Nonlinear Output Regulation with Time-varying or Nonlinear Exosystems

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In this thesis, we investigate the global robust output regulation problem for nonlinear systems subject to time-varying or nonlinear exosystems.

Output regulation problem, also known as servomechanism problem, is one of the central topics in control theory. The control objective is to design a feedback control law for the given plant so as to achieve asymptotic tracking for a class of reference signals and asymptotic rejection for a class of disturbance signals while maintaining the stability of closed-loop system. The reference or the disturbance signals are assumed to be generated from a dynamical system called the exosystem. Normally, the exosystem is a linear autonomous system, e.g. a harmonic oscillator, and the exogenous signals represent step or ramp signals, or sinusoidal signals contains finite number of harmonics. The extensions of the exosystem, from linear to nonlinear, autonomous to non-autonomous, significantly enlarge the categories of the exogenous signals, and more importantly, such extensions motivate the development of the output regulation theory in both scientific research and practical application.

Paying special attention to the appearance of time-varying or nonlinear exosystems, our research is mainly conducted under the general framework for tackling the output regulation problem. In general, first we convert the output regulation problem of the original plant into the stabilization problem of the augmented system which is composed of the plant and the designed internal model. Second, we achieve the global stabilization of the augmented system by robust and adaptive control approaches, according to both parameter uncertainty and dynamic uncertainty in either plant or the exosystem.

One of the crucial issues in output regulation problem is the design of the appropriate internal model. Internal model is a dynamical compensator which possesses an essential ability of generating all possible steady-state input information asymptotically, and it

should not only lead to a well-defined augmented system but also ensure the stabilizability of the augmented system. Besides, stabilization techniques for the augmented system should also be carefully chosen to meet the needs in different scenarios, e.g. the time-varying settings. Efforts are put on both sides throughout the thesis.

The main contributions of the thesis are outlined as follows.

1. A framework for handling the robust output regulation problem for general time-varying nonlinear systems subject to time-varying exosystem is proposed. Especially, certain existence conditions of a time-varying internal model is given, and problem conversion can be achieved.

   As an application of this framework, we give the solvability conditions of the output regulation problem for the time-varying nonlinear systems in output feedback form. Further, when parameter uncertainties occurred in the time-varying exosystem, we solve the corresponding adaptive robust output regulation problem resorting to some adaptive control methods. These results can also be applied to the time-varying nonlinear systems in lower triangular form.

2. The global robust output regulation problem for nonlinear systems subject to nonlinear exosystem is considered. A new class of internal models is introduced which relaxes the existence conditions of the former one. Also, this class of internal models has the merit that it is zero input globally asymptotically stable which greatly facilitates the global stabilization of the augmented system.

   Compared with the existing results, the new method solves the global robust output regulation problem without restrictions on the initial conditions or trajectory bounds of the exosystems, and the bound of the parameter uncertainties of the plant is not necessarily known. Moreover, utilizing the Nussbaum gain technique, the unknown control direction case can also be handled by modifying the control law.

3. The theoretical results have been applied to several practical control problems, such as the global disturbance rejection problem for FitzHugh-Nagumo model with Mathieu equation, the synchoniztion of periodically-forced pendulum with Rayleigh equation, etc..

摘 要

本文针对受时变或非线性外部系统影响的非线性控制系统, 研究了其全局鲁棒输出调节问题.

输出调节问题是控制理论中的一个核心问题. 输出调节问题的控制目标是对给定的系统模型设计一种反馈控制律, 使受控系统的输出实现对一类参考信号的渐近跟踪以及对一类干扰信号的渐近抑制, 同时保持闭环系统的稳定性. 上述的参考信号或干扰信号统称为外部信号, 并经外部系统生成. 外部系统通常是不受控制作用的线性时不变系统, 例如谐振振子. 而外部信号往往是阶跃信号, 斜坡信号或含有有限项谐波的弦信号. 本文尝试将外部系统从线性时不变系统推广到时变或非线性系统. 考虑到系统的时变或非线性特性, 上述对外部系统的推广不仅将明显地扩展外部信号的种类, 更重要的将在某种程度上促进输出调节问题的理论研究和实际应用.

针对提出的问题, 我们的研究将在输出调节问题的一般求解框架下展开. 简而言之, 我们首先设计一类称之为内模的动态补偿器, 受控系统连同内模被统称为增广系统. 同时, 我们将把受控系统的输出调节问题转化为针对增广系统的镇定问题. 随后, 考虑到增广系统所具有的参数不确定性和结构不确定性, 我们将利用鲁棒控制和自适应控制的方法, 解决增广系统的全局镇定问题.

由上述一般求解框架可以看出, 解决输出调节问题的一个关键是设计合适的内模. 作为一类动态补偿器, 内模应能够渐进生成所需稳态输入信息, 而作为增广系统的一部分, 内模应能够保证增广系统的可镇定性. 一个合适的内模必须兼顾上述两点要求. 同时, 输出调节问题的解决还有赖于针对增广系统的特性, 选取适合的镇定方法. 对此两方面的研究和阐述将贯穿论文始终.

本文的主要结论概括如下:

i) 针对受时变外部系统影响的非线性时变系统, 我们提出了其输出调节问题一般求解框架. 在给出时变内模存在条件和设计方法的基础上, 我们可相应地将输出调节问题转化为对增广系统的镇定问题.

在此求解框架下, 我们首先解决了一类具有输出反馈结构的非线性时变系统的输出调节问题. 进而, 我们考虑了时变外部系统存在参数不确定的情况. 通过改进时变内模并借助自适应控制的思想, 相应的输出调节问题可以获得解决. 上述的设计方法可以推广到具有下三角结构的非线性时变系统的输出调节问题中.

ii) 针对在非线性外部系统影响下的非线性系统的输出调节问题, 我们提出了一类新的内模设计方法. 经设计获得的内模避免了原有结果的若干存在性限制, 同时, 此类内模在零输入情况下具有全局渐进稳定的特性, 明显有利于增广系统的全局镇定问题的解决.

相应地, 我们解决了一类具有输出反馈结构的非线性系统的全局输出调节问题. 通过和原有结果的比较, 这种设计方法避免了对外部系统初始条件或轨迹范围的限制, 同时,

相应的控制器不需要借助受控系统的参数不确定性的信息.

iii) 上述两类问题的理论研究结果可以应用于若干系统的实际控制问题中. 例如, 我们考虑了当Mathieu方程做为时变外部系统时，FitzHugh-Nagumo模型的全局干扰抑制问题，又或当Rayleigh方程做为非线性外部系统时, 周期外力作用下的单摆和Rayleigh方程的同步控制问题.

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Yang Xi
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To My Mentor

.. All Sisyphus' silent joy is contained therein.
His fate belongs to him. His rock is a thing.
Likewise, the absurd man, when he contemplates his torment, silences all the idols...
There is no sun without shadow, and it is essential to know the night.
The absurd man says yes and his efforts will henceforth be unceasing.
If there is a personal fate, there is no higher destiny,
or at least there is, but one which he concludes is inevitable and despicable.
For the rest, he knows himself to be the master of his days ..

Albert Camus, "The Myth of Sisyphus"

Contents

List of Figures

Chapter 1

Introduction

1.1 Literature overview

The *output regulation problem*, also known as the *servomechanism problem*, is concerned with the design of a feedback control law for the given plant, such that the closed-loop system achieves asymptotic tracking for a class of reference signals and asymptotic rejection for a class of disturbance signals while maintaining the stability. A distinguishing feature of the output regulation problem is that the prescribed reference or disturbance signals are not required to be perfectly known as long as they are generated by a known autonomous dynamical system called the *exosystem*. The formulation and solution of the output regulation problem thus allow for the entire classes of signals to be tracked or to be rejected, and account for dynamical or static uncertainties of the given plant.

Output regulation problem is one of the central topics in control theory and application. The rigorous formulation of the problem in state-space description can be dated back to mid 1970s. Francis, Davison and Wonham et al. has considered the problem for linear time-invariant systems [18] [22] [23] etc. The solvability conditions have been shown to be equivalent to the solvability of a pair of linear matrix equations, known as the regulator equations, by Francis [23], or to be characterized as the property of the transmission polynomials of the composite system which contains the plant and the exosystem by Hautus [26]. An important outcome of these researches is the generalization of *internal model principle* given by Francis and Wonham [21] [22]. The internal model principle exhibits the essential fact that any feedback controller which solves the problem must incorporate the suitable copy (copies) of the exosystem to reproduce the feedforward information which is required to keep the regulated error output identically zero. The internal model principle can also be characterized by frequency domain approach [3] [11]. For the summary of linear output regulation problem, see [96] for instance.

Since 1980s, the researches on the output regulation problem have been extended from linear to nonlinear time-invariant systems. Early results have been concerned with the modification of internal model principle in the nonlinear settings, and for the case where the exogenous signals are constant signals, see Hepburn, Wonham, Huang, Rugh and Benendetto et al. [2] [27] [28] [29] [30]. Later, Isidori and Byrnes have considered the problem for the known nonlinear plant with time-varying exogenous signal, which is generated from a neutrally stable exosystem, in the tread-setting work [42]. The center manifold theory enables [42] to show the necessary condition for the solvability of problem is the solvability of a set of partial differential equations known as *nonlinear regulator equations*. The nonlinear regulator equations can be considered as the counterpart of the regulator equations (Sylvester equations) given Francis [23] in the nonlinear settings. The solution of such nonlinear regulator equations characterizes the steady-state response of the systems, and also provides a feedforward control information which is necessary to offset the steady-state regulated error. Based on these, both state and error output feedback control laws can be synthesized to achieve asymptotic tracking and disturbance rejection while securing the local asymptotic stability of the closed-loop systems. However, the construction of the controller/regulator explicitly depends on the plant parameters therefore it has no robustness with respect to the parameter variations.

A breakthrough in handling parameter uncertainties of the plant has been achieved by Huang, who first suggested that the linear internal model principle fails in nonlinear systems because the steady-state regulated error in the nonlinear settings is not a linear function of the exogenous signals. Based on this, Huang has elaborated in the seminal work [31] [33] a systematic solution to the robust nonlinear output regulation problem for the case where the solution of nonlinear regulator equations is polynomial in the exogenous signals, and showed the possibility to design an internal model generating all the exogenous signals and their higher order harmonics so as to achieve robust output regulation. This key idea has also been presented by Khalil [57] and Priscoli [87] independently.

In the past two decades, considerable research efforts have been paid to the robust nonlinear output regulation problem with semi-global or global stability. In the original formulation [42], only local asymptotic stability can be achieved for the closed-loop system.

For this purpose, the Lyapunov linearization methods can be used. When semi-global or global stability requirement is imposed, the situation becomes much more complicated. In order to achieve the goal, restriction on the structural properties of the nonlinear plant is required. A major step forward in this direction has been achieved by Khalil [57] where the semi-global robust output regulation problem for a class of feedback linearizable systems with no zero dynamics has been considered. Isidori [44] has extended Khalil's work to a class of lower triangular systems. Then Khalil [58] has extended his work [57] to a class of systems whose zero dynamics possesses certain properties of input-to-state stability. Later on, this condition has been relaxed to global asymptotical stability of the zero dynamics of the corresponding error systems by Serrani et al. [101]. It is worth noting that the solution of the regulation problem does not require the asymptotical stability of the zero dynamics as first shown by Huang [34], but simply a "non-resonance" condition between zero dynamics and exosystem.

The output regulation problem with global stability has been solved for a class of output feedback systems in [100], a class of strict feedback systems in [79], and a class of lower triangular systems in [64], to name a few. Huang and Chen have established a general framework for tackling the output regulation problem in [37], which offers great flexibility to systematically design the regulator to achieve global stability. Several classes of typical nonlinear systems was considered under this general framework, e.g. lower triangular systems [13], output feedback systems [15] or feedforward systems [12].

The latest progress on the nonlinear output regulation problem, in the personal point of view, lies in several prospects. One of them is the so called "non-equilibrium theory" which has been generalized to the case of nonlinear systems that do not necessarily possess equilibria [7] [8]. For instance, the zero dynamics of the controlled plant does not required to have a (global) asymptotically stable equilibrium, instead, the zero dynamics together with the exosystem is assumed to have a compact attractor which is locally exponentially stable. As the application, some considerations have been given to the possibly non-minimum phase nonlinear plants in a series of papers [10] [68] [91]. Another progress regarding the transient response of the regulated plant has been given by in [67] [80] [102] [103] [109]. The technique used in these publications is firstly introduced in the sliding model control framework and then by using Lyapunov redesign. Based on the conditional servocompensator and high-gain observer, zero regulated error output is achieved without degrading the transient response of the controlled plant. Last but not the least, the recognition of the observability property of the designed internal model and its role in achieving output regulation have been clarified in [16], [47], [69], [72] [113]. In such philosophy, the design of the internal model is considered, partially, to be inherited from the design of observer. It is worth noting that the most promising results on the design of nonlinear internal model, with regard to the observability property of the corresponding internal model (steady-state generator), can be categorized consequently. For instance, the linearly observable steady-state generator with nonlinear output [37], steady-state generator in the observability canonical form [9], and steady-state generator in the adaptive observer form [89] [90]. In this tread, serial results have also been given in [48] [70] [71] [73].

Vigorous activities of research in output regulation problem have been witnessed, and quite a few surveys and monographs have been published [4] [6] [36] [46] [85]. Besides the aforementioned outcomes, we would like to mention a few specific topics: nonlinear output regulation with uncertain linear time-invariant exosystem has been studied in [66] [83] [84] [101] [121], the practical output regulation has studied in [32] [71] [115], output regulation with a switched linear internal model has been studied in [94], output regulation of nonlinear time-delay system has studied in [25].

Recently, some attention has been given to the output regulation problem of time-varying systems and nonlinear output regulation problem subject to nonlinear exosystem. The output regulation of linear time-varying system has been considered in [39] [99] [122] [123] [124] which means either the plant or the exosystem is a linear time-varying system. In particular, [122] and [123] presents an internal model based design approach for linear periodic systems. The results of nonlinear output regulation with a nonlinear exosystem can be found in [9] [14] [20] [90] [117] etc.. A crucial issue when dealing with the nonlinear exosystem is how to give the testable condition for the existence of an appropriate internal model.

1.2 Thesis outline

The contributions of the thesis are mainly on expanding the categories of plant and exosystem for nonlinear output regulation problem. Normally, the exosystem is linear time-invariant and it generates step or ramp or sinusoidal signals. The extension from linear to nonlinear, autonomous to non-autonomous of the exosystem significantly will enlarge the types of the exogenous signals, and more importantly, such extension requires new design methodology for the internal model so as the overall controller.

In fact, the design of the internal model and the stabilization of the augmented system are two essential issues in output regulation problem, and they are strongly interlaced. Due to the different settings of our problem, investigation on these two issues acts as the main theme through the thesis. To summarize the progress we have made: first, a framework for handling the global robust output regulation problem for general time-varying nonlinear plants subject to time-varying exosystem is proposed. Whether the time-varying exosystem contains uncertain parameters or not, an appropriate time-varying internal model can be constructed under certain conditions, and the output regulation problem for the plant can be converted into the stabilization problem of a nonlinear time-varying augmented system. The global stabilization of the augmented system can also be achieved under certain assumptions. Second, global robust output regulation problem for nonlinear plants subject to nonlinear exosystem is considered. A new class of internal models is introduced, which relaxes the existence condition for the internal model in comparison with the former results. Also, this class of internal models has its merit that it is zero input globally asymptotically stable which greatly facilitates the global stabilization of the augmented system. These theoretical results can be applied to several practical control problems, such as the global disturbance rejection problem of FitzHugh-Nagumo model subject to Mathieu equation, or the synchronization of periodically-forced pendulum with Rayleigh equation.

The rest of the thesis are organized as follows.

Chapter 2: Some fundamental concepts, control techniques and useful lemmas in nonlinear control theory are reviewed, and the general framework for handling the output regulation problem is summarized. These contents will be used in subsequent chapters.

Chapter 3: A framework for handling global robust output regulation for general time-varying nonlinear systems subject to time-varying exosystem is proposed. As the application of the framework, the output regulation problem for the time-varying nonlinear systems in output feedback form is solved.

Chapter 4: The output regulation problem with uncertain time-varying exosystem is further considered by modifying the framework proposed in Chapter 4. A generalized time-varying internal model is provided. Combined with some adaptive control techniques, the adaptive robust output regulation problem is solved.

Chapter 5: In the presence of nonlinear exoystems, a new class of internal models is introduced. Comparing with the former results, the existence condition for the internal model is relaxed, and the internal model itself greatly facilitates the global stabilization of the augmented system. The global robust output regulation problem for a class of nonlinear systems in the strict output feedback form is studied.

Chapter 6: For nonlinear systems in the general output feedback form, along the line of Chapter 5, an internal model can be given to deal with the nonlinear exosystem. Unlike the input-filter based design in Chapter 5, an observer-based controller is proposed to solve the global robust output regulation problem.

Chapter 7: Some concluding remarks are given. The possible further works are also presented.

Chapter 2

Background and Preliminaries

In this chapter, we review some fundamental concepts, control techniques for nonlinear control systems and time-varying systems, and we summarize the general framework for tackling the output regulation problem proposed by Huang [36]. Some useful lemmas are also given. These fundamentals will be helpful and be referred to in the subsequent chapters. The materials shown in this chapter can be found in many textbooks, monographs and papers on linear and nonlinear control systems, for instance, Chen [11], Freeman and Kokotovic [24], Isidori [43] [45], Kailath [53], Khalil [59], Kokotovic and Arcak [60], Kristic, Kanellakopoulos, and Kokotovic [62], Marino and Tomer [74], Rugh [95], Sastry [97], Slotine and Li [110], to name a few.

2.1 Fundamentals of nonlinear systems

Consider the nonlinear system described by

$$\dot{x} = f\big(x(t), t\big), \quad x(t_0) = x_0 \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $t \in [t_0, \infty)$, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. t_0 is the initial time and x_0 is the initial state. The components of x and f can be represented by $x = \mathrm{col}(x_1, \cdots, x_n)$ and $f = \mathrm{col}(f_1, \cdots, f_n)$ respectively.

If the function $f(x, t)$ does not depend on time t explicitly, system (2.1) will reduce to

$$\dot{x} = f(x), \quad x(t_0) = x_0 \tag{2.2}$$

The dynamic systems in the form of (2.1) are called *non-autonomous system* while (2.2) are called *autonomous system*.

A constant vector $x = x^*$ is said to be the *equilibrium point* of (2.1) if

$$f(x^*, t) = 0, \quad \forall t \geq t_0$$

i.e. whenever the state of the system starts at x^*, it will remain at x^* for all future time. Without loss of generality, we assume $x^* = 0$ is the equilibrium point of system (2.1).

Throughout this chapter, we assume that $f(x, t)$ is piecewise continuous in t and *locally Lipschitz* in x, i.e.

$$\|f(x,t) - f(y,t)\| \leq L\|x - y\| \tag{2.3}$$

where L is a constant called *Lipschitz constant*, and (2.3) holds for all (x, t) and (y, t) in some open neighborhood of (x_0, t_0). Under this assumption, given any x_0, there exists some $t_1 > t_0$ and a unique continuous function $x : [t_0, t_1] \to \mathbb{R}^n$ satisfying equation (2.1) with $x(t_0) = x_0$. The local solution $x(t)$ of system (2.1) over the time interval $[t_0, t_1]$ is also called the state trajectory or system state of (2.1).

Definition 2.1. ([59] Def.4.4, Def.4.5)
The equilibrium point $x = 0$ of (2.1) is

- ***Stable*** *if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, t_0) > 0$ such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0 \geq 0 \tag{2.4}$$

- ***Unstable*** *if it is not stable.*

- ***Uniform stable*** *if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$, independent of t_0, such that (2.4) is satisfied.*

- ***Asymptotically stable*** *if it is stable and there is a positive constant $c = c(t_0)$ such that $x(t) \to 0$ as $t \to \infty$, for all $x(t_0) < 0$.*

- ***Uniformly asymptotically stable*** *if it is uniformly stable and there is a positive constant c, independent of t_0, such that for all $\|x(t_0)\| < 0$, $x(t) \to 0$ as $t \to \infty$, uniformly in t; that is, for each $\eta > 0$, there is $T = T(\eta) > 0$ such that*

$$\|x(t)\| < \eta, \quad \forall t \geq t_0 + T(\eta), \ \forall \|x(t_0)\| < c \tag{2.5}$$

- ***Globally uniformly asymptotically stable*** *if it is uniformly stable, $\delta(\varepsilon)$ can be chosen to satisfy $\lim_{\delta(\varepsilon)} = \infty$, and, for each pair of positive number η and c, there is $T = T(\eta, c) > 0$ such that*

$$\|x(t)\| < \eta, \quad \forall t \geq t_0 + T(\eta, c), \ \forall \|x(t_0)\| < c \tag{2.6}$$

- ***Exponentially stable*** *if there exist positive constants c, k, and λ such that*

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall \|x(t_0)\| < c \tag{2.7}$$

- ***Global exponentially stable*** *if (2.7) is satisfied for any initial state $x(t_0)$.*

∎

The equivalent definitions of the stability can also be characterized by comparison functions.

Definition 2.2. ([59] Lemma 4.5)
The equilibrium point $x = 0$ of (2.1) is

- ***Uniformly stable*** *if and only if there exist a class $\mathcal{K}$ function α and a positive constant c, independent of t_0, such that*

$$\|x(t)\| \leq \alpha\big(\|x(t_0)\|\big), \quad \forall t \geq t_0 \geq 0, \ \forall \|x(t_0)\| < c \tag{2.8}$$

- ***Uniformly asymptotically stable*** *if and only if there exist a class $\mathcal{KL}$ function β and a positive constant c, independent of t_0, such that*

$$\|x(t)\| \leq \beta\big(\|x(t_0)\|, t - t_0\big), \quad \forall t \geq t_0 \geq 0, \ \forall \|x(t_0)\| < c \tag{2.9}$$

- ***Globally uniformly asymptotically stable*** *if and only if (2.9) holds for any initial state $x(t_0)$.*

∎

By Lyapunov direct method, some stability criterions can be given. We notate first

- A function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is said to be *positive definite* if $V(0) = 0$ and $V(0) > 0$, $\forall x \neq 0$. If $V(0) = 0$ and $V(0) \geq 0$, $\forall x \neq 0$, $V(x)$ is said to be *positive semidefinite.*
- A function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is said to be *radially unbounded* if $V(x) \to \infty$ as $\|x\| \to \infty$.

- A function $V(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite if $V(t,x) \geq W_1(x)$ for some positive definite function $W_1(x)$. $V(t,x)$ is said to be positive semidefinite if $V(t,x) \geq 0$. $V(t,x)$ is said to be radially unbounded if $V(t,x) \geq W_1(x)$ for some positive definite function $W_1(x)$ and $W_1(x)$ is radially unbounded, $V(t,x)$ is said to be *decrescent* if $V(t,x) \leq W_2(x)$ for some positive definite function $W_2(x)$.

the we have

Theorem 2.1. **([59] Theorem 4.8)** *Let $x = 0$ be an equilibrium point for (2.46) and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : [0,\infty) \times D \to \mathbb{R}$ be a continuously differentiable function such that*

$$W_1(x) \leq V(t,x) \leq W_2(x) \tag{2.10}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0 \tag{2.11}$$

$\forall t \geq 0$ and $\forall x \in D$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on D. Then, $x = 0$ is uniformly stable. ∎

Theorem 2.2. **([59] Theorem 4.9)** *Suppose the assumptions of Theorem 2.1 are satisfied with inequality (2.11) strengthened to*

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} f(t,x) \leq -W_3(x)$$

$\forall t \geq 0$ and $\forall x \in D$, where $W_3(x)$ is a continuous positive definite functions on D. Then, $x = 0$ is uniformly asymptotically stable.

If $D = \mathbb{R}^n$ and $W_1(x)$ is radially unbounded, then $x = 0$ is globally uniformly asymptotically stable. ∎

2.2 Stabilization of nonlinear control systems

Consider the nonlinear control system described by

$$\dot{x} = f(x,u), \quad x \in \mathbb{R}^n,\ u \in \mathbb{R} \tag{2.12}$$

where $f(0,0) = 0$, and $f(x,u)$ is a smooth function of x and u. The control objective is to design a feedback control law $u = \alpha(x)$ such that the equilibrium point $x = 0$ of the closed-loop system

$$\dot{x} = f\big(x, \alpha(x)\big) \tag{2.13}$$

is globally asymptotically stable, i.e. considering $V(x)$ as a Lyapunov function candidate, where $V(x)$ is a positive definite, continuous differentiable function, its derivative along the trajectory of (2.13) should satisfy $\dot{V}(x) \leq -W(x)$, where $W(x)$ is a positive definite function. Therefore $\alpha(x)$ is required to guarantee that

$$\frac{\partial V(x)}{\partial x} f\big(x, \alpha(x)\big) \leq -W(x), \quad \forall x \in \mathbb{R}^n \tag{2.14}$$

In such scenario, roughly speaking, $V(x)$ can be referred to as the *control Lyapunov function*. In [1], Artstein showed that the existence of a control Lyapunov function is equivalent to global asymptotic stabilizability of the control system (2.12), i.e., under the control $\alpha(x)$, the equilibrium point of (2.13) is global asymptotic stable. (See [62] Section 2.1.2 for details)

By virtue of the control Lyapunov function, we can utilize the backstepping technique [62] to globally stabilize nonlinear systems in certain structural forms. It is well known that backstepping technique is a recursive design methodology, the essential idea of backstepping is to consider the specified state variables as the "virtual controls" in each step so as to design certain intermediate control law to stabilize the relevant subsystems in the "top-to-bottom" manner. For instance, as shown in [62] Section 2.3.1, the nonlinear control system in the strict feedback form

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= f_1(x_1, x_2) + g_2(x_1, x_2)x_3 \\
&\vdots \\
\dot{x}_{n-1} &= f_{n-1}(x_1, \cdots, x_{n-1}) + g_{n-1}(x_1, \cdots, x_{n-1})x_n \\
\dot{x}_n &= f_n(x_1, \cdots, x_n) + g_n(x_1, \cdots, x_n)u
\end{aligned} \tag{2.15}$$

can be globally stabilized with a state feedback control law $u(x_1, \cdots, x_n)$ under certain conditions.

In the presence of uncertain parameters, the adaptive backstepping and tuning functions are employed to achieve the stabilization. For example, consider the nonlinear control system in the parametric strict feedback form

$$\begin{aligned}
\dot{x}_1 &= x_2 + f_1^T(x_1)\theta \\
\dot{x}_2 &= x_3 + f_2^T(x_1, x_2)\theta \\
&\vdots \\
\dot{x}_{n-1} &= x_n + f_{n-1}^T(x_1, \cdots, x_{n-1})\theta \\
\dot{x}_n &= \beta(x)u + f_n^T(x)\theta
\end{aligned} \tag{2.16}$$

where θ is a vector of uncertain constant parameters. In each step of the recursive design procedure, we use the estimation $\hat{\theta}$ of θ to get the virtual control law based on the adaptive control Lyapunov function, and meanwhile we adapt $\hat{\theta}$ with certain update law $\dot{\hat{\theta}} = \tau(x_i)$ where $\tau(x_i)$ is also a recursively designed function called the "tuning function". The global stabilization of system (2.16) with a state feedback control law is shown in [62] Section 4.2.

The backstepping and tuning functions are the classical nonlinear design tools. However, considering the facts that nonlinear systems may contain both dynamical and static uncertainties, and some of the state variables may not be able to utilize for feedback design, the aforementioned methods shall be modified and some new ideas shall be introduced to achieve the stabilization of nonlinear control systems.

2.3 Fundamentals of time-varying systems

A major part of thesis is considering the time-varying control systems. Some definitions and preliminaries are given in this section for further convenience.

2.3.1 Polynomial differential operator

Consider an ordinary differential equation with time-varying coefficients as follows.

$$y^{(n)} - a_{n-1}(t)y^{(n-1)} - \cdots - a_1(t)y^{(1)} - a_0(t)y = u \tag{2.17}$$

where $y : \mathbb{R}_+ \to \mathbb{R}$, $u : \mathbb{R}_+ \to \mathbb{R}$, $y^{(i)} \stackrel{\text{def}}{=} \frac{d^i y}{dt^i}$, and $a_i(t)$, $i = 0, 1, \cdots, n-1$ are sufficiently smooth, uniformly bounded functions.

The state-space representation of equation (2 17) can be shown in the following form

$$x = \begin{bmatrix} 0 & 1 & & 0 \\ & & & \\ 0 & 0 & & 1 \\ a_0(t) & a_1(t) & & a_{n-1}(t) \end{bmatrix} x + \begin{bmatrix} 0 \\ \\ 0 \\ 1 \end{bmatrix} u, \quad x = \begin{bmatrix} y \\ y^{(1)} \\ \\ y^{(n-1)} \end{bmatrix} \tag{2 18}$$

$$y = \begin{bmatrix} 1 & 0 & & 0 \end{bmatrix} x$$

Alternatively, by introducing the following definition, we can express equation (2 17) in another manner

Definition 2.3. Left Polynomial Differential Operator

An LTV left Polynomial Differential Operator (PDO) of degree n is defined by

$$P_l(s,t) = a_n(t)s^n + a_{n-1}(t)s^{n-1} + \quad + a_1(t)s + a_0(t) \tag{2 19}$$

where $s \overset{def}{=} \frac{d}{dt}(\ )$, $a_i(t)$, $i = 0, 1, \quad , n-1, n$ are sufficiently smooth, uniformly bounded functions, $a_n(t) \neq 0$ for some $t \in \mathbb{R}_+$ And when $a_n(t) \equiv 1$ for all $t \in \mathbb{R}_+$, $P_l(s,t)$ is called the monic left PDO ▪

By the definition left PDO, the I/O property of (2 17) or (2 18) can be simplified in the form of

$$P_{m\,l}(s,t)[y] = u, \quad P_{m\,l}(s,t) = s^n - a_{n-1}(t)s^{n-1} - \quad - a_1(t)s - a_0(t) \tag{2 20}$$

Obviously, $P_{m\,l}(s,t)$ is a monic left PDO

Compared with (2 18), another state-space description of linear time-varying systems often occurs as follows

$$z = \begin{bmatrix} b_{n-1}(t) & 1 & & 0 \\ & & & \\ b_1(t) & 0 & & 1 \\ b_0(t) & 0 & & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ \\ 0 \\ 1 \end{bmatrix} u \tag{2 21}$$

$$y = \begin{bmatrix} 1 & 0 & & 0 \end{bmatrix} z$$

To characterize the I/O property of system (2 21), we further define right PDO in the parallel manner with Definition 2 3

Definition 2.4. Right Polynomial Differential Operator
An LTV right Polynomial Differential Operator (PDO) of degree n is defined by

$$P_r(s,t) = s^n b_n(t) + s^{n-1} b_{n-1}(t) + \cdots + s b_1(t) + b_0(t) \tag{2.22}$$

where $s \stackrel{def}{=} \frac{d}{dt}(\cdot)$, $b_i(t)$, $i = 0, 1, \cdots, n-1, n$ *are sufficiently smooth, uniformly bounded functions,* $b_n(t) \neq 0$ *for some* $t \in \mathbb{R}_+$. *And when* $b_n(t) \equiv 1$ *for all* $t \in \mathbb{R}_+$, $P_r(s,t)$ *is referred to as a monic right PDO.* ▪

It can be verified that, along the trajectory of (2.21)

$$y = z_1, \quad y^{(1)} = b_{n-1}(t)y + z_2, \quad y^{(2)} = \frac{d}{dt}\big(b_{n-1}(t)y\big) + b_{n-2}(t)y + z_3, \cdots$$
$$y^{(n)} = \frac{d^{n-1}}{dt^{n-1}}\big(b_{n-1}(t)y\big) + \cdots + \frac{d}{dt}\big(b_1(t)y\big) + b_0(t)y + u$$

Clearly, for system (2.21), the I/O property can be shown by

$$P_{m\,r}(s,t)[y] = u, \quad P_{m\,r}(s,t) = s^n - s^{n-1}b_{n-1}(t) - \cdots - sb_1(t) - b_0(t)$$

where $P_{m\,r}(s,t)$ is a monic right PDO.

Remark 2.1. Definitions 2.3 and 2.4 are slightly modified from Definitions 2.1 and 2.2 in [114]. As shown in [114], we may extend the definitions of the left PDO or the right PDO to admit piecewise continuous parameters $a_i(t)$ or $b_i(t)$ respectively. ▪

In definitions 2.3 and 2.4, it is noticeable that the sufficient smooth property of $a_i(t)$ or $b_i(t)$ is required. We could further show that, under certain addition conditions, an identical I/O expression can be realized either in the form of (2.19) or (2.22), which also implies the dynamical system (2.18) can be transformed into (2.21). The coefficients of these two forms are correlated, i.e. $a_i(t)$ can be determined from $b_i(t)$ and vice versa.

Such relationships are insufficiently addressed in [114] by giving the operator identity $a(t)s = sa(t) - \dot{a}(t)$ only. For clarification and self-sustained, we can show the following result.

Proposition 2.1. *A monic PDO of degree n can be written either in the left form or the right form, if* $a_i(t)$, $b_i(t)$, $i = 0, 1, \ldots, n-1$, *are sufficiently smooth functions of time,* $a_i(t)$, $b_i(t)$ *and their kth derivatives,* $k = 1, \ldots, l$ *with l sufficiently large, are all uniformly* *bounded over $[0, +\infty)$. Moreover, the coefficients $a_i(t)$ can be determined from $b_i(t)$, and v.v. as follows*

$$a_i(t) = \sum_{j=i}^{n-1} C_j^i b_j^{(j-i)}(t) \tag{2.23}$$

$$b_{n-i}(t) = \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^j a_{n-i+j}^{(j)}(t) \tag{2.24}$$

where C_n^k is the binomial coefficient, the number of $k-$combination from n elements, $a_i^{(k)}(t)$ and $b_i^{(k)}(t)$ are the kth derivative of $a_i(t)$ and $b_i(t)$ respectively. ∎

Proof: Consider the left monic PDO corresponds to the system expression (2.18), we note that

$$y^{(n)} = a_{n-1}(t)y^{(n-1)} + a_{n-2}(t)y^{(n-2)} + \cdots + a_1(t)y^{(1)} + a_0(t)y + u \tag{2.25}$$

On the contrary, the right monic PDO corresponds to the system expression (2.21), and

$$y^{(n)} = \big(b_{n-1}(t)y\big)^{(n-1)} + \big(b_{n-2}(t)y\big)^{(n-2)} + \cdots + \big(b_1(t)y\big)^{(1)} + b_0(t)y + u \tag{2.26}$$

where $\big(b(t)y\big)^{(i)} = \frac{d^i}{dt^i}\big(b(t)y\big)$.

To show (2.23), first notice the fact that for any sufficiently smooth function $b(t) \in \mathbb{R}$

$$\big(b(t)y\big)^{(i)} = \sum_{k=0}^{i} C_i^k b^{(k)}(t) y^{(i-k)}, \quad i = 1, 2, \cdots \tag{2.27}$$

By using (2.27), equation (2.26) can be expressed as

$$\begin{aligned} y^{(n)} &= \sum_{j=0}^{n-1} \big(b_j(t)y\big)^{(j)} + u = \sum_{j=0}^{n-1} \Big(\sum_{k=0}^{j} C_j^k b_j^{(k)}(t) y^{(j-k)}\Big) + u \\ &\overset{i=j-k}{=} \sum_{j=0}^{n-1} \Big(\sum_{i=0}^{j} C_j^{j-i} b_j^{(j-i)}(t) y^{(i)}\Big) + u = \sum_{i=0}^{n-1} \Big(\sum_{j=i}^{n-1} C_j^{j-i} b_j^{(j-i)}(t)\Big) y^{(i)} + u \end{aligned} \tag{2.28}$$

On the other hand, equation (2.25) can be expressed as $y^{(n)} = \sum_{i=0}^{n-1} a_i(t)y^{(i)} + u$. Compare the right hand sides between (2.25) and (2.28) and match the coefficient of the ith derivative of y in (2.28) with those of the ith derivative of y in (2.25) gives

$$a_i(t) = \sum_{j=i}^{n-1} C_j^{j-i} b_j^{(j-i)}(t) = \sum_{j=i}^{n-1} C_j^i b_j^{(j-i)}(t)$$

since $b_i(t)$ and its kth derivatives, $k = 1, \dots, l$ with l sufficiently large, are all uniformly bounded over $[0, +\infty)$, so $a_i(t)$ is uniformly bounded over $[0, +\infty)$.

To show (2.24). we will utilize the following fact

$$a(t)y^{(i)} = \sum_{k=0}^{i} C_i^k (-1)^k \left(a^{(k)}(t)y\right)^{(i-k)} \tag{2.29}$$

The proof of (2.29) is given by induction.
First, it can be seen that $a(t)y^{(1)} = \left(a(t)y\right)^{(1)} - a^{(1)}(t)y$.
Next, suppose (2.29) holds, then

$$\begin{aligned}
a(t)y^{(i+1)} &= \left(a(t)y^{(i)}\right)^{(1)} - a^{(1)}(t)y^{(i)} \\
&= \sum_{k=0}^{i} C_i^k(-1)^k \left(a^{(k)}(t)y\right)^{(i-k+1)} - \sum_{k=0}^{i} C_i^k(-1)^k\left(a^{(k+1)}(t)y\right)^{(i-k)} \\
&= C_i^0(-1)^0\left(a^{(0)}(t)y\right)^{(i+1)} + \cdots + C_i^m(-1)^m\left(a^{(m)}(t)y\right)^{(i-m+1)} + \cdots + C_i^i(-1)^i\left(a^{(i)}(t)y\right)^{(1)} \\
&\quad - C_i^0(-1)^0\left(a^{(1)}(t)y\right)^{(i)} - \cdots - C_i^{m-1}(-1)^{m-1}\left(a^{(m)}(t)y\right)^{(i-m+1)} - \cdots - C_i^i(-1)^i\left(a^{(i+1)}(t)y\right)
\end{aligned}$$

since

$$\begin{aligned}
C_i^0(-1)^0\left(a^{(0)}(t)y\right)^{(i+1)} &= C_{i+1}^0(-1)^0\left(a^{(0)}(t)y\right)^{(i+1)} \\
-C_i^i(-1)^i\left(a^{(i+1)}(t)y\right) &= C_{i+1}^{i+1}(-1)^{i+1}\left(a^{(i+1)}(t)y\right) \\
C_i^m(-1)^m - C_i^{m-1}(-1)^{m-1} &= C_{i+1}^m(-1)^m
\end{aligned}$$

so

$$a(t)y^{(i+1)} = \sum_{m=0}^{i+1} C_{i+1}^m(-1)^m\left(a^{(m)}(t)y\right)^{(i+1-m)}$$

By induction, we verify that (2.29) holds.

By using (2.29), equation (2.25) can be expressed as

$$\begin{aligned}
y^{(n)} &= \sum_{j=0}^{n-1} a_j(t)y^{(j)} + u = \sum_{j=0}^{n-1}\left(\sum_{k=0}^{j} C_j^k(-1)^k\left(a_j^{(k)}(t)y\right)^{(j-k)}\right) \\
&\overset{i=j-k}{=} \sum_{j=0}^{n-1}\sum_{i=0}^{j} C_j^{j-i}(-1)^{j-i}\left(a_j^{(j-i)}(t)y\right)^{(i)} + u = \sum_{i=0}^{n-1}\sum_{j=i}^{n-1}\left(C_j^{j-i}(-1)^{j-i}a_j^{(j-i)}(t)y\right)^{(i)} + u
\end{aligned} \tag{2.30}$$

On the other hand, equation (2.26) can be expressed as $y^{(n)} = \sum_{i=0}^{n-1}\left(b_i(t)y\right)^{(i)} + u$. Compare the right hand sides between (2.26) and (2.30) gives

$$b_i(t) = \sum_{j=i}^{n-1} C_j^{j-i}(-1)^{j-i}a_j^{(j-i)}(t) \tag{2.31}$$

since $a_i(t)$ and its kth derivatives, $k = 1, \dots, l$ with l sufficiently large, are all uniformly bounded over $[0, +\infty)$, so $b_i(t)$ is uniformly bounded over $[0, +\infty)$.

(2.31) is equivalent to (2.24), i.e. $b_{n-i}(t) = \sum_{j=0}^{i-1}(-1)^j C_{n-i+j}^j a_{n-i+j}^{(j)}(t)$. As a matter of fact, in (2.24), define $n - i = k$, then $i = n - k$ and

$$b_k(t) = \sum_{j=0}^{n-k-1} (-1)^j C_{k+j}^j a_{k+j}^{(j)}(t)$$

define $k + j = m$, then $j = m - k$ and

$$b_k(t) = \sum_{m=k}^{n-1} (-1)^{m-k} C_m^{m-k} a_m^{(m-k)}(t)$$

which is actually (2.31). ◇

2.3.2 Controllability, observability and canonical forms

Consider the following linear time-varying system

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x \end{aligned} \tag{2.32}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, $y \in \mathbb{R}^m$ are the state, input and output respectively, $A(t)$, $B(t)$, $C(t)$ are matrices of appropriate order compatible with x, u, y and are assumed to be at least continuous on $\mathbb{R}$.

The complete solution of (2.32) with initial state x_0 at t_0 is given by

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau, \quad t \geq t_0$$

where $\Phi(t, \tau)$ is the *state-transition matrix* associated with $A(t)$ and is defined by $\Phi(t, \tau) = X(t)X^{-1}(\tau)$, where $X(t)$ is any $n \times n$ matrix solution of the homogeneous system $\dot{x} = A(t)x$ with $\det X(t) \neq 0$, $\forall t$. $X(t)$ is called the *fundamental matrix* for $A(t)$.

Next, we clarify the "controllability" of system (2.32).

Recall the definitions given by Kalman [54], for system (2.32), a state x_0 is said to be *controllable at time* t_0 if there exists a control function $u(t)$, depending on x_0, t_0 and being defined over some finite closed interval $[t_0,\ t_1]$, such that $x(t_1) = 0$ If this is true for every state x_0, we say that (2.32) is *completely controllable at time* t_0. If this is true for every t_0, we say simply that (2.32) is *completely controllable*. In particular, (2.32) is completely controllable at time t_0 if and only if the *controllability gramian* $M(t_0, t_1)$

$$M(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, \tau) B(\tau) B^T(\tau) \Phi^T(t_0, \tau) d\tau \tag{2.33}$$

is nonsingular for some $t_1 > t_0$.

The definition of *uniformly completely controllable* for system (2.32) is also given by Kalman [54]. We are more interested in the case when $A(t), B(t), C(t)$ are uniformly bounded. As in [106], (2.32) with uniformly bounded coefficients is uniformly completely controllable if there exists some $\delta > 0$ such that for all t, $M(t-\delta, t) \geq \alpha_1(\delta) > 0$, where $\alpha_1(\delta)$ is a positive constant determined solely by δ.

The observability of system (2.32) can be defined in the dual manner. Especially, the *observability gramian* is given by

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi^T(\tau, t_1) C^T(\tau) C(\tau) \Phi^T(\tau, t_1) d\tau \tag{2.34}$$

By using the terminologies given by Kailath [53], we say a SISO time-varying system similar to (2.18) is in the *observability canonical form* with the triple $\big(A(t), b(t), c(t)\big)$, where $b(t)$ being some sufficiently smooth vector, and

$$A(t) = \left[\begin{array}{c|c} 0 & I_{n-1} \\ \hline a_0(t) & a_1(t) \ \cdots \ a_{n-1}(t) \end{array}\right], \quad c(t) = \left[\begin{array}{c|c} 1 & 0 \ \cdots \ 0 \end{array}\right]$$

and a SISO time-varying system similar to (2.21) is in the *observer canonical form* with the triple $\big(F(t), g(t), h(t)\big)$, where $g(t)$ being some sufficiently smooth vector, and

$$F(t) = \left[\begin{array}{c|c} \begin{array}{c} b_{n-1}(t) \\ \vdots \end{array} & I_{n-1} \\ \hline b_0(t) & 0 \end{array}\right], \quad h(t) = \left[\begin{array}{c|c} 1 & 0 \ \cdots \ 0 \end{array}\right]$$

System in either of these two canonical forms with bounded coefficients is uniformly completely observable.

Remark 2.2. These concepts and criterions are also shown in several publications with some differences in terminologies or definitions, e.g. Kalman [55] [56], Silverman et. al. [104] [105] [108], and Weiss [116] to name a few, and also in some classical textbooks like Chen [11], Kailath [53] and Rugh [95]. To maintain the coherence, we follow those given by Kalman and Silverman. ▪

2.4 The general framework for tackling output regulation problem

In this section, we will review some essentials of nonlinear output regulation problem. For consistency of the thesis, we formulate the problem and introduce the general framework for handling the problem based on [36]. This general framework includes three steps. First, the concept of steady-state generator is introduced. Steady-state generator is some dynamic system which can produce partial or whole solution of the regulator equations virtually. Second, the internal model is defined based on the steady-state generator. Attaching the internal model to the given plant yields the augmented system. More importantly, the stabilizability of the augmented system implies the solvability of the output regulation problem of the original plant. Third, different techniques are implemented due to the structural properties of the augmented system, so that the stabilization of the augmented system is achieved. It can be seen that this framework provides a systematical and flexible design procedure for solving global robust nonlinear output regulation problem.

Consider the nonlinear plant described by

$$\begin{aligned} \dot{x} &= f(x, u, v, w) \\ \dot{e} &= h(x, u, v, w) \end{aligned} \tag{2.35}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $e \in \mathbb{R}^p$ is the error output, $w \in \mathbb{R}^{n_w}$ denotes the uncertain constant parameters of the plant, and $v \in \mathbb{R}^{n_v}$ represents the exogenous signals which is generated by the following system

$$\dot{v} = a(v) \tag{2.36}$$

It is assumed that all functions in (2.35) and (2.36) are globally defined, sufficiently smooth and satisfy $f(0,0,0,w) = 0$ and $h(0,0,0,w) = 0$ for all $w \in \mathbb{R}^{n_w}$. Also, we assume that

Assumption 2.1. *The equilibrium of exosystem (2.36) at $v = 0$ is Lyapunov stable, and all the eigenvalues of $\frac{\partial a}{\partial v}(0)$ have zero real parts.* ▪

Briefly, the global robust output regulation problem can be stated as follows: to design a feedback control law in the form of

$$\begin{aligned} u &= u_k(e, \zeta, x) \\ \dot{\zeta} &= g_k(e, \zeta, x) \end{aligned} \tag{2.37}$$

where both u_k and g_k are sufficiently smooth function vanishing at the origin, such that

- the solution of the closed-loop system composed of (2 35) and (2.37) exists and is bounded for all initial states (x_0, ζ_0, v_0) and all $w \in \mathbb{R}^{n_w}$;
- the error output e asymptotically approaches zero, i.e. $\lim_{t\to\infty} e = 0$.

To solve the problem, a standard assumption is posed in the first place.

Assumption 2.2. *There exists globally defined, sufficiently smooth functions* $\mathbf{x}(v, w)$ *and* $\mathbf{u}(v. w)$ *with* $\mathbf{x}(0,0) = 0$ *and* $\mathbf{u}(0,0) = 0$ *satisfying the following equations*

$$\begin{aligned} \frac{\partial \mathbf{x}(v,w)}{\partial v} a(v) &= f\big(\mathbf{x}(v,w), \mathbf{u}(v,w), v, w\big) \\ 0 &= h\big(\mathbf{x}(v,w), \mathbf{u}(v,w), v, w\big) \end{aligned} \tag{2.38}$$

for all $v \in \mathbb{R}^{n_v}$ *and* $w \in \mathbb{R}^{n_w}$. ▪

Equations (2.38) are called the *regulator equations.* It is known that the solvability of the regulator equations is a necessary condition for the solvability of the output regulation problem. The solution of the regulator equation exhibits the steady-state behaviors $\mathbf{x}(v, w)$ for the *composite system* (2.35) (2.36) when asymptotic regulation achieves, and also provides the necessary steady-state feedforward information $\mathbf{u}(v, w)$ necessary for the controller to achieve asymptotic regulation. Nevertheless, $\mathbf{u}(v, w)$ cannot be used directly due the uncertain quantities (v, w). We need to introduce the concept of steady-state generator and internal model to asymptotically provide the information of $\mathbf{u}(v, w)$.

Definition 2.5. Generator ([36] Def.6.1)
Let $F : V \times W \to \mathbb{R}^l$, *where* V *and* W *are some open neighborhoods of the origin of* $\mathbb{R}^{n_w}$ *and* $\mathbb{R}^{n_w}$, *respectively, and* l *is some integer, be a smooth function vanishing at the origin The function* F *is said to have a generator if, for somo integer* s, *there exists a triple* $\{\theta, \alpha, \beta\}$, *where* $\theta : V \times W \to \mathbb{R}^s$, $\alpha . \mathbb{R}^s \to \mathbb{R}^s$, *and* $\beta : \mathbb{R}^s \to \mathbb{R}^l$ *are sufficiently smooth functions vanishing at the origin, such that, for all trajectories* $v(t) \in V$ *of the exosystem (2.36) and all* $w \in W$,

$$\begin{aligned} \frac{d\theta(v,w)}{dt} &= \alpha\big(\theta(v,w)\big) \\ F(v,w) &= \beta\big(\theta(v,w)\big) \end{aligned}$$

If $V = \mathbb{R}^{n_v}$, $W = \mathbb{R}^{n_w}$, *then the triple* $\{\theta, \alpha, \beta\}$ *is called a global generator of* $F(v, w)$.

Let the triple $\{\theta, \alpha, \beta\}$ *be a (global) generator of* $F(v, w)$. *If, in addition, the linearization of the pair* $\{\beta(\theta), \alpha(\theta)\}$ *at the origin is observable, the the triple* $\{\theta, \alpha, \beta\}$ *is called a linearly observable (global) generator of* $F(v, w)$ ▪

Definition 2.6. Steady-State Generator ([36] Def.6.2)
Let g_o $\mathbb{R}^{n+m} \to \mathbb{R}^l$ be a mapping for some integer $1 \le l \le n+m$. Under Assumption 2.1 and 2.2, the composite system *(2.35)-(2.36) is said to have a (global) steady-state generator with output $g_o(x,u)$ if the function $g_o\big(\mathbf{x}(v,w),\mathbf{u}(v,w)\big)$ has a (global) generator. The composite system is said to have a (global) steady-state generator with output $g_o(x,u)$ with linear observability if the function $g_o\big(\mathbf{x}(v,w),\mathbf{u}(v,w)\big)$ has a (global) generator with linear observability.* ▪

It can be seen that when $g_o(x,u) = \mathrm{col}(x,u)$, the steady-state generator reproduces the whole solution of regulator equations, and when $g_o(x,u) = u$, it reproduces partial solution. Without loss of generality, we assume $g_o(x,u) = \mathrm{col}(x_1,\cdots,x_d,u)$ in the following, where $0 \le d \le n$, and correspondingly, we call

$$\begin{aligned} \frac{d\theta(v,w)}{dt} &= \alpha\big(\theta(v,w)\big) \\ g_o\big(\mathbf{x}(v,w),\mathbf{u}(v,w)\big) &= \beta\big(\theta(v,w)\big) \end{aligned} \tag{2.39}$$

a partial (full) steady-state state generator if $0 < d \le n$ or a steady-state input generator if $d = 0$.

Definition 2.7. Internal Model ([36] Def.6.6)
Under Assumption 2.1 and 2.2, suppose the composite system (2.35) (2.36) has a (global) steady-state generator with output $g_o(x,u)$ Let $\gamma : \mathbb{R}^{s+d+m+p} \to \mathbb{R}^s$ be a sufficiently smooth function vanishing at the origin. Then we call the following system

$$\dot{\eta} = \gamma\big(\eta, g_o(x,u), e\big) \tag{2.40}$$

an internal model of the composite system with output $g_o(x,u)$ if

$$\gamma\Big(\theta(v,w), g_o\big(\mathbf{x}(v,w),\mathbf{u}(v,w)\big), 0\Big) = \alpha\big(\theta(v,w)\big)$$

(For convenience, we always use the notation $\gamma(\eta,x,u,e)$ instead of $\gamma\big(\eta,g_o(x,u),e\big)$) ▪

Now we are ready to show the "problem conversion", i.e., to convert the output regulation problem for the original plant (2.35) into the stabilization problem for the augmented system.

Attaching the internal model (2.40) to the given plant yields the following *augmented system*

$$\begin{aligned}\dot{x} &= f(x, u, v, w)\\ \dot{\eta} &= \gamma(\eta, x, u, e)\\ e &= h(x, u, v, w)\end{aligned} \tag{2.41}$$

Performing on (2.41) the following coordinate and input transformation

$$\begin{aligned}\bar{x}_i &= x_i - \beta_i(\eta), \quad i = 1, \cdots, d\\ \bar{x}_i &= x_i - \mathbf{x}_i(v, w), \quad i = d+1, \cdots, n\\ \bar{\eta} &= \eta - \theta\\ \bar{u} &= u - \left[\ \beta_{d+1}(\eta), \cdots, \beta_{d+m}(\eta)\ \right] = u - \beta_u(\eta)\end{aligned}$$

gives a new system denoted by

$$\begin{aligned}\dot{\bar{x}} &= \bar{f}(\bar{x}, \bar{\eta}, \bar{u}, v, w)\\ \dot{\bar{\eta}} &= \bar{\gamma}(\bar{x}, \bar{\eta}. \bar{u}, v, w)\\ e &= \bar{h}(\bar{x}, \bar{\eta}. \bar{u}, v, w)\end{aligned} \tag{2.42}$$

where

$$\begin{aligned}\bar{f}(\bar{x}, \bar{\eta}, \bar{u}, v, w) &= f_i(x, u, v, w) - \frac{\partial \beta_i(\eta)}{\partial \eta}\gamma(\eta, x, u, e), \quad i = 1, \cdots, d\\ \bar{f}(\bar{x}, \bar{\eta}, \bar{u}, v, w) &= f_i(x, u, v, w) - f_i\big(\mathbf{x}(v, w), \mathbf{u}(v, w), v, w\big), \quad i = d+1, \cdots, m\\ \bar{\gamma}(\bar{x}, \bar{\eta}, \bar{u}, v, w) &= \gamma(\eta, x, u, e) - \alpha\big(\theta(v, w)\big)\\ \bar{h}(\bar{x}, \bar{\eta}, \bar{u}, v, w) &= h(x, u, v, w)\end{aligned}$$

For system (2.42), the origin $(\bar{x}, \bar{\eta}) = (0, 0)$ is the equilibrium point of the unforced augmented system, and at the origin, the error output is identically zero. This argument can be verified by showing that, for system (2.42) the following hold

$$\begin{aligned}\bar{f}(0, 0, 0, v, w) &= 0\\ \bar{\gamma}(0, 0, 0, v, w) &= 0\\ \bar{h}(0, 0, 0, v, w) &= 0\end{aligned} \tag{2.43}$$

for all trajectories $v(t) \in V \subset \mathbb{R}^{n_v}$ of the exosystem, and all $w \in W \subset \mathbb{R}^{n_w}$.

Proposition 2.2. *The output regulation problem of the original plant (2.35) can be converted into the stabilization problem of the augmented system (2.42), i.e., if there exists a control law*

$$\begin{aligned} \bar{u} &= u_s(\bar{x}_1, \cdots, \bar{x}_d, \xi, e) \\ \dot{\xi} &= g_s(\bar{x}_1, \cdots, \bar{x}_d, \xi, e) \end{aligned} \tag{2.44}$$

where $\xi \in \mathbb{R}^{n_\xi}$ and u_s and g_s are sufficiently smooth functions vanishing at the origin, such that, the solution of the closed-loop system composed of (2.42) and (2.44) exists and is bounded over $[0, \infty)$, and the equilibrium point of the closed-loop system is globally asymptotically stable for any initial states and all $v \in V$ and $w \in W$. Then the following control law

$$\begin{aligned} u &= \beta_u(\eta) + u_s\big(x_1 - \beta_1(\eta), \cdots, x_d - \beta_d(\eta), \xi, e\big) \\ \dot{\eta} &= \gamma(\eta, x, u, e) \\ \dot{\xi} &= g_s\big(x_1 - \beta_1(\eta), \cdots, x_d - \beta_d(\eta), \xi, e\big) \end{aligned} \tag{2.45}$$

solves the global robust output regulation problem for the original plant (2.35) subject to exosystem (2.36). ∎

2.5 Some useful lemmas

In this section, some lemmas which will be frequently cited in the forthcoming chapters are introduced.

Lemma 2.1. Barbalat's Lemma ([59] Lemma 8.2)
Let $\phi : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function on $[0, \infty)$. Suppose that $\lim_{t\to\infty} \int_0^\infty \phi(\tau)d\tau$ exists and is finite. Then, $\phi(t) \to 0$ as $t \to \infty$. ∎

Corollary 2.1.
If the differentiable function $\phi(t)$ has a finite limit as $t \to \infty$, and is such that $\ddot{\phi}(t)$ exists and is bounded, then $\dot{\phi}(t) \to 0$ as $t \to \infty$. ∎

Theorem 2.3. LaSalle-Yoshizawa Theorem ([62] Theorem A.8)
Consider the nonautonomous system

$$\dot{x} = f(x, t) \tag{2.46}$$

where $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^n$ is locally Lipschitz in x uniformly in t

Let $x = 0$ be an equilibrium point of (2.46). Let $V . \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be a continuously differentiable function such that

$$\gamma_1(|x|) \leq V(x,t) \leq \gamma_1(|x|)$$
$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x.t) \leq -W(x) \leq 0$$

$\forall t \geq 0$, $\forall x \in \mathbb{R}^n$, where γ_1 and γ_2 are class $\mathcal{K}_\infty$ functions and W is a continuous function. Then, all solutions of (2.46) are globally uniformlyy bounded and satisfy

$$\lim_{t\to\infty} W(x) = 0$$

In addition, if $W(x)$ is positive definite, then the equilibrium $x = 0$ is globally uniformly asymptotically stable. ∎

Lemma 2.2. Changing Supply Functions Technique ([111])
Consider the nonautomous control system

$$\dot{x} = f(x,u,t) \tag{2.47}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$. Suppose there exists a continuously differentiable ($\mathcal{C}^1$) function $U(x,t)$ satisfying $\underline{\alpha}(\|x\|) \leq U(x,t) \leq \bar{\alpha}(\|x\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$ and $\bar{\alpha}(\cdot)$, such that, along the trajectory of system (2.47)

$$\frac{dU(x,t)}{dt} \leq -\alpha(\|x\|) + \delta(u) \tag{2.48}$$

for some class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ satisfying $\lim_{s\to 0^+}\left(\alpha^{-1}(s^2)/s\right) < \infty$, and some smooth positive definite function $\delta(u)$.
Then given any smooth function $\Delta(x) > 0$. there exists a continuously differentiable function $V(x,t)$ satisfying $\underline{\alpha}_1(\|x\|) \leq V(x,t) \leq \bar{\alpha}_1(\|x\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_1(\cdot)$ and $\bar{\alpha}_1(\cdot)$, such that, along the trajectory of system (2.47)

$$\frac{dV(x,t)}{dt} \leq -\Delta(x)\|x\|^2 + \delta_1(u)u^2 \tag{2.49}$$

for some smooth positive function $\delta_1(u)$. ∎

Corollary 2.2.
Consider the uncertain nonlinear system

$$\dot{x} = f(x,y,t,\mu) \tag{2.50}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $\mu \in \Sigma \subset \mathbb{R}^{n_\mu}$ with Σ being any compact subset is some uncertain parameters. Suppose there exists a $\mathcal{C}^1$ function $V(x,t)$ satisfying $\underline{\alpha}(\|x\|) \leq V(x,t) \leq \bar{\alpha}(\|x\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$ and $\bar{\alpha}(\cdot)$, such that, along the trajectory of system (2.50), $\frac{dV(x,t)}{dt} \leq -\alpha(\|x\|)+\delta\gamma(y)$, where $\alpha(\cdot)$ is some known $\mathcal{K}_\infty$ function satisfying $\lim_{s\to 0^+}\left(\alpha^{-1}(s^2)/s\right) < \infty$, δ is some unknown positive constant, and $\gamma(\cdot)$ is some known smooth positive definite function.

Then given any smooth function $\Delta(x) > 0$, there exists a $\mathcal{C}^1$ function $\bar{V}(x,t)$ satisfying $\underline{\alpha}_1(\|x\|) \leq \bar{V}(x,t) \leq \bar{\alpha}_1(\|x\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_1(\cdot)$ and $\bar{\alpha}_1(\cdot)$, such that, along the trajectory of system (2 50), $\frac{d\bar{V}(x,t)}{dt} \leq -\Delta(x)\|x\|^2 + \bar{\delta}\bar{\gamma}(y)y^2$, where $\bar{\delta}$ is some unknown constant, and $\bar{\gamma}(\cdot)$ is some known smooth positive function ▪

Lemma 2.3. ***Dominating Functions' Inequality*** *(*[36] **Lemma 7.8, 7.9***)*
Let $f : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ be $\mathcal{C}^1$ function, where $\mu \in \Sigma$ with Σ being a compact subset of $\mathbb{R}^p$ Then there exist smooth functions $F_1 \cdot \mathbb{R}^m \to \mathbb{R}$ and $F_2 : \mathbb{R}^n \to \mathbb{R}$ such that

$$|f(x,y,\mu)| \leq F_1(x) + F_2(y) \tag{2.51}$$

for all $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $\mu \in \Sigma$.

If $f(x,y,\mu)$ satisfies $f(0,0,\mu) = 0$ for all $\mu \in \Sigma$, then $F_1(x)$ and $F_2(y)$ satisfies $F_1(0) = 0$ and $F_2(0) = 0$. ▪

Corollary 2.3. *By Taylor theorem, there exist a constant c, smooth positive definite functions $\phi_x(\cdot)$ and $\phi_y(\cdot)$ such that, for Lemma 2.3, $F_1(x) \leq c_1\|x\|\phi_x(x)$ for all $x \in \mathbb{R}^m$, $F_2(y) \leq c_2\|y\|\phi_y(y)$ for all $x \in \mathbb{R}^n$ (2 51) thus turns into*

$$|f(x,y,\mu)| \leq c\big(\|x\|\phi_x(x) + \|y\|\phi_y(y)\big) \tag{2.52}$$

for all $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $\mu \in \Sigma$

If the bound of compact set Σ is known, the constant c can be determined, (2.52) thus turns into

$$|f(x,y,\mu)| \leq \big(\|x\|\phi_x(x) + \|y\|\phi_y(y)\big) \tag{2.53}$$

Moreover, we can further show that from (2.52),

$$f^2(x,y,\mu) \leq p\big(\|x\|^2\bar{\phi}_x(x) + \|y\|^2\bar{\phi}_y(y)\big) \tag{2.54}$$

where $\bar{\phi}_x(x) > 0$ and $\bar{\phi}_y(y) > 0$ are some known smooth functions, $p > 0$ is some unknown constant. ▪

□ **End of chapter.**

Chapter 3

Output Regulation of Time-Varying Nonlinear Systems

In this chapter, we address the global robust output regulation problem for time-varying nonlinear systems subject to time-varying exosystems. Along the line of [36], some modifications are made due to the time-varying properties of the systems, and a framework for handling such kind of problem is introduced. Further, the output regulation problem is solved for a class of time-varying output feedback systems with a time-varying exosystem.

3.1 Introduction

The output regulation problem has been extensively studied since the 1970s for both linear time-invariant systems [18] [22] [23], and nonlinear time-invariant systems [5] [33] [42] [44] [57] etc..

Recently, some attention has been given to the linear time-varying systems [39] [99] [122] [123] [124] meaning either the plant or the exosystem is a time-varying system. [39] summarizes the main results regarding the output regulation for linear time-invariant systems as given in [96], and extends them to the linear time-varying systems in a parallel manner, and shows the solvability of output regulation problem for linear time-varying systems relies on the solvability of a differential Sylvester equation, in contrast with the solvability of a Sylvester equation in the time-invariant scenario. [122] and [123] show the similar solvability condition, and, in particular, present an internal model based design approach for linear periodic systems.

The time-varying systems exhibit distinct properties in comparison with the time-invariant systems. Even in the linear settings, more rigorous classifications and specifications on "controllability" or "observability" are posed for time-varying systems than time-invariant systems as shown in Section 2.3. Moreover, time-varying systems provides richer categories of signals. When a finite-dimensional linear time-invariant systems serving as the exosystem in the nonlinear output regulation problem, i.e. $\dot{v} = A_1 v$, it is usually assumed that all eigenvalues of A_1 are simple with zero real parts, so the periodic solutions of the exosystem are simple sinusoidal signals which contain finite number of harmonics. For instance, consider the constant harmonic oscillator

$$\dot{v}_1 = \omega v_2, \quad \dot{v}_2 = -\omega v_1$$

where ω is some constant, then for any initial condition (v_{10}, v_{20}), the solutions are

$$v_1(t) = \cos(\omega t)v_{10} + \sin(\omega t)v_{20}, \quad v_2(t) = -\sin(\omega t)v_{10} + \cos(\omega t)v_{20}$$

On the contrary, when time-varying systems serving as the exosystem, they produce periodic or aperiodic solutions not merely the simple sinusoidal signals. For example, consider a periodically time-varying oscillator,

$$\dot{v}_1 = \sigma \sin(t)v_2, \quad \dot{v}_2 = -\sigma \sin(t)v_1$$

where σ is some constant, then for any initial conditions (v_{10}, v_{20}), the solutions are

$$v_1(t) = \cos\big(\sigma \cos t\big)v_{10} - \sin\big(\sigma \cos t\big)v_{20}, \quad v_2(t) = \sin\big(\sigma \cos t\big)v_{10} + \cos\big(\sigma \cos t\big)v_{20}$$

Obvious, the exogenous signal contains infinite number of harmonics.

As a consequence, the time-varying nature of the system distinguishes the output regulation problem from time-invariant case for at least two reasons. First, it is difficult to characterize the existence condition for a (time-varying) internal model, second, special requirement is in need to achieve global stabilization or regulation.

3.2 A framework for handling the problem

In this section, we first describe the output regulation problem for time-varying nonlinear systems. The basic assumption for solving the problem is posed, and the concepts like steady-state generator and internal model are introduced. Next we show the output regulation problem can be converted into a stabilization problem of the time-varying nonlinear augmented system which is composed of the plant and the internal model. Further, we give the existence condition for the steady-state generator, so that the appropriate internal model can be designed based upon such steady-state generator.

3.2.1 Problem descriptions and problem conversion

Consider the general time-varying nonlinear plant described by

$$\begin{aligned} \dot{x} &= f(t, x, u, v, w) \\ e &= h(t, x, v, w) \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state, $e \in \mathbb{R}^m$ is the regulated error output, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^{n_w}$ denotes the constant uncertain parameters of the plant, and $v \in \mathbb{R}^{n_v}$ is the exogenous signal representing a class of reference and/or disturbance which is generated by a time-varying exosystem

$$\dot{v} = a(t, v) \tag{3.2}$$

The functions $f(t, x, u, v, w)$, $h(t, x, v, w)$ and $a(t, v)$ are assumed to be sufficiently smooth in their arguments satisfying $f(t, 0, 0, 0, w) = 0$, $h(t, 0, 0, w) = 0$, and $a(t, 0) = 0$. Also, it is assumed that the solution of the time-varying exosystem (3.2) exists and is bounded for all $t \geq t_0 \geq 0$ and for all v_0.

The control objective is to find the output feedback control law in the following form

$$\begin{aligned} u &= u_K(t, \zeta, e) \\ \dot{\zeta} &= g_K(t, \zeta, e) \end{aligned} \tag{3.3}$$

where $u_K(t, \zeta, e)$ and $g_K(t, \zeta, e)$ are sufficiently smooth functions vanishing at $(\zeta, e) = (0, 0)$, such that, for any initial time $t_0 \geq 0$, any initial condition (x_0, v_0, ζ_0), and any constant parameters $w \in \mathbb{R}^{n_w}$,

- the solution of the closed-loop system composed of (3.1), (3.2) and (3.3) exists and is bounded over $[\ t_0, +\infty\ )$;
- the regulated error output e uniformly asymptotically approaches zero.

Like the nonlinear time-invariant case, we first pose a standard assumption.

Assumption 3.1. *There exist globally defined sufficiently smooth functions* $\mathbf{x}(t,v,w)$, $\mathbf{u}(t,v,w)$ *with* $\mathbf{x}(t,0,w)=0$ *and* $\mathbf{u}(t,0,w)=0$ *for all* $w\in\mathbb{R}^{n_w}$ *and all* $t\geq t_0\geq 0$, *such that*

$$\begin{aligned}\mathcal{L}_{a(t,v)}\mathbf{x}(t,v,w) &= f\big(t,\mathbf{x}(t,v,w),\mathbf{u}(t,v,w),v,w\big)\\ 0 &= h\big(t,\mathbf{x}(t,v,w),v,w\big)\end{aligned} \tag{3.4}$$

where $\mathcal{L}_{a(t,v)}\mathbf{x}(t,v,w)=\frac{\partial\mathbf{x}(t,v,w)}{\partial t}+\frac{\partial\mathbf{x}(t,v,w)}{\partial v}a(t,v)$ *for all* $(t,v,w)\in\mathbb{R}^+\times\mathbb{R}^{n_v}\times\mathbb{R}^{n_w}$. ▪

Remark 3.1. In analogy to the time-invariant case, we call equations (3.4) the (time-varying) regulator equations associated with the composite system (3.1) (3.2), and the solution of the regulator equations (3.4) provides a steady-state input $\mathbf{u}(t,v,w)$ under which the closed-loop system has a steady-state trajectory $\mathbf{x}(t,v,w)$ at which the steady-state error output is identically zero.

However, since $\mathbf{u}(t,v,w)$ depends on the unmeasurable quantities as v,w, it cannot be used for feedback control directly. Instead, we need to reproduce it by the dynamics called steady-state generator which is independent of the model uncertainty w and the exogenous signal v. The concept of the (time-varying) steady-state input generator is introduced as follows. ▪

Definition 3.1. Steady-State Input Generator
Under Assumption 3.1, the composite system (3.1)-(3.2) is said to have a steady-state input generator if, for some integer l, *there exists a triple* $\{\vartheta,\alpha,\beta\}$, *where* $\vartheta:\mathbb{R}^+\times\mathbb{R}^{n_v}\times\mathbb{R}^{n_w}\to\mathbb{R}^l$, $\alpha:\mathbb{R}^+\times\mathbb{R}^l\to\mathbb{R}^l$, *and* $\beta:\mathbb{R}^+\times\mathbb{R}^l\to\mathbb{R}^m$ *are sufficiently smooth functions satisfying* $\vartheta(t,0,w)=0$ *for all* $t\geq 0$, *such that,*

$$\begin{aligned}\frac{d\vartheta(t,v,w)}{dt} &= \alpha\big(t,\vartheta(t,v,w)\big)\\ \mathbf{u}(t,v,w) &= \beta\big(t,\vartheta(t,v,w)\big)\end{aligned} \tag{3.5}$$

for all $v\in\mathbb{R}^{n_v}$, $w\in\mathbb{R}^{n_w}$. ▪

In the case that both $\alpha(t,\vartheta)$ and $\beta(t,\vartheta)$ are linear in ϑ, i.e., there exist continuously differential matrices $\Phi(t)$ and $\Gamma(t)$ such that

$$\alpha(t,\vartheta)=\Phi(t)\vartheta,\quad \beta(t,\vartheta)=\Gamma(t)\vartheta$$

(3.5) reduces to

$$\begin{aligned} \frac{d\vartheta(t,v,w)}{dt} &= \Phi(t)\vartheta(t,v,w) \\ \mathbf{u}(t,v,w) &= \Gamma(t)\vartheta(t,v,w) \end{aligned} \tag{3.6}$$

we call (3.6) a linear steady-state input generator.

Corresponding to a steady-state input generator, we can further introduce the concept of the internal model as follows.

Definition 3.2. Internal Model
Suppose the composite system (3.1) (3.2) has a steady-state input generator (3.5). We call the following system

$$\dot{\eta} = \gamma(t,\eta,u) \tag{3.7}$$

where $\gamma(t,\eta,u)$ is a sufficiently smooth function for all $(t,\eta,u) \in \mathbb{R}^+ \times \mathbb{R}^l \times \mathbb{R}^m$, an internal model with output u if

$$\gamma\big(t,\vartheta(t,v,w),\mathbf{u}(t,v,w)\big) = \alpha\big(t,\vartheta(t,v,w)\big) \tag{3.8}$$

∎

Remark 3.2. A steady-state input generator itself can be viewed as an internal model. However, this particular internal model cannot make the augmented system stabilizable because it does not affect the given plant. An internal model should be carefully conceived so that the augmented system is stabilizable in a desirable sense.

For time-invariant systems, there are extensive studies on the construction of the various internal models [5] [9] [33] [37] [84] [88]. For time-varying systems, this issue has not been adequately addressed. However, if a linear steady-state generator of the form (3.6) has the property that there exist sufficiently smooth matrices $F(t)$ and $G(t)$ such that

$$\Phi(t) = F(t) + G(t)\Gamma(t) \tag{3.9}$$

then it can be verified that

$$\dot{\eta} = F(t)\eta + G(t)u \tag{3.10}$$

is an internal model with output u. In particular, if $F(t)$ is a constant Hurwitz matrix, then (3.10) reduces to the so-called canonical linear internal model proposed in [123]. ∎

Attaching the internal model (3.7) to the given plant (3.1) yields the following augmented system

$$\begin{aligned} \dot{x} &= f(t, x, u, v, w) \\ \dot{\eta} &= \gamma(t, \eta, u) \\ e &= h(t, x, v, w) \end{aligned} \tag{3.11}$$

Performing on (3.11) the following coordinate and input transformation

$$\begin{aligned} \bar{x} &= x - \mathbf{x}(t, v, w) \\ \bar{\eta} &= \eta - \vartheta(t, v, w) \\ \bar{u} &= u - \beta(t, \eta) \end{aligned} \tag{3.12}$$

gives a new system denoted by

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(t, \bar{x}, \bar{\eta}, \bar{u}, \mu) \\ \dot{\bar{\eta}} &= \bar{\gamma}(t, \bar{x}, \bar{\eta}, \bar{u}, \mu) \\ e &= \bar{h}(t, \bar{x}, \bar{\eta}, \mu) \end{aligned} \tag{3.13}$$

where $\mu = (v, w)$ and

$$\begin{aligned} \bar{f} &= f(t, x, u, v, w) - f\big(t, \mathbf{x}(t, v, w), \mathbf{u}(t, v, w), v, w\big) \\ \bar{\gamma} &= \gamma(t, \eta, u) - \alpha(t, \vartheta) \\ \bar{h} &= h(t, x, v, w) \end{aligned} \tag{3.14}$$

It can be easily verified that system (3.13) satisfies

$$\begin{aligned} 0 &= \bar{f}(t, 0, 0, 0, \mu) \\ 0 &= \bar{\gamma}(t, 0, 0, 0, \mu) \\ 0 &= \bar{h}(t, 0, 0, \mu) \end{aligned} \tag{3.15}$$

for all $t \geq t_0 \geq 0$, all trajectories $v(t) \in \mathbb{R}^{n_v}$ and all $w \in \mathbb{R}^{n_w}$.

Proof: From the expressions of the coordinate and input transformation (3.12), it is evident when $\bar{x} = \bar{\eta} = \bar{u} = 0$, we have

$$x = \mathbf{x}(t, v, w), \quad \eta = \vartheta(t, v, w), \quad u = \mathbf{u}(t, v, w)$$

Under these conditions, comparing the plant (3.1) and the regulator equations (3.4), we could find their right-hand sides are identical, thus the first equation of (3.15) holds directly, so does the third one.

Under the same circumstances, it can be seen that

$$\begin{aligned}\dot{\bar{\eta}} &= \dot{\eta} - \dot{\vartheta} = \gamma(t, \eta, u) - \alpha(t, \vartheta) \\ &= \gamma\big(t. \vartheta(t, v, w), \mathbf{u}(t, v, w)\big) - \alpha\big(t, \vartheta(t. v, w)\big)\end{aligned}$$

Resorting to the definition of internal model (3.8), it can be guaranteed that $\dot{\bar{\eta}} = 0$.

Thus the origin $(\bar{x}, \bar{\eta}) = (0, 0)$ is an equilibrium point of the unforced augmented system (3.13) for all trajectories of the exosystem. ◇

Remark 3.3. As shown above, the augmented system (3.13) has the property that the origin $(\bar{x}, \bar{\eta}) = (0, 0)$ is the equilibrium point of the unforced augmented system for all trajectories of the exosystem, and, at the origin, the error output e is identically zero. Therefore, if we can find an output feedback control law of the form

$$\begin{aligned}\bar{u} &= u_S(t, \xi, e) \\ \dot{\xi} &= g_S(t, \xi, e)\end{aligned} \tag{3.16}$$

where $u_S(t, \xi, e)$ and $g_S(t, \xi, e)$ are sufficiently smooth functions vanishing at $(\xi, e) = (0, 0)$ such that (3.16) globally asymptotically stabilize the equilibrium point of the augmented system (3.13), then the following control law

$$\begin{aligned}u &= \beta(t, \eta) + u_S(t, \xi, e) \\ \dot{\eta} &= \gamma(t, \eta, u) \\ \dot{\xi} &= g_S(t, \xi, e)\end{aligned} \tag{3.17}$$

solves the output regulation problem of the original plant (3.1). ▪

Till now, we have shown that, like the time-invariant case, the output regulation problem of the time-varying systems can also be converted into a stabilization problem of an augmented system composed of the given plant and the internal model if an appropriate internal model exists. However, for time-varying systems, the issue of the existence of the internal model may be much less tractable than the time-invariant case. So far. the existence issue of internal model is only studied for the special case where both the plant and the exosystem are linear periodic [122] [123].

3.2.2 On the existence and design of time-varying internal model

Next, we will propose an existence condition for the steady-state input generator in the time-varying settings. Such condition can be viewed as an extension of the one given in

[37]. Consequently, an appropriate internal model can be defined based on the steady-state input generator. A special case of this steady-state input generator will lead to the canonical linear internal model proposed in [123]. (For further convenience, we assume $m = 1$ for the rest of this chapter.)

Definition 3.3.
Let $X(t, v, w)$ $\mathbb{R}^+ \times \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \to \mathbb{R}$ be a sufficiently smooth function with $v(t)$ generated by the exosystem (3.2). If $X(t, v, w)$ satisfies a linear differential equation of the following form

$$\frac{d^l X}{dt^l} - a_{l-1}(t)\frac{d^{l-1}X}{dt^{l-1}} - \cdots - a_1(t)\frac{dX}{dt} - a_0(t)X = 0 \tag{3.18}$$

where $a_i(t) : \mathbb{R}^+ \to \mathbb{R}$, $i = 0, \ldots, l-1$, are sufficiently smooth functions of time, then $X(t, v, w)$ is said to be in the kernel of the left monic polynomial differential operator $P(s, t)$ of degree l,

$$P(s,t) = s^l - a_{1-1}(t)s^{l-1} - \cdots - a_1(t)s - a_0(t)$$

where $s \stackrel{def}{=} \frac{d}{dt}(\cdot)$. ∎

Assumption 3.2. *There exist a sufficiently smooth function $X(t, v, w)$ which is in the kernel of some left monic polynomial differential operator of degree l, and a continuously differentiable function $\Gamma : \mathbb{R}^+ \times \mathbb{R}^l \to \mathbb{R}$ such that*

$$\mathbf{u}(t, v, w) = \Gamma\left(t, X, \dot{X}, \cdots, \frac{d^{l-1}X}{dt^{l-1}}\right)$$

where $\mathbf{u}(t, v, w)$ is the solution of the regulator equations (3.4). ∎

Proposition 3.1. *Under Assumption 3.2, there exists a steady-state generator of the form (3.5) with*

$$\alpha\big(t, \vartheta(t, v, w)\big) = \Phi(t)\vartheta(t, v, w)$$
$$\beta\big(t, \vartheta(t, v, w)\big) = \Gamma\big(t, \vartheta(t, v, w)\big)$$

where

$$\vartheta(t, v, w) = \begin{bmatrix} X(t, v, w) \\ \dot{X}(t, v, w) \\ \vdots \\ \frac{d^{l-1}X(t,v,w)}{dt^{l-1}} \end{bmatrix}, \quad \Phi(t) = \left[\begin{array}{c|ccc} 0 & & I_{l-1} & \\ \hline a_0(t) & a_1(t) & & a_{l-1}(t) \end{array}\right]$$

∎

The proof is straightforward and is omitted.

Remark 3.4. Assumption 3.2 is an obvious extension of the condition in [37] which deals with nonlinear time-invariant systems. A special case of Assumption 3.2 is that the function $\mathbf{u}(t, v, w)$ itself is in the kernel of some left monic polynomial differential operator of degree l. Then Assumption 3.2 reduces to the following.

Assumption 3.3. *There exists an integer l such that along the trajectory of exosystem, for all $(t, v, w) \in \mathbb{R}^+ \times \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$, the function $\mathbf{u}(t, v, w)$ satisfies*

$$\frac{d^l \mathbf{u}}{dt^l} = a_{l-1}(t) \frac{d^{l-1}\mathbf{u}}{dt^{l-1}} + \cdots + a_1(t) \frac{d\mathbf{u}}{dt} + a_0(t)\mathbf{u}$$

where $a_i(t)$, $i = 0, 1, \ldots, l-1$ are sufficiently smooth functions of time. ▪

Under Assumption (3.3), we can always take $\Gamma(t, \vartheta) = \Gamma\vartheta$ where $\Gamma = [\,1 \;\; 0 \;\; \cdots \;\; 0\,]$, i.e., we can obtain a linear steady-state input generator as follows.

$$\begin{aligned} \frac{d\vartheta(t, v, w)}{dt} &= \Phi(t)\vartheta(t, v, w) \\ \mathbf{u}(t, v, w) &= \Gamma\vartheta(t, v, w) \end{aligned} \tag{3.19}$$

In what follows, we say that the triple $\{\vartheta, \Phi(t), \Gamma\}$ constitutes a linear steady-state input generator. ▪

Remark 3.5. Existence of internal models for nonlinear output regulation problem has been extensively studied. The first condition is given in [33] where a linear internal model is constructed under the assumption that $\mathbf{u}(v, w)$ is a polynomial in v and the exosystem is linear time-invariant. Later a condition similar to (3.19) is proposed in [5]. It is shown in [35] that the condition proposed in [5] is equivalent to the condition that $\mathbf{u}(v, w)$ is a polynomial in v and the exosystem is linear. Recent results on the existence of nonlinear internal models can be found in [9] [37] [88]. More recently, condition similar to (3.19) has been given in [123] w.r.t the linear periodic systems. However, in the time-varying settings, condition (3.19) may not be satisfied even if the solution $\mathbf{u}(v, w)$ of the regulator equations is a polynomial and the exosystem is linear. ▪

It is shown in Remark 3.2 that it is possible to obtain a canonical linear internal model of the form (3.9) once a linear steady-state generator of the form (3.6) is available. In fact, since the pair (Φ, Γ) is in the observability canonical form [53], by a Lyapunov transformation $\tau = N_o^{-1}(t)\vartheta$, (3.19) can be transformed into the observer canonical form as follows.

$$\begin{aligned} \frac{d\tau(t,v,w)}{dt} &= \Phi_o(t)\tau(t,v,w) \\ \mathbf{u}(t,v,w) &= \Gamma_o\tau(t,v,w) \end{aligned} \tag{3.20}$$

where

$$\Phi_o(t) = \left[\begin{array}{c|c} b_{l-1}(t) & \\ \vdots & I_{l-1} \\ b_1(t) & \\ \hline b_0(t) & 0 \ \cdots \ 0 \end{array}\right], \quad \Gamma_o = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

and $N_o(t)$ is the observability matrix of the pair $(\Phi_o(t), \Gamma_o)$. The relation between (3.19) and (3.20) holds as

$$\dot{N}_o^{-1}(t)N_o(t) + N_o^{-1}(t)\Phi(t)N_o(t) = \Phi_o(t), \quad \Gamma = \Gamma_o$$

and the coefficients $b_i(t)$ can be obtained from $a_i(t)$ as given in Section 2.3.

$$b_{l-i}(t) = \sum_{j=0}^{i-1}(-1)^j C_{l-i+j}^j a_{l-i+j}^{(j)}(t) \quad i = 1, \cdots, l \tag{3.21}$$

where $a_{l-i+j}^{(j)}(t)$ denotes the jth derivative of $a_{l-i+j}(t)$, and C_{l-i+j}^j denotes the number of distinct combinations of order j from $l - i + j$ elements.

By assuming $b_i(t)$ are uniformly bounded, and let $\Phi_o(t) = \Phi_b + b(t)\Gamma_o$ where

$$\Phi_b = \begin{bmatrix} 0 & & \\ \vdots & I_{l-1} & \\ 0 & \cdots & 0 \end{bmatrix}, \quad b(t) = \begin{bmatrix} b_{l-1}(t) \\ \vdots \\ b_0(t) \end{bmatrix} \tag{3.22}$$

Also, let $L_0 = (\ l_{l-1}, \cdots, l_0\ )^T$ be such that $F = \Phi_b - L_0\Gamma_o$ is Hurwitz, and let $G(t) = L_0 + b(t)$. It can be verified that $\Phi_o(t) = F + G(t)\Gamma_o$. Thus, by Remark 3.2, the following linear time-varying system

$$\dot{\eta} = F\eta + G(t)u \tag{3.23}$$

is an internal model corresponding to the steady-state input generator (3.20).

3.3 Applications for time-varying output feedback systems

Having described the framework for handling the output regulation problem for time-varying nonlinear systems in the last section, we can see that the successful implementation of the framework depends on two issues. First, the existence of a desirable internal model, and second, the stabilization of the augmented system (3.13). As neither of these two issues is tractable even for a general time-invariant system, in this section we will focus on a class of time-varying nonlinear systems in the output feedback form.

3.3.1 System with unity relative degree

Consider the following SISO time-varying nonlinear system.

$$\begin{aligned} \dot{z} &= f(t, z, y, v, w) \\ \dot{y} &= g(t, z, y, v, w) + b(w)u \\ e &= y - q(t, v, w) \end{aligned} \tag{3.24}$$

where $(z, y) \in \mathbb{R}^n \times \mathbb{R}$ is the state, $u \in \mathbb{R}$ is the control input, $e \in \mathbb{R}$ is the error output, $b(w) > 0$, and other notations follow those of (3.1). System (3.24) is called time-varying output feedback systems with unity relative degree. The output regulation problem for the special case of (3.24) where the functions $f(t, z, y, v, w)$, $g(t, z, y, v, w)$, $q(t, v, w)$ and $a(t, v)$ do not explicitly depend on the time t is studied in [120].

For system (3.24), Assumption 3.1 can be reduced to the follows.

Assumption 3.4. *There exists a globally defined sufficiently smooth function* $\mathbf{z} : \mathbb{R}^+ \times \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \to \mathbb{R}^n$ *with* $\mathbf{z}(t, 0, w) = 0$ *such that*

$$\mathcal{L}_{a(t,v)}\mathbf{z}(t, v, w) = f\big(t, \mathbf{z}(t, v, w), q(t, v, w), v, w\big)$$

for all $(t, v, w) \in \mathbb{R}^+ \times \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$. ▪

Under Assumption 3.4, denote $\mathbf{y}(t, v, w) = q(t, v, w)$, and let $\mathbf{u}(t, v, w) = b^{-1}(w) \Big(\mathcal{L}_{a(t,v)}\mathbf{y}(t, v, w) - g\big(t, \mathbf{z}(t, v, w), \mathbf{y}(t, v, w), v, w\big)\Big)$. Then $\mathbf{z}(t, v, w)$, $\mathbf{y}(t, v, w)$, $\mathbf{u}(t, v, w)$ are the solution of the regulator equations associated with composite system (3.24)-(3.2).

Further we assume that the steady-state input $\mathbf{u}(t, v, w)$ satisfies Assumption 3.3. Then we can find a triple $\big\{\vartheta, \Phi(t), \Gamma\big\}$ that defines the linear steady-state input generator

(3.19). Moreover, under the Lyapunov transformation $\tau = N_o^{-1}(t)\vartheta$, (3.19) can be transformed into observer canonical form (3.20). Thus, we can obtain a linear internal model in the form of (3.23). As a result, we can put the augmented systems as follows.

$$\begin{aligned}\dot{z} &= f(t, z, y, v, w)\\ \dot{\eta} &= F\eta + G(t)u\\ \dot{y} &= g(t, z, y, v, w) + b(w)u\end{aligned} \tag{3.25}$$

Performing the coordinate and input transformation

$$\begin{aligned}\bar{z} &= z - \mathbf{z}(t, v, w), \quad e = y - \mathbf{y}(t, v, w)\\ \hat{\eta} &= \eta - N_o^{-1}\vartheta = \eta - \tau, \quad \bar{u} = u - \Gamma_o\eta\end{aligned} \tag{3.26}$$

gives

$$\begin{aligned}\dot{\bar{z}} &= \bar{f}(t, \bar{z}, e, \mu)\\ \dot{\hat{\eta}} &= \hat{f}_2(t, \hat{\eta}, \bar{u}, \mu)\\ \dot{e} &= \bar{g}(t, \bar{z}, \hat{\eta}, e, \mu) + b(w)(\bar{u} + \Gamma_o\hat{\eta})\end{aligned} \tag{3.27}$$

where $\mu = (v, w)$ and

$$\begin{aligned}\bar{f}(t, \bar{z}, e, \mu) &= f\big(t, \bar{z} + \mathbf{z}(t, v, w), e + q(t, v, w), v, w\big) - f\big(t, \mathbf{z}(t, v, w), q(t, v, w), v, w\big)\\ \bar{g}(t, \bar{z}, e, \mu) &= g\big(t, \bar{z} + \mathbf{z}(t, v, w), e + q(t, v, w), v, w\big) - g\big(t, \mathbf{z}(t, v, w), q(t, v, w), v, w\big)\\ \hat{f}_2(t, \hat{\eta}, \bar{u}, \mu) &= F\hat{\eta} + G(t)(\bar{u} + \Gamma_o\hat{\eta})\end{aligned}$$

Since (3.27) is not in lower triangular form, performing another transformation

$$\bar{\eta} = \hat{\eta} - b^{-1}(w)G(t)e \tag{3.28}$$

gives

$$\begin{aligned}\dot{\bar{z}} &= \bar{f}(t, \bar{z}, e, \mu)\\ \dot{\bar{\eta}} &= F\bar{\eta} + \bar{f}_2(t, \bar{z}, e, \mu)\\ \dot{e} &= \bar{g}_e(t, \bar{z}, \bar{\eta}, e, \mu) + b(w)\bar{u}\end{aligned} \tag{3.29}$$

where, for simplicity, $\bar{g}$ stands for $\bar{g}(t, \bar{z}, e, \mu)$ and

$$\begin{aligned}\bar{f}_2(t, \bar{z}, e, \mu) &= b^{-1}(w)\left(FG(t)e - \dot{G}(t)e - G(t)\bar{g}\right)\\ \bar{g}_e(t, \bar{z}, e, \bar{\eta}, \mu) &= \bar{g} + b(w)\Gamma_o\bar{\eta} + \Gamma_o G(t)e\end{aligned}$$

System (3.29) takes the same form as system (13) in [120] except that the functions $\bar{f}$, $\bar{f}_2$, and $\bar{g}_e$ depend on t explicitly. To stabilize system (3.29), we need two additional assumptions.

Assumption 3.5. *For any compact subset* $\Sigma \subset \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$*, there exists a* $\mathcal{C}^1$ *function* $V_{\bar{z}}$ *satisfying* $\underline{\alpha}(\|\bar{z}\|) \leq V_{\bar{z}}(t, \bar{z}) \leq \bar{\alpha}(\|\bar{z}\|)$ *for some class* $\mathcal{K}_\infty$ *functions* $\underline{\alpha}(\cdot)$ *and* $\bar{\alpha}(\cdot)$*, and for any* $\mu \in \Sigma$*, along the trajectory of the subsystem* $\dot{\bar{z}} = \bar{f}(t, \bar{z}, e, \mu)$*,*

$$\dot{V}_{\bar{z}} \leq -\alpha(\|\bar{z}\|) + \delta\gamma(e)$$

where $\alpha(\cdot)$ *is some known class* $\mathcal{K}_\infty$ *function and is locally quadratic, i.e.,* $\lim_{s\to 0^+} \sup(\alpha^{-1}(s^2)/s) < \infty$*,* δ *is some unknown positive constant, and* $\gamma(\cdot)$ *is a known smooth positive definite function.* ▪

Assumption 3.6. *There exist a bounded function* $\chi : \mathbb{R} \to \mathbb{R}^\kappa$ *for some* $\kappa > 0$*, and smooth functions* $\tilde{f}_2(\bar{z}, e, \chi)$ *and* $\tilde{g}_e(\bar{z}, \bar{\eta}, e, \chi)$ *vanishing at* $(\bar{z}, \bar{\eta}, e) = 0$ *such that,*

$$\begin{aligned} \|\bar{f}_2(t, \bar{z}, e, \mu)\| &\leq \|\tilde{f}_2(\bar{z}, e, \chi)\| \\ |\bar{g}_e(t, \bar{z}, \bar{\eta}, e, \mu)| &\leq |\tilde{g}_e(\bar{z}, \bar{\eta}, e, \chi)| \end{aligned} \tag{3.30}$$

for all $t \geq 0$*, and any* $\mu \in \Sigma$*.* ▪

Remark 3.6. Assumption 3.5 is obtained by slightly modifying Assumption 3 of [120] to accommodate the time-varying case. Under Assumptions 3.5, by changing supply functions technique [111], for any smooth function $\Delta(\bar{z}) \geq 0$, there exists a $\mathcal{C}^1$ function $V_{\bar{z}1}(t, \bar{z})$ satisfying $\underline{\alpha}_1(\|\bar{z}\|) \leq V_{\bar{z}1}(t, \bar{z}) \leq \bar{\alpha}_1(\|\bar{z}\|)$ for some class $\mathcal{K}_\infty$ function $\underline{\alpha}_1(\cdot)$ and $\bar{\alpha}_1(\cdot)$, such that along the trajectory of $\bar{z}$ subsystem, $\dot{V}_{\bar{z}1} \leq -\Delta(\bar{z})\|\bar{z}\|^2 + \delta_1\gamma_1(e)e^2$ for some positive number δ_1 and some known continuous function $\gamma_1(\cdot) \geq 1$. ▪

Remark 3.7. Assumption 3.6 is introduced to take care of the fact that both $\bar{f}_2$ and $\bar{g}_e$ are time-varying. In fact, by Lemma 7.8 in [36], under Assumption 3.6, for all $\bar{z} \in \mathbb{R}^n$, $e \in \mathbb{R}$ and $t \geq 0$, $\bar{f}_2$ satisfies

$$\|\bar{f}_2\| \leq \|\tilde{f}_2\| \leq c_1(\psi_{\bar{z}}(\bar{z})\|\bar{z}\| + \psi_{e1}(e)|e|) \tag{3.31}$$

$$|\bar{g}_e|^2 \leq c_g(\psi_g(Z)\|Z\|^2 + \psi_e(e)e^2) \tag{3.32}$$

where $c_1, c_g > 0$ are some real constants, $Z = \text{col}(\bar{z}, \bar{\eta})$, $\psi_{\bar{z}}(\cdot) \geq 1$, $\psi_{e1}(\cdot) \geq 1$, $\psi_g(\cdot) \geq 1$, $\psi_e(\cdot) \geq 1$ are some continuous functions. Under Assumptions 3.5 and 3.6, let $V_1(t, Z) = dV_{\bar{z}1} + 2\bar{\eta}^T P_1 \bar{\eta}$ where P_1 is the positive definite matrix satisfying $P_1 F + F^T P_1 = -I$ and

d is some positive constant Then it can be shown that along the trajectory of $(\bar{z}, \bar{\eta})$ subsystem,

$$\dot{V}_1 \leq -\|Z\|^2 + \delta_2 \gamma_2(e) \tag{3 33}$$

where δ_2 is some positive number and $\gamma_2(\cdot)$ is some known smooth positive definite function

Assumption 3 6 is not restrictive as it may appear to be It is satisfied if t enters the functions $\bar{f}_2$ and $\bar{g}_e$ through some bounded functions such as sinusoidal functions In particular, it is always satisfied if the given system is a periodic system Also, similar bounded conditions are commonly used in the control problem of time-varying nonlinear systems [49] [50] [61] [63] ▪

By (3 33), we can obtain the following results

Theorem 3.1. *Under Assumptions 3 3, 3 4, 3 5 and 3 6, there exists a smooth function $\rho(e) \geq 1$ such that the following control law*

$$\begin{aligned} \dot{\eta} &= F\eta + G(t)u \\ u &= -k\rho(e)e + \Gamma_o \eta \\ \dot{k} &= \rho(e)e^2 \end{aligned} \tag{3 34}$$

solves the robust output regulation problem of plant (3 24) with exosystem (3 2) ▪

Proof: Under the stated assumptions, (3 33) holds Applying the changing supply functions technique to (3 33) shows that, for any smooth function $\Delta(Z) > 0$, there exists a $\mathcal{C}^1$ function V_Z satisfying $\underline{\alpha}_2(\|Z\|) \leq V_Z(t, Z) \leq \bar{\alpha}_2(\|Z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_2(\cdot)$ and $\bar{\alpha}_2(\cdot)$, suc that along the trajectory of $(\bar{z}, \bar{\eta})$ subsystem

$$\dot{V}_Z \leq -\Delta(Z)\|Z\|^2 + \delta_e \gamma_e(e)e^2 \tag{3 35}$$

for some positive number δ_e and some known continuous function $\gamma_e(\cdot) \geq 1$

Let

$$V = V_Z + \frac{1}{2}e^2 + \frac{1}{2}b(k - \bar{k})^2 \tag{3 36}$$

where $\bar{k}$ is some positive constant Then using (3 32), we have, along the trajectory of the augmented system (3 29),

$$\begin{aligned} \dot{V} &\leq -\Delta(Z)\|Z\|^2 + \delta_e \gamma_e(e)e^2 + \frac{1}{2}|\bar{g}_e|^2 + \frac{1}{2}e^2 - bk\rho(e)e^2 + bk(k - \bar{k}) \\ &\leq -\left(\Delta(Z) - \frac{c_g}{2}\psi_g(Z)\right)\|Z\|^2 + \left(\delta_e\gamma_e(e) + \frac{c_g}{2}\psi_e(e) + \frac{1}{2} - b\bar{k}\rho(e)\right)e^2 \end{aligned} \tag{3 37}$$

By choosing $\Delta(Z) \geq 1 + \frac{c_g}{2}\psi_g(Z)$, $\rho(e) \geq \max(\gamma_e(e), \psi_e(e), 1)$ and $\bar{k} \geq b^{-1}(\delta_e + \frac{c_g}{2} + \frac{1}{2})$, we have

$$\dot{V} \leq -\|Z\|^2 \tag{3.38}$$

From (3.36) to (3.38) and by LaSalle-Yoshizawa Theorem, it can be concluded that the trajectory of the closed-loop system from any initial state is bounded and $\lim_{t\to\infty} Z = 0$. The boundedness of e and $\dot{e}$ implies $\dot{k}$ is bounded and uniformly continuous. By Barbalat's lemma, $\lim_{t\to\infty} \dot{k} = 0$, which implies $\lim_{t\to\infty} e = 0$. Then by Remark 3.3, control law (3.34) solves the robust output regulation problem. ◇

3.3.2 System with relative degree ≥ 2

Consider the following SISO time-varying nonlinear system.

$$\begin{aligned}
\dot{z} &= f(t, z, y, v, w) \\
\dot{x}_i &= x_{i+1} + g_i(t, z, y, v, w) \quad i = 1, \cdots, r-1 \\
\dot{x}_r &= b(w)u + g_r(t, z, y, v, w) \\
y &= x_1 \\
e &= y - q(t, v, w)
\end{aligned} \tag{3.39}$$

where $z \in \mathbb{R}^{n-r}$ and $x = \text{col}(x_1, \cdots, x_r) \in \mathbb{R}^r$ are the states, $r \geq 2$, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, $e \in \mathbb{R}$ is the error output, $w \in \mathbb{R}^{n_w}$ represents the uncertain constant parameters, $b(w) > 0$, and $v \in \mathbb{R}^{n_v}$ represents the exogenous signals generated by (3.2).

Compared with (3.24), system (3.39) is called the time-varying nonlinear system in output feedback form with relative degree $r \geq 2$. All functions in (3.39) are supposed to be globally defined. sufficiently smooth and satisfying $f(t, 0, 0, 0, w) = 0$, $g_i(t, 0, 0, 0, w) = 0$ and $q(t, 0, w) = 0$ for all $w \in \mathbb{R}^{n_w}$.

Due to the structure of system (3.39), the satisfaction of Assumption 3.4 also ensures the solution of the regulator equations associated with the composite system (3.39) (3.2) exists. We denote $\mathbf{y} = \mathbf{x}_1(t, v, w) = q(t, v, w)$, and the rest can be solved recursively as follows.

$$\begin{aligned}
\mathbf{x}_{i+1}(t, v, w) &= \mathcal{L}_{a(t,v)}\mathbf{x}_i(t, v, w) - g_i\big(t, \mathbf{z}(t, v, w), q(t, v, w), v, w\big) \quad i = 1, \cdots, r-1 \\
\mathbf{u}(t, v, w) &= b^{-1}(w)\Big(\mathcal{L}_{a(t,v)}\mathbf{x}_r(t, v, w) - g_r\big(t, \mathbf{z}(t, v, w), q(t, v, w), v, w\big)\Big)
\end{aligned}$$

Also, we suppose Assumption 3.3 is satisfied. Then as introduced in Section 3.2, an internal model in the form of (3.23) can be built as

$$\dot{\eta} = F\eta + G(t)u \tag{3.40}$$

where $\eta \in \mathbb{R}^l$ is the state vector, F is a constant Hurwitz matrix.

Attaching the internal model (3.40) to the plant (3.39) gives

$$\begin{aligned}
\dot{z} &= f(t, z, y, v, w) \\
\dot{\eta} &= F\eta + G(t)u \\
\dot{x}_i &= x_{i+1} + g_i(t, z, y, v, w) \quad i = 1, \cdots, r-1 \\
\dot{x}_r &= b(w)u + g_r(t, z, y, v, w) \\
y &= x_1 \\
e &= y - q(t, v, w)
\end{aligned} \tag{3.41}$$

Performing the input and coordinate transformation on (3.41)

$$\begin{aligned}
\bar{z} &= z - \mathbf{z}(t, v, w) \\
\bar{x}_i &= x_i - \mathbf{x}_i(t, v, w) \\
\hat{\eta} &= \eta - N_o^{-1}(t)\vartheta = \eta - \tau \\
\bar{u} &= u - \Gamma_o \eta
\end{aligned} \tag{3.42}$$

where $\Gamma_o = [\ 1,\ 0,\ \cdots,\ 0\ ]_{1\times l}$, gives

$$\begin{aligned}
\dot{\bar{z}} &= \bar{f}(t, \bar{z}, e, v, w) \\
\dot{\hat{\eta}} &= F\hat{\eta} + G(t)(\bar{u} + \Gamma_o\hat{\eta}) \\
\dot{\bar{x}}_i &= \bar{x}_{i+1} + \bar{g}_i(t, \bar{z}, e, v, w) \quad i = 1, \cdots, r-1 \\
\dot{\bar{x}}_r &= b(w)(\bar{u} + \Gamma_o\hat{\eta}) + \bar{g}_r(t, \bar{z}, e, v, w)
\end{aligned} \tag{3.43}$$

where $\bar{x}_1 = e$ and

$$\begin{aligned}
\bar{f}(\cdot) &= f(t, \bar{z} + \mathbf{z}(t, v, w), e + q(t, v, w), v, w) - f\big(t, \mathbf{z}(t, v, w), q(t, v, w), v, w\big) \\
\bar{g}_i(\cdot) &= g_i(t, \bar{z} + \mathbf{z}(t, v, w), e + q(t, v, w), v, w) - g_i\big(t, \mathbf{z}(t, v, w), q(t, v, w), v, w\big)
\end{aligned}$$

The stabilization of augmented system (3.43) will lead us to solve the output regulation problem for the original plant (3.39).

Due to the presence of the internal model, system (3.43) is not in any standard form of nonlinear system. To make the stabilization problem more tractable, illuminated by [51], we perform a coordinate transformation on the $\hat{\eta}$ dynamics

$$\bar{\eta} = \hat{\eta} - c_r(t)\bar{x}_r - \cdots - c_1(t)\bar{x}_1 \tag{3.44}$$

where the coefficients $c_i(t)$ are to be determined, and we have the following proposition.

Proposition 3.2. *Under (3.44), by choosing*

$$c_r(t) = b^{-1}(w)G(t) \text{ and } c_i(t) = Fc_{i+1}(t) - \dot{c}_{i+1}(t),\ i = r-1, \cdots, 1$$

system (3.43) can be transformed into the following form

$$\begin{aligned}
\dot{\bar{z}} &= \bar{f}(t, \bar{z}, e, v, w) \\
\dot{\bar{\eta}} &= F\bar{\eta} + \bar{g}_0(t, \bar{z}, e, v, w) \\
\dot{\bar{x}}_i &= \bar{x}_{i+1} + \bar{g}_i(t, \bar{z}, e, v, w) \quad i = 1, \cdots, r-1 \\
\dot{\bar{x}}_r &= b(w)\Big(\bar{u} + \Gamma_o\big(\bar{\eta} + \sum_{i=1}^{r} c_i(t)\bar{x}_i\big)\Big) + \bar{g}_r(t, \bar{z}, e, v, w)
\end{aligned} \tag{3.45}$$

where $\bar{g}_0(t, \bar{z}, e, v, w) = \big(Fc_1(t) - \dot{c}_1(t)\big)\bar{x}_1 - \sum_{i=1}^{r} c_i(t)\bar{g}_i(t, \bar{z}, e, v.w)$, *and* $\bar{x}_1 = e$. ∎

Proof: Under transformation (3.44), we see first

$$\begin{aligned}
\dot{\bar{\eta}} = \dot{\hat{\eta}} - \sum_{i=1}^{r} \frac{d\big(c_i(t)\bar{x}_i\big)}{dt} &= F\hat{\eta} + G(t)(\bar{u} + \Gamma_o\hat{\eta}) - \sum_{i=1}^{r} \frac{d\big(c_i(t)\bar{x}_i\big)}{dt} \\
&= F\hat{\eta} + G(t)(\bar{u} + \Gamma_o\hat{\eta}) - \sum_{i=1}^{r-1} \frac{d\big(c_i(t)\bar{x}_i\big)}{dt} \\
&\quad - \dot{c}_r(t)\bar{x}_r - c_r(t)\big(b(w)(\bar{u} + \Gamma_o\hat{\eta}) + \bar{g}_r(t, \bar{z}, e, v, w)\big)
\end{aligned}$$

By choosing $c_r(t) = b^{-1}(w)G(t)$, the items $\bar{u}$ and $\hat{\eta}$ are eliminating from the above equation, i.e.

$$\dot{\bar{\eta}} = F\hat{\eta} - \sum_{i=1}^{r-1} \frac{d\big(c_i(t)\bar{x}_i\big)}{dt} - \dot{c}_r(t)\bar{x}_r - c_r(t)\bar{g}_r(t, \bar{z}, e, v, w)$$

Further, by replacing $\hat{\eta}$ with $\bar{\eta} + c_r(t)\bar{x}_r + \cdots + c_1(t)\bar{x}_1$, it can be seen that

$$\begin{aligned}\dot{\bar{\eta}} &= F\big(\bar{\eta} + \sum_{i=1}^{r} c_i(t)\bar{x}_i\big) - \sum_{i=1}^{r-1} \frac{d\big(c_i(t)\bar{x}_i\big)}{dt} - \dot{c}_r(t)\bar{x}_r - c_r(t)\bar{g}_r(t, \bar{z}, e, v, w)\\ &= F\big(\bar{\eta} + \sum_{i=1}^{r-1} c_i(t)\bar{x}_i\big) + Fc_r(t)\bar{x}_r - \sum_{i=1}^{r-2} \frac{d\big(c_i(t)\bar{x}_i\big)}{dt} - \dot{c}_{r-1}(t)\bar{x}_{r-1}\\ &\quad - c_{r-1}(t)\big(\bar{x}_r + \bar{g}_{r-1}(t, \bar{z}, e, v, w)\big) - \dot{c}_r(t)\bar{x}_r - c_r(t)\bar{g}_r(t, \bar{z}, e, v, w)\end{aligned}$$

To render the coefficient of $\bar{x}_r$ being zero, obviously, by choosing $c_{r-1}(t) = Fc_r(t) - \dot{c}_r(t)$, it can be seen that

$$\begin{aligned}\dot{\bar{\eta}} &= F\big(\bar{\eta} + \sum_{i=1}^{r-1} c_i(t)\bar{x}_i\big) - \sum_{i=1}^{r-2} \frac{d\big(c_i(t)\bar{x}_i\big)}{dt} - \dot{c}_{r-1}(t)\bar{x}_{r-1}\\ &\quad - c_{r-1}(t)\bar{g}_{r-1}(t, \bar{z}, e, v, w) - c_r(t)\bar{g}_r(t, \bar{z}, e, v, w)\end{aligned}$$

Repeat the aforementioned procedure, all coefficients $c_i(t)$ can be determined recursively from $i = r - 2$ to $i = 1$ as $c_i(t) = Fc_{i+1}(t) - \dot{c}_{i+1}(t)$. Finally, by noticing that $\bar{x}_1 = e$, the $\bar{\eta}$ dynamics can be expressed as

$$\begin{aligned}\dot{\bar{\eta}} &= F\bar{\eta} + \big(Fc_1(t) - \dot{c}_1(t)\big)\bar{x}_1 - \sum_{i=1}^{r} c_i(t)\bar{g}_i(t, \bar{z}, e, v, w)\\ &\stackrel{\text{def}}{=} F\bar{\eta} + \bar{g}_0(t, \bar{z}, e, v, w)\end{aligned}$$

Also, due to the transformation (3.44), the $\bar{x}_r$ dynamics of (3.43) turns to be

$$\dot{\bar{x}}_r = b(w)\Big(\bar{u} + \Gamma_o\big(\bar{\eta} + \sum_{i=1}^{r} c_i(t)\bar{x}_i\big)\Big) + \bar{g}_r(t, \bar{z}, e, v, w)$$

Together, system is exactly in the form of (3.45). ◇

For system (3.45), we denote $\bar{x} = \text{col}(\ \bar{x}_1,\ \cdots,\ \bar{x}_r\ )$, $s_i(t) = b(w)\Gamma_o c_i(t)$, $i = 1, \cdots, r$ and $\bar{g}' = \text{col}\big(\ \bar{g}_1(\cdot),\ \cdots,\ \bar{g}_{r-1}(\cdot)\big)$, then the $\bar{x}$ subsystem of (3.45) can be expressed in the following compact form

$$\dot{\bar{x}} = \left[\begin{array}{c|c} 0 & I_{r-1} \\ \hline s_1(t) & s_2(t)\ \cdots\ s_r(t) \end{array}\right]\bar{x} + \left[\begin{array}{c} 0_{1\times(r-1)} \\ \hline b(w) \end{array}\right]\bar{u} + \left[\begin{array}{c} \bar{g}'(t, \bar{z}, e, v, w) \\ \hline \bar{g}_r(t, \bar{z}, e, v, w) + b(w)\Gamma_o\bar{\eta} \end{array}\right] \quad (3.46)$$

For the linear part of system (3.46), i.e.

$$\begin{aligned}\dot{\bar{x}} &= \left[\begin{array}{c|c} 0 & I_{r-1} \\ \hline s_1(t) & s_2(t)\ \cdots\ s_r(t) \end{array}\right]\bar{x} + \left[\begin{array}{c} 0_{1\times(r-1)} \\ \hline b(w) \end{array}\right]\bar{u}\\ e &= \bar{x}_1\end{aligned} \quad (3.47)$$

since $s_i(t)$ are time-varying variables, like the transformation between (3.19) and (3.20), there exists a well-defined time-varying coordinate transformation $\chi = T(t)\bar{x}$ such that (3.47) can be transformed into the following observer canonical form

$$\begin{aligned} \dot{\chi} &= \left[\begin{array}{c|c} d_r(t) & \\ \vdots & I_{r-1} \\ \hline d_1(t) & 0 \end{array}\right] \chi + \left[\begin{array}{c} 0 \\ \vdots \\ \hline b(w) \end{array}\right] \bar{u} \\ e &= \chi_1 \end{aligned} \tag{3.48}$$

Performing another transformation $\xi = b^{-1}(w)\chi$, and we have the following proposition.

Proposition 3.3. *There exists a well-defined coordinate transformation $\xi = b^{-1}(w)T(t)\bar{x}$, such that system (3.45) can be transformed into the following form*

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(t, \bar{z}, e, v, w) \\ \dot{\bar{\eta}} &= F\bar{\eta} + \bar{g}_0(t, \bar{z}, e, v, w) \\ \dot{\xi}_i &= \xi_{i+1} + d_{r-i+1}(t)\xi_1 + \tilde{g}_i(t, \bar{z}, e, v, w) \quad i = 1, \cdots, r-1 \\ \dot{\xi}_r &= (\bar{u} + \Gamma_o\bar{\eta}) + d_1(t)\xi_1 + \tilde{g}_r(t, \bar{z}, e, v, w) \end{aligned}$$

where $\tilde{g}_i(\cdot)$ denotes the form which $\bar{g}_i(\cdot)$ takes after transformation. Moreover, by noting that $\xi_1 = b^{-1}(w)e$, the above system can be turned into the following compact form

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(t, \bar{z}, e, v, w) \\ \dot{\bar{\eta}} &= F\bar{\eta} + G_0(t, \bar{z}, e, v, w) \\ \dot{\xi}_i &= \xi_{i+1} + G_i(t, \bar{z}, e, v, w) \quad i = 1, \cdots, r-1 \\ \dot{\xi}_r &= (\bar{u} + \Gamma_o\bar{\eta}) + G_r(t, \bar{z}, e, v, w) \end{aligned} \tag{3.49}$$

where $G_0(t, \bar{z}, e, v, w) = \bar{g}_0(t, \bar{z}, e, v, w)$ and $G_i(t, \bar{z}, e, v, w) = d_{r-i+1}(t)\xi_1 + \tilde{g}_i(t, \bar{z}, e, v, w)$. ■

To stabilize the transformed augmented system (3.49) by output feedback, we follow the stabilization method exhibited in [86].

First, we introduce an observer-like dynamics to estimate the state of ξ subsystem of (3.49)

$$\begin{aligned} \dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + \lambda_i(e - \hat{\xi}_1) \quad i = 1, \cdots, r-1 \\ \dot{\hat{\xi}}_r &= \bar{u} + \lambda_r(e - \hat{\xi}_1) \end{aligned} \tag{3.50}$$

The parameters λ_i are chosen to ensure the matrix A_o being Hurwitz, where

$$A_o = \left[\begin{array}{c|c} -\lambda_1 & \\ \vdots & I_{r-1} \\ \hline -\lambda_r & 0 \end{array}\right]$$

The estimation error of ξ is denoted by $\tilde{\xi} = \xi - \hat{\xi}$, and it is not hard to verify that the dynamics of $\tilde{\xi}$ satisfies

$$\begin{aligned} \dot{\tilde{\xi}}_i &= \tilde{\xi}_{i+1} - \lambda_i(e - \hat{\xi}_1) + G_i(t, \bar{z}, e, v, w) \\ &= -\lambda_i\tilde{\xi}_1 + \tilde{\xi}_{i+1} - \lambda_i(e - \xi_1) + G_i(t, \bar{z}, e, v, w) \quad i = 1, \cdots, r-1 \\ \dot{\tilde{\xi}}_r &= \Gamma_o\bar{\eta} - \lambda_r(e - \hat{\xi}_1) + G_r(t, \bar{z}, e, v, w) \\ &= -\lambda_r\tilde{\xi}_1 + \Gamma_o\bar{\eta} - \lambda_r(e - \xi_1) + G_r(t, \bar{z}, e, v, w) \end{aligned}$$

which can be put into the following compact form

$$\dot{\tilde{\xi}} = A_o\tilde{\xi} + B\Gamma_o\bar{\eta} + G(t, \bar{z}, e, v, w) - \lambda\big(e - b^{-1}(w)e\big) \tag{3.51}$$

where $B = \text{col}(\ 0,\ \cdots,\ 0,\ 1\ )$, $G(t, \bar{z}, e, v, w) = \text{col}\big(\ G_1(\cdot),\ \cdots,\ G_r(\cdot)\ \big)$, and $\lambda = \text{col}(\ \lambda_1,\ \cdots,\ \lambda_r\ )$.

Now the problem turns to be similar to the one studied in [119]. By noting $e = b(w)\xi_1$, replacing ξ_i by its estimation $\hat{\xi}_i$, $i = 2, \cdots, r$, and attaching (3.51) to (3.49), we have the following system

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(t, \bar{z}, e, v, w) \\ \dot{\bar{\eta}} &= F\bar{\eta} + G_0(t, \bar{z}, e, v, w) \\ \dot{\tilde{\xi}} &= A_o\tilde{\xi} + B\Gamma_o\bar{\eta} + G(t, \bar{z}, e, v, w) - \lambda\big(1 - b^{-1}(w)\big)e \\ \dot{e} &= b(w)\big(\hat{\xi}_2 + \tilde{\xi}_2 + G_1(t, \bar{z}, e, v, w)\big) \\ \dot{\hat{\xi}}_2 &= \hat{\xi}_3 + \lambda_2(e - \hat{\xi}_1) \\ &\ \vdots \\ \dot{\hat{\xi}}_r &= \bar{u} + \lambda_r(e - \hat{\xi}_1) \end{aligned} \tag{3.52}$$

It can be seen system (3.52) is in the lower triangular form, viewing $(\bar{z}, \bar{\eta}, \tilde{\xi})$ as the inverse dynamics. Till now, both e and $\hat{\xi}_i$, $i = 1, \cdots, r$ are available for feedback design.

To global stabilize (3.52), certain ISS property of its inverse dynamics is required. Specifically, we suppose Assumption 3.5 hold

Also, by taking care of the the fact that $G_i(\cdot)$ is time-varying, we introduce the following assumption similar to Assumption 3.6 in the first place.

Assumption 3.7. *There exist bounded function $\chi : \mathbb{R} \to \mathbb{R}^{\kappa}$ for some $\kappa > 0$, and smooth functions $\tilde{G}_i(\bar{z}, e, \chi)$ vanishing at $(\bar{z}, e) = 0$ such that for all $t \geq 0$, and any $(v, w) \in \Sigma \subset \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$,*

$$|G_i(t, \bar{z}, e, v, w)| \leq |\tilde{G}_i(\bar{z}, e, \chi)|$$

where $i = 0, 1, \cdots, r$. ▪

Now notice the fact that both F and A_o are Hurwitz matrices, then under Assumptions 3.5 and 3.7, we arrive at the following proposition.

Proposition 3.4. *For the following system*

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(t, \bar{z}, e. v, w) \\ \dot{\bar{\eta}} &= F\bar{\eta} + G_0(t, \bar{z}, e, v, w) \\ \dot{\tilde{\xi}} &= A_o\tilde{\xi} + B\Gamma_o\bar{\eta} + G(t, \bar{z}, e, v, w) - \lambda\big(e - b^{-1}(w)e\big) \end{aligned} \tag{3.53}$$

with the notation $Z = col(\ \bar{z}, \bar{\eta}, \tilde{\xi}\ )$, there exists a $\mathcal{C}^1$ function $U_Z(t, Z)$ satisfying $\underline{\alpha}_Z(\|Z\|) \leq U_Z(t, Z) \leq \bar{\alpha}_Z(\|Z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_Z(\cdot)$ and $\bar{\alpha}_Z(\cdot)$, such that, for any $(v, w) \in \Sigma$, along the trajectory of system (3.53),

$$\dot{U}_Z(t, Z) \leq -\|Z\|^2 + \delta\gamma(e) \tag{3.54}$$

holds for some positive constant δ and some smooth positive definite function $\gamma(\cdot)$.

Further, by changing supply functions technique, for any smooth function $\Delta(Z) \geq 0$, there exists a $\mathcal{C}^1$ function $V_Z(t, Z)$ satisfying $\underline{\alpha}'_Z(\|Z\|) \leq V_Z(t, Z) \leq \bar{\alpha}'_Z(\|Z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}'_Z(\cdot)$ and $\bar{\alpha}'_Z(\cdot)$, such that, for any $(v, w) \in \Sigma$, along the trajectory of system (3.53), the following holds

$$\dot{V}_Z(t, Z) \leq -\Delta(Z)\|Z\|^2 + \bar{\delta}\bar{\gamma}(e)e^2 \tag{3.55}$$

for some unknown positive constant $\bar{\delta}$ and some known smooth function $\bar{\gamma}(\cdot) \geq 1$. ▪

To stabilize system (3.52) we will utilize the backstepping design. For this purpose, we introduce some notations first. Denote $\hat{\xi}_{r+1} = \bar{u}$, and for $i = 3, \cdots, r$ let

$$\begin{aligned}
w_i &= \hat{\xi}_{i+1} - \kappa_i, \quad K_i = \frac{\partial \kappa_i}{\partial k}\dot{k}, \quad E_i = \frac{\partial \kappa_i}{\partial e} \\
\kappa_1 &= -k\rho(e)e, \quad \dot{k} = \rho(e)e^2 \\
\kappa_2 &= -2w_1 - \lambda_2(e - \hat{\xi}_1) + K_1 + E_1\hat{b}\hat{\xi}_2 - \frac{1}{2}w_1E_1^2 \\
\kappa_i &= -w_{i-2} - w_{i-1} - \lambda_i(e - \hat{\xi}_1) + K_{i-1} + E_{i-1}\hat{b}\hat{\xi}_2 - \frac{1}{2}w_{i-1}E_{i-1}^2 + \sum_{j=1}^{i-1}\frac{\partial \kappa_{i-1}}{\partial \hat{\xi}_j}\dot{\hat{\xi}}_j + \frac{\partial \kappa_{i-1}}{\partial \hat{b}}\dot{\hat{b}} \\
\phi_2 &= w_1E_1\hat{\xi}_2 \\
\phi_i &= w_{i-1}E_{i-1}\hat{\xi}_2 + \phi_{i-1}
\end{aligned}$$

Then we arrive at the following theorem.

Theorem 3.2. *Under the Assumptions 3.5 and 3.7, there exists a control law*

$$\bar{u} = \kappa_r, \quad \dot{k} = \rho(e)e^2, \quad \dot{\hat{b}} = -\phi_r \tag{3.56}$$

where $\rho(e) > 1$ is some smooth function, such that the trajectory of the closed-loop system composed of (3.52) and (3 56) is bounded over $[0, \infty)$, and $\lim_{t\to\infty} e = 0$ ∎

Proof: For simplicity, we denote $b = b(w)$ and $G_0 = G_0(t, \bar{z}, e, v, w)$, $\hat{b}$ denotes the estimation of $b(w)$, and $\tilde{b} = b(w) - \hat{b}$ denotes the estimation error.

In the 1st step, define $V_1 = \frac{1}{2}e^2$, then

$$\begin{aligned}
\dot{V}_1 = e\dot{e} &= eb(\kappa_1 + w_1 + \tilde{\xi}_2 + G_1) \\
&\leq -bk\rho(e)e^2 + w_1^2 + \Pi_0
\end{aligned}$$

where $\Pi_0 = eb(\tilde{\xi}_2 + G_1) + \frac{1}{4}b^2e^2$.

In the 2nd step, define $V_2 = V_1 + \frac{1}{2}w_1^2 + \frac{1}{2}\tilde{b}^2$, then

$$\begin{aligned}
\dot{V}_2 &= \dot{V}_1 + w_1\dot{w}_1 + \tilde{b}\dot{\tilde{b}} = \dot{V}_1 + w_1(\dot{\hat{\xi}}_2 - \dot{\kappa}_1) - \tilde{b}\dot{\hat{b}} \\
&\le -bk\rho(e)e^2 + w_1^2 + \Pi_0 + w_1\big(\kappa_2 + w_2 + \lambda_2(e - \hat{\xi}_1) - \frac{\partial \kappa_1}{\partial e}\dot{e} - \frac{\partial \kappa_1}{\partial k}\dot{k}\big) - \tilde{b}\dot{\hat{b}} \\
&= -bk\rho(e)e^2 + w_1^2 + \Pi_0 + w_1\big(\kappa_2 + w_2 + \lambda_2(e - \hat{\xi}_1) - K_1 - E_1(\hat{b} + \tilde{b})(\hat{\xi}_2 + \tilde{\xi}_2 + G_1)\big) - \tilde{b}\dot{\hat{b}} \\
&\le -bk\rho(e)e^2 + w_1^2 + \Pi_0 + w_1\big(\kappa_2 + w_2 + \lambda_2(e - \hat{\xi}_1) - K_1\big) \\
&\quad - w_1E_1\hat{b}\hat{\xi}_2 - w_1E_1\tilde{b}\hat{\xi}_2 + \frac{1}{2}(w_1E_1)^2 + \frac{1}{2}b^2(\tilde{\xi}_2 + G_1)^2 - \tilde{b}\dot{\hat{b}} \\
&= -bk\rho(e)e^2 + \Pi_0 + w_1w_2 + w_1\big(w_1 + \kappa_2 + \lambda_2(e - \hat{\xi}_1) - K_1 - E_1\hat{b}\hat{\xi}_2 + \frac{1}{2}w_1E_1^2\big) \\
&\quad - \tilde{b}(\dot{\hat{b}} + w_1E_1\hat{\xi}_2) + \frac{1}{2}b^2(\tilde{\xi}_2 + G_1)^2
\end{aligned}$$

since

$$\begin{aligned}
\kappa_2 &= -2w_1 - \lambda_2(e - \hat{\xi}_1) + K_1 + E_1\hat{b}\hat{\xi}_2 - \frac{1}{2}w_1E_1^2 \\
\phi_2 &= w_1E_1\hat{\xi}_2
\end{aligned}$$

it is ready to show that

$$\dot{V}_2 \le -bk\rho(e)e^2 + w_1w_2 - w_1^2 - \tilde{b}(\dot{\hat{b}} + \phi_2) + \Pi_2$$

where $\Pi_2 = \Pi_1 + \Pi_0$ and $\Pi_1 = \frac{1}{2}b^2(\tilde{\xi}_2 + G_1)^2$.

In the 3rd step, define $V_3 = V_2 + \frac{1}{2}w_2^2$, then

$$\begin{aligned}
\dot{V}_3 &\le -bk\rho(e)e^2 + w_1w_2 - w_1^2 - \tilde{b}(\dot{\hat{b}} + \phi_2) + \Pi_2 + w_2(\dot{\hat{\xi}}_3 - \dot{\kappa}_2) \\
&= -bk\rho(e)e^2 + w_1w_2 - w_1^2 - \tilde{b}(\dot{\hat{b}} + \phi_2) + \Pi_2 \\
&\quad + w_2\big(\kappa_3 + w_3 + \lambda_3(e - \hat{\xi}_1) - \frac{\partial \kappa_2}{\partial e}\dot{e} - \frac{\partial \kappa_2}{\partial k}\dot{k} - \frac{\partial \kappa_2}{\partial \hat{\xi}_1}\dot{\hat{\xi}}_1 - \frac{\partial \kappa_2}{\partial \hat{\xi}_2}\dot{\hat{\xi}}_2 - \frac{\partial \kappa_2}{\partial \hat{b}}\dot{\hat{b}}\big) \\
&= -bk\rho(e)e^2 - w_1^2 - \tilde{b}(\dot{\hat{b}} + \phi_2) + \Pi_2 + w_2w_3 \\
&\quad + w_2\big(w_1 + \kappa_3 + \lambda_3(e - \hat{\xi}_1) - E_2(\hat{b} + \tilde{b})(\hat{\xi}_2 + \tilde{\xi}_2 + G_1) - K_2 - \frac{\partial \kappa_2}{\partial \hat{b}}\dot{\hat{b}} - \sum_{j=1}^{2}\frac{\partial \kappa_2}{\partial \hat{\xi}_j}\dot{\hat{\xi}}_j\big)
\end{aligned}$$

by using κ_3 and ϕ_3, it is easy to show that

$$\dot{V}_3 \le -bk\rho(e)e^2 + w_2w_3 - w_1^2 - w_2^2 - \tilde{b}(\dot{\hat{b}} + \phi_3) + \Pi_3$$

where $\Pi_3 = \Pi_2 + \Pi_1$.

Similarly, in the ith step, define $V_i = V_{i-1} + \frac{1}{2}w_{i-1}^2$, then

$$\dot{V}_i \leq -bk\rho(e)e^2 + w_{i-1}w_i - \sum_{j=1}^{i-1} w_j^2 - \tilde{b}(\dot{\hat{b}} + \phi_i) + \Pi_i$$

where $\Pi_i = \Pi_{i-1} + \Pi_1$

In the rth step, define $V_r = V_{r-1} + \frac{1}{2}w_{r-1}^2$, then

$$\dot{V}_r \leq -bk\rho(e)e^2 + w_{r-1}w_r - \sum_{j=1}^{r-1} w_j^2 - \tilde{b}(\dot{\hat{b}} + \phi_r) + \Pi_r$$

where $\Pi_r = \Pi_{r-1} + \Pi_1 = \Pi_0 + (r-1)\Pi_1$.

Setting $w_r = 0$ and $\dot{\hat{b}} = -\phi_r$, then $\bar{u} = \kappa_r$ and

$$\dot{V}_r \leq -bk\rho(e)e^2 - \sum_{j=1}^{r-1} w_j^2 + \Pi_r$$

Notice the fact

$$\begin{aligned}
\Pi_r &= \Pi_0 + (r-1)\Pi_1 \\
&= eb(\tilde{\xi}_2 + G_1) + \frac{1}{4}b^2e^2 + \frac{r-1}{2}b^2(\tilde{\xi}_2 + G_1)^2 \\
&\leq eb(\tilde{\xi}_2 + G_1) + \frac{1}{4}b^2e^2 + (r-1)b^2(\tilde{\xi}_2^2 + G_1^2)
\end{aligned}$$

Under Assumption 3.7 and by Lemma 7.8 of [36], we have

$$\begin{aligned}
&|G_1(t, \bar{z}, e, v, w,)| \leq |\tilde{G}_1(\bar{z}, e, \chi)| \leq c\big(\phi_{\bar{z}}(\bar{z})\|\bar{z}\| + \phi_e(e)|e|\big) \\
\Rightarrow\quad &|G_1|^2 \leq c'\big(\hbar_1'(\bar{z})\|\bar{z}\|^2 + \hbar_2'(e)e^2\big)
\end{aligned}$$

where c, c' are some positive constants, $\phi_{\bar{z}}(\cdot)$, $\phi_e(\cdot)$, $\hbar_1'(\cdot)$ and $\hbar_2'(\cdot)$ are some known smooth functions. Similarly, we could further show the upper bound of $|\Pi_r|$ as

$$\begin{aligned}
|\Pi_r| &\leq eb(\tilde{\xi}_2 + G_1) + \frac{1}{4}b^2e^2 + (r-1)b^2(\tilde{\xi}_2^2 + G_1^2) \\
&\leq \frac{1}{2}b^2e^2 + \frac{1}{2}\tilde{\xi}_2^2 + \frac{1}{2}b^2e^2 + \frac{1}{2}G_1^2 + \frac{1}{4}b^2e^2 + (r-1)b^2(\tilde{\xi}_2^2 + G_1^2) \\
&= \big(\frac{1}{2} + (r-1)b^2\big)\tilde{\xi}_2^2 + \big(\frac{1}{2} + (r-1)b^2\big)G_1^2 + \frac{5b^2}{4}e^2 \\
&\leq \big(\frac{1}{2} + (r-1)b^2\big)\tilde{\xi}_2^2 + \big(\frac{1}{2} + (r-1)b^2\big)c'\big(\hbar_1'(\bar{z})\|\bar{z}\|^2 + \hbar_2'(e)e^2\big) + \frac{5b^2}{4}e^2 \\
&\leq c_\pi\big(\hbar_Z(Z)\|Z\|^2 + \hbar_e(e)e^2\big)
\end{aligned}$$

where c_π is some positive constant, and $\hbar_Z(\cdot)$, $\hbar_e(\cdot)$ are known positive functions.

Consequently, we have

$$\dot{V}_r \leq -bk\rho(e)e^2 - \sum_{j=1}^{r-1} w_j^2 + c_\pi\big(\hbar_Z(Z)\|Z\|^2 + \hbar_e(e)e^2\big) \tag{3.57}$$

Finally, from Proposition 3.4 where V_Z is well posed as (3.55), we further define

$$V(t, Z, e, k, \hat{\xi}_1, \cdots, \hat{\xi}_r, \tilde{b}) = V_Z(t, Z) + V_r + \frac{b}{2}(k - \bar{k})^2$$

where $\bar{k}$ is a constant to be specified later. Its derivative can be calculated as follows

$$\begin{aligned}\dot{V} &\leq -\Delta(Z)\|Z\|^2 + \bar{\delta}\bar{\gamma}(e)e^2 - bk\rho(e)e^2 - \sum_{j=1}^{r-1} w_j^2 \\ &\quad + c_\pi\big(\hbar_Z(Z)\|Z\|^2 + \hbar_e(e)e^2\big) + (k - \bar{k})\rho(e)e^2 \\ &= -\big(\Delta(Z) - c_\pi\hbar_Z(Z)\big)\|Z\|^2 + \big(\bar{\delta}\bar{\gamma}(e) + c_\pi\hbar_e(e) - b\bar{k}\rho(e)\big)e^2 - \sum_{j=1}^{r-1} w_j^2\end{aligned}$$

By choosing $\Delta(Z) \geq c_\pi\hbar_Z(Z) + 1$, $\rho(e) \geq \max\big(\bar{\gamma}(e), \hbar_e(e), 1\big)$ and $\bar{k} \geq (\bar{\delta} + c_\pi)/b$, we have

$$\dot{V} \leq -\|Z\|^2 - \sum_{j=1}^{r-1} w_j^2$$

which shows the states of closed-loop system composed of system (3.52) and controller (3.56) are bounded over $t \in [0, +\infty)$, especially k is bounded. Moreover, since e and $\dot{e}$ are bounded, $\ddot{k}$ exists and is bounded, so $\dot{k} = \rho(e)e^2$ is uniformly continuous. By Barbalat's Lemma, it can be concluded $\dot{k} \to 0$ as $t \to \infty$, which implies $e \to 0$ as $t \to \infty$. ◇

Recall the internal model (3.23) and the observer-like dynamics (3.50), Theorem 3.2 leads us directly to the solution of output regulation problem for the original plant (3.39) with time-varying exosystem (3.2).

Corollary 3.1. *Under Assumptions 3.3, 3.4, 3.5, 3.7, the following control law*

$$\begin{aligned}\dot{\eta} &= F\eta + G(t)u \\ \dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + \lambda_i(e - \hat{\xi}_1) \quad i = 2, \cdots, r-1 \\ \dot{\hat{\xi}}_r &= (u - \Gamma_o\eta) + \lambda_r(e - \hat{\xi}_1) \\ u &= \kappa_r + \Gamma_o\eta \\ \dot{k} &= \rho(e)e^2, \quad \dot{\hat{b}} = -\phi_r\end{aligned} \tag{3.58}$$

solves the global output regulation problem for the original plant (3.39) with time-varying exosystem (3.2). ∎

3.4 Examples

In this section, we will show the effectiveness of the proposed design by the following two examples.

Example 3.1. ∎

Consider the controlled FitzHugh-Nagumo model described by the following equations.

$$\begin{aligned} \dot{z}_1 &= -\varepsilon_1 z_1 + \varepsilon_1 e \\ \dot{z}_2 &= -\varepsilon_2 z_2 - \varepsilon_2 e \\ \dot{y} &= y - \frac{1}{3}y^3 - z_1 + z_2 - d(t) + u \\ e &= y \end{aligned} \tag{3.59}$$

where $z_i \in \mathbb{R}$, $i = 1, 2$, $y \in \mathbb{R}$ is the output, $u \in \mathbb{R}$ is the control input, $d(t)$ represents the external disturbance, and ε_i, $i = 1, 2$ are positive uncertain parameters.

FitzHugh-Nagumo model is usually used to model the qualitative behavior of the neurons or to demonstrate the bursting mechanism in excitable systems [93] [98]. Equations (3.59) are taken from [120] Eq.(28). Obviously, (3.59) is in the output feedback form with the unity relative degree (3 24).

The disturbance signal is denoted by $d(t) = r\cos(2t)v_1$, where r is some unknown constant, and v_1 is supposed to be generated from the following equations.

$$\begin{aligned} \dot{v}_1 &= v_2 \\ \dot{v}_2 &= \big(-a + 2q\cos(2t)\big)v_1 \end{aligned} \tag{3 60}$$

where (a, q) are some known constant parameters.

Equations (3.60) are referred as Mathieu equation, and it is the most widely known and most extensively treated periodically time-varying system [92]. And when $q = 0$, Mathieu equaiton will reduce to the constant harmonic oscillator. Figure 3.1 shows the dynamics of the Mathieu equation.

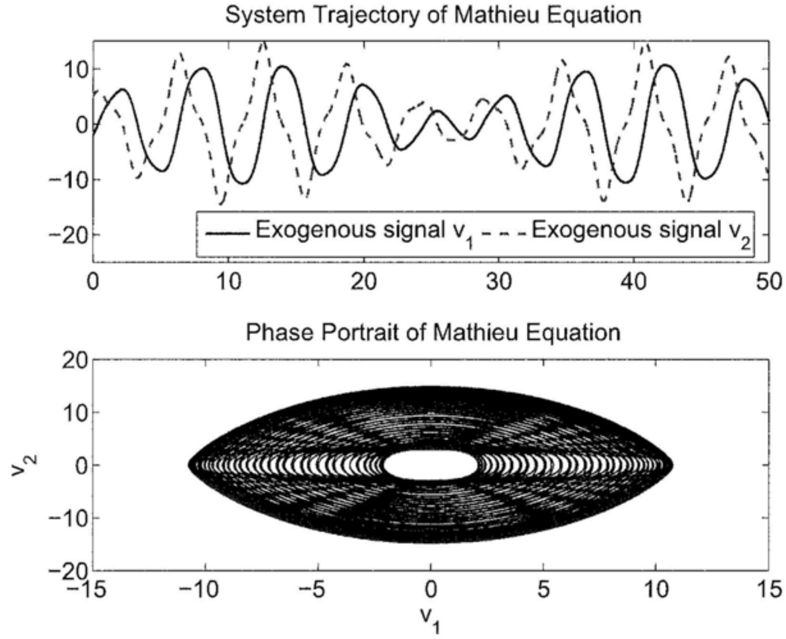

Figure 3.1: Dynamics of Mathieu Equation

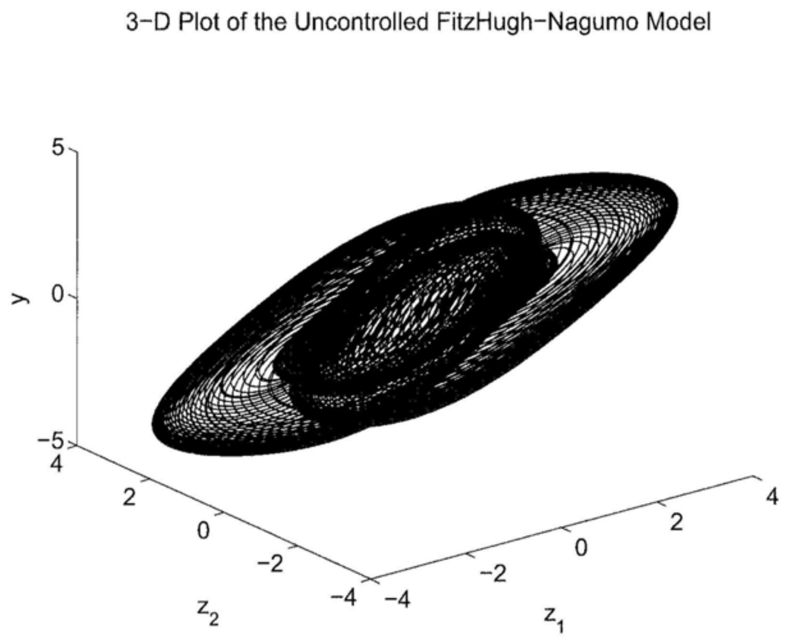

Figure 3.2: Uncontrolled FitzHugh-Nagumo Model under Disturbance

It can be seen from Figure 3.2 that the uncontrolled FitzHugh-Nagumo model ($u = 0$)

may exhibits chaotic behaviors under the disturbance $d(t)$.

Our control objective is to achieve globally disturbance rejection for FitzHugh-Nagumo model (3.59).

The solution of the corresponding regulator equations is given by

$$\mathbf{z}_1(t,v,w) = \mathbf{z}_2(t,v,w) = 0, \quad \mathbf{y}(t,v,w) = 0 \quad \mathbf{u}(t,v,w) = r\cos(2t)v_1$$

so Assumption 3.4 is satisfied.

It can be verified that

$$\frac{d^4\mathbf{u}}{dt^3} = a_2(t)\frac{d^2\mathbf{u}}{dt^2} + a_1(t)\frac{d\mathbf{u}}{dt} + a_0(t)\mathbf{u}$$

where

$$a_2(t) = -8 - 2a + 2q\cos(2t), \quad a_1(t) = -16q\sin(2t)$$
$$a_0(t) = -16 + 8a - a^2 - 32q\cos(2t) + 2aq\cos(2t)$$

Thus, Assumption 3.3 is satisfied. So we can obtain the steady-state input generator in observability canonical form (3.19) with the pair

$$\Phi(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ a_0(t) & a_1(t) & a_2(t) & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

Using the Lyapunov transformation $\tau = N_0^{-1}(t)\vartheta$, we can also obtain the steady-state input generator in observer canonical form (3.20) with the pair

$$\Phi_o(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ b_2(t) & 0 & 1 & 0 \\ b_1(t) & 0 & 0 & 1 \\ b_0(t) & 0 & 0 & 0 \end{bmatrix}, \quad \Gamma_o = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

where

$$b_2(t) = -8 - 2a + 2q\cos(2t), \quad b_1(t) = -8q\sin(2t)$$
$$b_0(t) = -16 + 8a - a^2 - 8q\cos(2t) + 2aq\cos(2t)$$

Let Φ_b and $b(t)$ be defined as in (3.22) and choose $L_0 = (\ 15,\ 35,\ 50,\ 24\ )^T$. Then, we can obtain the canonical internal model in the form of (3.23) as follows

$$\dot{\eta} = F\eta + G(t)u \tag{3.61}$$

where

$$F = \begin{bmatrix} -15 & 1 & 0 & 0 \\ -35 & 0 & 1 & 0 \\ -50 & 0 & 0 & 1 \\ -24 & 0 & 0 & 0 \end{bmatrix}, \quad G(t) = \begin{bmatrix} 15 \\ 27 - 2a + 2q\cos(2t) \\ 50 - 8q\sin(2t) \\ 8 + 8a - a^2 - 8q\cos(2t) + 2aq\cos(2t) \end{bmatrix}$$

Attaching the internal model (3.61) to the plant and performing the coordinate transformations (3.26) and (3.28), we have the augmented system in the form (3.29)

$$\begin{aligned} \dot{\bar{z}}_1 &= -\varepsilon_1 \bar{z}_1 + \varepsilon_1 e \\ \dot{\bar{z}}_2 &= -\varepsilon_2 \bar{z}_2 - \varepsilon_2 e \\ \dot{\bar{\eta}} &= F\bar{\eta} + \bar{f}_2 \\ \dot{e} &= \bar{g}_e + \bar{u} \end{aligned} \tag{3.62}$$

where $\bar{g} = e - \frac{1}{3}e^3 - \bar{z}_1 + \bar{z}_2$, $\bar{f}_2 = FG(t)e - \dot{G}(t)e - G(t)\bar{g}$ and $\bar{g}_e = \bar{g} + \Gamma_o\bar{\eta} + \Gamma_o G(t)e$.

Using Lyapunov function $V_{\bar{z}} = 0.5\bar{z}_1^2 + 0.5\bar{z}_2^2$, it can be seen

$$\begin{aligned} \dot{V}_{\bar{z}} &= (-\varepsilon_1 \bar{z}_1^2 + \varepsilon_1 \bar{z}_1 e) + (-\varepsilon_2 \bar{z}_2^2 - \varepsilon_2 \bar{z}_2 e) \\ &\le (-0.5\varepsilon_1 \bar{z}_1^2 + 0.5\varepsilon_1 e^2) + (-0.5\varepsilon_2 \bar{z}_2^2 + 0.5\varepsilon_2 e^2) \\ &\le -0.5\varepsilon_{\min}\|\bar{z}\|^2 + \varepsilon_{\max} e^2 \end{aligned}$$

where $\varepsilon_{\min} = \min(\varepsilon_1, \varepsilon_2)$ and $\varepsilon_{\max} = \max(\varepsilon_1, \varepsilon_2)$. Thus Assumption 3.5 is satisfied.

Finally, Assumption 3.6 is also satisfied since both $\bar{f}_2$ and $\bar{g}_e$ are periodic. Thus the output regulation problem of the FitzHugh-Nagumo model is solvable. We can obtain a control law in the form (3.34) with $\rho(e) = 1 + e^4$.

Simulation is conducted with the initial conditions $(z_0, y_0) = (\ 0.6589,\ -1.3279,\ 2.2439\ )$ for the plant, $v_0 = (\ -2,\ 5\ )$ for the exosystem, and 0 for the controller. The unknown parameters are chosen as $(\varepsilon_1, \varepsilon_2, r) = (\ 5,\ 1.3,\ -2.1\ )$, and the parameters of the exosystem are $(a, q) = (\ 1.6,\ -0.6\ )$.

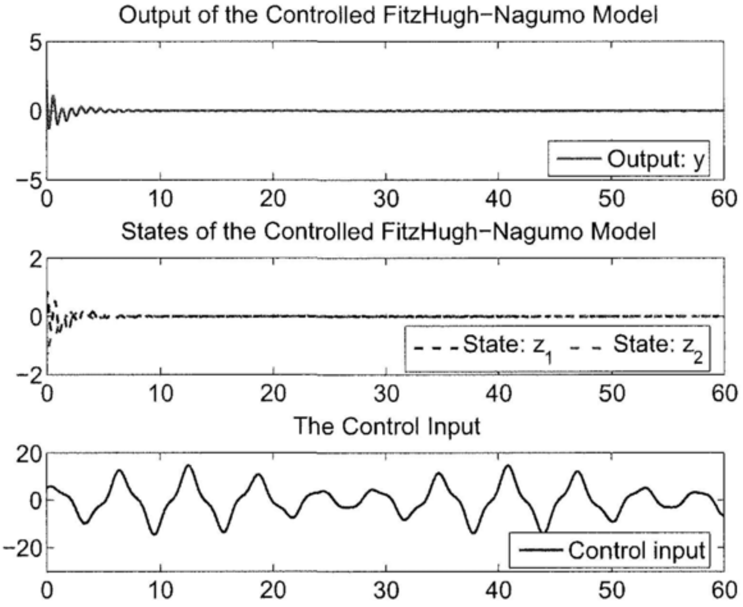

Figure 3.3: Closed-loop Response of FitzHugh-Nagumo Model

Fig 3.3 shows the response of the closed-loop system. It can be seen that the objectives of the robust output regulation are achieved. ◇

Example 3.2. ▪

Consider the following time-varying nonlinear system in the output feedback form with relative degree 2.

$$\begin{aligned}\dot{z} &= -z - z^3 + w_1 e \sin(t) \\ \dot{x}_1 &= x_2 + w_2 y z \\ \dot{x}_2 &= w_3 u + 2q\cos(2t)y + v_2 z^2 \\ y &= x_1, \quad e = y - v_1\end{aligned} \tag{3.63}$$

where (w_1, w_2, w_3) are uncertain constant parameters, and to be specific, $w_3 > 0$.

The exogenous signal is supposed to be generated from the Mathieu equation

$$\dot{v}_1 = v_2, \quad \dot{v}_2 = \big(-a + 2q\cos(2t)\big)v_1$$

The solution of the corresponding regulator equations are

$$\mathbf{z}(t,v,w) = 0, \quad \mathbf{x}_1(t,v,w) = v_1, \quad \mathbf{x}_2(t,v,w) = v_2, \quad \mathbf{u}(t,v,w) = -\frac{a}{w_3}v_1$$

for further convenience, we denote $r = -\frac{a}{w_3}$, where r is a unknown constant parameter.

It can be verified that Assumption 3.3 is satisfied with

$$\frac{d^2\mathbf{u}}{dt^2} = \big(-a + 2q\cos(2t)\big)\mathbf{u}$$

so consequently, an internal model in the form of (3.23) can be designed as follows.

$$\dot{\eta} = F\eta + G(t)u \tag{3.64}$$

where

$$F = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}, \quad G(t) = \begin{bmatrix} 3 \\ 2 - a + 2q\cos(2t) \end{bmatrix}$$

The design procedure is followed those introduced in Section 3.3.2. Especially, as mentioned in Proposition 3.3, the transformation $\xi = b^{-1}(w)T(t)\bar{x}$ is defined as

$$\xi_1 = w_3^{-1}\bar{x}_1, \quad \xi_2 = w_3^{-1}(-3\bar{x}_1 + \bar{x}_2)$$

and the observer-like dynamics (3.50) is chosen to be

$$\dot{\hat{\xi}}_1 = \hat{\xi}_2 + 2(e - \hat{\xi}_1), \quad \dot{\hat{\xi}}_2 = \bar{u} + (e - \hat{\xi}_1)$$

Also, Assumption 3.5 and 3.7 can be verified. The overall controller can be designed in the form of (3.58) with $\rho(e) = 1 + e^6$. The simulation is performed with initial conditions $(z_0, x_{10}, x_{20}) = (\ 3,\ 1.2,\ -2.5\ )$ for the plant, $v_0 = (\ -2,\ 1\ )$ for the exosystem. The certain parameters of exosystem are $a = 1.6$, $q = -0.6$, and the uncertain parameters of the plant is chosen to be $w1 = 3.3$, $w2 = -2$, $w3 = 1.2$. The simulation results are shown by Figure 3.4 and 3.5.

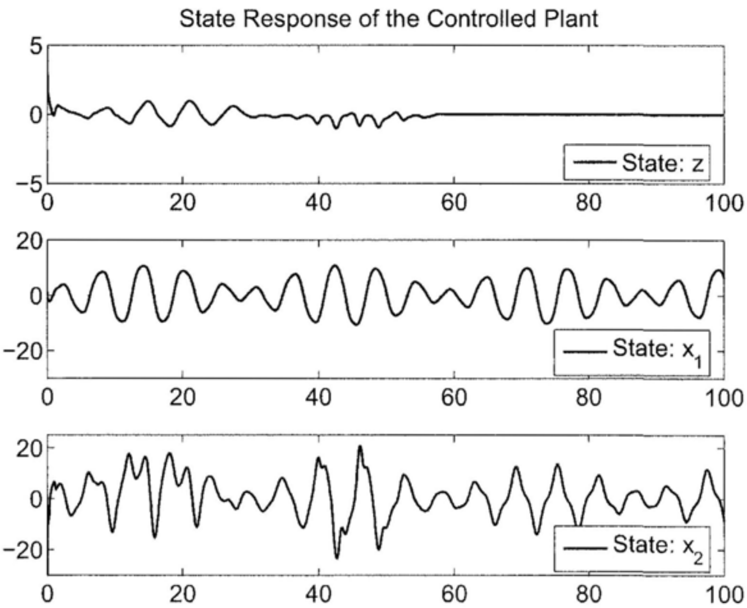

Figure 3.4: State Response of the Closed-loop System

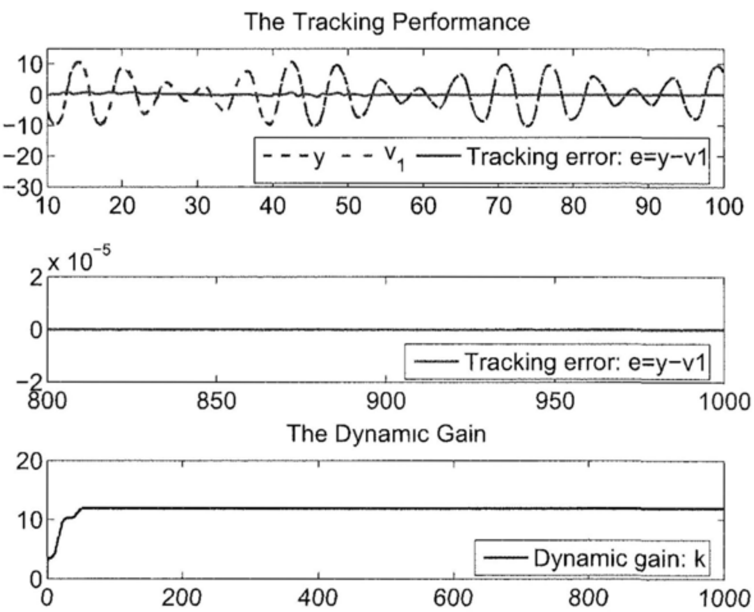

Figure 3.5: Tracking Performance and the Dynamic Gain

3.5 Conclusion

In this chapter, we have presented a framework for handling the robust output regulation problem for time-varying nonlinear systems. This framework can be viewed as an extension of framework proposed in [37]. Due to the time-varying settings, the characterization of the steady-state generator and internal model is modified, and the existence conditions are also given. As the illustration of the framework, we have established the solvability of the problem for the class of time-varying nonlinear systems in the output feedback form. This approach can also be extended to nonlinear plant in the lower triangular form.

□ **End of chapter.**

Chapter 4

Adaptive Output Regulation with Uncertain Time-Varying Exosystems

In this chapter, we will further address the global robust output regulation problem for time-varying nonlinear systems in the presence of uncertain time-varying exosystems. Under the framework introduced in the last chapter, we witness the "robust" issue reflects on the fact that the nonlinear plant is allowed to contain both statical and dynamical uncertainties. And especially, when the time-varying exosystem contains no uncertainties, a class of time-varying internal models is introduced to solve the problem. However, the appearance of uncertainties in the exosystem may jeopardize the control design as shown in the last section, and the former design of the internal model is not practicable.

To cope with the uncertainties of the time-varying exosystem, some modifications will be made based on the aforementioned framework. A generalized internal model will be designed under additional assumptions. By utilizing some adaptive control techniques, we will show that adaptive robust output regulation problem for time-varying nonlinear system is also solvable.

4.1 Problem descriptions and preliminaries

Consider the time-varying nonlinear plant

$$\begin{aligned} \dot{x} &= f(t, x, u, v, w) \\ e &= h(t, x, v, w) \end{aligned} \tag{4.1}$$

The system descriptions follow those given for (3.1).

The exogenous signal $v \in \mathbb{R}^{n_v}$ is supposed to be generated from some uncertain time-varying exosystem. In this scenario, (3.2) becomes

$$\dot{v} = a(t, v, \sigma) \tag{4.2}$$

where $\sigma \in \mathbb{S} \subset \mathbb{R}^{n_\sigma}$ represents the constant uncertain parameters, with $\mathbb{S}$ being some subset of $\mathbb{R}^{n_\sigma}$. It is assumed that $a(t, v, \sigma)$ is sufficiently smooth and satisfies $a(t, 0, \sigma) = 0$. Also, it is assumed that the solution of the time-varying exosystem (4.2) exists and is bounded for all $t \geq t_0 \geq 0$, all initial conditions v_0, and all $\sigma \in \mathbb{S}$.

The control objective of the output regulation problem is to find the dynamic output feedback control law in the form of (3.3), i.e.

$$u = u_K(t, \zeta, e), \quad \dot{\zeta} = g_K(t, \zeta, e) \tag{4.3}$$

such that for any initial time $t_0 \geq 0$, any initial condition (x_0, v_0, ζ_0), and any constant parameters $(w, \sigma) \in \mathbb{W} \times \mathbb{S}$, where w denotes the uncertainties of the plant (4.1), with $\mathbb{W}$ and $\mathbb{S}$ bing some subset of $\mathbb{R}^{n_w}$ and $\mathbb{R}^{n_\sigma}$ respectively,

- the solution of the closed-loop system composed of (4.1), (4.2) and (4.3) exists and is bounded over $[t_0, +\infty)$;
- the regulated error output e uniformly asymptotically approaches zero.

In the past decade, output regulation problem with uncertain exosystem haven been considered. For instance, linear time-invariant plant with uncertain linear time-invariant exosystem has been studied in [77] [78], nonlinear time-invariant plant with uncertain linear time-invariant exosystem has been studied in [19] [66] [84] [101], and linear periodically time-varying plant with linear periodically time-varying exosystem has been studied in [123].

Considering the framework for tackling the output regulation problem for time-varying nonlinear system introduced in the last section, some definitions and assumptions need to be reformulated to account for the occurrence of uncertain parameters σ in the time-varying exosystem. First, the standard assumption is proposed.

Assumption 4.1. *There exist globally defined sufficiently smooth functions* $\mathbf{x}(t, v, w, \sigma)$, $\mathbf{u}(t, v, w, \sigma)$ *with* $\mathbf{x}(t, 0, w, \sigma) = 0$ *and* $\mathbf{u}(t, 0, w, \sigma) = 0$ *for all* $(w, \sigma) \in \mathbb{R}^{n_w} \times \mathbb{R}^{n_\sigma}$ *and all*

$t \geq t_0 \geq 0$, *such that*

$$\begin{aligned} \mathcal{L}_{a(t,v,\sigma)}\mathbf{x}(t,v,w,\sigma) &= f\big(t,\mathbf{x}(t,v,w,\sigma),\mathbf{u}(t,v,w,\sigma),v,w\big) \\ 0 &= h\big(t,\mathbf{x}(t,v,w,\sigma),v,w\big) \end{aligned} \tag{4.4}$$

for all $(t,v,w,\sigma) \in \mathbb{R}^+ \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_\sigma}$, *where* $\mathcal{L}_{a(t,v,\sigma)}\mathbf{x}(t,v,w,\sigma) = \frac{\partial \mathbf{x}(t,v,w,\sigma)}{\partial t} + \frac{\partial \mathbf{x}(t,v,w,\sigma)}{\partial v}a(t,v,\sigma)$ ∎

Equations (4.4) are the regulator equations, and the solutions $\mathbf{x}(t,v,w,\sigma)$ and $\mathbf{u}(t,v,w,\sigma)$ are the steady-state and steady-state input respectively. Since the uncertainties σ appear in either $\mathbf{x}(t,v,w,\sigma)$ or $\mathbf{u}(t,v,w,\sigma)$, the definitions of steady-state input generator and internal model need to be redefined consequently.

Definition 4.1. Steady-State Input Generator

Let $\mathbf{u}(t,v,w,\sigma)$ *be the solution of the regulator equations (4.4). The composite system (4.1) (4.2) is said to have a steady-state input generator if, for some integer* l, *there exists a triple* $\{\vartheta,\alpha,\beta\}$, *where* $\vartheta: \mathbb{R}^+ \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_\mu} \times \mathbb{R}^{n_\sigma} \to \mathbb{R}^l$, $\alpha: \mathbb{R}^+ \times \mathbb{R}^{n_\sigma} \times \mathbb{R}^l \to \mathbb{R}^l$, *and* $\beta: \mathbb{R}^+ \times \mathbb{R}^{n_\sigma} \times \mathbb{R}^l \to \mathbb{R}^m$ *are sufficiently smooth functions satisfying* $\vartheta(t,0,w,\sigma) = 0$ *for all* $t \geq 0$, *such that, the following holds*

$$\begin{aligned} \frac{d\vartheta(t,v,w,\sigma)}{dt} &= \alpha(t,\sigma,\vartheta(t,v,w,\sigma)) \\ \mathbf{u}(t,v,w,\sigma) &= \beta(t,\sigma,\vartheta(t,v,w,\sigma)) \end{aligned} \tag{4.5}$$

for all $v \in \mathbb{R}^n$ *and* $(w,\sigma) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_\sigma}$ ∎

Definition 4.2. Internal Model

Suppose the composite system (4.1) (4.2) has a steady-state input generator (4.5). We call the following system

$$\eta = \gamma(t,\eta,u) \tag{4.6}$$

an internal model with output u *if*

$$\gamma\big(t,\vartheta(t,v,w,\sigma),\mathbf{u}(t,v,w,\sigma)\big) = \alpha\big(t,\sigma,\vartheta(t,v,w,\sigma)\big)$$

∎

Remark 4.1. A distinct feature of the steady-state input generator (4.5) posed here from the one defined by (3.5) is that (4.5) is allowed to contain uncertain parameters σ. Also, it is worth noting that the internal model is a certain dynamical system which is not allowed to contain any uncertainties. These conflicting facts between the steady-state input generator and the internal model require us to reconsider the possible conditions under which the internal model can be constructed from the steady-state input generator, and how the internal model asymptotically reproduces the ideal steady-state input information $\mathbf{u}(t, v, w, \sigma)$. These issues will be discussed in the forthcoming section. ▪

Next we will restate the problem conversion to maintain the integrity of this chapter.

Similar to the procedures introduced in Section 3.2, attaching the internal model (4.6) to the given plant (4.1) yields the augmented system, and performing the coordinate and input transformation

$$\bar{x} = x - \mathbf{x}(t, v, w, \sigma), \quad \bar{\eta} = \eta - \vartheta(t, v, w, \sigma), \quad \bar{u} = u - \beta(t, \eta, \sigma) \tag{4.7}$$

so that the augmented system turns into

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(t, \bar{x}, \bar{\eta}, \bar{u}, \mu) \\ \dot{\bar{\eta}} &= \bar{\gamma}(t, \bar{x}, \bar{\eta}, \bar{u}, \mu) \\ e &= \bar{h}(t, \bar{x}, \bar{\eta}, \mu) \end{aligned} \tag{4.8}$$

where $\mu = (v, w, \sigma)$ and

$$\begin{aligned} \bar{f} &= f(t, x, u, v, w) - f\big(t, \mathbf{x}(t, v, w, \sigma), \mathbf{u}(t, v, w, \sigma), v, w\big) \\ \bar{\gamma} &= \gamma(t, \eta, u) - \alpha(t, \sigma, \vartheta) \\ \bar{h} &= h(t, x, v, w) \end{aligned} \tag{4.9}$$

It can be verified that system (4.8) also satisfies

$$\begin{aligned} 0 &= \bar{f}(t, 0, 0, 0, \mu) \\ 0 &= \bar{\gamma}(t, 0, 0, 0, \mu) \\ 0 &= \bar{h}(t, 0, 0, \mu) \end{aligned} \tag{4.10}$$

This means the origin $(\bar{x}, \bar{\eta}) = (0, 0)$ is the equilibrium point of the unforced augmented system for all trajectories of the exosystem, and, at the origin, the error output e is identically zero. Thus, as mentioned in Remark 3.3, if we can find an output feedback control law of the form

$$\bar{u} = u_S(t, \xi, e), \quad \dot{\xi} = g_S(t, \xi, e) \tag{4.11}$$

where $u_S(t, \xi, e)$ and $g_S(t, \xi, e)$ are sufficiently smooth functions vanishing at $(\xi, e) = (0, 0)$, that globally stabilize the equilibrium point of the augmented system (4.8), then the following control law

$$
\begin{aligned}
u &= \beta(t,\sigma,\eta) + u_S(t,\xi,e) \\
\dot{\eta} &= \gamma(t,\eta,u) \\
\dot{\xi} &= g_S(t,\xi,e)
\end{aligned}
\tag{4.12}
$$

solves the output regulation problem of the original system (4.1) for all trajectories $v \in \mathbb{R}^{n_v}$ and $(w,\sigma) \in \mathbb{R}^{n_w} \times \mathbb{R}^{n_\sigma}$.

Remark 4.2. It is noticeable that control law (4.12) is not directly implementable since the function $\beta(t,\sigma,\eta)$ still contains the unknown parameters σ. Nevertheless, as will be shown in the next section, if the function $\beta(t,\sigma,\eta)$ is linearly parameterized in some unknown parameter vector, it is possible to further introduce some adaptive techniques to solve the problem. Particularly, a new internal model compared with (3.23) will be synthesized in the first place. ▪

4.2 The generalized time-varying internal model

As we emphasized in Remarks 4.1 and 4.2, in the presence of the uncertain exosystem, the existence condition for the internal model should be carefully discussed. In the primary step, the concept and the assumption introduced in Section 3.2 to characterize the steady-state input generator will be modified into the followings

Definition 4.3.
Let $X(t,v,w,\sigma)$ be a smooth function with $v(t)$ generated by the exosystem (4.2). If $X(t,v,w,\sigma)$ satisfies a linear differential equation of the following form

$$
\frac{d^l X}{dt^l} - a_{l-1}(t,\sigma)\frac{d^{(l-1)}X}{dt^{l-1}} - \cdots - a_1(t,\sigma)\frac{dX}{dt} - a_0(t,\sigma)X = 0 \tag{4.13}
$$

where $a_i(t,\sigma)$, $i = 0,1,\ldots,l-1$ are smooth functions of time, then $X(t,v,w,\sigma)$ is said to be in the kernel of the left monic polynomial differential operator of degree l

$$
P(s,t,\sigma) = s^l - a_{l-1}(t,\sigma)s^{l-1} - \cdots - a_1(t,\sigma)s - a_0(t,\sigma)
$$

where $s \overset{def}{=} \frac{d}{dt}(\cdot)$. ▪

Assumption 4.2. *There exists an integer l such that along the trajectory of exosystem (4.2), for all $(t,w,\sigma) \in \mathbb{R}^+ \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_\sigma}$, the function $\mathbf{u}(t,v,w,\sigma)$ is in the kernel of some left monic polynomial differential operator of degree l, i.e.*

$$
\frac{d^l \mathbf{u}}{dt^l} = a_{l-1}(t,\sigma)\frac{d^{l-1}\mathbf{u}}{dt^{l-1}} + \cdots + a_1(t,\sigma)\frac{d\mathbf{u}}{dt} + a_0(t,\sigma)\mathbf{u} \tag{4 14}
$$

where $a_i(t,\sigma)$, $i = 0, \ldots, l-1$ are sufficiently smooth functions of time. ▪

Under Assumption 4.2, we can easily obtain a linear steady-state input generator as follows.

$$\begin{aligned} \frac{d\vartheta(t,v,w,\sigma)}{dt} &= \Phi(t,\sigma)\vartheta(t,v,w,\sigma) \\ \mathbf{u}(t,v,w,\sigma) &= \Gamma\vartheta(t,v,w,\sigma) \end{aligned} \tag{4.15}$$

where

$$\vartheta(t,v,w,\sigma) = \mathrm{col}\big(\ \mathbf{u}(t,v,w,\sigma) \quad \tfrac{d\mathbf{u}(t,v,w,\sigma)}{dt} \quad \cdots \quad \tfrac{d^{l-1}\mathbf{u}(t,v,w,\sigma)}{dt^{l-1}}\ \big)$$

$$\Phi(t,\sigma) = \left[\begin{array}{c|c} 0 & I_{l-1} \\ \hline a_0(t,\sigma) & a_1(t,\sigma) \quad \ldots \quad a_{l-1}(t,\sigma) \end{array}\right], \quad \Gamma = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array}\right]$$

We say that the triple $\{\vartheta, \Phi(t,\sigma), \Gamma\}$ constitutes a linear steady-state input generator.

And once again, since the pair $(\Phi(t,\sigma), \Gamma)$ is in the observability canonical form, there exists a Lyapunov transformation $\tau = N_o^{-1}(t,\sigma)\vartheta$ such that (4.15) can be transformed into the observer canonical form.

$$\begin{aligned} \frac{d\tau(t,v,w,\sigma)}{dt} &= \Phi_o(t,\sigma)\tau(t,v,w,\sigma) \\ \mathbf{u}(t,v,w,\sigma) &= \Gamma_o\tau(t,v,w,\sigma) \end{aligned} \tag{4.16}$$

where

$$\Phi_o(t,\sigma) = \left[\begin{array}{c|c} b_{l-1}(t,\sigma) & \\ \vdots & I_{l-1} \\ \hline b_0(t,\sigma) & 0 \end{array}\right], \quad \Gamma_o = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array}\right]$$

and the coefficients $b_i(t,\sigma)$ can be solved recursively from $a_i(t,\sigma)$, which is similar to (3.21)

$$b_{l-i}(t,\sigma) = \sum_{j=0}^{i-1}(-1)^j C_{l-i+j}^{j} a_{l-i+j}^{(j)}(t,\sigma) \quad i = 1, \cdots, l \tag{4.17}$$

The triple $\{\tau, \Phi_o(t,\sigma), \Gamma_o\}$ of (4.16) is also a steady-state input generator. With the notions of polynomial differential operator introduced in Section 2.3, it is ready to show that $\mathbf{u}(t,v,w,\sigma)$ is in the kernal of some right monic PDO $P_{m.r}(s,t,\sigma)$ of degree l, where

$$P_{m.r}(s,t,\sigma) = s^l - s^{l-1}b_{l-1}(t,\sigma) - \cdots - sb_1(t,\sigma) - b_0(t,\sigma)$$

Here we propose assumption on (4.16).

Assumption 4.3. $b_i(t,\sigma)$, $i = 0, \cdots, l-1$ *are uniformly bounded, analytic functions* ▪

Remark 4.3. We say $b_i(t,\sigma)$ an "analytic" function means $b_i(t,\sigma)$ has its power series expansions valid for all t. These requirements on $b_i(t,\sigma)$ will facilitate the proof of Proposition 4.1. ▪

The next step is to derive a time-varying internal model corresponding to (4.16) To achieve so, we need to introduce some more concepts. Compared with Definition 4.1, we state

Definition 4.4.
Let the triple $\{\vartheta, \alpha, \beta\}$ *be a steady-state input generator of the composite system (4.1)-(4 2). If there exist a continuously differential matrix* $M(t) : \mathbb{R}^+ \to \mathbb{R}^{q\times l}$ *with* $q \geq l$, *and two sufficiently smooth functions* $\hat{\alpha} : \mathbb{R}^+ \times \mathbb{R}^{n_\sigma} \times \mathbb{R}^q \to \mathbb{R}^q$, *and* $\hat{\beta} : \mathbb{R}^+ \times \mathbb{R}^{n_\sigma} \times \mathbb{R}^q \to \mathbb{R}^m$ *such that*

$$\begin{aligned} \dot{M}(t)\vartheta + M(t)\alpha(t,\sigma,\vartheta) &= \hat{\alpha}\big(t,\sigma,M(t)\vartheta\big) \\ \beta(t,\sigma,\vartheta) &= \hat{\beta}\big(t,\sigma,M(t)\vartheta\big) \end{aligned} \tag{4.18}$$

then the transformation $\tau = M(t)\vartheta$ *is called a* generalized Lyapunov transformation *on the triple* $\{\vartheta, \alpha, \beta\}$, *and the triple* $\{\tau, \hat{\alpha}, \hat{\beta}\}$ *is called a steady-state generator of the composite system (4.1) (4.2) obtained from* $\{\vartheta, \alpha, \beta\}$ *through the generalized Lyapunov transformation.* ▪

Remark 4.4. If $q = l$ and $M(t)$ is nonsingular for all $t \in \mathbb{R}$, then the generalized Lyapunov transformation becomes an ordinary one. If both α and β are linear in ϑ, i.e., the steady-state input generator (4.5) takes the form

$$\begin{aligned} \frac{d\vartheta(t,v,w,\sigma)}{dt} &= \Phi(t,\sigma)\vartheta(t,v,w,\sigma) \\ \mathbf{u}(t,v,w,\sigma) &= \Gamma(t,\sigma)\vartheta(t,v,w,\sigma) \end{aligned} \tag{4.19}$$

then (4 18) reduces to

$$\begin{aligned} \dot{M}(t) + M(t)\Phi(t,\sigma) &= \hat{\Phi}(t,\sigma)M(t) \\ \Gamma(t,\sigma) &= \hat{\Gamma}(t,\sigma)M(t) \end{aligned} \tag{4.20}$$

for some continuously differential matrices $\hat{\Phi}(t,\sigma)$, $\hat{\Gamma}(t,\sigma)$. ▪

Remark 4.5. Similar to Remark 3 5, it is can be noted that if a linear steady-state generator of the form (4 19) has the property that there exist continuously differentiable matrices $F(t)$ and $G(t)$ such that

$$\Phi(t,\sigma) = F(t) + G(t)\Gamma(t,\sigma) \tag{4.21}$$

then

$$\dot{\eta} = F(t)\eta + G(t)u \tag{4.22}$$

is an internal model with output u. And if $F(t)$ is a constant Hurwitz matrix, then (4.22) reduces to (3.23) which is the so-called canonical linear internal model. ∎

Recall (4.16) and let $\Phi_o(t,\sigma) = \Phi_b + b(t,\sigma)\Gamma_o$, where

$$\Phi_b = \left[\begin{array}{c|c} 0 & \\ \vdots & I_{l-1} \\ \hline 0 & \cdots \ 0 \end{array}\right], \quad b(t,\sigma) = \begin{pmatrix} b_{l-1}(t,\sigma) \\ \vdots \\ b_0(t,\sigma) \end{pmatrix} \tag{4.23}$$

Let $L_0 = \mathrm{col}(\ l_{l-1},\ \cdots,\ l_0\ )$ be such that $F_o = \Phi_b - L_0\Gamma_o$ is Hurwitz matrix, and let $G_o(t,\sigma) = L_0 + b(t,\sigma)$. It can be verified that $\Phi_o(t,\sigma) = F_o + G_o(t,\sigma)\Gamma_o$.

Thus, by Remark 4.5, if σ is known, the following linear time-varying system

$$\dot{\eta} = F_o\eta + G_o(t,\sigma)u \tag{4.24}$$

is an internal model corresponding to the steady-state input generator (4.16). Till now, the derivation of internal model (4.24) is similar to those given for internal model (3.23).

However, since σ is unknown, (4.24) cannot be an internal model. The following additional assumption on the function $b(t,\sigma)$ is made to guarantee the existence of an appropriate internal model without uncertain parameters σ.

Assumption 4.4. *For each $i = 0, 1, \ldots, l-1$, there exists a smooth vector-valued function $\beta(t) : \mathbb{R} \to \mathbb{R}^{\rho}$ and vector-valued functions $\theta_i(\sigma) : \mathbb{R}^{n_\sigma} \to \mathbb{R}^{\rho}$, such that $b_i(t,\sigma) = \theta_i^T(\sigma)\beta(t)$.* ∎

Assumption 4.4 indicates for each $b_i(t,\sigma)$, the uncertain parameters and the time-varying functions are separable. Under Assumption 4.4, let

$$\Theta(\sigma) = \mathrm{col}(\ \theta_{l-1}^T(\sigma)\ \ \cdots\ \ \theta_0^T(\sigma)\ )$$

Then $b(t,\sigma) = \Theta(\sigma)\beta(t)$ so that $G_o(t,\sigma) = L_0 + \Theta(\sigma)\beta(t)$, i.e. $G_o(t,\sigma)$ is linearly parameterized with respect to the uncertain parameter function $\Theta(\sigma)$.

We will now construct an internal model through a generalized Lyapunov transformation. For this purpose, let $\beta_0(t) = 1$ and denote $L_i(\sigma) = \text{col}(\theta_{(l-1)i}, \cdots, \theta_{0i})$, and $\beta(t) = \text{col}(\beta_1(t), \cdots, \beta_\rho(t))$ where $\beta_i(t)$ is a scalar function, we have

$$G_o(t,\sigma) = L_0\beta_0(t) + L_1(\sigma)\beta_1(t) + \cdots + L_\rho(\sigma)\beta_\rho(t) \tag{4.25}$$

Also, let

$$F_g = \begin{bmatrix} F_o^T & 0 & 0 & 0 \\ 0 & F_o^T & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & F_o^T \end{bmatrix}, \quad G_g(t) = \begin{bmatrix} \Gamma_o^T\beta_0(t) \\ \Gamma_o^T\beta_1(t) \\ \vdots \\ \Gamma_o^T\beta_\rho(t) \end{bmatrix}$$

$$H_g(\sigma) = \begin{bmatrix} L_0^T & L_1^T(\sigma) & \cdots & L_\rho^T(\sigma) \end{bmatrix}$$

$$\hat{\Phi}(t,\sigma) = F_g + G_g(t)H_g(\sigma)$$

We have the following result.

Proposition 4.1. *There exists a continuously differentiable matrix $M(t,\sigma) \in \mathbb{R}^{l(\rho+1)\times l}$ with constant rank l, such that*

$$\begin{aligned} \dot{M}(t,\sigma) + M(t,\sigma)\Phi_o(t,\sigma) &= \big(F_g + G_g(t)H_g(\sigma)\big)M(t,\sigma) \\ \Gamma_o &= H_g(\sigma)M(t,\sigma) \end{aligned} \tag{4.26}$$

■

Proof:

Consider the following system representation with the triple $\big(F_g, G_g(t), H_g(\sigma)\big)$

$$\dot{x} = F_g x + G_g(t)u, \quad y = H_g(\sigma)x \tag{4.27}$$

and

$$\dot{\eta} = F_o\eta + G_o(t,\sigma)u, \quad y = \Gamma_o\eta \tag{4.28}$$

where the triple $\big(F_o, G_o(t,\sigma), \Gamma_o\big)$ is defined by (4.24).

Recall Assumption 4.3, $\big(F_o, G_o(t,\sigma), \Gamma_o\big)$ are all uniformly bounded and analytical functions, then (4.28) is a constant rank system representation [108]. Notice system (4.28) is in the observer canonical form, so it is completely observable, and the designed parameters L_0 can be chosen to ensure it is completely controllable. According to Theorem 24 in [108], (4.28) is the minimal realization of the impulse response matrix $H(t,\tau)$ which is given by

$$H(t,\tau) = \Gamma_o\Phi(t,\tau)G_o(\tau,\sigma), \quad \Phi(t,\tau) = e^{F_o(t-\tau)}$$

By the expression of $G_o(t,\sigma)$ in (4.25), it further shows

$$H(t,\tau) = \Gamma_o\Phi(t,\tau)L_0\beta_0(\tau) + \Gamma_o\Phi(t,\tau)L_1(\sigma)\beta_1(\tau) + \cdots + \Gamma_o\Phi(t,\tau)L_\rho(\sigma)\beta_\rho(\tau) \quad (4.29)$$

since $\beta_i(\cdot)$ are scalar functions, $\Gamma_o\Phi(t,\tau)L_i(\sigma)\beta_i(\tau) = L_i^T(\sigma)\Phi^T(t,\tau)\Gamma_o^T\beta_i(\tau)$, so correspondingly, (4.29) turns into

$$H(t,\tau) = L_0^T\Phi^T(t,\tau)\Gamma_o^T\beta_0(\tau)+L_1^T(\sigma)\Phi^T(t,\tau)\Gamma_o^T\beta_1(\tau)+\cdots+L_\rho^T(\sigma)\Phi^T(t,\tau)\Gamma_o^T\beta_\rho(\tau) \quad (4.30)$$

where $\Phi^T(t,\tau) = e^{F_o^T(t-\tau)}$. Obviously, system (4.27) is a diagonal realization of the impulse response matrix $H(t,\tau)$ given by (4.30).

$\left(F_g, G_g(t), H_g(\sigma)\right)$ are uniformly bounded and analytic as the construction shows, so system (4.27) is also constant rank system representation. By making use of the Canonical Structure Theorem ([108] Theorem23), (4.27) is algebraically equivalent to the system representation with the triple $\left(\bar{F}(t,\sigma), \bar{G}(t,\sigma), \bar{H}(t,\sigma)\right)$ which takes the following form

$$\bar{F}(t,\sigma) = \begin{bmatrix} \bar{F}_{11} & 0 & \bar{F}_{13} \\ \bar{F}_{21} & \bar{F}_{22} & \bar{F}_{23} \\ 0 & 0 & \bar{F}_{33} \end{bmatrix}, \quad \bar{G}(t,\sigma) = \begin{bmatrix} \bar{G}_1 \\ \bar{G}_2 \\ 0 \end{bmatrix}, \quad \bar{H}(t,\sigma) = \begin{bmatrix} \bar{H}_1 & 0 & \bar{H}_3 \end{bmatrix}$$

where $\left(\bar{F}_{11}, \bar{G}_1, \bar{H}_1\right)$ is completely controllable and completely observable, with constant rank and has the same impulse response matrix as $\left(F_g, G_g(t), H_g(\sigma)\right)$.

By saying "algebraically equivalent" it means there is a continuous differentiable matrix $P(t,\sigma)$, with $\det P \neq 0$ for all t such that

$$\begin{aligned} \bar{F}(t,\sigma) &= P(t,\sigma)F_gP^{-1}(t,\sigma) + \dot{P}(t,\sigma)P^{-1}(t,\sigma) \\ \bar{G}(t,\sigma) &= P(t,\sigma)G_g(t) \\ \bar{H}(t,\sigma) &= H_g(\sigma)P^{-1}(t,\sigma) \end{aligned} \quad (4.31)$$

According to [108] Theorem 24 and 25, $\left(\bar{F}_{11}, \bar{G}_1, \bar{H}_1\right)$ is also a minimal realization of impulse response matrix $H(t,\tau)$, so it is algebraically equivalent to the minimal realization $\left(F_o, G_o(t,\sigma), \Gamma_o\right)$, i.e. there exists a continuous differentiable matrix $T_1(t,\sigma)$, with $\det T_1 \neq 0$ for all t such that

$$\bar{F}_{11} = T_1F_oT_1^{-1} + \dot{T}_1T_1^{-1}, \quad \bar{G}_1 = T_1G_o(t,\sigma), \quad \bar{H}_1 = \Gamma_oT_1^{-1}$$

Since $\Phi_o(t,\sigma) = F_o + G_o(t,\sigma)\Gamma_o$, it can be verified that

$$\dot{T}_1 + T_1\Phi_o(t,\sigma) = \left(\bar{F}_{11} + \bar{G}_1\bar{H}_1\right)T_1, \quad \Gamma_o = \bar{H}_1T_1 \quad (4.32)$$

Also, define $T_2(t,\sigma)$ as the solution of the following differential equation

$$T_2 + T_2\Phi_o(t,\sigma) = F_{22}T_2 + \left(\bar{F}_{21} + G_2H_1\right)T_1 \tag{4 33}$$

and further define $T(t,\sigma) = \mathrm{col}\left(T_1(t,\sigma), T_2(t,\sigma), 0\right)$, then by (4 32) and (4 33), $T(t,\sigma)$ is solution of

$$\begin{aligned} T(t,\sigma) + T(t,\sigma)\Phi_o(t,\sigma) &= \left(\bar{F}(t,\sigma) + \bar{G}(t,\sigma)\bar{H}(t,\sigma)\right)T(t,\sigma) \\ \Gamma_o &= \bar{H}(t,\sigma)T(t,\sigma) \end{aligned} \tag{4 34}$$

Define $M(t,\sigma) = P^{-1}(t,\sigma)T(t,\sigma)$, by (4 31) and $T(t,\sigma) = P(t,\sigma)M(t,\sigma)$, equations (4 34) turns to be

$$\begin{aligned} P(t,\sigma)M(t,\sigma) + P(t,\sigma)M(t,\sigma)\Phi_o(t,\sigma) &= P(t,\sigma)\left(F_g + G_g(t)H_g(\sigma)\right)M(t,\sigma) \\ \Gamma_o &= H_g(\sigma)M(t,\sigma) \end{aligned}$$

which shows $M(t,\sigma)$ is the solution of the equations (4 26) ◇

Remark 4.6. The requirements on $b_i(t,\sigma)$ proposed in Assumptions 4 3 and 4 4 are essential for the proof Without the property of constant rank system representation, there exists no algebraical equivalence between different representations of the same impulse response matrix, and the Canonical Structure Theorem will not be applicable In [123], similar results are proved for the case that $\Phi_o(t,\sigma)$ and $G_g(t)$ are periodic time-varying functions ▪

Proposition 4.2. *Let $\zeta = M(t)(t,\sigma)\tau$, then the triple $\{\zeta, \hat{\Phi}(t,\sigma), H_g(\sigma)\}$ is also a steady-state input generator of the composite system (4 1) (4 2) by Remark 4 4 And by Remark 4 5, the following system is an internal model*

$$\eta = F_g\eta + G_g(t)u \tag{4 35}$$

corresponding to the steady-state input generator $\left\{\zeta, \hat{\Phi}(t,\sigma), H_g(\sigma)\right\}$ ▪

Internal model (4 35) is the generalized internal model in comparison with (4 24) It is evident that (4 35) contains no uncertain parameters so that we can use it as part of the overall controller And (4 35) maintains the I/O property of (4 24) since they share the same impulse response matrix, so (4 35) reproduces the information of steady-state input asymptotically by its output $H_g(\sigma)\eta$ It is obvious that $H_g(\sigma)$ contains uncertainties σ, so some adaptive control techniques will be introduced to estimate $H_g(\sigma)$, and by tuning its estimation $\hat{H}_g(\sigma)$, the information of steady-state input can be given by $\hat{H}_g(\sigma)\eta$ As an application of this design methodology, in the next section we will show the solvability for a class of time-varying nonlinear systems

4.3 Adaptive robust output regulation problem

Consider the time-varying nonlinear systems in the output feedback form with unity relative degree (3.24) subject to uncertain time-varying exosystem (4.2), the composite system is

$$\begin{aligned}\dot{z} &= f(t, z, y, v, w)\\ \dot{y} &= g(t, z, y, v, w) + b(w)u\\ \dot{v} &= a(t, v, \sigma)\\ e &= y - q(t, v, w)\end{aligned} \tag{4.36}$$

In this scenario, Assumption 3.4 reduces to

Assumption 4.5. *There exists a globally defined smooth function* $\mathbf{z} : \mathbb{R}^+ \times \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \times \mathbb{R}^{n_\sigma} \to \mathbb{R}^n$ *with* $\mathbf{z}(t, 0, w, \sigma) = 0$ *such that*

$$\mathcal{L}_{a(t,v,\sigma)}\mathbf{z}(t, v, w, \sigma) = f(t, \mathbf{z}(t, v, w, \sigma), q(t, v, w), v, w)$$

for all $(t, v, w, \sigma) \in \mathbb{R}^+ \times \mathbb{R}^{n_v} \times \mathbb{W} \times \mathbb{S}$. ▪

Under Assumption 4.5, we have $\mathbf{y}(t, v, w) = q(t, v, w)$, and

$$\mathbf{u}(t, v, w, \sigma) = b^{-1}(w)\Big(\mathcal{L}_{a(t,v,\sigma)}\mathbf{y}(t, v, w) - g\big(t, \mathbf{z}(t, v, w, \sigma), \mathbf{y}(t, v, w), v, w\big)\Big)$$

Then $\mathbf{z}(t, v, w, \sigma)$, $\mathbf{y}(t, v, w)$, $\mathbf{u}(t, v, w, \sigma)$ are the solution of the regulator equations associated with composite system (4.36).

Further we assume the function $\mathbf{u}(t, v, w, \sigma)$ satisfies Assumption 4.2. Then after some manipulations, we can obtain a generalized internal model in the form of (4.35). Attaching the internal model to the given plant (4.36) and performing the coordinate and input transformation

$$\begin{aligned}\bar{z} &= z - \mathbf{z}(t, v, w, \sigma), \quad e = y - \mathbf{y}(t, v, w)\\ \bar{\eta} &= \eta - \zeta - b^{-1}(w)G_g(t)e, \quad \bar{u} = u - H_g(\sigma)\eta\end{aligned} \tag{4.37}$$

gives

$$\begin{aligned}\dot{\bar{z}} &= \bar{f}(t, \bar{z}, e, \mu)\\ \dot{\bar{\eta}} &= F_g\bar{\eta} + \bar{f}_2(t, \bar{z}, e, \mu)\\ \dot{e} &= \bar{g}_e(t, \bar{z}, \bar{\eta}, e, \mu) + b(w)\bar{u}\end{aligned} \tag{4.38}$$

where $\mu = (v, w, \sigma)$ and

$$\begin{aligned}\bar{f}(t, \bar{z}, e, \mu) &= f(t, \bar{z} + \mathbf{z}, e + q, v, w) - f(t, \mathbf{z}, q, v, w)\\ \bar{g}(t, \bar{z}, e, \mu) &= g(t, \bar{z} + \mathbf{z}, e + q, v, w) - g(t, \mathbf{z}, q, v, w)\\ \bar{f}_2(t, \bar{z}, e, \mu) &= b^{-1}(w)\big(F_g G_g(t)e - \dot{G}_g(t)e - G_g(t)\bar{g}\big)\\ \bar{g}_e(t, \bar{z}, e, \bar{\eta}, \mu) &= \bar{g} + b(w)H_g(\sigma)\bar{\eta} + H_g(\sigma)G_g(t)e\end{aligned}$$

System (4.38) takes the same form as system (3.29) except the control $\bar{u} = u - H_g(\sigma)\eta$ depends on the unknown parameter vector $H_g(\sigma)$. Therefore, we will adopt the adaptive control technique to handle this, and the rest of the control design is similar to those given in Section 3.3.1. The main result is shown as follows.

Theorem 4.1. *Under Assumptions 3.5, 3.6, 4.2 4.5, there exist a smooth function $\rho(e) \geq 1$ and real number $\gamma > 0$ such that the following adaptive control law*

$$\begin{aligned}\dot{\eta} &= F_g\eta + G_g(t)u\\ \dot{\hat{H}} &= -\gamma\eta^T e\\ u &= -k\rho(e)e + \hat{H}\eta\\ \dot{k} &= \rho(e)e^2\end{aligned} \tag{4.39}$$

where $\hat{H}$ denotes the estimation of $H_g(\sigma)$, solves the adaptive robust output regulation problem for the composite system (4.36). ∎

Proof: Under the stated assumptions, denote $Z = \text{col}(\bar{z}, \bar{\eta})$, it can be shown that along the trajectory of $(\bar{z}, \bar{\eta})$ subsystem, there exists a $\mathcal{C}^1$ function $V_1(t, \bar{Z})$ satisfying $\underline{\alpha}_1(\|Z\|) \leq V_1(t, \bar{Z}) \leq \bar{\alpha}_1(\|Z\|)$ for some class $\mathcal{K}_\infty$ function $\underline{\alpha}_1(\cdot)$ and $\bar{\alpha}_1(\cdot)$, such that

$$\dot{V}_1 \leq -\|Z\|^2 + \delta_1\gamma_1(e) \tag{4.40}$$

where δ_1 is some positive number and $\gamma_1(\cdot)$ is some known smooth positive definite function.

Applying the changing supply functions technique to (4.40) shows that, for any smooth function $\Delta(Z) > 0$, there exists a $\mathcal{C}^1$ function V_Z satisfying $\underline{\alpha}_2(\|Z\|) \leq V_Z(t, Z) \leq \bar{\alpha}_2(\|Z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_2(\cdot)$ and $\bar{\alpha}_2(\cdot)$, such that along the trajectory of $(\bar{z}, \bar{\eta})$ subsystem

$$\dot{V}_Z \leq -\Delta(Z)\|Z\|^2 + \delta_e\gamma_e(e)e^2$$

for some positive number δ_e and some known smooth continuous function $\gamma_e(\cdot) \geq 1$.

Let $\tilde{H} = H_g(\sigma) - \hat{H}$ and let

$$V = V_Z + \frac{1}{2}e^2 + \frac{1}{2}b(k - \bar{k})^2 + \frac{b}{2\gamma}\tilde{H}\tilde{H}^T \tag{4.41}$$

where $\bar{k}$ is some positive constant, then it can be shown that along the trajectory of the closed-loop system composed of augmented system (4.38) and the control law (4.39),

$$\begin{aligned} \dot{V} &\leq -\Delta(Z)\|Z\|^2 + \delta_e\gamma_e(e)e^2 + \frac{1}{2}|\bar{g}_e|^2 + \frac{1}{2}e^2 \\ &\quad - bk\rho(e)e^2 + b\dot{k}(k - \bar{k}) - b(\eta^T e + \frac{\dot{\hat{H}}}{\gamma})\tilde{H}^T \\ &\leq -\big(\Delta(Z) - \frac{c_g}{2}\psi_g(Z)\big)\|Z\|^2 + \big(\delta_e\gamma_e(e) + \frac{c_g}{2}\psi_e(e) + \frac{1}{2} - b\bar{k}\rho(e)\big)e^2 \end{aligned} \tag{4.42}$$

By choosing

$$\Delta(Z) \geq 1 + \frac{c_g}{2}\psi_g(Z), \quad \rho(e) \geq \max\big(\gamma_e(e), \psi_e(e), 1\big), \quad \bar{k} \geq b^{-1}\big(\delta_e + \frac{c_g}{2} + \frac{1}{2}\big)$$

it gives

$$\dot{V} \leq -\|Z\|^2 \tag{4.43}$$

From (4.41) and (4.43), the trajectory of the closed-loop system from any initial state is bounded, and by LaSalle-Yoshizawa Theorem $\lim_{t\to\infty} \|Z\| = 0$. The boundedness of e and $\dot{e}$ implies $\dot{k}$ is bounded and uniformly continuous. By Barbalat's lemma, $\lim_{t\to\infty} \dot{k} = 0$, which implies $\lim_{t\to\infty} e = 0$. So control law (4.39) solves the adaptive robust output regulation problem. ◇

4.4 Examples

Consider again the global disturbance rejection problem for the FitzHugh-Nagumo model

$$\begin{aligned} \dot{z}_1 &= -\varepsilon_1 z_1 + \varepsilon_1 e \\ \dot{z}_2 &= -\varepsilon_2 z_2 - \varepsilon_2 e \\ \dot{y} &= y - \frac{1}{3}y^3 - z_1 + z_2 - d(t) + u \\ e &= y \end{aligned} \tag{4.44}$$

The system descriptions are followed those given in Example 3.1, and particularly, $d(t)$ represents the external disturbance generated from the following uncertain periodically time-varying oscillator

$$\begin{aligned}\dot{v}_1 &= \sigma \sin t v_2 \\ \dot{v}_2 &= -\sigma \sin t v_1 \\ d(t) &= r_1 v_1 + r_2 v_2\end{aligned} \tag{4.45}$$

where σ, r_i, $i = 1, 2$ are some unknown constant parameters. Denote $\mathbb{W} = \{(\varepsilon_1, \varepsilon_2, r_1, r_2) | \varepsilon_1 > 0, \varepsilon_2 > 0, r_1 \in \mathbb{R}, r_2 \in \mathbb{R}\}$, and $\mathbb{S} = \{\sigma | \sigma \in \mathbb{R}\}$.

The solution of (4.45) with the initial condition (v_{10}, v_{20}) is

$$\begin{aligned}v_1(t) &= \cos\big(\sigma \cos t\big) v_{10} - \sin\big(\sigma \cos t\big) v_{20} \\ v_2(t) &= \sin\big(\sigma \cos t\big) v_{10} + \cos\big(\sigma \cos t\big) v_{20}\end{aligned}$$

and thus is bounded for all $t \geq t_0 \geq 0$, and all σ. The dynamics of periodical oscillator is given in Figure 4.1 with $v_0 = (\ 0.2,\ \ -1.5\ )$ and $\sigma = 1.5$.

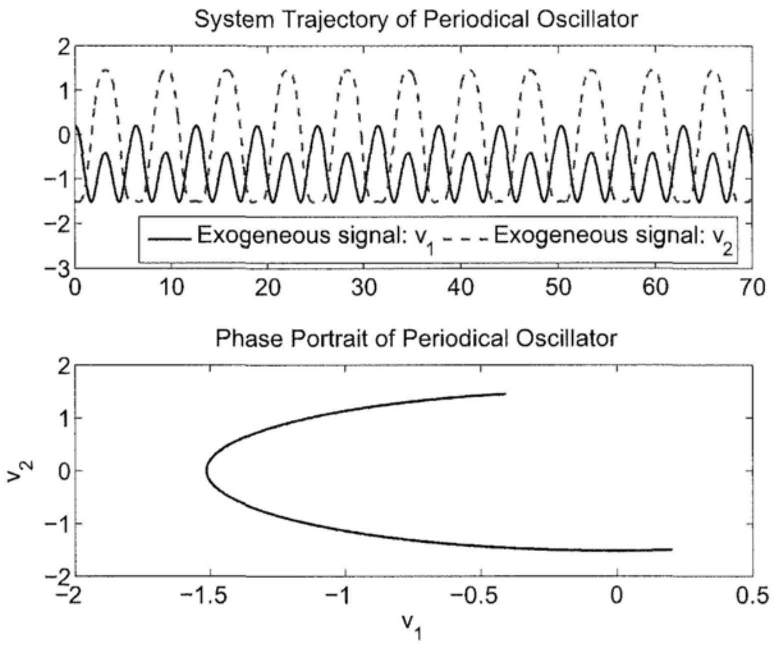

Figure 4.1: Dynamics of Periodical Oscillator

The solution of the corresponding regulator equations is given by $\mathbf{z}_1 = \mathbf{z}_2 = 0$, $\mathbf{y} = 0$ and $\mathbf{u} = r_1 v_1 + r_2 v_2$, so Assumption 4.5 is satisfied. It is ready to verify that $\mathbf{u}$ satisfies

$$\frac{d^3\mathbf{u}}{dt^3} = a_0(t, \sigma)\mathbf{u} + a_1(t, \sigma)\frac{d\mathbf{u}}{dt}$$

where $a_0(t, \sigma) = -3\sigma^2 \sin t \cos t$, $a_1(t, \sigma) = -1 - \sigma^2 \sin^2 t$. Thus, Assumption 4.2 is satisfied.

Consequently, it can be found that a steady-state input generator in the form of (4.16) is with $\Phi_o(t,\sigma) \in \mathbb{R}^{3\times 3}$, and

$$b_2(t,\sigma) = 0, \quad b_1(t,\sigma) = -1 - \sigma^2 \sin^2 t, \quad b_0(t,\sigma) = -\sigma^2 \sin t \cos t$$

Obviously, by denoting $\beta(t) = \text{col}(\ 1,\ \sin^2 t,\ \sin t \cos t\ )$, we have $\theta_2 = 0$, $\theta_1 = \text{col}(\ -1,\ -\sigma^2,\ 0\ )$, $\theta_0 = \text{col}(\ 0,\ 0,\ -\sigma^2\ )$, so Assumption 4.4 is satisfied.

Moreover, it can be verified that other assumptions of Theorem 4.1 also hold. Thus the adaptive robust output regulation problem of system (4.44) is solvable by a control law of the form (4.39) with $\rho(e) = 1 + e^4$, and the internal model constructed in the form of (4.35) with

$$\begin{aligned} F_g &= \text{block diag}(\ F_o^T,\ F_o^T,\ F_o^T,\ F_o^T\ ) \\ G_g(t) &= \text{col}(\ \Gamma_o^T,\ \Gamma_o^T,\ \sin^2 t\Gamma_o^T,\ \sin t \cot t\Gamma_o^T) \\ H_g(\sigma) &= \text{col}\big(\ L_0,\ L_1(\sigma),\ L_2(\sigma),\ L_3(\sigma)\ \big)^T \end{aligned}$$

where

$$\begin{aligned} L_0 &= \text{col}(6,11,6), \quad L_1(\sigma) = \text{col}(0,-1,0) \\ L_2(\sigma) &= \text{col}(0,-\sigma^2,0), \quad L_3(\sigma) = \text{col}(0,0,-\sigma^2) \end{aligned}$$

Some simulation results are provided here with the initial conditions for the plant as $(z_0, y_0) = \big(\ 0.6589, -1.3279, 0.2439\ \big)$, $v_0 = \big(\ 0.2, -0.1\ \big)$ for the exosystem, and 0 for the controller. The designed parameter γ is chosen to be 10. The uncertain parameters are chosen as $\varepsilon_1 = 0.5$, $\varepsilon_2 = 0.5$, $r_1 = 0.3$, $r_2 = -2$ for the plant, and $\sigma = 1.5$ for the exosystem.

Figure 4.2 shows the uncontrolled FitzHugh-Nagumo model may exhibit chaotic behaviors under the disturbance $d(t)$ which is generated from the uncertain periodical oscillator. Figure 4.3 shows by the adaptive control law in the form of (4.39), the global adaptive robust disturbance rejection is achieved.

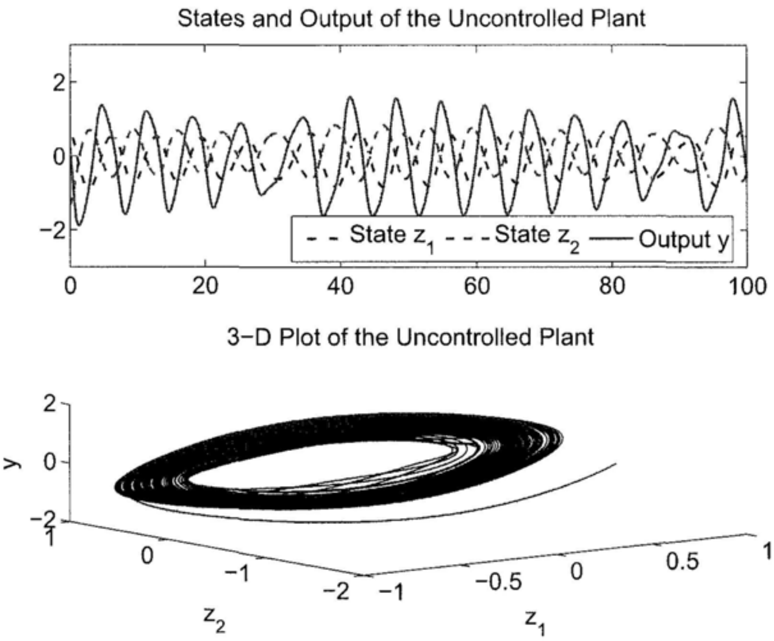

Figure 4.2: Uncontrolled FitzHugh-Nagumo Model under Disturbance

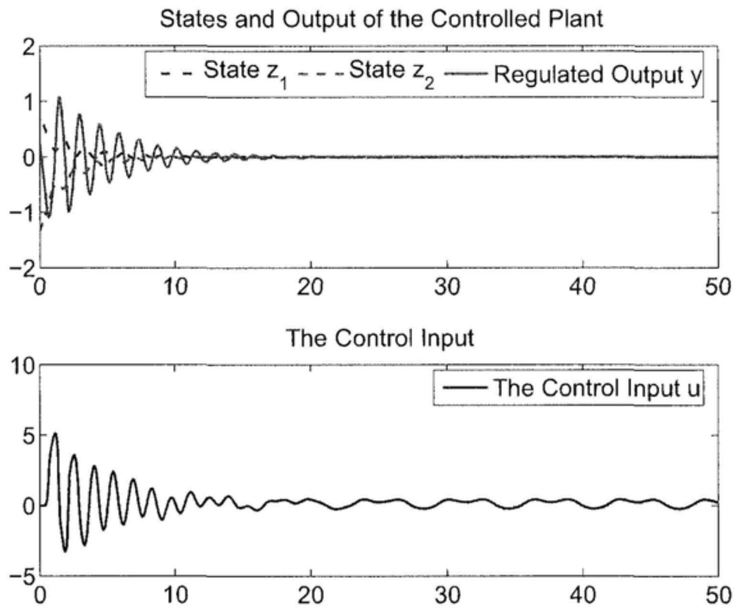

Figure 4.3: Closed-loop Response of FitzHugh-Nagumo Model

4.5 Conclusion

In this chapter, we extend the framework proposed in the last chapter, and further consider the output regulation problem for time-varying nonlinear systems subject to some uncertain time-varying exosystem.

With respect to the uncertain exosystem, a generalized steady-state input generator is introduced in the first place, and a generalized internal model can be constructed correspondingly. The output of the internal model is linearly parameterized in some unknown parameter vector, so combining with some adaptive control techniques we can solve the adaptive robust output regulation problem. As an illustration, we give the solvability of the problem for the time-varying nonlinear system in the output feedback systems with unity relative degree. This design approach can also be extended to the higher relative degree case and lower triangular systems.

□ **End of chapter.**

Chapter 5

Nonlinear Output Regulation with Nonlinear Exosystems I

The global robust output regulation problem for nonlinear plants subject to nonlinear exosystems has been a challenging problem. One of the main difficulties lies in finding a suitable internal model.

In this chapter, we first propose a new class of internal models which are amenable to the output regulation problem of nonlinear systems subject to nonlinear exosystems. Then we utilize this class of internal models to solve the global robust output regulation problem for nonlinear systems in strict output feedback form with a nonlinear exosystem. Both of the theoretical analysis and the numerical examples shows the improvement are achieved compared with the former results.

5.1 Introduction

Consider the robust output regulation problem for the nonlinear plant described by

$$\begin{aligned} \dot{x} &= f(x, u, v, w) \\ e &= h(x, u, v, w) \end{aligned} \tag{5.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $e \in \mathbb{R}^m$ is the error output, $w \in \mathbb{R}^{n_w}$ denotes the uncertain constant parameters of the plant, and $v \in \mathbb{R}^{n_v}$ represents the exogenous signal which is generated by the following autonomous system

$$\dot{v} = a(v) \tag{5.2}$$

All functions in (5.1) and (5.2) are supposed to be globally defined, sufficiently smooth and satisfy $f(0,0,0,w) = 0$, $h(0,0,0,w) = 0$ for all $w \in \mathbb{R}^{n_w}$.

The control objective is to design an output feedback control law, such that for any $w \in \mathbb{R}^{n_w}$, any v_0, and any initial condition of the closed-loop system, the solution of the closed-loop system exsits and is bounded, and the error output approaches zero asymptotically.

It is known that the aforementioned robust output regulation problem can be handled by internal model design. As illustrated in the former chapters, this design methodology consists of two steps. In the first step, a dynamical compensator called internal model is synthesized. The composition of the given plant and the internal model yields the augmented system. The internal model has the property that the stabilization solution of the augmented system will lead to the output regulation solution of the original plant. Thus the second step is to stabilize the augmented system.

The key to the success of this design methodology is the existence of an appropriate internal model, which not only leads to a well defined augmented system but also ensures the stabilizability of the augmented system. Indeed, finding the appropriate internal model has been the central issue in the research of the output regulation problem over the past two decades. When the exosystem is linear, several existence conditions have been given in [5] [33] [35]. In particular, it is shown in [33] that if the solution of the regulator equations associated with the given plant and the exosystem is polynomial, then there exists a linear internal model. An advantage of a linear internal model is that it leads to a simpler augmented system than a nonlinear internal model would.

Nevertheless, when the exosystem is nonlinear, the solvability of the output regulation problem becomes much more complicated for at least two reasons. First, few testable conditions for the existence of the internal model are available even if the solution of the regulator equations is polynomial. Second, the nonlinearity of the exosystem invariably leads to a nonlinear or time-varying internal model. Thus the stabilization of the augmented system becomes less tractable. In our opinion, so far the only testable existence condition for the internal model is given in [14] which leads to an internal model of the form $\dot{\eta} = M(v)\eta + Nu$ where $M(v)$ is some square matrix and N is some column vector. There is no guarantee that the system $\dot{\eta} = M(v)\eta$ is globally asymptotically stable unless $v(t)$ is sufficiently small. This fact complicates the task of the global stabilization of the augmented system. Consequently, in [14], only the local version of the robust output regulation problem has been studied. Recently, using the internal model of [14], some attempts have been made on tackling the global robust output regulation for the class of output feedback systems [20] [112] [117] [125]. However, their results rely on some quite restrictive conditions.

In this chapter, we will propose another class of internal models of the form $\dot{\eta} = M\eta + N(v)u$ where M is some constant Hurwitz matrix and $N(v)$ is some column vector. An existence condition for such internal model is also given. Some example shows that this form of internal model may exist even if the internal model proposed in [14] does not. An advantage of this internal model is that it is zero input globally asymptotically stable, i.e., the linear system $\dot{\eta} = M\eta$ is asymptotically stable. This fact will greatly facilitate the global stabilization of the augmented system associated with the output feedback system. In particular, applying our result to the example in [125] will lead to the global solution of the problem.

5.2 Problem descriptions and preliminaries

In this section, we will summarize the framework for dealing with output regulation problem with nonlinear exosystem in [14]. Some standard assumptions are listed first.

Assumption 5.1. *For any initial condition v_0, the solution of (5.2) exists and is bounded over $t \in [0, \infty)$.* ▪

Assumption 5.2. *There exist globally defined sufficiently smooth functions $\mathbf{x}(v, w)$ and $\mathbf{u}(v, w)$ with $\mathbf{x}(0, w) = 0$ and $\mathbf{u}(0, w) = 0$, such that the following holds*

$$\begin{aligned} \frac{\partial \mathbf{x}(v,w)}{\partial v} a(v) &= f\big(\mathbf{x}(v,w), \mathbf{u}(v,w), v, w\big) \\ 0 &= h\big(\mathbf{x}(v,w), \mathbf{u}(v,w), v, w\big) \end{aligned} \tag{5.3}$$

for all $(v, w) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$. ▪

Assumption 5.3. *There exist three sufficiently smooth functions $\theta(v, w)$, $\alpha(\theta, v)$ and $\beta(\theta, v)$ vanishing at the origin, such that the following holds*

$$\begin{aligned} \frac{d\theta(v,w)}{dt} &= \alpha\big(\theta(v,w), v\big) \\ \mathbf{u}(v,w) &= \beta\big(\theta(v,w), v\big) \end{aligned} \tag{5.4}$$

for all $(v, w) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$. ▪

Remark 5.1. Equations (5.3) are the regulator equations. The solution of the regulator equations provides the necessary steady-state behavior for the controller to achieve asymptotic regulation. However, the solution cannot be directly used by the controller because it depends on the uncertain parameter w. Assumption 5.3 further guarantees that the solution of the regulator equations can be generated by an autonomous system independent of the uncertain parameter w. The triple $\{\theta, \alpha(\theta, v), \beta(\theta, v)\}$ is called a *(generalized) steady-state input generator* of the composite system (5.1)–(5.2). In particular, when both $\alpha(\theta, v)$ and $\beta(\theta, v)$ are linear in θ, i.e. there exist sufficiently smooth matrices $\Phi(v)$ and $\Psi(v)$ such that $\alpha(\theta, v) = \Phi(v)\theta$, $\beta(\theta, v) = \Psi(v)\theta$, (5.4) will reduced to

$$\frac{d\theta(v,w)}{dt} = \Phi(v)\theta(v,w), \quad \mathbf{u}(v,w) = \Psi(v)\theta(v,w) \tag{5.5}$$

which is called a linear steady-state input generator. ▪

The notion of the steady-state input generator will further lead to the definition of the internal model.

Definition 5.1.
Under Assumptions 5.1, 5.2 and 5.3, if there exists a sufficiently smooth function $\gamma(\eta, u, v)$ vanishing at the origin such that

$$\gamma\big(\theta(v,w), \mathbf{u}(v,w), v\big) = \alpha\big(\theta(v,w), v\big)$$

holds for for all $(v, w) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$, then the following system

$$\dot{\eta} = \gamma(\eta, u, v) \tag{5.6}$$

is called a (generalized) internal model *with output u.* ▪

Remark 5.2. The (generalized) internal model has the same asymptotic property as the (generalized) steady-sate generator. Moreover, notice that v appears in the internal model, so generally speaking, the overall controller can be termed as the dynamic output feedback with feedforward controller ([36] Chapter 5). ▪

Attaching the internal model (5.6) to the given plant (5.1) yields the following augmented system

$$\begin{aligned} \dot{x} &= f(x, u, v, w) \\ \dot{\eta} &= \gamma(\eta, u, v) \\ e &= h(x, u, v, w) \end{aligned} \tag{5.7}$$

Performing on (5.7) the following coordinate and input transformation

$$\bar{x} = x - \mathbf{x}(v, w), \quad \bar{u} = u - \beta(\eta, v), \quad \bar{\eta} = \eta - \theta(v, w)$$

gives a new system denoted by

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(\bar{x}, \bar{\eta}, \bar{u}, \mu) \\ \dot{\bar{\eta}} &= \gamma(\bar{x}, \bar{\eta}, \bar{u}, \mu) \\ e &= \bar{h}(\bar{x}, \bar{\eta}, \bar{u}, \mu) \end{aligned} \tag{5.8}$$

where $\mu = (v, w)$. As usual, it is ready to verify that, for all $(v, w) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$,

$$\begin{aligned} 0 &= \bar{f}(0, 0, 0, \mu) \\ 0 &= \gamma(0, 0, 0, \mu) \\ 0 &= \bar{h}(0, 0, 0, \mu) \end{aligned}$$

Consequently, we have the following result.

Proposition 5.1. *If there is a feedback control law*

$$\begin{aligned} \bar{u} &= u_S(\xi, e) \\ \dot{\xi} &= g_S(\xi, e) \end{aligned} \tag{5.9}$$

where $u_S(\xi, e)$ and $g_S(\xi, e)$ are sufficiently smooth functions vanishing at $(\xi, e) = (0, 0)$, such that (5.9) globally stabilizes the equilibrium point of the augmented system (5.8), then the following control law

$$\begin{aligned} u &= \beta(\eta, v) + u_S(\xi, e) \\ \dot{\eta} &= \gamma(\eta, u, v) \\ \dot{\xi} &= g_S(\xi, e) \end{aligned}$$

solves the robust output regulation problem for plant (5.1). ∎

As a result, we have converted the output regulation problem for the plant (5.1) into the stabilization problem for the augmented system (5.8). Once the stabilization problem is solvable, the aforementioned output regulation problem can also be solved.

5.3 On the existence and design of internal model

As we have already seen, the success of the aforementioned method depends on the existence of an appropriate internal model which not only produces the steady-state input information, but also makes the augmented system globally stabilizable as exemplified in the general framework. The existence of the internal model indeed relies in turn on the existence of the steady-state generator.

The existence condition of the internal model was first given in [33] under the assumption that the solution of the regulator equations is a polynomial. Later another condition was given in [5] which requires the solution of the regulator equations to satisfy the following equation

$$\frac{d^l}{dt^l}\mathbf{u}(v,w) = a_0\mathbf{u}(v,w) + a_1\frac{d}{dt}\mathbf{u}(v,w) + \cdots + a_{l-1}\frac{d^{l-1}}{dt^{l-1}}\mathbf{u}(v,w) \tag{5.10}$$

where l is some integer and a_i are some constant real numbers. And it was further shown in [35] that, when the exosystem is linear, condition (5.10) is equivalent to the condition that $\mathbf{u}(v,w)$ is a polynomial, and both of these conditions lead to a linear internal model.

Nevertheless, when the exosystem is nonlinear, condition (5.10) is not equivalent to the condition that $\mathbf{u}(v,w)$ is a polynomial any more. In [14], a more complicated condition for the existence of the steady-state generator is given and is summarized as follows. For convenience, we will assume $m=1$ for the rest of this chapter.

First, we put $a(v)$ in the following form

$$a(v) = A_1 v + \sum_{k=2}^{K} A_k v a_k(v) \tag{5.11}$$

for some integer $K \geq 2$ and some matrices $A_i \in \mathbb{R}^{n_v \times n_v}$. The functions $a_k(v)\mathbb{R}^{n_v} \to \mathbb{R}$ are sufficiently smooth and satisfying $a_k(0)=0$.

Assume $\mathbf{u}(v,w)$ is a polynomial in v. By Lemma 3.1 of [14], there exists a set of real numbers a_i, $i=1,\cdots,r$, such that

$$\mathcal{L}^r_{A_1 v}\mathbf{u} = a_0\mathbf{u} + a_1\mathcal{L}_{A_1 v}\mathbf{u} + \cdots + a_{r-1}\mathcal{L}^{r-1}_{A_1 v}\mathbf{u} \tag{5.12}$$

Moreover, assume there exist some matrices Φ_k satisfying

$$\frac{\partial \vartheta(v,w)}{\partial v} A_k v = \Phi_k \vartheta(v,w) \quad k = 2,\cdots,K \tag{5.13}$$

where $\vartheta(v,w) = \mathrm{col}(\mathbf{u}, \mathcal{L}_{A_1 v}\mathbf{u}, \cdots, \mathcal{L}^{r-1}_{A_1 v}\mathbf{u})$. Then let

$$\Phi = \left[\begin{array}{c|ccc} 0 & & I_{l-1} & \\ \hline a_0 & a_1 & \cdots & a_r \end{array}\right], \quad \Gamma = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array}\right]$$

and $\phi(v) = \sum_{k=2}^{K} \Phi_k a_k(v)$, it can be verified that the following system

$$\frac{d\vartheta}{dt}(v,w) = \big(\Phi + \phi(v)\big)\vartheta(v,w), \quad \mathbf{u} = \Gamma\vartheta(v,w) \tag{5.14}$$

is a (generalized) steady-state input generator with output u. Corresponding to this steady-state input generator, an internal model can be constructed as follows

$$\dot{\eta} = \big(M + T\phi(v)T^{-1}\big)\eta + Nu \tag{5.15}$$

where (M, N) is any controllable pair with $M \in \mathbb{R}^{r\times r}$ being Hurwitz, $N \in \mathbb{R}^{r\times 1}$, and T is the nonsingular solution of the Sylvester equation

$$T\Phi = MT + N\Gamma$$

Such solution T always exists since (Φ, Γ) is observable.

As mentioned in Section 5.1, the result of [14] may have two drawbacks. First, the condition (5.13) may be restrictive as will be shown in Example 5.1. Second, the internal model (5.15) is not zero input asymptotically stable which hinders the global stabilization of the augmented system. In view of these two facts, we generalize condition (5.10) to the following one.

Assumption 5.4. *There exist some integer l and sufficiently smooth scalar functions $a_i(v)$, $i = 1, \cdots, l$, such that*

$$\frac{d^l\mathbf{u}}{dt^l} = a_0(v)\mathbf{u} + a_1(v)\frac{d\mathbf{u}}{dt} + \cdots + a_{l-1}(v)\frac{d^{l-1}\mathbf{u}}{dt^{l-1}} \tag{5.16}$$

where $\mathbf{u}$ stands for $\mathbf{u}(v, w)$. ∎

Under Assumption 5.4, let

$$\vartheta = \mathrm{col}\left(\mathbf{u}, \frac{d\mathbf{u}}{dt}\cdots, \frac{d^{l-1}\mathbf{u}}{dt^{l-1}}\right)$$

$$\Phi(v) = \left[\begin{array}{c|c} 0 & I_{l-1} \\ \hline a_0(v) & a_1(v) \ \cdots \ a_{l-1}(v) \end{array}\right], \quad \Gamma = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array}\right]$$

it can be verified that

$$\frac{d\vartheta}{dt} = \Phi(v)\vartheta(v,w), \quad \mathbf{u} = \Gamma\vartheta(v,w) \tag{5.17}$$

Thus, the triple $\{\vartheta, \Phi(v), \Gamma\}$ constitutes a (generalized) linear steady-state input generator with output u.

We will now show that condition (5.16) may be satisfied by some example when condition (5.13) cannot be satisfied.

Example 5.1. ▪

Consider the following nonlinear plant

$$\begin{aligned} \dot{z} &= -z - z^3 + e \\ \dot{y} &= u + z^2 y - wy + v_2 v_3 \\ e &= y - v_1 \end{aligned}$$

where w is an uncertain constant, and the exogenous signal $v(t)$ is generated by the following nonlinear exosystem

$$\dot{v} = A_1 v + A_2 v a_2(v), \quad \text{where } a_2(v) = v_3,$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The steady-state input is $\mathbf{u}(v, w) = wv_1$, and condition (5.12) holds with $\mathcal{L}^1_{A_1 v}\mathbf{u} = a_0 \mathbf{u}$, $a_0 = 0$, and $\vartheta(v, w) = \mathbf{u}(v, w) = wv_1$. It can be seen that (5.13) holds only if we could find Φ_2 such that

$$\frac{\partial \vartheta(v, w)}{\partial v} A_2 v = \Phi_2 \vartheta(v, w) \quad \longrightarrow \quad wv_2 = \Phi_2 w v_1$$

Obviously, Φ_2 does not exist, so the method introduced in [14] can not proceed.

However, we can see that Assumption 5.4 holds with

$$\frac{d^3 \mathbf{u}}{dt^3} = (-3 v_3 v_4)\mathbf{u} + (-1 - v_3^2)\frac{d\mathbf{u}}{dt}$$

Thus a steady-state input generator of the form (5.17) exists. ◇

What makes Assumption 5.4 more interesting is that it guarantees the existence of an internal model which is zero input globally asymptotically stable as shown by the following result.

Theorem 5.1. *Under Assumptions 5.1, 5.2 and 5.4, given any Hurwitz matrix $M \in \mathbb{R}^{l \times l}$, there exists a column vector $N(v) \in \mathbb{R}^{l \times 1}$ such that the following system*

$$\dot{\eta} = M\eta + N(v)u \tag{5.18}$$

is an internal model with output u for (5.1). ▪

Proof: Let us first show that there exist sufficiently smooth scalar functions $b_0(v)$, $b_1(v)$, $\cdots$, $b_{l-1}(v)$ and an l dimensional sufficiently smooth vector function $\tau(v,w)$ such that

$$\frac{d\tau}{dt} = \Phi_o(v)\tau, \quad \mathbf{u} = \Gamma_o\tau \tag{5.19}$$

where

$$\Phi_o(v) = \left[\begin{array}{c|c} b_{l-1}(v) & \\ \vdots & I_{l-1} \\ \hline b_0(v) & 0 \end{array}\right], \quad \Gamma_o = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array}\right]$$

In other words, system (5.19) is a state space realization of the differential equation (5.16). For this purpose, denote $\tau = [\ \tau_1 \ \cdots \ \tau_l\ ]^T$. Then (5.19) leads to the following sequence of equations

$$\begin{aligned} \mathbf{u} &= \tau_1 \\ \frac{d\mathbf{u}}{dt} &= b_{l-1}(v)\mathbf{u} + \tau_2 \\ \frac{d^2\mathbf{u}}{dt^2} &= \frac{d}{dt}\big(b_{l-1}(v)\mathbf{u}\big) + b_{l-2}(v)\mathbf{u} + \tau_3 \\ &\vdots \\ \frac{d^{l-1}\mathbf{u}}{dt^{l-1}} &= \frac{d^{l-2}}{dt^{l-2}}\big(b_{l-1}(v)\mathbf{u}\big) + \frac{d^{l-3}}{dt^{l-3}}\big(b_{l-2}(v)\mathbf{u}\big) + \cdots + b_1(v)\mathbf{u} + \tau_l \end{aligned} \tag{5.20}$$

and another differential equation in $\mathbf{u}(v,w)$ of order l as follows

$$\frac{d^l\mathbf{u}}{dt^l} = \frac{d^{l-1}}{dt^{l-1}}\big(b_{l-1}(v)\mathbf{u}\big) + \cdots + \frac{d}{dt}\big(b_1(v)\mathbf{u}\big) + b_0(v)\mathbf{u} \tag{5.21}$$

Let $b^{(i)}(v) = \frac{d^i}{dt^i}b(v)$ with $\dot{v} = a(v)$, and C_n^i the number of distinct combinations of order i from n elements. Then we have, for $n = 1, 2, \cdots$,

$$\frac{d^n}{dt^n}\big(b(v)\mathbf{u}\big) = \sum_{i=0}^{n} C_n^i b^{(i)}(v)\frac{d^{n-i}}{dt^{n-i}}\mathbf{u} \tag{5.22}$$

Using (5.22) on the right hand side of (5.21) and matching the coefficients of the ith derivative of $\mathbf{u}(v,w)$ in (5.21) with those of the ith derivative of $\mathbf{u}(v,w)$ in (5.16) gives

$$b_{l-i}(v) = \sum_{j=0}^{i-1}(-1)^j C_{l-i+j}^j a_{l-i+j}^{(j)}(v) \quad i = 1, \cdots, l \tag{5.23}$$

Thus, with $b_i(v)$ given by (5.23) and $\tau = [\ \tau_1 \ \cdots \ \tau_l\ ]^T$ obtained from (5.20), system (5.19) is indeed a state space realization of the differential equation (5.16).

Now note that the matrix $\Phi_o(v)$ can be written in the form of $\Phi_o(v) = \Phi_b + b(v)\Gamma_o$ where

$$\Phi_b = \left[\begin{array}{c|c} 0 & \\ \vdots & I_{l-1} \\ \hline 0 & 0 \end{array}\right], \quad b(v) = \left[\begin{array}{c} b_{l-1}(v) \\ \vdots \\ \hline b_0(v) \end{array}\right]$$

Since (Φ_b, Γ_o) is an observable pair, for any controllable pair (M, N) with $M \in \mathbb{R}^{l\times l}$ being Hurwitz, $N \in \mathbb{R}^{l\times 1}$, the Sylvester equation

$$T\Phi_b = MT + N\Gamma_o$$

admits a unique nonsingular solution T [84].

Let $\theta(v, w) = T\tau(v, w)$, then,

$$\dot{\theta} = T\Phi_o(v)T^{-1}\theta = \Phi_\theta(v)\theta, \quad \mathbf{u} = \Gamma_o T^{-1}\theta = \Psi\theta \tag{5.24}$$

where $\Phi_\theta(v) = T\Phi_o(v)T^{-1}$ and $\Psi = \Gamma_o T^{-1}$, that is, the triple $\{\theta, \Phi_\theta(v), \Psi\}$ constitutes another linear steady-state input generator.

Let $N(v) = N + Tb(v)$. Then we can show (5.18) is the internal model corresponding to (5.24). In fact,

$$\begin{aligned} \dot{\theta} &= M\theta + N(v)\Psi\theta = \Big(M + \big(N + Tb(v)\big)\Gamma_o T^{-1}\Big)\theta \\ &= T\big(\Phi_b + b(v)\Gamma_o\big)T^{-1}\theta = T\Phi_o(v)T^{-1}\theta \end{aligned} \tag{5.25}$$

Thus, by Definition 5.1, (5.18) is a (generalized) internal model with output u corresponding to (5.24). ◇

Remark 5.3. If all $a_i(v)$ are independent of v, $b(v)$ and hence $N(v)$ are also independent of v. Thus the internal model (5.18) reduces to the canonical linear internal model proposed in [84]. What makes (5.18) interesting compared with (5.15) is that the internal model is zero input globally asymptotically stable. ▪

5.4 Global output regulation for strict output feedback systems

In this section, we will apply the aforementioned internal model to solve the global robust output regulation problem for nonlinear systems in strict output feedback form subject to a nonlinear exosystem (5.2).

Consider the following nonlinear plants

$$
\begin{aligned}
\chi &= \bar{F}(w)\chi + \bar{G}(y, v, w) + \bar{D}_1(v, w) + g(w)u \\
y &= \bar{H}(w)\chi + \bar{K}(y, v, w) + \bar{D}_2(v, w) \\
e &= y - q(v, w)
\end{aligned} \tag{5 26}
$$

where $\text{col}(\chi, y) \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}$ is the output, $q(v, w) \in \mathbb{R}$ is the reference trajectory, $e \in \mathbb{R}$ is the error output, $u \in \mathbb{R}$ is the input, $w \in \mathbb{R}^{n_w}$ represents the uncertain constant parameters, and all the functions in (5 26) are supposed to be sufficiently smooth and satisfy $\bar{G}(0, v, w) = 0$, $\bar{K}(0, v, w) = 0$, $\bar{D}_i(0, w) = 0$, $i = 1, 2$, $q(0, w) = 0$

When the exosystem is linear, global robust output regulation problem of the system (5 26) is studied in [15] [100], and when the exosystem is nonlinear, the same problem is considered in [125] However, the solution in [125] can only be obtained for sufficiently small initial condition v_0 Here we will provide a global solution for the same problem as in [125] with arbitrarily large initial condition v_0 For this purpose, let us first make the following assumption

Assumption 5.5. *The system (5 26) has a uniform relative degree $r \geq 2$, i e , for all $w \in \mathbb{R}^{n_u}$, $\bar{H}(w)g(w) = \bar{H}(w)\bar{F}(w)g(w) = \quad = \bar{H}(w)\bar{F}^{r-3}(w)g(w) = 0$ and $\bar{H}(w)\bar{F}^{r-2}(w)g(w) \neq 0$* ▪

Under the above assumption, like in [15] [100] [125], we can attach a dynamic filter to (5 26)

$$
\begin{aligned}
x_i &= -\lambda_i x_i + x_{i+1} \quad i = 1, \quad , r \\
u &= x_{i+1}
\end{aligned} \tag{5 27}
$$

with $\lambda_i > 0$, and perform the following change of coordinate

$$z = \chi - D(w)x - h(w)y$$

on (5 26) and (5 27) to obtain the following extended system

$$
\begin{aligned}
z &= F(w)z + G(y, v, w) + D_1(v, w) \\
y &= H(w)z + K(y, v, w) + D_2(v, w) + b(w)x_1 \\
x_i &= -\lambda_i x_i + x_{i+1} \quad i = 1, \quad , r \\
e &= y - q(v, w), \quad u = x_{r+1}
\end{aligned} \tag{5 28}
$$

where $z \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$ and $x_i \in \mathbb{R}$, and $b(w) = \bar{H}(w)\bar{F}^{r-2}(w)g(w) \neq 0$. Other functions are defined in [125]. It can be seen that if the output regulation problem of the system (5.28) is solvable, the same problem of the system (5.26) is also solvable.

The extended system (5.28) is now in lower triangular form where both error output e and the filtered input x_i are available for feedback design.

Three more assumptions are needed.

Assumption 5.6. *For all $w \in \mathbb{R}^{n_w}$, $F(w)$ is Hurwitz.* ▪

Assumption 5.7. *There exists a globally defined sufficiently smooth function $\mathbf{z}(v,w)$ with $\mathbf{z}(0,w) = 0$ such that the following holds*

$$\frac{\partial \mathbf{z}(v,w)}{\partial v} a(v) = F(w)\mathbf{z}(v,w) + G\big(q(v,w), v, w\big) + D_1(v,w)$$

for all $(v,w) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$. ▪

Under Assumptions 5.5 5.6 5.7, let

$$\begin{aligned}
\mathbf{y}(v,w) &= q(v,w) \\
\mathbf{x}_1(v,w) &= \frac{1}{b(w)}\Big(\mathcal{L}_a \mathbf{y}(v,w) - H(w)\mathbf{z}(v,w) - K\big(\mathbf{y}(v,w), v, w\big) - D_2(v,w)\Big) \\
\mathbf{x}_i(v,w) &= \mathcal{L}_a \mathbf{x}_{i-1}(v,w) + \lambda_{i-1}\mathbf{x}_{i-1}(v,w), \quad i = 2, \cdots, r \\
\mathbf{u}(v,w) &= \mathcal{L}_a \mathbf{x}_r(v,w) + \lambda_r \mathbf{x}_r(v,w)
\end{aligned}$$

where $\mathcal{L}_a \mathbf{y}(v,w)$ and $\mathcal{L}_a \mathbf{x}_i(v,w)$ represent the Lie derivative of $\mathbf{y}(v,w)$ and $\mathbf{x}_i(v,w)$ along $a(v)$, respectively.

Denote $\mathbf{u}(v,w) = \mathbf{x}_{r+1}(v,w)$. Then it can be seen that $\mathbf{z}(v,w)$, $\mathbf{y}(v,w)$, $\mathbf{x}_i(v,w)$, $i = 1, \cdots, r+1$, constitutes the global solution of the regulator equations.

Assumption 5.8. *There exist some integer l and sufficiently smooth scalar functions $a_i(v)$, $i = 1, \cdots, l$ such that*

$$\frac{d^l \mathbf{x}_1}{dt^l} = a_0(v)\mathbf{x}_1 + a_1(v)\frac{d\mathbf{x}_1}{dt} + \cdots + a_{l-1}(v)\frac{d^{l-1}\mathbf{x}_1}{dt^{l-1}}$$

where $\mathbf{x}_1$ stands for $\mathbf{x}_1(v,w)$. ▪

Under Assumptions 5.8, we can find a (generalized) steady-state input generator in the form of (5.25) with x_1 as output as follows.

$$\begin{aligned}
\dot{\theta}(v,w) &= \Phi_\theta(v)\theta(v,w) = \big(M + N(v)\Psi\big)\theta(v,w) \\
\mathbf{x}_1(v,w) &= \Psi\theta(v,w)
\end{aligned} \tag{5.29}$$

Correspondingly, an internal model with output x_1 of the form (5.18) can be constructed as follows.

$$\dot{\eta} = M\eta + N(v)x_1 \tag{5.30}$$

Now let

$$\begin{aligned}
\beta_1(\theta) &= \Psi\theta \\
\beta_2(\theta, v) &= \frac{\partial \beta_1(\theta)}{\partial \theta}\Phi_\theta(v)\theta + \lambda_1\beta_1(\theta) \\
\beta_3(\theta, v) &= \frac{\partial \beta_2(\theta, v)}{\partial \theta}\Phi_\theta(v)\theta + \frac{\partial \beta_2(\theta, v)}{\partial v}a(v) + \lambda_2\beta_2(\theta, v) \\
\beta_i(\theta, v) &= \frac{\partial \beta_{i-1}(\theta, v)}{\partial \theta}\Phi_\theta(v)\theta + \frac{\partial \beta_{i-1}(\theta, v)}{\partial v}a(v) + \lambda_{i-1}\beta_{i-1}(\theta, v) \quad i = 4, \cdots, r+1
\end{aligned}$$

Attaching the internal model (5.30) to the given plant (5.28), and performing the input and coordinate transformation

$$\begin{aligned}
\bar{z} &= z - \mathbf{z}(v, w), \quad e = y - q(v, w) \\
\bar{\eta} &= \eta - \theta(v, w), \quad \bar{x}_1 = x_i - \beta_1(\eta) \\
\bar{x}_i &= x_i - \beta_i(\eta, v) \quad i = 2, \cdots, r+1
\end{aligned} \tag{5.31}$$

leads to the following augmented system

$$\begin{aligned}
\dot{\bar{z}} &= F(w)\bar{z} + \tilde{G}(e, v, w) \\
\dot{\bar{\eta}} &= M\bar{\eta} + N(v)(\Psi\bar{\eta} + \bar{x}_1) \\
\dot{e} &= H(w)\bar{z} + \tilde{K}(e, v, w) + b(w)(\Psi\bar{\eta} + \bar{x}_1) \\
\dot{\bar{x}}_1 &= -\lambda_1\bar{x}_1 + \bar{x}_2 - \frac{\partial \beta_1(\eta)}{\partial \eta}N(v)\bar{x}_1 \\
\dot{\bar{x}}_i &= -\lambda_i\bar{x}_i + \bar{x}_{i+1} - \frac{\partial \beta_i(\eta, v)}{\partial \eta}N(v)\bar{x}_1 \quad i = 2, \cdots, r
\end{aligned} \tag{5.32}$$

where

$$\begin{aligned}
\tilde{G}(e, v, w) &= G\big(e + q(v, w), v, w\big) - G\big(q(v, w), v, w\big) \\
\tilde{K}(e, v, w) &= K\big(e + q(v, w), v, w\big) - K\big(q(v, w), v, w\big)
\end{aligned}$$

Performing another transformation

$$\tilde{\eta} = \bar{\eta} - N(v)b^{-1}(w)e \tag{5.33}$$

on (5.32) turns the augmented system into the following lower triangular form

$$\begin{aligned}
\dot{\bar{z}} &= F(w)\bar{z} + \tilde{G}(e, v, w) \\
\dot{\tilde{\eta}} &= M\tilde{\eta} + f(\bar{z}, e, v, w) \\
\dot{e} &= b(w)\bar{x}_1 + f_0(\bar{z}, \tilde{\eta}, e, v, w) \\
\dot{\bar{x}}_1 &= -\lambda_1\bar{x}_1 + \bar{x}_2 - \frac{\partial\beta_1(\eta)}{\partial\eta}N(v)\bar{x}_1 \\
\dot{\bar{x}}_i &= -\lambda_i\bar{x}_i + \bar{x}_{i+1} - \frac{\partial\beta_i(\eta, v)}{\partial\eta}N(v)\bar{x}_1 \quad i = 2, \cdots, r
\end{aligned} \tag{5.34}$$

where $\bar{u} = \bar{x}_{r+1}$ and

$$\begin{aligned}
f(\bar{z}, e, v, w) &= MN(v)b^{-1}(w)e - N^{(1)}(v)b^{-1}(w)e - N(v)b^{-1}(w)\big(H(w)\bar{z} + \tilde{K}(e, v, w)e\big) \\
f_0(\bar{z}, \tilde{\eta}, e, v, w) &= H(w)\bar{z} + \tilde{K}(e, v, w) + b(w)\Psi\big(\tilde{\eta} + N(v)b^{-1}(w)e\big)
\end{aligned}$$

System (5.34) can be put in the following more standard form

$$\begin{aligned}
\dot{Z} &= F_0(Z, e, v, w) \\
\dot{e} &= f_0(Z, e, v, w) + b(w)\bar{x}_1 \\
\dot{\bar{x}}_i &= f_i(e, v, \bar{x}_1, \bar{x}_i, \eta) + \bar{x}_{i+1} \quad i = 1, \cdots, r
\end{aligned} \tag{5.35}$$

where $Z = \mathrm{col}(\bar{z}, \tilde{\eta})$ and

$$\begin{aligned}
F_0(Z, e, v, w) &= \begin{bmatrix} F(w)\bar{z} + \tilde{G}(e, v, w) \\ M\tilde{\eta} + f(\bar{z}, e, v, w) \end{bmatrix} \\
f_i(e, v, \bar{x}_1, \bar{x}_i, \eta) &= -\big(\lambda_i\bar{x}_i + \frac{\partial\beta_i(\eta, v)}{\partial\eta}N(v)\bar{x}_1\big)
\end{aligned}$$

Since $F(w)$ is Hurwitz for all w due to Assumption 5.6, and M is also a Hurwitz matrix, by Lemma 3.1 of [120], there exists a $\mathcal{C}^1$ function $U(Z)$ satisfying $\underline{\alpha}_Z(\|Z\|) \leq U(Z) \leq \bar{\alpha}_Z(\|Z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_Z(\cdot)$ and $\bar{\alpha}_Z(\cdot)$, such that for any $(v, w) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$, along the trajectory of Z subsystem, the following holds

$$\dot{U}(Z) \leq -\|Z\|^2 + \delta\gamma(e) \tag{5.36}$$

for some known smooth positive definite function $\gamma(\cdot)$ and some positive constant δ depending on w and v_0. As a result, by the standard backstepping technique, it is possible to show that (5.35) can be globally stabilized.

More precisely, let

$$\begin{aligned}
&w_i = \bar{x}_i - \kappa_i, \quad E_i = \frac{\partial \kappa_i}{\partial e}, \quad K_i = \frac{\partial \kappa_i}{\partial k}\dot{k} \\
&\kappa_1 = -k\rho(e)e, \quad \dot{k} = \rho(e)e^2 \\
&\kappa_2 = -2w_1 - f_1 + K_1 - \frac{1}{2}w_1 E_1^2 + \hat{b}E_1\bar{x}_1 \\
&\phi_2 = -w_1 E_1 \bar{x}_1
\end{aligned}$$

and for $i = 2, \cdots, r$, let

$$\begin{aligned}
\kappa_{i+1} &= -w_{i-1} - w_i - f_i + K_i - \frac{1}{2}w_i E_i^2 + \hat{b}E_i\bar{x}_1 + \sum_{j=1}^{i-1}\frac{\partial \kappa_i}{\partial \bar{x}_j}\dot{\bar{x}}_j + \frac{\partial \kappa_i}{\partial \hat{b}}\dot{\hat{b}} + \frac{\partial \kappa_i}{\partial \eta}\dot{\eta} + \frac{\partial \kappa_i}{\partial v}\dot{v} \\
\phi_{i+1} &= -w_i E_i \bar{x}_1 + \phi_i
\end{aligned}$$

where b denotes $b(w)$, $\hat{b}$ denotes the estimation of $b(w)$, $\tilde{b} = b - \hat{b}$. We have the following main result.

Theorem 5.2. *Under Assumptions 5.1 and 5.5 5.8, there exists a control law of the following form*

$$\begin{aligned}
u &= \kappa_{r+1}(e, x_1, \cdots, x_r, \eta, v) + \beta_{r+1}(\eta, v) \\
\dot{\eta} &= M\eta + N(v)x_1 \\
\dot{k} &= \rho(e)e^2 \\
\dot{\hat{b}} &= \phi_{r+1}(e, x_1, \cdots, x_r, \eta, v)
\end{aligned} \tag{5.37}$$

such that the trajectory of the closed-loop system composed of (5.28) and (5.37) exists and is bounded over $t \in [0, \infty)$, and the error output e tends to zero asymptotically. ∎

Proof: Let us first show, under Assumptions 5.1 and 5.5 5.8, there exists a control law of the form

$$\bar{u} = \kappa_{r+1}, \quad \dot{k} = \rho(e)e^2, \quad \dot{\hat{b}} = \phi_{r+1} \tag{5.38}$$

where $\rho(e)$ is a sufficiently smooth function, such that the trajectory of the closed-loop system composed of (5.35) and (5.38) exists and is bounded over $t \in [0, \infty)$, and $\lim_{t\to\infty} e = 0$.

For this purpose, note that, due to (5.36), by changing supply functions technique, given any smooth function $\Delta(Z) > 0$, there exists a $\mathcal{C}^1$ function V_Z satisfying $\underline{\alpha}_Z(\|Z\|) \leq V_Z \leq \bar{\alpha}_Z(\|Z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_Z(\cdot)$ and $\bar{\alpha}_Z(\cdot)$, such that along the trajectory of Z-subsystem of (5.35), the following holds

$$\dot{V}_Z \leq -\Delta(Z)\|Z\|^2 + \bar{\delta}\bar{\gamma}(e)e^2 \tag{5.39}$$

for some unknown positive constant $\bar{\delta}$ and some known smooth function $\bar{\gamma}(e) \geq 1$.

Let $V_e = \frac{1}{2}e^2$. Then

$$\dot{V}_e = e(f_0 + b(w_1 - k\rho(e)e)) \leq -bk\rho(e)e^2 + w_1^2 + \Pi_e, \quad \Pi_e = ef_0 + \frac{1}{4}b^2e^2$$

Let $V_1 = V_e + \frac{1}{2}w_1^2 + \frac{1}{2}\tilde{b}^2$, then

$$\begin{aligned}
\dot{V}_1 &= \dot{V}_e + w_1\dot{w}_1 - \tilde{b}\dot{\hat{b}} \\
&\leq -bk\rho(e)e^2 + w_1^2 + \Pi_e + w_1\big(f_1 + w_2 + \kappa_2 - \dot{\kappa}_1\big) - \tilde{b}\dot{\hat{b}} \\
&= -bk\rho(e)e^2 + w_1^2 + \Pi_e + w_1\big(f_1 + w_2 + \kappa_2 - K_1\big) - w_1E_1(f_0 + b\bar{x}_1) - \tilde{b}\dot{\hat{b}} \\
&\leq -bk\rho(e)e^2 + w_1^2 + \Pi_e + w_1\big(f_1 + w_2 + \kappa_2 - K_1\big) \\
&\quad + \frac{1}{2}(w_1E_1)^2 + \frac{1}{2}f_0^2 - (\hat{b} + \tilde{b})w_1E_1\bar{x}_1 - \tilde{b}\dot{\hat{b}} \\
&= -bk\rho(e)e^2 + \Pi_e + \frac{1}{2}f_0^2 + \tilde{b}\big(-w_1E_1\bar{x}_1 - \dot{\hat{b}}\big) + w_1w_2 \\
&\quad + w_1\big(w_1 + f_1 + \kappa_2 - K_1 + \frac{1}{2}w_1E_1^2 - \hat{b}E_1\bar{x}_1\big)
\end{aligned}$$

using the expressions of κ_2, ϕ_2 gives

$$\dot{V}_1 \leq -bk\rho(e)e^2 - w_1^2 + w_1w_2 + \tilde{b}\big(\phi_2 - \dot{\hat{b}}\big) + \Pi_1, \quad \Pi_1 = \Pi_e + \Pi_0, \ \Pi_0 = \frac{1}{2}f_0^2$$

Let $V_2 = V_1 + \frac{1}{2}w_2^2$, then

$$\begin{aligned}
\dot{V}_2 &= \dot{V}_1 + w_2\dot{w}_2 \\
&\leq -bk\rho(e)e^2 - w_1^2 + w_1w_2 + \tilde{b}\big(\phi_2 - \dot{\hat{b}}\big) + \Pi_1 + w_2(f_2 + w_3 + \kappa_3 - \dot{\kappa}_2)
\end{aligned}$$

since κ_2 depends on $(e, \bar{x}_1, \eta, v, \hat{b}, k)$, the derivative of κ_2 can be expressed as

$$\dot{\kappa}_2 = E_2\dot{e} + \frac{\partial\kappa_2}{\partial\bar{x}_1}\dot{\bar{x}}_1 + \frac{\partial\kappa_2}{\partial\eta}\dot{\eta} + \frac{\partial\kappa_2}{\partial v}\dot{v} + \frac{\partial\kappa_2}{\partial\hat{b}}\dot{\hat{b}} + K_2$$

thus we have

$$\begin{aligned}\dot V_2 &\le -bk\rho(e)e^2 - w_1^2 + w_1w_2 + \tilde b\big(\phi_2 - \dot{\hat b}\big) + \Pi_1 \\ &\quad + w_2\Big(f_2 + w_3 + \kappa_3 - E_2(f_0 + b\bar x_1) - \frac{\partial \kappa_2}{\partial \bar x_1}\dot{\bar x}_1 - \frac{\partial \kappa_2}{\partial \eta}\dot\eta - \frac{\partial \kappa_2}{\partial v}\dot v - \frac{\partial \kappa_2}{\partial \hat b}\dot{\hat b} - K_2\Big) \\ &\le -bk\rho(e)e^2 - w_1^2 + w_1w_2 + \tilde b\big(\phi_2 - \dot{\hat b}\big) + \Pi_1 \\ &\quad + \frac{1}{2}(w_2E_2)^2 + \frac{1}{2}f_0^2 - (\hat b + \tilde b)w_2E_2\bar x_1 \\ &\quad + w_2\Big(f_2 + w_3 + \kappa_3 - \frac{\partial \kappa_2}{\partial \bar x_1}\dot{\bar x}_1 - \frac{\partial \kappa_2}{\partial \eta}\dot\eta - \frac{\partial \kappa_2}{\partial v}\dot v - \frac{\partial \kappa_2}{\partial \hat b}\dot{\hat b} - K_2\Big)\end{aligned}$$

using the expressions of κ_3, ϕ_3 gives

$$\begin{aligned}\dot V_2 &\le -bk\rho(e)e^2 - w_1^2 - w_2^2 + w_2w_3 + \tilde b\big(- w_2E_2\bar x_1 + \phi_2 - \dot{\hat b}\big) + \Pi_2 \\ &= -bk\rho(e)e^2 - \sum_{j=1}^{2} w_j^2 + w_2w_3 + \tilde b\big(\phi_3 - \dot{\hat b}\big) + \Pi_2, \quad \Pi_2 = \Pi_1 + \Pi_0 = \Pi_e + 2\Pi_0\end{aligned}$$

Let $V_i = V_{i-1} + \frac{1}{2}w_i^2$ and repeat the above procedure gives

$$\dot V_i \le -bk\rho(e)e^2 - \sum_{j=1}^{i} w_j^2 + w_iw_{i+1} + \tilde b\big(\phi_{i+1} - \dot{\hat b}\big) + \Pi_i, \quad \Pi_i = \Pi_e + i\Pi_0$$

Set $w_{r+1} = 0$ so that $\kappa_{r+1} = \bar x_{r+1} = \bar u$. Under the control law (5.38), we have

$$\dot V_r \le -bk\rho(e)e^2 - \sum_{j=1}^{r} w_j^2 + \Pi_r \tag{5.40}$$

Next, let $\tilde k = k - \bar k$ with $\bar k$ being a positive constant to be specified later, and define

$$V = V_Z + V_r + \frac{1}{2}b\tilde k^2 \tag{5.41}$$

by (5.39) and (5.40),

$$\begin{aligned}\dot V &\le -\Delta(Z)\|Z\|^2 + \bar\delta\bar\gamma(e)e^2 - bk\rho(e)e^2 - \sum_{j=1}^{r} w_j^2 + \Pi_r + bk\rho(e)e^2 - b\bar k\rho(e)e^2 \\ &= -\Delta(Z)\|Z\|^2 - \sum_{j=1}^{r} w_j^2 + \bar\delta\bar\gamma(e)e^2 - b\bar k\rho(e)e^2 + \Pi_r\end{aligned}$$

Since $\Pi_r = \Pi_e + r\Pi_0 = ef_0 + \frac{1}{4}b^2e^2 + r(\frac{1}{2}f_0^2)$ and $f_0(Z, e, v, w)$ satisfies $f_0(0, 0, v, w) = 0$, using Lemma 7.8 of [36] and Taylor Theorem, it is possible to show that

$$|\Pi_r| \le \phi_Z(Z)\|Z\|^2 + p_1\phi_e(e)e^2$$

where $p_1 > 0$ is a constant, and $\phi_Z(Z)$, $\phi_e(e)$ are some smooth functions with $\phi_e(e)$ known.

Now choosing

$$\rho(e) \geq \max\left(\bar{\gamma}(e), \phi_e(e)\right), \quad \Delta(Z) \geq \phi_Z(Z) + 1, \quad \bar{k} \geq (\bar{\delta} + p_1)/b$$

gives

$$\dot{V} \leq -\|Z\|^2 - \sum_{j=1}^{r} w_j^2$$

This implies the trajectory of the closed-loop system composed of (5.35) and (5.38) is bounded over $t \in [0, \infty)$. Since e and $\dot{e}$ are bounded and $\dot{k} = \rho(e)e^2$, $\ddot{k}$ exists and is bounded, so that $\dot{k}$ is uniformly continuous. By Barbalat's Lemma, $\dot{k} \to 0$ as $t \to \infty$, which implies $e \to 0$ as $t \to \infty$.

The proof of Theorem 5.2 is completed by noting Proposition 5.1. ◇

Remark 5.4. The above control law is good for the case where $b(w) > 0$ for all w. The derivation of the control law is quite similar to that shown in [65]. A similar control law can be obtained for the case where $b(w) < 0$ for all w. The case where the sign of $b(w)$ is unknown can also be handled by introducing the Nussbaum gain technique [82] as detailed in [65]. ▪

Remark 5.5. In the above control law, k is called the dynamic gain [40] [41] which is introduced to account for the case where w and v_0 are arbitrary. If the w and v_0 belong to some known compact subsets, respectively, there is no need to employ the dynamic gain technique. It suffices to use a sufficiently large static gain k determined by the boundaries of the compact subsets. ▪

5.5 Examples

Example 5.2. ▪

Consider the example given in [125].

$$\begin{aligned} \dot{z} &= -z + 2wv_2y + D_1(v, w) \\ \dot{y} &= wz - v_2y + x_1 + D_2(v, w) \\ \dot{x}_1 &= -x_1 + u \\ e &= y - v_1 \end{aligned} \tag{5.42}$$

where

$$D_1(v,w) = v_1^2 + 2(1-w)v_1v_2$$
$$D_2(v,w) = v_1v_2 + v_2 - wv_1^2 - v_1$$

The exosystem is Van der Pol oscillator

$$\dot{v}_1 = v_2$$
$$\dot{v}_2 = -v_1 + (1-v_1^2)v_2$$

System (5.42) is already in the form (5.26). A state feedback control law was given in [125] that can solve the output regulation problem of (5.26) under the assumption that $|v_1| < 2.7072$ and $|w| < 1$. Here, using our approach, we will solve the output regulation problem of (5.42) for any w and any v.

As derived in [125], the solution of the regulator equations is

$$\mathbf{z}(v,w) = v_1^2, \quad \mathbf{y}(v,w) = v_1, \quad \mathbf{x}_1(v,w) = v_1, \quad \mathbf{u}(v,w) = v_1 + v_2$$

Since $\mathbf{x}_1(v,w) = v_1$, it can be verified that $\mathbf{x}_1(v,w)$ satisfies

$$\frac{d^2}{dt^2}\mathbf{x}_1 = (-1 - v_1v_2)\mathbf{x}_1 + \frac{d}{dt}\mathbf{x}_1 \tag{5.43}$$

So Assumption 5.8 is satisfied.

The steady-state generator in the form (5.19) can be obtained with $\Gamma_o = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and

$$\Phi_0 = \begin{bmatrix} 1 & 1 \\ -1 - v_1v_2 & 0 \end{bmatrix}$$

which leads to the following internal model

$$\dot{\eta} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}\eta + \begin{bmatrix} 3 \\ -v_1v_2 \end{bmatrix}x_1 \tag{5.44}$$

Attaching (5.44) to the plant (5.42), and performing the transformation (5.31) and (5.33) gives the following augmented system

$$\dot{\bar{z}} = -\bar{z} + 2wv_2e$$
$$\dot{\tilde{\eta}} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}\tilde{\eta} + f(\bar{z}, e, v, w)$$
$$\dot{e} = \bar{x}_1 + f_0(\bar{z}, \tilde{\eta}, e, v, w)$$
$$\dot{\bar{x}}_1 = -4\bar{x}_1 + \bar{u}$$

where

$$f(\bar{z}, e, v, w) = \begin{bmatrix} -6 - v_1 v_2 + 3v_2 \\ -3 - v_1^2 + v_2^2 - 2v_1 v_2 - v_1^3 v_2 \end{bmatrix} e - \begin{bmatrix} 3w \\ -wv_1 v_2 \end{bmatrix} \bar{z}$$

$$f_0(\bar{z}, \tilde{\eta}, e, v, w) = w\bar{z} - v_2 e + \tilde{\eta}_1 + 3e$$

By Theorem 5.2, a control law of the form (5.37) can be obtained as follows

$$\begin{aligned} u &= \kappa_2 + \beta_2(\eta, v) \\ \dot{\eta} &= M\eta + N(v)x_1 \\ \dot{k} &= \rho(e)e^2 \\ \beta_2(\eta, v) &= 2\eta_1 + \eta_2 \end{aligned}$$

with

$$\begin{aligned} w_1 &= \bar{x}_1 - \kappa_1, \quad E_1 = \frac{\partial \kappa_1}{\partial e}, \quad K_1 = \frac{\partial \kappa_1}{\partial k}\dot{k} \\ \kappa_1 &= -k\rho(e)e \\ \kappa_2 &= -2w_1 + 4\bar{x}_1 + K_1 - \frac{1}{2}w_1 E_1^2 + E_1 \bar{x}_1 \\ \dot{k} &= \rho(e)e^2 \end{aligned}$$

where $\rho(e) = 1 + e^4$.

Simulation is conducted with the initial conditions for the plant being $(z_0, y_0, x_{10}) = (0.5, 0.2, -0.5)$ (as used in [125]), and the initial conditions for the controller being zero. Note that, unlike [125], where $|v_1| < 2.7072$ and $|w| \leq 1$ are required, our control law works for any initial condition v_0 and any w. Figure 5.1 5.3 show the simulation results with $v_0 = (4, -4)$, $w = 1.5$.

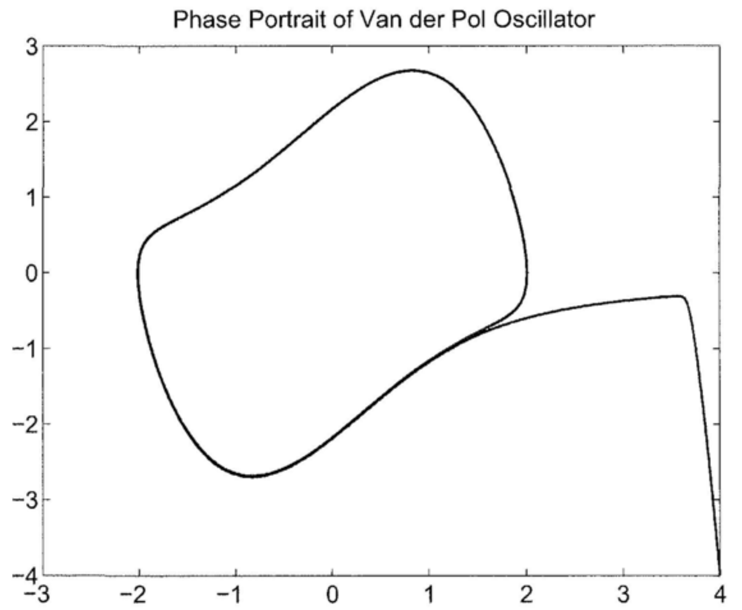

Figure 5.1: Phase Portrait of Van der Pol Oscillator

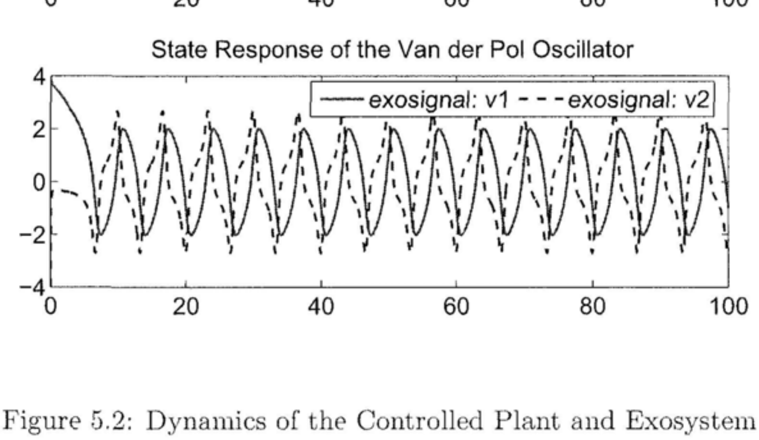

Figure 5.2: Dynamics of the Controlled Plant and Exosystem

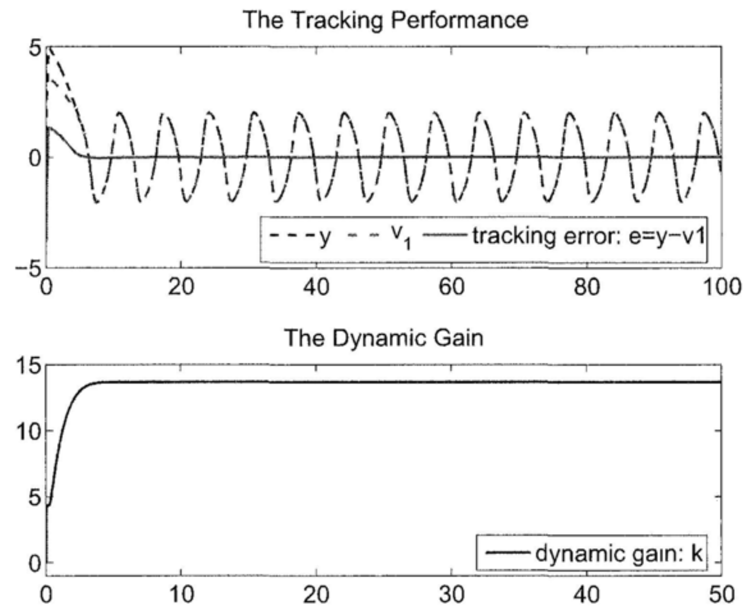

Figure 5.3: Tracking Performance and the Dynamic Gain

5.6 Conclusion

In this chapter, we have first proposed a new class of internal models for the output regulation problem for nonlinear systems with nonlinear exosystems. An advantage of this class of the internal models over the existing ones is that it reduces to a linear time-invariant stable system which is globally asymptotically stable when the input is set to zero. This advantage has been taken to give a complete solution to the global robust output regulation problem for nonlinear systems in strict output feedback form with a nonlinear exosystem.

□ **End of chapter.**

Chapter 6

Nonlinear Output Regulation with Nonlinear Exosystems II

In this chapter, we will further consider the global robust output regulation problem for nonlinear systems in general output feedback form with a nonlinear exosystem.

6.1 Problem descriptions and preliminaries

Consider the global robust output regulation problem for the following nonlinear system

$$
\begin{aligned}
\dot{z} &= f(z, y, v, w) \\
\dot{x}_i &= x_{i+1} + g_i(z, y, v, w) \quad i = 1, \cdots, r-1 \\
\dot{x}_r &= b(w)u + g_r(z, y, v, w) \\
y &= x_1, \quad e = y - q(v, w)
\end{aligned}
\tag{6.1}
$$

with a nonlinear exosystem in the form of (5.2), i.e.

$$\dot{v} = a(v) \tag{6.2}$$

where $z \in \mathbb{R}^{n-r}$, $x = \mathrm{col}(x_1, \cdots, x_r) \in \mathbb{R}^r$ are the states, $e \in \mathbb{R}$ is the error output. $w \in \mathbb{R}^{n_w}$ represents the uncertain parameters of the plant, $b(w) > 0$ and the relative degree $r \geq 2$. It is supposed that the state trajectory $v(t)$ generated by (6.2) exists and is bounded over $t \in [0, \infty)$. The functions $f(z, y, v, w)$ and $g_i(z, y, v, w)$ are globally defined sufficiently smooth and satisfying $f(0, 0, 0, w) = 0$, $g_i(0, 0, 0, w) = 0$ for all $w \in \mathbb{R}^{n_w}$.

The control objective is to design an output feedback control law, such that, for any $w \in \mathbb{R}^{n_w}$, any v_0, and any initial condition of the closed-loop system, the solution of the closed-loop system exists and is bounded, and the error output approaches zero asymptotically.

Remark 6.1. A well-known "output feedback form" of nonlinear systems is given by [62] [74]

$$\begin{aligned} \dot{\zeta} &= A_o\zeta + bu + \theta^T(w)\varphi(y) \\ y &= C_o\zeta \end{aligned} \tag{6.3}$$

where

$$A_o = \left[\begin{array}{c|c} 0 & \\ \vdots & I_{n-1} \\ \hline 0 & \cdots\ 0 \end{array}\right], \quad C_o = \left[\begin{array}{c|c} 1 & 0\ \cdots\ 0 \end{array}\right]$$

$$b = \text{col}(\ 0, \cdots, 0, b_m, \cdots, b_1, b_0\ ), \quad r = n - m$$

As shown in [52], system (6.3) can be transformed into

$$\begin{aligned} \dot{z} &= F(w)z + g_0(y, w) \\ \dot{x}_i &= x_2 + g_i(y, w) \quad i = 1, \cdots, r-1 \\ \dot{x}_r &= b_m u + z_1 + g_r(y, w) \\ y &= x_1 \end{aligned} \tag{6.4}$$

Obviously, system (6.4) is a subclass of system (6.1), so is (6.3). Especially, when (6.3) is minimum phase, i.e., $b_m s^m + \cdots + b_1 s + b_0$ is a Hurwitz polynomial, the corresponding system (6.4) is minimum phase too, i.e., the $F(w) \in \mathbb{R}^{m\times m}$ is a Hurwitz matrix.

The output feedback system (5.26) we studied in the last chapter can also be transformed into (6.3) by a (global) change of coordinates under certain conditions given in [75] [76]. This implies (5.26) is also a subclass of system (6.1).

So we call system (6.1) the *general output feedback system*, and comparatively, system (5.26) or (6.3) are termed as *strict output feedback system*.

The output regulation problem for strict output feedback system with a linear exosystem has been considered in numerous papers, e.g. [15] [19] [100]. In these researches, it is always assumed that the zero dynamics of the corresponding system is a linear stable system, i.e. by assuming that the matrix $F(w)$ in (6.4) is Hurwitz for all w. However, the zero dynamics of system (6.1) is not linear in z. ▪

Remark 6.2. As shown in Section 5.4, the structural property of (5.26) leads us to a dynamic filter based approach. By extending the given plant (5.26) with a input filter, an *extended system* (5.28) is achieved. Attaching the internal model to the extended system gives the *augmented extended system*, and thus problem conversion is achieved. It is remarkable that the designed internal model is with output x_1, which is a filtered input. However, such design has some drawbacks. It can be seen that, if the steady-state input $\mathbf{u}(v,w)$ is a polynomial in v and the exosystem is linear time-invariant, it is quite conspicuous that $\mathbf{x}_1(v,w)$ is also a polynomial in v due to the linear structure of the input filter. But when the nonlinear exosystem occurs, it may be difficult to figure out the filtered input in most cases, thus the internal model with output x_1 is not accessible. We will illustrate this fact by the following example. ▪

Example 6.1. ▪

Consider the following nonlinear system in the form of (5.26) with relative degree 2,

$$\begin{aligned} \dot{x} &= x - 2y + 10u + wv_1y + (9v_1 - 10wv_1^2 - v_1^2v_2) \\ \dot{y} &= x - 2y + (9v_2 + 20v_1) \\ e &= y - 10v_1 \end{aligned} \tag{6.5}$$

the exosystem is Van der Pol oscillator

$$\dot{v}_1 = v_2, \quad \dot{v}_2 = -v_1 + (1 - v_1^2)v_2$$

For further convenience, we denote the solution of the corresponding regulator equations by $\mathbf{x} = v_2$, $\mathbf{y} = 10v_1$ and $\mathbf{u} = v_1$.

Attaching the linear input-filter $\dot{\xi}_1 = -\xi_1 + u$ to the given plant and applying the coordinate transformation $z = x - 10\xi_1 - 2y$ gives the extended system

$$\begin{aligned} \dot{z} &= -z + wv_1y + (-31v_1 - 18v_2 - 10wv_1^2 - v_1^2v_2) \\ \dot{y} &= z + 10\xi_1 + (9v_2 + 20v_1) \\ \dot{\xi}_1 &= -\xi_1 + u \\ e &= y - 10v_1 \end{aligned} \tag{6.6}$$

However, unlike (6.5), now it is difficult to solve the corresponding regulator equations with respect to (6.6), especially along the trajectory of Van der Pol oscillator, we don't have the solution $\Xi_1(v,w)$ in a compact form corresponding to

$$\dot{\Xi}_1(v,w) = -\Xi_1(v,w) + \mathbf{u}(v,w) = -\Xi_1(v,w) + v_1$$

So the internal model (5.30) is not applicable.

Alternatively, if we perform the coordinate transformation

$$y_1 = y, \quad y_2 = y - x$$

it can be seen that (6.5) turns into

$$\begin{aligned} \dot{y}_1 &= -y_1 - y_2 + (9v_2 + 20v_1) \\ \dot{y}_2 &= -10u - wv_1y_1 + (11v_1 + 9v_2 + 10wv_1^2 + v_1^2v_2) \\ e &= y_1 - 10v_1 \end{aligned} \tag{6.7}$$

it is evident that system (6.7) is in the form of (6.1) and the internal model (5.18) with output u can be designed, and the output regulation problem for the original plant (6.5) is still solvable based on the design method introduced later. ◇

Remark 6.3. Inspired by the idea introduced in [38], we aim to use an alternative approach to solve the problem. Generally speaking, we design an internal model with output u in the first place, thus we have an augmented system after the input and coordinate transformation. Next we extend the augmented system with some observer-based dynamics so we have the *extended augmented system*, and finally we solve the global robust stabilization problem for the extended augmented system. ▪

To solve the output regulation problem for the general output feedback system (6.1), some standard assumptions are also in need.

Assumption 6.1. *There exists a globally defined smooth function* $\mathbf{z}(v, w)$ *with* $\mathbf{z}(0, w) = 0$ *such that the following holds*

$$\dot{\mathbf{x}}(v, w) = f\big(\mathbf{z}(v, w), q(v, w), v, w\big)$$

for all $(v, w) \in \mathbb{R}^{n_v} \times \mathbb{W}$. ▪

Assumption 6.2. *There exist some integer* l *and sufficiently smooth scalar functions* $a_i(v)$, $i = 1, \cdots, l$, *such that*

$$\frac{d^l\mathbf{u}}{dt^l} = a_0(v)\mathbf{u} + a_1(v)\frac{d\mathbf{u}}{dt} + \cdots + a_{l-1}(v)\frac{d^{l-1}\mathbf{u}}{dt^{l-1}} \tag{6.8}$$

where $\mathbf{u}$ *stands for* $\mathbf{u}(v, w)$. ▪

And as shown before, due to the structure of system (6.1), the satisfaction of Assumption 6.1 ensures the solution of the regulator equations exist

$$\begin{aligned}
&\mathbf{z}(v,w), \quad \mathbf{x}_1(v,w) = q(v,w)\\
&\mathbf{x}_{i+1}(v,w) = \dot{\mathbf{x}}_i(v,w) - g_i\big(\mathbf{z}(v,w), q(v,w), v, w\big), \quad i = 1, \cdot, r-1\\
&\mathbf{u}(v,w) = b^{-1}(w)\Big(\dot{\mathbf{x}}_r(v,w) - g_r\big(\mathbf{z}(v,w), q(v,w), v, w\big)\Big)
\end{aligned}$$

Also the satisfaction of Assumption 6.2 ensures the internal model in the form of (5.18) exist as $\dot{\eta} = M\eta + N(v)u$.

6.2 Global output regulation for general output feedback systems

In this section, we further pursue the global robust output regulation problem for the general output feedback nonlinear systems.

Attaching the internal model to the given plant (6.1), and performing the input and coordinate transformations

$$\begin{aligned}
&\bar{z} = z - \mathbf{z}(v,w), \quad \bar{x}_i = x_i - \mathbf{x}_i(v,w),\\
&\bar{u} = u - \Psi\eta, \quad \hat{\eta} = \eta - \theta(v,w)
\end{aligned} \tag{6.9}$$

where Ψ and θ are defined in (5.25), we have the following augmented system

$$\begin{aligned}
\dot{\bar{z}} &= \bar{f}(\bar{z}, e, v, w)\\
\dot{\bar{x}}_i &= \bar{x}_{i+1} + \bar{g}_i(\bar{z}, e, v, w), \quad i = 1, \cdots, r-1\\
\dot{\bar{x}}_r &= (\bar{u} + \Psi\hat{\eta}) + \bar{g}_r(\bar{z}, e, v, w)\\
\dot{\hat{\eta}} &= M\hat{\eta} + N(v)(\bar{u} + \Psi\hat{\eta})
\end{aligned} \tag{6.10}$$

where $\bar{x}_1 = e$ and

$$\begin{aligned}
\bar{f}(\cdot) &= f\big(\bar{z} + \mathbf{z}, e + q(v,w), w\big) - f\big(\mathbf{z}, q(v,w), w\big)\\
\bar{g}_i(\cdot) &= g_i\big(\bar{z} + \mathbf{z}, e + q(v,w), w\big) - g_i\big(\mathbf{z}, q(v,w), w\big)
\end{aligned}$$

It has been shown that the stabilization of augmented system (6.10) will lead to solvability of the output regulation problem of the original plant (6.1).

As the method shown in Section 3.3.2, we could perform a coordinate transformation on $\hat{\eta}$,

$$\bar{\eta} = \hat{\eta} - c_r(v)\bar{x}_r - \cdots - c_1(v)\bar{x}_1 \tag{6.11}$$

where $c_r(v) = b^{-1}(w)N(v)$, $c_i(v) = Mc_{i+1}(v) - c_{i+1}^{(1)}(v)$, $i = r-1, r-2, \cdots, 1, 0$, so that system (6.10) can be transformed into the following form

$$\begin{aligned}
\dot{\bar{z}} &= \bar{f}(\bar{z}, e, v, w) \\
\dot{\bar{\eta}} &= M\bar{\eta} + \bar{g}_0(\bar{z}, e, v, w) \\
\dot{\bar{x}}_i &= \bar{x}_{i+1} + \bar{g}_i(\bar{z}, e, v, w) \quad i = 1, \cdots, r-1 \\
\dot{\bar{x}}_r &= b(w)\Big(\bar{u} + \Psi\big(\bar{\eta} + \sum_{i=1}^{r} c_i(v)\bar{x}_i\big)\Big) + \bar{g}_r(\bar{z}, e, v, w)
\end{aligned} \tag{6.12}$$

where $\bar{g}_0(\bar{z}, e, v, w) = c_0(v)\bar{x}_1 - \sum_{i=1}^{r} c_i(v)\bar{g}_i(\bar{z}, e, v, w)$ and $\bar{x}_1 = e$.

For system (6.12), denote $s_i(v) = b(w)\Psi c_i(v)$ and $\bar{g}' = \text{col}\big(\bar{g}_1(\cdot), \cdots, \bar{g}_{r-1}(\cdot)\big)$, then the $\bar{x}$ subsystem can be expressed in the following compact form

$$\dot{\bar{x}} = \left[\begin{array}{c|c} 0 & I_{r-1} \\ \hline s_1(v) & \cdots \; s_r(v) \end{array}\right]\bar{x} + \left[\begin{array}{c} 0 \\ \hline b(w) \end{array}\right]\bar{u} + \left[\begin{array}{c} \bar{g}'(\bar{z}, e, v, w) \\ \hline \bar{g}_r(\bar{z}, e, v, w) + b(w)\Psi\bar{\eta} \end{array}\right] \tag{6.13}$$

with $\bar{x}_1 = e$. Notice the linear part of (6.13) is in the form of (5.17), thus in light of the transformation between (5.17) and (5.19), we can show that, there exists a well-defined coordinate transformation denoted by $\xi = T_2(v, w)\bar{x}$, such that system (6.12) can be transformed into the following form

$$\begin{aligned}
\dot{\bar{z}} &= \bar{f}(\bar{z}, e, v, w) \\
\dot{\bar{\eta}} &= M\bar{\eta} + \bar{g}_0(\bar{z}, e, v, w) \\
\dot{\xi}_i &= d_{r-i+1}(v)\xi_1 + \xi_{i+1} + \tilde{g}_i(\bar{z}, e, v, w) \quad i = 1, \cdots, r-1 \\
\dot{\xi}_r &= d_1(v)\xi_1 + \bar{u} + \tilde{g}_r(\bar{z}, e, v, w) + \Psi\bar{\eta}
\end{aligned}$$

where $\tilde{g}_i(\cdot)$ corresponds to the nonlinear terms $\bar{g}_i(\cdot)$ after transformation. And especially, $\xi_1 = b^{-1}(w)\bar{x}_1 = b^{-1}(w)e$, and $\tilde{g}_1(\bar{z}, e, v, w) = b^{-1}(w)\bar{g}_1(\bar{z}, e, v, w)$.

The above system can be formed into the following compact form

$$\begin{aligned}
\dot{\bar{z}} &= \bar{f}(\bar{z}, e, v, w) \\
\dot{\bar{\eta}} &= M\bar{\eta} + G_0(\bar{z}, e, v, w) \\
\dot{\xi}_i &= \xi_{i+1} + G_i(\bar{z}, e, v, w) \quad i = 1, \cdots, r-1 \\
\dot{\xi}_r &= \bar{u} + G_r(\bar{z}, e, v, w) + \Psi\bar{\eta}
\end{aligned} \tag{6.14}$$

where $G_0(\bar{z}, e, v, w) = \bar{g}_0(\bar{z}, e, v, w)$ and $G_i(\bar{z}, e, v, w) = d_{r-i+1}(v)\xi_1 + \tilde{g}_i(\bar{z}, e, v, w)$. Especially, $G_1(\bar{z}, e, v, w) = d_r(v)\xi_1 + \tilde{g}_0(\cdot) = b^{-1}(w)\big(d_r(v)e + \bar{g}_1(\bar{z}, e, v, w)\big)$.

Till now, the stabilization problem of the transformed augmented system (6.14) can be further pursued by dynamic output feedback. The forthcoming analysis shares some points in common with those given in Section 3.3.2.

Firstly, we introduce an observer-like dynamics to estimate the states of ξ subsystem,

$$\begin{aligned} \dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + \lambda_i(e - \hat{\xi}_1) \quad i = 1, \cdots, r-1 \\ \dot{\hat{\xi}}_r &= \bar{u} + \lambda_r(e - \hat{\xi}_1) \end{aligned} \tag{6.15}$$

The parameters λ_i are chosen to ensure the matrix A_o being Hurwitz, where

$$A_o = \left[\begin{array}{c|c} -\lambda_1 & \\ \vdots & I_{r-1} \\ \hline -\lambda_r & 0 \end{array}\right]$$

The estimation error is denoted by $\tilde{\xi} = \xi - \hat{\xi}$, and it is not hard to verify that $\tilde{\xi}$ satisfies

$$\dot{\tilde{\xi}} = A_o\tilde{\xi} + B\Psi\bar{\eta} + G(\bar{z}, e, v, w) - \lambda\big(e - b^{-1}(w)e\big) \tag{6.16}$$

where $B = \text{col}(0, \cdots, 0, 1)$, $G(\bar{z}, e, v, w) = \text{col}\big(G_1(\cdot), \cdots, G_r(\cdot)\big)$ and $\lambda = \text{col}(\lambda_1, \cdots, \lambda_r)$.

Secondly, we replace ξ_i by its estimation $\hat{\xi}_i$, $i = 2, \cdots, r$, $\xi_1 = e$, and attach (6.16) to (6.14), so we have the following system expression

$$\begin{aligned} \dot{\bar{z}} &= \bar{f}(\bar{z}, e, v, w) \\ \dot{\bar{\eta}} &= M\bar{\eta} + G_0(\bar{z}, e, v, w) \\ \dot{\tilde{\xi}} &= A_o\tilde{\xi} + B\Psi\bar{\eta} + G(\bar{z}, e, v, w) - \lambda\big(e - b^{-1}(w)e\big) \\ \dot{e} &= b(w)\big(\hat{\xi}_2 + \tilde{\xi}_2 + G_1(\bar{z}, e, v, w)\big) \\ &\overset{\text{def}}{=} b(w)\hat{\xi}_2 + G_1'(\bar{z}, \tilde{\xi}, e, v, w) \\ \dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + \lambda_i(e - \hat{\xi}_1) \quad i = 2, \cdots, r-1 \\ \dot{\hat{\xi}}_r &= \bar{u} + \lambda_r(e - \hat{\xi}_1) \end{aligned} \tag{6.17}$$

It can be seen system (6.17) is in the lower triangular form by viewing $\big(\bar{z}, \bar{\eta}, \tilde{\xi}\big)$ as the inverse dynamics. And till now, both e and $\hat{\xi}_i$, $i = 1, \cdots, r$ are available for feedback design.

Thirdly, we introduce the following assumption to further proceed.

Assumption 6.3. *For any compact set $\Sigma \subset \mathbb{R}^{n_v} \times \mathbb{R}^{n_w}$, there exists a $\mathcal{C}^1$ function $V_{\bar{z}}(\bar{z})$ satisfying $\underline{\alpha}_{\bar{z}}(\|\bar{z}\|) \leq V_{\bar{z}}(\bar{z}) \leq \bar{\alpha}_{\bar{z}}(\|\bar{z}\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{\bar{z}}(\cdot)$ and $\bar{\alpha}_{\bar{z}}(\cdot)$ so that along the trajectory of $\bar{z}$ subsystem of (6.17), the following inequality holds for any $(v, w) \in \Sigma$*

$$\dot{V}_{\bar{z}}(\bar{z}) \leq -\alpha(\|\bar{z}\|) + \delta_e \gamma_e(e)$$

where $\alpha(\cdot)$ is some known class $\mathcal{K}_\infty$ function satisfying $\lim_{s\to 0^+} \sup(\alpha^{-1}(s^2)/s) < \infty$, $\gamma_e(\cdot)$ is a known smooth positive definite function and δ_e is some unknown positive constant. ▪

Assumption 6.3 ensures the $\bar{z}$ subsystem is RISS w.r.t state $\bar{z}$ and input e. And the equilibrium point $\bar{z} = 0$ of the undriven system $\dot{\bar{z}} = \bar{f}(\bar{z}, 0, v, w)$ is locally exponentially stable if $\underline{\alpha}_{\bar{z}}(\cdot)$ and $\bar{\alpha}_{\bar{z}}(\cdot)$ are locally quadratic. Under Assumption 6.3, by changing supply functions technique, for any smooth function $\Delta(\bar{z}) \geq 0$, there exists a $\mathcal{C}^1$ function $\bar{V}_{\bar{z}}(\bar{z})$ satisfying $\underline{\alpha}'_{\bar{z}}(\|\bar{z}\|) \leq \bar{V}_{\bar{z}}(\bar{z}) \leq \bar{\alpha}'_{\bar{z}}(\|\bar{z}\|)$ such that, for any $(v, w) \in \Sigma$, along the trajectory of $\bar{z}$ subsystem, $\dot{\bar{V}}_{\bar{z}}(\bar{z}) \leq -\Delta(\bar{z})\|\bar{z}\|^2 + \bar{\delta}_e \bar{\gamma}_e(e)e^2$ holds for some known smooth function $\bar{\gamma}_e(\cdot) \geq 1$ and some unknown positive constant $\bar{\delta}_e$.

Moreover, since both M and A_o are Hurwitz matrices, we can see that for the inverse dynamics of system (6.17)

$$\begin{aligned}
\dot{\bar{z}} &= \bar{f}(\bar{z}, e, v, w) \\
\dot{\bar{\eta}} &= M\bar{\eta} + G_0(\bar{z}, e, v, w) \\
\dot{\tilde{\xi}} &= A_o\tilde{\xi} + B\Psi\bar{\eta} + G(\bar{z}, e, v, w) - \lambda\big(e - b^{-1}(w)e\big)
\end{aligned} \tag{6.18}$$

with the notation $Z = \text{col}\big(\bar{z}, \bar{\eta}, \tilde{\xi}\big)$, there exists a $\mathcal{C}^1$ function $U(Z)$ satisfying $\underline{\alpha}_Z(\|Z\|) \leq U(Z) \leq \bar{\alpha}_Z(\|Z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_Z(\cdot)$ and $\bar{\alpha}_Z(\cdot)$, such that for any $(v, w) \in \Sigma$, along the trajectory of (6.18) the following holds

$$\dot{U}(Z) \leq -\|Z\|^2 + \delta\gamma(e)$$

for some known smooth positive definite function $\gamma(\cdot)$ and some unknown positive constant δ.

Then by using changing supply functions technique again, for any smooth function $\Delta(Z) \geq 0$, there exists a $\mathcal{C}^1$ function $V_Z(Z)$ satisfying $\underline{\alpha}'_Z(\|Z\|) \leq V_Z(Z) \leq \bar{\alpha}'_Z(\|Z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}'_Z(\cdot)$ and $\bar{\alpha}'_Z(\cdot)$, such that for any $(v, w) \in \Sigma$, along the trajectory of (6.18) the following holds

$$\dot{V}_Z(Z) \leq -\Delta(Z)\|Z\|^2 + \bar{\delta}\bar{\gamma}(e)e^2 \tag{6.19}$$

for some known smooth function $\bar{\gamma}(\cdot) \geq 1$ and some unknown positive constant $\bar{\delta}$.

Finally, we utilize backstepping design to achieve stabilization. For this purpose, some notations are introduced. For $i = 1, \cdots, r$, denote

$$
\begin{aligned}
w_i(e, k, \hat{\xi}_1, \cdots, \hat{\xi}_i) &= \hat{\xi}_{i+1} - \kappa_i(e, k, \hat{\xi}_1, \cdots, \hat{\xi}_i), \quad \hat{\xi}_{r+1} = \bar{u} \\
K_i &= \frac{\partial \kappa_i}{\partial k}\dot{k}, \quad E_i = \frac{\partial \kappa_i}{\partial e} \\
\kappa_1(e, k) &= -k\rho(e)e, \quad \dot{k} = \rho(e)e^2 \\
\kappa_2(e, k, \hat{\xi}_1, \hat{\xi}_2) &= -2w_1 - \lambda_2(e - \hat{\xi}_1) + K_1 + \hat{b}E_1\hat{\xi}_2 - \frac{1}{2}w_1E_1^2 \\
\phi_2 &= w_1E_1\hat{\xi}_2 \\
\kappa_3(e, k, \hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3) &= -w_1 - w_2 - \lambda_3(e - \hat{\xi}_1) + K_2 + \hat{b}E_2\hat{\xi}_2 - \frac{1}{2}w_2E_2^2 + \frac{\partial \kappa_2}{\partial \hat{\xi}_1}\dot{\hat{\xi}}_1 + \frac{\partial \kappa_2}{\partial \hat{\xi}_2}\dot{\hat{\xi}}_2 \\
\phi_3 &= w_2E_2\hat{\xi}_2 + \phi_2 \\
\kappa_i(e, k, \hat{\xi}_1, \cdots, \hat{\xi}_i) &= -w_{i-2} - w_{i-1} - \lambda_i(e - \hat{\xi}_1) + K_{i-1} + \hat{b}E_{i-1}\hat{\xi}_2 - \frac{1}{2}w_{i-1}E_{i-1}^2 + \sum_{j=1}^{i-1}\frac{\partial \kappa_{i-1}}{\partial \hat{\xi}_j}\dot{\hat{\xi}}_j \\
\phi_i &= w_{i-1}E_{i-1}\hat{\xi}_2 + \phi_{i-1}
\end{aligned}
$$

Then we arrive at the following result.

Theorem 6.1. *Under Assumption 6.3, there exists a dynamic feedback control law*

$$
\begin{aligned}
\bar{u} &= \kappa_r(e, k, \hat{\xi}_1, \cdots, \hat{\xi}_r) \\
\dot{k} &= \rho(e)e^2
\end{aligned}
\tag{6.20}
$$

so that the trajectory of closed-loop system composed of (6.17) and (6.20) exists and is bounded over $t \in [0, \infty)$ and the error output $\lim_{t\to\infty} e = 0$. ∎

Proof: In the 1st step, define $V_1(e) = \frac{1}{2}e^2$, and denote $b(w)$ with b for convenience, then we have

$$
\begin{aligned}
\dot{V}_1 &= e\big(b(\kappa_1 + w_1) + \tilde{\xi}_2 + G_1'\big) \\
&\leq -bk\rho(e)e^2 + w_1^2 + e\big(\frac{1}{4}e + G_1'\big)
\end{aligned}
$$

denote $e\big(\frac{1}{4}e + G_1'\big) = \Pi_0$, then

$$\dot{V}_1 \leq -k\rho(e)e^2 + w_1^2 + \Pi_0$$

In the 2nd step, define $V_2(e, w_1, \tilde{b}) = V_1(e) + \frac{1}{2}w_1^2 + \frac{1}{2}\tilde{b}^2$, then we have

$$\begin{aligned}
\dot{V}_2 &= \dot{V}_1 + w_1\big(\dot{\hat{\xi}}_2 - \dot{\kappa}_1\big) - \tilde{b}\dot{\hat{b}} \\
&\leq -bk\rho(e)e^2 + w_1^2 + \Pi_0 + w_1\big(\kappa_2 + w_2 + \lambda_2(e - \hat{\xi}_1) - E_1(b\hat{\xi}_2 + G_1') - K_1\big) - \tilde{b}\dot{\hat{b}} \\
&= -bk\rho(e)e^2 + w_1^2 + \Pi_0 + w_1\big(\kappa_2 + w_2 + \lambda_2(e - \hat{\xi}_1) - K_1\big) - w_1E_1(b\hat{\xi}_2 + G_1') - \tilde{b}\dot{\hat{b}} \\
&\leq -bk\rho(e)e^2 + w_1^2 + \Pi_0 + w_1\big(\kappa_2 + w_2 + \lambda_2(e - \hat{\xi}_1) - K_1\big) - (\hat{b} + \tilde{b})w_1E_1\hat{\xi}_2 \\
&\quad + \frac{1}{2}(w_1E_1)^2 + \frac{1}{2}(G_1')^2 - \tilde{b}\dot{\hat{b}} \\
&= -bk\rho(e)e^2 + \Pi_0 + w_1w_2 + w_1\big(w_1 + \kappa_2 + \lambda_2(e - \hat{\xi}_1) - K_1 - \hat{b}E_1\hat{\xi}_2 + \frac{1}{2}w_1E_1^2\big) \\
&\quad - \tilde{b}\big(w_1E_1\hat{\xi}_2 + \dot{\hat{b}}\big) + \frac{1}{2}(G_1')^2
\end{aligned}$$

denote $\frac{1}{2}(G_1')^2 = \Pi_1$, $\Pi_2 = \Pi_0 + \Pi_1$, and use κ_2, ϕ_2 gives

$$\dot{V}_2 \leq -bk\rho(e)e^2 - w_1^2 + w_1w_2 - \tilde{b}\big(\phi_2 + \dot{\hat{b}}\big) + \Pi_2$$

In the 3rd step, define $V_3(e, w_1, w_2, \tilde{b}) = V_2(e, w_1, \tilde{b}) + \frac{1}{2}w_2^2$, use the similar arguments we have

$$\begin{aligned}
\dot{V}_3 &\leq -bk\rho(e)e^2 - w_1^2 + w_1w_2 - \tilde{b}\big(\phi_2 + \dot{\hat{b}}\big) + \Pi_2 + w_2\big(\dot{\hat{\xi}}_3 - \dot{\kappa}_2\big) \\
&\leq -bk\rho(e)e^2 - w_1^2 + w_1w_2 - \tilde{b}\big(\phi_2 + \dot{\hat{b}}\big) + \Pi_2 \\
&\quad + w_2\big(\kappa_3 + w_3 + \lambda_3(e - \hat{\xi}_1) - E_2(b\hat{\xi}_2 + G_1') - K_2 - \frac{\partial \kappa_2}{\partial \hat{\xi}_1}\dot{\hat{\xi}}_1 - \frac{\partial \kappa_2}{\partial \hat{\xi}_2}\dot{\hat{\xi}}_2\big) \\
&= -bk\rho(e)e^2 - w_1^2 - \tilde{b}\big(w_2E_2\hat{\xi}_2 + \phi_2 + \dot{\hat{b}}\big) + \Pi_2 + \frac{1}{2}(G_1')^2 \\
&\quad + w_2\big(w_1 + \kappa_3 + \lambda_3(e - \hat{\xi}_1) - \hat{b}E_2\hat{\xi}_2 - K_2 - \frac{\partial \kappa_2}{\partial \hat{\xi}_1}\dot{\hat{\xi}}_1 - \frac{\partial \kappa_2}{\partial \hat{\xi}_2}\dot{\hat{\xi}}_2 + \frac{1}{2}w_2E_2^2\big) + w_2w_3 \\
&\leq -bk\rho(e)e^2 - w_1^2 - w_2^2 + w_2w_3 - \tilde{b}\big(\phi_3 + \dot{\hat{b}}\big) + \Pi_3
\end{aligned}$$

where $\Pi_3 = \Pi_2 + \Pi_1$.

In the ith step, define $V_i(e, w_1, \cdots, w_{i-1}, \tilde{b}) = V_{i-1}(e, w_1, \cdots, w_{i-2}, \tilde{b}) + \frac{1}{2}w_{i-1}^2$, then

$$\dot{V}_i \leq -bk\rho(e)e^2 - \sum_{j=1}^{i-1} w_j^2 + w_{i-1}w_i - \tilde{b}\big(\phi_i + \dot{\hat{b}}\big) + \Pi_i$$

where $\Pi_i = \Pi_{i-1} + \Pi_1$.

In the rth step, define $V_r(e, w_1, \cdots, w_{r-1}, \tilde{b}) = V_{r-1}(e, w_1, \cdots, w_{r-2}, \tilde{b}) + \frac{1}{2}w_{r-1}^2$, then

$$\dot{V}_r \leq -bk\rho(e)e^2 - \sum_{j=1}^{r-1} w_j^2 + w_{r-1}w_r - \tilde{b}(\phi_r + \dot{\hat{b}}) + \Pi_r$$

where $\Pi_r = \Pi_{r-1} + \Pi_1 = \Pi_0 + (r-1)\Pi_1$.

Setting $w_r = 0$, i.e. $\bar{u} = \kappa_r(e, k, \hat{\xi}_1, \cdots, \hat{\xi}_r)$, and $\dot{\hat{b}} = -\phi_r$, we have

$$\dot{V}_r \leq -bk\rho(e)e^2 - \sum_{j=1}^{r-1} w_j^2 + \Pi_r$$

Notice

$$\begin{aligned} \Pi_r &= \Pi_0 + (r-1)\Pi_1 = e\big(\frac{1}{4}e + G_1'\big) + (r-1)\big(\frac{1}{2}(G_1')^2\big) \\ G_1' &= \tilde{\xi}_2 + G_1(\bar{z}, e, v, w) \\ G_1 &= b^{-1}(w)\big(d_r(v)e + \bar{g}_1(\bar{z}, e, v, w)\big) \end{aligned}$$

it is easy to verify that for the smooth functions $G_1'(\bar{z}, \tilde{\xi}, e, v, w)$ and $\Pi_r(Z, e, v, w)$, $G_1'(0, 0, 0, v, w) = 0$, and $\Pi_r(0, 0, v, w) = 0$ for any $(v, w) \in \Sigma$.

Then by Lemma 7.8 of [36], we can show that

$$|G_1'|^2 \leq c'\big(\hbar_1'(\bar{z})\|\bar{z}\|^2 + \hbar_2'(e)e^2\big)$$

for some positive constant c', some known smooth functions $\hbar_1'(\cdot) \geq 1$, $\hbar_2'(\cdot) \geq 1$. Further the upper bound of $|\Pi_r|$ can be estimated as

$$|\Pi_r| \leq c_\pi\big(\hbar_Z(Z)\|Z\|^2 + \hbar_e(e)e^2\big)$$

where c_π is some positive constant, and $\hbar_Z(\cdot)$, $\hbar_e(\cdot)$ are some known positive definite functions. Consequently, we have

$$\dot{V}_r \leq -bk\rho(e)e^2 - \sum_{j=1}^{r-1} w_j^2 + c_\pi\big(\hbar_Z(Z)\|Z\|^2 + \hbar_e(e)e^2\big) \tag{6.21}$$

As shown in (6.19) where V_Z is well posed, we further define

$$V = V_Z(Z) + V_r(e, w_1, \cdots, w_{r-1}, \tilde{b}) + \frac{1}{2}(k - \bar{k})^2$$

where $\bar{k}$ is a constant to be specified later.

The derivative of V is

$$\begin{aligned}\dot{V} &\leq -\Delta(Z)\|Z\|^2 + \bar{\delta}\bar{\gamma}(e)e^2 - bk\rho(e)e^2 - \sum_{j=1}^{r-1} w_j^2 \\ &\quad + c_\pi\big(\hbar_Z(Z)\|Z\|^2 + \hbar_e(e)e^2\big) + (k-\bar{k})\rho(e)e^2 \\ &= -\big(\Delta(Z) - c_\pi\hbar_Z(Z)\big)\|Z\|^2 + \big(\bar{\delta}\bar{\gamma}(e) + c_\pi\hbar_e(e) - b\bar{k}\rho(e)\big)e^2 - \sum_{j=1}^{r-1} w_j^2\end{aligned}$$

Letting $\Delta(Z) \geq c_\pi\hbar_Z(Z) + 1$, $\rho(e) \geq \max\{\bar{\gamma}(e),\ \hbar_e(e)\}$ and $\bar{k} \geq \frac{1}{b}(\bar{\delta} + c_\pi)$, we have

$$\dot{V} \leq -\|Z\|^2 - \sum_{j}^{r-1} w_j^2$$

which implies the states of closed-loop system composed of system (6.17) and controller (6.20) are bounded over $t \in [0, +\infty)$, especially k is bounded. Moreover, since e and $\dot{e}$ are bounded, $\ddot{k}$ exists and is bounded, so $\dot{k} = \rho(e)e^2$ is uniformly continuous. By Barbalat's Lemma, it can be concluded $\dot{k} \to 0$ as $t \to \infty$, which implies $e \to 0$ as $t \to \infty$. ◇

Recall the internal model (5.18) and the observer-like dynamics (6.15), as mentioned in Proposition 5.1, Theorem 6.1 leads us directly to the solution of output regulation problem for the original plant (6.1) with the uncertain reference trajectory generated by nonlinear dynamics (5.2).

Corollary 6.1. *Under Assumptions 6.1, 6.2 and 6.3, by using the following dynamic output feedback control law*

$$\begin{aligned}\dot{\eta} &= M\eta + N(v)u \\ \dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + \lambda_i(e - \hat{\xi}_1) \quad i = 2, \cdots, r-1 \\ \dot{\hat{\xi}}_r &= (u - \Psi\eta) + \lambda_r(e - \hat{\xi}_1) \\ u &= \kappa_r(e, k, \hat{\xi}_1, \cdots, \hat{\xi}_r) + \Psi\eta \\ \dot{k} &= \rho(e)e^2\end{aligned} \tag{6.22}$$

the trajectory of closed-loop system composed of (6.1) and (6.22) exists and is bounded over $t \in [0, \infty)$ and tracking error e approaches zero asymptotically as $t \to \infty$. ▪

6.3 Examples

In this section, we consider some examples to show the effectiveness of the proposed design. Our method gives the global results compared with the former ones.

Example 6.2. ▪

Consider the numerical example given in [14], where the plant is described by

$$\begin{aligned} x_1 &= x_1 + w_1 e^2 + x_2 \\ x_2 &= (1 + w_2)x_1 + \sin(w_3 e x_2) + u \\ e &= x_1 - v_1 \end{aligned} \tag{6.23}$$

and the exosystem is the Van der Pol oscillator

$$v_1 = v_2, \quad v_2 = -v_1 + (1 - v_1^2)v_2 \tag{6.24}$$

In [14], the output regulation problem is solved for sufficiently small initial state of the closed-loop system and the exosystem and sufficiently small uncertain parameters (w_1, w_2, w_3), so the result is local. We intend to achieve the global results.

Notice $\sin(w_3 e x_2)$ is a sinusoidal function, by using the Taylor expansion we can express $\sin(w_3 e x_2) = g(t)e$, where $g(t)$ is some uncertain time-varying function. It can be seen that system (6.23) is now in the form of (6.1) with no inverse dynamics z.

The solution of the corresponding regulator equations is given as follows

$$\begin{aligned} \mathbf{x}_1(v, w) &= v_1, \quad \mathbf{x}_2(v, w) = -v_1 + v_2 \\ \mathbf{u}(v, w) &= -v_2 - v_1 + (1 - v_1^2)v_2 - (1 + w_2)v_1 \end{aligned} \tag{6.25}$$

As mentioned in [14], we denote $\mathbf{u}(v, w) = \mathbf{u}_0(v) + \mathbf{u}_1(v, w)$ with $\mathbf{u}_0(v) = -v_2 - v_1 + (1 - v_1^2)v_2$ and $\mathbf{u}_1(v, w) = -(1 + w_2)v_1$. By letting $u = \mathbf{u}_0 + u_1$, we introduce an auxiliary system

$$\begin{aligned} x_1 &= x_1 + w_1 e^2 + x_2 \\ x_2 &= (1 + w_2)x_1 + g(t)e + \mathbf{u}_0(v) + u_1 \\ e &= x_1 - v_1 \end{aligned} \tag{6.26}$$

if there is a control u_1 solving the output regulation problem for the auxiliary system (6.26), then the control law $u = u_1 + \mathbf{u}_0$ solves the output regulation problem for the original plant (6.23).

It can be verified that

$$\frac{d^2\mathbf{u}_1(v, w)}{dt^2} = -\mathbf{u}_1(v, w) + (1 - v_1^2)\frac{d\mathbf{u}_1(v, w)}{dt}$$

Consequently, the steady-state input generator of the form 5.19 can be derived as follows.

$$\dot{\tau}_1 = (1 - v_1^2)\tau_1 + \tau_2$$
$$\dot{\tau}_2 = (-1 + 2v_1 v_2)\tau_1$$

and the internal model with output u_1 can be designed as follows.

$$\dot{\eta} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \eta + \begin{bmatrix} 3 - v_1^2 \\ 2v_1 v_2 \end{bmatrix} u_1 \stackrel{\text{def}}{=} M\eta + N(v)u_1 \tag{6.27}$$

Attaching the internal model (6.27) to the auxiliary system (6.26) and performing the following input and coordinate transformation

$$\bar{x}_1 = x_1 - \mathbf{x}_1, \quad \bar{x}_2 = x_2 - \mathbf{x}_2. \quad \bar{\eta} = \eta - \tau, \quad \bar{u}_1 = u_1 - \Psi\eta$$

the resulting augmented system is

$$\begin{aligned} \dot{\bar{\eta}} &= M\hat{\eta} + N(v)(\bar{u}_1 + \Psi\hat{\eta}) \\ \dot{\bar{x}}_1 &= \bar{x}_2 + (e + w_1 e^2) \\ \dot{\bar{x}}_2 &= (\bar{u}_1 + \Psi\hat{\eta}) + \big(1 + w_2 + g(t)\big)e \end{aligned}$$

Further, performing the input transformation (6.11) as $\bar{\eta} = \hat{\eta} - c_2(v)\bar{x}_2 - c_1(v)\bar{x}_1$, where

$$c_2(v) = N(v),\ c_1(v) = MN(v) - N^{(1)}(v)$$

the augmented system can be formed into

$$\begin{aligned} \dot{\bar{\eta}} &= M\bar{\eta} + \bar{g}_0(e, v, w) \\ \dot{\bar{x}}_1 &= \bar{x}_2 + (e + w_1 e^2) \\ \dot{\bar{x}}_2 &= \bar{u}_1 + \Psi\big(\bar{\eta} + c_1(v)\bar{x}_1 + c_2(v)\bar{x}_2\big) + \big(1 + w_2 + g(t)\big)e \end{aligned}$$

The transformation $\xi = T_2(v, w)\bar{x}$ is given as $\xi_1 = \bar{x}_1$, $\xi_2 = \bar{x}_2 - d_1(v)\bar{x}_1$, with $d_1(v) = -2v_1 v_2$, and by denoting $d_0(v) = 3 - v_1^2 - 2v_2^2 - 2v_1(-v_1 + (1 - v_1^2)v_2)$, we have

$$\begin{aligned} \dot{\bar{\eta}} &= M\bar{\eta} + \bar{g}_0(e. v, w) \\ \dot{\xi}_1 &= d_1(v)\xi_1 + \xi_2 + (e + w_1 e^2) \\ \dot{\xi}_2 &= d_0(v)\xi_1 + \bar{u}_1 + \Psi\bar{\eta} + \bar{g}_2(e, v, w) \end{aligned}$$

where $\xi_1 = \bar{x}_1 = e$ and $\bar{g}_2(e, v, w) = \big(1 + w_2 + g(t)\big)e - d_1(v)(e + w_1 e^2)$.

Then we can show that the following control law in the form of (6.22) solve the global output regulation problem for (6.23).

$$\begin{aligned}
\dot{\eta} &= M\eta + N(v)u_1 \\
\dot{\hat{\xi}}_1 &= \hat{\xi}_2 + 2(e - \hat{\xi}_1) \\
\dot{\hat{\xi}}_2 &= (u_1 - \Psi\eta) + (e - \hat{\xi}_1) \\
u_1 &= \kappa_2(e, k, \hat{\xi}_1, \hat{\xi}_2) + \Psi\eta, \quad u = u_1 + \mathbf{u}_c \\
\dot{k} &= \rho(e)e^2
\end{aligned}$$

where

$$\begin{aligned}
w_1 &= \hat{\xi}_2 - \kappa_1(\cdot) \\
\kappa_1(\cdot) &= -k\rho(e)e \\
\kappa_2(\cdot) &= -2w_1 - (e - \hat{\xi}_1) + E_1\hat{\xi}_2 + K_1 - w_1 E_1^2
\end{aligned}$$

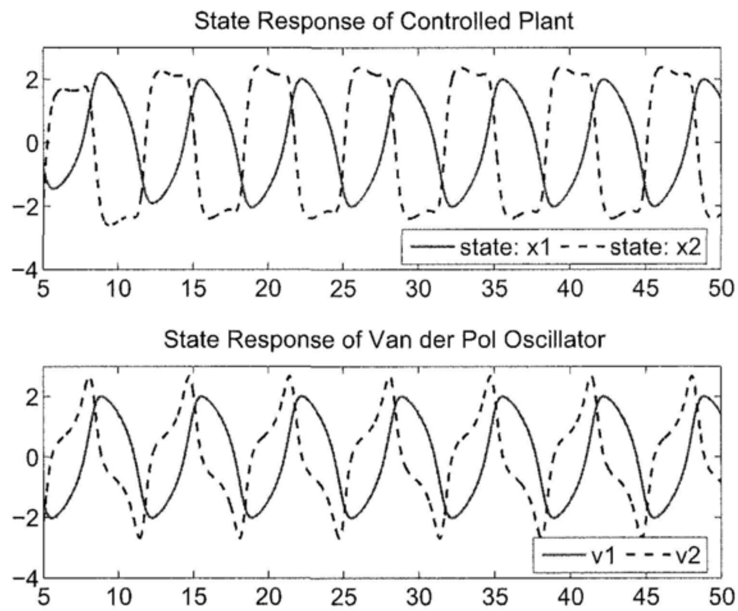

Figure 6.1: Dynamics of the Controlled Plant and Exosystem

In [14], the uncertain parameters of the plant is chosen as $w_1 = 0.1$, $w_2 = -0.2$, $w_3 = 0.3$, and the initial states of the plant and the exosystem is taken with $x_0 = (-1, 0.2)$, $v_0 = (0.1, 0)$. For simulation, we arbitrarily choose, no need to be sufficiently small, the initial states for the plant and exosystem as $x_0 = (3, -2, 1)$, $v_0 = (-2, 4)$, and $w_1 = 3, 5,$

$w_2 = 2$, $w_3 = -5$, and initial states for the controller are set to be zero. The results are shown in Fig 6.1 6.2. ◇

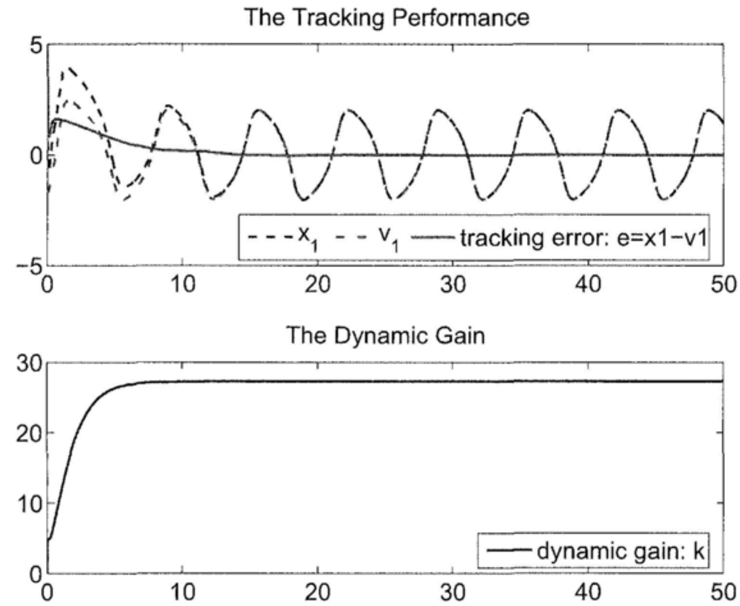

Figure 6.2: Tracking Performance and the Dynamic Gain

Example 6.3. ▪

Consider the synchronization problem of periodically forced pendulum with Rayleigh equation.

The dynamics of periodically forced pendulum under control can be described by

$$\ddot{\theta} = -\gamma\dot{\theta} - \sin\theta + a\cos(\omega t) + u$$

The Rayleigh equation, as typical a nonlinear system as Van der Pol oscillator is, is given by

$$\ddot{x} - \mu(1 - \dot{x}^2)\dot{x} + x = 0$$

It should be noted that the Rayleigh equation can be obtained from Van der Pol oscillator $\ddot{y} - \mu(1-y^2)\dot{y} + y = 0$ by the substitution $y = \sqrt{3}\dot{x}$. Some examples with Rayleigh equation as exosystem have been considered in [20] and [112]. The dynamics of Rayleigh equation is shown in Figure 6.3.

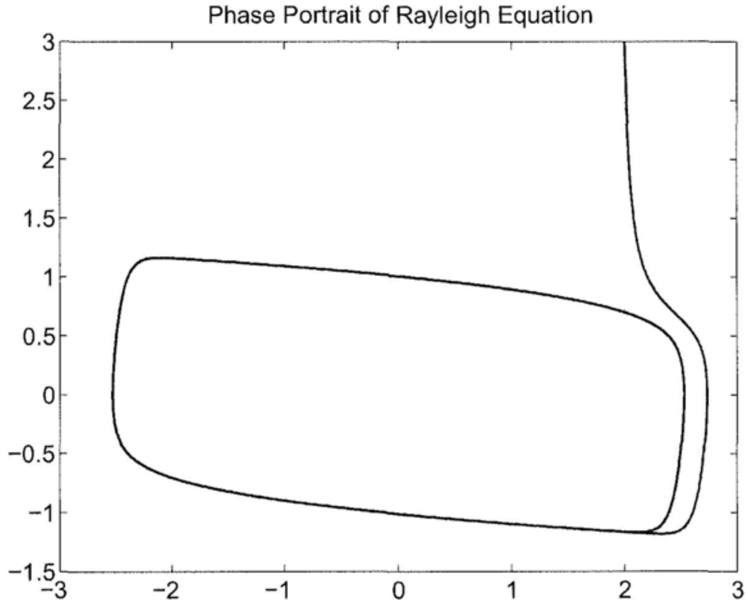

Figure 6.3: Phase Portrait of Rayleigh Equation

The composite system of periodically forced pendulum and Rayleigh equation is as follows.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 - \gamma x_2 + a\cos(\omega t) + u \\ \dot{v}_1 &= v_2 \\ \dot{v}_2 &= -v_1 + \mu(1 - v_2^2)v_2 \end{aligned} \tag{6.28}$$

To be specified, $\cos(\omega t)$ is certain signal, μ is some certain parameter, and γ is some uncertain parameter. The control objective is to render the output of the periodically forced pendulum $y = x_1$ to track v_1 asymptotically.

Performing a coordinate transformation $y_1 = x_1$ and $y_2 = x_2 + \gamma x_1$, (6.28) turns into the general output feedback form with relative degree $r = 2$ as follows.

$$\begin{aligned} \dot{y}_1 &= y_2 - \gamma y_1 \\ \dot{y}_2 &= u - \sin y_1 + a\cos(\omega t) \\ \dot{v}_1 &= v_2 \\ \dot{v}_2 &= -v_1 + \mu(1 - v_2^2)v_2 \end{aligned} \tag{6.29}$$

The solution of the regulator equations are

$$\mathbf{y}_1(v,w) = v_1, \quad \mathbf{y}_2(v,w) = \gamma v_1 + v_2, \quad \mathbf{u}(v,w) = \mathbf{u}_0(v) + \mathbf{u}_1(v,w)$$
$$\mathbf{u}_0(v) = -v_1 + \mu(1-v_2^2)v_2 + \sin v_1 - a\cos(\omega t), \quad \mathbf{u}_1(v,w) = \gamma v_2$$

Like the former example, by letting $u = \mathbf{u}_0 + u_1$ we introduce an auxiliary system

$$\begin{aligned} \dot{y}_1 &= y_2 - \gamma y_1 \\ \dot{y}_2 &= u_1 - \sin y_1 + a\cos(\omega t) + \mathbf{u}_0 \end{aligned} \tag{6.30}$$

Since $\mathbf{u}_1 = \gamma v_2$, it can be verified that

$$\frac{d^2\mathbf{u}_1}{dt^2} = a_0(v)\mathbf{u}_1 + a_1(v)\frac{d\mathbf{u}_1}{dt}$$

where

$$a_0(v) = -1 + \mu(-2v_2)\big(-v_1 + \mu(1-v_2^2)v_2\big), \quad a_1(v) = \mu(1-v_2^2)$$

so the steady-state input generator in observer canonical form is

$$\begin{aligned} \dot{\tau}_1 &= \mu(1-v_2^2)\tau_1 + \tau_2 \\ \dot{\tau}_2 &= -\tau_1 \end{aligned}$$

and the internal model with output u_1 can be designed as

$$\dot{\eta} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}\eta + \begin{bmatrix} 2+\mu(1-v_2^2) \\ 0 \end{bmatrix} u_1 \stackrel{\text{def}}{=} M\eta + N(v)u_1$$

Attaching the internal model to the auxiliary system and performing the transformations

$$\begin{aligned} \bar{y}_1 &= y_1 - \mathbf{y}_1(v,w) = e, \quad \bar{y}_2 = y_2 - \mathbf{y}_2(v,w) \\ \hat{\eta} &= \eta - \tau, \quad \bar{u}_1 = u_1 - \Psi\eta \end{aligned}$$

we have the following augmented system

$$\begin{aligned} \dot{\bar{y}}_1 &= \bar{y}_2 - \gamma e \\ \dot{\bar{y}}_2 &= \bar{u}_1 + \Psi\hat{\eta} + \sin v_1 - \sin(v_1 + e) \\ \dot{\hat{\eta}} &= M\hat{\eta} + N(v)(\bar{u}_1 + \Psi\hat{\eta}) \end{aligned} \tag{6.31}$$

Performing the coordinate transformation on $\hat{\eta}$,

$$\bar{\eta} = \hat{\eta} - c_2(v)\bar{y}_2 - c_1(v)\bar{y}_1$$

with $c_2(v) = N(v)$, $c_1(v) = Mc_2(v) - c_2^{(1)}(v)$, $c_0(v) = Mc_1(v) - c_1^{(1)}(v)$, we have

$$\begin{aligned}
\dot{\bar\eta} &= M\bar\eta + \bar g_0(e, v, w) \\
\dot{\bar y}_1 &= \bar y_2 + \bar g_1(e, v, w) \\
\dot{\bar y}_2 &= \bar u_1 + \Psi\big(\bar\eta + c_2(v)\bar y_2 + c_1(v)\bar y_1\big) + \bar g_2(e, v, w)
\end{aligned}$$

where $\bar g_1 = -\gamma e$, $\bar g_2 = \sin v_1 - \sin(v_1 + e)$, and $\bar g_0 = c_0(v)e - c_1(v)\bar g_1 - c_2(v)\bar g_2$.

And by transformation $\xi = T(v)\bar y$, where $\xi_1 = \bar y_1$, $\xi_2 = -d_2(v)\bar y_1 + \bar y_2$, we have

$$\begin{aligned}
\dot{\bar\eta} &= M\bar\eta + \bar g_0(e, v, w) \\
\dot\xi_1 &= \xi_2 + d_2(v)\xi_1 + \tilde g_1 \\
\dot\xi_2 &= \bar u_1 + \Psi\bar\eta + d_1(v)\xi_1 + \tilde g_2
\end{aligned}$$

with

$$\begin{aligned}
d_2(v) &= 2 + \mu(1 - v_2^2), \\
d_1(v) &= -4 - 2\mu(1 - v_2^2) - 2\mu(-2v_2)(-v_1 + \mu(1 - v_2^2)v_2) \\
\tilde g_1 &= \bar g_1, \quad \tilde g_2 = \bar g_2 - d_2(v)\bar g_1
\end{aligned}$$

After introducing the observer-like dynamics to estimate ξ, we have the system in the following form

$$\begin{aligned}
\dot{\bar\eta} &= M\bar\eta + G_0(e, v, w) \\
\dot{\tilde\xi} &= A_0\tilde\xi + B\Psi\eta + G(e, v, w) \\
\dot e &= \hat\xi_2 + \tilde\xi_2 + G_1(e, v, w) \\
\dot{\hat\xi}_1 &= \hat\xi_2 + \lambda_1(e - \hat\xi_1) \\
\dot{\hat\xi}_2 &= \bar u_1 + \lambda_2(e - \hat\xi_2)
\end{aligned} \tag{6.32}$$

and finally, the following control law in the form of (6.22)

$$\begin{aligned}
\dot\eta &= M\eta + N(v)u_1 \\
\dot{\hat\xi}_1 &= \hat\xi_2 + 2(e - \hat\xi_1) \\
\dot{\hat\xi}_2 &= (u_1 - \Psi\eta) + (e - \hat\xi_1) \\
u_1 &= \kappa_2(e, k, \hat\xi_1, \hat\xi_2) + \Psi\eta, \quad u = u_1 + \mathbf{u}_0 \\
\dot k &= \rho(e)e^2
\end{aligned} \tag{6.33}$$

where

$$\begin{aligned} w_1 &= \hat{\xi}_2 - \kappa_1(\cdot) \\ \kappa_1(\cdot) &= -k\rho(e)e \\ \kappa_2(\cdot) &= -2w_1 - (e - \hat{\xi}_1) + E_1\hat{\xi}_2 + K_1 - w_1 E_1^2 \end{aligned}$$

solves the synchronization problem.

For simulation, we take $\gamma = 0.22$, $\omega = 1$, $a = 2.7$ for the periodically forced pendulum, and $\mu = 5$ for the Rayleigh Equation. The initial states for the plant and exosystem are randomly chosen as $x_0 = (\ 1.2, -5\ )$, $v_0 = (\ 2, 3\ )$, and initial states for the controller are all taken zero. The results are shown in Fig6.4–6.5.

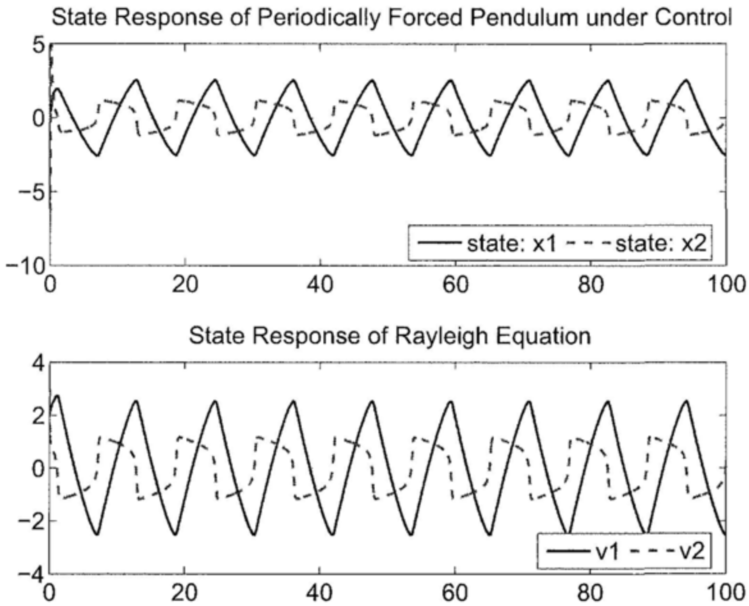

Figure 6.4: Dynamics of Periodically Forced Pendulum and Rayleigh Equation

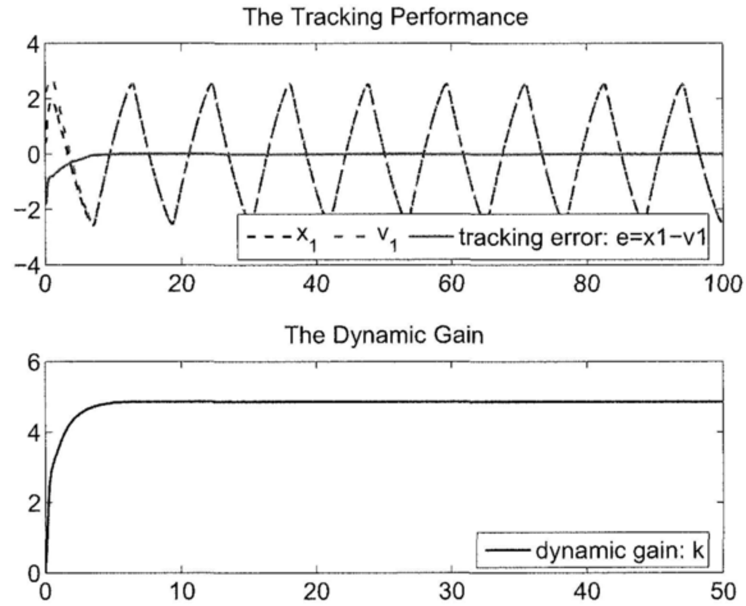

Figure 6.5: Tracking Performance and the Dynamic Gain

6.4 Conclusion

In this section, we further investigate the nonlinear output regulation problem with a nonlinear exosystem. The nonlinear system we considered in this chapter is in the general output feedback form, has been extended to a more general output feedback form, and an observer-based approach is adopted to solve the problem.

□ **End of chapter.**

Chapter 7

Conclusions

In this thesis, under the general framework for handling nonlinear output regulation problem, we have investigated: i) the global robust output regulation problem for time-varying nonlinear systems subject to time-varying exosystem; and ii) the global robust output regulation problem for nonlinear systems subject to nonlinear exosystem.

It has been witnessed that the design of internal model plays the crucial role in achieving the goal of output regulation. An internal model possesses an essential ability of generating all possible steady-state input information asymptotically which is needed to enforce the regulated error output identically zeros, also it ensures the stabilizability of the augmented system which is composed of the original plant and the internal model itself. The design of the internal model and the (global) stabilization of the augmented system are strongly interlaced.

The concentration on the internal model design runs throughout the thesis. In either problems mentioned above, the internal models are all carefully conceived due to the time-varying settings or nonlinearity property of the exosytem. It has been shown that internal model is based on the dynamics called steady-state generator, and the existence conditions for such steady-state generator are rigorously characterized in both scenarios, thus the design of an appropriate internal model is applicable. Moreover, the internal model is zero input globally asymptotically stable which great facilitates the global output regulation. In addition, in the time-varying settings, a generalized internal model in proposed to deal with the uncertain exosytem case.

The stabilization of the augmented system, on the other hand, requires certain techniques due to the specified system structural properties. We mainly focus on the nonlinear system in the output feedback form. In the time-varying case, certain conditions are proposed to handle the occurrence of time-variant terms in plant and the exosystem. And in the time-invariant case, a comparative study between filter-based and observer-based design is conducted.

The thesis will be closed with several prospects of future research.

- For the adaptive output regulation with uncertain time-varying exosystem, a relevant issue on the asymptotically estimation of the uncertain parameters of the exosystem is not fully addressed. Usually, this issue is referred to as the "parameter convergence" in the terminology of adaptive control, and it is commonly known that the parameter convergence relates closely to the "persistent excitation" condition (PE condition) of certain signals inside the closed-loop system.

  The former work [66] links "minimal internal model" property to the PE condition of state vector of steady-state generator, and shows the satisfactory results on the parameter convergence when the linear time-invariant exosystem contains uncertain parameters. For time-varying exosystem, this is worthy of further investigation. The preliminary tests and examples fully show the parameter convergence achieves. A reasonable explanation is the steady-state input, which is asymptotically generated from the internal model, contains sufficient rich frequency contents so that the state of the generalized internal model (or the generalized steady-state generator) is persistent excitation.

  The adaptive output regulation problem can be partially regarded as adaptive control and identification of time-varying systems which is still a challenging problem and only few publications addressed on this topic [81] [118].

- For output regulation with nonlinear exosystem, the presented control strategy combines dynamic output feedback and feedforward control, which render the problem tractable. The ultimate aim is to use dynamic output feedback only. This requires further characterization of the existence condition of internal model which probably turns to be a nonlinear one. Under such hypothetical circumstance, the stabilizability of the augmented system becomes more crucial. Especially the dynamics of the transformed internal model is a nonlinear one, and it together with the transformed inverse dynamics of the original plant is a combined one, which may contain unique or multiple or none equilibriums. To stabilize the augmented system with such inverse dynamics, we need further inversigate some advanced techniques.

□ **End of chapter.**

Bibliography

[1] Z. Artstein, "Stabilization with relaxed controls", *Nonlinear Analysis: Theory, Methods & Applications*, Vol. 7, No. 11, pp. 1163–1173, 1983.

[2] M.D. Di Benedetto, "Synthesis of an internal model for non-linear output regulation", *International Journal of Control*, Vol. 45, No. 3, pp. 1023 1034, 1987.

[3] G. Bengtsson, "Output regulation and internal model a frequency domain approach", *Automatica*, Vol. 13, No. 4, pp. 333 345, 1977.

[4] C.I. Byrnes, F. Delli Priscoli, and A. Isidori, "Output Regulation of Uncertain Nonlinear Systems", *Systems & Control: Foundations & Applications*, Birkhäuser, Boston, 1997.

[5] C.I. Byrnes, F. Delli Priscoli, A. Isidori, and W. Kang, "Structually stable output regulation of nonlinear systems", *Automatica*, Vol. 33, No. 3, pp. 369 385, 1997.

[6] C.I. Byrnes and A. Isidori, "Output regulation for nonlinear systems: an overview", *International Journal of Robust and Nonlinear Control*, Vol. 10, No. 5. pp. 323 337, 2000.

[7] C.I. Byrnes and A. Isidori, "Limit sets, zero dynamics, and internal models in the problem of nonlinear output regulation", *IEEE Transactions on Automatic Control*, Vol. 48, No. 10, pp. 1712–1723, 2003.

[8] C.I. Byrnes, A. Isidori, and L. Praly, "On the asymptotic properties of a system arising in non-equilibrium theory of output regulation", Mittag-Leffler Institute, Stockholm, Sweden, 2003.

[9] C.I. Byrnes and A. Isidori, "Nonlinear internal models for output regulation", *IEEE Transactions on Automatic Control*, Vol. 49, No. 12, pp. 2244 2247, 2004.

[10] F. Celani, A. Isidori, and L. Marconi, "A reduction paradigm for output regulation", *International Journal of Robust and Nonlinear Control*, Vol. 18, No. 7, pp. 756 781, 2008.

[11] C.T. Chen, *Linear System Theory and Design (1st Edition)*, Holt, Rinehart and Winston, New York, 1984.

[12] T. Chen and J. Huang, "Global robust output regulation by state feedback for strict feedforward systems", *IEEE Transactions on Automatic Control*, Vol. 54, No. 9, pp. 2157 2163, 2009.

[13] Z. Chen and J. Huang, "Global robust servomechanism problem of lower trigular systems in the general case", *Systems & Control Letters*, Vol. 52, No. 3-4, pp. 209 220, 2004.

[14] Z. Chen and J. Huang, "Robust output regulation with nonlinear exosystems", *Automatica*, Vol. 41, No. 8, pp. 1447–1454, 2005.

[15] Z. Chen and J. Huang, "Global robust output regulation for output feedback systems", *IEEE Transactions on Automatic Control*, Vol. 50, No. 1, pp. 117–121, 2005.

[16] Z. Chen, "On observability of nonlinear steady-state generators", *Automatica*, Vol. 46, No. 10, pp. 1712 1718, 2010.

[17] D. Cheng, T.J. Tarn, and S.K. Spurgeon, "On the design of output regulators for nonlinear systems", *Systems & Control Letters*, Vol. 43, No. 3, pp. 167–179, 2001.

[18] E.J. Davison, "The robust control of a servomechanism problem for linear time-variant multivariable systems", *IEEE Transactions on Automatic Control*, Vol. 21, No. 1, pp. 25–34, 1976.

[19] Z. Ding, "Universal disturbance rejection for nonlinear systems in output feedack form", *IEEE Transactions on Automatic Control*, Vol. 48, No. 7, pp. 1222 1226, 2003.

[20] Z. Ding, "Output regulation of uncertain nonlinear systems with nonlinear exosystems", *IEEE Transactions on Automatic Control*, Vol. 51, No. 3, pp. 498–503, 2006.

[21] B.A. Francis and W.M. Wonham, "The internal model principle for linear multivariable regulators", *Applied Mathematics & Optimization*, Vol. 2, No. 2, pp. 170 194, 1975.

[22] B A Francıs and W M Wonham, "The ınternal model prıncıple of control theory", *Automatıca*, Vol 12, No 5, pp 457 465, 1976

[23] B A Francıs, "The lınear multıvarıable regulator Problem", *SIAM Jorunal of Control and Optımızatıon*, Vol 15, No 3, pp 486 505, 1977

[24] R A Freeman and P V Kokotovıc, *Robust Nonlınear Control Desıgn State-Space and Lyapunov Technıques*, *Systems & Control Foundatıons & Applıcatıons*, Bırkhauser, Boston, 1996

[25] E Frıdman, "Output regulatıon of nonlınear systems wıth delay", *Systems & Control Letters*, Vol 50, No 2, pp 81 93, 2003

[26] M Hautus, "Lınear matrıx equatıons wıth applıcatıons to the regulator problem", *Outıls and Modeles Mathematıque pour l'Auutomatıque, l'Analyse de Systems et al Traıtement du Sıgnal*, I D Landau (Eds), pp 399 412, 1983

[27] J S A Hepburn and W M Wonham, "Structurally stable nonlınear regulatıon wıth step ınputs", *Mathematıcal Systems Theory*, Vol 17, No 1, pp 319 333, 1984

[28] J S A Hepburn and W M Wonham, "Error feedback and ınternal models on dıfferentıable manıfolds", *IEEE Transactıons on Automatıc Control*, Vol 29, No 5, pp 397 403, 1984

[29] J S A Hepburn and W M Wonham, "The semıstable-center-unstable manıfold near a crıtıcal element", *Journal of Mathematıcal Analysıs and Applıcatıons*, Vol 103, No 2, pp 321-1331, 1984

[30] J Huang and W J Rugh, "On a nonlınear multıvarıable servomechanısm problem", *Automatıca*, Vol 26, No 6, pp 963 972, 1990

[31] J Huang and C-F Lın, "On a robust nonlınear servomechanısum problem", *Proceedıngs of 30th IEEE Conference on Decısıon and Control*, pp 2529 2530, 1991

[32] J Huang and W J Rugh, "An approxımatıon method for the nonlınear servomechanısum problem", *IEEE Transactıons on Automatıc Control*, Vol 37, No 9, pp 1395 1398, 1992

[33] J Huang and C-F Lın, "On a robust nonlınear servomechanısm problem", *IEEE Transactıons on Automatıc Control*, Vol 39, No 7, pp 1510 1513, 1994

[34] J Huang, "Asymptotic tracking of a nonminimum phase nonlinear system with non-hyperbolic zero dynamics", *IEEE Transactions on Automatic Control*, Vol 45, No. 3, pp. 542–546, 2000.

[35] J. Huang, "Remarks on the robust output regulation problem for nonlinear systems", *IEEE Transactions on Automatic Control*, Vol. 46, No. 12, pp. 2028–2031, 2001.

[36] J. Huang, *Nonlinear Output Regulation: Theory and Application*, SIAM, Philadelphia, 2004.

[37] J. Huang and Z. Chen, "A general framework for tackling the output regulation problem", *IEEE Transactions on Automatic Control*, Vol. 49, No. 12, pp. 2203–2218, 2004.

[38] J. Huang, "An alternative approach to global robust output regulation of output feedback systems", *Journal of System Science and Complexity*, Vol. 20, No. 2, pp. 235–242, 2007.

[39] A. Ichikawa and H. Katayama, "Output regulation of time-varying systems", *Systems & Control Letters*, Vol. 55, No. 12, pp. 999–1005, 2006.

[40] A. Ilchmann and E.R. Ryan, "Universal λ-tracking for nonlinearly perturbed systems in the presence of noise", *Automatica*, Vol. 30. No. 2, pp. 337–346, 1994.

[41] A. Ilchmann and E.R. Ryan, "High-gain control without identification: a survey", *GAMM-Mitteilungen*, Vol. 31, No. 1, pp. 115-125, 2008.

[42] A. Isidori and C.I. Byrnes, "Output regulation of nonlinear systems", *IEEE Transactions on Automatic Control*, Vol. 35, No. 2, pp. 131–140, 1990.

[43] A. Isidori, *Nonlinear Control Systems, (3rd Edition)*, Springer-Verlag, N.Y., 1995.

[44] A. Isidori, "A remark on the problem of semiglobal nonlinear output rcgualtion", *IEEE Transactions on Automatic Control*, Vol. 42, No. 12, pp. 1734–1738, 1997.

[45] A. Isidori, *Nonlinear Control Systems, Volume II*, Springer-Verlag, N.Y., 1999.

[46] A. Isidori, L. Marconi, and A. Serrani, "Robust Autonomous Guidance: An Internal Model Approach", *Advances in Industrial Control*, Springer, 2003.

[47] A. Isidori and L. Marconi, "Asymptotic analysis and observer design in the theory of nonlinear output regulation", *Nonlinear Observers and Applications*, G. Besancon (Eds.), *Lecture Notes in Control and Information Sciences*, Vol. 363, pp. 181 210, 2007.

[48] A. Isidori, L. Praly, and L. Marconi, "About the existence of locally Lipschitz output feedback stabilizers for nonlinear systems", *SIAM Jorunal of Control and Optimization*, Vol. 48, No. 5, pp. 3389–3402, 2010.

[49] Z.P. Jiang and L. Praly, "Design of robust adaptive controllers for nonlinear systems with dynamic uncertainties", *Automatica*, Vol. 34, No. 7, pp. 825-840, 1998.

[50] Z.P. Jiang, "A combined backstepping and small-gain approach to adaptive output feedback control", *Automatica*, Vol. 35, No. 6, pp. 1131-1139, 1999.

[51] Z.P. Jiang and M. Arcak, "Robust global stabilization with ignored input dynamics: An input-to-state stability (ISS) small-gain approach", *IEEE Transactions on Automatic Control*, Vol. 46, No. 9, pp. 1411 1415, 2001.

[52] Z.P. Jiang, I. Mareels, D.J. Hill, and J. Huang, "A unifying framework for global regulation via nonlinear output feedback: From ISS to iISS", *IEEE Transactions on Automatic Control*, Vol. 49, No. 4, pp. 549–562, 2004.

[53] T. Kailath, *Linear Systems*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1980.

[54] R.E. Kalman, "Contributions to the theory of optimal control", *Boletin de la Sociedad Matematica Mexicana*, Vol. 5, pp. 102–119, 1960.

[55] R.E. Kalman, "Canonical structure of linear dynamical systems", *Proceedings of the National Academy of Science of the United States of America*, Vol. 48, No. 4, pp.596-600, 1962.

[56] R.E. Kalman, "Mathematical description of linear dynamical systems", *SIAM Journal Series A Control*, Vol. 1, No. 2, pp. 128-151, 1968.

[57] H.K. Khalil, "Robust servomechanism output feedback controllers for feedback linearizable systems", *Automatica*, Vol. 30, No. 10, pp. 1587–1599, 1994.

[58] H.K. Khalil, "On the design of robust servomechansums for minimum phase nonlinear systems", *International Journal of Robust and Nonlinear Control*, Vol. 10, No. 5, pp. 339 361, 2000.

[59] H.K. Khalil, *Nonlinear Systems (3rd Edition)*, Prentice Hall, Upper Saddle River, N.J., 2002.

[60] P.V. Kokotovic and M. Arcak, "Constructive nonlinear control: a historical perspective", *Automatica*, Vol. 37, No. 5, pp. 637–662, 2001.

[61] P. Krishnamurthy and F. Khorrami, "Dynamic high-gain scaling: State and output feedback with application to systmes with ISS appended dynamics driven by all states", *IEEE Transactions on Automatic Control*, Vol. 49, No. 12, pp. 2219 2239, 2004.

[62] M. Kristic, I. Kanellakopoulos, and P.V. Kokotovic, *Nonlinear and Adaptive Control Design*, John Wiley & Sons. Inc., 1995.

[63] H. Lei and W. Lin, "Universal adaptive control of nonlinear systems with unknown growth rate by output feedback", *Automatica*, Vol. 42, No. 10, pp. 1783 1789, 2006.

[64] H. Lei and W. Lin, "Error feedback and a servomechanism problem for uncertain noninear systems", *IEEE Transactions on Automatic Control*, Vol. 51, No. 7, pp. 1184–1189, 2006.

[65] L. Liu and J. Huang, "Global robust output regulation of output feedback systems with unknown high-frequency gain sign", *IEEE Transactions on Automatic Control*, Vol. 51, No. 4, pp. 625–631, 2006.

[66] L. Liu, Z. Chen, and J. Huang, "Parameter convergence and minimal internal model with an adaptive output regulation problem", *Automatica*, Vol. 45, No. 5, pp. 1206–1311, 2009.

[67] K. Ma and H.K. Khalil, "On the transient response of a nonlinear output regulator", *IEEE Transactions on Automatic Control*, Vol. 55, No. 6, pp. 1455 1460, 2010.

[68] L. Marconi, A. Isidori, and A. Serrani, "Non-resonance conditions for uniform observability in the problem of nonlinear output regulation", *Systems & Control Letters*, Vol. 53, No. 3 4, pp. 281-298, 2004.

[69] L. Marconi and A. Isidori, "A unifying approach to the design of nonlinear output regulators", *Advances in Control Theory and Applications*, C. Bonivento et al. (Eds.), *Lecture Notes in Control and Information Sciences*, Vol. 353, pp. 185 200, 2007.

[70] L. Marconi, L. Praly, and A. Isidori, "Output regulation via nonlinear Luenberger observers", *SIAM Jorunal of Control and Optimization*, Vol. 45, No. 6, pp. 2277–2298, 2007.

[71] L. Marconi and L. Praly, "Uniform practical nonlinear output regulation", *IEEE Transactions on Automatic Control*, Vol. 53, No. 5, pp. 1184 1202, 2008.

[72] L. Marconi and L. Praly, "Essential and redundant internal models in nonlinear output regulation", *Analysis and Design of Nonlinear Coontrol Systems*, A. Astolfi and L. Marconi (Eds.), pp. 263 283, Springer-Verlag Berlin Heidelberg, 2008.

[73] L. Marconi and L. Praly, " Robust asymptotic stabilization of nonlinear systems with non-hyperbolic zero dynamics", *IEEE Transactions on Automatic Control*, Vol. 55, No. 4, pp. 907–921, 2010.

[74] R. Marino and P. Tomei, *Nonlinear Control Design: Geometric, Adaptive and Robust*, Prentice Hall, 1995.

[75] R. Marino and P. Tomei, "Global adaptive observers for nonlinear systems via filtered tranformations", *IEEE Transactions on Automatic Control*, Vol. 37, No. 8, pp. 1239-1245, 1992.

[76] R. Marino and P. Tomei, "Global adaptive output-feedback control of nonlinear sytems, Part I: linear parameterization", *IEEE Transactions on Automatic Control*, Vol. 38, No. 1, pp. 17-32, 1993.

[77] R. Marino and G.L. Santosuosso, "Regulation of linear systems with unknonw exosystems of uncertain order", *IEEE Transactions on Automatic Control*, Vol. 52, No. 2, pp. 352–359, 2007.

[78] R. Marino and P. Tomei, "Output regulation for linear minimum phase systems with unknown order exosystem", *IEEE Transactions on Automatic Control*, Vol. 52, No. 10, pp. 2000-2005, 2007.

[79] R. Marino and C.M. Verrelli, "A global state feedback output regulaing control for uncertain systems in strict feedback form", *Systems & Control Letters*, Vol. 58, No. 9, pp. 682 690, 2009.

[80] A. Memon and H.K. Khalil, "Output regulation of nonlinear systems using conditional servocompensators", *Automatica*, Vol. 46, No. 7, pp. 1119 1128, 2010.

[81] K.S. Narendra and Z. Tian, "Adaptive identification and control of linear periodic systems", *Proceedings of the 45th IEEE Confernece on Decision and Control*, pp. 465 470, 2006.

[82] R.D. Nassbaum, "Some remarks on a conjecture in parameter adaptive control", *Systems & Control Letters*, Vol. 3, No. 5, pp. 243–246, 1983.

[83] V.O. Nikiforov, "Adaptive servomechanisum controller with an implicit reference modle", *International Journal of Control*, VOl. 68, No. 2, pp. 277 286, 1997.

[84] V.O. Nikiforov, "Adaptive non-linear tracking with complete compensation of unknown disturbances", *European Journal of Control*, Vol. 4, No. 2, pp. 132–139, 1998.

[85] A. Pavlov, N. van de Wouw, and H. Nijmeijer, *Uniform Output Regulation of Nonlinear Systems: A Convergent Dynamics Approach*, *Systems & Control: Foundations & Applications*, Birkhäuser, Boston, 2006.

[86] L. Praly and Z.P. Jiang, "Stabilization by output feedback for systems with ISS inverse dynamics", *Systems & Control Letters*, Vol. 21, No. 1, pp. 133 139, 1993.

[87] F. Delli Priscoli, "Robust tracking for polynomial plants", *Proceedings of European Control Conferenece*, pp. 369–373, 1993.

[88] F. Delli Priscoli, "Output regulation with nonlinear internal models", *Systems & Control Letters*, Vol. 53, No. 3–4, pp. 177–185, 2004.

[89] F. Delli Priscoli, L. Marconi, and A. Isidori, "A new approach to adaptive nonlinear regulation", *SIAM Jorunal of Control and Optimization*, vol. 45, No. 5, pp. 829 855, 2006.

[90] F. Delli Priscoli, L. Marconi, and A. Isidori, "Adaptive observers as nonlinear internal models", *Systems & Control Letters*, Vol. 55, No. 8, pp. 640 649, 2006.

[91] F. Delli Priscoli, A. Isidori, and L. Marconi, "A dissipativity-based approach to output regulation of non-minimum-phase systems", *Systems & Control Letters*, Vol. 58, No. 8, pp. 584–591, 2009.

[92] J.A. Richards, *Analysis of Periodically Time-Varying Systems*, Springer-Verlag, 1983.

[93] J. Rinzel, "A formal classification of bursting mechanisms in excitable systems", *Mathematical Topics in Population Biology, Morphogenesis and Neurosciences*, E.

Teramoto and M yamagutı (Eds), *Lecture Notes ın Bıomathematıcs*, Vol 71, pp 267 281, 1987

[94] C Rossı, A Tıllı, and M Tonıato, "Asymptotıc trackıng of non-sınusoıdal perıodıc refernces a swıtched lınear ınternal model approach", *Proceedıngs of the 17th World Congress The Internatıonal Federatıon of Automatıc Control*, pp 3895 3900, 2008

[95] W J Rugh, *Lınear System Theory*, Pıentıce-Hall, Inc , Englewood Clıffs, N J , 1993

[96] A Saberı, A A Stoorvogel, and P Sannutı, *Control of Lınear Systems wıth Regulatıon and Input Constraınts, Communıcatıons and Control Engıneerıng*, Sprınger-Verlag, Berlın, 2000

[97] S Sastry, *Nonlınear Systems Analysıs, Stabılıty, and Control*, Sprınger-Verlag New York, Inc , 1999

[98] A Scott (Eds) *Encyclopedıa of Nonlınear Scıence*, Taylor & Francıs Books, Inc , 2005

[99] H Shım, J S Kım, H Kım, and J Back, "A note on the dıfferentıal regulator equatıon for non-mınımum phase lınear systems wıth tıme-varyıng exosystems", *Automatıca*, Vol 46, No 3, pp 605 609, 2010

[100] A Serranı and A Isıdorı, "Global robust output regulatıon for a class of nonlınear systems", *Systems & Control Letters*, Vol 39, No 2, pp 133–139, 2000

[101] A Serranı, A Isıdorı, and L Marconı, "Semıglobal nonlınear output regualtıon wıth adaptıve ınternal models", *IEEE Transactıons on Automatıc Control*, Vol 46, No 8, pp 1178-1194, 2001

[102] S Seshagırı and H K Khalıl, "Robust output feedback regulatıon of mınımum-phase nonlınear systems usıng condıtıonal ıntegrators", *Automatıca*, Vol 41, No 1, pp 43–54, 2005

[103] S Seshagırı and H K Khalıl, "Robust output feedback regulatıon of mınımum-phase nonlınear systems usıng condıtıonal servocompensators", *Internatıonal Journal of Robust and Nonlınear Control*, Vol 15, No 2, pp 83–102, 2005

[104] L M Sılverman and B D O Anderson, "Controllabılıty and observabılıty ın tıme-varıable lınear systems", *SIAM Journal on Control*, Vol 5, No 1, pp 64 73, 1967

[105] L.M. Silverman and B.D.O. Anderson, "Controllability, observability and stability of linear systems", *SIAM Journal on Control*, Vol. 6, No 1, pp. 121-130, 1968.

[106] L.M. Silverman, "Synthesis of impulse response matrices by internally stable and passive realizations", *IEEE Transactions on Circuit Theory*, Vol. 15, No. 3, pp. 238 245, 1968.

[107] L.M. Silverman and H.E. Meadows, "Equivalent realizations of linear systems", *SIAM Jorunal on Applied Mathematics*, Vol 17, No. 2, pp. 393 408, 1969.

[108] L.M. Silverman, "Realization of linear dynamical systems", *IEEE Transactions on Automatic Control*, Vol. 16, No. 6, pp. 554–567, 1971.

[109] A. Singh and H.K. Khalil, "Regulation of nonlinear systems using conditional integrators", *International Journal of Robust and Nonlinear Control*, Vol. 15, No. 8, pp. 339 362, 2005.

[110] J.J.E. Slotine and W. Li, *Applied Nonlinear Control*, Prentice-Hall, Englewood Cliffs, New Jersey, 1991.

[111] E.D. Sontag and A. Teel, "Changing supply function in input/state stable systems", *IEEE Transactions on Automatic Control*, Vol. 40, No. 8, pp. 1476 1478, 1995

[112] W. Sun, "Global robust output regulation for a class of output feedback systems with nonlinear exosystems and its application", *Proceedings of the 8th World Congress on Intelligent Control and Automation*, pp. 1635–1640, 2010.

[113] V. Sundarapandian, "A relation between the output regulation and observer design for nonlinear systems", *Applied Mathematics Letters*, Vol. 16, No. 2, pp. 235–242, 2003.

[114] K.S. Tsakalis and P.A. Ioannou, *Linear Time-Varying Systems. Control and Application*, Prentice Hall, 1995.

[115] J. Wang, J. Huang, and S.T.T. Yau, "Practical output regulation based on the universal approximation theorem", *International Journal of Robust and Nonlinear Control*, Vol. 10, No. 5, pp. 439 456, 2000.

[116] L. Weiss, "On the structure theory of linear differential systems", *SIAM Journal on Control*, Vol. 6, No. 4, pp. 659–680, 1968.

[117] Z.R. Xi and Z. Ding, "Global adaptive output regulation of a class of nonlinear systems with nonlinear exosystem", *Automatica*, Vol. 43, No. 1, pp. 143–149, 2007.

[118] J.X. Xu, "A new periodic adaptive control for time-varying parameters with known periodicity", *IEEE Transactions on Automatic Control*, Vol. 49, No. 4, pp. 579–583, 2004.

[119] D. Xu and J. Huang, "Output regulation for a class of nonlinear systems using the observer based output feedback control", *Proceedings of the 2009 American Control Conference*, pp. 5327–5332, 2009.

[120] D. Xu and J. Huang, "Robust adaptive control of a class of nonlinear systems and its applications", *IEEE Transactions on Circuits and Systems I: Regular Papers*, Vol. 57, No. 3, pp. 691–702, 2010.

[121] X. Ye and J. Huang, "Decentralized adaptive output regulation for a class of large-scale nonlinear systems", *IEEE Transactions on Automatic Control*, Vol. 48, No. 2, pp. 276–281, 2003.

[122] Z. Zhang and A. Serrani, "The linear periodic output regulation problem", *Systems & Control Letters*, Vol. 55, No. 7, pp. 518–529, 2006.

[123] Z. Zhang and A. Serrani, "Adaptive robust output regulation of uncertain linear periodic systems", *IEEE Transactions on Automatic Control*, Vol. 54, No. 2, pp. 266–278, 2009.

[124] Z. Zhang and Z.X. Sun, "A novel internal model-based tracking control for a class of linear time-varying systems", *Transcations of the ASME Jorunal of Dynamic Systems, Measurement, and Control*, Vol. 132, No. 1, 011004, 2010.

[125] G. Zhou, "Restricted global robust output regulation for output feedback systems with nonlinear exosystems", *Proceedings of the 2007 IEEE International Conference on Control and Automation*, pp. 3259-3264, 2007.

Biography

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Journal:

1. X. Yang and J. Huang, "New results on robust output regulation of nonlinear systems with a nonlinear exosystem", submitted to *International Journal of Robust and Nonlinear Control*

2. X. Yang and J. Huang, "Output regulation of time-varying nonlinear systems", *Asian Journal of Control*, accepted

Conference:

1. X. Yang and J. Huang, "A New Internal Model with its Application to the Robust Output Regulation of Nonlinear Systems with a Nonlinear Exosystem", submitted to *18th IFAC World Congress, IFAC 2011*

2. X. Yang and J. Huang, "A framework for nonlinear output regulation for time-varying uncertain systems", *the 49th IEEE Conference on Decision and Control, 2010*, accepted.

3. X. Yang and J. Huang, "Output regulation of time-varying nonlinear systems", *Proceedings of the 29th Chinese Control Conference*, pp. 5833–5838, 2010.