A Study of System Efficiencies Through Game Theory and Optimization

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Abstract

Game theory is a common tool in modeling human decisions and strategies under various decision environments, with the foundation being the fact that a joint decision from all the individuals (we shall adhere to the term players hereafter) will impact on each other's well being. In this thesis, we shall study how the behaviors of the players affect the performance of the whole system, and shall introduce some measurements to quantify the influence on the system performance.

The first part is devoted to the study of the loss of system efficiency caused by selfish behavior of the players. We use the notion of the *Price of Anarchy* and consider two different but intrinsically related game settings to address the issue. One is to consider the cost incurred to the players due to the usage of some shared resources, modeled as the links of a network. Suppose that there are K players and each of them must achieve a given throughput. Furthermore, the unit cost on each link is affine linear in the total flow. Then the price of anarchy for the game can be upper bounded by $(3K + 1)/(2K + 2)$. The second model is a generalization of Cournot oligopolistic competition, in which the players utilize some shared resources to produce some commodities to sell. Again, suppose there are K players, and the unit costs of the shared resources and the selling prices of the products are all affine linear functions in the amount of demand and supply respectively. Then the price of anarchy is shown to be lower bounded by $1/K$.

In the second part, we turn to the consequence of greediness of the players. In a dynamic decision-making process, system inefficiency may be caused by the unwise use of the resources due to the myopia of the decision makers. The loss of efficiency is measured by a ratio termed the *Price of Myopia*: the value at the greedy solution divided by the optimal value. A specific setting is studied to illustrate the point. Furthermore, we consider the combined effect of selfishness and myopia under a game framework and introduce a new notion, the *Price of Isolation* to quantify the matter. Some bounds for the price of isolation are established in a dynamic setting of the previous two models.

In the third part, we investigate the influence of cooperation and altruistic behavior of players. The incentive of the players to cooperate, and the impact of cooperation on the members of the coalition and on the whole system are analyzed. We consider a model of resource competition game and find that the system will benefit from the cooperation of players at the expense of some individual members in the coalition. A measurement termed the *Price of Socialism* is introduced to characterize how much any individual will need to sacrifice for the social optimum. We obtain a tight bound for the price of socialism for our particular model.

摘要

博弈论是一门研究多人决策行为的理论，其核心在于参与者的决策行为交叉影响，进而影响整个系统的状态。该论文针对此类现象进行量化研究，在某些具体设定下，分析了可能出现的最差状况。

首先，我们研究了完全自利行为可能导致的后果。我们采用无政府混乱代价作为量度，并就此分析了两种模型：交通运输模型和古诺寡头垄断模型。在交通运输模型中，假定有K个参与者，每个参与者需要从指定起点经一给定网络运送一定数量的货物至指定终点，并承担相应费用。如果网络中每条链接上的单位费用随此链接的总使用量线性增长，那么此设定下无政府混乱代价将不超过$(3K+1)/(2K+1)$。而在古诺寡头垄断模型中，我们对模型进行了推广，加入了生产资源的考虑。同样，假定市场有K个生产者，他们在获取资源以及销售产品两方面同时进行竞争。如果每单位资源花费随着需求线性增长，而产品价格随着供给线性下降，那么过度竞争可能导致无政府混乱代价达到$1/K$。

其次，短视也是此论文中考虑的因素之一。我们考虑在一个动态的设定中，现在的决定可能会影响将来的状况，由此对之前的两个模型重新进行了研究。此时，决策者的短视可能导致资源过度开发或市场恶性竞争，因此降低系统的效率。为此，我们引进短视的代价作为量度，并在一些具体设定之下，得到了此量度的取值范围。更进一步，我们综合了自利与短视两个因素的影响，并由此定义了孤立的代价。同样，我们也得到了类似的取值范围。

此外，我们还考虑了博弈中合作的影响以及参与者的牺牲行为。这一部分主要探讨了参与者合作的动机与合作对联盟内外成员的影响。我们考虑了一个资源竞争博弈，并且发现在此设定下，合作可能有利于整个系统，但有损于参与合作者。因此，为了达成联盟，某些参与者可能需要有所牺牲。我们引进了合作的代价来计量参与者为整个系统的合作所需付出的代价，并且得到了在资源竞争博弈中此量度的一个紧的上界。

Acknowledgement

I have been very fortunate to have the opportunity to learn from a number of outstanding scholars at the Chinese University of Hong Kong during my graduate studies. First and foremost, I wish to express my gratitude to my supervisor Professor Shuzhong Zhang for his guidance and constant support during the past four years. I am also deeply indebted to my colleague Professor Simai He for his kindness and useful discussions. I would also like to thank all the members in Zhang's group for their useful suggestions in the group meetings. I thank Professor Duan Li, Anthony So and Yinyu Ye for serving on my committee.

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Finally, I want to thank my parents and my family for their love and support which help me overcome the difficulties encountered during the process of completing this thesis. This thesis is dedicated to them.

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Chapter 1

Introduction

In society, the behaviors of a group of individuals may affect each other's well being. Studying these behaviors and analyzing their influences have been the central theme of research in the field of game theory and behavioral economics. There are many fruitful results in the literature and many interesting problems remain to be solved. For an individual, being self-interested seems rational, however, it is well known that the 'rational' behavior of each individual can collectively result in a much lowered overall performance of the entire system, eventually undermining the well-beings of all the individuals involved. This phenomenon is pervasive in social, political, economical, and even biological systems, and a suitable terrain for its study can be found in non-cooperative game theory. For instance, the notion of *Prisoner's Dilemma* is devised to illustrate precisely this point. Selfish behavior can hurt the self whose interest it sets out to promote in the first place. The wisdom contained in the previous statement may be profound; however, as such it appears to be no more than a pure philosophical thought. Now if selfish behavior may cause inefficiency, how bad can it be? Furthermore, are there ways to avoid the inefficiency caused and improve the system performance? More generally, by how much will some specific types of behaviors affect the individuals and the society as a whole?

This thesis is devoted to a study about the effect of individual behaviors on the system performance in some specific settings. We study a variety of transportation and Cournot game models, and focus on the consequences of three different behavior codes of the decision maker: selfishness, greediness and altruism. Our aim is to estimate the largest-possible impact, through bounding some relevant measures.

1.1 Motivating Examples

First of all, let us consider some examples relevant to our studies.

1.1.1 Pigou's Example

The following example, named after an economist Pigou, was first introduced in 1920. It shows that selfish behavior by noncooperative players in a game may result in a loss of efficiency for the whole system.

Example 1.1.1 Consider a network shown in Figure 1.1, which consists of a source node s, a destination node d, and two links between s and d. Suppose there are a population of users and each of them chooses between the two links to get from s to d. Some transportation costs will be incurred due to the usage of the links and the cost on each link is given by a function of the total flow through it. Assume that the upper link has a cost function $c_1(f_1) = f_1$, and the lower link has a constant cost function $c_2(f_2) = 1$, where f_1 and f_2 denote the total flow through the upper and the lower link respectively. In other words, the cost on the upper link will increase as the link gets more congested, while the lower link is immune to congestion. Suppose that each user is selfish and only interested in minimizing his/her own transportation cost. Viewing the users as players, this is a noncooperative game problem. We shall use the term '*transportation game*' to specify this type of game setting which involves a network with some transportation cost and players with corresponding transportation tasks.

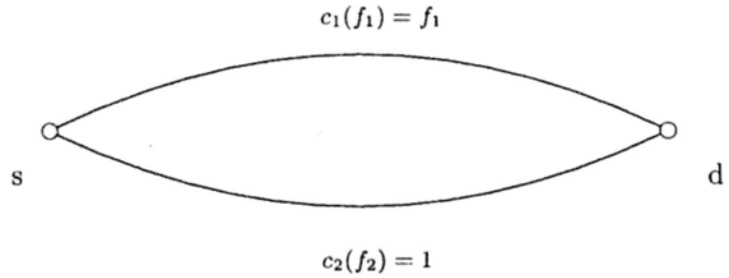

Figure 1.1: Pigou's example

Here we assume that each user is infinitesimal, but in total they form one unit of flow together. When it reaches the Nash equilibrium, at which no player could lower its cost by changing the decision of itself, all the players will choose the upper link, observing that the upper link will always be cheaper than the lower link if the total flow through the upper link is less than 1. Consequently, the total cost incurred to all the players at the Nash equilibrium will be 1. However, the optimal distribution solution is splitting the total flow equally between the two links. The cost for half of the one unit of flow will be 1/2, and for the other half, the cost will be 1. Then the total cost at the optimal solution is 3/4, which is less than the cost at the Nash equilibrium.

Thus, this shows that selfish behavior may undermine the overall performance of the whole system.

1.1.2 A Triangular Example

In Pigou's example, the inefficiency is introduced to the whole system. However, the next example shows that selfish behavior will not necessarily be good even for the self.

Example 1.1.2 Consider a transportation game with the network shown in Figure 1.2. We label the four links $\overrightarrow{AB}, \overrightarrow{BA}, \overrightarrow{AC}, \overrightarrow{BC}$ by 1, 2, 3, 4, respectively. Assume the unit

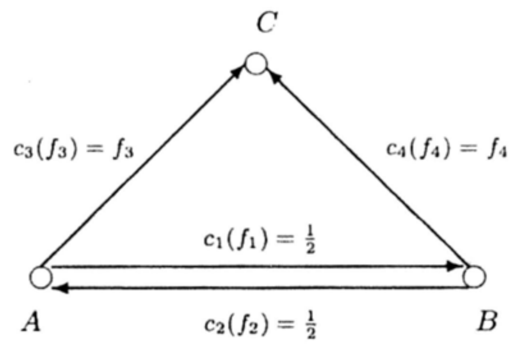

Figure 1.2: A triangular example

cost functions on the links are given by

$$c_1(f_1) = c_2(f_2) = \frac{1}{2}, \; c_3(f_3) = f_3 \;\text{ and }\; c_4(f_4) = f_4.$$

Different from Pigou's example, suppose there are two players, each controlling a strictly positive (as opposed to negligible) amount of flow to operate. The first player is required to transport one unit of flow from A to C, while the second player needs to transport one unit of flow from B to C. Furthermore, suppose the flow could be split in transportation through the network. Then it is clear that the optimal transportation plan for the whole system is to separate the two units of flow equally into two independent routes: the required flow of the first player goes from A to C and the one of the second player goes from B to C, yielding a transportation cost of 1 for each player. Nevertheless, the (unique) Nash equilibrium is for the first player to transport a flow of 3/4 from A

to C, and another flow of 1/4 from A via B to C, and symmetrically the same for the second player. In this case, each of the two players will incur a cost of 9/8, which is more than the cost at the optimal solution.

This demonstrates that the outcome of selfish behavior and local optimizer may be detrimental to all the participants of the system.

1.1.3 Braess's Paradox and a Relevant Variant

It may become complicated to improve the system in the case that the participants are a group of uncoordinated individuals with conflicting interest. The following example, the so-called Braess's Paradox given by Braess in [13], shows that adding some resources to the system could reduce overall performance, which is quite counter-intuitive.

Example 1.1.3 Again, we consider a transportation game with the network shown in Figure 1.3(a). We number the four links $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{BD}, \overrightarrow{CD}$ by 1, 2, 3, 4, respectively.

Figure 1.3: Braess's Paradox

Assume the unit cost functions on the links are given by

$$c_1(f_1) = f_1,\ c_2(f_2) = c_3(f_3) = 1 \ \text{ and } \ c_4(f_4) = f_4.$$

Suppose there are a population of infinitesimal players, who in total form one unit of flow and need to get from Node A to Node D through the network. By symmetry, if each player wishes to minimize the transportation cost, half of the total flow will choose the route $A \to B \to D$ and the other half will choose the route $A \to C \to D$. Then the cost for each player will be 3/2 and hence the total cost will be 3/2.

Now suppose that the operator of the network wants to add a costless link between B and C to connect the two links which have lower unit costs to improve the transportation situation. The new network is shown in Figure 1.3(b). What will happen to the players?

Note that the costs on Link $\overrightarrow{AB}$ and Link $\overrightarrow{CD}$ will be less than the ones on Link $\overrightarrow{AC}$ and Link $\overrightarrow{BD}$ only if the amount of flow through them is less than 1. Since the new link is free, the half of the total flow through Link $\overrightarrow{AC}$ will be motivated to make use of the less costly Link $\overrightarrow{AB}$ and the new link. In this case, all of the players will use the route $A \to B \to C \to D$ and the cost will be 2, which is larger than the original one.

Braess's Paradox shows that the addition of some intuitively helpful links in the network could lead to a worse result. This phenomenon is due to the interactions between uncoordinated selfish behavior and the underlying network. Then what if the players cooperate with each other? Of course, the overall performance should be better. Nonetheless, in case the cost is nontransferable, we can modify Braess's Paradox a little in the following example to show that there may not exist a distribution mechanism such that everyone can benefit from the cooperation; in other words, while one is better off, someone else is always worse off.

Example 1.1.4 Given the augmented network in Braess's Paradox shown in Figure 1.4, similar to the triangular example, we consider the players who are not infinitesimal and control a certain amount of flow. Specifically, suppose there are 2 players in the game and label them by Player 1 and Player 2. Moreover, Player 1 and Player 2 need to ship 1/10 and 9/10 units of flow from A to D, respectively. The cost functions for the five links are kept the same as in Braess's Paradox.

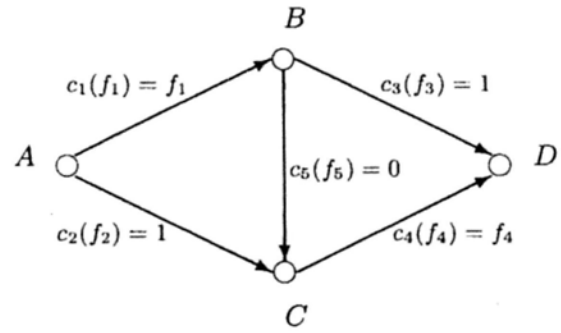

Figure 1.4: A variant of Braess's Paradox

Denote a transportation plan for Player $k\,(k = 1, 2)$ by a vector $x^k \in \mathbb{R}^5$, whose $l-$component represents the amount of flow of Player k through Link $l\,(l = 1, 2, \ldots, 5)$. Then the flow vectors at the Nash equilibrium are $x^1 = (1/10, 0, 0, 1/10, 1/10)$ and $x^2 = (9/20, 9/20, 9/20, 9/20, 0)$, under which Player 1 needs to incur a cost of $(\frac{1}{10} + \frac{9}{20}) \times 2 \times \frac{1}{10} = \frac{11}{100}$. When they cooperate, the optimal flow should be $x^* = (1/2, 1/2, 1/2, 1/2, 0)$, which implies that Link $\overrightarrow{BC}$ (and hence the route $A \to B \to C \to D$) would never be used in the optimal transportation plan. Then Player 1 will incur a cost of $(1+\frac{1}{2}) \times \frac{1}{10} = \frac{3}{20}$ no matter which one between the two routes left is chosen. In other words, there exists no allocation plan such that Player 1 can benefit from the cooperation.

1.2 Our Contributions

1.2.1 Selfishness and Bounding the Price of Anarchy

Selfish behavior is the first subject to be studied in this thesis. We shall use the notion of the *Price of Anarchy* (PoA for brevity), which was first introduced in Koutsoupias and Papadimitriou [39] to quantify the inefficiency caused. Two different but related models are considered to address the issue.

The first one is a transportation game, which could be viewed as an extension of the setting in Pigou's example. We consider a structure whereby the cost for a user (player) is due to the utilization of some shared resources, modeled as the links of a network. Each user has a given amount of flow to ship from user-specific source to user-specific destination and may split the flow through the paths between the two nodes and the objective is to minimize the transportation cost. In the context of road traffic networks, for example, a user can be viewed as a transportation company which needs to ship a flow of vehicles. For telecommunication network, players corresponding to those users who are to send their signals through a shared network. The games that we consider involve K players, instead of infinitesimally small individual traffic participants. We focus on some special cost functions and give an upper bound for the PoA. Specifically, we assume that the *unit* cost on each link is affine linear in the total flow. In that case, the PoA of the game is shown to be upper bounded by $\frac{3K+1}{2K+2}$. If all the players have the same Origin-Destination (OD for brevity) pair and the same desired throughput, the PoA of the game is upper bounded by $\frac{4K^2}{3K^2+2K-1}$ and it is tight.

Moreover, we investigate the impact of selfishness in another model, which could be viewed as a generalization of the classical Cournot oligopolistic competition model, or, from a different angle, the Wardrop type routing model. In this setting, the producers face the same market and choose their own production levels which affect the sales prices, to maximize their own profit. The competition among the producers may result in the congestion effect: the more commodities produced, the less unit revenue one can expect. In particular, we also take the resource competition into consideration and suppose that there are K players, who compete for the usage of resources as well as the sales of the end-products. Moreover, the unit costs of the shared resources and the selling prices of the products are assumed to be affine linear functions in the consumption/production quantities. We show that the PoA in this case is lower bounded by $1/K$, and this bound is essentially tight, which manifests the harsh nature of the competitive market for the producers.

1.2.2 Greediness and Bounding the Price of Myopia

Greedy attitude is the second issue we address. At first, we consider the case in which the decision maker is myopic in a dynamic decision problem, *i.e.*, he/she focuses on the current state and disregards the future outcome. The results of the greedy solution and the optimal one are compared, and the ratio between these two values is used to define a notion of the *Price of Myopia* (PoM for brevity). To make it more explicit, we consider a specific dynamic process over T stages, whereby in each stage a prescribed throughput must be achieved. Assume that the unit cost on each link is affine linear in the flow on the link and the unit cost from the previous stage. The objective is to achieve the total throughput at minimum cost. In that case the PoM for the transportation problem is shown to be at most 4.

Then we present an integrated study on the loss of system efficiencies caused by the selfish behavior of the players participating in the system (modeled by a noncooperative game) *and* the greedy attitude of the players (modeled by the 'rolling-horizon' type of strategies in the dynamic decision-making process). The loss of efficiency is measured by a new ratio termed as the *Price of Isolation* (PoI for brevity): the total social value of the worst selfish and greedy solution divided by the optimal total social value. We extend the two specific problems discussed previously to a dynamic setting to address the issue in our study. The first model is an extension of the static transportation game, where K players compete on a given set of resources to get their jobs done at minimum cost, over T stages. The unit cost of each resource in each stage is assumed to be an affine linear function in the total demand in this stage and the cost level in the previous stage. We show that the tight PoI in this minimization game is in the interval [2, 4.4865). The second model is a dynamic profit maximization game, where K producers compete on a given set of resources to produce a set of goods, and compete to sell them in the market, over T stages. The unit costs of the resources and the unit sale prices of the finished goods are assumed to be affine linear functions, similar to the assumptions made in the first model. We prove that the tight PoI of this maximization game is in the order of $\frac{1}{KT}$. We also discuss possible interpretations of these results from the viewpoint of management science. Since the assumptions in the model are fairly simple and standard, the results warn against the fast deterioration of the social value when many selfish players struggle to maximize profits over an extended period of time.

1.2.3 Altruism and Bounding the Price of Socialism

Our work also includes a study on altruistic behavior under a cooperative game structure. The notion of the *Price of Socialism* (PoS for brevity) is introduced to characterize at most how much a player needs to sacrifice for the system optimal solution. Furthermore, we consider a specific resource competition game in this part and find that it is equivalent to a transportation game with parallel network. In order to utilize the structures of the network to analyze the influence of the cooperation among the players, we mainly focus on the transportation game instead. The incentive of the players to cooperate, the impact of cooperation on the members of the coalition and on the whole system are discussed in this part. It is shown that the system may benefit from the cooperation of players while some members in a coalition may be harmed. Some monotonicity property has been established through carrying out parameter sensitivity analysis. As a consequence, we get some bounds for both the PoA and the PoS for the game. It is also shown that these bounds are tight.

1.3 Comparison to Previous Work

Behavior studies and inefficiency analysis have formed a sizable of literature in game theory in these decades. In this section we give a brief review of the most relevant results in the references to locate our contributions.

The fact that selfish behavior could have undesirable consequences has been realized and discussed for a long time already. However, Koutsoupias and Papadimitriou [39] may be the first publication to quantify this issue in mathematical terms. They introduced the notion of *price of anarchy*, which is defined as the ratio between the social value of the worst Nash equilibrium solution and the best possible social value as a measurement for the degree of damage caused by the selfish behavior of the individuals.

The traffic and routing game models were among the very first to be studied under this light. For the Wardrop model introduced by Wardrop in [67], in which infinite number of players each controlling infinitesimal amount of flow compete to use a network, Roughgarden [61, 62] showed that the PoA in that case is upper bounded by $\frac{(p+1)^{(p+1)/p}}{(p+1)^{(p+1)/p}-1} = \Theta(\frac{p}{\ln p})$ when the cost function on each link is polynomial with the degree less than or equal to p. Specifically, when the cost functions are restricted to be affine linear, the PoA will be no more than $\frac{4}{3}$. For a variant of Wardrop model, in which finitely many players are considered and each of them controls a certain amount of traffic, Awerbuch, Azar and Epstein [7], and Christodoulou and Koutsoupias [14] showed that the PoA in the atomic case, namely the flow is unsplittable, is upper bounded by $\frac{5}{2}$ when the cost function on each link is affine linear. In the nonatomic case, Cominetti, Correa and Stier-Moses [15] obtained an upper bound for the PoA if the cost function satisfies some convexity conditions. Some recent results on PoA in the transportation game can be found in [64, 57, 45].

The landscape changes dramatically when the study extends to another classical non-cooperative game framework — the Cournot competition game (cf. [18]). Immorlica, Markakis and Piliouras [33] studied coalition formation in a dynamic setting of Cournot oligopolies and proved that the PoA under their notion of stability is bounded by $\Theta(K^{2/5})$, where K is the number of players. Kluberg and Perakis [34] extended the classic Cournot model and bounded the PoA by means of the market power parameters and the number of players and products.

Following the notion of price of anarchy, several similar measures have been introduced to quantify the gap between social optimality and Nash equilibrium solutions in literature. Shakkottai et al. [66] studied revenue-maximizing pricing by a service provider in a communication network and defined the *price of simplicity* as the ratio between revenues from simple pricing rules to the maximum possible revenues. Later, Zhu and Basar [68] introduced a notion of *price of information* to compare game performances under different information structures and characterized it for a class of scalar linear quadratic differential games. Grossklags, Johnson and Christin [26] introduced a notion of the *price of uncertainty* to measure the relative payoff of an expert user of a security game under complete information to the one under incomplete information. In all of these works with different application domains, the central underlying idea is to characterize the impact of various types of behaviors and attitudes.

1.4 Tips for Reading this Thesis

1.4.1 Prerequisites

First of all, the reader is expected to be familiar with the basic concepts in game theory, such as players, strategy sets, Nash equilibria, static/dynamic games and so on. All these concepts can be found in Gibbons [25], and Osborne and Rubinstein [52]. We also will use some computation game models to address our objective issues, such as transportation game, Cournot competition game. Our favorite reference for this material is a book [50] which is written by a group of researchers in the field of computational game theory and introduces several types of normal algorithmic game models. Furthermore, we occasionally need to make use of some techniques and results in the theory of optimization including the basic of linear and nonlinear programming. Being familiar with some basic operations and notations of matrices are prerequisites as well. A standard introduction to these topics can be found in Luenberger and Ye [42]. Also, the reader can find in Cottle, Pang, and Stone [17] some useful conclusions about *Linear Complementary Problem* (LCP) that are used in the thesis.

1.4.2 Organization

This thesis is divided into six chapters, including the current one and the conclusion at the end. We present the organization as follows.

The next two chapters deal with the same topic, concerning selfishness and the price of anarchy. Chapter 2 will be dedicated to discussing the price of anarchy in a static transportation game. Our focus in this part is to consider the case where a finitely many noncooperative players interact, each with a positive and *splittable* flow to operate on a general network. We assumes that the unit cost on each link is affine linear in the total flow on the link, and each player must achieve a given throughput. The objective is to achieve the throughput at the minimum cost. Besides, the result is sharpened under a more specific setting. A generalized Cournot competition model is considered in Chapter 3. We confine ourselves to the case where the competition will affect the unit cost of the shared resources, as well as the price of products, in an affine linear manner. It is proved that the PoA is actually lower bounded by the inverse of the number of players. Furthermore, an example is given to show that this bound is essentially tight, which is a clear warning against uncoordinated and selfish behaviors in such settings.

Greediness is taken into consideration in Chapter 4. In Section 4.1.1 we first focus on a pure dynamic transportation decision problem to study the inefficiency caused due to myopia and introduce the price of myopia. Then, in Section 4.2.1 we extend the transportation problem to a game framework involving a number of players and investigate the combined effect of selfishness and greediness, introducing the price of isolation. In Section 4.2.2, an extended dynamic model of the Cournot production competition is considered and the measure is bounded if the selling prices and the resource costs (respectively) at each stage are dependent on the supply and demand (respectively) in an affine linear fashion.

In Chapter 5, we turn to the cooperation among the players, which can be thought of as the other aspect of the game. Another measure called the price of socialism is proposed and our motivation to consider this issue is discussed in Section 5.1.1. In the subsequent part, the setting of the model considered in this chapter is introduced. Before presenting our main conclusions, we give a brief review for some related references in 5.1.3. In Section 5.2, we confine ourselves to the case in which the unit cost functions are affine linear again. Some monotonicity is established with respect to the coordination of the players. The property also suggests that the PoS and the PoA in this setting can be bounded by constants. They are presented in Section 5.2.2 and Section 5.3, respectively.

Finally, the last chapter is devoted to a summary of the ideas and the results introduced in this thesis and some discussions for the future work.

1.5 Bibliographic Notes

Most of the work reported in this thesis appears in research papers [28, 29, 30], all of which are joint work with Simai He and Shuzhong Zhang.

Chapter 2

The Price of Anarchy in a Transportation Game

2.1 Introduction

2.1.1 An Oligopolistic Transportation Game

In this chapter, we investigate a generalization of transportation game considered in Pigou's example. Specifically, given a directed graph $G = (V; L)$ with the set of nodes V, and the set of arcs L (interchangeably we also use the terminology *link* as synonym of *arc* in this thesis), we assume that $|V| = n$, $|L| = m$. Note that multiple parallel links are allowed but no self-loop exists. Let us denote $A \in \mathbb{R}^{n\times m}$ to be the node-to-arc incidence matrix.[1]

[1]Each row of A represents a node and each column of A represents an arc. For an arc connecting node i to node j, the corresponding column in A will have all 0 elements except for the i-th element, where it is $+1$, and the j-th element, where it is -1.

Suppose that there are K players in the game. Each player wishes to transport a given amount of commodity from the given origin (node) to the given destination (node), through the paths on the network. (Splitting of the commodity is allowed). We denote the origin-destination (OD) pairs for all the players to be $\{s^1, d^1\}, \{s^2, d^2\}, \ldots, \{s^K, d^K\}$. Let r denote a vector in $\mathbb{R}^K$, where the component r^k represents the amount of commodity that Player k needs to transport. The transportation plan of Player k will be given by a vector $x^k \in \mathbb{R}^m$, which indicates the flow on each link. Clearly, a feasible flow is given by the constraints $Ax^k = r^k\delta_{s^k} - r^k\delta_{d^k}$, where the notation δ_i signifies the unit vector in $\mathbb{R}^n$ whose i-th component is one while all others are zero.

For each link l, we denote the total flow on the link to be $f_l = \sum_{k=1}^K x_l^k$. Moreover, let us denote the *unit* cost for the flow on the link l to be a function $c_l : f_l \hookrightarrow c_l(f_l)$. Therefore, the data (G, r, c) specifies an instance of the non-cooperative routing game that is of interest to us in this thesis. Indeed, for Player k, given the decisions of other players (conventionally denoted as x^{-k}) is to minimize his/her own transportation cost given as:

$$C^k(x^k, x^{-k}) = \sum_{l \in L} x_l^k c_l(f_l).$$

Naturally, given the decisions of all the players, the social cost is a simple summation: $\mathrm{SC}(x) = \sum_{k=1}^K C^k(x^k, x^{-k})$. Let y denote the flow when the game reaches a Nash equilibrium; i.e. a solution at which no player will be able to improve his/her situation unilaterally. At the same time, let us denote x to be the socially optimal solution – the solution that minimizes the social cost function $\mathrm{SC}(x)$ over all feasible solutions. The price of anarchy is defined as:

$$\mathrm{PoA} = \frac{\mathrm{SC}(y)}{\mathrm{SC}(x)}.$$

2.1.2 Related Work

The research of routing game problem has been started since 1950's. The first such model was due to Wardrop [67] (1952), and the aim was to study the traffic formation and congestions. In the Wardrop model each player controls an infinitesimal amount of flow, and the number of players is infinite. This introduces a natural notion of equilibrium, to be distinguished from the familiar Nash equilibrium. Later, Beckmann *et al.* [11] showed that a flow at the Wardrop equilibrium can be expressed as an optimal solution of a certain convex program, thus established its existence and uniqueness.

The interest for the Wardrop type selfish traffic routing model was amplified due to a curious example, now famously known as the Braess paradox [13]: adding one more link (resource) to the network can in fact make everyone worse off, in the sense that each and every player in the game will find his/her cost increased because of the added resource. Structures of the network have been therefore studied so as to avoid such pathological properties; see Korilis, Lazar and Orda [38]. Moreover, Azouzi, Altman and Pourtallier [9] gave specific guidelines to avoid the Braess paradox type of situations when upgrading the network. Besides the network topology design, Korilis, Lazar and Orda [37] also considered how to allocate link capacities such that the resulting Nash equilibria will be efficient, so as to avoid the paradoxical property. Akamatsu [1] focused on the dynamic framework using the notion of *dynamic user equilibrium*, where the Braess paradox may also arise. Later, Akamatsu and Heydecker [2] presented a generalization of the model. Recently, Lin and Lo [41] identified some "good" congestions in the dynamic setting, meaning that it can actually help eliminate the negative impact due to the Braess paradox at the equilibrium. In other words, keeping certain congestions would help.

Generally speaking, the selfish behavior may socially be very inefficient. To explicitly quantify this matter, Koutsoupias and Papadimitriou [39] introduced the notion of the *Price of Anarchy* (PoA), which is the ratio between the social (total sum) value of the *worst* Nash equilibrium solution and the optimal social value. If the objective of the game is minimization, then PoA is no less than 1 — the larger the worse. Similarly, if the objective of the game is maximization, then PoA is no more than 1 — the smaller the worse. It turns out that the PoA for the general routing game (whose objective is minimization) can be arbitrarily large. However, in the Wardrop case, Roughgarden [61] gave an upper bound for the PoA where the cost on each link is a polynomial function. Later, in [62] Roughgarden showed that the worst case occurs on a very simple network and hence the upper bound is tight, and in fact holds for virtually any network topology. Recently, Roughgarden [64] (also cf. Correa, Schulz and Stier-Moses [16]) proposed a new approach to obtain an upper bound for the price of anarchy. The main point is that if a player adheres to the socially optimal strategy while other players adopt the selfish Nash strategies, then this player will of course be hurt due to the altruist behavior. However, if the damage can be controlled by a combinations of the social value at the Nash equilibrium and the true optimal social solution value, then the PoA can be controlled as well. We shall use this technique to bound the PoA for our models.

An extension of the Wardrop model is to assume that there are only a finite number of players, each controlling a positive amount of traffic. In a sense, as the number of the players tends to infinity, the model then asymptotically assimilates the Wardrop one. In the case that the flow for each player is not splittable, Rosenthal [59] introduced a special class of the noncooperative game, known as the *congestion game*, and showed that the pure Nash equilibrium exists. Mavronicolas and Spirakis [47] gave tight bounds on the PoA, which is no worse than $O(\ln n/\ln\ln n)$ if the network consists of parallel links and each player employs a mixed strategy which is a probability distribution over the links with the objective to minimize the expected cost, where n is the number of players. Awerbuch, Azar, Epstein [7], and Christodoulou and Koutsouplas [14] showed that if the unit cost function is affine linear then the PoA is upper bounded by $\frac{5}{2}$. Gairing, Monien and Tiemann [24] considered the player specific latency functions and some potential functions, obtaining some upper and lower bounds on the PoA. Aland *et al.* [3] established the bounds to the case with polynomial cost functions. If the flow is splittable, Orda, Rom and Shimkin [51] proved the uniqueness of the Nash equilibrium of a two-node multi-link system under some convexity conditions; however, they gave an example to show that the uniqueness fails for general networks. Dumauf and Gairing [19] used the notion of the Wardrop equilibrium to obtain upper and lower bounds on the PoA provided that the cost functions are polynomial. Cominetti, Correa and Stier-Moses [15] obtained an upper bound for the PoA in this setting if the cost function satisfies some convexity conditions.

2.2 Bounding the PoA with Affine Linear Cost Functions

We shall consider the case where the unit cost function is affine linear in the total flow; that is, the unit cost function on Link l is $c_l(f_l) = a_l f_l + b_l$, where $a_l, b_l \geq 0$. Then, Player k will face the following optimization problem:

$$\begin{aligned} (P_k) \quad \min \quad & \textstyle\sum_{l \in L}(a_l f_l + b_l)x_l^k \\ \text{s.t.} \quad & Ax^k = r^k\delta_{s^k} - r^k\delta_{d^k}, \\ & x^k \geq 0. \end{aligned}$$

Replacing f_l with $\sum_{k=1}^{K} x_l^k$, the above problem for Player k is a convex quadratic program, in which the decision vector is x^k:

$$\begin{aligned} (P_k') \quad \min \quad & \textstyle\sum_{l\in L} \left\{ b_l x_l^k + a_l \left[\left(\sum_{i\neq k} x_l^i \right) x_l^k + (x_l^k)^2 \right] \right\} \\ \text{s.t.} \quad & Ax^k = r^k \delta_{s^k} - r^k \delta_{d^k}, \\ & x^k \geq 0. \end{aligned}$$

To conclude the existence and the uniqueness of the Nash equilibrium under this setting, we are going to derive an LCP characterization system for the solution at Nash equilibrium. Let $y^k \in \mathbb{R}^m$ be the Lagrangian multiplier associated with the equality constraint $Ax^k = r^k \delta_{s^k} - r^k \delta_{d^k}$. The Karush-Kuhn-Tucker optimality condition for (P_k') is:

$$\begin{cases} Ax^k = r^k \delta_{s^k} - r^k \delta_{d^k}, \; x^k \geq 0 \\ b_l + a_l \sum_{i=1}^{K} x_l^i + a_l x_l^k + (A^{\mathrm{T}} y^k)_l - s_l^k = 0, \text{ for } \; l = 1, ..., m \\ s_l^k \geq 0, \; x_l^k s_l^k = 0, \text{ for } \; l = 1, ..., m. \end{cases}$$

Denote s^k to be the vector whose l-th component is s_l^k, $l = 1, ..., m$. A Nash equilibrium for the transportation game is attained if and only if each player attains the optimum simultaneously; i.e., for all $k = 1, ..., K$, we have

$$\begin{cases} Ax^k = r^k \delta_{s^k} - r^k \delta_{d^k} \\ b + \mathrm{Diag}(a) \sum_{i=1}^{K} x^i + \mathrm{Diag}(a) x^k + A^{\mathrm{T}} y^k - s^k = 0 \\ x^k \geq 0, \; s^k \geq 0, \; (x^k)^{\mathrm{T}} s^k = 0, \end{cases}$$

where $\mathrm{Diag}(a)$ is the diagonal matrix whose l-th diagonal is a_l, $l = 1, ..., m$.

We can explicitly write the KKT optimality condition using the block matrix notation. Let x (respectively y, and s, and R) be the vector consisting of $x^1, ..., x^K$ (respectively $y^1, ..., y^K$, and $s^1, ..., s^K$, and $(r^1\delta_{s^1} - r^1\delta_{d^1}), ..., (r^K\delta_{s^K} - r^K\delta_{d^K})$) by stacking sequentially on top of each other. The equations for the Nash equilibrium solutions are:

$$(NE)\begin{cases}(I_K \otimes A)x = R \\ e \otimes b + (E_K \otimes \mathrm{Diag}(a))x + (I_K \otimes \mathrm{Diag}(a))x + (I_K \otimes A)^{\mathrm{T}}y - s = 0 \\ x \geq 0,\ s \geq 0,\ x^{\mathrm{T}}s = 0,\end{cases}$$

where '$\otimes$' stands for the Kronecker product between two matrices, e is the (K by 1) all-one vector, E_K is the (K by K) all-one matrix, and I_K is the (K by K) identity matrix. The so-expressed Nash equilibrium is a mixed linear complementarity problem, and it specifies an equivalent condition for an point to be at Nash equilibrium. Since (P'_k) is a convex quadratic program for each given k, (NE) is a *necessary and sufficient* condition for a solution to be at Nash equilibrium. It follows from the properties of a convex program that the cost C^k is continuously dependent on the parameters R.

As (NE) is now posed as a mixed LCP, let us quote a well-known property of for the *monotone* LCP. Consider

$$(LCP)\begin{cases}s = q + Mx + L^{\mathrm{T}}y \\ Lx = b \\ x \geq 0,\ s \geq 0,\ x^{\mathrm{T}}s = 0,\end{cases}$$

where the dimensions of all the matrices are assumed to be compatible. The problem (LCP) is called *monotone* if $(\Delta x)^{\mathrm{T}}M\Delta x \geq 0$ for all Δx satisfying $L\Delta x = 0$; that is, $M + M^{\mathrm{T}}$ is a positive semidefinite matrix over the null space of L. Also, the mixed LCP problem (LCP) is *feasible* if there exist x and y satisfying

$$q + Mx + L^{\mathrm{T}}y \geq 0,\ x \geq 0,\ \text{and } Lx = b.$$

The following result is adapted from Theorems 3.1.2 and 3.1.7 in Cottle, Pang and Stone [17].

Theorem 2.2.1 *Suppose that (LCP) is monotone and is feasible. Then (LCP) has a solution, and the solution is unique in the sense that there is an index set α such that x is a solution to (LCP) if and only if x is feasible and the support of x is in α.*

Indeed as we shall see below that the mixed LCP problem arising from the Nash equilibrium of the minimum cost transportation game is monotone. For any Δx satisfying

$$(I_K \otimes A)\Delta x = 0,$$

and the corresponding

$$\Delta s = (E_K \otimes \mathrm{Diag}(a))\Delta x + (I_K \otimes \mathrm{Diag}(a))\Delta x + (I_K \otimes A)^{\mathrm{T}}\Delta y,$$

we have

$$\begin{aligned}(\Delta x)^{\mathrm{T}}\Delta s &= (\Delta x)^{\mathrm{T}}(E_K \otimes \mathrm{Diag}(a))\Delta x + (\Delta x)^{\mathrm{T}}(I_K \otimes \mathrm{Diag}(a))\Delta x \\ &= (\Delta x)^{\mathrm{T}}((E_K + I_K) \otimes \mathrm{Diag}(a))\Delta x \geq 0,\end{aligned}$$

because $(E_K + I_K) \otimes \mathrm{Diag}(a)$ is positive semidefinite. This shows that (NE) is a monotone mixed LCP. Clearly, it is also feasible by noting $b \geq 0$. It follows from Theorem 2.2.1 that (NE) has a unique Nash solution.

Summarizing, we have the following theorem:

Theorem 2.2.2 *For a transportation game (G, r, c) with affine linear cost, the Nash equilibrium is unique and when observing the cost of each player at the Nash equilibrium as function of the input variables (r, c), it is continuous.*

The remaining analysis is to derive a definite bound on the PoA for the routing game. As it turns out, our result can be viewed as an extension of the celebrated 4/3 bound on the PoA due to Roughgarden and Tardos [61]. To put the picture in perspective, let us first state the following conclusion:

Lemma 2.2.3 *Let us denote y to be the solution at the Nash equilibrium. Suppose that Player k changes his/her strategy from y^k to any other feasible flow x^k, while all other players' strategies remain unchanged. Then, the cost for Player k will increase by at least $\sum_{l\in L} a_l(x_l^k - y_l^k)^2$; that is*

$$C^k(x^k, y^{-k}) - C^k(y^k, y^{-k}) \geq \sum_{l\in L} a_l(x_l^k - y_l^k)^2.$$

Proof. Denote y to be a flow vector at Nash equilibrium, and y^k is the solution of Player k, whose cost is $C^k(\,\cdot\,, y^{-k})$ where other players' strategy is denoted as y^{-k}. Then

$$\begin{aligned}
& C^k(x^k, y^{-k}) - C^k(y^k, y^{-k}) - \sum_{l\in L} a_l(x_l^k - y_l^k)^2 \\
= & \sum_{l\in L} \left\{ \left[a_l(x_l^k + \sum_{i\neq k} y_k^i) + b_l \right] x_l^k - \left[a_l(y_l^k + \sum_{i\neq k} y_k^i) + b_l \right] y_l^k - a_l(x_l^k - y_l^k)^2 \right\} \\
= & \sum_{l\in L} \left(2a_l y_l^k + a_l \sum_{i\neq k} y_k^i + b_l \right) (x_l^k - y_l^k) \geq 0.
\end{aligned}$$

The last inequality is due to the convexity of function $C^k(\,\cdot\,, y^{-k})$, and the fact that $2a_l y_l^k + a_l \sum_{i\neq k} y_k^i + b_l$ is the derivative of $C^k(\,\cdot\,, y^{-k})$ at the minimum point y^k. □

With this lemma in mind, following a similar argument as in Roughgarden [64], we then get an upper bound for the price of anarchy:

Theorem 2.2.4 *In the flow transportation game on a general network, suppose the unit cost is affine linear in the total flow value, then the price of anarchy is upper bounded by $\frac{3K+1}{2K+2}$.*

Proof. Let x and y denote the solutions at the social optimum and a Nash equilibrium, respectively. According to Lemma 4.1.1, to get an upper bound for the price of anarchy, we just need to find such a (λ, μ) pair with $\lambda > 0$ and $0 < \mu < 1$, such that

$$\sum_k \left[C^k(x^k, y^{-k}) - \sum_l a_l (x_l^k - y_l^k)^2 \right] \leq \lambda\, \mathrm{SC}(x) + \mu\, \mathrm{SC}(y). \tag{2.1}$$

Note that if (4.1) holds, then by Lemma 4.1.1 we have

$$\begin{aligned} \mathrm{SC}(y) &= \sum_k C^k(y^k, y^{-k}) \leq \sum_k \left[C^k(x^k, y^{-k}) - \sum_l a_l (x_l^k - y_l^k)^2 \right] \\ &\leq \lambda\, \mathrm{SC}(x) + \mu\, \mathrm{SC}(y). \end{aligned}$$

Thus the price of anarchy can be bounded as: $\mathrm{PoA} = \frac{\mathrm{SC}(y)}{\mathrm{SC}(x)} \leq \frac{\lambda}{1-\mu}$.

Let us now turn to the search of (λ, μ) to satisfy (4.1). Denote f^x (respectively, f^y) to be the total flow on the links when the game attains the optimum (respectively, Nash equilibrium), i.e.,

$$f_l^x = \sum_{k=1}^K x_l^k, \qquad \left(\text{respectively, } f_l^y = \sum_{k=1}^K y_l^k \right),$$

and substitute the explicit form of the cost functions in (4.1), then the intended inequality becomes:

$$\begin{aligned} &\sum_k \sum_l \left[a_l (x_l^k + f_l^y - y_l^k) x_l^k + b_l x_l^k - a_l (x_l^k - y_l^k)^2 \right] \\ \leq\ & \lambda \sum_l \left[a_l (f_l^x)^2 + b_l f_l^x \right] + \mu \sum_l \left[a_l (f_l^y)^2 + b_l f_l^y \right]. \end{aligned}$$

Notice that the order of summation can be interchanged, and so the left hand side of the inequality above can be rewritten as:

$$LHS = \sum_l \left[a_l f_l^x f_l^y + a_l \sum_k [y_l^k (x_l^k - y_l^k)] + b_l f_l^x \right].$$

Rearranging the items on the right hand side, we have

$$RHS = \sum_l \left[a_l \lambda (f_l^x)^2 + a_l \mu (f_l^y)^2 + b_l \lambda f_l^x + b_l \mu f_l^y \right].$$

We regroup the terms and get

$$\begin{aligned}&RHS - LHS\\ =\ & \sum_l a_l \left[\lambda(f_l^x)^2 + \mu(f_l^y)^2 - f_l^x f_l^y - \sum_k [y_l^k(x_l^k - y_l^k)]\right] + \sum_l b_l \left[(\lambda - 1)f_l^x + \mu f_l^y\right].\end{aligned}$$

Note that the above is a summation on the index l. To ensure $RHS - LHS \geq 0$ (which leads to (4.1)), we need only to establish the inequality for each link, namely

$$\lambda(f_l^x)^2 + \mu(f_l^y)^2 - f_l^x f_l^y - \sum_k [y_l^k(x_l^k - y_l^k)] \geq 0 \text{ and } (\lambda - 1)f_l^x + \mu f_l^y \geq 0 \text{ for all } l. \quad (2.2)$$

By requiring $\lambda \geq 1, \mu \geq 0$, the second part of (2.2) holds trivially.

Now, by the Cauchy-Schwartz inequality we have

$$\sum_{k=1}^{K} (\frac{1}{2}x_l^k - y_l^k)^2 \geq \frac{1}{K}(\frac{1}{2}f_l^x - f_l^y)^2,$$

which leads to

$$\begin{aligned}\sum_k [y_l^k(x_l^k - y_l^k)] &= \frac{1}{4}\sum_k (x_l^k)^2 - \sum_{k=1}^{K}(\frac{1}{2}x_l^k - y_l^k)^2\\ &\leq \frac{1}{4}\sum_k (x_l^k)^2 - \frac{1}{4K}(f_l^x)^2 - \frac{1}{K}(f_l^y)^2 + \frac{1}{K}f_l^x f_l^y \qquad (2.3)\\ &\leq \left(\frac{1}{4} - \frac{1}{4K}\right)(f_l^x)^2 - \frac{1}{K}(f_l^y)^2 + \frac{1}{K}f_l^x f_l^y. \qquad (2.4)\end{aligned}$$

Therefore,

$$\begin{aligned}&\lambda(f_l^x)^2 + \mu(f_l^y)^2 - f_l^x f_l^y - \sum_k [y_l^k(x_l^k - y_l^k)]\\ \geq\ & \left(\lambda - \frac{1}{4} + \frac{1}{4K}\right)(f_l^x)^2 + \left(\mu + \frac{1}{K}\right)(f_l^y)^2 - \left(\frac{1}{K} + 1\right) f_l^x f_l^y.\end{aligned}$$

To ensure the above to be always nonnegative (hence the first part of (2.2)), it suffices to have (λ, μ) satisfy

$$4\left(\lambda - \frac{1}{4} + \frac{1}{4K}\right)\left(\mu + \frac{1}{K}\right) \geq \left(1 + \frac{1}{K}\right)^2.$$

Hence to derive an upper bound for price of anarchy, we are naturally led to the following optimization problem:

$$\begin{aligned} \min \quad & \frac{\lambda}{1-\mu} \\ \text{s.t.} \quad & 4(\lambda - \tfrac{1}{4} + \tfrac{1}{4K})(\mu + \tfrac{1}{K}) \geq (1 + \tfrac{1}{K})^2 \\ & \lambda \geq 1, \ \mu \geq 0. \end{aligned}$$

The above problem can be explicitly solved, and the optimal value $\frac{3K+1}{2K+2}$ is attained at the optimal solution $\lambda = 1$, $\mu = \frac{K-1}{3K+1}$. □

According to the Cauchy-Schwartz Inequality, the equality holds in (2.3) if and only if all x_l^k's are equal and all y_l^k's are equal. However, the equality holds in (2.4) only if $x_l^{k_0} = f_l$ for some k_0 and $x_l^k = 0$ for $k \neq k_0$. It means that the equalities can not hold simultaneously, hence the bound cannot be tight. It remains an interesting open problem to find the tight upper bound for the PoA in this general case.

2.2.1 The Bound for the PoA when All the Players are Identical

Although it is in general not known how to improve the upper bound for the PoA, we are able to get a tight bound when all the players have the same loads and the same pair of OD. In other words, it is possible to get the tight bound for the PoA when all the players are identical. The result is formalized in the next theorem.

Theorem 2.2.5 *In the routing game with K identical players, the price of anarchy is bounded above by $\frac{4K^2}{3K^2+2K-1}$, which is tight, meaning that there is a specific instance of the game in which the PoA is precisely this ratio.*

Proof. Since the players are assumed to be identical, in the optimal solution we can simply choose $x_l^k = f_l^x/K$ for all k and l. It follows from (4.2) that

$$
\begin{aligned}
&RHS - LHS \\
&= \sum_l a_l \left\{ \lambda (f_l^x)^2 + \mu (f_l^y)^2 - f_l^x f_l^y - \sum_k \left[y_l^k \left(\frac{f_l^x}{K} - y_l^k \right) \right] \right\} + \sum_l b_l \left[(\lambda - 1) f_l^x + \mu f_l^y \right] \\
&= \sum_l a_l \left[\lambda (f_l^x)^2 + \mu (f_l^y)^2 - \left(1 + \frac{1}{K}\right) f_l^x f_l^y + \sum_k (y_l^k)^2 \right] + \sum_l b_l \left[(\lambda - 1) f_l^x + \mu f_l^y \right] \\
&\geq \sum_l a_l \left[\lambda (f_l^x)^2 + \left(\mu + \frac{1}{K}\right) (f_l^y)^2 - \left(1 + \frac{1}{K}\right) f_l^x f_l^y \right] + \sum_l b_l \left[(\lambda - 1) f_l^x + \mu f_l^y \right],
\end{aligned}
$$

where in the last inequality we used the fact that $\sum_k (y_l^k)^2 \geq \frac{1}{K}(f_l^y)^2$.

Similar as before, to obtain the best possible bound we are led to consider the optimization problem:

$$
\begin{aligned}
\min \quad & \tfrac{\lambda}{1-\mu} \\
\text{s.t.} \quad & 4\lambda(\mu + \tfrac{1}{K}) \geq (1 + \tfrac{1}{K})^2 \\
& \lambda \geq 1, \ \mu \geq 0
\end{aligned}
$$

which indeed has an optimal solution $\lambda = 1$, $\mu = \frac{(K-1)^2}{4K^2}$, and thus

$$\text{PoA} \leq \frac{4K^2}{3K^2 + 2K - 1}.$$

□

To show that the bound in Theorem 2.2.5 is tight, let us consider the following example.

Example 2.2.6 Consider a transportation game with a two-node-two-link network shown in Figure 2.1. Suppose that the unit cost functions are given by:

$$c_1(f_1) = 1, \ c_2(f_2) = \frac{K}{K+1} f_2.$$

Suppose there are K identical players and $r^1 = \cdots = r^K = \frac{1}{K}$. Then, in the Nash

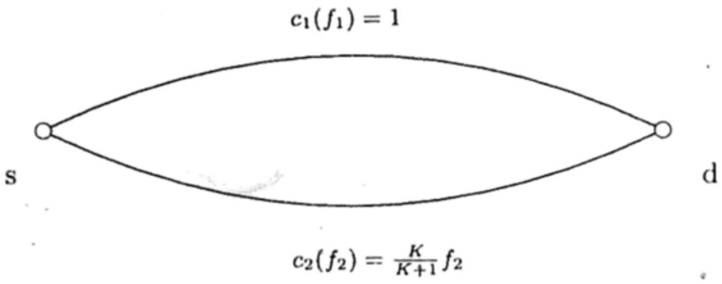

Figure 2.1: An example showing that the bound $\frac{4K^2}{3K^2+2K-1}$ is tight

solution, all the players will choose the lower link, yielding a total social cost of $\frac{K}{K+1}$. A simple calculation shows that the social optimal solution is that each player transports a flow of $\frac{K+1}{2K^2}$ along the lower link, and transports a flow of $\frac{K-1}{2K^2}$ along the upper link, yielding a total social cost of $\frac{3K-1}{4K}$. Therefore, the PoA in this case is exactly $\frac{4K^2}{3K^2+2K-1}$.

One may view Theorem 2.2.5 as a generalization of the famous 4/3 bound for the Wardrop model, where K is infinity. Also, it is interesting to note that if the players are not identical then $\frac{4K^2}{3K^2+2K-1}$ cannot be an upper bound for the general case as discussed in Theorem 2.2.4; see the example below.

Example 2.2.7 Consider a transportation game with a triangular network shown in Figure 2.2. The unit cost functions on $\overrightarrow{AC}, \overrightarrow{BC}$ are $c_l(f_l) = f_l$, and on $\overrightarrow{AB}, \overrightarrow{BA}$ are $\frac{1}{2}$. Suppose there are two players with $s^1 = A$, $d^1 = C$; $s^2 = B$, $d^2 = C$, and the required transportation tasks $r^1 = r^2 = 1$. In this case, the social optimal solution would be to separate the two routes of transportation: Player 1 goes from A to C and Player 2 goes from B to C, yielding the total social cost of 2. The (unique) Nash equilibrium in this case is for Player 1 to transport a flow of 3/4 from A to C, and another flow of 1/4 from A via B to C, and symmetrically the same for Player 2. In this case, the

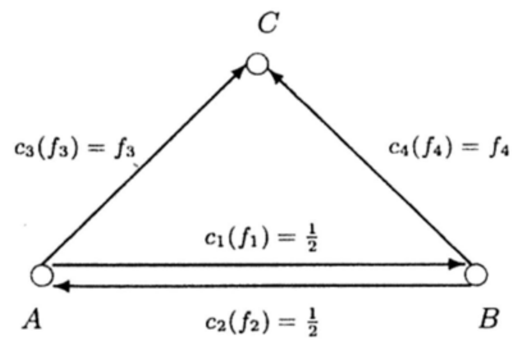

Figure 2.2: An example showing that $\frac{4K^2}{3K^2+2K-1}$ fails to be a bound generally

total social cost is 9/4. Therefore, for this instance the price of anarchy is

$$\text{PoA} = \frac{\frac{9}{4}}{2} = \frac{9}{8} > \frac{16}{15} = \frac{4 \cdot 2^2}{3 \cdot 2^2 + 2 \cdot 2 - 1}.$$

2.3 Bounding the PoA in an Affine Cost Sharing Transportation Game

In this section we shall consider another variant of the transportation game. Our discussion is motivated by the following consideration. It is common in economics to assume that the total cost occurred to a facility is affine linearly dependent on the total usage, the first part being the fixed cost and the second part being the variable cost. In the context of network, the total cost occurred to a link l, under this consideration, is $a_l f_l + b_l$. Suppose that the rule of the game is to let all the participants share the cost according to their respective (proportional) usage of the facility. Then, the cost for Player k on link l, assuming his/her usage of the link is x_l^k, is $(a_l f_l + b_l)\frac{x_l^k}{f_l} = (a_l + b_l/f_l)x_l^k$. In other words, this consideration leads to the game in which the unit cost function is $a_l + b_l/f_l$, instead of affine linear.

Let us consider the routing problem (the total flow being transported is not a part of decision). The optimization problem faced by Player k is:

$$
\begin{array}{lll}
(Q_k) & \min & \sum_{l\in\{l|x_l^k>0\}}(a_l+b_l/f_l)x_l^k \\
 & \text{s.t.} & Ax^k = r^k\delta_{s^k} - r^k\delta_{d^k},\ x^k \geq 0.
\end{array}
$$

Note that the objective function above is lower semi-continuous[2], and the feasible region is compact. Therefore, the optimal solution of (Q_k) exists. In particular, the objective for Player k is

$$
\begin{aligned}
\sum_{l\in\{l|x_l^k>0\}}(a_l+b_l/f_l)x_l^k &= \sum_{l\in\{l|x_l^k>0\}} a_l x_l^k + \sum_{l\in\{l|x_l^k>0\}} \frac{b_l x_l^k}{\sum_{j\neq k} x_l^j + x_l^k} \\
&= \sum_{l\in\{l|x_l^k>0\}} a_l x_l^k + \sum_{l\in\{l|x_l^k>0\}} \left(b_l - \frac{b_l \sum_{j\neq k} x_l^j}{\sum_{j\neq k} x_l^j + x_l^k} \right),
\end{aligned}
$$

which is in fact concave in x^k, for the fixed x^{-k}. This implies that each player will naturally choose to transport along a *single path*. Moreover, for the fixed x^{-k}, the path is nothing but the shortest path on the network from s^k to d^k, where the weight on the link l is $c_l(r^k; x_l^{-k})$, for each $l \in L$.

As before, we may consider the social optimal solution which minimizes the total cost of all the players. Such optimal solution exists, due to the lower semi-continuity of the objective. Next, let confine ourselves to a special case where all the players have the same origin and destination for the transportation. In this case, the social optimal solution is indeed easy to find: it is equivalent to finding the shortest path for a single user (with the demand aggregated as $r^1 + \cdots + r^K$). In other words, the social optimal solution in this case is to transport all the commodities along a single path, which can be found in polynomial-time by solving a shortest path problem. Moreover, the social

[2]The discontinuity only occurs at the points where $x_l^k = 0$ for some l. Since $c_l = 0$ if $x_l^k = 0$ and $c_l > 0$ if $x_l^k \neq 0$, the lower semi-continuity follows.

optimal solution is, in this case, a Nash equilibrium.

Proposition 1 *If the OD pairs for all the players are identical, the social optimal solution is a Nash equilibrium.*

Proof. Suppose that there is a player, say Player 1, who will be better off deviating from the social optimal solution of transporting along the path p. Now fix the strategies of all other players. Suppose Player 1 is better off transporting a flow of volume r^1 along another path p′ from source s to destination d. This means that

$$\sum_{l \in \mathrm{p}'} (a_l r^1 + b_l) \leq \sum_{l \in \mathrm{p}} \left[a_l \left(\sum_{k=1}^{K} r^k \right) + b_l \right] \frac{r^1}{\sum_{k=1}^{K} r^k},$$

or

$$\sum_{l \in \mathrm{p}'} \left(a_l + \frac{b_l}{r^1} \right) \leq \sum_{l \in \mathrm{p}} \left(a_l + \frac{b_l}{\sum_{k=1}^{K} r^k} \right).$$

Hence,

$$\sum_{l \in \mathrm{p}'} \left(a_l + \frac{b_l}{\sum_{k=1}^{K} r^k} \right) \leq \sum_{l \in \mathrm{p}} \left(a_l + \frac{b_l}{\sum_{k=1}^{K} r^k} \right),$$

which leads to that transporting all flows along p′ is simply better than p, contradicting to the optimality of p. This in turn shows that in the social optimal solution, no player will have any incentive to deviate; it is actually a Nash equilibrium. □

Proposition 1 shows that finding the social optimal solution, is polynomially solvable, and moreover it is automatically a Nash equilibrium. Therefore, for this game, the so-called *Price of Stability*, defined as the ratio between the best social value among the Nash equilibria and the optimal social value, is equal to 1. However, the Nash equilibrium is not necessarily unique in this case, due to the lack of convexity of the objective function. As for the PoA, which is defined as the ratio between the *worst* social value among the Nash equilibrium solutions and that of the optimal social value, may still be larger than 1. However there is a tight upper bound for the PoA, as we show below.

Theorem 2.3.1 *Consider the flow transportation game with K players. Suppose that the total cost on each link is affine linear in the total flow, to be shared by the players proportional to the usage. Furthermore, suppose that all the players have the same origin and destination. Then, the price of anarchy in the game is upper bounded by K, and this bound can be attained.*

Proof. The proof for Proposition 1 in fact suggests that any Nash equilibrium must be so that all the players share a path. Indeed, if in a Nash equilibrium there are two different paths being used, since all the players have the same OD pair, then it follows that the marginal cost along these two paths must be identical. Switching from either one of them and combining reduces the marginal cost. In this game, there is an incentive to combine due to the economy of scale effect. Hence, any Nash equilibrium will occur on a single path.

Now let $\mathbf{p}^{\mathrm{n}}$ denote a path all players use at a Nash equilibrium, and $\mathbf{p}^*$ denote the social optimal path. Then the price of anarchy is

$$\mathrm{PoA} = \frac{\left[\sum_{l\in\mathbf{p}^{\mathrm{n}}}\left(a_l + b_l/\sum_k r^k\right)\right]\sum_k r^k}{\left[\sum_{l\in\mathbf{p}^*}\left(a_l + b_l/\sum_k r^k\right)\right]\sum_k r^k} = \frac{\sum_{l\in\mathbf{p}^{\mathrm{n}}}\left(a_l\sum_k r^k + b_l\right)}{\sum_{l\in\mathbf{p}^*}\left(a_l\sum_k r^k + b_l\right)}, \tag{2.5}$$

where r^k is the required traffic rate of player k. Since $\mathbf{p}^{\mathrm{n}}$ is the path under Nash equilibrium, we have

$$\sum_{l\in\mathbf{p}^{\mathrm{n}}}\left(a_l\sum_k r^k + b_l\right)\frac{r^i}{\sum_k r^k} \le \sum_{l\in\mathbf{p}^*}\left(a_l r^i + b_l\right) \quad \text{for all } i = 1, \ldots, K.$$

Summing over i, we have

$$\sum_{l\in \mathbf{p}^n}\left(a_l\sum_k r^k+b_l\right)\le \sum_{l\in \mathbf{p}^*}\left(a_l\sum_k r^k+Kb_l\right)\le K\sum_{l\in \mathbf{p}^*}\left(a_l\sum_k r^k+b_l\right).$$

Hence, by the above relation it follows from (2.5) the desired inequality

$$\text{PoA}=\frac{\sum_{l\in \mathbf{p}^n}\left(a_l\sum_k r^k+b_l\right)}{\sum_{l\in \mathbf{p}^*}\left(a_l\sum_k r^k+b_l\right)}\le K.$$

□

The tightness of the bound is shown in the following example:

Example 2.3.2 Suppose we are give a two-node-two-link network shown in Figure 2.3 and the cost functions of the two links are given by

$$c_1(f_1)=1,\ \text{and}\ \ c_2(f_2)=\frac{1}{f_2}.$$

Suppose there are K identical players and $r^1=\cdots=r^K=1$. In this example, all the

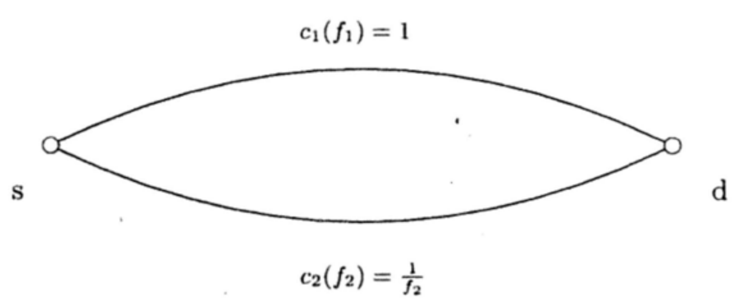

Figure 2.3: An example showing that the bound K is tight

players may use the upper link, and all the players may as well use the lower link, and both are indeed the Nash equilibria, with the later being socially optimal. The first mentioned equilibrium has a social cost of K, while the second equilibrium has a social cost of 1. Therefore, the price of anarchy is precisely K in this example.

It is still an open problem to figure out whether or not a bound exists for the PoA when the players are not identical (with different OD pairs) in this game.

Chapter 3

The Price of Anarchy in a Cournot Game

3.1 Introduction

3.1.1 A Generalized Cournot Competition Model

The classic Cournot competition game was first formulated by Cournot in [18]. It describes a game structure in which finitely many companies compete on the sales of the products they will produce, which they jointly and independently decide. Here, we consider a generalized model of the Cournot competition game. In addition to the competition in the market in terms of the sales, we shall also consider the competition on the resources. Specifically, we consider the case where finitely many noncooperative players compete for a set of shared resources, and then sell the produced products in the market. We take the sum of the profit of all the players as the social value and our aim is to study the loss of the social efficiency due to the competition. In particular, we investigate the Nash equilibrium of the network game involving K players and obtain some bounds on the PoA in such games.

Suppose there are m given types of resources that are used to produce n types of products. The relationship between the resources and the products can be characterized by a technology matrix, denoted by M. Namely, if one uses a nonnegative vector $x \in \mathbb{R}_+^m$ to denote the usage of resources and $v \in \mathbb{R}_+^n$ to denote the quantity of products being produced from the resource, we then have $Mv \le x$. Suppose there are K producers (viewed as players in the framework of a game) who are in the business of making use of the resources to produce the products and earn their profits via the sales of the products. On the one hand, the cost of each type of resource increases as the usage of the resource increases. For resource l, we denote player k's usage of the resource as x_l^k, and the total usage of the resource as $f_l = \sum_{k=1}^K x_l^k$. Then, the *unit* cost for the usage of l is given by a function $c_l : f_l \hookrightarrow c_l(f_l)$. On the other hand, players earn profits from sales of the products. However, the price of each type of product is dependent on the amount of supply of similar products in the market. Suppose that the amount of products produced by player k is v^k, and v^{-k} is the decision of other players. The price vector of that player k is assumed to be a vector $p^k(v^k, v^{-k}) \in \mathbb{R}_+^n$. Note the technology matrix for player k is M_k. To maximize the profit, player k shall consider the following optimization problem:

$$
\begin{aligned}
(P_k) \quad & \max \quad V^k(v^k, x^k; v^{-k}, x^{-k}) = (v^k)^{\mathrm{T}} p^k(v^k, v^{-k}) - (x^k)^{\mathrm{T}} c^k(x^k, x^{-k}) \\
& \text{s.t.} \quad M_k v^k \le x^k,\ x^k \ge 0, v^k \ge 0.
\end{aligned}
$$

Naturally, given the decisions of all the players, the social value is a simple summation: $\mathrm{SV}(v, x) = \sum_{k=1}^K V^k(v^k, x^k; v^{-k}, x^{-k})$. Let us denote (w, y) to be the solution when the game reaches a Nash equilibrium; i.e. a solution at which no player will be able to improve his/her situation unilaterally. At the same time, let us denote (v, x) to be the socially optimal solution – the solution that maximizes the social function SV over all feasible solutions. The price of anarchy is defined as: $\text{PoA} = \frac{\text{SV}(w,y)}{\text{SV}(v,x)}$.

3.1.2 Related Work

Since the Cournot model was introduced in [18], it has become a classical economic tool for companies competing on the amount of output they will produce. A lot of empirical studies demonstrates that the vicious competition among the producers may result in the destruction of the industry. The PoA is employed as a quantitative measure to estimate the inefficiency and is well studied in the literature. We focus here on the oligopolistic Cournot setting, in which finitely many suppliers face the same market and choose their own production levels which affect the sales prices, to maximize their own profit. The competition among the producers may result in the congestion effect: the more commodities produced, the less unit revenue one can expect. Guo and Yang [27] employed the total surplus of consumers and producers as the social welfare and achieved bounds on the PoA which are dependent on the market shares and demand, and the number of players. Immorlica, Markakis and Piliouras [33] studied coalition formation in a dynamic setting of Cournot oligopolies and proved that the PoA under their notion of stability is bounded by $\Theta(K^{2/5})$, where K is the number of players. Based on the work [57] by Perakis, which considered the loss of efficiency if the cost is asymmetric in the traffic equilibria, Kluberg and Perakis [34] extended the classic Cournot model to an asymmetric case for both substitute and complement products and bounded the PoA through the market power parameters and the number of players and products.

3.2 Bounding the PoA with Affine Linear Price/Cost Functions

In this section we shall consider the case where the unit costs for the usage of resources are affine linear in the total usages, and the unit selling prices are also affine linear in the total supply. In particular, suppose that the total usage of Resource l is f_l, then the *unit cost* of Resource l is $c_l(f_l) = a_l f_l + b_l$, where a_l, b_l are some nonnegative constant parameters. Moreover, we assume the n types of products are uncorrelated, however the same type of products produced by different players are substitute to each other. Suppose Player k produces v^k while other players produce v^{-k}, then the price for Product j applicable to Player k is $p^k(v^k, v^{-k})_j = q_j^k - \sum_{i=1}^K \gamma_j^{ki} v_j^i$, where q_j^k and γ_j^{ki} are nonnegative constants. In matrix notation we may write $p^k(v^k, v^{-k}) = q^k - \sum_{i=1}^K \Gamma^{ki} v^i$, where Γ^{ki} is a diagonal matrix, for the fixed k, i. Technically we assume that $(\gamma_j^{ki})^2 \le \gamma_j^{kk}\gamma_j^{ii}$ for all $1 \le i, k \le K$ and $1 \le j \le m$. (One may interpret this condition as to say that the effect on the price due to the actions of the other players is less significant than the effect of the action of oneself). Moreover, we also assume that $\gamma_j^{kk} > 0$ for all j, k, meaning that one's production of a certain product will affect his/her own sales of the same product. Player k will face the following optimization problem:

$$
\begin{array}{lll}
(\bar{P}_k) & \max & \sum_{j=1}^n (q_j^k - \sum_{i=1}^K \gamma_j^{ki} v_j^i) v_j^k - \sum_{l=1}^m (a_l f_l + b_l) x_l^k \\
 & \text{s.t.} & M_k v^k \le x^k,\ x^k \ge 0, v^k \ge 0.
\end{array}
$$

Since the prices are naturally nonnegative, simple estimation shows that for any optimal solution we have $0 \le v_j^k < q_j^k/\gamma_j^{kk}$ for all j, k, and $x_l^k < \frac{(q_j^k)^2}{b_l \gamma_j^{kk}}$ if $b_l > 0$ and $x_l^k < \sqrt{\frac{(q_j^k)^2}{a_l \gamma_j^{kk}}}$ if $a_l > 0$. For any free resource l (i.e. the item with $a_l = 0$ and $b_l = 0$), the usage of the resource will have no impact on the profit but it may affect the constraints on v^k. However, since v^k is already bounded, we may without loss of generality assume that the free resources are also bounded. Due to the boundedness of the solutions, the existence of a Nash equilibrium follows from the concavity of objective function (when the actions of other players are fixed) and the convexity of feasible regions of the players. Letting z^k be the Lagrangian dual variable for the constraint $M_k v^k - x^k \leq 0$, and s^k and t^k be the dual variables for the constraint $x^k \geq 0$ and $v^k \geq 0$ respectively, the overall optimality condition (or the equilibrium condition) is the following LCP system:

$$\begin{cases} M_k v^k - x^k \leq 0,\ z^k \geq 0, \\ (z^k)^{\mathrm{T}}(M_k v^k - x^k) = 0, \\ \mathrm{Diag}(a)\sum_{i=1}^K x^i + \mathrm{Diag}(a)x^k - z^k - s^k = -b, \\ \sum_{i=1}^K \Gamma^{ki} v^i + \Gamma^{kk} v^k + M_k^{\mathrm{T}} z^k - t^k = q^k, \\ (x^k)^{\mathrm{T}} s^k = 0,\ x^k \geq 0,\ s^k \geq 0, \\ (v^k)^{\mathrm{T}} t^k = 0,\ v^k \geq 0,\ t^k \geq 0, \end{cases}$$

where $\mathrm{Diag}(a)$ is the diagonal matrix whose l-th diagonal is a_l, $l = 1, ..., m$.

Lemma 3.2.1 *At a Nash equilibrium of the extended Cournot competition game with solution (w, y), the profit of Player k in the equilibrium equals to $\sum_{j=1}^n \gamma_j^{kk}(w_j^k)^2 + \sum_{l=1}^m a_l(y_l^k)^2$.*

Proof. Since each player at the Nash equilibrium attains optimality given the strategies of the others, by the KKT condition we have

$$\begin{aligned} &(z^k)^{\mathrm{T}} M_k w^k = (z^k)^{\mathrm{T}} y^k, \\ &q^k - \sum_{i=1}^K \Gamma^{ki} w^i = \Gamma^{kk} w^k + M_k^{\mathrm{T}} z^k - t^k, \\ &\mathrm{Diag}(a)\sum_{i=1}^K y^i + \mathrm{Diag}(a) y^k + b - s^k = z^k. \end{aligned}$$

Hence the profit for Player k is

$$
\begin{aligned}
& \left(q^k - \sum_{i=1}^{K} \Gamma^{ki} w^i\right)^{\mathrm{T}} w^k - \left(\sum_{i=1}^{K} \mathrm{Diag}(a) y^i + b\right)^{\mathrm{T}} y^k \\
= \; & (\Gamma^{kk} w^k + M_k^{\mathrm{T}} z^k - t^k)^{\mathrm{T}} w^k - \left(\sum_{i=1}^{K} \mathrm{Diag}(a) y^i + b\right)^{\mathrm{T}} y^k \\
= \; & (w^k)^{\mathrm{T}} \Gamma^{kk} w^k + (z^k)^{\mathrm{T}} y^k - \left(\sum_{i=1}^{K} \mathrm{Diag}(a) y^i + b\right)^{\mathrm{T}} y^k \\
= \; & (w^k)^{\mathrm{T}} \Gamma^{kk} w^k + \left(\mathrm{Diag}(a) \sum_{i=1}^{K} y^i + \mathrm{Diag}(a) y^k + b - s^k\right)^{\mathrm{T}} y^k \\
& - \left(\mathrm{Diag}(a) \sum_{i=1}^{K} y^i + b\right)^{\mathrm{T}} y^k \\
= \; & (w^k)^{\mathrm{T}} \Gamma^{kk} w^k + (y^k)^{\mathrm{T}} \mathrm{Diag}(a) y^k \\
= \; & \sum_{j=1}^{n} \gamma_j^{kk} (w_j^k)^2 + \sum_{l=1}^{m} a_l (y_l^k)^2 .
\end{aligned}
$$

□

Lemma 3.2.2 *Denote (v, x) and (w, y) to be the solutions at the social optimum and at a Nash equilibrium respectively. At the Nash equilibrium (w, y), suppose that player k switches to the strategy to (v^k, x^k) while all other players' strategies remain unchanged. Then, the profit of player k will decrease by at least an amount of $\sum_{j=1}^{n} \gamma_j^{kk} (v_j^k - w_j^k)^2 + \sum_{l=1}^{m} a_l (x_l^k - y_l^k)^2$; that is,*

$$
V^k(w^k, y^k; w^{-k}, y^{-k}) - V^k(v^k, x^k; w^{-k}, y^{-k}) \geq \sum_{j=1}^{n} \gamma_j^{kk} (v_j^k - w_j^k)^2 + \sum_{l=1}^{m} a_l (x_l^k - y_l^k)^2 .
$$

Proof. By the definition of the Nash equilibrium, (w^k, y^k) is maximal for player k's profit function $V^k(\cdot, \cdot; w^{-k}, y^{-k})$, assuming the other players' strategies are fixed as

(w^{-k}, y^{-k}). Therefore,

$$
\begin{aligned}
&V^k(w^k, y^k; w^{-k}, y^{-k}) - V^k(v^k, x^k; w^{-k}, y^{-k}) - \sum_{j=1}^{n} \gamma_j^{kk}(v_j^k - w_j^k)^2 - \sum_{l=1}^{m} a_l(x_l^k - y_l^k)^2 \\
&= (q^k)^{\mathrm{T}}(w^k - v^k) \\
&\quad + \sum_{j=1}^{n} \left[\sum_{i \neq k} \gamma_j^{ki} w_j^i v_j^k + \gamma_j^{kk}(v_j^k)^2 - \sum_{i \neq k} \gamma_j^{ki} w_j^i w_j^k - \gamma_j^{kk}(w_j^k)^2 - \gamma_j^{kk}(v_j^k - w_j^k)^2 \right] \\
&\quad - \sum_{l=1}^{m} \left\{ \left[a_l(y_l^k + \sum_{i \neq k} y_k^i) + b_l \right] y_l^k - \left[a_l(x_l^k + \sum_{i \neq k} y_k^i) + b_l \right] x_l^k + a_l(x_l^k - y_l^k)^2 \right\} \\
&= \sum_{j=1}^{n} \left(q_j^k - \sum_{i \neq k} \gamma_j^{ki} w_j^i - 2\gamma_j^{kk} w_j^k \right)(w_j^k - v_j^k) - \sum_{l=1}^{m} \left(2a_l y_l^k + a_l \sum_{i \neq k} y_k^i + b_l \right)(y_l^k - x_l^k) \\
&= \nabla V^k(w^k, y^k; w^{-k}, y^{-k})^{\mathrm{T}} \begin{pmatrix} w^k - v^k \\ y^k - x^k \end{pmatrix} \geq 0,
\end{aligned}
$$

where the last inequality is due to the fact that (w^k, y^k) is maximal. (Note that the inequality because if x^* is the maximal point of a concave function $q(x)$ over a convex set S, then $\nabla q(x^*)^{\mathrm{T}}(x^* - x) \geq 0$ for all $x \in S$). □

Before proceeding, let us note the following property concerning monotone matrices.

Lemma 3.2.3 *Consider a matrix $\Omega = (\alpha^{ij})_{K \times K}$. Suppose that the diagonal elements of Ω are nonnegative, and $(\alpha^{ij})^2 \leq \alpha^{ii}\alpha^{jj}$ for all $1 \leq i, j \leq K$. Then,*

$$
\bar{\Omega} := \begin{bmatrix} (K-1)\alpha^{11} & -|\alpha^{12}| & \dots & -|\alpha^{1K}| \\ \dots & \dots & \dots & \dots \\ -|\alpha^{K1}| & -|\alpha^{K2}| & \dots & (K-1)\alpha^{KK} \end{bmatrix}
$$

is monotone.

Proof. Take any $\xi = (\xi^1, \ldots, \xi^K)^{\mathrm{T}} \in \mathbb{R}^K$. We have

$$\begin{aligned}
\xi^{\mathrm{T}}\bar{\Omega}\xi &= (\xi^1,\ldots,\xi^K)\begin{bmatrix} (K-1)\alpha^{11} & -|\alpha^{12}| & \ldots & -|\alpha^{1K}| \\ \ldots & \ldots & \ldots & \ldots \\ -|\alpha^{K1}| & -|\alpha^{K2}| & \ldots & (K-1)|\alpha^{KK}| \end{bmatrix}\begin{pmatrix} \xi^1 \\ \vdots \\ \xi^K \end{pmatrix} \\
&= \sum_{i=1}^{K}\left[(K-1)\alpha^{ii}(\xi^i)^2 - \sum_{j<i}(|\alpha^{ij}| + |\alpha^{ji}|)\,\xi^i\xi^j\right] \\
&= \sum_{i=1}^{K}\sum_{j<i}\left[\alpha^{ii}(\xi^i)^2 - (|\alpha^{ij}| + |\alpha^{ji}|)\,\xi^i\xi^j + \alpha^{jj}(\xi^j)^2\right] \\
&\geq \sum_{i=1}^{K}\sum_{j<i}\left(\sqrt{\alpha^{ii}}\,|\xi^i| - \sqrt{\alpha^{jj}}\,|\xi^j|\right)^2 \geq 0.
\end{aligned}$$

Therefore, $\bar{\Omega}$ is a monotone matrix as asserted. □

Theorem 3.2.4 *The price of anarchy in the extended Cournot game with K players, is lower bounded by $\frac{1}{K}$.*

Proof. As before, let (v,x) and (w,y) denote the solutions at the social optimum and at a Nash equilibrium respectively. According to Lemma 4.1.1, to get a lower bound for the price of anarchy, it suffices to find a (λ,μ) pair with $\lambda > 0$ and $\mu > -1$, such that

$$\begin{aligned}
&\sum_{k=1}^{K}\left[V^k(v^k,x^k;w^{-k},y^{-k}) + \sum_{j=1}^{n}\gamma_j^{kk}(v_j^k - w_j^k)^2 + \sum_{l=1}^{m}a_l(x_l^k - y_l^k)^2\right] \\
\geq\ & \lambda\,\mathrm{SV}(v,x) - \mu\,\mathrm{SV}(w,y). \qquad (3.1)
\end{aligned}$$

Note that if (4.1) holds, then by Lemma 4.1.1 we have

$$\begin{aligned}
\mathrm{SV}(w,y) &= \sum_{k=1}^{K}V^k(w^k,y^k;w^{-k}y^{-k}) \\
&\geq \sum_{k=1}^{K}\left[V^k(v^k,x^k;w^{-k},y^{-k}) + \sum_{j=1}^{n}\gamma_j^{kk}(v_j^k - w_j^k)^2 + \sum_{l=1}^{m}a_l(x_l^k - y_l^k)^2\right] \\
&\geq \lambda\,\mathrm{SV}(v,x) - \mu\,\mathrm{SV}(w,y).
\end{aligned}$$

Thus the price of anarchy can be bounded as: PoA $= \frac{\mathsf{SV}(w,y)}{\mathsf{SV}(v,x)} \geq \frac{\lambda}{1+\mu}$.

Let us now turn to searching (λ, μ) to satisfy (4.1). Denote f^x (respectively, f^y) to be the total usage of the resources when the game attains the social optimum (respectively, Nash equilibrium); i.e., $f_l^x = \sum_{k=1}^K x_l^k$, (respectively, $f_l^y = \sum_{k=1}^K y_l^k$). Substitute the explicit form of the cost functions in (4.1), and apply Lemma 3.2.1, then the intended inequality (4.1) becomes:

$$\sum_{k=1}^{K}\sum_{j=1}^{n}\left[q_j^k v_j^k - \sum_{i\neq k}\gamma_j^{ki} w_j^i v_j^k - \gamma_j^{kk}(v_j^k)^2 + \gamma_j^{kk}(v_j^k - w_j^k)^2\right]$$
$$-\sum_{k=1}^{K}\sum_{l=1}^{m}\left[a_l(x_l^k + f_l^y - y_l^k)x_l^k + b_l x_l^k - a_l(x_l^k - y_l^k)^2\right] - \lambda\sum_{k=1}^{K}\sum_{j=1}^{n}(q_j^k v_j^k - \sum_{i=1}^{K}\gamma_j^{ki} v_j^i v_j^k)$$
$$+\lambda\sum_{l=1}^{m}\left[a_l(f_l^x)^2 + b_l f_l^x\right] + \mu\sum_{k=1}^{K}\left[\sum_{j=1}^{n}\gamma_j^{kk}(w_j^k)^2 + \sum_{l=1}^{m}a_l(y_l^k)^2\right] \geq 0. \tag{3.2}$$

Observe that the left-hand side of the inequality (3.2) can be regrouped into two parts: Part I (see (3.3)) and Part II (see (3.6)) to be introduced below:

$$\text{`Part I'} = \lambda\sum_{l=1}^{m}\left[a_l(f_l^x)^2 + b_l f_l^x\right] + \mu\sum_{l=1}^{m}\sum_{k=1}^{K}a_l(y_l^k)^2$$
$$-\sum_{l=1}^{m}\left[a_l f_l^x f_l^y + a_l\sum_{k=1}^{K}[y_l^k(x_l^k - y_l^k)] + b_l f_l^x\right]. \tag{3.3}$$

Let us set $\lambda = 1$ and $\mu = K - 1$, and the above can be further written as

$$\begin{aligned}\text{`Part I'} &= \sum_{l=1}^{m}a_l\left[(f_l^x)^2 + (\mu+1)\sum_{k=1}^{K}(y_l^k)^2 - f_l^x f_l^y - \sum_{k=1}^{K}x_l^k y_l^k\right]\\ &= \sum_{l=1}^{m}a_l(f_l^x - f_l^y)^2 + \sum_{l=1}^{m}a_l\left[(\mu+1)\sum_{k=1}^{K}(y_l^k)^2 - (f_l^y)^2 + f_l^x f_l^y - \sum_{k=1}^{K}x_l^k y_l^k\right].\end{aligned} \tag{3.4}$$

For any $1 \leq l \leq m$ we have $f_l^x f_l^y - \sum_{k=1}^K x_l^k y_l^k = (\sum_{k=1}^K x_l^k)(\sum_{k=1}^K y_l^k) - \sum_{k=1}^K x_l^k y_l^k \geq 0$, since all x_l^k's and y_l^k's are nonnegative. Moreover, $(\mu+1)\sum_{k=1}^K (y_l^k)^2 - (f_l^y)^2 \geq 0$ when $\mu = K - 1$, due to the Cauchy-Schwartz inequality. Therefore,

$$\text{`Part I'} \geq \sum_{l=1}^{m}a_l(f_l^x - f_l^y)^2. \tag{3.5}$$

Now that $\lambda = 1$ and $\mu = K - 1$, the second part of (3.2) becomes

$$\begin{aligned}
\text{`Part II'} &= \sum_{k=1}^{K}\sum_{j=1}^{n}\left[-\sum_{i\neq k}\gamma_j^{ki}w_j^i v_j^k + \gamma_j^{kk}(w_j^k - v_j^k)^2\right] + \sum_{k=1}^{K}\sum_{j=1}^{n}\sum_{i\neq k}\gamma_j^{ki}v_j^i v_j^k \\
&\quad + (K-1)\sum_{k=1}^{K}\sum_{j=1}^{n}\gamma_j^{kk}(w_j^k)^2 \\
&= \sum_{k=1}^{K}\sum_{j=1}^{n}\left[\gamma_j^{kk}(v_j^k - w_j^k)^2 + \sum_{i\neq k}\gamma_j^{ki}(v_j^i - w_j^i)v_j^k + (K-1)\gamma_j^{kk}(w_j^k)^2\right],
\end{aligned} \tag{3.6}$$

or equivalently,

$$\begin{aligned}
\text{`Part II'} &= \sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{i=1}^{K}\gamma_j^{ki}(v_j^i - w_j^i)(v_j^k - w_j^k) + \sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{i\neq k}\gamma_j^{ki}v_j^i w_j^k \\
&\quad + \sum_{j=1}^{n}\left[(K-1)\sum_{k=1}^{K}\gamma_j^{kk}(w_j^k)^2 - \sum_{k=1}^{K}\sum_{i\neq k}\gamma_j^{ki}w_j^i w_j^k\right].
\end{aligned} \tag{3.7}$$

Observing the independence among the commodities, the terms in (3.7) can be bounded for each j: the second term is nonnegative since $\gamma_j^{ki}, v_j^i, w_j^k \geq 0$; the sum of the third and forth terms, is nonnegative, thanks to Lemma 3.2.3. Thus, we have

$$\text{`Part II'} \geq \sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{i=1}^{K}\gamma_j^{ki}(v_j^i - w_j^i)(v_j^k - w_j^k). \tag{3.8}$$

Combining the two inequalities (3.5) and (3.8), we have

$$\begin{aligned}
\text{`Part I'} + \text{`Part II'} &\geq \sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{i=1}^{K}\gamma_j^{ki}(v_j^i - w_j^i)(v_j^k - w_j^k) + \sum_{l=1}^{m}a_l(f_l^x - f_l^y)^2 \\
&= -\begin{pmatrix} v - w \\ x - y \end{pmatrix}^{\mathrm{T}} \nabla^2\mathrm{SV}(v, x)\begin{pmatrix} v - w \\ x - y \end{pmatrix}.
\end{aligned}$$

Notice that (v, x) is the social maximum and that (w, y) is a feasible solution; therefore, this term is nonnegative due to the second order optimality condition. It means that if we let $\lambda = 1$ and $\mu = K - 1$, then (3.2) holds (consequently (4.1) holds), which implies that $\mathrm{PoA} \geq \frac{1}{K}$. □

3.2.1 Tightness of the Bound and Numerical Test

The lower bound in the previous section may at first appear to be quite loose. However, it is essentially tight, as our next example shows.

Example 3.2.5 Assume all the resources are free and there is only one kind of commodity to produce. Suppose that there are K identical players, with the price $p^k = 1 + \frac{1}{K} - \sum_{i=1}^{K} v^i$ for all k, where v^i is the production level of player i. Note that in the Nash equilibrium, each player will produce $\frac{1}{K}$. The profit for each player is $\left(1+\frac{1}{K}\right) \cdot \frac{1}{K} - \frac{1}{K} = \frac{1}{K^2}$, and the total social profit is $\frac{1}{K}$. One can easily compute that the social optimal solution is to produce a total of $\frac{1}{2}\left(1+\frac{1}{K}\right)$ units of product. This yields the total social value of $\frac{1}{4}\left(1+\frac{1}{K}\right)^2$. Thus, the price of anarchy is PoA $= \frac{\frac{1}{K}}{\frac{1}{4}\left(1+\frac{1}{K}\right)^2} = \frac{4K}{K^2+2K+1}$. This example shows that the bound on PoA in Theorem 3.2.4 is essentially tight when K is large.

The tightness of the bound suggests that the outcome at the Nash equilibrium gets increasingly inefficient as the number of players increases in the game: the PoA is inversely proportional to the number of players in the game.

This conclusion is also confirmed from the computational experiments. We have done some numerical simulation tests in a series of Cournot games with the following setting: Suppose that all the resources are free and there is only one type of commodities to produce in the game. The data set for the mutual impact factors is generated randomly following a normal distribution. Note that we need to use the absolute value of the random numbers generated to preserve the impact factors nonnegative.

We let the number of players vary from 1 to 100 and find the average PoA in 1,000 random examples for each fixed number of players; and then plot the graph of the number of players and the corresponding average PoA, which is shown in Figure 3.1.

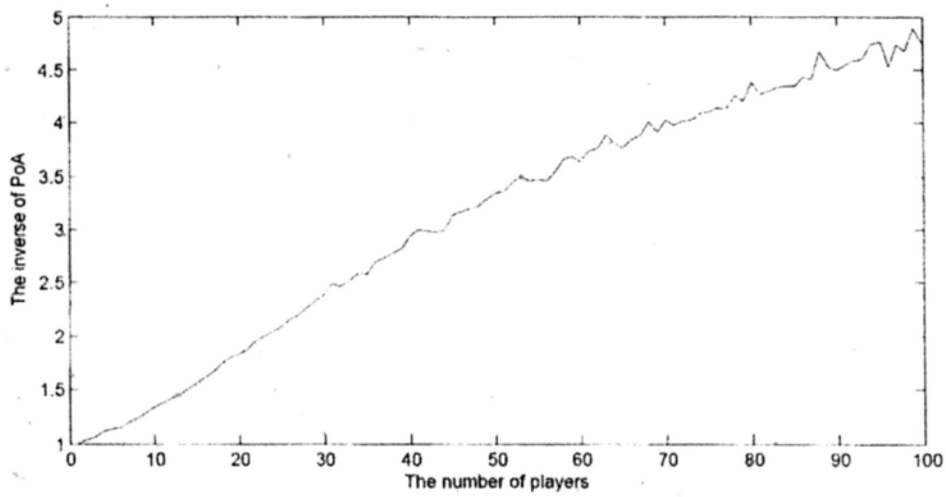

Figure 3.1: Simulation result indicating the relationship between the PoA and the number of players

From the figure above, it can be concluded that the *average* PoA, in comparison with the *worst* case that we have discussed so far, also clearly displays an inverse proportionality to the number of players.

On the other hand, the result also in a certain sense helps to illustrate how the 'market force' comes into existence, from the viewpoint of consumers.

Chapter 4

The Price of Isolation

Other from selfish behavior, myopic (or greedy) attitude also typically contributes to the system efficiency loss, in a dynamic decision process. There are a variety of models in the literature to characterize the situation. For instance, Sandal and Steinshamn [65] considered a harvesting game in a dynamic framework of Cournot competition and discussed several cases where losses occur due to the myopic behavior (cf. [46, 40] and the references therein). The so-called effort games in certain dynamic framework were studied as well; e.g. Bachrach, Zuckerman and Rosenschein [10] estimated the impact of myopia in effort games and proved some bounds for the measure introduced in the paper. Along a different line, dynamic decision making is closely related to learning in a game framework (see e.g. [48]).

The aim of this part is to study the *combined* impact of *selfish* behavior in noncooperative game and *myopic* (or *greedy*) attitude in dynamic decision making process in an integrated framework. In a literal sense, selfishness is a form of isolation (disconnectedness) in space, and myopia is a form of isolation (disconnectedness) in time. These two phenomena are intrinsically related; however, they are not identical (or symmetric) in the technical sense, as we shall see later in the analysis. For ease of referencing, the new measure will be generically called the *Price of Isolation* (PoI). Specifically, for a game with multiple players played over multiple stages, suppose that each player at each given stage of decision is both myopic and selfish, and if a Nash solution exists for each player at each stage (meaning that no player can unilaterally improve his/her solution at each given stage), then the PoI is defined to be the ratio between the total social value of the Nash solutions and the optimal total social value. To showcase the afore-introduced notion of PoI, we shall investigate two quite general forms of dynamic competitive game models, taking the splittable transportation game and Cournot oligopoly competition game as the special cases.

4.1 Dynamic Setting and the Price of Myopia

4.1.1 The Price of Myopia

At first we want to characterize the myopia in a quantitative way in a dynamic process. Consider a dynamic decision making process spanning over a period of stages, where the state in one stage depends on the action of the current stage and the state of the previous stage. We are interested in the two extremes of possible decision policies: the greedy policy and the dynamically optimal policy. To compare the difference of the outcomes, we introduce a quantity, to be called the *Price of Myopia* (PoM), which is defined as the ratio between the value of the myopic (or greedy) solution and the value of the dynamically optimal solution. Obviously this ratio will vary from one problem instance to the other. Following the same practice as for the definition of the worst-case approximation ratios, we shall call the PoM for a class of dynamic decision problem instances as the worst PoM in the instance belonging to this class. In other words, for a given class of dynamic decision problems $\mathscr{P}$, if the objective in the problem is nonnegative and is minimization, then the PoM for $\mathscr{P}$ is defined as:

$$\text{PoM}_{\mathscr{P}} = \sup_{P \in \mathscr{P}} \left\{ \frac{\text{the value of the greedy solution for } P}{\text{the value of dynamically optimal solution for } P} \right\},$$

which is always in the interval $[1, \infty]$: the larger value the heavier system efficiency loss. If the objective is nonnegative and is maximization, then

$$\text{PoM}_{\mathscr{P}} = \inf_{P \in \mathscr{P}} \left\{ \frac{\text{the value of the greedy solution for } P}{\text{the value of dynamically optimal solution for } P} \right\},$$

which is always in the interval $[0, 1]$: the smaller value the heavier system efficiency loss. Note that in many cases, PoM can be 0 (or ∞). Consider for instance the textbook example of dynamic fishing problem. Suppose the fish population in a pond is Q, and one may decide to take out half of the fish to sale or let the fish regenerate: in the first case the fish population in the next stage will regenerate back to Q and the sales profit will be λQ, while in the second case the fish population will be $2Q$ in the next stage. In this simple case, we assume that the initial fish population Q_0, the profit rate $\lambda > 0$, and the number of stages T are the parameters (or data) of the problem in class $\mathscr{P}$. Suppose that the objective is to maximize the total profit (ignoring the time discounting factor). Then, the PoM is at least $\frac{T}{2^T}$ for a given problem instance, and so $\text{PoM}_{\mathscr{P}}$ is $\frac{T}{2^T}$ if T is regarded as a fixed constant for all instances in $\mathscr{P}$, and $\text{PoM}_{\mathscr{P}}$ is 0 if the parameter T is a part of the input parameter in the problem class $\mathscr{P}$.

4.1.2 The PoM in a Dynamic Transportation Problem

In this subsection, we shall consider a nontrivial example to illustrate the afore introduced notion of PoM. Consider a transportation model involving T decision stages. In each stage, we are required to transport a given amount of commodity through a given network with a prescribed OD pair. Suppose that the commodity is splittable. We view the arcs in the network as the resources and assume them to be fixed, while the formation of the network may depend on the decision epoch t. Specifically, for Stage t, denote the given network to be $G_t = (V_t, L; A_t)$, with the set of nodes V_t, the set of arcs L and the node-to-arc incidence matrix[1] A_t. We assume multiple parallel links between nodes are allowed but no self-loop exists. Furthermore, we assume that $|V_t| = n_t$, $|L| = m$ and $A_t \in \mathbb{R}^{n_t \times m}$.

We denote the required OD pair in Stage t to be $\{s_t, d_t\}$. Let r_t denote the amount of commodity that needs to be transported in Stage t. The transportation plan in Stage t will be given by a vector $f_t \in \mathbb{R}^{m_t}$, whose components indicate the amount of the flows on the links. Let $f = (f_1^{\mathrm{T}}, \ldots, f_T^{\mathrm{T}})^{\mathrm{T}}$ denote the whole transportation plan consisting of the plans over the entire period of T stages. Clearly, a feasible plan is given by the constraints $A_t f_t = r_t \delta_{s_t} - r_t \delta_{d_t}$, where the notation δ_i signifies the unit vector in $\mathbb{R}^{n_t}$ whose i-th component is one while all others are zero.

Moreover, a transportation cost will be incurred for a flow on each link. We assume the costs will depend on the current flow and on the cost of the previous stage. Specifically, let us denote the *unit* cost for the flow on Link l in Stage t to be a function $c_{lt}(f_{lt}, c_{l,t-1})$,

[1]Each row of A_t represents a node and each column of A_t represents an arc. For an arc connecting node i to node j, the corresponding column in A_t will have all 0 elements except for the i-th element, where it is $+1$, and the j-th element, where it is -1.

where f_{lt} is the flow on Link l in Stage t and $c_{l,t-1}$ is the unit cost on Link l in Stage $t-1$. Therefore, the data (G, r, c) specifies an instance of the dynamic decision problem that is of interest to us in this section. Indeed, given the costs in previous stages, the transportation cost in Stage t is defined as:

$$C_t = \sum_{l \in L} f_{lt} c_{lt}(f_{lt}, c_{l,t-1}).$$

Naturally, the total cost over the entire period considered is a simple summation: $\mathrm{TC}(f) = \sum_{t=1}^{T} C_t(f)$. To quantify the impact of the myopic attitude, we shall investigate two special transportation plans: the myopic one denoted by f^M, in which for all t, f_t^M minimizes the cost C_t in Stage t given the costs in previous stages; the optimal one denoted by f^*, which minimizes the total cost TC over all feasible solutions. The so-called *Price of Myopia* (PoM) is now:

$$\mathrm{PoM} = \frac{\mathrm{TC}(f^M)}{\mathrm{TC}(f^*)}.$$

We shall consider the case where the unit cost function is affine linear in the current flow and the unit cost of the previous stage; that is, the unit cost function on Link l is $c_{lt}(f_{lt}, c_{l,t-1}) = a_l f_{lt} + b_{lt} + \alpha_{t-1} c_{l,t-1}$, where $a_l, b_{lt} \geq 0$ and $c_{l0} = 0$ for all l and t. The parameter α_{t-1} represents the impact of c_{t-1} on c_t. We further assume that $0 \leq \alpha_t \leq 1$ for all t. Based on the dynamic recursion formula of the costs, it can be deduced that $c_{lt} = a_l f_{lt} + b_{lt} + \sum_{\tau=1}^{t-1} (\Pi_{h=\tau}^{t-1} \alpha_h)(a_l f_{l\tau} + b_{l\tau})$. Denote

$$L_\alpha := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_1 & 1 & 0 & \cdots & 0 \\ \alpha_1\alpha_2 & \alpha_2 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Pi_{h=1}^{T-1}\alpha_h & \Pi_{h=2}^{T-1}\alpha_h & \Pi_{h=3}^{T-1}\alpha_h & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{T \times T}.$$

Then the total transportation cost can be written in matrix notations:

$$\mathrm{TC}(f) = f^{\mathrm{T}}\left(L_\alpha \otimes \mathrm{Diag}(a)\right) f + b^{\mathrm{T}} f,$$

where '$\otimes$' stands for the Kronecker product between two matrices, $\mathrm{Diag}(a)$ is the diagonal matrix whose l-th diagonal entries are a_l, for $l = 1, ..., m$, and b is the long vector $(b_{11}, \ldots, b_{m1}, \ldots, b_{1T}, \ldots, b_{mT})^{\mathrm{T}}$. Therefore the optimal transportation plan f^* should be the solution of the following optimization problem:

$$\begin{array}{lll} (P^*) & \min_f & f^{\mathrm{T}}\left(L_\alpha \otimes \mathrm{Diag}(a)\right) f + b^{\mathrm{T}} f \\ & \text{s.t.} & A_t f_t = r_t \delta_{s_t} - r_t \delta_{d_t}, \text{ for } t = 1, \ldots, T, \\ & & f \geq 0. \end{array}$$

On the other hand, for the myopic transportation plan, f_t^M is the solution of the following optimization problem in Stage t:

$$\begin{array}{lll} (P_t^M) & \min_{f_t} & \sum_{l \in L}(a_l f_{lt} + b_{lt} + \alpha_{t-1} c_{l,t-1}^M) f_{lt} \\ & \text{s.t.} & A_t f_t = r_t \delta_{s_t} - r_t \delta_{d_t},\ f_t \geq 0. \end{array}$$

Here $c_{l,t-1}^M$ is the corresponding cost in Stage $t-1$ on Link l in the myopic plan. Since $a_l \geq 0$, the above problem for each stage is a convex quadratic program and hence the existence and the uniqueness (in the sense of costs on the links) of the myopic solution are implied. Regarding the cost in the solution, we have the following result:

Lemma 4.1.1 *In the myopic solution, suppose that we change the plan in Stage t from f_t^M to any other feasible plan f_t, while the plans in all other stages remain unchanged. Then, we have*

$$C_t(f_t, c_{t-1}^M) - C_t(f_t^M, c_{t-1}^M) \geq \sum_{l \in L} a_l (f_{lt} - f_{lt}^M)^2.$$

Proof. Due to the properties of convex quadratic programming,

$$\begin{aligned}
& C_t(f_t, c_{t-1}^M) - C_t(f_t^M, c_{t-1}^M) - \sum_{l \in L} a_l (f_{lt} - f_{lt}^M)^2 \\
= & \sum_{l \in L} \left[(a_l f_{lt} + b_{lt} + \alpha_{t-1} c_{l,t-1}) f_{lt} - \left(a_l f_{lt}^M + b_{lt} + \alpha_{t-1} c_{l,t-1}\right) f_{lt}^M - a_l (f_{lt} - f_{lt}^M)^2 \right] \\
= & \sum_{l \in L} \left(2 a_l f_{lt}^M + b_{lt} + \alpha_{t-1} c_{l,t-1}\right) (f_{lt} - f_{lt}^M) \geq 0.
\end{aligned}$$

Note that the vector $2\,\mathrm{Diag}(a)\, f_t^M + b_t + \alpha_{t-1} \cdot c_{t-1}$ is the derivative of $C_t(\,\cdot\,, c_{t-1}^M)$ at the minimum point f_t^M. □

Following a similar argument as in Roughgarden [64] (with some necessary modifications), we get an upper bound for the price of myopia:

Theorem 4.1.2 *In the dynamic transportation decision problem, suppose that the unit cost on each link is affine linear in the total flow value and the previous cost on the link, then the price of myopia is upper bounded by* 4.

Proof. According to Lemma 4.1.1, to get an upper bound for the price of myopia it suffices to find a pair (λ, μ) with $\lambda > 0$ and $0 < \mu < 1$, such that

$$\sum_{t=1}^{T} \left[C_t(f_t^*, c_{t-1}^M) - \sum_{l \in L} a_l (f_{lt}^* - f_{lt}^M)^2 \right] \leq \lambda\, \mathrm{TC}(f^*) + \mu\, \mathrm{TC}(f^M). \tag{4.1}$$

Note that if (4.1) holds, then by Lemma 4.1.1 we have

$$\begin{aligned}
\mathrm{TC}(f^M) &= \sum_{t=1}^{T} C_t(f_t^M, c_{t-1}^M) \leq \sum_{t=1}^{T} \left[C_t(f_t^*, c_{t-1}^M) - \sum_{l \in L} a_l (f_{lt}^* - f_{lt}^M)^2 \right] \\
&\leq \lambda\, \mathrm{TC}(f^*) + \mu\, \mathrm{TC}(f^M).
\end{aligned}$$

Thus the price of myopia can be bounded as:

$$\mathrm{PoM} = \frac{\mathrm{TC}(f^M)}{\mathrm{TC}(f^*)} \leq \frac{\lambda}{1-\mu}.$$

Let us now turn to the search of (λ, μ) to satisfy (4.1). Substituting the expression of the cost functions in (4.1), the intended inequality becomes:

$$\begin{aligned}
& \sum_{t=1}^{T}\sum_{l\in L}\left[(a_l f_{lt}^* + b_{lt} + \alpha_{t-1}c_{l,t-1}^M)f_l^* - a_l(f_{lt}^* - f_{lt}^M)^2\right] \\
= & \sum_{t=1}^{T}\sum_{l\in L}\left[(2a_l f_{lt}^M + b_{lt} + \alpha_{t-1}c_{l,t-1}^M)f_l^* - a_l(f_{lt}^M)^2\right] \\
= & (f^*)^{\mathrm{T}}\left((I_T + L_\alpha)\otimes \mathrm{Diag}(a)\right)f^M + b^{\mathrm{T}}f^* - (f^M)^{\mathrm{T}}\left(I_T \otimes \mathrm{Diag}(a)\right)f^M \\
\leq & \lambda\left[(f^*)^{\mathrm{T}}(L_\alpha \otimes \mathrm{Diag}(a))f^* + b^{\mathrm{T}}f^*\right] + \mu\left[(f^M)^{\mathrm{T}}(L_\alpha\otimes \mathrm{Diag}(a))f^M + b^{\mathrm{T}}f^M\right],
\end{aligned}$$

where I_T is the T by T identity matrix. Regrouping the terms, the above is equivalent to

$$\begin{aligned}
& \lambda(f^*)^{\mathrm{T}}(L_\alpha\otimes \mathrm{Diag}(a))f^* + (f^M)^{\mathrm{T}}((I_T + \mu L_\alpha)\otimes \mathrm{Diag}(a))f^M \\
& -(f^*)^{\mathrm{T}}\left((I_T + L_\alpha)\otimes \mathrm{Diag}(a)\right)f^M + b^{\mathrm{T}}\left[(\lambda - 1)f^* + \mu f^M\right] \geq 0. \qquad (4.2)
\end{aligned}$$

By requiring $\lambda \geq 1, \mu \geq 0$, the linear part of the left hand side of (4.2) is obviously nonnegative. It will be sufficient to ensure that the quadratic term is also nonnegative. Denote $f_l = (f_{l1}, f_{l2}, \dots, f_{lT})^{\mathrm{T}}$, and we have

$$\begin{aligned}
& \lambda(f^*)^{\mathrm{T}}(L_\alpha\otimes \mathrm{Diag}(a))f^* + (f^M)^{\mathrm{T}}((I_T + \mu L_\alpha)\otimes \mathrm{Diag}(a))f^M \\
& -(f^*)^{\mathrm{T}}\left((I_T + L_\alpha)\otimes \mathrm{Diag}(a)\right)f^M \\
= & \sum_{l\in L} a_l\left[\lambda(f_l^*)^{\mathrm{T}}L_\alpha f_l^* + (f_l^M)^{\mathrm{T}}(I_T + \mu L_\alpha)f_l^M - (f_l^*)^{\mathrm{T}}(I_T + L_\alpha)f_l^M\right].
\end{aligned}$$

Note that the above is a summation over index l. To ensure (4.2), we need only to establish the inequality for each link, namely,

$$\lambda(f_l^*)^{\mathrm{T}}L_\alpha f_l^* + (f_l^M)^{\mathrm{T}}(I_T + \mu L_\alpha)f_l^M - (f_l^*)^{\mathrm{T}}(I_T + L_\alpha)f_l^M \geq 0.$$

Since $f_l^* \geq 0$ and $f_l^M \geq 0$, we have $(f_l^*)^{\mathrm{T}}I_T f_l^* \leq (f_l^*)^{\mathrm{T}}L_\alpha^{\mathrm{T}}f_l^*$ and $(f_l^M)^{\mathrm{T}}I_T f_l^M \geq \frac{1}{T}(f_l^M)^{\mathrm{T}}E_T f_l^M \geq \frac{1}{T}(f_l^M)^{\mathrm{T}}L_\alpha f_l^M$. It is sufficient to show that

$$\lambda(f_l^*)^{\mathrm{T}}L_\alpha f_l^* + \left(\mu + \frac{1}{T}\right)(f_l^M)^{\mathrm{T}}L_\alpha f_l^M - (f_l^*)^{\mathrm{T}}(L_\alpha + L_\alpha^{\mathrm{T}})f_l^M \geq 0. \qquad (4.3)$$

Denote

$$\Lambda_\alpha := \begin{pmatrix} \lambda(L_\alpha + L_\alpha^{\mathrm{T}}) & -(L_\alpha + L_\alpha^{\mathrm{T}}) \\ -(L_\alpha + L_\alpha^{\mathrm{T}}) & (\mu + \frac{1}{T})(L_\alpha + L_\alpha^{\mathrm{T}}) \end{pmatrix} = \begin{pmatrix} \lambda & -1 \\ -1 & \mu + \frac{1}{T} \end{pmatrix} \otimes (L_\alpha + L_\alpha^{\mathrm{T}}).$$

Clearly, (4.3) is equivalent to $\begin{pmatrix} f^* \\ f^M \end{pmatrix}^{\mathrm{T}} \Lambda_\alpha \begin{pmatrix} f^* \\ f^M \end{pmatrix} \geq 0$. Observe that

$$L_\alpha^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\alpha_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\alpha_2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -\alpha_{T-1} & 1 \end{pmatrix}.$$

Therefore $L_\alpha^{-1} + (L_\alpha^{-1})^{\mathrm{T}}$ is positive semidefinite when $0 \leq \alpha_t \leq 1$ for $1 \leq t \leq T-1$. So L_α^{-1} (as well as L_α) is monotone. Thus, if $\lambda(\mu + \frac{1}{T}) \geq 1$, Λ_α will be the Kronecker product of two positive semidefinite matrices and hence positive semidefinite itself (cf. [32]), and then (4.3) holds for all f^* and f^M. Thus, the required conditions boils down to choosing (λ, μ) such that

$$\mu + \frac{1}{T} - \frac{1}{\lambda} \geq 0 \text{ with } \lambda \geq 1 \text{ and } \mu \geq 0.$$

While satisfying the above relations, we shall minimize $\frac{\lambda}{1-\mu}$, leading to the choice $\lambda = \frac{2T}{T+1}$ and $\mu = \frac{T-1}{2T}$. Summarizing all the above, we have shown

$$\text{PoM} \leq \frac{2T/(T+1)}{1-(T-1)/(2T)} = \frac{4T^2}{(T+1)^2} \leq 4.$$

The theorem is proven. □

The above bound is probably not tight, and it remains a challenge to find the *tight* bound on the PoM for this model. Moreover, we know that a *lower* bound for the PoM is 2, as the following example shows:

Example 4.1.3 Suppose that we are again given a parallel network, which consists of two nodes denoted by s and d and two links between them. In the dynamic transportation problem, T stages (assume T is even) are considered and the OD pair in each stage is always $\{s, d\}$. Assume the required throughput in each stage is 1. Furthermore, the

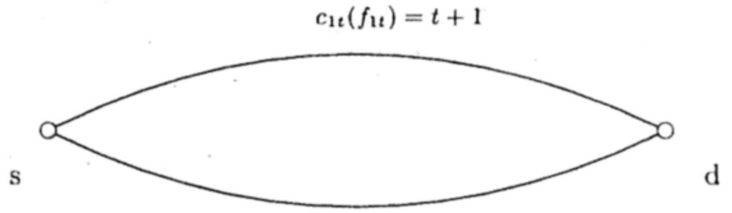

Figure 4.1: An example implying a lower bound for the PoM

cost on the first link in Stage t is $t+1$, while the cost on the other link in Stage t is given by $c_t = f_t + c_{t-1}$ and $c_0 = 0$. In this case, the myopic decision will use the second link all the time, which incurs a total cost of $\frac{T(T+1)}{2}$. On the other hand, if we use the first link for the first $T/2$ stages and then the second link for the remaining $T/2$ stages, then the total cost will be $2 \times \frac{T/2(T/2+1)}{2} + \frac{T}{2} = \frac{T^2}{4} + T$. Thus, the PoM is at least

$$\text{PoM} \geq \frac{T^2/2 + T/2}{T^2/4 + T} = 2 - \frac{3T/2}{T^2/4 + T},$$

which tends to 2 as T becomes large.

Corollary 4.1.4 *If T is taken as a fixed parameter of the dynamic transportation decision problem, then*

$$\frac{2T+2}{T+4} \leq \text{PoM}_{\mathscr{P}} \leq \frac{4T^2}{(T+1)^2}.$$

If T is an input parameter of the dynamic transportation decision problem, then $2 \leq \text{PoM}_{\mathscr{P}} \leq 4$.

4.2 Combination of Selfishness and Myopia and the Price of Isolation

4.2.1 The PoI in a Dynamic Transportation Game

In this section, we shall extend the dynamic transportation problem to a game framework, in which a finite number of self-interested players are facing the same dynamic decision processes and the decision of each player may affect the cost structures of each other. The network G_t in each stage is shared by all the players, and the tasks for the players may differ. First of all, the OD pair and required throughput of the players are different: for Player k in Stage t the OD is $\{s_t^k, d_t^k\}$ and the required amount of transportation is r_t^k. Furthermore, we denote the transportation plan of Player k to be a vector $x_t^k \in \mathbb{R}^m$ and a feasible flow is given by the constraints $A_t x_t^k = r_t^k \delta_{s_t^k} - r_t^k \delta_{d_t^k}$. For Link l, we denote the total flow on it in Stage t to be $f_{lt} = \sum_{k=1}^K x_{lt}^k$. We still focus on the case that the unit costs are affine linearly dependent on the current congestion and the previous cost level. That is, by denoting the total flow and the corresponding unit cost on Link l at Stage t to be f_{lt} and c_{lt}, we have $c_{lt} = a_l f_{lt} + b_{lt} + \alpha_{t-1} c_{l,t-1}$, where $a_l, b_{lt} \geq 0$, $0 \leq \alpha_t \leq 1$ and $c_{l0} = 0$ for all l and t. Then in Stage t, for Player k, given the decisions of other players (denoted as x_t^{-k}) and the cost in previous stage c_{t-1}, the objective is to minimize his/her own transportation cost:

$$C_t^k(x_t^k, x_t^{-k}; c_{t-1}) = \sum_{l \in L} x_{lt}^k c_{lt}(f_{lt}, c_{l,t-1}) = \sum_{l \in L} (a_l f_l + b_{lt} + \alpha_{t-1} c_{l,t-1}) x_{lt}^k.$$

Again, given the decisions of all the players over the entire period, the total social cost is a simple summation: $\mathrm{SC}(x) = \sum_{k=1}^{K} \sum_{t=1}^{T} C_t^k(x_t^k; x_t^{-k})$. In this case, the *myopic* and *selfish* Player k will face the following optimization problem at Stage t:

$$
\begin{aligned}
(P_t^k) \quad \min \quad & C_t^k(x_t^k, x_t^{-k}; c_{t-1}) = \textstyle\sum_{l\in L}(a_l f_l + b_{lt} + \alpha_{t-1} c_{l,t-1}) x_{lt}^k \\
\text{s.t.} \quad & A_t x_t^k = r_t^k \delta_{s_t^k} - r_t^k \delta_{d_t^k}, \\
& x_t^k \geq 0.
\end{aligned}
$$

Replacing f_{lt} with $\sum_{k=1}^{K} x_{lt}^k$, the above problem is a convex quadratic program, in which the decision vector is x_t^k:

$$
\begin{aligned}
\min \quad & \textstyle\sum_{l\in L} \left\{ (b_{lt} + \alpha_{t-1} c_{l,t-1}) x_{lt}^k + a_l \left[\left(\sum_{i\neq k} x_{lt}^i \right) x_{lt}^k + (x_{lt}^k)^2 \right] \right\} \\
\text{s.t.} \quad & A_t x_t^k = r_t^k \delta_{s_t^k} - r_t^k \delta_{d_t^k}, \\
& x_t^k \geq 0.
\end{aligned}
$$

The Myopic Nash Equilibrium

Let us introduce a solution, called myopic Nash equilibrium, where no myopic player will be able to improve his/her current situation unilaterally. To analyze its existence and the uniqueness, we characterize the conditions underlying the myopic Nash equilibrium by a particular Linear Complementarity (LCP) formulation. Let $z_t^k \in \mathbb{R}^n$ be the Lagrangian multiplier associated with the equality constraint $A_t x_t^k = r_t^k \delta_{s_t^k} - r_t^k \delta_{d_t^k}$. The Karush-Kuhn-Tucker optimality condition for (P_t^k) is:

$$
\begin{cases}
A_t x_t^k = r_t^k \delta_{s_t^k} - r_t^k \delta_{d_t^k} \\
b_t + \mathrm{Diag}(a) \sum_{i=1}^{K} x_t^i + \mathrm{Diag}(a) x_t^k + \alpha_{t-1} c_{t-1} + A_t^{\mathrm{T}} z_t^k - s_t^k = 0 \\
(x_t^k)^{\mathrm{T}} s_t^k = 0,\ x_t^k \geq 0,\ s_t^k \geq 0.
\end{cases}
$$

A myopic Nash equilibrium for the dynamic transportation game is attained if and only if each player at each stage attains the optimum; i.e., the system of equations above holds for all $k = 1, ..., K$ and $t = 1, ..., T$. Combining the form of c_{lt} we can explicitly write the KKT optimality condition using the block matrix notation. Let x^k denote the Player k's decision vector over the whole period (similarly for z^k, s^k, r^k). We use x (respectively z, and s, and R) to denote the prolonged vector consisting of of $x^1, \ldots, x^K$ by sequentially stacking them on top of each other with x^1 on the very top (respectively $z^1, ..., z^K$, and $s^1, ..., s^K$, and $(r^1\delta_{s^1} - r^1\delta_{d^1}), ..., (r^K\delta_{s^K} - r^K\delta_{d^K})$). The equations for the myopic Nash equilibrium solution are:

$$(MNE)\begin{cases} (I_K \otimes A)x = R \\ e_K \otimes b + (E_K \otimes (L_a \otimes \text{Diag}(a))x + (I_{KT} \otimes \text{Diag}(a))x + (I_K \otimes A)^{\mathrm{T}}z - s = 0 \\ x \geq 0,\ s \geq 0,\ x^{\mathrm{T}}s = 0, \end{cases}$$

where e_K is the (K by 1) all-one vector, E_K is the (K by K) all-one matrix, I_{KT} (I_K) are the KT by KT (K by K) identity matrices, and A is a diagonal block partitioned matrix with A_t as diagonal block entry. The so-expressed myopic Nash equilibrium is a mixed linear complementarity problem, and it specifies an equivalent condition for myopic Nash equilibrium. Since (P_t^k) is a convex quadratic program for each given k and t, (MNE) is also a *necessary and sufficient* condition for a solution to be at Nash equilibrium. It follows from the properties of a convex program that the cost C_t^k is continuously dependent on the parameters R.

Similar with the arguments on the existence and the uniqueness of Nash equilibrium of static transportation games in Chapter 2, we can apply the theory about LCP to get similar conclusions about the myopic Nash equilibrium. Indeed as we shall see below that the mixed LCP problem arising from the myopic Nash equilibrium of the dynamic transportation game is monotone. For any Δx satisfying

$$(I_K \otimes A)\Delta x = 0,$$

and the corresponding

$$\Delta s = (E_K \otimes (L_\alpha \otimes \mathrm{Diag}(a))\Delta x + (I_{KT} \otimes \mathrm{Diag}(a))\Delta x + (I_K \otimes A)^{\mathrm{T}} \Delta z,$$

we have

$$\begin{aligned}(\Delta x)^{\mathrm{T}} \Delta s &= (\Delta x)^{\mathrm{T}} (E_K \otimes (L_\alpha \otimes \mathrm{Diag}(a)))\Delta x + (\Delta x)^{\mathrm{T}} (I_{KT} \otimes \mathrm{Diag}(a))\Delta x \\ &= (\Delta x)^{\mathrm{T}} ((E_K \otimes L_\alpha + I_{KT}) \otimes \mathrm{Diag}(a))\Delta x \geq 0, \qquad (4.4)\end{aligned}$$

because $(E_K \otimes L_\alpha + I_{KT}) \otimes \mathrm{Diag}(a)$ is positive semidefinite. This shows that (MNE) is a monotone mixed LCP. Clearly, it is also feasible by noting $b \geq 0$. It follows from Theorem 2.2.1 that (MNE) has a unique myopic Nash solution.

An Upper Bound for the PoI

For a dynamic game, let us introduce here the notion of the price of isolation as the ratio between the total objective values of the worst myopic Nash solution and the dynamically optimal social value. If the problem reduces to a single player (the dynamic decision model) then the PoI reduces to the PoM as we discussed in Section 4.1.1; if the problem is static, then the PoI reduces to the PoA. In our discussion, let us denote y to be the solution where each player is myopic and selfish (i.e. y_t^k is optimal to C_t^k for all k and t), and denote x to be the dynamic optimal solution for the whole system (x is the minimum for SC over the feasible region). Then the price of isolation is $\mathrm{PoI} = \frac{\mathrm{SC}(y)}{\mathrm{SC}(x)}$. Our subsequent analysis is similar to the static case. First, we have the following result regarding the difference of the solutions at a myopic Nash equilibrium and at the optimum.

Lemma 4.2.1 *Suppose the game attains a myopic Nash Equilibrium and let us denote y to be the solution. Suppose that Player k changes his/her strategy at the Stage t from*

y_t^k to any other feasible flow x_t^k, while the strategies of this player at the other stages and all other players' strategies remain unchanged. Then, the cost for Player k will increase by at least $\sum_{l\in L} a_l(x_{lt}^k - y_{lt}^k)^2$; that is

$$C_t^k(x_t^k, y_{-t}^k; y^{-k}) - C^k(y_t^k, y_{-t}^k; y^{-k}) \geq \sum_{l\in L} a_l(x_{lt}^k - y_{lt}^k)^2.$$

Proof. According to the definition of the notion of myopic Nash equilibrium, y_t^k is the optimal strategy of the myopic Player k, whose cost at Stage t is $C_t^k(\,\cdot\,, y_{-t}^k; y^{-k})$. Then

$$\begin{aligned}
& C_t^k(x_t^k, y_{-t}^k; y^{-k}) - C_t^k(y_t^k, y_{-t}^k; y^{-k}) - \sum_{l\in L} a_l(x_{lt}^k - y_{lt}^k)^2 \\
= & \sum_{l\in L}\left\{\left[a_l(x_{lt}^k + \sum_{i\neq k} y_{lt}^i) + b_{lt} + \sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)(b_{l\tau} + a_l f_{l\tau}^y)\right] x_{lt}^k \right. \\
& \left. - \left[a_l f_{lt}^y + b_{lt} + \sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)(b_{l\tau} + a_l f_{l\tau}^y)\right] y_{lt}^k - a_l(x_{lt}^k - y_{lt}^k)^2\right\} \\
= & \sum_{l\in L}\left(2a_l y_{lt}^k + a_l\sum_{i\neq k} y_{lt}^i + b_{lt} + \sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)(b_{l\tau} + a_l f_{l\tau}^y)\right)(x_{lt}^k - y_{lt}^k) \geq 0.
\end{aligned}$$

The last inequality is due to the convexity of function $C_t^k(\,\cdot\,, y_{-t}^k; y^{-k})$, and the fact that $2a_l y_{lt}^k + a_l\sum_{i\neq k} y_{lt}^i + b_{lt} + \sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)(b_{l\tau} + a_l f_{l\tau}^y)$ is the derivative of $C_t^k(\,\cdot\,, y_{-t}^k; y^{-k})$ at the minimum point y_t^k. □

We shall establish an upper bound for the PoI based on the following observation about the copositive matrix:

Lemma 4.2.2 *Suppose that all the entries of a monotone matrix L are nonnegative, i.e. $L + L^{\mathrm{T}} \succeq 0$ and $L \geq 0$. Then, for any positive semidefinite (symmetric) matrix M_1 and any nonnegative matrix M_2, we have that the Kronecker product $\Lambda = (M_1 + M_2)\otimes L$ is copositive.*

Proof. To prove the matrix Λ is copositive, it suffices to show $\Lambda + \Lambda^{\mathrm{T}}$ is copositive.

$$\begin{aligned}
\Lambda + \Lambda^{\mathrm{T}} &= ((M_1 + M_2) \otimes L) + ((M_1 + M_2) \otimes L)^{\mathrm{T}} \\
&= M_1 \otimes L + (M_1 \otimes L)^{\mathrm{T}} + M_2 \otimes L + (M_2 \otimes L)^{\mathrm{T}} \\
&= M_1 \otimes (L + L^{\mathrm{T}}) + M_2 \otimes L + (M_2 \otimes L)^{\mathrm{T}}.
\end{aligned}$$

The last equality follows from the symmetry of M_1. Noting that $M_1 \succeq 0$ and $L + L^{\mathrm{T}} \succeq 0$, we have $M_1 \otimes (L + L^{\mathrm{T}}) \succeq 0$. Moreover, since $M_2 \geq 0$ and $L \geq 0$, both $M_2 \otimes L \geq 0$ and $(M_2 \otimes L)^{\mathrm{T}} \geq 0$. Therefore, $\Lambda + \Lambda^{\mathrm{T}}$ is the sum of a positive semidefinite matrix and a nonnegative matrix, hence copositive, and so is Λ. □

By applying the similar scheme as in Section 4.1.1, we have the following upper bound for the price of isolation:

Theorem 4.2.3 *For the dynamic transportation game, the price of isolation is upper bounded by 4.4865.*

Proof. Let x and y be the social optimum and a myopic Nash equilibrium respectively. We now set out to look for (λ, μ) with $\lambda > 0$ and $0 < \mu < 1$, such that

$$\sum_{k=1}^{K} \sum_{t=1}^{T} \left[C_t^k(x_t^k, y_{-t}^k; y^{-k}) - \sum_{l \in L} \sum_{t=1}^{T} a_l (x_{lt}^k - y_{lt}^k)^2 \right] \leq \lambda \, \mathrm{SC}(x) + \mu \, \mathrm{SC}(y). \tag{4.5}$$

If (4.5) is shown, then according to Lemma 4.2.1 and Lemma 4.2.1 we will have

$$\begin{aligned}
\mathrm{SC}(y) &= \sum_{k=1}^{K} \sum_{t=1}^{T} C_t^k(y^k; y^{-k}) \leq \sum_k \left[C^k(x_t^k, y_{-t}^k; y^{-k}) - \sum_{l \in L} \sum_{t=1}^{T} a_l (x_{lt}^k - y_{lt}^k)^2 \right] \\
&\leq \lambda \, \mathrm{SC}(x) + \mu \, \mathrm{SC}(y).
\end{aligned}$$

Thus the price of isolation can be bounded as:

$$\mathrm{PoI} = \frac{\mathrm{SC}(y)}{\mathrm{SC}(x)} \leq \frac{\lambda}{1 - \mu}. \tag{4.6}$$

Substituting the exact forms of the cost functions in (4.5), the intended inequality becomes:

$$\sum_{k=1}^{K}\sum_{t=1}^{T}\sum_{l\in L}\left[a_l(x_{lt}^k + f_{lt}^y - y_{lt}^k)x_{lt}^k + b_{lt}x_{lt}^k + \sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)(b_{l\tau} + a_l f_{l\tau}^y)x_{lt}^k - a_l(x_{lt}^k - y_{lt}^k)^2\right]$$
$$\leq \lambda\sum_{t=1}^{T}\sum_{l\in L}\left[a_l(f_{lt}^x)^2 + b_{lt}f_{lt}^x + \sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)(b_{l\tau} + a_l f_{l\tau}^x)f_{lt}^x\right]$$
$$+\mu\sum_{t=1}^{T}\sum_{l\in L}\left[a_l(f_{lt}^y)^2 + b_{lt}f_{lt}^y + \sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)(b_{l\tau} + a_l f_{l\tau}^y)f_{lt}^y\right].$$

Notice that the order of summation can be interchanged, so the left hand side of the inequality above can be rewritten as:

$$LHS = \sum_{l\in L}\sum_{t=1}^{T}\left[\sum_{k=1}^{K} a_l(x_{lt}^k - y_{lt}^k)y_{lt}^k + b_{lt}f_{lt}^x + a_l f_{lt}^y f_{lt}^x + \sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)(b_{l\tau} + a_l f_{l\tau}^y)f_{lt}^x\right].$$

We regroup the terms on both the left and the right hand sides and obtain

$$RHS - LHS$$
$$= \sum_{l\in L}\sum_{t=1}^{T} a_l\left[\lambda(f_{lt}^x)^2 + \mu(f_{lt}^y)^2 - f_{lt}^x f_{lt}^y + \sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)\left(\lambda f_{l\tau}^x f_{lt}^x + \mu f_{l\tau}^y f_{lt}^y - f_{l\tau}^y f_{lt}^x\right)\right.$$
$$\left. - \sum_{k=1}^{K} y_{lt}^k(x_{lt}^k - y_{lt}^k)\right] + \sum_{l\in L}\sum_{t=1}^{T}\left(b_{lt} + \sum_{\tau=1}^{t}(\Pi_{h=\tau}^{t-1}\alpha_h)b_{l\tau}\right)\left[(\lambda-1)f_{lt}^x + \mu f_{lt}^y\right]. \quad (4.7)$$

By requiring $\lambda \geq 1, \mu \geq 0$, we have

$$(\lambda - 1)f_{lt}^x + \mu f_{lt}^y \geq 0 \text{ for all } l \text{ and } t. \quad (4.8)$$

Thus the second summation part of (4.7) is nonnegative because b_{lt}'s and α_τ's are nonnegative. Let us now pay attention to first summation part of (4.7). Again, observing that this is a summation over the index l, so it suffices to establish the following inequality for each link:

$$\sum_{t=1}^{T}\left[\lambda(f_{lt}^x)^2 + \mu(f_{lt}^y)^2 - f_{lt}^x f_{lt}^y + \sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)\left(\lambda f_{l\tau}^x f_{lt}^x + \mu f_{l\tau}^y f_{lt}^y - f_{l\tau}^y f_{lt}^x\right)\right.$$
$$\left. - \sum_{k=1}^{K} y_{lt}^k(x_{lt}^k - y_{lt}^k)\right] \geq 0.$$

Note that

$$\sum_{k=1}^{K} y_{lt}^k (x_{lt}^k - y_{lt}^k) = \frac{1}{4}\sum_{k=1}^{K}(x_{lt}^k)^2 - \sum_{k=1}^{K}(\frac{1}{2}x_{lt}^k - y_{lt}^k)^2 \le \frac{1}{4}\sum_{k=1}^{K}(x_{lt}^k)^2 \le \frac{1}{4}(f_{lt}^x)^2.$$

Therefore, we only need to establish the following inequality

$$\sum_{t=1}^{T}\left[\left(\lambda - \frac{1}{4}\right)(f_{lt}^x)^2 + \mu(f_{lt}^y)^2 - f_{lt}^x f_{lt}^y + \sum_{\tau=1}^{t-1}(\Pi_{h=\tau}^{t-1}\alpha_h)(\lambda f_{l\tau}^x f_{lt}^x + \mu f_{l\tau}^y f_{lt}^y - f_{l\tau}^y f_{lt}^x)\right] \ge 0,$$

which can be rewritten in matrix notation:

$$(f_l^x)^{\mathrm{T}}\left(\lambda L_\alpha - \frac{1}{4}I_T\right)f_l^x + \mu(f_l^y)^{\mathrm{T}}L_\alpha f_l^y - (f_l^x)^{\mathrm{T}}L_\alpha f_l^y \ge 0. \tag{4.9}$$

Since $f_l^x \ge 0$, we have $(f_l^x)^{\mathrm{T}} I_T f_l^x \le (f_l^x)^{\mathrm{T}} L_\alpha f_l^x$. It is sufficient to show the matrix

$$\begin{pmatrix} (\lambda - \frac{1}{4})L_\alpha & -L_\alpha \\ 0 & \mu L_\alpha \end{pmatrix} = \begin{pmatrix} \lambda - \frac{1}{4} & -1 \\ 0 & \mu \end{pmatrix} \otimes L_\alpha$$

is copositive. Recall that L_α is monotone and observe that

$$\begin{pmatrix} \lambda - \frac{1}{4} & -1 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} \lambda - \frac{1}{4} & -1 \\ -1 & \mu \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Therefore, we can apply Lemma 4.2.2 here and only require the matrix $\begin{pmatrix} \lambda - \frac{1}{4} & -1 \\ -1 & \mu \end{pmatrix}$ to be positive semidefinite, which is $\left(\lambda - \frac{1}{4}\right)\mu \ge 1$.

Combining the condition obtained from (4.8), we shall minimize $\frac{\lambda}{1-\mu}$, leading to the choice $\lambda = \frac{5+2\sqrt{5}}{4} = 2.368$ and $\mu = 0.4722$.

Finally, we have shown (4.6) and hence

$$\text{PoI} \le \frac{2.368}{1 - 0.4722} = 4.4865.$$

□

Since the dynamic transportation decision problem is a special case of the dynamic transportation game, Example 4.1.3 also applies here. So we have:

Corollary 4.2.4 *If T is an input parameter of the dynamic transportation decision problem, then* $2 \leq \mathrm{PoM}_{\mathscr{P}} \leq 4.4865$.

4.2.2 The PoI in a Dynamic Cournot Oligopoly Game

Next we turn to a profit maximization model, which can be viewed as an extension of the Cournot oligopoly competition game. Suppose that we have m given types of resources, which can be used to produce n types of goods. As for how the producers actually deploy the resources and turn them into the final goods depend on the infrastructure available to them, as well as the status of their knowhow's. In our model, this relationship is characterized by a pair of technology matrices, denoted by (M^k, N^k) for Producer k. Namely, if Producer k uses a nonnegative vector $x^k \in \mathbb{R}^m_+$ to denote the usage of resources and $v^k \in \mathbb{R}^n_+$ to denote the quantity of goods produced from the resources, then the constraint $N^k v^k \leq M^k x^k$ holds. The producers in the game use the resources to produce goods and then attempt to sell the goods in the market. The competition among the producers are twofold: first, the cost for the use of the shared resources are subject to competition; second, and the goods produced by different producers are substitutable and the prices of the goods are also subject to the competition in the market place. The static version of this model is considered in previous chapters, where the price of anarchy is lower bounded by the inverse of the number of producers.

In this section, we consider a dynamic version of the game. Suppose there are K producers competing over T decision stages. The prices of the goods and the costs for using the resources for each player are affected by two factors: the historical decisions of the previous stage and the decisions of the current stage. Specifically, we use p^k_{jt} and c^k_{lt} to denote the selling price of Goods j and the unit cost of using Resource l for

Player k at Stage t, respectively. Suppose that the amount of Goods j produced by Player k at Stage t is v_{jt}^k, and v_{jt}^{-k} is the decision of all other players except Player k. Similarly, denote the usage of Resource l for Producer k at Stage t to be x_{lt}^k and the other producers' decision to be x_{lt}^{-k}. For convenience, in the sequel we use the following vector notations for v (similar for all other variables and parameters with these three indices): $v_t^k := (v_{1t}^k, v_{2t}^k, \ldots, v_{nt}^k)^{\mathrm{T}}, v_{(j)}^k := (v_{j1}^k, v_{j2}^k, \ldots, v_{jT}^k)^{\mathrm{T}}$, and $v_{jt} := (v_{jt}^1, v_{jt}^2, \ldots, v_{jt}^K)^{\mathrm{T}}$. Then, the sales prices for Producer k at Stage t is assumed to be a vector $p_t^k(v_t^k, v_t^{-k}; p_{t-1}^k) \in \mathbb{R}_+^n$, which can be viewed as a function of the amount of current supply, v_t^k and v_t^{-k}, and the prices at the previous stage, p_{t-1}^k. Similarly, the unit cost of the resources is a vector $c_t^k(x_t^k, x_t^{-k}; c_{t-1}^k) \in \mathbb{R}_+^m$, which is dependent on the current total demand x_t and the cost level inherited from the previous stage.

Suppose that the technology-matrix pair for Producer k is (M^k, N^k). To maximize the profit at Stage t, the selfish and greedy Producer k considers the following optimization problem:

$$
\begin{array}{lll}
(P_t^k) & \max & V_t^k(v_t^k, x_t^k; v_t^{-k}, x_t^{-k}) = (v_t^k)^{\mathrm{T}} p_t^k(v_t^k, v_t^{-k}; p_{t-1}^k) - (x_t^k)^{\mathrm{T}} c_t^k(x_t^k, x_t^{-k}; c_{t-1}^k) \\
 & \text{s.t.} & N^k v_t^k \le M^k x_t^k,\ x_t^k \ge 0,\ v_t^k \ge 0.
\end{array}
$$

Naturally, the social value is defined to be the summation of the profits of all the producers over the entire period: $\mathrm{SV}(v, x) = \sum_{k=1}^K \sum_{t=1}^T V_t^k(v_t^k, x_t^k; v_t^{-k}, x_t^{-k})$.

Let (w, y) denote a Nash solution, and (v, x) denote the socially optimal solution. The PoI can be expressed as: $\mathrm{PoI} = \frac{\mathrm{SV}(w,y)}{\mathrm{SV}(v,x)}$.

As in the last section, we now confine ourselves to the affine linear case, in which $p_t^k(v_t^k, v_t^{-k}; p_{t-1}^k)$ and $c_t^k(x_t^k, x_t^{-k}; c_{t-1}^k)$ are both affine linear functions. To be precise, we suppose that Producer k produces v^k while other producers produce v^{-k}, then the price of the Product j for Producer k at Stage t can be written as $p_{jt}^k = q_{jt}^k -$

$\sum_{i=1}^{K} \gamma_j^{ki} v_{jt}^i + \rho_{j,t-1} p_{j,t-1}^k$, where q_{jt}^k and γ_j^{ki} are positive parameters, $0 \le \rho_{j,t} \le 1$ and $p_{j,0}^k = 0$. The parameter γ_j^{ki} reflects the impact of the sales of Product j by Player i on the selling price for Player k, and $\rho_{j,t}$ reflects the price dynamics. We assume $q_{jt}^k \ge 0$, and $\gamma_j^{ki} \ge 0$ since the price is deceasing in the amount of supply. Moreover, for each given product j, we assume that $(\gamma_j^{ki})^2 \le \gamma_j^{kk} \gamma_j^{ii}$ for all $1 \le i, k \le K$ and $1 \le j \le m$.

Also, for the unit cost of using Resource l at Stage t, $c_{lt}^k = b_{lt}^k + \sum_{i=1}^{K} \beta_l^{ki} x_{lt}^i + \alpha_{l,t-1} c_{l,t-1}^k$, where b_{lt}^k, β_l^{ki} are nonnegative constants and $0 \le \alpha_{l,t} \le 1$ for all l and t. Then Producer k at Stage t will face the following optimization problem:

$$
\begin{aligned}
(\tilde{P}_t^k) \quad \max \quad & \textstyle\sum_{j=1}^{n} (q_{jt}^k - \sum_{i=1}^{K} \gamma_j^{ki} v_{jt}^i + \rho_{j,t-1} p_{j,t-1}^k) v_{jt}^k \\
& - \textstyle\sum_{l=1}^{m} (b_{lt}^k + \sum_{i=1}^{K} \beta_l^{ki} x_{lt}^i + \alpha_{l,t-1} c_{l,t-1}^k) x_{lt}^k \\
\text{s.t.} \quad & N^k v_t^k \le M^k x_t^k, \\
& x_t^k \ge 0, v_t^k \ge 0.
\end{aligned}
$$

The existence of the myopic Nash equilibrium is guaranteed due to the boundedness of the solutions of the problem above (see also the argument in Chapter 3). Letting z_t^k be the Lagrangian dual variable for the constraint $M^k x_t^k - N^k v_t^k \ge 0$, s_t^k be the dual variable for the constraint $x_t^k \ge 0$ and u_t^k be the dual variable for the constraint $v_t^k \ge 0$, the overall optimality condition (or the equilibrium condition) is the following monotone LCP system:

$$
\begin{cases}
b_{lt}^k + \sum_{i=1}^{K} \beta_l^{ki} x_{lt}^i + \alpha_{l,t-1} c_{l,t-1}^k + \beta_l^{kk} x_{lt}^k - ((M^k)^{\mathrm{T}} z_t^k)_l - s_{lt}^k = 0, \\
q_{jt}^k - \sum_{i=1}^{K} \gamma_j^{ki} v_{jt}^i + \rho_{j,t-1} p_{j,t-1}^k - \gamma_j^{kk} v_{jt}^k - ((N^k)^{\mathrm{T}} z_t^k)_j + u_{jt}^k = 0, \\
(z_t^k)^{\mathrm{T}} (M^k x_t^k - N^k v_t^k) = 0, \\
M^k x_t^k - N^k v_t^k \ge 0, z_t^k \ge 0, \\
(x_t^k)^{\mathrm{T}} s_t^k = 0,\ x_t^k \ge 0,\ s_t^k \ge 0, \\
(v_t^k)^{\mathrm{T}} u_t^k = 0,\ v_t^k \ge 0,\ u_t^k \ge 0.
\end{cases}
$$

Lemma 4.2.5 *At a myopic Nash equilibrium of an instance of the Cournot oligopoly competition with solution (w, y), the profit of Player k at Stage t in the equilibrium equals to $\sum_{j=1}^{n} \gamma_j^{kk}(w_{jt}^k)^2 + \sum_{l=1}^{m} \beta_l^{kk}(y_{lt}^k)^2$.*

Proof. Since each player at the myopic Nash equilibrium attains optimality, given the strategies of the others, by the KKT condition we have

$$q_{jt}^k - \sum_{i=1}^{K} \gamma_j^{ki} w_{jt}^i + \rho_{j,t-1} p_{j,t-1}^k = \gamma_j^{kk} w_{jt}^k + ((N^k)^{\mathrm{T}} z_t^k)_j - u_{jt}^k,$$

and

$$b_{lt}^k + \sum_{i=1}^{K} \beta_l^{ki} y_{lt}^i + \alpha_{l,t-1} c_{l,t-1}^k = -\beta_l^{kk} y_{lt}^k + ((M^k)^{\mathrm{T}} z_t^k)_l + s_{lt}^k.$$

Hence the profit for Producer k is

$$\begin{aligned}
& \sum_{j=1}^{n}(q_{jt}^k - \sum_{i=1}^{K} \gamma_j^{ki} w_{jt}^i + \rho_{j,t-1} p_{j,t-1}^k) w_{jt}^k - \sum_{l=1}^{m}(b_{lt}^k + \sum_{i=1}^{K} \beta_l^{ki} y_{lt}^i + \alpha_{l,t-1} c_{l,t-1}^k) y_{lt}^k \\
= & \sum_{j=1}^{n}(\gamma_j^{kk} w_{jt}^k + ((N^k)^{\mathrm{T}} z_t^k)_j - u_{jt}^k) w_{jt}^k - \sum_{l=1}^{n}(-\beta_l^{kk} y_{lt}^k + ((M^k)^{\mathrm{T}} z_t^k)_l + s_{lt}^k) y_{lt}^k \\
= & \sum_{j=1}^{n} \gamma_j^{kk}(w_{jt}^k)^2 + \sum_{l=1}^{m} \beta_l^{kk}(y_{lt}^k)^2 + \sum_{j=1}^{n} w_{jt}^k (N^k)^{\mathrm{T}} z_t^k)_j - \sum_{l=1}^{n} y_{lt}^k ((M^k)^{\mathrm{T}} z_t^k)_l \\
= & \sum_{j=1}^{n} \gamma_j^{kk}(w_{jt}^k)^2 + \sum_{l=1}^{m} \beta_l^{kk}(y_{lt}^k)^2 + (z_t^k)^{\mathrm{T}}(N^k w_t^k - M^k y_t^k) \\
= & \sum_{j=1}^{n} \gamma_j^{kk}(w_{jt}^k)^2 + \sum_{l=1}^{m} \beta_l^{kk}(y_{lt}^k)^2.
\end{aligned}$$

□

Lemma 4.2.6 *Denote (v, x) and (w, y) to be the solutions at the social optimum and at a myopic Nash equilibrium respectively. At the myopic Nash equilibrium (w, y), suppose that Producer k switches his/her strategy at Stage t to (v_t^k, x_t^k) while keeping his/her strategies in other stages and all other producers' strategies over the entire period unaltered. Then, the profit of Producer k at Stage t will decrease by at least an* *amount of* $\sum_{j=1}^{n}\gamma_j^{kk}(v_{jt}^k - w_{jt}^k)^2 + \sum_{l=1}^{m}\beta_l^{kk}(x_{lt}^k - y_{lt}^k)^2$*; that is,*

$$V_t^k(w_t^k, y_t^k; w_t^{-k}, y_t^{-k}) - V_t^k(v_t^k, x_t^k; w_t^{-k}, y_t^{-k}) \geq \sum_{j=1}^{n}\gamma_j^{kk}(v_{jt}^k - w_{jt}^k)^2 + \sum_{l=1}^{m}\beta_l^{kk}(x_{lt}^k - y_{lt}^k)^2.$$

Proof. By the definition of the myopic Nash equilibrium, (w_t^k, y_t^k) attains maximum for Producer k's profit at Stage t, assuming his/her strategy in other stages is (w_{-t}^k, y_{-t}^k) and all other producers' strategies are fixed as (w^{-k}, y^{-k}). Since the strategies of Producer k in the previous stages have been fixed, p_{t-1} and c_{t-1} are not changed while Producer k's strategy in Stage t is changed, therefore

$$\begin{aligned}
& V_t^k(w_t^k, y_t^k; w_t^{-k}, y_t^{-k}) - V_t^k(v_t^k, x_t^k; w_t^{-k}, y_t^{-k}) - \sum_{j=1}^{n}\gamma_j^{kk}(v_{jt}^k - w_{jt}^k)^2 \\
& \quad - \sum_{l=1}^{m}\beta_l^{kk}(x_{lt}^k - y_{lt}^k)^2 \\
= \; & (q_t^k + \mathrm{Diag}(\rho_{t-1})p_{t-1}^k)^{\mathrm{T}}(w_t^k - v_t^k) \\
& + \sum_{j=1}^{n}\left[\sum_{i\neq k}\gamma_j^{ki}w_{jt}^i v_{jt}^k + \gamma_j^{kk}(v_{jt}^k)^2 - \sum_{i\neq k}\gamma_j^{ki}w_{jt}^i w_{jt}^k - \gamma_j^{kk}(w_{jt}^k)^2 - \gamma_j^{kk}(v_{jt}^k - w_{jt}^k)^2\right] \\
& - (b_t^k + \mathrm{Diag}(\alpha_{t-1})c_{t-1}^k)^{\mathrm{T}}(y_t^k - x_t^k) \\
& + \sum_{l=1}^{m}\left[\sum_{i\neq k}\beta_l^{ki}y_{lt}^i x_{lt}^k + \beta_l^{kk}(x_{lt}^k)^2 - \sum_{i\neq k}\beta_l^{ki}y_{lt}^i y_{lt}^k - \beta_l^{kk}(y_{lt}^k)^2 - \beta_l^{kk}(x_{lt}^k - y_{lt}^k)^2\right] \\
= \; & \sum_{j=1}^{n}\left(q_{jt}^k - \sum_{i\neq k}\gamma_j^{ki}w_{jt}^i - 2\gamma_j^{kk}w_{jt}^k + \rho_{j,t-1}p_{j,t-1}^k\right)(w_{jt}^k - v_{jt}^k) \\
& - \sum_{l=1}^{m}\left(b_{lt}^k + \sum_{i\neq k}\beta_l^{ki}y_{lt}^i + 2\beta_l^{kk}y_{lt}^k + \alpha_{l,t-1}c_{l,t-1}^k\right)(y_{lt}^k - x_{lt}^k) \\
= \; & \nabla V_t^k(w_t^k, y_t^k; w_t^{-k}, y_t^{-k})^{\mathrm{T}}\begin{pmatrix} w_t^k - v_t^k \\ y_t^k - x_t^k \end{pmatrix} \geq 0,
\end{aligned}$$

where the last inequality is due to the fact that (w_t^k, y_t^k) is maximal. □

Theorem 4.2.7 *Under our assumptions above, the PoI of the dynamic Cournot oligopolistic competition game is lower bounded by* $\frac{1}{KT}$.

Proof. Similar as our analysis for the transportation model, to get a lower bound for the price of isolation, it suffices to find a (λ, μ) pair with $\lambda > 0$ and $\mu > -1$, such that

$$\begin{aligned}&\sum_{t=1}^{T}\sum_{k=1}^{K}\left[V_t^k(v_t^k, x_t^k; w_t^{-k}, y_t^{-k}) + \sum_{j=1}^{n}\gamma_j^{kk}(v_{jt}^k - w_{jt}^k)^2 + \sum_{l=1}^{m}\beta_l^{kk}(x_{lt}^k - y_{lt}^k)^2\right]\\ \geq\ & \lambda\,\mathrm{SV}(v,x) - \mu\,\mathrm{SV}(w,y). \end{aligned} \tag{4.10}$$

Then, by Lemma 4.2.6, we can bound the price of isolation as well:

$$\mathrm{PoI} = \frac{\mathrm{SV}(w,y)}{\mathrm{SV}(v,x)} \geq \frac{\lambda}{1+\mu}.$$

Here we use the similar matrix notations as before. Let

$$L_{\rho_j} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \rho_{j1} & 1 & \cdots & 0 \\ \rho_{j1}\rho_{j2} & \rho_{j2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \Pi_{h=1}^{T-1}\rho_{jh} & \Pi_{h=2}^{T-1}\rho_{jh} & \cdots & 1 \end{pmatrix},$$

and

$$L_{\alpha_l} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \alpha_{l1} & 1 & \cdots & 0 \\ \alpha_{l1}\alpha_{l2} & \alpha_{l2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \Pi_{h=1}^{T-1}\alpha_{lh} & \Pi_{h=2}^{T-1}\alpha_{lh} & \cdots & 1 \end{pmatrix}.$$

According to the dynamics of p_{jt}^k and c_{lt}^k, we may write the dynamic equation in the vector form as:

$$p_{(j)}^k = -\sum_{i=1}^{K}\gamma_j^{ki}L_{\rho_j}v_{(j)}^i + L_{\rho_j}q_{(j)}^k, \quad c_{(l)}^k = \sum_{i=1}^{K}\beta_l^{ki}L_{\alpha_l}x_{(l)}^i + L_{\alpha_l}b_{(l)}^k$$

Substitute the explicit form of each term in (4.10), and apply Lemma 4.2.5. Then the intended inequality (4.10) becomes:

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{j=1}^{n}\Big[(v^k_{(j)})^{\mathrm{T}}L_{\rho_j}q^k_{(j)} - \sum_{i=1}^{K}\gamma_j^{ki}(v^k_{(j)})^{\mathrm{T}}L_{\rho_j}w^i_{(j)} - \gamma_j^{kk}(v^k_{(j)})^{\mathrm{T}}(v^k_{(j)} - w^k_{(j)})\\
&\quad +\gamma_j^{kk}(v^k_{(j)} - w^k_{(j)})^{\mathrm{T}}(v^k_{(j)} - w^k_{(j)})\Big]\\
&\quad -\sum_{k=1}^{K}\sum_{l=1}^{m}\Big[(x^k_{(l)})^{\mathrm{T}}L_{\alpha_l}b^k_{(l)} + \sum_{i=1}^{K}\beta_l^{ki}(x^k_{(l)})^{\mathrm{T}}L_{\alpha_l}y^i_{(l)} + \beta_l^{kk}(x^k_{(l)})^{\mathrm{T}}(x^k_{(l)} - y^k_{(l)})\\
&\quad -\beta_l^{kk}(x^k_{(l)} - y^k_{(l)})^{\mathrm{T}}(x^k_{(l)} - y^k_{(l)})\Big]\\
\geq\ & \lambda\sum_{k=1}^{K}\sum_{j=1}^{n}\Big[(v^k_{(j)})^{\mathrm{T}}L_{\rho_j}q^k_{(j)} - \sum_{i=1}^{K}\gamma_j^{ki}(v^k_{(j)})^{\mathrm{T}}L_{\rho_j}v^i_{(j)}\Big] - \lambda\sum_{k=1}^{K}\sum_{l=1}^{m}\Big[(x^k_{(l)})^{\mathrm{T}}L_{\alpha_l}b^k_{(l)}\\
&\quad + \sum_{i=1}^{K}\beta_l^{ki}(x^k_{(l)})^{\mathrm{T}}L_{\alpha_l}x^i_{(l)}\Big] - \mu\sum_{t=1}^{T}\sum_{k=1}^{K}\Big[\sum_{j=1}^{n}\gamma_j^{kk}(w^k_{jt})^2 + \sum_{l=1}^{m}\beta_l^{kk}(y^k_{lt})^2\Big].
\end{aligned}$$

To simplify, we let $\lambda = 1$. Then, the desired inequality becomes:

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{k=1}^{K}\Big[\sum_{i=1}^{K}\gamma_j^{ki}(v^k_{(j)})^{\mathrm{T}}L_{\rho_j}(v^i_{(j)} - w^i_{(j)}) - \gamma_j^{kk}(w^k_{(j)})^{\mathrm{T}}(v^k_{(j)} - w^k_{(j)}) + \mu\gamma_j^{kk}(w^k_{(j)})^{\mathrm{T}}(w^k_{(j)})\Big]\\
+&\sum_{l=1}^{m}\sum_{k=1}^{K}\Big[\sum_{i=1}^{K}\beta_l^{ki}(x^k_{(l)})^{\mathrm{T}}L_{\alpha_l}(x^i_{(l)} - y^i_{(l)}) - \beta_l^{kk}(y^k_{(l)})^{\mathrm{T}}(x^k_{(l)} - y^k_{(l)}) + \mu\beta_l^{kk}(y^k_{(l)})^{\mathrm{T}}y^k_{(l)}\Big] \geq 0.
\end{aligned} \tag{4.11}$$

Next, we set $\mu = KT - 1$. Then, the first term on the left hand side of (4.11) can be rewritten as

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{i=1}^{K}\gamma_j^{ki}(v^k_{(j)} - w^k_{(j)})^{\mathrm{T}}L_{\rho_j}(v^i_{(j)} - w^i_{(j)}) + \sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{i=1}^{K}\gamma_j^{ki}(w^k_{(j)})^{\mathrm{T}}(L_{\rho_j} - I_T)v^i_{(j)}\\
+&\sum_{j=1}^{n}\sum_{k=1}^{K}\Big[KT\gamma_j^{kk}(w^k_{(j)})^{\mathrm{T}}(w^k_{(j)}) - \sum_{i=1}^{K}\gamma_j^{ki}(w^k_{(j)})^{\mathrm{T}}L_{\rho_j}w^i_{(j)}\Big].
\end{aligned} \tag{4.12}$$

Note that all of γ_j^{kk}, w^k_{jt} and v^k_{jt} are nonnegative, and $L_{\rho_j} - I_T \geq 0$, then

$$\sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{i=1}^{K}\gamma_j^{ki}(w^k_{(j)})^{\mathrm{T}}(L_{\rho_j} - I_T)v^i_{(j)} \geq 0.$$

Furthermore, $L_{\rho_j} - E_T \leq 0$ (i.e. $L_{\rho_j} - E_T$ is nonpositive componentwise), where E_T

denotes the $T \times T$ all-one matrix. Thus,

$$\begin{aligned}
& \sum_{k=1}^{K}\left[KT\gamma_j^{kk}(w_{(j)}^k)^{\mathrm{T}}(w_{(j)}^k) - \sum_{i=1}^{K}\gamma_j^{ki}(w_{(j)}^k)^{\mathrm{T}}L_{\rho_j}w_{(j)}^i\right] \\
\geq & \sum_{k=1}^{K}\left[KT\gamma_j^{kk}(w_{(j)}^k)^{\mathrm{T}}(w_{(j)}^k) - \sum_{i=1}^{K}\gamma_j^{ki}(w_{(j)}^k)^{\mathrm{T}}E_T w_{(j)}^i\right] \\
= & (KT-1)\sum_{t=1}^{T}\sum_{k=1}^{K}\gamma_j^{kk}(w_{jt}^k)^2 - \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{\tau=1}^{t-1}\sum_{i\neq k}\gamma_j^{ki}w_{j\tau}^i w_{jt}^k \\
\geq & \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{\tau<t}\sum_{i<k}\left[\gamma_j^{kk}(w_{jt}^k)^2 - (\gamma_j^{ik}+\gamma_j^{ki})w_{j\tau}^i w_{jt}^k + \gamma_j^{ii}(w_{j\tau}^i)^2\right] \\
\geq & \sum_{t=1}^{T}\sum_{k=1}^{K}\sum_{\tau<t}\sum_{i<k}\left(\sqrt{\gamma_j^{kk}}(w_{jt}^k) - \sqrt{\gamma_j^{ii}}(w_{j\tau}^i)\right)^2 \geq 0.
\end{aligned}$$

Therefore, we have

$$\text{The term (4.12)} \geq \sum_{j=1}^{n}\sum_{k=1}^{K}\sum_{i=1}^{K}\gamma_j^{ki}(v_{(j)}^k - w_{(j)}^k)^{\mathrm{T}}L_{\rho_j}(v_{(j)}^i - w_{(j)}^i).$$

Similarly, it can also be shown that

$$\text{The second term in (4.11)} \geq \sum_{l=1}^{m}\sum_{k=1}^{K}\sum_{i=1}^{K}\beta_l^{ki}(x_{(l)}^k - y_{(l)}^k)^{\mathrm{T}}L_{\alpha_l}(x_{(l)}^i - y_{(l)}^i).$$

To summarize, the left hand side of (4.11) is no less than

$$\sum_{k=1}^{K}\sum_{i=1}^{K}\left[\sum_{j=1}^{n}\gamma_j^{ki}(v_j^k - w_j^k)^{\mathrm{T}}L_{\rho_j}(v_j^i - w_j^i) + \sum_{l=1}^{m}\beta_l^{ki}(x_l^k - y_l^k)^{\mathrm{T}}L_{\alpha_l}(x_l^i - y_l^i)\right]. \quad (4.13)$$

Let

$$v_{(j)} = \left((v_{(j)}^1)^{\mathrm{T}}, (v_{(j)}^2)^{\mathrm{T}}, \ldots, (v_{(j)}^K)^{\mathrm{T}}\right)^{\mathrm{T}}, v = \left((v_{(1)})^{\mathrm{T}}, (v_{(2)})^{\mathrm{T}}, \ldots, (v_{(m)})^{\mathrm{T}}\right)^{\mathrm{T}},$$

(similarly for w, x, y) and denote $\Gamma_j = (\gamma_j^{ki})_{K\times K}$, $\Xi_l = (\beta_l^{ki})_{K\times K}$, then the term (4.13) can be rewritten as

$$(v-w)^{\mathrm{T}}\mathrm{Diag}\left((L_{\rho_j}\otimes\Gamma_j)\right)(v-w) + (x-y)^{\mathrm{T}}\mathrm{Diag}\left((L_{\alpha_l}\otimes\Xi_l)\right)(x-y),$$

where Diag$(\cdot)$ denotes the diagonal block matrix, whose diagonal block entries are the corresponding matrices. Furthermore, the second order derivative matrix for the social profit function is

$$\nabla^2 \mathrm{SV}(v,x) = \mathrm{sym}\left(\left(\begin{array}{cc} \mathrm{Diag}\left((L_{\rho_j} \otimes \Gamma_j)\right) & 0 \\ 0 & \mathrm{Diag}\left((L_{\alpha_l} \otimes \Xi_l)\right) \end{array}\right)\right).$$

Here 'sym' signifies the symmetric operation. Notice that (v,x) is the social maximum and that (w,y) is a feasible solution, and so

$$(v-w)^{\mathrm{T}}\mathrm{Diag}\left((L_{\rho_j} \otimes \Gamma_j)\right)(v-w) + (x-y)^{\mathrm{T}}\mathrm{Diag}\left((L_{\alpha_l} \otimes \Xi_l)\right)(x-y) \geq 0,$$

due to the second order optimality condition for (v,x). Putting all the pieces together, we have shown that that (4.11) holds when $\lambda = 1$ and $\mu = KT - 1$. Consequently, we have $\mathrm{PoI} \geq \frac{1}{KT}$. □

This lower bound is essentially tight, as is shown by the example below.

Example 4.2.8 Suppose there is only one kind of resource available to produce one kind of commodity. The technology-matrix-pairs for all the players are the same (M,N) and assume $M = N = (1)$. The unit cost function for each play to use the only resource is $c_t^k(x) = \sum_{i=1}^K x_t^i + c_{t-1}^k(x)$, with $c_0^k = 0$ and there is no competition for the price. Suppose that there are K identical players, with the price $p_t^k = 1 + p_{t-1}^k(v)$, with $p_1^k = 1 + \frac{1}{K}$ for all k. In this setting, there is only one myopic Nash equilibrium, at which each player will use $\frac{1}{K}$ amount of resources to produce the commodity to receive a profit of $\frac{1}{K^2}$ at each stage. Thus the total social profit over the whole period is $\frac{T}{K}$. On the other hand, consider a feasible strategy for the players: each player uses $\frac{1}{2K}$ amount of resources to produce the commodity at each stage. Then the profit for each player at stage t will be $\frac{t}{2} + \frac{1}{K}$. The total social profit will be larger than $K \cdot \sum_{t=1}^T \frac{t}{2} \cdot \frac{1}{2K} = \frac{T(T+1)}{8}$. Then the price of isolation in this game is at least $\frac{T/K}{T(T+1)/8} = \frac{8}{K(T+1)} = O(\frac{1}{KT})$.

Summarizing, we have:

Corollary 4.2.9 *Under our assumptions, if T and K are taken fixed parameters of the dynamic Cournot oligopoly game, then*

$$\frac{1}{KT} \leq \mathrm{PoI}_{\mathscr{G}} \leq \frac{8}{K(T+1)}.$$

Chapter 5

The Price of Socialism

5.1 Introduction

5.1.1 The Price of Socialism

In contrast to the inefficiency caused by the isolated behaviors of players discussed before, we now turn to the cooperation among the players in this chapter. Compared to the noncooperative environment, we assume that coordination to some extent is allowed among the players. Generally speaking, some players may benefit from the cooperation. However, as the example in Chapter 1 shows, if the profit or cost are nontransferable, it might be impossible for all players to be better off, in terms of attaining more profits or less costs, through coordination than under the Nash equilibrium. In other words, some players may need to make a sacrifice under the coordination framework. To capture the altruistic behavior in the cooperation, we introduce the notion of *Price of Socialism* (PoS) as a measure of the change of benefit or loss to a player due to the cooperation, based on his/her payoff at Nash equilibrium.

Note that if the utility/cost is transferable, it makes no sense to discuss the altruism because there will always exist a compensation distribution plan such that everyone benefits as long as the coalition gets better from the cooperation. Therefore, we confine ourselves to the nontransferable case.

Again, we are interested in the two solutions: the Nash equilibrium and the social optimum. To estimate how much the players may need to sacrifice, we are to compare the difference of the outcomes at Nash equilibrium and the social optimum for each player, and choose the worst one as a benchmark. Specifically, for a given setting, if the objective in the problem is nonnegative and is minimization, the price of socialism is defined as:

$$\text{PoS} = \sup_{k \in \text{ the set of players}} \left\{ \frac{\text{the maximal cost incurred to } k \text{ at the social optimum}}{\text{the minimal cost incurred to } k \text{ at Nash equilibrium}} \right\},$$

If the objective is nonnegative and is maximization,

$$\text{PoS} = \inf_{k \in \text{ the set of players}} \left\{ \frac{\text{the minimal profit gained by } k \text{ at the social optimum}}{\text{the maximal value gained by } k \text{ at Nash equilibrium}} \right\}.$$

Notice that the PoS could take any nonnegative value and 1 is a watershed indicating the different impact of cooperation. Take the case in which the objective is minimization as an example. If $0 \leq \text{PoS} < 1$, it is implied that each player will benefit from the cooperation — the smaller the PoS is, the more benefit the players may obtain. If $\text{PoS} > 1$, at least one player needs to make a sacrifice for system optimum — the larger the PoS is, the more the player needs to sacrifice. If $\text{PoS} = 1$, then the Nash equilibrium is also the social optimum.

5.1.2 A Resource Competition Game

We shall consider a specific example to illustrate the PoS explicitly and introduce the setting in details here. Suppose that there are K users competing for a sort of resource, which is located in m suppliers. The demand of User k for the resource is denoted by r^k, which could be fulfilled *splitably* from the m suppliers. The price of the resource at each supplier, or the unit cost from the viewpoint of the users, is dependent on the total demand of K users for the resource in this supplier. Then the users need to minimize the costs which are paid for the resource to the suppliers, and fulfill the demand. Let the decision of Player k be a vector $x^k \in \mathbb{R}^m_+$, whose l-th component represents the amount of resource Player k obtains from the l-th supplier. Clearly, to fulfill the demand of itself, the feasible strategy solution must satisfy $\sum_{l=1}^m x_l^k = r^k$. For each supplier l, we define the total demand as $f_l = \sum_{k=1}^K x_l^k$. The price of the resource at the l-th supplier is given by a function $c_l : f_l \hookrightarrow c_l(f_l)$. Then (K, m, r, c) specifies an instance of the resource competition game problem.

In the instance (K, m, r, c), if considering the Nash equilibrium, Player k is to find a feasible strategy solution to minimize its total cost function, which is defined as

$$C^k(x^k, x^{-k}) = \sum_{l=1}^{m} x_l^k c_l(f_l).$$

We define the social cost as the simple sum of all players: $\mathrm{SC}(x) = \sum_{k=1}^K C^k(x^k, x^{-k})$. Here we assume the function $xc_l(x)$ is convex for each l. Then it can be concluded that a Nash equilibrium exists in this type of game, noting that the feasible strategy set for each player is a bounded and convex set. Let x^{NE} denote the flow when the game reaches the Nash equilibrium, at which each player minimizes its cost given the others' strategies. Let x^* denote the minimal solution of the function $\mathrm{SC}(x)$ over the feasible field of x. Clearly, x^{NE} and x^* are different vectors in general. Similar with the previous chapters, the difference from the angle of the whole system can be quantified by the price of anarchy, which is defined by the ratio between the social cost at the two states, i.e.,

$$\text{PoA} = \frac{SC(x^{NE})}{SC(x^*)}.$$

However, for individuals, to investigate the impact of cooperation and altruistic behavior, we consider the price of socialism defined as:

$$\text{PoS} = \sup_k \frac{C^k(x^*)}{C^k(x^{NE})}.$$

Equivalent to Transportation Game with Parallel Network

Before discussing those measurements introduced above, we observe that the resource competition game can be reduced to a transportation game with a parallel network equivalently. For an instance of resource competition game (K, m, r, c), we can construct an equivalent transportation game as follows:

First, assume that the underlying network in the transportation game consists of two nodes, denoted by s and d, and m links between them. The total demand for each supplier can be viewed as the total flow through the corresponding link in the transportation game, and the unit cost on Link l is given by the function $c_l(f_l)$, where f_l is the amount of the flow through the link. Finally, the competitive users in resource competition problem can be viewed as the players in transportation game and each of them has the same OD pair (from s to d) and the required throughput task given by r^k. Consequently, to make the problem more explicit, we focus on the transportation game with a parallel network, instead of the resource competition problem in the following sections.

5.1.3 Related Work

The altruistic behavior attracts attention of many researchers in recent years. Fotakis, Kontogiannis and Spirakis [23] considered a coalitional congestion model among atomic players in the transportation game. They found many similarities of the model with the noncooperative case. Furthermore, they established the existence of Nash equilibria in the (even unrelated) parallel links setting. Hoefer and Skopalik [31] considered a kind of altruism called ξ-altruistic behavior in atomic congestion games and derived some results about the existence of Nash equilibria in that sense and the convergence of sequential best response dynamics. Later, Azad, Altman and El-Azouzi [8] introduced a parameter to characterize the extent of cooperation, which ranges from the fully noncooperative behavior and the partially cooperative to the fully cooperative behavior, and even more, the fully altruistic behavior. They found a similar case with Braess Paradox in presence of players' cooperation. Altuism is also discussed in the context of economics, as well as telecommunication networks. For more details, we refer to [35] for a survey.

As introduced later, we need to apply some sensitivity analysis with respect to the parameters in the model in order to derive some monotonicity property. This method has also been well developed in literature. Dafermos and Nagurney [20] investigated the variational inequality (VI) associated with the equilibrium (cf. Chapter 3 of [53]) and showed that the traffic equilibrium pattern depends continuously upon the assigned travel demands and travel cost functions. Furthermore, they analyzed the direction of the change in the traffic pattern and the travel costs incurred as a result of the changes in the travel cost functions and the travel demands. Patriksson and Rockafellar [55] studied the computational issues in the sensitivity analysis. Later, Patriksson [56] gave a characterization for the existence of a directional derivative in terms of the variables at equilibrium (e.g. the flow on the links, the least travel cost, the demand, etc.) along a given direction, and demonstrated how to compute such quantities. In this chapter we shall draw on some of these tools for our analysis.

5.2 Bounding the PoS with Affine Linear Cost

5.2.1 The Effect of Coordination

Now we focus on the transportation game with parallel network and confine ourselves to the case that the unit cost on each link is affine linear in the flow. Again, suppose that the unit cost on Link l is given by $c_l(f_l) = a_l f_l + b_l$, in which $a_l, b_l \geq 0$. Then Player k will face the following optimization problem:

$$\begin{array}{lll} (P_k) & \min & \sum_{l=1}^m (a_l f_l + b_l) x_l^k \\ & \text{s.t.} & \sum_{l=1}^m x_l^k = r^k,\ x^k \geq 0. \end{array}$$

According to the KKT condition, when Player k attains to the optimum, we have

$$\begin{cases} \sum_{l\in L} x_l^k = r^k, \\ a_l(f_l + x_l^k) + b_l - q_l^k - s_l^k = 0, \\ x_l^k \geq 0,\ s_l^k \geq 0,\ x_l^k s_l^k = 0, \quad \text{for all } l. \end{cases}$$

It follows that

$$\begin{cases} q^k = a_l(f_l + x_l^k) + b_l, & \text{if } x_l^k > 0 \\ q^k \leq a_l f_l + b_l, & \text{if } x_l^k = 0. \end{cases}$$

The quantity q^k could be thought of as the marginal cost of Player k. Denote the set of links which Player k uses to be $S^k = \{l | x_l^k > 0\}$. Without lost of generality we assume that $a_l > 0$ for all $l = 1, 2, \ldots, m$. Also, we can rearrange the indices and assume that

$r^1 \geq r^2 \geq \ldots \geq r^K$, $b_1 \leq b_2 \leq \ldots \leq b_m$. Since $x_l^k = \left[\frac{q^k - b_l}{a_l} - f_l\right]_+$ for all k and l,

$$r^k = \sum_{l=1}^{m} x_l^k = \sum_{l=1}^{m} \left[\frac{q^k - b_l}{a_l} - f_l\right]_+,$$

which is monotonically increasing with respect to q^k. It is implied that $q^1 \geq q^2 \geq \ldots \geq q^K$. Consequently,

$$x_l^1 \geq x_l^2 \geq \ldots \geq x_l^K, \text{ for all } l = 1, 2, \ldots, m, \text{ and } S^1 \supseteq S^2 \supseteq \ldots \supseteq S^K.$$

Here the mergence of the required throughput of the players could be thought of as a form of coordination. Then with reference to this kind of cooperation, we have the following monotonicity results:

Theorem 5.2.1 *If we denote the cost of Player k when the game attains the (unique) Nash equilibrium, as function $C^k(r)$ of flow vector r, if $r^{k_1} > r^{k_2}$, then for any $0 \leq t \leq r^{k_2}$, the following holds true:*

$$\mathrm{SC}(r + t\delta_{k_1} - t\delta_{k_2}) \leq \mathrm{SC}(r),$$

moreover,

$$C^k(r + t\delta_{k_1} - t\delta_{k_2}) \leq C^k(r) \text{ for } k \neq k_1, k_2,$$

where δ_j is the all zero vector with the exception that of value 1 at the j-th entry.

Notice that in the theorem the case where $t = r^{k_2}$ represents when the two players agree to merge together to form a new player to play this competitive routing game.

To prove the theorem, we aim to determine the sign of the directional derivatives and make use of parametric analysis of the linear complementary problem deduced from the KKT condition of the original problem. Since the rigorous proof is too lengthy and may somewhat mislead the main theme of the thesis, we defer it to Appendix A.

The theorem implies that if any two players choose to partially cooperate (e.g., the player with bigger required throughput seizes more throughput from the player with smaller one), all the other players in the game will benefit, *i.e.* get less cost from the cooperation, and the social cost decreases at the same time. However, it says nothing about the change of the cost for the two cooperating players. It is not provable using our proof idea of directional derivatives that two players always have the incentive to cooperate. Notice that for Player k_1 and k_2, whose cooperation we consider, it only makes sense to consider the total cost by these two players, because the flow demand of them changes but the total flow demand inside this subgroup does not change. We can construct a simple example to show that the directional derivative of their total cost with respect to the change of parameter may be positive as follows:

Example 5.2.2 Consider a transportation game with a network consisting of two n-odes and two links. Suppose the unit cost functions on the two links are given by

$$c_1(f_1) = f_1, \quad \text{and} \quad c_2(f_2) = 2.$$

Suppose there are three players in this game and the required throughput is given by

$$r^1 = 1, \quad r^2 = 1, \quad \text{and} \quad r^3 = 1/2.$$

Suppose that $k_1 = 2, k_2 = 3$, *i.e.*, keep Player 1 fixed and let Player 2 and Player 3 cooperate partially, then it can be calculated that the directional derive of $C^{k_1} + C^{k_2}$ is positive, which means that the cost of the coalition consisting of Player 2 and Player 3 may even increase due to the cooperation.

5.2.2 Bounding the Price of Socialism

The arguments in last section suggests that some players in the coalition might be hurt from the cooperation under some payoff distribution mechanism. However, the following proposition shows that no matter what the distribution way is, the cost for an arbitrary player in the grand coalition is at most twice than the cost for him/her at Nash equilibrium.

Proposition 5.2.3 *In the resource competition game, as well as the transportation game with parallel network, if the unit cost functions are affine linear, the price of socialism will be upper bounded by 2.*

Proof. To complete the proof, we only need to show that for any player, the maximal cost incurred in a social optimal solution will be always less than twice of the cost at the Nash equilibrium.

Now choose a player k arbitrarily. At first, we divide all players into two groups denoted by S^k and S^{-k}, where $S^k = \{k\}$ and S^{-k} consists of all other players except Player k. We consider the case in which the players in S^{-k} cooperate together to form a coalition. According to the conclusions on monotonicity of the costs incurred to the players in our framework, Player k can benefit from the cooperation of other players. In other words, less cost for Player k will be incurred compared to at the Nash equilibrium.

Then consider the case that Player k join the coalition at the final step and all the players form a grand coalition to optimize the social cost. We will establish the bound for the PoS through estimating how much Player k may sacrifice in this step. Before that, we have the following two observations:

First, Player k will benefit from this move if he/she gets the best allocation in the cost distribution at the social optimum. That is, the links with less costs are assigned to Player k. Then Player k will get less cost than before joining the coalition. Otherwise, it will be contradictory with the social optimum.

Second, the ratio between the costs incurred to Player k at the worst allocation and the best one for him/her is at most 2. Note that at the social optimum, the marginal costs on all the links have the same value, *i.e.*,

$$2a_{l_1}f_{l_1} + b_{l_1} = 2a_{l_2}f_{l_2} + b_{l_2} \quad \text{for any} \quad l_1, l_2.$$

Therefore,

$$\begin{aligned} a_{l_1}f_{l_1} + b_{l_1} &\leq 2a_{l_1}f_{l_1} + b_{l_1} = 2a_{l_2}f_{l_2} + b_{l_2} \\ &\leq 2(a_{l_2}f_{l_2} + b_{l_2}) \quad \text{for any} \quad l_1, l_2. \end{aligned}$$

It means that the unit cost on the link with the highest cost is at most twice than the unit cost on the one with the lowest cost. Thus, the cost for Player k at the social optimum will be bounded by twice of the cost when he/she gets the best allocation.

Based on the arguments above, we use C^k to denote the cost incurred to Player k and have the following relationship:

$$\begin{aligned} & C^k(\text{at the social optimum}) \\ \leq\ & 2C^k(k \text{ obtains the best allocation at the social optimum}) \\ \leq\ & 2C^k(\text{before } k \text{ joining the coalition}) \\ \leq\ & 2C^k(\text{at the Nash equilibrium}). \end{aligned}$$

From the arbitrariness of k, we can finally conclude that PoS ≤ 2. □

This bound is tight for the PoS in the transportation game with parallel networks, as is shown in the following example:

Example 5.2.4 Consider a two-node-two-link network shown in Figure 5.1. Assume that the upper link has a cost function $c_1(f_1) = f_1$, and the lower link has a constant cost function $c_2(f_2) = 1$, where f_1 and f_2 denote the total flow through the upper and the lower link respectively. Suppose that there are two players, Player 1 and 2 with different required throughput, $r^1 = \epsilon$, and $r^2 = 1 - \epsilon$.

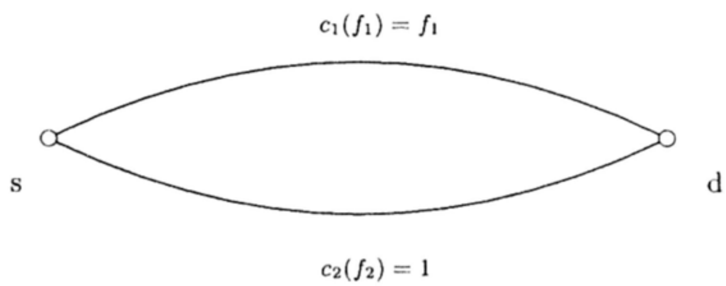

Figure 5.1: An example showing that the upper bound for the PoS is tight

Suppose ϵ is small enough. At the (unique) Nash equilibrium, Player 1 will always choose the upper link, while Player 2 will assign his/her transportation task to the two links equally. In this case, the cost for Player 1 is $\epsilon(1+\epsilon)/2$. When these two players cooperate, the social optimum is the same as Pigou's example, in which the one unit of transportation flow is assigned to the two links equally. We consider the worst allocation scenario for Player 1. His/her ϵ amount of flow may be assigned to the lower link in the social optimum, then the cost for Player 1 will be ϵ. Thus, the price of socialism in this setting is

$$\text{PoS} = \frac{\epsilon}{\epsilon(1+\epsilon)/2} = \frac{2}{1+\epsilon}.$$

When $\epsilon \to 0$, 2 is a bound to be attained.

5.3 Revisit the Price of Anarchy

According to Theorem 5.2.1 on monotonicity in last section, we can find the worst scenario for the price of anarchy in this type of game:

Corollary 5.3.1 *In the transportation game with the parallel network, suppose the unit cost functions are affine linear, then the price of anarchy attains to the maximum when all the players have exactly the same required throughput.*

Consider the worst case when all the players have the same required throughput, then by symmetry, we can conclude that all players have the same support set, the same flow vector and the same marginal cost when the game reaches the Nash equilibrium. Recall that we have derived the tight upper bound of the PoA for the transportation game with identical players in Theorem 2.2.5 in Chapter 2, then together with the corollary above, we can obtain an upper bound for the PoA in the transportation game with parallel network, as well as the resource competition game:

Theorem 5.3.2 *In the transportation game with the parallel network, as well as the resource competition game, suppose that there are K players and the unit cost functions are affine linear, then the PoA is upper bounded by* $\frac{4K^2}{3K^2+2K-1}$.

The tightness of this bound follows from Example 2.2.7 in Chapter 2 as well.

5.4 More Discussion about Monotonicity

The key of deriving our bounds on the PoS and the PoA in the transportation game with parallel network is the monotonicity of the cost with respect to the parameter of the required throughput. However, it is shown that the monotonicity of social cost may not hold any more for the transportation game with general networks in the following examples.

Example 5.4.1 Considering the directed graph given in Figure 5.2, there are 3 players in the transportation game and each has one unit of flow to transport. Both Player 1 and Player 2 have the same starting node A and terminal node C, while Player 3 starts at node B and ends at node C. We label the four links $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{BC}$ by 1, 2, 3, respectively. The unit cost functions on these links are given by

$$c_1(f_1) = 0, c_2(f_2) = f_2, \quad \text{and} \quad c_3(f_3) = 3f_3.$$

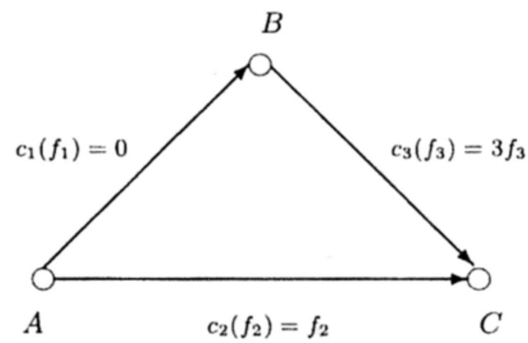

Figure 5.2: An example showing that monotonicity fails to hold.

Then in this example: If there is no cooperation between Player 1 and Player 2, both of them would only use Link $\overrightarrow{AC}$ and the corresponding social cost at the unique Nash equilibrium is 7; however, if Player 1 and 2 cooperate, they would let $\frac{1}{4}$ amount of the flow go along the path $A \to B \to C$, and the remaining $\frac{7}{4}$ amount of flow use Link $\overrightarrow{AC}$, then the corresponding social cost becomes 31/4, which is greater than the cost before.

The conclusion fails to hold even if we strengthen the assumption that all players have the same OD pair, as shown in the following example:

Example 5.4.2 Consider a transportation game with a directed network given in Figure 5.3. Suppose that there are three players and all of them have the same origin node A and the same destination node D. Moreover, the required throughput for these three players are given as follows:

$$r^1 = 1/10, \quad r^2 = 1/10, \quad \text{and } r^3 = 3/5.$$

We label the four links $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{BD}, \overrightarrow{CD}, \overrightarrow{BC}$ by 1, 2, 3, 4, 5, respectively. The unit cost functions on these links are given by

$$c_1(f_1) = 2 + 10f_1, \; c_2(f_2) = 4 + f_2, \; c_3(f_3) = 9 + 20f_3,$$

$$c_4(f_4) = 1 + 6f_4, \quad \text{and} \quad c_5(f_5) = 2 + 10f_5.$$

In this case: If there is no cooperation among all these three players, it can be easily

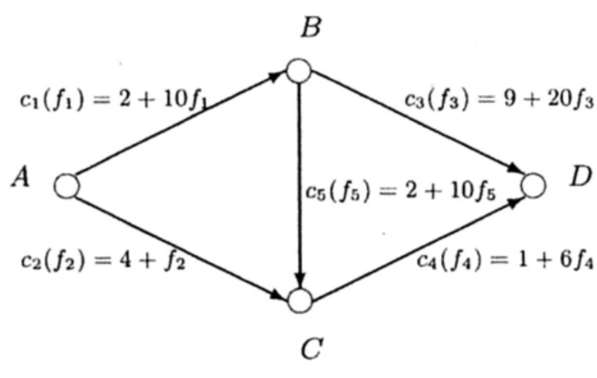

Figure 5.3: An example showing that monotonicity fails to hold even if OD pairs are the same.

computed that the social cost at the unique Nash equilibrium is 8.2491; however, if Player 1 and Player 2 choose to cooperate, while Player 3 is still an independent player, the corresponding social cost becomes 8.2497, which is greater than the one before.

However, with the assumption that all players have the same OD pairs, the transportation game with serial parallel network has some similar properties to the one with parallel network.

Note that V is the set of vertices. For any $v_1, v_2 \in V$, we use P to denote the path and $\mathscr{P}(v_1, v_2)$ to denote the set of paths between v_1 and v_2. Let q_l^k and q_P^k be the marginal cost of Player k on Link l and Path P, respectively. Clearly, $q_P^k = \sum_{l \in P} q_l^k$.

From the LCP system (or equivalently, the KKT condition), each player should have the same marginal cost on any path between two vertices, or mathematically,

$$q_{P_1}^k = q_{P_2}^k \text{ for any } P_1, P_2 \in \mathscr{P}(v_1, v_2) \text{ for } v_1, v_2 \in V.$$

Then for each player k, we can define the marginal cost with respect to v_1 and v_2 and denote it as $q^k(v_1, v_2)$. Similarly with the case with parallel network, by rearranging the indices of players according to the magnitude of demand flow, we can establish the monotonicity of $q^k(v_1, v_2)$ with respect to k, which is the key in the establishment of monotonicity. Since the idea is similar, we do not intend to present more details here.

Chapter 6

Conclusions and Future Work

In this thesis, we presented an integrated study on several measurements in order to characterize the consequences of some common types of behaviors for many real-life operations. More specifically, we presented some lower and upper bounds for such measurements, which may be constant or dependent on the parameters of the specific setting.

In particular, we first focus on the *selfishness* and employ a well established notion termed the price of anarchy as a measure. We investigated two types of framework for this study and some bounds are obtained under the affine cost/price framework. The first one is a cost minimization model, where each player attempts to fulfill a desired throughput over time at a minimum cost, while competing for the deployment of a shared set of resources whose costs increase as the competition intensifies. A typical instance is a transportation game where the transporters compete to use the roads to complete their service orders. The results about the transportation game can be viewed as a generalization of the results in [61], by allowing the number of players to be any finite number. Furthermore, in contrast to the familiar congestion game, we allow the flow to split through the network. The PoA is always bounded by 3/2, regardless the structure of the network in the game. The second model describes a profit maximization game where K producers compete on a set of shared resources to produce a common set of goods which their compete to sell in a common market. This model could be thought of an extension of the classical Cournot oligopolistic game. The fact that the inverse of the worst PoA is linearly dependent on the number of players shows that the Cournot competition game has a quite different nature: the market inefficiency is a much more important issue since the PoA may be very bad in that case.

Next we consider *greedy* nature of the decision makers, which is in many cases responsible for the deterioration of the overall system performance. We call this combined effect '*isolation*' here. In our particular context, by a somewhat loose usage of the word 'isolation' we refer to some type of disconnectedness of a decision maker within the system, in terms of the lack of coordination with other players, as well as the lack of coordination with oneself over time. In plain language, the failure to coordinate with other players is also called *selfishness*; the failure to coordinate with oneself over time is known as *myopia* or *greediness*. Our aim here is to understand to what extent such usual behaviors *combined* can damage the performance of the whole system. The measurement for the loss of system efficiency is called the price of isolation.

Both transportation game and Cournot oligopoly competition game are extended to a dynamic setting, over a period of T stages. We proved that if the unit costs are affine linear functions then the tight PoI lies in the interval [2, 4.4865). Assuming the unit costs of the resources and the unit prices of the goods are all affine linear functions, we prove that the tight PoI is in the order of $O\left(\frac{1}{KT}\right)$. The first model shows that the loss of the system efficiency is in a constant order, meaning that the cost of the selfish and greedy attitude will not deteriorate the system performance much further over time, even with many players and many stages played in the game. The second model, however, depicts a different picture. It says that as more profit-driven producers struggle to survive in the market, competing for a limited set of resources and on the sale prices as well, then the overall profit margin diminishes quickly (inversely proportional to the multiplicity of the number of players and the number of stages in the game).

The insights gained from our study may help to shed some light to understand the nature of the competitive market: *How strong will the market force be? What are the possible outcomes of a competitive market without regulation? When will the need for management and regulation arise?* Clearly, much more research efforts will be needed to fully understand those important issues.

Appendix A

Proof of Theorem 5.2.1

This appendix provides a rigorous proof for Theorem 5.2.1 in Chapter 5. To prove the theorem, we need to begin with some properties of the support sets and marginal costs at the Nash equilibrium, then proceed to the parametric analysis and the directional derivatives of the costs with respect to the change of the required throughput.

The following lemma gives the conditions under which a series of sets and scalars could be used to construct a Nash equilibrium in a transportation game with some fixed parameters.

Lemma A.0.3 *Given K sets $S^1 \supseteq S^2 \supseteq \ldots \supseteq S^K$, and K nonnegative scalars $q^1 \geq q^2 \geq \ldots \geq q^K$, they are corresponding to the support sets and marginal costs under the Nash equilibrium of a given transportation game if for $k = 1, 2, \ldots, K$,*

$$(k+1)q^k - Q^k > b_l \geq (k+1)q^{k+1} - Q^k,$$

where $Q^k = \sum_{i=1}^{k} q^i$ and $q^{K+1} = \frac{Q^K}{K+1}$. The required throughput of Player k is given by

$$\frac{Q^k}{k+1} - \frac{Q^{k-1}}{k} = g^k(r).$$

Here

$$g^k(r) = \frac{\sum_{i>k} r^i + (k+1)r^k + B^k}{k(k+1)A^k},$$

whose value only depends on r and the support set S^k.

Proof. Due to the monotonicity of the support sets, there are exactly k players using Link l for $l \in S^k - S^{k+1}$ and $f_l = \sum_{i=1}^k x_l^i$. Furthermore, $f_l + x_l^i = \frac{q^i - b_l}{a_l}$ for $i \leq k$. Take the sum of the equations for $i = 1, 2, \ldots, k$, we get

$$(k+1)f_l = \frac{Q^k - kb_l}{a_l} \quad \text{for all } l \in S^k - S^{k+1}. \tag{A.1}$$

Then we have the following equality

$$x_l^i = \left[\frac{(k+1)q^i - Q^k - b_l}{(k+1)a_l}\right]_+ \quad \text{for } l \in S^k - S^{k+1}. \tag{A.2}$$

Consequently, for any $i > k$ and $l \in S^k - S^{k+1}$, $(k+1)q^i - Q^k - b_l \leq 0$. Furthermore, $x_l^i - x_l^k = \frac{q^i - q^k}{a_l}$ for all $l \in S^k$ and $i \leq k$. Taking the sum of $f_l + x_l^k$ for $l \in S^k$, we have

$$\begin{aligned}
\sum_{l \in S^k} \frac{q^k - b_l}{a_l} &= \sum_{l \in S^k} (f_l + x_l^k) \\
&= \sum_{l \in S^k} \left[x_l^k + \sum_{i>k} x_l^i + \sum_{i \leq k} (x_l^i + \frac{q^i - q^k}{a_l}) \right] \\
&= \sum_{l \in S^k} \left[(k+1)x_l^k + \sum_{i>k} x_l^i + \frac{Q^k - kq^k}{a_l} \right] \\
&= \sum_{i>k} r^i + (k+1)r^k + \sum_{l \in S^k} \frac{Q^k - kq^k}{a_l}.
\end{aligned}$$

Therefore,

$$\sum_{i>k} r^i + (k+1)r^k = \sum_{l \in S^k} \frac{(k+1)q^k - Q^k - b_l}{a_l} = \sum_{l \in S^k} \frac{kQ^k - (k+1)Q^{k-1} - b_l}{a_l}. \tag{A.3}$$

Finally, we get the difference equation:

$$\frac{Q^k}{k+1} - \frac{Q^{k-1}}{k} = g^k(r).$$

□

For convenience let $A^k = \sum_{l \in S^k} \frac{1}{a_l}$ and $B^k = \sum_{l \in S^k} \frac{b_l}{a_l}$. Note that for any $l \in S^j - S^{j+1}$ and $j < k$,

$$(k+1)q^k - Q^k = (k+1)q^k - Q^j - (Q^k - Q^j) \leq (k+1)q^k - Q^j - (k-j)q^k = (j+1)q^k - Q^j \leq b_l.$$

For any $l \in S^i - S^{i+1}$ with $i > k$,

$$(k+1)q^k - Q^k = (k+1)q^k - Q^i + (Q^i - Q^k) \geq (k+1)q^i - Q^i + (i-k)q^i = (i+1)q^i - Q^i > b_l.$$

Consequently, $b_l > b_{l'}$ for any $l \in S^{k-1} - S^k$ and $l' \in S^k - S^{k+1}$.

By checking the KKT condition, we can show that for given the support sets $S^1 \supseteq S^2 \supseteq \ldots \supseteq S^K$, due to the difference equation,

$$Q^k = (k+1) \sum_{i=1}^{k} g^i(r). \tag{A.4}$$

And reversely, if the calculated q^i's are in decreasing order and for all $l \in S^k - S^{k+1}$, we have $(k+1)q^k - Q^k > b_l \geq (k+1)q^{k+1} - Q^k$, then the KKT condition is satisfied. Therefore, we have the following conclusion:

Proposition A.0.4 *Given the support sets $S^1 \supseteq S^2 \supseteq \ldots \supseteq S^K$, the condition for this support set to be indeed corresponding to the unique Nash equilibrium is that, for any $k = 1, 2, \ldots, K$ and $l \in S^k - S^{k+1}$,*

$$(k^2 + 2k)g^{k+1}(r) - \textstyle\sum_{i=1}^{k} g^i(r) \leq b_l < (k^2 + k)g^k(r),$$

$$(k+2)g^{k+1}(r) \leq kg^k(r).$$

Also, we can establish the monotonicity of $\frac{\sum_{l \in S^k}(a_l f_l + b_l)/a_l}{\sum_{l \in S^k} 1/a_l}$ and $\frac{\sum_{l \in S^k}(2a_l f_l + b_l)/a_l}{\sum_{l \in S^k} 1/a_l}$:

Lemma A.0.5 *For any $k > i$,*

$$\begin{aligned}\frac{\sum_{l\in S^k}(a_l f_l + b_l)/a_l}{\sum_{l\in S^k} 1/a_l} &\leq \frac{\sum_{l\in S^i}(a_l f_l + b_l)/a_l}{\sum_{l\in S^i} 1/a_l},\\ \frac{\sum_{l\in S^k}(2a_l f_l + b_l)/a_l}{\sum_{l\in S^k} 1/a_l} &\geq \frac{\sum_{l\in S^i}(2a_l f_l + b_l)/a_l}{\sum_{l\in S^i} 1/a_l}.\end{aligned}$$

Proof. For any $l_1 \in S^i - S^{i+1}$ and $l_2 \in S^k - S^{k+1}$, from $l_1 \notin S^k$, we have

$$a_{l_1} f_{l_1} + b_{l_1} \geq q^k = a_{l_2}(f_{l_2} + x_{l_2}^k) + b_{l_2} \geq a_{l_2} f_{l_2} + b_{l_2}.$$

Since $l_1 \notin S^{i+1}, l_2 \in S^k$, we have $(i+1)q^{i+1} - Q^i \leq b_{l_1}$, and $(k+1)q^k - Q^k \geq b_{l_2}$.

Furthermore, according to the monotonicity of q^k,

$$\left[Q^k - (k-1)q^k\right] - \left[Q^i - (i-1)q^{i+1}\right] = \sum_{j=i+1}^{k} q^j + (i-1)q^{i+1} - (k-1)q^k \geq 0,$$

i.e.,

$$Q^k - (k-1)q^k \geq Q^i - (i-1)q^{i+1}.$$

Therefore,

$$\begin{aligned}2a_{l_2} f_{l_2} + b_{l_2} &= \frac{2Q^k - (k-1)b_{l_2}}{k+1} \geq \frac{2Q^k - (k-1)[(k+1)q^k - Q^k]}{k+1}\\ &= Q^k - (k-1)q^k \geq Q^i - (i-1)q^{i+1}\\ &= \frac{2Q^i - (i-1)[(i+1)q^{i+1} - Q^i]}{i+1}\\ &\geq \frac{2Q^i - (i-1)b_{l_1}}{i+1} = 2a_{l_1} f_{l_1} + b_{l_1}.\end{aligned}$$

Due to the arbitrariness of k and i, for all $l_1 \in S^i - S^k, l_2 \in S^k$, we have $a_{l_2} f_{l_2} + b_{l_2} \leq a_{l_1} f_{l_1} + b_{l_1}$ and $2a_{l_2} f_{l_2} + b_{l_2} \geq 2a_{l_1} f_{l_1} + b_{l_1}$. Consider $\frac{\sum_{l\in S^k}(a_l f_l + b_l)/a_l}{\sum_{l\in S^k} 1/a_l}$ and $\frac{\sum_{l\in S^k}(2a_l f_l + b_l)/a_l}{\sum_{l\in S^k} 1/a_l}$ as the weighted average,

$$\begin{aligned}a_{l_1} f_{l_1} + b_{l_1} &\geq \frac{\sum_{l\in S^k}(a_l f_l + b_l)/a_l}{\sum_{l\in S^k} 1/a_l},\\ 2a_{l_1} f_{l_1} + b_{l_1} &\leq \frac{\sum_{l\in S^k}(2a_l f_l + b_l)/a_l}{\sum_{l\in S^k} 1/a_l}.\end{aligned}$$

The inequalities hold for any $l_1 \in S^i - S^k$, consequently, for $i < k$,

$$\frac{\sum_{l\in S^i}(a_l f_l + b_l)/a_l}{\sum_{l\in S^i} 1/a_l} \geq \frac{\sum_{l\in S^k}(a_l f_l + b_l)/a_l}{\sum_{l\in S^k} 1/a_l},$$
$$\frac{\sum_{l\in S^i}(2a_l f_l + b_l)/a_l}{\sum_{l\in S^i} 1/a_l} \leq \frac{\sum_{l\in S^k}(2a_l f_l + b_l)/a_l}{\sum_{l\in S^k} 1/a_l},$$

Because

$$q^k A^k - \sum_{l\in S^k} \frac{a_l f_l + b_l}{a_l} = \sum_{l\in S^k} \frac{q^k - a_l f_l - b_l}{a_l} = \sum_{l\in S^k} x_l^k = r^k,$$

we can also conclude that

$$q^i - \frac{r^i}{A^i} \geq q^k - \frac{r^k}{A^k} \text{ for all } i < k.$$

□

To study the change of costs for players when two players cooperate with each other, we start by looking at the subgradient of costs when the two players partially cooperate with each other, e.g., the player with smaller flow demand "gives" ϵ-small demand to the player with bigger flow demand. Note that the cost for Player k is $C^k = \sum_{l\in S^k}(a_l f_l + b_l)x_l^k$. Substitute (A.1) and (A.2) into it,

$$C^k = \sum_{i=k}^{K}\left[\sum_{l\in S^i - S^{i+1}} \frac{Q^i + b_l}{i+1} \cdot \frac{(i+1)q^k - Q^i - b_l}{(i+1)a_l}\right].$$

According to the conclusions obtained in Chapter 2, C^k is continuous with respect to r clearly.

For given support sets $S = \{S^i, i = 1, 2, \ldots, K, S^1 \supseteq S^2 \supseteq \ldots \supseteq S^K\}$, define the corresponding "feasible" set $r(S)$ of r by the two conditions in Proposition A.0.4. Denote the closure of $r(S)$ to be $R(S)$. Since all the conditions are linear with respect to r, for a fixed S, we know that $R(S)$ is a polyhedron. There are only finitely many of them since the choice of support sets are finite, and in the interior of each of these polyhedrons, all the values can be fully differentiated with respect to r. Before that we shows that it is the only case in which r moves locally inside the interior of each $R(S)$ in the following lemma:

Lemma A.0.6 *Suppose $X_i, i = 1, 2, \dots, n$ are n closed polyhedrons in $\mathbb{R}^K$ whose union is the whole space, e.g., $\bigcup_{i=1}^n X_i = \mathbb{R}^K$, and $F(x)$ is a continuous value function on $\mathbb{R}^K$ which is differentiable in the interior X_i° of each X_i. For any direction d, if the directional derivative $\nabla_d F(x) \leq 0$ for all $x \in X_i^\circ$, then $F(x + td) \leq F(x)$ for any $x \in \mathbb{R}^K$ and $t \geq 0$.*

Proof. If both x and $x + td, t \geq 0$ are in the interior of the same X_i, then by the directional derivative we know that $F(x + td) \leq F(x)$. Therefore, by continuity the property holds if both x and $x+td$ are in the same X_i. For any $x \in \mathbb{R}^k$, $t > 0$, and $s > 0$, we choose a v such that $\|v\| \leq s$ randomly and uniformly. For each facet of any X_i, the line $[x+v, x+v+td]$ lies in the facet with probability 0. Because there are at most finitely many facets, with probability 1 the line cross each facet only once. Therefore there exists a vector v with $\|v\| \leq s$, such that the line $[x + v, x + v + td]$ cross the boundary of all X_i's at most finitely many times. For each fraction of the line, the directional property holds, therefore it holds for the whole line, that is, $F(x) \geq F(x + td)$. □

Based on the above lemma, we only need to worry about the directional derivatives in the interior of each $R(S)$ for given S. Taking the direction $d = \delta_{k_1} - \delta_{k_2}$, we consider the directional derivatives, with direction d. We denote the directional derivative of Player k's cost with respect to d to be $\nabla_d C^k$ and the directional derivative of g^k to be

$\nabla_d g^k$. Notice that

$$\nabla_d g^k = \begin{cases} \frac{1}{(k_1+1)A^{k_1}}, & \text{if } k = k_1 \\ -\frac{1}{k(k+1)A^k}, & \text{if } k_1 < k < k_2 \\ -\frac{1}{k_2 A^{k_2}}, & \text{if } k = k_2 \\ 0, & \text{if else} \end{cases} \tag{A.5}$$

Hence,

$$\begin{aligned} \nabla_d C^k &= \sum_{i=k}^{K} \sum_{l \in S^i - S^{i+1}} \sum_{j=1}^{i} \nabla_d g^j \cdot \frac{(i+1)q^k - Q^i - b_l}{(i+1)a_l} \\ &+ \sum_{i=k}^{K} \sum_{l \in S^i - S^{i+1}} \frac{Q^i + b_l}{(i+1)a_l} \cdot \left((k+1)\sum_{j=1}^{k} \nabla_d g^j - k \sum_{j=1}^{k-1} \nabla_d g^j - \sum_{j=1}^{i} \nabla_d g^j \right). \end{aligned}$$

Regroup the items in the equation above,

$$\begin{aligned} \nabla_d C^k &= \sum_{i<k} \nabla_d g^i \left(q^k \sum_{l \in S^k} 1/a_l - \sum_{j=k}^{K} \sum_{l \in S^j - S^{j+1}} \frac{Q^j + b_l}{(j+1)a_l} \right) \\ &+ \nabla_d g^k \left(q^k \sum_{l \in S^k} 1/a_l + (k-1) \sum_{j=k}^{K} \sum_{l \in S^j - S^{j+1}} \frac{Q^j + b_l}{(j+1)a_l} \right) \\ &+ \sum_{i>k} \nabla_d g^i \left(q^k \sum_{l \in S^i} 1/a_l - 2 \sum_{j=i}^{K} \sum_{l \in S^j - S^{j+1}} \frac{Q^j + b_l}{(j+1)a_l} \right). \end{aligned}$$

Substitute (A.1) in,

$$\begin{aligned} \nabla_d C^k = & \textstyle\sum_{i<k} \nabla_d g^i \left(q^k A^k - \sum_{l \in S^k} \frac{a_l f_l + b_l}{a_l} \right) + \nabla_d g^k \left(q^k A^k + (k-1) \sum_{l \in S^k} \frac{a_l f_l + b_l}{a_l} \right) \\ & + \textstyle\sum_{i>k} \nabla_d g^i \left(q^k A^i - 2 \sum_{l \in S^i} \frac{a_l f_l + b_l}{a_l} \right). \end{aligned} \tag{A.6}$$

Because

$$q^k A^k - \sum_{l \in S^k} \frac{a_l f_l + b_l}{a_l} = \sum_{l \in S^k} \frac{q^k - a_l f_l - b_l}{a_l} = \sum_{l \in S^k} x_l^k = r^k.$$

we can rewrite Equation (A.6) as

$$\nabla_d C^k = \sum_{i<k} \nabla_d g^i r^k + \nabla_d g^k (k q^k A^k - (k-1) r^k) + \sum_{i>k} \nabla_d g^i (q^k A^i - 2 q^i A^i + 2 r^i). \tag{A.7}$$

With the derivatives obtained above, we now start to prove Theorem (5.2.1) by looking at the directional derivative.

Directional Derivative of the Social Cost

Note that the social cost could be rewritten as:

$$\mathrm{SC} = \sum_{k=1}^{K} \sum_{l \in S^k - S^{k+1}} a_l f_l^2 + b_l f_l = \sum_{k=1}^{K} \sum_{l \in S^k - S^{k+1}} \frac{Q^k - kb_l}{(k+1)a_l} \cdot \frac{Q^k + b_l}{k+1}.$$

Hence the directional derivative of the total social cost is

$$\begin{aligned}
\nabla_d \mathrm{SC} &= \sum_{k=1}^{K} \sum_{l \in S^k - S^{k+1}} \sum_{i=1}^{k} \nabla_d g^i \cdot \frac{Q^k + b_l}{(k+1)a_l} + \sum_{k=1}^{K} \sum_{l \in S^k - S^{k+1}} \sum_{i=1}^{k} \nabla_d g^i \cdot \frac{Q^k - kb_l}{(k+1)a_l} \\
&= \sum_{k=1}^{K} \sum_{l \in S^k - S^{k+1}} \sum_{i=1}^{k} \nabla_d g^i \cdot \frac{2Q^k - (k-1)b_l}{(k+1)a_l} \\
&= \sum_{k=1}^{K} \sum_{l \in S^k - S^{k+1}} \sum_{i=1}^{k} \nabla_d g^i \cdot \left[\frac{b_l}{a_l} + \frac{2Q^k - 2kb_l}{(k+1)a_l} \right].
\end{aligned}$$

Note that according to (A.1), $\frac{2Q^k - 2kb_l}{(k+1)a_l} = 2f_l$ and then

$$\nabla_d \mathrm{SC} = \sum_{k=1}^{K} \sum_{l \in S^k - S^{k+1}} \sum_{i=1}^{k} \nabla_d g^i (2a_l f_l + b_l)/a_l \, .$$

Regroup the items, we get

$$\nabla_d \mathrm{SC} = \sum_{k=1}^{K} \nabla_d g^k \sum_{l \in S^k} (2a_l f_l + b_l)/a_l = \sum_{k=k_1}^{k_2} \nabla_d g^k A^k \frac{\sum_{l \in S^k} (2a_l f_l + b_l)/a_l}{\sum_{l \in S^k} 1/a_l} \, .$$

With Lemma A.0.5,

$$\begin{aligned}
\nabla_d \mathrm{SC} &= \nabla_d g^{k_1} A^{k_1} \frac{\sum_{l \in S^{k_1}} (2a_l f_l + b_l)/a_l}{\sum_{l \in S^{k_1}} 1/a_l} + \sum_{k=k_1+1}^{k_2} \nabla_d g^k A^k \frac{\sum_{l \in S^k} (2a_l f_l + b_l)/a_l}{\sum_{l \in S^k} 1/a_l} \\
&\le \nabla_d g^{k_1} A^{k_1} \frac{\sum_{l \in S^{k_1}} (2a_l f_l + b_l)/a_l}{\sum_{l \in S^{k_1}} 1/a_l} + \sum_{k=k_1+1}^{k_2} \nabla_d g^k A^k \frac{\sum_{l \in S^{k_1}} (2a_l f_l + b_l)/a_l}{\sum_{l \in S^{k_1}} 1/a_l} \\
&= \sum_{k=k_1}^{k_2} \nabla_d g^k A^k \cdot \frac{\sum_{l \in S^{k_1}} (2a_l f_l + b_l)/a_l}{\sum_{l \in S^{k_1}} 1/a_l} = 0.
\end{aligned}$$

Directional Derivatives of Individual Players

We investigate the consequence of (partial) cooperation to the other players. The players who are not in the coalition could be divided into three cases:

Case 1: when $k < k_1$. Note that it is followed from (A.5) that $\sum_{i=k_1}^{k_2} \nabla_d g^{i'} A^i = 0$ and $\nabla_d g^{i'} = 0$ for all $i < k_1$ or $i > k_2$. Then

$$\begin{aligned}
\nabla_d C^k &= \sum_{i=k_1}^{k_2} \nabla_d g^i (q^k A_i - 2q^i A^i + 2r^i) \\
&= q^k \sum_{i=k_1}^{k_2} \nabla_d g^i A^i - 2 \sum_{i=k_1}^{k_2} \nabla_d g^i \sum_{l \in S^i} (q^i A^i - r^i) = -2 \sum_{i=k_1}^{k_2} \nabla_d g^i A^i (q^i - r^i / A^i) \\
&= 2 \sum_{i=k_1+1}^{k_2-1} \frac{1}{i(i+1)} [(q^i - \frac{r^i}{A^i}) - (q^{k_1} - \frac{r^{k_1}}{A^{k_1}})] + 2 \frac{1}{k_2} [(q^{k_2} - \frac{r^{k_2}}{A^{k_2}}) - (q^{k_1} - \frac{r^{k_1}}{A^{k_1}})] \\
&\leq 0.
\end{aligned}$$

Case 2: When $k > k_2$,

$$\begin{aligned}
\nabla_d C^k &= \sum_{i=k_1}^{k_2} \nabla_d g^i r^k = \nabla_d g^{k_1} A^{k_1} \frac{r^k}{A^{k_1}} + \sum_{i=k_1+1}^{k_2} \nabla_d g^i A^i \frac{r^k}{A^k} \\
&\leq \nabla_d g^{k_1} A^{k_1} \frac{r^k}{A^{k_1}} + \sum_{i=k_1+1}^{k_2} \nabla_d g^i A^i \frac{r^k}{A_1^k} = 0.
\end{aligned}$$

Case 3: when $k_1 < k < k_2$. First consider the special case that $k = k_1 + 1$:

$$\begin{aligned}
\nabla_d C^k &= \nabla_d g^{k_1} \left(q^{k_1+1} A^{k_1+1} - \sum_{l \in S^{k_1+1}} \frac{a_l f_l + b_l}{a_l} \right) + \nabla_d g^{k_1+1} \left(q^{k_1+1} A^{k_1+1} \right. \\
&\quad \left. + k \sum_{l \in S^{k_1+1}} \frac{a_l f_l + b_l}{a_l} \right) + \sum_{i > k_1+1} \nabla_d g^{i'} \left(q^{k_1+1} A^i - 2 \sum_{l \in S^i} \frac{a_l f_l + b_l}{a_l} \right) \\
&= q^{k_1+1} \sum_{i=k_1}^{k_2} \nabla_d g^i A^i + q^{k_1+1} \nabla_d g^{k_1} (A^{k_1+1} - A^{k_1}) \frac{r^k}{A^{k_1}} - \nabla_d g^{k_1} \sum_{l \in S^{k_1+1}} \frac{a_l f_l + b_l}{a_l} \\
&\quad + k_1 \nabla_d g^{k_1+1} \sum_{l \in S^{k_1+1}} \frac{a_l f_l + b_l}{a_l} - 2 \sum_{i > k_1+1} \nabla_d g^i \sum_{l \in S^i} \frac{a_l f_l + b_l}{a_l} \\
&\leq 0 + q^{k_1+1} \nabla_d g^{k_1} (A^{k_1+1} - A^{k_1}) \frac{r^k}{A^{k_1}} - \nabla_d g^{k_1} \sum_{l \in S^{k_1+1}} (a_l f_l + b_l)/a_l \\
&\quad + (k_1 \nabla_d g^{k_1+1} A^{k_1+1} - 2 \sum_{i > k_1+1} \nabla_d g^i A^i) \frac{\sum_{l \in S^{k_1+1}} (a_l f_l + b_l)/a_l}{A^{k_1+1}} \\
&= \frac{q^{k_1+1}}{(k_1+1) A^{k_1}} (A^{k_1+1} - A^{k_1}) + (-\frac{1}{A^{k_1}} + \frac{1}{A^{k_1+1}}) \sum_{l \in S^{k_1+1}} \frac{a_l f_l + b_l}{(k_1+1) a_l} \\
&= \frac{A^{k_1+1} - A^{k_1}}{A^{k_1+1} A^{k_1}} (q^{k_1+1} A^{k_1+1} - \sum_{l \in S^{k_1+1}} \frac{a_l f_l + b_l}{(k_1+1) a_l}) \\
&= \frac{A^{k_1+1} - A^{k_1}}{A^{k_1+1} A^{k_1}} r^{k_1+1} \leq 0, \quad \text{noting that } A^{k_1+1} \leq A^{k_1}.
\end{aligned}$$

When $k > k_1 + 1$, notice that the definition of $\nabla_d C^k$ is dependent on the parameters k_1 and k_2, thus we can consider it as a function of k_1 and k_2, which can be written as $\nabla_d C^k(k_1, k_2)$. We also write the directional gradient of g^k as a function of k_1 and k_2, denoted by $\nabla_d g^k(k_1, k_2)$. Note that $\nabla_d g^i(k_1, k_2) = \nabla_d g^i(k_1 + 1, k_2)$ for all $i \neq k_1$ and $i \neq k_1 + 1$ by (A.5), hence the change of $\nabla_d C^{k'}$ is

$$\begin{aligned}
& \nabla_d C^k(k_1 + 1, k_2) - \nabla_d C^k(k_1, k_2) \\
= \; & \nabla_d g^{k_1+1}(k_1 + 1, k_2) r^k - \nabla_d g^{k_1+1}(k_1, k_2) r^k - \nabla_d g^{k_1}(k_1, k_2) r^k \\
= \; & r^k \left(\frac{1}{(k_1 + 2)A^{k_1+1}} + \frac{1}{(k_1 + 1)(k_1 + 2)A^{k_1+1}} - \frac{1}{(k_1 + 1)A^{k_1}} \right) \\
= \; & \frac{r^k}{k_1 + 1} \left(\frac{1}{A^{k_1+1}} - \frac{1}{A^{k_1}} \right) \geq 0.
\end{aligned}$$

It is followed that

$$\nabla_d C^k(k_1, k_2) \leq \nabla_d C^k(k_1 + 1, k_2) \leq \ldots \leq \nabla_d C^k(k - 1, k_2) \leq 0.$$

From the three cases above, we know that when two players cooperate with each other (partially or fully), any other player would benefit from this cooperation. It also completes the proof of the theorem.

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