future research direction. For the single-stage inventory-pricing model, the extension to the stationary infinite horizon can be accomplished by following the standard approach (see Federgruen and Heching, 1999).

- **Markov-Modulated Demand Model:** As elaborated in Song and Zipkin (1993), many randomly changing environmental factors, such as fluctuating economic conditions and uncertain market conditions in different stages of a product life-cycle, can have a major effect on demand. For such situations, the Markov chain approach provides a natural and flexible alternative for modeling the demand process. This analysis is presented in Appendix 1.6.2. As expected, the optimal policy is state-dependent. With the same logic, the results can also be extended to the case with Markovian variable ordering cost.

• **Other Possible Extensions:** First, for the inventory-pricing model, we consider only the additive demand form. A nature question is whether $(r, Q, p)$ remains optimal for the multiplicative or more general demand form. Our notion of $Q$-jump-convexity may be useful, but is not immediately applicable to tackling the multiplicative or more general demand form. We leave it for our future research. Second, all of our models contain only one system state, i.e., the inventory level. If other issues are considered, such as information updating, then these models may lead to multiple system states. A natural future research direction would be to explore how $Q$-jump-convexity can be extended from one dimension to multiple dimensions. Third, lot sizing with uncertain yields is an important issue in production/manufacturing systems. It would be interesting to see if the method developed in this chapter can be extended to structural analysis of a random yield model with batch ordering. Finally, one of the limitations of the models analyzed in this chapter is the lack of capacity constraints. Incorporating capacity constraints of inventory replenishment would therefore be another possible extension. In the case of single-stage models without fixed costs, we expect that the optimal inventory policy would become a modified $(r, Q)$ policy, i.e., if ordering, then the firm orders according to the $(r, Q)$ requirement if possible; otherwise, it orders to full capacity.

## 1.6 Appendix.

### 1.6.1 Proofs

PROOF OF LEMMA 1.3. Parts (a) and (b) follow directly from the definition of Q-jump-$K$-convexity.

Part (c). Suppose that $f_1(x)$ and $f_2(x)$ are $Q$-jump-$K_1$-convex and $Q$-jump-$K_2$-convex, respectively. Then, by the definition, for any $x \in \mathbb{Z}$, $z \in \mathbb{Z}^+$, and $b \in \mathbb{Z}^+/\{0\}$,

$$f_1(x + zQ) + K_1 \geq f_1(x) + \frac{z}{b}[f_1(x) - f_1(x - bQ)],$$

and

$$f_2(x + zQ) + K_2 \geq f_2(x) + \frac{z}{b}[f_2(x) - f_2(x - bQ)].$$

Letting $f(x) = f_1(x) + f_2(x)$ and combing above two inequalities, we obtain that for any $x \in \mathbb{Z}$, $z \in \mathbb{Z}^+$, and $b \in \mathbb{Z}^+/\{0\}$,

$$f(x + zQ) + (K_1 + K_2) \geq f(x) + \frac{z}{b}[f(x) - f(x - bQ)],$$

which results in that $f(x)$ is $Q$-jump-$(K_1 + K_2)$-convex.

Part (d). Without loss of generality, suppose that the distribution of $w$ can be characterized by $P(w = i) = \lambda_i$ for $i = 0, 1, 2, ..., W$, where $W$ is a upper bound and $\sum_{i=0}^{W} \lambda_i = 1$. Then, $G(y) = \sum_{i=0}^{W} \lambda_i v(y - i)$. Next, we prove that $G(y)$ satisfies $G(y+zQ)+K \geq G(y)+\frac{z}{b}\{G(y)-G(y-bQ)\}$ for any $y \in \mathbb{Z}$, $z \in \mathbb{Z}^+$, and $b \in \mathbb{Z}^+/\{0\}$. Note the fact that for any $i = 0, 1, 2, ..., W$, $\lambda_i v(y + zQ - i) + \lambda_i K \geq \lambda_i v(y - i) + \frac{z}{b}\{\lambda_i v(y - i) - \lambda_i v(y - bQ - i)\}$, due to the $Q$-jump-$K$-convexity of $v(x)$. Combing these $W$ inequalities, we obtain the desired result.

Part (e). We first prove the necessity. By the $Q$-jump-$K$-convexity of $f(x)$, for any $y, a \in \mathbb{Z}$, $z \in \mathbb{Z}^+$, and $b \in \mathbb{Z}^+/\{0\}$, we have

$$f(a+(y+z)Q)+K \geq f(a+yQ)+\frac{z}{b}\{f(a+yQ)-f(a+(y-b)Q)\},$$

which implies that $g^a(y+z)+K \geq g^a(y)+\frac{z}{b}\{g^a(y)-g^a(y-b)\}$.

For the sufficiency, we just need to prove that if for any $x \in \mathbb{Z}$, $z \in \mathbb{Z}^+$, and $b \in \mathbb{Z}^+/\{0\}$, $f(x+zQ)+K \geq f(x)+\frac{z}{b}\{f(x)-f(x-bQ)\}$. By the $K$-convexity of $g^a(y)$, we have for any $y \in \mathbb{Z}$, $z \in \mathbb{Z}^+$, and $b \in \mathbb{Z}^+/\{0\}$,

$$g^a(y+z)+K \geq g^a(y)+\frac{z}{b}\{g^a(y)-g^a(y-b)\},$$

which implies

$$f(a+(y+z)Q)+K \geq f(a+yQ)+\frac{z}{b}\{f(a+yQ)-f(a+(y-b)Q)\}.$$

Then, the result holds by letting $x = a+yQ$. □

PROOF OF LEMMA 1.6. For any $z_1 \geq z_2$, we need to prove $f(z_1+Q_2)-f(z_1) \geq f(z_2+Q_2)-f(z_2)$. Because $Q_2 = mQ_1$ for some integer $m$, we have

$$f(z_1+Q_2)-f(z_1)=\sum_{i=0}^{m-1} f(z_1+Q_1+iQ_1)-f(z_1+iQ_1),$$

and

$$f(z_2+Q_2)-f(z_2)=\sum_{i=0}^{m-1} f(z_2+Q_1+iQ_1)-f(z_2+iQ_1).$$

By the strong $Q_1$-jump-convexity of $f(x)$, we can obtain that for any $i = 0, 1, ..., m-1$, $f(z_1+Q_1+iQ_1)-f(z_1+iQ_1) \geq f(z_2+Q_1+iQ_1)-f(z_2+iQ_1)$. Therefore, $f(z_1+Q_2)-f(z_1) \geq f(z_2+Q_2)-f(z_2)$. □

PROOF OF LEMMA 1.7. Define $d(y) = \min\{d : d \in \arg\min_{d\in\mathbf{D}}\{\eta(d)+\gamma(y-d)\}\}$. Then, for any $z \in \mathbb{Z}$ and $d(z) \in \mathbf{D}$, we have

$$\Gamma(z) = \eta(d(z)) + \gamma(z - d(z)). \tag{1.23}$$

Next, we prove by discussing two cases that for any $z_1 \in \mathbb{Z}$, $z_2 \in \mathbb{Z}$ and $z_1 \geq z_2$,

$$\Gamma(z_1 + Q) - \Gamma(z_1) \geq \Gamma(z_2 + Q) - \Gamma(z_2).$$

Case 1: $z_1 - d(z_1 + Q) \leq z_2 - d(z_2)$. In this case, $d(z_1 + Q) \geq d(z_2) + z_1 - z_2$, which, by the convexity of $\eta(x)$ and $z_1 \geq z_2$, implies

$$\eta(d(z_1 + Q)) - \eta(d(z_1 + Q) + z_2 - z_1) \geq \eta(d(z_2) + z_1 - z_2) - \eta(d(z_2)). \tag{1.24}$$

Note that $d(z_1 + Q), d(z_2) \in \mathbf{D}$. As $d(z_1 + Q) \geq d(z_2) + z_1 - z_2 \geq d(z_2)$, then $d(z_2) + z_1 - z_2 \in \mathbf{D}$. Therefore, by the definition of $d(z_1)$, we have

$$\begin{aligned}\Gamma(z_1) &= \eta(d(z_1)) + \gamma(z_1 - d(z_1)) \\ &\leq \eta(d(z_2) + z_1 - z_2) + \gamma(z_2 - d(z_2)).\end{aligned} \tag{1.25}$$

Also note that $d(z_1 + Q) \geq d(z_1 + Q) + z_2 - z_1 \geq d(z_2)$, which implies $d(z_1 + Q) + z_2 - z_1 \in \mathbf{D}$. Similarly, by the definition of $d(z_2 + Q)$, we can obtain

$$\begin{aligned}\Gamma(z_2 + Q) &= \eta(d(z_2 + Q)) + \gamma(z_2 + Q - d(z_2 + Q)) \\ &\leq \eta(d(z_1 + Q) + z_2 - z_1) + \gamma(z_1 + Q - d(z_1 + Q)).\end{aligned} \tag{1.26}$$

By the definition of $\Gamma(z_1 + Q)$ and $\Gamma(z_2)$ (referring to (1.23)), (1.24), (1.25), and

(1.26), the following inequalities hold.

$$\begin{aligned}\Gamma(z_1+Q)-\Gamma(z_1)-\Gamma(z_2+Q)+\Gamma(z_2) &\geq \eta(d(z_1+Q))-\eta(d(z_2)+z_1-z_2)\\ &\quad -\eta(d(z_1+Q)+z_2-z_1)+\eta(d(z_2))\\ &\geq 0.\end{aligned}$$

Case 2: $z_1-d(z_1+Q) > z_2-d(z_2)$. In this case, $z_1+Q-d(z_1+Q) \geq z_2+Q-d(z_2)$, which, by the strong $Q$-jump-convexity of $\gamma(x)$, implies

$$\gamma(z_1+Q-d(z_1+Q))-\gamma(z_1-d(z_1+Q)) \geq \gamma(z_2+Q-d(z_2))-\gamma(z_2-d(z_2)). \tag{1.27}$$

By the definition of $d(z_1)$, we have

$$\begin{aligned}\Gamma(z_1) &= \eta(d(z_1))+\gamma(z_1-d(z_1))\\ &\leq \eta(d(z_1+Q))+\gamma(z_1-d(z_1+Q)).\end{aligned} \tag{1.28}$$

Similarly, by the definition of $d(z_2+Q)$, we can obtain

$$\begin{aligned}\Gamma(z_2+Q) &= \eta(d(z_2+Q))+\gamma(z_2+Q-d(z_2+Q))\\ &\leq \eta(d(z_2))+\gamma(z_2+Q-d(z_2)).\end{aligned} \tag{1.29}$$

By the definition of $\Gamma(z_1+Q)$ and $\Gamma(z_2)$ (referring to (1.23)), (1.27), (1.28), and (1.29), the following inequalities hold.

$$\begin{aligned}\Gamma(z_1+Q)-\Gamma(z_1)-\Gamma(z_2+Q)+\Gamma(z_2) &\geq \gamma(z_1+Q-d(z_1+Q))-\gamma(z_1-d(z_1+Q))\\ &\quad -\gamma(z_2+Q-d(z_2))+\gamma(z_2-d(z_2))\\ &\geq 0.\end{aligned}$$

Therefore, $\Gamma(y)$ is also strong $Q$-jump-convex. □

PROOF OF COROLLARY 1.1. (a). Letting $\hat{d}_t = y_t - d_t$, then $J_t(y_t, \hat{d}_t) = -R(y_t - \hat{d}_t) + c_t y_t + E\{h_t(\hat{d}_t - \beta_t) + \alpha v_{t+1}(\hat{d}_t - \beta_t)\}$. By the convexity of $-R(x)$, $J_t(y_t, \hat{d}_t)$ is submodular in $y_t$ and $\hat{d}_t$. Then, the result thus follows from Topkis (1998, Theorem 2.4.3 and Lemma 2.8.1).

(b). Define

$$\tilde{R}(d_t) = -R(d_t) + c_t d_t$$

and

$$\omega(y_t - d_t) = E\{h_t(y_t - d_t - \beta_t) + \alpha v_{t+1}(y_t - d_t - \beta_t)\} + c_t(y_t - d_t).$$

Then, $U(y_t, d_t) = \tilde{R}(d_t) + \omega(y_t - d_t)$. By the definition of $d(z_1)$ and $d(z_1 + Q)$, we have

$$-\tilde{R}(d(z_1)) + \omega(z_1 - d(z_1)) \leq -\tilde{R}(d(z_1 + Q)) + \omega(z_1 - d(z_1 + Q)),$$
$$-\tilde{R}(d(z_1 + Q)) + \omega(z_1 + Q - d(z_1 + Q)) \leq -\tilde{R}(d(z_1)) + \omega(z_1 + Q - d(z_1)).$$

Combing them, we can obtain

$$\omega(z_1 + Q - d(z_1)) - \omega(z_1 - d(z_1)) \geq \omega(z_1 + Q - d(z_1 + Q)) - \omega(z_1 - d(z_1 + Q)).$$

By the strong $Q$-jump-convexity of $\omega(x)$, $d(z_1) \leq d(z_1 + Q)$. □

### 1.6.2 Markov-Modulated Demand

We only consider the single-stage case. Suppose the demand process to be driven by an exogenous Markov chain, i.e., the state of the Markov chain in a period determines the demand distribution in that period. We show that batch-based policies with state-dependent order-up-to levels are optimal for the system.

Specifically, the demand process is driven by a discrete-time Markov chain $\mathbf{W} = \{W(t), t \geq 0\}$ which has $w$ states and is time homogeneous. Let $I = \{1, 2, ..., w\}$ be the state space of $\mathbf{W}$ and $\rho_{i_1,i_2}$, $i_1, i_2 \in \mathbf{W}$, be the one-step transition probability from state $i_1$ to $i_2$.

For clarity, we assume that the events in each period occur in the following sequence. At the beginning of each period, (1) the state of the Markov chain $\mathbf{W}$ is observed; (2) a replenishment order, if any, is placed; (3) the order is received from the outside supplier; and (4) demand arrives during the period and at the end of the period, holding and backorder costs are assessed.

Using the principle of optimality, we can write the following dynamic program for the above problem. For each $t = 1, 2, ..., T$, we have

$$v_t(x_t, i) = -c_t x_t + \inf_{y_t \in \mathcal{A}(x_t)} \{\delta(y_t - x_t) \cdot K + J_t(y_t, i)\}, \tag{1.30}$$

where $K$ is the fixed cost, and

$$J_t(y_t, i) = L_t(y_t, i) + c_t y_t + \alpha E[v_{t+1}(y_t - D_t, i_{t+1}) | i_t = i].$$

In the following, we first consider the case without a fixed ordering cost (i.e., $K = 0$), and then with $K > 0$, respectively.

**Zero Fixed Cost**

We now show that the optimal policy is of a $(r, Q)$ structure with state-dependent reorder levels.

**Theorem 1.6.** (a) *For any* $t = 1, 2, ..., T$, $J_t(y_t, i)$ *and* $v_t(x_t, i)$ *are strong Q-jump-convex for any given state* $i$.

(b) *The state-dependent* $(r(i), Q)$ *policy is optimal.*

**Proof.** We prove the results by induction. For $t = T + 1$, $v_{T+1} = 0$ and the results clearly hold. Assume $v_{t+1}(x, i)$ to be strong $Q$-jump-convex for any given $i$. By Lemma 1.2 parts (b) and (d), $\omega_{t+1}(y_t, i) = \alpha E[v_{t+1}(y_t - D_t, i_{t+1})|i_t = i]$ is also strong $Q$-jump-convex for any given $i$. Then, with the same logic of Theorem 1.1, we can obtain the desired results. □

**Positive Fixed Cost**

In this case, we prove that a state-dependent $(\mathbf{s}, \mathbf{S})_Q$ policy is optimal by using $Q$-jump-$K$-convexity. When $Q = 1$, Sethi and Cheng (1997) establish the optimality of state dependent $(s, S)$ policy. Their results are built upon $K$-convexity. As for all $x \in [j]_Q$, the respective functions are $K$-convex in our setting, their analysis can also carry over.

**Theorem 1.7.** (a) *For any* $t = 1, 2, ..., T$, $J_t(y_t, i)$ *and* $v_t(x_t, i)$ *are Q-jump-K-convex for any given state* $i$.

(b) *For any* $t = 1, 2, ..., T$, *there exist* $s_t^j(i)$ *and* $S_t^j(i)$ *with* $s_t^j(i) \leq S_t^j(i)$, *and* $s_t^j(i), S_t^j(i) \in [j]_Q$, *where* $j = 1, 2, ..., Q$, *such that the state-dependent* $(\mathbf{s}(i), \mathbf{S}(i))_Q$ *policy is optimal.*

**Proof.** The proof is similar to that of Theorem 1.2 and thus omitted.

# Chapter 2

# Inventory System with Quantity-Dependent Setup Cost

## 2.1 Introduction

We consider a periodic-review, stochastic, inventory-control system where the fixed order/setup cost depends on the size of each order. In particular, there is a prespecified order-quantity limit, $C$, based on which the setup cost can take one of two values, $K_1$ or $K_2$. If the order quantity is within the specified limit, then the fixed cost is $K_1$; otherwise it is $K_2$. The ordering cost also includes a variable component that is linear in quantity. Mathematically, the cost of ordering quantity $z$ is given by

$$c(z) = K_1 1[0 < z \leq C] + K_2 1[z > C] + cz, \tag{2.1}$$

where $K_1$ and $K_2$ are the two fixed setup costs, $c$ is the unit purchase cost, and $1[.]$ is the indicator function with value one if the statement in brackets is true and zero otherwise. In this chapter, we concentrate on the case where $0 \leq K_1 \leq K_2$.

Some examples of the specific cost structure can be found in transportation and production contracts with order-quantity restrictions. In such contracts, it is usually required that the buyer commits to a certain order volume in a given period, and deviations can be penalized with increased ordering costs. For example, if satisfying an order beyond the contract volume would necessitate a new setup or an emergency shipment for the producer, this additional cost can be reflected in the buyer's ordering cost. Faced with uncertain demand and implications of exceeding the quota, the buyer then needs to carefully control the inventory decisions. Chao and Zipkin (2008) and Henig et al. (1997) study inventory-control models motivated from similar applications in practice. The former article considers a case where the fixed cost $K$ is incurred only if the order quantity exceeds a certain threshold, $R$, and there is no fixed cost otherwise, i.e., $c(z) = K1[z > R] + cz$. Clearly, our order-cost function provides a generalization as we can select $K_1 = 0$ to obtain their model. In a similar spirit, the cost structure in Henig et al. (1997) includes a linear purchase cost which is only incurred when the order size is larger than $R$, however no fixed cost is analyzed.

For better intuition on the cost structure, consider the following examples with which the authors are familiar from practice. A firm in the aluminum profile industry produces customized profiles for desk and chair manufacturers in a make-to-stock fashion and deliver them in short notice. A typical product first goes through a continuous production line and then is put into a furnace for solution heat treatment before a finishing process. A bottleneck is related to the heat treatment, which is confined by the capacity of the furnace equipment. The firm currently has two furnaces with different capacities, one being larger than the other but necessitating a higher setup cost. According to each period's production-order size, the company utilizes the appropriate furnace and incurs the corresponding setup cost. Other examples come from the electricity market and air/sea cargo carrier industry, where the service providers discourage the use of resources beyond contract capacity by monetarily penalizing overuse during peak periods. More such examples can be found in Gupta (1994) and Lippman (1969).

Our order-cost function captures the characteristics of order-size dependent fixed costs that are observed in practice. However, the complex structure brings analytical challenges. More specifically, the function given in (2.1) is neither convex nor concave, hence the existing results with general convex or concave ordering costs (Porteus 2002) are not directly applicable to our setting. Correspondingly, we develop a new methodological concept to characterize the optimal policy in the presence of order-size dependent fixed setup cost. To explore structural results, we first tackle a special case of the problem with the condition $K_1 \leq K_2 \leq 2K_1$. We introduce a concept called $C-(K1, K2)$-convexity and show that it is preserved in the dynamic programming recursions. This result enables us to partially characterize the optimal ordering policy. The optimal policy is defined using five critical points and consists of different ordering decisions in five regions of the initial inventory level. Next, we consider the general case with the condition $K_1 \leq K_2$ and employ a concept called strong $K$-convexity to derive the preservation and optimality results parallel to the special case. In the general model, the optimal policy becomes more complex with less explicit decisions in some of the decision regions. This motivates us to construct a heuristic policy that is easy to implement. Our computational experiments show that the heuristic policy performs very close to the optimal policy.

Our analysis of the general case extends the work by Chao and Zipkin (2008) and the structure of the optimal policy for our model has some similarities to theirs. However, since the authors analyze a less complex cost structure, their optimal policy is a bit simpler and can be obtained from our optimal policy by setting $K_1 = 0$. Furthermore, the definitions of some critical points and the construction of the heuristic policy are different in our analysis. In the special case, the condition $K_1 \leq K_2 \leq 2K_1$ leads to an order-cost function that is subadditive, which implies that $c(z_1+z_2) \leq c(z_1)+c(z_2)$ for all $z_1, z_2 \geq 0$, where $z_i$ is the ordering quantity. Lippman (1969) discusses an inventory control problem where the order-cost function is nondecreasing and subadditive. Although our model with the condition $K_1 \leq K_2 \leq 2K_1$ is a special case of Lippman's, we not only use a different method to analyze the problem, but also provide a more complete characterization of the optimal policy, which also facilitates constructing a more efficient heuristic policy.

When $K_2 \to \infty$ and $C \leq \infty$, our model can address the case with finite ordering capacity. Periodic-review inventory systems with finite ordering capacity and fixed setup costs have been investigated by Gallego and Scheller-Wolf (2000), Shaoxiang and Lambrecht (1996), and Shaoxiang (2004). The first two articles tackle the problem in a finite horizon scenario and provide partial characterizations of the optimal policy. Shaoxiang and Lambrecht (1996) show that the optimal policy is not generally of $(s, S)$ type and rather follows an $X$-$Y$ band structure, while no complete characterization is given within the $X$ and $Y$ bounds. Gallego and Scheller-Wolf (2000) provide further results towards more explicit characterization by using a notion that the authors call $CK$-convexity. Different from these works, the fixed cost in our analysis is not constant but depends on the order size. Correspondingly, to partially characterize the optimal policy when $K_1 \leq K_2 \leq 2K_1$, we introduce a different concept called $C$-$(K_1, K_2)$-convexity, which takes root in $CK$-convexity (Gallego and Scheller-Wolf, 2000) and $(K_1, K_2)$-convexity (Ye and Duenyas, 2007). In the general case, we use strong $K$-convexity, a restricted version of strong $CK$-convexity due to Gallego and Scheller-Wolf (2000), to prove our optimality results.

We also consider a more general setting, multiple fixed costs, with the following ordering cost structure

$$c(z) = \sum_{i=1}^{n} K_i 1[C_i < z \leq C_{i+1}] + cz, \tag{2.2}$$

where $K_{i+1} \geq K_i$ and $C_{i+1} \geq C_i$.

However, we numerically show that the optimal policy sometimes is very complex. As a special case, we study the trucking problem, where $K_{i+1} - K_i = K$, $C_{i+1} - C_i = C$, and $n \to +\infty$. Such a problem can be observed when the same setup activities with the identical fixed cost are required, when the order quantity exceeds some threshold points. For example, a Hong Kong based herbal tea manufacturer bottles its drinks outside of Hong Kong but maintains a warehouse locally. The company ships its products from the bottling plant using a heterogenous fleet of trucks, which are charged at different fixed costs per truck plus a flat rate per kilogram shipped. The cost structure is such that the fixed portion of the ordering cost increases as the size of an order increases. In this case, $C$ represents the capacity of a truck and $K$ the cost of its use. Iwaniec (1979) provides the conditions under which the full-load ordering policy is optimal: if the inventory is below some critical point $\theta$, order the smallest number of full vehicle loads to raise the inventory level just above $\theta$; otherwise, do not order. These conditions are, however, restrictive and difficult to verify (or can be verified only numerically). Here, we relax these conditions and explore the structure of optimal policies.

Our research contributes to the periodic-review, stochastic, inventory-control systems literature by analyzing an order-size dependent fixed cost, which is motivated from industry practices. Analytically, we contribute by introducing a new class of functions, and extending or redeveloping several existing results in the literature using different techniques. As inventory costs continue to represent significant operating expenses for companies, we believe our research findings will be valuable in efficient management of these systems.

## 2.2 The Model

Consider a firm which manages a single-product, periodic-review inventory system facing stochastic demand over a finite time horizon of $T$ periods. At the beginning of each period $t$, the firm may place an order, in which case an ordering cost as shown in (2.1) is incurred: An order of size $C$ or lower is available for a fixed setup cost $K_1$, however, ordering larger than $C$ requires a higher setup cost $K_2$. Once placed, the order arrives instantaneously and is then used as part of the on-hand inventory to satisfy the random demand. Any unsatisfied demand is fully backordered, and holding/backorder costs are assessed at the end of a given period. We assume that all costs in future periods are discounted by $\alpha \leq 1$ and demands in consecutive periods are independently distributed.

For period $t = 1, 2, ..., T$, let

$x_t$ = inventory level at the start of period $t$, before an order is placed.

$y_t$ = inventory level after any order is placed, but before demand is realized.

$D_t$ = nonnegative demand in period $t$.

$L(y_t)$ = one-period expected holding/backorder cost with inventory level $y_t$.

We assume that $L(y_t)$ is convex and $\lim_{|y_t|\to\infty} L(y_t) = \infty$. Let $f_t(x_t)$ be the total expected cost when the initial inventory in period $t$ is $x_t$ and the optimal ordering policy is employed in the remaining $t$ periods. The dynamic programming recursion for $f_t(x_t)$ can be written as

$$f_t(x_t) = -cx_t + \inf_{y_t \geq x_t} \{K_1 1[x_t < y_t \leq x_t + C] + K_2 1[y_t > x_t + C] + G_t(y_t)\}, \tag{2.3}$$

where $G_t(y_t) = cy_t + L(y_t) + \alpha E[f_{t-1}(y_t - D_t)]$. The firm's objective is to determine the policy that returns $f_T(x)$ for all $x$. We assume the boundary conditions $f_0(x) = 0$. For convenience, we also assume that $c = 0$ (Veinott and Wagner 1965).

We analyze the problem in two cases. In the first case, we assume that the condition $K_1 \leq K_2 \leq 2K_1$ holds, which allows us to exploit a new type of convexity property to derive some structural results. In practice, such a condition might correspond to situations where a higher order quantity results in a larger setup cost, but the difference is limited with the smaller setup cost. For example, dispatching a high capacity truck at a high setup cost can be more cost effective than dispatching two identical, smaller-capacity trucks each with lower setup costs. Our analysis in the second case considers the general condition $K_1 \leq K_2$.

## 2.3 The Case with Condition $K_1 \leq K_2 \leq 2K_1$

In this section, we consider a special case with the condition $K_1 \leq K_2 \leq 2K_1$. Our analytical results in this case are facilitated by a notion that we call $C$-$(K_1, K_2)$-convexity. In the following, we first formally define $C$-$(K_1, K_2)$-convexity and derive some preliminary results for the analysis of the optimal policy.

### 2.3.1 Preliminary Results: $C$-$(K_1, K_2)$-Convexity

Many different inventory models with setup costs have been studied in the literature. According to the structure of the model, different kinds of convexity (concavity) properties have been employed to characterize the optimal policy. To analyze our problem, we introduce a new class of functions that are $C$-$(K_1, K_2)$-convex and show a preservation property, under which optimality results are derived. To connect this new concept with other related notions in the literature, we first provide a general definition using a property that we refer to as $\sigma(K)$-convexity.

**Definition 2.1.** *A real-valued function $G$ is called $\sigma(K)$-convex for $K \geq 0$, if for all $y$, $0 < b < \infty$, and $z \in [0, \infty)$,*

$$G(y+z) + \sigma(K) \geq G(y) + \frac{z}{b}\{G(y) - G(y-b)\}.$$

For convenience, we write the variables in one dimensional form. This definition can easily be extended to the multidimensional case. Note that, $\sigma(K)$-convexity corresponds to mere convexity when $\sigma(K) = 0$, $K$-convexity when $\sigma(K) = K$ (Scarf, 1960), symmetric-$K$-convexity when $\sigma(K) = \max\{\frac{z}{b}, 1 - \frac{z}{b}\}K$ (Chen and Simchi-Levi 2004a), $(K_1, K_2)$-convexity recently proposed by Ye and Duenyas (2007) when $\sigma(K) = (1-\frac{z}{b})K_1 + \frac{z}{b}K_2 - \min\{\frac{z}{b}, 1-\frac{z}{b}\}\min\{K_1, K_2\}$, and weak $(K_1, K_2)$-convexity when $\sigma(K) = (1-\frac{z}{b})K_1 + \frac{z}{b}K_2$ (Semple, 2007).

As a general procedure, we try to find a right $\sigma(K)$ to define a class of functions with appealing properties, which we then use to characterize the optimal policy.

To tackle our model, we define a particular $\sigma(K)$-convexity, which has similarities to the $(K_1, K_2)$-convexity of Ye and Duenyas (2007).

**Definition 2.2.** *A real-valued function $G$ is called $C$-$(K_1, K_2)$-convex for $K_1$, $K_2 \geq 0$, if for all $y$, $0 \leq a < \infty$, $0 < b < \infty$, and $z \in [0, \infty)$,*

$$G(y+z) + \sigma_C(K_1, K_2) \geq G(y) + \frac{z}{b}\{G(y-a) - G(y-a-b)\},$$

*where*

$$\sigma_C(K_1, K_2) = \begin{cases} K_1 & z \in [0, C], \\ K_2 & z > C. \end{cases} \tag{2.4}$$

See Figure 2.1 for an illustration of $C$-$(K_1, K_2)$-convexity. In the graph, let $A = (y-a-b, G(y-a-b))$, $B = (y-a, G(y-a))$, and $R = (y, G(y))$. Further, for $C \geq z_1 \geq 0$ let $E = (y+z_1, K_1 + G(y+z_1))$, and for $z_2 > C$ let $F = (y+z_2, K_2 + G(y+z_2))$. Note that the points are selected such that $A$ and $B$ lie on the left of $y$, $E$ lies between $y$ and $y+C$, and $F$ lies on the right of $y+C$. Geometrically, $C$-$(K_1, K_2)$-convexity means that the two lines drawn from any point $R$ connecting points $R$ and $E$ and connecting points $R$ and $F$ both have larger slopes than a line connecting any two points $A$ and $B$ behind $y$. Note that when $K_2 \rightarrow +\infty$, $C$-$(K_1, K_2)$-convexity reduces to $CK$-convexity (Gallego and Scheller-Wolf, 2000).

Using the definition, we show some properties of $C$-$(K_1, K_2)$-convex functions, which will be useful in our analysis.

**Lemma 2.1.** (a) *A convex function is also a $C$-$(0, 0)$-convex function.*

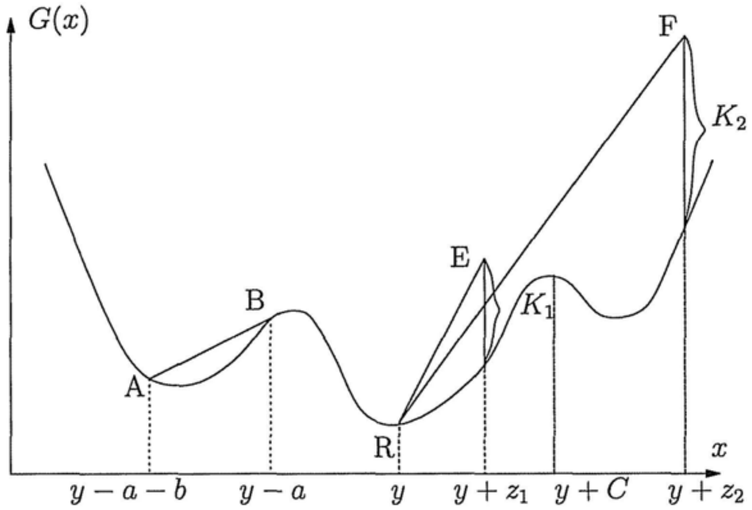


Figure 2.1: Geometric Illustration of a $C$-$(K_1, K_2)$-convex Function

(b) *If $f$ is $C$-$(K_1, K_2)$-convex and $\gamma$ is a positive scalar, then $\gamma f$ is $C$-$(\gamma K_1, \gamma K_2)$-convex.*

(c) *If $f$ is $C$-$(K_1, K_2)$-convex, then it is also $C$-$(K_1', K_2')$-convex for any $K_1' \geq K_1$ and $K_2' \geq K_2$.*

(d) *The sum of a $C$-$(K_1, K_2)$-convex function and a $C$-$(K_1', K_2')$-convex function is $C$-$(K_1 + K_1', K_2 + K_2')$-convex.*

(e) *If $v$ is $C$-$(K_1, K_2)$-convex, $\phi$ is the probability density of a positive random variable, and $G(y) = \int_0^{+\infty} v(y-\xi)\phi(\xi)d\xi$, then $G$ is also $C$-$(K_1, K_2)$-convex.*

The stated results imply preservation properties for $C$-$(K_1, K_2)$-convexity under some common operators and are relatively straightforward to show. Next, we prove that $C$-$(K_1, K_2)$-convexity can be preserved under a minimization operator. This nontrivial result will play a central role in deriving the structural properties of the optimal cost functions.

**Theorem 2.1.** *Suppose $G(x)$ is a $C$-$(K_1, K_2)$-convex function and $K_1$, $K_2$ are nonnegative constants such that $K_1 \leq K_2 \leq 2K_1$. Then, $f(x) = \min_{y \geq x}\{K_1 1[x < y \leq x + C] + K_2 1[y > x + C] + G(y)\}$, is also $C$-$(K_1, K_2)$-convex.*

In proving the theorem, it turns out that the condition $K_2 \leq 2K_1$ is necessary. (Otherwise, we refer to part (a) of case (VI) in the proof, where the assumptions $f(x-a) = K_2 + G(x-a-b+\mu')$ and $z \in [0, C]$ imply $\sigma_C(K_1, K_2) = K_1$. Then, by setting $K_1 = 0$ and $K_2 > 0$, we find that $K_1 + G(x+z+\mu) - f(x) + \sigma_C(K_1, K_2) = G(x+z+\mu) - f(x) \leq 0$, which contradicts with the definition of $C$-$(K_1, K_2)$-convexity.)

### 2.3.2 Analysis of the Optimal Policy

To characterize the optimal policy, we first investigate the structure of objective function. Let us define

$$J_t(x) = \min_{0 \leq z \leq C}\{K_1 1(z > 0) + G_t(x + z)\}, \text{ and} \tag{2.5}$$

$$V_t(x) = \inf_{y \geq x}\{K_2 1(y > x) + J_t(y)\}. \tag{2.6}$$

Then, we have the following result:

**Lemma 2.2.** *For any $K_2 \geq K_1$, $f_t(x) = V_t(x)$.*

Lemma 2.2 specifies an alternative representation of the objective function $f_t(x)$, which facilitates easier analysis in the rest of this chapter.

Having established the technical results, we proceed to characterize the optimal policy by using some critical points. We first verify the continuity of the functions $f_t(x_t)$ and $G_t(x_t)$, which makes the critical points well-defined.

**Theorem 2.2.** *For any $K_2 \geq K_1$, $f_t(x_t)$ and $G_t(x_t)$ are continuous functions for any $t = 0, 1, ..., T$.*

**Definition 2.3.** *Given the non-negative constants $C$, $K_1$ and $K_2$, let us define:*

$$S = \arg\inf G_t(x);$$

$$s_1 = \inf\{x|G_t(x) \leq K_1 + G_t(S)\};$$

$$s_2 = \inf\{x|G_t(x) \leq K_2 + G_t(S)\}.$$

Note that we drop the subscript $t$ from the definition of critical points for notational simplicity. The critical points $s_1$ and $s_2$ can be interpreted as the points below which ordering lower and larger than $C$ up-to $S$, respectively, dominates ordering nothing. With Theorem 2.2 on hand, the continuity can allow the critical points to take the corresponding values, e.g., $G_t(s_1) = K_1 + G_t(S)$. We can use $s_1$, $s_2$, and $S$ to prove the following lemma.

**Lemma 2.3.** *If $G_t(x)$ is a $C$-$(K_1, K_2)$-convex function, then*

(i) $G_t(y) + K_1 \geq G_t(x)$ *for any* $x + C > y > x > s_1$;

(ii) $G_t(y) + K_2 \geq G_t(x)$ *for any* $y > x > s_2$;

(iii) $s_2 \leq s_1 \leq S$;

(iv) $G_t(x)$ *is non-increasing in* $(-\infty, s_2)$;

(v) $J_t(S) = G_t(S)$.

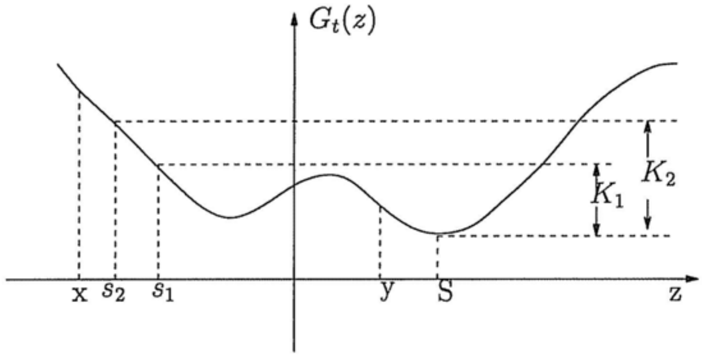


Figure 2.2: Illustration of Critical Points in Definition 3

Please refer to Figure 2.2 to better visualize the critical points. The preceding lemma reveals some optimality results. For example, when $x > s_1$, it is more costly to place an order of any size than not to order, hence "order nothing" is the optimal decision in this region. However, the given critical points are not sufficient to characterize the optimal policy completely in all regions. For example, when $s_2 < x \leq s_1$, one could think that it is optimal to order up to $S$ if $S - x \leq C$, and when $x \leq s_2$, it is optimal to order up to $S$ if $S - x > C$. However, this may not be true. For illustration, we reconsider the example in Figure 2.2 and concentrate on the points $x$ and $y$. Given the initial inventory level $x \leq s_2$, notice that ordering up to $y$, instead of $S$ may result in a lower cost, i.e., $G_t(y) + K_1$ as opposed to $G_t(S) + K_2$, provided that $y - x \leq C$ and $S - x > C$. Consequently, we need to define additional critical points to characterize the structure of the optimal policy.

**Definition 2.4.** *Given non-negative constants $C$, $K_1$ and $K_2$, and $C$-$(K_1, K_2)$-*

*convex functions* $G_t(x)$ *and* $J_t(x)$*, let us define*

$$s = \inf\{x | J_t(x) \le K_2 + J_t(S)\};$$

$$s' = \min\{S - C, s_1\};$$

$$s'' = \inf\{x | s \le x \le s', J_t(x) < K_1 + G_t(x + C)\}.$$

The time indices of the critical points are again suppressed for notational convenience. Here, $s$ is the point below which ordering larger than $C$ up-to $S$ dominates ordering no more than $C$ and $s''$ is the point below which ordering $C$ dominates ordering less than $C$. The following lemma, together with Lemma 2.3, will be vital in partially characterizing the optimal policy.

**Lemma 2.4.** *If* $G_t(x)$ *is a* $C$*-*$(K_1, K_2)$*-convex function and* $J_t(x)$ *and* $G_t(x)$ *satisfy Equation (2.5), then*

(i) $J_t(y) + K_2 \ge J_t(x)$ *for any* $y > x > s$*;*

(ii) $s + C \le S$*;*

(iii) $s \le s'' \le s' \le s_1$*;*

(iv) $J_t(x) \ge K_1 + G_t(x + C)$ *for any* $s \le x < s''$*.*

We use the following five critical points $(s, s'', s', s_1, S)$ to characterize the optimal policy. The decision area is now divided into the following five regions:

$$(-\infty, s), \quad [s, s''), \quad [s'', s'), \quad [s', s_1), \quad [s_1, +\infty).$$

Theorem 2.3 establishes our main result in characterizing the optimal policy. For illustration, we show the critical points and decision areas in Figure 2.3, where we also summarize the optimal ordering decisions.

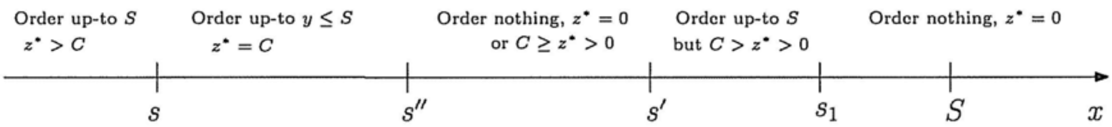


Figure 2.3: Optimal Policy for the Case where $K_1 \leq K_2 \leq 2K_1$

**Theorem 2.3.** *(a) $G_t(x)$ and $f_t(x)$ are $C$-$(K_1, K_2)$-convex for all $t$.*

*(b) For each $t = 1, 2, ...T$, there exists an optimal policy that can be characterized by the points $s \leq s'' \leq s' \leq s_1 \leq S$ in the following way:*

(i) *Order up to $S$, when $x \leq s$;*

(ii) *Order exactly $C$, when $s \leq x \leq s''$;*

(iii) *Order no more than $C$, when $s'' \leq x \leq s'$;*

(iv) *Order up to $S$, when $s' \leq x \leq s_1$;*

(v) *Order nothing when $x \geq s_1$.*

**Proof.** (a) We show the proof by induction. First, we assume that $f_{t-1}(x)$ is $C$-$(K_1, K_2)$-convex. Then, it follows from parts (b), (c) and (e) of Lemma 2.1 that $\alpha E[f_{t-1}(y - D)]$ is $C$-$(\alpha K_1, \alpha K_2)$, and thus $C$-$(K_1, K_2)$-convex. From the properties of $C$-$(K_1, K_2)$-convexity in parts (a), and (d) of the same lemma, we can easily verify that $G_t(y)$ is $C$-$(K_1, K_2)$-convex. We can now apply the result in Theorem 2.1 and conclude that $f_t(x)$ is $C$-$(K_1, K_2)$-convex.

(b) We divide the decision area into five regions (Figure 2.3):

$$(-\infty, s), \quad [s, s''), \quad [s'', s'), \quad [s', s_1), \quad [s_1, +\infty),$$

and prove that the given policy is optimal in each region.

(i) $x \in (-\infty, s)$: From the definition of $s$ and part (v) of Lemma 2.3, for any $x \leq s$, we have $J_t(x) \geq J_t(S) + K_2 = G_t(S) + K_2$, which implies that ordering larger than $C$ up-to $S$ dominates ordering no more than $C$. Since $s < S - C$ by part (ii) of Lemma 2.4, ordering larger than $C$ up-to $S$ is a feasible action, and thus it is optimal in this region.

(ii) $x \in [s, s'')$: From part (iv) of Lemma 2.4, it is clear that ordering less than $C$ is not better than ordering exactly $C$. Further, it follows from part (i) of the same lemma that ordering no more than $C$ always dominates ordering larger than $C$. Hence, it is optimal to order exactly $C$.

(iii) $x \in [s'', s')$: Similar to the previous case, it follows from part (i) of Lemma 2.4 that ordering no more than $C$ always dominates ordering larger than $C$. However, in this case, the optimal decision is not always to order $C$, since $G_t(x)$ may or may not be decreasing in this subinterval. Correspondingly, we know that a quantity between 0 and $C$ must be ordered, but we cannot determine its exact value.

(iv) $x \in [s', s_1)$: We analyze this region in two scenarios. In the first scenario, $S - C \geq s_1$, i.e. $s' = s_1$, and then this region becomes empty; otherwise, we consider the second scenario, $s' = S - C$, which implies that given the initial inventory level $x$ within the region, the postorder inventory level can reach $S$ by ordering no more than $C$. From the definition of $s_1$, for any $x < s_1$, we have $G_t(x) \geq G_t(S) + K_1$. Furthermore, since $S$ is the global minimizer of $G_t(x)$ and $K_1 \leq K_2$, we have $G_t(S) + K_1 \leq G_t(x) + K_2$ for any $x$. Consequently, the optimal decision in this region is to order up to $S$.

(v) $x \in [s_1, +\infty)$: From parts (i) and (ii) of Lemma 2.3 and Equation 2.4, it follows that $G_t(x+z) + \sigma_C(K_1, K_2) \geq G_t(x)$, and the optimal decision is clearly not to order. □

Note that, when $K_1 = K_2$, our model reduces to the classical inventory model with a fixed cost, for which the optimal policy is well known to be of $(s, S)$ type. We can show that our optimal policy takes an $(s, S)$ form if we can verify that $s_1 = s$ when $K_1 = K_2$. From Lemma 2.4, we know $s_1 \geq s$, hence it suffices to prove that $s_1 \leq s$, which, by the definition of $s$, is equivalent to proving that for any $x < s_1$, $J_t(x) \geq J_t(S) + K_2$. By the definition of $J_t(x)$, we have

$$\begin{aligned} J_t(x) &= \min\{G_t(x), K_2 + \min_{0<z\leq C} G_t(x+z)\}, \\ &\geq \min\{G_t(x), K_2 + G_t(S)\}, \\ &\geq K_2 + G_t(S), \end{aligned}$$

where the first inequality follows from the definition of $S$ and the last inequality follows from Lemma 2.3(v) and that $x < s_1$. Hence, we can conclude that $s_1 \leq s$ and obtain the desired result.

## 2.4 The Case with Condition $K_1 \leq K_2 \leq 2K_1$ and Heuristic Policy

In this section, we remove the condition $K_2 \leq 2K_1$ and analyze the general case where the two setup costs only satisfy $K_2 \geq K_1$. Our analysis so far has shown that $C$-$(K_1, K_2)$-convexity may not be preserved under this general condition. (Please refer to the discussion after Theorem 2.1 for details.) To derive some results regarding the optimal policy in this case, we use a different concept called strong $K$-convexity, which is a less restricted form of $C$-$(K_1, K_2)$-convexity. As a result, the characterization of the optimal policy becomes less explicit and more complex in some regions of the state space.

### 2.4.1 Preliminary Results: Strong $K$-Convexity

We start with the definition of strong $K$-convexity.

**Definition 2.5.** *A real-valued function $G$ is called strong $K$-convex for $K \geq 0$, if for all $y$, $0 \leq a < \infty$, $0 < b < \infty$, and $z \in [0, \infty)$,*

$$G(y+z) + K \geq G(y) + \frac{z}{b}\{G(y-a) - G(y-a-b)\}.$$

Strong $K$-convexity is a special form of strong $CK$-convexity with $C \to \infty$, which was introduced by Gallego and Scheller-Wolf (2000). Clearly, strong $K$-convexity is $K$-convexity and $C$-$(K_1, K_2)$-convexity is strong $K_2$-convexity, if $K_2 \geq K_1$; however, the reverse may not hold. For completeness, we summarize some properties of strong $K$-convex functions that are useful for our analysis.

**Lemma 2.5.** (a) *If $f$ is convex, then $f$ is strong $K$-convex.*

(b) *If $f$ is strong $K$-convex and $\gamma > 0$, then $\gamma f$ is strong $\gamma K$-convex.*

(c) *If $f$ is strong $K_1$ convex, then it is also strong $K_2$ convex for any $K_2 \geq K_1$.*

(d) *The sum of a strong $K_1$-convex function and a strong $K_2$-convex function is strong $(K_1 + K_2)$-convex. function.*

(e) *If $f$ is strong $K$-convex and $X$ is a positive random variable, then $g(x) := Ef(x - X)$ is strong $K$-convex.*

**Proof.** The proof is similar to that of Lemma 2.1. □

In Theorem 2.4, we show a preservation property of strong $K$-convexity, which will enable us to prove that the cost functions are strong $K_2$-convex and partially characterize the optimal policy in the next section.

**Theorem 2.4.** *Suppose $G_t(x)$ is a strong $K_2$-convex function, and $K_1$, $K_2$ are nonnegative constants such that $K_1 \leq K_2$. Then, $f_t(x) = \min_{y \geq x}\{K_1 1[x < y \leq x + C] + K_2 1[y > x + C] + G_t(y)\}$ is also strong $K_2$-convex.*

**Proof.** Referring to the proof of Theorem 2.1, the result directly follows from the fact that $K_2 \geq \sigma_C(K_1, K_2)$. □

### 2.4.2 Analysis of the Optimal Policy

Before we proceed, we note that Lemma 2.2 is valid under the general case, i.e., $f_t(x) = V_t(x)$, where $V_t(x)$ is defined as in Equation (2.6).

We describe the optimal policy using the critical points $(s, s'', s', s_1, S)$ that were introduced in Section 2.3. First, we describe some properties of $G_t(x)$, $J_t(x)$, and $f_t(x)$ with respect to these critical points.

**Lemma 2.6.** *If $G_t(x)$ is a strong $K_2$-convex function, and $J_t(x)$ and $G_t(x)$ satisfy Equation (2.5), then*

(i) $G_t(y) + K_2 \geq G_t(x)$ *for any* $y > x > s_2$*;*

(ii) $G_t(x)$ *is non-increasing in* $(-\infty, s_2)$ *and* $G_t(S) = J_t(S)$*;*

(iii) $J_t(y) + K_2 \geq J_t(x)$ *for any* $y > x > s$*;*

(iv) $J_t(x) \geq K_1 + G_t(x + C)$ *for any* $s \leq x < s''$*.*

(v) $s \leq s'' \leq s' \leq s_1 \leq S$ *and* $s + C \leq S$*.*

Lemma 2.6 verifies that the results in Lemmas 2.3 and 2.4 continue to hold in the general case, except Lemma 2.3(i), which results in a different ordering decision in the region $[s_1, +\infty)$. The characterization of the optimal policy for all regions is given in the following theorem.

**Theorem 2.5.** *(a) $G_t(x)$ and $f_t(x)$ are strong $K_2$-convex for all $t$.*

*(b) For each* $t = 1, 2, ...T$*, there exists an optimal policy that can be characterized by the points* $s \leq s'' \leq s' \leq s_1 \leq S$ *in the following way:*

(i) *Order up to* $S$*, when* $x \leq s$*;*

(ii) *Order exactly* $C$*, when* $s \leq x \leq s''$*;*

(iii) *Order no more than* $C$*, when* $s'' \leq x \leq s'$*;*

(iv) *Order up to* $S$*, when* $s' \leq x \leq s_1$*;*

(v) *Order no more than* $C$ *when* $x \geq s_1$.

**Proof.** (a) We show the proof by induction. First, we assume that $f_{t-1}(x)$ is strong $K_2$-convex. Then, it follows from parts (b), (c) and (e) of Lemma 2.5 that $\alpha E[f_{t-1}(y-D)]$ is strong $\alpha K_2$-convex, and thus strong $K_2$-convex. From the properties of strong $K$-convexity in parts (a) and (d) of the same lemma, we can verify strong $K_2$-convexity of $G_t(y)$ and, applying Theorem 2.4, we conclude that $f_t(x)$ is strong $K_2$-convex.

(b) All cases except (v) can be proved along the same line of arguments as in the proof of Theorem 2.3. In case (v), Theorem 2.3 uses the result in Lemma 2.3(i) which no longer holds. Hence, it is sufficient to consider the last region $x \geq s_1$ to complete the proof. Note that, Lemma 2.6(i) can still guarantee ordering less than $C$ dominates ordering larger than $C$. However, we can not determine the exact value of the optimal order quantity. □

Excluding the last region, the optimal policy in the general case is identical to that in the special case. In particular, for a given initial inventory $x \in [s_1, \infty)$, the policy under the condition $K_1 \leq K_2 \leq 2K_1$ indicates "ordering nothing" as the optimal decision, whereas the corresponding decision in the general case is less clear. This is indeed expected since the condition under the special case is stronger than $K_1 \leq K_2$ and facilitates tighter results due to $C$-$(K1, K2)$-convexity.

Note that, when we assume $K_1 = 0$, our model reduces to that studied by Chao and Zipkin (2008). In their analysis, the characterization of the optimal policy is as shown in Figure 2.4. In our model, setting $K_1 = 0$ leads to the equality $s_1 = S$, which implies that $s' = S - C$. Correspondingly, we can see that our policy is identical to theirs with $s'' = u$. Similar to a result in the authors' study, we can easily verify using induction that the optimal cost function $f_t(x)$ is decreasing in $C$.

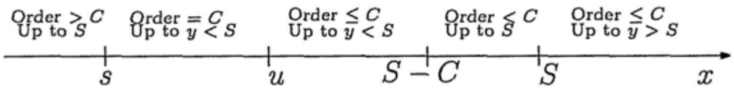


Figure 2.4: Optimal Policy given by Chao and Zipkin (2008)

### 2.4.3 A Heuristic Policy

In Theorem 2.5, we have provided a characterization of the optimal policy. However, the policy is not explicit regarding the optimal ordering decisions in the regions where the initial inventory is between $s''$ and $s'$ or is larger than $s_1$. This complication is mainly due to the fact that $G_t(x)$ may not be well-behaved in the corresponding regions. In particular, $G(x)$ may not be unimodal, i.e., decreasing in $(-\infty, S)$ and increasing in $(S, +\infty)$, and even if it is, we may still not know whether $G_t(x)$ decreases sufficiently low that ordering exactly $C$ becomes a better decision than ordering nothing. Consequently, implementing the optimal policy requires exhaustive search procedures, which can be undesirable from a practical perspective. Hence, we turn our attention to development of a heuristic policy that simplifies the decisions in the corresponding regions. To construct an easily-implementable heuristic policy, we assume that $G_t(x)$ is unimodal. Under this assumption, it is clearly optimal to order nothing in $[s_1, \infty)$. Furthermore, letting the unit penalty/shortage cost per-period be $p$, we note that the marginal profit of adding one unit to the on-hand inventory can be assumed to be $p$, when the inventory level is low. Thus, if $pC \geq K_1$, it is reasonable to assume that $G_t(x+C) + K_1 \leq G_t(x)$, which together with the monotonicity of $G_t(x)$ implies that ordering exactly $C$ is optimal; otherwise, ordering nothing is optimal. Based on the preceding assumptions, we propose the following heuristic policy:

(i) When the initial inventory level is less than $s$, order up to $S$.

(ii) When the initial inventory level is between $s$ and $s''$, order exactly $C$.

(iii) When the initial inventory level is between $s''$ and $s'$, order exactly $C$, if $pC \geq K_1$; order nothing, otherwise.

(iv) When the initial inventory level is between $s'$ and $s_1$, order up to $S$.

(v) When the initial inventory level is above $s_1$, order nothing.

We test the performance of the proposed heuristic policy with some numerical instances. Our computational experiment is designed as follows. We assume that the one-period cost function takes the form $L(y) = hE[(y-D)^+] + pE[(D-y)^+]$, where $h$ and $p$ are the unit holding and penalty costs per period, with values $h = 4$, and $p = 2$ or $8$. The rest of the parameters are selected as follows: $\alpha = 0.9$, $T = 10$, $C = 2, 10$, and $20$, $K_1 = 10$, and $K_2 = 10, 20, 50$, and $200$. In the base case, we represent the demand using Poisson distribution, with mean values $\lambda = 10, 20$, and $30$. The state space is taken as the interval $[-200, 300]$.

Let $f_t^H$ and $f_t^O$ be the discounted cost of heuristic and optimal policy, respectively. To evaluate the performance of the heuristic, we use a measure based on relative error, which is defined as follows.

$$E_i = \max_{x\in[-200,300]}(f_t^H - f_t^O)/f_t^O.$$

Similar to Chao and Zipkin (2008), we take the maximum relative error over all possible states of $x$.

Table 1 summarizes the results. Since the optimal policy is only partially characterized, we obtain the optimal costs through complete enumerations. We also note that our assumption on the unimodal structure of $G_t(x)$ actually holds in most of the cases that we tested. From the results, it appears that the heuristic policy is optimal in majority of the cases, with less than 3% maximum relative errors. The cases with small $p$ and $C$ values or large $K_2$ values seem to have the worst performances.

In alternative settings, we test binomial and symmetric triangle demand distributions. In the first case, $D$ follows the distribution function $B(x|n,q) = C_n^x q^x(1-q)^{n-x}$, where $E[D] = nq$ and $Var(D) = nq(1-q)$. In the numerical instances, we keep the mean demand fixed, i.e., $E[D] = nq = 20$, and test various levels of variance, by altering $q$. Table 2.2 shows the results. We can see that the heuristic policy performs quite well with the largest error around 1%. For the case with symmetric Triangle distribution, we test instances with the lower limit set at zero and the upper limit at the values 10, 30, 50 and 100. Similarly, the heuristic policy gives near-optimal solutions. For space restrictions, we omit the computational results for this case.

## 2.5 A Special Case of the General Setting: Multiple Identical Fixed Costs

In this section, we consider a more general model where the ordering cost depends on multiple volumes denoted by $C_i$ with a corresponding fixed cost $K_i$, $i = 1, 2, ..., n$. Without loss of generality, we assume $C_i > C_j$ for any $i > j$. Further, we let $C_1 = 0$ and $C_{n+1} = +\infty$. In detail, when the order quantity is between $C_i$ and $C_{i+1}$, the fixed cost is $K_i$. Similarly, we assume that $K_i$ have the increasing property, i.e. $K_i \geq K_j$ for any $i > j$. Then, our problem can be formulated as a dynamic programme by:

$$f_t(x_t) = \inf_{y_t \geq x_t} \{\sum_{i=1}^{n} K_i 1[x_t + C_i < y_t \leq x_t + C_{i+1}] + G_t(y_t)\}, \tag{2.7}$$

where $G_t(y_t) = L(y_t) + \alpha E[f_{t-1}(y_t - D_t)]$.

However, it is difficult to deal with such a general problem. First, from a technical viewpoint, the conditions that guarantee the preservation of $C$-$(K_1, K_2)$-convexity seem very restrictive, although the definition can be easily extended to the multiple setup costs case. Second, the policy is indeed very complex in some cases. Consider the following example: Demand in each period has a Binomial distribution with $\lambda = 27$ and $q = 0.75$. The discount factor is 0.9. There are linear holding and backlog costs with rates $h = 1$ and $p = 8$. The fixed costs are $K_1 = 20$, $K_2 = 40$, and $K_3 = 60$ and the capacities are $C_2 = 10$ and $C_3 = 40$, respectively. Consider $T = 6$. The following table shows the optimal policy for this example. We can see that in some areas, for example, $(-16, -11]$, $(-6, -3]$, and (14, 17], it is optimal to order up-to certain local minimums.

| | | | | | |
|---|---|---|---|---|---|
| $x_t \in$ | $(-\infty, -21]$ | $(-21, -16]$ | $(-16, -11]$ | $(-11, -6]$ | $(-6, -3]$ |
| | order up-to 44 | order exactly 40 | order up-to 24 | order exactly 40 | order up-to 34 |
| $x_t \in$ | $(-3, 4]$ | $(4, 9]$ | $(9, 14]$ | $(14, 17]$ | $(17, +\infty)$ |
| | order exactly 40 | order up-to 44 | order exactly 10 | order up-to 24 | order nothing |

In the following, we consider a special case, the trucking problem, studied by Iwaniec (1979), with $C_{i+1} - C_i = C$, $K_{i+1} - K_i = K$, and $n \to +\infty$. In this case, $C$ represents the capacity of a truck and $K$ the cost of its use.

Let $J_t^0(x) = G_t(x)$, and define

$$J_t^i(x) = \min_{0 \le z \le C} \{K1(z > 0) + J_t^{i-1}(x + z)\}, \quad i = 1, 2, .... \tag{2.8}$$

We may interpret $J_t^i(x)$ as the minimal cost of using at most $i$ trucks in period $t$ with the beginning inventory level $x$.

Then, we can re-express $f_t(x)$ and have the following result:

**Lemma 2.7.** *For any $t = 1, 2, ..., T$, $f_t(x) = \lim_{i \to +\infty} J_t^i(x)$.*

**Proof.** For $i = 1, 2, ...$, we define

$$\Lambda^i(x) = \min_{1 \le j \le i} \left\{K_j + \min_{x + C_j < y \le x + C_{j+1}} G_t(y)\right\}.$$

Clearly, $f_t(x) = \min\{G_t(x), \Lambda^{+\infty}(x)\}$. We first prove the recursion property of $\Lambda^i(x)$:

$$\Lambda^{i+1}(x) = \min\{\Lambda^i(x), K + \min_{x < y \le x + C} \Lambda^i(y)\}. \tag{2.9}$$

Note that

$$\begin{aligned} K + \min_{x<y\le x+C} \Lambda^{\imath}(y) &= K + \min_{x<y\le x+C} \min_{1\le\jmath\le\imath} \left\{ K_\jmath + \min_{y+C_\jmath<z\le y+C_{\jmath+1}} G_t(z) \right\} \\ &= K + \min_{1\le\jmath\le\imath} \min_{x<y\le x+C} \left\{ K_\jmath + \min_{y+C_\jmath<z\le y+C_{\jmath+1}} G_t(z) \right\} \\ &= \min_{1\le\jmath\le\imath} \left\{ K_{\jmath+1} + \min_{x+C_\jmath<z\le y+C_{\jmath+2}} G_t(z) \right\}. \end{aligned}$$

Since $K_{\jmath+1} \ge K_\jmath$, we have

$$\begin{aligned} \min\{\Lambda^{\imath}(x), K + \min_{x<y\le x+C} \Lambda^{\imath}(y)\} &= \min\{\Lambda^{\imath}(x), K_{\imath+1} + \min_{x+C_{\imath+1}<y\le x+C_{\imath+2}} G_t(y)\} \\ &= \Lambda^{\imath+1}(x). \end{aligned}$$

Next, we prove by induction that for any $\imath = 1, 2, ...,$

$$J_t^{\imath}(x) = \min\left\{G_t(x), \Lambda^{\imath}(x)\right\}.$$

The result clearly holds for $\imath = 1$. Suppose it holds for $\imath = \jmath$. Then, we can obtain

$$\begin{aligned} J_t^{\jmath+1}(x) &= \min\left\{ J_t^{\jmath}(x), K + \min_{x<y\le x+C} J_t^{\jmath}(y) \right\} \\ &= \min\{G_t(x), \Lambda^{\jmath}(x), K + \min_{x<y\le x+C} G_t(y), K + \min_{x<y\le x+C} \Lambda^{\jmath}(y)\} \\ &= \min\left\{ G_t(x), \Lambda^{\jmath}(x), K_{\jmath+1} + \min_{x<y\le x+C_{\jmath+2}} G_t(y) \right\} \\ &= \min\left\{G_t(x), \Lambda^{\jmath+1}(x)\right\}, \end{aligned}$$

where the third equality follows from (2.9).

Therefore, $f_t(x) = \min\{G_t(x), \Lambda^{+\infty}(x)\} = \lim_{\imath\to+\infty} J_t^{\imath}(x)$. □

Our analysis is facilitated by the definition of $(C, K)$-convexity, which is due to Shaoxiang (2004), who in turn extends Gallego and Scheller-Wolf (2000).

**Definition 2.6.** *A real-valued function $G$ is called $(C,K)$-convexity, if for all $y$, $0 \le a < \infty$, and $0 < b < \infty$, $\forall z \in [0,C]$,*

$$G(y+z)+K \ge G(y)+\frac{z}{b}\{G(y-a)-G(y-a-b)\},$$

*and*

$$(G(y-a-C)-G(y-a)-K)/C \ge (G(y)-G(y+z)-K)/z\}.$$

We list some of properties of $(C,K)$-convexity, while for their proofs, refer to Shaoxiang (2004).

**Lemma 2.8.** (a) *A convex function is also a $(C,0)$-convex function.*

(b) *If $f$ is $(C,K)$-convex and $\gamma$ is a positive scalar, then $\gamma f$ is $(C,\gamma K)$-convex.*

(c) *If $f$ is $(C,K)$-convex, then it is also $(C,K')$-convex for any $K' \ge K$.*

(d) *The sum of a $(C,K_1)$-convex function and a $(C,K_2)$-convex function is $(C,K_1+K_2)$-convex.*

(e) *If $v$ is $(C,K)$-convex, $\phi$ is the probability density of a positive random variable, and $G(y)=\int_0^{+\infty} v(y-\xi)\phi(\xi)d\xi$, then $G$ is also $(C,K)$-convex.*

**Lemma 2.9.** *Suppose $G(x)$ is $(C,K)$-convex, then there exists $Y$, such that*

(i) *$G(x)$ is a decreasing function for $x \le Y$;*

(ii) *$G(x-C) \ge G(x)+K$ for $x \le Y$;*

(iii) *$G(x) \le G(x+z)+K$, for $x > Y$ and $0 < z \le C$.*

The $(C, K)$-convexity is used to solve the traditional capacity inventory model. In that case, Property (i) and (ii) imply that it is optimal to order full capacity when the inventory level is below $Y-C$; and property (iii) indicates that when the inventory level is above $Y$, it is optimal not to order. However, when the inventory level is between $Y-C$ and $Y$, the optimal decision cannot be characterized. In fact, as Shaoxiang(2004) shows numerically, in such an interval, the ordering pattern will be different from problem to problem.

Let $S = \arg\inf G_t(x)$ and $y^*(x_t)$ be the optimal order up-to level of Problem (2.7) with the initial inventory level $x_t$.

**Theorem 2.6.** *For any* $t = 1, 2, ..., T$,

(i) $f_t(x)$, $G_t(x)$ *and* $J_t^i(x)$ *are* $(C, K)$*-convex, where* $i = 1, 2, ...;$

(ii) *there exists* $Y_t$ *such that* $y^*(x_t) = x_t$*, when* $x_t \geq Y_t$*;* $y^*(x_t) = y^*(x_t + C)$*, when* $x_t < Y_t - C$*;* $0 \leq y^*(x_t) \leq C$*, when* $Y_t - C \leq x_t < Y_t$

**Proof.** (i) We prove the results by induction and suppose that $f_{t-1}(x)$ is $(C, K)$-convex. We first prove that $J_t^i(x)$, $i = 1, 2, ...$ are $(C, K)$-convex. By Lemma 2.8 and $(C, K)$-convexity of $f_{t-1}(x)$, $J_t^0(x) = G_t(x)$ is also $(C, K)$-convex. Note that Problem (2.8) is a capacity inventory problem with setup cost $K$. Referring to Shaoxiang (2004), $(C, K)$-convexity can be preserved under minimization of Problem (2.8). Therefore, $J_t^1(x)$ is $(C, K)$-convex. With the same logic, we can obtain that $J_t^i(x)$, $i = 1, 2, ...,$ are $(C, K)$-convex. By Lemma 2.7, $f_t(x)$ is also $(C, K)$-convex.

(ii) Because $G_t(x)$ is $(C,K)$-convex, by Lemma 2.9 (iii), we have that $G_t(x_t) \leq G_t(x_t + mC + z) + mK$, for any $x_t \geq Y_t$, nonnegative integer $m$ and $0 < z \leq C$, which implies that it is optimal not to order, when $x_t \geq Y_t$, and $0 \leq y^*(x_t) \leq C$, when $Y_t - C \leq x_t < Y_t$. Please refer to Shaoxiang (2004) for the detail definition of $Y_t$.

Lemma 2.9 (i) and (ii) imply that for any $x_t < Y_t - C$,

$$\min\{G_t(x_t), K + \inf_{x_t < y_t \leq x_t + C} G_t(y_t)\} = K + G_t(x_t + C).$$

Therefore, for any $x_t < Y_t - C$, we can obtain

$$\begin{aligned}
f_t(x_t) &= \inf_{y_t \geq x_t} \left\{ \sum_{i=1}^{+\infty} K_i 1[x_t + C_i < y_t \leq x_t + C_{i+1}] + G_t(y_t) \right\} \\
&= \min\{G_t(x_t), K + \inf_{x_t < y_t \leq x_t + C} G_t(y_t), \\
&\inf_{y_t \geq x_t + C} \sum_{i=2}^{+\infty} K_i 1[x_t + C_i < y_t \leq x_t + C_{i+1}] + G_t(y_t)\} \\
&= \min\left\{ K + G_t(x_t + C), \inf_{y_t \geq x_t + C} \sum_{i=2}^{+\infty} K_i 1[x_t + C_i < y_t \leq x_t + C_{i+1}] + G_t(y_t) \right\} \\
&= K + \min\left\{ G_t(x_t + C), \inf_{y_t \geq x_t + C} \sum_{i=2}^{+\infty} K_{i-1} 1[x_t + C_i < y_t \leq x_t + C_{i+1}] + G_t(y_t) \right\} \\
&= K + f_t(x_t + C),
\end{aligned}$$

which implies $y^*(x_t) = y^*(x_t + C)$. □

Theorem 2.6 indicates that it is optimal to order nothing, when the inventory level is above $Y_t$ and the optimal actions associated with two starting inventory levels below $Y_t$, are the same, provided that these starting inventory levels differ by a multiple of $C$. That is, it suffices to find the optimal order up-to level $y^*(x_t)$, for $Y_t - C \leq x_t < Y_t$. Moreover, for any $Y_t - C \leq x_t < Y_t$, $0 \leq y^*(x_t) \leq C$.

Therefore, the optimal action $y^*(x_t)$ in any period $t$ can be restricted to the set $(Y_t - C, Y_t + C]$ for any starting inventory level $x_t \leq Y_t$.

Let $\lceil \cdot \rceil$ be the largest integer smaller than or equal to the argument inside.

Theorem 2.6 also implies that when the initial inventory level $x_t < Y_t - C$, first order at least $\lceil \frac{Y_t - x_t}{C} \rceil$ full-truck amount to raise the inventory level up-to $[Y_t - C, Y_t)$ and then, it is possible to partially use another truck.

## 2.6 Conclusion

In most of the earlier studies on periodic-review inventory problems, the fixed setup cost has been assumed to be invariant of the order size. However, in real systems there can be situations where the fixed cost is dependent on the order size. For example, contractual agreements in some industries such as transportation and production usually reflect cases where the buyers bear higher fixed ordering costs when they exceed a specified contract volume $C$. Motivated from these applications, we analyze an inventory control problem where the firm incurs a fixed cost of $K_1$ when the order quantity is less than or equal to $C$ and a fixed cost of $K_2 \geq K_1$ otherwise.

We first tackle the problem in a special case with the condition $K_1 \leq K_2 \leq 2K_1$. Our analysis in this case is based on a new technical concept that we introduce, i.e., $C$-$(K_1, K_2)$-convexity. In particular, we prove that the optimal cost functions are $C$-$(K_1, K_2)$-convex, which facilitates a characterization of the optimal policy. Although the characterization is partial, as is the case in several other studies analyzing order-size dependent fixed costs, our results show a reasonably simple optimal policy, which is defined using five critical points and has some resemblance to the classical $(s, S)$ type policies. In the general case, the $C$-$(K_1, K_2)$-convexity property no longer holds, and to partially characterize the optimal policy, we rely on another notion called strong $K$-convexity. We also develop a heuristic method to simplify the policy in the general case and conduct numerical experiments to test the performance, which appears highly effective.

The ordering cost function with two fixed setup costs provides generalizations of some existing results in the literature. When $K_1 = K_2$, our model reduces to the classical inventory problem analyzed by Scarf (1960). In the general case, if we assume that $K_1 = 0$ then our model reduces to that of Chao and Zipkin (2008), and we verify that our results extend theirs. Our analysis redevelops an earlier result by Lippman (1969), which was derived for inventory problems with a subadditive ordering cost function such as ours in the special case $K_1 \leq K_2 \leq 2K_1$. To redevelop the author's result, we use a different technical property, i.e., $C$-$(K_1, K_2)$-convexity, which facilitates a more complete characterization of the optimal policy and brings advantages in implementation. We devise the notion of $C$-$(K_1, K_2)$-convexity inspired from some other structural properties in the literature; specifically, $CK$-convexity by Gallego and Scheller-Wolf (2000) and $(K_1, K_2)$-convexity by Ye and Duenyas (2007). Strong $K$-convexity property that we utilize in the general case is originated from a relevant concept called strong $CK$-convexity that was introduced by Gallego and Scheller-Wolf (2000).

For more than two fixed costs, we consider a special case, i.e., trucking problem. We explore the structure of the optimal policy. The analysis is facilitated by the notion of $(C, K)$-convexity introduced by Gallego and Scheller-Wolf (2000) and Shaoxiang (2004).

One extension of our research could be to consider the complementary case with $K_1 \geq K_2$. Our preliminary analysis has shown that some of the current results no longer hold under this alternate setting and a new method should be developed to analyze the optimal policy. In another extension, multiple thresholds on the order size can be introduced which will lead to multiple setup costs and potentially further analytical challenges.

## 2.7 Appendix

**Proof of Lemma 2.1**. Parts (b), (c), and (d) directly follow from the definition of $C$-$(K_1, K_2)$-convexity.

(a) Assume $G(x)$ is a convex function, which implies that for all $0 \leq a < \infty$ and $b > 0$,

$$G(y) - G(y-b) \geq G(y-a) - G(y-a-b).$$

By the definition of convexity, we have:

$$G(y+z) - G(y) - \frac{z}{b}\{G(y) - G(y-b)\} \geq 0,$$

$$G(y+z) - G(y) - \frac{z}{b}\{G(y-a) - G(y-a-b)\} \geq 0.$$

The last inequality implies that $G(x)$ is $C$-$(0, 0)$-convex.

(e) For each $x \leq y$ and $0 \leq \lambda \leq 1$,

$$\begin{aligned} G(y+z) &= \int_0^{+\infty} v(y+z-\xi)\phi(\xi)d\xi \\ &\geq \int_0^{+\infty} [v(y-\xi) + \frac{z}{b}(v(y-a-\xi) - v(y-a-b-\xi)) - \sigma_C(K_1, K_2)]\phi(\xi)d\xi \\ &= G(y) + \frac{z}{b}\{G(y-a) - G(y-a-b)\} - \sigma_C(K_1, K_2). \end{aligned}$$

Hence, $G$ is also $C$-$(K_1, K_2)$-convex. □

**Proof of Theorem 2.1.** For notational simplicity, we define

$$\Delta = f(x+z) - f(x) - \frac{z}{b}\{f(x-a) - f(x-a-b)\} + \sigma_C(K_1, K_2)$$

To show that $f(x)$ is $C$-$(K_1, K_2)$-convex, it suffices to show that $\Delta \geq 0$. We do so by considering the nine different cases for the pair of values $f(x+z)$ and $f(x-a-b)$. In doing so, we will replace $f(x)$ and $f(x-a)$ by suitable upper bounds.

Case I: $f(x+z) = G(x+z)$ and $f(x-a-b) = G(x-a-b)$

We can write $f(x) \leq G(x)$ and $f(x-a) \leq G(x-a)$. With these inequalities, the result then follows from the $C$-$(K_1, K_2)$-convexity of $G$.

Case II: $f(x+z) = G(x+z)$ and $f(x-a-b) = K_1 + G(x-a-b')$ for some $b - C \leq b' < b$. Then,

$$\Delta = G(x+z) - f(x) - \frac{z}{b}\{f(x-a) - K_1 - G(x-a-b')\} + \sigma_C(K_1, K_2)$$

We consider two subcases.

(a) Assume first that $f(x-a) \leq K_1 + G(x-a-b')$. Then, it suffices to show that $G(x+z) - f(x) + \sigma_C(K_1, K_2) \geq 0$. Using the definition of $\sigma_C(K_1, K_2)$, we can verify that for any $z_1 \in (0, C]$ and $z_2 \in (C, +\infty)$, $f(x) \leq G(x + z_1) + K_1$ and $f(x) \leq G(x + z_2) + K_2$. Hence, the result follows.

(b) Assume now that $f(x - a) \geq K_1 + G(x - a - b')$. From the definition of $f(x)$, we have $K_1 + G(x - a) \geq f(x - a)$, which implies that

$$G(x - a) - G(x - a - b') \geq f(x - a) - K_1 - G(x - a - b') > 0.$$

Using this relation and the inequalities $G(x) \geq f(x)$ and $0 < b' < b$, respectively, we can verify the first two inequalities in

$$\begin{aligned}
\Delta &\geq G(x + z) - G(x) - \frac{z}{b}\{G(x - a) - G(x - a - b')\} + \sigma_C(K_1, K_2) \\
&\geq G(x + z) - G(x) - \frac{z}{b'}\{G(x - a) - G(x - a - b')\} + \sigma_C(K_1, K_2) \\
&\geq 0.
\end{aligned}$$

The final inequality follows from the $C$-$(K_1, K_2)$-convexity of $G$.

<u>Case III:</u> $f(x + z) = G(x + z)$, and $f(x - a - b) = K_2 + G(x - a - b + \mu')$ for some $\mu' > C$. We consider two subcases.

(a) If $f(x - a) \leq K_2 + G(x - a - b + \mu')$, the result follows from similar line of arguments used in part (a) of Case II.

(b) If $f(x - a) > K_2 + G(x - a - b + \mu')$, then due to the condition $K_1 < K_2$, we have $x - a - b + \mu' < x - a$, i.e., $\mu' < b$. This can also be seen by assuming on the contrary that $x - a < x - a - b + \mu' \leq x - a + C$ or $x - a - b + \mu' > x - a + C$. In the former case, $f(x - a) \leq K_1 + G(x - a - b + \mu') \leq K_2 + G(x - a - b + \mu')$, which contradicts with the assumption in this subcase. In the latter case, it is clear that $f(x - a) \leq K_2 + G(x - a - b + \mu')$, which also contradicts with the assumption. Next, letting $b' = b - \mu'$, we have

$$\begin{aligned}\Delta &= G(x+z) - f(x) - \frac{z}{b}\{f(x-a) - K_2 - G(x-a-b+\mu')\} + \sigma_C(K_1, K_2)\\ &\geq G(x+z) - G(x) - \frac{z}{b}\{G(x-a) - K_2 - G(x-a-b+\mu')\} + \sigma_C(K_1, K_2)\\ &\geq G(x+z) - G(x) - \frac{z}{b'}\{G(x-a) - G(x-a-b')\} + \sigma_C(K_1, K_2) + \frac{z}{b'}K_2\\ &\geq 0, \end{aligned} \tag{2.10}$$

where the first inequality follows from $f(x) \leq G(x)$ and $f(x-a) \leq G(x-a)$, the second inequality follows from $0 < b' = b - \mu' < b$, and the last inequality follows from the $C$-$(K_1, K_2)$-convexity of $G$.

Case IV: $f(x+z) = K_1 + G(x+z+\mu)$, for some $\mu \in [0, C]$ and $f(x-a-b) = G(x-a-b)$. Using the inequalities $f(x) \leq K_1 + G(x+\mu)$ and $f(x-a) \leq G(x-a)$, we can write

$$\begin{aligned}\Delta &\geq K_1 + G(x+z+\mu) - K_1 - G(x+\mu) - \frac{z}{b}\{G(x-a) - G(x-a-b)\} + \sigma_C(K_1, K_2)\\ &= G(x'+z) - G(x') - \frac{z}{b}\{G(x'-a') - G(x'-a'-b)\} + \sigma_C(K_1, K_2)\\ &\geq 0, \end{aligned} \tag{2.11}$$

where $x' = x + \mu$, $a' = a + \mu$ and the last inequality holds due to the $C$-$(K_1, K_2)$-convexity of $G$.

Case V: $f(x+z) = K_1 + G(x+z+\mu)$ and $f(x-a-b) = K_1 + G(x-a-b+\mu')$ for some $\mu, \mu' \in [0, C]$. Let $\tilde{G}(x+z) = K_1 + G(x+z+\mu)$ and $\tilde{G}(x-a-b) = K_1 + G(x-a-b+\mu')$. Then

$$\Delta = \tilde{G}(x+z) - f(x) - \frac{z}{b}\{f(x-a) - \tilde{G}(x-a-b)\} + \sigma_C(K_1, K_2)$$

If $f(x-a) > K_1 + G(x-a-b+\mu')$, then the analysis is similar to part (b) of Case III. Otherwise, $f(x-a) \leq K_1 + G(x-a-b+\mu')$, and there are two subcases to consider.

(a) If $\mu + z \leq C$, then given the initial inventory level $x$, the cost function can reach the point $x+\mu+z$ with setup cost $K_1$. Thus, $\tilde{G}(x+z) = K_1 + G(x+z+\mu) \geq f(x)$. Furthermore, $f(x-a) \leq \tilde{G}(x-a-b)$, hence we have $\Delta \geq 0$.

(b) If $\mu + z > C$, then

$$\begin{aligned}\Delta &\geq K_1 + G(x+z+\mu) - (K_1 + G(x+C)) \\ &\quad - \frac{z'}{b}\{K_1 + G(x-a+\mu') - K_1 - G(x-a-b+\mu')\} + \sigma_C(K_1, K_2) \\ &= G(x'+z') - G(x') - \frac{z'}{b}\{G(x'-a') - G(x'-a'-b)\} + \sigma_C(K_1, K_2) \\ &\geq 0,\end{aligned}$$

where $x' = x + C$, $z' = z + \mu - C \geq 0$ and $a' = a + C = \mu' \geq 0$.

Case VI: $f(x+z) = K_1 + G(x+z+\mu)$ and $f(x-a-b) = K_2 + G(x-a-b+\mu')$ for some $\mu \in [0, C]$, and $\mu' \in [C, \infty]$. We consider two subcases.

(a) If $f(x-a) \leq K_2 + G(x-a-b+\mu')$, then it suffices to show that $K_1 + G(x+z+\mu) - f(x) + \sigma_C(K_1, K_2) \geq 0$. Using the inequalities $\sigma_C(K_1, K_2) \geq K_1$ and our assumption $2K_1 \geq K_2$, we can verify that $K_1 + G(x+z+\mu) - f(x) + \sigma_C(K_1, K_2) \geq 2K_1 + G(x+z+\mu) - f(x) \geq K_2 + G(x+z+\mu) - G(x+\mu) \geq 0$, hence the result follows.

(b) If $f(x-a) > K_2 + G(x-a-b+\mu')$, since $K_1 < K_2$, then $x-a-b+\mu' < x-a$, i.e., $\mu' < b$. This can also be verified by contradiction similar to the analysis in part (b) of Case III. Then,

$$\begin{aligned}
\Delta &= K_1 + G(x+z+\mu) - f(x) - \frac{z}{b}\{f(x-a) - K_2 - G(x-a-b+\mu')\} + \sigma_C(K_1, K_2) \\
&\geq G(x+z+\mu) - G(x+\mu) - \frac{z}{b}\{G(x-a) - K_2 - G(x-a-b+\mu')\} + \sigma_C(K_1, K_2) \\
&\geq G(\tilde{x}+z) - G(\tilde{x}) - \frac{z}{b'}\{G(\tilde{x}-a-\mu) - G(\tilde{x}-a-b'-\mu)\} + \sigma_C(K_1, K_2) + \frac{z}{b'}K_2 \\
&\geq 0,
\end{aligned}$$

where $\tilde{x} = x + \mu$, $b' = b - \mu'$, the second inequality follows from $G(x-a) \geq f(x-a) > K_2 + G(x-a-b+\mu')$ and $0 < b' \leq b$, and the last inequality follows from the $C$-$(K_1, K_2)$-convexity of $G$.

Case VII: $f(x+z) = K_2 + G(x+z+\mu)$ and $f(x-a-b) = G(x-a-b)$ for some $\mu \in [C, \infty]$. Letting $\tilde{G}(x+z) = K_2 + G(x+z+\mu)$ and $\tilde{G}(x-a-b) = G(x-a-b)$, this case can be analyzed similar to Case IV.

Case VIII: $f(x+z) = K_2 + G(x+z+\mu)$ and $f(x-a-b) = K_1 + G(x-a-b+\mu')$ for some $\mu \in [C, \infty]$ and $\mu' \in [0, C]$. Letting $\tilde{G}(x+z) = K_2 + G(x+z+\mu)$ and $\tilde{G}(x-a-b) = K_1 + G(x-a-b+\mu')$, this case can be analyzed similar to Case IV.

Case IX: $f(x+z) = K_2 + G(x+z+\mu)$ and $f(x-a-b) = K_2 + G(x-a-b+\mu')$ for some $\mu, \mu' \in [C, \infty]$. Letting $\tilde{G}(x+z) = K_2 + G(x+z+\mu)$ and $\tilde{G}(x-a-b) = K_2 + G(x-a-b+\mu')$, this case can be analyzed similar to Case III. □

**Proof of Lemma 2.2.** From (2.3), we can reexpress $f_t(x)$ as:

$$f_t(x) = \min\{G_t(x), K_1 + \inf_{x<y\leq x+C} G_t(y), K_2 + \inf_{y>x+C} G_t(y)\}.$$

Note also that,

$$\begin{aligned}
V_t(x) &= \min\{J_t(x), K_2 + \inf_{y>x} J_t(y)\} \\
&= \min\{G_t(x), K_1 + \inf_{x<y\leq x+C} G_t(y), K_2 + \inf_{y>x} J_t(y)\} \\
&= \min\{G_t(x), K_1 + \inf_{x<y\leq x+C} G_t(y), K_2 + \inf_{y>x+C} G_t(y)\}
\end{aligned}$$

where the last equality follows from the fact that $\inf_{y>x} J_t(y) = \inf_{y>x} G_t(y)$ and $K_2 \geq K_1$. Hence, we obtain identical expressions for $f_t(x)$ and $V_t(x)$ and the result follows. □

**Proof of Theorem 2.2.** We prove the result by induction. First, we assume that $f_{t-1}(x_t)$ is continuous. Then, $G_t(x_t)$ is clearly a continuous function, i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $|x_t - y_t| \leq \delta$, $|G_t(x_t) - G_t(y_t)| < \varepsilon$.

Without loss of generality, we assume $y_t > x_t$ and consider the following cases.

(a) $f_t(x_t) < f_t(y_t)$. There exists $\mu \geq 0$ such that $f_t(x_t) = K(\mu) + G_t(x_t + \mu)$. Then,

$$\begin{aligned}
f_t(y_t) - f_t(x_t) &= f_t(y_t) - K(\mu) - G_t(x_t + \mu) \\
&\leq K(\mu) + G_t(y_t + \mu) - K(\mu) - G_t(x_t + \mu) \\
&= G_t(y_t + \mu) - G_t(x_t + \mu) \\
&< \varepsilon,
\end{aligned}$$

where we define $K(\mu)$ such that $K(\mu) = 0$, if $\mu = 0$, $K(\mu) = K_1$, if $0 < \mu \leq C$, and $K(\mu) = K_2$, otherwise.

(b) $f_t(y_t) < f_t(x_t)$. There exists $\mu \geq 0$ such that $f_t(y_t) = K(\mu) + G_t(y_t + \mu)$.

Then,

$$\begin{aligned} f_t(x_t) - f_t(y_t) &= f_t(x_t) - K(\mu) - G_t(y_t + \mu) \\ &\leq K(\mu) + G_t(x_t + \mu) - K(\mu) - G_t(y_t + \mu) \\ &= G_t(x_t + \mu) - G_t(y_t + \mu) \\ &< \varepsilon. \end{aligned}$$

Therefore, we have $|f_t(y_t) < f_t(x_t)| < \varepsilon$. □

**Proof of Lemma 2.3.** First note from the definition of $C$-$(K_1, K_2)$-convexity that for all $x$, $z \geq 0$, $0 \leq a < \infty$, and $0 < b < \infty$, we have:

$$G_t(x + z) + \sigma_C(K_1, K_2) \geq G_t(x) + \frac{z}{b}\{G_t(x - a) - G_t(x - a - b)\}.$$

(i) For $s_1 < x < y < x + C$, we take $a = 0, b = x - s_1$, and let $z = y - x$. Then, $z \in [0, C)$ and the following inequality holds:

$$G_t(y) + K_1 \geq G_t(x) + \frac{z}{b}\{G_t(x) - G_t(s_1)\}.$$

By the definition of $s_1$ and Theorem 2.2, we have $G_t(s_1) = K_1 + G_t(S)$ and thus, by the above inequality,

$$(1 + \frac{z}{b})(G_t(y) + K_1) \geq (1 + \frac{z}{b})G_t(x),$$

which implies that $G_t(y) + K_1 \geq G_t(x)$.

(ii) For $s_2 < x < y$, we take $a = 0, b = x - s_2$, and let $z = y - x$. Then, the following inequality holds for any $z$ in $[0, C)$ or $[C, +\infty)$:

$$G_t(y) + \sigma_C(K_1, K_2) \geq G_t(x) + \frac{z}{b}\{G_t(x) - G_t(s_2)\}.$$

By the definition of $s_2$ and Theorem 2.2, we have $G_t(s_2) = K_2 + G_t(S)$. Note that $\sigma_C(K_1, K_2) \leq K_2$. Thus, by the above inequality, we have

$$(1 + \frac{z}{b})(G_t(y) + K_2) \geq (1 + \frac{z}{b})G_t(x),$$

which implies that $G_t(y) + K_2 \geq G_t(x)$.

(iii) The results follow directly from the definitions of $s_1$, $s_2$ and $S$.

(iv) We take $x + z = S$ and $x = s_2$. Then for any $s_2 \geq x_1 \geq x_2$,

$$\begin{aligned}
\frac{S - s_2}{x_1 - x_2}\{G_t(x_1) - G_t(x_2)\} &\leq G_t(S) + \sigma_C(K_1, K_2) - G_t(s_2) \\
&\leq G_t(S) + K_2 - G_t(s_2) \\
&= 0,
\end{aligned}$$

where the first inequality follows from the definition of $C$-$(K_1, K_2)$-convexity, the second from that $\sigma_C(K_1, K_2) \leq K_2$, and the equality from the definition of $s_2$.

(v) Referring to Equation (2.5), the result follows from the fact that $S$ is a minimizer of $G_t(x)$ □

**Proof of Lemma 2.4.** (i) We first prove that $s_2 - C \leq s \leq s_2$. By Equation (2.5) and Lemma 2.3(v), we have $J_t(s_2) \leq G_t(s_2) \leq K_2 + G_t(S) = K_2 + J_t(S)$, which implies that $s \leq s_2$. Note that for any $x < s_2 - C$, $J_t(x) \geq \min\{G_t(x), K_1 + \min_{x<y\leq x+C} G_t(y)\} > K_2 + J_t(S)$, where the second inequality holds due to the fact $y \leq x + C < s_2$. Hence, by the definition of $s$, we have $s_2 - C \leq s$. Now, we are ready to prove the result: $J_t(y) + K_2 \geq J_t(x)$ for any $y > x > s$ by considering three cases.

Case 1: $y > x > s_2$. Let $q(y)$ be the optimal solution of (2.5). Then, we have

$$\begin{aligned} J_t(x) &\leq K_1 1(q(y) > 0) + G_t(x + q(y)) \\ &\leq K_1 1(q(y) > 0) + G_t(y + q(y)) + K_2 \\ &= J_t(y) + K_2, \end{aligned}$$

where the second inequality follows from Lemma 2.3(ii) and $y + q(y) > x + q(y) > s_2$.

Case 2: $s_2 \geq y \geq x > s$. In this case, if $y + q(y) \geq s_2$, then $J_t(x) \leq K_1 + G_t(s_2) \leq K_1 + G_t(y + q(y)) + K_2 = J_t(y) + K_2$. If $y + q(y) < s_2$, $J_t(x) \leq K_1 + G_t(y + q(y)) \leq K_1 + J_t(y) \leq J_t(y) + K_2$, where the first inequality follows from the fact that $y + q(y) - x \leq s_2 - (s_2 - C) = C$.

Case 3: $y \geq s_2 \geq x > s$. In this case, we prove the result by contradiction and assume that $J_t(x) > J_t(y) + K_2$. Then, for any $x \geq \rho > s$,

$$\begin{aligned} J_t(x) &= \min_{0 \leq z \leq C} \{K_1 1(z > 0) + G_t(x + z)\}, \\ &\leq \min_{0 \leq z \leq C} \{K_1 1(z > 0) + G_t(\rho + z)\}, \\ &= J_t(\rho), \end{aligned}$$

where the inequality follows from Lemma 2.3 (iv) and $s_2 \geq x \geq \rho$.

Therefore, for any $x \geq \rho > s$, $J_t(\rho) > J_t(x) > J_t(y) + K_2 \geq J_t(S) + K_2$, which contradicts with the definition of $s$.

(ii) Let us define $s_3 = \inf\{x | G_t(x) \leq K_2 - K_1 + G_t(S)\}$. Clearly, $s_3 \leq S$. Then, for any $x \in [s_3 - C, s_3]$, we have

$$J_t(x) \leq K_1 + G_t(s_3) \leq K_2 + G_t(S) = K_2 + J_t(S),$$

where the first inequality follows from (2.5), the second from the definition of $s_3$, and the equality is due to part (v) of Lemma 2.3. By the definition of $s$, we have $s \le s_3 - C \le S - C$.

(iii) Here we show the proof for the result $s_1 \ge s$. Other inequalities hold directly from the definitions of the critical points and part (ii) of this lemma. Since $G_t(s_1) \le K_1 + G_t(S)$, we have $J(s_1) \le G_t(s_1) \le K_1 + G_t(S) \le K_2 + J_t(S)$ which implies by the definition of $s$ that $s_1 \ge s$.

(iv) The result follows from the definition of $s''$. □

**Proof of Lemma 2.6.** First note from the definition of strong $K_2$-convexity that for all $y$, $0 \le a < \infty$, $0 < b < \infty$, and $z \in [0, \infty)$,

$$G(y+z) + K \ge G(y) + \frac{z}{b}\{G(y-a) - G(y-a-b)\}.$$

(i) We take $y > s_2$, $b = y - s_2$, $a = 0$, and $z > 0$. Then, the following inequality holds:

$$G_t(y+z) + K_2 \ge G_t(y) + \frac{z}{b}\{G_t(y) - G_t(s_2)\}.$$

By the definition of $s_2$ and Theorem 2.2, we have $G_t(s_2) = K_2 + G_t(S) \le K_2 + G_t(y+z)$ and thus, by the above inequality,

$$(1+\frac{z}{b})(G_t(y+z) + K_2) \ge (1+\frac{z}{b})G_t(y),$$

which implies that $G_t(y+z) + K_1 \ge G_t(y)$.

(ii) Then for any $s_2 \ge x_1 \ge x_2$, take $y + z = S$, $y = x_1$, $a = 0$, and $y - b = x_2$

and we have

$$\frac{S - x_2}{x_1 - x_2}\{G_t(x_1) - G_t(x_2)\} \leq G_t(S) + K_2 - G_t(x_1)$$

$$\leq 0,$$

where the first inequality follows from the definition of strong $K_2$-convexity, and the second from the definition of $s_2$.

By the definition of $S$, it is clear that $G_t(S) = J_t(S)$.

(iii) The proof is similar to that of Lemma 2.4(i).

(iv) The results follow directly from the definition of $s$ and $s''$.

(v) The results follow directly from the definitions of the critical points. □

| $\lambda$ | $p$ | $C$ | $K_2$ | $Error$ | $\lambda$ | $p$ | $C$ | $K_2$ | $Error$ | $\lambda$ | $p$ | $C$ | $K_2$ | $Error$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 2 | 2 | 10 | 0 | 20 | 2 | 2 | 10 | 0 | 30 | 2 | 2 | 10 | 0 |
| 10 | 2 | 2 | 20 | 0 | 20 | 2 | 2 | 20 | 0 | 30 | 2 | 2 | 20 | 0 |
| 10 | 2 | 2 | 50 | 0 | 20 | 2 | 2 | 50 | 0 | 30 | 2 | 2 | 50 | 0 |
| 10 | 2 | 2 | 200 | 0.0281 | 20 | 2 | 2 | 200 | 0 | 30 | 2 | 2 | 200 | 0 |
| 10 | 2 | 10 | 10 | 0 | 20 | 2 | 10 | 10 | 0 | 30 | 2 | 10 | 10 | 0 |
| 10 | 2 | 10 | 20 | 0 | 20 | 2 | 10 | 20 | 0 | 30 | 2 | 10 | 20 | 0 |
| 10 | 2 | 10 | 50 | 0 | 20 | 2 | 10 | 50 | 0 | 30 | 2 | 10 | 50 | 0 |
| 10 | 2 | 10 | 200 | 0 | 20 | 2 | 10 | 200 | 0 | 30 | 2 | 10 | 200 | 0 |
| 10 | 2 | 20 | 10 | 0 | 20 | 2 | 20 | 10 | 0 | 30 | 2 | 20 | 10 | 0 |
| 10 | 2 | 20 | 20 | 0 | 20 | 2 | 20 | 20 | 0 | 30 | 2 | 20 | 20 | 0 |
| 10 | 2 | 20 | 50 | 0 | 20 | 2 | 20 | 50 | 0 | 30 | 2 | 20 | 50 | 0 |
| 10 | 2 | 20 | 200 | 0 | 20 | 2 | 20 | 200 | 0 | 30 | 2 | 20 | 200 | 0 |
| 10 | 8 | 2 | 10 | 0 | 20 | 8 | 2 | 10 | 0 | 30 | 8 | 2 | 10 | 0 |
| 10 | 8 | 2 | 20 | 0 | 20 | 8 | 2 | 20 | 0.0012 | 30 | 8 | 2 | 20 | 0.0059 |
| 10 | 8 | 2 | 50 | 0.0068 | 20 | 8 | 2 | 50 | 0.0007 | 30 | 8 | 2 | 50 | 0.0038 |
| 10 | 8 | 2 | 200 | 0.0075 | 20 | 8 | 2 | 200 | 0.0093 | 30 | 8 | 2 | 200 | 0.0087 |
| 10 | 8 | 10 | 10 | 0 | 20 | 8 | 10 | 10 | 0 | 30 | 8 | 10 | 10 | 0 |
| 10 | 8 | 10 | 20 | 0 | 20 | 8 | 10 | 20 | 0 | 30 | 8 | 10 | 20 | 0 |
| 10 | 8 | 10 | 50 | 0 | 20 | 8 | 10 | 50 | 0 | 30 | 8 | 10 | 50 | 0 |
| 10 | 8 | 10 | 200 | 0 | 20 | 8 | 10 | 200 | 0 | 30 | 8 | 10 | 200 | 0 |
| 10 | 8 | 20 | 10 | 0 | 20 | 8 | 20 | 10 | 0 | 30 | 8 | 20 | 10 | 0 |
| 10 | 8 | 20 | 20 | 0 | 20 | 8 | 20 | 20 | 0 | 30 | 8 | 20 | 20 | 0 |
| 10 | 8 | 20 | 50 | 0 | 20 | 8 | 20 | 50 | 0 | 30 | 8 | 20 | 50 | 0 |
| 10 | 8 | 20 | 200 | 0 | 20 | 8 | 20 | 200 | 0 | 30 | 8 | 20 | 200 | 0 |

Table 2.1: Numerical Results: Poisson Distribution

| $q$ | $p$ | $C$ | $K_2$ | *Error* | $q$ | $p$ | $C$ | $K_2$ | *Error* | $q$ | $p$ | $C$ | $K_2$ | *Error* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.25 | 2 | 2 | 10 | 0 | 0.5 | 2 | 2 | 10 | 0 | 0.75 | 2 | 2 | 10 | 0 |
| 0.25 | 2 | 2 | 20 | 0 | 0.5 | 2 | 2 | 20 | 0 | 0.75 | 2 | 2 | 20 | 0 |
| 0.25 | 2 | 2 | 50 | 0 | 0.5 | 2 | 2 | 50 | 0 | 0.75 | 2 | 2 | 50 | 0 |
| 0.25 | 2 | 2 | 200 | 0 | 0.5 | 2 | 2 | 200 | 0.0004 | 0.75 | 2 | 2 | 200 | 0.0016 |
| 0.25 | 2 | 10 | 10 | 0 | 0.5 | 2 | 10 | 10 | 0 | 0.75 | 2 | 10 | 10 | 0 |
| 0.25 | 2 | 10 | 20 | 0 | 0.5 | 2 | 10 | 20 | 0 | 0.75 | 2 | 10 | 20 | 0 |
| 0.25 | 2 | 10 | 50 | 0 | 0.5 | 2 | 10 | 50 | 0 | 0.75 | 2 | 10 | 50 | 0 |
| 0.25 | 2 | 10 | 200 | 0 | 0.5 | 2 | 10 | 200 | 0 | 0.75 | 2 | 10 | 200 | 0 |
| 0.25 | 2 | 20 | 10 | 0 | 0.5 | 2 | 20 | 10 | 0 | 0.75 | 2 | 20 | 10 | 0 |
| 0.25 | 2 | 20 | 20 | 0 | 0.5 | 2 | 20 | 20 | 0 | 0.75 | 2 | 20 | 20 | 0 |
| 0.25 | 2 | 20 | 50 | 0 | 0.5 | 2 | 20 | 50 | 0 | 0.75 | 2 | 20 | 50 | 0 |
| 0.25 | 2 | 20 | 200 | 0 | 0.5 | 2 | 20 | 200 | 0 | 0.75 | 2 | 20 | 200 | 0 |
| 0.25 | 8 | 2 | 10 | 0 | 0.5 | 8 | 2 | 10 | 0 | 0.75 | 8 | 2 | 10 | 0 |
| 0.25 | 8 | 2 | 20 | 0 | 0.5 | 8 | 2 | 20 | 0 | 0.75 | 8 | 2 | 20 | 0 |
| 0.25 | 8 | 2 | 50 | 0 | 0.5 | 8 | 2 | 50 | 0 | 0.75 | 8 | 2 | 50 | 0 |
| 0.25 | 8 | 2 | 200 | 0.0090 | 0.5 | 8 | 2 | 200 | 0.0089 | 0.75 | 8 | 2 | 200 | 0.0080 |
| 0.25 | 8 | 10 | 10 | 0 | 0.5 | 8 | 10 | 10 | 0 | 0.75 | 8 | 10 | 10 | 0 |
| 0.25 | 8 | 10 | 20 | 0 | 0.5 | 8 | 10 | 20 | 0 | 0.75 | 8 | 10 | 20 | 0 |
| 0.25 | 8 | 10 | 50 | 0 | 0.5 | 8 | 10 | 50 | 0 | 0.75 | 8 | 10 | 50 | 0.0094 |
| 0.25 | 8 | 10 | 200 | 0 | 0.5 | 8 | 10 | 200 | 0 | 0.75 | 8 | 10 | 200 | 0 |
| 0.25 | 8 | 20 | 10 | 0 | 0.5 | 8 | 20 | 10 | 0 | 0.75 | 8 | 20 | 10 | 0 |
| 0.25 | 8 | 20 | 20 | 0 | 0.5 | 8 | 20 | 20 | 0 | 0.75 | 8 | 20 | 20 | 0 |
| 0.25 | 8 | 20 | 50 | 0 | 0.5 | 8 | 20 | 50 | 0 | 0.75 | 8 | 20 | 50 | 0 |
| 0.25 | 8 | 20 | 200 | 0 | 0.5 | 8 | 20 | 200 | 0 | 0.75 | 8 | 20 | 200 | 0 |

Table 2.2: Numerical Results: Binomial Distribution

# Chapter 3

# Inventory System with Warm/Cold States

## 3.1 Introduction and Literature Review

We consider a periodic-review production/inventory control problem in which the current period's setup cost varies depending on the quantity produced/ordered in the previous period. Specifically, we analyze the following scenario. At the beginning of each period, the production/ordering process is considered to be in one of two states: "warm" or "cold". The warm state corresponds to the case where an amount over a threshold value $R$ has been produced/ordered in the previous period, and the cold state corresponds to the case otherwise. If the process begins with a warm state, then there is no fixed setup cost for any amount of production/ordering in the current period; otherwise, a fixed cost $K$ is incurred. More formally, the cost for producing/ordering a quantity $z$ is given by

$$c(z,q) = K\delta(z)\delta(R-q) + cz, z \geq 0, \tag{3.1}$$

where $q$ is the quantity produced/ordered in the previous period, $c$ is the unit production/purchase cost, and $\delta(x) = 1$ if $x > 0$ and 0 otherwise. We refer to (3.1) as the order-cost function and use the words "production" and "ordering" interchangeably in the rest of this chapter.

The main motivation for the described cost structure comes from applications in some manufacturing environments. For example, Robinson and Sahin (2001) mention that manufacturers in the food, chemical, and pharmaceutical industries may sometimes deliberately increase the output on a given day to save the next day's equipment cleanup and inspection fees. Toy and Berk (2006) give examples from other industries such as glass, steel, and ceramic, which often include processes with highly expensive setup costs. In these settings, continued production in one period allows the manufacturer to avoid setup activities such as shutdowns/startups and cooling/reheating in the next period, and it can be cost effective even when the excess output has to be recycled and reproduced in some cases.

Although manufacturing settings provide intuitive reasons for such cost structures, it is not difficult to think of similar schemes that could be applicable in purchasing environments. For example, ordering larger than a threshold quantity in a given period could render a large portion of the next period's customs inspection/clearance activities unnecessary. In vendor-buyer settings, the specific cost structure can be captured through some incentives. For instance, the vendor may offer to eliminate some setup activities or reimburse the associated costs in the following period if the buyer's current order quantity exceeds a specified threshold quantity. Similar incentives can also be designed with rebates given in a qualifying period to be exercised in the next period as also mentioned in Toy and Berk (2006).

Systems with the concepts of warm/cold states have been investigated by Berk et al. (2008) and Toy and Berk (2006) in the context of dynamic lot-sizing. In both studies, finite capacity is assumed, and the process can be kept warm by producing larger than a threshold value and by incurring a warming cost over unused capacity. If the next period starts with a warm state, then only a minor setup is needed; otherwise a major setup is performed at a higher cost. In the first article, the authors analyze the case where both setup costs can take positive values, and provide theoretical results on the optimal production plan by assuming deterministic demand and no shortages. The second article provides an extension with possible lost sales but assumes zero minor setup cost, similar to our model, and maintains the deterministic demand structure. A relevant work that considers lost sales in lot-sizing is by Aksen et al. (2003). In their analysis, fixed setup cost parameters are time-variant but they are independent of the production quantity, hence state-independent, and demand is assumed known. Agra and Constantino (1999) consider state-dependent setup costs, which can be minor or major (including also a start up cost) depending on whether a setup was done in the previous period or not. No threshold restriction or capacity limit is imposed and demand is assumed to be known with the unsatisfied part backlogged. Our model differs from these studies mainly in that we analyze decisions in a stochastic demand setting. Although this allows us to address the problem in more realistic scenarios, it also complicates the analysis and necessitates some simplifying assumptions such as infinite capacity and zero fixed setup cost in the warm state. We also assume that no additional cost is incurred for ordering beyond the threshold quantity except the variable unit-purchase cost, i.e., warming cost is zero. These assumptions appear to hold more easily in procurement settings than production settings, in which capacity limits can be restrictive and warming costs could be nonzero. For example, a process can be kept warm onto the next period by utilizing costly undertime and overtime options, which facilitate producing larger quantities or slowing down the process for continued production. Such deliberate production rate adjustments have been studied by Gallego (1993) and Moon et al. (1991) in the context of economic lot scheduling.

Stochastic inventory problems have been extensively studied in the literature, dating back to the work by Scarf (1960). Earlier results for the classical periodic-review problem assume convex ordering costs and rely on the $K$-convexity of the optimal cost functions to prove that the optimal policy is of $(s, S)$ type: an order size up to the target level $S$ is placed if the inventory level is less than the reorder level $s$; nothing is ordered otherwise. Some extensions of this work include generalizations of the cost-order function. For example, under the assumption that the demand densities are Pólya or Uniform, Porteus (1971) and (1972) show that a generalized $(s, S)$ policy (with multiple reorder and target levels) is optimal when the ordering cost is concave. To prove this result, a new class of functions is defined with the property of quasi-$K$-convexity, which is also employed in our analysis. We make the same assumptions on the demand distribution as in Porteus (1971) and (1972) and partially characterize the optimal policy, which in part carries some properties of the $(s, S)$ structure. In a more recent study, Chao and Zipkin (2008) work on an extension where the fixed component of the ordering cost is incurred only if the order quantity exceeds a threshold value, while the variable component is assumed linear. The authors find that this structure complicates the optimal policy, but a partial characterization can be derived using the $K$-convexity of the optimal cost functions. Our order-cost function is different in that the fixed cost depends on the order quantity in the *previous* period rather than the current period and is incurred when the order quantity *does not* exceed the threshold value. Furthermore, our results require the $K$-convexity property be generalized to quasi-$K$-convexity, which helps us derive a partial characterization of the optimal policy similar to their work.

Ordering costs of general forms appear in problems with multi-sourcing options. Fox et al. (2006) analyze a case where items can be procured from two suppliers with different fixed and variable costs. The resulting order-cost function is piecewise linear and concave, and a reduced form of generalized $(s, S)$ policy is shown to be optimal under the assumption of log-concave demand. Hua et al. (2009) analyze a similar problem but introduce limits on order sizes. This leads to an order-cost function that is neither concave nor convex. In their analysis, the optimal cost functions are shown to be quasi-convex and the structure of the optimal policy is fully characterized under certain conditions. We derive our structural results relying on the quasi-$K$-convexity of the cost functions, and we use the more restrictive quasi-convexity property to develop a heuristic policy. One technical difficulty in applying the concept of quasi-$K$-convexity to the stochastic inventory problems is that some preservation properties need to be proved under complex operations. Recently, Chen et al. (2010) show that quasi-$K$-concavity can be preserved under an optimization operation, which is used to characterize the optimal policy for a joint inventory control and pricing problem with a concave ordering cost and backlogged excess demand. In our analysis, we find a new preservation property of quasi-$K$-convexity under a minimization operation and build upon this property to derive our results for the inventory control problem with state-dependent ordering cost and lost sales.

In particular, we study a production/inventory control problem with an order-cost function as defined in (3.1). We analyze the problem in a single-item periodic-review setting with stochastic demand and lost sales, and the objective is to find the optimal policy that minimizes the total cost over a finite horizon. The classical lost-sales inventory models with and without setup cost are special cases of our model by assuming $R = +\infty$ and $R = 0$, respectively. We prove some structural results on the optimal cost functions using the concept of quasi-$K$-convexity, and use these to provide a partial characterization of the optimal policy. Based on this analysis, we propose some heuristic policies that are easy to implement and highly effective. This chapter contributes to the literature by analyzing a warm/cold process-state-dependent fixed setup cost under stochastic demand, which can address relations between production/ordering decisions in two successive periods.

## 3.2 The Model and Preliminaries

We consider a firm's ordering decisions over a finite time horizon of $T$ periods. At the beginning of period $t$, the firm has an initial stock level $x_t$ and needs to decide on the order-up-to inventory level $y_t$ ($\geq x_t$), or equivalently the order quantity $q_t = y_t - x_t$. Then, the random demand $D_t$ is realized and satisfied, while the excess part is considered as lost sales. We assume that demands in different periods are independent random variables.

Three types of costs are assessed: ordering, inventory holding, and penalty cost of unsatisfied demand or shortage. The cost of ordering includes a variable component with the unit-purchase cost given by $c$ and a fixed component $K$, which is incurred only when the process is in a cold state as shown by (3.1). Without loss of generality, we set the unit-purchase cost to zero. The one-period holding/penalty cost, $h(x)$, is incurred at the end of the period as an inventory holding cost when $x > 0$ and a shortage cost otherwise. Consistent with most of the relevant literature, we assume that $h(x)$ is convex, minimized at $x = 0$, and that $\lim_{|x|\to\infty} h(x) = \infty$. The expected one-period holding and shortage cost is conveniently written in terms of the inventory level after ordering in period $t$ and is denoted with $L(y_t)$

Since the total cost in a period depends on the process state as well as the current inventory level, we use a two-dimensional vector $(x_t, q_{t-1})$ to represent the system state Here, $q_{t-1}$ is the order quantity in period $t-1$ and identifies whether the process is in a warm or cold state at the start of period $t$ Let $0 < \alpha \leq 1$ denote the discount factor Then, given states $x_t$ and $q_{t-1}$, we define $\pi_t(x_t, q_{t-1})$ to be the optimal total expected discounted cost from period $t$ through the end of the planning horizon $T$ The firm's objective is to determine the ordering policy that minimizes the total expected cost over the planning horizon Then, the firm's optimization problem is given by the following dynamic programming formulation

$$\pi_t(x_t, q_{t-1}) = \min_{y_t \geq x_t} \left\{ K\delta(q_t)\delta(R - q_{t-1}) + L(y_t) + \alpha E[\pi_{t+1}((y_t - D_t)^+, q_t)] \right\} \tag{3 2}$$

We assume the boundary conditions $\pi_{T+1}(x_{T+1}, q_T) = 0$ The two-dimensional system state is fairly difficult to deal with, thus we re-express $\pi_t(x_t, q_{t-1})$, $t = 1, 2, \quad, T$ as

$$\pi_t(x_t, q_{t-1}) = \begin{cases} f_{t,1}(x_t) & q_{t-1} \in [0, R), \\ f_{t,2}(x_t) & q_{t-1} \geq R, \end{cases} \tag{3 3}$$

where the functions $f_{t,i}(x_t)$, $i = 1, 2$, are defined as the optimal cost functions corresponding to a cold and warm process state, respectively We again assume $f_{T+1,i}(x_{T+1}) = 0$, for $i = 1, 2$ For a given period with cold state, the firm may order larger or smaller than $R$, and incur a fixed cost for any positive quantity

In the former case, the system state is transformed from cold to warm in the next period, while it stays cold in the latter. When a period starts in a warm state, similar decisions are made but no fixed cost is incurred. Maintaining the one-dimensional representation, the recursive expressions for $f_{t,1}(x_t)$ and $f_{t,2}(x_t)$ are then given by

$$f_{t,1}(x_t) = \min\left\{ \min_{y_t \geq x_t + R} \{K + g_{t,2}(y_t)\}, \min_{x_t \leq y_t < x_t + R} \{K\delta(y_t - x_t) + g_{t,1}(y_t)\} \right\}, \tag{3.4}$$

and

$$f_{t,2}(x_t) = \min\left\{ \min_{y_t \geq x_t + R} g_{t,2}(y_t), \min_{x_t \leq y_t < x_t + R} g_{t,1}(y_t) \right\}, \tag{3.5}$$

and $g_{t,i}(y_t)$, $i = 1, 2$ is defined as $g_{t,i}(y_t) = L(y_t) + \alpha E\tilde{f}_{t+1,i}(y_t - D_t)$, where

$$\tilde{f}_{t+1,i}(z) = \begin{cases} f_{t+1,i}(0) & z \leq 0, \\ f_{t+1,i}(z) & z > 0. \end{cases} \tag{3.6}$$

We start our analysis with the definition of a quasi-$K$-convex function, which is due to Porteus (1972).

**Definition 3.1.** *A function $f : R \to R$ is quasi-$K$-convex if any $x \leq y$ and $0 \leq \theta \leq 1$ imply that*

$$f(\theta x + (1 - \theta)y) \leq \max\left[f(x), K + f(y)\right].$$

A notion closely related with quasi-K-convexity is defined next. First, note that a function $f$ is said to be non-$K$-decreasing on $X$ if for all $x, y \in X$ satisfying $x \leq y$ and $K \geq 0$, $f(x) \leq K + f(y)$. Then, a function $f : R \to R$ is (nontrivially) quasi-$K$-convex if there exists a scalar $a$ such that $f$ is quasi-$K$-convex with changeover $a$, i.e., $f$ is decreasing on $(-\infty, a]$ and non-$K$-decreasing on $[a, +\infty)$ (Porteus (1972)).

Note the following properties of quasi-$K$-convex functions (Porteus (2002)):

**Lemma 3.1.** (a) *If $f$ is convex or $K$-convex, then $f$ is quasi-$K$-convex.*

(b) *If $f$ is quasi-$K$-convex and $\gamma > 0$, then $\gamma f$ is quasi-$\gamma K$-convex.*

(c) *The sum of a quasi-$K_1$-convex function with changeover at $b$ and a quasi-$K_2$-convex function with changeover at $b$ is quasi-$(K_1 + K_2)$-convex with changeover at $b$.*

(d) *If $f$ is quasi-$K$-convex with changeover at $a$ and $X$ is a positive Pólya or uniformly distributed random variable, then there exists $b \geq a$ such that $g(x) := Ef(x - X)$ is quasi-$K$-convex with changeover at $b$.*

The stated properties are useful since they show certain conditions under which quasi-$K$-convexity is preserved with some common operators. Part (d) suggests that quasi-$K$-convexity is preserved under expectation for Pólya or Uniform distributions. However, no result exists to indicate that the property would extend for other distributions. This leads to the following assumption in our analysis.

**Assumption 3.1.** *Demand in any period follows a positive Pólya or Uniform distribution.*

As stated in Porteus (2002), Pólya distributions have useful smoothing properties and they consist solely of translations and convolutions of exponentials, reflected exponentials, and normals. We also note that quasi-$K$-convexity is not necessarily preserved for the minimum of two functions that are quasi-$K$-convex. In Lemma 3.2, we prove that the property holds under certain conditions, obtaining the first preliminary result towards investigating the structural properties of the optimal cost functions.

**Lemma 3.2.** *Suppose $J_1(x)$ and $J_2(x)$ are both quasi-$K_1$-convex functions such that for any $x \in \mathcal{R}$, $J_2(x) \geq J_1(x)$. Then, for any $0 \leq K_2 \leq K_1$,*

$$J(x) = \min \left\{ \min_{y \geq x+R} \{K_2 + J_1(y)\}, \min_{x+R > y \geq x} \{K_2 \delta(y-x) + J_2(y)\} \right\}$$

*is also quasi-$K_1$-convex and non-$K_1$-decreasing in $\mathbb{R}$.*

**Proof.** We let $S_1$ and $S_2$ denote the minimizers of $J_1(x)$ and $J_2(x)$, respectively. (If they are not unique, we take the largest one.) Then, we have either $K_2 + J_1(S_1) \leq J_2(S_2)$ or $K_2 + J_1(S_1) > J_2(S_2)$.

Case I: Suppose $K_2 + J_1(S_1) \leq J_2(S_2)$. We prove the quasi-$K_1$-convexity of $J(x)$ by showing that the function is quasi-$K_1$-convex with a changeover at $S_1 - R$. First, we analyze the interval $[-\infty, S_1 - R]$. For any $x \leq S_1 - R$, $\min_{y \geq x+R}\{K_2 + J_1(y)\} = K_2 + J_1(S_1)$ and hence, we need to compare $K_2 + J_1(S_1)$ and $\min_{x+R>y\geq x}\{K_2\delta(y-x) + J_2(y)\}$ to determine $J(x)$. Note that $\min_{x+R>y\geq x}\{K_2\delta(y-x) + J_2(y)\} > J_2(S_2)$, because $S_2$ is a minimizer of $J_2(x)$. Further, since $K_2 + J_1(S_1) \leq J_2(S_2)$, it follows that $J(x) = K_2 + J_1(S_1)$, which is a constant.

Next, we analyze the interval $[S_1 - R, \infty]$. To establish the result that $J(x)$ is non-$K_1$-decreasing on $[S_1 - R, \infty]$, we show that any $x_1, x_2 \in [S_1 - R, \infty]$

satisfying $x_1 \leq x_2$ imply $J(x_1) \leq J(x_2)+K_1$. At the point $x = x_2$, we have either $J(x_2) = \min_{y \geq x_2+R}\{K_2 + J_1(y)\}$ or $J(x_2) = \min_{x_2+R>y\geq x_2}\{K_2\delta(y - x_2) + J_2(y)\}$.

In the first subcase, $J(x_1) \leq \min_{y\geq x_1+R}\{K_2 + J_1(y)\} \leq \min_{y\geq x_2+R}\{K_2 + J_1(y)\} = J(x_2) \leq J(x_2)+K_1$, where the first inequality follows from the definition of $J(x)$, the second inequality is due to fewer choices since $x_1 \leq x_2$, and the third equality holds since $J(x_2) = \min_{y\geq x_2+R}\{K_2 + J_1(y)\}$ in this case.

In the second subcase, $J(x_2) = \min_{x_2+R>y\geq x_2}\{K_2\delta(y-x_2)+J_2(y)\}$. It suffices to consider the case where $x_2$ is the minimizer of $J_2(y)$ on $[x_2, x_2 + R)$, since this will imply the result for any other $x^*$ minimizing $J_2(y)$ in this region. Because $x_2$ is the minimizer of $J_2(y)$ on $[x_2, x_2 + R)$, we have $J(x_2) = J_2(x_2)$. Given $x_1 \leq x_2$, we have either $x_1 + R \leq x_2$ or $x_1 + R > x_2$. In the former case, we have $J(x_1) \leq \min_{y\geq x_1+R}\{K_2 + J_1(y)\} \leq K_2 + J_1(x_2) \leq K_1 + J_2(x_2) = K_1 + J(x_2)$, where the first and second inequalities follow from the definition of $J(x)$ and minimization, respectively, the third inequality follows from the assumptions stated in the lemma, and the last equality holds since $J(x_2) = J_2(x_2)$ in this case. Using similar arguments for the case with $x_1+R > x_2$, we can verify that $J(x_1) \leq \min_{x_1+R>y\geq x_1}\{K_2\delta(y - x_1) + J_2(y)\} \leq K_2 + J_2(x_2) \leq K_1 + J_2(x_2) = K_1 + J(x_2)$. Hence, we can conclude that $J(x)$ is quasi-$K_1$-convex under Case I.

Case II: Suppose $K_2 + J_1(S_1) > J_2(S_2)$. Then, it follows that the minimizer of $J(x)$ is $S_2$, i.e., $J(S_2) = J_2(S_2)$. In this case, we prove the quasi-$K_1$-convexity of $J(x)$ by showing that there exists a changeover point $b \leq S_2$ such that the function is decreasing (constant) in $(-\infty, b]$ and non-$K_1$-decreasing in $[b, +\infty]$.

We first show that $J(x)$ is non-$K_1$-decreasing in $[S_2, +\infty)$. Let $x_1, x_2 \in [S_2, +\infty)$ such that $x_1 \leq x_2$. At the point $x = x_2$, we have either $J(x_2) = \min_{x_2+R>y\geq x_2}\{K_2\delta(y-x_2)+J_2(y)\}$ or $J(x_2) = \min_{y\geq x_2+R}\{K_2+J_1(y)\}$. In the first case, we consider the case where $x_2$ is the minimizer of $J_2(y)$ on $[x_2, x_2+R)$ (similar to the analysis in Case I), and find $J(x_2) = J_2(x_2)$. Then, we have $J(x_1) \leq J_2(x_1) \leq J_2(x_2) + K_1 = J(x_2) + K_1$, where the second inequality follows from the quasi-$K_1$-convexity of $J_2(x)$. In the second case, $J(x_2) = \min_{y\geq x_2+R}\{K_2 + J_1(y)\}$. Then, by using the definition of $J(x)$ and the relationship $x_1 \leq x_2$, we can verify that $J(x_1) \leq \min_{y\geq x_1+R}\{K_2 + J_1(y)\} \leq \min_{y\geq x_2+R}\{K_2 + J_1(y)\} = J(x_2) \leq J(x_2) + K_1$, and the non-$K_1$-decreasing property holds.

Next, we show the existence of the changeover point $b$. Let $a$ be the smallest number such that $K_2 + J_1(S_1) > J_2(a)$. The existence of a number $a < S_2$ is assured by $K_2 + J_1(S_1) > J_2(S_2)$. Thus, for any $x \leq a$, $K_2 + J_1(S_1) \leq J_2(x)$.

We consider two cases: $a \leq S_1 - R$ and $a > S_1 - R$.

If $a \leq S_1 - R$, then we set $b = a$. Notice that for any $x \leq b(= a)$, $K_2 + J_1(S_1) \leq J_2(x)$, hence it follows that $J(x) = K_2 + J_1(S_1)$, that is, $J(x)$ is a constant in $(-\infty, b]$. Moreover, for any $x_1, x_2 \in [b, S_2]$ and $x_1 \leq x_2$, it follows from the definition of $J(x)$ and the quasi-$K_1$-convexity of $J_2(x)$ that $J(x_1) \leq J_2(x_1) \leq K_1 + J_2(S_2) = K_1 + J(S_2) \leq K_1 + J(x_2)$, where the last inequality holds, because $S_2$ is the minimizer of $J(x)$. Hence, $J(x)$ is non-$K_1$-decreasing on $[b, S_2]$.

If $a > S_1 - R$, then we set $b = S_1 - R$. Then, for any $x \leq b = S_1 - R < a$, $K_2 + J_1(S_1) \leq J_2(x)$. Hence, it follows that $J(x) = K_2 + J_1(S_1)$, i.e., a constant in

$(-\infty, b]$. For any $x \in [b, S_2]$, if $x \leq S_2 - R$, then $J(x) \leq J_1(S_2) + K_2 \leq J_2(S_2) + K_1 = K_1 + J(S_2)$; otherwise, $J(x) \leq K_2 + J_2(S_2) \leq J_2(S_2) + K_1 = K_1 + J(S_2)$. So, $J(x)$ is non-$K_1$-decreasing in $[b, S_2]$.

Given that $J(x)$ is non-$K_1$-decreasing on $[b, S_2]$ and $[S_2, +\infty]$, we can easily show that it is non-$K_1$-decreasing on $[b, +\infty]$: for any $z_1 \in [b, S_2]$ and $z_2 \in [S_2, +\infty)$, we have $J(z_2) + K_1 \geq J(S_2) + K_1 \geq J(z_1)$, which follows since $S_2$ is a minimizer and $J(x)$ is non-$K_1$-decreasing on $[b, S_2]$.

Thus far, we have proved the quasi-$K_1$-convexity. Notice that in all cases, $J(x)$ is a constant on the left of the changeover point and non-$K_1$-decreasing on the right. Hence, we can conclude that $J(x)$ is non-$K_1$-decreasing in $\mathbb{R}$. □

In our next preliminary result, we compare the total cost functions depending on the starting state status of the process. In part (i), we show that, with the same inventory level, the total cost with a cold state is always larger than that with a warm state, but it is less than the total cost with a warm state plus the setup cost. Part (ii) shows that with a higher inventory level, the sum of the total cost with a warm state and setup cost is always larger than the total cost with a cold state. Part (iii) implies that both $f_{t,1}(x)$ and $f_{t,1}(x)$ are non-$K$-decreasing in $\mathbb{R}$. It is clear that these properties will also hold for $\tilde{f}_{t,i}(x)$.

**Lemma 3.3.** *For any $t = 1, 2, ..., T$, $f_{t,1}$ and $f_{t,2}$ satisfy the following relationships:*

(i) *for any $x \in \mathbb{R}$, $f_{t,2}(x) + K \geq f_{t,1}(x) \geq f_{t,2}(x)$,*

(ii) *for any $x_2 > x_1$, $f_{t,2}(x_2) + K \geq f_{t,1}(x_1)$,*

(iii) *for any* $x_2 > x_1$, $f_{t,i}(x_2) + K \geq f_{t,i}(x_1)$.

**Proof.** We prove parts (i) and (ii) by considering two separate cases: (a) $f_{t,2}(x) = \min_{y \geq x+R}\{g_{t,2}(y)\}$, and (b) $f_{t,2}(x) = \min_{x+R>y\geq x}\{g_{t,1}(y)\}$.

(i) For case (a), we have $f_{t,1}(x) \leq \min_{y\geq x+R}\{K + g_{t,2}(y)\} = K + f_{t,2}(x)$. For case (b), $f_{t,1}(x) \leq \min_{x+R>y\geq x}\{K\delta(y-x)+g_{t,1}(y)\} \leq K+\min_{x+R>y\geq x}\{g_{t,1}(y)\} = K + f_{t,2}(x)$. The second inequality in the lemma can be easily verified. From the results of part (i), we can also conclude that $g_{t,1}(x) \geq g_{t,2}(x)$.

(ii) For case (a), $f_{t,1}(x_1) \leq \min_{y\geq x_1+R}\{K+g_{t,2}(y)\} \leq \min_{y\geq x_2+R}\{K+g_{t,2}(y)\} = K + f_{t,2}(x_2)$, where the second inequality follows from that $x_2 > x_1$. For case (b), let $f_{t,2}(x_2) = g_{t,1}(y(x_2))$ where $x_2 + R > y(x_2) \geq x_2$. If $y(x_2) \leq x_1 + R$, then $f_{t,1}(x_1) \leq K + g_{t,1}(y(x_2)) = K + f_{t,2}(x_2)$. If $y(x_2) > x_1 + R$, then $\min_{y\geq x_1+R}\{K + g_{t,1}(y)\} \leq \min_{x_2+R>y\geq x_2}\{K + g_{t,1}(y)\} = K + f_{t,2}(x_2)$. From part (i), we have $g_{t,1}(x) \geq g_{t,2}(x)$ which implies that $\min_{y\geq x_1+R}\{K + g_{t,2}(y)\} \leq \min_{y\geq x_1+R}\{K + g_{t,1}(y)\} \leq K + f_{t,2}(x_2)$. Thus, we have $f_{t,1}(x_1) \leq \min_{y\geq x_1+R}\{K + g_{t,2}(y)\} \leq K + f_{t,2}(x_2)$.

(iii) Again, consider two cases: $f_{t,1}(x_2) = \min_{y>x_2+R}\{K+g_{t,2}(y)\}$ and $f_{t,1}(x_2) = \min_{x_2+R\geq y\geq x_2}\{K\delta(y - x) + g_{t,1}(y)\}$. In the first case, $f_{t,1}(x_1) \leq \min_{y>x_1+R}\{K + g_{t,2}(y)\} \leq \min_{y>x_2+R}\{K + g_{t,2}(y)\} = f_{t,1}(x_2)$. In the second case, we have

$$
\begin{aligned}
f_{t,1}(x_1) &\leq K + \min\left\{\min_{y>x_1+R}\{g_{t,2}(y)\}, \min_{x_1+R\geq y\geq x_1}\{g_{t,1}(y)\}\right\} \\
&\leq K + \min_{x_2+R\geq y\geq x_2}\{K\delta(y-x) + g_{t,1}(y)\} \\
&\leq K + f_{t,1}(x_2),
\end{aligned}
$$

where the second inequality follows due to the fact that $g_{t,2}(x) \leq g_{t,1}(x)$ from part (i) and since $x_1 \leq x_2$. Using similar arguments, we can also prove that $f_{t,2}(x_1) \leq K + f_{t,2}(x_2)$. □

We are now ready to prove our main results on the structural property of the optimal cost functions.

**Theorem 3.1.** *For any $t = 1, 2, ..., T$, $f_{t,i}$ and $g_{t,i}$, $i = 1, 2$, are both quasi-$K$-convex.*

**Proof.** We prove the results by induction. For $t = T+1$, since $f_{T+1,i}(x) = 0$, it is clear that $f_{T+1,i}(x)$, $i = 1, 2$, are quasi-$K$-convex. Assume that it is true for period $t+1$, i.e., $f_{t+1,1}(x)$ and $f_{t+1,2}(x)$ are quasi-$K$-convex. Lemma 3.3(iii) implies that $f_{t+1,i}(x)$, $i = 1, 2$, are non-$K$-decreasing in $\mathbb{R}$. Hence, by the definition of $\tilde{f}_{t+1,i}(x)$, $\tilde{f}_{t+1,i}(x)$ is nonincreasing on $(-\infty, 0]$ and non-$K$-decreasing on $[0, +\infty)$, that is, it is quasi-$K$-convex with a changeover point $a_i = 0, i = 1, 2$. As $h(x)$ is a convex function with a minimizer $x^* = 0$, $h(x)$ is also quasi-$K$-convex with a changeover point $a_h = 0$ and $K = 0$. From parts (c) and (d) of Lemma 3.1, $g_{t,1}$ and $g_{t,2}$ are quasi-$K$-convex. Moreover, by Lemma 3.3(i), we have $g_{t,1} \geq g_{t,2}$. Therefore, by Lemma 3.2, both $f_{t,1}$ and $f_{t,2}$ are quasi-$K$-convex which completes the proof. □

Before we proceed, we make some remarks about the sensitivity of the preceding results to the model assumptions. We assume in our analysis that the demands follow Pólya or Uniform distribution. Under this assumption, we show that the optimal cost functions possess the special structure of quasi-$K_1$-convexity, which is preserved under the minimization function shown in Lemma 3.2. Recall a related concept called $K_1$-convexity, which suggests that a real-valued function $f$ is $K_1$-convex for $K_1 \geq 0$, if for any $x \leq y$, and $\lambda \in [0,1]$, $f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) + \lambda K_1$ Simchi-Levi et al. (2004). When $K_1 = 0$, we have convexity and Lemma 3.2 trivially holds. If one could show for any $K_1$ that $K_1$-convexity is preserved in Lemma 3.2, then our assumption on the demand distribution would not be needed. Unfortunately, by constructing a counterexample (Example 1) we can verify that $K_1$-convexity is not preserved, and correspondingly, we are not able to relax the restriction on the demand distribution.

The second remark is about the unsatisfied demand, which is assumed to be lost in our analysis. Alternatively, if backorders are allowed the quasi-$K$-convexity of the optimal cost functions may no longer hold. This can be seen from Theorem 3.1, where $f_{t+1,1}(x)$ and $f_{t+1,2}(x)$ may not be non-increasing in $(-\infty, 0]$. This implies that the changeover points of $f_{t+1,1}(x)$ and $f_{t+1,2}(x)$ may be below zero and hence are not necessarily consistent with that of $h(x)$. Consequently, $g_{t+1,i}$ may not be quasi-$K$-convex.

**Example 3.1.** In Lemma 3.2, let $K_2 = K_1$. Suppose $J_1(x)$ and $J_2(x)$ have the same minimizer denoted by $S$. Then, by assumption $J_1(S) < J_2(S)$. Let $s_1 = \min\{x | J_1(x) = J_1(S) + K_1\}$ and $s_2 = \min\{x | J_2(x) = J_2(S) + K_1\}$. First, pick up $R$ such that $S - R < \min\{s_1, s_2\}$. Then for any $x \leq S - R$, $J(x) = K_1 + J_1(S)$. Next, suppose $J_1(x)$ and $J_2(x)$ are both strictly increasing in $(S, +\infty)$ and $K_1 + J_1(S+R) > J_2(S)$, then $J(S) = J_2(S)$. Finally, note that by construction $S - s_2 < R$, and thus $J(s_2) = \min\{J_1(s_2 + R) + K_1, K_1 + J_2(S)\}$. Since we assumed that $J_1(x)$ is strictly increasing in $(S, +\infty)$, we have $J_1(s_2+R) \geq J_2(S)$. Consequently, $J(s_2) = K_1 + J_2(S)$. Now we can pick up three points $x = S - R$, $s_2$, and $S$ with the values $J(x) = K_1 + J_1(S)$, $K_1 + J_2(S)$ and $J_2(S)$, respectively. Recall that a function $f(x)$ is $K_1$-convex if, for any $x_1 \leq x_2$ and $\lambda \in (0, 1]$, $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)(f(x_2) + K)$. For $K$-convexity to hold in this example, we need to show that for any $\lambda \in (0, 1]$, $J(s_2) \leq \lambda J(S - R) + (1 - \lambda)(J(S) + K_1)$, which implies $J_2(S) \leq \lambda J_1(S) + (1-\lambda)J_2(S)$. However, since we have $J_1(S) < J_2(S)$, we have $J_2(S) > \lambda J_1(S) + (1-\lambda)J_2(S)$, which contradicts with $K_1$-convexity.

Having established the preliminary theoretical results, we are now ready to explore the structure of the optimal policy. As also mentioned by Chao and Zipkin (2008), when the ordering cost function takes complex forms such as in our model, the optimal policy becomes complicated and even impossible to fully characterize, nevertheless, partial characterizations can be developed, which we tackle next.

## 3.3 Partial Characterization of the Optimal Policy

In this section, we investigate the optimal ordering/production policy for the warm/cold process. First, we define some critical points that will be useful to show several properties comparing $g_{t,i}(x)$, $i = 1, 2$.

**Definition 3.2.** *Given non-negative constants $R$, $K$, functions $g_{t,i}(x)$, $i = 1, 2$,* *and* $L(x)$, *we define:*

$$S_L = \arg\inf L(x)$$

$$S_\imath = \arg\inf g_{t,\imath}(x),\ i = 1, 2$$

$$s_\imath = \inf\{x | g_{t,\imath}(x) < K + g_{t,\imath}(S_\imath)\},\ i = 1, 2$$

$$s_3 = \inf\{x | g_{t,1}(x) < K + g_{t,2}(S_2)\}$$

$$s_4 = \inf\{x \geq S_2 - R | g_{t,1}(x) < K + \min_{y \geq x+R} g_{t,2}(y)\}$$

$$s_5 = \inf\{s_1 \geq x \geq S_1 - R | g_{t,1}(S_1) < \min_{y \geq x+R} g_{t,2}(y)\}$$

For notational convenience, we suppress the time index for the critical points. Note that the set used to define $s_5$ can be empty; in such cases $\inf \emptyset$ refers to $s_1$. Obviously, there is a correspondence between $s_\imath$ and the reorder level, and between $S_\imath$ and the order-up-to level. For an example interpretation of the critical points, notice that $s_5$ corresponds to the first point between $S_1 - R$ and $s_1$ such that ordering larger than $R$ fails to dominate ordering less than $R$ up-to $S_1$. The following lemma reveals several results using the critical points and the relationships among them.

**Lemma 3.4.** *For any* $t = 1, 2, ..., T$, *and* $\imath = 1, 2$, *we have*

(i) *For any* $x \leq s_\imath$, $g_{t,\imath}(x) \geq K + g_{t,\imath}(S_\imath)$, *and for any* $y > x > s_\imath$, $g_{t,\imath}(x) \leq K + g_{t,\imath}(y)$,

(ii) *For any* $x \leq s_3$, $g_{t,1}(x) \geq K + g_{t,2}(S_2)$; *for any* $S_2 - R \leq x \leq s_4$, $g_{t,1}(x) \geq K + \min_{y>x+R} g_{t,2}(y)$; *for any* $S_1 - R \leq x < s_5$, $g_{t,1}(S_1) \geq \min_{y>x+R} g_{t,2}(y)$, *and for any* $x \geq s_5$, $g_{t,1}(S_1) < \min_{y>x+R} g_{t,2}(y)$,

(iii) *For any* $x_2 > x_1 \geq S_L$, $g_{t,1}(x_1) \leq K + g_{t,2}(x_2)$,

(iv) $s_1 \leq s_3 < \min\{S_1, S_2, S_L\}$, $s_4 \leq s_3$, *and if* $S_2 - R \in [S_1 - R, s_1]$, *then* $s_5 > S_2 - R$.

**Proof.** Parts (i) and (ii) follow from the definitions of quasi-$K$-convexity and the critical points.

(iii) Since $L(x)$ is increasing on $x \geq S_L$, $L(x_2) \geq L(x_1)$. Then, using Lemma 3.3(ii), we have $g_{t,1}(x_1) = L(x_1) + \alpha E\tilde{f}_{t+1,1}(x_1 - D_t) \leq L(x_2) + \alpha E\tilde{f}_{t+1,2}(x_2 - D_t) + K \leq g_{t,2}(x_2) + K$.

(iv) By Lemma 3.3(i), $g_{t,2}(x) \leq g_{t,1}(x)$ which implies $g_{t,2}(S_2) \leq g_{t,1}(S_1)$, because $S_2$ is a minimizer of $g_{t,2}(x)$. Thus, we have $g_{t,1}(s_3) < K + g_{t,2}(S_2) \leq K + g_{t,1}(S_1)$, which by definition, implies $s_1 \leq s_3$.

By Lemma 3.3(i), $g_{t,1}(S_i) < K + g_{t,2}(S_2)$. Thus, by the definition, $S_i > s_3$ for $i = 1, 2$.

If $S_L \geq S_2$, clearly $s_3 < S_2 \leq S_L$. Otherwise by Lemma 3 (i), $g_{t,1}(S_L) < K + g_{t,2}(S_2)$, which implies $s_3 < S_L$.

By definition, $g_{t,1}(s_3) < K + g_{t,2}(S_2) \leq K + \min_{y \geq x+R} g_{t,2}(y)$, which implies $s_4 \leq s_3$.

As $g_{t,1}(S_1) < g_{t,2}(S_2)$ and $S_2 - R \in [S_1 - R, s_1]$, we have $s_5 > S_2 - R$. □

With Lemma 3.4, we have established some order relationships among the critical points. Unfortunately, we can not describe all such relationships, e.g., between $s_3$ and $S_2 - R$, $s_1$ and $S_i - R$, and $S_1$ and $S_2$. In fact, we can construct examples by setting the proper value for $R$ and show that different relationships can exist. This is especially easy to see considering that the critical points $s_1$, $s_3$, and $S_i$, $i = 1, 2$, in period $T$, are independent of $R$. Thus, we need to investigate different cases in terms of the relative values of these critical points. Clearly, the optimal policy depends on the current state and hence, will be discussed separately under two different scenarios: Cold (Scenario I) and Warm (Scenario II). Before we show our main result in Theorem 3.2, we further define $s_6 = \inf\{S_1 \geq x \geq S_1 - R | g_{t,1}(S_1) < \min_{y \geq x+R} g_{t,2}(y)\}$. Again, $\inf \emptyset$ indicates $S_1$. Let $x$ be the initial inventory level at the beginning of period $t$.

**Theorem 3.2.** *Beginning with a cold (warm) state, the optimal policy is partially characterized as Scenario I (Scenario II) and shown in Table 3.1 (3.2).*

**Proof.** In the following, we first show the result for the most complex case in the cold state (A4), for which the optimal policy is illustrated in Figure 3.1. (The proofs for the other cases follow a similar logic.) Next, we prove the result for the the warm state.

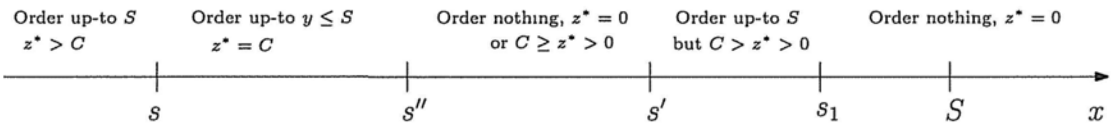


Figure 3.1: Optimal Policy for Case (A4)

In case (A4), we first show that for any $x \in (-\infty, S_2 - R]$, it is optimal to order up to $S_2$ and for any $x \in (S_L, +\infty)$, it is optimal to order nothing.

By Lemma 3.4 (ii), for any $x \leq S_2 - R < s_3$, $g_{t,2}(S_2) + K \leq g_{t,1}(x)$ which implies that ordering up-to $S_2$ dominates ordering nothing. Since for any $x$,

Table 3.1: Optimal Policy for Scenario I (Cold State)

| Case | Inventory Level, $x$ | Optimal Decision |
|---|---|---|
| $s_3 \leq S_2 - R$ | $x \leq s_3$ | order up-to $S_2$ |
| (A1) | $s_3 < x \leq S_L$ | order larger than $R$ or nothing |
| | $x > S_L$ | order nothing |
| $s_3 > S_2 - R$ | $x \leq S_2 - R$ | order up-to $S_2$ |
| and | $S_2 - R < x \leq s_4$ | order larger than $R$ |
| $s_1 \leq S_2 - R$ | $s_4 < x \leq S_L$ | order larger than $R$ or nothing |
| (A2) | $x > S_L$ | order nothing |
| $s_3 > S_2 - R$ | $x \leq S_2 - R$ | order up-to $S_2$ |
| and | $S_2 - R < x \leq s_5$ | order larger than $R$ |
| $s_1 > S_2 - R \geq S_1 - R$ | $s_5 < x \leq s_1$ | order up-to $S_1$ |
| (A3) | $s_1 < x \leq S_L$ | order larger than $R$ or nothing |
| | $x > S_L$ | order nothing |
| $s_3 > S_2 - R$ | $x \leq S_2 - R$ | order up-to $S_2$ |
| and | $S_2 - R < x \leq s_5$ | order larger than $R$ |
| $s_1 \geq S_1 - R > S_2 - R$ | $s_5 < x \leq s_1$ | order up-to $S_1$ |
| (A4) | $s_1 < x \leq s_7$[a] | order larger than $R$ |
| | $s_7 < x \leq S_L$ | order larger than $R$ or nothing |
| | $x > S_L$ | order nothing |
| $s_3 > S_2 - R$ | $x \leq S_2 - R$ | order up-to $S_2$ |
| and | $S_2 - R < x \leq s_7$ | order larger than $R$ |
| $S_1 - R > s_1 > S_2 - R$ | $s_7 < x \leq S_L$ | order larger than $R$ or nothing |
| (A5) | $x > S_L$ | order nothing |

[a] $s_7 = \max\{s_1, s_4\}$

$g_{t,2}(S_2) \leq g_{t,1}(x)$, then $g_{t,2}(S_2) + K \leq g_{t,1}(x) + K$ which implies that ordering up-to $S_2$ dominates ordering less than $R$, when $x \leq S_2 - R$. Hence, for any $x \in (-\infty, S_2 - R]$, ordering up-to $S_2$ is a feasible action and dominates any other action which proves its optimality.

Also by Lemma 3.4 (i) and (iii), for any $y > x \geq S_L > s_1$, $g_{t,1}(x) \leq K + g_{t,1}(y)$ and $g_{t,1}(x) \leq K + g_{t,2}(y)$ which is sufficient to conclude that it is optimal to order nothing.

We have proved the results in the regions, $(-\infty, S_2 - R]$ and $[S_L, +\infty)$. Next, other regions are discussed.

For any $x \in (S_2 - R, S_1 - R]$, since $x < s_1$, $\min_{x \leq y \leq x+R}\{K\delta(y-x) + g_{t,1}(y)\} \geq K + g_{t,1}(S_1)$. Then, $\min_{y > x+R}\{K + g_{t,2}(x)\} \leq K + g_{t,2}(S_1) \leq K + g_{t,1}(S_1) \leq \min_{x \leq y \leq x+R}\{K\delta(y-x) + g_{t,1}(y)\}$ where the first inequality follows from $x + R < S_1$ and the second from $g_{t,2}(x) \leq g_{t,1}(x)$. Thus, ordering larger than $R$ is optimal.

For any $x \in [S_1 - R, s_5]$, by Lemma 3.4 (ii), ordering larger than $R$ dominates ordering up to $S_1$, which dominates ordering nothing due to the relation that $S_1 - R \leq x < s_1$. Hence, it is optimal to order larger than $R$.

Table 3.2: Optimal Policy for Scenario II (Warm State)

| | |
|---|---|
| $x \leq S_2 - R$ | order up-to $S_2$ |
| $S_2 - R < x \leq s_6$ | order larger than $R$ |
| $s_6 < x \leq S_1$ | order up-to $S_1$ |
| $x > S_1$ | order less or larger than $R$ |

Consequently, for any $x \in (S_2 - R, s_5]$, it is optimal to order larger than $R$.

For any $x \in (s_5, s_1]$, it is clear that ordering up-to $S_1$ dominates ordering nothing. By Lemma 3.4 (ii), ordering up-to $S_1$ dominates ordering larger than $R$ since $x > s_5$. Hence, it is optimal to order up-to $S_1$.

For any $x \in [s_1, \max\{s_1, s_4\}]$, it is not optimal to order up-to $S_1$, as $x \geq s_1$. By Lemma 3.4 (ii), it is optimal to order larger than $R$.

For any $x \in [\max\{s_1, s_4\}, S_L]$, as before, it is not optimal to order up-to $S_1$. However, it may be optimal to order larger than $R$ or order nothing.

Next, we analyze the warm scenario by considering two different cases: $S_1 > S_2$ or $S_1 \leq S_2$. First, it is clearly optimal to order up to $S_2$, when the inventory level is below $S_2 - R$. Second, if $S_1 \leq S_2$, i.e. $S_1 - R \leq S_2$, by the definition of $s_6$, it is optimal to order larger than $R$ when the inventory is between $S_2 - R$ and $s_6$ and order up-to $S_1$ when the inventory is between $s_6$ and $S_1$. Now, we consider the case where $S_1 > S_2$. When the inventory is between $S_2 - R$ and $S_1 - R$, ordering larger than $R$ up-to $S_1$ always dominates both ordering less than $R$ and ordering nothing, as $g_2(x) \leq g_1(x)$ and $S_1$ is the minimizer of $g_1(x)$. Then, since it is optimal to order larger than $R$ when the inventory is between $S_1 - R$ and $s_6$, it is also optimal to order larger than $R$ when the inventory level is between $S_2 - R$ and $s_6$.

For any $x \in [s_6, S_1]$, by the definition of $s_6$, ordering up-to $S_1$ dominates other actions. □

Although the policy is discussed in many different cases depending on the critical points, only one of these cases will occur in any given period and the values of the critical points can be calculated efficiently with a dynamic programming procedure. For an illustration of the optimal policy, consider the following example. Suppose that the holding and shortage costs are linear, i.e., $L(y) = hE[(y-D)^+] + pE[(D-y)^+]$ where $h = 4$ and $p = 10$, and that the demand follows a uniform distribution within the interval [0, 20]. We set the fixed cost $K = 40$ and the threshold value $R = 20.$, and consider period $T = 30$. The optimal policy requires comparison of the functions $g_{t,i}$, $i = 1, 2$, which we plot in Figure 3.2. We can compute the following critical points: $S_1 = 19$, $S_2 = 17$, $s_1 = 10$, $s_2 = 9$, $s_3 = 14$, $s_4 = 8$, $s_5 = 5$. If the current process is warm, the optimal policy is readily available from Scenario II. If the current process is cold, then we first determine the specific case under Scenario I by comparing the values of the critical points. In this example, we can see that $s_3 > S_2 - R$ and $s_1 \geq S_1 - R \geq S_2 - R$, hence case (A4) occurs. Finally, we implement the corresponding policy as a function of the initial inventory level.

The optimal policy can be greatly simplified under some special cases. For example, when $R = 0^+$, our setting reduces to the scenario studied in Agra and Constantino (1999), and it follows from Lemma 3.4 (iv) that the only feasible case in the cold state is (A1). Note that the total optimal discounted cost function $\pi_t(x_t, q_{t-1})$ depends on the threshold value $R$. In Theorem 3.3, we show that $\pi_t(x_t, q_{t-1})$ is decreasing in $R$.

**Theorem 3.3.** *The total discounted cost function $\pi_t(x_t, q_{t-1})$ is decreasing in $R$.*

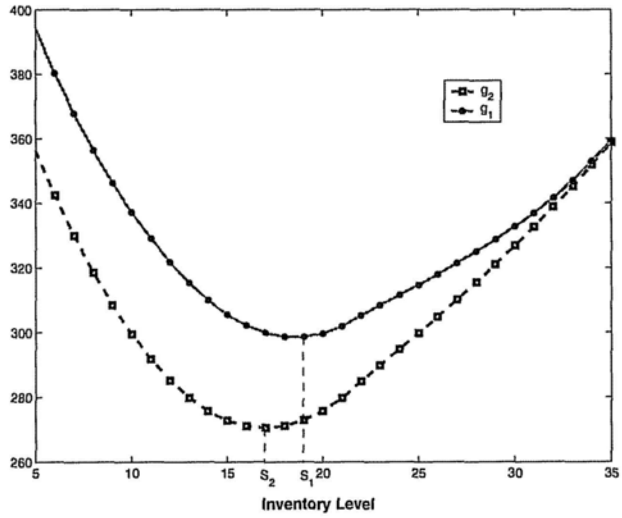


Figure 3.2: Illustration of functions $g_{t,i}$, $i = 1, 2$.

**Proof.** We prove the result by induction. It is clearly true for period $T + 1$, because $\pi_{T+1}(x_{T+1}, q_{T-1}) = 0$. Assume the result holds for period $t + 1$, i.e., $\pi_{t+1}(x_{t+1}, q_t)$ is decreasing in $R$. Referring Equation (3.2), both the first term and the last term in the bracket are decreasing in $R$ and other terms are independent of $R$. Thus, $\pi_t(x_t, q_{t-1})$ is also decreasing in $R$, which completes the proof. □

Thus far, we have determined the optimal ordering decision in many of the possible intervals of $x$, however, we have not been able to give a full characterization in others. For example, when $S_2 - R < x \leq s_5$ in case (A4), the optimal decision indicates that the order size should be larger than $R$ without specifying the exact value, which must be found by solving $\min_{y>x+R}\{K + g_{t,2}(x)\}$. Note that $g_{t,2}(x)$ may or may not be decreasing on this subinterval, thus it may not be optimal to order exactly $R$ units. The theoretical results are useful to understand the structure of the optimal policy. From a practical point of view, however, the optimal policy might be difficult to implement since multiple sets of different actions are required depending on the specific relations between the critical points which are computed in each period. Correspondingly, we concentrate our efforts on developing heuristic policies.

## 3.4 Heuristic Policies and Numerical Results

In this section, we propose three heuristic policies and compare their performances with a numerical study. We aim to develop simple and effective heuristic methods that firms can use in practice. We also generate insights on the performance of the heuristic policies as a function of the model parameters $K$ and $R$, and the initial inventory level.

The first heuristic is based on the structure of the optimal policy that has been characterized in Section 3.3. Our main goal in developing the optimality-based heuristic, which we refer as OB policy, is to obtain close-to-optimal performance and could trade off with implementation challenges. Correspondingly, it is more appropriate for firms with significantly high inventory costs. The next heuristic is a myopic (MO) policy, which is widely used in practice due to its easy implementation. Under such a policy, the firm only focuses on the single-period problem at each period, hence ignores the impact of its decisions on the future costs. The last heuristic that we study is based on a structural assumption on the optimal cost functions and simplifies the ordering decisions to base stock policies. We refer to this heuristic as the generalized base stock (GBS) policy. In the following, we first explain each heuristic policy in more detail, and next present the results from our numerical investigations.

### 3.4.1 Heuristic Policies

**Optimality-Based Policy:** The OB policy is inspired by the partial characterization of the optimal policy given in Table 3.1 and 3.2. More specifically, we construct the heuristic by approximating the decision "order larger than $R$ or nothing" with "order nothing", and the decision "order larger than $R$ " with "order exactly $R$." When the decision is not clear as in the last interval of the warm state, we choose to "order nothing" treating the critical point $S_1$ as the order-up-to point. Ordering decisions are further simplified by merging some intervals with different optimal decisions into one with a common ordering decision. Table 3.3 and 3.4 summarizes our proposed heuristic method.

While further simplifications of the ordering decisions are possible and might be desirable from a practical perspective, this will most likely degrade the heuristic performance. On the other hand, since the main motivation for the OB heuristic is to obtain close-to-optimal performance and the implementation issues are assumed secondary, we believe that our proposed method is effective under such managerial concerns.

**Myopic Policy:** The MO policy can be adopted to our model as follows. First, let us define $s_L = \inf\{x | L(x) < K + L(S_L)\}$, where $L(x)$ is the expected one-period holding and shortage cost when the current period has the initial inventory

Table 3.3: Heuristic Policy for Scenario I (Cold State)

| Case | Inventory Level | Ordering Decision |
|---|---|---|
| $s_3 \leq S_2 - R$ | $x \leq s_3$ | order up-to $S_2$ |
| | $x > s_3$ | order nothing |
| $s_3 > S_2 - R$ | $x \leq S_2 - R$ | order up-to $S_2$ |
| and | $S_2 - R < x \leq s_4$[b] | order exactly $R$ |
| $s_1 \leq S_2 - R$ | $s_4 < x$ | order nothing |
| $s_3 > S_2 - R$ | $x \leq S_2 - R$ | order up-to $S_2$ |
| and | $S_2 - R < x \leq s_5$[b] | order exactly $R$ |
| $s_1 > S_2 - R \geq S_1 - R$ | $s_5 < x \leq s_1$ | order up-to $S_1$ |
| | $s_1 < x$ | order nothing |
| $s_3 > S_2 - R$ | $x \leq S_2 - R$ | order up-to $S_2$ |
| and | $S_2 - R < x \leq s_5$ | order exactly $R$ |
| $s_1 \geq S_1 - R > S_2 - R$ | $s_5 < x \leq s_1$ | order up-to $S_1$ |
| | $s_1 < x \leq s_7$ | order exactly $R$ |
| | $s_7 < x$ | order nothing |
| $s_3 > S_2 - R$ | $x \leq S_2 - R$ | order up-to $S_2$ |
| and | $S_2 - R < x \leq s_7$ | order exactly $R$ |
| $S_1 - R > s_1 > S_2 - R$ | $s_7 < x$ | order nothing |

[b] For easier computation we redefine $s_4 = \inf\{x \geq S_2 - R | g_{t,1}(x) < K + g_{t,2}(x + R + 1)\}$, and $s_5 = \inf\{s_1 \geq x \geq S_1 - R | g_{t,1}(S_1) < g_{t,2}(x + R + 1)\}$.

$x$. In each period, depending on the system state and the current inventory level, the heuristic implements the following decisions. When the system is in a cold state, if $x \leq s_L$, it is optimal to order up to $S_L$; otherwise nothing should be ordered. When the system is in a warm state, then it is optimal to order up to $S_L$ whenever the inventory level falls below $S_L$. While the main advantage of the MO policy is the easiness in implementation, it also provides us benchmark results to compare our OB policy. As we present in the numerical section, the OB policy outperforms the MO policy with significant cost differences, which implies that exploring the structure of the optimal policy pays off in generating cost savings.

Generalized Base Stock Policy: We develop the GBS policy by making some assumptions on the structural properties of the cost functions $g_{t,i}$, $i = 1, 2$, which have been shown to be quasi-$K$-convex. We first remind a concept called quasi-convexity: A function $f$ is quasi-convex if for any $x$ and $y$, and $0 \leq \lambda \leq 1$, $f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$. Obviously, any function that is quasi-convex is also quasi-$K$-convex, but the opposite is not necessarily true. In the GBS

Table 3.4: Heuristic Policy for Scenario II (Warm State)

| | |
|---|---|
| $x \leq S_2 - R$ | order up-to $S_2$ |
| $S_2 - R < x \leq s_6$ | order exactly $R$ |
| $s_6 < x \leq S_1$ | order up-to $S_1$ |
| $x > S_1$ | order nothing |

heuristic, we assume that $g_{t,i}$, $i = 1, 2$ are quasi-convex. With this assumption, we are assured that the cost functions cannot be increasing and then decreasing, which can be used to verify the following results:

$$\min_{y_t \geq x_t + R} g_{t,2}(y_t) = \begin{cases} g_{t,2}(S_2) & x_t + R \leq S_2, \\ g_{t,2}(x_t + R) & otherwise. \end{cases} \tag{3.7}$$

and

$$\min_{x_t \leq y_t \leq x_t + R} g_{t,1}(y_t) = \begin{cases} g_{t,1}(x_t + R) & x_t + R < S_1, \\ g_{t,1}(S_1) & x_t \leq S_1 \leq x_t + R, \\ g_{t,1}(x_t) & S_1 < x_t. \end{cases} \tag{3.8}$$

As previously defined, $S_1$ and $S_2$ are the minimum points of $g_{t,i}$, $i = 1, 2$, respectively. The GBS policy is implemented as follows. In each period, we first determine the optimal policy under the hypothesis that the process state in the following period must be warm or cold. Next, we compare the costs corresponding to each of these cases and select the one with the lower cost. The optimal policy is then implemented accordingly. We can derive the optimal order decisions by using Equations (3.7) and (3.8). If the process is assumed to be kept warm in the next period, the current period's order quantity must be larger than $R$, and the relevant optimization problems from Equations (3.4) and (3.5) are $\min_{y_t \geq x_t + R} K + g_{t,2}(y_t)$ and $\min_{y_t \geq x_t + R} g_{t,2}(y_t)$. Then, it follows from (3.7) that a base stock policy with safety stock $S_2$ is optimal irrelevant of the process state in the current period. If the next period's process state is hypothesized to be cold, then the optimal ordering decision depends on the current process state. In particular, if the process is in a warm state then we solve the following optimization problem: $\min_{x_t \le y_t \le x_t+R} g_{t,1}(y_t)$. From (3.8), we can verify that the optimal policy suggests to order $R$, whenever the inventory level is less than $S_1 - R$, to order up-to $S_1$, whenever the inventory level is between $S_1 - R$ and $S_1$, and to order nothing, whenever the inventory level is larger than $S_1$. If the process is currently in a cold state, then we solve $\min_{x_t \le y_t < x_t+R}\{K\delta(y_t - x_t) + g_{t,1}(y_t)\}$, by comparing the cost functions $g_{t,1}(x_t)$ and $K + \min_{x_t \le y_t \le x_t+R} g_{t,1}(y_t)$. The optimization problem in the latter case can be solved by using the expressions in (3.8).

From an implementation perspective, the GBS policy is more advantageous compared to the OB policy since it requires computing only two critical points $S_1$ and $S_2$ in each period. Among all heuristic policies, the MO policy is the easiest to implement since minimization is done for a single period. Next, we evaluate the performance of the heuristic policies.

### 3.4.2 Numerical Results

This section presents results from our numerical study. In our computational experiments, the one-period expected cost function is assumed linear in the ending inventory, i.e., $L(y) = hE[(y-D)^+] + pE[(D-y)^+]$ where $h$ and $p$ are the per-unit per-period holding and shortage cost, respectively. We compare the performance of the heuristics with the optimal policy using the maximal relative error over all possible states, similar to Chao and Zipkin (2008). More specifically, if we let $f^H_{t,i}$ and $f^O_{t,i}$ be the discounted cost of heuristic and optimal policy, respectively, then the relative percentage error of the heuristic policy is defined as

$$E_i^H = \max_{x\in[0,500]} \left[(f_{t,i}^H - f_{t,i}^O)/f_{t,i}^O\right] \times 100\%,$$

where $i = C, W$ denotes the cold and warm initial states and the superscript $H = \{OB, MO, GBS\}$ represents the type of the heuristic policy. Clearly, a zero relative error implies the heuristic to be optimal. As we only partially characterize the optimal policy, the minimal cost $f_{t,i}^O$ is obtained through complete enumerations. We select the base parameters as follows: $T = 10$, $R = 5, 20, 50$, and $K = 10, 20, 50, 200$. The discount factor is set as $\alpha = 0.9$. We let the unit holding cost $h = 4$, and the unit shortage cost $p = 2$ or $p = 8$, reflecting two situations with less/more costly shortages than inventory leftovers.

We first report the performance of the OB policy. Although our results on the characterization of the optimal policy require the assumption that the demands follow a Pólya or Uniform distribution, we test Poisson distribution as well as the Uniform distribution. For the Uniform demand, we draw the demand values from the interval $[0, B]$ where $B = 20, 50$ and 100. In the case of Poisson demand distribution, the mean demand $\lambda$ is tested at values 10, 20, and 30. The total number of instances tested for each demand distribution is 72. We observed that the OB policy is in fact optimal in almost all cases except in two instances with Poisson demands where it is very close to the optimal. Overall, the maximum relative error among all cases tested is less than 0.6%, hence the heuristic policy performs quite satisfactorily. (The complete numerical results are available in the Supplementary Appendix.) The results also reveal that the cases with smaller $p$

and larger $R$ or $K$ values lead to suboptimal results. For these instances, we also observe that $E_C$ is larger than $E_W$, implying that the heuristic performance can be affected from the initial process state. We performed additional experiments with the symmetric Triangle distribution by setting its lower limit to zero and upper limit to 10, 30, 50 and 100. Similarly, the results indicate that the heuristic policy is optimal for almost all of the cases.

The main strength of the OB heuristic is that it performs quite close to the optimal policy even when the demand assumption is relaxed. Given its near-optimal performance, however, we are not able to derive further insights on the heuristic performance as a function of model parameters. Overall, we recommend the OB policy as an effective method to solve the periodic-review inventory problem for the warm/cold process in which the ordering cost function takes a special form and the inventory costs are substantial.

Next, we evaluate the performance of the MO and GBS policies and compare them with the OB policy. From the base parameters, we select $U[0, 20]$ as the demand distribution. Table 3.5 summarizes the maximum relative errors of the MO and GBS policies for different values of $p$, $K$ and $R$. The OB policy is optimal in all of these instances. We can see that the MO policy performs significantly worse than the OB and GBS policies, with the maximum relative errors as high as 97.90%. This implies that it can be very costly to ignore the future cost functions when making decisions for a given period. We also observe that the relative errors of the MO policy are increasing in $K$ and decreasing in $R$. Unlike the OB policy, the MO policy ignores the future cost and in particular, the process state in the following period. Thus, the main cost difference between the MO and OB policies comes from the future cost, which is a function of the process state. As $K$ increases, the future cost difference between the warm and cold states also becomes larger, and hence the cost of the MO policy increases. On the other hand, as $R$ increases, the system is kept in the cold state more frequently, which makes the MO policy perform closer to the OB policy. The results also show that the relative errors of the MO policy are generally higher with an initial warm state than a cold state. Under the MO policy, the critical points are stationary over periods and the order-up-to level $S_L$ is the same for the warm and cold states. In the given instances, we can numerically find $S_2 \leq S_1 \leq S_L$ implying that the minimum point in the warm state is more further away from $S_L$, hence possibly resulting in a worse situation for the warm state. In the case of the GBS heuristic, we can see that the relative errors are ranging from 1.83% to 8.01% where the best performance appears for large $R$, $K$ and small $p$ values, however we do not observe any monotonicity properties. Different from the MO policy, the relative errors under the GBS policy are not sensitive to the process state.

To derive further insights on the performance of the MO and GBS heuristics, we investigate the impact of the starting inventory level in period 1. Figures 3.3 and 3.4 show the maximum relative error as a function of initial inventory level assuming that the system starts in a cold state and that $R = 20$ and $K = 20$. Roughly speaking, the relative error is first increasing and then decreasing in the

| $p$ | $R$ | $K$ | $E_C^{MO}$ | $E_W^{MO}$ | $E_C^{GBS}$ | $E_W^{GBS}$ | $p$ | $R$ | $K$ | $E_C^{MO}$ | $E_W^{MO}$ | $E_C^{GBS}$ | $E_W^{GBS}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 5 | 10 | 11.37 | 11.95 | 4.17 | 4.17 | 8 | 5 | 10 | 5.47 | 5.72 | 4.56 | 4.56 |
| 2 | 5 | 20 | 25.19 | 30.53 | 4.45 | 4.45 | 8 | 5 | 20 | 11.37 | 11.95 | 4.87 | 4.87 |
| 2 | 5 | 50 | 51.20 | 62.33 | 4.38 | 4.38 | 8 | 5 | 50 | 30.02 | 38.07 | 5.49 | 5.48 |
| 2 | 5 | 200 | 97.39 | 94.67 | 3.58 | 3.58 | 8 | 5 | 200 | 73.02 | 93.60 | 5.29 | 5.29 |
| 2 | 20 | 10 | 6.99 | 7.70 | 4.68 | 4.68 | 8 | 20 | 10 | 6.30 | 6.48 | 5.32 | 5.31 |
| 2 | 20 | 20 | 12.66 | 14.95 | 3.60 | 3.60 | 8 | 20 | 20 | 6.99 | 7.70 | 5.81 | 5.81 |
| 2 | 20 | 50 | 30.71 | 32.43 | 4.51 | 4.50 | 8 | 20 | 50 | 15.12 | 18.37 | 5.69 | 5.69 |
| 2 | 20 | 200 | 97.90 | 87.75 | 3.14 | 3.14 | 8 | 20 | 200 | 56.16 | 66.89 | 4.34 | 4.34 |
| 2 | 50 | 10 | 6.00 | 6.59 | 5.08 | 5.08 | 8 | 50 | 10 | 9.20 | 9.49 | 7.93 | 7.93 |
| 2 | 50 | 20 | 11.25 | 13.14 | 3.70 | 3.70 | 8 | 50 | 20 | 12.16 | 13.84 | 8.01 | 8.01 |
| 2 | 50 | 50 | 23.51 | 31.40 | 1.85 | 1.83 | 8 | 50 | 50 | 13.17 | 15.77 | 5.15 | 5.15 |
| 2 | 50 | 200 | 88.78 | 66.83 | 2.65 | 2.65 | 8 | 50 | 200 | 37.59 | 40.23 | 3.27 | 3.27 |

Table 3.5: Maximum Relative Errors for the MO and the GBS Policies

starting inventory level for the MO policy. When the inventory level is low, the optimal policy requires an order size up to $S_2$. However, in the MO policy, low inventory levels trigger an ordering decision to bring the inventory level to $S_L$. In this range, the relative error is constant around 6%. When the inventory levels are in the intermediate range, the performance of the MO policy worsens significantly due to its non forward-looking behavior. In particular, although it is optimal to order larger than $R$ to switch from the cold state to the warm, the MO policy does not consider the future cost function and leads to an order size less than $R$ failing to recognize the cost savings by switching. As the initial inventory level increases, both the optimal policy and the MO heuristic place orders less frequently, and

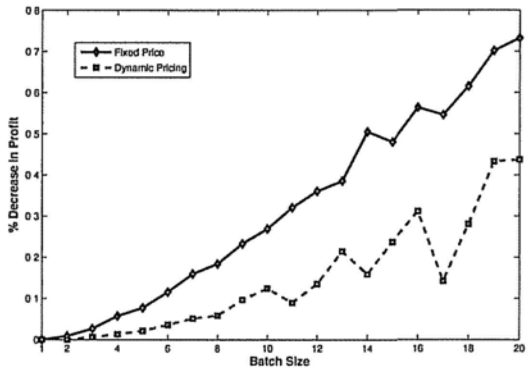


Figure 3.3: Relative Errors for MO Policy.

correspondingly, the relative error decreases and converges to zero in the higher range of the interval. Figure 3.4 shows the results for the GBS policy. Unlike the MO policy, the GBS heuristic is optimal when the starting inventory level is low. This is due to the reason that both the optimal policy and the GBS policy decide to order up to $S_2$ for the low levels of inventory. In the other regions, the relative errors follow an approximately similar pattern to that in the MO policy.

## 3.5 Concluding Remarks

Most manufacturing and purchasing systems require certain setup activities prior to production or ordering, which are generally carried out at a cost. These setup costs are sometimes so substantial that firms may look for ways to reduce or eliminate certain activities for cost savings. For example, some manufacturing firms deliberately produce quantities exceeding a threshold value, i.e., keep the system warm in a given period, to avoid some of the setup activities in the next

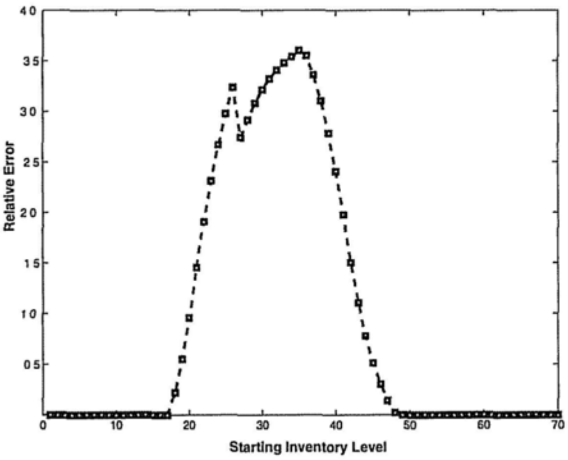


Figure 3.4: Relative Errors for GBS Policy.

period. The firm's production decisions under such settings have been investigated by some researchers assuming demand is known. In this chapter, we tackle the problem in a stochastic-demand setting.

More specifically, we analyze a firm's production/ordering decisions over a finite-horizon where the firm incurs a fixed setup cost in a period only when the previous period's production/order quantity is less than the threshold value $R$. We present a dynamic programming formulation of the problem and provide some structural results on the cost functions. One important concept that we use in our analysis is called quasi-$K$-convexity. Since quasi-$K$-convexity is not necessarily preserved under minimization, the analysis of the problem becomes nontrivial, and we show several intermediary results which help us partially characterize the optimal policy. The optimal policy can be complex or undesirable for some firms, hence we also develop some easier and effective heuristic policies.

A numerical investigation is presented to evaluate the heuristic performances. An optimality-based heuristic policy provides near-optimal solutions under several demand distributions, significantly outperforming a myopic policy that has advantages in implementation. A heuristic policy that trades off well the implementation issues and solution performance is developed by assuming some structural properties on the cost functions and is based on generalized base stock policies. The numerical experiments show less than 10% maximum relative errors compared to the optimal policy.

Some extensions of our work might be useful. In this direction, we analyzed the case of an infinite horizon model. (The analysis is provided in the Supplementary Appendix.) We find that the optimal policy for the infinite horizon case inherits that of finite horizon case. Typically the optimal policy can be simplified in the infinite horizon case. However, in the current model, we are unable to eliminate any of the cases that exist in the finite horizon case. Our work can be extended to incorporate capacity limit per period, $C > R$, which might be of practical relevance. In that setting, a heuristic policy that combines our results with those of Shaoxiang (2004) may be effective. Another extension could be to include a non-zero minor setup cost when the process is in the warm state rather than assuming that setup cost is zero. Understandably, this would lead to an even more complicated optimal policy, however, an efficient heuristic policy may be developed to guide the production/inventory managers in their decisions.

## 3.6 Appendix

### 3.6.1 Numerical Results

The numerical results are shown in Tables 3 6 and 3 7

### 3.6.2 Extension to the Infinite Horizon Case

Assume the demands for different periods to be i i d We show that the $T$-period optimal cost function $f_{t,1}$ and $f_{t,2}$ both converge uniformly to finite-valued functions $f_1$ and $f_2$, respectively The optimal policy for the infinite horizon problem inherits the structure of the $T$-period problem

Define $\hat{S}_L = \min\{x > S_L \quad L(x) \geq L(S_L) + K\}$ The following lemma shows that $\hat{S}_L$ is a upper bound of $S_{t,1}$ and $S_{t,2}$

**Lemma 3.5.** *For any $t = 1, 2, \quad , T$, $\max\{S_{t,1}, S_{t,2}\} \leq \hat{S}_L$*

**Proof.** To prove Lemma 3 5, it suffices to show that $g_{t,i}(x) \geq g_{t,i}(S_L)$ for all $x \geq \hat{S}_L > S_L$ Applying part (iii) of Lemma 3 3, we can obtain

$$
\begin{aligned}
g_{t,i}(x) - g_{t,i}(S_L) &= L(x) - L(S_L) + \alpha E[\tilde{f}_{t+1,i}(x - D) - \tilde{f}_{t+1,i}(S_L - D)] \\
&\geq L(x) - L(S_L) - \alpha K \\
&\geq L(\hat{S}_L) - L(S_L) - \alpha K \\
&\geq 0
\end{aligned}
$$

This ends the proof □

**Theorem 3.4.** *The optimal cost functions $f_{t,1}$ and $f_{t,2}$ in the $T$-period problem converge uniformly to functions $f_1$ and $f_2$ in any finite interval*

**Proof.** Denote by $y_t(x)$ the optimal inventory level after ordering when the initial inventory level is $x$ By Lemma 8, it is easy to prove that there exists a constant $M$ such that $y_t(x) \in [0, M]$ for all $x \in [0, M]$ independent of $t$

| $B$ | $p$ | $R$ | $K$ | $E_C^{OB}$ | $E_W^{OB}$ | $B$ | $p$ | $R$ | $K$ | $E_C^{OB}$ | $E_W^{OB}$ | $B$ | $p$ | $R$ | $K$ | $E_C^{OB}$ | $E_W^{OB}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 20 | 2 | 5 | 10 | 0 | 0 | 50 | 2 | 5 | 10 | 0 | 0 | 100 | 2 | 5 | 10 | 0 | 0 |
| 20 | 2 | 5 | 20 | 0 | 0 | 50 | 2 | 5 | 20 | 0 | 0 | 100 | 2 | 5 | 20 | 0 | 0 |
| 20 | 2 | 5 | 50 | 0 | 0 | 50 | 2 | 5 | 50 | 0 | 0 | 100 | 2 | 5 | 50 | 0 | 0 |
| 20 | 2 | 5 | 200 | 0 | 0 | 50 | 2 | 5 | 200 | 0 | 0 | 100 | 2 | 5 | 200 | 0 | 0 |
| 20 | 2 | 20 | 10 | 0 | 0 | 50 | 2 | 20 | 10 | 0 | 0 | 100 | 2 | 20 | 10 | 0 | 0 |
| 20 | 2 | 20 | 20 | 0 | 0 | 50 | 2 | 20 | 20 | 0 | 0 | 100 | 2 | 20 | 20 | 0 | 0 |
| 20 | 2 | 20 | 50 | 0 | 0 | 50 | 2 | 20 | 50 | 0 | 0 | 100 | 2 | 20 | 50 | 0 | 0 |
| 20 | 2 | 20 | 200 | 0 | 0 | 50 | 2 | 20 | 200 | 0 | 0 | 100 | 2 | 20 | 200 | 0 | 0 |
| 20 | 2 | 50 | 10 | 0 | 0 | 50 | 2 | 50 | 10 | 0 | 0 | 100 | 2 | 50 | 10 | 0 | 0 |
| 20 | 2 | 50 | 20 | 0 | 0 | 50 | 2 | 50 | 20 | 0 | 0 | 100 | 2 | 50 | 20 | 0 | 0 |
| 20 | 2 | 50 | 50 | 0 | 0 | 50 | 2 | 50 | 50 | 0 | 0 | 100 | 2 | 50 | 50 | 0 | 0 |
| 20 | 2 | 50 | 200 | 0 | 0 | 50 | 2 | 50 | 200 | 0 | 0 | 100 | 2 | 50 | 200 | 0 | 0 |
| 20 | 8 | 5 | 10 | 0 | 0 | 50 | 8 | 5 | 10 | 0 | 0 | 100 | 8 | 5 | 10 | 0 | 0 |
| 20 | 8 | 5 | 20 | 0 | 0 | 50 | 8 | 5 | 20 | 0 | 0 | 100 | 8 | 5 | 20 | 0 | 0 |
| 20 | 8 | 5 | 50 | 0 | 0 | 50 | 8 | 5 | 50 | 0 | 0 | 100 | 8 | 5 | 50 | 0 | 0 |
| 20 | 8 | 5 | 200 | 0 | 0 | 50 | 8 | 5 | 200 | 0 | 0 | 100 | 8 | 5 | 200 | 0 | 0 |
| 20 | 8 | 20 | 10 | 0 | 0 | 50 | 8 | 20 | 10 | 0 | 0 | 100 | 8 | 20 | 10 | 0 | 0 |
| 20 | 8 | 20 | 20 | 0 | 0 | 50 | 8 | 20 | 20 | 0 | 0 | 100 | 8 | 20 | 20 | 0 | 0 |
| 20 | 8 | 20 | 50 | 0 | 0 | 50 | 8 | 20 | 50 | 0 | 0 | 100 | 8 | 20 | 50 | 0 | 0 |
| 20 | 8 | 20 | 200 | 0 | 0 | 50 | 8 | 20 | 200 | 0 | 0 | 100 | 8 | 20 | 200 | 0 | 0 |
| 20 | 8 | 50 | 10 | 0 | 0 | 50 | 8 | 50 | 10 | 0 | 0 | 100 | 8 | 50 | 10 | 0 | 0 |
| 20 | 8 | 50 | 20 | 0 | 0 | 50 | 8 | 50 | 20 | 0 | 0 | 100 | 8 | 50 | 20 | 0 | 0 |
| 20 | 8 | 50 | 50 | 0 | 0 | 50 | 8 | 50 | 50 | 0 | 0 | 100 | 8 | 50 | 50 | 0 | 0 |
| 20 | 8 | 50 | 200 | 0 | 0 | 50 | 8 | 50 | 200 | 0 | 0 | 100 | 8 | 50 | 200 | 0 | 0 |

Table 3.6: Maximum Relative Errors for the OB Policy: Uniform Demand

| $\lambda$ | $p$ | $R$ | $K$ | $E_C^{OB}$ | $E_W^{OB}$ | $\lambda$ | $p$ | $R$ | $K$ | $E_C^{OB}$ | $E_W^{OB}$ | $\lambda$ | $p$ | $R$ | $K$ | $E_C^{OB}$ | $E_W^{OB}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 2 | 5 | 10 | 0 | 0 | 20 | 2 | 5 | 10 | 0 | 0 | 30 | 2 | 5 | 10 | 0 | 0 |
| 10 | 2 | 5 | 20 | 0 | 0 | 20 | 2 | 5 | 20 | 0 | 0 | 30 | 2 | 5 | 20 | 0 | 0 |
| 10 | 2 | 5 | 50 | 0 | 0 | 20 | 2 | 5 | 50 | 0 | 0 | 30 | 2 | 5 | 50 | 0 | 0 |
| 10 | 2 | 5 | 200 | 0 | 0 | 20 | 2 | 5 | 200 | 0 | 0 | 30 | 2 | 5 | 200 | 0 | 0 |
| 10 | 2 | 20 | 10 | 0 | 0 | 20 | 2 | 20 | 10 | 0 | 0 | 30 | 2 | 20 | 10 | 0 | 0 |
| 10 | 2 | 20 | 20 | 0 | 0 | 20 | 2 | 20 | 20 | 0 | 0 | 30 | 2 | 20 | 20 | 0 | 0 |
| 10 | 2 | 20 | 50 | 0 | 0 | 20 | 2 | 20 | 50 | 0 | 0 | 30 | 2 | 20 | 50 | 0 | 0 |
| 10 | 2 | 20 | 200 | 0 | 0 | 20 | 2 | 20 | 200 | 0 | 0 | 30 | 2 | 20 | 200 | 0 | 0 |
| 10 | 2 | 50 | 10 | 0 | 0 | 20 | 2 | 50 | 10 | 0 | 0 | 30 | 2 | 50 | 10 | 0 | 0 |
| 10 | 2 | 50 | 20 | 0 | 0 | 20 | 2 | 50 | 20 | 0 | 0 | 30 | 2 | 50 | 20 | 0 | 0 |
| 10 | 2 | 50 | 50 | 0 | 0 | 20 | 2 | 50 | 50 | 0 | 0 | 30 | 2 | 50 | 50 | 0 | 0 |
| 10 | 2 | 50 | 200 | 2.7e-003 | 0 | 20 | 2 | 50 | 200 | 0.61 | 0.054 | 30 | 2 | 50 | 200 | 0 | 0 |
| 10 | 8 | 5 | 10 | 0 | 0 | 20 | 8 | 5 | 10 | 0 | 0 | 30 | 8 | 5 | 10 | 0 | 0 |
| 10 | 8 | 5 | 20 | 0 | 0 | 20 | 8 | 5 | 20 | 0 | 0 | 30 | 8 | 5 | 20 | 0 | 0 |
| 10 | 8 | 5 | 50 | 0 | 0 | 20 | 8 | 5 | 50 | 0 | 0 | 30 | 8 | 5 | 50 | 0 | 0 |
| 10 | 8 | 5 | 200 | 0 | 0 | 20 | 8 | 5 | 200 | 0 | 0 | 30 | 8 | 5 | 200 | 0 | 0 |
| 10 | 8 | 20 | 10 | 0 | 0 | 20 | 8 | 20 | 10 | 0 | 0 | 30 | 8 | 20 | 10 | 0 | 0 |
| 10 | 8 | 20 | 20 | 0 | 0 | 20 | 8 | 20 | 20 | 0 | 0 | 30 | 8 | 20 | 20 | 0 | 0 |
| 10 | 8 | 20 | 50 | 0 | 0 | 20 | 8 | 20 | 50 | 0 | 0 | 30 | 8 | 20 | 50 | 0 | 0 |
| 10 | 8 | 20 | 200 | 0 | 0 | 20 | 8 | 20 | 200 | 0 | 0 | 30 | 8 | 20 | 200 | 0 | 0 |
| 10 | 8 | 50 | 10 | 0 | 0 | 20 | 8 | 50 | 10 | 0 | 0 | 30 | 8 | 50 | 10 | 0 | 0 |
| 10 | 8 | 50 | 20 | 0 | 0 | 20 | 8 | 50 | 20 | 0 | 0 | 30 | 8 | 50 | 20 | 0 | 0 |
| 10 | 8 | 50 | 50 | 0 | 0 | 20 | 8 | 50 | 50 | 0 | 0 | 30 | 8 | 50 | 50 | 0 | 0 |
| 10 | 8 | 50 | 200 | 0 | 0 | 20 | 8 | 50 | 200 | 0 | 0 | 30 | 8 | 50 | 200 | 0 | 0 |

Table 3.7: Maximum Relative Errors for the OB Policy: Poisson Demand

Note the fact that for any constant $c_1$, $c_2$, $c_3$ and $c_4$, we have

$$|\min\{c_1, c_2\} - \min\{c_3, c_4\}| \leq \min\{|c_1 - c_3|, |c_2 - c_3|, |c_2 - c_3|, |c_2 - c_4|\}.$$

Thus, we can obtain

$$\begin{aligned}
\max_{0\leq x\leq M} |f_{t,i}(x) - f_{t+1,i}(x)| &\leq \alpha \left| \int_0^{+\infty} [\tilde{f}_{t+1,i}(y_t - \xi) - \tilde{f}_{t+2,i}(y_{t+1} - \xi)]\phi(\xi)d\xi \right| \\
&\leq \alpha \max_{0\leq x\leq M} \int_0^{+\infty} |\tilde{f}_{t+1,i}(y_t - \xi) - \tilde{f}_{t+2,i}(y_{t+1} - \xi)|\phi(\xi)d\xi \\
&\leq \alpha \max_{0\leq x\leq M} |f_{t+1,i}(x) - f_{t+2,i}(x)|
\end{aligned}$$

where the fist inequality follows from the above fact and the second from that $y_t(x) \in [0, M]$ for all $x \in [0, M]$ independent of $t$.

Since $M$ can be chosen arbitrarily large, we have shown that $f_{t,i}(x)$ converges monotonely and uniformly for all $x$ in any finite interval. The functions $f_{t,i}(x)$ are continuous and converge uniformly, thus the limit function $f_i(x)$ is also continuous. □

Let functions $f_1(x)$ and $f_2(x)$ represent the minimum total expected discounted cost of the infinite horizon when the current process state is cold and warm, respectively. It follows from the theory of Markov decision processes that

$$f_1(x_t) = \min\{\min_{y>x+R}\{K+L(y)+\alpha Ef_2((y-D)^+)\}, \min_{x+R\geq y\geq x}\{k\delta(y-x)+L(y)+\alpha Ef_1((y-D)^+)\}\}$$

and

$$f_2(x_t) = \min\{\min_{y>x+R}\{L(y) + \alpha Ef_2((y - D)^+)\}, \min_{x+R\geq y\geq x}\{L(y) + \alpha Ef_1((y - D)^+)\}\}.$$

The convergence guarantees that the optimal policy for the infinite horizon case inherits that of finite horizon case. Typically in the infinite horizon case, the optimal policy can be simplified. However, in the current model, we are unable to eliminate any of the cases that exist in the finite horizon case.

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