Some Studies on Viscous Fluids

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Abstract

In this thesis, we study several issues involving incompressible viscous fluids with the slip boundary conditions and the motions of fluid-solid interactions.

In the first part,we study the issue of the inviscid limit of the incompressible Navier-Stokes equations on the general smooth domains for completely slip boundary conditions. We verify an asymptotic expansion which involves a weak amplitude boundary layer with the same thickness as in the Prantle's theory. We improve the better regularity for the boundary layer and obtain the uniform L^p-estimates $(3 < p \leq 6)$ of the remainder. Then we improved these estimates to H^1-estimates. It is shown that the viscous solution converges to the solution of Euler equation in $C([0,T];H^1(\Omega))$ as the viscosity tends to zero.

In the second part, we consider the non-stationary problems of a class of non-Newtonian fluid which is a power law fluid with $p > \dfrac{3n}{n+2}$ in the half space with slip boundary conditions. We present the local pressure estimate with the Navier's slip boundary conditions. Using these estimates and an $L^\infty-$ truncation method, we can obtain that this system has at least one required weak solution.

Finally, we investigate the motion of a general form rigid body with smooth boundary by an incompressible perfect fluid occupying $\mathbb{R}^3$. Due to the domain occupied by the fluid depending on the time, this problem can be transformed into a new systems of the fluid in a fixed domain by the frame attached with the body. With the aid of Kato-Lai's theory, we construct a sequence of successive solutions to this problem in some unform time interval. Then by a fixed point argument, we have proved that the existence, uniqueness and persistence of the regularity for the solutions of original fluid-structure interaction problem.

摘　要

本論文研究了有關具有滑動邊界的不可壓流體和流體與固體相互作用的主題.

在論文的第一部分, 我們主要研究的是粘性消失極限問題. 利用Navier-Stokes方程的強解漸近展開式, 研究其具有與Prandl'方程的相同寬度的邊界層. 通過研究邊界層的方程來提高邊界層的正則性. 當$3 < p \leq 6$時, 我們得到了展開式餘項具有L^p一致有界. 於是, 我們能得到H^1的估計,從而我得到了Navier-Stokes方程的強解在空間$C([0,T];H^1(\Omega))$ 收驗於Euler方程的強解.

在論文第二部分, 我們得到具有滑動邊界壓力的局部估計, 利用正則化粘性項, 構造逼近方程的解, 然後, 利用L^∞-截斷的方法, 我們證明了逼近解收驗非牛頓流體方程的弱解.

論文的最後部分, 我們處理理想流體與剛性固體的相互作用方程,我們利用Kato-Lai 理論和不動點方法. 得到了此方程的局部光滑解的存在唯一性.

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Contents

Introduction

Many fluid motions are characterized by Stoke's law

$$S = \nu D(u)$$

where S is the stress tensor, $D(u)$ is the symmetric velocity gradient, i.e.

$$D_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Assume that ν is a constant, we obtain the following Navier-Stokes equations.

The continuity equation (or mass conservation equation)

$$\partial_t \rho + \text{div } (\rho u) = 0, \quad \text{in } \Omega \times (0, T]$$

and the conservation of momentum equation

$$\rho \partial_t u + (\rho u \cdot \nabla) u - \mu \Delta u - (\mu + \lambda) \nabla \text{div } u + \nabla p = f, \quad \text{in } \Omega \times (0, T].$$

Here ρ and u denote the density and the velocity field of fluid. respectively. p is the pressure and f is the external force. Ω is the domain occupied by the fluid.

A fluid is said to be incompressible if the volume of any quantity of the fluid remains invariant. For such a fluid,

$$\text{div } u = 0.$$

If we further assume that the initial state is homogeneous, which means

$$\rho(x, 0) = \rho_0 = const.,$$

then

$$\rho(x,t) = \rho_0.$$

Hence, the system describing the motion of an incompressible homogeneous Newtonian fluid is the following incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla(p/\rho_0) = f/\rho_0, & \text{in } \Omega\times(0,T], \\ \text{div } u = 0, \quad \text{in } \Omega\times[0,T], \\ u(t=0,x) = u_0(x), \quad \text{in } \Omega, \end{cases} \tag{0.1}$$

where $\nu = \mu/\rho_0 > 0$ is called the kinematic viscosity coefficient. For simplicity of presentation, in what follows we will set $\rho_0 = 1$ and omit the external force.

Traditionally, u satisfy the non-slip boundary condition for the Navier-Stokes equations, which is

$$u = 0 \qquad \text{on } \partial\Omega\times[0,T]. \tag{0.2}$$

J. Leray and E. Hopf constructed the famous Leray-Hopf weak solution to the Navier-Stokes system, by the Galerkin method. After J. Leray's pioneering work [76], there have been vast literature on the well-posedness of (0.1) and (0.2). For the system (0.1), there are two main difficulties. One is that it is not a standard parabolic system. The other one is that the system has strong nonlinearity.

For the 2D case, the strong nonlinearity can be eliminated by the dissipation. The global weak solution was proved to be strong and unique, see [76]. Actually, it is a classical solution when $t > 0$.

When the spatial dimension $d = 3$, the problem of well-posedness is much more involved. From the experience of dealing with semilinear parabolic equations, it is found that the effort to eliminate the nonlinearity only by the dissipation term is invalid. For general large initial data, strong solutions were derived locally. In 1962, H. Fujita and T. Kato [46] obtained a unique local strong solution in $W^{s,2}$-space. The proof was based on the theory of analytic semigroups and fractional powers of the generator. For the non-Hilbert cases, T. Kato [66] constructed a local strong solution $u \in C([0,T]; L^3(\mathbb{R}^3)) \cap C((0,T]; W^{1,3}(\mathbb{R}^3))$, by making use of the $L^p - L^q$ estimates of the semigroup $\{e^{-tA}\}_{t\geq 0}$ generated by the Stokes operator A. For initial data $u_0 \in L^p(\Omega)$, $p > 3$, unique local strong solutions were also derived, please refer to [40, 52, 53]. However, whether the local strong solutions blow up in finite time or remain smooth is yet unknown. While the initial data u_0 satisfies that $\|u_0\|_{L^3(\mathbb{R}^3)}$ is sufficiently small, then unique global solution was derived in [66]. Global well-posedness also holds for axis-symmetric initial data without swirl. The same results as above hold for the bounded domain and exterior domain, see [52, 53, 60, 72]. For more information about the Navier-Stokes system, please refer to [77] and references therein.

When the viscosity vanishes, i.e. $\nu = 0$, the system (0.1) becomes the Euler equations. If the domain Ω has a boundary, then the common boundary condition is the slip boundary condition,

$$u \cdot \boldsymbol{n} = 0, \qquad \text{on } \partial\Omega, \tag{0.3}$$

here $\boldsymbol{n}$ is the unit outer normal at the boundary. The boundary condition (0.3) means that the fluid does not across the boundary.

Global well-posedness for the 2D Euler equations has been basically solved. T. Kato [64] got a unique global classical solution by the Picard theorem in Banach spaces. The same result was previously derived by Ebin and Marsden [39] using infinite dimensional differential geometry. If the initial data is less regular, a class of global weak solutions were constructed. Given initial data u_0 with $\omega_0 = \text{curl } u_0 \in L^p(\mathbb{R}^2)$, $1 < p \leq \infty$, a global weak solution was derived. In particular, uniqueness can be guaranteed by the assumption that ω_0 is bounded [120]. For more rough initial data, some global existence results can be found in [32, 109]. The fundamental reason for global well-posedness in 2D is that the vorticity $\omega = \text{curl } u$ satisfies the following transport equation,

$$\partial_t \omega + (u \cdot \nabla)\omega = 0.$$

However, the vorticity equation will lose the beautiful structure when $d = 3$. It's hard to predict the global well-posedness of 3D Euler equations. A local classical solution was derived by Kato [65] and the solution was shown to be unique. Then Beale, Kato and Majda [4] proved global existence provided that the vorticity remains bounded. The same results as those for the whole space hold for more general domain Ω, see [24, 39, 43, 69].

In the thesis, we study one of the issue that is vanishing viscosity limits for Navier-Stokes equations . In the case where no physical boundaries, if the the ideal Euler system is sufficiently regular, the solutions can be approximated by the ones to Navier-Stokes equations, we can see [27, 28, 39, 65, 67, 105]. However, in the case where there are physical boundaries, this problem is a challenging problem due to the formation of boundary layers. The problem of the classical non-slip boundary condition was formally derived by Prandtl in [94], in which it was derived that the boundary layer can be described by an initial-boundary problem for a nonlinear degenerate parabolic-elliptic couple system, which is called the Prandtl's equations. Under monotonic assumption on the velocity of outflow, Oleinik and her collaborators established the local existence of smooth solutions for boundary value problem of Prandtl's equation in the surveyed monography [90]. The existence and uniqueness of the weak solutions for the Prandtl's equations was established by Xin, Zhang [119] (see [118]). In [96], Sammartino and Caflisch obtained the local existence of the analytic solutions to the Prandtl's equations, and a rigorous theory on the boundary layer in incompressible fluids with analytic data in the frame of the abstract Cauchy-Kowaleskaya theory.

The usual non-slip assumption was not always accept from experimental facts. In [87], Navier first proposal the slip with friction boundary condition i.e. the tangential velocity is propositional to the tangential component of the viscous stress, which is now called Navier boundary condition. This boundary was rigorously justified as the effective boundary conditions for flows over rough boundaries, we can refer [61].

There already have been many interesting results on the vanishing limit of solutions to (0.1) for Navier boundary conditions. For 2D case, Yudovich [120] and Lions, P.L. [77]studied the vanishing viscosity limit for the incompressible Navier-Stokes equation with Navier's boundary conditions, more precisely, $u \cdot n = 0$ $\mathrm{curl} u = 0$ on boundary $\partial\Omega$. For the Navier friction conditions, Clopeau, et.al. [26], Lopes Filho, Nussenveig Lopes and Planas [42] obtained that the solution u^ν to (0.1) converges to the solution u^0 of Euler equations in $L^\infty(0,T;L^2(\mathbb{R}^2_+))$ under assuming initial vorticity is uniformly bound. For 3D case, Iftimie and Planas [57] have further studied the small viscosity limit for the anisotropic viscosities in half space, with the fixed horizontal and the vertical tends to zero.

Recently, Wang, X. P., Wang, Y.G. and Xin, Z. [111] study asymptotic behavior of solutions to (0.1) with Navier boundary conditions for variational slip length. While in [58], Iftimie and Sueur study the boundary layer of the solution to (0.1) with Navier boundary condition for fixed slip length in both $2D$ and $3D$. They obtain the Leray's weak solution of Navier-Stokes equation converges to the smooth one of Euler equation in the space $L^\infty(0,T;L^2(\Omega))$ and gave almost sharp convergent rate. N. Masmoudi [85] prove the solutions to (0.1) uniformly converges to the one of Euler equation in the spatial and time variable by the the frame of conormal Sobolev space. However, it is difficult to obtain the convergence in higher order, even in H^1 as mentioned in [57], but one can not obtain the convergence in H^2. For flat boundary with the following completely slip boundary conditions,

$$u \cdot n = 0 \;\; \mathrm{curl} u \times n = 0 \;\; \text{on } \partial\Omega \times [0,T]. \tag{0.4}$$

Xiao and Xin [116] the convergence in H^2 holds on the flat boundary. Later there are many authors obtain the L^p theory and $W^{k,p}$ inviscid limit in [13, 14], and in [13], they also obtain that the convergence always holds in global time for $2D$ case.

As clarified in [13] the model introduced in [116] seems to work just in presence of a domain with a flat boundary. This is due to the fact that certain surface integral are identically zero in the flat case. In the general case these surface integrals, besides involving lower order term, are not easily handled. Consequently, the study of the vanishing limit in a general domain under boundary conditions 0.4 represents a challenging open problem, see [14]. The difficulties and the interest for this problem is also emphasized in the work of Wang, X. P., Wang, Y.G. and Xin, Z. [111].

Our *main new contribution* is the viscosity limit problem of the Navier-Stokes's equations with the completely slip boundary conditions in 3D general domain, solving one of the questions left open in [13, 116]. But it is different from [17], here we do not need the Euler equation satisfies the boundary conditions (0.4). In [17], the Euler is over-determined as noted in [117], the tangential of the vorticity is not zero even if the initial datum satisfies the boundary conditions (0.4) for general boundary. Therefore, it is interesting to study the strong solution of Euler equation with only boundary condition (0.3) is approximated from the strong solutions to (0.1) with completely slip boundary conditions (0.4).

In Chapter 2, we borrow the equation of the boundary layer from [58] and improve the regularity of the boundary layer. We obtain the strong solution to (0.1) with boundary conditions (0.4) converges to the one of Euler system in $C([0,T];H^1(\Omega))$, provided that initial velocity is regular enough.

There are phenomena that can not be described by Navier-Stokes equations, such as rod-climibing or Weisenberg effect, normal stress effect and earth's mantle dynamics and so on. In the study of these models, scientist have use nonlinear versions of the constitutive law, one can refer to [49] for details. In this context, there are some models , that viscous force can be effective functions of the shear rate $|D(u)|$.

The mentioned class of models as above, i.e. $\nu = \nu(|D(u)|)$ belong to power-law ansatz to model certain non-Newtonian behavior of the fluid flows, and they are frequently used engineering literature. We can refer the book by Bird, Armstrong and Hassager [19] and the survey paper due to Málek and Rajagopal [81].

The mathematical analysis of these models started with the work of Ladyžhenska [73], [74], [75]. She investigated the well-posedness of the initial boundary value problem with non-slip boundary conditions, associated with the stress tensor

$$S = (\nu_0 + \nu_1 |D(u)|^{p-2}) D(u) \tag{0.5}$$

with positive constants ν_0, ν_1 and $p > 1$. In 1969, J.L. Lions [78] proved some existence results for $p-$Laplacian equation with $p \geq 1 + \dfrac{2n}{n+2}$ and the uniqueness for $p \geq \dfrac{n+2}{n}$ under no-slip boundary conditions. In those papers, the authors applied the properties of monotone operator and Minty trick theory for the stress tensor satisfies the strict monotonicity and coercivity.

Over these years, Ladyžhenska's and Lions' work were improved and in several directions by differen authors. In particular, for the steady problem, there are several results proving existence of weak solution in bounded domain [34, 44, 45], interior regularity [1, 86] and recently regularity up to boundary for the Dirichlet problem [6–12, 30, 31, 98]. Concerning the time-evolution Dirichlet problem in a 3D domain, J. Málek, J. Nečas, and M. Růžička [80] study the weak solution for $p \geq 2$. Later, L. Diening et.al have a recent advances on the existence of weak solutions in [115] for $p > \dfrac{8}{5}$ and in [35] for $p > \dfrac{6}{5}$. There are also many papers dealing with regularity of for evolution Dirichlet boundary problems and we refer instance to [2, 3, 9–12, 21, 22]. In the three-dimensional cube with space periodic boundary conditions, there is a lot of literature for the well-posedness of this model, we refer to the monograph [79] and papers [18, 33].

However, there are not too many results for non-Newtonian fluid with Navier type slip boundary conditions. In [6,39], the authors investigated the regularity of steady flows with shear-dependent viscosity on the slip boundary conditions. M. Bulíček, J.Málek and K.R. Rajagopal [23] obtained the weak solution for the evolutionary generalized Navier-stokes-like system of pressure and shear-dependent viscosity on the Navier-type slip boundary conditions in the bounded domain.

In Chapter 3, we consider unsteady flows of an incompressible non-Newtonian fluid described by the system

$$\begin{cases} \partial_t u - \mathrm{div} S(D(u)) + (u \cdot \nabla)u + \nabla \pi = f, & \text{in } \mathbb{R}^3_+ \times (0,T) \\ \nabla \cdot u = 0, & \text{in } \mathbb{R}^3_+ \times (0,T), \\ u|_{t=0} = u_0(x), & \text{in } \mathbb{R}^3_+, \end{cases} \tag{0.6}$$

where u is the velocity, π is pressure and f is the force, and $S(\cdot)$ is stress tensor and induced by a $p-$potential, for example, (0.5) is induced by a potential function. We can impose the following slip boundary conditions

$$u \cdot n|_{x_3=0} = 0, \quad ((S(D(u)) \cdot n) - (n \cdot S(D(u)) \cdot n)n)|_{x_3=0} = 0. \tag{0.7}$$

In fact, this problem corresponds to the free boundary problem for the non-Newtonian fluids with free surface supposed invariable.

Use the different method as before, we overcome the two main difficulties which come from the unbounded domain and the nonlinearity of stress tensor. We regularize the convection term and obtain approximated solutions. We can prove these solutions are regular enough. Then by L^∞-truncation method, we obtain the approximated solutions converges to the weak solution of the problem (0.6)with slip boundary conditions (0.7) in half space.

The last part is devoted to the study of the so-called "fluid-rigid body" system. Many physical phenomena involve the interactions between moving structures and fluids. An interesting problem is the motion of a rigid body immersed in a incompressible fluid. The motion of the fluid is governed by the classical Euler or Navier-Stokes equations, depending on the viscosity of the fluid. And the motion of the rigid body consisting of a translation part and a rotation part, is ruled by the conservation of linear and angular momentum. The force exerted on the rigid body is from the the fluid.

In the chapter 4, we investigate the motion of a rigid body immersed by an incompressible perfect fluid. That is, the behavior of the fluid is described by the Euler equations, while the motion of the rigid body conforms to the Newton's law. Assume that both the fluid and the rigid body are homogeneous. The domain occupied by the solid at the time is $\mathcal{O}(t)$, and $\Omega(t) = \mathbb{R}^3 \setminus \overline{\mathcal{O}(t)}$ is the domain occupied by the fluid. Suppose $\mathcal{O}(0) = \mathcal{O}$ and $\Omega(0) = \Omega$ share a smooth boundary $\partial\mathcal{O}$(or $\partial\Omega$). The equations modeling the dynamics of the system read(see also [95])

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = f, \qquad \text{in } \Omega(t) \times [0,T], \tag{0.8}$$

$$\text{div } u = 0, \qquad \text{in } \Omega(t) \times [0,T], \tag{0.9}$$

$$u \cdot \boldsymbol{n} = (h' + \omega \times (x - h(t))) \cdot \boldsymbol{n}, \qquad \text{on } \partial\Omega(t) \times [0,T], \tag{0.10}$$

$$\lim_{|x| \to \infty} u(x,t) = u_\infty, \tag{0.11}$$

$$mh'' = \int_{\partial\Omega(t)} p\boldsymbol{n} \mathrm{d}\sigma + f_{rb}, \qquad \text{in } [0,T], \tag{0.12}$$

$$(J\omega)' = \int_{\partial\Omega(t)} (x - h(t)) \times p\boldsymbol{n} \mathrm{d}\sigma + T_{rb}, \qquad \text{in } [0,T], \tag{0.13}$$

$$u(x,0) = u_0(x) \qquad x \in \Omega, \tag{0.14}$$

$$h(0) = 0 \in \mathbb{R}^3, \ h'(0) = l_0 \in \mathbb{R}^3, \ \omega(0) = \omega_0 \in \mathbb{R}^3. \tag{0.15}$$

In the above system, u and p are the velocity field and the pressure of the fluid respectively. f is the external force field applied to the fluid. f_{rb} and T_{rb} denote the external force and the external torque of the rigid body respectively. m is the mass, J is the inertia matrix moment related to the mass center of the solid. Suppose the density of the rigid body is ρ, then

$$m = \int_{\mathcal{O}(t)} \rho dx = \int_{\mathcal{O}} \rho dx,$$

and

$$[J(t)]_{kl} = \int_{\bar{\mathcal{O}}(t)} \rho \left[|x - h(t)|^2 \delta_{kl} - (x - h(t))_k (x - h(t))_l \right] dx.$$

Here $h(t)$ denotes the position of the mass center of the rigid body and δ_{kl} is the Kronecker symbol. $\omega(t)$ is the angular velocity of the rigid body. $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega(t)$. Assume that the center of $\mathcal{O}$ is the origin, i.e.,

$$\int_{\mathcal{O}} y dy = 0 \in \mathbb{R}^3.$$

For the case that the fluid is viscous, there have been many results over the last two decades. The global existence of weak solutions of the above system was proved by [62] and [97]. For the case that the fluid-rigid body system occupies a bounded domain, the existence of weak solutions has been treated by many mathematicians, see [36, 37, 51, 55, 56, 83].

If the rigid body is a disk in $\mathbb{R}^2$, T. Takahashi and M.Tucsnak [106] showed the existence and uniqueness of global strong solutions. Later, P.C. Santiago and T. Takahashi [103] extended the result to general rigid body case in $\mathbb{R}^2$. They also proved the local existence and uniqueness of strong solutions in $\mathbb{R}^3$.

It seems that much fewer results for the perfect fluid-rigid body problem were obtained. When the solid is of C^1 and piecewise C^2, and the fluid fills in $\mathbb{R}^2$, a unique global classical solution was obtained under some assumption on the initial vorticity in [92]. A global weak solution was constructed in [113] when the initial data belongs to $W^{1,p}$, $p > \frac{4}{3}$. Recently, C. Roiser and L. Roiser [95] proved the local existence of $W^{s,2}$-strong solutions for $d \geq 2$, $s \geq [d/2] + 2$ and the solid is a ball. The key idea is to make use of the Kato-Lai theory, which was originated in [68].

In the chapter 4, we plan to extend the result of [95] to a more general setting. We will deal with the case that the solid is of general form. The main idea is also the Kato-Lai theory. The difference comes from coordinates transformation. In our proof, different from [95], we apply another coordinates transformation to fix the boundary. This kind of transformation which coincides with the motion in a neighborhood of the solid and becomes identity when far away from it, has been used by [59, 103].

Chapter 1

Preliminaries

In this chapter, we will give some definitions and recall some fundamental inequalities and lemmas to be used in the thesis.

1.1 Notational conventions and function spaces

In this thesis, C is always an unspecified constant that may vary from line to line. If C depends on some special parameters $x_1, \cdots, x_k$, we write $C(x_1, \cdots, x_k)$.

For vector-valued functions $u = (u_1, u_2, \cdots, u_d), v = (v_1, v_2, \cdots, v_d)$ of $\mathbb{R}^d$, define

$$(u \cdot \nabla)v = \sum_{i=1}^{d} u_i \partial_i v, \quad \nabla u : \nabla v = \sum_{i,j=1}^{d} \partial_i u_j \partial_i v_j,$$

and

$$\text{div } u = \sum_{i=1}^{d} \partial_i u_i, \quad D(u) = \frac{(\nabla u) + (\nabla u)^T}{2}.$$

For the vector-valued function u of $\mathbb{R}^2$, define

$$\text{curl } u = \partial_1 u_2 - \partial_2 u_1,$$

and denote the vector $(-u_2, u_1)$ by $u^\perp$.

While for the vector-valued function u of $\mathbb{R}^3$, define

$$\text{curl } u = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1).$$

In this thesis, $B_r(y)$ will denote the open ball in $\mathbb{R}^d$ centered at y and with radius r.

Suppose Ω is a domain in $\mathbb{R}^d$, let $\Omega_r = \Omega \cap B_r(0)$, $Q_T = \Omega \times (0, T)$, and denote the closure by $\overline{\Omega}$. $B(\overline{\Omega})$ $(B(\overline{Q_T}))$ is the Banach space of all continuous and bounded functions defined on Ω $(\overline{Q_T})$, endowed with the L^∞ norm. The Hölder space $C^\lambda(\overline{\Omega})$ $(C^\lambda(\partial\Omega))$ is the space of all the functions $\omega \in B(\overline{\Omega})$ $(B(\partial\Omega))$, which are uniformly Hölder continuous in y with exponent λ on $\overline{\Omega}$ $(\partial\Omega)$.

$L^p(\Omega), W^{k,p}(\Omega)(1 \leq p \leq +\infty)$ are the usual Sobolev spaces. $[L^p(\Omega)]^d, [W^{k,p}(\Omega)]^d$ are the corresponding Sobolev spaces with elements being vector-valued functions. They equiv the norm $\|\cdot\|_p$ and $\|\cdot\|_{k,p}$ respectively. In many cases, we do not distinguish the vector-valued functions and scalar-valued functions very strictly. In particular, denote $W^{k,2}(\Omega)$ by H^k. Define $W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

Sometimes we need the homogeneous Sobolev spaces. For $1 < p < \infty$,

$$D^{1,p}(\Omega) = \{u \in L^1_{loc}(\Omega) : \ \nabla u \in L^p(\Omega)\},$$

with the seminorm

$$|u|_{D^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)}.$$

If we identify the two functions $u_1, u_2 \in D^{1,p}(\Omega)$ whenever $|u_1 - u_2|_{D^{1,p}(\Omega)} = 0$, i.e., u_1 and u_2 differ by a constant, we denote the quotient space by $\dot{D}^{1,p}(\Omega)$, with the norm $|\cdot|_{D^{1,p}(\Omega)}$. In the following text, without any confusion, we do not distinguish the elements in $D^{1,p}(\Omega)$ and $\dot{D}^{1,p}(\Omega)$. $D_0^{1,p}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ in the seminorm of $D^{1,p}(\Omega)$.

Since we deal with the incompressible flow, spaces consisting of divergence free functions are needed. $C_{0,\sigma}^\infty(\Omega) = \{u \in C_0^\infty(\Omega) : \ \ \text{div } u = 0 \ \text{ in } \Omega\}$. And $L^2_\sigma(\Omega)$ is defined as

$$L^2_\sigma(\Omega) = \{u \in L^2(\Omega) : \ u = 0 \ \ \text{div } u = 0 \text{ in } \Omega\}.$$

$$D(\mathbb{R}^3_+) = \{u \in C_0^\infty(\overline{\mathbb{R}^3_+}) : u_3 = 0 \text{ on } x_3 = 0\},$$
$$V_p(\mathbb{R}^3_+) = \overline{\{u \in D(\mathbb{R}^3_+) : \nabla \cdot u = 0\}}^{\|\nabla \cdot\|_{L^p}},$$
$$V = \{u \in V_2 : \int_{\mathbb{R}^3_+} |\nabla u\|^p \mathrm{d}x < \infty\}, \ \ H = \overline{D(\mathbb{R}^3_+)}^{\|\cdot\|_{L^2}}.$$

We define by $(W^{k,m,l,p}(\Omega), \|\cdot\|_{k,m,l,p})$ the anisotropic Sobolev spaces and norm as follows: for $k, m, l \in \mathbf{N}, \ \ p \geq 1$

$$W^{k,m,l,p}(\Omega \times \mathbf{R}_+) = \Big\{g(x,z) \in L^p(\Omega \times R_+) : (1+z^{2k})^{\frac{1}{p}} \partial_x^\alpha \partial_z^\beta g(x,z) \in L^p(\Omega \times R_+) \ , \\ |\alpha| \leq m, \beta \in \mathbf{N} \cup \{0\}\}$$

with the norm

$$\|g\|^p_{k,m,l,p} = \sum_{|\alpha| \leq m, |\beta| \leq l} \iint_{\Omega \times \mathbf{R}_+} (1+z^{2k}) |\partial_x^\alpha \partial_z^\beta g(x,z)|^p \mathrm{d}x\mathrm{d}z.$$

When $p = 2$, for simplicity, denote $W^{k,m,l,2}(\Omega)$ as $H^{k,m,l}(\Omega)$.

In Chapter 4, without special claim, O is a bounded C^2-smooth domain in $\mathbb{R}^3$, and Ω is its exterior domain, $\Omega = \mathbb{R}^3 \setminus \overline{O}$. And the center of O is the origin.

$$m = \int_O dy.$$

Define

$$J = (J_{kl}) = \int_O (|y|^2 \delta_{kl} - y_k y_l) dy.$$

1.2 Elementary inequalities

We start with the Young's inequality.

Theorem 1.2.1 (Young's inequality) *Let* $1 < p, q < \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. *For any positive number* a *and* b, *it holds that*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

The Young's inequality yields immediately the well-known Hölder's inequality.

Theorem 1.2.2 (Hölder's inequality) *Given Ω an arbitrary domain in $\mathbb{R}^d$. Assume $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. If $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, then we have*

$$\int_\Omega |uv| dx \leq \|u\|_{L^p(\Omega)} \cdot \|v\|_{L^q(\Omega)}.$$

Thus the interpolation inequality is shown.

Theorem 1.2.3 (Interpolation inequality) *Assume $1 \leq s, r, t \leq \infty$ and*

$$\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}.$$

Suppose $u \in L^s(\Omega) \cap L^t(\Omega)$. Then $u \in L^r(\Omega)$, and

$$\|u\|\|_{L^r(\Omega)} \leq \|u\|^\theta_{L^s(\Omega)} \cdot \|u\|^{1-\theta}_{L^t(\Omega)}. \tag{1.2.1}$$

One more general of the interpolation inequality is the following one.

Theorem 1.2.4 (General interpolation inequality) *Let Ω be a $C^{0,1}$ domain and $u \in W^{k,p}(\Omega)$. Then for any $0 < |\beta| < k$,*

$$\|D^\beta u\|_{L^p(\Omega)} \leq C\|u\|^{1-\theta}_{L^p(\Omega)} \cdot \|u\|^\theta_{W^{k,p}(\Omega)},$$

where $\theta = \frac{|\beta|}{k}$, and $C = C(k, \Omega)$.

The interpolation inequality is closely related to the Sobolev embedding theorem.

Theorem 1.2.5 (Sobolev embedding theorem) *Let Ω be a $C^{0,1}$ bounded domain in $\mathbb{R}^d$. Then,*

(1) if $kp < d$, the space $W^{k,p}(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$, $p^* = dp/(d-kp)$, and compactly embedded in $L^q(\Omega)$ for any $q < p^*$;

(2) if $0 \leq m < k - \frac{d}{p} < m+1$, the space $W^{k,p}(\Omega)$ is continuously embedded in $C^{m,\beta}(\overline{\Omega})$ for any $\beta < \alpha$.

For functions in $W^{1,p}(\Omega)$ with some special homogeneous properties, there are Poincaré's inequalities.

Theorem 1.2.6 (Poincaré's inequalities) *Let Ω be a bounded, connected open subset of $\mathbb{R}^d$ with a C^1 boundary $\partial\Omega$. Assume $1 \leq p \leq \infty$. Then for each function $u \in W^{1,p}(\Omega)$, then there exists a constant C, depending only on d, p, Ω, such that*

$$\|u - (u)_\Omega\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)},$$

where $(u)_\Omega$ =average of u over Ω.

For each $u \in W_0^{1,p}(\Omega)$, there exists a constant $\tilde{C}$, depending only on d, p, Ω, such that

$$\|u\|_{L^p(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)}.$$

Theorem 1.2.7 (Trace theorem) *Assume Ω is bounded and $\partial\Omega$ is C^1 then there exists a bounded linear operator $T: W^{1,p}(\Omega) \to L^p(\partial\Omega)$, such that*

(1) $Tu = u|_{\partial\Omega}$, if $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$;

(2) $\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)} \leq C\|u\|_{L^p(\Omega)}^{1-\frac{1}{p}}\|u\|_{W^{1,p}(\Omega}^{\frac{1}{p}}$ for each $u \in W^{1,p}(\Omega)$ with the constant C depending only on p and Ω.

The following famous Gronwall's Lemma will be used frequently in this thesis.

Theorem 1.2.8 (Gronwall's Lemma) *a) (Differential Version) Let us assume h, r are integrable on (a, b) and nonnegative a.e. in (a, b). Further assume that $y \in C([a, b])$ and $y' \leq L^1(a, b)$ and that the following inequality is satisfied·*

$$y'(t) \leq h(t) + r(t)y(t) \text{ for a.a.} t \in (a, b).$$

Then

$$y(t) \leq \left[y(a) + \int_a^t h(s)\exp\left(-\int_a^s r(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]\exp\left(\int_a^t r(s)\mathrm{d}s\right), \quad t \in [a, b].$$

b) (Integral Form) Let us assume h is continuous on $[a,b]$, *r is integrable on* (a,b) *and nonnegative a.e. in* (a,b). *Further assume that* $y \in C([a,b])$ *satisfies the following inequality:*

$$y(t) \leq h(t) + \int_a^t r(s)y(s)\mathrm{d}s \text{ for a.a.} t \in (a,b).$$

Then

$$y(t) \leq h(t) + \int_a^t h(s)r(s)\exp\left(\int_s^t r(\tau)\mathrm{d}\tau\right)\mathrm{d}s, \quad t \in [a,b].$$

c) (Local Version) Let $T, \alpha, c_0 > 0$ *be given constants and let* $h \in L(0,T)$ *with* $h \geq 0$ *a.e. in* $[0,T]$, *for nonnegative function* $y \in C^1([0,T])$ *satisfy*

$$y'(t) \leq h(t) + c_0 y(t)^{1+\alpha} \text{ for a.a. } t \in (0,T).$$

Let $t_0 \in [0,T]$ *be such that* $\alpha c_0 H(t_0)^\alpha t_0 < 1$, *where*

$$H(t) = f(0) + \int_0^t h(s)\mathrm{d}s.$$

Then for all $t \in [0,t_0]$ *there holds*

$$f(t) \leq H(t) + H(t)\left((1 - \alpha c_0 H(t)^\alpha t)^{-\frac{1}{\alpha}} - 1\right).$$

1.3 Fundamental lemmas

When we are studying the existence of weak solutions to some partial differential equation, one often makes use of a theorem, which is the following Lax-Milgram theorem.

Definition 1.3.1 *Suppose H is a Hilbert space and B is a bilinear form on H. B is called bounded if there exists a constant K such that*

$$|B(x,y)| \leq K\|x\| \cdot \|y\|,$$

for $x, y \in H$.

B is called coercive if there exists a number $\nu > 0$, such that

$$B(x,x) \geq \nu \|x\|^2,$$

for all $x \in H$.

Theorem 1.3.2 (Lax-Milgram theorem) *Let B be a bounded, coercive bilinear form on a Hilbert space H. Then for every bounded linear functional F on H, there exists a unique element $f \in H$ such that*

$$B(x,f) = F(x) \quad \textit{for all } x \in H.$$

The proof can be found in many books, for reference see [77].

The next lemma from [99] which is a more general result of the famous Aubin-Lions Lemma.

Theorem 1.3.3 (Aubin-Lions lemma) *Let $X \hookrightarrow B \hookrightarrow Y$ be three Banach spaces with compact imbedding $X \hookrightarrow\hookrightarrow Y$. Further, let there exist $0 < \theta < 1$, and $M > 0$, such that*

$$\|v\|_B \leq M \|v\|_X^{1-\theta} \cdot \|v\|_Y^{\theta}, \quad \textit{for all } v \in X.$$

Denote for $T > 0$,

$$W(0,T) := W^{s_0,r_0}(0,T;X) \cap W^{s_1,r_1}(0,T;Y),$$

with

$$s_0, s_1 \in \mathbb{R}, \quad r_0, r_1 \in [1, +\infty],$$
$$s_\theta := (1-\theta)s_0 + \theta s_1, \quad \frac{1}{r_\theta} := \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad s^* := s_\theta - \frac{1}{r_\theta}.$$

Assume that $s_\theta > 0$ and F is a bounded set in $W(0,T)$. If $s^* \leq 0$, then F is relatively compact in $L^p(0,T;B)$ for all $1 \leq p < p^* := -1/s^*$. If $s^* > 0$, then F is relatively compact in $C([0,T];B)$.

Chapter 2

Asymptotic analysis for the 3D Navier-Stokes equations with vorticity boundary conditions on non-flat boundaries

In this chapter, we consider the approximated problem of the solutions for the inviscid incompressible fluid from Navier-Stokes equation in three dimensional general smooth domain. The boundary conditions of the viscous fluid is described by the vorticity slip boundary conditions.

2.1 Introduction

In this chapter, we consider the vanishing viscosity limit problem from the Navier-Stokes flow to Euler equations. The Naiver-Stokes equation is

$$\begin{cases} \partial_t u^\nu - \nu\Delta u^\nu + (u^\nu \cdot \nabla)u^\nu + \nabla\pi^\nu = 0, & \text{in } \ \Omega\times(0,T) \\ \nabla\cdot u^\nu = 0, & \text{in } \ \Omega\times(0,T), \end{cases} \tag{2.1.1}$$

with the boundary conditions

$$u^\nu \cdot n = 0, \quad \text{curl} u^\nu \times n = 0 \quad \text{on } \ \partial\Omega\times(0,T), \tag{2.1.2}$$

and initial velocity

$$u^\nu|_{t=0} = u_0(x), \quad \text{in } \ \Omega.$$

Here the unknowns are the velocity $u^\nu(t,x)$ and the scalar pressure $\pi^\nu(t,x)$, $u_0(x)$ is the given initial velocity and the Euler equations reads

$$\begin{cases} \partial_t u^0 + (u^0\cdot\nabla)u^0 + \nabla\pi^0 = 0, & \text{in } \ \Omega\times(0,T) \\ \nabla\cdot u^0 = 0, & \text{in } \ \Omega\times(0,T), \\ u^0\cdot n = 0, & \text{on } \ \partial\Omega\times(0,T), \\ u^0|_{t=0} = u_0(x), & \text{in } \ \Omega. \end{cases} \tag{2.1.3}$$

The main proposal of this chapter is to investigate the asymptotic expansion of the strong solution to (2.1.1) with complete slip boundary conditions (2.1.2) and we obtain vanishing viscosity limit results..

As in [58], we apply the following formally expansion

$$u^\nu(t,x) = u^0(t,x) + \sqrt{\nu}u^c(t,x,\frac{z}{\sqrt{\nu}}) + O(\nu),$$

where u^c is a smooth profile which is fastly decreasing in its last variable.

More precisely, we introduce a smooth function $\varphi \in C^\infty(\mathbb{R}^3;\mathbb{R})$ such that in a neighborhood Λ of $\partial\Omega$, one has that $\Omega\cap\Lambda = \{\varphi > 0\}\cap\Lambda$, $\Omega^c\cap\Lambda = \{\varphi <$

$0\} \cap \Lambda$, $\partial\Omega \cap \Lambda = \{\varphi = 0\} \cap \Lambda$ and normalized such that $|\nabla\varphi| = 1$ for all $x \in \Lambda$. This implies that φ is the distance between x and $\partial\Omega$ for $x \in \Lambda$. Without restriction, we assume that $\Lambda = \{x \in \Omega : \varphi(x) < \eta\}$ for a small number $\eta > 0$. We define a smooth extension of the normal unit vector n inside Ω by taking $n = \nabla\varphi$.

As in [58, 111], we take the following ansatz:

$$\begin{aligned} u^\nu(t,x) &= u^0(t,x) + \sqrt{\nu}u^c(t,x,\frac{\varphi(x)}{\sqrt{\nu}}) + \nu v(t,x,\frac{\varphi(x)}{\sqrt{\nu}}) + \nu R^\nu(t,x); \\ \pi^\nu(t,x) &= \pi^0(t,x) + \sqrt{\nu}p(t,x,\frac{\varphi(x)}{\sqrt{\nu}}) + \nu q(t,x,\frac{\varphi(x)}{\sqrt{\nu}}) + \nu\kappa(t,x). \end{aligned} \tag{2.1.4}$$

Plugging (2.1.4) into (2.1.1), we can obtain that

$$\begin{aligned} &\partial_z u^c \cdot n = 0; \\ &\mathrm{div}_x u^c = -\partial_z v \cdot n. \end{aligned} \tag{2.1.5}$$

As in the argument in [111], it is easy to see that $p \equiv 0$.

Slight modify the proof in [58], we can prove that if $u^c(t,x,0)\cdot n(x) = 0$ and u^c satisfies the following equations

$$\partial_t u^c - \partial_z^2 u^c + \frac{u^0\cdot n}{\varphi(x)} z\partial_z u^c + (u^0\cdot\nabla u^c + u^c\cdot\nabla u^0)\times n = 0, \tag{2.1.6}$$

then $u^c \cdot n = 0$, for all $(t,x,z) \in (0,T)\times\Omega\times\mathbb{R}_+$.

Therefore, we can infer (u^c, q, v) satisfies the following system

$$\begin{cases} \partial_t u^c - \partial_z^2 u^c + \dfrac{u^0\cdot n}{\varphi(x)} z\partial_z u^c + (u^0\cdot\nabla u^c + u^c\cdot\nabla u^0)\times n = 0, \\ (u^0\cdot\nabla u^c + u^c\cdot\nabla u^0)\cdot n = \partial_z q, \\ v = n\displaystyle\int_z^\infty \mathrm{div}_x u^c(t,x,\eta)\mathrm{d}\eta, \end{cases} \tag{2.1.7}$$

with boundary and initial conditions

$$\begin{cases} u^c\cdot n = 0, \ \ \partial_z u^c \times n = \nabla\times u^0 \times n, \ \text{ on } \ z = 0; \\ u^c(0,x,z) = 0. \end{cases} \tag{2.1.8}$$

For the flat boundary, it is easy to prove that this profile vanishes and hence uniform H^3 or $W^{2,p}, p > 3$ estimates have been obtained. While the boundary is not flat, the conditions (2.1.2) are related to the curvature of $\partial\Omega$, hence the profile u^c does not vanish. Along the procedure derived by Iftimie and Sueur, we obtain the following proposition:

Proposition 2.1.1 *There exists a unique pair (u^c, q) satisfies the system (2.1.7) with the following regularity for the boundary layer*

$$u^c \in L^\infty(0,T; H^{k,2,0}) \cap L^2(0,T; H^{k,2,1})$$

for all $k \in \mathbf{N}$ and $\partial_z u^c \in L^\infty((0,T) \times \Omega \times \mathbb{R}_+)$.

Moreover, u^c vanishes for x outside the neighborhood Λ and $u^c \cdot n = 0$ for all $(t,x,z) \in (0,T) \times \Omega \times \mathbb{R}_+$. Consequently, we have the limitation

$$\sup_{t\in[0,T]} \|u^\nu - u^0\|_{L^2(\Omega)} \leq C\nu^{\frac{3}{4}},$$

provided that initial velocity $u_0 \in H^3(\Omega)$.

Since the system (2.1.7) of the boundary layer is linear one. Then we prove the higher regularity of u^c for time t and last variable. We improve the L^p uniformly bound of the remainder to more general exponent $3 < p \leq 6$. Therefore, we can expect to prove the estimates of the remainder for higher order derivative. Although it is not easy to obtain the uniformly bound of the remainder's derivative, one can bound H^1-norm of the remainder by $O(\nu^{-\frac{1}{2}})$. Thanks to the asymptotic expansion, we obtain the strong solution to (2.1.1) with boundary conditions (2.1.2) converges to the one of Euler system in $C([0,T]; H^1(\Omega))$, provided that initial velocity is regular enough.

Now, we sate our main results as follows.

Theorem 2.1.2 *Let $u_0 \in H^s$ for $s \geq 5$, be a divergence free vector field satisfies the boundary conditions (2.1.2), assume that u^ν is the strong solution of Navier-Stokes equations, with initial velocity u_0. Let u^0 is the smooth solution of the*

Euler equations with the same initial data. The above boundary layer profile u^c as in Proposition 2.1.1, satisfies the following regularity for $p > 2$

$$
\begin{aligned}
&u^c \in L^\infty(0,T;W^{k,s,0,p});\\
&u^c \in C([0,T];H^{k,s-2,1}) \cap L^\infty(0,T;H^{k,s-2,2} \cap H^{k,s-1,1}) \cap L^2(0,T;H^{k,s-2,3});\\
&\partial_t u^c \in L^\infty(0,T;H^{k,s-2,0}) \cap L^2(0,T;H^{k,s-2,1} \cap H^{k,s-1,0}).
\end{aligned} \tag{2.1.9}
$$

Consequently, There exists $\nu_0 > 0$ small enough, such that for all $0 < \nu \leq \nu_0$, we have

$$\sup_{t\in[0,T]} \|u^\nu - u^0\|_{L^p(\Omega)} \leq C\nu^{\frac{1}{2}+\frac{1}{2p}}, \ \sup_{t\in[0,T]} \|R^\nu\|_p \leq C$$

For all $3 < p \leq 6$, and C is independent of ν.

From the above theorem, we can obtain the following results.

Theorem 2.1.3 *Under the same assumption in theorem 2.1.2, then there exists $\nu_0 > 0$ small enough, such that for all $0 < \nu \leq \nu_0$, we have*

$$\sup_{0\leq t\leq T} \|u^\nu - u^0\|_{H^1(\Omega)} \leq C_1\nu^{\frac{1}{4}}, \|u^\nu - u^0 - \sqrt{\nu}u^c\|_{L^2(0,T,H^2(\Omega))} \leq C_2,$$

Where C_i is independent of ν, $i = 1, 2$

Remark 2.1.4 *In the proposition 2.1.1, the time interval is the maximum existence time interval of the strong solution to Euler equations. However, in Theorem 2.1.2 and 2.1.3, the time interval is the interval of the existence maximum intervals of the strong solutions to both the Navier-Stokes and Euler equations, does not depend on the viscosity.*

2.2 Preliminaries

We now state some lemmas which will be used in this chapter.

Lemma 2.2.1 *Let $u \in W^{s,p}(\Omega)$ be a vector-value function. Then for $s \geq 1$*

$$\|u\|_{s,p} \leq C\left(\|\nabla \times u\|_{s-1,p} + \|\text{div}u\|_{s-1,p} + \|u\cdot n\|_{s-\frac{1}{p},\partial\Omega} + \|u\|_{s-1,p}\right).$$

Proof: See [24, 116]. □

Lemma 2.2.2 *Let be given $u \in W^{1,p}(\Omega),\ 1 < p < +\infty$, then there exists $C > 0$ such that*

$$|u|_p \leq C|\nabla u|_p,$$

for all u such that $u \cdot n|_{\partial\Omega} = 0,\ or\ u \times n|_{\partial\Omega} = 0$ Moreover,

$$|\nabla u|_p \leq C(|\nabla \times u|_p + \|\mathrm{div} u\|_p),$$

for all u such that $u \cdot n|_{\partial\Omega} = 0,\ or\ u \times n|_{\partial\Omega} = 0$

Proof: See [17, 116]. □

Lemma 2.2.3 *Let u be a smooth enough function such that $u \cdot n|_{\partial\Omega} = 0, \mathrm{curl} u \times n|_{\partial\Omega} = 0$. Then $\omega = \mathrm{curl} u$ satisfies the following equality on $\partial\Omega$*

$$-\frac{\partial \omega}{\partial n}\omega = (\epsilon_{1jk}\epsilon_{1i\gamma} + \epsilon_{2jk}\epsilon_{2i\gamma} + \epsilon_{3jk}\epsilon_{3i\gamma})\omega_j\omega_i\partial_k n_\gamma$$

In particular,

$$-\int_\Omega \triangle\omega \cdot \omega \mathrm{d}x \leq \int_\Omega |\nabla\omega|^2 \mathrm{d}x + C\int_{\partial\Omega} |\omega|^2 \mathrm{d}x$$

where ϵ_{ijk} denotes the totally anti-symmetric tensor such that $(\varphi \times \psi)_i = \epsilon_{ijk}\varphi_j\psi_k$.

Proof: Directly compute or refer [5]. □

Lemma 2.2.4 (Hardy's Inequality) *If $\Omega \subset \mathbb{R}^n, n \geq 2$ is a bounded lipschitz domain, then*

$$\int_\Omega \frac{|u(x)|^p}{d(x,\partial\Omega)^{p-\beta}}\mathrm{d}x \leq C\int_\Omega \frac{|\nabla u|^p}{d(x,\partial\Omega)^{-\beta}}\mathrm{d}x,\ \ \forall u \in C_0^\infty(\Omega),$$

for all $\beta < p - 1$, where $d(x, \partial\Omega)$ is the distance between x and $\partial\Omega$

Proof: See [88] □

The following theorems show the existence of strong solutions for Euler equations and Navier-Stokes equations respectively.

Theorem 2.2.5 *Assume that Ω is a regular bounded open set of $\mathbb{R}^3$. Let m and p be given, $p \geq 1$, $m > 1+\dfrac{3}{p}$. Then for each u_0, if $u_0 \in W^{m,p}(\Omega)$, $\mathrm{div} u_0 = 0, u_0 \cdot n = 0$ on $\partial\Omega$, there exist $\bar{T} \leq T$ and a unique functions u^0 and π on $(0, \bar{T})$, such that $u^0 \in C([0,\bar{T}]; W^{m,p}(\Omega)) \cap C^1([0,\bar{T}]; W^{m-1,p}(\Omega))$, $\pi^0 \in L^\infty(0,\bar{T}; W^{m+1,p}(\Omega))$ and satisfying Euler equations(2.1.3).*

Proof: See [107] or [24]. □

In fact, we can check the proof of theorem 6.3 and theorem 7.1 in [116], we can find the existence of Leray's weak solution for the whole time interval and the existence and uniqueness of strong solutions in the $\nu-$independent time interval hold true for general smooth boundary. Next we state the existence theorem of strong solutions as follows

Theorem 2.2.6 *Let $u_0 \in H^1(\Omega)$ be divergence free with the boundary conditions (2.1.2). Then there is a time $T^* = T^*(u_0) > 0$ such that the problem (2.1.1) and (2.1.2) with initial velocity u_0 has a unique strong solution of u^ν on the interval $[0, T^*)$ satisfying*

$$
\begin{aligned}
& u^\nu \in C([0,T^*); H^1(\Omega)) \cap L^2(0,T^*; H^2(\Omega)), \\
& \partial_t u^\nu \in L^2(0,T^*; L^2(\Omega)), \\
& \|u^\nu\|_1 \to \infty \ \text{as} \ t \to T^* \ \text{if} \ T^* < \infty.
\end{aligned}
$$

Let us stress that this time $T = \min\{\bar{T}, T^*\}$ is from now on assume to finite and fixed.

2.3 Estimates of boundary layer

In this section, we will show the estimates for the first boundary layer including L^p-estimates in (x, z) and the estimates of time regularity.

The following lemmas will be frequently applied in the rest of paper.

Lemma 2.3.1 *There exists a constant C independent of ν such that for all $v \in L^p_z(\mathbb{R}_+; W^{2,p}_x(\Omega)) = W^{0,2,0,p}(\Omega \times \mathbb{R}_+)$, $p > 1$ which vanishes for x outside the neighborhood Λ of $\partial\Omega$,*

$$\|v(x, \frac{\varphi(x)}{\sqrt{\nu}})\|_p \leq C\nu^{\frac{1}{2p}}\|v\|_{0,1,0,p}, \tag{2.3.1}$$

where $\varphi(x)$ is a smooth function defined as in the previous section.

Proof: The proof of (2.3.1) is the same as Lemma 3 in [58]. We outline the main estimates here.

$$\int_\Omega v^p(x, \frac{\varphi(x)}{\sqrt{\nu}})\mathrm{d}x = \int_0^\eta \int_{\partial\Omega} v^p(\delta - sn(\delta)), \frac{s}{\sqrt{\nu}}\gamma_s(\delta)\mathrm{d}s,$$

where $\gamma_s(\delta)$ is the Jacobian of the transformation $\delta \to \delta - sn(\delta)$. Then

$$\int_\Omega v^p(x, \frac{\varphi(x)}{\sqrt{\nu}})\mathrm{d}x = \sqrt{\nu}\int_0^\eta \int_{\partial\Omega} v^p(\delta, \sqrt{\nu}z. z)\gamma_{\sqrt{\nu}z}(\delta)\mathrm{d}\delta,$$

$$v(\delta, s, z) := v^p(\delta - sn(\delta), z)$$

Since $\partial\Omega$ is smoothly compact and η is small, we have

$$0 < min\{\gamma_s(\delta); 0 < s < \eta, \delta \in \partial\Omega\} \leq max\{\gamma_s(\delta); 0 < s < \eta, \delta \in \partial\Omega\} < +\infty \tag{2.3.2}$$

Therefore

$$V(z) = \int_{\partial\Omega} v^p(\delta, \sqrt{\nu}z, z)\gamma_{\sqrt{\nu}z}(\delta)\mathrm{d}\delta \leq C\int_{\partial\Omega} sup_{0<s<\eta} v^p(\delta, s, z)\mathrm{d}\delta.$$

For each $(\delta, z) \in \partial\Omega \times (0, \infty)$, Sobolev embedding $W^{1,p}(0, \eta) \hookrightarrow L^\infty(0, \eta)$ and the above inequality implies that

$$V(z) \leq C\int_\Omega (v^p + (\partial_n v)^p)(x, z)\mathrm{d}x.$$

Thus

$$\|v(x, \frac{\varphi(x)}{\sqrt{\nu}})\|_p \leq C\nu^{\frac{1}{2p}}\|v\|_{L^p_z(R_+;W^{1,p}_x(\Omega))}.$$

The proof is completed. □

Lemma 2.3.2 *Let* $u_0 \in W^{s+1,p}(\Omega)$ *with* $u_0 \cdot n = 0$, *then* $f(x,t) = \dfrac{u^0 \cdot n}{\varphi(x)} \in C([0,T]; W^{s,p}(\Omega)) \cap C^1([0,T]; W^{s-1,p}(\Omega))$, *where* u^0 *is the smooth solution of Euler equations (2.1.3).*

Proof: For all tangential derivatives, we use the Lemma 2.2.4. The other derivatives use the argument in Lemma 4 of [58]. In fact, in the proof of lemma 4 of [58], we only require that $u^0 \cdot n = 0$ on the boundary. □

The following proposition shows the L^p-estimates of the high order derivative for x-variable of the boundary layer u^c.

Proposition 2.3.3 *Let* $3 < p < \infty$. *If* $u_0 \in W^{m+2,p}(\Omega)$ *with* $\nabla \cdot u_0 = 0$, *and* $u_0 \cdot n = 0, \mathrm{curl} u_0 \times n = 0$ *on the boundary* $\partial\Omega$, *then*

$$u^c \in L^\infty(0,T; W^{k,m,0,p}(\Omega \times \mathbb{R}_+)).$$

Proof: We verify it by induction. Set $g(x,t) = \mathrm{curl} u^0 \times n$, then

$$g \in C([0,T]; W^{m+2,p}(\Omega)) \cap C^1([0,T]; W^{m+1,p}(\Omega)).$$

At first, we prove it is true when $|\alpha| = 0$. Multiply (2.1.6) by $(1+z^{2k})|u^c|^{p-2}u^c$ and integrate in x and z to obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|u^c|^p\mathrm{d}x\mathrm{d}z + \iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})u^c \cdot \nabla u^0|u^c|^{p-2}u^c\mathrm{d}x\mathrm{d}z +$$
$$\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})u^0 \cdot \nabla_x u^c|u^c|^{p-2}u^c\mathrm{d}x\mathrm{d}z + \iint_{\Omega\times\mathbb{R}_+}(z+z^{2k+1})f \cdot \partial_z u^c|u^c|^{p-2}u^c\mathrm{d}x\mathrm{d}z$$
$$-\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_z^2 u^c|u^c|^{p-2}u^c\mathrm{d}x\mathrm{d}z = 0.$$

Since $\nabla \cdot u^0 = 0$ and $u^0 \cdot n = 0$ on the $\partial\Omega$, the third term on the left hand side vanishes. Integrate by part with respect to z to the last term on the same side and the fact that the boundary condition (2.1.7) yields

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|u^c\|_{k,0,0,p} + \iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_z u^c|^2|u^c|^{p-2}\mathrm{d}x\mathrm{d}z \\
&= -\iint_{\Omega\times\mathbb{R}_+} 2kz^{2k-1}\partial_z u^c|u^c|^{p-2}u^c\mathrm{d}x\mathrm{d}z - \iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})u^c\cdot\nabla u^0|u^c|^{p-2}u^c\mathrm{d}x\mathrm{d}z \\
&\quad -\frac{1}{p}\iint_{\Omega\times\mathbb{R}_+}(1+2kz^{2k+1})f|u^c|^p\mathrm{d}x\mathrm{d}z + \int_\Omega g(x,t)|u^c(x,t,0)|^{p-2}|u^c(x,t,0)|\mathrm{d}x \\
&= \mathbb{I}_1+\mathbb{I}_2+\mathbb{I}_3+\mathbb{I}_4.
\end{aligned} \tag{2.3.3}$$

Young's inequality implies that

$$|\mathbb{I}_1| \le \varepsilon\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_z u^c|^2|u^c|^{p-2}\mathrm{d}x\mathrm{d}z + \iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|u^c|^p\mathrm{d}x\mathrm{d}z \tag{2.3.4}$$

Due to the regularity of u^0 and f, we can deduce that

$$|\mathbb{I}_2|+|\mathbb{I}_3| \le C\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|u^c|^p\mathrm{d}x\mathrm{d}z \tag{2.3.5}$$

Now we check the last term as follows:

$$\begin{aligned}
|\mathbb{I}_4| &\le \|g\|_p\left(\int_\Omega |u^c(x,t,0)|^p\mathrm{d}x\right)^{\frac{p-1}{p}} \le C\|g\|_p\left(\int_{\Omega\times\mathbb{R}_+}\partial_z u^c|u^c|^{p-2}u^c\mathrm{d}x\mathrm{d}z\right)^{\frac{p-1}{p}} \\
&\le C\|g\|_p\left(\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_z u^c|^2|u^c|^{p-2}\mathrm{d}x\mathrm{d}z\right)^{\frac{p-1}{2p}}\left(\iint_{\Omega\times\mathbb{R}_+}|u^c|^{p-2}\mathrm{d}x\mathrm{d}z\right)^{\frac{p-1}{2p}} \\
&\le \varepsilon\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_z u^c|^2|u^c|^{p-2}\mathrm{d}x\mathrm{d}z + C\|g\|_p^p + C\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|u^c|^p\mathrm{d}x\mathrm{d}z.
\end{aligned} \tag{2.3.6}$$

Then, put (2.3.4), (2.3.5) and (2.3.6) into (2.3.3) and choose $\varepsilon = \frac{1}{2}$ to obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|u^c\|_{k,0,0,p}^p \le C\|u^c\|_{k,0,0,p}^p + C.$$

By Gronwall's Lemma, we get

$$\sup_{(0,T)}\|u^c\|_{k,0,0,p} \le C.$$

Assume that when $|\alpha| = s \le m-1$, we have that $u^c \in L^\infty(0,T;W^{k,s,0,p}(\Omega \times \mathbb{R}_+))$, when $u_0 \in W^{s+2,p}$. Next we verify when $s=m$, $u^c \in L^\infty(0,T;W^{k,s,0,p}(\Omega \times \mathbb{R}_+))$ is also true.

To this end, we apply the operator ∂_x^α to (2.1.6) with $|\alpha| = m$, multiplied by $(1+z^{2k})|\partial_x^\alpha u^c|^{p-2}\partial_x^\alpha u^c$ to the both sides of this equation and integrate in x and z to obtain

$$\frac{1}{p}\frac{d}{dt}\|u^c\|_{k,m,0,p} = \iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_z^2\partial_x^\alpha u^c|\partial_x^\alpha u^c|^{p-2}\partial_x^\alpha u^c \mathrm{d}x\mathrm{d}z$$
$$-\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_x^\alpha[(u^c\cdot\nabla u^0 + u^0\cdot\nabla_x u^c)\times n]|\partial_x^\alpha u^c|^{p-2}\partial_x^\alpha u^c \mathrm{d}x\mathrm{d}z \quad (2.3.7)$$
$$-\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_x^\alpha(fz\partial_z u^c)|\partial_x^\alpha u^c|^{p-2}\partial_x^\alpha u^c \mathrm{d}x\mathrm{d}z = I_1+I_2+I_3$$

We estimate I_1. Integrate by part with respect to z to get

$$I_1 = -\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})(\partial_z\partial_x^\alpha u^c)|\partial_x^\alpha u^c|^{p-2}\partial_z\partial_x^\alpha u^c \mathrm{d}x\mathrm{d}z$$
$$-\iint_{\Omega\times\mathbb{R}_+}2kz^{2k-1}(\partial_z\partial_x^\alpha u^c)|\partial_x^\alpha u^c|^{p-2}\partial_z\partial_x^\alpha u^c \mathrm{d}x\mathrm{d}z$$
$$+\int_\Omega \partial_z(\partial_x^\alpha u^c)|\partial_x^\alpha u^c|^{p-2}\partial_x^\alpha u^c|_{z=0}\mathrm{d}x$$
$$= -\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_z\partial_x^\alpha u^c|^2|\partial_x^\alpha u^c|^{p-2}\mathrm{d}x\mathrm{d}z - I_{1_1} + I_{1_2}$$

Due to Young inequality, we can infer that

$$|I_{1_1}| = \varepsilon\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_z\partial_x^\alpha u^c|^2|\partial_x^\alpha u^c|^{p-2}\mathrm{d}x\mathrm{d}z + C\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_x^\alpha u^c|^p \mathrm{d}x\mathrm{d}z$$

Since $\partial_z(\partial_x^\alpha u^c)|_{z=0} = \partial_x^\alpha g(x,t)$, by the same argument in estimates of $\mathbb{I}_4$, we can conclude that

$$|I_{1_2}| \le \varepsilon\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_z\partial_x^\alpha u^c|^2|\partial_x^\alpha u^c|^{p-2}\mathrm{d}x\mathrm{d}z$$
$$+C\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_x^\alpha u^c|^p\mathrm{d}x\mathrm{d}z + C\|\partial_x^\alpha g(x,t)\|_p^p.$$

Then the estimate for I_1 is

$$I_1 \le -(1-2\varepsilon)\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_z\partial_x^\alpha u^c|^2|\partial_x^\alpha u^c|^{p-2}\mathrm{d}x\mathrm{d}z + C\|u^c\|_{k,m,0,p}^p + C. \quad (2.3.8)$$

Now we bound I_2. By direct calculation, we have the following formula

$$\partial_x^\alpha(u \times n) = (\partial_x^\alpha u) \times n + D_x^{|\alpha|-1}(u), \tag{2.3.9}$$

where $D_x^\alpha(u)$ denotes a linear combination of components of u and derivatives with respect to x of order $\leq |\alpha|$ of such components with coefficients components of n and derivatives of n. Thanks to (2.3.9), we infer that

$$\begin{aligned}
I_2 = &\iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})\{[\partial_x^\alpha(u^c\cdot u^0 + u^0\cdot\nabla_x u^c)]\times n \\
&+ D_x^{m-1}(u^c\cdot\nabla u^0 + u^0\cdot\nabla_x u^c)\}\cdot|\partial_x^\alpha u^c|^{p-2}\partial_x^\alpha u^c \mathrm{d}x\mathrm{d}z \\
= &\iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})\{[\partial_x^\alpha(u^c\cdot u^0 + u^0\cdot\nabla_x u^c)] + D_x^{m-1}(u^c\cdot\nabla u^0 + u^0\cdot\nabla_x u^c) \\
&- [\partial_x^\alpha(u^c\cdot\nabla u^0 + u^0\cdot\nabla_x u^c)\cdot n]n\}|\partial_x^\alpha u^c|^{p-2}\partial_x^\alpha u^c \mathrm{d}x\mathrm{d}z = J_1 + J_2 + J_3.
\end{aligned} \tag{2.3.10}$$

First, Let recall the Leibniz' formula

$$D^\alpha(uv) = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}D^\beta u D^{\alpha-\beta}v,$$

where $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$, $\alpha = (\alpha_1\cdots\alpha_n)$ and $\dbinom{\alpha}{\beta} = \dfrac{\alpha!}{\beta!(\alpha-\beta)!}$, $\beta\leq\alpha$ means $\beta_i\leq\alpha_i (i = 1,\cdots,n)$. we can calculate the term J_1 by above formula

$$\begin{aligned}
J_1 &= \iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})[\partial_x^\alpha(u^c\cdot u^0 + u^0\cdot\nabla_x u^c)]|\partial_x^\alpha u^c|^{p-2}\partial_x^\alpha u^c \mathrm{d}x\mathrm{d}z \\
&= \iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\left(\partial_x^\beta u^c\cdot\nabla\partial_x^{\alpha-\beta}u^0 + \partial_x^\beta u^0\cdot\partial_x^{\alpha-\beta}u^c\right)|\partial_x^\alpha u^c|^{p-2}\partial_x^\alpha u^c \mathrm{d}x\mathrm{d}z \\
&= J_{11} + J_{12}.
\end{aligned}$$

Since the terms of $\beta\geq 1$, we have

$$\begin{aligned}
&\left|\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\partial_x^\beta u^c\cdot\nabla\partial_x^{\alpha-\beta}u^0|\partial_x^\alpha u^c|^{p-2}\partial_x^\alpha u^c \mathrm{d}x\mathrm{d}z\right| \\
&\qquad\leq C\|\nabla\partial_x^{\alpha-\beta}u^0\|_{L^\infty}\|(1+z^{2k})^{\frac{1}{p}}u^c\|_{|\beta|,p}\|(1+z^{2k})^{\frac{1}{p}}\partial_x^\alpha u^c\|_p^{p-1} \\
&\qquad\leq C\|u^0\|_{m+3,p}^p\|(1+z^{2k})^{\frac{1}{p}}u^c\|_{|\beta|,p}^p + \|(1+z^{2k})^{\frac{1}{p}}\partial_x^\alpha u^c\|_p^p.
\end{aligned}$$

For the terms of $\beta = 0$, we get

$$\begin{aligned}
&\left| \iint_{\Omega\times\mathbb{R}_+} (1+z^{2k}) \sum_{\beta<\alpha} \binom{\alpha}{\beta} \partial_x^\beta u^c \cdot \nabla \partial_x^{\alpha-\beta} u^0 |\partial_x^\alpha u^c|^{p-2} \partial_x^\alpha u^c \mathrm{d}x\mathrm{d}z \right| \\
&\qquad \le C\| \cdot \nabla \partial_x^\alpha u^0\|_{2p} \|(1+z^{2k})^{\frac{1}{p}} u^c\|_{2p} \|(1+z^{2k})^{\frac{1}{p}} \partial_x^\alpha u^c\|_p^{p-1} \\
&\qquad \le C\|u^0\|_{m+3,p}^p \|(1+z^{2k})^{\frac{1}{p}} u^c\|_{1,p}^p + \|(1+z^{2k})^{\frac{1}{p}} \partial_x^\alpha u^c\|_p^p,
\end{aligned}$$

since $2p \le \dfrac{3p}{3-p}$.

Since $u^0 \cdot n = 0$ on the boundary and $\mathrm{div} u^0 = 0$, we know that

$$\left| \iint_{\Omega\times\mathbb{R}_+} (1+z^{2k}) u^0 \cdot \nabla \partial_x^\alpha u^c |\partial_x^\alpha u^c|^{p-2} \partial_x^\alpha u^c \mathrm{d}x\mathrm{d}z \right| = 0.$$

Next, we estimate the other terms of J_{12} for $\beta \neq 0$ as follow

$$\begin{aligned}
&\left| \iint_{\Omega\times\mathbb{R}_+} (1+z^{2k}) \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial_x^\beta u^0 \cdot \nabla \partial_x^{\alpha-\beta} u^c |\partial_x^\alpha u^c|^{p-2} \partial_x^\alpha u^c \mathrm{d}x\mathrm{d}z \right| \\
&\le C\|\partial_x^\beta u^0\|_{m+3,p}^p \|(1+z^{2k})^{\frac{1}{p}} u^c\|_{m-|\beta|+1,p}^p + \|(1+z^{2k})^{\frac{1}{p}} \partial_x^\alpha u^c\|_p^p.
\end{aligned}$$

Therefore, we can conclude that

$$|J_1| \le C\|(1+z^{2k})^{\frac{1}{p}} u^c\|_{|\beta|,p}^p + C\|(1+z^{2k})^{\frac{1}{p}} u^c\|_{m-|\beta|+1,p}^p + \|(1+z^{2k})^{\frac{1}{p}} \partial_x^\alpha u^c\|_p^p.$$

We note that

$$\begin{aligned}
|J_3| &= \left| \iint_{\Omega\times\mathbb{R}_+} (1+z^{2k}) [\partial_x^\alpha (u^c \cdot u^0 + u^0 \cdot \nabla_x u^c) \cdot n] n |\partial_x^\alpha u^c|^{p-2} \partial_x^\alpha u^c \mathrm{d}x\mathrm{d}z \right| \\
&= \Bigg| \iint_{\Omega\times\mathbb{R}_+} (1+z^{2k}) \Bigg[\Bigg(\sum_{\beta\le\alpha} \binom{\alpha}{\beta} \Big(\partial_x^\beta u^c \cdot \nabla \partial_x^{\alpha-\beta} u^0 \\
&\quad + \partial_x^\beta u^0 \cdot \partial_x^{\alpha-\beta} u^c \Big) \Bigg) \cdot n \Bigg] n |\partial_x^\alpha u^c|^{p-2} \partial_x^\alpha u^c \mathrm{d}x\mathrm{d}z \Bigg| \\
&\le C|J_1| + \iint_{\Omega\times\mathbb{R}_+} (1+z^{2k}) \left[\left(u^0 \cdot \nabla (\partial_x^\alpha u^c) \cdot n \right) \right] n |\partial_x^\alpha u^c|^{p-2} \partial_x^\alpha u^c \mathrm{d}x\mathrm{d}z \\
&\le C|J_1| + J_{31}
\end{aligned}$$

Since $u^c \cdot n = 0$, then $\partial_x^\alpha u^c \cdot n = -D_x^{m-1}(u^c)$ and compute the term J_{31} as follow

$$\begin{aligned}
J_{31} = &\iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})(u^0\cdot\nabla(\partial_x^\alpha u^c\cdot n)|\partial_x^\alpha u^c|^{p-2}(\partial_x^\alpha u^c\cdot n) \\
&+ u^0\cdot\nabla n\cdot u^c D_x^{m-1}u^c|\partial_x^\alpha u^c|^{p-2})\mathrm{d}x\mathrm{d}z \\
= &\iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})(u^0\cdot\nabla D_x^{m-1}u^c|\partial_x^\alpha u^c|^{p-2}(\partial_x^\alpha u^c\cdot n) \\
&+ u^0\cdot\nabla n\cdot u^c D_x^{m-1}u^c|\partial_x^\alpha u^c|^{p-2})\mathrm{d}x\mathrm{d}z \\
\le &\, C|J_1|
\end{aligned}$$

Use the Leibniz' formula, we easily obtain $J_2 \le C|J_1|$. Thus we have

$$I_2 \le C\|(1+z^{2k})^{\frac{1}{p}}u^c\|_{|\beta|,p}^p + C\|(1+z^{2k})^{\frac{1}{p}}u^c\|_{m-|\beta|+1,p}^p + \|(1+z^{2k})^{\frac{1}{p}}\partial_x^\alpha u^c\|_p^p \tag{2.3.11}$$

Use the Leibniz' formula again we know that

$$\begin{aligned}
|I_3| = &\left|-\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})z\left(\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\partial_x^\beta f\partial_x^{\alpha-\beta}\partial_z u^c\right)|\partial_x^\alpha u^c|^{p-2}\partial_x^\alpha u^c\mathrm{d}x\mathrm{d}z\right| \\
\le &\, C - \iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})zf\partial_z|\partial_x^\alpha u^c|^p\mathrm{d}x\mathrm{d}z \\
&+ \left|-\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})z\left(\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\partial_x^\beta f\partial_z\partial_x^{\alpha-\beta}u^c\right)|\partial_x^\alpha u^c|^{p-2}\partial_x^\alpha u^c\mathrm{d}x\mathrm{d}z\right| \\
:= &\, J_{31} + J_{32}
\end{aligned}$$

By the regularity of f and integrating by parts for z, we have

$$\begin{aligned}
|J_{31}| \le &\, C\iint_{\Omega\times\mathbb{R}_+} z(1+z^{2k})\partial_z|\partial_x^\alpha u^c|^p\mathrm{d}x\mathrm{d}z \\
\le &\, C\iint_{\Omega\times\mathbb{R}_+}(1+(2k+1)z^{2k})|\partial_x^\alpha u^c|^p\mathrm{d}x\mathrm{d}z \\
\le &\, C\|u^c\|_{k,m,0,p}^p.
\end{aligned}$$

And with integrating by parts, the regularity of $\partial_x^\beta f$ and Young's inequality imply

$$\begin{aligned}
J_{32} \leq & \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\iint_{\Omega\times\mathbb{R}_+}(1+(2k+1)z^{2k})\partial_x^\beta f\partial_x^{\alpha-\beta}u^c|\partial_x^\alpha u^c|^{p-2}\partial_x^\alpha u^c \mathrm{d}x\mathrm{d}z \\
& - \iint_{\Omega\times\mathbb{R}_+} z(1+z^{2k})\partial_x^\beta f\partial_x^{\alpha-\beta}u^c|\partial_x^\alpha u^c|^{p-2}\partial_x^\alpha\partial_z u^c \mathrm{d}x\mathrm{d}z \\
& \leq C(\|u^c\|_{k+\left[\frac{p}{2}\right]+1,m-|\beta|,0,p} + C\|u^c\|_{k,m,0,p} \\
& + \varepsilon\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_x^\alpha\partial_z u^c|^2|\partial_x^\alpha u^c|^{p-2}\mathrm{d}x\mathrm{d}z
\end{aligned}$$

Therefore, choose $\varepsilon = \dfrac{1}{3}$, we have for $|\beta| > 0$

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|u^c\|_{k,m,0,p}^p \leq & C(\|u^c\|_{k,m-|\beta|,0,p}^p + \|u^c\|_{k+\left[\frac{p}{2}\right]+1,m-|\beta|,0,p}^p + \|u^c\|_{k,|\beta|,0,p}^p \\
& + C\|u^c\|_{k,m,0,p}^p.
\end{aligned} \tag{2.3.12}$$

By the assumption and Gronwall's Lemma, we finished this proof □

Remark 2.3.4 *In this proof, we require that $u_0 \in W^{s+2,p}(\Omega)$ and $p > 3$. It is easy to know that if $u_0 \in W^{s+3,p}(\Omega)$ with $p \geq 2$, then the above proposition holds. However, when $p = 2$, we can prove that if $u_0 \in H^{m+1}(\Omega)\cap L_\sigma^2(\Omega)$ then $u^c \in L^\infty(0,T;H^{k,m,0}(\Omega\times\mathbb{R}_+))\cap L^2(0,T;H^{k,m,1}(\Omega\times\mathbb{R}_+))$. In fact, For $m \leq 2$, we can refer the proof in [58]. However, if $m > 2$, the proof is not difficult along the proof of H_x^2-estimates in [58] or see the following proof.*

It is difficult to get L^p-estimates for $(t,z)-$variable, but we get the following estimates which are better regularity with respect to (t,z)-variable than [58].

Lemma 2.3.5 *If $u_0 \in H^{m+1}(\Omega)\cap L_\sigma^2(\Omega)$, $m \geq 2$, with $u_0\cdot n|_{\partial\Omega} = 0$, then $u^c \in L^\infty(0,T;H^{k,m,1}(\Omega\times\mathbb{R}_+))$ and $\partial_t u^c \in L^2(0,T;H^{k,m,0}(\Omega\times\mathbb{R}_+))$*

Proof: Apply the operator ∂_x^α with $|\alpha| = m$ to the equation (2.1.6) to get

$$\partial_t\partial_x^\alpha u^c - \partial_z^2 u^c + \partial_x^\alpha(fz\partial_z u^c) + \partial_x^\alpha[(u^c\cdot\nabla u^0 + u^0\nabla u^c)\times n] = 0. \tag{2.3.13}$$

Multiply (2.3.13) by $(1+z^{2k})\partial_t\partial_x^\alpha u^c$ and integrate on $\Omega\times\mathbb{R}_+$ we have

$$\begin{aligned}0=&\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_t\partial_x^\alpha u^c|^2\mathrm{d}x\mathrm{d}z-\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_t\partial_x^\alpha u^c\partial_z^2u^c\mathrm{d}x\mathrm{d}z\\&+\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_t\partial_x^\alpha u^c\partial_x^\alpha(fz\partial_zu^c)\mathrm{d}x\mathrm{d}z\\&+\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_t\partial_x^\alpha u^c\partial_x^\alpha[(u^c\cdot\nabla u^0+u^0\nabla u^c)\times n]\\&=\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_t\partial_x^\alpha u^c|^2\mathrm{d}x\mathrm{d}z+K_1+K_2+K_3.\end{aligned}$$

We will focus the estimate of K_1. In fact, integral by parts respect z, we have

$$\begin{aligned}K_1=&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_x^\alpha\partial_zu^c|^2\mathrm{d}x\mathrm{d}z\\&+\iint_{\Omega\times\mathbb{R}_+}2kz^{2k-1}\partial_z\partial_x^\alpha u^c\cdot\partial_t\partial_x^\alpha u^c\mathrm{d}x\mathrm{d}z-\int_\Omega\partial_z\partial_x^\alpha u^c(x,t,0)\partial_t\partial_x^\alpha u^c(x,t,0)\mathrm{d}x\\&\le\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_x^\alpha\partial_zu^c|^2\mathrm{d}x\mathrm{d}z+C\|\partial_z\partial_x^\alpha u^c\|_{k,0,0}^2+\varepsilon\|\partial_t\partial_x^\alpha u^c\|_{k,0,0}^2\\&-\int_\Omega\partial_x^\alpha g\cdot\partial_t\partial_x^\alpha u^c(x,t,0)\mathrm{d}x.\end{aligned}$$

For last term on the right, we can estimate as follows

$$\begin{aligned}&-\int_\Omega\partial_x^\alpha g\cdot\partial_t\partial_x^\alpha u^c(x,t,0)\mathrm{d}x=-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\partial_x^\alpha g\partial_x^\alpha u^c|_{z=0}\mathrm{d}x+\int_\Omega\partial_t\partial_x^\alpha g\partial_x^\alpha u^c|_{z=0}\mathrm{d}x\\&=-\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega\times\mathbb{R}_+}\partial_x^\alpha g\partial_z\partial_x^\alpha u^c\mathrm{d}x\mathrm{d}z-\iint_{\Omega\times\mathbb{R}_+}\partial_t\partial_x^\alpha g\partial_z\partial_x^\alpha u^c\mathrm{d}x\mathrm{d}z\\&\le-\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega\times\mathbb{R}_+}\partial_x^\alpha g\partial_z\partial_x^\alpha u^c\mathrm{d}x\mathrm{d}z+\|\partial_z\partial_x^\alpha u^c\|_{k,0,0}\|\partial_t\partial_x^\alpha g(1+z^{2k})^{-\frac{1}{2}}\|_{L^2(\Omega\times\mathbb{R}_+)}.\end{aligned}$$

Therefore, we can obtain the following estimate

$$\begin{aligned}K_1\le&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega\times\mathbb{R}_+}\left((1+z^{2k})|\partial_x^\alpha\partial_zu^c|^2-2\partial_x^\alpha g\partial_z\partial_x^\alpha u^c\right)\mathrm{d}x\mathrm{d}z\\&+C\|\partial_z\partial_x^\alpha u^c\|_{k,0,0}^2+\varepsilon\|\partial_t\partial_x^\alpha u^c\|_{k,0,0}^2+\|\partial_t\partial_x^\alpha g(1+z^{2k})^{-\frac{1}{2}}\|_{L^2(\Omega\times\mathbb{R}_+)}^2.\end{aligned}\tag{2.3.14}$$

We are going to estimate K_2 by Leibniz' Formula,

$$\begin{aligned}K_2=&\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_t\partial_x^\alpha u^c\partial_x^\alpha(fz\partial_zu^c)\mathrm{d}x\mathrm{d}z\\&=\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_t\partial_x^\alpha u^c\partial_x^\alpha(fz\partial_zu^c)\mathrm{d}x\mathrm{d}z\\&=\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_t\partial_x^\alpha u^cz\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\partial_x^\beta f\partial_x^{\alpha-\beta}\partial_zu^c\mathrm{d}x\mathrm{d}z.\end{aligned}$$

We will estimate the right terms divided by several cases.

In case of $\beta = 0$, we have

$$\iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})\partial_t\partial_x^\alpha u^c z f \partial_x^\alpha \partial_z u^c \mathrm{d}x\mathrm{d}z$$
$$\leq |f|_{L^\infty(\Omega)} \|(1+z^{2k})^{\frac{1}{2}}\partial_t\partial_x^\alpha u^c\|_2 \|(1+z^{2(k+1)})^{\frac{1}{2}}\partial_z\partial_x^\alpha u^c\|_2$$

In the case of $|\beta| = 1$, we obtain

$$\iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})\partial_t\partial_x^\alpha u^c z \sum_{\beta=1} \partial_x^\beta f \partial_x^{\alpha-\beta}\partial_z u^c \mathrm{d}x\mathrm{d}z$$
$$\leq C\int_{\mathbb{R}_+} (1+z^{2k}) z \|\partial_t\partial_x^\alpha u^c\|_2 \|\partial_x^\beta f\|_6 \|\partial_x^{\alpha-\beta}\partial_z u^c\|_3 \mathrm{d}z$$
$$\leq C\|(1+z^{2k})^{\frac{1}{2}}\partial_t\partial_x^\alpha u^c\|_2 \|f\|_{1,2} \|(1+z^{2(k+2)})^{\frac{1}{2}}\partial_z\partial_x^{\alpha-\beta}u^c\|_2^{\frac{1}{2}} \|(1+z^{2k})^{\frac{1}{2}}\nabla_x\partial_z\partial_x^{\alpha-\beta}u^c\|_2^{\frac{1}{2}}$$

here use the interpolation inequality $\|u\|_3 \leq \|u\|_2^{\frac{1}{2}}\|u\|_{1,2}^{\frac{1}{2}}$.

In the case of $|\beta| \geq 2$, we get

$$\iint_{\Omega\times\mathbb{R}_+} (1+z^{2k})\partial_t\partial_x^\alpha u^c z \sum_{\beta\geq 2} \partial_x^\beta f \partial_x^{\alpha-\beta}\partial_z u^c \mathrm{d}x\mathrm{d}z$$
$$\leq C\int_{\mathbb{R}_+} (1+z^{2k}) z \|\partial_t\partial_x^\alpha u^c\|_2 \|\partial_x^\beta f\|_2 \|\partial_x^{\alpha-\beta}\partial_z u^c\|_\infty \mathrm{d}z$$
$$\leq C\|(1+z^{2k})^{\frac{1}{2}}\partial_t\partial_x^\alpha u^c\|_2 \|f\|_{\beta,2} \|(1+z^{2(k+2)})^{\frac{1}{2}}\nabla_x\partial_z\partial_x^{\alpha-\beta}u^c\|_2^{\frac{1}{2}} \|(1+z^{2k})^{\frac{1}{2}}\nabla_x^2\partial_z\partial_x^{\alpha-\beta}u^c\|_2^{\frac{1}{2}}$$

here apply the interpolation inequality $\|u\|_\infty \leq \|u\|_{1,2}^{\frac{1}{2}}\|u\|_{2,2}^{\frac{1}{2}}$.

Therefore, we know that

$$K_2 \leq C(\|u^c\|_{k+1,m,1}^2 + \|u^c\|_{k+1,m-1,1}^2 + \|u^c\|_{k,m,1}^2) + \varepsilon\|\partial_t\partial_x^\alpha u^c\|_{k,0,0}^2. \tag{2.3.15}$$

Use the same argument above and the proof of I_2 in Proposition 2.3.3, we easily obtain that

$$K_3 \leq C(\|u^c\|_{k,m-1,1}^2 + \|u^c\|_{k,m,1}^2) + \varepsilon\|\partial_t\partial_x^\alpha u^c\|_{k,0,0}^2. \tag{2.3.16}$$

Since Remark 2.3.4, we know that $\|u^c\|_{L^2(0,T;H^{k,m,1}(\Omega\times\mathbb{R}_+)}^2 \leq C$. By the Gronwall's lemma and Hölder inequality and choose $\varepsilon = \dfrac{1}{6}$, we conclude that

$$\|\partial_t u^c\|_{L^2(0,T;H^{k,m,0}(\Omega\times\mathbb{R}_+))} \leq C, \ \ \|u^c\|_{L^\infty(0,T;H^{k,m,1}(\Omega\times\mathbb{R}_+))} \leq C.$$

The proof is completed. □

In fact, we can improve the regularity of $\partial_t u^c$ for the equations (2.1.6) if initial data is regular enough.

Lemma 2.3.6 *If $u_0 \in H^{m+2}(\Omega) \cap L^2_\sigma(\Omega), m \geq 2$ with $u_0 \cdot n|_{\partial\Omega} = 0$, then*

$$\partial_t u^c \in L^\infty(0,T; H^{k,m,0}(\Omega \times \mathbb{R}_+)) \cap L^2(0,T; H^{k,m,1}(\Omega \times \mathbb{R}_+)).$$

Consequently,

$$u^c \in L^\infty(0,T; H^{k,m,2}(\Omega \times \mathbb{R}_+)) \cap L^2(0,T; H^{k,m,3}(\Omega \times \mathbb{R}_+)).$$

Proof: Apply $\partial_t \partial_x^\alpha,\ |\alpha| = m$ to (2.1.6), we have

$$\partial_t(\partial_t\partial_x^\alpha u^c) - \partial_z^2(\partial_t\partial_x^\alpha u^c) + \partial_t\partial_x^\alpha(fz\partial_z u^c) + \partial_t\partial_x^\alpha[(u^0\cdot\nabla u^c + u^c\cdot\nabla u^0)\times n] = 0. \quad (2.3.17)$$

Multiply (2.3.17) by $(1+z^{2k})\partial_t\partial_x^\alpha u^c$ and integral over $\Omega \times \mathbb{R}_+$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_t\partial_x^\alpha u^c|^2\mathrm{d}x\mathrm{d}z - \iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_z^2(\partial_t\partial_x^\alpha u^c)\partial_t\partial_x^\alpha u^c\mathrm{d}x\mathrm{d}z\\
&\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_t\partial_x^\alpha(fz\partial_z u^c)\partial_t\partial_x^\alpha u^c + \mathrm{d}x\mathrm{d}z +\\
&\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_t\partial_x^\alpha[(u^0\cdot\nabla u^c + u^c\cdot\nabla u^0)\times n]\partial_t\partial_x^\alpha u^c\mathrm{d}x\mathrm{d}z = 0.
\end{aligned} \quad (2.3.18)$$

First, we give the estimates when $|\alpha| = 0$, thus

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_t u^c|^2\mathrm{d}x\mathrm{d}z - \iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_z^2(\partial_t u^c)\partial_t u^c\mathrm{d}x\mathrm{d}z\\
&\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_t(fz\partial_z u^c)\partial_t u^c\mathrm{d}x\mathrm{d}z +\\
&\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_t[(u^0\cdot\nabla u^c + u^c\cdot\nabla u^0)\times n]\partial_t u^c\mathrm{d}x\mathrm{d}z = 0.
\end{aligned} \quad (2.3.19)$$

Since

$$\begin{aligned}
&-\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_z^2(\partial_t u^c)\partial_t u^c\mathrm{d}x\mathrm{d}z = \iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_z\partial_t u^c|^2\mathrm{d}x\mathrm{d}z\\
&+2k\iint_{\Omega\times\mathbb{R}_+}z^{2k-1}\partial_z\partial_t u^c\partial_t u^c\mathrm{d}x\mathrm{d}z - \int_\Omega \partial_t\partial_z u^c\partial_t u^c|_{z=0}\mathrm{d}x.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_z\partial_t u^c|^2\mathrm{d}x\mathrm{d}z\\
&\le C\|(1+z^{2k})^{\frac{1}{2}}\partial_z\partial_t u^c\|_2\|(1+z^{2k})^{\frac{1}{2}}\partial_t u^c\|_2-\iint_{\Omega\times\mathbb{R}_+}\partial_t g\partial_t\partial_z u^c\mathrm{d}x\mathrm{d}z\\
&\le C\|(1+z^{2k})^{\frac{1}{2}}\partial_z\partial_t u^c\|_2\|(1+z^{2k})^{\frac{1}{2}}\partial_t u^c\|_2+C\|(1+z^{2k})^{\frac{1}{2}}\partial_t u^c\|_2\|\partial_t g(1+z^{2k})^{-\frac{1}{2}}\|_2.
\end{aligned}$$

We can calculate about the third term on the left side of (2.3.19) as follows

$$\begin{aligned}
&\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_t(fz\partial_z u^c)\partial_t u^c\mathrm{d}x\mathrm{d}z\\
&=\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_t fz\partial_z u^c\partial_t u^c\mathrm{d}x\mathrm{d}z\\
&+\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})fz\partial_z\partial_t u^c\partial_t u^c\mathrm{d}x\mathrm{d}z\\
&\le C\|\partial_t f\|_{L^\infty(\Omega)}\|(1+z^{2k+2})^{\frac{1}{2}}\partial_z u^c\|_2\|(1+z^{2k})^{\frac{1}{2}}\partial_t u^c\|_2\\
&+C\|f\|_{L^\infty(\Omega)}\|(1+z^{2k+2})^{\frac{1}{2}}\partial_t u^c\|_2\|\partial_t\partial_z u^c(1+z^{2k})^{\frac{1}{2}}\|_2\\
&\le C\|\partial_t u^c\|^2_{k+1,0,0}+\|u^c\|^2_{k,0,1}+\varepsilon\|\partial_t\partial_z u^c\|^2_{k,0,0}.
\end{aligned}$$

Similar argument in Proposition 2.3.3 and above proof, It is easy to see that

$$\begin{aligned}
&\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\partial_t[(u^0\cdot\nabla u^c+u^c\cdot\nabla u^0)\times n]\partial_t u^c\mathrm{d}x\mathrm{d}z\\
&=\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})\left[(\partial_t u^0\cdot\nabla u^c+u^0\cdot\nabla\partial_t u^c+\partial_t u^c\cdot\nabla u^0+u^c\nabla\partial_t u^0)\times n\right]\partial_t u^c\mathrm{d}x\mathrm{d}z\\
&=C(\|\partial_t u^c\|^2_{k,0,0}+\|u^c\|^2_{k,1,0}).
\end{aligned}$$

By the Gronwall's Lemma and the $\partial_t u^c\in L^2(0,T;H^{k+1,0,0}(\Omega\times\mathbb{R}_+))$ from the previous lemma, we have

$$\sup_{0\le t\le T}\|\partial_t u^c(t)\|^2_{k,0,0}+\int_0^T\iint_{\Omega\times\mathbb{R}_+}(1+z^{2k})|\partial_t\partial_z u^c|^2\mathrm{d}x\mathrm{d}z\mathrm{d}t\le C+\lim_{t\to 0}\|\partial_t u^c(t)\|^2_{k,0,0}. \tag{2.3.20}$$

By induction, as in argument of Proposition 2.3.3, we can obtain

$$\sup_{0\le t\le T}\|\partial_t u^c(t)\|^2_{k,m,0}+\|\partial_t u^c|^2_{L^2(0,T;H^{k,m,1}(\Omega\times\mathbb{R}_+))}\le C+\lim_{t\to 0}\|\partial_t u^c(t)\|^2_{k,m,0}. \tag{2.3.21}$$

Since $u^c(x,0,z)=0$, thus $\partial_x^\alpha\partial_z^j u^c(x,0,z)=0$, for all α, j. Therefore, take limit to both side of (2.3.13) as $t\to 0$, we can get $\partial_t\partial_x^\alpha u^c(x,t,z)\to 0$ in a.e. $\Omega\times\mathbb{R}_+$ as $t\to 0$. Therefore, we can know that last terms of the right side of (2.3.20) and (2.3.21) are vanish.

Multiply (2.1.6) by $\partial_z^2 u^c$, using Hölder inequality and Lemma 2.3.5, we get that

$$u^c\in L^\infty(0,T;H^{k,m,2}(\Omega\times\mathbb{R}_+)).$$

Apply ∂_z to (2.1.6) and inner product by $\partial_z^3 u^c$, by Hölder inequality and Lemma 2.3.5, we also know that

$$u^c\in L^2(0,T;H^{k,m,3}(\Omega\times\mathbb{R}_+)).$$

□

2.4 The proof of Theorem 2.1.2

In this section, we give the L^p uniform bound of this remainder and give the proof of theorem 2.1.2. With the same arguments as [58], we obtain that the remainder R^ν solves the following equation:

$$\begin{aligned}
&\partial_t R^\nu-\nu\triangle R^\nu+u^\nu\cdot\nabla R^\nu+R^\nu\cdot\nabla u^0+\sqrt{\nu}R^\nu\cdot n\partial_z v+R^\nu\cdot n\partial_z u^c+\sqrt{\nu}R^\nu\cdot\nabla_x u^c\\
&=-\partial_t v+\triangle u^0+\sqrt{\nu}\triangle_x u^c+2n\cdot\nabla_x\partial_z u^c+\nu\triangle_x[v(x,\frac{\varphi(x)}{\sqrt{\nu}})]-u^\nu\cdot\nabla_x v-v\cdot\nabla u^0\\
&-\frac{1}{\sqrt{\nu}}u^0\cdot n\partial_z v-\sqrt{\nu}v\cdot n\partial_z v-v\cdot n\partial_z u^c-u^c\cdot\nabla_x u^c+\triangle\varphi\cdot\partial_z u^c-\sqrt{\nu}v\cdot\nabla_x u^c\\
&+\nabla_x q+\nabla_x\kappa:=R.H.S. \qquad\qquad \text{in }\ \Omega,\\
&\text{div}R^\nu=-\text{div}_x v(t,x,\frac{\varphi(x)}{\sqrt{\nu}}) \qquad\qquad \text{in }\ \Omega.
\end{aligned}\tag{2.4.1}$$

With the boundary condition

$$R^\nu \cdot n(x) + v(t,x,0)\cdot n(x) = 0, \qquad \text{for } \ x \in \partial\Omega \tag{2.4.2}$$

$$\mathrm{curl} R^\nu \times n + \frac{1}{\sqrt{\nu}}\mathrm{curl}_x u^c(t,x,0) \times n + \mathrm{curl}_x v(t,x,0)\times n = 0, \ \text{for } \ x \in \partial\Omega. \tag{2.4.3}$$

And the initial data of R^ν is

$$R^\nu(0,x) = 0, \qquad\qquad \text{for } \ x \in \Omega. \tag{2.4.4}$$

Denote $b(t,x) = \frac{1}{\sqrt{\nu}}u^c(t,x,0) + v(t,x,0)$.

In the sequel, we always use the following anisotropic Sobolev embedding.

Lemma 2.4.1 *Let $U(x,z)$ be a sufficiently regular function defined on $\Omega \times \mathbb{R}_+$, if $2 \le p < \infty$, $m \ge \frac{3}{2} - \frac{3}{p}$ and if $p = \infty$, $m > \frac{3}{2}$, Then*

$$\|U(x, \frac{\varphi(x)}{\sqrt{\nu}})\|_p \le C\|U\|_{1,m,1}. \tag{2.4.5}$$

Proof: In fact,

$$\begin{aligned}
&\|U(x, \frac{\varphi(x)}{\sqrt{\nu}})\|_p \le \|\|\partial_z U\|_{L^1_z(\frac{\varphi(x)}{\sqrt{\nu}},\infty)}\|_{L^p(\Omega)} \\
&\le \|\|\partial_z U\|_{L^1_z(\mathbb{R}_+)}\|_{L^p(\Omega)} \le \|\|\partial_z U\|_{L^p(\Omega)}\|_{L^1_z(\mathbb{R}_+)} \\
&\le \|(1+z)\partial_z U\| L^2_z(\mathbb{R}_+); L^p(\Omega)\|\|(1+z)^{-1}\|_{L^2(\mathbb{R}_+)} \\
&\le C\|U\|_{1,m,1},
\end{aligned}$$

here we used the Sobolev embedding $H^m \hookrightarrow L^p(\Omega)$. □

In this section, we need the following Helmholtz-Weyl decomposition of the space $L^p(\Omega)$:

Lemma 2.4.2 *Let the space $G_p(\Omega) = \{u \in L^p(\Omega) : u = \nabla q, q \in W^{1,p}(\Omega)\}$ and $J_p(\Omega) = \{u \in L^p(\Omega) : \mathrm{div} u = 0, u\cdot n = 0\}$, then*

$$L^p(\Omega) = G_p(\Omega) \oplus J_p(\Omega)$$

and the projections of an arbitrary vector field $u(x)$ to the above subspaces are defined by the formulas

$$\begin{aligned}\mathbb{P}_G u &= -\nabla \int_\Omega \nabla_y N(x,y)\cdot u(y)dy\\ \mathbb{P}_J u &= u + \nabla \int_\Omega \nabla_y N(x,y)\cdot u(y)dy,\end{aligned} \tag{2.4.6}$$

where $N(x,y)$ is the Green's functions with Neumann boundary conditions, with the following estimates:

$$\|\mathbb{P}_G u\|_{l,p} + \|\mathbb{P}_J u\|_{l,p} \le C\|u\|_{l,p}$$

where $l < r$ if $\partial\Omega \in C^{r+1}$ and $u \in W^{l,p}(\Omega)$. When $p = 2$, $G_2(\Omega) \perp J_2(\Omega)$.

Proof: See [102]. □

For simplicity, we denote by $\mathbb{P}$ the projector on the space $J_G(\Omega)$, and decompose $R^\nu = \mathbb{P}R^\nu + (I-\mathbb{P})R^\nu)$ and show first that $(I-\mathbb{P})R^\nu$ is bounded in $W^{1,p}(\Omega)$ independently of ν.

Lemma 2.4.3 *The family $(I-\mathbb{P})R^\nu$ is bounded in $L^\infty(0,T;W^{1,p}(\Omega))$ with $1 < p \le 6$, that is,*

$$\|(I-\mathbb{P})R^\nu\|_{1,p} \le C\|u^c\|_{1,3,0}.$$

Proof: It is easy to obtain this conclusion by using the standard L^p estimates of elliptic equations, or refer [58]. □

Now we estimate the bound of $\|\mathbb{P}R^\nu\|_p$ independently of ν. In order to avoid estimating the unknown pressure term $\nabla\kappa$, we need to multiply the equation of R^ν (2.4.1), by $\mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)$. From Lemma 2.4.2, we know that

$$\begin{aligned}\mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu) &= |\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu + \nabla \int_\Omega \nabla_y N(x,y)\cdot |\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu dy\\ &:= |\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu + \nabla Q.\end{aligned}$$

From the Lemma 2.4.2 we have the following properties

$$\begin{aligned} &\|\nabla Q\|_s \leq C\|\mathbb{P}R^\nu\|_{(p-1)s}^{p-1}, \forall 1 < s < \infty, \\ &\|\nabla^2 Q\|_{\frac{p}{p-1}} \leq C\|\nabla\left(|\mathbb{P}R^\nu|^{\frac{p}{2}}\right)\|_2\|\mathbb{P}R^\nu\|_p^{\frac{p-2}{2}}. \\ &\frac{\partial Q}{\partial n} = 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.4.7}$$

Integrate in x, to obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbb{P}R^\nu\|_p^p - \nu\int_\Omega \triangle R^\nu \cdot (|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu + \nabla Q)\mathrm{d}x \leq \sum_{k=1}^{19} B_k, \tag{2.4.8}$$

where

$$\begin{aligned}
B_1 &= -\int_\Omega u^\nu \cdot \nabla R^\nu \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, & B_2 &= -\int_\Omega R^\nu \cdot \nabla u^0 \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, \\
B_3 &= \sqrt{\nu}\int_\Omega R^\nu \cdot n\partial_z v \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, & B_4 &= -\int_\Omega R^\nu \cdot n\partial_z u^c \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, \\
B_5 &= -\sqrt{\nu}\int_\Omega R^\nu \cdot \nabla_x u^c \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, & B_6 &= -\int_\Omega \partial_t v \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, \\
B_7 &= \int_\Omega \triangle u^0 \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, & B_8 &= \sqrt{\nu}\int_\Omega \triangle_x u^c \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, \\
B_9 &= \int_\Omega 2n \cdot \nabla_x\partial_z u^c \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, & B_{10} &= \nu\int_\Omega \triangle_x[v(x, \frac{\varphi(x)}{\sqrt{\nu}}] \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, \\
B_{11} &= -\int_\Omega u^\nu \cdot \nabla_x v \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, & B_{12} &= -\int_\Omega v \cdot \nabla u^0 \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, \\
B_{13} &= -\frac{1}{\sqrt{\nu}}\int_\Omega u^0 \cdot n\partial_z v \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, & B_{14} &= -\sqrt{\nu}\int_\Omega v \cdot n\partial_z v \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, \\
B_{15} &= -\int_\Omega v \cdot n\partial_z u^c \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, & B_{16} &= -\int_\Omega u^c \cdot \nabla_x u^c \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, \\
B_{17} &= \int_\Omega \triangle\varphi \cdot \partial_z u^c \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, & B_{18} &= -\sqrt{\nu}\int_\Omega v \cdot \nabla_x u^c \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x, \\
B_{19} &= \int_\Omega (\nabla_x q) \cdot (|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu + \nabla Q)\mathrm{d}x.
\end{aligned}$$

We have to estimate each of the terms in (2.4.8). We first deal with the Laplacian term.

$$
\begin{aligned}
&-\nu\int_\Omega \triangle R^\nu\cdot(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu+\nabla Q)\mathrm{d}x\\
&=-\nu\int_\Omega\triangle(\mathbb{P}R^\nu)|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu-\nu\int_\Omega\triangle((I-\mathbb{P})R^\nu)|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu\mathrm{d}x\\
&-\nu\int_\Omega\triangle(\mathbb{P}R^\nu)\nabla Q\mathrm{d}x-\nu\int_\Omega\triangle((I-\mathbb{P})R^\nu)\nabla Q\mathrm{d}x\\
&:=d_1+d_2+d_3+d_4
\end{aligned}
$$

By integration by part, it is easy to obtain

$$
\begin{aligned}
d_1+d_2&=\nu\int_\Omega\nabla(R^\nu)|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu\mathrm{d}x-\nu\int_{\partial\Omega}(\nabla R^\nu\cdot n)\mathbb{P}R^\nu)|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu\mathrm{d}\sigma\\
&=\nu\int_\Omega\nabla(R^\nu)|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu\mathrm{d}x\\
&-\nu\int_{\partial\Omega}\left((\mathrm{curl}b\times n)\mathbb{P}R^\nu)|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu+|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu\cdot\nabla n\cdot\mathbb{P}R^\nu\right)\mathrm{d}\sigma\\
&=d_{1_1}+d_{1_2}
\end{aligned}
$$

By the Young's inequality, we have

$$
\begin{aligned}
d_{1_1}&=\nu\int_\Omega\nabla(\mathbb{P}R^\nu)|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu\mathrm{d}x+\nu\int_\Omega\nabla((I-\mathbb{P})R^\nu)|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu\mathrm{d}x\\
&\geq\nu\frac{2(2p-3)}{p^2}\int_\Omega\left|\nabla|\mathbb{P}R^\nu|^{\frac{p}{2}}\right|\mathrm{d}x-C\|\nabla(I-\mathbb{P})R^\nu\|_p^p-C\|\mathbb{P}R^\nu\|_p^p.
\end{aligned}
$$

Since $\mathrm{curl}R^\nu\times n=\mathrm{curl}b\times n$, thus

$$
\begin{aligned}
d_{1_2}&\leq\nu\int_{\partial\Omega}\left((\mathrm{curl}b\times n)|\mathbb{P}R^\nu|^{p-1}+|\mathbb{P}R^\nu|^p+|I-\mathbb{P}R^\nu||\mathbb{P}R^\nu|^{p-1}\right)\mathrm{d}\sigma\\
&=\nu\|\mathrm{curl}b\|_{p,\partial\Omega}\|\mathbb{P}R^\nu\|_{p,\partial\Omega}^{p-1}+\|\mathbb{P}R^\nu\|_{p,\partial\Omega}^p+\|I-\mathbb{P}R^\nu\|_{p,\partial\Omega}\|\mathbb{P}R^\nu\|_{\partial,\partial\Omega}^{p-1}
\end{aligned}
$$

Since $\sqrt{\nu}\|\mathrm{curl}b\|_{p,\partial\Omega}\leq C\|u^c\|_{1,4,0}+\|u^c\|_{1,3,1}$ and Theorem 1.2.7, thus

$$
\begin{aligned}
&\nu\|\mathrm{curl}b\|_{p,\partial\Omega}\|\mathbb{P}R^\nu\|_{p,\partial\Omega}^{p-1}\leq\nu^{\frac{p}{2}}\|\mathrm{curl}b\|_{p,\partial\Omega}^p+\nu^{\frac{p'}{2}}\|\mathbb{P}R^\nu\|_{p,\partial\Omega}^p\\
&\leq C+\nu^{\frac{p'}{2}}\||\mathbb{P}R^\nu|^{\frac{p}{2}}\|_{2,\partial\Omega}^2\\
&\leq C+\nu^{\frac{p'}{2}}\||\mathbb{P}R^\nu|^{\frac{p}{2}}\|_2\|\nabla|\mathbb{P}R^\nu|^{\frac{p}{2}}\|_2\\
&\leq C+\|\mathbb{P}R^\nu\|_p^p+\nu^{p'}\|\nabla|\mathbb{P}R^\nu|^{\frac{p}{2}}\|_2^2.
\end{aligned}
$$

Therefore, by Lemma 2.4.3, we know that

$$d_{1_2}\leq C\|\mathbb{P}R^\nu\|_p^p+\varepsilon\nu\|\nabla|\mathbb{P}R^\nu|^{\frac{p}{2}}\|_2^2+C.$$

Integral by parts and the formula $\mathrm{curl}(\mathrm{curl}u) = \nabla(\mathrm{div}u) - \Delta u$. Thus by (2.4.7), we get

$$\begin{aligned} d_3 + d_4 &= \nu \int_\Omega \mathrm{curl}(\mathrm{curl}\mathbb{P}R^\nu) \cdot \nabla Q \mathrm{d}x - \nu \int_\Omega \mathrm{div}(I - \mathbb{P})R^\nu \cdot \nabla Q \\ &= -\nu \int_{\partial\Omega} (\mathrm{curl}b \times n) \nabla Q \mathrm{d}x + \nu \int_\Omega \mathrm{div}(I - \mathbb{P})R^\nu \Delta Q \\ &\leq \|\mathrm{curl}b \times n\|_{p,\partial\Omega} \|\nabla Q\|_{p',\partial\Omega} + \|\mathrm{div}(I - \mathbb{P})R^\nu\|_p \|\Delta Q\|_{p'} \\ &\leq C\|\mathbb{P}R^\nu\|_p^p + \varepsilon\nu\|\nabla|\mathbb{P}R^\nu|^{\frac{p}{2}}\|_2^2 + C \end{aligned}$$

Now we estimate the each term of the left side one by one,

Estimate of B_1:

$$\begin{aligned} B_1 &= -\int_\Omega (u^\nu - u^0)\nabla R^\nu \cdot (|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu + \nabla Q)\mathrm{d}x \\ &\quad - \int_\Omega u^0 \nabla R^\nu \cdot (|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu + \nabla Q)\mathrm{d}x \\ &= B_{1_1} + B_{1_2}. \end{aligned}$$

By the regularity of u^0, (2.4.7) and Lemma 2.4.2, it is easy to see that

$$B_{1_2} \leq C + C\|\mathbb{P}R^\nu\|_p^p.$$

We need estimate the term B_{1_1} in details, we have

$$\begin{aligned} |B_{1_1}| &= -\int_\Omega (u^\nu - u^0)\nabla R^\nu \cdot |\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu - \int_\Omega (u^\nu - u^0)\nabla R^\nu \nabla Q \mathrm{d}x \\ &:= B_{1_{11}} + B_{1_{12}}. \end{aligned}$$

By the expansion, one can obtain

$$\begin{aligned} B_{1_{11}} &\leq (\sqrt{\nu}\|u^c\|_\infty + \nu\|v\|_\infty)\|(I - \mathbb{P})R^\nu\|_{1,p}\|\mathbb{P}R^\nu\|_p^{p-1} + \nu\|\mathbb{P}R^\nu\|_{\frac{p^2}{p-1}}^p + C \\ &\leq \nu\|\nabla|\mathbb{P}R^\nu|^{\frac{p}{2}}\|_2^{\frac{3}{p}}\|\mathbb{P}R^\nu\|_p^{\frac{2p-3}{2}} + C\|\mathbb{P}R^\nu\|_p^p + C \\ &\leq C\|\mathbb{P}R^\nu\|_p^p + \varepsilon\nu\|\nabla|\mathbb{P}R^\nu|^{\frac{p}{2}}\|_2^2 + C. \end{aligned}$$

Now we estimate the term $B_{1_{12}}$ as follows,

$$\begin{aligned}
B_{1_{12}} &= \int_\Omega \operatorname{div}((u^\nu - u^0)\otimes \nabla \mathbb{P}R^\nu)\cdot \nabla Q \mathrm{d}x \\
&+ \int_\Omega (\sqrt{\nu}u^c + \nu v + \nu R^\nu)\cdot \nabla (I-\mathbb{P})R^\nu \cdot \nabla Q \mathrm{d}x \\
&\le \int_\Omega (\sqrt{\nu}u^c + \nu v + \nu R^\nu)\otimes \mathbb{P}R^\nu : \nabla^2 Q \mathrm{d}x + (\sqrt{\nu}\|u^c\|_\infty + \nu\|v\|_\infty)\|(I-\mathbb{P})R^\nu\|_{1,p}\|\nabla Q\|_{p'} \\
&+ \nu|\mathbb{P}R^\nu|_{\frac{p^2}{p-1}}\|(I-\mathbb{P})R^\nu\|_{1,p}\|\nabla Q\|_{\frac{p^2}{(p-1)^2}} + \|(I-\mathbb{P})R^\nu\|_\infty \|\nabla (I-\mathbb{P})R^\nu\|_p\|\nabla Q\|_{p'} \\
&\le C\|\mathbb{P}R^\nu\|_p^p + \varepsilon\nu\|\nabla|\mathbb{P}R^\nu|^{\frac{p}{2}}\|_2^2 + C\nu\|\mathbb{P}R^\nu\|^{p\frac{p-1}{p-3}}.
\end{aligned}$$

Therefore, we can obtain that

$$B_1 \le C\|\mathbb{P}R^\nu\|_p^p + C\nu\|\mathbb{P}R^\nu\|^{p\frac{p-1}{p-3}} + C + \varepsilon\nu\|\nabla|\mathbb{P}R^\nu|^{\frac{p}{2}}\|_2^2. \tag{2.4.9}$$

Estimate of B_2+B_4:

Since the regularity of u^0 and the uniform bound of $\partial_z u^c$, use the estimate (2.4.7) we easily get

$$|B_2 + B_4| \le \|\mathbb{P}R^\nu\|_p^p + C. \tag{2.4.10}$$

Estimate of $B_3 + B_5$:

$$\begin{aligned}
|B_3 + B_5| &\le |\sqrt{\nu}\int_\Omega (|\mathbb{P}R^\nu||\nabla_x u^c| + \sqrt{\nu}\int_\Omega |(I-\mathbb{P})R^\nu||\nabla_x u^c|\mathbb{P}(|\mathbb{P}R^\nu|^{p-2}|\mathbb{P}R^\nu)\mathrm{d}x \\
&:= \bar{B}_1 + \bar{B}_2
\end{aligned} \tag{2.4.11}$$

Since when $p > 3$, we know that $\|\nabla_x u^c\|_\infty \le C\|u^c\|_{1,3,1}$ and $\|\nabla u^c\|_p \le C\|u^c\|_{1,2,1}$

$$\|(I-\mathbb{P})R^\nu\|_{2p} \le C\|\nabla (I-\mathbb{P})R^\nu\|_{1,p} \le C.$$

Use the estimate (2.4.7) once more, we have

$$\begin{aligned}
\bar{B}_1 &\le C\sqrt{\nu}\|\mathbb{P}R^\nu\|_{2p}\|\mathbb{P}R^\nu\|_p^{p-1} \\
&\le C\sqrt{\nu}\|\nabla|\mathbb{P}R^\nu|^{\frac{p}{2}}\|_2^{\frac{2}{p}}\|\mathbb{P}R^\nu\|_p^p \\
&\le C\nu^{\frac{p}{2}}\|\nabla|\mathbb{P}R^\nu|^{\frac{p}{2}}\|_2^2 + C\|\mathbb{P}R^\nu\|_p^p. \\
\bar{B}_2 &\le C\sqrt{\nu}\|(I-\mathbb{P})R^\nu\|_\infty\|\mathbb{P}R^\nu\|_p^{p-1}\|\nabla u^c\|_p \\
&\le C\|\mathbb{P}R^\nu\|_p^p + C\nu^p.
\end{aligned}$$

Plugging the above two inequalities into (2.4.11) to obtain

$$|B_3 + B_5| \le C\|\mathbb{P}R^\nu\|_p^p + C\nu^{\frac{p}{2}}\|\nabla|\mathbb{P}R^\nu|^{\frac{p}{2}}\|_2^2 + C. \qquad (2.4.12)$$

Estimate of B_6:

We apply equations (2.1.7) and (2.1.6) to write

$$\begin{aligned} -B_6 =& \int_\Omega \int_{\varphi(x)/\sqrt{\nu}}^\infty \partial_t \mathrm{div}_x u^c(x,z)\mathrm{d}z \cdot n \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x \\ =& \int_\Omega \int_{\varphi(x)/\sqrt{\nu}}^\infty \mathrm{div}_x\{[u^c(x,z)\cdot\nabla u^0 + u^0\cdot\nabla_x u^c(x,z)]_{tan}\}\mathrm{d}z \cdot n \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x \\ &+ \int_\Omega \int_{\varphi(x)/\sqrt{\nu}}^\infty \mathrm{div}_x[fz\cdot\partial_z u^c(x,z)]\mathrm{d}z \cdot n \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x \qquad (2.4.13) \\ &- \int_\Omega \int_{\varphi(x)/\sqrt{\nu}}^\infty \mathrm{div}_x\partial_z^2 u^c(x,z)\mathrm{d}z \cdot n \cdot \mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)\mathrm{d}x \\ &:= B_{6_1} + B_{6_2} + B_{6_3}. \end{aligned}$$

Thanks to the regularity of $u^c(x,t,z)$ and $u^0(x,t)$, by (2.4.7) we bound that

$$|B_{6_1}| \le C\|\mathbb{P}R^\nu\|_p^p + C\|u^c\|_{1,3,0}^p.$$

And with the help of the regularity of f and $\partial_x\partial_z u^c$, we have

$$|B_{6_2}| \le \|\mathbb{P}R^\nu\|_p^p + C\|u^c\|_{1,2,1}^p.$$

Next, we estimate B_{6_3}

$$|B_{6_3}| \le C\|\mathrm{div}_x\partial_z u^c|_{z=\varphi(x)/\sqrt{\nu}}\|_2\|\mathbb{P}R^\nu\|_p^{p-1} \le C\|u^c\|_{1,2,2}^p + \|\mathbb{P}R^\nu\|_p^p.$$

Together with the above estimates of $B_{6_1}, B_{6_2}, B_{6_3}$, we have that

$$|B_6| \le C\|\mathbb{P}R^\nu\|_p^p + C. \qquad (2.4.14)$$

Estimate of $B_7 + B_8 + B_9$:

Utilizing (2.4.7), the regularity of u^c and u^0. It is easy to see that

$$|B_7 + B_8 + B_9| \le C\|\mathbb{P}R^\nu\|_p^p + C + \sqrt{\nu}\|u^c\|_{1,2,0}^2 + \|u^c\|_{1,2,2}^2 + C. \qquad (2.4.15)$$

Estimate of B_{10}:

Since

$$\begin{aligned}B_{10} &\leq |\nu\int_{\Omega}\nabla_x[v(x,\frac{\varphi(x)}{\sqrt{\nu}})]:(|\mathbb{P}R^\nu|^{p-2}\nabla\mathbb{P}R^\nu+\nabla^2 Q)\mathrm{d}x| \\ &+\nu|\int_{\partial\Omega}\partial_n(v(t,x,\frac{\varphi}{\sqrt{\nu}}))\cdot(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu+\nabla Q)\mathrm{d}\sigma \\ &:= B_{10_1}+B_{10_2}.\end{aligned} \tag{2.4.16}$$

Due to Young's inequality, we deduce that

$$\begin{aligned}|B_{10_1}| &\leq C\int_{\Omega}|\nabla[v(x,\frac{\varphi(x)}{\sqrt{\nu}})]|^2|\mathbb{P}R^\nu|^{p-2}\mathrm{d}x+\nu^2\int_{\Omega}|\nabla\mathbb{P}R^\nu|^2|\mathbb{P}R^\nu|^{p-2}\mathrm{d}x \\ &+\nu\|\nabla[v(x,\frac{\varphi(x)}{\sqrt{\nu}})]\|_p\|\nabla^2 Q\|_{p'}\overset{(2.4.7)}{\leq} C\|\mathbb{P}R^\nu\|_p^p+\varepsilon\nu\|\nabla|\mathbb{P}R^\nu|^{\frac{p}{2}}\|_2^2.\end{aligned}$$

Since $\partial_n[v(x,\varphi(x)/\sqrt{\nu})] = \partial_n[\overline{v}(x,\varphi(x)/\sqrt{\nu})]\cdot n+\overline{v}(x,\varphi(x)/\sqrt{\nu})\cdot\partial_n n(x)$ and $\mathbb{P}(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu)$ is tangent to the boundary, we get that

$$\begin{aligned}B_{10_2} &= \nu\int_{\partial\Omega}\overline{v}(x,0)\partial_n n(x)|(\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu+\nabla Q)\mathrm{d}x \\ &\leq \nu\int_{\Omega}\mathrm{div}[(\overline{v}(x,0)\partial_n n(x))n(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu+\nabla Q)]\mathrm{d}x \\ &\leq \nu\int_{\Omega}\nabla[(\overline{v}\cdot\partial_n n(x))n(x)]|(\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu+\nabla Q)\mathrm{d}x \\ &\leq C\|\mathbb{P}R^\nu\|_p^p+C\nu^p\end{aligned}$$

In conclusion

$$|B_{10}|\leq C\|\mathbb{P}R^\nu\|_p^p+\varepsilon\nu\|\nabla|\mathbb{P}R^\nu|^{\frac{p}{2}}\|_2^2+C\nu^p+C. \tag{2.4.17}$$

Estimate of B_{11}: Since $\|u^c\|_{1,2,0,2p}\leq C\|u^c\|_{1,4,1}$, thus

$$\begin{aligned}|B_{11}| &\leq \int_{\Omega}(u^0+\sqrt{\nu}u^c+\nu v+\nu R^\nu)\cdot\nabla_x v\cdot(|\mathbb{P}R^\nu|^{p-2}\mathbb{P}R^\nu+\nabla Q)\mathrm{d}x \\ &\leq C\|u^0\|_\infty\|\mathbb{P}R^\nu\|_p^{p-1}\|\nabla_x v\|_p+\sqrt{\nu}\|u^c\|_{2p}\|\nabla_x v\|_{2p}\|\mathbb{P}R^\nu\|_p^{p-1} \\ &+\nu\|v\|_{2p}\|\nabla_x v\|_{2p}\|\mathbb{P}R^\nu\|_p^{p-1}+\nu\|R^\nu\|_{2p}\|\nabla_x v\|_{2p}\|\mathbb{P}R^\nu\|_p^{p-1} \\ &\leq (\|u^0\|+\|u^c\|_{1,3,0}+\sqrt{\nu}\|u^c\|_{1,3,1}+\|u^c\|_{1,2,0,2p})^p \\ &+\nu\|\mathbb{P}R^\nu\|_{2p}\|u^c\|_{1,2,0,2p}\|\mathbb{P}R^\nu\|_p^{p-1}+C\|\mathbb{P}R^\nu\|_p^p \\ &\leq \varepsilon\nu\|\nabla|\mathbb{P}R^\nu|^{\frac{p}{2}}\|_2^2+\|\mathbb{P}R^\nu\|_p^p+C_2.\end{aligned} \tag{2.4.18}$$

Estimate of $B_{12}+\sum_{\imath=14}^{18}B_\imath$:

Apply the uniform bound of $\partial_z u^c$ the regularity of u^0 and u^c, we have

$$\left|B_{12}+\sum_{\imath=14}^{18}B_\imath\right|\le C\|\mathbb{P}R^\nu\|_p^p+C \tag{2.4.19}$$

Estimate of B_{13}: It is easy to see that $\|u^c\|_{[\frac{p}{2}]+1,2,0,p}\le C$ with $p>3$, since $H^5(\Omega)\hookrightarrow W^{4,p}(\Omega)$

$$\begin{aligned}|B_{13}|&=-\int_\Omega\frac{u^0\cdot n}{\varphi}(z\mathrm{div}_x u^c)|_{z=\frac{\varphi}{\sqrt{\nu}}}\cdot n\cdot(|\mathbb{P}R^\nu|^{p-2}PR^\nu+\nabla Q)\mathrm{d}x\\&\le\|(1+z)\mathrm{div}_x u^c|_{z=\varphi/\sqrt{\nu}}\|_p\|\mathbb{P}R^\nu\|_p^{p-1}\\&\le C\nu^{\frac{1}{2p}}\|u^c\|_{[\frac{p}{2}]+1,2,0,p}\|\mathbb{P}R^\nu\|_p^{p-1}\\&\le C\|\mathbb{P}R^\nu\|_p^p+C\nu^{\frac{1}{2}}.\end{aligned} \tag{2.4.20}$$

To estimate the term B_{19}, since

$$\nabla_x q=-\int_z^\infty((\nabla_x u^c(z))\cdot\nabla u^0+u^0\cdot(\nabla_x^2u^c(z)))\cdot n+(u^c\cdot\nabla u^0+u^0\cdot\nabla_x u^c)\cdot\nabla_x n\mathrm{d}z.$$

Then

$$\|\nabla_x q\|_p\le C\|u^c\|_{L_z^1(R_+,W^{2,p}(\Omega))}\le C\|u^c\|_{1,3,1}.$$

Therefore

$$B_{19}\le C\|\mathbb{P}R^\nu\|_p^p+C_6. \tag{2.4.21}$$

Plugging all the estimates of $B_\imath(\imath=1,\cdots,19),d_\imath(i=1,2,3,4)$ into (2.4.8), and choose ε very small, we can deduce that

$$\frac{d}{dt}\|\mathbb{P}R^\nu\|_p^p+c_0\nu\|\nabla|\mathbb{P}R^\nu|^{\frac{p}{2}}\|_2^2\le C\|\mathbb{P}R^\nu\|_p^p+C+\nu\|\mathbb{P}R^\nu\|_p^{p\frac{p-1}{p-3}}. \tag{2.4.22}$$

Since the number $\dfrac{p-1}{p-3}>1$, by the part (c) of Gronwall's Lemma, we know that there exist a small $0<\nu_0<1$, such that for all $0<\nu\le\nu_0$, and

$$\sup_{0\le t\le T}\|\mathbb{P}R^\nu\|_p^p+c_0\nu\int_0^T\|\nabla|\mathbb{P}R^\nu|^{\frac{p}{2}}\|_2^2\mathrm{d}t\le C.$$

The proof of Theorem 2.1.2 is completed.

2.5 The proof of Theorem 2.1.3

In this section we consider the H^1-estimates for the remainder R^ν and give the proof of theorem 2.1.3. Since the boundary conditions of R^ν is not homogenous, so Lemma 2.2.3 is not used in this estimate. To this end, we can define an new function on the $\Omega \times (0,T)$ related R^ν as follows

$$R(t,x) = R^\nu(t,x) + b(t,x),$$

Then we have the following conditions

$$\begin{aligned}
&\mathrm{div} R(t,x) = -\mathrm{div}_x v(t,x,\frac{\varphi(x)}{\sqrt{\nu}}) + \mathrm{div} b(t,x), && \text{in } \Omega,\\
&R\cdot n = 0, && \text{on } \partial\Omega,\\
&\mathrm{curl} R \times n = 0, && \text{on } \partial\Omega,\\
&R(0,x) = 0 && \text{in } \Omega.
\end{aligned}$$

Therefore, it is trivial to verify that R satisfies the following equations

$$\begin{aligned}
&\partial_t R - \nu\Delta R + u^\nu\cdot\nabla R + R\cdot\nabla u^0 + \sqrt{\nu} R\cdot n\partial_z v + R\cdot n\partial_z u^c + \sqrt{\nu}R\cdot\nabla_x u^c\\
&= R.H.S + \partial_t b - \nu\Delta b + u^\nu\nabla b + b\nabla u^0 + \sqrt{\nu} b\cdot n\partial_z v + b\cdot n\partial_z u^c + \sqrt{\nu} b\cdot\nabla_x u^c.
\end{aligned} \tag{2.5.1}$$

By the L^p bound of R^ν, we know that if $0 < \nu \le \nu_0$

$$\|R^\nu\|_{L^\infty(0,T;L^p(\Omega))} \le C,\ 3 < p \le 6 \tag{2.5.2}$$

$$\|b\|_{L^\infty(0,T;H^1)} \le C\nu^{-\frac{1}{2}}, \tag{2.5.3}$$

$$\|R\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathrm{div} R\|_{L^\infty(0,T;L^2(\Omega))} \le C\nu^{-\frac{1}{2}}. \tag{2.5.4}$$

From (2.5.2)-(2.5.4) and Lemma 2.2.2, we know that

$$\begin{aligned}
&\|R\|_{1,2} \le \|\mathrm{curl} R\|_2 + C\nu^{-\frac{1}{2}},\\
&\|R\|_{2,2} \le \|\nabla\mathrm{curl} R\|_2 + C\nu^{-\frac{1}{2}}.
\end{aligned}$$

Therefore, it suffices to estimate the bound of $\|\mathrm{curl} R\|_{L^\infty(0,T;L^2(\Omega))}$.

Apply the operator curl to (2.5.1) and set $\mathrm{curl} R = \omega$, then we have

$$\begin{aligned}&\partial_t\omega - \nu\Delta\omega + \mathrm{curl}(u^\nu\cdot\nabla R) + \mathrm{curl}(R\cdot\nabla u^0) + \\ &\sqrt{\nu}\mathrm{curl}(R\cdot n\partial_z v) + \mathrm{curl}(R\cdot n\partial_z u^c) + \sqrt{\nu}\mathrm{curl}(R\cdot\nabla_x u^c) \\ &= \mathrm{curl} R.H.S + \mathrm{curl}\partial_t b - \nu\Delta\mathrm{curl} b + \mathrm{curl}(u^\nu\cdot\nabla b) + \mathrm{curl}(b\cdot\nabla u^0) \\ &+ \sqrt{\nu}\mathrm{curl}(b\cdot n\partial_z v) + \mathrm{curl}(b\cdot n\partial_z u^c) + \sqrt{\nu}\mathrm{curl}(b\cdot\nabla u^c).\end{aligned} \tag{2.5.5}$$

Multiply (2.4.9) by ω and integral on Ω, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^2 - \nu\int_\Omega \Delta\omega\cdot\omega\mathrm{d}x = \int_\Omega \mathrm{curl}(R.H.S.)\cdot\omega\mathrm{d}x + \sum_{i=1}^{12} E_i. \tag{2.5.6}$$

First, we estimate the Laplacian term on the left hand side. Due to Lemma 2.2.3 and Theorem 1.1.7, we have

$$\begin{aligned}&-\nu\int_\Omega \Delta\omega\cdot\omega\mathrm{d}x = \nu\int_\Omega |\nabla\omega|^2\mathrm{d}x - \nu\int_{\partial\Omega}(n\cdot\nabla\omega)\cdot\omega\mathrm{d}\sigma \\ &\geq (1-\varepsilon)\nu\int_\Omega |\nabla\omega|^2\mathrm{d}x - C\nu\|\omega\|_2^2.\end{aligned} \tag{2.5.7}$$

Next we bound the term on the right hand side of (2.5.6) one by one.

$$\begin{aligned}&E_1 = \int_\Omega \mathrm{curl}(u^\nu\cdot\nabla R)\cdot\omega\mathrm{d}x = \int_\Omega \mathrm{curl}((u^\nu - u^0)\cdot\nabla R)\cdot\omega\mathrm{d}x + \int_\Omega \mathrm{curl}(u^0\cdot\nabla R)\cdot\omega\mathrm{d}x \\ &= \sqrt{\nu}\int_\Omega \mathrm{curl}(u^c\cdot\nabla R)\omega\mathrm{d}x + \nu\int_\Omega \mathrm{curl}(v\cdot\nabla R)\omega\mathrm{d}x + \nu\int_\Omega \mathrm{curl}(R^\nu\nabla R)\omega\mathrm{d}x \\ &+ \int_\Omega \mathrm{curl}(u^0\cdot\nabla R)\cdot\omega\mathrm{d}x := \mathbb{J}_1 + \mathbb{J}_2 + \mathbb{J}_3 + \mathbb{J}_4.\end{aligned}$$

Since $\omega\times n = 0$ on $\partial\Omega$, integral by parts, then

$$\begin{aligned}&|\mathbb{J}_1| = \sqrt{\nu}|\int_\Omega (u^c\cdot\nabla R)\mathrm{curl}\omega\mathrm{d}x| \leq \sqrt{\nu}\|u^c(t,x,\frac{\varphi(x)}{\sqrt{\nu}})\|_{L^\infty(\Omega)}\|\nabla R\|_2\|\mathrm{curl}\omega\|_2 \\ &\leq \|u^c\|_{1,2,1}^2\|\omega\|_2^2 + C\|u^c\|_{1,2,1}^2\nu^{-1} + \frac{1}{3}\varepsilon\nu\|\mathrm{curl}\omega\|_2^2. \\ &|\mathbb{J}_2| \leq \nu\|v\|_6\|\nabla R\|_3\|\mathrm{curl}\omega\|_2 \leq \nu\|v\|_6\|\nabla R\|_2^{\frac{1}{2}}\|\nabla\omega\|_2^{\frac{3}{2}} \\ &\leq C\nu\|u^c\|_{1,2,0}^4\|\omega\|_2^2 + C + \frac{1}{3}\varepsilon\nu\|\nabla\omega\|_2^2, \\ &|\mathbb{J}_4| = \int_\Omega \mathrm{curl}u^0\nabla R\cdot\omega\mathrm{d}x + \int_\Omega u^0\cdot\nabla\omega\omega\mathrm{d}x \\ &= \|\mathrm{curl}u^0\|_\infty\|\nabla R\|_2\|\omega\|_2 \leq \|\mathrm{curl}(u^0\|_\infty + 1)\|\omega\|_2^2 + \|\mathrm{curl}u^0\|_\infty^2\nu^{-1}.\end{aligned}$$

It is more complicated to estimate the term J_3, from(2.5.2) for $p = 4$,

$$
\begin{aligned}
&|\mathbb{J}_3| \leq \nu|\int_\Omega R^\nu \cdot \nabla R \mathrm{curl}\omega \mathrm{d}x| \leq \nu\|R^\nu\|_4\|\nabla R\|_4\|\mathrm{curl}\omega\|_2\\
&\leq \nu\|\nabla R^\nu\|_2^{\frac{1}{4}}\|\nabla\omega\|_2^{\frac{7}{4}} + \nu^{\frac{5}{8}}\|\nabla R^\nu\|_2^{\frac{1}{4}}\|\nabla\omega\|_2\\
&\leq C\nu\|\omega + \nabla b\|_2^{\frac{1}{4}}\|\nabla\omega\|_2^{\frac{7}{4}} + \nu^{\frac{5}{8}}\|\omega + \nabla b\|_2^{\frac{1}{4}}\|\nabla\omega\|_2\\
&\leq \frac{1}{3}\varepsilon\nu\|\nabla\omega\|_2^2 + C\|\omega\|_2^2 + C.
\end{aligned}
$$

Since Lemma 2.3.3-Lemma 2.3.6 and $0 < \nu \leq \nu_0$, therefore, we obtain the estimate of E_1 as follows

$$
E_1 \leq C\|\omega\|_2^2 + \varepsilon\nu\|\nabla\omega\|_2^2 + C + C\nu^{-1}. \tag{2.5.8}
$$

It is easy to estimate the term E_2:

$$
\begin{aligned}
|E_2| &= |\int_\Omega \mathrm{curl}(R \cdot \nabla u^0) \cdot \omega \mathrm{d}x| \leq \left|\int_\Omega \omega \cdot \nabla u^0 \omega \mathrm{d}x\right| + \left|\int_\Omega R \cdot \nabla \mathrm{curl} u^0 \cdot \omega \mathrm{d}\right|\\
&\leq \|\nabla u^0\|_\infty\|\omega\|_2^2 + C\|R\|_6\|\nabla\mathrm{curl}u^0\|_3\|\omega\|_2\\
&\leq C(\|\nabla u^0\|_\infty + \|u^0\|_{3,2})\|\omega\|_2^2 + C\|u^0\|_{3,2}^2\nu^{-1},\\
&\leq C\|\omega\|_2^2 + C\nu^{-1}.
\end{aligned} \tag{2.5.9}
$$

Let me compute the term E_3,

$$
\begin{aligned}
E_3 &= \sqrt{\nu}\int_\Omega \mathrm{curl}(R \cdot n\partial_z v)\mathrm{d}x\\
&= \sqrt{\nu}\left(\int_\Omega R \cdot n\mathrm{curl}\partial_z v\omega \mathrm{d}x + \int_\Omega \nabla(R \cdot n) \times \partial_z v\omega \mathrm{d}x\right)\\
&:= \mathbb{K}_1 + \mathbb{K}_2.
\end{aligned}
$$

Now we estimate $\mathbb{K}_1, \mathbb{K}_2$ respectively

$$
\begin{aligned}
&|\mathbb{K}_2| = \sqrt{\nu}\int_\Omega \nabla(R \cdot n) \times \partial_z v\omega \mathrm{d}x\\
&\leq \sqrt{\nu}\|\nabla(R \cdot n)\|_2\|\partial_z v\|_6\|\omega\|_3 \leq \sqrt{\nu}C\|\omega\|_2^{\frac{1}{2}}\|\nabla(R \cdot n)\|_2\|\nabla\omega\|_2^{\frac{1}{2}}\|u^c\|_{1,2,1}\\
&\leq C\nu^{-\frac{1}{2}} + C\|\omega\|_2^2 + \frac{1}{2}\varepsilon\nu\|\nabla\omega\|_2^2.
\end{aligned}
$$

$$\begin{aligned}|\mathbb{K}_1| &\leq \sqrt{\nu}\left|\int_\Omega R\cdot n\mathrm{curl}_x\partial_z v\omega \mathrm{d}x\right| + \sqrt{\nu}\left|\int_\Omega R\cdot n\partial_z^2 v\times n\omega\mathrm{d}x\right| \\ &= \sqrt{\nu}\left|\int_\Omega R\cdot n\mathrm{curl}_x(n\mathrm{div}_x u^c)\omega\mathrm{d}x\right| + \sqrt{\nu}\left|\int_\Omega R\cdot n\partial_z\mathrm{div}_x u^c\times n\omega\mathrm{d}x\right| \\ &\leq \sqrt{\nu}\|R\|_6\|\mathrm{curl}_x(n\mathrm{div}_x u^c)\|_2\|\omega\|_3 + \sqrt{\nu}\|R\|_6\|n\times\partial_z\mathrm{div}_x u^c)\|_2\|\omega\|_3 \\ &\leq C\sqrt{\nu}\|\nabla R\|_2(\|\mathrm{curl}_x(n\mathrm{div}_x u^c)\|_2 + \|n\times\partial_z\mathrm{div}_x u^c)\|_2)\|\omega\|_2^{\frac{1}{2}}\|\nabla\omega\|_2^{\frac{1}{2}} \\ &\leq C + C\|\omega\|_2^2 + \frac{1}{2}\varepsilon\nu\|\nabla\omega\|_2^2.\end{aligned}$$

Hence, we can obtain that

$$E_3 \leq C + C\nu^{-\frac{1}{2}} + C\|\omega\|_2^2 + \varepsilon\nu\|\nabla\omega\|_2^2. \tag{2.5.10}$$

We now calculate the the integral E_4

$$\begin{aligned}E_4 &= \int_\Omega \mathrm{curl}(R\cdot n\partial_z u^c)\omega\mathrm{d}x \\ &= \int_\Omega \nabla(R\cdot n)\times\partial_z u^c\omega\mathrm{d}x + \int_\Omega R\cdot n\mathrm{curl}_x\partial_z u^c\omega\mathrm{d}x \\ &+ \frac{1}{\sqrt{\nu}}\int_\Omega R\cdot n(\partial_z^2 u^c\times n)\cdot\omega dx.\end{aligned}$$

Thus, since $\|R\|_{L^\infty(0,T,L^4(\Omega))} \leq C$, we have

$$\begin{aligned}|E_4| &\leq \|\partial_z u^c\|_\infty\|\nabla(R\cdot n)\|_2\|\omega\|_2 + \|R\|_6\|\mathrm{curl}_x\partial_z u^c\|_3\|\omega\|_2 \\ &+ \left|\frac{1}{\sqrt{\nu}}\int_\Omega R\cdot n\partial_z^2 u^c\cdot(n\times\omega)\mathrm{d}x\right| \\ &\leq C\|\omega\|_2^2 + C\nu^{-1} + \frac{1}{\sqrt{\nu}}\left|\int_\Omega R\cdot n\partial_z^2 u^c|\cdot(n\times\omega)\mathrm{d}x\right| \qquad (2.5.11) \\ &\leq C\|\omega\|_2^2 + C\nu^{-1} + \frac{1}{\sqrt{\nu}}\|R\cdot n\|_4\|\partial_z^2 u^c\|_4\|\omega\|_2 \\ &\leq C\|\omega\|_2^2 + C\nu^{-1} + \nu^{-1}(\|u^c\|_{1,1,3}^2 + 1)\end{aligned}$$

Integral by parts, we obtain

$$\begin{aligned}|E_5| &= \left|-\sqrt{\nu}\int_\Omega R\cdot\nabla_x u^c\mathrm{curl}\omega\mathrm{d}x\right| \leq \sqrt{\nu}\|R\|_6\|\nabla_x u^c\|_3\|\mathrm{curl}\omega\|_2 \\ &\leq \sqrt{\nu}\|\nabla_x u^c\|_{1,2,1}\|\nabla R\|_2\|\mathrm{curl}\omega\|_2 \leq C\nu^{-1} + \|\omega\|_2^2 + \varepsilon\nu\|\nabla\omega\|_2^2.\end{aligned}$$

Since $\partial_t b(x,t) = \partial_t v(t,x,0) + \frac{1}{\sqrt{\nu}}\partial_t u^c(t,x,0)$, hence

$$\begin{aligned}\|\partial_t \mathrm{curl} b(x,t)\|_2 &= \|\partial_t \mathrm{curl}_x v(t,x,0) + \frac{1}{\sqrt{\nu}}\partial_t \mathrm{curl}_x u^c(t,x,0)\|_2\\ &\le \|\partial_t \mathrm{curl}_x v(t,x,0)\|_2 + \frac{1}{\sqrt{\nu}}\|\partial_t \mathrm{curl}_x u^c(t,x,0)\|_2\\ &\le C\|\partial_t u^c\|_{1,2,0} + \frac{1}{\sqrt{\nu}}\|\partial_t u^c\|_{1,1,1}\end{aligned}$$

Therefore, we can bound E_6 as follows,

$$|E_6| \le \|\partial_t \mathrm{curl} b(x,t)\|_2\|\omega\|_2 \le (\|\partial_t u^c\|_{1,2,0}^2 + \nu^{-1}\|\partial_t u^c\|_{1,1,1}^2) + \|\omega\|_2^2. \tag{2.5.12}$$

Integral by parts, then $E_7 = \nu\int_\Omega \Delta_x b\cdot \mathrm{curl}\omega \mathrm{d}x$. Whence

$$\begin{aligned}|E_7| &\le \nu\|\Delta b(x,t)\|_2\|\mathrm{curl}\omega\|_2 \le \nu(\|\Delta_x v(t,x,0)\|_2 + \frac{1}{\sqrt{\nu}}\|\Delta_x u^c(t,x,0)\|_2)\|\mathrm{curl}\omega\|_2\\ &\le C(\nu\|u^c\|_{1,3,0}^2 + \|u^c\|_{1,2,1}^2) + \varepsilon\nu\|\nabla\omega\|_2^2.\end{aligned} \tag{2.5.13}$$

As the argument in the estimate of E_1, one follows that

$$\begin{aligned}|E_8| &= \left|\int_\Omega \mathrm{curl}((u^\nu - u^0)\cdot\nabla b)\cdot\omega \mathrm{d}x + \int_\Omega \mathrm{curl}(u^0\cdot\nabla b)\cdot\omega \mathrm{d}x\right|\\ &\le \left|-\int_\Omega((u^\nu - u^0)\cdot\nabla b)\cdot \mathrm{curl}\omega \mathrm{d}x\right| + \|u^0\|_{1,\infty}\|b\|_{H^2}\|\omega\|_2\\ &\le \left|\sqrt{\nu}\int_\Omega u^c\cdot\nabla b\cdot \mathrm{curl}\omega \mathrm{d}x\right| + \nu\left|\int_\Omega v\cdot\nabla b\cdot \mathrm{curl}\omega \mathrm{d}x\right|\\ &+ \nu\left|\int_\Omega R^\nu\cdot\nabla b\cdot \mathrm{curl}\omega \mathrm{d}x\right| + \|u^0\|_{1,\infty}\|b\|_{H^2}\|\omega\|_2\\ &\le \sqrt{\nu}\|u^c\|_\infty\|\nabla b\|_2\|\mathrm{curl}\omega\|_2 + \nu\|v\|_6\|\nabla b\|_3\|\mathrm{curl}\omega\|_2\\ &+ \|u^0\|_{1,\infty}\|b\|_{H^2}\|\omega\|_2 + \nu\left|\int_\Omega R^\nu\cdot\nabla b\cdot \mathrm{curl}\omega \mathrm{d}x\right|\\ &\le \varepsilon\nu\|\mathrm{curl}\omega\|_2^2 + C(\|u^c\|_\infty^2\|\nabla b\|_2^2\\ &+ \|v\|_6^2\|\nabla b\|_3^2) + \|u^0\|_{1,\infty}\|b\|_{H^2}\|\omega\|_2 + \nu\left|\int_\Omega R^\nu\cdot\nabla b\cdot \mathrm{curl}\omega \mathrm{d}x\right|\end{aligned}$$

We estimate the last term of right side above,

$$\begin{aligned}\nu\left|\int_\Omega R^\nu\cdot\nabla b\cdot \mathrm{curl}\omega \mathrm{d}x\right| &\le \nu\|R^\nu\|_4\|\nabla b\|_4\|\mathrm{curl}\omega\|_2\\ &\le \varepsilon\nu\|\nabla\omega\|_2^2 + C.\end{aligned}$$

Since $\|b\|_\infty + \|b\|_{H^2} \le C\|u^c\|_{1,2,1}\nu^{-\frac{1}{2}}$, thus

$$|E_8| \le \varepsilon\nu\|\nabla\omega\|_2^2 + C\|\omega\|_2^2 + C\nu^{-1} \tag{2.5.14}$$

It is easy to verify that E_9 as follows

$$|E_9| = \left|\int_\Omega (\text{curl}b \cdot \nabla u^0 - b \cdot \nabla\text{curl}u^0) \cdot \omega \text{d}x\right| \le C\|\omega\|_2^2 + C\nu^{-1} \tag{2.5.15}$$

Since $u^c \cdot n = 0$, we easily obtain

$$\begin{aligned} |E_{10} + E_{11} + E_{12}| \le & \left|\sqrt{\nu}\int_\Omega \bar{v}(t,x,0)\text{div}_x u^c)\text{curl}\omega\text{d}x + \int_\Omega \bar{v}(t,x,0)\partial_z u^c\text{curl}\omega\text{d}x \right.\\ & \left. +\sqrt{\nu}\int_\Omega b \cdot \nabla_x u^c \cdot \text{curl}\omega\text{d}x\right| \\ & \le \varepsilon\nu\|\nabla\omega\|_2^2 + C(\|u^c\|_{1,2,1}^4 + 1)\nu^{-1}. \end{aligned} \tag{2.5.16}$$

It remains to estimate the term $\int_\Omega \text{curl}(R.H.S) \cdot \omega\text{d}x$. Set

$$\int_\Omega \text{curl}(R.H.S) \cdot \omega\text{d}x = \sum_{i=1}^{14} D_i.$$

Where

$$\begin{aligned}
D_1 &= -\int_\Omega \text{curl}(\partial_t v) \cdot \omega\text{d}x, & D_2 &= \int_\Omega \text{curl}\triangle u^0 \cdot \omega\text{d}x,\\
D_3 &= \sqrt{\nu}\int_\Omega \text{curl}[\triangle_x u^c] \cdot \omega\text{d}x, & D_4 &= \int_\Omega \text{curl}(2n \cdot \nabla_x\partial_z u^c) \cdot \omega\text{d}x,\\
D_5 &= \nu\int_\Omega \text{curl}(\triangle_x[v(x, \frac{\varphi(x)}{\sqrt{\nu}}]) \cdot \omega\text{d}x, & D_6 &= -\int_\Omega \text{curl}(u^\nu \cdot \nabla_x v) \cdot \omega\text{d}x,\\
D_7 &= -\int_\Omega \text{curl}(v \cdot \nabla u^0) \cdot \omega\text{d}x, & D_8 &= -\frac{1}{\sqrt{\nu}}\int_\Omega (u^0 \cdot n\partial_z v) \cdot \omega\text{d}x,\\
D_9 &= -\sqrt{\nu}\int_\Omega \text{curl}(v \cdot n\partial_z v) \cdot \omega\text{d}x, & D_{10} &= -\int_\Omega \text{curl}(v \cdot n\partial_z u^c) \cdot \omega\text{d}x,\\
D_{11} &= -\int_\Omega \text{curl}(u^c \cdot \nabla_x u^c) \cdot \omega\text{d}x, & D_{12} &= \int_\Omega \text{curl}(\triangle\varphi \cdot \partial_z u^c) \cdot \omega\text{d}x,\\
D_{13} &= -\sqrt{\nu}\int_\Omega \text{curl}(v \cdot \nabla_x u^c) \cdot \omega\text{d}x, & D_{14} &= \frac{1}{\sqrt{\nu}}\int_\Omega (\partial_z\nabla_x q \times n) \cdot \omega\text{d}x.
\end{aligned}$$

As in the argument in the estimate of E_6, we can infer

$$|D_1| \leq C(\|\partial_t u^c\|_{1,2,0}^2)\nu^{-\frac{1}{2}} + C\|\omega\|_2^2 \tag{2.5.17}$$

Integral by parts and the regularity of u^0, we have the following estimates,

$$\begin{aligned}
|D_2| &= \left|\int_\Omega \text{curl}\Delta u^0 \cdot \omega\right| \leq C + \|\omega\|_2^2, \\
|D_3| &= \sqrt{\nu}\left|\int_\Omega \Delta_x u^c \text{curl}\omega \text{d}x\right| \leq \sqrt{\nu}\|\Delta_x u^c\|_2\|\text{curl}\omega\|_2 \\
&\leq C\|u^c\|_{0,2,0}^2 + \varepsilon\nu\|\text{curl}\omega\|_2^2
\end{aligned} \tag{2.5.18}$$

Since $\text{curl}_x[2n\cdot\nabla_x\partial_z u^c] = 2\text{curl}n\cdot\nabla_x\partial_z u^c - 2n\cdot\nabla_x\text{curl}_x u^c - 2n\cdot\nabla_x(\partial_z^2 u^c \times n)\frac{1}{\sqrt{\nu}}$, thus

$$|D_4| \leq C\|u^c\|_{0,1,2}^2\nu^{-\frac{1}{2}} + \varepsilon\nu\|\nabla\omega\|_2^2. \tag{2.5.19}$$

Since

$$\begin{aligned}
\Delta_x\left[h(x, \frac{\varphi(x)}{\sqrt{\nu}})\right] &= \Delta_x h(x, \frac{\varphi(x)}{\sqrt{\nu}}) + 2\frac{n(x)}{\sqrt{\nu}}\cdot\nabla_x\partial_z h(x, \frac{\varphi(x)}{\sqrt{\nu}}) + \frac{\Delta\varphi}{\sqrt{\nu}}\partial_z h(x, \frac{\varphi(x)}{\sqrt{\nu}}) \\
&\quad + \frac{1}{\nu}\partial_z^2 h(x, \frac{\varphi(x)}{\sqrt{\nu}})
\end{aligned}$$

Therefore, we can estimate the D_5 to obtain

$$\begin{aligned}
|D_5| &\leq \nu\left|\int_\Omega \Delta_x\left[v(t,x,\frac{\varphi}{\sqrt{\nu}})\right]\text{curl}\omega\text{d}x\right| \leq \nu\left|\int_\Omega\left(\Delta_x v(t,x,\frac{\varphi}{\sqrt{\nu}})\right.\right. \\
&+ 2\frac{n(x)}{\sqrt{\nu}}\cdot\nabla_x\partial_z v(t,x,\frac{\varphi(x)}{\sqrt{\nu}}) + \frac{\Delta\varphi}{\sqrt{\nu}}\partial_z v(t,x,\frac{\varphi(x)}{\sqrt{\nu}}) \\
&\left.\left.+\frac{1}{\nu}\partial_z^2 v(t,x,\frac{\varphi(x)}{\sqrt{\nu}})\right)\text{curl}\omega\text{d}x\right| \\
&\leq C\|u^c\|_{1,3,0}^2 + \frac{1}{2}\varepsilon\nu\|\nabla\omega\|_2^2 + \left|\int_\Omega n\text{div}_x\partial_z u^c\cdot\text{curl}\omega\text{d}x\right| \\
&\leq C(\|u^c\|_{1,3,0}^2 + \|u^c\|_{0,1,1}^2\nu^{-\frac{1}{2}}) + \varepsilon\nu\|\nabla\omega\|_2^2.
\end{aligned} \tag{2.5.20}$$

By Theorem 2.1.2, it is easy to see that

$$|D_6| = \left|\int_\Omega \text{curl}(u^\nu\cdot\nabla_x v)\cdot\omega\text{d}x\right| \leq C\|u^c\|_{1,3,1}^2 + \varepsilon\nu\|\text{curl}\omega\|_2^2. \tag{2.5.21}$$

Directly compute, it is easy to see that

$$|D_7| \leq C\|\omega\|_2. \tag{2.5.22}$$

From the regularity of f and integral by parts, we have

$$\begin{aligned} |D_8| &= \left| \int_\Omega \frac{u^0 \cdot n}{\varphi} (z\partial_z v)|_{z=\frac{\varphi(x)}{\sqrt{\nu}}} \cdot \mathrm{curl}\omega \mathrm{d}x \right| \\ &\leq \|f\|_\infty \|u^c\|_{2,1,0} \|\mathrm{curl}\omega\|_2 \leq C\nu^{-\frac{1}{2}} + \varepsilon\nu\|\nabla\omega\|_2^2. \end{aligned} \tag{2.5.23}$$

Integral by parts and the uniform bound of $\partial_z u^c$ and $\Delta\varphi$, we also obtain that

$$|\sum_{i=9}^{13} D_i| \leq C\nu^{-\frac{1}{2}} + C + \varepsilon\nu\|\mathrm{curl}\omega\|_2^2, \tag{2.5.24}$$

where C depends only on $\|u^c\|_{L^\infty(0,T;H^{1,3,1})}$. Next, we must estimate the last term D_{14}. Since $\mathrm{curl}[\nabla_x q] = \partial_z \nabla_x q \times n$, thus, we have

$$\begin{aligned} |D_{14}| &= \frac{1}{\sqrt{\nu}} \left| \int_\Omega (\partial_z \nabla_x q \times n) \cdot \omega \mathrm{d} \right| \leq \|\nabla_x \partial_z q\|_2 \|\omega\|_2 \\ &\leq \frac{1}{\sqrt{\nu}} \|\nabla \left((u^0 \cdot \nabla u^c + u^c \cdot \nabla u^0) \cdot n \right) \|_2 \|\omega\|_2 \\ &\leq C\nu^{-\frac{1}{2}} \|u^c\|_{0,2,0}^2 + \|\omega\|_2^2. \end{aligned} \tag{2.5.25}$$

In addition, by the Lemma 2.3.3-Lemma 2.3.6, we know the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_2^2 + (1-13\varepsilon)\nu \iint_{\Omega\times\mathbb{R}_+} |\nabla\omega|^2 \mathrm{d}x \leq C\|\omega\|_2^2 + C(\|\partial_t u^c\|_{1,1,1}^2 + \|u^c\|_{1,1,3}^2 + 1)\nu^{-1}. \tag{2.5.26}$$

Since$\|\partial_t u^c\|_{1,1,1}^2 + \|u^c\|_{1,1,3}^2 + 1$ is bound in $L^1(0,T)$, by the part (b) of Gronwall's Lemma, for the $0 < \nu_0 \leq 1$ as in section, such that for all $\nu \in (0, \nu_0)$, we have

$$\|R^\nu\|_{1,2} \leq \|R\|_{1,2} + \|b\|_{1,2} \leq C\nu^{-\frac{1}{2}}.$$

Since

$$\nabla_x[u^c(t,x,\frac{\varphi(x)}{\sqrt{\nu}})] = \nabla_x u^c(t,x,\frac{\varphi(x)}{\sqrt{\nu}}) + \frac{n(x)}{\sqrt{\nu}} \otimes \partial_z u^c(t,x,\frac{\varphi(x)}{\sqrt{\nu}}),$$

and $v(t,x,z) = n\int_z^\infty \mathrm{div}_x u^c(t,x,\eta)\mathrm{d}\eta$. Thus from Lemma 2.4.1, we know that

$$\|\nabla_x[v(t,x,\frac{\varphi}{\sqrt{\nu}})]\|_2 \leq \|u^c\|_{1,2,0} + \nu^{-\frac{1}{4}}\|u^c\|_{0,2,0}.$$

Consequently, we have got for any $0 < \nu \leq \nu_0$

$$\begin{aligned} &\sup_{0\leq t\leq T} \|u^\nu - u^0\|_{1,2} \leq C\nu^{\frac{1}{4}}, \\ &\int_0^T \|u^\nu - u^0 - \sqrt{\nu}u^{\mathrm{c}}\|_{2,2}^2 \mathrm{d}t \leq C. \end{aligned} \tag{2.5.27}$$

Thus the Theorem 2.1.3 is proved.

Chapter 3

The existence of weak solution for a class of non-Newtonian fluid with slip boundary on the half space

In this chapter,we consider the non-stationary problems of a class of non-Newtonian fluid which is a power law fluid with $p > \dfrac{3n}{n+2}$ in the half space with slip boundary conditions. We construct the approximation solutions to some auxiliary problem by regularizing the convection term. Using the difference method, we improve the regularity of weak solution to this regularized problem. Furthermore, the existence of approximation solutions is obtained by these regularity estimates and Galerkin method. Applying the the local pressure estimate with the Navier's type slip boundary conditions, and an $L^{\infty}-$ truncation method, we can prove the sequence of approximation solution converges to the required weak solution for origin system

3.1 Introduction

In this chapter, we consider unsteady flows of an incompressible fluid described by the system

$$\begin{cases} \partial_t u - \mathrm{div} S(D(u)) + (u \cdot \nabla)u + \nabla \pi = f, & \text{in } \mathbb{R}^3_+ \times (0,T) \\ \nabla \cdot u = 0, & \text{in } \mathbb{R}^3_+ \times (0,T), \\ u|_{t=0} = u_0(x), & \text{in } \mathbb{R}^3_+, \end{cases} \tag{3.1.1}$$

where u is the velocity, π is pressure and f is the force, and $S(\cdot)$ is nonlinear stress tensor, which defined in Definition 3.1.1. We can impose the following slip boundary conditions

$$u \cdot n|_{x_3=0} = 0, \quad ((S(D(u)) \cdot n) - (n \cdot S(D(u)) \cdot n)n)|_{x_3=0} = 0. \tag{3.1.2}$$

In fact, this problem corresponds to the free boundary problem for the non-Newtonian fluids with free surface supposed invariable.

In this problem (3.1.1), there are two main difficulties. At first, allowing for a non-constant viscosity always brings new complications to analysis, as the equation now becomes nonlinear in the leading term and the boundary conditions are not clear. Other obstacle comes from the domain is not bounded. In the unbounded domain, some required compact results do not always hold. In these years, there are few results in unbounded domain to non-Newtonian fluid. For example, M.Pokorný consider the existence of weak solution for the Cauchy problem in [93] and P. Galdi et.al(see [50], [84], [16]) obtained the existence theorems of the steady flows for shear-rate liquids in exterior two-dimensional domains. Here, we show how to generalize these existence results to half space in $\mathbb{R}^3$.

To overcome these difficulties, we begin to investigate the regularization of the convection term to obtain a new system. Fix a positive number $\varepsilon > 0$ and cut the mirror reflection of the vector u by a characterized function on the ball

$\{x \in \mathbb{R}^3 : |x| \leq \frac{1}{\varepsilon}\}$. Then we can find a sequence of smooth functions$\{u_\varepsilon\}$ which is divergence-free and converges to the function u in suitable function spaces. Therefore, for any $\varepsilon > 0$, we will get the following system

$$\begin{cases} \partial_t u - \mathrm{div} S(D(u)) + (u_\varepsilon \cdot \nabla)u + \nabla \pi = f, & \text{in } \mathbb{R}^3_+ \times (0,T) \\ \nabla \cdot u = 0, & \text{in } \mathbb{R}^3_+ \times (0,T), \\ u|_{t=0} = u_0(x), & \text{in } \mathbb{R}^3_+. \end{cases} \tag{3.1.3}$$

Hence, the convection term of the system above becomes more regular. It is convenient to obtain the regularity, uniqueness and existence of the solutions for above system.

To avoid the complication of the boundary conditions, we choose the stress tensor induced by a $p-$potential as the following definition:

Let $M^{n\times n}$ be the vector space of all symmetric $n \times n$ matrices $\xi = (\xi_{ij})$. We equip $M^{n\times n}$ with scalar product $\xi : \eta = \sum_{i,j=1}^{n} \xi_{ij}\eta_{ij}$ and norm $|\xi| = (\xi : \eta)^{\frac{1}{2}}$.

Definition 3.1.1 *Let $p > 1$ and let $F : \mathbb{R}_+ \bigcup\{0\} \to \mathbb{R}_+ \bigcup\{0\}$ be a convex function, which is C^2 on the $\mathbb{R}_+ \bigcup\{0\}$, such that $F(0) = 0$, $F'(0) = 0$. Assume that the induced function $\Phi : M^{n\times n} \to \mathbb{R}_+ \bigcup\{0\}$, defined through $\Phi(B) = F(|B|)$, satisfies*

$$\sum_{jklm} (\partial_{jk}\partial_{lm}\Phi)(B)C_{jk}C_{lm} \geq \gamma_1(1 + |B|^2)^{\frac{p-2}{2}}|C|^2, \tag{3.1.4}$$

$$|(\nabla^2_{n\times n}\Phi)(B)| \leq \gamma_2(1 + |B|^2)^{\frac{p-2}{2}} \tag{3.1.5}$$

*for all $B, C \in M^{n\times n}$ with constants $\gamma_1, \gamma_2 > 0$. Such a function F, resp. Φ, is called a $p-$**potential**.*

We define the extra stress S induced by F, resp. Φ, by

$$S(B) = \nabla^2_{n\times n}\Phi(B) = F'(|B|)\frac{B}{|B|}$$

for all $B \in M^{n\times n} \setminus \{\mathbf{0}\}$ From (3.1.4), (3.1.5) and $F'(0) = 0$, it easy to know that S can be continuously extended by $S(\mathbf{0}) = \mathbf{0}$.

As in the [33] and [79], one can obtain from (3.1.4) and (3.1.5) the following properties of S.

Theorem 3.1.2 *There exist constants $c_1, c_2 > 0$ independent of γ_1, γ_2 such that for all $B, C \in M^{n\times n}$ there holds*

$$S(\mathbf{0}) = \mathbf{0}, \tag{3.1.6}$$

$$\sum_{i,j}(S_{ij}(B) - S_{ij}(C))(B_{ij} - C_{ij}) \geq c_1\gamma_1(1 + |B|^2 + |C|^2)^{\frac{p-2}{2}}|B - C|^2,$$

$$\sum_{i,j} S_{ij}(B)B_{ij} \geq c_1\gamma_1(1 + |B|^2)^{\frac{p-2}{2}}|B|^2, \tag{3.1.7}$$

$$|S(B) - S(C)| \leq c_2\gamma_2(1 + |B|^2 + |C|^2)^{\frac{p-2}{2}}|B - C|, \tag{3.1.8}$$

$$|S(B)| \leq c_2\gamma_2(1 + |B|^2)^{\frac{p-2}{2}}|B|. \tag{3.1.9}$$

Therefore, we will obtain the equivalent conditions:

$$u_3|_{x_3=0} = \left.\frac{\partial u_i}{\partial x_3}\right|_{x_3=0} = 0 \ (i = 1, 2). \tag{3.1.10}$$

From these conditions, we extend the external force term f and initial velocity u_0 to whole space by mirror reflection method and change (3.1.3) into a Cauchy problem. Hence we show the existence, Uniqueness and regularity of the solutions in (3.1.3). By the local Minty method, we prove the following existence theorem to problem (3.1.1) with (3.1.2), or (3.1.10).

Theorem 3.1.3 *Let $u_0 \in V_p \cap H$ with boundary conditions (3.1.10) and S is induced by a $p-$potential function from Definition 3 1.1. Then there exists a weak solution $u \in L^p(0, T; V_p) \cap L^\infty(0, T; H)$ to system (3.1.1) with(3.1.2).*

3.2 Preliminaries

In this section, we will give some assumptions, function spaces and definitions for weak solutions. We will show the Korn's type inequality for unbounded domain and construction of the basis with boundary conditions (3.1.10).

Definition 3.2.1 *Let $\frac{6}{5} \leq p < \infty$, under the assumption of Definition 3 1.1. Let $f \in L^2(0,T;H)$ or $f \in L^p(0,T;V_p^*)$, which is the dual space of V_p, and $u_0 \in H$ with $\nabla \cdot u_0 = 0$ in the sense of distribution. A vector function $u \in L^\infty(0,T;H) \cap L^p(0,T;V_p)$ is called a weak solution to (3.1.1) if the following identity*

$$
\begin{aligned}
-\int_Q (u \cdot \partial_t \phi) \mathrm{d}x\mathrm{d}t + & \int_Q (S(x,t,D(u)) - u \otimes u) : D(\phi) \mathrm{d}x\mathrm{d}t \\
& = \int_Q f \cdot \phi \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^3_+} u_0 \cdot \phi(0) \mathrm{d}x
\end{aligned}
\tag{3.2.1}
$$

holds for all $\phi \in C^\infty(\overline{\mathbb{R}^3_+} \times [0,T])$ with $\mathrm{div}\phi = 0$, $\phi_3|_{x_3=0} = 0$, and supp$\phi \subset \overline{\mathbb{R}^3_+} \times [0,T)$.

In this chapter, we will consider the following auxiliary problem

$$
\begin{cases}
\partial_t u - \mathrm{div} S(D(u)) + (u \cdot \nabla) u + \nabla \pi = f, & \text{in } \Omega_R \times (0,T) \\
\nabla \cdot u = 0, & \text{in } \Omega_R \times (0,T), \\
u_3 = 0, \quad \dfrac{\partial u_1}{\partial x_3} = \dfrac{\partial u_2}{\partial x_3} = 0 & \text{on } \Gamma^1_R \times (0,T), \\
u = 0 & \text{on } \Gamma^2_R \times (0,T), \\
u|_{t=0} = u_0(x), & \text{in } \Omega_R.
\end{cases}
\tag{3.2.2}
$$

Where $\Gamma^1_R = \overline{\Omega_R} \cap \{x_3 = 0\}$ and $\Gamma^2_R = \overline{\Omega_R} \cap \partial B_R$. The corresponding definition of weak solution in (3.2.2) is following,

Definition 3.2.2 *Let $\frac{6}{5} \leq p < \infty$, under the assumption of Definition 3.1.1 Let $f \in L^2(Q_R)$ or $f \in L^2(0,T;(V_p(\Omega_R))^*)$, where $(V_p(\Omega_R))^*$ is the dual space of $V_p(\Omega_R)$, and $u_0 \in L^2(\Omega_R)$ with $\nabla \cdot u_0 = 0$ in the sense of distribute. A vector function $u \in L^\infty(0,T;L^2(\Omega_R)) \cap L^p(0,T;V_p(\Omega_R))$ is called a weak solution to (3.1.1) if the following identity*

$$
\begin{aligned}
-\int_{Q_R} (u \cdot \partial_t \xi) \mathrm{d}x\mathrm{d}t + & \int_{Q_R} (S(x,t,D(u)) - u \otimes u) : D(\xi) \mathrm{d}x\mathrm{d}t \\
& = \int_{Q_R} f \cdot \xi \mathrm{d}x\mathrm{d}t + \int_{\Omega_R} u_0 \cdot \xi(0) \mathrm{d}x
\end{aligned}
\tag{3.2.3}
$$

holds for all $\xi \in C^{\infty}(\overline{\Omega_R} \times [0,T])$ *with* $\text{div}\xi = 0$, $\xi_3|_{x_3=0} = 0$, *and* $\text{supp}\xi \subset \overline{\Omega_R} \times [0,T)$.

Next, we show the Korn's type inequality and construct the basis in $W^{2,2}(\Omega_R)$ with boundary conditions (3.1.10).

Lemma 3.2.3 *There exists a constant C depending only on p such that*

$$\|\nabla u\|_p \leq C\|D(u)\|_p$$

for all $u \in C_0^{\infty}(\overline{\mathbb{R}_+^3})$.

Proof: Use the method of [50], since $\mathbb{R}_+^3$ satisfies the condition of Theorem 3-2 in [70]. □

Now we consider the following problem

$$\begin{cases} -\Delta u + \nabla p = f, & \text{in } \Omega_R \\ \nabla \cdot u = 0, & \text{in } \Omega_R, \\ u_3 = 0, \quad \dfrac{\partial u_1}{\partial x_3} = \dfrac{\partial u_2}{\partial x_3} = 0 & \text{on } \Gamma_R^1, \\ u = 0 & \text{on } \Gamma_R^2, \end{cases} \tag{3.2.4}$$

Definition 3.2.4 *By a weak solution of the problem (3.2.4) we mean a function* $u(x) \in V_2(\Omega_R)$ *such that*

$$(D(u), D(v)) = (f, v), \ \forall v \in V_2(\Omega_R)$$

Theorem 3.2.5 *Assume that (f, v) is a linear functional on the space $V_2(\Omega_R)$. There exists a unique weak solution $u \in V_2(\Omega_R)$ to problem (3.2.4)*

Proof: See [82], or [100]. □

The following lemma concerns the well-posedness of problem (3.2.4) in $V_2(\Omega_R) \cap W^{2,2}(\Omega_R)$. The proof can be found in [82].

Lemma 3.2.6 *Assume $f \in H(\Omega_R)$, then there exists a unique solution $(u(x), p(x))$ to problem (3.2.4) such that $u \in V_2(\Omega_R) \cap W^{2,2}(\Omega_R)$. Moreover, the following estimates hold:*

$$\begin{aligned}
&\|D(u)\|_{L^2(\Omega_R)} \leq C\|f\|_{H(\Omega_R)} \\
&\|\nabla^2 u\|_{L^2(\Omega_R)} + \|\nabla p\|_{L^2(\Omega_R)} \leq C\left(\|f\|_{H(\Omega_R)} + \|u\|_{V_2(\Omega_R)}\right) \qquad (3.2.5)\\
&\|\nabla u\|_{L^3(\Omega_R)} \leq C\left(\|f\|^{\frac{1}{2}}_{H(\Omega_R)}\|\nabla u\|^{\frac{1}{2}}_{L^2(\Omega_R)} + \|u\|_{V_2(\Omega_R)}\right),
\end{aligned}$$

where C is independent of u, f.

With the aid of previous lemma, one can prove the following proposition

Proposition 3.2.7 *The eigenvalue problem*

$$\begin{cases}
-\Delta u + \nabla p = \lambda u, & in \ \ \Omega_R \\
\nabla \cdot u = 0, & in \ \ \Omega_R, \\
u_3 = 0, \quad \dfrac{\partial u_1}{\partial x_3} = \dfrac{\partial u_2}{\partial x_3} = 0 & on \ \ \Gamma^1_R, \\
u = 0 & on \ \ \Gamma^2_R,
\end{cases}$$

$\lambda \in \mathbb{R}$, $u \in V_2(\Omega_R)$ admits a denumberable positive eigenvalue $\{\lambda_i\}$ clustering at infinity Moreover, the corresponding eigenfunctions $\{a_i\}$ are in $W^{2,2}(\Omega_R)$, and associate pressure fields $p_i \in W^{1,2}(\Omega_R)$. Finally, $\{a_i\}$ are orthogonal and complete in $H(\Omega_R)$ and $V_2(\Omega_R)$

Proof: The mapping $A: \ \ f \longrightarrow u$ defined by lemma 3.2.6 is linear and continuous from $H(\Omega_R)$ onto $V_2(\Omega_R)$, into $W^{1,2}(\Omega_R)$ Since Ω_R is bounded, by Rellich Theorem, we know that $W^{1,2}(\Omega_R) \hookrightarrow L^2(\Omega_R)$ is compact. It is easy to know that operator A is a positive symmetric and self-adjoint operator on $L^2(\Omega_R)$. Therefore, A possess an sequence of eigenfunctions a_i:

$$Aa_i = \lambda_i a_i \ k \geq 0, \lambda_i > 0, \ \lambda_i \to \infty \text{ as } k \to \infty$$

$$(a_i, a_j)_{L^2} = \delta_{i,j}, \ \ (D(a_i), D(a_j))_{L^2} = \lambda_k \delta_{i,j}.$$

By lemma 3.2.6, we can get for each i, there exists p_i with the estimates (3.2.5). □

3.3 The construction of approximate solutions

To obtain a sequence of approximate solution for the problem (3 1.1), fix $\varepsilon > 0$ and define a cutoff function ψ_ε by

$$\psi_\varepsilon(x) = \begin{cases} 1 & \text{if } |x| < \dfrac{1}{2\varepsilon}; \\ 0, & \text{elsewhere.} \end{cases}$$

Define a reflection as follows;

$$u^*(x) = \begin{cases} (u_1(x_1. x_2, x_3), u_2(x_1, x_2, x_3), u_3(x_1, x_2, x_3)) & \text{if } x_3 \geq 0; \\ (u_1(x_1, x_2, -x_3), u_2(x_1, x_2, -x_3), -u_3(x_1, x_2, -x_3), & \text{if } x_3 < 0. \end{cases} \tag{3.3.1}$$

Therefore, we have obtain the following proposition

Proposition 3.3.1 *Let $u \in L^p(0,T; V_p(\mathbb{R}^3_+)) \cap L^\infty(0,T; H)$. J_η be the standard modifier on $\mathbb{R}^3$, denote $u_{\varepsilon,\eta} = J_\eta * \mathbb{P}(\psi_\varepsilon u^*)$, where $\mathbb{P}$ is the projector on the $W^{1,2}(\Omega_R)$ to $V_2(\Omega_R)$, then $u_{\varepsilon,\eta} \in C_0^\infty(\mathbb{R}^3)$ with $u_{\varepsilon,\eta} \cdot n|_{x_3=0} = 0$ and $\int_{\mathbb{R}^3_+} \operatorname{div}(u_{\varepsilon,\eta} \otimes u) \cdot u = 0$. Moreover, there exists a subsequence $u_{\eta(\varepsilon)} = u_{\varepsilon,\varepsilon} \to u$ as $\varepsilon \to 0$ in the $L^p(0,T; V_p(\mathbb{R}^3_+))$ for all $p > 1$, and $\|u_{\eta(\varepsilon)}\|_{W^{2\,2}} \leq C\frac{1}{\varepsilon^2}\|u\|_2$, where C is independent of ε.*

Proof: From the properties of the modification and Cantors diagonalization argument, this proposition holds provided that $\mathbb{P}(\psi_\varepsilon u^*) \to u$ in the space $L^p(0,T; V_p(\mathbb{R}^3_+))$ as $\varepsilon \to 0$. It is clear from the proof of lemma 3.4 in [82]. □

Now we will study the following problem

$$\begin{cases} \partial_t u - \operatorname{div} S(D(u)) + (u_{\eta(\varepsilon)} \cdot \nabla)u + \nabla\pi = f, & \text{in } \mathbb{R}^3_+ \times (0,T) \\ \nabla \cdot u = 0, & \text{in } \mathbb{R}^3_+ \times (0,T), \\ u_3 = 0, \quad \dfrac{\partial u_1}{\partial x_3} = \dfrac{\partial u_2}{\partial x_3} = 0 & \text{on } \{x_3 = 0\} \times (0,T), \\ u|_{t=0} = u_0(x), & \text{in } \mathbb{R}^3_+ \end{cases} \tag{3.3.2}$$

The goal of this section is to prove the existence of a strong solution to the problem (3.3.2) provided that $\varepsilon > 0$ is fixed. Now we give the definition of the strong solution for the problem (3.3.2) as follows

Definition 3.3.2 *We say a couple $(u,\pi) = (u^\varepsilon, \pi^\varepsilon)$ is a strong solution to problem (3.3.2) if*

$$\begin{aligned} &u \in L^\infty(0,T;W^{1,2}_{\text{loc}}(\overline{\mathbb{R}^3_+})) \cap L^p(0,T;W^{2,p}_{\text{loc}}(\overline{\mathbb{R}^3_+})) \cap L^p(0,T;V_p) \cap L^\infty(0,T;H) \\ &\frac{\partial u}{\partial t} \in L^2(0,T;L^2_{\text{loc}}(\overline{\mathbb{R}^3_+})); \pi \in L^{p'}(0,T;L^{p'}_{\text{loc}}(\overline{\mathbb{R}^3_+})). \end{aligned} \tag{3.3.3}$$

where $p' = \dfrac{p}{p-1}$ and satisfies the weak formulation

$$\begin{aligned} \int_{\mathbb{R}^3_+} \frac{\partial u}{\partial t}\varphi \mathrm{d}x + \int_{\mathbb{R}^3_+} S(D(u)) : D(\varphi)\mathrm{d}x + \int_{\mathbb{R}^3_+} (u_{\eta(\varepsilon)} \cdot \nabla) u \cdot \varphi \mathrm{d}x \\ = \int_{\mathbb{R}^3_+} \pi \mathrm{div}\varphi \mathrm{d}x + \int_{\mathbb{R}^3_+} f\varphi \mathrm{d}x \end{aligned} \tag{3.3.4}$$

holds for all $\varphi \in C_0^\infty(\overline{\mathbb{R}^3_+})$ and almost all $t \in (0,T)$, at same time, the boundary conditions hold in the sense of trace.

At first, we provide some definitions and recall a well-known result. Given any bounded domain $\Omega \subset \overline{\mathbb{R}^3_+}$

Definition 3.3.3 *For any $\Omega' \subset\subset \Omega$, we put $\delta(\Omega', \Omega) = \mathrm{dist}(\Omega', \partial\Omega \setminus \{\mathrm{x}_3 = 0\})$*

Definition 3.3.4 *For any $\Omega' \subset\subset \bar{\Omega}$, and $g : \Omega \longrightarrow \mathbb{R}^3$ we set*

$$(\Delta_{\lambda,k} g)(x) = g(x + \lambda e_k) - g(x), \; x \in \Omega', 0 < \lambda < \delta(\Omega', \Omega), k = 1 \cdots 3$$

where e_1, e_2, e_3 is the canonical base of $\mathbb{R}^3$. We shall omit the dependence on where the meaning is clear.

For a give second order tensor D, set $Sym(D) = (D_{ij} + D_{ji})$.

Lemma 3.3.5 *For any $u \in W^{1,p}(\Omega)$ and $0 < |\lambda| < \delta(\Omega', \Omega)$ it holds*

$$\|\Delta_{\lambda,k} u\|_{p,\Omega'} \le |\lambda| \|u_{,k}\|_{p,\Omega}.$$

The following theorem and lemma show that the regularity and uniqueness of the weak solutions to the problem (3.3.2).

Theorem 3.3.6 *Let $\frac{9}{5} < p < 2, f \in L^{p'}(Q_T)$, $u_0 \in V \cap H$ satisfies the boundary conditions (3.1.10) , and S given by a p-potential from Definition 3.1.1. If for any $\varepsilon > 0$, $u \in L^p(0,T;V_p) \cap L^\infty(0,T;H)$ is the weak solution for problem (3.3.2), then this solution is also a unique strong solution to problem (3.3.2) such that*

$$\|u\|_{L^\infty(0,T;W^{1,2}(\Omega'))\cap L^p(0,T;W^{2,p}(\Omega'))} \leq C(\frac{1}{\varepsilon}, |\Omega'|, u_0, f, T), \tag{3.3.5}$$

$$\int_0^T \int_{\Omega'} (1+|D(u)|)^{p-2} |\nabla D(u)|^2 \mathrm{d}x\mathrm{d}t \leq C(\frac{1}{\varepsilon}, |\Omega'|, u_0, f, T). \tag{3.3.6}$$

$$\|\frac{\partial u}{\partial t}\|^2_{L^2((0,T)\times\Omega')} + \|\Phi(D(u))\|_{L^\infty(0,T;L^1(\Omega'))} \leq C(\frac{1}{\varepsilon}, \delta(\Omega',\Omega), |\Omega'|, u_0, f, T). \tag{3.3.7}$$

Proof: We extent the u_0 and force term f to the whole space by the reflection defined in (3.3.1). Denote these functions by u_0^*, f^*.

We begin to consider the Cauchy problem as follows

$$\begin{cases} \partial_t v - \mathrm{div} S(D(v)) + (v_{\eta(\varepsilon)} \cdot \nabla) v + \nabla \pi = f^*, & \text{in } \mathbb{R}^3 \times (0,T) \\ \nabla \cdot v = 0, & \text{in } \mathbb{R}^3 \times (0,T), \\ v|_{t=0} = u_0^*(x), & \text{in } \mathbb{R}^3. \end{cases} \tag{3.3.8}$$

From [93], There exists a weak solution $u \in L^p(0,T;V_p(\mathbb{R}^3)) \cap L^\infty(0,T;L^2(\mathbb{R}^3))$ to problem (3.3.8), then $u \in L^{\frac{5p}{3}}((0,T)\times\mathbb{R}^3)$. If $p > \frac{9}{5}$ then $p' < \frac{5p}{3}$, and $u_{\eta(\varepsilon)} \otimes u \in L^{p'}((0,t)\times B_a)$, since $u_{\eta(\varepsilon)} \in L^\infty$. From the proof of Theorem 2.6 in [115], we know that

$$\|\pi\|_{L^{p'}((0,t)\times B_a)} = C(\|u\|_{L^{p'}((0,t)\times B_a)} + \|S + f^*\|_{L^{p'}((0,t)\times B_a)} + 1)$$

where C only depends on p, T, f, u_0, a.

For any set $\Omega' \subset\subset \Omega$. and for any $\rho > 0$ such that $0 < \rho < \delta(\Omega', \Omega)$, Set $\Omega_\rho = \{x \in \Omega; \mathrm{dist}(x, \Omega') < \rho\}$. Now fix $r < \frac{1}{4}\delta(\Omega',\Omega)$, there exists a ball B_a, such that $\Omega \subset B_a$ Let us choose a cut-off function η such that $\eta \equiv 1$ on Ω_r, $\eta \equiv 0$ in $\mathbb{R}^3 \setminus \Omega_{2r}$, $0 \leq \eta \leq 1$ and $|\nabla \eta| < \dfrac{C}{r}$, $|\nabla^2 \eta| < \dfrac{C}{r^2}$ in Ω_{2r}, where the constant C depends only on the geometry of $\partial\Omega$.

If $|\lambda| < r$, it results that $\Delta_{-\lambda}(\eta^2 \Delta_\lambda u) \in L^2((0,T)\times\Omega_{3r}) \cap L^p(0,T;W_0^{1,p}(\Omega_{3r}))$, but it is not divergence free. Take $\Delta_{-\lambda}(\eta^2\Delta_\lambda u)$ as a test function in the equation $(3.3.8)_1$, we obtain

$$\begin{aligned}&\langle u_t, \Delta_{-\lambda}(\eta^2\Delta_\lambda u)\rangle + (S(D(u)), \Delta_{-\lambda}(\eta^2\Delta_\lambda u)) + (u_{\eta(\varepsilon)}\cdot\nabla u, \Delta_{-\lambda}(\eta^2\Delta_\lambda u))\\ &= (\pi, \operatorname{div}(\Delta_{-\lambda}(\eta^2\Delta_\lambda u))) + (f, \Delta_{-\lambda}(\eta^2\Delta_\lambda u)).\end{aligned}$$

Set

$$\begin{aligned}J_1 &= \langle u_t, \Delta_{-\lambda}(\eta^2\Delta_\lambda u)\rangle,\\ J_2 &= (S(D(u)), \Delta_{-\lambda}(\eta^2\Delta_\lambda u)),\\ J_3 &= (u_{\eta(\varepsilon)}\cdot\nabla u, \Delta_{-\lambda}(\eta^2\Delta_\lambda u)),\\ J_4 &= (\pi, \operatorname{div}(\Delta_{-\lambda}(\eta^2\Delta_\lambda u))),\\ J_5 &= (f, \Delta_{-\lambda}(\eta^2\Delta_\lambda u)).\end{aligned}$$

Clearly, $J_1 = \dfrac{\mathrm{d}}{\mathrm{d}t}\|\eta\Delta_\lambda u\|^2_{L^2(\Omega_{2r})}$. Let $I_\lambda(u) = \displaystyle\int_{\Omega_{2r}}(1+|D(u)(x+\lambda e_k)|+|D(u)|)^{p-2}|\eta\Delta_\lambda D(u)|^2\mathrm{d}x$.

$$\begin{aligned}J_2 &= 2\int_{\Omega_{3r}} S(D(u))\Delta_{-\lambda}sym(\Delta_\lambda u\otimes\eta\nabla\eta)\mathrm{d}x - \int_{\Omega_{2r}}\eta^2(\Delta_\lambda(S(D(u)))\Delta_\lambda D(u))\mathrm{d}x\\ &:= J_{21} - J_{22}.\end{aligned}$$

Since (3.1.6),

$$\begin{aligned}&J_{22} \geq C_1\gamma_1\int_{\Omega_{2r}}(1+|D(u)(x+\lambda e_k)|+|D(u)|)^{p-2}|\eta\Delta_\lambda D(u)|^2\mathrm{d}x = C_1\gamma_1 I_\lambda(u),\\ &|J_{21}| \leq c\|S(D(u))\|_{L^{p'}(\Omega_{3r})}\|\Delta_{-\lambda}sym(\Delta_\lambda u\otimes\eta\nabla\eta)\|_{L^p(\Omega_{3r})},\end{aligned}$$

Thus

$$\begin{aligned}&\|\Delta_{-\lambda}sym(\Delta_\lambda u\otimes\eta\nabla\eta)\|_{L^p(\Omega_{3r})} \leq |\lambda|\|\nabla sym(\Delta_\lambda u\otimes\eta\nabla\eta)\|_{L^p(\Omega_{3r})},\\ &\leq |\lambda|\left[\||\nabla\eta| sym(\Delta_\lambda u\otimes\nabla\eta)|\|_{L^p(\Omega_{3r})} + \|\eta sym(\Delta_\lambda\nabla u_{x_k}\otimes\nabla\eta)\|_{L^p(\Omega_{3r})}\right.\\ &\left.+\|sym(\Delta_\lambda u\otimes\nabla\eta_{x_k})|\|_{L^p(\Omega_{3r})}\right]\\ &\leq 4C\frac{\lambda^2}{r^2}\|\nabla u\|_{L^p(\Omega_{3r})} + \frac{2C|\lambda|}{r}\left(\int_{\Omega_{2r}}|\eta\nabla(\Delta_\lambda u)|^p\mathrm{d}x\right)^{\frac{1}{p}}.\end{aligned}$$

Since $\eta\nabla(\Delta_\lambda u) = \nabla(\eta\Delta_\lambda u) - (\nabla\eta)\cdot\Delta_\lambda u$, thus

$$\left(\int_{\Omega_{2r}} |\eta\nabla(\Delta_\lambda u)|^p \mathrm{d}x\right)^{\frac{1}{p}} \le \frac{C|\lambda|}{r}\|\nabla u\|_{L^p(\Omega_{3r})} + C_p\left(\int_{\Omega_{2r}} |\eta D(\Delta_\lambda u)|^p \mathrm{d}x\right)^{\frac{1}{p}}$$

However, by the Hölder's inequality we have

$$\left(\int_{\Omega_{2r}} |\eta D(\Delta_\lambda u)|^p \mathrm{d}x\right)^{\frac{1}{p}} \le I_\lambda(u)^{\frac{1}{2}}\left(\int_{\Omega_{2r}} (1 + |D(u)(x+\lambda e_k)| + |D(u)|)^p \mathrm{d}x\right)^{\frac{2-p}{2}}.$$

Hence

$$\begin{aligned}
J_2 &\le \|S(D(u))\|_{L^{p'}(\Omega_{3r})}\left(\frac{C\lambda^2}{r^2}\|\nabla u\|_{L^p(\Omega_{3r})} + CI_\lambda(u)^{\frac{1}{2}}(1 + \|\nabla u\|^{\frac{2-p}{2}}_{L^p(\Omega_{3r})})\right) - C\gamma_1 I_\lambda(u)\\
&\le (1 + \|\nabla u\|^{p-1}_{L^p(\Omega_{3r})})\left(\frac{C\lambda^2}{r^2}\|\nabla u\|_{L^p(\Omega_{3r})} + CI_\lambda(u)^{\frac{1}{2}}(1 + \|\nabla u\|^{\frac{2-p}{2}}_{L^p(\Omega_{3r})})\right) - C\gamma_1 I_\lambda(u)\\
&\le \lambda^2 C(|\Omega_{3r}|, r, \epsilon)\left(1 + \|\nabla u\|^p_{L^p(\Omega_{3r})}\right) - (C_1\gamma_1 - \epsilon)I_\lambda(u).
\end{aligned}$$

$$\begin{aligned}
|J_5| &= |(f, \Delta_{-\lambda}(\eta^2\Delta_\lambda u))| \le \|f\|_{L^{p'}(\Omega_{3r})}|\lambda|\|\nabla(\eta^2\Delta_\lambda u\|_{L^p(\Omega_{2r})}\\
&\le C(r,p)\lambda^2\|f\|_{L^{p'}(\Omega_{3r})}\|\nabla u\|_{L^p(\Omega_{3r})} + |\lambda|\|\eta^2\Delta_\lambda \nabla u\|_{L^p(\Omega_{2r})}\\
&\le |\lambda| I_\lambda(u)^{\frac{1}{2}}(1 + \|\nabla u\|^{\frac{2-p}{2}}_{L^p(\Omega_{3r})})\|f\|_{L^{p'}(\Omega_{3r})} + C(\frac{1}{r^2} + \frac{1}{r})\lambda^2\|f\|_{L^{p'}(\Omega_{3r})}\|\nabla u\|_{L^p(\Omega_{3r})}\\
&\le C(p, |\Omega_{3r}|)\lambda^2\left(\|f\|^{p'}_{L^{p'}(\Omega_{3r})} + \|\nabla u\|^p_{L^p(\Omega_{3r})} + 1\right) + \epsilon I_\lambda(u).
\end{aligned}$$

From the estimates for pressure, divergence-free and the method above, we have

$$\begin{aligned}
|J_4| &\le 2\left|\int_{\Omega_{3r}} \pi\Delta_\lambda(\eta\eta_i\Delta_\lambda u_i)\mathrm{d}x\right|\\
&\le C|\lambda|\|\pi\|_{L^{p'}(\Omega_{3r})}\|\frac{\partial}{\partial x_k}(\eta\eta_{x_i}\Delta_\lambda u_i)\|_{L^p(\Omega_{3r})}\\
&\le C\frac{1}{r^2}|\lambda|^2\|\pi\|_{L^{p'}(\Omega_{3r})}\|\nabla u\|_{L^p(\Omega_{3r})} + \frac{C|\lambda|}{r}|\lambda|^2\|\pi\|_{L^{p'}(\Omega_{3r})}\|\eta\nabla\Delta_\lambda u\|_{L^p(\Omega_{2r})}\\
&\le C\frac{1}{r^2}|\lambda|^2\left(\|\pi\|^{p'}_{L^{p'}(\Omega_{3r})} + \|\nabla u\|^p_{L^p(\Omega_{3r})} + 1\right) + \epsilon I_\lambda(u)
\end{aligned}$$

Now we estimate the term J_3. In fact,

$$\begin{aligned}
J_3 &= \int_{\Omega_{3r}} \eta^2\Delta_\lambda u_{\eta(\varepsilon)}\cdot\nabla u\Delta_\lambda u\mathrm{d}x - 2\int_{\Omega_{3r}} \eta u_{\eta(\varepsilon)}\cdot\nabla\eta|\Delta_\lambda u|^2\mathrm{d}x\\
&:= J_{31} + J_{32}
\end{aligned}$$

Since $\frac{9}{5} < p \leq 2$, then $2 < q = \dfrac{6p}{5p-6} < \dfrac{3p}{3-p} = p^*$, by the Hölder inequality,we have

$$J_{31} = \|\eta\Delta_\lambda u_{\eta(\varepsilon)}\|_{L^6(\Omega_{3r})}\|\nabla u\|_{L^p(\Omega_{3r})}\|\eta\Delta_\lambda u\|_{L^q(\Omega_{3r})}$$
$$\leq C|\lambda|\|\nabla u\|_{L^p(\Omega_{3r})}\|\nabla u_{\eta(\varepsilon)}\|_{L^6(\Omega_{3r})}\|\eta\Delta_\lambda u\|_{L^q(\Omega_{3r})}$$
$$\leq C|\lambda|\|\nabla u\|_{L^p(\Omega_{3r})}\|u_{\eta(\varepsilon)}\|_{W^{2,2}(\Omega_{3r})}\|\eta\Delta_\lambda u\|_{L^q(\Omega_{3r})}$$
$$\leq C|\lambda|\frac{1}{\varepsilon^2}\|u\|_{L^\infty(0,T;H)}\|\nabla u\|_{L^p(\Omega_{3r})}\|\eta\Delta_\lambda u\|_{L^q(\Omega_{3r})}$$
$$J_{32} \leq \frac{C}{r}\|u_{\eta(\varepsilon)}\|_{L^6(\Omega_{3r})}\|\eta\Delta_\lambda u\|_{L^q(\Omega_{3r})}\|\Delta_\lambda u\|_{L^p(\Omega_{3r})}$$
$$\leq C|\lambda|\frac{1}{r\varepsilon^2}\|u\|_{L^\infty(0,T;H)}\|\nabla u\|_{L^p(\Omega_{3r})}\|\eta\Delta_\lambda u\|_{L^q(\Omega_{3r})}.$$

It is implies

$$|J_3| \leq C(\frac{1}{r}, |\Omega_{3r}|)|\lambda|\frac{1}{\varepsilon^2}\|u\|_{L^\infty(0,T;H)}\|\nabla u\|_{L^p(\Omega_{3r})}\|\eta\Delta_\lambda u\|_{L^q(\Omega_{3r})}$$

Since $2 < q < p^*$, from the following interpolation inequalities

$$\|u\|_{L^q} \leq \|u\|_{L^{p^*}}^{\theta}\|u\|_{L^2}^{1-\theta};$$
$$\|u\|_{L^q} \leq \|u\|_{L^{p^*}}^{\theta_1}\|u\|_{L^p}^{1-\theta_1},$$

where $\theta = \dfrac{6-2p}{5p-6}, \theta_1 = \dfrac{12-5p}{2p}$. Therefore, we have

$$\begin{aligned}|J_3| \quad &\leq C(\frac{1}{r}, |\Omega_{3r}|)|\lambda|\frac{1}{\varepsilon^2}\|u\|_{L^\infty(0,T;H)}\|\nabla u\|_{L^p(\Omega_{3r})}\|\eta\Delta_\lambda u\|_{L^p(\Omega_{3r})}^{(1-\alpha)(1-\theta_1)} \\ &\qquad \|\eta\Delta_\lambda u\|_{L^2(\Omega_{2r})}^{(1-\theta)\alpha}\|\eta\Delta_\lambda u\|_{L^{p^*}(\Omega_{3r})}^{\alpha\theta+(1-\alpha\theta_1)} \\ \leq \quad &\frac{c}{\varepsilon^2}\|u\|_{L^\infty(0,T;H)}|\lambda|^{1+Q_3}\|\nabla u\|_{L^p(\Omega_{3r})}^{1+Q_3}\|\eta\Delta_\lambda u\|_{L^2(\Omega_{2r})}^{Q_1}\|\eta\Delta_\lambda u\|_{L^{p^*}(\Omega_{2r})}^{Q_2}\end{aligned}$$

where

$$Q_1 = (1-\theta)\alpha = \frac{7p-12}{5p-6}\alpha,\ 0 < \alpha < 1,$$
$$Q_2 = \alpha\theta + (1-\alpha\theta_1) = \frac{6-2p}{5p-6}\alpha + \frac{12-5p}{2p}(1-\alpha)$$
$$Q_3 = (1-\alpha)(1-\theta_1) = \frac{7p-12}{2p}(1-\alpha).$$

Since

$$\|\eta\Delta_\lambda u\|_{L^{p^*}(\Omega_{2r})} \leq C\|\nabla(\eta\Delta_\lambda u)\|_{L^p(\Omega_{2r})} \leq C\|D(\eta\Delta_\lambda u)\|_{L^p(\Omega_{2r})} \leq CI_\lambda(u)^{\frac{1}{2}}(1+|\nabla u|_{L^p(\Omega_{2r})}^{\frac{2-p}{2}}).$$

It infer that

$$|J_3| \le \frac{C}{\varepsilon^2}\|u\|_{L^\infty(0,T;H)}|\lambda|^{1+Q_3}\|\nabla u\|^{1+Q_3}_{L^p(\Omega_{3r})}\|\eta\Delta_\lambda u\|^{Q_1}_{L^2(\Omega_{2r})}I_\lambda(u)^{\frac{1}{2}Q_2}\left(1+|\nabla u|^{\frac{2-p}{2}Q_2}_{L^p(\Omega_{2r})}\right)$$
$$\le \frac{C}{\varepsilon^2}\|u\|_{L^\infty(0,T;H)}|\lambda|^{1+Q_3}\left(1+\|\nabla u\|^{1+Q_3+\frac{2-p}{2}Q_2}_{L^p(\Omega_{3r})}\right)\|\eta\Delta_\lambda u\|^{Q_1}_{L^2(\Omega_{2r})}I_\lambda(u)^{\frac{1}{2}Q_2}$$

By Young's Inequality, we have

$$|J_3| \le \frac{C}{\varepsilon^2}\|u\|_{L^\infty(0,T;H)}|\lambda|^{(1+Q_3)\delta'}\left(1+\|\nabla u\|^{(1+Q_3+\frac{2-p}{2}Q_2)\delta'}_{L^p(\Omega_{3r})}\right)\|\eta\Delta_\lambda u\|^{Q_1\delta'}_{L^2(\Omega_{2r})}+\epsilon I_\lambda(u).$$

Then choose δ and δ' satisfy the following identities

$$\frac{Q_2}{2}\delta=1,\ (1+Q_3+\frac{2-p}{2}Q_2)\delta'=p,\ \text{and}\ \frac{1}{\delta}+\frac{1}{\delta'}=1.$$

From these identities, we can obtain $\alpha = \dfrac{(5p-6)(2-p)}{7p-12}$. Since $p > \dfrac{9}{5}$,thus $0<\alpha<1$, $(1+Q_1+Q_3)\delta'=2$ and $\dfrac{Q_1\delta'}{2}<1$. Therefore,

$$|J_3| \le \frac{C}{\varepsilon^2}\|u\|_{L^\infty(0,T;H)}|\lambda|^{(1+Q_3)\delta'}\left(1+\|\nabla u\|^p_{L^p(\Omega_{3r})}\right)\|\eta\Delta_\lambda u\|^{Q_1\delta'}_{L^2(\Omega_{2r})}+\epsilon I_\lambda(u).$$

Combined these relations about $J_1,\cdots,J_5$, we can conclude that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\Delta_\lambda u\|^2_{L^2(\Omega_{2r})}+(C_1\gamma_1-4\epsilon)I_\lambda(u)\\&\le C\lambda^2\left(1+\|\nabla u\|^p_{L^p(\Omega_{3r})}+\|\pi\|^{p'}_{L^{p'}(\Omega_{3r})}+\|f\|^{p'}_{L^{p'}(\Omega_{3r})}\right)\\&+\frac{C}{\varepsilon^2}\|u\|_{L^\infty(0,T;H)}|\lambda|^{(1+Q_3)\delta'}\left(1+\|\nabla u\|^p_{L^p(\Omega_{3r})}\right)\|\eta\Delta_\lambda u\|^{Q_1\delta'}_{L^2(\Omega_{2r})}\end{aligned}\tag{3.3.9}$$

Since $(1+Q_1+Q_3)\delta'=2$ and set

$$\begin{aligned}r(t)&=\frac{C}{\varepsilon^2}\|u\|_{L^\infty(0,T;H)}\left(1+\|\nabla u\|^p_{L^p(\Omega_{3r})}\right),\\h(t)&=C\left(1+\|\nabla u\|^p_{L^p(\Omega_{3r})}+\|\pi\|^{p'}_{L^{p'}(\Omega_{3r})}+\|f\|^{p'}_{L^{p'}(\Omega_{3r})}\right).\end{aligned}\tag{3.3.10}$$

where C does not depend on λ, u. Hence (3.3.9) can be rewritten

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\frac{\eta\Delta_\lambda u}{\lambda}\|^2_{L^2(\Omega_{2r})}+(C_1\gamma_1-4\epsilon)\frac{I_\lambda(u)}{\lambda^2}\\&\le h(t)+r(t)\|\frac{\eta\Delta_\lambda u}{\lambda}\|^{Q_1\delta'}_{L^2(\Omega_{2r})}.\end{aligned}$$

Since $p > \dfrac{9}{5}$,it is easy to know that $\dfrac{Q_1\delta'}{2} < 1$. Whence by Young's inequality, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\frac{\eta\Delta_\lambda u}{\lambda}\|^2_{L^2(\Omega_{2r})} + (C_1\gamma_1 - 4\epsilon)\frac{I_\lambda(u)}{\lambda^2}\\ &\leq (h(t) + r(t)) + r(t)\|\frac{\eta\Delta_\lambda u}{\lambda}\|^2_{L^2(\Omega_{2r})}.\end{aligned}$$

by the Gronwall's Lemma, we have

$$\begin{aligned}&\|\frac{\eta\Delta_\lambda u}{\lambda}\|^2_{L^2(\Omega_{2r})}(t) + (C_1\gamma_1 - 4\epsilon)\int_0^t \frac{I_\lambda(u)}{\lambda^2}\\ &\leq \left[\|\frac{\eta\Delta_\lambda u}{\lambda}\|^2_{L^2(\Omega_{2r})}(0) + \int_0^t h(s)\exp\left(-\int_0^s r(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]\exp\int_0^t r(s)\mathrm{d}s.\end{aligned}$$

Assume that $u_0 \in V$, then we have for all $t \in [0,T]$

$$\begin{aligned}&\|\frac{\eta\Delta_\lambda u}{\lambda}\|^2_{L^2(\Omega_{2r})}(t) + (C_1\gamma_1 - 4\epsilon)\int_0^t \frac{I_\lambda(u)}{\lambda^2}\\ &\leq \left[\|\nabla u_0^*\|^2_{L^2(\mathbb{R}^3_+)} + \int_0^1 (h(s) + r(s))\exp\left(-\int_0^s r(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]\exp\int_0^t r(s)\mathrm{d}s.\end{aligned} \tag{3.3.11}$$

Choose $\epsilon = \dfrac{C_1\gamma_1}{8}$ and from (3.3.10),(3.3.11), we have for any $t \in [0,T]$

$$\|\nabla u\|^2_{L^2(\Omega_r)}(t) + \int_0^t I(u) \leq C. \tag{3.3.12}$$

where $I(u) = \int_{\Omega_r}(1 + 2|D(u)|)^{p-2}|D(\nabla u)|^2\mathrm{d}x$ and C depend on $\varepsilon, p, a, u_0, f, T, r$. Since

$$\int_0^t \|\nabla^2 u\|^p_{L^p(\Omega_r)}\mathrm{d}x \leq \int_0^t I(u) + \int_0^t \|\nabla u\|^p_{L^p(\Omega_r)}$$

for all $t \in [0,T]$, thus $\|u\|_{L^p(0,T;W^{2,p}(\Omega_r))} \leq C$. Multiply $(3.3.8)_1$ by $\eta^2 u_t$ and integral on the Ω_{2r}, we have

$$\begin{aligned}&\|\eta u_t\|^2_{L^2(\Omega_{3r})}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\|\eta^2\Phi(D(u))\|_{L^1(\Omega_{3r})}\\ &\leq 2\int_{\Omega_{3r}}|\nabla\eta S(D(u))\eta u_t|\mathrm{d}x + \int_{\Omega_{3r}} u_{\eta(\varepsilon)}\cdot\nabla u\eta^2 u_t\mathrm{d}x\\ &+ 2\int_{\Omega_{3r}}\pi\nabla\eta\cdot\eta u_t\mathrm{d}x + \int_{\Omega_{3r}} f\cdot\eta^2 u_t\mathrm{d}x.\end{aligned} \tag{3.3.13}$$

Since $\frac{1}{2}+\frac{1}{p'}<1$, by Hölder inequality, one can

$$\int_{\Omega_{3r}} |\nabla\eta S(D(u))\eta u_t|\mathrm{d}x \le \|\nabla\eta\|_{L^q(\Omega_{3r})}\|S(D(u))\|_{L^{p'}(\Omega_{3r})}\|\eta u_t\|_{L^2(\Omega_{3r})}$$
$$\int_{\Omega_{3r}} u_{\eta(\varepsilon)}\cdot\nabla u\eta^2 u_t\mathrm{d}x \le C\|u\|_{L^\infty(0,T;H)}\|\eta\nabla u\|_{L^2(\Omega_{2r})}\|\eta u_t\|_{L^2(\Omega_{3r})}$$
$$\int_{\Omega_{3r}} \pi\nabla\eta\cdot\eta u_t\mathrm{d}x + \int_{\Omega_{3r}} f\cdot\eta^2 u_t\mathrm{d}x \le C\|\nabla\eta+\eta\|_{L^q(\Omega_{3r})}\|\pi+f\|_{L^{p'}(\Omega_{3r})}\|\eta u_t\|_{L^2(\Omega_{3r})}.$$

By Young's inequality, we can obtain that

$$\|\eta u_t\|^2_{L^2(\Omega_{3r}\times(0,t))} + \|\eta^2\Phi(D(u))\|_{L^\infty(0,t;L^1(\Omega_{3r}))} \le K$$

where K depend only on $p, a, u_0, f, r, \|\nabla u\|_{L^p(\mathbb{R}^3\times(0,T))}$.

In fact, from this proof, we can see that the bound depends on the measure of Ω and Ω'. Hence, we have if the radius of the ball B is fixed, then $\|\nabla^2 u\|_{L^p(0,T;L^p(B))} \le C$, therefore, we can see that $u \in C^\gamma(B)$ for almost $t \in [0,T]$. We use the following argument (see [50]) to know that $u(x,t)\to 0$ for almost$t \in (0,T)$, as $|x|\to\infty$. Let the radius B be one, suppose that there exist $\varepsilon > 0$ and a sequence $\{x_n\}\subset\mathbb{R}^3$ with $\lim\limits_{n\to\infty}|x_n|\to\infty$, such that for almost $t\in(0,T)$, $u(x_n,t)\ge\epsilon$. By the continuity of $u(x,t)$, then we get that if $|x-x_n|\le\delta=\min\{1,(\frac{\varepsilon}{2C})^{\frac{1}{\gamma}}\}$ then $u(x,t)\ge\frac{\epsilon}{2}$. Without loss of generality, we assume that $|x_i-x_j|>2$ provided that $i\ne j$ thus

$$\int_0^T(\int_{\mathbf{R}^3}|u|^{p^*}\mathrm{d}x)^{\frac{p}{p^*}}\mathrm{d}t \ge \sum_j\int_0^T(\int_{B(x_j)}|u|^{p^*}\mathrm{d}x)^{\frac{p}{p^*}}\mathrm{d}t = +\infty.$$

and this contradiction the fact $u\in L^p(0,T;V_p(\mathbb{R}^3))$.

From the fact $u(x,t)\to 0$ a.e. $t\in(0,T)$, as $|x|\to\infty$ and estimate (3.3.12), we can conclude that the weak solution is unique. Actually, assume that u, v are that weak solutions of problem (3.3.8), then set $e=u-v$ and multiply the difference of the equations of u and v by e. After integral on the $\mathbb{R}^3$, we have

$$(\partial_t e, e) + (S(D(u))-S(D(v)), D(u)-D(v)) + (u_{\eta(\varepsilon)}\cdot\nabla u - v_{\eta(\varepsilon)}\cdot\nabla v, e) = 0.$$

This reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2_{L^2(\mathbf{R}^3)} + (S(D(u)) - S(D(v)), D(u) - D(v)) = |(e_{\eta(\varepsilon)} \cdot \nabla u, e)| \quad (3.3.14)$$

Since $u_{\eta(\varepsilon)}, v_{\eta(\varepsilon)} \in C_0^\infty(\mathbb{R}^3)$ and the support of $u_{\eta(\varepsilon)}, v_{\eta(\varepsilon)}$ are contained in $B_{\frac{1}{\varepsilon}}$. Hence, from the inequality (4.2.16), we have that

$$|(e_{\eta(\varepsilon)} \cdot u, e)| \le |e_{\eta(\varepsilon)}|_{L^\infty} \|\nabla u\|_{L^2(B_{\frac{1}{\varepsilon}})} \|e\|_{L^2(\mathbb{R}^{\not{\varepsilon}})} \le C\|\nabla u\|_{L^\infty(0,T;L^2(B_{\frac{1}{\varepsilon}}))} \frac{1}{\varepsilon} \|e\|^2_{L^2(\mathbb{R}^{\not{\varepsilon}})}.$$

Use the Gronwall's inequality in (3.3.13), we have $e = 0$, i.e. $u = v$. Define

$$u^* = (u_1(x_1, x_2, -x_3), u_2(x_1, x_2, -x_3), -u_3(x_1, x_2, -x_3)$$
$$\pi^* = \pi(x_1, x_2, -x_3), \ \forall x \in \mathbb{R}^3.$$

Then by the method in [15], the couple (u^*, π^*) is also a solution to problem (3.3.4) a.e. in $\mathbb{R}^3_+ \times (0,T)$. Hence by the uniqueness, we know that $u^* = u$.

From the regularity of u and Sobolev imbedding theorems $u(x)$ is a continuous function on B_r for any $r > 0$, so that from $u_3(x) = -u_3(x_1, x_2, -x_3)$ we know that $u_3|_{x_3=0} = 0$. Analogously $u_i(x) = u_i(x_1, x_2, -x_3)$, $\forall x \in \mathbb{R}^3 (i = 1, 2)$, satisfy the conditions (3.1.10) in the sense of trace.

By takeing into account the properties of regularity of $u(x)$, and Theorem 7.1 of [15], we know the solution $u(t)$ to problem (4.2.4) is simply the restriction of $u^*(t)$ to the half-space $\mathbb{R}^3_+$, and $\|\nabla u^*\|_{L^p(\mathbb{R}^3\times(0,T))} \le C\|\nabla u\|_{L^p(\mathbb{R}^3_+\times(0,T))}$, $\|u^*\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \le C\|u\|_{L^\infty(0,T;H)}$. Since u is a weak solutions to problem (3.3.2), therefore, we can obtain the following estimates

$$\|u\|^2_{L^\infty(0,T;H)} + \|u\|^p_{L^p(0,T;V_p)} \le C(\|u_0\|^2_H + \|f\|^{p'}_{L^{p'}(\mathbb{R}^3)\times(0,T)}), \quad (3.3.15)$$

from these estimates, we know that C, K depend on $a, r, |\Omega'|, p, u_0, f$, do not depend on u. The theorem is completely proved. □

By the minor modification of the proof in theorem above, we can obtain the regularity results in the case $p \ge 2$.

Lemma 3.3.7 *Let $p \geq 2, f \in L^{p'}(Q_T) \cap L^2(Q_T)$, $u_0 \in V \cap H$ satisfies the boundary conditions (3.1.10) , and S given by a p-potential from Definition 3.1.1. If for any $\varepsilon > 0$, $u \in L^p(0,T;V_p) \cap L^\infty(0,T;H)$ is the weak solution for problem (3.3.2), then this solution is also a unique strong solution to problem (3.3.2) such that*

$$\|u\|_{L^\infty(0,T,W^{1,2}(\Omega')) \cap L^2(0,T,W^{2,2}(\Omega'))} \leq C(\frac{1}{\varepsilon}, \delta(\Omega',\Omega), |\Omega'|, u_0, f, T),$$

$$\int_0^T \int_{\Omega'} (1 + |D(u)|)^{p-2} |\nabla D(u)|^2 \mathrm{d}x\mathrm{d}t \leq C(\frac{1}{\varepsilon}, \delta(\Omega',\Omega), |\Omega'|, u_0, f, T).$$

$$\|\frac{\partial u}{\partial t}\|^2_{L^2((0,T)\times\Omega')} + \|\Phi(D(u))\|_{L^\infty(0,T,L^1(\Omega'))} \leq C(\frac{1}{\varepsilon}, \delta(\Omega',\Omega), |\Omega'|, u_0, f, T)$$

$$\|\pi\|_{L^2((0,T)\times\Omega')} \leq C(\frac{1}{\varepsilon}, \delta(\Omega',\Omega), |\Omega'|, u_0, f, T)$$

From these regularity estimates presented in Theorem 3.3.6 and lemma 3.3.7, we can obtain the existence of unique weak solution to problem (3.3.2) stated by the following theorem.

Theorem 3.3.8 *Let $p > \frac{9}{5}, f \in L^{p'}(Q_T) \cap L^2(Q_T), u_0 \in V \cap H$, and S given by a p-potential from Definition 1 Then for any $\varepsilon > 0$, there exists a unique weak solution for problem (3.3.2) $u_\varepsilon \in L^p(0,T;V_p) \cap L^\infty(0,T;H)$ and satisfies the inequality (3.3.15), there, the constant C is independent of ε.*

Proof: From the proof in Theorem 3.3.6, it is easy to see that the weak solution is unique. We will use standard Galerkin method to prove its existence.

Let

$$\mathbb{R}^3_+ = \bigcup_{R=1}^{\infty} \Omega_R = \bigcup_{R=1}^{\infty} \{x \in \mathbb{R}^3_+ : |x| \leq R\}.$$

Fix $R > 0$, we consider the auxiliary problem (3.2.2) for the initial $u_0^R = P(\chi_{\Omega_R}(x) u_0(x))$ and external force term $f^R = \chi_{\Omega_R}(x) f$. As in [82], we know that $u_0^R \to u_0$ in $V \cap H$, as $R \to \infty$ and$f^R \to f$ in $L^{p'}(Q)$.

Choose the sequence $\{a_k^R\}$ is the eigenvector of the operator A as in Proposition 3.2.7, then $\{a_k^R\}$ is a basis $W^{2,2}(\Omega_R) \cap V_2(\Omega_R)$. We look for the weak solution to (3.2.3) of the form

$$u_m^R(x,t) = \sum_{k=1}^{m} c_{k,m}^R(t) a_k^R(x).$$

For simplicity, in the clear meaning setting, we omit the superscript R. Therefore, $c_{k,m}(t)$ solve the following system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_m, a_k) + (S(D(u_m)), D(a_k)) - (u_{\eta(\varepsilon),m} \otimes u_m, \nabla a_k) = (f, a_k). \qquad (3.3.16)$$

Due to the continuity of $S, u_{\eta(\varepsilon),m}$, the local-in-time existence follows from Caratheodory theory. The global-in-time existence will be established by the following a-priori estimates.

Multiply the equations (3.3.16) by $c_{k,m}$, then sum over k and integral on $(0,t)$. We easily obtain

$$\|u_m\|_{L^2(\Omega_R)}^2 + \int_0^t (S(D(u_m)), D(u_m)) = (f, u_m) + \|u_0\|_{L^2(\Omega_R)}. \qquad (3.3.17)$$

Hence

$$\sup_{0\le t\le T} \|u_m\|_{L^2(\Omega_R)}^2(t) + \|u_m\|_{L^p(0,T;V_p(\Omega_R))}^p \le \|u_0\|_H^2 + \|f\|_{L^{p'}(\mathbb{R}^3)\times(0,T)}^{p'} := M. \qquad (3.3.18)$$

From (3.3.16) and (3.3.18), we infer that $\|u_m'\|_{L^{p'}(0,T;(V_p(\Omega_R))^*)} \le C(R, \varepsilon, M)$.

It follows from these estimates, and Aubin-Lions Lemma, we have

$$u_m' \rightharpoonup u', \quad \text{weakly in } L^{p'}(0,T;V_p(\Omega_R)^*); \qquad (3.3.19)$$

$$u_m \overset{*}{\rightharpoonup} u, \quad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega_R)); \qquad (3.3.20)$$

$$u_m \rightharpoonup u, \quad \text{weakly in } L^p(0,T;V_p(\Omega_R)),; \qquad (3.3.21)$$

$$u_m \longrightarrow u, \quad \text{strongly in } L^q(0,T;L^q(\Omega_R)) \ q \in [1, \frac{5p}{3}); \qquad (3.3.22)$$

$$S(D(u_m^R)) \rightharpoonup \tilde{S}^R, \quad \text{weakly in } L^{p'}(\Omega_R \times (0,T)). \qquad (3.3.23)$$

From these properties, it is easy to see that

$$\langle u_t, a_k \rangle + (\tilde{S}^R, D(a_k)) - (u_{\eta(\varepsilon)} \cdot \nabla a_k, u) = (f, a_k). \qquad (3.3.24)$$

Multiplying both sides of (3.3.24) by $c_{k,m}$ and summing over k we find

$$\langle u_t, u_m \rangle + (\tilde{S}^R, D(u_m)) - (u_{\eta(\varepsilon)} \cdot \nabla u_m, u) = (f, u_m). \tag{3.3.25}$$

Let us pass to the limit for $m \to \infty$ in to this relation. By the convergence properties (3.3.19) and (3.3.21), we know that as $m \to \infty$

$$\int_0^T \langle u_t, u_m \rangle \to \int_0^T \langle u_t, u \rangle = \|u\|^2_{L^2(\Omega_R)} - \|u_0\|^2_{L^2(\Omega_R)},$$
$$\int_0^T (f, u_m) \to \int_0^T (f, u), \quad \int_0^T (\tilde{S}^R, D(u_m)) \to \int_0^T (\tilde{S}^R, D(u)).$$

Since

$$(u_{\eta(\varepsilon)} \cdot \nabla u_m, u) - (u_{\eta(\varepsilon)} \cdot \nabla u, u) = \int_{\Omega_R} (u_{\eta(\varepsilon)} \otimes u) \cdot \nabla (u_m - u) \mathrm{d}x$$

From (3.3.21) and (3.3.22),we know that $u_{\eta(\varepsilon)} \otimes u \in L^{p'}(\Omega_R \times (0,T))$ whenever $p > \frac{9}{5}$. Hence $\int_0^T (u_{\eta(\varepsilon)} \cdot \nabla u_m, u) \to 0$ as $m \to \infty$, since $(u_{\eta(\varepsilon)} \cdot \nabla u, u) = 0$. Subtracting(3.3.25) by (3.3.17), passing to limit as $m \to \infty$, we get

$$\lim_{m\to\infty} \int_0^T (S(D(u_m^R)), D(u_m))\mathrm{d}t = \int_0^T (\tilde{S^R}, D(u))\mathrm{d}t \tag{3.3.26}$$

By the monotonicity property (3.1.7), we can write the following inequality

$$\int_0^T (S(D(u_m^R)) - S(D(\Psi)), (D(u_m) - D(\Psi)))\mathrm{d}t \geq 0, \tag{3.3.27}$$

For any $\Psi \in L^p(0,T; V_p(\Omega))$. Thus, pass to the limit as m goes to infinity into this relation and using (3.3.21), (3.3.23)and (3.3.26), we have

$$\int_0^T (\tilde{S}^R - S(D(\Psi)), D(u) - D(\Psi))\mathrm{d}t \geq 0, \tag{3.3.28}$$

For all $\Psi \in L^p(0,T; V_p(\Omega))$, Take $\Psi = u - \epsilon\varphi$, $\epsilon > 0$ and $\varphi \in L^p(0,T; V_p(\Omega))$, we have that

$$\int_0^T (\tilde{S}^R - S(D(u - \epsilon\varphi)), D(\varphi))\mathrm{d}t \geq 0,$$

Letting $\epsilon \to 0$ and using the continuity of S, we arrived at

$$\int_0^T (\tilde{S}^R - S(D(u)), D(\varphi))\mathrm{d}t \geq 0, \forall \varphi \in L^p(0,T; V_p(\Omega))$$

Choose $-\varphi$ in place of φ, we get

$$\int_0^T (\tilde{S}^R - S(D(u)), D(\varphi))\mathrm{d}t \le 0, \forall \varphi \in L^p(0,T;V_p(\Omega))$$

This implies that $\tilde{S}^R = S(D(u))$ a.e. $\Omega_R \times (0,T)$. Thus the existence of weak solution u^R to problem (3.2.2) is proved.

Next we must consider the limits as R tend to ∞. Now we choose a sequence of real number $\{R_N : N \in \mathbf{N}\}$ increasing to infinity. We set $u_N = u^{R_N}$ and extend u_N to zero outside Ω_{R_N} to obtain a function still denote $u_N \in L^\infty(0,T;H) \cap L^p(0,T;V_p)$ and satisfies the following a-priori estimate

$$\sup_{0\le t\le T} \|u_N\|_H(t) + \|u_N\|_{L^p(0,T;V_p)} \le \|u_0\|_H^2 + \|f\|^{p'}_{L^{p'}(\mathbb{R}^3\times(0,T)} := K. \tag{3.3.29}$$

Clearly, K is independent of N and ε. Take $R_N > \dfrac{1}{\varepsilon}$ large enough, since $u^N_{\eta(\varepsilon)} \in C_0^\infty(B_{\frac{1}{\varepsilon}})$, then $u_{\eta(\varepsilon),N} \otimes u_N \in L^{p'}(\mathbb{R}^3_+ \times (0,T))$. From (3.3.29) and interpolation inequality we have $\|u_{\eta(\varepsilon),N} \otimes u_N\|_{L^{p'}(\mathbb{R}^3_+\times(0,T))} \le C(\dfrac{1}{\varepsilon}, K)$. let $Y = V_p$, Therefore, from the equations (3.3.25) we obtain

$$\|u'_N\|_{L^{p'}(0,T;Y^*)} \le \|S(D(u_N))\|_{L^{p'}(Q)} + \|u_{\eta(\varepsilon),N} \otimes u_N\|_{L^2(Q)} \le C(\frac{1}{\varepsilon}, K).$$

By the Aubin-Lions Lemma, we know that there exists a subsequence $u_{N_k} \to u$ in $L^p(\Omega_R \times (0,T))$, thus $u_{N_k} \to u$ a.e. in $\mathbb{R}^3_+ \times [0,T]$.

We choose $\phi \in C_0^\infty(\overline{\mathbb{R}^3_+} \times [0,T))$ with $\mathrm{div}\phi = 0$, $\phi_3|_{x_3=0} = 0$, and $\mathrm{supp}\phi \subset \overline{\mathbb{R}^3_+} \times [0,T)$. There exists a number $K = K(\phi) > 0$ such that $G = \mathrm{supp}\phi \subsetneqq \Omega_{N_K} \times [0,T)$ for $k > K(\phi)$. From the formula (3.2.2). we can obtain the following identity

$$\begin{aligned} &-\int_Q (u_{N_k} \cdot \partial_t \phi)\mathrm{d}x\mathrm{d}t + \int_Q S(x,t,D(u_{N_k})) : D(\phi)\mathrm{d}x\mathrm{d}t \\ &-\int_Q (u_{\eta(\varepsilon),N_k} \otimes u_{N_k}) : D(\phi)\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^3_+} u_0 \cdot \phi(0)\mathrm{d}x. \end{aligned} \tag{3.3.30}$$

From the estimate (3.3.29), we know that $u_{N_k} \rightharpoonup u$ in $L^2(Q)$, $S(x,t,D(u_{N_k}) \rightharpoonup \tilde{S}$ in the space $L^{p'}(Q)$ and $|u_{\eta(\varepsilon),N_k} \otimes u_{N_k}|_{L^2(Q)} \le C\dfrac{1}{\varepsilon^2}\|u_0\|_H^2$, by Vitali's theorem, we have as $k \to \infty$

$$\int_Q (u_{N_k} \cdot \partial_t \phi) \mathrm{d}x\mathrm{d}t \to \int_Q (u \cdot \partial_t \phi) \mathrm{d}x\mathrm{d}t,$$
$$\int_Q (u_{N_k} \otimes u_{N_k}) : D(\phi) \mathrm{d}x\mathrm{d}t \to \int_Q (u_{\eta(\varepsilon)} \otimes u) : D(\phi) \mathrm{d}x\mathrm{d}t.$$

From these convergence and the formula (3.3.30), we know that

$$\int_Q (S(x,t,D(u_{N_k})) : D(\phi) \mathrm{d}x\mathrm{d}t \to \int_Q (\tilde{S} : D(\phi) \mathrm{d}x\mathrm{d}t.$$

One need to check that $S(x,t,D(u)) = \tilde{S}$ a.e. in $\mathbb{R}^3_+ \times [0,T]$. It suffice to prove that

$$\int_Q (S(x,t,D(u_{N_k})) : D(\phi) \mathrm{d}x\mathrm{d}t \to \int_Q (S(D(u)) : D(\phi) \mathrm{d}x\mathrm{d}t.$$

Indeed, observing that for any $R_N > R_{N_K}$ the solutions u_N satisfies the hypotheses of Theorem 3.3.6 and lemma 3.3.7 with $\Omega = \Omega_{N_K}$ and fixed a set Ω' such that $G \subset \Omega' \times (0,T) \subset\subset \Omega \times (0,T)$, we get that $u_N \in L^p(0,T;W^{2,p}(\Omega'))$ and

$$\|u'_{N_k}\|_{L^2((0,T)\times\Omega')} + \|\nabla u_{N_k}\|_{L^\infty(0,T;L^2(\Omega'))}$$
$$+ \|\nabla^2 u_{N_k}\|_{L^p(\Omega'\times(0,T))} \leq C(\frac{1}{\varepsilon}, \delta(\Omega',\Omega), |\Omega'|, u_0, f, T), \text{ in the case } \frac{9}{5} < p < 2;$$
$$\|u'_{N_k}\|_{L^2((0,T)\times\Omega')} + \|\nabla u_{N_k}\|_{L^\infty(0,T;L^2(\Omega'))}$$
$$+ \|\nabla^2 u_{N_k}\|_{L^2(\Omega'\times(0,T))} \leq C(\frac{1}{\varepsilon}, \delta(\Omega',\Omega), |\Omega'|, u_0, f, T), \text{ in the case } p \geq 2.$$

From the boundedness above and the Aubin-Lions lemma we obtain that

$$\nabla u_{N_k} \to \nabla u \text{ in } L^p(\Omega' \times (0,T)), \quad \nabla u_{N_k} \to \nabla u \text{ a.e. in } G.$$

Therefore, $S(x,t,D(u_{N_k})) \to S(x,t,D(u))$ a.e. in G, then by the Vitali's theorem, we get as $k \to \infty$

$$\int_Q ((S(x,t,D(u_{N_k})) - (S(D(u))) : D(\phi) \mathrm{d}x\mathrm{d}t =$$
$$\int_G ((S(x,t,D(u_{N_k})) - (S(D(u))) : D(\phi) \mathrm{d}x\mathrm{d}t \to 0.$$

Whence, this proves the theorem. □

3.4 The proof of Theorem 3.1.3

In this section, for brevity, assume that $f = 0$, we will give the proof of theorem 3.1.3.

To this end, we still need a lemma which presents the local estimate of the pressure with slip boundary conditions.

Lemma 3.4.1 *Let $G \subset\subset \mathbb{R}^3_+$, $S \in L^r(G \times [0,T])^{n^2} (r > 1)$ with $(S \cdot n)_\tau|_{x_3=0} = 0$ and $u \in L^\infty(0,T; L^r(G))^n$ with $\nabla \cdot u = 0$ (in the sense of distribution) Suppose that*

$$-\int_0^T \int_G \eta(t)' u \cdot \varphi \mathrm{d}x\mathrm{d}t + \int_0^T \int_G \eta(t) S : \nabla\varphi \mathrm{d}x\mathrm{d}t = 0 \tag{3.4.1}$$

holds for all $\varphi \in C_0^\infty(\bar{G})$ with $\nabla \cdot \varphi = 0$, $\varphi \cdot n|_{x_3=0} = 0$ and $\eta \in C_0^\infty(0,T)$ Then there exists a unique function $\pi \in L^\infty(0,T; L^r(G))$ with $\int_G \pi(x,t)\mathrm{d}x = 0$, such that

$$-\int_G (u(t) - u(0)) \cdot \varphi \mathrm{d}x + \int_G S : \nabla\varphi \mathrm{d}x = \int_G \pi(t) \nabla \cdot \varphi \mathrm{d}x \tag{3.4.2}$$

holds for all $\varphi \in C_0^\infty(\bar{G})$ with $\varphi \cdot n|_{x_3=0} = 0$. Moreover,

$$\|\pi\|_{L^\infty(0,T,L^r(G))} \leq C(\|u\|_{L^\infty(0,T,L^r(G))} + \|S\|_{L^r(G\times[0,T])}), \tag{3.4.3}$$

where C dependent only G, r and T.

Proof: Let $\alpha(t) = \int_G u \cdot \varphi \mathrm{d}x$ and $\beta(t) = \int_G S : \nabla\varphi \mathrm{d}x$ for $\varphi \in C_0^\infty(\bar{G})$ with $\varphi \cdot n|_{x_3=0} = 0$. From the formula (3.3.1) and Fubini's Theorem, one yields

$$-\int_0^T \alpha\eta' \mathrm{d}t = \int_0^T \beta\eta \mathrm{d}t.$$

Since $S \in L^r(G \times [0,T])$, thus $\alpha \in W^{1,r}(0,T)$ and $\frac{\mathrm{d}}{\mathrm{d}t}(\alpha) = -\beta$. By the Sobolev Imbedding theorem, we know that α is continuous in the interval $[0,T]$ and

$$\int_G (u(t) - u(0)) \cdot \varphi \mathrm{d}x + \int_0^T \int_G S : \nabla\varphi \mathrm{d}x = 0$$

, denote $\tilde{S}(t) = \int_0^t S(s)\mathrm{d}s,\ t \in [0,T]$ By the Fubini's Theorem,

$$\int_G (u(t) - u(0)) \cdot \varphi \mathrm{d}x + \int_G \tilde{S}(t) : \nabla\varphi \mathrm{d}x = 0,$$

From the Proposition 1 in [50], there exists a unique function $\pi(t) \in L^r(G)$ with $\int_G \pi(t)\mathrm{d}x = 0$, such that for any $\varphi \in C_0^\infty(\bar{G})$ with $\varphi \cdot n|_{x_3=0} = 0$ such that

$$-\int_G (u(t) - u(0)) \cdot \varphi \mathrm{d}x + \int_G \tilde{S} : \nabla\varphi \mathrm{d}x = \int_G \pi(t)\nabla \cdot \varphi \mathrm{d}x. \tag{3.4.4}$$

In additions, there exists a constant $C > 0$, depending only on r, n, and G (see [6])such that

$$\|\pi(t)\|_{L^r(G)} \le C(\|u(t) - u(0)\|_{L^r(G)} + \|\tilde{S}\|_{L^r(G)}).$$

Choose any $v \in L^{r'}(G)$, there exists a function $\psi \in D^1(G)$ such that

$$\nabla \cdot \psi = v - \frac{1}{|G|}\int_G v\mathrm{d}x = v - v_G$$

see [20]. Thus (3.4.4) implies

$$\int_G \pi(t)v\mathrm{d}x = \int_G \pi(t)(v - v_G)\mathrm{d}x = -\int_G (u(t) - u(0)) \cdot \psi \mathrm{d}x + \int_G \tilde{S} : \nabla\psi \mathrm{d}x.$$

Since the function on the right side above is continuous, so also the function on the left. Consequently, $\pi(t) \in L^\infty(0, T; L^r(G))$. By the Hölder's inequality, one easily verifies the following a-priori estimate

$$\|\pi\|_{L^\infty(0,T;L^r(G))} \le C(\|u\|_{L^\infty(0,T;L^r(G))} + \|S\|_{L^r(G\times[0,T])}),$$

□

Remark 3.4.2 *From this proof, we can conclude other estimates for local pressure π as follows*

$$\|\pi\|_{L^r(G\times(0,T))} \le C(\|u\|_{L^r(G\times(0,T))} + \|S\|_{L^r(G\times[0,T])}).$$

The proof of Theorem 3.1.3: Let $u_\varepsilon \in L^p(0, T; V_p(\mathbb{R}^3_+)) \cap L^\infty(0, T; H)$ denote the weak solution to the approximate system (4.1.12) from the Theorem

3.3.8, and we have there exists a constant $K > 0$ only depending on the initial data,

$$\|u_\varepsilon\|^2_{L^\infty(0,T;H)} + \|u_\varepsilon\|^p_{L^p(0,T;V_p)} \leq K. \tag{3.4.5}$$

By (3.1.9) and interpolation inequalities, we obtain that

$$\|S(\cdot, D(u_\varepsilon)\|_{L^{p'}(Q)} \leq K, \tag{3.4.6}$$

$$\|u_\varepsilon\|_{L^{\frac{p(n+2)}{n}}(Q)} \leq K, \tag{3.4.7}$$

where $\frac{1}{p'} + \frac{1}{p} = 1$. From the boundedness (3.4.5)-(3.4.7) and Proposition 2, there exists a sequence of $\{\varepsilon_k\}$ with $\varepsilon_k \to 0$ as $k \to \infty$ and functions $u \in L^p(0,T;V_p(\mathbb{R}^3_+)) \cap L^\infty(0,T;H)$ and $\tilde{S} \in L^{p'}(Q)$ such that as $k \to \infty$

$$\begin{aligned} &u_{\varepsilon_k} \overset{*}{\rightharpoonup} u, && \text{weakly*in } \ L^\infty(0,T;H) \\ &u_{\varepsilon_k} \rightharpoonup u, && \text{weakly in } \ L^p(0,T;V_p(\mathbb{R}^3_+)), \\ &S(x,t,D(u_{\varepsilon_k}) \rightharpoonup \tilde{S} && \text{weakly } \ L^{p'}(Q), \\ &u_{\eta(\varepsilon_k)} \otimes u_{\varepsilon_k} \rightharpoonup u \otimes u, && \text{in } \ L^{\frac{p(n+2)}{2n}}(Q). \end{aligned} \tag{3.4.8}$$

Then the identity

$$-\int_Q (u \cdot \partial_t \phi) \mathrm{d}x\mathrm{d}t + \int_Q (\tilde{S} - u \otimes u) : D(\phi) \mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^3_+} u_0 \cdot \phi(0) \mathrm{d}x \tag{3.4.9}$$

holds for any $\phi \in C_0^\infty(\overline{\mathbb{R}^3_+} \times [0,T))$ with $\nabla \cdot \phi = 0$ and $\phi \cdot n|_{x_3=0} = 0$. For simplicity, sometimes we denote $u_k = u_{\varepsilon_k}$, $S_k = S(x,t,D(u_{\varepsilon_k}))$. To end this proof, we must prove that $\tilde{S} = S(x,t,D(u))$ a.e. in $\mathbb{R}^3_+ \times [0,T]$. As in the [79], it suffices to prove that as $k \to \infty$

$$\int_{G\times[\delta,T-\delta]} (S(x,t,D(u_k)) - S(x,t,D(u))) : (D(u_k) - D(u)) \mathrm{d}x\mathrm{d}t \to 0 \tag{3.4.10}$$

for all bounded compact set $G \subset \overline{\mathbb{R}^3_+}$ and any $0 < \delta < \frac{T}{2}$.

If (3.4.10) holds, for any cutoff function $\Psi \in C_0^\infty(\overline{\mathbb{R}^3_+} \times (0,T))$, then

$$\begin{aligned}
&\int_Q (\tilde{S} - S(x,t,D(v))) : (D(u) - D(v))\Psi \mathrm{d}x\mathrm{d}t, \\
&= \int_Q (\tilde{S} - S(x,t,D(u_k))) : (D(u) - D(v))\Psi \mathrm{d}x\mathrm{d}t \\
&- \int_Q (S(x,t,D(u_k)) - S(x,t,D(u))) : (D(u_k) - D(u))\Psi \mathrm{d}x\mathrm{d}t \\
&+ \int_Q (S(x,t,D(u_k)) - S(x,t,D(v))) : (D(u_k) - D(v))\Psi \mathrm{d}x\mathrm{d}t \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

By the (3.4.8) and (3.4.10), we know $I_i \to 0 (i = 1, 2)$ as $k \to \infty$. From (3.1.7), it shows that $I_3 \geq 0$. By local Minty Trick theorem (see the appendix of [115]), we know $\tilde{S} = S(x,t,D(u))$ a.e. in $\mathbb{R}^3_+ \times (0,T)$.

Next, we will prove that (4.3.9) by some $L^\infty-$truncation method. As in [44] or [23], Let $g^k = |\nabla u_k|^p + |\nabla u|^p + (|S_k| + |S(x,t,D(u))|)(|D(u_k) + D(u)|)$. The following lemma shows that the properties of g^k on $\mathbb{R}^3_+ \times (0,T)$, its proof can be found in [44] for steady case and in [23] for the unsteady case.

Lemma 3.4.3 *$\eta > 0$, there exists $L \leq \dfrac{\eta}{K}$ and there is a subsequence $\{u_k\}_{k=1}^\infty$ and sets $E^k = \{(x,t) \in \mathbb{R}^3_+ \times (0,T) : L^2 \leq |u_k - u| \leq L\}$ such that*

$$\int_{E^k} g^k \mathrm{d}x\mathrm{d}t \leq \eta. \tag{3.4.11}$$

Denote $Q^k = \{(x,t) \in \mathbb{R}^3_+ \times (0,T) : |u_k - u| \leq L\}$ and $\psi^k = (u_k - u)\left(1 - \min\left(\dfrac{|u_k - u|}{L}, 1\right)\right)$. Then we can prove that the following proposition described the required properties of ψ^k.

Proposition 3.4.4 *(1) $\psi^k \in L^p(0,T;V_p) \cap L^\infty(0,T;H)$ and*

$$\|\psi^k\|_{L^\infty(\mathbb{R}^3_+ \times (0,T))} \leq L;$$

(2) $\psi^k \rightharpoonup 0$ in $L^p(0,T;V_p(\mathbb{R}^3_+)$;

(3) $\psi^k \to 0$ *in* $L^s(0,T;L^s_{\rm loc}(\mathbb{R}^3_+)$ *for all* $1 \le s < \infty$*;*

(4) $|{\rm div}\psi^k| \le \left|\frac{1}{L}(u^k-u)\cdot\nabla|u^k-u|\chi_{Q^k}\right|$*; and*

$$|{\rm div}\psi^k|_{L^p(\mathbb{R}^3_+\times(0,T))} \le C\eta, \ \ |\nabla\psi^k|_{L^p(\mathbb{R}^3_+\times(0,T))} \le C\eta,$$

where C *is independent of* k *and* χ_{Q^k} *denotes the characteristic function of the set* Q^k*;*

Proof: It is easy to see that (1) and (2) hold. By the simple calculation and (3.4.11), we know that (4) hold. Now we check that (3). Indeed, since $V_p(\mathbb{R}^3_+) \hookrightarrow L^p(G)$ is compact for all $G \subset\subset \overline{\mathbb{R}^3_+}$, it follows that $\psi^k \to 0$ in $L^p(0,T;L^p(G))$ and there exists a subsequence which denote $\psi^k \to 0$ a.e. in $G\times(0,T) := G_T$. Therefore, for $p < s < \infty$,

$$\int_{G_T}|\psi^k|^s{\rm d}x{\rm d}t \le \|\psi^k\|^{s-p}_{L^\infty(\mathbb{R}^3_+\times(0,T))}\int_{G_T}|\psi^k|^s{\rm d}x{\rm d}t \le L\|\psi^k\|^p_{L^p(G_T)} \le o(1).$$

For $1 \le s \le p$, we have

$$\int_{G^\delta_T}|\psi^k|^s{\rm d}x{\rm d}t \le \left(\int_{G_T}|\psi^k|^p{\rm d}x{\rm d}t\right)^{\frac{s}{p}}\left|G^\delta_T\right|^{1-\frac{s}{p}} \le K\left|G^\delta_T\right|^{1-\frac{s}{p}}.$$

Let $\left|G^\delta_T\right| < \delta$ small enough, by Vitali's theorem we have $\psi^k \to 0$ in $L^s(0,T;L^s_{\rm loc}(\mathbb{R}^3_+)$ for all $1 \le s \le p$. Hence, we proved (3). □

Consider the following problem

$$\begin{cases} \Delta z^k = {\rm div}\psi^k, & \text{in } \mathbb{R}^3_+; \\ \dfrac{\partial z^k}{\partial n} = 0, & \text{on } \{x_3 = 0\}, \\ z^k(x) \to 0, & \text{as } |x| \to \infty. \end{cases}$$

Then there exists a unique solutions with the form (see [82],or [101])

$$z^k(x,t) = \int_{\mathbb{R}^3_+} N(x,y){\rm div}\psi^k(y,t){\rm d}y \ \forall x \in \mathbb{R}^3_+,$$

where

$$N(x,y) = \frac{1}{|x-y|} - \frac{1}{|x-y^*|}, \ \text{with} y^* = (y_1,y_2,-y_3), \ \forall y \in \mathbb{R}^3_+.$$

By the theorems on the singular integrals (see [104]), we have the following estimates

$$\begin{aligned} \|\nabla z^k\|_{L^s(G_T)} &\le C(s,G,T)\|\psi^k\|_{L^s(G_T)}; \\ \|\nabla^2 z^k\|_{L^p(Q)} &\le C(p)\|\mathrm{div}\psi^k\|_{L^p(Q)} \le C\eta. \end{aligned} \tag{3.4.12}$$

For any bounded with smooth boundary set $G \subset \overline{\mathbb{R}^3_+}$, we can choose another set $G \subset\subset\ G' \subset\subset \overline{\mathbb{R}^3_+}$, and any positive number $\delta > 0$. Define smooth functions $\tau \in C_0^\infty(\frac{\delta}{2}, T - \frac{\delta}{2})$ and $\zeta \in C_0^\infty(\overline{G'})$ such that $0 \le \tau \le 1$ in $(\frac{\delta}{2}, T-\frac{\delta}{2})$ and $\tau \equiv 1$ in $(\delta, T-\delta)$, $0 \le \zeta \le 1$ in G' and $\zeta \equiv 1$ in G. Let $\varphi^k = \tau\zeta(\psi^k - \nabla z^k)$, then $\varphi^k \in C_0^\infty((0,T); W_0^{1,p}(\mathbb{R}^3_+))$.

Use the Lemma 3.4.1 for $\tilde{S}_k$ and $\|S_k - u_{\eta(k)} \otimes u_k\|_{L^r(G'\times(0,T))} \le C$, where C only depends on G', G, p, and $r = \min\{\frac{p}{p-1}, p\frac{5}{6}\} < 2$, we have there exist π^k with vanish mean value such that

$$\begin{aligned} &\int_{G'_T} (S(x,t,D(u_k)) - \tilde{S}) : \nabla\xi \mathrm{d}x\mathrm{d}t \\ &= -\int_{G'_T} (u_k - u, \partial_t\xi)\mathrm{d}x\mathrm{d}t \\ &- \int_{G'_T} (u_{\varepsilon(k)} \otimes u_k - u\otimes u)\nabla\xi \mathrm{d}x\mathrm{d}t \\ &+ \int_{G'_T} (\pi^k - \pi)(\nabla\cdot\xi)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.4.13}$$

for all $\xi \in C_0^\infty(\overline{G'} \times (0,T))$ with $\xi \cdot n|_{x_3=0} = 0$ holds and the estimate $\|\pi^k - \pi\|_{L^\infty(0,T;L^r(G'))} \le C$.

From Remark 3.4.2 and (3.2.1), it follows that

$$\|(u_k)'\|_{L^r(0,T;W^{-1,r})} \le C,$$

where C does not only depend on k. By Aubin-Lions Lemma, we easily obtain that $u_k \to u$ a.e. in $G \times (\delta, T-\delta) := G_{\delta T}$.

Take φ^k as a test function in (3.4.13) , we have

$$\begin{aligned}
&\int_0^T S(x,t,D(u_k)) : \nabla\varphi^k \mathrm{d}x\mathrm{d}t \\
&= \int_Q \tilde{S} : \nabla\varphi^k \mathrm{d}x\mathrm{d}t - \int_0^T \langle u_k' - u', \varphi^k\rangle \mathrm{d}t \\
&- \int_Q \nabla\cdot(u_{\varepsilon(k)} \otimes u_k - u\otimes u)\varphi^k \mathrm{d}x\mathrm{d}t \\
&+ \int_Q (\pi^k - \pi)(\nabla\cdot\varphi^k)\mathrm{d}x\mathrm{d}t \\
&:= D_1 + D_2 + D_3 + D_4.
\end{aligned} \tag{3.4.14}$$

From proposition 3.4.4, we can know that $D_1 \to 0$ as $k \to \infty$. Since $\|\nabla\cdot(u_{\varepsilon(k)} \otimes u_k)\|_{L^\sigma(G_T')} = \|u_{\varepsilon(k)}\cdot\nabla u_k\|_{L^\sigma(G_T')} \le C$, where $\sigma = \dfrac{5}{8}p > 1$ provided that $p > \dfrac{8}{5}$ and C is independent of k. Therefore,

$$\begin{aligned}
D_3 &\le \left|\nabla\cdot(u_{\varepsilon(k)}\otimes u_k - u\otimes u)(\psi^k - \nabla z^k)\right| \mathrm{d}x\mathrm{d}t \\
&\le C\|\psi^k - \nabla z^k\|_{L^{\sigma'}(G_T')} = o(1) \text{ as } k \to \infty.
\end{aligned}$$

From the argument above, it is easy to see that

$$D_4 \le \|\pi^k - \pi\|_{L^r(G_T')}\|\nabla\cdot\varphi^k\|_{L^{r'}(G_T')} = o(1) \text{ as } k\to\infty.$$

In fact, since $\nabla\cdot\varphi^k = \tau\nabla\zeta\cdot(\psi^k - \nabla z^k)$, thus from Proposition 3.4.4(3) and (3.4.12), we have

$$\|\nabla\cdot\varphi^k\|_{L^{r'}(G_T')} = \|\tau\nabla\zeta\cdot(\psi^k - \nabla z^k)\|_{L^{r'}(G_T')} \le C\|\psi^k - \nabla z^k\|_{L^{r'}(G_T')} \to 0 \text{ as } k \to \infty.$$

Now estimate the term D_2, Let $w = u_k - u \in L^p(0,T;V_p)$, then by lemma 3.4 [82] there exist a sequence $w_n \in C^\infty(0,T;D(\mathbb{R}^3_+))$ with $\mathrm{div} w_n = 0$, and such that $w_n' \to w'$ in $L^r(0,T;W^{-1,r}(G'))$ and $w_n \to w$ in $L^p(0,T;V_p)$. One have

$$\begin{aligned}
|D_2| &= \int_0^T \langle u_k' - u', \tau\zeta(\psi^k - \nabla z^k)\rangle \mathrm{d}t \\
&= \lim_{n\to\infty}\int_0^T \langle w_n', \tau\zeta(w_n(1-\min(\frac{|w_n|}{L},1)) - \nabla z^k)\rangle \mathrm{d}t = D_{2_1} + D_{2_2}.
\end{aligned}$$

Let

$$F_n(x,t) = \begin{cases} |w_n|^2(1 - \frac{2}{3}\frac{|w_n|}{L}) & \text{if } |w_n| < L; \\ \frac{1}{3}L^2 & \text{if } |w_n| \geq L. \end{cases}$$

Therefore, we know that

$$\begin{aligned} D_{2_1} &= \lim_{n\to\infty} \int_0^T \int_\Omega w_n' \tau\zeta(w_n(1 - \min(\frac{|w_n|}{L}, 1)))\mathrm{d}\mathrm{d}t \\ &= \lim_{n\to\infty} \int_0^T \int_\Omega (\tau\zeta F_n(x,t))'\mathrm{d}x\mathrm{d}t - \lim_{n\to\infty} \int_0^T \int_\Omega \tau'\zeta F_n(x,t)\mathrm{d}x\mathrm{d}t \\ &\leq 0 + C|G_T'|L^2. \end{aligned}$$

Since $\mathrm{div} w_n = 0$, so

$$\begin{aligned} D_{2_2} &= -\lim_{n\to\infty} \int_0^T \int_\Omega (w_n'\tau\zeta\nabla z^k)\mathrm{d}x\mathrm{d}t = \lim_{n\to\infty} \int_0^T \int_\Omega (w_n'\tau\nabla\zeta(z^k - \bar{z}^k))\mathrm{d}x\mathrm{d}t \\ &\leq \lim_{n\to\infty} \|w_n'\|_{L^r(0,T;W^{-1,r}(G'))} \|\nabla\zeta(z^k - \bar{z}^k))\|_{L^{r'}(0,T;W^{1,r'}(G'))} \\ &\leq \lim_{n\to\infty} \|w_n'\|_{L^r(0,T;W^{-1,r}(G'))} \|\nabla(z^k)\|_{L^{r'}(0,T\times G')} \\ &\leq C\|(u^k)' - u'\|_{L^r(0,T;W^{-1,r}(G'))} \|\psi^k\|_{L^{r'}(0,T\times G')} \\ &= o(1). \end{aligned}$$

So far, we can conclude that

$$\int_0^T \int_{\mathbb{R}^3_+} S(x,t,D(u_k)) : D(\varphi^k)\mathrm{d}x\mathrm{d}t \leq o(1) + C\eta, \text{ as } k \to \infty.$$

It following that

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3_+} S(x,t,D(u_k)) : D(\psi^k)\tau\zeta\mathrm{d}x\mathrm{d}t \\ &\leq \int_0^T \int_{\mathbb{R}^3_+} S(x,t,D(u_k)) : D(\nabla z^k)\tau\zeta\mathrm{d}x\mathrm{d}t \\ &- \int_0^T \int_{\mathbb{R}^3_+} S(x,t,D(u_k)) : ((\psi^k - \nabla z^k) \otimes \nabla\zeta)\tau\mathrm{d}x\mathrm{d}t + o(1) + C\eta \end{aligned}$$

Since $\left|\int_0^T \int_{\mathbb{R}^3_+} S(x,t,D(u_k)) : D(\nabla z^k)\tau\zeta\mathrm{d}x\mathrm{d}t\right| \leq K\|\nabla^2 z^k\|_{L^p(Q)} \leq C\eta$ from (4.12), and

$$\left|-\int_0^T \int_{\mathbb{R}^3_+} S(x,t,D(u_k)) : ((\psi^k - \nabla z^k) \otimes \nabla\zeta)\tau\mathrm{d}x\mathrm{d}t\right| \leq CK\|\psi^k - \nabla z^k\|_{L^p(G_T')} = o(1).$$

Therefore, we have

$$\int_0^T \int_{\mathbb{R}^3_+} S(x,t,D(u_k)) \quad D(\psi^k)\tau\zeta \mathrm{d}x\mathrm{d}t \le C\eta + o(1) \text{ as } k \to \infty,$$

where C is independent of k, η

On the other hand,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3_+} S(x,t,D(u_k)) \quad D(\psi^k)\tau\zeta \mathrm{d}x\mathrm{d}t \\
&= \int_{Q^k} S(x,t,D(u_k)) \quad D(u_k-u)\left(1-\min\left(\frac{|u_k-u|}{L},1\right)\right)\tau\zeta \mathrm{d}x\mathrm{d}t \\
&+ \int_{Q^k} S(x,t,D(u_k)) \quad \mathrm{sym}\left((\frac{u_k-u}{L})\otimes\nabla|u_k-u|\right)\tau\eta \mathrm{d}x\mathrm{d}t \\
&= \int_{Q^k} (S(x,t,D(u_k))-S(x,t,D(u))) \quad D(u_k-u)\tau\zeta \mathrm{d}x\mathrm{d}t \\
&+ \int_{Q^k} S(x,t,D(u)) \quad D(u_k-u)\tau\zeta \mathrm{d}x\mathrm{d}t \\
&- \int_{Q^k} S(x,t,D(u_k)) \quad D(u_k-u)\min\left(\frac{|u_k-u|}{L},1\right)\tau\zeta \mathrm{d}x\mathrm{d}t \\
&+ \int_{Q^k} S(x,t,D(u_k)) \quad \mathrm{sym}\left((\frac{u_k-u}{L})\otimes\nabla|u_k-u|\right)\tau\eta \mathrm{d}x\mathrm{d}t \\
&= J_1+J_2+J_3+J_4,
\end{aligned}$$

where $\mathrm{sym}\left((\frac{u^k-u}{L})\otimes\nabla|u_k-u|\right)$ is the symmetric part of $(\frac{u_k-u}{L})\otimes\nabla|u^k-u|$ Clearly, since $D(u_k) \rightharpoonup D(u)$ in $L^p(Q)$, thus $J_2 \to 0$ as $k \to \infty$ As in (4)of proposition 3 4 4, we can compute that

$$\begin{aligned}
|J_3|+|J_4| \quad &\le \int_{E^k} |S(x,t,D(u_k)) \quad D(u_k-u)|\mathrm{d}x\mathrm{d}t \\
&+L\int_{Q^k\setminus E^k} |S(x,t,D(u_k)) \quad D(u_k-u)|\mathrm{d}x\mathrm{d}t \\
&\le \int_{E^k} g^k \mathrm{d}x\mathrm{d}t + L\int_{Q^k\setminus E^k} g^k \mathrm{d}x\mathrm{d}t \le \eta + KL \\
&\le C\eta
\end{aligned}$$

Consequently,

$$\int_{Q^k} (S(x,t,D(u_k))-S(x,t,D(u))) \quad D(u_k-u)\tau\zeta \mathrm{d}x\mathrm{d}t \le o(1)+C\eta, \text{ as } k\to\infty \tag{3 4 15}$$

Since $u_k \to u$ a.e. in $G \times (\delta, T-\delta)$ and G, δ are arbitrary, thus we know that $u_k \to u$ a.e. in Q. Hence, choose a sequence which still denote $\{u_k\}$ satisfying $|G'_T \setminus Q^k| \leq 2^{-k}$, for all $k \in \mathbf{N}$. Thus there exists $k_0 \in \mathbf{N}$ such that $2^{-k_0} < \eta$, and one can find

$$\sum_{k=k_0+1}^{\infty} |G'_T \setminus Q^k| \leq 2^{-k_0} < \eta.$$

Setting $M = \bigcup_{k=k_0+1}^{\infty} (G'_T \setminus Q^k)$, from (3.4.5)-(3.4.6) we have

$$\int_M (S(x,t,D(u_k)) - S(x,t,D(u))) : D(u_k - u)\tau\zeta \mathrm{d}x\mathrm{d}t \leq C\eta. \tag{3.4.16}$$

Whence (3.4.15) and (3.4.16) infers that

$$\int_{G'_T} (S(x,t,D(u_k)) - S(x,t,D(u))) : D(u_k - u)\tau\zeta \mathrm{d}x\mathrm{d}t \leq o(1) + C\eta, \text{ as } k \to \infty. \tag{3.4.17}$$

From (3.1.7), it implies

$$\int_{G_{\delta T}} (S(x,t,D(u_k)) - S(x,t,D(u))) : D(u_k - u)\mathrm{d}x\mathrm{d}t \to 0, \text{ as } k \to \infty.$$

This proves the main theorem.

Remark 3.4.5 *We can assume that $f \in L^{p'}(0,T;V^*) \cap L^2(Q)$. Then all estimates (3.4.5)-(3.4.7) also depend on $\|f\|_{L^{p'}(0,T;V^*)\cap L^2(Q)}$. In the proof of (3.4.10), we must estimate the term $\int_0^T \langle f, \varphi^k \rangle \mathrm{d}t$. Indeed it is easy to obtain it as follows;*

$$\begin{aligned}\int_0^T \langle f, \varphi^k \rangle \mathrm{d}t \ &\leq \int_0^T \langle f, \tau\zeta(u_k - u)\rangle \mathrm{d}t - \int_0^T \langle f, \tau\zeta(u_k - u)\min\left(\frac{u_k - u}{L}, 1\right)\rangle \mathrm{d}t \\ &\quad - \int_0^T \langle f, \tau\zeta\nabla z^k \rangle \mathrm{d}t.\end{aligned}$$

The first term on the right vanishes as $k \to \infty$, while the second term is estimated analogously as J_3 and J_4. Finally, the third term is small thanks to (3.4.12). Therefore, we obtain

$$\int_0^T \langle f, \varphi^k \rangle \mathrm{d}t \leq o(1) + c\eta.$$

It shows that this term can not change the statement of the main theorem.

Remark 3.4.6 *If* $p \geq \frac{12}{5}$, *use the the theorem 7.2 of [15], we extend the basis* a_k *to whole space. Then apply the argument in [33], we can obtain the weak solution is a unique global-in-time strong solution for the problem (3.1.1).*

However, if $\frac{9}{5} < p < \frac{12}{5}$, *use the basis obtained above and the argument in [33], one can get a unique local-in-time strong solution to the problem (3.1.1), but we cannot know whether the weak solution is the exact strong solution.*

Chapter 4

Smooth solutions for motion of a rigid body of general form in an incompressible perfect fluid

In this chapter, we investigate the motion of a general form rigid body with smooth boundary by an incompressible perfect fluid occupying $\mathbb{R}^3$. Due to the domain occupied by the fluid depending on the time, this problem can be transformed into a new systems of the fluid in a fixed domain by the frame attached with the body. With the aid of Kato-Lai's theory, we construct a sequence of successive solutions to this problem in some unform time interval. Then by a fixed point argument, we have proved that the existence, uniqueness and persistence of the regularity for the solutions of original fluid-structure interaction problem.

4.1 Introduction

In this chapter, we investigate the motion of a rigid body immersed in an incompressible perfect fluid. The behavior of the fluid is described by the Euler equations, while the motion of the rigid body conforms to the Newton's law. Assume that both the fluid and the rigid body are homogeneous. The domain occupied by the solid at the time is $\mathcal{O}(t)$, and $\Omega(t) = \mathbb{R}^3 \setminus \overline{\mathcal{O}(t)}$ is the domain occupied by the fluid. Suppose $\mathcal{O}(0) = \mathcal{O}$ and $\Omega(0) = \Omega$ share a smooth boundary $\partial\mathcal{O}$(or $\partial\Omega$). The equations modeling the dynamics of the system has been in the introduction, see (0.8)-(0.15)

To solve such a problem, we must fix the region occupied by the fluid. For simplicity, we assume that $f = 0, u_\infty = 0, f_{rb} = 0$ and $T_{rb} = 0$.

Generally, it is natural to adapt a ideal by attaching the coordinates system to the rigid body. Let $Q(t)$ be a rotation matrix associated with the angular velocity $\omega(t)$ of the rigid body, which is the solution of the following initial value problem:

$$\begin{cases} \dfrac{\mathrm{d}Q(t)}{dt} = A(\omega(t))Q(t) \\ Q(0) = Id. \end{cases} \tag{4.1.1}$$

Here

$$A(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix},$$

and Id is the identity matrix. Then the domain $\mathcal{O}(t)$ is defined by

$$\mathcal{O}(t) = \{Q(t)y + h(t) : y \in \mathcal{O}(0)\}.$$

Set

$$\begin{aligned} x &= Q(t)y + h(t), & \bar{u}(y,t) &= Q(t)^T u(Q(t)y + h(t), t),\\ \bar{p}(y,t) &= p(Q(t)y + h(t), t), & \bar{h}(t) &= \int_0^t Q(s)^T h'(s)ds,\\ \bar{J} &= J(0), & \bar{\omega}(y,t) &= Q(t)^T \omega, \end{aligned}$$

where $Q(t)$ is given in (4.1.1) and $Q(t)^T$ is the transpose of $Q(t)$.

After the transformation, an equivalent system is obtainded as follows:

$$\frac{\partial \bar{u}}{\partial t} + [(\bar{u} - \bar{h}' - \bar{\omega} \times y) \cdot \nabla]\bar{u} + \bar{\omega} \times \bar{u} + \nabla \bar{p} = 0, \quad \text{in } \Omega \times [0,T], \tag{4.1.2}$$

$$\text{div } \bar{u} = 0, \quad \text{in } \Omega \times [0,T], \tag{4.1.3}$$

$$\bar{u} \cdot \boldsymbol{n} = (\bar{h}' + \bar{\omega} \times y) \cdot \boldsymbol{n}, \quad \text{on } \partial\Omega \times [0,T], \tag{4.1.4}$$

$$m\bar{h}'' = \int_{\partial\Omega} \bar{p}\boldsymbol{n} d\Gamma - m\bar{\omega}(t) \times \bar{h}'(t), \quad \text{in}[0,T], \tag{4.1.5}$$

$$\bar{J}\bar{\omega}' = \int_{\partial\Omega} y \times \bar{p}\boldsymbol{n} d\Gamma + \big(\bar{J}\bar{\omega}(t)\big) \times \bar{\omega}(t), \quad \text{in}[0,T], \tag{4.1.6}$$

$$\bar{u}(y,0) = u_0, \quad y \in \Omega, \tag{4.1.7}$$

$$\bar{h}(0) = 0,\ \bar{h}'(0) = l_0,\ \bar{\omega}(0) = \omega_0. \tag{4.1.8}$$

The new problem is a fixed boundary problem now. However, there is a term $[\bar{\omega} \times y) \cdot \nabla)]\bar{u}$, whose coefficient become unbounded at large spatial distance. For the 2D case, the difficulty was overcome in [92] by assuming that u_0 belongs to some weighted space. However, the 3D case is much more complicated, since vorticity does not satisfy a transport equation any more. To avoid this term, we will use another change of variables. The new transformation coincides with $Q(t)y + h(t)$ in a neighborhood of the solid and becomes identity when far away from it.

More precisely, fix a pair of functions $(l(t), \omega(t))$, let

$$h(t) = \int_0^t l(s)ds, \quad t \in [0,T], \tag{4.1.9}$$

$$V(x,t) = l(t) + \omega(t) \times (x - h(t)),\ (x,t) \in \mathbb{R}^3 \times [0,T], \tag{4.1.10}$$

which is a rigid body movement.

Choose a smooth function $\xi : \mathbb{R}^3 \to \mathbb{R}$ with compact support such that $\xi(x) = 1$ in a neighborhood of $\bar{\mathcal{O}}$, and set

$$\psi(x,t) = \xi\left[Q(t)^T(x - h(t))\right].$$

Then introduce the functions W and Λ,

$$W(x,t) = \frac{1}{2}l(t) \times (x - h(t)) + \frac{|x - h(t)|^2}{2}\omega, \tag{4.1.11}$$

$$\Lambda(x,t) = \psi V + \nabla\psi \times W. \tag{4.1.12}$$

It is easy to check that Λ satisfies the following lemma.

Lemma 4.1.1 *(1) $\Lambda(x,t) = 0$, if x is far away from $\mathcal{O}(t)$;*

(2) $\Lambda(x,t) = h'(t) + \omega(t) \times (x - h(t))$ in $\mathcal{O}(t) \times [0,T]$;

(3) $\operatorname{div} \Lambda = 0$ in $\mathbb{R}^3 \times [0,T]$;

(4) For all $t \in [0,T], \Lambda(\cdot,t)$ is a $C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ function. Moreover, for every $s \in \mathbb{N}$, $\|\Lambda(\cdot,t)\|_{W^{s,\infty}(\mathbb{R}^3)} \leq C(\xi, s, T)(|l(t)| + |\omega(t)|)$;

(5) For all $x \in \mathbb{R}^3$, the function $\Lambda(x,\cdot)$ is in $C^1([0,T];\mathbb{R}^3)$, provided that $l, \omega \in C^0[0,T]$.

Next, consider the vector field $X(y,t)$ which satisfies

$$\begin{cases} \dfrac{\partial X(y,t)}{\partial t} = \Lambda(X(y,t),t), \quad t \in (0,T], \\ X(y,0) = y \in \mathbb{R}^3. \end{cases} \tag{4.1.13}$$

Lemma 4.1.2 *For every $y \in \mathbb{R}^3$, the initial-value problem (4.1.13) admits a unique solution $X(y,\cdot) : [0,T] \to \mathbb{R}^3$, which is a C^1 function on $[0,T]$. Moreover, the solution has the following properties:*

(1) For all $t \in [0,T]$, the mapping $y \mapsto X(y,t)$ is a C^∞ diffeomorphism from $\mathcal{O}$ onto $\mathcal{O}(t)$ and from Ω onto $\Omega(t)$.

(2) Denote by $Y(\cdot,t)$ the inverse of $X(\cdot,t)$. Then for every $x \in \mathbb{R}^3$ the mapping $t \mapsto Y(x,t)$ is C^1-continuous and satisfies the following initial value problem,

$$\begin{cases} \dfrac{\partial Y(x,s)}{\partial s} = -(J_X)^{-1}\Lambda(Y(x,s),s), \quad s \in (0,T], \\ Y(x,0) = x \in \mathbb{R}^3, \end{cases} \tag{4.1.14}$$

Where J_X is the Jacobian matrices of $X(y,t)$.

(3) For every $x, y \in \mathbb{R}^3$ and for every $t \in [0,T]$, the determinants of the Jacobian matrices J_X of $X(y,t)$ and J_Y of $Y(x,t)$ both equal to 1, i.e.,

$$\det(J_X(y,t)) = \det(J_Y(x,t)) = 1 \tag{4.1.15}$$

For the proof of Lemma 4.1.2, please refer to [103].

Let

$$\begin{array}{ll} x = X(y,t), & v(y,t) = J_Y(X(y,t),t)u(X(y,t),t), \\ q(y,t) = p(X(y,t),t). & H(t) = Q(t)^T h(t), \\ L(t) = Q(t)^T l(t), & R(t) = Q(t)^T \omega(t). \end{array} \tag{4.1.16}$$

Denote

$$\begin{cases} G(y,t) = \left(g^{ij}(y,t)\right) = \left(\displaystyle\sum_{k=1}^{3} \frac{\partial Y_i}{\partial x_k}(X(y,t),t)\frac{\partial Y_j}{\partial x_k}(X(y,t),t)\right), \\ g_{ij} = \displaystyle\sum_{k=1}^{3} \frac{\partial X_k}{\partial y_i}(y,t)\frac{\partial X_k}{\partial y_j}(y,t), \\ \Gamma_{i,j}^k(y,t) = \dfrac{1}{2}\displaystyle\sum_{l=1}^{3} g^{kl}\left\{\frac{\partial g_{il}}{\partial y_j} + \frac{\partial g_{jl}}{\partial y_i} - \frac{\partial g_{ij}}{\partial y_l}\right\} \end{cases} \tag{4.1.17}$$

Now one can transform the original system (0.8)-(0.15) into the following system, which is a fixed boundary problem, (see [103])

$$\frac{\partial v}{\partial t} + Mv + Nv + G \cdot \nabla q = 0, \qquad \text{in } \Omega \times [0,T], \tag{4.1.18}$$

$$\text{div } v = 0, \qquad \text{in } \Omega \times [0,T], \tag{4.1.19}$$

$$v(y,t) \cdot \boldsymbol{n} = (L(t) + R(t) \times y) \cdot \boldsymbol{n}, \qquad \text{on } \partial\Omega \times [0,T], \tag{4.1.20}$$

$$mL'(t) = \int_{\partial\Omega} q\boldsymbol{n} \mathrm{d}\sigma - mR(t) \times L(t), \qquad \text{in } [0,T], \tag{4.1.21}$$

$$\bar{J}R'(t) = \int_{\partial\Omega} y \times q\boldsymbol{n} \mathrm{d}\sigma + \bar{J}R(t) \times R(t), \quad \text{in } [0,T], \tag{4.1.22}$$

$$v(y,0) = u_0(y), \qquad y \in \Omega, \tag{4.1.23}$$

$$H(0) = 0, \quad L(0) = l_0, \quad R(0) = \omega_0. \tag{4.1.24}$$

where

$$\begin{cases} (Mv)_i = \displaystyle\sum_{j=1}^{3} \frac{\partial Y_j}{\partial t} \frac{\partial v_i}{\partial y_j} + \sum_{j,k=1}^{3} \left\{ \Gamma^i_{j,k} \frac{\partial Y_k}{\partial t} + \frac{\partial Y_i}{\partial x_k} \frac{\partial^2 X_k}{\partial t \partial y_j} \right\} v_j; \\ (Nv)_i = \displaystyle\sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial y_j} + \sum_{j,k=1}^{3} \Gamma^i_{j,k} v_j v_k; \\ (G \cdot \nabla q)_i = \displaystyle\sum_{j=1}^{3} g^{ij} \frac{\partial q}{\partial y_j} \end{cases} \tag{4.1.25}$$

Our main result is following theorem.

Theorem 4.1.3 *Suppose that* $s > \frac{5}{2}$, $u_0 \in H^s(\Omega)$ *and* $u_0 \cdot \boldsymbol{n} = (l_0 + \omega_0 \times y) \cdot \boldsymbol{n}$ *on* $\partial\Omega$. *Then there exist some* $T_0 > 0$ *and a solution* (v, q, L, R) *of (4.1.18)-(4.1.24) such that*

$$v \in C([0,T_0]; H^s(\Omega)), \ \nabla q \in C([0,T_0]; H^{s-1}(\Omega)),$$

and

$$L, R \in C^1([0,T_0]; \mathbb{R}^3).$$

Such a solution is unique up to an arbitrary function of t *which may be added to* q. *Furthermore,* T_0 *does not depend on* s.

4.2 Preliminaries

4.2.1 Kato-Lai Theory

In this section we receive briefly Kato-Lai theory and introduce some notations. One is referred to [68] for more details. Let V, H, X be three real separable Banach spaces. We say that the family $\{V, H, X\}$ is an admissible triplet if the following conditions hold.

(1) $V \subset H \subset X$, the inclusions being dense and continuous.

(2) H is a Hilbert space, with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H = (\cdot,\cdot)_H^{\frac{1}{2}}$.

(3) There is a continuous, nondegenerate bilinear form on $V \times X$, denote by $\langle\cdot,\cdot\rangle$, such that

$$\langle v, u\rangle = (v, u)_H, \text{ for all } v \in V \text{ and } u \in H. \tag{4.2.1}$$

Recall that the bilinear form $\langle v, u\rangle$ is continuous and nondegenerate when

$$|\langle v, u\rangle| \le C\|v\|_V\|u\|_X;$$

$$\langle v, u\rangle = 0 \text{ for all } u \in X \text{ implies } v = 0;$$

$$\langle v, u\rangle = 0 \text{ for all } v \in X \text{ implies } u = 0.$$

A map $A : [0,T] \times H \to X$ is said to be sequentially weakly continuous if $A(t_n, v_n) \rightharpoonup A(t, v)$ in X whenever $t_n \to t$ and $v_n \rightharpoonup v$ in H.

We are concerned with the Cauchy problem

$$\frac{\mathrm{d}v}{\mathrm{d}t} + A(t, v) = 0, \ t \ge 0, \ v(0) = v_0. \tag{4.2.2}$$

The Kato-Lai existence result for abstract evolution equations is as follows.

Theorem 4.2.1 *Let $\{v, H, X\}$ be an admissible triplet. Let A be a sequentially weakly continuous map from $[0,T] \times H$ into X such that*

$$\langle v, A(t, v)\rangle \ge \beta(\|v\|_H^2) \ for \ \ t \in [0,T], v \in V,$$

where $\beta(r) \geq 0$ is a continuous nondecreasing function for $r \geq 0$. Then for any $v_0 \in H$ there is a time $T_0 > 0, T \leq T_0$, and a solution v of (4.2.2) in the class

$$v \in C_w([0,T]; H) \cap C_w^1([0,T]; X).$$

Moreover, one has

$$\|v(t)\|_H^2 \leq \gamma(t), \ t \in [0,T],$$

where γ solve the ODE

$$\gamma'(t) = 2\beta(\gamma(t)), \ \gamma(0) = \|v_0\|_H^2.$$

4.2.2 Admissible triplet and some properties for coefficients

We'd like to construct the admissible triplet and show some useful properties for the coefficients which in the(4.1.18)-(4.1.24).

Suppose $\mathcal{S}$ is a domain in $\mathbb{R}^3$. $L^2(\mathcal{S})$ is the space of L^2-integrable functions with the standard inner product $(\cdot,\cdot)_{L^2(\mathcal{S})}$. By the way, we will not distinguish the scalar function spaces and the corresponding vector-valued function spaces.

s is a nonnegative integer, then

$$H^s(\mathcal{S}) = \{u \in L^2(\mathcal{S}): \ D^\alpha u \in L^2(\mathcal{S}), \ \ \forall \alpha, s.t. \ |\alpha| \leq s\},$$

with the inner product

$$(u,v)_{H^s(\mathcal{S})} = \sum_{|\alpha| \leq s} (D^\alpha u, D^\alpha v)_{L^2(\mathcal{S})}$$

and the homogeneous Sobolev space given in Chapter 1.

Let $B_R(x)$ denote the ball centered at x and with the radius R. $\Omega_R := \Omega \cap B_R(0)$. Let $\rho = \dfrac{m}{|\mathcal{O}|}$, where $|\mathcal{O}|$ stands for the volume of $\mathcal{O}$. Hence ρ is the density of the solid. Let $\tilde{X} = L^2(\mathbb{R}^3)$ be endowed with the inner product,

$$(u,v)_{\tilde{X}} = \int_\Omega u(x) \cdot v(x) dx + \rho \int_{\mathcal{O}} u(x) \cdot v(x) dx.$$

Define

$$\tilde{X}_* = \{u \in \tilde{X} : \text{ div } u = 0 \text{ in } \mathbb{R}^3, \ \exists\, l, \omega \in \mathbb{R}^3, s.t., \ u = l + \omega \times y \text{ in } \mathcal{O}\},$$

which is a closed subspace of $\tilde{X}$.

Remark 4.2.2 *For every $u \in \tilde{X}_*$, and suppose $u = l + \omega \times y$ on $\mathcal{O}$, in fact, l and ω are uniquely determined by the vectors. The fact has been proved, see [25] or [112]. In what follows, we will denote l, ω by l_u, ω_u.*

Let $H_s = \{u \in \tilde{X} : \ u|_\Omega \in H^s(\Omega)\}$ be endowed with the scalar product

$$(u, v)_{H_s} = (u, v)_{H^s(\Omega)} + \rho(u, v)_{L^2(\mathcal{O})}.$$

V_s is the space of functions $v \in H_s$ such that $v|_\Omega$ belongs to $\mathcal{D}(A)$, where A is the elliptic operator $Af = \sum_{|\alpha| \leq s} (-1)^{|\alpha|} \partial^{2\alpha} f$ with Neumann boundary conditions, and $\mathcal{D}(A) \subseteq H^{2s}(\Omega)$. V_s is endowed with the scalar product

$$(u, v)_{V_s} = (u, v)_{H^{2s}(\Omega)} + \rho(u, v)_{L^2(\mathcal{O})}.$$

As in [95], we introduce a bilinear form on $V_s \times \tilde{X}$:

$$\langle v, u \rangle = \left(\sum_{|\alpha| \leq s} (-1)^{|\alpha|} \partial^{2\alpha} v, u \right)_{L^2(\Omega)} + \rho(v, u)_{L^2(\mathcal{O})}.$$

It was proved in [95] that the triplet$\{\tilde{X}, H_s, V_s\}$ is admissible.

Lemma 4.2.3 *Let $G_1^2 = \{u \in L^2(\mathbb{R}^3) : \ u = \nabla q_1, \ q_1 \in L^1_{\text{loc}}(\mathbb{R}^3)\}$, and*

$$G_2^2 = \Big\{u \in L^2(\mathbb{R}^3) : \text{ div } u = 0 \ \textit{in}\ \mathbb{R}^3, \ u = \nabla q_2 \ \textit{in}\ \Omega, q_2 \in L^1_{\text{loc}}(\Omega), \\ u = \phi \ \textit{in}\ \mathcal{O}, \ \textit{and} \ \int_{\mathcal{O}} \phi \times y \mathrm{d}y = \int_{\partial\mathcal{O}} q_2 \boldsymbol{n} \times y \, \mathrm{d}\sigma \Big\}.$$

Then (1) $\tilde{X}_*, G_1^2$ and G_2^2 are mutually orthogonal and

$$L^2(\mathbb{R}^3) = \tilde{X}_* \oplus G_1^2 \oplus G_2^2.$$

It means that for every $u \in L^2(\mathbb{R}^3)$,

$$u(y) = \begin{cases} u_1 + \nabla q_1 + \nabla q_2, & y \in \Omega \\ u_1 + \nabla q_1 + \phi, & y \in \mathcal{O} \end{cases} \in \tilde{X}_* \oplus G_1^2 \oplus G_2^2. \tag{4.2.3}$$

Suppose $u_1 = l_{u_1} + \omega_{u_1} \times y$ in $\mathcal{O}$, then there exists some constant C independent of u, such that

$$|l_{u_1}| + |\omega_{u_1}| \leq C\|u\|_{L^2(\mathbb{R}^3)}. \tag{4.2.4}$$

(2)Define the projector $\mathbb{P}$ which maps $L^2(\mathbb{R}^3)$ to $\tilde{X}_*$. In fact, $\mathbb{P}$ maps H_s into H_s continuously for any $s \geq 0$.

Proof: (1) has been proved in [112]. Now we verify that (2) holds. For every $u \in H_s (s \geq 0)$, it suffices to prove that

$$\|\nabla q = \nabla q_1 + \nabla q_2\|_{H^s(\Omega)} \leq C\|u\|_{H_s},$$

with some C independent of u.

In fact, q satisfies the following equations:

$$\begin{cases} \Delta q = \text{div } u, & \text{in } \Omega, \\ \dfrac{\partial q}{\partial \boldsymbol{n}} = u \cdot \boldsymbol{n} - (l_{u_1} + \omega_{u_1} \times y) \cdot \boldsymbol{n}, & \text{on } \partial\Omega, \end{cases} \tag{4.2.5}$$

Let

$$\varphi = \nabla \times \left[\frac{1}{2}\xi\left(l_{u_1} \times y - \omega_{u_1}|y|^2\right)\right],$$

where ξ is a cut-off function defined in the previous section. Clearly, div $\varphi = 0$ in Ω and $\varphi \cdot \boldsymbol{n} = (l_{u_1} + \omega_{u_1} \times y) \cdot \boldsymbol{n}$ on $\partial\Omega$ Therefore, (4.2.5) can be rewritten,

$$\begin{cases} \Delta q = \text{div } (u - \varphi), & \text{in } \Omega, \\ \dfrac{\partial q}{\partial \boldsymbol{n}} = (u - \varphi) \cdot \boldsymbol{n}, & \text{on } \partial\Omega. \end{cases} \tag{4.2.6}$$

The solution to the system (4.2.6) is closely related to the Helmoltz-Weyl decomposition. As proved in [47],

$$\begin{aligned}\|\nabla q\|_{H^s(\Omega)} &\leq C\|u-\varphi\|_{H^s(\Omega)} \\ &\leq C\left(\|u\|_{H^s(\Omega)}+\|\varphi\|_{H^s(\Omega)}\right) \\ &\leq C\left(\|u\|_{H^s(\Omega)}+|l_{u_1}|+|\omega_{u_1}|\right) \\ &\leq C\|u\|_{H_s},\end{aligned} \tag{4.2.7}$$

which completes the proof of Lemma 4.2.2. □

The following lemma is to give the bounds of the coefficients which appear in the system (4.1.18)-(4.1.24).

Lemma 4.2.4 *Assume that v is a function in $L^\infty(0,T;\tilde{X}_*)$ and s is a nonnegative integer. Suppose there exists $M_*>0$, such that $\|v\|_{L^\infty(0,T,\tilde{X})}\leq M_*$. Let $\Lambda, X, Y, g_{ij}, g^{ij}, \Gamma$ be defined as in the previous section by replacing l,ω by l_v,ω_v. Then for every $t\in[0,T]$, the following estimates hold:*

$$\|J_X(\cdot,t)\|_{W^{s,\infty}(\mathbb{R}^3)}\leq C(s,T,M_*),\ \|J_Y(X(\cdot,t),t)\|_{W^{s,\infty}(\mathbb{R}^3)}\leq C(s,T,M_*), \tag{4.2.8}$$

$$\|\Lambda(X(\cdot,t),t)\|_{W^{s,\infty}(\mathbb{R}^3)}\leq C(s,T,M_*),\ \|g_{ij}(\cdot,t)\|_{W^{s,\infty}(\mathbb{R}^3)}\leq C(s,T,M_*), \tag{4.2.9}$$

$$\|g^{ij}(\cdot,t)\|_{W^{s,\infty}(\mathbb{R}^3)}\leq C(s,T,M_*),\ \|G^{-1}(\cdot,t)\|_{W^{s,\infty}(\mathbb{R}^3)}\leq C(s,T,M_*), \tag{4.2.10}$$

$$\|\Gamma(\cdot,t)\|_{W^{s,\infty}(\mathbb{R}^3)}\leq C(s,T,M_*), \tag{4.2.11}$$

where G^{-1} is the inverse of G.

Proof: For every $j=1,2$ or 3, let $z(y,t)=\dfrac{\partial X}{\partial y_j}$,

$$\begin{cases}\dfrac{\partial z(y,t)}{\partial t}=\dfrac{\partial\Lambda}{\partial x}(X(y,t),t)\cdot z(y,t),\\ z(y,0)=\boldsymbol{e}_j,\end{cases}$$

where $\boldsymbol{e}_j$ is the j-th vector of the basis of $\mathbb{R}^3$. Then

$$z(y,t)=\boldsymbol{e}_j+\int_0^t\frac{\partial\Lambda}{\partial x}(X(y,t),t)\cdot z(y,s)\mathrm{d}s. \tag{4.2.12}$$

It follows from Gronwall's lemma that $|z(y,t)| \le C(T, M_*)$.

Since $\det(J_X) = 1$, then $J_Y = (J_X^{ij})$, where J_X^{ij} is the cofactor of J_X. Hence

$$|J_Y(X(\cdot,t),t)| \le C(T, M_*).$$

Furthermore,

$$|G(\cdot,t)| \le C(T, M_*), \quad |G^{-1}(\cdot,t)| \le C(T, M_*)$$

Denote $D^\beta = \dfrac{\partial^\beta}{\partial y^\beta}$. From (4.2.12) and Leibniz' formula, one can deduce that

$$D^\alpha z = \int_0^t \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \left(\frac{\partial \Lambda}{\partial x}\right) D^{\alpha-\beta} z \mathrm{d}s, \ |\alpha| \le s$$

Following the preceding process, one can get the estimates (4.2.8)-(4.2.11). □

Next lemma is about the Lipschitz continuity of the coefficients with respect to v.

Lemma 4.2.5 *Assume that the assumptions of Lemma 4.2.3 hold for v^i, $i = 1, 2$. Let $l(t) = l_{v^1}(t) - l_{v^2}(t)$, $\omega(t) = \omega_{v^1}(t) - \omega_{v^2}(t)$, $X = X^1 - X^2$, $Y = Y^1 - Y^2$ $\Lambda(y,t) = \Lambda(X^1(y,t),t) - \Lambda(X^2(y,t),t)$, $G = (g^{ij}) = (g^{ij,1} - g^{ij,2})$, $g_{ij} = g_{ij}^1 - g_{ij}^2$, $G^{-1} = (G^1)^{-1} - (G^2)^{-1}$, and $\Gamma_{m,k}^j = \Gamma_{m,k}^{j,1} - \Gamma_{m,k}^{j,2}$. Then for every $t \in [0,T]$,*

$$\|X(\cdot,t)\|_{W^{s,\infty}(\mathbb{R}^3)} \le C(s,T,M_*) \left(\|l\|_{L^\infty(0,T)} + \|\omega\|_{L^\infty(0,T)}\right), \tag{4.2.13}$$

$$\|Y(\cdot,t)\|_{W^{s,\infty}(\mathbb{R}^3)} \le C(s,T,M_*) \left(\|l\|_{L^\infty(0,T)} + \|\omega\|_{L^\infty(0,T)}\right), \tag{4.2.14}$$

$$\|\Lambda(\cdot,t)\|_{W^{s,\infty}(\mathbb{R}^3)} \le C(s,T,M_*) \left(\|l\|_{L^\infty(0,T)} + \|\omega\|_{L^\infty(0,T)}\right), \tag{4.2.15}$$

$$\|g_{ij}(\cdot,t)\|_{W^{s,\infty}(\mathbb{R}^3)} \le C(s,T,M_*) \left(\|l\|_{L^\infty(0,T)} + \|\omega\|_{L^\infty(0,T)}\right), \tag{4.2.16}$$

$$\|g^{ij}(\cdot,t)\|_{W^{s,\infty}(\mathbb{R}^3)} \le C(s,T,M_*) \left(\|l\|_{L^\infty(0,T)} + \|\omega\|_{L^\infty(0,T)}\right), \tag{4.2.17}$$

$$\|G^{-1}(\cdot,t)\|_{W^{s,\infty}(\mathbb{R}^3)} \le C(s,T,M_*) \left(\|l\|_{L^\infty(0,T)} + \|\omega\|_{L^\infty(0,T)}\right), \tag{4.2.18}$$

$$\|\Gamma(\cdot,t)\|_{W^{s,\infty}(\mathbb{R}^3)} \le C(s,T,M_*) \left(\|l\|_{L^\infty(0,T)} + \|\omega\|_{L^\infty(0,T)}\right). \tag{4.2.19}$$

Proof: From the initial problem (4.1.1), we infer the following estimate

$$\|Q^1(t) - Q^2(t)\| \le C(T_0, M_0) \sup_{[0,t]} |\omega^1(s) - \omega^2(s)|. \tag{4.2.20}$$

By simple calculate we have

$$\|V^1 - V^2\|_{L^\infty(\Omega_t)} \le C(T_0, M_0) \sup_{[0,t]} (|l^1(s) - l^2(s)| + |\omega^1(s) - \omega^2(s)|), \tag{4.2.21}$$

$$\|W^1 - W^2\|_{L^\infty(\Omega_t)} \le C(T_0, M_0) \sup_{[0,t]} (|l^1(s) - l^2(s)| + |\omega^1(s) - \omega^2(s)|), \tag{4.2.22}$$

where V^i, W^i are defined by (4.1.10),(4.1.11). Since $\Lambda(x,t) = \psi V + \nabla\psi \times W$, thus for

$$\begin{aligned}
&\Lambda(x,t) = \Lambda^1(x,t) - \Lambda^2(x,t) = \\
&\xi(Q^{1T}(t)(x - h^1(t)))V^1 + Q^{1T}(t)\nabla\xi(Q^{1T}(t)(x - h^1(t)))W^1 - \\
&\xi(Q^{2T}(t)(x - h^2(t)))V^2 - Q^{2T}(t)\nabla\xi(Q^{2T}(t)(x - h^2(t)))W^2.
\end{aligned}$$

From (4.2.20)-(4.2.22), we have (4.2.15).

$$\begin{cases} X'(t) = \Lambda^1(X^1(t), t) - \Lambda^2(X^2(t), t) \\ X(0) = 0. \end{cases}$$

It follows that

$$X(t) = \int_0^t (\Lambda^1(X^1(s), s) - \Lambda^1(X^2(s), s) + \Lambda(X^2(s), s))\mathrm{d}s,$$

$$|X(t)| \le \int_0^t |\Lambda^1(X^1(s), s) - \Lambda^1(X^2(s), s)|\mathrm{d}s + \int_0^t \Lambda(X^2(s), s))\mathrm{d}s. \tag{4.2.23}$$

Since

$$|\Lambda^1(X^1(s), s) - \Lambda^1(X^2(s), s)| \le |\frac{\partial \Lambda^1}{\partial x}||X(t)|,$$

$$\int_0^t \Lambda(X^2(s), s))\mathrm{d}s \le C(M_0, T_0) \sup_{[0,t]} (|l(s)| + |\omega(s)|).$$

Apply Gronwall's Inequality to (4.2.20), we have

$$|X(t)| \le C(T_0, M_0) \sup_{[0,t]} (|l| + |\omega|).$$

Since $\dfrac{\partial X}{\partial y_i}(y,t) = \displaystyle\int_0^t \left(\frac{\partial \Lambda^1(X^1,s)}{\partial x} \frac{\partial X^1}{\partial y_i}(y,s) - \frac{\partial \Lambda^2(X^2,s)}{\partial x} \frac{\partial X^2}{\partial y_i}(y,s) \right) \mathrm{d}s$, use the above argument and inequality (4.2.15), then

$$|\frac{\partial X}{\partial y_i}| \leq C(T_0, M_0) \sup_{[0,t]}(|l| + |\omega|).$$

By iterating the same process, we can easily obtain that (4.2.13) and (4.2.16). From lemma 4.1.2 (2), we have the inequalities (4.2.14) and (3.14), and thus (4.2.18) and (4.2.19) also holds. □

4.3 H^s-estimates of ∇q

In the following text, $s > s_0 = \dfrac{5}{2}$. Given a function $v \in L^\infty(0,T; H_s \cap \tilde{X}_*)$, which satisfics that $\|v\|_{L^\infty(0,T;H_{s_0})} \leq M_0$. We shall give the H^s-estimates of ∇q, a solution to the following system,

$$\begin{cases} \operatorname{div}\left(\displaystyle\sum_{j=1}^{3} g^{ij} \frac{\partial q}{\partial y_j} \right) = -\operatorname{div}(Mv + Nv), & \text{in } \Omega, \\ \displaystyle\sum_{i,j=1}^{3} g^{ij} \frac{\partial q}{\partial y_j} n_i + \left(\frac{1}{m} \int_{\partial\Omega} q\boldsymbol{n} \mathrm{d}\sigma \right) \cdot \boldsymbol{n} + \left(\bar{J}^{-1} \int_{\partial\Omega} y \times q\boldsymbol{n} \mathrm{d}\sigma \right) \times y \cdot \boldsymbol{n} & \\ = -(Mv + Nv) \cdot \boldsymbol{n} + \omega_v \times l_v \cdot \boldsymbol{n} - \left[\bar{J}^{-1}(\bar{J}\omega_v \times \omega_v) \right] \times y \cdot \boldsymbol{n}, & \text{on } \partial\Omega. \end{cases} \tag{4.3.1}$$

Here g^{ij}, Mv and Nv are given as in (4.1.17) and (4.1.25), replacing h', ω by l_v, ω_v.

For every fixed $t \in [0,T]$, the matrix $G = (g^{ij}) = J_Y J_Y^T$, so G is positive definite. Denote $\lambda_i(y,t) > 0, (i = 1,2,3)$ the eigenvalues of the matrix (g^{ij}). Since $\det(g^{ij}) = 1$, thus $\prod_{i=1}^{3} \lambda_i = 1$ and $\sum_{i=1}^{3} \lambda_i = \sum_{i=1}^{3} g^{ii} > 0$. Let $\gamma_0 = \sup_{y\in\mathbb{R}^3} |g^{ii}|$, then we have $3\gamma_0 \geq \lambda_i \geq \dfrac{1}{(3\gamma_0)^2}$ for every $i = 1,2,3$. By virtue of Lemma 4.2.3, there exist constants $C_1(T, M_0)$ and $C_2(T, M_0)$,

$$C_1(T, M_0) \leq |\lambda_i| \leq C_2(T, M_0), \quad i = 1,2,3.$$

Next, we shall use the Lax-Milgram theorem to prove the existence of the solutions of (4.3.1), and then we give H^s-estimate of this solution. For simplicity, the vector-valued functions l_v, ω_v are denoted by l, ω respectively.

Set a bilinear form B and a linear functional F on $\dot{D}^{1,2}(\Omega)$ as follows, for every $\eta, q \in \dot{D}^{1,2}(\Omega)$,

$$\begin{aligned} B(q,\eta) = &\sum_{i=1}^{3}\left(\sum_{j=1}^{3} g^{ij}\frac{\partial q}{\partial y_j}, \frac{\partial \eta}{\partial y_i}\right)_{L^2(\Omega)} + \frac{1}{m}\left(\int_{\partial\Omega} q\boldsymbol{n}\mathrm{d}\sigma\right)\cdot\left(\int_{\partial\Omega}\xi\boldsymbol{n}\mathrm{d}\sigma\right) \\ &+ \left(\bar{J}^{-1}\int_{\partial\Omega} y\times q\boldsymbol{n}\mathrm{d}\sigma\right)\cdot\left(\int_{\partial\Omega} y\times\eta\boldsymbol{n}\mathrm{d}\sigma\right). \\ F(\eta) = &-\int_{\Omega}(Mv+Nv)\cdot\nabla\eta dy + \int_{\partial\Omega}(\omega_v\times l_v)\cdot\eta\boldsymbol{n}\mathrm{d}\sigma \\ &- \int_{\partial\Omega}\left[\bar{J}^{-1}(\bar{J}\omega_v\times\omega_v)\right]\times y\cdot\eta\boldsymbol{n}\mathrm{d}\sigma. \end{aligned}$$

Note that

$$\begin{aligned} B(q,q) = &\sum_{i=1}^{3}\left(\sum_{j=1}^{3} g^{ij}\frac{\partial q}{\partial y_j}, \frac{\partial q}{\partial y_i}\right)_{L^2(\Omega)} + \frac{1}{m}\left(\int_{\partial\Omega} q\boldsymbol{n}\mathrm{d}\sigma\right)^2 + \bar{J}w\cdot w \\ &\geq C_1(T, M_0)\|\nabla q\|^2_{L^2(\Omega)} + mL_1^2 + \bar{J}w\cdot w, \end{aligned} \tag{4.3.2}$$

where $L_1 = \frac{1}{m}\left(\int_{\partial\Omega} q\boldsymbol{n}\mathrm{d}\sigma\right),\ w = \bar{J}^{-1}\int_{\partial\Omega} y\times q\boldsymbol{n}\mathrm{d}\sigma.$

$\bar{J}$ is also a positive definite matrix. then there exists some constant $a > 0$ such that

$$a^{-1}|w|^2 \leq \bar{J}w\cdot w \leq a|w|^2.$$

Combining the above inequality and (4.3.2), one gets that B is coercive.

On the other hand,

$$\left|\sum_{i=1}^{3}\left(\sum_{j=1}^{3} g^{ij}\frac{\partial q}{\partial y_j}, \frac{\partial \eta}{\partial y_i}\right)_{L^2(\Omega)}\right| \leq \|G\|_{L^\infty(\mathbb{R}^3)}\|\nabla q\|_{L^2(\Omega)}\|\nabla\eta\|_{L^2(\Omega)}.$$

Then along the line of the proposition 3.3.1 in [113], one can easily verify that the bilinear form B is bounded.

Now we turn to the functional F.

$$\begin{aligned}
&\left|-\int_{\Omega}(Mv+Nv)\cdot\nabla\eta dx\right| \\
&\leq \|Mv+Nv\|_{L^2(\Omega)}\cdot\|\nabla\eta\|_{L^2(\Omega)} \\
&\leq C\left(\|\Lambda\|_{W^{1,\infty}(\Omega)}+\|J_Y\|_{L^\infty(\Omega)}+\|J_X\|_{L^\infty(\Omega)}+\|v\|_{L^\infty(\Omega)}\right. \\
&\qquad \left.+\|\Gamma\|_{L^\infty(\Omega)}\right)\cdot\|v\|_{H^1(\Omega)}\cdot\|\nabla\eta\|_{L^2(\Omega)} \\
&\leq C(T,M_0)\|\nabla\eta\|_{L^2(\Omega)},
\end{aligned} \tag{4.3.3}$$

$$\begin{aligned}
&\left|\int_{\partial\Omega}\{\omega\times l-[\bar{J}^{-1}(\bar{J}\omega\times\omega)]\times y\}\cdot\eta\boldsymbol{n}\mathrm{d}\sigma\right| \\
&\leq C(|\omega||l|+|\omega|^2)\int_{\partial\Omega}|\eta|\mathrm{d}\sigma \\
&\leq C(\Omega)(|\omega||l|+|\omega|^2)\|\eta\|_{L^2(\partial\Omega)} \\
&\leq C(T,M_0,\Omega)\|\nabla\eta\|_{L^2(\Omega)}.
\end{aligned} \tag{4.3.4}$$

From the above estimates, it follows that F is bounded. By Lax-Milgram Theorem, there exists a unique $q\in\dot{D}^{1,2}(\Omega)$ such that

$$B(q,\eta)=F(\eta),\ \forall\ \eta\in\dot{D}^{1,2}(\Omega).$$

Furthermore,

$$\|\nabla q\|_{L^2(\Omega)}\leq C(T,M_0,\Omega). \tag{4.3.5}$$

Let

$$L_1=\frac{1}{m}\int_{\partial\Omega}q\boldsymbol{n}d\sigma,\quad w=\bar{J}^{-1}\int_{\partial\Omega}y\times q\boldsymbol{n}d\sigma.$$

Then according to (4.3.5),

$$|L_1|\leq C(T,M_0,\Omega),\quad |w|\leq C(T,M_0,\Omega). \tag{4.3.6}$$

Now we consider the Neumann system which is equivalent to (4.3.1),

$$\begin{cases}
\operatorname{div}\left(\displaystyle\sum_{j=1}^{3}g^{ij}\frac{\partial q}{\partial y_j}\right)=-\operatorname{div}(Mv+Nv), & \text{in }\Omega, \\
\displaystyle\sum_{i,j=1}^{3}g^{ij}\frac{\partial q}{\partial y_j}n_i=-(Mv+Nv)\cdot n-(\omega\times l)\cdot\boldsymbol{n} & \\
\quad+\left[\bar{J}^{-1}(\bar{J}\omega\times\omega)\right]\times y\cdot\boldsymbol{n}-L_1\cdot\boldsymbol{n}-(w\times y)\cdot\boldsymbol{n}, & \text{on }\partial\Omega.
\end{cases} \tag{4.3.7}$$

To estimate $\|\nabla q\|_{H^s(\Omega)}$, one needs to estimate the terms $\|\text{div}(Mv+Nv)\|_{H^{s-1}(\Omega)}$ and $\|-(Mv+Nv)\cdot \boldsymbol{n}\|_{H^{s-\frac{1}{2}}(\partial\Omega)}$.

$$\begin{aligned}
I_1 &:= \left\|\text{div}\left(\left(\frac{\partial Y}{\partial t}+v\right)\cdot\nabla v\right)\right\|_{H^{s-1}(\Omega)}\\
&= \left\|\sum_{i,j=1}^{3}\frac{\partial}{\partial y_i}\left(\frac{\partial Y_j}{\partial t}+v_j\right)\frac{\partial v_i}{\partial y_j}\right\|_{H^{s-1}(\Omega)}\\
&\le \left\|\sum_{i,j,k=1}^{3}\frac{\partial^2 Y_j}{\partial t\partial x_k}\frac{\partial X_k}{\partial y_i}\frac{\partial v_i}{\partial y_j}\right\|_{H^{s-1}(\Omega)}+\left\|\sum_{i,j=1}\frac{\partial v_j}{\partial y_i}\frac{\partial v_i}{\partial y_j}\right\|_{H^{s-1}(\Omega)}\\
&\le C\left(|\Lambda|_{W^{s,\infty}(\mathbb{R}^3)}\,|J_X|_{W^{s-1,\infty}(\mathbb{R}^3)}\,\|\nabla v\|_{H^{s-1}(\Omega)}+\|\nabla v\|_{H^{s-1}(\Omega)}\|v\|_{H^{s_0}(\Omega)}\right).
\end{aligned}$$

$$\begin{aligned}
I_2 &:= \left\|\sum_{i=1}^{3}\frac{\partial}{\partial y_i}\left(\sum_{j,k=1}^{3}\left\{\Gamma^i_{j,k}\frac{\partial Y_k}{\partial t}+\frac{\partial Y_k}{\partial x_k}\frac{\partial^2 X_k}{\partial t\partial y}\right\}v_j+\sum_{j,k=1}^{3}\Gamma^i_{j,k}v_jv_k\right)\right\|_{H^{s-1}(\Omega)}\\
&\le C\Big[\left(\|\Gamma\|_{W^{s,\infty}(\mathbb{R}^3)}\|\Lambda\|_{W^{s,\infty}(\mathbb{R}^3)}+\|J_Y\|_{W^{s,\infty}(\mathbb{R}^3)}\|\Lambda\|_{W^{s,\infty}(\mathbb{R}^3)}\right)\|v\|_{H^s(\Omega)}\\
&\quad+\|\Gamma\|_{W^{s,\infty}(\mathbb{R}^3)}\|v\|_{H^s(\Omega)}\|v\|_{H^{s_0}(\Omega)}\Big].
\end{aligned}$$

Hence,

$$\|\text{div}(Mv+Nv)\|_{H^{s-1}(\Omega)}\le C(T,M_0)\|v\|_{H_s}. \tag{4.3.8}$$

Denote

$$I_3=\sum_{i=1}^{3}n_i\left(\sum_{j,k=1}\left\{\Gamma^i_{j,k}\frac{\partial Y_k}{\partial t}+\frac{\partial Y_k}{\partial x_k}\frac{\partial^2 X_k}{\partial t\partial y}\right\}v_j+\sum_{j,k=1}\Gamma^i_{j,k}v_jv_k\right).$$

Then

$$-(Mv+Nv)\cdot\boldsymbol{n}=\left(\frac{\partial Y}{\partial t}+v\right)\cdot\nabla v\cdot\boldsymbol{n}+I_3.$$

$$\begin{aligned}
\|I_3\|_{H^{s-\frac{1}{2}}(\partial\Omega)} &\le C\|I_3\|_{H^s(\Omega)}\\
&\le C\big[\left(\|\Gamma\|_{W^{s,\infty}(\mathbb{R}^3)}\|\Lambda\|_{W^{s,\infty}(\mathbb{R}^3)}+\|J_Y\|_{W^{s,\infty}(\mathbb{R}^3)}\|\Lambda\|_{W^{s,\infty}(\mathbb{R}^3)}\right)\\
&\qquad\cdot\|v\|_{H^s(\Omega)}+\|\Gamma\|_{W^{s,\infty}(\mathbb{R}^3)}\,\|v\|_{H^s(\Omega)}\,\|v\|_{H^{s_0}(\Omega)}\big]\\
&\le C(T,M_0)\|v\|_{H_s}.
\end{aligned} \tag{4.3.9}$$

To estimate $\left\|(\frac{\partial Y}{\partial t}+v)\cdot\nabla v\cdot \boldsymbol{n}\right\|_{H^{s-\frac{1}{2}}(\partial\Omega)}$, we shall use the method in [24, 95]. Since $(\frac{\partial Y}{\partial t}+v)\cdot \boldsymbol{n}=0$ on $\partial\Omega$, one can easily get that

$$\left\|(\frac{\partial Y}{\partial t}+v)\cdot\nabla v\cdot n\right\|_{H^{s-\frac{1}{2}}(\partial\Omega)}\leq C\left(\|\Lambda\|^2_{W^{s,\infty}(\mathbb{R}^3)}+1\right)\left(1+\|v\|_{H^s(\Omega)}\right). \quad (4.3.10)$$

Combining (4.3.9) and (4.3.10), one has

$$\|-(Mv+Nv)\cdot\boldsymbol{n}\|_{H^{s-\frac{1}{2}}(\partial\Omega)}\leq C(T,M_0)(1+\|v\|_{H_s}). \quad (4.3.11)$$

The other terms can be estimated as follows:

$$\|\left[\bar{J}^{-1}(\bar{J}\omega\times\omega)\right]\times y\cdot\boldsymbol{n}\|_{H^{s-\frac{1}{2}}(\partial\Omega)}\leq C(\Omega,R)|\omega|^2\leq C(T,M_0), \quad (4.3.12)$$

$$\|L_1\cdot\boldsymbol{n}\|_{H^{s-\frac{1}{2}}(\partial\Omega)}\leq C|L_1|\leq C(T,M_0), \quad (4.3.13)$$

$$\|(\omega\times l)\cdot\boldsymbol{n}\|_{H^{s-\frac{1}{2}}(\partial\Omega)}\leq C(\Omega,R)|\omega||l|\leq C(T,M_0), \quad (4.3.14)$$

$$\|(w\times y)\cdot\boldsymbol{n}\|_{H^{s-\frac{1}{2}}(\partial\Omega)}\leq C(\Omega,R)|w|\leq C(T,M_0). \quad (4.3.15)$$

Choose some $r>0$ such that $\operatorname{supp}(\xi)\subset B_{\frac{r}{2}}$, and a cut-off function ξ_1,

$$\xi_1(y)=\begin{cases}1, & \text{if } |y|\leq 2r,\\ 0, & \text{if } |y|\geq 3r.\end{cases}$$

Hence, $p_1 = \xi_1 q$ solves the following equation

$$\begin{cases} \operatorname{div}\left(\sum_{j=1}^{3} g^{ij}\frac{\partial p_1}{\partial y_j}\right) = -\xi_1 \operatorname{div}(Mv+Nv) + \sum_{i,j=1}^{3} \frac{\partial g^{ij}}{\partial y_i}\frac{\partial \xi_1}{\partial y_j} q \\ \qquad + \sum_{i,j=1}^{3} g^{ij}\left(\frac{\partial \xi_1}{\partial y_j}\frac{\partial q}{\partial y_i} + \frac{\partial \xi_1}{\partial y_i}\frac{\partial q}{\partial y_j}\right), & \text{in } B_{4r}\setminus\mathcal{O}, \\ \sum_{i,j=1}^{3} g^{ij}\frac{\partial p_1}{\partial y_j} n_i = -(Mv+Nv)\cdot\boldsymbol{n} - \omega\times l\cdot\boldsymbol{n} \\ \qquad + \left[\bar{J}^{-1}(\bar{J}\omega\times\omega)\right]\times y\cdot\boldsymbol{n} - L_1\cdot\boldsymbol{n} - w\times y\cdot\boldsymbol{n}, & \text{on } \partial\mathcal{O}, \\ \sum_{i,j=1}^{3} g^{ij}\frac{\partial p_1}{\partial y_j} n_i = 0, & \text{on } \partial B_{4r}. \end{cases} \tag{4.3.16}$$

By virtue of the regularity theory for elliptic equations [100],

$$\begin{aligned} \|p_1\|_{H^{s+1}(B_{4r}\setminus\mathcal{O})} \le{}& h_1\left(\|G\|_{W^{s,\infty}(\mathbb{R}^3)}, (3\gamma_0)^2\right)\left(\left\|\sum_{i,j=1}^{3}\frac{\partial g^{ij}}{\partial y_i}\frac{\partial \xi_1}{\partial y_j}q\right\|_{H^{s-1}(B_{4r}\setminus\mathcal{O})}\right. \\ &+ \|\xi_1\operatorname{div}(Mv+Nv)\|_{H^{s-1}(B_{4r}\setminus\mathcal{O})} + \left\|\sum_{i,j=1}^{3} g^{ij}\left(\frac{\partial \xi_1}{\partial y_j}\frac{\partial q}{\partial y_i} + \frac{\partial \xi_1}{\partial y_i}\frac{\partial q}{\partial y_j}\right)\right\|_{H^{s-1}(B_{4r}\setminus\mathcal{O})} \\ &+ \|-(Mv+Nv)\cdot\boldsymbol{n}\|_{H^{s-\frac{1}{2}}(\partial\Omega)} + \left\|\bar{J}^{-1}(\bar{J}\omega\times\omega)\cdot\boldsymbol{n}\right\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \\ &\left.+\|(L_1 + w\times y)\cdot\boldsymbol{n}\|_{H^{s-\frac{1}{2}}(\partial\Omega)} + \|p_1\|_{L^2(B_{4r}\setminus\mathcal{O})}\right), \end{aligned} \tag{4.3.17}$$

where $h_1(\cdot,\cdot)$ can be chosen an increasing function with respect to both variables. In fact,

$$\begin{aligned} &\left\|\sum_{i,j=1}^{3}\frac{\partial g^{ij}}{\partial y_i}\frac{\partial \xi_1}{\partial y_j}q\right\|_{H^{s-1}(B_{4r}\setminus\mathcal{O})} + \left\|\sum_{i,j=1}^{3} g^{ij}\left(\frac{\partial \xi_1}{\partial y_j}\frac{\partial q}{\partial y_i} + \frac{\partial \xi_1}{\partial y_i}\frac{\partial q}{\partial y_j}\right)\right\|_{H^{s-1}(B_{4r}\setminus\mathcal{O})} \\ &\le C\left(\|G\|_{W^{s,\infty}(\mathbb{R}^3)}\cdot\|q\|_{H^{s-1}(B_{3r}\setminus B_{2r})} + \|G\|_{W^{s-1,\infty}(\mathbb{R}^3)}\cdot\|\nabla q\|_{H^{s-1}(B_{3r}\setminus B_{2r})}\right). \end{aligned} \tag{4.3.18}$$

Combining the above estimates, one gets

$$\begin{aligned} &\|q\|_{H^{s+1}(B_{2r}\setminus\mathcal{O})} \\ \le{}& C(T, M_0, r)\left(\|q\|_{H^{s-1}(B_{3r}\setminus B_{2r})} + \|q\|_{L^2(B_{3r}\setminus\mathcal{O})} + 1\right) \end{aligned} \tag{4.3.19}$$

Choose some particular q such that

$$\int_{B_{4r}\backslash\mathcal{O}} q(y)dy = 0.$$

It is reasonable, since q is still a solution to (4.3.7) after added by any constant. By Poincaré's inequality,

$$\begin{aligned}\|q\|_{H^{s+1}(B_{2r})} &\leq C(T, M_0, r)\left(\|q\|_{H^s(B_{3r}\backslash B_{2r})} + \|\nabla q\|_{L^2(B_{4r}\backslash\mathcal{O})} + 1\right) \\ &\leq C(T, M_0, r, \mathcal{O})\left(\|q\|_{H^s(B_{3r}\backslash B_{2r})} + 1\right).\end{aligned} \tag{4.3.20}$$

It implies that high order regularity of q can be controlled by the lower order regularity. Therefore, using this method by choosing appropriate r, we can get that for every $R > 0$, such that $\mathcal{O} \subseteq B_{\frac{R}{2}}$,

$$\|\nabla q\|_{H^s(\Omega_R)} \leq C(T, M_0, R)(1 + \|v\|_{H_s}). \tag{4.3.21}$$

Fix some R big enough. Choose some smooth cut-off function ξ_2, such that

$$\xi_2(y) = \begin{cases} 0, & \text{if } |y| \leq \dfrac{3}{4}R, \\ 1, & \text{if } |y| \geq R. \end{cases}$$

Since $g^{ij} = \delta_{ij}$ outside $B_{\frac{R}{2}}$, hence

$$\text{div}(G \cdot \nabla q) = \Delta q.$$

Let $p_2 = \xi_2 q$, then

$$\Delta p_2 = \xi_2(-\text{div}(Mv + Nv)) + \nabla\xi_2\nabla q + \Delta\xi_2 q := \tilde{f}. \tag{4.3.22}$$

Therefore,

$$\|\nabla p_2\|_{H^s(\mathbb{R}^3)} \leq C\|\tilde{f}\|_{H^{s-1}(\mathbb{R}^3)}. \tag{4.3.23}$$

$\tilde{f}$ is estimated as follows,

$$\begin{aligned}&\|\tilde{f}\|_{H^{s-1}(\mathbb{R}^3)} \\ &\leq C\left(\|\text{div}(Mv + Nv)\|_{H^{s-1}(\Omega)} + \|\nabla q\|_{H^{s-1}(\frac{R}{2}\leq|y|\leq R)} + \|q\|_{H^{s-1}(\frac{R}{2}\leq|y|\leq R)}\right) \\ &\leq C(T, M_0, R, \mathcal{O})(1 + \|v\|_{H_s}).\end{aligned} \tag{4.3.24}$$

Hence

$$\|\nabla q\|_{H^s(\mathbb{R}^3\setminus B_R)} \le C(T, M_0, R, \mathcal{O})(1 + \|v\|_{H_s}). \tag{4.3.25}$$

(4.3.21) and (4.3.25) give that

$$\|\nabla q\|_{H^s(\Omega)} \le C(T, M_0)(1 + \|v\|_{H_s}). \tag{4.3.26}$$

4.4 Construction of approximate solutions

In this section, we will construct a sequence of approximate solutions. First, for $v^n(t) \in H_s$, denote $l^n(t) = l_{\mathbb{P}v^n(t)}$, $\omega^n(t) = \omega_{\mathbb{P}v^n(t)}$. Solving the following initial value problem

$$\begin{cases} \dfrac{\mathrm{d}Q^n(t)}{\mathrm{d}t} \left(Q^n\right)^T(t) y = \left(Q^n(t)\omega^n(t)\right) \times y, \\ Q(0) = Id. \end{cases} \tag{4.4.1}$$

One can get a solution $Q^n(t)$.

Define

$$\psi^n = \xi\left((Q^n)^T(t)(x - h^n(t))\right), \ h^n = \int_0^t Q^n(s) l^n(s) \mathrm{d}s,$$

and

$$V^n = Q^n(t)l^n(t) + Q^n(t)\omega^n(t) \times (x - h^n(t)),$$
$$W^n = Q^n(t)l^n(t) \times (x - h^n(t)) + \frac{|x - h^n(t)|^2}{2} Q^n(t)\omega^n(t),$$

where ξ is a cut-off function given in section 1.

Let $\Lambda^n = \psi^n V^n + \nabla\psi^n W^n$. Hence one can define $X^n(\cdot, t)$, $\Lambda^n(X^n(\cdot, t), t)$, $g^{ij,n}$, g^n_{ij}, $\Gamma^{i,n}_{jk}$, M^n, N^n given in (4.1.17) and (4.1.25). Suppose that q^n is the solution to the following system,

$$\begin{cases} \operatorname{div}\left(\displaystyle\sum_{j=1}^3 g^{ij,n} \frac{\partial q^n}{\partial y_j}\right) = -\operatorname{div}(M^n \mathbb{P}v^n + N^n \mathbb{P}v^n), & \text{in } \Omega \\ \displaystyle\sum_{i,j=1}^3 g^{ij,n} \frac{\partial q^n}{\partial y_j} n_i + \frac{1}{m}\left(\int_{\partial\Omega} q^n \boldsymbol{n} \mathrm{d}\sigma\right) \cdot \boldsymbol{n} + \left[\bar{J}^{-1} \int_{\partial\Omega} y \times q^n \boldsymbol{n} \mathrm{d}\sigma\right] \times y \cdot \boldsymbol{n} & \\ = -(M^n \mathbb{P}v^n + N^n \mathbb{P}v^n) \cdot \boldsymbol{n} - \omega^n \times l^n \cdot \boldsymbol{n} + \left[\bar{J}^{-1}(\bar{J}\omega^n \times \omega^n)\right] \times y \cdot \boldsymbol{n}, & \text{on } \partial\Omega. \end{cases} \tag{4.4.2}$$

Now define an operator $A^n(t,v)$ as in [95] to use Kato-Lai Theory,

$$A^n(t,v) = \mathbf{1}_\Omega\left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1}\right)\cdot\nabla v - \mathcal{Q}\left[\mathbf{1}_\Omega\left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1}\right)\cdot\nabla\mathbb{P}v\right] +$$
$$\mathbb{P}\left[\mathbf{1}_\Omega\left(\sum_{j,k=1}^{3}\left\{\Gamma_{j,k}^{\cdot,n-1}\frac{\partial Y_k^{n-1}}{\partial t} + \frac{\partial Y^{n-1}}{\partial x_k}\frac{\partial^2 X_k^{n-1}}{\partial t\partial y_j}\right\}(\mathbb{P}v)_j^{n-1} + \sum_{j,k=1}\Gamma_{j,k}^{\cdot,n-1}(\mathbb{P}v)_j^{n-1}(\mathbb{P}v)_k^{n-1}\right)\right] +$$
$$\mathbb{P}\left[\mathbf{1}_\Omega\left(\sum_{j=1}^{3} g^{\,j,n-1}\frac{\partial q^{n-1}}{\partial y_j}\right)\right] + \mathbb{P}\left[\mathbf{1}_\mathcal{O}\left(\omega^{n-1}\times l^{n-1} - \bar{J}^{-1}(\bar{J}\omega^{n-1}\times\omega^{n-1})\times y\right)\right]$$
$$+\,\mathbb{P}\left[\mathbf{1}_\mathcal{O}\left(-\frac{1}{m}\int_{\partial\Omega} q^{n-1}\boldsymbol{n}\mathrm{d}\sigma - \left(\bar{J}^{-1}\int_{\partial\Omega} y\times q^{n-1}\boldsymbol{n}\mathrm{d}\sigma\right)\times y\right)\right], \tag{4.4.3}$$

where the operator $\mathcal{Q} = I - \mathbb{P}$.

Consider the following Cauchy problem,

$$\begin{cases} v_t^n + A^n(t,v^n) = 0, \\ v^n(0) = v_0, \end{cases} \tag{4.4.4}$$

where $v_0 \in H_s \cap \tilde{X}_*$.

For simplicity, denote

$$(F^{n-1}v)_i = \left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1}\right)\cdot\nabla v_i,\ (G^{n-1}\cdot\nabla q^{n-1})_i = \sum_{j=1}^{3} g^{ij,n-1}\frac{\partial q^{n-1}}{\partial y_j},$$
$$(E^{n-1})_i = \mathbf{1}_\Omega\left[\left(\sum_{j,k=1}\left\{\Gamma_{j,k}^{i,n-1}\frac{\partial Y_k^{n-1}}{\partial t} + \frac{\partial Y_i^{n-1}}{\partial x_k}\frac{\partial^2 X_k^{n-1}}{\partial t\partial y_j}\right\}(\mathbb{P}v^{n-1})_j\right.\right.$$
$$\left.\left.+\sum_{j,k=1}\Gamma_{j,k}^{i,n-1}(\mathbb{P}v^{n-1})_j(\mathbb{P}v^{n-1})_k\right)\right],$$
$$(L^{n-1})_i = \left[\mathbf{1}_\mathcal{O}\left(\frac{1}{m}\omega^{n-1}\times l^{n-1} - \bar{J}^{-1}(\bar{J}\omega^{n-1}\times\omega^{n-1})\times y\right)_i\right]$$
$$-\left\{\mathbf{1}_\mathcal{O}\left[\frac{1}{m}\int_{\partial\Omega} q^{n-1}n_i\mathrm{d}\sigma - \left((\bar{J}^{-1}\int_{\partial\Omega} y\times q^{n-1}\boldsymbol{n}\mathrm{d}\sigma)\times y\right)_i\right]\right\},$$

and assume $M_0 = 2\|v_0\|_{H_{s_0}}$ and $M^* = 2\|v_0\|_{H_s}$, and there exists some $T > 0$ such that for all $k < n$,

$$\|v^k\|_{L^\infty(0,T,H)} \le M^*.$$

At first, we prove when $s = s_0$, there exists a $T_0 > 0$, $\|v^k\|_{L^\infty(0,T_0;H)} \leq M_0$. Therefore, we have

$$\begin{aligned}|(v, A^n(t,v))| &\leq |(\mathbf{1}_\Omega F^{n-1}v, v)_H| + |(\mathcal{Q}(\mathbf{1}_\Omega F^{n-1}\mathbb{P}v), v)_H| + |(\mathbb{P}E^{n-1}v, v)_H| \\ &\quad + |(\mathbb{P}(\mathbf{1}_\Omega G^{n-1}\nabla q^{n-1}). v)_H| + |(\mathbb{P}L^{n-1}, v)_H| \\ &:= J_1 + J_2 + J_3 + J_4 + J_5\end{aligned}$$

Then we estimate them term by term. Staring from the easiest one,

$$\begin{aligned}J_5 &\leq \|\mathbb{P}L^{n-1}\|_{H_s}\|v\|_{H_s} \\ &\leq C\|L^{n-1}\|_{H_s}\|v\|_{H_s} \\ &\leq C\left(|L_1^{n-1}| + |w^{n-1}| + \frac{1}{m}|l^{n-1}||\omega^{n-1}| + |\omega^{n-1}|^2\right)\|v\|_{H_s} \\ &\leq C(T_0, M_0)\|v\|_{H_s}.\end{aligned} \tag{4.4.5}$$

By Lemma 4.2.2 and Lemma 4.2.3,

$$\begin{aligned}J_4 &\leq \|(\mathbb{P}(\mathbf{1}_\Omega G^{n-1}\nabla q^{n-1})\|_{H_s}\|v\|_{H_s} \\ &\leq C\|G^{n-1}\nabla q^{n-1}\|_{H^s(\Omega)}\|v\|_{H_s} \\ &\leq C(T_0, M_0)\|v\|_{H_s}.\end{aligned} \tag{4.4.6}$$

J_3 is also easy to estimate since there is no derivative of v or v^{n-1},

$$J_3 \leq C(T_0, M_0)\|v\|_H. \tag{4.4.7}$$

Now the most difficult terms J_1 and J_2 are left, since there is derivative of v or $\mathbb{P}v$. Since

$$J_1 = \left|\sum_{|\alpha|\leq s}\sum_{\alpha_1\leq\alpha}\sum_{i=1}^{3}\int_\Omega \partial^{\alpha_1}\left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1}\right)\cdot\nabla\partial^{\alpha-\alpha_1}v_i\partial^\alpha v_i dy\right| \tag{4.4.8}$$

When $\alpha_1 = (0,0,0)$, since

$$\operatorname{div}\left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1}\right) = 0, \quad \text{and} \quad \left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1}\right)\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega,$$

$$\int_\Omega\left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1}\right)\cdot\nabla\partial^\alpha v_i\partial^\alpha v_i dy = 0.$$

Therefore, we assume that $|\alpha_1| \geq 1$. Consequently,

$$\begin{aligned}
&\left\| \partial^{\alpha_1} \left(\frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1} \right) \cdot \nabla \partial^{\alpha_2} v_i \right\|_{L^2(\Omega)} \\
&\leq \left\| \partial^{\alpha_1} \frac{\partial Y^{n-1}}{\partial t} \cdot \nabla \partial^{\alpha_2} v_i \right\|_{L^2(\Omega)} + \left\| \partial^{\alpha_1} \mathbb{P}v^{n-1} \cdot \nabla \partial^{\alpha_2} v_i \right\|_{L^2(\Omega)} \\
&\leq C(T_0, M_0)\|v\|_{H_s} + \|\mathbb{P}v^{n-1}\|_{L^\infty(\Omega)} \|\nabla v\|_{H^{s-1}(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} \|\mathbb{P}v^{n-1}\|_{H^s(\Omega)} \\
&\leq C(T_0, M_0)\|v\|_{H_s}.
\end{aligned}$$

Hence,

$$J_1 \leq C(T_0, M_0)\|v\|_{H_s}^2. \tag{4.4.9}$$

$$\begin{aligned}
&J_2 \leq \|\mathcal{Q}[\mathbf{1}_\Omega F^{n-1}\mathbb{P}v]\|_{H^s(\Omega)} \|v\|_{H^s(\Omega)} + C\|F^{n-1}\mathbb{P}v]\|_{L^2(\Omega)} \|v\|_{L^2(\mathcal{O})} \\
&\|F^{n-1}\mathbb{P}v]\|_{L^2(\Omega)} \|v\|_{L^2(\mathcal{O})} \leq \left| \frac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1} \right|_{L^\infty(\Omega)} \|\mathbb{P}v\|_{H_{s_0}} \|v\|_{L^2(\mathcal{O})} \\
&\leq C(T_0, M_0)\|v\|_H^2.
\end{aligned}$$

To estimate $\|\mathcal{Q}[\mathbf{1}_\Omega F^{n-1}\mathbb{P}v]\|_{H^s(\Omega)}$, we can a function ϕ satisfies

$$\begin{cases}
\Delta\phi = \operatorname{div} \left(\dfrac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1} \right) \cdot \nabla \mathbb{P}v, \text{ in } \Omega \\
\dfrac{\partial \phi}{\partial n} = \left(\dfrac{\partial Y^{n-1}}{\partial t} + \mathbb{P}v^{n-1} \right) \cdot \nabla \mathbb{P}v \cdot n. \text{ on } \partial\Omega
\end{cases}$$

As in the procedure of estimating ∇q(or refer to [95]), we have

$$I_2 \leq C(T_0, M_0)(1 + \|v\|_H)\|v\|_H.$$

Therefore, we have

$$|(v, A^n(t, v))_H| \leq C(T_0, M_0)(1 + \|v\|_H^2). \tag{4.4.10}$$

By the Kato-Lai theory, there exists $T^n > 0$, and $v^n \in C_{\mathrm{w}}(0, T_n; H) \cap C_{\mathrm{w}}^1(0, T_n; X)$ satisfying the Cauchy problem (4.4.4), and we have

$$\|v^n(t)\|_H \leq \gamma(t), \tag{4.4.11}$$

where $\gamma(t)$ is given by

$$\gamma'(t) = C(T_0, M_0)(1+\gamma(t)), \quad \gamma(0) = \|v_0\|_H^2,$$

Suppose

$$|\gamma(t)| \le 4\|v_0\|_{H_s}^2 = (M_0)^2, \ \forall t \in [0, T_0].$$

Thus we know that $T_n = T_0$ which is independent of n.

Along the previous proof and the estimates of ∇q, for general $s \ge s_0$, it is easy to prove that

$$|(v, A^n(t,v))_H| \le C(T_0, \|v^{n-1}\|_{H_{s_0}})(1+\|v^{n-1}\|_H)(1+\|v\|_{H_{s_0}})(1+\|v\|_H), \tag{4.4.12}$$

where $C(\cdot, r)$ is bounded if r is bounded. Therefore, use the above argument, we obtain that there exists $T \le T_0$ that is independent of n such that

$$\|v^n\|_{L^\infty(0,T;H)} \le M^* \ \text{ for all } n.$$

For $n = 1$, we choose $v^0 = v_0$, $l^0 = l_0$, $\omega^0 = \omega_0$. Following the preceding process, one can construct a solution v^1 to (4.4.4). By iterating the same steps, a sequence of approximate solutions $\{v^n\}$ can be derived.

4.5 The convergence of approximated solutions

In this section, we show that $\{v^n\}$ converges to a solution of the system (4.1.18)-(4.1.24).

According to the estimates in the last section,

$$\|v^n\|_{L^\infty(0,T;H_s)} \le M^*, \tag{4.5.1}$$

$$\|\partial_t v^n\|_{L^\infty(0,T;H_s)} \le M_1, \tag{4.5.2}$$

$$\|\nabla q^n\|_{L^\infty(0,T;H^{s-1}(\Omega))} \le M_2. \tag{4.5.3}$$

Since $\mathbb{P}$ is a bounded operator on H_s, H_{s-1}, and it commutes with ∂_t, then

$$\|\mathbb{P}v^n\|_{L^\infty(0,T;H_s)} \leq M_3, \tag{4.5.4}$$

$$\|\partial_t\mathbb{P}v^n\|_{L^\infty(0,T;H_{s-1})} \leq M_4. \tag{4.5.5}$$

Hence there exists some function $v \in C_w([0,T];H_s)$ such that for any big R_0,

$$v^n \overset{*}{\rightharpoonup} v \text{ in } L^\infty(0,T;H_s), \tag{4.5.6}$$

$$\mathbb{P}v^n \overset{*}{\rightharpoonup} \mathbb{P}v \;\text{ in }\; L^\infty(0,T;H_s). \tag{4.5.7}$$

By the Aubin-Lion's lemma,

$$v^n \to v \;\text{ in }\; C([0,T];H^{s-1}(\Omega_{R_0}) \cap L^2(B_{R_0})), \tag{4.5.8}$$

$$\mathbb{P}v^n \to \mathbb{P}v \;\text{ in }\; C([0,T];H^{s-1}(\Omega_{R_0}) \cap L^2(B_{R_0})). \tag{4.5.9}$$

(4.5.9) implies that

$$\omega^n(t) \to \omega(t) \;\text{ in }\; C[0,T], \tag{4.5.10}$$

$$Q^n(t) \to Q(t) \;\text{ in }\; C[0,T], \tag{4.5.11}$$

and

$$l^n(t) \to l(t) = Q(t)l_{\mathbb{P}v} \;\text{ in }\; C[0,T]. \tag{4.5.12}$$

In fact, $\omega(t) = Q(t)\omega_{\mathbb{P}v}$.

While (4.5.3) tells that there exists some function q such that

$$\nabla q^n \overset{*}{\rightharpoonup} \nabla q \;\text{ in }\; C_w([0,T];H^{s-1}(\Omega)), \tag{4.5.13}$$

$$\int_{\partial\Omega} q^n \boldsymbol{n} d\Gamma \to \int_{\partial\Omega} q\boldsymbol{n} d\Gamma \;\text{ in }\; C[0,T], \tag{4.5.14}$$

$$\int_{\partial\Omega} y \times q^n \boldsymbol{n} d\Gamma \to \int_{\partial\Omega} y \times q\boldsymbol{n} d\Gamma \;\text{ in }\; C[0,T]. \tag{4.5.15}$$

In fact, q is a solution to the system (4.3.1). It can be seen by taking the limit of (4.4.2).

From all the convergence results (4.5.1)-(4.5.15) and Lemma 4.2.4, it follows that

$$\begin{cases} v_t + A(t,v) = 0, \\ v(0) = v_0, \end{cases} \tag{4.5.16}$$

where

$$\begin{aligned} A(t,v) &= \mathbf{1}_\Omega\left(\frac{\partial Y}{\partial t} + \mathbb{P}v\right)\cdot\nabla v - \mathcal{Q}\left[\mathbf{1}_\Omega\left(\frac{\partial Y}{\partial t} + \mathbb{P}v\right)\cdot\nabla\mathbb{P}v\right] \\ &+ \mathbb{P}\left[\mathbf{1}_\Omega\left(\sum_{\jmath,k=1}\left\{\Gamma_{\jmath,k}\frac{\partial Y_k}{\partial t} + \frac{\partial Y}{\partial x_k}\frac{\partial^2 X_k}{\partial t\partial y_\jmath}\right\}(\mathbb{P}v)_\jmath + \sum_{\jmath,k=1}\Gamma_{\jmath,k}(\mathbb{P}v)_\jmath(\mathbb{P}v)_k\right)\right] \\ &+ \mathbb{P}\left[\mathbf{1}_\Omega\left(\sum_{\jmath=1}^{3} g^\jmath\frac{\partial q}{\partial y_\jmath}\right)\right] + \mathbb{P}\left[\mathbf{1}_\mathcal{O}\left(\omega\times l + \bar{J}^{-1}(\bar{J}\omega\times\omega)\times y\right)\right] \\ &- \mathbb{P}\left[\mathbf{1}_\mathcal{O}\left(\frac{1}{m}\int_{\partial\Omega} q\boldsymbol{n}\mathrm{d}\sigma - \left(\bar{J}^{-1}\int_{\partial\Omega} y\times q\boldsymbol{n}\mathrm{d}\sigma\right)\times y\right)\right], \end{aligned}$$

Next, we shall prove that v is a solution of the systems (4.1.18)-(4.1.24). The proof starts with the observation that $v(t) = \mathbb{P}v(t)$, for all $t \in [0,T]$. In fact, Applying $\mathcal{Q}$ to each term in (4.5.16) and taking the inner product with $\mathcal{Q}v(t)$ in $\tilde{X}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\mathcal{Q}v(t)\|_{\tilde{X}}^2 + (\mathcal{Q}v(t), \mathcal{Q}A(t,v))_{\tilde{X}} = 0. \tag{4.5.17}$$

Note that $\operatorname{div}\left(\frac{\partial Y}{\partial t} + \mathbb{P}v\right) = 0$ in Ω and $\left(\frac{\partial Y}{\partial t} + \mathbb{P}v\right)\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, then

$$\begin{aligned} (\mathcal{Q}v(t), \mathcal{Q}A(t,v))_{\tilde{X}} &= \left(\mathcal{Q}v(t), \mathbf{1}_\Omega\left(\frac{\partial Y}{\partial t} + \mathbb{P}v\right)\cdot\nabla\mathcal{Q}v(t)\right)_{\tilde{X}} \\ &= \int_\Omega \mathcal{Q}v\cdot\left(\left(\frac{\partial Y}{\partial t} + \mathbb{P}v\right)\cdot\nabla\mathcal{Q}v(t)\right)\mathrm{d}y \\ &= 0 \end{aligned} \tag{4.5.18}$$

Since $v_0 = \mathbb{P}v_0$, it tells that $\mathcal{Q}v_0 = 0$. Hence, for every $t \in [0,T]$, $\mathcal{Q}v(t) = 0$. Therefore, (4.5.16) can be written as

$$\begin{aligned} &\frac{\partial v}{\partial t} + \mathbb{P}[\mathbf{1}_\Omega(Mv + Nv + G\cdot\nabla q)] + \\ &\mathbb{P}\left[\mathbf{1}_\mathcal{O}\left(\frac{1}{m}\omega\times l - \bar{J}^{-1}(\bar{J}\omega\times\omega)\times y\right)\right. \\ &\left. - \mathbf{1}_\mathcal{O}\left(\frac{1}{m}\int_{\partial\Omega} q\boldsymbol{n}\mathrm{d}\sigma - (\bar{J}^{-1}\int_{\partial\Omega} y\times q\boldsymbol{n}\mathrm{d}\sigma)\times y\right)\right] = 0 \end{aligned} \tag{4.5.19}$$

Taking the inner product in $\tilde{X}$ with a test function $\phi \in \tilde{X}_*$, one has

$$\begin{aligned}&\int_\Omega (v' + Mv + Nv + G \cdot \nabla q) \cdot \phi \mathrm{d}y + \\ &ml' \cdot l_\phi - \int_{\partial\Omega} q\boldsymbol{n} \mathrm{d}\sigma \cdot l_\phi + m(\omega \times l) \cdot l_\phi + \\ &\bar{J}\omega' \cdot \omega_\phi - \bar{J}\left[\bar{J}^{-1}(\bar{J}\omega \times \omega)\right] \cdot \omega_\phi - \bar{J}\left[\bar{J}^{-1}\int_{\partial\Omega} y \times q\boldsymbol{n}\mathrm{d}\sigma\right] \cdot \omega_\phi \\ &= 0.\end{aligned} \tag{4.5.20}$$

For every function $\phi \in C_0^\infty(\mathbb{R}^3)$, with $supp(\phi) \subseteq \Omega$, and div $\phi = 0$ in $\mathbb{R}^3$, (4.5.20) yields

$$\int_\Omega (v' + Mv + Nv + G \cdot \nabla q) \cdot \phi \mathrm{d}y = 0,$$

After the theory of Hodge's decomposition, there exists a function p such that $\nabla p \in L^\infty(0, T; H^{s-1}(\Omega))$ and

$$v' + Mv + Nv + G \cdot \nabla q + \nabla p = 0 \text{ in } \Omega \times [0, T]. \tag{4.5.21}$$

From the identification of q and (4.5.21), one knows that for every $t \in [0, T]$,

$$\begin{cases} \Delta p = 0, & \text{in } \Omega, \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = 0, & \text{on } \partial\Omega. \end{cases}$$

The above system has only constant solutions, thus

$$v' + Mv + Nv + G \cdot \nabla q = 0 \quad \text{in} \;\; \Omega \times [0, T]. \tag{4.5.22}$$

Now taking some test function $\phi(x) \in \tilde{X}$ such that $\phi(y) = l_\phi$ in $\mathcal{O}$, then

$$ml' \cdot l_\phi - \left(\int_{\partial\Omega} q\boldsymbol{n}\mathrm{d}\sigma\right) \cdot l_\phi + (m\omega \times l) \cdot l_\phi = 0.$$

Since l_ϕ is arbitrary,

$$ml' = \int_{\partial\Omega} q\boldsymbol{n} d\sigma - \omega \times l. \tag{4.5.23}$$

Similarly, taking some test function $\phi(x) \in \tilde{X}$ such that $\phi(y) = \omega_\phi \times y$ in $\mathcal{O}$, then

$$\bar{J}\omega' \cdot \omega_\phi - \bar{J}(\bar{J}^{-1}(\bar{J}\omega \times \omega)) \cdot \omega_\phi - \bar{J}(\bar{J}^{-1}\int_{\partial\Omega} y \times q\boldsymbol{n}\mathrm{d}\sigma) \cdot \omega_\phi = 0.$$

Thus

$$\bar{J}\omega' = \int_{\partial\Omega} y \times q\boldsymbol{n}\mathrm{d}\sigma + \bar{J}\omega \times \omega. \qquad (4.5.24)$$

4.6 Uniqueness and continuity with respect to time

In this section, we will use Lemma 4.2.4 to prove that the solution of the system (4.1.18)-(4.1.24) is unique and then get the continuity in H_s with respect to time.

Assume that there exist two solutions $v^1, v^2 \in C_{\mathrm{w}}([0,T]; H_s) \cap C^1_{\mathrm{w}}([0,T]; X)$ to the system (4.1.18)-(4.1.24) then

$$v^1_t + M^1 v^1 + N^1 v^1 + G^1 \nabla q^1 = 0, \quad \text{in} \quad \Omega \times [0,T], \qquad (4.6.1)$$

$$v^2_t + M^2 v^2 + N^2 v^2 + G^2 \nabla q^2 = 0, \quad \text{in} \quad \Omega \times [0,T]. \qquad (4.6.2)$$

Since G^1, G^2 are positive definite matrices, thus let $H^1 = (G^1)^{-1}$ and $H^2 = (G^2)^{-1}$. Multiplying (4.6.1) and (4.6.2) by H^1 and H^2 respectively, and denote $K = \max\{\|v^1\|_{L^\infty(0,T;H)}, \|v^2\|_{L^\infty(0,T;H)}\}$.

Subtracting the two equations and taking inner product in $L^2(\Omega)$ with function $v^1 - v^2$, then one gets

$$\begin{aligned} 0 &= (H^1 v^1_t - H^2 v^2_t, v^1 - v^2)_{L^2(\Omega)} + (\nabla q^1 - \nabla q^2, v^1 - v^2)_{L^2(\Omega)} \\ &\quad + (H^1(M^1 v^1 + N^1 v^1) - H^2(M^2 v^2 + N^2 v^2), v^1 - v^2)_{L^2(\Omega)} \\ &:= I_1 + I_2 + I_3, \end{aligned}$$

Denote $l_{v^1}, \omega_{v^1}, l_{v^2}, \omega_{v^2}$ by $l_1, \omega_1, l_2, \omega_2$ respectively. Since

$$\mathrm{div}(v^1 - v^2) = 0 \quad \text{in} \quad \Omega,$$

and for $i = 1, 2$,

$$\int_{\partial\Omega} q^i \boldsymbol{n}\mathrm{d}\sigma = m l_i' + m\omega_i \times l_i,$$

$$\int_{\partial\Omega} y \times q^i \boldsymbol{n}\mathrm{d}\sigma = \bar{J}\omega_i' - \bar{J}\omega_i \times \omega_i,$$

then

$$
\begin{aligned}
I_2 &= \int_\Omega \nabla(q^1-q^2)\cdot(v^1-v^2)\mathrm{d}y \\
&= \int_{\partial\Omega}(q^1-q^2)(v^1-v^2)\cdot \boldsymbol{n}\mathrm{d}\sigma \\
&= \int_{\partial\Omega}(q^1-q^2)(l_1-l_2)\cdot \boldsymbol{n}\mathrm{d}\sigma + \int_{\partial\Omega}(q^1-q^2)(\omega_1-\omega_2)\times y\cdot \boldsymbol{n}\mathrm{d}\sigma \\
&= m(l_1-l_2)'\cdot(l_1-l_2) + m(\omega_1\times l_1-\omega_2\times l_2)\cdot(l_1-l_2) \\
&\quad + \bar{J}(\omega_1-\omega_2)'\cdot(\omega_1-\omega_2) - (\bar{J}\omega_1\times\omega_1-\bar{J}\omega_2\times\omega_2)\cdot(\omega_1-\omega_2) \\
&= \frac{1}{2}m\frac{\mathrm{d}}{\mathrm{d}t}|l_1-l_2|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[(\bar{J}(\omega_1-\omega_2))\cdot(\omega_1-\omega_2)\right] \\
&\quad + m(\omega_1\times l_1-\omega_2\times l_2)\cdot(l_1-l_2) - (\bar{J}\omega_1\times\omega_1-\bar{J}\omega_2\times\omega_2)\cdot(\omega_1-\omega_2).
\end{aligned} \tag{4.6.3}
$$

We estimate the term I_1 as follows.

$$
\begin{aligned}
I_1 &= (H^1 v_t^1 - H^2 v_t^2, v^1-v^2)_{L^2(\Omega)} \\
&= \left(H^1(v^1-v^2)_t, v^1-v^2\right)_{L^2(\Omega)} + \left((H^1-H^2)v_t^2, v^1-v^2\right)_{L^2(\Omega)} \\
&:= J_1 + J_2.
\end{aligned} \tag{4.6.4}
$$

From the definition of G, we easily know that

$$
\begin{aligned}
J_1 &= \left(H^1(v^1-v^2)_t,\ v^1-v^2\right)_{L^2(\Omega)} \\
&= \left(J_{X^1}^T J_{X^1}(v^1-v^2)_t,\ v^1-v^2\right)_{L^2(\Omega)} \\
&= \left(J_{X^1}(v^1-v^2)_t,\ J_{X^1}(v^1-v^2)\right)_{L^2(\Omega)} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(J_{X^1}(v^1-v^2),\ J_{X^1}(v^1-v^2)\right)_{L^2(\Omega)} \\
&\quad - \left(\frac{\partial J_{X^1}}{\partial t}(v^1-v^2),\ J_{X^1}(v^1-v^2)\right)_{L^2(\Omega)}.
\end{aligned} \tag{4.6.5}
$$

Therefore,

$$
J_1 \ge \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|J_{X^1}(v^1-v^2)\right\|^2_{L^2(\Omega)} - C(T,K)\sup_{s\in[0,t]}\|v^1(s)-v^2(s)\|^2_{L^2(\Omega)}. \tag{4.6.6}
$$

$$
\begin{aligned}
|J_2| &\le C\|g^1_{ij} - g^2_{ij}\|_{L^\infty(\Omega)}\|v^1 - v^2\|_{L^2(\Omega)}\|v^2_t\|_{L^\infty[(0,T),L^2(\Omega)}\\
&\le C(T,K)\left(\sup_{s\in[0,t]}(|l_1 - l_2| + |\omega_1 - \omega_2|)\right)\sup_{s\in[0,t]}\|v^1(s) - v^2(s)\|_{L^2(\Omega)} \quad (4.6.7)\\
&\le C\sup_{s\in[0,t]}\|v^1(s) - v^2(s)\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \big((H^1 - H^2)(M^1v^1 + N^1v^1),\ v^1 - v^2\big)_{L^2(\Omega)}\\
&\quad + \big(H^2(M^1v^1 - M^2v^2 + N^1v^1 - N^2v^2),\ v^1 - v^2\big)_{L^2(\Omega)} \qquad (4.6.8)\\
&:= I_{31} + I_{32}.
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
|I_{31}| &\le \|H^1 - H^2\|_{L^\infty(\Omega)}\|M^1v^1 + N^1v^1\|_{L^2(\Omega)}\|v_1 - v_2\|_{L^2(\Omega)}\\
&\le C(T,K)\sup_{s\in[0,t]}[|l_1 - l_2| + |\omega_1 - \omega_2|]\cdot\|v^1 - v^2\|_{L^2(\Omega)} \qquad (4.6.9)\\
&\le C(T,K)\sup_{s\in[0,t]}\|v^1(s) - v^2(s)\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}
$$

$$
\begin{aligned}
I_{32} &= \big(H^2(M^1v^1 - M^2v^2 + N^1v^1 - N^2v^2),\ v^1 - v^2\big)_{L^2(\Omega)}\\
&= \left(H^2_{li}\left(\frac{\partial Y^1_j}{\partial t}\frac{\partial v^1_i}{\partial y_j} + v^1_j\frac{\partial v^1_i}{\partial y_j} - \frac{\partial Y^2_j}{\partial t}\frac{\partial v^2_i}{\partial y_j} - v^2_j\frac{\partial v^2_i}{\partial y_j},\right),\ v^1_l - v^2_l\right)_{L^2(\Omega)}\\
&\quad + \left(H^2_{li}\left(\left\{\Gamma^{i,1}_{jk}\frac{\partial Y^1_k}{\partial t} + \frac{\partial Y^1_i}{\partial x_k}\frac{\partial^2 X^1_k}{\partial t\partial y_j}\right\}v^1_j - \left\{\Gamma^{i,2}_{jk}\frac{\partial Y^2_k}{\partial t} + \frac{\partial Y^2_i}{\partial x_k}\frac{\partial^2 X^2_k}{\partial t\partial y_j}\right\}v^2_j\right),\ v^1_l - v^2_l\right)_{L^2(\Omega)}\\
&\quad + \big(H^2_{li}[\Gamma^{i,1}_{jk}v^1_jv^1_k - \Gamma^{i,2}_{jk}v^2_jv^2_k],\ v^1_l - v^2_l\big)_{L^2(\Omega)}\\
&:= J_{31} + J_{32} + J_{33}.
\end{aligned}
\qquad (4.6.10)
$$

As in Lemma 3.3, one can obtain

$$|J_{32}| \leq \|H^2\|_{L^\infty(\Omega)} \left\| \left(\left\{ (\Gamma^i_{jk}) \frac{\partial Y^1_k}{\partial t} v^1_j + \Gamma^{i,2}_{jk} \frac{\partial Y_k}{\partial t} v^1_j + \Gamma^{i,2}_{jk} \frac{\partial Y^2_k}{\partial t} (v^1_j - v^2_j) \right.\right.\right.$$
$$\left.\left.\left. + \frac{\partial Y_i}{\partial x_k} \frac{\partial^2 X^1_k}{\partial t \partial y_j} v^1_j + \frac{\partial Y^2_i}{\partial x_k} \frac{\partial^2 X_k}{\partial t \partial y_j} \right\} v^1_j + \frac{\partial Y^2_i}{\partial x_k} \frac{\partial^2 X^2_k}{\partial t \partial y_j} (v^1_j - v^2_j) \right) \right\|_{L^2(\Omega)} \|v^1 - v^2\|_{L^2(\Omega)}$$
$$\leq \|H^2\|_{L^\infty(\Omega)} \left(\|\Gamma^i_{jk}\|_{L^\infty(\Omega)} \left\| \frac{\partial Y^1_k}{\partial t} \right\|_{L^\infty(\Omega)} \||v^1\|_{L^2(\Omega)} + \|\Gamma^{i,2}_{jk}\|_{L^\infty(\Omega)} \left\| \frac{\partial Y_k}{\partial t} \right\|_{L^\infty(\Omega)} \|v^1\|_{L^2(\Omega)} \right.$$
$$+ |\Gamma^{i,2}_{jk}|_{L^\infty(\Omega)}| \left\| \frac{\partial Y^2_k}{\partial t} \right\|_{L^\infty(\Omega)} \|(v^1 - v^2)\|_{L^2(\Omega)}$$
$$\left\| \frac{\partial Y_i}{\partial x_k} \right\|_{L^\infty(\Omega)} \left\| \frac{\partial^2 X^1_k}{\partial t \partial y_j} \right\|_{L^\infty(\Omega)} \|v^1\|_{L^2(\Omega)} + \left\| \frac{\partial Y^2_i}{\partial x_k} \right\|_{L^\infty(\Omega)} \left\| \frac{\partial^2 X_k}{\partial t \partial y_j} \right\|_{L^\infty(\Omega)} \|v^1\|_{L^2(\Omega)}$$
$$\left. + \left\| \frac{\partial Y^2_i}{\partial x_k} \right\|_{L^\infty(\Omega)} \left\| \frac{\partial^2 X^2_k}{\partial t \partial y_j} \right\|_{L^\infty(\Omega)} \|v^1_j - v^2_j\|_{L^2(\Omega)} \right) \|v^1 - v^2\|_{L^2(\Omega)}$$
$$|J_{32}| \leq C \sup_{s \in [0,t]} \|v^1(s) - v^2(s)\|^2_{L^2(\mathbb{R}^3)}. \tag{4.6.11}$$

Similarly, one can get that

$$J_{33} \leq C \sup_{s \in [0,t]} \|v^1 - v^2\|^2_{L^2(\mathbb{R}^3)}(s). \tag{4.6.12}$$

It remains to estimate the last term J_{31}. In fact,

$$J_{31} = \left(J_{X^2} \left[(\frac{\partial Y^1}{\partial t} + v^1) \cdot \nabla v^1 - (\frac{\partial Y^2}{\partial t} + v^2) \cdot \nabla v^2 \right], J_{X^2}(v^1 - v^2) \right)_{L^2(\Omega)}$$
$$= \left(J_{X^2} \left[(\frac{\partial Y}{\partial t} + v^1 - v^2) \cdot \nabla v^1 + J_{X^2}(\frac{\partial Y^2}{\partial t} + v^2) \cdot \nabla (v^1 - v^2) \right], J_{X^2}(v^1 - v^2) \right)_{L^2(\Omega)}$$
$$|J_{31}| \leq C \left(\left| \frac{\partial Y}{\partial t} \right|_{L^\infty(\Omega_t)} |\nabla v^1| \|J_{X^2}(v^1 - v^2)\|_{L^2(\Omega)} + \right.$$
$$|J_{X^2}|_{L^\infty(\Omega_T)} \|(v^1 - v^2)\|_{L^2(\Omega)} \|J_{X^2}(v^1 - v^2)\|_{L^2(\Omega)}$$
$$+ \left| \left((\frac{\partial Y^2}{\partial t} + v^2) \cdot \nabla J_{X^2}(v^1 - v^2), J_{X^2}(v^1 - v^2) \right)_{L^2(\Omega)} \right|$$
$$\left. + \left| \frac{\partial Y^2}{\partial t} + v^2 \right|_{L^\infty(\Omega_T)} |\nabla J_{X^2}|_{L^\infty(\Omega_T)} \|(v^1 - v^2)\|_{L^2(\Omega)} \|J_{X^2}(v^1 - v^2)\|_{L^2(\Omega)} \right). \tag{4.6.13}$$

Since $\|J_{X^2}(v^1 - v^2)\|_{L^2(\Omega)} \leq C\|v^1 - v^2\|_{L^2(\Omega)}$, whence

$$|J_{31}| \leq C \sup_{[0,t]} \|v^1 - v^2\|_{L^2(\mathbb{R}^3)}. \tag{4.6.14}$$

From Lemma 4.2.4,we have the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|J_{X^1}(v^1 - v^2)\|^2_{L^2(\Omega)} + m\frac{\mathrm{d}}{\mathrm{d}t}|l_1 - l_2|^2 + \frac{\mathrm{d}}{\mathrm{d}t}[(\bar{J}(\omega_1 - \omega_2)) \cdot (\omega_1 - \omega_2)]$$
$$\leq C \sup_{[0,t]} \|v^1(s) - v^2(s)\|^2_{L^2(\mathbb{R}^3)}$$

It follows that

$$\|J_{X^1}(v^1 - v^2)\|^2_{L^2(\Omega)} + m|l_1 - l_2|^2 + (\bar{J}(\omega_1 - \omega_2)) \cdot (\omega_1 - \omega_2)$$
$$\leq C \int_0^t \sup_{s\in[0,\tau]} \|v^1(s) - v^2(s)\|^2_{L^2(\mathbb{R}^3)} \mathrm{d}\tau$$

Since X^1 is diffeomorphism and$(\bar{J}\omega) \cdot \omega \geq C|\omega|^2$, thus

$$\|v^1 - v^2\|^2_{L^2(\mathbb{R}^3)}(t) \leq C_0 \int_0^t \sup_{\tau\in[0,s]} \|v^1 - v^2\|^2_{L^2(\mathbb{R}^3)}(\tau)\mathrm{d}s \tag{4.6.15}$$

Assume that any $\theta \in [0, t]$, From (4.6.15), we have

$$\|v^1 - v^2\|^2_{L^2(\mathbb{R}^3)}(\theta) \leq C_0 \int_0^\theta \sup_{\tau\in[0,s]} \|v^1 - v^2\|^2_{L^2(\mathbb{R}^3)}(\tau)\mathrm{d}s$$
$$\leq \int_0^t \sup_{\tau\in[0,s]} \|v^1 - v^2\|^2_{L^2(\mathbb{R}^3)}(\tau)\mathrm{d}s$$

It shows that for any $t \in [0, T]$

$$\sup_{\tau\in[0,t]} \|v^1 - v^2\|^2_{L^2(\mathbb{R}^3)}(\tau) \leq C \int_0^t \sup_{\tau\in[0,s]} \|v^1 - v^2\|^2_{L^2(\mathbb{R}^3)}(\tau)\mathrm{d}s,$$

by Gronwall's inequality, $v^1 = v^2$ *a.e.* in $[0, T] \times \mathbb{R}^3$. Uniqueness, as in [95], implies that

$$v \in C([0, T]; H_s) \cap C^1([0, T]; H_{s-1}).$$

From (4.4.12), we know that

$$|(v, A(t, v))_H| \leq C(T_0, \|v\|_{H_{s_0}})(1 + \|v\|^2_{H_s}).$$

From the uniqueness and the continuity on the time, T can be extended to T_0 which does not depend on s.

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