Some Results on Steady States of the Thin-film Type Equation

ZHANG, Zhenyu

A Thesis Submitted in Partial Fulfilment
of the Requirements for the Degree of
Doctor of Philosophy
in
Mathematics

The Chinese University of Hong Kong
July 2011

Thesis/Assessment Committee

Professor WAN Yau Heng Tom (Chair)

Professor CHOU Kai Seng (Thesis Supervisor)

Professor NG Kung Fu (Committee Member)

Abstract

In this thesis we study the thin-film type equations in one spatial dimension. These equations arise from the lubrication approximation to the thin films of viscous fluids which is described by the Navier-Stokes equations with free boundary. From the structural point of view, they are fourth-order degenerate nonlinear parabolic equations, with principal term from diffusion and lower order term from external forces. In Chapter one we study the dynamics of the equations when the external force is given by a power law. Classification of steady states of this equation, which is important for the dynamics, was already known. Previous numerical studies show that there is a mountain pass scenario among the steady states. We shall provide a rigorous justification to these numerical results. As a result, a rather complete picture of the dynamics of the thin film is obtained when the power law is in the range $(1, 3)$. In Chapter two we turn to the special case of the equation where the external force is the gravity. This is important, but, unfortunately not a power law. We study and classify the steady states of this equation as well as compare their energy levels. Some numerical results are also present.

摘要

本論文中，我們主要研究了四階退化拋物型薄膜方程的穩定態和解的長時間形態。這類方程起源於由Navier - Stokcs方程描述的，自由邊界條件下對粘性流體薄膜的近似。在第一章中，我們關注一類特殊的長時間形態——山路現象，並給出山路現象以及異宿軌道的存在性證明。作為應用，描述了關於指數增長型薄膜方程的相應結論，這同時對於已有的數值模擬提供了理論上的解釋。其中，對於指數系數 $q \in (1,3)$ 的情況進行了著重分析，並詳細列表各類情況。

第二章中，我們研究了選取典型系數 $f'(x) = x^3/(x^3 + \lambda x^n)$ 的薄膜方程的穩定態。利用對週期性正穩定態的週期和面積的一些單調性結果，我們考慮穩定態的分類和數目問題。進一步地，我們也比較了不同類型穩定態之間的能量水平。最後則討論了一些數值模擬的結果。

Acknowledgement

There are a number of people without whom this thesis might not have been written, and to whom I am greatly indebted.

First and foremost, I am heartily thankful to my supervisor, Prof. Kai-Seng Chou. He has been the ideal thesis supervisor. His sage advice, insightful criticisms, and patient encouragement aided the writing of this thesis in innumerable ways.

I also wish to express my gratitude to Prof. Xian-Gao Liu of Fudan University, for his careful guidance during my studying there and continuous encouragement these years.

I should devote special thanks to my family, for their support and understanding.

Lastly but by no means least, I would like thank my friends, colleagues in CUHK, Ms, W.W. Ao, Mr. S.Z. Du, Mr. P.T. Jin, Mr. C.Y. Ng, Mr. W.H. Tse, Ms. L.P. Wang, Mr. W. Wang, Mr. X.L. Wang, Ms, W.F. Wo, Mr. C.J. Xie, Mr. J. Yang, Mr. W. Yang, Mr. W. Yao, Mr. C.M. Yip, Mr. C.L. Yu.

Contents

Introduction

Fourth-order degenerate parabolic equations of the form

$$h_t = -\nabla \cdot (a(h)\nabla\Delta h + b(h)\nabla h) \tag{1}$$

arise in the lubrication approximation to thin films of viscous fluids which are described by the Navier-Stokes equation with free boundary. The air/liquid interface is at height $z = h(x, y, t)$ and the liquid/solid interface is at $z = 0$. The coefficient $a(h)$ is positive in $(0, \infty)$ and vanishes at 0, reflecting surface tension effects. A typical choice is $a(h) = h^3 + \lambda h^n$ [21, 22, 24, 25, 32, 33] where $n \in (0, 3)$ and $\lambda \geq 0$. The coefficient of the second-order term $b(h)$ is a continuous function in $(0, \infty)$. It can reflect additional forces such as gravity $b(h) \sim h^3$ [23], van der Waals interactions $b(h) \sim h^m$, $m < 0$ [33, 20, 27, 42], or thermocapillary effects $b(h) \sim \frac{h^2}{(1+ch^2)^2}$ [34, 39], etc. One may find a wealthy information from surveys Oron et al [33] and Bertozzi [9] including many experimental, numerical and theoretic results. The analytic study of this equation was initiated in Bernis-Friedman [8] ($b \equiv 0$), Beretta et al [7], and Bertozzi-Pugh [12, 13] and now many results have been obtained.

Most attention so far has been focused on the one-dimensional case, where $h = h(x, t)$ and the liquid film is uniform in the y direction:

$$h_t = -(a(h)h_{xxx} + b(h)h_x)_x. \tag{2}$$

Usually, periodic boundary condition is imposed. Note that (2) is of conservation form, the area is preserved by the evolution

$$\int h(x,t)dx = \int h(x,0)dx. \tag{3}$$

Besides, one can check directly the following dissipation relation

$$\mathcal{E}(h(x,t)) + \int_0^t \int a(h)\big(h_{xx} + f(h)\big)_x^2 dxdt = \mathcal{E}(h(x,0)) \tag{4}$$

by introducing the energy associated to (2):

$$\mathcal{E}(h) = \frac{1}{2}\int h_x^2 dx - \int F(h)dx \tag{5}$$

The integrals in energy are over one period and

$$f'(x) = \frac{b(x)}{a(x)},\ F(x) = \int_0^x f(s)ds.$$

Although (2) is a degenerate equation, it is known that h preserves positivity under the assumption

$$0 < a(h) \leq const.\ h^n, \quad h \text{ small, for some } n \geq 3.5. \tag{6}$$

Moreover, under the further one-sided growth assumption

$$f(h) \leq const.\ (1 + |h|^q), \quad \text{for some } q < 3,$$

the solution starting from a positive. periodic initial function exists for all time. Letting $X(P,A)$ (resp. $X^+(P,A)$) be the subset of $H^1[-P/2, P/2]$ which consists of all (resp. positive) functions of period P and area A, (2) generates a flow, or more precisely a semi-flow, in $X^+(P,A)$.

The long time behavior of the flow is dictated by its associated energy and its steady states. The energy dissipation relation (4) suggests its steady state h be defined as an element in $X(P,A)$ where on each component of $\{h > 0\}$ there exists a constant c such that

$$h_{xx} + f(h) = c. \tag{7}$$

In general, steady states satisfying (7) can be divided into following three classes:

(i) Constant function $h \equiv A/P$, which clearly always exists.

(ii) Nonconstant positive steady state. It can been shown that such steady state must be periodic. Moreover, it is symmetric with respect to its maximum or minimum point ([28], Appendix A, B), whose minimal period is P/j, $j = 1, 2, \cdots$.

(iii) Droplet and configuration of droplets. A single droplet is a nonnegative steady state in an interval and vanishes outside. Its length may be shorter or equal to P. It is symmetric with respect to its midpoint and is strictly decreasing near its right endpoint. So the contact angle is well-defined. We may further group such single droplets into those with zero contact angle and with nonzero contact angle. A configuration of droplets is made up of at least two separate droplets whose interiors of their supports are mutually disjoint.

The abundance of steady states makes the study of the long time behavior of (2) different from other parabolic equations. the closest one being the Cahn-Hilliard equation. Among these steady states, three types of them are most relevant- the constant state, positive steady states of minimal period, and droplets with zero contact angle. They have better regularity; indeed, they belong to $H^2[-P/2, P/2]$.

Concerning steady states, there are two fundamental issues. First, the classification problem. namely, given $X(P, A)$, when does there exist a positive periodic steady state or a droplet or a figuration of droplets in it? If such steady state exists, is it unique? Or is it possible for two distinct steady states, for example, positive steady states have same minimal period and area? This problem has been studied extensively in a series of papers by Laugesen and Pugh [28]–[31] when $f(h)$ is a power law h^q. Based on the scaling property of the power law, it is solved to a large extent. As a typical case, for each $q \in [2, 3)$ and positive A, there exist P_d, P_c, $P_d < P_c$ such that a positive steady state of minimal period exists in $X(P, A)$ if and only if $P_d < P < P_c$, and it is unique whenever it exists. A droplet with zero contact angle exists in $X(P, A)$ if and only if $P \geq P_d$. Things become more delicate in some cases. For instance, when q belongs to a certain narrow range between 1.750 and 1.794, there is no droplet with zero contact angle, instead there are exactly two positive steady states of minimal period in $X(P, A)$ for some (P, A). After classifying the steady states, the next step is to investigate their stability. The linear and energy stability of the positive steady states are discussed in [29] and [30]. Then in [17], the energy stability of droplets and their configurations have been studied. Moreover, a comprehensive comparison of the energy levels of various steady states can be found in [30]. This being done, one moves closer to the study of the long time behavior of (2).

In Chapter 1, we focus on a special long time behavior, mountain pass scenario. It is guided by the extensive numerical works carried out in [31], where a detailed description of the mountain pass scenario among the positive steady state, constant state and the droplet with zero contact angle first observed in [30] is present. Let h_p, h_c and h_d be these steady states specified by their maxima being attained at the origin. For $q \in [2, 3)$, a perturbation of h_p of the form $h_p + \varepsilon\varphi$ for robust choices of φ leads to relaxation to h_c while the opposite perturbation $h_p - \varepsilon\varphi$ relaxes to h_d. They also observe there are hetroclinic orbits from h_p to h_c and to h_d.

We obtain general results on the long time behavior of (2) including proofs of the mountain pass scenario and the existence of heteroclinic orbits. We will describe some of these results by specifying to the power law in this introduction. First of all, it is necessary to focus on the set $X_e(P, A)$ consisting of even functions in $X(P, A)$. As f is independent of x, any translate of a (non-constant) steady state is again a steady state. However, when restricting to even functions, there are exactly two. We denote the positive steady state and droplet with zero contact angle whose minima are attained at the origin by h'_p and h'_d respectively.

Theorem A. *Consider* (2) *in the power law where* $q \in (1, 3)$. *For each* A *and* P *satisfying*

$$P^{3-q} < 4\pi^2 A^{1-q}, \tag{8}$$

all H^2−steady states in $X_e(P, A)$ are given among h_c, h_p, h_p', h_d and h_d' (provided exist). Any positive flow of (2) in $X_e(P, A)$ converges to one of these steady states uniformly as $t \to \infty$.

When the inequality in (8) is reversed, the constant state is no longer linearly stable in $X_e(P, A)$. Furthermore, there may be other H^2−steady states such as positive ones of non-minimal period as well as configurations of droplets with zero contact angle. The dynamics would be more and more complex as P increases for a fixed A.

Next we establish a mountain pass scenario among the steady states. Note that from the above discussion, for every A there exist P_d and $P_c, P_d < P_c$, such that h_p, h_d and h_c coexist in $X(P, A)$ if and only if $P \in (P_d, P_c)$ for $q \in [2, 3)$.

Theorem B. *Consider (2) in power law where $q \in [2, 3)$ and h_p, h_c and h_d coexist in $X_e(P, A)$. There is an open set U in $X_e(P, A)$ containing h_p such that the set $\{h \in U : \mathcal{E}(h) < \mathcal{E}(h_p)\}$, where $\mathcal{E}(h)$ is the energy of h, consists of two components U_1 and U_2. Any flow (2) starting from U_1 and U_2 uniformly converges to h_c and h_d respectively as $t \to \infty$. Consequently, there are heteroclinic orbits from h_p to h_c and to h_d.*

For q in a narrow range between 1.750 and 1.794, it is known there are two positive steady states of minimal period in the same $X_e(P, A)$, a mountain pass scenario will be established among these two steady states and the constant state. Besides, there are mountain pass scenarios among the constant state and two droplets in some cases, too.

A mountain pass scenario was also observed in the normalized curve shortening problem. A locally convex, immersed, closed plane curve γ which is a contracting self-similar solution of the curve shortening flow becomes a steady state for the normalized curve shortening problem. In Abresch-Langer [2] it was conjectured that a perturbation of the form $\gamma + \varepsilon$, $\varepsilon > 0$ small, starts a flow converging to a folded circle, while $\gamma - \varepsilon$ leads to the development of cusps in finite time. The conjecture was affirmatively settled in Au [5] and further studied in Wang [40] based on methods very different from those used here.

The chapter contains six sections. In Section 1.1 basic results are summarized, and a new result, Proposition 1.4, is proved. The significance of this proposition is that it shows the ω-limit set of any flow of (2) under (6) and (1.1.3) in $X^+(P, A)$ must contain an H^2-steady state. Essentially it enables us to avoid configurations of droplets with non-zero contact angle. Next, in Section 1.2 we adopt a min-max argument to prove the existence of an H^2-steady state h_m, a mountain pass solution. More precisely, h_m is characterized by

$$\mathcal{E}(h_m) = \inf_{\gamma(s)\in\Gamma} \max_s \mathcal{E}(\gamma(s)), \tag{9}$$

where Γ consists of all (continuous) paths between some fixed h_1 and h_2 in $X_e(P, A)$. In Section 4 a mountain pass scenario, Theorem 1.14, is proved under further assumptions on $h_i, i = 1, 2$, and the energy levels. In practise we will take h_i's to be "strict local minima" of the energy, see (1.3.3). Local properties necessary for the application of Theorem 1.14 are established in Section 3.

Ideas involved in the proof of Theorem 1.14 are very simple and could be sketched as follows. Assume that (a) the steady state in (9) is realized at some positive steady state h_m and (b) there are no other H^2-steady states with energy less than $\mathcal{E}(h_m)$ in $X_e(X, P)$ except h_c, h_d and h'_d. The crux is to construct an admissible path γ^* in Γ satisfying $\gamma^*(1/2) = h_m$ and $\mathcal{E}(\gamma^*(s)) < \mathcal{E}(h_m)$ otherwise. Then the flow starting at $\gamma^*(s)$, where $s \in (0, 1/2)$ is close to $1/2$, converges to h_c. For if not, it must converge to either h_d or h'_d by (b). But then the combined path from h_c to $\gamma^*(s)$ along γ^* and then from $\gamma^*(s)$ to h_d or h'_d along the flow constitute an admissible path with energy lower than $\mathcal{E}(h_m)$, a contradiction with the min-max characterization of h_m in (9). Similarly, flows starting from $\gamma^*(s)$ for $s \in (1/2, 1)$ sufficiently close to $1/2$ converge to h_d or h'_d. By the Morse lemma, the same property holds for points in U_1 and U_2 respectively.

In Section 1.5 we apply the general results to the power laws. Based on the fairly complete understanding of the classification and stability of the H^2-steady states as well as the global energy landscape, the dynamics of the flow will be discussed case by case in the range $(1,3)$. Theorems A and B as well as other results will be established and tabulated.

Further comments and some open questions concerning especially the power laws for q not belonging to $(1,3)$ can be found in Section 1.6. The difficulty extending our results to the range $q \in (-1,1)$ will be addressed. We also examine (6). So far the condition $n \geq 3.5$ has been imposed to preserve the positivity of the solution. The solution may touch down at 0 in finite time when n is less than 3.5 and then it is necessary to consider weak solutions.

The long time behavior of the solution of (2) for some f can be found in Bertozzi-Pugh [11], Tudorascu [38], Witelski et al [41] and Zhang [44]. In these works droplets with zero contact are not steady states. Asymptotic behavior of the solution to the Cauchy problem for some thin film type equations are studied in Carlen-Ulusoy [15] and Carrillo-Toscani [16]

In Chaper 2. we study the steady states of the following thin film type equation:

$$h_t = -\big((h^3 + \lambda h^n)h_{xxx} + h^3 h_x\big)_x \tag{10}$$

where $\lambda > 0$ and $n \in (0, 3)$. This equation applies to the case of thin viscous liquid film on the underside of a horizontal plane, where gravity acts as the main external force, e.g., the hanging droplet on the ceiling or on the bottom of a plate.

Under a special case that $\lambda = 0$ for (10), i.e., $a(h) = b(h) = h^3$, all steady states have the form

$$h = c + a\cos x + b\sin x,\ \ c > 0$$

constrained only by the requirement $a^2+b^2 \le c^2$, ensuring the nonnegativity. All these have a fixed minimal period 2π. Oron-Rosenau [34] and Yiantsios-Higgins [43] have studied this respectively. For stability, it is shown in [43] that a single zero contact angle droplet has the least energy among all configurations of zero contact droplets with the same area.

When $\lambda > 0$, unlike power law, the scaling property does not hold. The classification problem becomes more complicated. Let h be a positive steady state satisfying (7), where $f(0) = 0$,

$$f'(x) = \frac{x^3}{x^3 + \lambda x^n} = \frac{x^{3-n}}{x^{3-n} + \lambda}. \tag{11}$$

Multiplying (7) by h_x and integrating yields a conserved quantity,

$$\frac{1}{2}h_x^2 + F(h) - ch = d, \tag{12}$$

where $F(x) = \int_0^x f(s)ds$ and d is a constant. Denote the minimum value of h by α, then $d = F(\alpha) - c\alpha$. As f increases, we could define $h_c(\alpha)$ as the unique root of

$$f(h_c) = c. \tag{13}$$

Obviously, the admissible interval of α is $[0, h_c]$. When $\alpha = h_c$, $h \equiv h_c$ is a constant state. When $\alpha = 0$, h is a 0-droplet. Solving for the first derivative from (12),

$$h_x(x) = \pm\sqrt{2\big(F(\alpha) - c\alpha - F(h(x)) + ch(x)\big)}.$$

We invert and then integrate to obtain the inverse function $x = x(h)$ over a half-period.

$$x(h) = \frac{1}{\sqrt{2}}\int_\alpha^h \frac{dy}{\sqrt{F(\alpha) - c\alpha - F(y) + cy}}.$$

Here $h \in [\alpha, \beta]$,where β is the next root larger than α of the equation $F(x) - cx = F(\alpha) - c\alpha$. Obviously, the maximum value of h is β, and $h_x(x) > 0$ from heights α rising to β. Denote the minimal period and area of h by $P(\alpha, c)$ and $A(\alpha, c)$ respectively. We have

$$P(\alpha, c) = \sqrt{2}\int_\alpha^\beta \frac{dx}{\sqrt{F(\alpha) - c\alpha - F(x) + cx}}. \tag{14}$$

The area $A(\alpha, c)$ under a period is also found by integrating the inverse function,

$$A(\alpha, c) = 2\int_0^{\frac{P}{2}} h(x)dx = 2\int_\alpha^\beta h x_h(h)dh$$
$$= \sqrt{2}\int_\alpha^\beta \frac{xdx}{\sqrt{F(\alpha) - c\alpha - F(x) + cx}}. \tag{15}$$

If h is a zero contact angle droplet, i.e., $\alpha = 0$. The constant d in (12) vanishes. Analogically, the length and area of droplet h are given by (14) and (15) with $\alpha = 0$,

$$P(0, c) = \sqrt{2}\int_0^\beta \frac{dx}{\sqrt{cx - F(x)}} \tag{16}$$

and

$$A(0, c) = \sqrt{2}\int_0^\beta \frac{xdx}{\sqrt{cx - F(x)}}. \tag{17}$$

We will see that equations (14) and (15) are footstones of Chapter 2.

Now, consider the mapping $(\alpha, c) \mapsto (P(\alpha, c), A(\alpha, c))$ where (α, c) in the admissible set

$$\Sigma \equiv \{(\alpha, c) : \alpha \in (0, h_c), c \geq 0\}. \tag{18}$$

Setting

$$\Gamma_c \equiv \{(\alpha, c) : \alpha = h_c(c), c > 0\}, \tag{19}$$

and

$$\Gamma_d \equiv \{(0, c) : c > 0\}, \tag{20}$$

the boundary of the admissible set $\partial\Sigma$ is given by $\Gamma_c \cup \Gamma_d \cup \{(0, 0)\}$. $P(\alpha, c)$ and $A(\alpha, c)$, $(\alpha, c) \in \Gamma_d$, correspond the length and area of a zero contact angle droplet respectively. When $(\alpha, c) \in \Sigma$, $P(\alpha, c)$ and $A(\alpha, c)$ correspond the minimal period and area of a nonconstant positive steady state. Now the classification problem could be restated as: Given an arbitrary choice of (P_0, A_0), does the mapping have any preimage $(\alpha, c) \in \Sigma$? If yes, is the preimage unique?

Our main results in this chapter are:

Theorem C. *Any positive steady state of* (2) *satisfying* (11) *has its minimal period larger than* 2π. *Moreover, for any* $P_0 > 2\pi$,

$$A_{P_0} \equiv \{A(\alpha, c) : (\alpha, c) \in \Sigma, P(\alpha, c) = P_0\} \tag{21}$$

is a closed interval $[A_1, A_2]$, $A_1 < A_2$. *When* $n \geq 2$, *for any* $A_0 > 0$,

$$P_{A_0} \equiv \{P(\alpha, c) : (\alpha, c) \in \Sigma, A(\alpha, c) = A_0\} \tag{22}$$

is also a closed interval $[P_1, P_2]$, $P_1 < P_2$.

Theorem D. *There exists* $n^* \in [\frac{12}{5}, 3)$ *such that when* $n \geq n^*$, *the number of positive steady state of* (2) *and* (11) *in given* $X(P, A)$ *is finite. When* $n < n^*$, *the number of positive steady state is at most infinitely countable.*

Chapter 2 is divided into five sections. In section 2.1, we build some basic results such as monotonicity and limiting properties of $P(\alpha, c)$ and $A(\alpha, c)$, and use them to prove Theorem C. In section 2.2, after studying some properties of $E(\alpha, c)$, we give the proof of Theorem D. Some propositions about comparison of energy levels will be given in section 2.3. In section 2.4, numerical simulations are present.

Chapter 1

A Mountain Pass Scenario and Heteroclinic Orbits

1.1 Basic Results

Consider the initial value problem for the thin film type equation

$$\begin{cases} h_t + \Big[h^n(h_{xx} + f(h))_x\Big]_x = 0, \quad n > 0, \\ \\ h(\cdot, 0) = h_0. \end{cases} \tag{1.1.1}$$

Here the initial function is given and the solution is sought in the space

$$X(P, A) = \left\{ h \in H^1[-P/2, P/2] : \ h(-P/2) = h(P/2), \quad h \geq 0, \quad \int h = A \right\},$$

where P and A are positive numbers and the integral is over $[-P/2, P/2]$. The set $X(P, A)$ is endowed with the H^1-norm. Every function in $X(P, A)$ can be extended to an H^1_{loc}-function of period P. We set

$$X^+(P, A) = \{h \in X(P, A) : \ h > 0\},$$

and

$$X_e(P, A) = \{h \in X(P, A) : \ h(-x) = h(x), \ x \in [-P/2, P/2]\}.$$

We assume that f is smooth on $(0,\infty)$ and satisfies

$$f(z) \le C_1(1+z^q), \quad \forall z>0, \quad \text{for some } C_1 \text{ and } q\in(0,3). \tag{1.1.2}$$

Later a stronger condition will be imposed

$$f'(z) \le C_2(1+z^{q-1}), \quad \forall z>0, \quad \text{for some } C_2 \text{ and } q\in(0,3). \tag{1.1.3}$$

For h_0 in $X^+(P,A)$, it is easy to show that (1.1.1) has a unique, positive solution $h(t)$ for small time. The divergence structure of the equation implies that the area is conserved under the flow, so $h(t)$ belongs to $X^+(P,A)$. To study the long time behavior of the flow, the following two relations are essential. The first is the mentioned energy dissipation relation (4). Another is entropy inequality. Letting the entropy of h be

$$\mathcal{I}(h) = \begin{cases} \dfrac{1}{(2-n)(1-n)}\displaystyle\int h^{2-n}, & n\neq 1,2, \\ \displaystyle\int \log h, & n=2, \\ \displaystyle\int h(\log h-1), & n=1, \end{cases}$$

we have

$$\mathcal{I}(h(t)) + \int_0^t\!\!\int \left(h_{xx}^2 - f'(h)h_x^2\right) = \mathcal{I}(h_0), \tag{1.1.4}$$

provided h_0 is positive or $\mathcal{I}(h_0)$ is finite. As every H^1-function is Hölder continuous with exponent $1/2$, it follows from this relation that (1.1.1) preserves positivity as long as $n\ge 4$. In fact, it could be improved to $n\ge 3.5$. The proof follows the analogous one for the $f=0$ case, see [10].

From (1.1.2), F satisfies the estimate

$$F(z) \le C_3(1+z^{q+1}), \quad z\ge 0, \quad q\in(0,3).$$

Using Gagliardo-Nirenberg interpolation inequality, we have

$$\int F(h) \le C_3 A + C A^{\frac{q+3}{3}} \left(\int h_x^2 \right)^{\frac{q}{3}} .$$

As $q/3 < 1$, one can show that the following results hold .

Proposition 1.1. *Consider the energy where f satisfies* (1.1.2).

(a) For any h in $X(P, A)$, its H^1-norm is bounded if and only if its energy is bounded.

(b) There is a minimizer for

$$\inf\{\mathcal{E}(h) : \quad h \in X(P, A)\}.$$

Proposition 1.2. *For each $h_0 \in X^+(P, A)$, problem* (1.1.1) *under* (1.1.2) *and $n \ge 3.5$ has a unique global solution which is positive for all $t > 0$.*

We note the following useful estimate.

Proposition 1.3. *Let $h(t), t \ge 0$, be a solution of* (1.1.1) *in $X(P, A)$. Then*

$$|h(x, t) - h(x, s)| \le C|t - s|^{\frac{1}{8}}, \quad t, s \ge 0,$$

for some constant C depending only on the upper bound of the H^1-norm of the solution.

This proposition is proved in [8] without the f-term, and the proof extends with minor changes to include f, see [13].

For a solution of (1.1.1) under (1.1.2), as its H^1-norm is controlled by its energy, the estimate in Proposition 1.3 holds with some constant depending on the upper bound of its energy.

Propositions 1.1–1.3 extend to more general equation

$$h_t + \Big[a(h) \big(h_{xx} + f(h) \big)_x \Big]_x = 0, \tag{1.1.5}$$

where a is a positive, smooth function on $(0, \infty)$ satisfying

$$a(z) \le C z^n, \quad z \text{ near } 0, \text{ for some } n \ge 3.5. \tag{1.1.6}$$

In particular, (1.1.4) still holds where the expression of the entropy can be found in [8]. We refer to [8, 12, 13] for details and further discussion.

As discussed in Introduction, positive steady states, droplets with zero contact angle and configurations of droplets with zero contact angle exhaust all steady states which are also in $H^2_{\text{loc}}(\mathbb{R})$. Others steady states only belong to $H^1_{\text{loc}}(\mathbb{R})$.

A standard argument based on (5) shows that any global solution in $X(P, A)$ whose existence is asserted in Proposition 1.2 subconverges weakly and uniformly to some steady state. In general, the steady state lies in $H^1[-P/2, P/2]$ only. However, the following result shows that flows in $X^+(P, A)$ have better regularity.

Proposition 1.4. *Consider* (1.1.5) *under* (1.1.3) *and* (1.1.6). *Any global solution* $h(t)$ *of* (1.1.5) *in* $X^+(P, A)$ *contains a subsequence* $\{h(t_k)\}$ *converging to an* H^2-*steady state* h_* *in* $X(P, A)$ *with* $\mathcal{E}(h_*) = \inf_t \mathcal{E}(h(t))$.

We call a number c a *critical value of the energy* if there exists an H^2–steady state in $X(P, A)$ whose energy is equal to c.

Proof. Let us first look at the entropy relation (1.1.4). By (1.1.3), we have

$$\begin{aligned}\int f'(h)h_x^2 &\leq C_2 \int \left(1 + h^{q-1}\right) h_x^2 \\ &= C_2 \int h_x^2 - \frac{C_2}{q} \int h^q h_{xx} \\ &\leq C_2 \int h_x^2 + \frac{C_2}{q} \left(\varepsilon \int h_{xx}^2 + C_\varepsilon \int h^{2q}\right).\end{aligned}$$

By choosing ε satisfying $C_2\varepsilon/q = 1/2$, we deduce from (1.1.4) that

$$\int_0^t \int \left(h_{xx}^2 - C^*\right) \leq 2\mathcal{I}(h_0), \quad \forall t > 0,$$

where C^* depends on the H^1-norm of the solution which is uniformly bounded. Setting

$$g(t) = \int \left(h_{xx}^2 - C^*\right),$$

we rewrite this estimate as

$$\int_0^t g^+(s)ds \le \int_0^t g^-(s)ds + 2\mathcal{I}(h_0), \tag{1.1.7}$$

where g^+ and g^- are respectively the positive and negative parts of g.

Given k, we find from the energy relation some large s_k such that

$$\int_{s_k}^t \int a(h)\Big(h_{xx} + f(h)\Big)_x^2 \le \frac{1}{k}, \quad \forall t > s_k. \tag{1.1.8}$$

If

$$\int_0^t g^+(s)ds \le 2\mathcal{I}(h_0) + N, \ \forall t > 0,$$

where $N = C^*(s_k + 1)$, then

$$\int_0^\infty g^+(s)ds \le 2\mathcal{I}(h_0) + N,$$

which implies that the measure of the set $\{t \in [T, T+1] : g^+(t) \le 1\}$ tends to 1 as T tends to ∞. Together with (1.9), we can find some $t_k \ge s_k$ such that

$$\int a\big(h(t_k)\big)\Big(h_{xx}(t_k) + f(h(t_k))\Big)_x^2 < \frac{1}{k}, \tag{1.1.9}$$

and $g^+ \le 1$ at t_k. The last inequality means $\|h_{xx}\|_2^2 \le 1 + C^*P$. On the other hand, in case there is some T such that

$$\int_0^T g^+(s)ds > 2\mathcal{I}(h_0) + N,$$

by (1.1.7)

$$\int_0^T g^-(s)ds > N.$$

It follows from this estimate and $g^- \le C^*$ that the measure of the set $I = \{t \in [0, T] : \ g^-(t) > 0\}$ satisfies $C^*|I| > N$, hence $|I| > s_k + 1$ by our choice of N. We conclude that the set $J = \{t \in [s_k, T] : \ g^-(t) > 0\}$ has measure greater than or equal to 1. Together with (1.1.8) we find some $t_k \ge s_k$ such that (1.1.9) and $g^- > 0$, that's, $\|h_{xx}\|_2^2 \le C^*$, hold at t_k. Finally by a standard argument, $\{h(t_k)\}$ subconverges to an H^2-steady state in $X(P, A)$.

□

Proposition 1.5. *Assumptions as in Proposition 1.4, suppose further that* $h_* \in X_e(P, A)$ *is positive and there are only finitely many* H^2*-steady states on the same energy level. Then* $h(t)$ *converges to* h_* *in* $X_e(P, A)$.

Proof. If $h(t)$ subconverges but does not converge to h_*, by Proposition 1.4 we can find some $\rho_0 > 0$ and $s_j, s_j \to \infty$ such that $h(s_j) \in B_{2\rho_0} \setminus B_{\rho_0}(h_*)$ for all j. Here ρ_0 is so small that all functions in $B_{4\rho_0}(h_*)$ have a positive uniform lower bound and there are no other H^2-steady states in $B_{4\rho_0}(h_*)$ except h_*. By Proposition 1.3, there is a $\tau_0 > 0$, such that $h(s_j + \tau) \in B_{3\rho_0} \setminus B_{\rho_0/2}(h_*)$ for all $\tau \in [0, \tau_0]$. As

$$\int_{s_j}^{s_j+\tau_0} \int a(h)\big(h_{xx} + f(h)\big)_x^2 \longrightarrow 0, \quad \text{as } j \to \infty,$$

we can extract a subsequence $h(s_j + \tau_j), \tau_j \in [0, \tau_0]$, converging to some H^2-steady state in $B_{4\rho_0} \setminus B_{\rho_0/2}(h_*)$, contradiction holds. □

1.2 Mountain Pass Solutions

There is always a steady state of (1.1.5) in each $X(P, A)$, namely, the constant state. On the other hand, from Proposition 1.1 another steady state arises as a minimizer of the energy over $X(P, A)$. This steady state may or may not coincide with the constant state. And this is easily checked by using linear stability. In fact, as a minimizer is never linearly unstable and the stability of the constant state is easily verified via linearization, see Proposition 1.10 below. Once the constant state is found to be linearly unstable, the minimizer provides us with a second steady state. In many cases they are droplets with zero contact angle, see [17].

In the study of some semilinear elliptic problems where the zero function is a trivial solution, the well-known result of Ambrosetti-Rabinowitz [3] finds a positive solution by the mountain pass lemma. We will borrow the same idea to obtain a non-constant steady state for (1.1.5). The general setting of the mountain pass lemma is on a Banach space, but we need to work on $X(P,A)$, which carries the structure of a cone rather than a space. Instead of generalizing the lemma to a cone, we utilize equation (1.1.5) (under (1.1.6)) which has the advantage of area and positivity preserving to replace the pseudo-gradient vector field involved in the proof of the lemma.

Let h_1 and h_2 in $X(P,A)$. Although in practise we usually take them to be the constant state and a droplet with zero contact angle, it is not necessary to specify them when formulating the result. Set

$$\Gamma = \Big\{\gamma \in C\big([0,1]; X(P,A)\big): \quad \gamma(0) = h_1, \gamma(1) = h_2, \ \gamma(s) > 0, s \in (0,1)\Big\}.$$

It is clear that Γ is non-empty.

Theorem 1.6. *Consider* (1.1.5) *where a and f satisfy* (1.1.6) *and* (1.1.3) *respectively. Suppose that there exists $\varepsilon_0 > 0$ such that*

$$\max_{s\in[0,1]} \mathcal{E}(\gamma(s)) \geq \max\big\{\mathcal{E}(h_1), \mathcal{E}(h_2)\big\} + \varepsilon_0 \tag{1.2.1}$$

for all $\gamma \in \Gamma$. Then

$$c = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} \mathcal{E}(\gamma(s))$$

is a critical value of the energy.

For $\gamma \in \Gamma$, let $\gamma(t,s)$ be the solution of (1.1.5) using $\gamma(s)$ as the initial value. Although for each t, the curve $\gamma(\cdot,t)$ is continuous in $X(P,A)$, its endpoints may vary. In order to obtain an admissible path, we need to modify it.

Lemma 1.7. *Let $d \in (c_1, c)$, where $c_1 = \max\{\mathcal{E}(h_1), \mathcal{E}(h_2)\}$, be a non-critical* *value of the energy. For each* $s \in [0,1]$, *define*

$$T(s) = \begin{cases} 0, & \mathcal{E}(\gamma(s)) \le d, \\ t, & \mathcal{E}(\gamma(s)) > d \text{ and } \mathcal{E}(\gamma(t,s)) = d, \\ \infty, & \mathcal{E}(\gamma(t,s)) > d, \quad \forall t > 0. \end{cases}$$

Then $T(s)$ *is continuous when it is finite and the path* $\widetilde{\gamma}$ *given by*

$$\widetilde{\gamma}(t,s) = \begin{cases} \gamma(t,s), & \mathcal{E}(\gamma(t,s)) > d, \\ \gamma(T(s),s), & \mathcal{E}(\gamma(t,s)) \le d, \end{cases}$$

belongs to Γ *for every* $t > 0$.

Proof. First we claim that

$$T(s_k) \to T(s), \text{ as } s_k \to s,$$

when $T(s)$ is finite. For, as d is non-critical, for any $\varepsilon > 0$,

$$\gamma(T(s) + \varepsilon, s) < d.$$

It follows that for all sufficiently large k, $\gamma(T(s) + \varepsilon, s_k) < d$, so

$$T(s_k) \le T(s) + \varepsilon,$$

and

$$\overline{\lim_{k\to\infty}} T(s_k) \le T(s).$$

Similarly,

$$\underline{\lim_{k\to\infty}} T(s_k) \ge T(s),$$

holds.

It is clear that $\widetilde{\gamma}(t,s) \in \Gamma$ if we can show that $\widetilde{\gamma}(t,s)$ is continuous at each $s \in (0,1)$. This is trivial when $\mathcal{E}(\gamma(t,s)) > d$. Let's consider $\mathcal{E}(\gamma(t,s)) < d$ and $\mathcal{E}(\gamma(t,s)) = d$ separately.

When $\mathcal{E}(\gamma(t,s)) < d$, $\mathcal{E}(\gamma(t,s_k)) < d$ for all s_k sufficiently close to s, hence

$$\widetilde{\gamma}(t,s_k) = \gamma(T(s_k),s_k) \to \gamma(T(s),s) = \widetilde{\gamma}(t,s)$$

as $s_k \to s$.

When $\mathcal{E}(\gamma(t,s)) = d$, we have $t = T(s)$. If $\mathcal{E}(\gamma(t,s_k)) > d$,

$$\widetilde{\gamma}(t,s_k) = \gamma(t,s_k) = \gamma(T(s),s_k) \to \gamma(T(s),s) = \widetilde{\gamma}(t,s)$$

as $s_k \to s$. If $\mathcal{E}(\gamma(t,s_k)) \le d$, then $T(s_k) \le T(s)$, and

$$\widetilde{\gamma}(t,s_k) = \gamma(T(s_k),s_k) \to \gamma(T(s),s) = \widetilde{\gamma}(t,s)$$

as $s_k \to s$. □

Lemma 1.8. *If c is non-critical, there exists $\delta_0 > 0$ such that $[c-\delta_0, c+\delta_0]$ contains no critical values.*

Proof. Suppose not, there exists $\{c^k\} \to c$ where each c^k is a critical value. Let h_k be an H^2-steady state with $\mathcal{E}(h_k) = c^k$. The existence of h_k is ensured by Proposition 1.4.

For each $h = h_k$, we have

$$h_{xx} + f(h) = c_j \text{ on } I_j$$

where $\{I_j\}$ are disjoint open intervals in $[-P/2, P/2]$, so that $\{h > 0\} = \bigcup I_j$. As the contact angle is zero, by integration over I_j,

$$\int_{I_j} h_{xx} + \int_{I_j} f(h) = c_j|I_j|,$$

which implies

$$|c_j| \le C_1$$

where C_1 depends on the uniform bound on h. Next, squaring the equation and then integrating over I_j, we have

$$\int_{I_j} h_{xx}^2 + 2\int_{I_j} h_{xx} f(h) + \int_{I_j} f^2(h) = c_j^2|I_j|.$$

After using Cauchy-Schwarz inequality to the middle term on the left hand side of this inequality. we have

$$\frac{1}{2}\int_{I_j} h_{xx}^2 \le C_1^2|I_j| + \int_{I_j} f^2(h)$$

and a uniform H^2-bound on h_k is obtained by summing up all I_j. By passing to a subsequence, we attain by a standard argument that there is an H^2-steady state on the energy level c, contradiction holds. □

Proof of Theorem 1.6. Suppose on the contrary that c is not a critical value. By Lemma 1.8, there is no critical value in $[c-\delta_0. c+\delta_0]$ for some δ_0. Pick $\gamma \in \Gamma$ such that

$$\mathcal{E}(\gamma(s)) < c + \delta_0,$$

which is made possible by the definition of c, and consider $\gamma(t, s)$. By Proposition 1.4, for each s, there exists a finite t such that $\mathcal{E}(\gamma(t, s)) \le c - \delta_0$. By a compactness argument, $T(s)$ is seen to be bounded on $[0, 1]$. Taking $d = c - \delta_0$ in Lemma 1.7, $\gamma(T(s). s) = \widetilde{\gamma}(t. s) \in \Gamma$ and

$$\mathcal{E}(\widetilde{\gamma}(t, s)) \le c - \delta_0,$$

contradiction holds. Hence c must be a critical value of the energy. □

The translate of any steady state is again a steady state. This fact poses a technical problem in the proof of Theorem 1.14 below, as at one point we need to apply the Morse lemma where the steady state must be isolated. To cope with this difficulty we will confine to even functions. As the flow (1.1.5) preserves even functions, this restriction does not cause new complications. Let

$$\Gamma_e = \{\gamma \in \Gamma : \quad \gamma(s) \in X_e(P, A)\},$$

where h_1 and h_2 are even. We have

Theorem 1.9. *Consider* (1.1.5) *under* (1.1.3) *and* (1.1.6). *Suppose that* h_1 *and* h_2 *are in* $X_e(P, A)$ *and there exists* $\varepsilon_0 > 0$ *such that*

$$\max_{s\in[0,1]} \mathcal{E}(\gamma(s)) \geq \max\left\{\mathcal{E}(h_1), \mathcal{E}(h_2)\right\} + \varepsilon_0 \tag{1.2.2}$$

for all $\gamma \in \Gamma_e$. *Then*

$$c' = \inf_{\gamma\in\Gamma_e} \max_{s\in[0,1]} \mathcal{E}(\gamma(s))$$

is a critical value of the energy.

1.3 Local Properties of Steady States

We will apply Theorem 1.9 by taking h_1 and h_2 to be the constant state, a linearly stable positive state or a droplet with zero contact angle. In order to meet condition (1.2.2) we are led to studying the local properties of these steady states.

Consider the eigenvalue problem

$$\begin{cases} \varphi_{xx} + f'(h)\varphi = -\lambda\varphi + c \\ \\ \varphi \in H, \end{cases} \tag{1.3.1}$$

where h is a steady state in $X(P, A)$, c is a constant, and

$$H = \left\{\varphi \in H^1[-P/2, P/2] :\ \varphi(-P/2) = \varphi(P/2), \int \varphi = 0\right\}.$$

Define the Rayleigh quotient by

$$\mathcal{R}(\varphi) = \frac{\displaystyle\int \left(\varphi_x^2 - f'(h)\varphi^2\right)}{\displaystyle\int \varphi^2}.$$

According to the min-max principle for eigenvalue problems, for instance, in Bandle [6], the eigenvalues of (1.3.1) are given by

$$\lambda_1 = \min\big\{\mathcal{R}(\varphi) : \varphi \neq 0 \text{ in } H\big\},$$

and

$$\lambda_k = \max_{\psi_1,\ldots,\psi_{k-1}\neq 0}\ \min_{\varphi\perp\psi_1,\ldots,\psi_{k-1}} \big\{\mathcal{R}(\varphi) : \varphi \neq 0 \text{ in } H\big\}.$$

Denote the corresponding normalized k-th eigenfunction by φ_k, $\|\varphi_k\|_{L^2} = 1$. It is known that $\{\varphi_k\}$ forms a complete orthonormal set in

$$E = \left\{\varphi \in L^2[-P/2, P/2] :\ \varphi(-P/2) = \varphi(P/2), \int \varphi = 0\right\}.$$

Similarly, we may consider the eigenvalue problem

$$\begin{cases} \varphi_{xx} + f'(h)\varphi = -\lambda\varphi + c \\ \\ \varphi \in H_e, \end{cases} \tag{1.3.2}$$

where

$$H_e = \big\{\varphi \in H : \varphi(-x) = \varphi(x)\big\}$$

and obtain eigenvalues λ_k' with corresponding normalized eigenfunctions φ_k'. All these eigenfunctions form a complete orthonormal set in $\{\varphi \in E :\ \varphi(-x) = \varphi(x)\}$.

Let h be a positive steady state in $X(P, A)$. By Taylor's theorem, for each $\varphi \in H$, $h + \varepsilon\varphi \in X^+(P, A)$, ε small,

$$\begin{aligned} &\mathcal{E}(h+\varepsilon\varphi) \\ =\ & \mathcal{E}(h) + \frac{\varepsilon^2}{2}\int\left(\varphi_x^2 - f'(h)\varphi^2\right) - \int\left(F(h+\varepsilon\varphi) - F(h) - \varepsilon f(h)\varphi - \frac{\varepsilon^2}{2}f'(h)\varphi^2\right) \\ \geq\ & \mathcal{E}(h) + \frac{\varepsilon^2}{2}\int\left(\varphi_x^2 - f'(h)\varphi^2\right) - C_1\varepsilon^3\int\varphi^3. \end{aligned}$$

for some constant C_1. It is clear that the sign of the ε^2-term is important. We call the steady state *linearly unstable* in $X(P, A)$ (resp. $X_e(P, A)$) if λ_1 (resp. λ_1') is negative. On the other hand, the function $\varphi = h_x \in H$ satisfies $\varphi_{xx} + f'(h)\varphi = 0$, so 0 is always an eigenvalue with h_x as an odd eigenfunction for (1.3.1) when the steady state is non-constant. In view of this, we call h *linearly stable* in $X(P, A)$ (resp. $X_e(P, A)$) if $\lambda_1 = 0$ and is of simple multiplicity (resp. $\lambda_1' > 0$). Note that our definitions do not depend on the linearization of (1.1.5).

The following result is immediate [29].

Proposition 1.10. *The constant state $h_c \in X(P, A)$ is linearly stable if and only if $f'(h_c)P^2 < 4\pi^2$. When this holds, for each sufficiently small ρ, there exists $\varepsilon > 0$ such that*

$$\mathcal{E}(h_c + \varphi) \geq \mathcal{E}(h_c) + \varepsilon, \ \forall \varphi \in H, \quad \|\varphi\| = \rho.$$

Proposition 1.11. *A positive steady state $h \in X_e(P, A)$ is linearly stable implies that for each sufficiently small ρ, there exists $\varepsilon > 0$ such that*

$$\mathcal{E}(h + \varphi) \geq \mathcal{E}(h) + \varepsilon, \ \forall \varphi \in H_e, \quad \|\varphi\| = \rho.$$

Proof. Since h is linearly stable,

$$\begin{aligned}
\int \varphi_x^2 &= \int \left(\varphi_x^2 - f'(h)\varphi^2\right) + \int f'(h)\varphi^2 \\
&\leq \int \left(\varphi_x^2 - f'(h)\varphi^2\right) + C \int \varphi^2 \\
&\leq \int \left(\varphi_x^2 - f'(h)\varphi^2\right) + \frac{C}{\lambda_1'} \int \left(\varphi_x^2 - f'(h)\varphi^2\right) \\
&= \left(1 + \frac{C}{\lambda_1'}\right) \int \left(\varphi_x^2 - f'(h)\varphi^2\right).
\end{aligned}$$

Furthermore, we have

$$\int \varphi^3 \leq P \sup |\varphi|^3 \leq P^{5/3} \|\varphi\|^3.$$

Consequently, for $\varphi \in H_e$, $\|\varphi\| = 1$, there exists a positive constants C_2 such that

$$\mathcal{E}(h+\varepsilon\varphi) \geq \mathcal{E}(h) + C_2\varepsilon^2 - C_1 P^{5/3}\varepsilon^3,$$

and the proposition follows. Here C_1 appears in the Taylor's expansion above.

□

Now we consider the droplet with zero contact angle h_d in $X_e(P, A)$. Assume that its span P_0 (the length of its support) satisfies $P_0 \leq P$. Then h_d is an even function which vanishes at $\pm P/2$ with zero derivatives.

Proposition 1.12. *Consider* (1.3.1) *where* $f \in C^1[0,\infty)$, $f > 0$ *on* $(0,\infty)$ *satisfying*

(a) $f' > 0$,

(b) $\lim_{z\to 0} \dfrac{f(z)}{z^q} \to a > 0$, $\lim_{z\to\infty} \dfrac{f(z)}{z^p} \to b > 0$, *for some* $p, q \in (1,3)$,

(c) $f'(z) \geq Cz^{q-1}$ *near* 0,

(d) $zf'(z) - 3f(z)$ *is negative and strictly monotone.*

Let $h_d \in X_e(P, A)$ *with span* P_0 *less than* P. *For each sufficiently small* ρ, *there exists some* $\varepsilon > 0$ *such that*

$$\mathcal{E}(h) \geq \mathcal{E}(h_d) + \varepsilon, \quad \forall h \in X_e(P, A),\ \|h - h_d\|_\infty = \rho.$$

Remark 1.1 The function $f(z) = z^q$, $q \in (1,3)$, satisfies conditions (a)-(d) in this proposition. More generally, let $f(z) = az^q + bz^r + z^p$, $a > 0, 1 < q < r < p < 3$. Then (a)-(d) are fulfilled under the following conditions: For $r = (1-\lambda)p + \lambda q$.

$$\begin{cases} a(3-q) > (1-\lambda)|b|(3-r), \\ 3-p > \lambda|b|(3-r), \\ a(3-q)q > (1-\lambda)|b|(3-r)r. \\ (3-p)p > \lambda|b|(3-r)r. \end{cases}$$

In particular, b could be taken to be negative.

Proof of Proposition 1.12. We claim that there is a small ρ_0 such that h_d is the unique minimum of the energy in $\{h \in X_e(P, A) : \|h - h_d\|_\infty \le \rho_0\}$. This is sufficient for the proof of the proposition. For, if not, there exists $\{h_k\}$, $\|h_k - h_d\|_\infty = \rho_1$ for some positive $\rho_1 \le \rho_0$, such that $\mathcal{E}(h_k) \to \mathcal{E}(h_d)$. We can find a subsequence, still denoted by $\{h_k\}$, such that it weakly converges to some h^* in $X_e(P, A)$. But then $\|h^* - h_d\|_\infty = \rho_1$ and $\mathcal{E}(h^*) = \mathcal{E}(h_d)$, contradicting that h_d is a strict local minimum of the energy.

To show that h_d is a strict local minimum in sup-norm, assume on the contrary that there exists $\{h_k\} \to h_d$ uniformly, $\mathcal{E}(h_k) \le \mathcal{E}(h_d)$, $h_k \in X_e(P, A)$. Let $h_k = h_d + \varepsilon_k\varphi_k$, $\varphi_k = (h_k - h_d)/\|h_k - h_d\|_\infty$, $\varepsilon_k = \|h_k - h_d\|_\infty > 0$, $\varepsilon_k \to 0$ as $k \to \infty$. Let h_d satisfy $h_{dxx} + f(h_d) = c,\ c > 0$. We have

$$
\begin{aligned}
0 \ \ge\ & \mathcal{E}(h_d + \varepsilon_k\varphi_k) - \mathcal{E}(h_d) \\
=\ & -\varepsilon_k \int_{\{h_d>0\}} \Big(h_{dxx} + f(h_d)\Big)\varphi_k + \frac{\varepsilon_k^2}{2}\int \Big(\varphi_{k,x}^2 - f'(h_d)\varphi_k^2\Big) \\
& -\varepsilon_k^2 \iint_0^1 \Big(f'(h_d + s\varepsilon_k\varphi_k) - f'(h_d)\Big)(1-s)\varphi_k^2\, ds \\
=\ & c\varepsilon_k \int_{\{h_d=0\}} \varphi_k + \frac{\varepsilon_k^2}{2}\int \Big(\varphi_{kx}^2 - f'(h_d)\varphi_k^2\Big) \\
& -\varepsilon_k^2 \iint_0^1 \Big(f'(h_d + s\varepsilon_k\varphi_k) - f'(h_d)\Big)(1-s)\varphi_k^2\, ds
\end{aligned}
$$

Since $f'(h_d + s\varepsilon_k\varphi_k) \to f'(h_d)$ uniformly as $\varepsilon_k \to 0$, by dividing the above inequality by ε_k we have

$$
\begin{aligned}
0 \ \ge\ & \varepsilon_k^{-1}\big(\mathcal{E}(h_d + \varepsilon_k\varphi_k) - \mathcal{E}(h_d)\big) \\
\ge\ & c\int_{\{h_d=0\}} \varphi_k + \frac{\varepsilon_k}{2}\int \Big(\varphi_{k,x}^2 - f'(h_d)\varphi_k^2\Big) + o(\varepsilon_k). \qquad (1.3.3)
\end{aligned}
$$

By passing to a subsequence if necessary, $\{\varphi_k\}$ converges to some even φ_1 uniformly. In particular, $\|\varphi_1\|_\infty = 1$. As $c > 0$ and $\varphi_k \ge 0$ on $\{h_d = 0\}$, we may dispose the first term in the right hand side of (1.3.2), then divide both sides by

ε_k and pass to limit to obtain

$$0 \geq \int \left(\varphi_{1x}^2 - f'(h_d)\varphi_1^2\right). \tag{1.3.4}$$

Note that $\varphi_1(\pm P_0/2) = 0$.

On the other hand, consider the problem

$$\Lambda = \inf \left\{\mathcal{R}(\varphi) : \varphi \subset V\right\}$$

where

$$V = \left\{\varphi \in H_0^1[-P_0/2, P_0/2] : \quad \int \varphi = 0\right\},$$

where $[-P_0/2, P_0/2]$ is the support of h_d. Under assumptions (a)-(c) it is known that $\Lambda = 0$ by theorem 2.2 in [17]. As φ_1 belongs to V, (1.3.4) shows that it is a minimizer of this problem. Taking first variation, we see that $\varphi_{1xx} + f'(h_d)\varphi_1 =$ *constant*, in other words, φ_1 is an even eigenfunction of the zero eigenvalue for the eigenvalue problem

$$\begin{cases} \varphi_{xx} + f'(h_d)\varphi = -\mu\varphi + c \\ \\ \varphi \in V. \end{cases}$$

However, according to proposition 2.3 in [17], the zero eigenspace is spanned by the odd function dh_d/dx under (d). This contradiction shows that h_d must be a strict local minimum in sup-norm, and the proposition follows.

□

The local properties of the constant state and droplets just established have the following implication on the long time behavior of the flow (1.1.5). See Proposition 1.5 for the corresponding result for positive steady states.

Proposition 1.13. *Let $h_* \in X_e(P, A)$ satisfy, for each sufficiently small ρ, there exists an $\varepsilon > 0$ such that*

$$\mathcal{E}(h) \geq \mathcal{E}(h_*) + \varepsilon, \quad \|h - h_*\| = \rho \quad (\textit{resp.}\ \|h - h_*\|_\infty = \rho). \tag{1.3.5}$$

Suppose $h(t)$ is a global solution of (1.1.5) in $X_e(P, A)$ such that $\{h(t_k)\}$ converges to h_* in H^1-norm for some $t_k \to \infty$. Then $h(t)$ converges to h_* in H^1-norm (resp. in sup-norm) as $t \to \infty$.

Proof. By (1.3.5), for each $\rho > 0$ we can find some t_j such that

$$\|h(t_j) - h_*\| < \rho, \text{ and } \mathcal{E}(h(t_j)) < \inf\left\{\mathcal{E}(h) : \|h - h_*\| = \rho\right\}.$$

As the energy decreases in time, $\mathcal{E}(h(t)) \le \mathcal{E}(h(t_j)) < \inf\left\{\mathcal{E}(h) : \|h - h_*\| = \rho\right\}$ for all $t \ge t_j$. It follows that $h(t)$ cannot escape from the ball $B_\rho(h_*)$, that is, $\|h(t) - h_*\| < \rho$, $\forall t \ge t_j$. We conclude that $h(t)$ tends to h_* in H^1-norm. The same proof still works when the H^1-norm is replaced by the sup-norm. Simply observe that $\underline{\lim}_j \mathcal{E}(h(t_j)) \le \mathcal{E}(h_*)$ as $h(t_j)$ tends to h_* uniformly. □

1.4 Mountain Pass Scenario

Let us return to the mountain pass solution obtained in Section 1.2. Now we take h_1 to be the constant state h_c and h_2 the even droplet h_d. We denote the other even droplet whose minimum is attained at 0 by h'_d.

Theorem 1.14. *Assume the followings:*

(a) For each sufficiently small ρ, there exists some $\varepsilon > 0$ such that

$$\mathcal{E}(h) \ge \mathcal{E}(h_c) + \varepsilon, \ \forall h \in X_c(P, A), \ \|h - h_c\| = \rho,$$

(b) For each sufficiently small ρ, there exists some $\varepsilon > 0$ such that

$$\mathcal{E}(h) \ge \mathcal{E}(h_d) + \varepsilon, \ \forall h \in X_e(P, A), \ \|h - h_d\|_\infty = \rho,$$

(c) The number

$$c' = \inf_{\gamma \in \Gamma_e} \max_s \mathcal{E}(\gamma(s))$$

is attained at some positive H^2- steady state h_m in $X_e(P, A)$ whose maximum is attained at 0;

(d) *For some $\delta > 0$, there are no H^2-steady states in $X_e(P, A)$ with energy less than $c' + \delta$ except h_m, h'_m, h_c, h_d and h'_d.*

Then there exists an open set U containing h_m such that

$$U \cap \{h \in X_e(P, A) : \mathcal{E}(h) < \mathcal{E}(h_m)\}$$

consists of exactly two components U_1 and U_2. All flows of (1.1.5) *under* (1.1.6) *and* (1.1.3) *starting from U_1 converge to h_c in H^1-norm, while those starting from U_2 converge to h_d or h'_d uniformly as t goes to ∞. Consequently, there are heteroclinic orbits from h_m to h_c and to h_d or h'_d.*

Proof of Theorem 1.14. The proof of this theorem is based on the construction of a special path γ^* in Γ_e which satisfies $\mathcal{E}(\gamma^*(s)) < c'$, $s \neq 1/2$ and $\gamma^*(1/2) = h_m$. In the first step of the construction, consider $B_{2\rho_0}(h_m) \equiv \{h \in X_e(P, A) : \|h - h_m\| < 2\rho_0\}$ where ρ_0 is so small that (i) h_m is the only H^2-steady state in $B_{2\rho_0}(h_m)$ and (ii) all h in $B_{2\rho_0}(h_m)$ have a uniform positive lower bound. We will restrict it further as we proceed. As a first step, we construct a path p_1 starting at h_c and ending in $B_{2\rho_0} \setminus B_{\rho_0/2}(h_m)$ which satisfies $\mathcal{E}(p_1(s)) < c'$ and a path p_2 from h_d or h'_d into $B_{2\rho_0} \setminus B_{\rho_0/2}(h_m)$ satisfying $\mathcal{E}(p_2(s)) < c'$.

For each k, there exists $\gamma = \gamma^k \in \Gamma_e$, $\mathcal{E}(\gamma_k(s)) < c' + 1/k$. Solve (1.1.5) using $\gamma(s)$ as initial data to get $\gamma(t, s)$ which belongs to Γ_e for each t. Consider the set

$$S = \big\{s \in [0, 1] : \mathcal{E}(\gamma(t, s)) > c', \quad \forall t \geq 0\big\}.$$

It is clear that S is non-empty and compact in $(0, 1)$. Let

$$s_1 = \inf S, \quad \text{and } s_2 = \sup S.$$

Then $\mathcal{E}(\gamma(t, s_i)) \geq c'$, $i = 1, 2$, for all t and hence $\gamma(t, s_1)$ tends to h_m or h'_m by Proposition 1.5. If it is h'_m, we translate the path so that it converges to h_m (h_c is invariant under translation). In any case we may assume $\gamma(t, s_1)$ converges to

h_m. Let $\gamma_1(s)$ be the restriction of $\gamma(s)$ on $[0, s_1]$. As $\gamma_1(t, s_1) \to h_m$ for $t \to \infty$, without loss of generality we may assume for each t there exists some $s(t) \in (0, s_1)$ such that $\gamma_1(t, s(t)) \in \partial B_{\rho_0}(h_m) \equiv \{h \in X_e(P, A) : \|h - h_m\| = \rho_0\}$.

Note that γ_1 depends on k. By Proposition 1.3, we fix a τ_0 such that $\gamma_1(t + \tau, s(t)) \in B_{3\rho_0/2} \setminus B_{\rho_0/2}(h_m)$ for all $\tau \in [0, \tau_0]$. We claim that there exist some k_0, t_0 and τ_1 such that

$$\mathcal{E}\big(\gamma_1^{k_0}(t_0 + \tau_1, s(t_0))\big) < c'. \tag{1.4.1}$$

For, if (1.4.1) does not hold, then $\mathcal{E}\big(\gamma_1(t+\tau, s(t))\big) \geq c$ for all $\tau \in [0, \tau_0]$. Therefore,

$$\int_t^{t+\tau_0}\int a(\gamma_1)\Big(\gamma_{1xr} + f(\gamma_1)\Big)_x^2 \leq \mathcal{E}\big(\gamma_1(t, s(t))\big) - \mathcal{E}\big(\gamma_1(t + \tau_0, s(t))\big) < \frac{1}{k}.$$

As $\gamma_1(t + \tau, s(t)), \tau \in [0, \tau_0]$, have a uniform positive lower bound, by parabolic regularity we can extract a subsequence from $\gamma_1(t + \tau, s(t))$ which converges in H^1-norm to some H^2-steady state in $B_{2\rho_0} \setminus B_{\rho_0/2}(h_m)$. By (d) this is impossible, hence (1.4.1) must hold.

We take $p_1(s) = \gamma_1^{k_0}(t_0 + \tau_1, s)$, $s \in [0, \alpha]$, $\alpha \equiv s(t_0)$. Then $p_1(0) = h_c$, $p_1(\alpha) \in B_{2\rho_0} \setminus B_{\rho_0/2}(h_m)$, and $\mathcal{E}(p_1(s(\alpha)) < c'$. Furthermore, by our choice of s_1, for each $s \in [0, \alpha)$, there exists some finite t such that $\mathcal{E}(p_1(t, s)) < c'$.

Similarly, $\gamma(t, s_2)$ converges to h_m or h'_m. When it is the former, we can obtain as above a continuous path p_2 from $[\beta, 1]$, $\beta > \alpha$, to $X_e(P, A)$ satisfying $p_2(1) = h_d$, $p_2(\beta) \in B_{2\rho_0} \setminus B_{\rho_0/2}(h_m)$, $\mathcal{E}(p_2(\beta)) < c'$, and, for each $s \in (\beta, 1]$, there exists some t such that $\mathcal{E}(p_2(t, s)) < c'$. When $\gamma(t, s_2)$ converges to h'_m, by translating the path from x to $x + P/2$, the translated path now converges to h_m at s_2 and ends at h'_d. The same construction then yields a curve p_2 satisfying $p_2(1) = h'_d$. □

Next, we need to connect $p_2(\beta)$ to $p_1(\alpha)$ inside $B_{2\rho_0}(h_m)$ below the energy level c' (except at h_m). For this purpose we need to use the infinite dimensional Morse lemma. There are several versions of this lemma, here we follow Hofer [26].

Lemma 1.15. *Let $\Phi \in C^2(U, \mathbb{R})$ where U is an open set in the Hilbert space $\mathcal{H}$ containing 0. Assume that 0 is the only critical point of Φ in U and $D\Phi(\varphi)$ is of the form "identity+compact operator" for $\varphi \in U$. Let $\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^0 \oplus \mathcal{H}^+$ be the canonical decomposition via the spectral resolution with respect to $D^2\Phi(0)$. Then there exists a local origin-preserving homeomorphism η from an open subset of U to U and a C^1-map α from a 0-neighborhood of $\mathcal{H}^0$ into $\mathcal{H}^+ \oplus \mathcal{H}^-$ such that*

$$\Phi\big(\eta(u)\big) = -\frac{1}{2}\|x\|^2 + \frac{1}{2}\|z\|^2 + \Phi\big(\alpha(y) + y\big). \quad u = x + y + z. \tag{1.4.2}$$

A useful consequence is, as $\Phi(\alpha(y) + y)$ is continuous and vanishes at $y = 0$, for each small $\rho > 0$, there exists a ρ' such that

$$\big|\Phi(\alpha(y) + y)\big| < \frac{\rho^2}{16}, \quad \forall y, \ \|y\| < \rho'.$$

Letting

$$C = \big\{(x, y, z) : \|x\| + \|z\| < \rho, \|y\| < \rho'\big\},$$

where ρ and ρ' are so small that C is contained in U, the set

$$A = \left\{(x, y, z) \in C : -\frac{1}{2}\|x\|^2 + \frac{1}{2}\|z\|^2 + \Phi(\alpha(y) + y) < 0\right\}$$

is path-connected when $\dim \mathcal{H} \geq 2$, and,

$$A_+ = \left\{(x, y, z) \in C : -\frac{1}{2}\|x\|^2 + \frac{1}{2}\|z\|^2 + \Phi(\alpha(y) + y) < 0, x > 0\right\},$$

$$A_- = \left\{(x, y, z) \in C : -\frac{1}{2}\|x\|^2 + \frac{1}{2}\|z\|^2 + \Phi(\alpha(y) + y) < 0, x < 0\right\}$$

are path-connected when $\mathcal{H}^-$ is one dimensional.

To see this, simply observe when $\dim \mathcal{H}^- \geq 2$, all "$y$-slices" in A are path-connected and, as the "vertical path" $\big(x, ty_1 + (1-t)y_2, 0\big), t \in [0, 1]$, where x is any point satisfying $\|x\| \in (\rho/2, \rho)$, connects the y_1-slice to y_2-slice in A. Similarly. one can show that $A_\pm$ are path-connected when $\dim \mathcal{H}^- = 1$.

As a consequence of this lemma we have

Lemma 1.16. *Under the setting in Lemma 1.15, there is an open set U containing 0 such that*

$$\{u \in U : \Phi(u) < 0\}$$

has at most two components.

We will use this lemma in the following way. We take

$$\mathcal{H} = \left\{\varphi \in H^1[-P/2, P/2] : \quad \varphi(-P/2) = \varphi(P/2), \int \varphi = 0\right\}$$

and

$$\Phi(\varphi) = \mathcal{E}(h_m + \varphi) - \mathcal{E}(h_m).$$

Then

$$D\Phi(\varphi)\psi = \int \varphi_x \psi_x - \int \Big(f(h_m + \varphi) - f(h_m)\Big)\psi$$

and

$$D^2\Phi(0)\varphi = \int \varphi_x^2 - \int f'(h_m)\varphi^2.$$

Now Lemma 1.16 applies to yield an open set U containing h_m such that

$$\big\{h \in U : \mathcal{E}(h) < \mathcal{E}(h_m)\big\}$$

has at most two components.

Proof of Theorem 1.14 Continued. Now, we would like to connect p_1 and p_2 inside $U \cap \{h \in B_{2\rho_0}(h_m) : \ \mathcal{E}(h) < \mathcal{E}(h_m)\}$. First we restrict ρ_0 further so that $B_{2\rho_0}(h_m)$ is contained in U. Indeed, we may simply identify U with $B_{2\rho_0}(h_m)$. Then $p_1(\alpha)$ can be connected to h_m by a path p_3 in $U \cap \{\mathcal{E}(h) < \mathcal{E}(h_m)\}$ with energy less than c' except at h_m. There is a similar path p_4 for $p_2(\beta)$ in $U \cap \{\mathcal{E}(h) < \mathcal{E}(h_m)\}$. By putting the paths p_1-p_4 together we obtain, after a rescaling so that α and β go to 1/4 and 3/4 respectively, a path γ in Γ_e such that $\mathcal{E}(\gamma(s)) < c'$, $s \in [1/4, 3/4] \setminus \{1/2\}$ and $\gamma(1/2) = h_m$. Since the set $\{s \in [0,1] \setminus (1/4, 3/4) :$

$\mathcal{E}(\gamma(s)) \geq c'\}$ is compact, there exists a finite T such that $\mathcal{E}(\gamma(T, s)) < c'$ except at $s = 1/2$. Finally, $\gamma(T, \cdot)$ is our desired path γ^*.

We observe that the set $U \cap \{\mathcal{E}(h) < \mathcal{E}(h_m)\}$ has exactly two components. For, if this is not true, in the above construction one can connect $p_1(\alpha)$ to $p_2(\beta)$ by a single path p_5 inside $U \cap \{\mathcal{E}(h) < \mathcal{E}(h_m)\}$ without touching h_m. Then the corresponding path γ^* would have energy less than c' everywhere, contradicting the definition of c'.

Let U_1 and U_2 be the components of U containing $\gamma^*(1/4)$ and $\gamma^*(3/4)$ respectively. As $\mathcal{E}(\gamma^*(1/4)) < c'$ and $h(t)$, the flow starting at $\gamma^*(1/4)$, must converge to h_c, h_d or h_d' by (d). We claim that it must be h_c. For, otherwise it would tend to h_d or h_d' along a sequence of time, by Propositions 1.4 and 1.13 it converges to h_d or h_d' in H^1−norm. Nevertheless, then the path from h_c to $\gamma^*(1/4)$ along γ^* and then along $\gamma^*(t, 1/4)$ would give rise a path in Γ_e whose energy is always less than c', contradicting the definition of c'. The same situation holds for all h_0 in U_1, for each of them can be connected to $\gamma^*(1/4)$ by a path below the energy level c'. For points in U_2, a similar reasoning shows that flows starting at them converge to either h_d or h_d'.

We point out that once there is a flow starting from h_0 in U_2 which converges to h_d (resp. to h_d'), all flows starting from U_2 converge to h_d (resp. to h_d'). For, by Proposition 1.12, the sets $\{h_0 \in U_2 :$ the flow starting at h_0 converges to $h_d\}$ and $\{h_0 \in U_2 :$ the flow starting at h_0 converges to $h_d'\}$ are open. They are complement to each other, so they are closed, too.

To complete the proof, we construct a heteroclinic orbit from h_m to h_d (assuming flows from U_2 converge to h_d). A similar construction yields a heteroclinic orbit from h_m to h_c.

Let $\{h_k\}$ be a sequence converging to h_m with energy lower than c'. By Proposition 1.4, the flow $h_k(t)$ starting at h_k converges to h_d as $t \to \infty$. Define

$$t_k = \sup\{t : h_k(t) \in \partial B_{\rho_0}(h_m)\},$$

the last time h_k hits the boundary of $B_{\rho_0}(h_m)$. Since h_m is a steady state and $h_k(t)$ depends on the initial data continuously, $t_k \to \infty$ as $k \to \infty$.

Let

$$\widetilde{h}_k(t) = h_k(t + t_k), \; t \in [-t_k, \infty).$$

Then $\widetilde{h}_k(0) = h_k(t_k) \in \partial B_{\rho_0}(h_m)$ has a uniform, positive lower bound. By the entropy relation

$$\mathcal{I}(\widetilde{h}_k(t)) + \int_0^t\!\!\int \left(\widetilde{h}_{k,xx}^2 - C\widetilde{h}_k^2\right) \le \mathcal{I}(\widetilde{h}_k(0)),$$

$\mathcal{I}(\widetilde{h}_k(t)) \le C_1 + C_2 t$ independent of k for some constants C_1 and C_2. For each T, there exists a constant C_T independent of k such that

$$\widetilde{h}_k(t) \ge C_T > 0, \; t \in [-T, T].$$

We deduce from parabolic regularity that $\{\widetilde{h}_k\}$ admits uniform bounds of all order in $[-T, T] \times [-P/2, P/2]$. By taking a diagonal subsequence (still denoted by $\widetilde{h}_k$), we obtain a flow $h_\infty(t)$ of (1.1.5), $t \in (-\infty, \infty)$ such that

$$\begin{cases} \widetilde{h}_k \to h_\infty \text{ smoothly on each } [-T, T], \\ \\ h_\infty(0) \in \partial B_{\rho_0}(h_m) \end{cases}$$

By assumption (d), Propositions 1.4 and 1.13, $h_\infty(t)$ converges to h_c, h_d or h'_d uniformly as $t \to \infty$. It cannot be h_c, for then there would be an admissible path below the energy level c'. If it converges to h'_d, by Proposition 1.13, there exists a T_1 such that $h_\infty(t)$ would stay in some n'd of h'_d for all $t \ge T_1$. But then it means for all large k, $h_k(t + t_k)$ would stay in the same n'd for all $t \ge T_1$, which is impossible. Thus $h_\infty(t)$ must converge to h_d as $t \to \infty$.

To complete the proof we show that $h_\infty(t) \to h_m$ as $t \to -\infty$. Let

$$s_k = \inf\{t : h_k(t) \in \partial B_{\rho_0}(h_m)\},$$

the first time h_k hits the boundary of $B_{\rho_0}(h_m)$. Also let

$$X_1(k) = \{t : h_k(t) \in \partial B_{\rho_0/2}(h_m)\},$$

and

$$X_2(k) = \{t : h_k(t) \in \partial B_{2\rho_0}(h_m)\}.$$

We claim that there exist some $\varepsilon_0 > 0$ and k_0 such that

$$\mathcal{E}(h_k(t')) + \varepsilon_0 \leq \mathcal{E}(h_k(t)), \quad t \in X_1(k),\ t' \in X_2(k),\ k \geq k_0. \tag{1.4.3}$$

For, if not, we can find τ_k and τ'_k in $X_1(k)$ and $X_2(k)$ respectively such that

$$\int_{\tau_k}^{\tau'_k}\int a(h_k)\big(h_{kxx} + f(h_k)\big)_x^2 = \mathcal{E}(h_k(\tau_k)) - \mathcal{E}(h_k(\tau'_k)) \to 0$$

as $k \to \infty$. Moreover, τ_k and τ'_k can be chosen so that $h_k(t), t \in (\tau_k, \tau'_k)$, belongs to $B_{2\rho_0} \setminus B_{\rho_0/2}(h_m)$. By Proposition 1.4, $|\tau_k - \tau'_k| \geq \tau_0$ for some positive τ_0 independent of k. Arguing as in the proof of Proposition 1.5, there is an H^2-steady state in $B_{2\rho_0} \setminus B_{\rho_0/2}(h_m)$, but this is impossible by assumption, so (1.4.3) holds. To complete the proof let us observe by (1.4.3) that $h_k(t), t \in (s_k, t_k)$, is confined to the ball $B_{3\rho_0/2}(h_m)$ for all $k \geq k_0$. Since $h_k(t),\ t \in (0, s_k]$, is also contained in the same ball, we conclude that $h_\infty(t),\ t < 0$, belongs to $B_{2\rho_0}(h_m)$ and hence converges to h_m as $t \to -\infty$.

□

Remark 1.2 If the mountain pass solution h_m is linearly unstable, this theorem shows that it must have exactly one negative eigenvalue for (1.3.2) at h_m. From the expansion of $\varphi \in H_e$ with respect to the eigenfunctions of (1.3.2), it is easy to see that if $h_m + \varepsilon\varphi \in U_1$ for small $\varepsilon > 0$ if and only if $h_m - \varepsilon\varphi \in U_2$.

Remark 1.3 This theorem does not assert that flows starting from U_2 converges to h_d. Conceivably it may converge to h'_d. Nevertheless, it is clear from the above proof that this could not happen if there exists a path connecting points in U_2 to

h_d under the energy level c'. We will construct such paths in the case of power laws to be discussed in Section 1.5.

Remark 1.4 Theorem 1.14 remains valid when h_d is replaced by h_p, a linearly stable, positive H^2-steady state in $X_e(P, A)$. By Proposition 1.10, assumption (b) can be replaced by a corresponding assumption on h_p.

Remark 1.5 The following variant of Theorem 1.14 holds. Let h_0 be a non-constant H^2-steady state in $X_e(P, A)$ satisfying, for each sufficiently small ρ, there exists $\varepsilon > 0$ such that

$$\mathcal{E}(h) \geq \mathcal{E}(h_0) + \varepsilon, \ \forall h \in X_e(P, A), \ \|h - h_0\| = \rho \ \ (\text{or } \|h - h_0\|_\infty = \rho). \quad (1.4.4)$$

Consider

$$\Gamma = \{\gamma \in C([0,1]; X_e(P, A)) : \ \gamma(0) = h_0, \gamma(1) = h_0'\}.$$

Assume that $c' = \inf_{\gamma \in \Gamma} \max_s \mathcal{E}(\gamma(s))$ is attained at some H^2-steay state h_m and there are no other H^2-steady states on the energy levels between c' and $\mathcal{E}(h_0)$ except h_m, h_m', h_0 and h_0. Then there exists an open set U containing h_m such that $\{\mathcal{E}(h) < \mathcal{E}(h_m)\}$ consists of two components U_1 and U_2. Flows starting from U_1 and U_2 converge uniformly to h_0 and h_0' respectively as $t \to \infty$.

Now we give a criterion for when a positive mountain pass solution is a steady state of minimal period.

Proposition 1.17. *Under the assumptions of Theorem 1.14, suppose that $c' = \mathcal{E}(h_m)$ is realized at a positive steady state h_m in $X_e(P, A)$. Suppose that h_m is of minimal period P/k and is linearly unstable in $[-P/2k, P/2k]$. Then k must be 1.*

Proof. Let φ be an eigenfunction of the problem (1.3.2) in $[-P/2k, P/2k]$, $k \geq 2$, with a negative first eigenvalue λ_1'. Regarding it as an element in $[-P/2, P/2]$, λ_1' is also an eigenvalue of (1.3.2) in this interval. In the following we will show that λ_1' is not the first eigenvalue on $[-P/2, P/2]$. That means $\Lambda = \inf\{\mathcal{R}(\varphi) :$

$\varphi \in H_e\}$ is also a negative eigenvalue. As there are two negative eigenvalues for (1.3.2), the set $\{h \in B_{2\rho_0}(h_m) : \mathcal{E}(h) < \mathcal{E}(h_m)\}$ is path-connected, contradiction holds. This forces $k = 1$.

Since φ is an even function and has zero mean, it must vanish at least once in $(0, P/2k)$, so there are at least two zeros of φ on $(0, P/2)$. Denote the first and the second ones by a and b. Define $\varphi_1(x) = \varphi(x)$ for $x \in [0, a]$ and 0 elsewhere and $\varphi_2(x) = \varphi(x)$ for $x \in [a, b]$ and 0 elsewhere. We consider the function $\psi = \alpha\varphi_1 + \beta\varphi_2$ where α and β are chosen so that $\psi \in H_e$ with $\|\psi\|_{L^2} = 1$. We have

$$\mathcal{R}(\psi) = \lambda_1' - c\left(\alpha^2 \int_0^a \varphi + \beta^2 \int_a^b \varphi\right).$$

In case the second term on the right hand side of this expression is equal to zero, $\mathcal{R}(\psi) = \lambda_1'$. As ψ vanishes identically on $[b, P/2]$, by elliptic regularity it cannot be a minimizer of the eigenvalue problem (1.3.2) and so Λ is another negative eigenvalue. On the other hand, if the second term is non-zero, by switching ψ to $-\psi$ if necessary, we may assume that it is negative. Again we have $\Lambda \leq \mathcal{R}(\psi) < \lambda_1'$, the same conclusion holds.

□

1.5 Power Laws

The power law is the special case of (2) given by

$$h_t + \left[h^n\left(h_{xx} + \frac{h^q}{q}\right)_x\right]_x = 0, \tag{1.5.1}$$

(and $f(z) = \log z$ when $q = 0$) where it is always assumed $n \geq 3.5$ and $q \in (-1, 3)$ in this section. One may consult [28] for the background on this model. For $q \neq 0$, the equation for the positive steady state is

$$h_{xx} + \frac{h^q - c}{q} = 0,$$

for some constant c. Integrating over a period shows that c is positive. For $q = 0$, the analogous equation is $h_{xx} + \log h - c = 0$. Setting $h(x) = \mu\tilde{h}(\lambda x)$, by suitably choosing μ and λ to turn c into 1, we can take $\tilde{h}$ to satisfy the equation

$$\tilde{h}_{xx} + \frac{\tilde{h}^q - 1}{q} = 0, \ \ q \neq 0,$$

(and $\tilde{h}_{xx} + \log\tilde{h} = 0$, when $q = 0$.) The steady state h belongs to $X(P, A)$ if and only if $\tilde{h} \in X(\tilde{P}, \tilde{A})$ where

$$P^{3-q}A^{q-1} = \tilde{P}^{3-q}\tilde{A}^{q-1}$$

by direct computation. Suppose now h is a positive steady state of minimal period P whose maximum is attained at the origin and (minimum at $\pm P/2$). The function $\hat{h}(x) = \tilde{h}(x - \tilde{P}/2)$ satisfies the initial value problem for the ordinary differential equation

$$\begin{cases} \hat{h}_{xx} + \dfrac{\hat{h}^q - 1}{q} = 0, \\ \hat{h}(0) = \alpha \in (0,1), \ \ \hat{h}_x(0) = 0. \end{cases} \tag{1.5.2}$$

Any solution $\hat{h}$ is periodic with minimum at α. We set

$$E_q(\alpha) = \hat{P}^{3-q}\hat{A}^{q-1}.$$

where $\hat{P}$ and $\hat{A}$ are respectively the minimal period and area of the solution $\hat{h}$ starting at α. When $\alpha = 0$, this gives a single droplet with zero contact angle whose span is given by $\hat{P}$, so $E_q(0)$ is also well-defined. It is not hard to see that $E_q(\alpha)$ tends to $4\pi^2$ as $\alpha \to 1$ [28]. We set $E_q(1) = 4\pi^2$.

It is clear that any solution $\hat{h}$ of this problem of minimal period $\hat{P}$ and area $\hat{A}$ determines a positive steady state of minimal period P in $X(P, A)$ if and only if P and A satisfying

$$\chi(P, A) = E_q(\alpha),$$

where we have set $\chi(P. A) = P^{3-q}A^{q-1}$. Using this relation. we know that there is a positive steady state of minimal period in $X(P, A)$ if and only if $\chi(P, A) \in$

$\{E_q(\alpha) : \alpha \in (0,1)\}$, and there is a single droplet with zero contact angle and span P in $X(P,A)$ if and only if $\chi(P,A) = E_q(0)$.

There is a formula expressing $E_q(0)$ in terms of the beta function we will use later, namely.

$$E_q(0) = \begin{cases} \frac{2}{q}(1+q)B\left(\frac{1}{2q},\frac{1}{2}\right)^{3-q} B\left(\frac{3}{2q},\frac{1}{2}\right)^{q-1}, & q > 0, \\ 4\sqrt{3}\pi, & q = 0, \\ \frac{2}{|q|}(1+q)B\left(\frac{1}{2|q|}+\frac{1}{2},\frac{1}{2}\right)^{3-q} B\left(\frac{3}{2|q|}+\frac{1}{2},\frac{1}{2}\right)^{q-1}, & q \in (-1,0), \end{cases} \tag{1.5.3}$$

see [28] and Abramowitz-Segun [1].

The behavior of $E_q(\alpha)$ is summarized in the following proposition.

Proposition 1.18. *There exist $q_1, q_2, q_3 \in (1,2)$, $q_1 < q_2 < q_3$, such that*

(a) For $q \in (-1,1) \cup [2,3)$, $E_q'(\alpha) > 0$, $\forall \alpha \in (0,1)$;

(b) For $q \in [q_3, 2)$, $E_q'(\alpha) > 0$, $\forall \alpha \in (0,1)$;

(c) For $q \in (1, q_1]$, $E_q'(\alpha) < 0$, $\forall \alpha \in (0,1)$; and

(d) For each $q \in (q_1, q_3)$, there exists $\alpha_q \in (0,1)$ such that $E_q'(\alpha) < 0$ for $\alpha \in (0, \alpha_q)$ and $E_q'(\alpha) > 0$ for $\alpha \in (\alpha_q, 1)$. Moreover, $E_q(0) > E_q(1)$ for $q \in (q_1, q_2)$, $E_q(0) < E_q(1)$ for $q \in (q_2, q_3)$ and $E_q(0) = E_q(1)$ at $q = q_2$.

Proposition 1.18 (a) is proved in [29], and (b), (c) and (d) are shown numerically valid in [30], see Figures 1.1-1.3. Here q_1, q_2 and q_3 are approximately given by 1.750, 1.768 and 1.794. Using this proposition, one can completely answer the question when there are positive steady states in $X(P,A)$. For instance, for $q \in [2,3)$, there is a positive steady state of minimal period in $X_e(P,A)$ if and only if $\chi(P, A)$ is equal to $E_q(\alpha)$ for some $\alpha \in (0, 1)$, and, as E_q is strictly increasing for all q in this range, it is unique. On the other hand, for $q \in (q_1, q_2)$, there is a unique positive steady state of minimal period in $X_e(P, A)$ if and only if $\chi(P, A)$ is equal to $E_q(\alpha_q)$ or in $[4\pi^2, E_q(0))$ and there are exactly two steady states of minimal period in $X_e(P, A)$ if and only if $\chi(P, A) \in (E_q(\alpha_q), 4\pi^2)$.

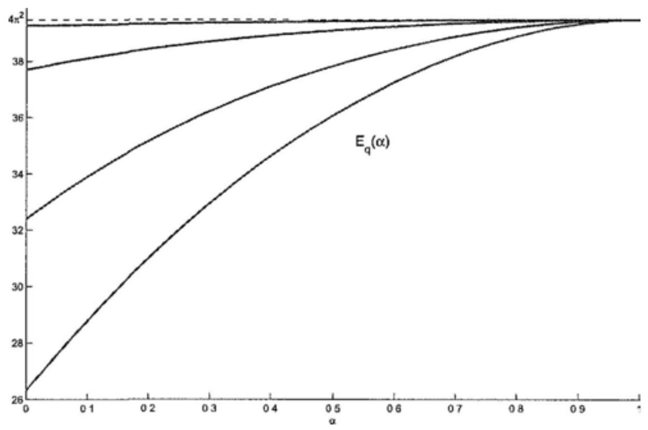

Figure 1.1: $E_q(\alpha)$, for $q = 1.8,\ 2,\ 2.5,\ 3$ from top.

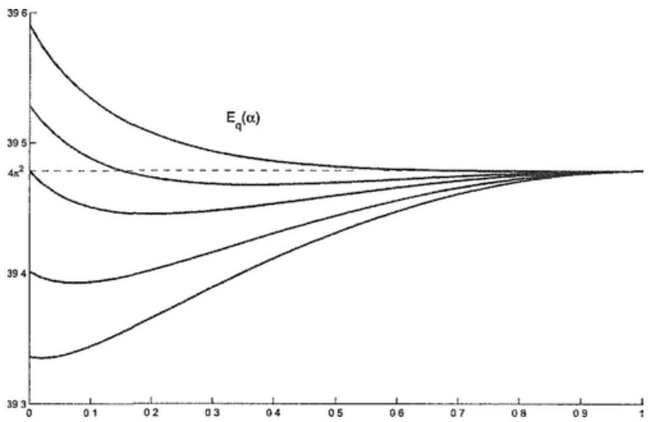

Figure 1.2: $E_q(\alpha)$, for $q = 1.75,\ 1.76,\ 1.768,\ 1.78,\ 1.79$ from top.

Besides positive steady states of minimal period and droplet with zero contact angle, there may be other H^2-steady states, that is, positive steady states of non-

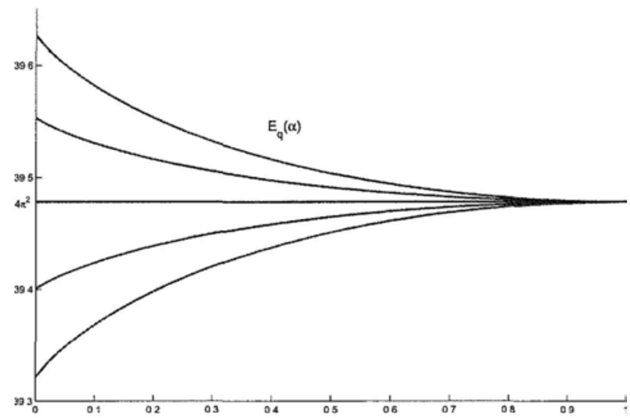

Figure 1.3: $E_q(\alpha)$, for $q = 1.02,\ 1.01,\ 1,\ 0.99,\ 0.98$ from top.

minimal period and configurations of droplets with zero contact angle, in the same $X_e(P, A)$. In the following we show, nevertheless, they cannot coexist with positive steady states of minimal period for $q \in (1, 3)$.

Proposition 1.19. *Consider $q \in (1, 3)$. Suppose there exists a positive steady state of minimal period in $X(P, A)$. There is no positive steady state of non-minimal period in $X(P, A)$.*

Proof. Consider $q \in (q_3, 3)$ first. By assumption there is a positive steady state of minimal period in $X(P, A)$, we can find some α' such that $\chi(P, A) = E_q(\alpha')$. By the same reasoning, there is an α'' such that $\chi(P/k, A/k) = E_q(\alpha'')$. We have $E_q(\alpha'') = k^{-2}E_q(\alpha')$. As E_q is strictly increasing,

$$E_q(0) < E_q(\alpha'') = \frac{E_q(\alpha')}{k^2} < \frac{4\pi^2}{k^2}.$$

As $k \geq 2$, we obtain

$$E_q(0) < \pi^2. \tag{1.5.4}$$

The plotting in Figure 1.4 using (1.5.3) shows that this is impossible. There is no positive steady state of non-minimal period in $X(P, A)$.

Similarly, for $q \in [q_1, q_3]$,

$$E_q(\alpha_q) \leq E_q(\alpha'') = \frac{E_q(\alpha')}{k^2} \leq \frac{1}{k^2}\max\{E_q(0), 4\pi^2\},$$

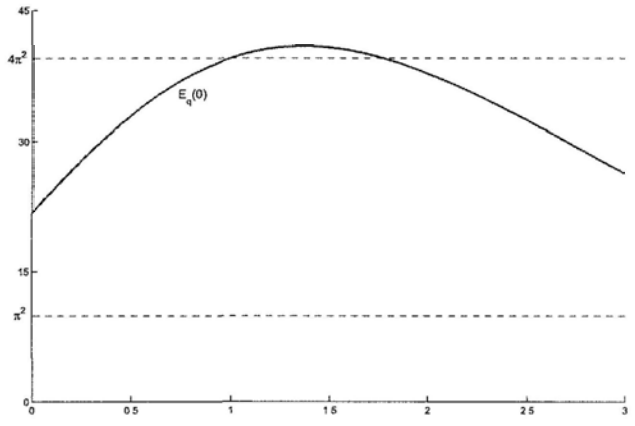

Figure 1.4: $E_q(0)$, for $q \in [0, 3]$

When $k \geq 2$, this is impossible by graph plotting, see Figure 1.5.

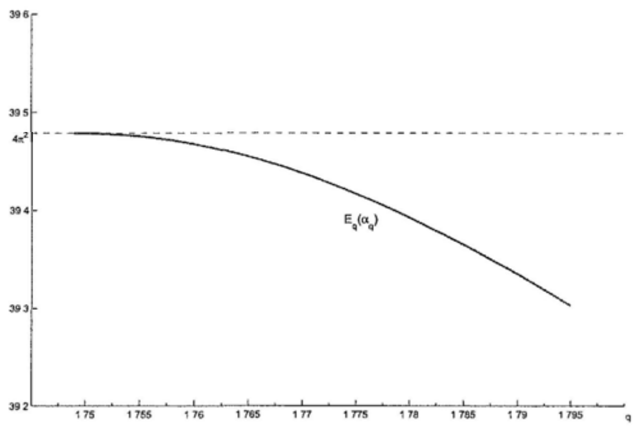

Figure 1.5: $E_q(\alpha_q)$, for $q \in (q_1, q_3)$

For $q \in (1, q_1)$, E_q is strictly decreasing. We have

$$4\pi^2 < E_q(\alpha'') = \frac{1}{k^2} E_q(\alpha') < \frac{1}{k^2} E_q(0).$$

If $k \geq 2$, $E_q(0) \geq 16\pi^2$. This is again impossible from Figure 1.4.

□

Proposition 1.20. *Consider $q \in (1,3)$. Suppose there exists a positive steady state of minimal period in $X(P,A)$. There is no configuration of droplets with zero contact angle in $X(P.A)$.*

Proof. For a configuration consisting of k many droplets with span P_j and area A_j respectively we have

$$\begin{cases} \chi(P_j, A_j) = E_q(0), j = 1, 2, \cdots, n, \\ \sum_{j=1}^{k} P_j \leq P, \\ \sum_{j=1}^{k} A_j = A. \end{cases}$$

It follows that

$$P \geq \left(E_q(0)\right)^{\frac{1}{3-q}} \sum_{j=1}^{k} A_j^{\frac{1-q}{3-q}}.$$

As $q \in (1,3)$, by convexity,

$$P \geq E_q(0)^{\frac{1}{3-q}} k \left(\frac{\sum_j^k A_j}{k}\right)^{\frac{1-q}{3-q}} = E_q(0)^{\frac{1}{3-q}} k^{\frac{2}{3-q}} A^{\frac{1-q}{3-q}},$$

which implies

$$\max\{4\pi^2, E_q(0)\} \geq \chi(P, A) \geq k^2 E_q(0).$$

When $k \geq 2$, we obtain again (1.5.4), which is impossible. Hence configurations of droplets with zero contact angle and positive steady states of minimal period do not coexist.

□

By Propositions 1.18–1.20 there are finitely many H^2-steady states in $X_e(P,A)$ for $q \in (1,3)$, all given among h_c, h_p, h_p', h_d and h_d'. Consequently, Theorem A holds in view of Propositions 1.5 and 1.13. Note that in order to apply these results, a restriction to $q > 1$ is needed.

From Proposition 1.10 we deduce the following stability result on the constant state.

Proposition 1.21. *The constant state $h_c \in X_e(P, A)$ is linearly stable if and only if*

$$\chi(P, A) < 4\pi^2.$$

Under this condition, for each sufficiently small ρ , there exists an $\varepsilon > 0$ such that

$$\mathcal{E}(h) \geq \mathcal{E}(h_c) + \varepsilon, \quad \forall h \in \partial B_\rho(h_c). \tag{1.5.5}$$

As an application of Proposition 3.3, we have the following stability result on droplets with zero contact angle.

Proposition 1.22. *Consider $q \in (1, 3)$. Let $h_d \in X_e(P, A)$ be a droplet whose span is less than P. For each sufficiently small ρ, there exists an $\varepsilon > 0$ such that*

$$\mathcal{E}(h) \geq \mathcal{E}(h_d) + \varepsilon, \quad \forall h \in X_e(P, A), \ \|h - h_d\|_\infty = \rho. \tag{1.5.6}$$

With these results at our disposal, we now discuss the dynamics of the positive global solutions of (1.5.1) in $X_e(P, A)$. We will consider four cases: $q \in [q_3, 3), (q_2, q_3), (q_1, q_2]$ and $(1, q_1]$, separately.

We begin with some general remarks. The graphs of the functions $\chi(P, A) = E_q(1)$ and $E_q(0)$ form two curves in the (P, A)–plane. When $q \in (1, 3)$ (resp. $(0, 1)$), they decrease (resp. increase) from ∞ (resp. 0) to 0 (resp. ∞) as $P \to \infty$. For every $A > 0$, the horizontal line passing A intersects these curves at (P_c, A) and (P_d, A) respectively. For $P < P_c$, $\chi(P, A) < \chi(P_c, A) = 4\pi^2$, hence the constant state h_c in $X_e(P, A)$ satisfies (1.5.5) by Proposition 1.21. On the other hand, for $P > P_d$, the droplet with zero contact angle and span P_d belongs to $X_e(P, A)$ and (1.5.6) holds by Proposition 1.22.

For $q \in [2, 3)$, E_q strictly increases in α by Proposition 1.18. Referring to Figure 1.6. the lower curve in this figure is the graph of $\chi(P, A) = E_q(0)$ and the upper curve is $\chi(P, A) = E_q(1)$. As α increases, the graphs $\chi(P, A) = E_q(\alpha)$

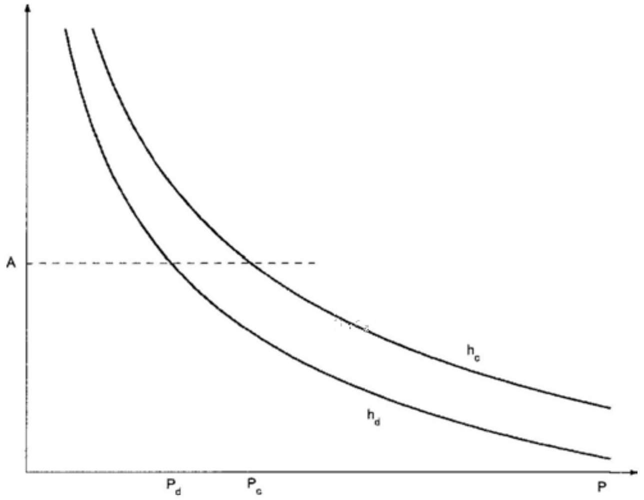

Figure 1.6: $q \in (q_3, 3]$. The upper curve is $\chi(P, A) = E_q(1)$ and the lower curve is $\chi(P, A) = E_q(0)$.

foliate the region bounded by these two curves. Every point (P, A) in the interior of this region corresponds to a positive steady state with minimal period in $X_e(P, A)$ (uniquely specified by its maximum attained at the origin). There are no steady states with minimal period in $X_e(P, A)$ when (P, A) does not belong to the interior of this region.

For $P \in (P_d, P_c)$, h_c, h_d, h_d', h_p, and h_p' coexist, and they exhaust all H^2-steady states according to Propositions 1.19 and 1.20. In view of Propositions 1.21 and 1.22. Theorem 1.14 applies to yield a mountain pass scenario among h_p, h_c and h_d (or h_d'). And there are heteroclinic orbits from h_p to h_c and to h_d (or h_d'). At $P = P_c$, the H^2-steady states are h_c, h_d and h_d'. As (1.5.6) holds for h_d and h_d', we may use the min-max scheme consisting of paths from h_d to h_d' to obtain a mountain pass scenario, see Remark 1.5. Since there are no H^2-steady states other than the constant state, the mountain pass solution must be h_c.

For $P \in (0, P_d)$, h_c is the only H^2-steady state in $X_e(P, A)$. It follows that any positive global solution (1.5.1) converges smoothly to h_c as $t \to \infty$. This is short wave stability. At $P = P_d$, there are three H^2-steady states given by h_c, h_d, and h'_d. It is known that $\mathcal{E}(h_c) < \mathcal{E}(h_d)$ ([30]), so h_c is the global minimizer of the energy. It can be shown that the first eigenvalue of problem (1.3.1) at h_d is negative as $E'_q(0) > 0$. In fact, the proof is identical to the proof of the same property for positive steady states of minimal period in [29]. When $q > 2$, a short proof is as follows. Observing that $f^{(3)}(z) = (q-1)(q-2)z^{q-3} > 0$,

$$\begin{aligned}\int \left(h_{dxx}^2 - f'(h_d)h_{dx}^2\right) &= -\int h_{dx}\left(h_{dxxx} + f'(h_d)h_{dx}\right)\\ &= \int f''(h_d)h_{dx}^2 h_{dxx} = -\frac{1}{3}\int f^{(3)}(h_d)h_{dx}^4 < 0.\end{aligned}$$

For ε small, the perturbation h_{dxx} is admissible as

$$h_d + \varepsilon h_{dxx} = \varepsilon c + h_d\left(1 - \frac{\varepsilon}{q}h_d^q\right) > 0.$$

By Theorem 1.14 there are heteroclinic orbits from h_d to h_c. Similarly, there are such orbits from h'_d to h_c. Theorem B holds in view of the above discussion.

By the numerically confirmed result in Proposition 1.18 (b), all results valid for $q \in [2,3)$ extend to $q \in [q_3, 2)$ except the assertion on the existence of heteroclinic orbits from h_d to h_c at $q = q_3$ and $P = P_d$. This is because in this limiting case $E'_q(0) = 0$ and the first eigenvalue of problem (1.3.1) at h_d is 0. We leave open the question whether h_d is a local minimum of the energy or not in this case.

The situation is illustrated in Table 1.1.

P	$(0, P_d)$	P_d	(P_d, P_c)	P_c
H^2 – steady states	h_c	h_c, h_d, h'_d	$h_c, h_d, h'_d, h_p, h'_p$	h_c, h_d, h'_d
Global minimizer	h_c	h_c	h_c or h_d, h'_d	h_d, h'_d
Heteroclinic orbits and MPS		$h_d \to h_c$ $h'_d \to h_c$	$h_c \leftarrow h_p \to h_d$ $h_c \leftarrow h'_p \to h'_d$	$h_d \leftarrow h_c \to h'_d$

Table 1.1: Dynamics of (1.5.1) in $X_e(P, A)$ for $q \in [q_3, 3)$. Here $h_1 \to h_2$ indicates a heteroclinic orbit from h_1 to h_2 and $h_1 \leftarrow h_2 \to h_3$ indicates a mountain pass scenario where h_2 is the mountain pass. Existence of a heteroclinic orbit from h_d or h'_d to h_c is not known at $q = q_3$ and $P = P_d$. Comparison of the energy levels between h_c and h_d is discussed in theorem 11 in [30].

For $q \in (q_2, q_3)$. according to Proposition 1.18(d), a positive steady state of minimal period only exist in the region bounded by $\chi(P, A) = E_q(\alpha_q)$ and $\chi(P, A) = E_q(1)$. The horizontal line passing A intersects $\chi(P, A) = E_q(\alpha_q)$ at P_q.

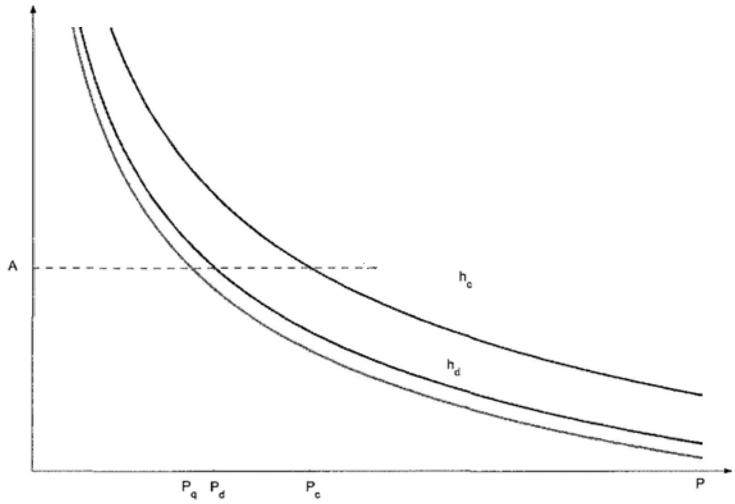

Figure 1.7: $q \in (q_2, q_3)$. The lower curve is given by $\chi(P, A) = E_q(\alpha_q)$. Points between $\chi = E_q(0)$ and $\chi = E_q(\alpha_q)$ admit two solutions h_{p1} and h_{p2}.

When $P < P_q$, h_c is the only H^2-steady state in $X_e(P. A)$, and every global solution of (1.5.1) converges smoothly to it as $t \to \infty$. For $P \in (P_q. P_d)$, there are exactly two positive steady states with minimal period (specified by maxima being attained at $x = 0$) h_1 and h_2, where $h_{p1}(0) > h_{p2}(0)$. By theorem 9 in [30], h_{p1} is energy stable under the numerically confirmed assumption that $\alpha \mapsto \alpha P(\alpha)^{2/q-1}$ strictly increases on $(0, 1)$. According to the definition of energy stability. for every non-zero $\varphi \in H^1[-P/2, P/2]$ with zero mean, there exists some ε_0 such that $\mathcal{E}(h_{p1} + \varepsilon\varphi) > \mathcal{E}(h_{p1})$, $\forall \varepsilon \in (0, \varepsilon_0)$. Clearly, it implies

$$\int (\varphi_x^2 - f'(h_{p1})\varphi^2) \geq 0, \quad \forall \varphi \in H^1[-P/2, P/2], \int \varphi = 0.$$

On the other hand, by theorem 10(a) in [29], the zero eigenspace of (1.3.1) is spanned by the odd function dh_{p1}/dx. It follows that

$$\inf\left\{\int\left(\varphi_x^2 - f'(h_{p1})\varphi^2\right):\quad \varphi \text{ is even },\int\varphi^2=1,\int\varphi=0\right\}>0,$$

which means that the first eigenvalue of (1.3.2) is positive. Using Proposition 1.11, we find some ρ and ε such that

$$\mathcal{E}(h)\geq\mathcal{E}(h_{p1})+\varepsilon,\quad \forall h\in\partial B_\rho(h_{p1}).$$

By Remark 1.4, we may apply Theorem 1.14 to conclude that h_{p2}, h_{p1} and h_c form a mountain pass scenario. When $P = P_q$, the H^2-steady states are h_c, h_p and h_p'. Letting $P \downarrow P_q$, we see that $\mathcal{E}(h_c) \leq \mathcal{E}(h_p)$. It is not hard to show that h_c is the global minimizer in $X_e(P, A)$. An extra argument is needed to show that h_p is not a local minimum in this case. Indeed, pick $P_k \downarrow P_q$ and note that h_{p2}^k converges to h_p in $H^1[-P_q/2, P_q/2]$. Choose a heteroclinic orbit $h^k(t)$ from h_{p2}^k to h_c^k where h_{p2}^k and h_c^k are respectively the unstable positive steady state of minimal period and constant state in $X_e(P_k, A)$. We claim that for any small $\rho > 0$, there is some positive δ such that

$$\inf_k \mathcal{E}\big(h^k(t_k)\big)+\delta\leq\mathcal{E}(h_{p2}^k)$$

where t_k is the first time $h^k(t)$ hitting $\partial B_\rho(h_{p2}^k)$. For, if on the contrary, there is a subsequence still denoted by $\{h^k\}$ satisfying $h^k(t_k) \in \partial B_\rho(h_{p2}^k)$ and

$$\mathcal{E}(h_{p2}^k)-\mathcal{E}\big(h^k(t_k)\big)\to 0^+,$$

as $k \to \infty$. For each t_k, trace back in time to get

$$s_k=\sup\big\{s<t_k : h^k(s)\in\partial B_{1/2\rho}(h_{p2}^k)\big\}.$$

By Proposition 1.3, fix a $\tau_0 > 0$ only depending on ρ so that $s_k + \tau_0 < t_k$ and

$$\int_{s_k}^{s_k+\tau_0}\int a(h^k)\big(h_{xx}^k+f(h^k)\big)_x^2\to 0,$$

as $k \to \infty$. Arguing as in the proof of Theorem 1.14, we can extract a subsequence of $\{h^k\}$ converging to some H^2-steady state in $B_{2\rho}\setminus B_{\rho/2}(h_p)$, contradiction holds.

Thus the claim holds. For any ρ small, letting $k \to \infty$ in $h^k(t_k)$, we get $h' \in \partial B_\rho(h_p)$ with $\mathcal{E}(h') + \delta \leq \mathcal{E}(h_p)$. This shows that h_p is not a local minimum, and a heteroclinic orbit from h_d to h_c can be constructed in the usual way.

When $P = P_d$, the H^2-steady states are h_c, h_d, h'_d, h_p, and h'_p. Viewing it as a limiting case $P \uparrow P_d$, h_{p1} and h_{p2} tend to h_d and h_p respectively, there is a mountain pass scenario among h_p, h_c and h_d. The case $P \in (P_d, P_c)$ is the same as in $q \in [2, 3)$ where a mountain scenario occurs among h_p, h_c and h_d (or h'_d). Finally, at $P = P_c$, the H^2-steady states are h_c, h_d and h'_d. Same as $q \in [2, 3)$, there is a mountain scenario among h_c, h_d and h'_d.

The situation is illustrated in Table 1.2.

P	$(0, P_q)$	P_q	(P_q, P_d)	$[P_d, P_c)$	P_c
H^2-steady states	h_c	h_c, h_p, h'_p	$h_c, h_{p_1}, h'_{p_1}, h_{p_2}, h'_{p_2}$	$h_c, h_d, h'_d, h_p, h'_p$	h_c, h_d, h'_d
Global minimizer	h_c	h_c	h_c or h_{p_1}, h'_{p_1}	h_c or h_d, h'_d	h_d, h'_d
Heteroclinic orbits and MPS		$h_p \to h_c$ $h'_p \to h_c$	$h_{p_1} \leftarrow h_{p_2} \to h_c$ $h'_{p_1} \leftarrow h'_{p_2} \to h_c$	$h_d \leftarrow h_p \to h_c$ $h'_d \leftarrow h'_p \to h_c$	$h_d \leftarrow h_c \to h'_d$

Table 1.2: Dynamics of (1.5.1) in $X_e(P, A)$ for $q \in (q_2, q_3)$.

A similar discussion covers the cases $(q_1, q_2]$ and $(1, q_1]$. We omit the details and simply refer to Tables 1.3 and 1.4.

P	$(0, P_q)$	P_q	(P_q, P_c)	$[P_c, P_d)$	P_d
H^2-steady states	h_c	h_c, h_p, h'_p	$h_c, h_{p_1}, h'_{p_1}, h_{p_2}, h'_{p_2}$	h_c, h_p, h'_p	h_c, h_d, h'_d
Global minimizer	h_c	h_c	h_c or h_{p_1}, h'_{p_1}	h_p, h'_p	h_d, h'_d
Heteroclinic orbits and MPS		$h_p \to h_c$ $h'_p \to h_c$	$h_{p_1} \leftarrow h_{p_2} \to h_c$ $h'_{p_1} \leftarrow h'_{p_2} \to h_c$	$h_p \leftarrow h_c \to h'_p$	$h_d \leftarrow h_c \to h'_d$

Table 1.3: Dynamics of (1.5.1) in $X_e(P, A)$ for $q \in (q_1, q_2]$

Let us now consider the following question. In the mountain pass scenario, for example, among h_p, h_c and h_d (or h'_d) when $q \in [q_3, 3)$, it is still not known

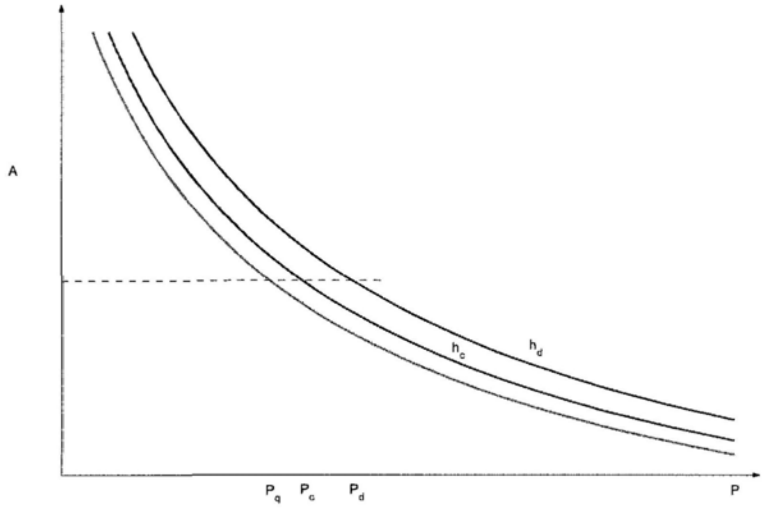

Figure 1.8: $q \in (q_1, q_2)$. The lower curve is given by $\chi(P, A) = E_q(\alpha_q)$. Points between $\chi = E_q(\alpha_q)$ and $\chi = E_q(1)$ admit two solutions h_{p1} and h_{p2}.

P	$(0, P_c)$	P_c	(P_c, P_d)	P_d
H^2 - steady states	h_c	h_c	h_c, h_p, h'_p	h_c, h_d, h'_d
Global minimizer	h_c	h_c	h_p, h'_p	h_d, h'_d
Heteroclinc orbits and MPS			$h_p \leftarrow h_c \rightarrow h'_p$	$h_d \leftarrow h_c \rightarrow h'_d$

Table 1.4: Dynamics of (1.5.1) in $X_e(P, A)$ for $q \in (1, q_1]$

whether the heteroclinic orbit from h_p goes to h_d or to h'_d. We would like to show that it must be h_d. As pointed out in Remark 1.3, this can be accomplished by showing there is a path from a point arbitrarily close to h_p to h_d lying below the energy level c'. We shall construct such a path explicitly below for the special case $q = 2$ and $\alpha = 1/2$. We explain why it is sufficient to ensure there are heteroclinic orbits from h_p to h_d for all $q \in (q_3, 3)$ and $\alpha \in (0, 1)$.

For each fixed P_0, the vertical line at P_0 intersects $\chi(P, A) = E_q(0)$ and $E_q(1)$ at A_d and A_c respectively. There is a positive steady state h_p in $X_e(P_0, A)$ if and only if $A \in (A_d, A_c)$. In fact, $A = \left(P_0^{q-3} E_q(\alpha)\right)^{1/(q-1)}$, so these steady states are

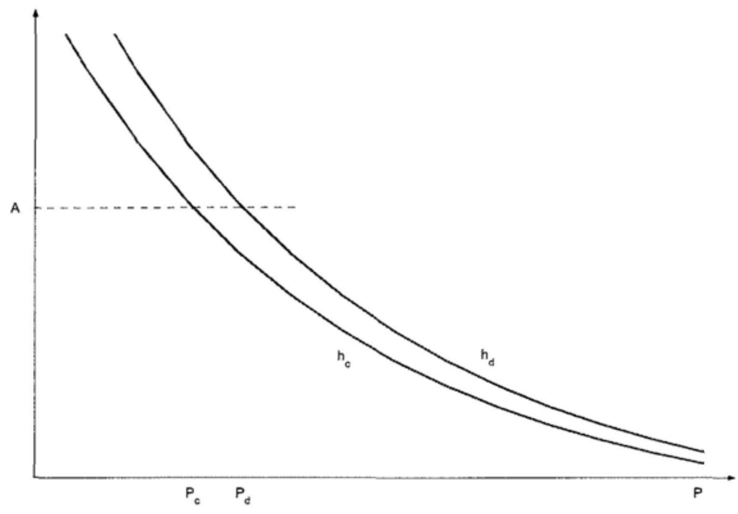

Figure 1.9: $q \in (1, q_1)$. The upper curve is $\chi(P, A) = E_q(0)$ and the lower curve is $\chi(P, A) = E_q(1)$.

in one-to-one correspondence with α. Each A determines P_d by a horizontal line as before, and there is a droplet h_d in $X_e(P_0, A)$ with span $P_d < P_0$. Both h_p and h_d are parametrized by α in $(0, 1)$. Let

$$S = \{\alpha \in (0, 1) : \text{ There is a heteroclinic orbit from } h_p \text{ to } h_d\}.$$

For h_{d_0} with $\alpha_0 \in S$, there exist some ρ_0 and ε_0 such that

$$\mathcal{E}(h) \geq \mathcal{E}(h_{d_0}) + \varepsilon_0, \quad \forall h \in X_e(P_0, A(\alpha_0)), \ \|h - h_{d_0}\|_\infty = \rho_0.$$

It is not hard to show that then for sufficiently small $\varepsilon > 0$, there exists δ such that for all h_d, $\|h_d - h_{d_0}\| < \delta$,

$$\mathcal{E}(h) \geq \mathcal{E}(h_d) + \frac{\varepsilon_0}{2}, \quad \forall h \in X_e(P_0, A(\alpha)), \ \|h - h_d\|_\infty = \rho_0.$$

Using this property, one can easily show that the set S is open in $(0, 1)$. The same reasoning shows that

$$S' = \{\alpha \in (0, 1) : \text{ There is a heteroclinic orbit from } h_p \text{ to } h_d'\}$$

is also open. It follows that both S and S' are open and closed, so $S = (0, 1)$ if there is a heteroclinic orbit from h_p to h_d for a particular α. By scaling P_0 to any other P, the result holds for in every $X_e(P, A)$.

It also extends to all other q in $(q_2, 3)$ by a continuity argument as long as h_p, h_d and h_c coexist.

Now. let us show that the line segment connecting h_p and h_d lies below the energy level c' when $q = 2$ and $\alpha = 1/2$. Consider $\gamma(s) = (1 - s)h_p + sh_d$, for $s \in [0, 1]$. We have

$$g(s) = \mathcal{E}(\gamma(s)) = \frac{1}{2}\int \Big((1-s)h_{px} + sh_{dx}\Big)^2 - \frac{1}{6}\int \Big((1-s)h_p + sh_d\Big)^3.$$

We would like to show that $g < c'$ on $(0, 1]$ at $\alpha = 1/2$. Observe that g is a cubic polynomial, so it has at most two critical points. As h_p is a steady state, $s = 0$ is a critical point for g. Furthermore, $g(1) = \mathcal{E}(h_d) < c' = \mathcal{E}(h_p) = g(0)$. Take $h_d(x) = \hat{h}(x; 0)$ and $h_p(x) = \lambda^{1/2}\hat{h}(\lambda^{1/4}x; \alpha)$ where $\hat{h}(x; \alpha)$ satisfies (1.5.2)($q = 2$) and λ is chosen so that h_p and h_d have the same area. By direct computation,

$$\begin{aligned}
g''(0) &= \int (h_d - h_p)_x^2 - \int h_p(h_d - h_p)^2 \\
&= -\int h_p(h_d - h_p)^2 + \int h_{pxx}(h_d - h_p) - \int h_{dxx}(h_d - h_p) \\
&= -\int h_p(h_d - h_p)^2 + \int \left(c - \frac{1}{2}h_p^2\right)(h_d - h_p) - \int_{\{h_d>0\}} \left(\frac{1}{2} - \frac{1}{2}h_d^2\right)(h_d - h_p) \\
&= -\frac{1}{2}\int (h_d - h_p)^3 - \frac{1}{2}\int_{\{h_d>0\}} (h_d - h_p). \qquad (1.5.7)
\end{aligned}$$

At $\alpha = 1/2$, we have $g''(0) = -0.2538 < 0$. It follows that g has a strict local maximum point at $s = 0$. In case $g \geq c'$ somewhere on $(0, 1]$, g must possess at least two more critical values, which is impossible. We conclude that the energy of the path $(1 - s)h_p + sh_d$ is less than c' for all s in $(0, 1]$.

A similar consideration shows that in the region $q \in (q_1, q_3)$ where h_{p1}, h_{p2} and h_c coexist, flows starting from U_2 converge uniformly to h_d if this is so for a particular q and α. Let $g(s) = \mathcal{E}(h_{p1} + s(h_{p2} - h_{p1}))$ where $c' = \mathcal{E}(h_{p1}) = g(0) > g(1) = \mathcal{E}(h_{p2})$. Apparently, $g'(0) = g'(1) = 0$. If $g''(0) < 0$ and $g > c'$ for some $0 < s < 1$, g' has at least three critical points, two local minima and one local maximum, in $(0, 1)$. However,

$$g^{(4)}(s) = (2-q)(q-1)\int \left(h_{p1} + s(h_{p2} - h_{p1})\right)^{q-3}(h_{p2} - h_{p1})^4 > 0, \qquad (1.5.8)$$

that is, g'' is convex. This is impossible.

By direct computation,

$$g''(0) = \frac{1}{q}\int \left(h_{p2} - h_{p1}\right)\left(h_{p2}^q - qh_{p1}^{q-1}h_{p2} + (q-1)h_{p1}^q\right).$$

Take $q = 1.77$, $\alpha_2 = 0.1$ and then $\alpha_1 = 0.2785$ so that $E_q(\alpha_1) = E_q(\alpha_2)$. Let

$$h_{p1}(x) = \lambda^{\frac{1}{q}}\hat{h}(\lambda^{\frac{q-1}{2q}}x; \alpha_1)$$

and $h_{p2}(x) = \hat{h}(x; \alpha_2)$, where λ is a constant chosen so that h_{p1} and h_{p2} have the same area. By theorem 9 in [30], $\mathcal{E}(h_{p1}) > \mathcal{E}(h_{p2})$. We have $g''(0) = -2.1697 \times 10^{-5} < 0$ (with error $\leq 10^{-7}$). From the above discussion, we conclude that the line segment connecting h_{p1} and h_{p2} lies below the energy level c'.

1.6 Further Comments

In the last section we studied the power law in the range $(1, 3)$ and a rather complete understanding of its dynamics is achieved. However, this has been done for $P \leq \max\{P_c, P_d\}$. What will happen when P goes beyond P_c and P_d? When P is not too far from $\max\{P_c, P_d\}$, the dynamics is the same as at $\max\{P_c, P_d\}$. Complications arise when P becomes larger, there may be other H^2-steady states in $X_e(P, A)$. These are positive steady states of non-minimal period and configurations of droplets with zero contact angle. While the former can be completely classification in principle from $E_q(\alpha)$, see Figures 1.6-1.10, there are too many configurations of droplets. The dynamics becomes more and more complex as P increases. Nevertheless, from [17] we know that these additional steady states are all energy unstable, so it leaves the droplets h_d and h'_d as the global energy minimizers in $X_e(P, A)$. Consequently, one may say that the most likely ultimate shape of the flow is a droplet with zero contact angle.

We will not discuss the linear case $q = 1$ where explicit solutions are known. What happens when $q \in (-1, 1)$? At a technical level we need $q > 0$ or $q > 1$ in Proposition 1.4 and Proposition 1.12 respectively. Nevertheless, numerical results in [31] indicate that the same mountain pass scenario holds as in the case $q \in [2, 3)$. Examining the situation more closely, these two cases have different features. When $q \in (-1, 1)$, configurations of droplets with zero contact angle coexist with single droplets with zero contact angle. Take a droplet with zero contact angle h_d with span 1 and area A_0. For any $\lambda > 0$, $h(x) = \lambda^{2/(q-1)}\tilde{h}(\lambda x)$, defines a droplet $\tilde{h}$ with zero contact angle whose span is λ and area μA_0, $\mu = \lambda^{(3-q)/(1-q)}$. We explain how to obtain configurations of droplets close to h_d in $X_e(P, A_0)$ for $P > 1$. Let h_1 be the droplet with span $\lambda < 1$ in $X_e(P, \mu A_0)$ where $\mu \in (0, 1)$ is very close to 1. It vanishes in $[-P/2, -\lambda/2] \cup [\lambda/2, P/2]$. Let h_2 and h_3 be the droplets with area equals to $(1 - \mu)A_0/2$ and whose maxima are attained at $(\lambda + P)/4$ and $-(\lambda + P)/4$ respectively. Then h_1, h_2 and h_3 form a configuration of droplets in $X_e(P, A_0)$. As $\mu \uparrow 1$, these configurations converge to h_d. In fact, a slightly more general construction shows that one can deploy many configurations of droplets in $X_e(P, A_0)$ as long as $P > 1$. The abundance of configurations of droplets lays an obstruction to the convergence to h_d. Speculating further, Sturmian theory tells us that the critical points of a solution of a second order parabolic equation do not increase in time [4]. In case this still works for (1.5.1), we do not have to worry about positive steady states of non-minimal period or configurations of droplets, for they have more critical points than a positive steady state of minimal period does. Results valid on $[2, 3)$ should be extendable to $(-1, 1)$ without much effort. Although numerical evidence shows that this is the case, an analytic proof seems to be remote.

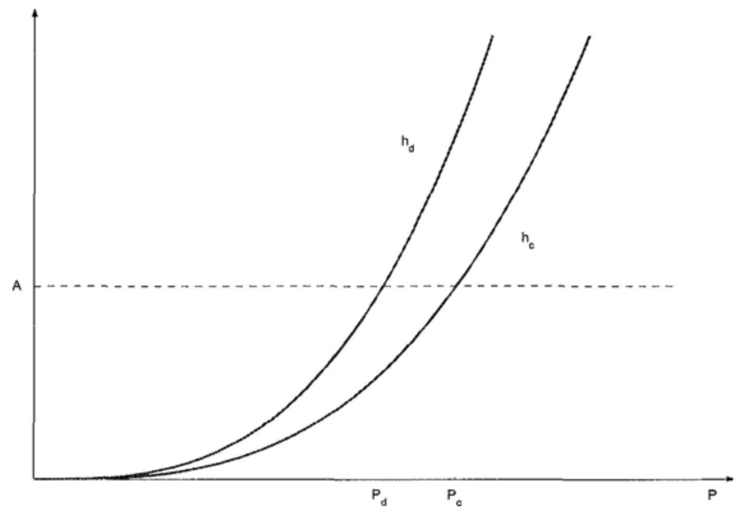

Figure 1.10: $q \in (-1, 1)$.

There are no droplets with zero contact angle when $q < -1$. In this case the solution of (1.5.1) may touch down at 0 and develop a singularity in finite time. In [19] an energy criterion for this to occur is given. When $q \geq 3$, the solution may blow up and touch down in finite time, see [13, 14, 37, 36] for related results.

So far $n \geq 3.5$ has been imposed in (1.1.6) to ensure positivity of the flow (1.1.5). When n is less than 3.5, a flow starting from a positive function may touch down in finite time. Although this fact has been established for $n \in (0, 0.5)$ [9] and [19], the most interesting cases $n = 1$ (thin-film) and $n = 3$ (Hele-Shaw cell) are still open. The range of the exponent n for which rupture occurs also varies with q in the power law. The reader is referred to [31] for many numerical results. Starting from [8, 9, 12, 13] and [18], people have constructed non-negative weak solutions in various cases. By using more precise entropy inequality, it is possible to establish a mountain pass scenario and the existence of heteroclinic orbits for these weak solutions for $n \geq 2$. Let us describe how this is done briefly.

In general, the weak solution is constructed by approximations. First, consider the regularized problem by replacing $a(z)$ in (1.1.5) by $a_\varepsilon(z)$ which behaves like z^4 near 0. For example, take it to be $z^4/(z^{4-n}+\varepsilon)$, $\varepsilon > 0$ small. when $a = z^n$. Denote the corresponding problem by $(1.1.5)'$. Under (1.1.3), $(1.1.5)'$ admits a global solution $h_\varepsilon(t)$ in $X_e^+(P, A)$ for the same positive initial value. It can be shown that there exists a sequence ε_j such that $h_{\varepsilon_j}(t)$ converges to some non-negative function $h(t)$ in H^1-norm for almost every t. The function $h(t) \in X_e(P, A)$, belonging to $L^\infty(0,\infty; H^1[-P/2, P/2]) \cap L^2_{\text{loc}}(0,\infty; H^2[-P/2, P/2])$, is called a weak solution to $(1.1.5)'$. The reader is referred to the above cited works for a precise description of the construction and various properties of the weak solution. Here we simply point out that the following energy and entropy estimates hold for a weak solution $h(t)$. For a.e. t,

$$\mathcal{E}(h(t)) + \int_0^t\!\!\int h^n \left|\left(h_{xx} + f(h)\right)_x\right|^2 \leq \mathcal{E}(h_0).$$

and

$$\begin{aligned} &\int h^{\frac{3}{2}-n+\delta}(t) + \sigma \int_0^t\!\!\int \left(h^{-\frac{1}{2}+\delta} h_{xx}^2 + h^{-\frac{5}{2}+\delta} h_x^4\right) \\ \leq\ & C_1 \int_0^t\!\!\int f^2(h) h^{-\frac{1}{2}+\delta} + C_2 \int h_0 + \int h_0^{\frac{3}{2}-n+\delta}, \end{aligned}$$

where σ, δ, C_1 and C_2 are positive constants, $\delta \in (0, \min\{3/2, n - 3/2\})$. These estimates are obtained from (2.2) and (2.3) in [18] by taking $\phi \equiv 1$ and $t_0 = 0$ (because $h_0 > 0$).

When $n > 2$, we can choose $\delta = 1/2$ in the above entropy estimate to obtain

$$\int_0^t\!\!\int h_{xx}^2 \leq C_1 \int_0^t\!\!\int f^2(h) + C_2 \int h_0 + \int h_0^{2-n}.$$

A similar estimate is available when $n > 3/2$ and $f(z) \leq C z^{1/4}$ near 0 (so that the first term on the right hand side of the above entropy estimate is bounded by a constant multiple of t). Using this, we can repeat the proof of Proposition 1.4 to get

Proposition 1.23. *Consider* (1.1.5) *under* (1.1.3) *and* $n > 2$ *or* $1/4 \leq q < 3$ *and* $n > 1.5$ *in the power law, any non-negative weak solution* $h(t)$ *starting from* $X^+(P, A)$ *contains a subsequence* $\{h(t_j)\}$ *converging to an* H^2*-steady state.*

Consequently we deduce

Theorem 1.24. *Under* (1.1.3) *and* $n > 2$ *in* (1.1.6) *or* $1/4 \leq q < 3$ *and* $n > 1.5$ *in the power law, Theorem 1.14 is valid for flows of global, non-negative weak solutions of* (1.1.5).

Proof. Let $\{h_{\varepsilon_j}(t)\}$ be a sequence of approximating solutions to the global weak solution $h(t)$. Theorem 1.14 holds for the flows h_{ε_j}. The components U_1 and U_2 depend only on h_m and are independent of ε_j. By Propositions 1.23, 1.10 and 1.12, for h_0 in U_1, $h(t)$ converges either to h_c, h_d or h'_d. In case $h(t)$ converges to h_d or h'_d in H^1, Proposition 1.12 implies that $h_{\varepsilon_j}(t)$ belongs to h_d or h'_d for all large j and t. This is impossible, so $h(t)$ must converge to h_c. Similarly, flows starting from U_2 converge to h_d or h'_d uniformly.

□

Most results in Section 5 can now be extended to flows comprising of non-negative weak solutions by this theorem.

Chapter 2

Classification and Energy Levels of Steady States for Viscous Thin-film Type Equation

In this chaper, we study the steady states of the following thin film type equation:

$$h_t = -\big((h^3 + \lambda h^n)h_{xxx} + h^3 h_x\big)_x \tag{2.0.1}$$

where $\lambda > 0$ and $n \in (0, 3)$. Assume h is a positive steady state satisfying

$$h_{xx} + f(h) = c$$

where $f(0) = 0$,

$$f'(x) = \frac{x^3}{x^3 + \lambda x^n} = \frac{x^{3-n}}{x^{3-n} + \lambda}. \tag{2.0.2}$$

Denote the minimum of h by α, the maximum by β. Then $F(\beta) - F(\alpha) = c(\beta - \alpha)$ where $F(x)$ is the primitive of $f(x)$ with $F(0) = 0$. The minimal period and area of h, $P(\alpha, c)$ and $A(\alpha, c)$ are given respectively by

$$P(\alpha, c) = \sqrt{2} \int_\alpha^\beta \frac{dx}{\sqrt{F(\alpha) - c\alpha - F(x) + cx}}, \tag{2.0.3}$$

and

$$A(\alpha, c) = \sqrt{2} \int_\alpha^\beta \frac{x\,dx}{\sqrt{F(\alpha) - c\alpha - F(x) + cx}}. \tag{2.0.4}$$

The admissible set of (α, c) is

$$\Sigma \equiv \{(\alpha, c) : \alpha \in (0, h_c), c \geq 0\}. \tag{2.0.5}$$

Where $f(h_c) = c$. Setting

$$\Gamma_c \equiv \{(\alpha, c) : \alpha = h_c(c), c > 0\}, \tag{2.0.6}$$

and

$$\Gamma_d \equiv \{(0, c) : c > 0\}, \tag{2.0.7}$$

the boundary of the admissible set $\partial\Sigma$ is given by $\Gamma_c \cup \Gamma_d \cup \{(0,0)\}$. $P(\alpha, c)$ and $A(\alpha, c)$, $(\alpha, c) \in \Gamma_d$, correspond the length and area of a zero contact angle droplet respectively. When $(\alpha, c) \in \Sigma$, $P(\alpha, c)$ and $A(\alpha, c)$ correspond the minimal period and area of a nonconstant positive steady state.

2.1 Monotonicity of $P(\alpha, c)$ and $A(\alpha, c)$

We try to investigate some basic properties, like monotonicity and limit values of $P(\alpha, c)$ and $A(\alpha, c)$. To simplify on calculations, we will change the variable first. Let $x = y(\beta - \alpha) + \alpha$. From (2.0.3) and (2.0.4), we have

$$P(\alpha, c) = \sqrt{2}(\beta - \alpha) \int_0^1 K(y)^{-\frac{1}{2}} dy \tag{2.1.1}$$

and

$$A(\alpha, c) = \sqrt{2}(\beta - \alpha) \int_0^1 K(y)^{-\frac{1}{2}} \big(y(\beta - \alpha) + \alpha\big) dy, \tag{2.1.2}$$

where

$$K(y) = yF(\beta) + (1 - y)F(\alpha) - F(y(\beta - \alpha) + \alpha). \tag{2.1.3}$$

We have

$$K''(y) = -(\beta - \alpha)^2 f'(y(\beta - \alpha) + \alpha) \leq 0. \tag{2.1.4}$$

From (12), it is known that $K(0) = K(1) = 0$, so $K(y) \geq 0$ on $[0, 1]$.

Proposition 2.1. *For $(\alpha, c) \in \Sigma$ or $\alpha = 0, c > 0$, $P_c(\alpha, c) < 0$.*

Proof. As $F(\beta) - F(\alpha) = c(\beta - \alpha)$,

$$\beta_c = \frac{\beta - \alpha}{f(\beta) - c} > 0. \tag{2.1.5}$$

From a direct computation,

$$P_c(\alpha, c) = \frac{\beta - \alpha}{\sqrt{2}(f(\beta) - c)} \int_0^1 K(y)^{-\frac{3}{2}} I(y) dy \tag{2.1.6}$$

where

$$I(y) = 2K(y) - y(\beta - \alpha)(f(\beta) - f(y(\beta - \alpha) + \alpha)). \tag{2.1.7}$$

As $I(y)$ vanishes at $y = 0, 1$ and its second derivative is given by

$$I''(y) = y(\beta - \alpha)^3 f''(y(\beta - \alpha) + \alpha), \tag{2.1.8}$$

where

$$f''(x) = \lambda(3 - n)\frac{x^{2-n}}{(x^{3-n} + \lambda)^2} \geq 0, \tag{2.1.9}$$

we conclude that $I(y)$ is convex and $I(y) < 0$ on $(0, 1)$. So $P_c(\alpha, c) < 0$. □

Proposition 2.2. *$A_c(0, c) > 0$ when $n \geq 1$, $c > 0$. For $\alpha > 0$, $A_c(\alpha, c) > 0$ when $n \geq 2$.*

Proof. From a direct computation,

$$A_c(\alpha, c) = \frac{\beta - \alpha}{\sqrt{2}(f(\beta) - c)} \int_0^1 K(y)^{-\frac{3}{2}} J(y) dy \tag{2.1.10}$$

where

$$\begin{aligned} J(y) &= y(\beta - \alpha)(y(\beta - \alpha) + \alpha)\big[f(y(\beta - \alpha) + \alpha) - f(\beta)\big] \\ &\quad + K(y)(2\alpha + 4y(\beta - \alpha)) \\ &= (y(\beta - \alpha) + \alpha)I(y) + 2y(\beta - \alpha)K(y) \end{aligned} \tag{2.1.11}$$

If $\alpha = 0$, $J(y) = y\beta(I(y) + 2K(y))$. By (2.1.4) and (2.1.8), its second derivative is given by

$$J''(y) = I''(y) + 2K''(y) = \beta^3 y f''(\beta y) - 2\beta^2 f'(\beta y)$$

When $n \geq 1$, the right hand side of this formula is negative as

$$xf''(x) - 2f'(x) = \frac{x^{3-n}}{(x^{3-n}+\lambda)^2}(-2x^{3-n} + \lambda(1-n)) \leq 0.$$

Both $I(y)$ and $K(y)$ vanish at $y = 0, 1$, so $J(y) \geq 0$ on $(0, 1)$. Hence $A_c(0, c) > 0$.

When $\alpha > 0$, we need a more careful estimate. Write $J(y)$ as two parts:

$$\begin{aligned} J(y) =& \big(y(\beta-\alpha)+\alpha\big)(I(y)+sK(y)) + \big((2-s)y(\beta-\alpha) - s\alpha\big)K(y) \\ \equiv& \big(y(\beta-\alpha)+\alpha\big)J_1(y) + J_2(y) \end{aligned}$$

where s is a constant in $[0, 2]$ depending on α, β. Using (2.1.8) and (2.1.4),

$$J_1''(y) = (\beta-\alpha)^2[y(\beta-\alpha)f''(y(\beta-\alpha)+\alpha) - sf'(y(\beta-\alpha)+\alpha)].$$

So that

$$\int_0^1 K^{\frac{3}{2}}(y)(y(\beta-\alpha)+\alpha)J_1(y)dy \geq 0$$

provided

$$(x-\alpha)f''(x) - sf'(x) \leq 0, \text{ for all } x \in [\alpha, \beta].$$

The above inequality is equivalent to

$$s \geq \max_{x\in[\alpha,\beta]} \frac{(x-\alpha)f''(x)}{f'(x)} = \max_{x\in[\alpha,\beta]} \frac{\lambda(3-n)(x-\alpha)}{\lambda x + x^{4-n}}. \tag{2.1.12}$$

We express this maximum explicitly.

$$\left(\frac{\lambda(3-n)(x-\alpha)}{\lambda x + x^{4-n}}\right)_x' = \frac{\lambda(3-n)}{(\lambda x + x^{4-n})^2}\big(\lambda\alpha + (4-n)\alpha x^{3-n} + (n-3)x^{4-n}\big)$$

Since

$$(\lambda\alpha + (4-n)\alpha x^{3-n} + (n-3)x^{4-n})_x' = (4-n)(3-n)x^{2-n}(\alpha - x) \leq 0$$

and $\lambda\alpha + \alpha^{4-n} > 0$, $\dfrac{\lambda(3-n)(x-\alpha)}{\lambda x + x^{4-n}}$ increases on $[\alpha, x_0]$ and then decreases on $[x_0, +\infty)$. The critical point x_0 satisfies

$$\lambda\alpha + (4-n)\alpha x_0^{3-n} + (n-3)x_0^{4-n} = 0.$$

Therefore,

$$\max_{x\in[\alpha,\beta]} \frac{(x-\alpha)f''(x)}{f'(x)} = \begin{cases} \dfrac{(\beta-\alpha)f''(\beta)}{f'(\beta)}, & \beta \le x_0 \\ \\ \dfrac{(x_0-\alpha)f''(x_0)}{f'(x_0)}, & \beta > x_0 \end{cases} \tag{2.1.13}$$

For the integration involving J_2, we need some estimates on $K(y)$. By (2.1.4) and (2.1.9),

$$(\beta-\alpha)^2 f'(\alpha)\frac{y(1-y)}{2} \le K(y) \le (\beta-\alpha)^2 f'(\beta)\frac{y(1-y)}{2}.$$

Then

$$\begin{aligned}\int_0^1 K(y)\ ^{\frac{1}{2}} y dy \ge& \frac{1}{\beta-\alpha} f'(\beta)^{-\frac{1}{2}} \int_0^1 \left(\frac{y(1-y)}{2}\right)^{-\frac{1}{2}} y dy \\ =& \frac{\pi}{\sqrt{2}(\beta-\alpha)} f'(\beta)^{-\frac{1}{2}}\end{aligned} \tag{2.1.14}$$

and similarly,

$$\begin{aligned}\int_0^1 K(y)^{-\frac{1}{2}} dy \le& \frac{1}{\beta-\alpha} f'(\alpha)^{-\frac{1}{2}} \int_0^1 \left(\frac{y(1-y)}{2}\right)^{-\frac{1}{2}} dy \\ =& \frac{\sqrt{2}\pi}{\beta-\alpha} f'(\alpha)^{-\frac{1}{2}}.\end{aligned} \tag{2.1.15}$$

Inserting above equations in the integral of J_2,

$$\begin{aligned}&\int_0^1 K(y)^{-3/2} J_2(y) dy \\ \ge& \frac{\pi}{\sqrt{2}(\beta-\alpha)} \left((2-s)(\beta-\alpha)f'(\beta)^{-\frac{1}{2}}. - 2s\alpha f'(\alpha)^{-\frac{1}{2}}\right)\end{aligned}$$

Therefore

$$\int_0^1 K(y)^{-3/2} J_2(y) dy > 0,$$

provided

$$s \le \frac{2(\beta-\alpha)f'(\beta)^{-\frac{1}{2}}}{(\beta-\alpha)f'(\beta)^{-\frac{1}{2}} + 2\alpha f'(\alpha)^{-\frac{1}{2}}}. \tag{2.1.16}$$

By (2.1.12) and (2.1.16), to show $A_c(\alpha, c) > 0$, it suffices to verify that

$$\frac{2(\beta-\alpha)f'(\beta)^{-\frac{1}{2}}}{(\beta-\alpha)f'(\beta)^{-\frac{1}{2}} + 2\alpha f'(\alpha)^{-\frac{1}{2}}} \ge \max_{x\in[\alpha,\beta]} \frac{(x-\alpha)f''(x)}{f'(x)}. \tag{2.1.17}$$

The function $(x-\alpha)f'(x)^{-\frac{1}{2}}$ is increasing as its derivative shows,

$$\begin{aligned}
&((x-\alpha)f'(x)^{-\frac{1}{2}})'_x\\
&=f'(x)^{-\frac{1}{2}}-\frac{1}{2}(x-\alpha)f'(x)^{-\frac{3}{2}}f''(x)\\
&=\frac{x^{2-n}}{2f'(x)^{\frac{3}{2}}(x^{3-n}+\lambda)^2}(2x^{4-n}+\lambda x(n-1)+\lambda\alpha(3-n))>0
\end{aligned}$$

Then,

$$\frac{2(x-\alpha)f'(x)^{-\frac{1}{2}}}{(x-\alpha)f'(x)^{-\frac{1}{2}}+2\alpha f'(\alpha)^{-\frac{1}{2}}}\text{ is increasing.}\tag{2.1.18}$$

When $n\geq 2$,

$$f'''(x)=\lambda(3-n)\frac{x^{1-n}}{(x^{3-n}+\lambda)^3}((n-4)x^{3-n}+(2-n)\lambda)<0.\tag{2.1.19}$$

So, by (2.1.19) and the increasing of $(x-\alpha)f'(x)^{-\frac{1}{2}}$, under the condition $n\geq 2$,

$$\begin{aligned}
&\frac{2(x-\alpha)f'(x)^{-\frac{1}{2}}}{(x-\alpha)f'(x)^{-\frac{1}{2}}+2\alpha f'(\alpha)^{-\frac{1}{2}}}-\frac{(x-\alpha)f''(x)}{f'(x)}\\
&=\frac{(x-\alpha)f'(x)^{-\frac{1}{2}}}{(x-\alpha)f'(x)^{-\frac{1}{2}}+2\alpha f'(\alpha)^{-\frac{1}{2}}}\left[2-f''(x)f'(x)^{-\frac{1}{2}}((x-\alpha)f'(x)^{-\frac{1}{2}}+2\alpha f'(\alpha)^{-\frac{1}{2}})\right]\\
&\geq\frac{2(x-\alpha)f'(x)^{-\frac{1}{2}}}{(x-\alpha)f'(x)^{-\frac{1}{2}}+2\alpha f'(\alpha)^{-\frac{1}{2}}}\left[1-\alpha f''(\alpha)f'(\alpha)\right]\\
&=\frac{2(x-\alpha)f'(x)^{-\frac{1}{2}}}{(x-\alpha)f'(x)^{-\frac{1}{2}}+2\alpha f'(\alpha)^{-\frac{1}{2}}}\cdot\frac{\alpha^{3-n}+(n-2)\lambda}{\alpha^{3-n}+\lambda}\geq 0.
\end{aligned}\tag{2.1.20}$$

In the case $\beta\leq x_0$, by (2.1.13),

$$\begin{aligned}
&\frac{2(\beta-\alpha)f'(\beta)^{-\frac{1}{2}}}{(\beta-\alpha)f'(\beta)^{-\frac{1}{2}}+2\alpha f'(\alpha)^{-\frac{1}{2}}}-\max_{x\in[\alpha,\beta]}\frac{(x-\alpha)f''(x)}{f'(x)}\\
&=\frac{2(\beta-\alpha)f'(\beta)^{-\frac{1}{2}}}{(\beta-\alpha)f'(\beta)^{-\frac{1}{2}}+2\alpha f'(\alpha)^{-\frac{1}{2}}}-\frac{(\beta-\alpha)f''(\beta)}{f'(\beta)}\geq 0.
\end{aligned}\tag{2.1.21}$$

In the case $\beta>x_0$, by (2.1.13) and (2.1.18),

$$\begin{aligned}
&\frac{2(\beta-\alpha)f'(\beta)^{-\frac{1}{2}}}{(\beta-\alpha)f'(\beta)^{-\frac{1}{2}}+2\alpha f'(\alpha)^{-\frac{1}{2}}}-\max_{x\in[\alpha,\beta]}\frac{(x-\alpha)f''(x)}{f'(x)}\\
&\geq\frac{2(x_0-\alpha)f'(x_0)^{-\frac{1}{2}}}{(x_0-\alpha)f'(x_0)^{-\frac{1}{2}}+2\alpha f'(\alpha)^{-\frac{1}{2}}}-\frac{(x_0-\alpha)f''(x_0)}{f'(x_0)}\geq 0.
\end{aligned}\tag{2.1.22}$$

We have proved that $A_c(\alpha,c)$ is increasing. □

To show the monotonicity of $P_\alpha(\alpha, c)$ and $A_\alpha(\alpha, c)$, we need the following lemma of Schaaf [35] for the Hamiltonian system:

$$\begin{cases} \dfrac{dx}{dt} = -G(y), \\ \dfrac{dy}{dt} = F(x). \end{cases}$$

Schaaf presents two sets of conditions and proves that the period of the solution $(x(t), y(t))$ is an increasing function of the initial data $x(0)$ if the first set is satisfied, and the period is decreasing if the second set is satisfied:

Lemma 2.3. *Let F and G be C^3 on an open interval J containing 0, with F and G vanishing at 0 and nowhere else, and with $F'(0) > 0$, $G'(0) > 0$. Assume either that $\varphi = F$ and $\phi = G$ both satisfies SA(ii) and SA(iii), or that they both satisfy condition SB(ii):*

SA(ii) $\phi'(x) > 0$, $x \in J$ implies $5\phi''(x)^2 - 3\phi'(x)\phi'''(x) > 0$.

SA(iii) $\phi'(x) = 0$, $x \in J$ implies $\phi(x)\phi''(x) < 0$.

SB(ii) $\phi'(x) \geq 0$, $x \in J$ implies $5\phi''(x)^2 - 3\phi'(x)\phi'''(x) \leq 0$.

Then there is a maximal interval $(0, \alpha^+) \subset J$, $\alpha^+ > 0$, such that any solution $(x(t), y(t))$ with initial data $x(0) = \alpha \in (0, \alpha^+)$ and $y(0) = 0$ is periodic with its orbit enclosing the fixed point $(0, 0)$. Let $P(\alpha)$ be the least period of this solution. Then P is differentiable on $(0, \alpha^+)$ and for all $\alpha \in (0, \alpha^+)$, $P'(\alpha) > 0$ if F and G satisfy SA(ii) and SA(iii), $P'(\alpha) < 0$ if F and G satisfy SB(ii).

Here we consider a special case, namely, the differential equation $\ddot{x} + \mu(x) = 0$. This equation can be written as a system with $F(x) = \mu(x)$ and $G(y) = y$. Although $G(y)$ does not satisfy condition SA(ii) or SB(ii), Schaaf [35] noted that in this special case, her proof remains valid as long as $F = \mu$ satisfies the hypotheses.

Proposition 2.4. *For $(\alpha, c) \in \Sigma$, $P_\alpha(\alpha, c) < 0$.*

Proof. After a translation, we may assume the steady state h attains its minimum at the origin, i.e. $h(0) = \alpha \in (0, h_c)$ and $h'(0) = 0$. Letting $\mu(x) = c - f(h_c - x)$ and $k = h_c - h$ where h_c is defined in (13), we obtain

$$k'' + \mu(k) = 0$$

with $k(0) = h_c - \alpha$. Since μ is C^3 on $J = (-\infty, h_c)$, $\mu(0) = c - f(h_c) = 0$ and $\mu'(0) = f'(h_c) > 0$, the lemma applies. Condition SA(iii) holds trivially as $\mu'(x) = f'(h_c - x) > 0$ on J. For condition SA(ii),

$$\begin{aligned} &5\mu''(x)^2 - 3\mu'(x)\mu'''(x) \\ =&5f''(h_c - x)^2 - 3f'(h_c - x)f'''(h_c - x) \\ =&\lambda(3-n)\frac{(h_c - x)^{4-2n}}{((h_c - x)^{3-n} + \lambda)^4}((9-2n)\lambda - 3(n-4)(h_c - x)^{3-n}), \end{aligned}$$

The condition $n < 3$ implies $5\mu''(x)^2 - 3\mu'(x)\mu'''(x) > 0$. So SA(ii) is satisfied, and $P_\alpha = -P'(h_c - \alpha) = -P'(k(0)) < 0$. □

Proposition 2.5. *For $(\alpha, c) \in \Sigma$, $A_\alpha(\alpha, c) > 0$ when $n \geq 1$.*

Proof. Define $V(x) = \int_0^x h(s)ds$. Since $V'(x) = h(x) \geq \alpha > 0$, the inverse V^{-1} exists. Let

$$k_*(y) = h_c^2 - h(V^{-1}(y))^2.$$

The minimal period of k_* is obviously $A(\alpha, c)$. The derivative of k_* is

$$k_*'(y) = -2h(V^{-1}(y))h'(V^{-1}(y))(V^{-1})'(y) = -2h'(V^{-1}(y)),$$

and so

$$k_*''(y) = -\frac{2h''(V^{-1}(y))}{h(V^{-1}(y))} = \frac{2(f(h(V^{-1}(y))) - c)}{h(V^{-1}(y))}.$$

Therefore, if we define

$$\mu_*(x) = \frac{2(c - f(\sqrt{h_c^2 - x}))}{\sqrt{h_c^2 - x}}, \tag{2.1.23}$$

k_* satisfies the equation

$$k_*'' + \mu_*(k_*) = 0$$

Note that $k_*(0) = h_c^2 - \alpha^2 > 0$, $\mu_*(0) = 0$ and

$$\mu_*'(0) = \frac{h_c f'(h_c) + c - f(h_c)}{h_c^3} = \frac{f'(h_c)}{h_c^2} > 0.$$

We could use the lemma again.

To simplify computation, setting $t = \sqrt{h_c^2 - x}$, $t_x = -\frac{1}{2t}$, then we have

$$\begin{aligned}
\mu_*'(x) &= \frac{1}{t^3}(tf'(t) - f(t) + c) > 0, \\
\mu_*''(x) &= -\frac{1}{2t^5}(t^2 f''(t) - 3tf'(t) + 3f(t) - 3c), \\
\mu_*'''(x) &= \frac{1}{4t^7}(t^3 f'''(t) - 6t^2 f''(t) + 15tf'(t) - 15f(t) + 15c).
\end{aligned}$$

By (11), (2.1.9) and (2.1.19).

$$\begin{aligned}
&5\mu_*''(x)^2 - 3\mu_*'(x)\mu_*'''(x) \\
=&\frac{1}{4t^{10}}(5t^4 f'''(t)^2 - 3t^2(tf'(t) - f(t) + c)(tf'''(t) + 4f''(t))) \\
=&\frac{\lambda(3-n)}{t^{6+n}(t^{3-n} + \lambda)^3}(nt^{3-n} + (6-n)\lambda)g(t)
\end{aligned}$$

where

$$g(t) = \frac{5\lambda(3-n)t^{4-n}}{(t^{3-n} + \lambda)(nt^{3-n} + (6-n)\lambda)} - 3(tf'(t) - f(t) + c).$$

The derivative of $g(t)$ is given by

$$\begin{aligned}
&g'(t) \\
=&\frac{2\lambda(3-n)t^{3-n}}{3(t^{3-n} + \lambda)^2(nt^{3-n} + (6-n)\lambda)^2}\big[n(n-5)t^{6-2n} + 3(n-5)(n-1)t^{3-n}\lambda \\
&+ (6-n)(1-n)\lambda^2\big].
\end{aligned}$$

When $n \geq 1$,

$$(3(n-5)(n-1))^2 - 4n(n-5)(6-n)(1-n) = 5(n-5)(n-1)(n-3)^2 \leq 0.$$

So $g'(t) \leq 0$, $g(t) \leq g(0) = -3c < 0$ and then $5\mu_*''(x)^2 - 3\mu_*'(x)\mu_*'''(x) < 0$. SB(ii) is satisfied, and we get $A_\alpha(\alpha, c) = -2\alpha A'(k_*(0)) > 0$. □

The next proposition is about the limiting properties of $P(\alpha, c)$ and $A(\alpha, c)$.

Proposition 2.6. *When* $\alpha \geq 0$,

$$\lim_{c\to\infty} P(\alpha, c) = 2\pi, \ \lim_{c\to\infty} \frac{A(\alpha, c)}{\beta - \alpha} = \pi.$$

When $\alpha > 0$,

$$\lim_{c\to h_c^{-1}(\alpha)} P(\alpha, c) = 2\pi f'(\alpha)^{-1/2}, \quad \lim_{c\to h_c^{-1}(\alpha)} A(\alpha, c) = 2\pi\alpha f'(\alpha)^{-1/2}.$$

Proof. Note that $f'(x) \to 1$ as $x \to +\infty$. By (2.1.3), for fixed α,

$$\begin{aligned}
&\lim_{c\to\infty} \frac{K(y)}{(\beta-\alpha)^2} = \lim_{\beta\to\infty} \frac{K(y)}{(\beta-\alpha)^2} \\
=&\lim_{\beta\to\infty} \frac{yF(\beta) + (1-y)F(\alpha) - F(y(\beta-\alpha)+\alpha)}{(\beta-\alpha)^2} \\
=&\lim_{\beta\to\infty} \frac{yf(\beta) - yf(y(\beta-\alpha)+\alpha)}{2(\beta-\alpha)} \\
=&\lim_{\beta\to\infty} \frac{1}{2}\big(yf'(\beta) - y^2 f'(y(\beta-\alpha)+\alpha)\big) \\
=&\frac{y(1-y)}{2}.
\end{aligned} \tag{2.1.24}$$

Thus by (2.1.1),

$$\begin{aligned}
\lim_{c\to\infty} P(\alpha, c) =& \lim_{\beta\to\infty} \sqrt{2}\int_0^1 \left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{1}{2}} dy \\
=&\sqrt{2}\int_0^1 \left(\frac{2}{y(1-y)}\right)^{\frac{1}{2}} dy = 2\pi.
\end{aligned} \tag{2.1.25}$$

By (2.1.2) and (2.1.24),

$$\begin{aligned}
\lim_{c\to\infty} \frac{A(\alpha, c)}{\beta-\alpha} =& \lim_{\beta\to\infty} \sqrt{2}\int_0^1 \left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{1}{2}} \frac{y(\beta-\alpha)+\alpha}{\beta-\alpha} dy \\
=&\sqrt{2}\int_0^1 \left(\frac{2}{y(1-y)}\right)^{\frac{1}{2}} y dy = \pi.
\end{aligned} \tag{2.1.26}$$

When $\alpha > 0$, $h_c^{-1}(\alpha)$ exists. Similarly, for fixed α, we have

$$\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)} \frac{K(y)}{(\beta-\alpha)^2} = \lim_{\beta\to\alpha} \frac{K(y)}{(\beta-\alpha)^2} \\
=&\lim_{\beta\to\alpha} \frac{yF(\beta)+(1-y)F(\alpha))-F(y(\beta-\alpha)+\alpha)}{(\beta-\alpha)^2} \\
=&\lim_{\beta\to\alpha} \frac{yf(\beta)-yf(y(\beta-\alpha)+\alpha)}{2(\beta-\alpha)} \\
=&\lim_{\beta\to\alpha} \frac{1}{2}\big(yf'(\beta)-y^2f'(y(\beta-\alpha)+\alpha)\big) \\
=&\frac{y(1-y)f'(\alpha)}{2}.
\end{aligned} \tag{2.1.27}$$

Then

$$\begin{aligned}
\lim_{c\to h_c^{-1}(\alpha)} P(\alpha,c) =& \lim_{\beta\to\alpha} \sqrt{2}\int_0^1 \left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{1}{2}} dy \\
=&\sqrt{2}\int_0^1 \left(\frac{2}{y(1-y)f'(\alpha)}\right)^{\frac{1}{2}} dy \\
=&2\pi f'(\alpha)^{-\frac{1}{2}}.
\end{aligned} \tag{2.1.28}$$

By (2.1.2) and (2.1.27),

$$\begin{aligned}
\lim_{c\to h_c^{-1}(\alpha)} A(\alpha,c) =& \lim_{\beta\to\alpha} \sqrt{2}\int_0^1 \left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{1}{2}} (y(\beta-\alpha)+\alpha)dy \\
=&\sqrt{2}\int_0^1 \left(\frac{2}{y(1-y)f'(\alpha)}\right)^{\frac{1}{2}} \alpha dy = 2\pi\alpha f'(\alpha)^{-\frac{1}{2}}.
\end{aligned} \tag{2.1.29}$$

□

With these monotonicity and Proposition 2.6, we can prove the Theorem C.

Proof of Theorem C. As $P_c(\alpha, c) < 0$ for $\alpha \geq 0$, for any $(\alpha, c) \in \Sigma$, $P(\alpha, c) > \lim_{c\to\infty} P(\alpha, c) = 2\pi$.

When $P_0 > 2\pi$, by Proposition 2.6, there is some $(\alpha_0, c_0) \in \Sigma$ such that $P(\alpha_0, c_0) = P_0$. By Propositions 2.1 and 2.4, $P_c(\alpha, c) < 0$ and $P_\alpha(\alpha, c) < 0$. From the implicit function theorem, there exists a unique continuous function $\alpha = \phi_{P_0}(c)$ on some interval (δ_1, δ_2) containing c_0 such that $\phi_{P_0}(c_0) = \alpha_0$ and

$P(\phi_{P_0}(c), c) = P_0$ on (δ_1, δ_2). Besides, $\phi'_{P_0}(c) < 0$, assume the maximal domain of $\phi_{P_0}(c)$ is (c_1, c_2). We show that c_2 must be finite. In fact, since $P_c(\alpha, c) < 0$ and $\lim_{c\to 0} P(0, c) = +\infty$, $\lim_{c\to\infty} P(0, c) = 2\pi$, there exists c_2^* such that $P(0, c_2^*) = P_0$. Then $P(\alpha, c) < P_0$ for any $c > c_2^*$ and $\alpha \geq 0$. Note that $\phi'_{P_0}(c) < 0$, so that $\phi_{P_0}(c_1) = h_c(c_1)$ and $\phi_{P_0}(c_2) = 0$, i.e. $(\phi_{P_0}(c_1), c_1) \in \Gamma_c$ and $(\phi_{P_0}(c_2), c_2) \in \Gamma_d$.

We claim that

$$\{(\alpha, c) \in \Sigma : P(\alpha, c) = P_0\} = \{(\phi_{P_0}(c), c) : c \in (c_1, c_2)\}. \tag{2.1.30}$$

In other words, $\{(\alpha, c) \in \Sigma : P(\alpha, c) = P_0\}$ consists of only one curve, the graph of the decreasing function $\phi_{P_0}(c)$, which will be called the P_0-curve in the following discussion.

If not, there are two different curves $\alpha = \phi(c)$, $c \in [c_1, c_2]$ and $\alpha = \widetilde{\phi}(c)$, $c \in [\tilde{c}_1, \tilde{c}_2]$ in $\{(\alpha, c) \in \Sigma : P(\alpha, c) = P_0\}$. $c_2 = \tilde{c}_2$ as $P_c(0, c) < 0$. Then we may find some $c_0 \in (\max\{c_1, \tilde{c}_1\}, c_2)$ such that $\phi(c_0) \neq \tilde{\phi}(c_0)$. However, $P_0 = P(\phi(c_0), c_0) = P(\tilde{\phi}(c_0), c_0)$, contradicting to Proposition 2.4.

Now, from continuity, $A_{P_0} = [A_1, A_2]$. We will prove that $A_1 < A_2$. Otherwise $A_1 = A_2$, meaning that for all $c \in [c_1, c_2]$,

$$\int_0^{P_0} \frac{d}{dc} h(\phi_{P_0}(c), c) dx = \frac{d}{dc} A_1 = 0. \tag{2.1.31}$$

On the other hand, from (7),

$$\frac{d}{dc} h''(\phi_{P_0}(c), c) + f'(h(\phi_{P_0}(c), c)) \frac{d}{dc} h(\phi_{P_0}(c), c) = 1.$$

Integrating above equation at $c = c_1$ over a period P_0, we have

$$f'(A_1/P_0) \int_0^{P_0} \frac{d}{dc} h(\phi_{P_0}(c), c_1) dx = P_0,$$

contradiction holds.

Similarly, with a further condition $n \geq 2$ stated in Propositions 2.2 and 2.5, we attain a same result on P_{A_0}. □

As a consequence of Theorem C, when $n \geq 2$, for given A, there exists $P^* > 2\pi$ such that if $P > P^*$, there exist no positive steady states with minimal period P and area A or single droplets with length P and area A. So the possible steady states are constant, non-minimal period positive steady states, 0-angle droplet whose length strictly lese than P, non-zero contact angle droplet or configuration of droplets. Among these, only constant state and single droplet may be energy stable, see [30] and [17]. To compare them. one need the result on [30], Theorem 10:

Lemma 2.7. *Let $\overline{h}, P > 0$. then for the constant steady state $h_{ss} = \overline{h}$, the eigenvalue*

$$\tau_1(h_{ss}) = \min_u \frac{\int_0^P (u')^2 - f'(h_{ss})u^2 dx}{\int_0^P u^2 dx},$$

where u is P-periodic function on $\mathbb{R}$ with $\int_0^P u dx = 0$ and $u \not\equiv 0$, is $\tau(\overline{h}) = (\frac{2\pi}{P})^2 - f'(\overline{h})$. The $\tau(\overline{h})$-eigenspace is spanned by $\sin(\frac{2\pi x}{P})$ and $\cos(\frac{2\pi x}{P})$.

(i) If $f'(\overline{h})P^2 > 4\pi^2$ or if $f'(\overline{h})P^2 = 4\pi^2$ and $f'''(\overline{h}) > 0$, then the constant steady state is energy unstable in the direction $\pm \sin(\frac{2\pi x}{P})$ and $\pm \cos(\frac{2\pi x}{P})$.

(ii) If $f'(\overline{h})P^2 < 4\pi^2$ or if $f'(\overline{h})P^2 = 4\pi^2$ and $f'''(\overline{h}) < 0$, then the constant steady state is energy stable with respect to zero-mean perturbations of period P.

Since $\overline{h} = A/P$,

$$f'(\overline{h})P^2 = \frac{A^3 P^{n-1}}{A^3 P^{n-3} + \lambda A^n} \to \infty \text{ as } P \to \infty.$$

When we take a larger P^* to make $f'(\frac{A}{P^*})P^{*2} > 4\pi^2$, the only possible stable steady state is a 0-angle droplet. Therefore we have

Corollary 2.8. *When $n \geq 2$, for given $A > 0$, there exists $P^* > 2\pi$ such that the only energy stable steady state in $X(P, A)$, $P > P^*$, is a 0-angle droplet.*

To conclude this section, we give two equations relating between $P_\alpha(\alpha, c)$, $P_c(\alpha, c)$ and $A_\alpha(\alpha, c)$, $A_c(\alpha, c)$. Define

$$H(y) = -(1-y)f(\alpha) - yf(\beta) + f(y(\beta - \alpha) + \alpha). \qquad (2.1.32)$$

then

$$P_\alpha + P_c \frac{f(\beta) - f(\alpha)}{\beta - \alpha} = \frac{\beta - \alpha}{\sqrt{2}} \int_0^1 K(y)^{-\frac{3}{2}} H(y) dy, \tag{2.1.33}$$

and

$$A_\alpha + A_c \frac{f(\beta) - f(\alpha)}{\beta - \alpha} = P + \frac{\beta - \alpha}{\sqrt{2}} \int_0^1 K(y)^{-\frac{3}{2}} H(y)(y(\beta - \alpha) + \alpha) dy \tag{2.1.34}$$

They are obtained from direct computations and we omit the details.

2.2 Counting the positive steady states

A natural question following Theorem C is, e.g., if $A_0 \in A_{P_0}$, how many corresponding steady states? Along the P_0-curve in (2.1.30), easy to see

$$\begin{aligned} \frac{d}{dc} A(\phi_{P_0}(c), c) =& A_\alpha(\phi_{P_0}(c), c)\phi'_{P_0}(c) + A_c(\phi_{P_0}(c), c) \\ =& \frac{1}{P_\alpha}(A_c P_\alpha - A_\alpha P_c)(\phi_{P_0}(c), c). \end{aligned} \tag{2.2.1}$$

So the uniqueness depends on the $(A_c P_\alpha - A_\alpha P_c)(\phi_{P_0}(c), c)$, say, if it does not change sign in (c_1, c_2), $A(\phi_{P_0}(c), c)$ is monotone in c. Obviously, such positive steady state must be unique. On the contrary, if it changes sign in (c_1, c_2), we can always find some A_0 so that at least two different steady states have same minimal period and area. By (2.1.33) and (2.1.34), we have the expression

$$\begin{aligned} &\frac{2(f(\beta) - c)}{(\beta - \alpha)^2}(A_c P_\alpha - A_\alpha P_c) \\ =& \int_0^1 K(y)^{-\frac{3}{2}} I(y) y dy \int_0^1 K(y)^{-\frac{3}{2}} H(y)(\beta - \alpha) dy \\ &- \int_0^1 K(y)^{-\frac{3}{2}} I(y) dy \int_0^1 K(y)^{-\frac{3}{2}} H(y) y (\beta - \alpha) dy \\ &+ 2 \int_0^1 K(y)^{-\frac{1}{2}} y dy \int_0^1 K(y)^{-\frac{3}{2}} H(y)(\beta - \alpha) dy \\ &- 2 \int_0^1 K(y)^{-\frac{1}{2}} dy \int_0^1 K(y)^{-\frac{3}{2}} I(y) dy. \end{aligned} \tag{2.2.2}$$

Denote it by $E(\alpha, c)$ for short.

Proposition 2.9. *Assume $n \in (0,3)$, $E(\alpha, c) < 0$ when c is large.*

Proof. We only prove the case $n \in (2,3)$ here. The other case, $n \in (0,2]$ is similar except for some technical modification. We put it in the appendix.

From $F(\beta) - F(\alpha) = c(\beta - \alpha)$,

$$\lim_{c\to\infty} \beta_c = \lim_{c\to\infty} \frac{\beta-\alpha}{f(\beta)-c} = \lim_{c\to\infty} \frac{\beta_c}{f'(\beta)\beta_c - 1}.$$

Since $f'(\beta) \to 1$ as $c \to \infty$,

$$\lim_{c\to\infty} \beta_c = \lim_{c\to\infty} \frac{\beta-\alpha}{f(\beta)-c} = 2. \tag{2.2.3}$$

By (2.1.9),

$$\lim_{x\to\infty} x^{4-n} f''(x) = \lim_{x\to\infty} \lambda(3-n)\frac{x^{6-2n}}{(x^{3-n}+\lambda)^2} = \lambda(3-n).$$

We have the following limits:

$$\begin{aligned}
&\lim_{c\to\infty} \frac{H(y)}{(\beta-\alpha)^{n-2}}\\
=&\lim_{\beta\to\infty} \frac{-yf'(\beta)+yf'(y(\beta-\alpha)+\alpha)}{(n-2)(\beta-\alpha)^{n-3}}\\
=&\lim_{\beta\to\infty} \frac{-yf''(\beta)+y^2f''(y(\beta-\alpha)+\alpha)}{(n-2)(n-3)(\beta-\alpha)^{n-4}}\\
=&\frac{\lambda}{n-2}(y-y^{n-2}),
\end{aligned} \tag{2.2.4}$$

and

$$\begin{aligned}
&\lim_{c\to\infty} \frac{I_1(y)}{(\beta-\alpha)^{n-1}}\\
=&\lim_{\beta\to\infty} \frac{yf(\beta)-yf(y(\beta-\alpha)+\alpha)-y(\beta-\alpha)(f'(\beta)-f'(y(\beta-\alpha)+\alpha))}{(n-1)(\beta-\alpha)^{n-2}}\\
=&\lim_{\beta\to\infty} \frac{-y(\beta-\alpha)(f''(\beta)-y^2f''(y(\beta-\alpha)+\alpha))}{(n-1)(n-2)(\beta-\alpha)^{n-3}}\\
=&\frac{\lambda(3-n)}{(n-1)(n-2)}(y^{n-1}-y).
\end{aligned} \tag{2.2.5}$$

Then by (2.1.24), it is easy to get

$$\begin{aligned}
&\lim_{c\to\infty}\int_0^1 K(y)^{-\frac{1}{2}}(\beta-\alpha)dy\\
=&\lim_{c\to\infty}\int_0^1\left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{1}{2}}dy\\
=&\int_0^1\left(\frac{y(1-y)}{2}\right)^{-\frac{1}{2}}dy\\
=&\sqrt{2}\pi,
\end{aligned}\tag{2.2.6}$$

and

$$\begin{aligned}
&\lim_{c\to\infty}\int_0^1 K(y)^{-\frac{1}{2}}(\beta-\alpha)ydy\\
=&\int_0^1\left(\frac{y(1-y)}{2}\right)^{-\frac{1}{2}}ydy\\
=&\frac{\sqrt{2}}{2}\pi.
\end{aligned}\tag{2.2.7}$$

From (2.1.24) and (2.2.5), we have

$$\begin{aligned}
&\lim_{c\to\infty}(\beta-\alpha)^{4-n}\int_0^1 K(y)^{-\frac{3}{2}}I(y)dy\\
=&\lim_{c\to\infty}\int_0^1\left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{3}{2}}\frac{I(y)}{(\beta-\alpha)^{n-1}}dy\\
=&\int_0^1\left(\frac{y(1-y)}{2}\right)^{-\frac{3}{2}}\left(\frac{\lambda(3-n)}{(n-1)(n-2)}(y^{n-1}-y)\right)dy\\
=&\frac{2\sqrt{2}\lambda(3-n)}{(n-1)(n-2)}\int_0^1(y(1-y))^{-\frac{3}{2}}(y^{n-1}-y)dy,
\end{aligned}\tag{2.2.8}$$

and

$$\begin{aligned}
&\lim_{c\to\infty}(\beta-\alpha)^{4-n}\int_0^1 K(y)^{-\frac{3}{2}}I(y)ydy\\
=&\lim_{c\to\infty}\int_0^1\left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{3}{2}}\frac{I(y)}{(\beta-\alpha)^{n-1}}ydy\\
=&\int_0^1\left(\frac{y(1-y)}{2}\right)^{-\frac{3}{2}}\left(\frac{\lambda(3-n)}{(n-1)(n-2)}(y^{n}-y^2)\right)dy\\
=&\frac{2\sqrt{2}\lambda(3-n)}{(n-1)(n-2)}\int_0^1(y(1-y))^{-\frac{3}{2}}(y^{n}-y^2)dy,
\end{aligned}\tag{2.2.9}$$

Similarly, from (2.1.24) and (2.2.4),

$$
\begin{aligned}
&\lim_{c\to\infty}(\beta-\alpha)^{4-n}\int_0^1 K(y)^{-\frac{3}{2}}H(y)(\beta-\alpha)dy\\
=&\lim_{c\to\infty}\int_0^1\left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{3}{2}}\frac{H(y)}{(\beta-\alpha)^{n-2}}dy\\
=&\int_0^1\left(\frac{y(1-y)}{2}\right)^{-\frac{3}{2}}\left(\frac{\lambda}{n-2}(y-y^{n-2})\right)dy\\
=&\frac{2\sqrt{2}\lambda}{n-2}\int_0^1 (y(1-y))^{-\frac{3}{2}}(y-y^{n-2})dy,
\end{aligned}
\tag{2.2.10}
$$

and

$$
\begin{aligned}
&\lim_{c\to\infty}(\beta-\alpha)^{4-n}\int_0^1 K(y)^{-\frac{3}{2}}H(y)(\beta-\alpha)ydy\\
=&\lim_{c\to\infty}\int_0^1\left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{3}{2}}\frac{H(y)}{(\beta-\alpha)^{n-2}}ydy\\
=&\int_0^1\left(\frac{y(1-y)}{2}\right)^{-\frac{3}{2}}\left(\frac{\lambda}{n-2}(y^2-y^{n-1})\right)dy\\
=&\frac{2\sqrt{2}\lambda}{n-2}\int_0^1 (y(1-y))^{-\frac{3}{2}}(y^2-y^{n-1})dy,
\end{aligned}
\tag{2.2.11}
$$

So with (2.1.6), (2.2.3) and (2.2.8),

$$
\lim_{c\to\infty}(\beta-\alpha)^{4-n}P_c \tag{2.2.12}
$$

$$
=-\frac{4\lambda(3-n)}{(n-1)(n-2)}\int_0^1 (y(1-y))^{-\frac{3}{2}}(y^{n-1}-y)dy \tag{2.2.13}
$$

is a negative finite number depending on n. Using (2.1.10), (2.1.11), (2.1.24), (2.2.8) and (2.2.9),

$$
\lim_{c\to\infty}A_c=2\pi. \tag{2.2.14}
$$

By (2.1.33), (2.2.10) and (2.2.13),

$$
\begin{aligned}
&\lim_{c\to\infty}(\beta-\alpha)^{4-n}P_\alpha\\
=&\frac{2\lambda}{n-2}\int_0^1 (y(1-y))^{-\frac{3}{2}}(y-y^{n-2})dy-\frac{4\lambda(3-n)}{(n-1)(n-2)}\int_0^1 (y(1-y))^{-\frac{3}{2}}(y^{n-1}-y)dy\\
=&\frac{2\lambda}{(n-1)(n-2)}\int_0^1 (y(1-y))^{-\frac{3}{2}}\left[(5-n)y-(n-1)y^{n-2}-2(3-n)y^{n-1}\right]dy.
\end{aligned}
\tag{2.2.15}
$$

This is also a negative finite number. By (2.1.34), (2.1.25), (2.2.10), (2.2.11) and (2.2.14),

$$\begin{aligned}&\lim_{c\to\infty} A_\alpha\\ =&\lim_{c\to\infty} P+\lim_{c\to\infty}\frac{\beta-\alpha}{\sqrt{2}}\int_0^1 K(y)^{-\frac{3}{2}}H(y)(y(\beta-\alpha)+\alpha)dy\\ &-2\lim_{c\to\infty}\frac{f(\beta)-f(\alpha)}{\beta-\alpha}A_c\\ =&2\pi-2\pi=0. \end{aligned}\tag{2.2.16}$$

Therefore,

$$\lim_{c\to\infty}(\beta-\alpha)^{4-n}(A_cP_\alpha-A_\alpha P_c)=2\pi\cdot\lim_{c\to\infty}(\beta-\alpha)^{4-n}P_\alpha<0. \tag{2.2.17}$$

□

Next we are going to check the sign of E when (α, c) is close to the boundary of Σ.

Lemma 2.10. *For any $\alpha > 0$,*

$$\begin{aligned}&\lim_{c\to h_c^{-1}(\alpha)} E(\alpha,c)\\ =&\frac{\lambda(3-n)}{12}\pi^2 f'(\alpha)^{-3}\frac{\alpha^{4-2n}}{(\alpha^{3-n}+\lambda)^4}\big[3(n-4)\alpha^{3-n}+\lambda(12-5n)\big].\end{aligned}$$

In particular, when $n\geq\frac{12}{5}$, $E(\alpha,c)<0$ as (α,c) is close to Γ_c.

Proof. Since $F(\beta)-F(\alpha)=c(\beta-\alpha)$, we have

$$\lim_{c\to h_c^{-1}(\alpha)}\beta_c=\lim_{c\to h_c^{-1}(\alpha)}\frac{\beta-\alpha}{f(\beta)-c}=\lim_{c\to h_c^{-1}(\alpha)}\frac{\beta_c}{f'(\beta)\beta_c-1}.$$

So

$$\lim_{c\to h_c^{-1}(\alpha)}\beta_c=\lim_{c\to h_c^{-1}(\alpha)}\frac{\beta-\alpha}{f(\beta)-c}=2f'(\alpha)^{-1}. \tag{2.2.18}$$

As in above proposition, we need limits of K, H and I. By (2.1.32),

$$
\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)} \frac{H(y)}{(\beta-\alpha)^2} \\
=&\lim_{\beta\to\alpha} \frac{-yf'(\beta)+yf'(y(\beta-\alpha)+\alpha)}{2(\beta-\alpha)} \\
=&\lim_{\beta\to\alpha} \frac{-yf''(\beta)+y^2f''(y(\beta-\alpha)+\alpha)}{2} \\
=&-\frac{y(1-y)}{2}f''(\alpha).
\end{aligned}
\tag{2.2.19}
$$

By (2.1.7),

$$
\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)} \frac{I(y)}{(\beta-\alpha)^3} \\
=&\lim_{\beta\to\alpha} \frac{yf(\beta)-yf(y(\beta-\alpha)+\alpha)-y(\beta-\alpha)(f'(\beta)-yf'(y(\beta-\alpha)+\alpha))}{3(\beta-\alpha)^2} \\
=&\lim_{\beta\to\alpha} \frac{-y(\beta-\alpha)(f''(\beta)-y^2f''(y(\beta-\alpha)+\alpha))}{6(\beta-\alpha)} \\
=&-\frac{y(1-y^2)}{6}f''(\alpha).
\end{aligned}
\tag{2.2.20}
$$

Then by (2.1.27),

$$
\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)} \int_0^1 K(y)^{-\frac{1}{2}}(\beta-\alpha)dy \\
=&\lim_{c\to h_c^{-1}(\alpha)} \int_0^1 \left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{1}{2}} dy \\
=&\int_0^1 \left(\frac{y(1-y)}{2}f'(\alpha)\right)^{-\frac{1}{2}} dy \\
=&\sqrt{2}\pi f'(\alpha)^{-\frac{1}{2}}.
\end{aligned}
\tag{2.2.21}
$$

Similarly,

$$
\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)} \int_0^1 K(y)^{-\frac{1}{2}}(\beta-\alpha)ydy \\
=&\int_0^1 \left(\frac{y(1-y)}{2}f'(\alpha)\right)^{-\frac{1}{2}} ydy \\
=&\frac{\sqrt{2}}{2}\pi f'(\alpha)^{-\frac{1}{2}}.
\end{aligned}
\tag{2.2.22}
$$

By (2.1.27) and (2.2.20), we have

$$\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)}\int_0^1 K(y)^{-\frac{3}{2}}I(y)dy\\
=&\lim_{c\to h_c^{-1}(\alpha)}\int_0^1 \left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{3}{2}}\frac{I(y)}{(\beta-\alpha)^3}dy\\
=&\int_0^1\left(\frac{y(1-y)}{2}f'(\alpha)\right)^{-\frac{3}{2}}\left(-\frac{y(1-y^2)}{6}f''(\alpha)\right)dy\\
=&-\frac{1}{3}f'(\alpha)^{-\frac{3}{2}}f''(\alpha)\left(\sqrt{2}\pi+\frac{\sqrt{2}}{2}\pi\right)\\
=&-\frac{\sqrt{2}}{2}\pi f'(\alpha)^{-\frac{3}{2}}f''(\alpha),
\end{aligned}\tag{2.2.23}$$

and

$$\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)}\int_0^1 K(y)^{-\frac{3}{2}}I(y)ydy\\
=&\int_0^1\left(\frac{y(1-y)}{2}f'(\alpha)\right)^{-\frac{3}{2}}\left(-\frac{y(1-y^2)}{6}f''(\alpha)\right)ydy\\
=&-\frac{1}{3}f'(\alpha)^{-\frac{3}{2}}f''(\alpha)\left(\frac{\sqrt{2}}{2}\pi+\frac{3\sqrt{2}}{8}\pi\right)\\
=&-\frac{7\sqrt{2}}{24}\pi f'(\alpha)^{-\frac{3}{2}}f''(\alpha).
\end{aligned}\tag{2.2.24}$$

By (2.1.27) and (2.2.19), we have

$$\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)}\int_0^1 K(y)^{-\frac{3}{2}}H(y)(\beta-\alpha)dy\\
=&\lim_{c\to h_c^{-1}(\alpha)}\int_0^1 \left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{3}{2}}\frac{H(y)}{(\beta-\alpha)^2}dy\\
=&\int_0^1\left(\frac{y(1-y)}{2}f'(\alpha)\right)^{-\frac{3}{2}}\left(-\frac{y(1-y)}{2}f''(\alpha)\right)dy\\
=&-\sqrt{2}\pi f'(\alpha)^{-\frac{3}{2}}f''(\alpha),
\end{aligned}\tag{2.2.25}$$

and

$$\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)}\int_0^1 K(y)^{-\frac{3}{2}}H(y)y(\beta-\alpha)dy\\
=&\int_0^1\left(\frac{y(1-y)}{2}f'(\alpha)\right)^{-\frac{3}{2}}\left(-\frac{y(1-y)}{2}f''(\alpha)\right)ydy\\
=&-\frac{\sqrt{2}}{2}\pi f'(\alpha)^{-\frac{3}{2}}f''(\alpha). \qquad (2.2.26)
\end{aligned}$$

Using (2.2.23), (2.2.24), (2.2.25) and (2.2.26), we obtain the limits

$$\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)}\int_0^1 K(y)^{-\frac{3}{2}}I(y)ydy\int_0^1 K(y)^{-\frac{3}{2}}H(y)(\beta-\alpha)dy\\
=&\frac{7}{12}\pi^2 f'(\alpha)^{-3}f''(\alpha)^2, \qquad (2.2.27)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)}\int_0^1 K(y)^{-\frac{3}{2}}I(y)dy\int_0^1 K(y)^{-\frac{3}{2}}H(y)y(\beta-\alpha)dy\\
=&\frac{1}{2}\pi^2 f'(\alpha)^{-3}f''(\alpha)^2. \qquad (2.2.28)
\end{aligned}$$

To calculate the values of the other two terms in $E(\alpha,c)$, write

$$\begin{aligned}
&\int_0^1 K(y)^{-\frac{1}{2}}ydy\int_0^1 K(y)^{-\frac{3}{2}}H(y)(\beta-\alpha)dy-\int_0^1 K(y)^{-\frac{1}{2}}dy\int_0^1 K(y)^{-\frac{3}{2}}I(y)dy\\
=&\left(\int_0^1 K(y)^{-\frac{1}{2}}ydy-\frac{\sqrt{2}\pi}{2(\beta-\alpha)}f'(\alpha)^{-\frac{1}{2}}\right)\int_0^1 K(y)^{-\frac{3}{2}}H(y)(\beta-\alpha)dy\\
&-\left(\int_0^1 K(y)^{-\frac{1}{2}}dy-\frac{\sqrt{2}\pi}{\beta-\alpha}f'(\alpha)^{-\frac{1}{2}}\right)\int_0^1 K(y)^{-\frac{3}{2}}I(y)dy\\
&+\frac{\sqrt{2}}{2}\pi f'(\alpha)^{-\frac{1}{2}}\int_0^1 K(y)^{-\frac{3}{2}}\left(H(y)-\frac{2I(y)}{\beta-\alpha}\right)dy \qquad (2.2.29)
\end{aligned}$$

Some limits of higher order are needed,

$$\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)}\frac{K(y)-\frac{y(1-y)}{2}(\beta-\alpha)^2 f'(\alpha)}{(\beta-\alpha)^3}\\
=&\lim_{\beta\to\alpha}\frac{yf''(\beta)-y^3 f''(y(\beta-\alpha)+\alpha)}{6}\\
=&\frac{y(1-y^2)}{6}f''(\alpha). \qquad (2.2.30)
\end{aligned}$$

By above results, we get

$$\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)}\int_0^1 K(y)^{-\frac{1}{2}}ydy-\frac{\sqrt{2}\pi}{2(\beta-\alpha)}f'(\alpha)^{-\frac{1}{2}}\\
=&\lim_{c\to h_c^{-1}(\alpha)}\int_0^1\left[K(y)^{-\frac{1}{2}}-\left(\frac{y(1-y)}{2}(\beta-\alpha)^2f'(\alpha)\right)^{-\frac{1}{2}}\right]ydy\\
=&-\frac{7\sqrt{2}}{48}\pi f''(\alpha)f'(\alpha)^{-\frac{3}{2}},
\end{aligned}\tag{2.2.31}$$

and

$$\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)}\int_0^1 K(y)^{-\frac{1}{2}}dy-\frac{\sqrt{2}\pi}{\beta-\alpha}f'(\alpha)^{-\frac{1}{2}}\\
=&\lim_{c\to h_c^{-1}(\alpha)}\int_0^1\left(K(y)^{-\frac{1}{2}}-(\frac{y(1-y)}{2}(\beta-\alpha)^2f'(\alpha))^{-\frac{1}{2}}\right)dy\\
=&-\frac{\sqrt{2}}{4}\pi f''(\alpha)f'(\alpha)^{-\frac{3}{2}}.
\end{aligned}\tag{2.2.32}$$

For the last limit, we have

$$\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)}\frac{H(y)(\beta-\alpha)-2I(y)}{(\beta-\alpha)^4}\\
=&\lim_{\beta\to\alpha}\frac{(\beta-\alpha)(yf^{(4)}(\beta)+(y^4-2y^5)f^{(4)}(y(\beta-\alpha)+\alpha))+4(y^3-y^4)f^{(3)}(y(\beta-\alpha)+\alpha)}{24}\\
=&\frac{y^3(1-y)}{6}f^{(3)}(\alpha).
\end{aligned}\tag{2.2.33}$$

Then

$$\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)}\left(\int_0^1 K(y)^{-\frac{1}{2}}ydy-\frac{\sqrt{2}\pi}{2(\beta-\alpha)}f'(\alpha)^{-\frac{1}{2}}\right)\int_0^1 K(y)^{-\frac{3}{2}}H(y)(\beta-\alpha)dy\\
=&-\frac{7\sqrt{2}}{48}\pi f''(\alpha)f'(\alpha)^{-\frac{3}{2}}\cdot\left(-\sqrt{2}\pi f'(\alpha)^{-\frac{3}{2}}f''(\alpha)\right)\\
=&\frac{7}{24}\pi^2f'(\alpha)^{-3}f''(\alpha)^2,
\end{aligned}\tag{2.2.34}$$

and

$$\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)}\left(\int_0^1 K(y)^{-\frac{1}{2}}dy-\frac{\sqrt{2}\pi}{\beta-\alpha}f'(\alpha)^{-\frac{1}{2}}\right)\int_0^1 K(y)^{-\frac{3}{2}}I(y)dy\\
=&-\frac{\sqrt{2}}{4}\pi f''(\alpha)f'(\alpha)^{-\frac{3}{2}}\cdot\left(-\frac{\sqrt{2}}{2}\pi f'(\alpha)^{-\frac{3}{2}}f''(\alpha)\right)\\
=&\frac{1}{4}\pi^2f'(\alpha)^{-3}f''(\alpha)^2.
\end{aligned}\tag{2.2.35}$$

and

$$\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)}\int_0^1 K(y)^{-\frac{3}{2}}\left(H(y)-\frac{2I(y)}{\beta-\alpha}\right)dy\\
=&\int_0^1\left(\frac{y(1-y)}{2}f'(\alpha)\right)^{-\frac{3}{2}}\left(\frac{y^3(1-y)}{6}f^{(3)}(\alpha)\right)dy\\
=&\frac{\sqrt{2}}{8}\pi f'(\alpha)^{-\frac{3}{2}}f^{(3)}(\alpha).
\end{aligned}\tag{2.2.36}$$

Putting all these limits together,

$$\begin{aligned}
&\int_0^1 K(y)^{-\frac{1}{2}}ydy\int_0^1 K(y)^{-\frac{3}{2}}H(y)(\beta-\alpha)dy-\int_0^1 K(y)^{-\frac{1}{2}}dy\int_0^1 K(y)^{-\frac{3}{2}}I(y)dy\\
=&\frac{7}{24}\pi^2 f'(\alpha)^{-3}f''(\alpha)^2-\frac{1}{4}\pi^2 f'(\alpha)^{-3}f''(\alpha)^2+\frac{\sqrt{2}}{2}\pi f'(\alpha)^{-\frac{1}{2}}\cdot\left(\frac{\sqrt{2}}{8}\pi f'(\alpha)^{-\frac{3}{2}}f^{(3)}(\alpha)\right)\\
=&\frac{1}{24}\pi^2 f'(\alpha)^{-3}f''(\alpha)^2+\frac{1}{8}\pi^2 f'(\alpha)^{-2}f^{(3)}(\alpha).
\end{aligned}\tag{2.2.37}$$

So

$$\begin{aligned}
&\lim_{c\to h_c^{-1}(\alpha)} E(\alpha,c)\\
=&\frac{7}{12}\pi^2 f'(\alpha)^{-3}f''(\alpha)^2-\frac{1}{2}\pi^2 f'(\alpha)^{-3}f''(\alpha)^2\\
&+2\cdot\left(\frac{1}{24}\pi^2 f'(\alpha)^{-3}f''(\alpha)^2+\frac{1}{8}\pi^2 f'(\alpha)^{-2}f^{(3)}(\alpha)\right)\\
=&\frac{1}{12}\pi^2 f'(\alpha)^{-3}\left(2f''(\alpha)^2+3f'(\alpha)f^{(3)}(\alpha)\right)\\
=&\frac{\lambda(3-n)}{12}\pi^2 f'(\alpha)^{-3}\frac{\alpha^{4-2n}}{(\alpha^{3-n}+\lambda)^4}\left[3(n-4)\alpha^{3-n}+\lambda(12-5n)\right].
\end{aligned}\tag{2.2.38}$$

In the last line, we have used (11),(2.1.9) and (2.1.19). Obviously, When $n\geq\frac{12}{5}$, the limit is negative for all α. In other words, when $n\geq\frac{12}{5}$, $E(\alpha,c)$ is negative when $(\alpha,c)\in\Sigma$ close to upper boundary Γ_c. □

Proposition 2.11.

$$\begin{aligned}
&\lim_{c\to 0}\beta^{5-n}E(0,c)\\
=&\frac{\pi\lambda(5-n)}{4-n}\left[\frac{(3-n)^2\Gamma(\frac{1}{8-2n})^2}{(4-n)\Gamma(\frac{5-n}{8-2n})^2}+2(5-n)(1-n)\frac{\Gamma(\frac{3}{8-2n})\Gamma(\frac{7-2n}{8-2n})}{\Gamma(\frac{7-n}{8-2n})\Gamma(\frac{3-n}{8-2n})}\right].
\end{aligned}\tag{2.2.39}$$

Numerically, the value is negative when $n\geq 2.205$.

Proof. First we have the following limiting behavior of $f'(x)$ and $f''(x)$,

$$\lim_{x\to 0}\frac{f'(x)}{x^{3-n}}=\lim_{x\to 0}\frac{1}{x^{3-n}+\lambda}=\frac{1}{\lambda}.$$

$$\lim_{x\to 0}\frac{f''(x)}{x^{2-n}}=\lim_{x\to 0}\frac{\lambda(3-n)}{(x^{3-n}+\lambda)^2}=\frac{3-n}{\lambda}.$$

When $\alpha=0$, $K(y)=yF(\beta)-F(y\beta)$. So we have

$$\begin{aligned}\lim_{c\to 0}\frac{K(y)}{\beta^{5-n}}&=\lim_{c\to 0}\frac{yf(\beta)-yf(y\beta)}{(5-n)\beta^{4-n}}\\&=\lim_{c\to 0}\frac{yf'(\beta)-y^2f'(y\beta)}{(5-n)(4-n)\beta^{3-n}}\\&=\frac{y-y^{5-n}}{(5-n)(4-n)\lambda},\end{aligned}\tag{2.2.40}$$

$$\begin{aligned}\lim_{c\to 0}\frac{I(y)}{\beta^{5-n}}&=\lim_{c\to 0}\frac{\beta y^3f''(y\beta)-\beta yf''(\beta)}{(5-n)(4-n)\beta^{3-n}}\\&=\frac{(3-n)(y^{5-n}-y)}{(5-n)(4-n)\lambda},\end{aligned}\tag{2.2.41}$$

and

$$\begin{aligned}\lim_{c\to 0}\frac{H(y)}{\beta^{4-n}}&=\lim_{c\to 0}\frac{yf'(y\beta)-yf'(\beta)}{(4-n)\beta^{3-n}}\\&=\frac{y^{4-n}-y}{(4-n)\lambda}.\end{aligned}\tag{2.2.42}$$

We could calculate the following integrals,

$$\begin{aligned}&\lim_{c\to 0}\beta^{\frac{5-n}{2}}\int_0^1 K(y)^{-\frac{1}{2}}dy\\=&((5-n)(4-n)\lambda)^{\frac{1}{2}}\int_0^1 y^{-\frac{1}{2}}(1-y^{4-n})^{-\frac{1}{2}}dy\\=&\left(\frac{\pi\lambda(5-n)}{4-n}\right)^{\frac{1}{2}}\frac{\Gamma(\frac{1}{8-2n})}{\Gamma(\frac{5-n}{8-2n})},\end{aligned}\tag{2.2.43}$$

$$\begin{aligned}&\lim_{c\to 0}\beta^{\frac{5-n}{2}}\int_0^1 K(y)^{-\frac{1}{2}}ydy\\=&((5-n)(4-n)\lambda)^{\frac{1}{2}}\int_0^1 y^{\frac{1}{2}}(1-y^{4-n})^{-\frac{1}{2}}dy\\=&\left(\frac{\pi\lambda(5-n)}{4-n}\right)^{\frac{1}{2}}\frac{\Gamma(\frac{3}{8-2n})}{\Gamma(\frac{7-n}{8-2n})},\end{aligned}\tag{2.2.44}$$

$$\begin{aligned}
&\lim_{c\to 0}\beta^{\frac{5-n}{2}}\int_0^1 K(y)^{-\frac{3}{2}}I(y)dy\\
=&-(3-n)((5-n)(4-n)\lambda)^{\frac{1}{2}}\int_0^1 y^{-\frac{1}{2}}(1-y^{4-n})^{-\frac{1}{2}}dy\\
=&-(3-n)\left(\frac{\pi\lambda(5-n)}{4-n}\right)^{\frac{1}{2}}\frac{\Gamma(\frac{1}{8-2n})}{\Gamma(\frac{5-n}{8-2n})},
\end{aligned}\tag{2.2.45}$$

$$\begin{aligned}
&\lim_{c\to 0}\beta^{\frac{5-n}{2}}\int_0^1 K(y)^{-\frac{3}{2}}I(y)ydy\\
=&-(3-n)((5-n)(4-n)\lambda)^{\frac{1}{2}}\int_0^1 y^{\frac{1}{2}}(1-y^{4-n})^{-\frac{1}{2}}dy\\
=&-(3-n)\left(\frac{\pi\lambda(5-n)}{4-n}\right)^{\frac{1}{2}}\frac{\Gamma(\frac{3}{8-2n})}{\Gamma(\frac{7-n}{8-2n})},
\end{aligned}\tag{2.2.46}$$

$$\begin{aligned}
&\lim_{c\to 0}\beta^{\frac{5-n}{2}}\int_0^1 K(y)^{-\frac{3}{2}}H(y)\beta dy\\
=&-(5-n)^{\frac{3}{2}}((4-n)\lambda)^{\frac{1}{2}}\int_0^1 y^{-\frac{1}{2}}(1-y^{4-n})^{-\frac{3}{2}}(1-y^{3-n})dy\\
=&-\left(\frac{\pi\lambda(5-n)^3}{(4-n)^3}\right)^{\frac{1}{2}}\left((3-n)\frac{\Gamma(\frac{1}{8-2n})}{\Gamma(\frac{5-n}{8-2n})}+(8-2n)\frac{\Gamma(\frac{7-2n}{8-2n})}{\Gamma(\frac{3-n}{8-2n})}\right).
\end{aligned}\tag{2.2.47}$$

and

$$\begin{aligned}
&\lim_{c\to 0}\beta^{\frac{5-n}{2}}\int_0^1 K(y)^{-\frac{3}{2}}H(y)\beta ydy\\
=&-(5-n)^{\frac{3}{2}}((4-n)\lambda)^{\frac{1}{2}}\int_0^1 y^{\frac{1}{2}}(1-y^{4-n})^{-\frac{3}{2}}(1-y^{3-n})dy\\
=&-\left(\frac{\pi\lambda(5-n)^3}{(4-n)^3}\right)^{\frac{1}{2}}\left(\frac{\Gamma(\frac{1}{8-2n})}{\Gamma(\frac{5-n}{8-2n})}+(1-n)\frac{\Gamma(\frac{3}{8-2n})}{\Gamma(\frac{7-n}{8-2n})}\right).
\end{aligned}\tag{2.2.48}$$

Finally,

$$\begin{aligned}
&\lim_{c\to 0}\beta^{5-n}E(0,c)\\
=&\frac{\pi\lambda(5-n)}{4-n}\left[\frac{(3-n)^2\Gamma(\frac{1}{8-2n})^2}{(4-n)\Gamma(\frac{5-n}{8-2n})^2}+2(5-n)(1-n)\frac{\Gamma(\frac{3}{8-2n})\Gamma(\frac{7-2n}{8-2n})}{\Gamma(\frac{7-n}{8-2n})\Gamma(\frac{3-n}{8-2n})}\right]
\end{aligned}$$

Calculating it numerically, when $n \geq 2.205$, it is negative. See Figure 2.1 below.

□

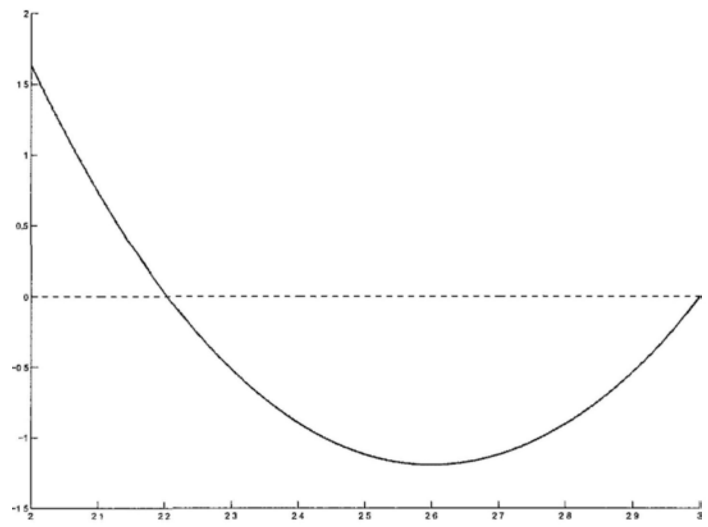

Figure 2.1: The term in bracket in (2.2.39) for $n \in (2, 3)$.

Lemma 2.12. *There exists some $n_0 < 3$ so that when $n \in (n_0, 3)$, $E(0, c) < 0$ for all $c > 0$.*

Proof. By Proposition 2.9 and Proposition 2.11, $E(0, c) < 0$ when c is small or large enough. So it suffices to show the result for c on a bounded interval (δ_1, δ_2).

As $\alpha = 0$,

$$K''(y) = -\beta^2 f'(\beta y) = -\beta^{5-n} y^{3-n} \frac{1}{(\beta y)^{3-n} + \lambda}$$
$$\leq -\beta^{5-n} y^{3-n} \frac{1}{\beta^{3-n} + \lambda} = -\beta^2 f'(\beta) y^{3-n}. \tag{2.2.49}$$

So

$$K(y) \geq \beta^2 f'(\beta) \frac{y - y^{5-n}}{(5-n)(4-n)}. \tag{2.2.50}$$

On the other hand,

$$K''(y) = -\beta^2 f'(\beta y) \geq -\beta^2 f'(\beta), \tag{2.2.51}$$

so

$$K(y) \leq \beta^2 f'(\beta) \frac{y - y^2}{2}. \tag{2.2.52}$$

By similar means,

$$\begin{aligned}H''(y) =&\beta^2 f''(\beta y) = \frac{\lambda(3-n)\beta^{4-n}y^{2-n}}{((\beta y)^{3-n}+\lambda)^2}\\ \geq&\frac{\lambda(3-n)\beta^{4-n}y^{2-n}}{(\beta^{3-n}+\lambda)^2} = \beta^2 f''(\beta)y^{2-n},\end{aligned} \tag{2.2.53}$$

so that

$$H(y) \leq \beta^2 f''(\beta)\frac{y^{4-n}-y}{(4-n)(3-n)}. \tag{2.2.54}$$

In the reverse direction,

$$\begin{aligned}H''(y) =&\beta^2 f''(\beta y) = \lambda(3-n)f'(\beta y)^2\beta^{n-2}y^{n-4}\\ \leq&\lambda(3-n)f'(\beta)^2\beta^{n-2}y^{n-4}\\ =&\beta^2 f''(\beta)y^{n-4},\end{aligned} \tag{2.2.55}$$

so

$$H(y) \geq \beta^2 f''(\beta)\frac{y^{n-2}-y}{(n-2)(n-3)}. \tag{2.2.56}$$

Similarly,

$$\begin{aligned}I''(y) =&\beta^3 y f''(\beta y) = \frac{\lambda(3-n)\beta^{5-n}y^{3-n}}{((\beta y)^{3-n}+\lambda)^2}\\ \geq&\frac{\lambda(3-n)\beta^{5-n}y^{3-n}}{(\beta^{3-n}+\lambda)^2} = \beta^3 f''(\beta)y^{3-n}.\end{aligned} \tag{2.2.57}$$

Hence,

$$I(y) \leq \beta^3 f''(\beta)\frac{y^{5-n}-y}{(5-n)(4-n)}. \tag{2.2.58}$$

In the reverse direction,

$$\begin{aligned}I''(y) =&\beta^3 y f''(\beta y) = \lambda(3-n)f'(\beta y)^2\beta^{n-1}y^{n-3}\\ \leq&\lambda(3-n)f'(\beta)^2\beta^{n-1}y^{n-3}\\ =&\beta^3 f''(\beta)y^{n-3},\end{aligned} \tag{2.2.59}$$

so

$$I(y) \geq \beta^3 f''(\beta)\frac{y^{n-1}-y}{(n-1)(n-2)}. \tag{2.2.60}$$

With these estimates, we have the following estimates:

$$\int_0^1 K^{-\frac{1}{2}}(y)dy \le \left(\frac{(5-n)(4-n)}{\beta^2 f'(\beta)}\right)^{\frac{1}{2}} \int_0^1 (y-y^{5-n})^{-\frac{1}{2}}dy$$
$$\equiv \beta^{-1} f'(\beta)^{-\frac{1}{2}} d_1(n), \tag{2.2.61}$$

$$\int_0^1 K^{-\frac{1}{2}}(y)ydy \ge \left(\frac{2}{\beta^2 f'(\beta)}\right)^{\frac{1}{2}} \int_0^1 (y-y^2)^{-\frac{1}{2}}ydy$$
$$\equiv \beta^{-1} f'(\beta)^{-\frac{1}{2}} d_2(n), \tag{2.2.62}$$

$$\int_0^1 K^{-\frac{3}{2}}(y)I(y)dy$$
$$\le \left(\frac{2}{\beta^2 f'(\beta)}\right)^{\frac{3}{2}} \frac{\beta^3 f''(\beta)}{(5-n)(4-n)} \int_0^1 (y-y^2)^{-\frac{3}{2}}(y^{5-n}-y)dy$$
$$= \frac{2\sqrt{2}}{(5-n)(4-n)} f''(\beta) f'(\beta)^{-\frac{3}{2}} \int_0^1 (y-y^2)^{-\frac{3}{2}}(y^{5-n}-y)dy$$
$$\equiv f''(\beta) f'(\beta)^{-\frac{3}{2}} d_3(n), \tag{2.2.63}$$

$$\int_0^1 K^{-\frac{3}{2}}(y)I(y)dy$$
$$\ge \left(\frac{(5-n)(4-n)}{\beta^2 f'(\beta)}\right)^{\frac{3}{2}} \frac{\beta^3 f''(\beta)}{(n-1)(n-2)} \int_0^1 (y-y^{(}5-n))^{-\frac{3}{2}}(y^{n-1}-y)dy$$
$$= \frac{(5-n)^{\frac{3}{2}}(4-n)^{\frac{3}{2}}}{(n-1)(n-2)} f''(\beta) f'(\beta)^{-\frac{3}{2}} \int_0^1 (y-y^{5-n})^{-\frac{3}{2}}(y^{n-1}-y)dy$$
$$\equiv f''(\beta) f'(\beta)^{-\frac{3}{2}} d_4(n), \tag{2.2.64}$$

$$\int_0^1 K^{-\frac{3}{2}}(y)I(y)ydy$$
$$\ge \frac{(5-n)^{\frac{3}{2}}(4-n)^{\frac{3}{2}}}{(n-1)(n-2)} f''(\beta) f'(\beta)^{-\frac{3}{2}} \int_0^1 (y-y^{5-n})^{-\frac{3}{2}}(y^{n-1}-y)ydy$$
$$\equiv f''(\beta) f'(\beta)^{-\frac{3}{2}} d_5(n), \tag{2.2.65}$$

$$\int_0^1 K^{-\frac{3}{2}}(y)H(y)\beta dy$$
$$\ge \frac{(5-n)^{\frac{3}{2}}(4-n)^{\frac{3}{2}}}{(n-2)(n-3)} f''(\beta) f'(\beta)^{-\frac{3}{2}} \int_0^1 (y-y^{5-n})^{-\frac{3}{2}}(y^{n-2}-y)dy$$
$$\equiv f''(\beta) f'(\beta)^{-\frac{3}{2}} d_6(n), \tag{2.2.66}$$

$$
\begin{aligned}
&\int_0^1 K^{-\frac{3}{2}}(y)H(y)\beta dy \\
\leq &\frac{2\sqrt{2}}{(4-n)(3-n)} f''(\beta)f'(\beta)^{-\frac{3}{2}} \int_0^1 (y-y^2)^{-\frac{3}{2}}(y^{4-n}-y)dy \\
\equiv &f''(\beta)f'(\beta)^{-\frac{3}{2}} d_7(n),
\end{aligned} \tag{2.2.67}
$$

and

$$
\begin{aligned}
&\int_0^1 K^{-\frac{3}{2}}(y)H(y)\beta y dy \\
\leq &\frac{2\sqrt{2}}{(4-n)(3-n)} f''(\beta)f'(\beta)^{-\frac{3}{2}} \int_0^1 (y-y^2)^{-\frac{3}{2}}(y^{4-n}-y)y dy \\
\equiv &f''(\beta)f'(\beta)^{-\frac{3}{2}} d_8(n).
\end{aligned} \tag{2.2.68}
$$

Note that all $d_i(n) = O(1)$, $i = 1, \cdots, 8$, as $n \to 3$. Putting things together,

$$
\begin{aligned}
E(0,c) \leq &f'(\beta)^{-3} f''(\beta)^2 \big[d_5(n)d_6(n) - d_3(n)d_8(n)\big] \\
&+ 2\beta^{-1} f'(\beta)^{-2} f''(\beta)\big[d_2(n)d_7(n) - d_1(n)d_4(n)\big],
\end{aligned} \tag{2.2.69}
$$

where $d_2(n)d_7(n) - d_1(n)d_4(n)$ turns negative as n close to 3, see Figure 2.2.

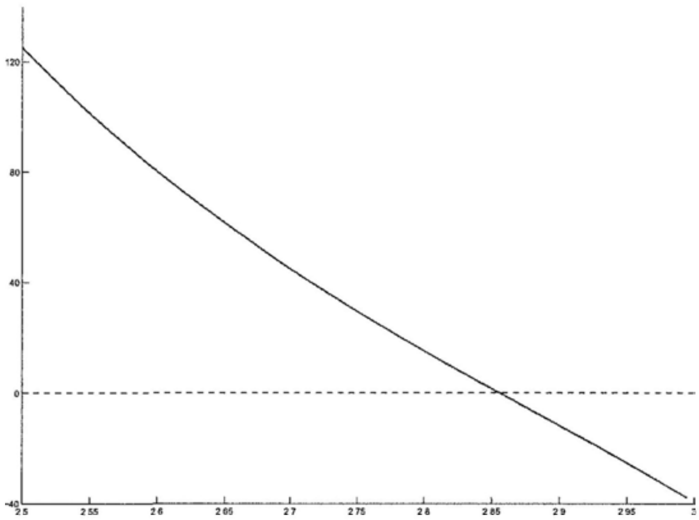

Figure 2.2: $d_2(n)d_7(n) - d_1(n)d_4(n)$ for $n \in (2.5, 3)$.

Since

$$f''(\beta) = \lambda(3-n)\frac{\beta^{2-n}}{\beta^{3-n}+\lambda} = (3-n)O(1) \text{ as } n \to 3. \tag{2.2.70}$$

there is some $n_0 < 3$ such that for $n \in (n_0, 3)$, (2.2.69) is negative on bounded interval (δ_1, δ_2). The lemma is proved. □

Now, we are ready to prove Theorem D.

Proof of Theorem D. Following the notations in proof of Theorem A. for any $P_0 > 2\pi$ and $A_0 \in A_{P_0}$, the number of minimal period steady states in $X(P_0, A_0)$ is exactly the cardinality of the set

$$\{c \in (c_1, c_2) : A(\phi_{P_0}(c), c) = A_0\}. \tag{2.2.71}$$

As both P and A are analytic in parameters α and c, the P_0-curve, $\alpha = \phi_{P_0}(c)$. is also analytic by the implicit function theorem. Thus, $A(\phi_{P_0}(c), c) - A_0$ is analytic in (c_1, c_2), too. It is known that the set of zeros of any analytic function contains no accumulation point unless the functiion is zero everywhere. By Theorem C, $A(\varphi_{P_0}(c), c) - A_0 \not\equiv 0$. So the number of steady states is at most infinitely countable.

Let $n^* = \max\{\frac{12}{5}, n_0\}$. When $n \geq n^*$, by Lemma 2.10, Lemma 2.12 and (2.2.1), neither of the two endpoints c_1, c_2 can be an accumulation point of (2.2.71). So there is only finitely many corresponding steady states.

As for non-minimal period steady states. they are in $X(P_0/k, A_0/k)$ for some integer k. Because minimal period of any positive steady state is larger than 2π. such k is bounded from above. □

2.3 Energy Levels of Steady States

In this section we investigate the phase space by comparing the values of the energy at the positive periodic, constant and 0-angle droplet steady states. Let $h(\alpha, c)(x)$ be the steady state of (7) satisfying (11) with minimum α. Then we have the monotonicity properties of energy.

Proposition 2.13. *The energy defined by* (5) *decreases in* α *and* c *for large* n. *Specifically, when* $n \geq 1$,

$$\frac{\partial}{\partial \alpha}\mathcal{E}(h) = -cA_\alpha(\alpha, c) + (c\alpha - F(\alpha))P_\alpha(\alpha, c) < 0 \tag{2.3.1}$$

and when $n \geq 2$,

$$\frac{\partial}{\partial c}\mathcal{E}(h) = -cA_c(\alpha, c) + (c\alpha - F(\alpha))P_c(\alpha. c) < 0. \tag{2.3.2}$$

Proof. As the energy is invariant under translation, we assume without loss of generality that h attains its minimum at the origin. Then

$$\begin{aligned}
&\frac{\partial}{\partial \alpha}\mathcal{E}(h) \\
=&\int_0^{P(\alpha,c)} h'\frac{\partial h'}{\partial \alpha}dx - \int_0^{P(\alpha,c)} f(h)\frac{\partial h}{\partial \alpha}dx + \frac{1}{2}h'(P(\alpha,c))P_\alpha - F(h(P(\alpha,c)))P_\alpha \\
=&-c\int_0^{P(\alpha,c)} \frac{\partial}{\partial \alpha}hdx - F(\alpha)P_\alpha.
\end{aligned} \tag{2.3.3}$$

On the other hand,

$$A_\alpha(\alpha, c) = \frac{\partial}{\partial \alpha}\int_0^{P(\alpha,c)} hdx = \int_0^{P(\alpha.c)} \frac{\partial}{\partial \alpha}hdx - \alpha P_\alpha. \tag{2.3.4}$$

Putting them together to get the equation in (2.3.1). To show monotonicity, we use Proposition 2.4, Proposition 2.5 and the fact $c\alpha - F(\alpha) \geq 0$. (2.3.2) can be proved similarly. □

Followings are the comparison results.

Proposition 2.14 (Constant vs Positive). *Assume* $n \geq 2$ *and* $E(\alpha, c) < 0$ *in* Σ. *Let* h_p *be the positive steady state in* $X(P_0, A_0)$ *and* $h_c \equiv A_0/P_0$. *Then* $\mathcal{E}(h_p) < \mathcal{E}(h_c)$.

Proof. Since $n \geq 2$, $A_c > 0$, $A_\alpha > 0$ and $P_c < 0$, $P_\alpha < 0$. So $(A/P)_c = (A_cP - P_cA)/P^2 > 0$ and similarly $(A/P)_\alpha > 0$. Thus, as in the proof of Theorem C, there exists an A_0/P_0-curve, $\alpha = \phi_{A_0/P_0}(c)$ such that $(A/P)(\phi_{A_0/P_0}(c), c) \equiv A_0/P_0$. Easy to see $\phi'_{A_0/P_0}(c) = (AP_c - A_cP)/(A_\alpha P - AP_\alpha) < 0$. There must be c_*. c^*, $c_* < c^*$ satisfying $h(\phi_{A_0/P_0}(c_*), c_*) = h_c$ and $h(\phi_{A_0/P_0}(c^*), c^*) = h_p$.

Along the A_0/P_0-curve, we have

$$\begin{aligned}\frac{d}{dc}\mathcal{E}(h(\phi_{P_0/A_0}(c), c)) =& - c(A_\alpha \phi'_{P_0/A_0}(c) + A_c) + (c\alpha - F(\alpha))(P_\alpha \phi'_{A_0/P_0}(c) + P_c) \\ =& \frac{A_\alpha P_c - A_c P_\alpha}{A_\alpha P - AP_\alpha}(c(P\alpha - A) - F(\alpha)P).\end{aligned}$$

Noting that $A > \alpha P$ and $E(\alpha, c) < 0$, it follows

$$\frac{d}{dc}\mathcal{E}(h(\phi_{P_0/A_0}(c), c)) < 0.$$

Therefore, $\mathcal{E}(h_c) > \mathcal{E}(h_p)$. □

Proposition 2.15 (0-angle droplet vs Positive). *Assume $n \geq 2$ and $E(\alpha, c) < 0$ in Σ. Let h_p, h_d be the positive steady state and 0-angle droplet in $X(P_0, A_0)$ respectively. Then $\mathcal{E}(h_p) < \mathcal{E}(h_c)$.*

Proof. When $n \geq 2$, the A_0-curve, $\alpha = \phi_{A_0}(c)$ exists with $\phi'_{A_0}(c) = -A_c/A_\alpha < 0$. There exist c_*, c^*, $c_* < c^*$ satisfying $h(\phi_{A_0}(c_*), c_*) = h_p$ and $h(\phi_{A_0}(c^*), c^*) = h_d$. Along the A_0-curve,

$$\begin{aligned}\frac{d}{dc}\mathcal{E}(h(\phi_{A_0}(c), c)) =& - c(A_\alpha \phi'_{A_0}(c) + A_c) + (c\alpha - F(\alpha))(P_\alpha \phi'_{A_0}(c) + P_c) \\ =& (c\alpha - F(\alpha))(A_\alpha P_c - A_c P_\alpha)/A_\alpha > 0.\end{aligned}$$

Therefore, $\mathcal{E}(h_d) > \mathcal{E}(h_p)$. □

Remarks: $E(\alpha, c) < 0$ holds numerically when $n \geq \frac{12}{5}$. see some evidence in the next section.

For positive vs positive case, we have no definite conclusion. Suppose $h^* = h(\alpha^*, c^*)$ and $h_* = h(\alpha_*, c_*)$ have same minimal period P_0 and area A_0. By Theorem C, (α^*, c^*) and (α_*, c_*) are in the graph of P_0-curve, $\{(\alpha, c) : \alpha = \phi_{P_0}(c)\}$. Along the P_0-curve,

$$\begin{aligned}\frac{d}{dc}\mathcal{E}(h(\phi_{P_0}(c), c)) &= -\, c(A_\alpha \phi'(c) + A_c) + (c\alpha - F(\alpha))(P_\alpha \phi'(c) + P_c) \\ &= -\, c\frac{d}{dc}A(\phi_{P_0}(c), c).\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{E}(h^*) - \mathcal{E}(h_*) &= -\int_{c_*}^{c^*} c\frac{d}{dc}A(\phi_{P_0}(c), c)dc \\ &= -\, cA\Big|_{c_*}^{c^*} + \int_{c^*}^{c_*} A(\phi_{P_0}(c), c)dc \\ &= \int_{c_*}^{c^*} (A(\phi_{P_0}(c), c) - A_0)dc.\end{aligned}$$

The sign of this difference is uncertain.

2.4 Numerical Computations

In this section we present some numerical work. In the following computations, we always take $\lambda = 1$. In fact, denoting (11) by $f'(x; \lambda)$, it is easy to see $f(x; \lambda) = \lambda^{3-n} f(\lambda^{n-3}x; 1)$. Then, defining $\hat{h}(x) = \lambda^{n-3}h(x)$ where h is the steady state, we have

$$\begin{cases}\hat{h}_{xx} + f(\hat{h}; 1) = \lambda^{n-3}c \\ \min \hat{h} = \lambda^{n-3}\alpha.\end{cases}$$

Thus $P(\lambda^{n-3}\alpha, \lambda^{n-3}c; 1) = P(\alpha, c; \lambda)$ and $A(\lambda^{n-3}\alpha, \lambda^{n-3}c; 1) = \lambda^{n-3}A(\alpha, c; \lambda)$. So specifying $\lambda = 1$ does not influence the monotonicity of A and P, and the sign of E, either.

In section 2.1, we have proven that $A(\alpha, c)$ is increasing in c when $n \geq 2$ and is increasing in α when $n \geq 1$. The numerical computations show that $A_c(\alpha, c)$ may be negative when $n < 2$. We have observed that when $n \leq 1.26$, $A_c(\alpha, c)$ turns negative as α and c are close to zero but not vanish, and when $n \geq 1.27$, $A_c(\alpha, c) > 0$. See Figures 2.3-2.6, where the x-axis and y-axis represent α and β respectively for simplifying computations. While $A_\alpha(\alpha, c)$ seems keeping positivity for $n \in (0, 1)$. See Figures 2.7, 2.8 below.

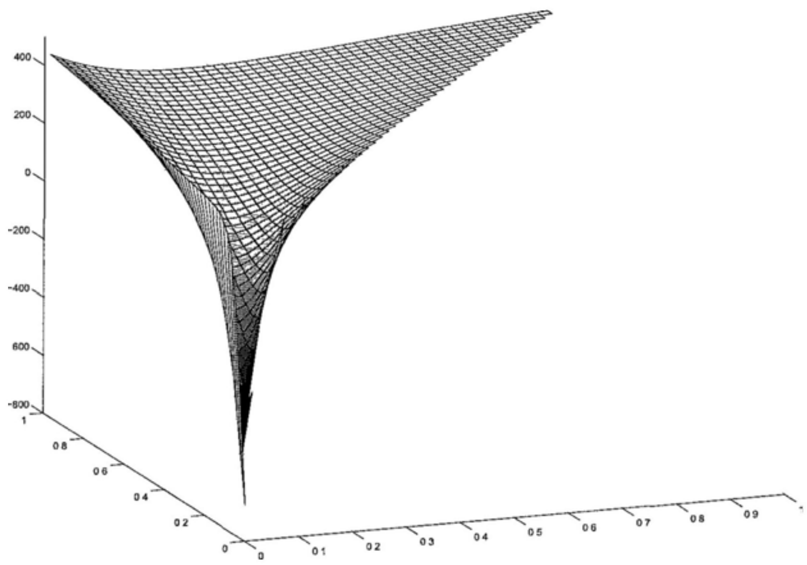

Figure 2.3: A_c for $n = 1.1$.It turns negative as α and β are very small.

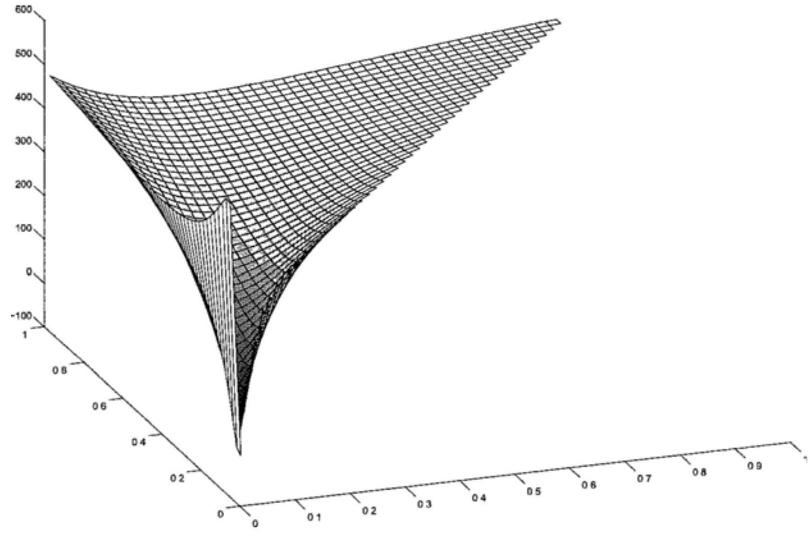

Figure 2.4: A_c for $n = 1.26$.

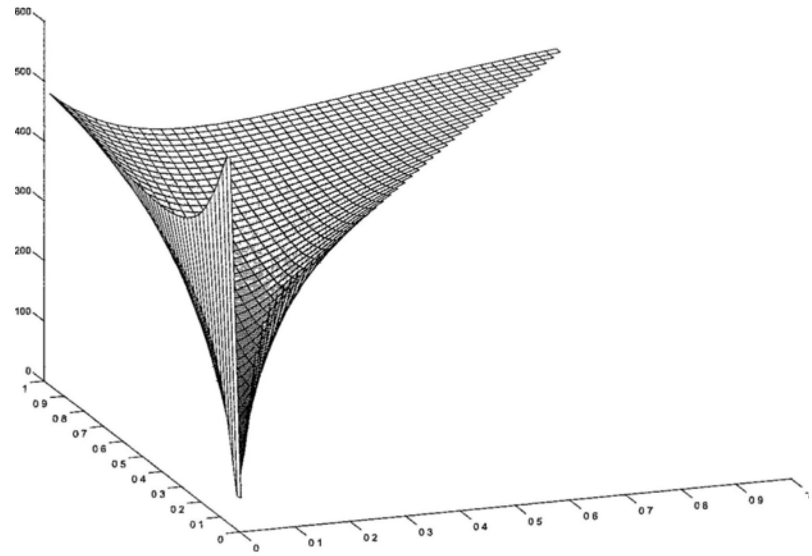

Figure 2.5: A_c for $n = 1.27$.

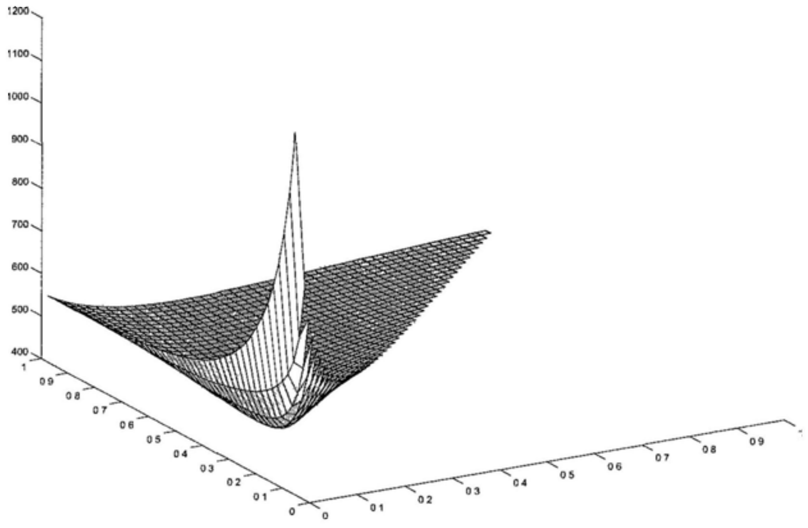

Figure 2.6: A_c for $n = 1.5$.

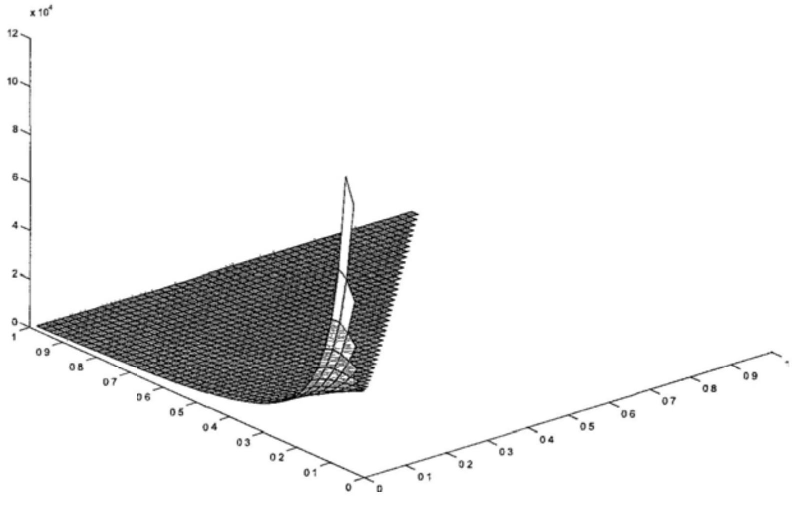

Figure 2.7: A_α for $n = 0.1$

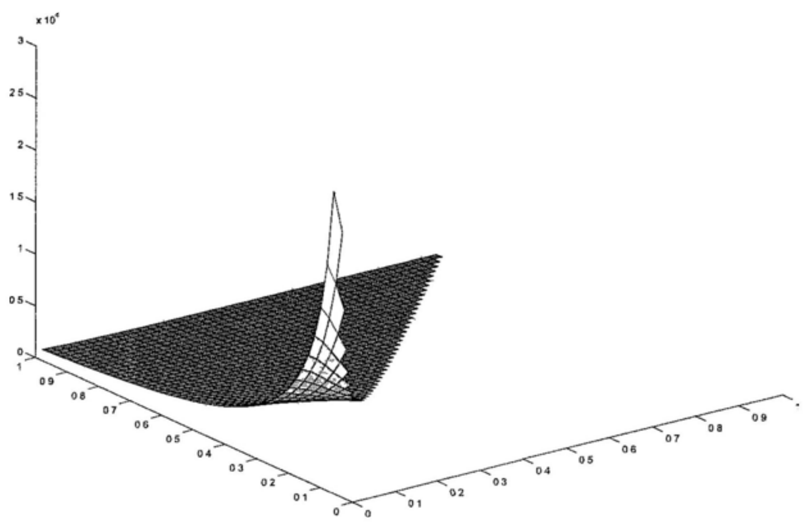

Figure 2.8: A_α for $n = 0.9$.

In the proof of Theorem D, we take $n^* = \min\{\frac{12}{5}, n_0\}$. The numerical evidence shows that $\frac{12}{5}$ is enough. When $n \le 2.204$, which obtained in Proposition 2.11, $E(0, c) \to +\infty$ as c goes to zero. When n is larger, say, $n \ge 2.4$, $E(0, c)$ seems increasing in c. See Figures 2.9, 2.10.

As explained in section 2.2, the sign changing or not of $E(\alpha, c)$ determines the uniqueness of the steady state. By Lemma 2.10, when $n < \frac{12}{5}$, $E(\alpha, c)$ is positive as c is close to $h_c^{-1}(\alpha)$ for some rather small α. Other the other hand, by Proposition 2.9, $E(\alpha, c)$ turns negative as c comes to infinity. So E must change sign, as Figure 2.11 shows ($n = 2.2$). When n goes larger, things are different. Our computations suggest that $E(\alpha, c) < 0$ when $\alpha \ge \frac{12}{5}$, but an analytical proof has not been built yet.

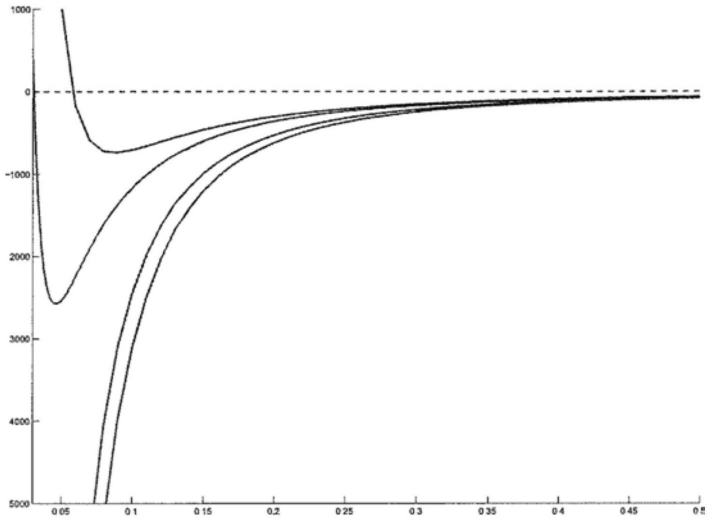

Figure 2.9: $E(0, c), c \in (0.03, 0.5)$ for $n = 2.15, 2.17, 2.25, 2.35$ from top.

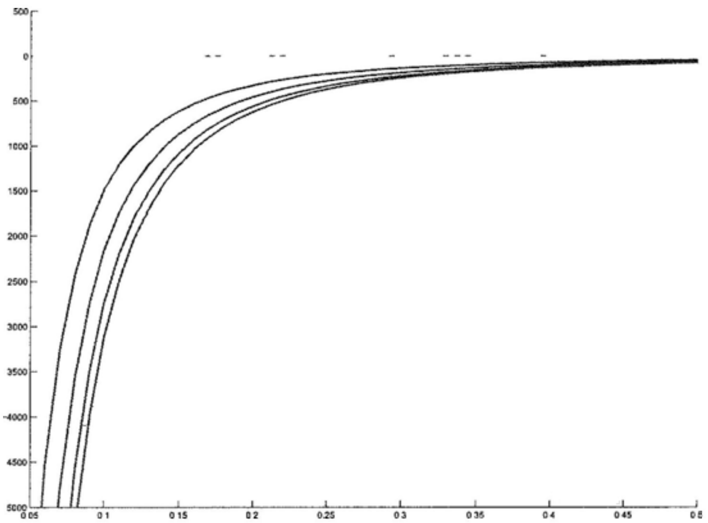

Figure 2.10: $E(0, c), c \in (0.05, 0.5)$ for $n = 2.75, 2.65, 2.55, 2.4$ from top.

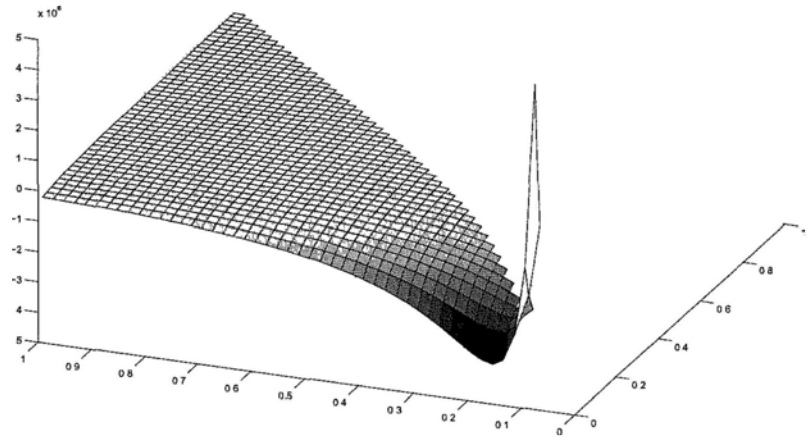

Figure 2.11: E for $n = 2.2$. It is positive for rather small α and β. The x-axis and y-axis represent α and β.

2.5 Appendix

Proof of Proposition 2.9 Continued. (ii) Case $n = 2$. By (2.1.9),

$$\lim_{x\to\infty} x^2 f''(x) = \lambda.$$

Then we have

$$\begin{aligned}
&\lim_{c\to\infty} \frac{H(y)}{\ln(\beta-\alpha)} \\
=&\lim_{\beta\to\infty} \frac{-yf'(\beta) + yf'(y(\beta-\alpha)+\alpha)}{(\beta-\alpha)^{-1}} \\
=&\lim_{\beta\to\infty} \frac{-yf''(\beta) + y^2 f''(y(\beta-\alpha)+\alpha)}{-(\beta-\alpha)^{-2}} \\
=&\lambda(y-1),
\end{aligned}$$

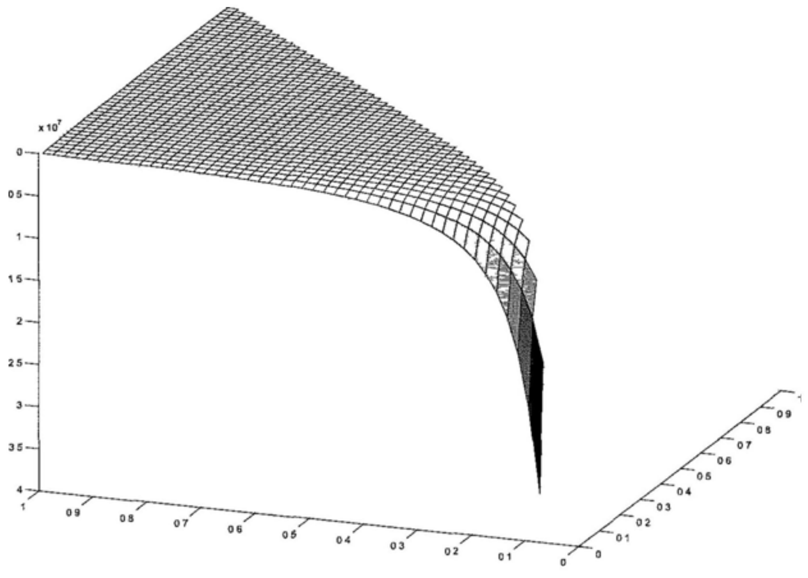

Figure 2.12: E for $n = 2\,4$.

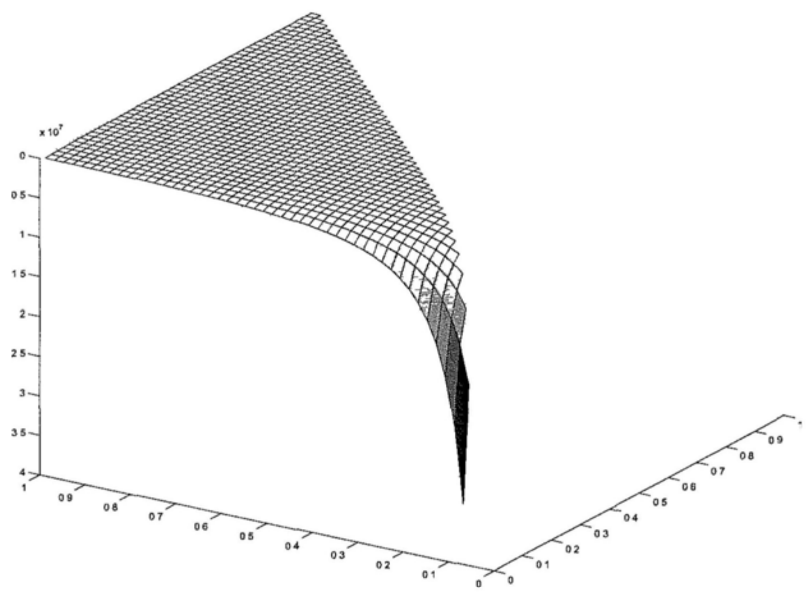

Figure 2.13: E for $n = 2\,5$.

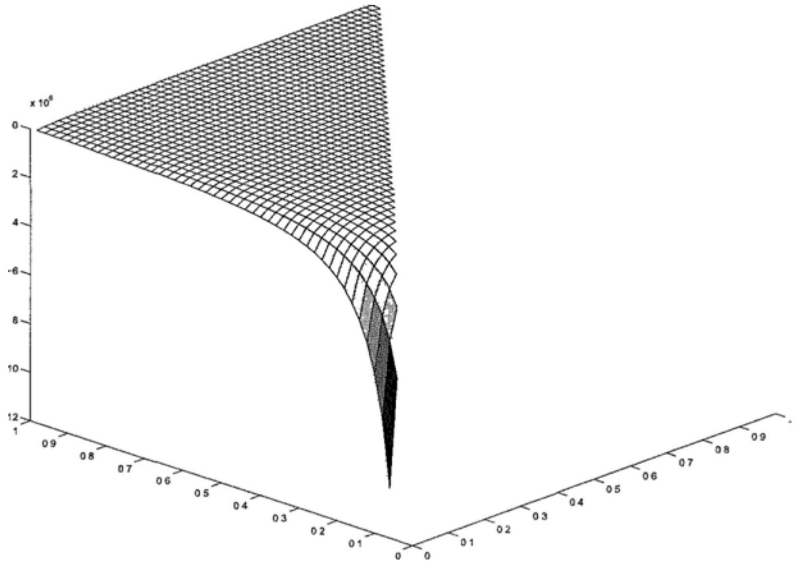

Figure 2.14: E for $n = 2.8$

and

$$\begin{aligned}
&\lim_{c\to\infty} \frac{I(y)}{(\beta-\alpha)\ln(\beta-\alpha)} \\
&= \lim_{\beta\to\infty} \frac{yf(\beta) - yf(y(\beta-\alpha)+\alpha) - y(\beta-\alpha)(f'(\beta) - yf'(y(\beta-\alpha)+\alpha))}{\ln(\beta-\alpha)+1} \\
&= \lim_{\beta\to\infty} \frac{-y(\beta-\alpha)(f''(\beta) - y^2 f''(y(\beta-\alpha)+\alpha))}{(\beta-\alpha)^{-1}} = 0.
\end{aligned}$$

Together with (2.1.24), it is easy to get

$$\lim_{c\to\infty} \frac{P_c(\beta-\alpha)^2}{\ln(\beta-\alpha)} = 0,$$

and

$$\lim_{c\to\infty} A_c = 2\pi$$

Using (2.1.33) and (2.1.34),

$$\lim_{c\to\infty} \frac{P_\alpha(\beta-\alpha)^2}{\ln(\beta-\alpha)} = -\infty,$$

and

$$\lim_{c\to\infty} A_\alpha = 0.$$

Therefore,

$$\lim_{c\to\infty}(A_c P_\alpha - A_\alpha P_c)\frac{(\beta-\alpha)^2}{\ln(\beta-\alpha)} = -\infty.$$

(iii) Case $n \in (1,2)$.

Since $f'(x) < 1$, $f(x) - x$ decreases on $(0,\infty)$. So $f(x) - x$ goes to either $-\infty$ or a negative finite number as $x \to \infty$. If the limit is $-\infty$, then we have

$$\begin{aligned}\lim_{x\to\infty}\frac{x^{n-2}}{f(x)-x} &= \lim_{x\to\infty}\frac{(n-2)x^{n-3}}{f'(x)-1}\\ &= \lim_{x\to\infty}\frac{(n-2)(n-3)x^{n-4}}{f''(x)}\\ &=\frac{2-n}{\lambda}.\end{aligned}$$

On the other hand, as $n - 2 < 0$,

$$\lim_{x\to\infty}\frac{x^{n-2}}{f(x)-x} = \frac{0}{-\infty} = 0.$$

Therefore,

$$\lim_{x\to\infty} f(x) - x = \gamma_0(n), \tag{2.5.1}$$

where γ_0 is a finite negative number depending on n.

When $y \in (0,1]$,

$$\begin{aligned}&\lim_{c\to\infty} H(y)\\ =&\lim_{\beta\to\infty}(y-1)f(\alpha) - y(f(\beta)-\beta) + f(y(\beta-\alpha)+\alpha) - y\beta\\ =&(y-1)f(\alpha) - y\gamma_0 + \gamma_0 - y\alpha + \alpha\\ =&(1-y)(\gamma_0 - f(\alpha) + \alpha) < 0.\end{aligned} \tag{2.5.2}$$

When $y = 0$, obviously, $H(y) \equiv 0$. As $n \in (1, 2)$, by (2.5.1) and (11)

$$\begin{aligned}
&\lim_{c\to\infty} \partial_\beta I(y)\\
=&\lim_{\beta\to\infty} yf(\beta) - yf(y(\beta-\alpha)+\alpha) - y(\beta-\alpha)(f'(\beta) - yf'(y(\beta-\alpha)+\alpha))\\
=&0. \qquad (2.5.3)
\end{aligned}$$

Note that to show (2.5.1), only $n < 2$ is needed. Hence both (2.5.2) and (2.5.3) hold for $n \in (0, 1]$. By (2.5.3), we have

$$\begin{aligned}
&\lim_{c\to\infty} \frac{I(y)}{(\beta-\alpha)^{n-1}}\\
=&\lim_{\beta\to\infty} \frac{yf(\beta) - yf(y(\beta-\alpha)+\alpha) - y(\beta-\alpha)(f'(\beta) - yf'(y(\beta-\alpha)+\alpha))}{(n-1)(\beta-\alpha)^{n-2}}\\
=&\lim_{\beta\to\infty} \frac{-y(\beta-\alpha)(f''(\beta) - y^2 f''(y(\beta-\alpha)+\alpha))}{(n-1)(n-2)(\beta-\alpha)^{n-3}}\\
=&\frac{\lambda(3-n)}{(n-1)(n-2)}(y^{n-1} - y). \qquad (2.5.4)
\end{aligned}$$

From (2.1.6),

$$\frac{P_c}{(\beta-\alpha)^{\frac{n}{2}-3}} = \frac{\beta-\alpha}{\sqrt{2}(f(\beta)-c)} \int_0^1 \left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{3}{2}} \frac{I(y)}{(\beta-\alpha)^{n-1}} \frac{1}{(\beta-\alpha)^{1-\frac{n}{2}}} dy.$$

Then with (2.5.4)

$$\lim_{c\to\infty} \frac{P_c}{(\beta-\alpha)^{\frac{n}{2}-3}} = 0.$$

By (2.1.10), (2.1.11), (2.1.24) and (2.5.4),

$$\lim_{c\to\infty} A_c = 2\sqrt{2}\int_0^1 \left(\frac{y(1-y)}{2}\right)^{-\frac{1}{2}} ydy = 2\pi. \qquad (2.5.5)$$

Using (2.1.33) and the above results,

$$\begin{aligned}
&\lim_{c\to\infty} \frac{P_\alpha}{(\beta-\alpha)^{\frac{n}{2}-3}}\\
=&\lim_{\beta\to\infty} \frac{1}{\sqrt{2}}(\beta-\alpha)^{1-\frac{n}{2}} \int_0^1 \left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{3}{2}} H(y)dy - \lim_{\beta\to\infty} \frac{P_c}{(\beta-\alpha)^{\frac{n}{2}-3}}\\
=&-\infty,
\end{aligned}$$

while by (2.1.34),(2.5.2), (2.5.4) and (2.5.5),

$$
\begin{aligned}
&\lim_{c\to\infty} A_\alpha \\
=&\lim_{c\to\infty} P + \lim_{c\to\infty} \frac{\beta-\alpha}{\sqrt{2}} \int_0^1 K(y)^{-\frac{3}{2}} H(y)(y(\beta-\alpha)+\alpha)dy - 2\pi \\
=&2\pi - 2\pi = 0.
\end{aligned}
$$

Therefore,

$$
\lim_{c\to\infty} \frac{A_c P_\alpha - A_\alpha P_c}{(\beta-\alpha)^{\frac{n}{2}-3}} = 2\pi(-\infty) = -\infty.
$$

(iv) Case $n = 1$. The analogues of (2.5.4) is

$$
\begin{aligned}
&\lim_{c\to\infty} \frac{I_1(y)}{\ln(\beta-\alpha)} \\
=&\lim_{\beta\to\infty} \frac{yf(\beta) - yf(y(\beta-\alpha)+\alpha) - y(\beta-\alpha)(f'(\beta) - yf'(y(\beta-\alpha)+\alpha))}{(\beta-\alpha)^{-1}} \\
=&\lim_{\beta\to\infty} \frac{y(\beta-\alpha)(f''(\beta) - y^2 f''(y(\beta-\alpha)+\alpha))}{(\beta-\alpha)^{-2}} \\
=&2\lambda(y-1).
\end{aligned}
$$

Then, from (2.1.6) again,

$$
\begin{aligned}
&\lim_{c\to\infty} \frac{P_c(\beta-\alpha)^{\frac{5}{2}}}{\ln(\beta-\alpha)} \\
=&\lim_{\beta\to\infty} \frac{\beta-\alpha}{\sqrt{2}(f(\beta)-c)} \int_0^1 \left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{3}{2}} \frac{I_1(y)}{\ln(\beta-\alpha)} \frac{1}{(\beta-\alpha)^{\frac{1}{2}}} dy \\
=&0.
\end{aligned}
$$

Similarly, from (2.1.10),

$$
\lim_{c\to\infty} A_c = 2\pi.
$$

Using (2.1.33), (2.1.34) and (2.5.2),

$$
\begin{aligned}
&\lim_{c\to\infty} \frac{P_\alpha(\beta-\alpha)^{\frac{5}{2}}}{\ln(\beta-\alpha)} \\
=&\lim_{\beta\to\infty} \frac{1}{\sqrt{2}}\frac{(\beta-\alpha)^{\frac{1}{2}}}{\ln(\beta-\alpha)} \int_0^1 \left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{3}{2}} H(y)dy \\
=&-\infty
\end{aligned}
$$

and

$$\lim_{c\to\infty} A_\alpha = 0.$$

Therefore,

$$\lim_{c\to\infty}(A_c P_\alpha - A_\alpha P_c)\frac{(\beta-\alpha)^{\frac{5}{2}}}{\ln(\beta-\alpha)} = 2\pi(-\infty) = -\infty.$$

(v) Case $n \in (0, 1)$. By (2.5.3),

$$\lim_{c\to\infty} \frac{I(y)}{\beta-\alpha} = 0.$$

From (2.1.6),

$$
\begin{aligned}
&\lim_{c\to\infty} P_c(\beta-\alpha)^2 \\
=&\lim_{c\to\infty} \frac{\beta-\alpha}{\sqrt{2}(f(\beta)-c)} \int_0^1 \left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{3}{2}} \frac{I(y)}{\beta-\alpha}dy \\
=&0.
\end{aligned}
$$

Similarly,

$$\lim_{c\to\infty} A_c = 2\pi.$$

By (2.1.33) and (2.1.34),

$$
\begin{aligned}
&\lim_{c\to\infty} P_\alpha(\beta-\alpha)^2 \\
=&\lim_{\beta\to\infty} \frac{1}{\sqrt{2}} \int_0^1 \left(\frac{K(y)}{(\beta-\alpha)^2}\right)^{-\frac{3}{2}} H(y)dy - \lim_{c\to\infty} P_c\frac{f(\beta)-f(\alpha)}{\beta-\alpha}(\beta-\alpha)^2 \\
=&-\infty,
\end{aligned}
$$

and

$$\lim_{c\to\infty} A_\alpha = 0.$$

Therefore,

$$\lim_{c\to\infty}(A_c P_\alpha - A_\alpha P_c)(\beta-\alpha)^{\frac{5}{2}} = 2\pi(-\infty) = -\infty.$$

$\square$

Bibliography

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables*, Dover, 1972.

[2] U. Abresch and J. Langer, The normalized curve shortening flow and homothetic solutions, *J. Differential Geom.* **23** (1986), 175–196.

[3] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349-381.

[4] S. Angenent, The zero set of a solution of a parabolic equation, *J. Reine Angew. Math.* **390** (1988), 79-96.

[5] T. K.-K Au, On the saddle point property of Abresch-Langer curves under the curve shortening flow, *Comm. Anal. Geom.* **18** (2010), 1-21.

[6] C. Bandle, *Isoperimetric Inequalities and Applications*, Pitman, Boston, Mass.-London. 1980.

[7] E. Beretta, M. Bertsch and R. Dal Passo, Nonnegative solutions of a fourth order nonlinear degenerate parabolic equation, *Arch. Rat. Mech. Anal.* **129** (1995), 175–200.

[8] F. Bernis and A. Friedman, Higher order nonlinear degenerate parabolic equations, *J. Differential Equations*, **83** (1990), 179–206.

[9] A. L. Bertozzi, The mathemaics of moving contact lines in thin liquid films, *Notices of The AMS*, **June/July** (1998), 689–697.

[10] A. L. Bertozzi, M. P. Brenner, T. F. Dupont and L. P. Kadanoff, Singularities and similarities in interface flow, in L. Sirovich, editor, *Trends and Perspectives in Applied Mathematics*, volume 100 of *Applied Mathematical Sciences*, 155–208. Springer–Verlag, New York, 1994.

[11] A. L. Bertozzi and M. C. Pugh, The lubrication approximation for thin viscous films: the moving contact line with a 'porous media' cut off of van der Waals interactions, *Nonlinearity,* **7**(1994), 1535–1564.

[12] A. L. Bertozzi and M. C. Pugh, The lubrication approximation for thin viscous films: regularity and long time behavior of weak solutions, *Comm. Pure Appl. Math.* **49** (1996), 85–123.

[13] A. L. Bertozzi and M. C. Pugh, Long–wave instabilities and saturation in thin film equations, *Comm. Pure Appl. Math.* **51** (1998), 625–661.

[14] A. L. Bertozzi and M. C. Pugh, Finite-time blow-up of solutions of some long-wave unstable thin film equations, *Indiana Univ. Math. J.* **49** (2000), 1323–1366.

[15] E. A. Carlen and S. Ulusoy, Asymptotic equipartition and long time behavior of solutions og a thin-film equation, *J. Differential Equations*, **241** (2007), 279–292.

[16] J. A. Carrillo and G. Toscani, Long-time asymptotics for strong solutions of the thin film equations, *Comm. Math. Phys.* **225** (2002), 551–571.

[17] K.-L. Cheung and K.-S. Chou, On the stability of single and multiple droplets for equations of thin film type, *Nonlinearity,* **23** (2010), 3003–3028.

[18] K.-S. Chou and S.-Z. Du, Estimates on the Hausdorff dimension of the rupture set of a thin film, *SIAM J. Math. Anal.* **40** (2008). 790–823

[19] K.-S. Chou and Y.-C. Kwong, Finite time rupture for thin films under Van der Waals forces, *Nonlinearity,* **20** (2007), 299–317.

[20] P. G. De Gennes, Wetting: statics and dynamics, Rev.Mod.Phys. **57(3)**(1985), 827-863.

[21] E. B. Dussan V, The moving contact line: the slip boundary condition, J.Fluid Mech. **77** (1976), 665-684.

[22] P. Ehrhard and S. H. Davis. Non-isothermal spreading of liquid drops on horizontal plates, J.Fluid Mech. **229**(1991), 365-388.

[23] P. Ehrhard, The spreading of hanging drops, J.Colloid Interf.Sci. **168**(1994), 242-246.

[24] H. P. Greanspan. On the motion of a small viscous droplet that wets a surface. J.Fluid Mech. **84**(1978) 125-143.

[25] P. J. Haley and M. J. Miksis, The effect of the contact line on droplet spreading, J.Fluid Mech. **223**(1991), 57-81.

[26] H. Hofer, A note on the topological degree at a critical point of mountain pass type, *Proc. Amer. Math. Soc.* **90** (1984), no. 2. 309–315.

[27] J. N. Israelachvill, *Intermolecular and surface forces*, 2nd ed. 1992, Academic Press.

[28] R. S. Laugesen and M. C. Pugh, Properties of steady states for thin film equations. *European J. Appl. Math.* **11** (2000), 293–351.

[29] R. S. Laugesen and M. C. Pugh, Linear stability of steady states for thin film and Cahn–Hilliard type equations, *Arch. Rat. Mech. Anal.* **154** (2000), 3–51.

[30] R. S. Laugesen and M. C. Pugh, Energy levels of steady states for thin film type equations, *J. Differential Equations*, **182** (2002), 377–415.

[31] R. S. Laugesen and M. C. Pugh, Heteroclinic orbits, mobility parameters and stability for thin film type equations, *Elect. J. Differential Equations*, **95** (2002), 1–29.

[32] P. Neogi and C. A.Miller, Spreading kinetics of a drop on a rough solid surface, J.Colloid Interf.Sci. **92**(1984) 338-349.

[33] A. Oron, S. H. Davis and S. G. Bankoff, Long–scale evolution of thin liquid films. *Rev. Modern Phys.* **69** (1997), 931–980.

[34] A. Oron and P. Rosenau, Formation of pattern induced by thermocapillarity and gravity, J.Phys. II, **2**(1992), 131-146.

[35] R. Schaaf, A class of Hamiltonian systems with increasing periods, J.Reine Angew. Math. **363**, 96-109.

[36] D. Slepčev, Linear stability of selfsimilar solutions of unstable thin-film equations. *Interfaces Free Bound.* **11** (2009), 375-398.

[37] D. Slepčev and M. C. Pugh, Selfsimilar blowup of unstable thin-film equations, *Indiana Univ. Math. J.* **54** (2005), 1697-1738.

[38] A. Tudorascu, Lubrication approximation for thin viscous films: asymptotic behavior of nonnegative solutions, *Comm. Partial Differential Equations*, **32** (2007), 1147-1172.

[39] S. J. VanHook, M. F. Schatz, J. B. Swift, W. D. McCormick and H. L. Swinney, Long-wavelength surface-tension-driven Bénard convection: experiment and theory, J.Fluid Mech. **345**(1997). 373-403.

[40] X.-L. Wang, The stability of m-fold circles in the curve shortening problem, *Manuscripta Math.* **134** (2011), 493-511.

[41] T. P. Witelski, A. J. Bernoff and A. L. Bertozzi, Blowup and dissipation in a critical-case unstable thin film equation, *European J. Appl. Math.* **15** (2004), 223-256.

[42] M. B. Williams and S. H. Davis, Nonlinear theory of film rupture, J.Colloid Interf.Sci. **90(1)**(1982), 220-228.

[43] S. G. Yiantsios and B. G. Higgins, Rayleigh-Taylor instability in thin viscous films, Phys.Fluids A, **1**(1989), 1484-1501.

[44] Y. Zhang, Counting the stationary states and the convergence to equilibrium for the 1-D thin film equation, *Nonlinear Anal.* **71** (2009), 1425-1437.