Some Topics on Compressible Navier-Stokes Equations

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Abstract

In this thesis, we study the motion of viscous compressible fluids in multi-space-dimension. The main concern is the global behavior of either weak or smooth solutions to various models with different physical backgrounds. We have obtained the following new results:

1. We establish the global existence and uniqueness of classical solutions to the half-space problem with the boundary condition proposed by Navier for the isentropic compressible Navier-Stokes equations in three spatial dimensions with smooth initial data which are of small energy but possibly large oscillations. The initial density is allowed to vanish and the spatial measure of the set of vacuum can be arbitrarily large.

2. We investigate a free boundary problem for compressible spherically symmetric Shallow water model with degenerate viscosity coefficients. For small perturbations to the stationary solution, we obtain the global existence and uniqueness of weak solutions and some uniform estimates with respect to time. Moreover, those solutions are shown to tend to the stationary solution as time goes to infinity.

3. We consider the vacuum free boundary problem of compressible Navier-Stokes-Poisson system with density-dependent viscosity. We obtain a local in time well-posedness of the strong solution in the spherically symmetric case.

摘 要

本論文主要研究高維粘性可壓縮流體的运動。我們主要關心的是Navier-Stokes 方程弱解惑者光滑解得全局性態。主要得到了以下幾個結果。

1．在Navier 邊界條件下，我們得到了半空間問題的全局光滑解得存在唯一性，並且允許初始密度真空。

2．對於球對稱的Shallow-Water 模型，如果初值是穩態解附近的小擾動，我們得到了弱解是存在唯一的。並且證明了當時間趨於無窮大時，弱解將逼近于穩態解。

3．我們研究密度依賴於粘性係數且帶自由邊界的可壓縮Navier-Stokes-Poisson 方程。在球對稱的假設條件下，我們證明了強解的局部適定性。

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Contents

Introduction

It is well-known that the motion of fluids in many cases are governed by the following famous compressible Navier-Stokes equations with constant viscous coefficients

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div}(\rho u) = 0, \\ \dfrac{\partial \rho u}{\partial t} + \mathrm{div}(\rho u \otimes u) + \nabla P = \mathrm{div} S, \\ \dfrac{\partial \rho E}{\partial t} + \mathrm{div}(\rho E u + P u) = \mathrm{div}(k \nabla T) + \mathrm{div}(S u) \end{cases} \tag{0.1}$$

where $k = k(T)$ is the thermal conductivity, S is the shear stress tensor

$$S = \mu(\nabla u + \nabla^t u) + \lambda(\mathrm{div} u) I,$$

μ and λ are shear and bulk viscous coefficients, respectively. These two coefficients satisfy the following physical constraints

$$\mu > 0, \quad \mu + \frac{N}{2}\lambda \geq 0.$$

If both heat conductivity and dissipation of mechanical energy are neglected in (0.1), then the entropy becomes constant along each particle path. This yields the following isentropic compressible Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div}(\rho u) = 0, \\ \dfrac{\partial \rho u}{\partial t} + \mathrm{div}(\rho u \otimes u) - \mu \Delta u - (\mu + \lambda) \nabla \mathrm{div} u + \nabla P(\rho) = 0, \end{cases} \tag{0.2}$$

where $\rho \in \mathbb{R}$, $u \in \mathbb{R}^n$, and $P \in \mathbb{R}$ denote density, velocity and pressure, respectively.

The behavior of the solution to (0.2) is closely related to the real world such as the water in the oceans and the air in the atmosphere. It displays an amazing range of phenomena from ordinary patterns to turbulent states. Moreover, (0.2) is a coupled hyperbolic-parabolic system which may be degenerate in the presence of vacuum. This important feature leads to great complexities and rich physical phenomena.

In the past several decades, significant progresses have been made for the system (0.2) not only for special data but also for general initial data, both in one-dimensional and multi-dimensional cases. For instance, the one-dimensional problem was addressed by Kazhikhov in [39] for the sufficiently smooth data, and by Serre in [61] and Hoff [29] for discontinuous initial data where the data were uniformly away from the vacuum. Concerning the global existence and the large-time behavior of solutions for sufficiently small data in multi-dimensional case, the system (0.2) (as well as the full compressible Navier-Stokes equations) is well-understood in the sense if the data are small perturbation of an uniform non-vacuum state, then there exists a (smooth or weak) solution which is time-asymptotically stable (see [49, 50, 51]). Later, Hoff generalized these results for the discontinuous initial data in a series of papers, see [29, 30, 31] and reference therein. Recently, Danchin in [15] obtained existence and uniqueness of global solutions in a functional space which is invariant by the natural scaling of the associated equations. However, for the large data, there are still many important open problems, such as, the existence of global solutions in the case of heat-conducting gases and the uniqueness of weak solutions. The first general result was obtained by Lions in [44], in which he used the method of weak convergence to obtain global weak solutions provided the specific heat ratio γ is appropriately large, for example, $\gamma \geq \frac{3N}{N+2}$, $N = 2, 3$. Later, this result was improved by Feireisl [23] for $\gamma > \frac{N}{2}$. It should be noted that the density is allowed to vanish initially. If the solution has certain symmetry, the global existence of weak solutions was obtained for any $\gamma > 1$ in [37, 39]. In addition, Hoff in [31] also obtained the global existence of weak solution for $\gamma > 1$ if the initial density and velocity were a general small perturbation of a non-vacuum state. There have been many generalizations of this results, see [23, 29, 31, 43, 44, 74] and references therein. Recently, under the additional assumptions that the viscosity coefficients μ and λ satisfy

$$\mu > \max\{4\lambda, -\lambda\},$$

and for the far field density away from vacuum ($\tilde{\rho} > 0$), Hoff [32] obtained a new type of global weak solutions with small energy which have extra regularity information compared with these large weak ones constructed by Lions [44] and Feireisl [23]. Furthermore, there are many studies on fluids in a fixed domain with various boundary conditions, see [44].

Once we obtain a weak solution, the natural question is about the regularity of this solution, i.e, when will the weak solution become strong or even classical? The partial regularity of two-dimensional periodic weak solutions to the isentropic compressible Navier-Stokes equations has been obtained by Desjardin in [16] under the condition that the density is bounded, where the quantity called effective viscous flux, defined as $F = (2\mu + \lambda)\text{div}u - P$, plays a key role to prove the global existence of weak solutions for the compressible Navier-Stokes equations in [31]. Moreover, the classical elliptic regularity estimate holds

$$\|\omega\|_{L^p(\Omega)} + \|\nabla F\|_{L^p(\Omega)} \leq C\|G\|_{L^p(\Omega)},$$

for $p \in (1, \infty)$ and $\Omega = \mathbb{R}^3$ or T^3, where $G = \rho(u_t + u \cdot \nabla u)$ is the convection term and $\omega = \nabla \times u$ is the vorticity. However, it is not easy to obtain the same estimate when Ω is a bounded domain because of the less boundary condition for ω and F. Therefore, it seems that some new techniques are needed to extend the result in [16] to the general bounded domains.

There are also a series of results about the existence of strong solutions. For instance, Solonnikov obtained in [68] a local existence of strong solutions with periodic non-vacuum data. If the initial density allows vacuum, it was shown recently in [12] that the Navier-Stokes equations admited a local strong solution as long as a suitable compatibility condition is satisfied initially. This result is also true for full compressible Navier-Stokes equations ([13]). Moreover, Kim and Choe [14] established a local classical solution in a bounded or unbounded domain Ω of $\mathbb{R}^3$. In their paper, the initial density does not need to be bounded below away from vacuum and may vanish in an open subset (vacuum) of Ω. Then a natural question is whether such solutions could be globally well defined. In general, one could not expect such general results due to Xin's blow up results in [73] where it is shown that in the case that the initial density has compact support, any smooth solution to the Cauchy problem of the non-barotropic compressible Navier-Stokes systems without heat conduction blows up in finite time for any space dimension and the same holds for the isentropic case at least in one-dimension and the symmetric 2-dimensional case [47]. Very recently, there is a surprising work by Huang, Li and Xin in [33], where they established the global existence and uniqueness of classical solutions to the 3-dimensional Cauchy problem for the isentropic compressible Navier-Stokes equations with smooth initial data which were of small energy but possibly large oscillation with constant state at far filed which could be either vacuum or non-vacuum. In addition, Luo in her Ph.D thesis [47] obtained similar results to the Cauchy problem for the isentropic compressible Navier-Stokes systems in 2-dimensional case. They also find that for spherically symmetric case, the local smooth solution $(\rho, u) \in C^1([0,T]; H^s)(s > 3)$ has to blow up in finite time with initial density having compact support. In chapter 2 of this thesis, we can also obtain the global well-posedness of classical solutions under the Navier-boundary condition for the half-space problem. This boundary condition was proposed by Navier and expressed the condition that the velocity an $\partial\Omega$ is proportional to the tangential component of the stress. This boundary condition for the flat-space case has been applied in a number of problems, usually for incompressible flows, see Arbogastand Lehr [5], Beavers and Joseph [25], Caflisch and Rubinstein [10] and Saffman [62] for example.

Besides the fixed boundary, the motion of free surfaces of fluids has important physical and engineering background, for example, the interface between fluids and vacuum, the interface between different fluids, etc. The free boundary problems of one-dimensional compressible Navier-Stokes equations were investigated in [2, 3], where the global existence of weak solutions was proved. Similar results were obtained by Okada and Makino [57] for the equations of spherically symmetric motion of viscous gases. Furthermore, the free boundary problem of the one-dimensional viscous gas expanding into the vacuum has been intensively studied, see [56, 57] and the references therein. In particular, in [46], Luo, Xin and Yang studied the regularity and the behavior of solutions near the interfaces between the gas and vacuum, and gave a quite precise description on growth rate of the free boundary.

However, it seems that (0.2) is not suitable to study fluids near vacuum. In general, there is no continuous dependence on the initial data for fluids with vacuum states, see [30]. Furthermore, it was proved in [73] that classical solutions will break down when the initial data had compact support. As pointed out in [45], the main reason for this came from the independence of the kinematic viscosity coefficient on the density.

To understand fluids behavior near vacuum, one can choose an alternative system for (0.1). In fact, if one derives the compressible Navier-Stokes equations from Boltzmann equation by exploiting Chapman-Enskog expansion up to the second order, as in [26], one can find that the viscosity is not constant but a function of the temperature. For isentropic flows, this dependence is translated to the dependence on the density by the law of Boyle and Gay-Lussac for ideal gas as discussed by Liu et al. [45].

In these cases, instead of (0.2), compressible Navier-Stokes equation is of the following form

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div}(\rho u) = 0, \\ \dfrac{\partial \rho u}{\partial t} + \mathrm{div}(\rho u \otimes u) + \nabla P - \mathrm{div}(\mu(\rho) D u) - \nabla(\lambda(\rho)\mathrm{div} u) = 0, \end{cases} \tag{0.3}$$

where $x \in \mathbb{R}^n$, $n = 2, 3$, $P(\rho) = A\rho^\gamma$, $\gamma > 1$, $A > 0$ are constants. $D(u) = \dfrac{\nabla u + \nabla^t u}{2}$ is the stress tensor, $\mu(\rho)$ and $\lambda(\rho)$ are Lamé viscosity coefficients. In particular, the viscous Saint–Venant system for shallow water is expressed exactly as (0.3) with $n = 2$, $\mu(\rho) = \rho$, $\lambda(\rho) = 0$ and $P(\rho) = \rho^2$. Shallow water equations are to describe vertically averaged flows in three–dimensional shallow domains in term of the mean velocity u and the variation of the depth ρ due to the free surface, which is widely used in geophysical flows. This equations were derived rigorously by Gerbeau-Perthame (see [24]). The global existence of weak solution with large aptitude to (0.3) remains to be carried under the Lion's framework of renormalized solutions [44] due to the new mathematical challenges encountered below. Indeed, the system of (0.3) is highly degenerate at vacuum because of the dependence of viscous coefficients on the flow density. This makes it very difficult to obtain the uniform a-priori estimate for the velocity and trace the particle pathes near vacuum regions. In particular, it is not known yet whether the vacuum states shall form or not for global (weak) solutions to (0.3) even if initial density is far from vacuum. In recent years, there are many studies for system (0.3) in both one-dimensional and higher- dimensional setting. Global smooth solutions for data close to equilibrium were established in [70]. Bresch, Desjardins, and Lin [6] showed the L^1 stability of weak solutions for the Korteweg system with the Korteweg stress tensor $\kappa\rho\nabla\triangle\rho$, and their result was later improved in [7] to include the case of vanishing capillarity ($\kappa = 0$) but with an additional quadratic friction term $r\rho|u|u$. Recently, Ansgar Jügel in [4] study the global existence of weak solution to compressible quantum Navier-

Stokes equations for large data. The model consists of the mass conservation equation and a momentum balance equation, including a nonlinear third-order differential operator, with the quantum Bohn potential. In their paper, a new entropy estimate was established in [6] which provided some high regularity for the density. Mellet and Vasseuer [52] proved the L^1 stability results of [6, 7] to the case $r = \kappa = 0$. Nevertheless, the global existence of weak solutions of the compressible Navier-Stokes equations with density-dependent viscosity (0.3) is still open in the multi-dimensional cases except for the spherical symmetric case, see [27]. The key issue now is how to construct approximate solutions satisfying the a priori estimates required in the L^1 stability analysis. It seems highly nontrivial to do so due to the degeneracy of viscosities near vacuum and the additional entropy inequality to be held in the construction of approximate solutions.

In contrast to higher dimensional case, there are fruitful studies for (0.3) in one-dimensional setting, where the system (0.3) reads as follows:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = (\mu(\rho) u_x)_x. \end{cases} \tag{0.4}$$

Suppose that $\mu = c\rho^\theta$ with c and θ being positive constants. When the initial density connects to vacuum with discontinuities, Makino, Liu, Xin and Yang obtained the local existence of weak solutions to Navier-Stokes equations with vacuum [45, 48]. The global existence and uniqueness of the weak solution when $0 < \theta < 1/3$ were obtained by Okada in [58]. Later, it was generalized to the cases for $0 < \theta < 1/2$ and $0 < \theta < 1$ in [76, 36] respectively. When heat-conducting effect are considered, Jiang in [35] proved the global existence of smooth solutions provided that $0 < \theta < 1/4$. The global solutions to (0.4) with discontinuous initial data were obtained by Fang and Zhang in [21]. Recently, if the initial density is bounded away from zero (no vacuum), Mellet and Vasseur proved the existence and uniqueness of the global strong solution in [53] for $0 < \theta < 1/2$.

The key estimate for all these results is the uniform positive lower bound of the density with respect to the construction of the approximate solution. Since these estimate implies that the second equation in (0.4) is uniformly parabolic in the fluid region, which yields existence, uniqueness of the solution and long time behavior. This is the key point to obtain the global existence of the solution to (0.4) when the initial data connects to vacuum discontinuously in [36, 58, 76].

If the density function connects to vacuum continuously, there is no positive lower bound for the density function and the viscosity coefficient vanishes at vacuum. This degeneracy in the viscosity coefficient gives rise to new difficulties for analysis because of the less regularizing effects on the solutions. A local existence result was obtained in [77] under the free boundary condition with $\theta > 1/2$, and global existence result in [78] for $0 < \theta < 2/9$ and in [72] for $0 < \theta < 1/3$. When the external force is constant, in [59], Okada obtained the global existence of the weak solution as long as $\theta \in (0, 5/37)$. It was circulated in [71] that Zhang and Fang obtained the global existence and uniqueness of the weak solution when the initial data was a small perturbation to the stationary solution as long as $\theta \in (0, \gamma - 1) \cap (0, \gamma/2]$, where $\gamma > 1$ is the adiabatic constant of polytropic gas. In their paper, the uniform bounds with respect to time of the density function were obtained. From this property, they showed that such a system did not develop vacuum states or concentration states in the domain for all time. Also, they estimated the upper bound of the velocity function uniformly in time and obtained one of the important features of this problem, that was, the interface separating the gas and vacuum propagated with finite speed. For 1-dimensional shallow water model, that is $\theta = 1$, $\gamma = 2$, Duan in [17] also obtained the global well-posedness of weak solutions. For $\mu(\rho) = \rho^\theta$ $(\theta > 1/2)$, Li, Li and Xin in [42] studied this case for both bounded spatial domains or periodic domains and showed that for any global entropy weak solution, any (possibly existing) vacuum state must vanish within finite time. The velocity (even if regular enough and well defined) blew up in finite time as the vacuum states vanish. Furthermore, after the vanishing of vacuum states, the global entropy weak solution became a strong solution. Recently, the Cauchy problem for one-dimensional compressible flows was investigated by Jiu, Xin in [38]. In this paper, two cases were considered. First, the initial density was assumed to be integrable on the whole real line. Second, the deviation of the initial density from a positive constant density was integrable on the whole real line. It was proved that for both cases, weak solutions existed globally in time. In particular, for the second case, the phenomena of vanishing of vacuum and blow-up of the solutions were presented and it was also shown that after the vanishing of vacuum states, the global weak solution became a unique strong one. These generalized the corresponding results in [42].

Meanwhile, there are some investigations on the large time behavior of solutions for the non-constant viscosity coefficient, for example, [64, 80] and the references therein. Under zero velocity boundary condition, A.A.Zlotnik in [80] studied the stabilization of symmetric solutions and the stabilization rate was evaluated. Later, the result was improved by Straškraba, Ivan in [66]. The one-dimensional fixed-free boundary problem with a non-monotone equation of state and self-gravitation was investigated in [19, 81], they showed that the kinetic energy tended to 0 and the specific volume $\frac{1}{\rho}$ tended to a stationary specific volume as time tended to infinity. But these results above strongly replied on the condition $\mu(\rho) \geq \underline{\rho} > 0$. Recently, Zhang and Fang in [71] obtained that the weak solution for the free boundary problem tended to the stationary one if $\theta \in (0, \gamma - 1) \cap (0, \gamma/2]$. In there paper, there was no uniform positive lower bound to the viscous coefficient $\mu(\rho)$. However, they cannot treat the case when $\theta = 1$, $\gamma = 2$. Duan in [17] extended their results to the shallow water case, that is, $\theta = 1$, $\gamma = 2$. In [38], Jiu and Xin also investigated the asymptotic behaviors of weak solutions for Cauchy problem. They proved that if the initial density

$\rho_0 \in L^1(\mathbb{R})$, then the density tended to 0 as $t \to \infty$. If there existed a positive constant $\bar{\rho}$ such that $\rho_0 - \bar{\rho} \in L^1(\mathbb{R})$, then the density tended to $\bar{\rho}$.

There are also very interesting investigations about free boundary value problems for the compressible Navier-Stokes equations with self-gravitation force taken into granted, refer to [18, 19, 34, 57, 63, 67, 79, 81, 82] and the references therein. Recently, Jang in [34] established the local in time well-posedness of strong solutions to the vacuum free boundary problem of the compressible Naver-Stokes-Poisson system in the spherically symmetric and isentropic motion. The main difficulty in their paper is to deal with the vacuum free boundary where the density vanished at certain rate, which makes the system degenerate along the boundary. Under the same framework of Jang, we can also obtain the local well-posedness of strong solution when viscosity coefficients depend on density. This result is outlined in chapter 4. For the spherically symmetric Naver-Stokes-Poisson system with density dependent viscosities, authors in [20, 82] obtained the existence, uniqueness and global behavior of the solution with a general mass force and a solid core when $\mu(\rho) \geq \underline{\mu} > 0$ and $\rho_0 \geq \underline{\rho} > 0$. Without the positive lower bounds on the viscous coefficients, Chen-Zhang [11] established the local existence and uniqueness of the solution when the solid core $r \geq a$. Under the small perturbation of stationary solution, Zhang-Fang in [79] obtained the global existence, uniqueness and asymptotical behavior of weak solutions with degenerate coefficients and without a solid core. But, in their paper, they cannot treat the case when $\theta = 1$, $\gamma = 2$, that is, Shallow water system. In this thesis, I will study the global well-posedness of weak solutions for spherically symmetric shallow-water model. By using of a new technology, we can obtain the similar result. The result is presented in chapter 3. Very recently, Guo-Li-Xin in [28] showed the spherically symmetric weak solutions with stress free boundary condition and arbitrarily large data existed globally in time. In particular, they also investigated the dynamics of global solutions. It was shown that the particle path is uniquely defined starting from any non-vacuum region away from the symmetry center, along which vacuum states should not form in any finite time. In addition, the free boundary will expand outward at an algebraic rate in time and the fluid density decays to zero almost everywhere away from the symmetry center as the time tends to infinity. Since $n \geq 2$ and the viscosity coefficient μ depends on ρ, the nonlinear term $(n-1)\frac{u}{r}\partial_x \mu$ makes the analysis significantly different from the one-dimensional case. It would be very interesting, challenging both physically and mathematically to study the full system without the symmetry assumption as a free boundary problem. In general case, no result is known for the compressible gas flow with the free boundary. We will leave them in future study.

In this thesis, we mainly investigate following problems for the compressible Navier-Stokes or Navier-Stokes-Poisson systems:

I. Global well-posedness of classical solutions to the compressible Navier-Stokes equations in a half-space.

This work is motivated by the three dimensional results of Huang, Li, Xin [33] and Hoff [32]. We prove the global existence of classical solutions in a half-space under the Navier boundary condition, that is,

$$(u^1(x), u^2(x), u^3(x)) = \beta(u^1_{x_3}(x), u^2_{x_3}(x), 0), \quad \text{for} \quad x \in \partial\mathbb{R}^3_+.$$

Concerning this result, there are a few remarks in order:

1. The Beal-Kato-Majda type inequality in Huang, Li, Xin [33] cannot be applied directly. This inequality holds only for the whole space. In order to deal with our problem, we need a new type estimate for the half-space, refer to [69].

2. The far field density $\tilde{\rho}$ can not be vacuum, that is $\tilde{\rho} > 0$. Since for the Navier boundary condition, we need to deal with some extra boundary terms, for instance, $\int (|u|^2|\nabla u| + |u||\nabla u|^2)dx$. Hence, we need to estimate $\|u\|_{L^p}$, $p > 2$. From the energy estimate, we only have $\int \rho|u|^2 dx \leq C$. However we can get the

L^2 norm of u by the following inequality:

$$\int \tilde{\rho}|u|^2 dx \leq \int |\rho - \tilde{\rho}||u|^2 dx + \int \rho |u|^2 dx.$$

3. Also, we can obtain the large time blow up behavior of the gradient of the density.

II. Global behavior of spherically symmetric compressible Navier-Stokes system with degenerate viscosity coefficients.

This result is motivated by Zhang and Fang in [79] where they consider the spherically symmetric Navier-Stokes-Poisson equations with degenerate viscosity coefficients and without a solid core. Under certain assumptions on the initial data, they obtain the global existence, uniqueness and large time behavior of weak solutions.

1. They can only deal with the case that $\theta \in (0, \gamma - 1) \cap (0, \frac{\gamma}{2}]$. In their paper, the uniform estimate of

$$\int_0^t \|u(\cdot, t)\|^2_{L^\infty([\frac{M}{2}, M])} ds \tag{0.5}$$

plays a crucial role. But their method to estimate (0.5) will fail when $\theta = 1,\ \gamma = 2$. In this thesis, although we cannot obtain the uniform estimate of $\int_0^t \|u(\cdot, t)\|^2_{L^\infty([\frac{M}{2}, M])} ds$, we have the following estimate:

$$\int_0^t \|u(\cdot, t)(M - x)^{\frac{1}{4}}\|^2_{L^\infty([\frac{M}{2}, M])} ds < \infty. \tag{0.6}$$

By using of (0.6), we can get the desired results.

2. This result can be regarded as a continuous work of my M.phil thesis, in which we consider the one-dimensional shallow-water model.

III. Local well-posedness of Navier-Stokes-Poisson equations

This work is motivated by Jang in [34], which concerns a local in time well-posedness of strong solution to the vacuum free boundary problem of the compressible Navier-Stokes-Poisson system in the spherically symmetric motion with constant viscous coefficient.

1. The main difficult is to deal with the vacuum free boundary where the density vanished at certain rate which made the system degenerate along the boundary.

2. Since the density will vanish at the boundary, so we need the decay behavior of the initial density near the boundary, that is

$$\rho_0(x) \sim (1-x)^\alpha, \quad \text{as} \quad x \sim 1, \quad 0 \leq \alpha \leq \frac{1}{2}.$$

Under the similar framework of Jang in [34], we can estimate the lower and upper bounds of density and show that it has the same decay behavior as the initial density near the boundary.

The thesis is organized as follows. In chapter 1, we give some preliminaries for the thesis, such as some basic inequalities, estimates for differential inequalities and elliptic regularity results. We prove a global well-posedness of classical solutions to compressible Navier-Stokes equation for the half-space in chapter 2. In chapter 3, global behavior of spherically symmetric shallow-water model will be described. We then show that such a system is stable under small perturbations. The chapter 4 is devoted to the local existence and uniqueness of strong solutions to the Navier-Stokes-Poisson equations. Finally, we will discuss some further works and future researches in chapter 5.

Chapter 1

Preliminaries

In this chapter, we list some elementary results which we will use later, such as some basic inequalities, estimates for differential inequalities and elliptic regularity results.

Definition 1.0.1 . *Assume U is an open subset of $\mathbb{R}^n$, and $1 \leq p \leq \infty$. If $f : U \to \mathbb{R}$ is measurable, we define*

$$\|f\|_{L^p(U)} := \begin{cases} (\int_U |f|^p)^{1/p}, & \text{if } 1 \leq p < \infty, \\ ess\sup_U |f|, & \text{if } p = \infty. \end{cases}$$

We define $L^p(U)$ to be the linear space of all measurable function $f : U \to \mathbb{R}$ for which $\|f\|_{L^p(U)} < \infty$. And

$$L^p_{loc}(U) = \{u : U \to \mathbb{R} \mid u \in L^p(V) \;\; for \;\; each \;\; V \subset\subset U\}.$$

Lemma 1.0.2 (Young's inequality)*Let $a > 0$, $b > 0$, $p > 1$, $q > 1$, and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

In particular, when $p = q = 2$, the above inequality also is called Cauchy inequality.

From the Lemma above, taking $a = \epsilon a$, $b = \epsilon^{-1} b$, and $p = q = 2$, we can easily obtain that:

Lemma 1.0.3 (Cauchy-Schwartz inequality) *Let $a > 0$, $b > 0$, $\epsilon > 0$, then we have*

$$ab \leq \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2.$$

Lemma 1.0.4 (Hölder's inequality) *Let $p > 1$, $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, if $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$, and*

$$\int_\Omega |f(x)g(x)|dx \leq \|f(x)\|_{L^p(\Omega)} \|g(x)\|_{L^q(\Omega)}.$$

Lemma 1.0.5 (Minkowski inequality) *Let $1 \leq p < \infty$, f, $g \in L^p(\Omega)$, then $f + g \in L^p(\Omega)$, and*

$$\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

Lemma 1.0.6 (Sobolev embedding theorem for bounded domain) *Let U be a bounded open subset of $\mathbb{R}^n$, and ∂U is C^1, let $u \in W^{k,p}(U)$, $k \in \mathbb{N}$, $1 \leq p < \infty$.*

(a) if $k < \frac{n}{p}$, then $u \in L^q(U)$, where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$, and $\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}(U)}$, where C depends only on k, p, n and U.

(b) if $k > \frac{n}{p}$, *then* $u \in C^{k-[\frac{n}{p}]-1,r}$, *where*

$$r = \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer,} \\ \text{any positive number} < 1, & \text{if } \frac{n}{p} \text{ is an integer,} \end{cases}$$

and $\|u\|_{C^{k-[\frac{n}{p}]-1,r}} \leq C\|u\|_{W^{k,p}(U)}$, *where* C *depends only on* k, p, r, n *and* U.

Lemma 1.0.7 (Interpolation inequality) *Assume* $1 \leq s, r, t \leq \infty$ *and*

$$\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}. \tag{1.0.1}$$

Suppose also $u \in L^s(\Omega) \cap L^t(\Omega)$. *Then* $u \in L^r(\Omega)$, *and*

$$\|u\|_{L^r(\Omega)} \leq \|u\|^{\theta}_{L^s(\Omega)} \|u\|^{1-\theta}_{L^t(\Omega)}. \tag{1.0.2}$$

The following well-known Gagliardo-Nirenberg inequality will be used later frequently (see [40]).

Lemma 1.0.8 (Gagliardo-Nirenberg) *For* $p \geq 2$, $q \in (1, \infty)$, *and* $r \in (2, \infty)$, *there exist some generic constant* $C > 0$ *which may depend on* q, r *such that for* $f \in H^1(\mathbb{R}^3)$ *and* $g \in L^q(\mathbb{R}^3) \cap D^{1,r}(\mathbb{R}^3)$, *we have*

$$\|f\|^p_{L^p} \leq C\|f\|^{\frac{6-p}{2}}_{L^2} \|\nabla f\|^{\frac{3p-6}{2}}_{L^2}, \tag{1.0.3}$$

$$\|g\|_{C(\overline{\mathbb{R}^3})} \leq C\|g\|^{\frac{q(r-3)}{3r+q(r-3)}} \|\nabla g\|^{\frac{3r}{3r+q(r-3)}}_{L^r}. \tag{1.0.4}$$

Lemma 1.0.9 (Gronwall's inequality (differential form)): *Let* $\eta(\cdot)$ *be a nonnegative, absolutely continuous function on* $[0, T]$, *which satisfies for a.e* t *the differential inequality*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi'(t), \tag{1.0.5}$$

where $\phi(t)$ *and* $\psi(t)$ *are nonnegative, summable functions on* $[0, T]$. *Then*

$$\eta(t) \leq e^{\int_0^t \phi(s)ds}[\eta(0) + \int_0^t \psi(s)ds], \tag{1.0.6}$$

for all $0 \leq t \leq T$.

In particular, if

$$\eta' \leq \phi\eta, \text{ on } [0,T], \text{ and } \eta(0) = 0, \tag{1.0.7}$$

then

$$\eta \equiv 0, \text{ on } [0,T].$$

Beside the differential form of Gronwall's inequality, we also have the following integral form of the Gronwall's inequality.

Lemma 1.0.10 (Gronwall's inequality (integral form)): *Let* $\xi(t)$ *be a non-negative, summable function on* $[0,T]$, *which satisfies for a.e* t *the integral inequality*

$$\xi(t) \leq C_1 \int_0^t \xi(s)ds + C_2, \tag{1.0.8}$$

for constants $C_1, C_2 \geq 0$. *Then*

$$\xi(t) \leq C_2(1 + C_1 t e^{C_1 t}), \tag{1.0.9}$$

for a.e $0 \leq t \leq T$.

In particular, if

$$\xi(t) \leq C_1 \int_0^t \xi(s)ds, \tag{1.0.10}$$

for a.e. $0 \leq t \leq T$, *then*

$$\xi(t) = 0. \quad a.e.$$

Lemma 1.0.11 . *Let the function* y *satisfy*

$$y' = g(y) + b'(t) \quad \text{on} \quad [0,T], \quad y(0) = y^0,$$

where $g \in C(R)$ *and* y, $b \in W^{1,1}(0,T)$. *If* $g(\infty) = -\infty$ *and*

$$b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1) \tag{1.0.11}$$

for all $0 \le t_1 < t_2 \le T$ *with some* $N_0 \ge 0$ *and* $N_1 \ge 0$*, then*

$$y(t) \le \max\{y^0, \bar{\xi}\} + N_0 < \infty \quad \text{on} \quad [0, T],$$

where $\bar{\xi}$ *is a constant such that*

$$g(\xi) \le -N_1 \quad \text{for} \quad \xi \ge \bar{\xi}. \tag{1.0.12}$$

Throughout this thesis, we adopt the following notations for the standard homogeneous and inhomogeneous Sobolev spaces.

$$D^{k,r}(\Omega) = \{u \in L^1_{loc}(\Omega) : \|\nabla^k u\|_{L^r} \le \infty\},$$

$$W^{k,r} = L^r \cap D^{k,r}, \quad H^k = W^{k,2}, \quad D^k = D^{k,2},$$

$$\|u\|_{D^{k,r}} = \|\nabla^k u\|_{L^r}.$$

We derive some regularity estimates for the so-called Láme system:

$$Lu = -\mu \triangle u - (\lambda + \mu)\nabla \text{div} u = F \quad \text{in} \quad \Omega, \tag{1.0.13}$$

where Ω is a bounded or unbounded domain in $\mathbb{R}^3$.

First we recall a famous elliptic theory due to S.Agmon, A.Douglis and L.Nirenberg.

Lemma 1.0.12 (Agmon-Douglis-Nirenberg [1]) *Assume that* Ω *is a bounded domain in* $\mathbb{R}^3$ *with smooth boundary, and let* $u \in W_0^{1,q}(\Omega)$ *be a weak solution of the system* (1.0.13)*, where* $1 < q < \infty$*. If* $F \in W^{k,q}(\Omega)$ *for* $k \ge 0$*, then* $u \in W^{k+2,q}(\Omega)$ *and*

$$\|u\|_{W^{k+2,q}(\Omega)} \le C\|F\|_{W^{k,q}(\Omega)}, \tag{1.0.14}$$

for some constant $C = C(q, \mu, \lambda, \Omega)$ *independent of* F*.*

Then, using the domain expansion and scaling technique, one can easily obtain

Lemma 1.0.13 (Choe,Kim [12]) *Let* Ω *be the whole space* $\mathbb{R}^3$*, the half space* $\mathbb{R}^3_+$*. If* $u \in D_0^1(\Omega)$ *is a weak solution of the system, then*

$$\|u\|_{D^{k+2,q}(\Omega)} \le C\|F\|_{W^{k,q}(\Omega)}, \tag{1.0.15}$$

for any $1 < q < \infty$*.*

Chapter 2

Global well-posedness of classical solutions to the compressible Navier-Stokes equations in a half-space

2.1 Main result

The motion of a viscous compressible barotropic fluid in half-space $\Omega = \{x \in \mathbb{R}^3 : x_3 > 0\}$ can be described by the Navier-Stokes equations

$$\rho_t + \operatorname{div}(\rho u) = 0, \quad \text{in} \quad \Omega \times (0, T), \tag{2.1.1}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + Lu + \nabla P = 0, \quad \text{in} \quad \Omega \times (0, T), \tag{2.1.2}$$

$$Lu = -\mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u, \quad P = P(\rho), \tag{2.1.3}$$

the initial boundary conditions are

$$(\rho, u)|_{t=0} = (\rho_0, u_0) \quad \text{in} \quad \Omega, \tag{2.1.4}$$

$$(u^1(x), u^2(x), u^3(x)) = \beta(u^1_{x_3}(x), u^2_{x_3}(x), 0), \quad \beta > 0, \quad \text{on} \quad \partial\Omega, \tag{2.1.5}$$

$$\rho(x,t) \longrightarrow \tilde{\rho}, \quad u(x,t) \longrightarrow 0 \quad as \quad |x| \longrightarrow \infty, \quad (x,t) \in \Omega \times (0,T). \tag{2.1.6}$$

Here we denote by ρ, P and u the unknown density, pressure and the velocity fields of the fluid respectively. The constants μ and λ are the viscosity coefficients. We suppose $P(\rho) = A\rho^\gamma$, $\gamma > 1$, $A > 0$ and $\mu > 0$, $3\mu + 2\lambda \geq 0$ so that $L = -\mu\Delta - (\lambda+\mu)\nabla\text{div}$ is a strongly elliptic operator.

In this chapter, we study the global well-posedness of classical solution for the initial boundary problem (2.1.1) – (2.1.6) with nonnegative initial densities.

Throughout this chapter, we will use the following simplified notations for the standard homogeneous and inhomogeneous Sobolev spaces.

$$L^r = L^r(\Omega), \quad D^{k,r} = \{u \in L^1_{loc}(\Omega) : |u|_{D^{k,r}} < \infty\}, \quad \|u\|_{D^{k,r}} \triangleq \|\nabla^k u\|_{L^r},$$

$$W^{k,r} = L^r \cap D^{k,r}, \quad H^k = W^{k,2}, \quad D^k = D^{k,2}, \quad D^1 = \{u \in L^6 | \|\nabla u\|_{L^2} < \infty\}.$$

The initial energy is defined as:

$$C_0 = \int \left(\frac{1}{2}\rho_0|u_0|^2 + G(\rho_0)\right)dx, \tag{2.1.7}$$

where G denotes the potential energy density given by

$$G(\rho) \triangleq \rho \int_{\tilde{\rho}}^{\rho} \frac{P(s) - P(\tilde{\rho})}{s^2} ds.$$

It is clear that

$$c_1(\bar{\rho},\tilde{\rho})(\rho-\tilde{\rho})^2 \leq G(\rho) \leq c_2(\bar{\rho},\tilde{\rho})(\rho-\tilde{\rho})^2, \quad \text{if} \quad \tilde{\rho} > 0, \quad 0 \leq \rho \leq \bar{\rho},$$

for positive constants $c_1(\bar{\rho},\tilde{\rho})$ and $c_2(\bar{\rho},\tilde{\rho})$.

The main results can be stated as follows:

Theorem 2.1.1 *For given numbers $M > 0$ (not necessarily small) and $\bar{\rho} \geq \tilde{\rho}+1$, $\tilde{\rho} > 0$, suppose that the initial data (ρ_0, u_0) satisfy*

$$0 \leq \inf \rho_0 \leq \sup \rho_0 \leq \bar{\rho}, \quad \|\nabla u_0\|^2_{L^2} \leq M, \tag{2.1.8}$$

$$u_0 \in D^1 \cap D^3, \quad (\rho_0 - \tilde{\rho}, P(\rho_0) - P(\tilde{\rho})) \in H^3, \tag{2.1.9}$$

and the compatibility condition

$$-\mu\triangle u_0 - (\mu+\lambda)\nabla \mathrm{div} u_0 + \nabla P(\rho_0) = \rho_0 g, \tag{2.1.10}$$

for some $g \in D^1$ *with* $\rho_0^{\frac{1}{2}} g \in L^2$. *Then there exists a positive constant* ϵ *depending on* μ, λ, A, γ, $\bar{\rho}$ *and* M *such that if*

$$C_0 \le \epsilon, \tag{2.1.11}$$

the half-space problem $(2.1.1)-(2.1.6)$ *has a unique global classical solution* (ρ, u) *satisfying for any* $0 < \tau < T < \infty$,

$$0 \le \rho(x,t) \le 2\bar{\rho}, \quad x \in \Omega, \quad t \ge 0, \tag{2.1.12}$$

$$\begin{cases} (\rho - \tilde{\rho}, P - P(\tilde{\rho})) \in C([0,T]; H^3), \\ u \in C([0,T]; D^1 \cap D^3) \cap L^2(0,T; D^4) \cap L^\infty(\tau, T; D^4), \\ u_t \in L^\infty(0,T; D^1) \cap L^2(0,T; D^2) \cap L^\infty(\tau,T; D^2) \cap H^1(\tau,T; D^1), \\ \sqrt{\rho} u_t \in L^\infty(0,T; L^2), \end{cases} \tag{2.1.13}$$

and the following large time behavior

$$\lim_{t\longrightarrow\infty} \int (|\rho - \tilde{\rho}|^q + \rho^{\frac{1}{2}}|u|^4 + |\nabla u|^2)(x,t)dx = 0, \tag{2.1.14}$$

for all $q \in (2, \infty)$.

Theorem 2.1.2 *In addition to the conditions of Theorem 2.1.1, assume further that there exists some point* $x_0 \in \Omega$ *such that* $\rho_0(x_0) = 0$. *Then the unique global classical solution* (ρ, u) *to* $(2.1.1)-(2.1.6)$ *obtained in Theorem 2.1.1 has to blow up as* $t \longrightarrow \infty$, *in the sense that for any* $r > 3$,

$$\lim_{t\longrightarrow\infty} \|\nabla\rho(\cdot,t)\|_{L^r} = \infty.$$

Remark 2.1.3 *The boundary condition can be replaced by*

$$(u^1(x), u^2(x), u^3(x)) = k(x)(u^1_{x_3}(x), u^2_{x_3}(x), 0),$$

where $k(x) \ge k_0 > 0$, *and* $k(x) \in W^{2,\infty}(\mathbb{R}^2)$.

Remark 2.1.4 *The solution obtained in Theorem 1.1.1 becomes a classical one for positive time. Although it has small energy, whose oscillations could be arbitrarily large.*

Remark 2.1.5 *It should be emphasized that in Theorem 1.1.1, the viscosity coefficients are only assume to satisfy the physical condition*

$$\mu > 0, \quad \mu + \frac{3}{2}\lambda \geq 0,$$

while the theory on weak energy solution in [32] requires additional assumptions.

2.2 Local existence and uniqueness of classical solutions

Using the same argument as in Kim, Choe [13] and the standard elliptic regularity results as in Agmon-Douglis-Nirenberg [1], we can obtain the following local existence and uniqueness of the classical solution.

Lemma 2.2.1 *For* $\tilde{\rho} > 0$, *assume that the initial data* $(\rho_0 \geq 0, u_0)$ *satisfy* $(2.1.9)-(2.1.10)$. *Then there exist a small time* T^* *and a unique classical solution* (ρ, u) *to the half-space problem* $(2.1.1) - (2.1.6)$ *such that*

$$\begin{cases} (\rho - \tilde{\rho}, P - P(\tilde{\rho})) \in C([0, T^*]; H^3), \\ u \in C([0, T^*]; D^1 \cap D^3) \cap L^2(0, T^*; D^4), \\ u_t \in L^\infty(0, T^*; D^1) \cap L^2(0, T^*; D^2), \quad \sqrt{\rho} u_t \in L^\infty(0, T^*; L^2), \\ \sqrt{\rho} u_{tt} \in L^2(0, T^*; L^2), \quad t^{\frac{1}{2}} u \in L^\infty(0, T^*; D^4), \\ t^{\frac{1}{2}} \sqrt{\rho} u_{tt} \in L^\infty(0, T^*; L^2), \quad t u_t \in L^\infty(0, T^*; D^3), \\ t u_{tt} \in L^\infty(0, T^*; D^1) \cap L^2(0, T^*; D^2). \end{cases} \tag{2.2.1}$$

2.3 A priori estimates

In this section, we will establish some a priori estimates for smooth solutions to the half-space problem (2.1.1) – (2.1.6) to extend the local classical solution obtained in Lemma 2.2.1. Let $T > 0$ be a fixed time and (ρ, u) be the smooth solution on $\Omega\times(0,T]$ in the class (2.2.1) with smooth initial data (ρ_0, u_0) satisfying (2.1.8) – (2.1.10). Now we define $\sigma(t) = \min\{1, t\}$ and

$$A_1(T) \triangleq \sup_{t\in[0,T]} (\sigma\|\nabla u\|_{L^2}^2) + \int_0^T \int \sigma\rho|\dot{u}|^2 dxdt,$$

$$A_2(T) \triangleq \sup_{t\in[0,T]} \sigma^3 \int \rho|\dot{u}|^2 dx + \int_0^T \int \sigma^3|\nabla\dot{u}|^2 dxdt,$$

and

$$A_3(T) \triangleq \sup_{t\in[0,T]} \|\nabla u\|_{L^2}^2,$$

where

$$\dot{f} \triangleq f_t + u\cdot\nabla f.$$

We have the following key a priori estimates on (ρ, u).

Proposition 2.3.1 *For given $M > 0$ and $\bar{\rho} \geq \tilde{\rho}+1$, assume that (ρ_0, u_0) satisfy (2.1.8) – (2.1.10). Then there exist positive constants ϵ and K both depending on μ, λ, $\tilde{\rho}$, A, γ, $\bar{\rho}$ and M such that if (ρ, u) is a smooth solution of (2.1.1) – (2.1.6) on $\Omega\times(0,T]$ satisfying*

$$\begin{cases} \sup\limits_{\Omega\times[0,T]} \rho \leq 2\bar{\rho}, \\ A_1(T) + A_2(T) \leq 2C_0^{\frac{1}{2}}, \\ A_3(\sigma(T)) \leq 3K, \end{cases} \tag{2.3.1}$$

the following estimates hold

$$\sup_{\Omega\times[0,T]} \rho \leq \frac{7}{4}\bar{\rho}, \quad A_1(T) + A_2(T) \leq C_0^{\frac{1}{2}}, \quad A_3(\sigma(T)) \leq 2K, \tag{2.3.2}$$

provided $C_0 \leq \epsilon$.

In the following, C denotes a generic positive constant depending on μ, λ, $\tilde{\rho}$, A, γ, $\bar{\rho}$ and M, and we write $C(\alpha)$ to emphasis that C depends on α.

Lemma 2.3.2 (Energy estimate) *Let* (ρ, u) *be a smooth solution of* (2.1.1) – (2.1.6) *with* $0 \le \rho(x,t) \le 2\bar{\rho}$. *Then there is a constant* $C(\bar{\rho})$ *such that*

$$\begin{aligned}&\sup_{0\le t\le T}\int\left(\frac{1}{2}\rho|u|^2+G(\rho)\right)dx+\int_0^T\int\left(\mu|\nabla u|^2+(\mu+\lambda)(\mathrm{div}u)^2\right)dxdt\\&+\beta^{-1}\int_0^T\int_{\partial\Omega}|u|^2dS_xdt\le C_0,\end{aligned}\tag{2.3.3}$$

$$A_1(T)\le CC_0+C\int_0^T\int\sigma|\nabla u|^3dxdt+C\int_0^T\int\sigma(|u|^2|\nabla u|+|u||\nabla u|^2)dxdt,\tag{2.3.4}$$

$$A_2(T)\le CC_0+CA_1(T)+\int_0^T\int\sigma^3[|u|^4+|\nabla u|^4+|\dot{u}||\nabla u||u|+|\dot{u}||\nabla u|^2]dxdt.\tag{2.3.5}$$

Proof: Multiplying the equation (2.1.1) by $G'(\rho)$ and the second equation by u^j and integrating, applying the far filed condition (2.1.4), one can obtain (2.3.3) easily.

For integer $m \ge 0$, multiplying (2.1.2) by $\sigma^m\dot{u}$, then integrating the resulting equality over Ω yields

$$\begin{aligned}\int\sigma^m\rho|\dot{u}|^2dx&=\int(-\sigma^m\dot{u}\nabla P+\mu\sigma^m\dot{u}\Delta u+(\lambda+\mu)\sigma^m\nabla\mathrm{div}u\dot{u})dx\\&=\sum_{i=1}^{3}M_i.\end{aligned}\tag{2.3.6}$$

Using (2.1.1) and integrating by parts leads to

$$\begin{aligned}
M_1 &= -\int \sigma^m \dot{u} \nabla P dx = \int \Big(\sigma^m (\mathrm{div} u)_t (P - P(\tilde{\rho})) - \sigma^m (u \cdot \nabla u) \nabla P \Big) dx \\
&= (\int \sigma^m \mathrm{div} u (P - P(\tilde{\rho})) dx)_t - m\sigma^{m-1}\sigma' \int \mathrm{div} u (P - P(\tilde{\rho})) dx \\
&+ \int \sigma^m (P' \rho (\mathrm{div} u)^2 - P(\mathrm{div} u)^2 + P \partial_i u^j \partial_j u^i) dx \\
&\leq (\int \sigma^m \mathrm{div} u (P - P(\tilde{\rho})) dx)_t + m\sigma^{m-1}\sigma' \|P - P(\tilde{\rho})\|_{L^2} \|\nabla u\|_{L^2} + C(\bar{\rho}) \|\nabla u\|_{L^2}^2 \\
&\leq (\int \sigma^m \mathrm{div} u (P - P(\tilde{\rho})) dx)_t + C(\bar{\rho}) \|\nabla u\|_{L^2}^2 + C(\bar{\rho}) m^2 \sigma^{2(m-1)} \sigma' C_0.
\end{aligned} \tag{2.3.7}$$

Integrating by parts implies

$$\begin{aligned}
M_2 &= \int \mu \sigma^m \triangle u \dot{u} dx \\
&= -\frac{\mu}{2} (\sigma^m \|\nabla u\|_{L^2}^2)_t + \frac{\mu}{2} m \sigma^{m-1} \sigma' \|\nabla u\|_{L^2}^2 - \mu \sigma^m \int (\partial_k u^j \partial_k (u^i \partial_i u^j)) dx \\
&+ \int_{\partial\Omega} \mu \sigma^m u_k^j \dot{u}^j N^k dS_x \\
&= -\frac{\mu}{2} (\sigma^m \|\nabla u\|_{L^2}^2)_t + \frac{\mu}{2} m \sigma^{m-1} \sigma' \|\nabla u\|_{L^2}^2 - \mu \sigma^m \int (\partial_k u^j \partial_k (u^i \partial_i u^j)) dx \\
&- \frac{1}{2} \mu \sigma^m \frac{d}{dt} \int_{\partial\Omega} \beta^{-1} |u|^2 dS_x - \sigma^m \mu \int_{\partial\Omega} \beta^{-1} u^j u^k u_k^j dS_x.
\end{aligned} \tag{2.3.8}$$

We need to estimate the boundary term $-\mu\sigma^m \int_{\partial\Omega} \beta^{-1} u^j u^k u_k^j dS_x$. We apply the fact that for $h \in (C^1 \cap W^{1,1})(\bar{\Omega})$,

$$\int_{\partial\Omega} h(x) dS = \int_{\Omega \cap \{0 \leq x_3 \leq 1\}} [h(x) + (x_3 - 1) h_{x_3}(x)] dx. \tag{2.3.9}$$

Since $j, k \in \{1, 2\}$, we can use (2.3.26) and integrating by parts in the x_1 and x_2 directions to obtain the bound

$$\mu \sigma^m \int_{\Omega} (|u|^2 |\nabla u| + |u| |\nabla u|^2) dx.$$

Hence

$$\begin{aligned}M_2 \le & -\frac{\mu}{2}(\sigma^m\|\nabla u\|_{L^2}^2)_t - \frac{1}{2}\mu\sigma^m(\int_{\partial\Omega}\beta^{-1}|u|^2dS_x)_t + Cm\sigma^{m-1}\|\nabla u\|_{L^2}^2 \\ & + C\int\sigma^m|\nabla u|^2dx + C\int\sigma^m|\nabla u|^3dx + C\int\sigma^m(|u|^2|\nabla u| + |\nabla u|^2|u|)dx,\end{aligned} \tag{2.3.10}$$

and similarly

$$\begin{aligned}M_3 = & -\frac{\mu+\lambda}{2}(\sigma^m\|\text{div}u\|_{L^2}^2)_t + \frac{m(\mu+\lambda)}{2}\sigma^{m-1}\sigma'\|\text{div}u\|_{L^2}^2 \\ & - (\lambda+\mu)\sigma^m\int \text{div}u\text{div}(u\cdot\nabla u)dx \\ \le & -\frac{\lambda+\mu}{2}(\sigma^m\|\text{div}u\|_{L^2}^2)_t + Cm\sigma^{m-1}\|\nabla u\|_{L^2}^2 + C\int\sigma^m|\nabla u|^3dx.\end{aligned} \tag{2.3.11}$$

Combining (2.3.7) $-$ (2.3.11), we have

$$\begin{aligned}& B'(t) + \int\sigma^m\rho|\dot{u}|^2dx + \mu\sigma^m\frac{d}{dt}\int_{\partial\Omega}\beta^{-1}|u|^2dS_x \\ & \le C(\bar{\rho})m^2\sigma^{2(m-1)}\sigma'C_0 + (Cm\sigma^{m-1} + C(\bar{\rho}))\|\nabla u\|_{L^2}^2 \\ & + C\sigma^m\int|\nabla u|^3dx + C\int\sigma^m(|u|^2|\nabla u| + |\nabla u|^2|u|)dx,\end{aligned} \tag{2.3.12}$$

where

$$\begin{aligned}B(t) \triangleq & \frac{\mu\sigma^m}{2}\|\nabla u\|_{L^2}^2 + \frac{\mu+\lambda}{2}\sigma^m\|\text{div}u\|_{L^2}^2 + \int\sigma^m\text{div}u(P - P(\tilde{\rho}))dx \\ \ge & \frac{\mu\sigma^m}{2}\|\nabla u\|_{L^2}^2 + \frac{\mu+\lambda}{2}\sigma^m\|\text{div}u\|_{L^2}^2 - C\sigma^mC_0^{1/2}\|\text{div}u\|_{L^2} \\ \ge & \frac{\mu}{4}\sigma^m\|\nabla u\|_{L^2}^2 + \frac{\mu+\lambda}{2}\sigma^m\|\text{div}u\|_{L^2}^2 - C\sigma^{2m}C_0.\end{aligned} \tag{2.3.13}$$

Integrating (2.3.13) over $[0, T]$, we obtain

$$\begin{aligned}& \sup_{0\le t\le T}\sigma^m\|\nabla u\|_{L^2}^2 + \int_0^T\int\sigma^m\rho|\dot{u}|^2dxds \\ & \le C(\bar{\rho})C_0 + \int_0^T\int\sigma^m|\nabla u|^3dxds + C\int\sigma^m(|u|^2|\nabla u| + |\nabla u|^2|u|)dx.\end{aligned}$$

For $m \ge 0$, multiplying $\sigma^m\dot{u}^j(\frac{\partial}{\partial t} + \text{div}(u\cdot))$ to $(2.1.2)^j$, summing with respect to

j, and integrating the resulting equation over Ω, we have

$$\begin{aligned}
&(\frac{\sigma^m}{2}\int\rho|\dot{u}|^2dx)_t-\frac{m}{2}\sigma^{m-1}\sigma'\int\rho|\dot{u}|^2dx\\
&=-\int\sigma^m\dot{u}^j[\partial_jP_t+\operatorname{div}(\partial_jPu)]dx+\mu\int\sigma^m\dot{u}^j[\triangle u_t^j+\operatorname{div}(u\triangle u^j)]dx\\
&+(\lambda+\mu)\int\sigma^m\dot{u}^j[\partial_t\partial_j\operatorname{div}u+\operatorname{div}(u\partial_j divu)]dx\\
&\triangleq\sum_{i=1}^3N_i.
\end{aligned}\tag{2.3.14}$$

Integrating by parts and using the equation (2.1.1), we have

$$\begin{aligned}
N_1&=-\int\sigma^m\dot{u}^j[\partial_jP_t+\operatorname{div}(\partial_jPu)]dx\\
&=\int\sigma^m[-P'\rho\operatorname{div}u\partial_j\dot{u}^j+\partial_k(\partial_j\dot{u}^ju^k)P-P\partial_j(\partial_k\dot{u}^ju^k)]dx\\
&\le C(\bar{\rho})\sigma^m\|\nabla u\|_{L^2}\|\nabla\dot{u}\|_{L^2}\le\delta\sigma^m\|\nabla\dot{u}\|_{L^2}^2+C(\bar{\rho},\delta)\sigma^m\|\nabla u\|_{L^2}^2.
\end{aligned}\tag{2.3.15}$$

$$\begin{aligned}
N_2&=\mu\int\sigma^m\dot{u}^j(\triangle u_t^j+\operatorname{div}(u\triangle u^j))dx\\
&=-\mu\int\sigma^m[|\nabla\dot{u}|^2+\partial_i\dot{u}^j\partial_ku^k\partial_iu^j-\partial_i\dot{u}^j\partial_iu^k\partial_ku^j-\partial_iu^j\partial_iu^k\partial_k\dot{u}^j]dx\\
&-\mu\int_{\partial\Omega}\sigma^m\beta^{-1}|\dot{u}|^2dS_x+\mu\int_{\partial\Omega}\sigma^m\beta^{-1}\dot{u}^ju_k^ju^kdS_x-\mu\int_{\partial\Omega}\sigma^m\partial_k\dot{u}^ju^k\partial_3u^jdS_x\\
&\le-\frac{3}{4}\mu\int\sigma^m|\nabla\dot{u}|^2dx+C\int\sigma^m|\nabla u|^4dx-\mu\int_{\partial\Omega}\sigma^m\beta^{-1}|\dot{u}|^2dS_x\\
&+C\mu\int\sigma^m[|u||\nabla u||\dot{u}|+|u||\nabla u||\nabla\dot{u}|+|\nabla u|^2|\dot{u}|]dx\\
&\le-\frac{3}{4}\mu\int\sigma^m|\nabla\dot{u}|^2dx+C\int\sigma^m|\nabla u|^4dx-\mu\int_{\partial\Omega}\sigma^m\beta^{-1}|\dot{u}|^2dS_x\\
&+\delta\sigma^m\|\nabla\dot{u}\|_{L^2}^2+C(\delta)\sigma^m\|\nabla u\|_{L^4}^4+C(\delta)\sigma^m\|u\|_{L^4}^4+C\mu\int\sigma^m[|u||\nabla u||\dot{u}|+|\nabla u|^2|\dot{u}|]dx,
\end{aligned}\tag{2.3.16}$$

where we have used

$$\int_{\partial\Omega}\sigma^m\beta^{-1}\dot{u}^ju_k^ju^kdS_x\le C\int_\Omega\sigma^m[|u||\nabla u||\dot{u}|+|u||\nabla u||\nabla\dot{u}|+|\nabla u|^2|\dot{u}|]dx,$$

and

$$\int_{\partial\Omega}\sigma^m\partial_k\dot{u}^ju^k\partial_3u^jdS_x\le C\int_\Omega\sigma^m|\nabla u||u||\nabla\dot{u}|dx,$$

the proof is similar to (2.3.8) and (2.3.26). Similarly,

$$N_3 \le -\frac{\mu+\lambda}{2}\int \sigma^m |\mathrm{div}\dot{u}|^2 dx + C\int \sigma^m |\nabla u|^4 dx. \tag{2.3.17}$$

Substituting (2.3.15) – (2.3.17) into (2.3.14), and choosing δ suitably small, it holds that

$$\begin{aligned}
&(\frac{\sigma^m}{2}\int \rho|\dot{u}|^2 dx)_t + \mu\int \sigma^m|\nabla \dot{u}|^2 dx + (\mu+\lambda)\int \sigma^m(\mathrm{div}\dot{u})^2 dx \\
&\le m\sigma^{m-1}\sigma'\int \rho|\dot{u}|^2 dx + C\sigma^m\|\nabla u\|_{L^4}^4 + C\sigma^m\|u\|_{L^4}^4 + C(\bar{\rho})\sigma^m\|\nabla u\|_{L^2}^2 \\
&+ C\mu\int \sigma^m(|u||\nabla u||\dot{u}| + |\nabla u|^2|\dot{u}|)dx.
\end{aligned} \tag{2.3.18}$$

Taking $m=3$ in (2.3.18) and using

$$3\int_0^T \sigma^2\sigma'\int \rho|\dot{u}|^2 dxdt \le CA_1(T),$$

we can obtain (2.3.5) after integrating (2.3.18) over $(0,T)$. □

If we denote

$$F \triangleq (2\mu+\lambda)\mathrm{div}u - P(\rho) + P(\tilde{\rho}), \quad \omega \triangleq \nabla\times u,$$

then we will have

$$\rho\dot{u}^j = F_{x_j} + \mu\omega^{j,k}_{x_k}, \quad \omega^{j,k} = u^j_{x_k} - u^k_{x_j}, \tag{2.3.19}$$

and the following Lemma.

Lemma 2.3.3 *There exists a constant $C = C(\bar{\rho})$ such that if (ρ,u) is a smooth solution to* (2.1.1) – (2.1.6) *on $\Omega\times[0,T]$, then for $0\le t\le T$,*

$$\int |u|^p dx \le C(\bar{\rho})\Big(C_0^{\frac{6-p}{4}}(\int |\nabla u|^2 dx)^{\frac{3p-6}{4}} + C_0^{\frac{6-p}{6}}(\int |\nabla u|^2 dx)^{\frac{p}{2}}\Big), \quad 2\le p\le 6, \tag{2.3.20}$$

$$\int |\nabla u|^p dx \le C\int (|F|^p + |\omega|^p + |P-P(\tilde{\rho})|^p)dx, \quad 1<p<\infty, \tag{2.3.21}$$

$$\int |(\nabla F|^p + |\nabla\omega|^p)dx \le C\int (|\rho\dot{u}|^p + |\nabla u|^p)dx, \quad 1<p<\infty, \tag{2.3.22}$$

$$\int(|F|^p+|\omega|^p)dx \le C(\tilde{\rho})[(\int \rho|\dot{u}|^2dx)^{\frac{3p-6}{4}}(\int(|\nabla u|^2+|P-P(\tilde{\rho})|^2)dx)^{\frac{6-p}{4}}$$
$$+(\int|\nabla u|^2dx)^{\frac{p}{2}}+(\int|P-P(\tilde{\rho})|^2dx)^{\frac{6-p}{4}}(\int|\nabla u|^2dx)^{\frac{3p-6}{4}}], \quad 2\le p\le 6, \tag{2.3.23}$$

$$\begin{aligned}\|\nabla u\|_{L^p} &\le \|\nabla u\|_{L^2}^{\frac{6-p}{2p}}\|\nabla u\|_{L^6}^{\frac{3p-6}{2p}}\\ &\le C(\tilde{\rho})\|\nabla u\|_{L^2}^{\frac{6-p}{2p}}(\|\rho\dot{u}\|_{L^2}+\|\nabla u\|_{L^2}\\ &+\|P-P(\tilde{\rho})\|_{L^2}+\|P-P(\tilde{\rho})\|_{L^6})^{\frac{3p-6}{2p}}, \quad 2\le p\le 6.\end{aligned} \tag{2.3.24}$$

Proof: From Gagliardo-Nirenberg inequality (1.0.3), we have

$$\|u\|_{L^p}^p \le C\|u\|_{L^2}^{\frac{6-p}{2}}\|\nabla u\|_{L^2}^{\frac{3p-6}{2}}. \tag{2.3.25}$$

Since

$$\tilde{\rho}\int|u|^2dx \le \int\rho|u|^2dx+(\int|\rho-\tilde{\rho}|^2dx)^{\frac{1}{2}}(\int|u|^4dx)^{\frac{1}{2}}, \tag{2.3.26}$$

making use of (2.3.25), we obtain

$$\int|u|^2dx \le C(\tilde{\rho})(C_0+C_0^{\frac{2}{3}}\int|\nabla u|^2dx),$$

using (2.3.25) again, we get (2.3.20).

Observing that u satisfies the elliptic boundary value problem

$$\begin{aligned}(\mu+\lambda)\triangle u^j &= ((\mu+\lambda)\mathrm{div}u-P(\rho))_{x_j}+(\mu+\lambda)(u^j_{x_k}-u^k_{x_j})_{x_k}+(P(\rho)-P(\tilde{\rho}))_{x_j}\\ &= F_{x_j}+(\mu+\lambda)\omega^{j,k}_{x_k}+(P(\rho)-P(\tilde{\rho}))_{x_j},\end{aligned} \tag{2.3.27}$$

$$\begin{cases} u^1_{x_3}=\beta^{-1}u^1, & \\ u^2_{x_3}=\beta^{-1}u^2, & u\in\partial\Omega,\\ u^3=0, & \end{cases} \tag{2.3.28}$$

by standard elliptic regularity results, we obtain (2.3.21).

In order to prove (2.3.22), we compute from the equation (2.1.2) that

$$\mu\triangle\omega^{j,k}=(\rho\dot{u}^j)_{x_k}-(\rho\dot{u}^k)_{x_j}. \tag{2.3.29}$$

Thus, if we let $H = \omega^{1,3} - \beta^{-1}u^1$, then $H = 0$ on $\partial\Omega$ by (2.1.4) and

$$\mu\triangle H = (\rho\dot{u}^j)_{x_k} - (\rho\dot{u}^k)_{x_j} - \mu\beta^{-1}\Delta u^1 \tag{2.3.30}$$

in Ω. The standard elliptic theory gives us the bounds for $\|\nabla H\|_{L^p}$ for $1 < p < \infty$ and therefore, we can obtain

$$\|\nabla\omega^{1,3}\|_{L^p} \leq C(\bar{\rho})(\|\rho\dot{u}\|_{L^p} + \|\nabla u\|_{L^p}), \quad 1 < p < \infty. \tag{2.3.31}$$

A similar argument applies to $\omega^{2,3}$. In order to obtain the estimate of $\omega^{1,2}$, we differentiate the $j = 1$ equation in (2.1.2) with respect to x_2, then reverse the indices and substract to get

$$\mu(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2})\omega^{1,2} = (\rho\dot{u}^1)_{x_2} - (\rho\dot{u}^2)_{x_1} + \mu(\omega^{2,3}_{x_3,x_1} - \omega^{1,3}_{x_3,x_2}). \tag{2.3.32}$$

Then we can estimate $\|\nabla_{x_1,x_2}\omega^{1,2}(\cdot,\cdot,x_3)\|_{L^p(\mathbb{R}^2)}$. Integrating this bound with respect to x_3, and applying (2.3.31), we get that $\|\nabla_{x_1,x_2}\omega^{1,2}\|_{L^p(\Omega)}$ is bounded by the right side of (2.3.31). Since

$$\omega^{1,2}_{x_3} = \omega^{1,3}_{x_2} - \omega^{2,3}_{x_1},$$

we prove the bound in (2.3.22) for ω. The bound for ∇F follows from the decomposition (2.3.19).

(2.3.23) follows from (2.3.22) for $p = 2$, Gagliardo-Nirenberg inequality and (2.3.3).

(2.3.24) is a direct result from interpolation inequality (1.0.2), (2.3.21) and (2.3.23).

□

Lemma 2.3.4 *Let (ρ, u) be a smooth solution of (2.1.1) − (2.1.6) with $0 \leq \rho(x,t) \leq 2\bar{\rho}$. Then there exist positive constants K and ϵ_0 both depending only on μ, λ, $\tilde{\rho}$, A, γ, $\bar{\rho}$ and M such that*

$$A_3(\sigma(T)) + \int_0^{\sigma(T)}\int \rho|\dot{u}|^2 dxdt \leq 2K, \tag{2.3.33}$$

provided $A_3(\sigma(T)) \le 3K$ *and* $C_0 \le \epsilon_0$.

Proof: Integrating (2.3.12) over $(0, \sigma(T))$, choosing $m = 0$, and using (2.3.13), one has

$$A_3(\sigma(T)) + \int_0^{\sigma(T)} \int \rho |\dot{u}|^2 dx dt \le C(\bar{\rho})(C_0 + M) + C(\bar{\rho}) \int_0^{\sigma(T)} \|\nabla u\|_{L^3}^3 dt + \int_0^{\sigma(T)} \int (|u|^2 |\nabla u| + |\nabla u|^2 |u|) dx dt. \tag{2.3.34}$$

It follows from (2.3.3) and (2.3.24) that

$$\begin{aligned} \int_0^{\sigma(T)} \|\nabla u\|_{L^3}^3 dt &\le C(\bar{\rho}) \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^{\frac{3}{2}} (\|\rho \dot{u}\|_{L^2}^{\frac{3}{2}} + C_0^{\frac{1}{4}} + \|\nabla u\|_{L^2}^{\frac{3}{2}} + C_0^{\frac{3}{4}}) \\ &\le \delta \int_0^{\sigma(T)} \int \rho |\dot{u}|^2 dx dt + C(\bar{\rho}, \delta) \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^6 dt + C(\bar{\rho}) C_0 \end{aligned}$$

which, together with (2.3.20), (2.3.34) and choosing δ small enough, we have

$$\begin{aligned} & A_3(\sigma(T)) + \int_0^{\sigma(T)} \int \rho |\dot{u}|^2 dx dt \le C(\bar{\rho}) \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^6 dt + C(\bar{\rho})(C_0 + M) \\ & + C(\bar{\rho}) \left(\int_0^{\sigma(T)} \|u\|_{L^4}^4 dt + \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 dt + \int_0^{\sigma(T)} \|\nabla u\|_{L^3}^3 dt + \int_0^{\sigma(T)} \|u\|_{L^3}^3 dt \right) \\ & \le C(\bar{\rho}) \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^6 dt + C(\bar{\rho})(C_0 + M) \le K + C(\bar{\rho}) C_0 [A_3(\sigma(T))]^2, \end{aligned}$$

for some positive constant K depending only on μ, λ, $\tilde{\rho}$, A, γ, $\bar{\rho}$ and M. By choosing $\epsilon_0 \triangleq (9C(\bar{\rho})K)^{-1}$, we finish the proof of (2.3.33). □

Lemma 2.3.5 *There exists a positive constant* $\epsilon_1(\mu, \lambda, \tilde{\rho}, A, \gamma, \bar{\rho}, M) \le \epsilon_0$ *such that if* (ρ, u) *is a smooth solution of* (2.1.1) – (2.1.6) *satisfying* (2.3.1) *for* K *as in Lemma 2.3.4, then*

$$A_1(T) + A_2(T) \le C_0^{\frac{1}{2}}, \tag{2.3.35}$$

provided $C_0 \le \epsilon_1$.

Proof: From Lemma 2.3.2 and Lemma 2.3.3, we have

$$
\begin{aligned}
&A_1(T)+A_2(T) \le C(\bar\rho)C_0 + C(\bar\rho)\int_0^T \sigma^3\|\nabla u\|_{L^4}^4 dt + C(\bar\rho)\int_0^T \sigma\|\nabla u\|_{L^3}^3 dt \\
&+ C(\bar\rho)\int_0^T \sigma^3\|u\|_{L^4}^4 dt + C(\bar\rho)\int_0^T\int \sigma^3[|u||\nabla u||\dot u| + |\nabla u|^2|\dot u|]dxdt \\
&+ C(\bar\rho)\int_0^T\int \sigma(|\nabla u||u|^2 + |\nabla u|^2|u|)dxdt.
\end{aligned}
\tag{2.3.36}
$$

From (2.3.21), we have

$$
\int_0^T \sigma^3\|\nabla u\|_{L^4}^4 dt \le C\int_0^T \sigma^3(\|F\|_{L^4}^4 + \|\omega\|_{L^4}^4)dt + \int_0^T \sigma^3\|P-P(\tilde\rho)\|_{L^4}^4 dt. \tag{2.3.37}
$$

It follows from (2.3.23) that

$$
\begin{aligned}
&\int_0^T \sigma^3(\|F\|_{L^4}^4 + \|\omega\|_{L^4}^4)dt \\
&\le C\int_0^T \sigma^3(\|\nabla u\|_{L^2} + \|P-P(\tilde\rho)\|_{L^2})\|\rho\dot u\|_{L^2}^3 dt + C(\bar\rho)\int_0^T \sigma^3\|\nabla u\|_{L^2}^4 dt \\
&+ C(\bar\rho)\int_0^T \sigma^3\|P-P(\tilde\rho)\|_{L^2}\|\nabla u\|_{L^2}^3 dt \\
&\le C(\bar\rho)\sup_{0\le t\le T}(\sigma^{\frac32}\|\sqrt\rho\dot u\|_{L^2}(\sigma^{\frac12}\|\nabla u\|_{L^2} + C_0^{1/2}))\int_0^T\int \sigma\rho|\dot u|^2 dxdt \\
&+ C(\bar\rho)C_0^{3/2}\sup_{0\le t\le T}(\sigma^{1/2}\|\nabla u\|_{L^2}) + C(\bar\rho)C_0\sup_{0\le t\le T}(\sigma\|\nabla u\|_{L^2}^2) \\
&\le C(\bar\rho)(A_1^{1/2}(T) + C_0^{1/2})A_2^{1/2}(T)A_1(T) + C(\bar\rho)C_0^{3/2}A_1(T)^{1/2} + C(\bar\rho)C_0A_1(T) \\
&\le C(\bar\rho)C_0,
\end{aligned}
\tag{2.3.38}
$$

provided $C_0 \le \epsilon_0$.

From (2.1.1), we have

$$
(P-P(\tilde\rho))_t + u\cdot\nabla(P-P(\tilde\rho)) + \gamma(P-P(\tilde\rho))\mathrm{div}u + \gamma P(\tilde\rho)\mathrm{div}u = 0. \tag{2.3.39}
$$

Multiplying (2.3.39) by $3(P-P(\tilde\rho))^2$ and integrating the resulting equality over

Ω, one can gets after using $\text{div}u = \frac{1}{2\mu+\lambda}(F + P - P(\tilde{\rho}))$ that

$$
\begin{aligned}
&\frac{3\gamma-1}{2\mu+\lambda}\|P - P(\tilde{\rho})\|_{L^4}^4 = -(\int (P - P(\tilde{\rho}))^3 dx)_t - \frac{3\gamma-1}{2\mu+\lambda}\int (P - P(\tilde{\rho}))^3 F dx \\
&- 3\gamma P(\tilde{\rho})\int (P - P(\tilde{\rho}))^2 \text{div}u dx \\
&\leq -(\int (P - P(\tilde{\rho}))^3 dx)_t + \delta\|P - P(\tilde{\rho})\|_{L^4}^4 + C_\delta\|F\|_{L^4}^4 + C_\delta\|\nabla u\|_{L^2}^2.
\end{aligned}
\tag{2.3.40}
$$

Multiplying (2.3.40) by σ^3 and integrating over $(0, T)$, choosing δ suitably small, one has

$$
\begin{aligned}
&\int_0^T \sigma^3\|P - P(\tilde{\rho})\|_{L^4}^4 dt \leq C \sup_{0\leq t\leq T}\|P - P(\tilde{\rho})\|_{L^3}^3 + C\int_0^{\sigma(T)}\|P - P(\tilde{\rho})\|_{L^3}^3 dt \\
&+ C(\bar{\rho})\int_0^T \sigma^3\|F\|_{L^4}^4 dt + C(\bar{\rho})C_0 \\
&\leq C(\bar{\rho})C_0,
\end{aligned}
\tag{2.3.41}
$$

where (2.3.38) has been used. Therefore, combining (2.3.37), (2.3.38) and (2.3.41), we have

$$
\int_0^T \sigma^3(\|\nabla u\|_{L^4}^4 + \|P - P(\tilde{\rho})\|_{L^4}^4)dt \leq C(\bar{\rho})C_0. \tag{2.3.42}
$$

Next, we will estimate the term $\int_0^T \sigma\|\nabla u\|_{L^3}^3 dt$. First, (2.3.42) implies that

$$
\int_{\sigma(T)}^T \int \sigma|\nabla u|^3 dx dt \leq \int_{\sigma(T)}^T \int (|\nabla u|^4 + |\nabla u|^2) dx dt \leq C(\bar{\rho})C_0, \tag{2.3.43}
$$

and from (2.3.24) and (2.3.33), one gets

$$\begin{aligned}
&\int_0^{\sigma(T)} \sigma\|\nabla u\|_{L^3}^3 dt \\
&\le C(\bar\rho)\int_0^{\sigma(T)} \sigma\|\nabla u\|_{L^2}^{3/2}(\|\rho\dot u\|_{L^2}^{3/2} + C_0^{1/4} + \|\nabla u\|_{L^2}^{3/2} + C_0^{3/4})dt \\
&\le C(\bar\rho)\int_0^{\sigma(T)} (\sigma^{1/4}\|\nabla u\|_{L^2}^{3/2})(\sigma\int \rho|\dot u|^2 dx)^{3/4} dt + \int_0^{\sigma(T)} \sigma\|\nabla u\|_{L^2}^3 dt + C(\bar\rho)C_0 \\
&\le C(\bar\rho)\sup_{t\in(0,\sigma(T)]}((\sigma\|\nabla u\|_{L^2}^2)^{1/4}\|\nabla u\|_{L^2}^{1/2})\int_0^{\sigma(T)} \|\nabla u\|_{L^2}^{1/2}(\sigma\int\rho|\dot u|^2 dx)^{3/4} dt \\
&+ \sup_{t\in(0,\sigma(T)]}(\sigma\|\nabla u\|_{L^2}^2)^{1/2}\int_0^{\sigma(T)} \sigma^{1/2}\|\nabla u\|_{L^2}^2 dt + C(\bar\rho)C_0 \\
&\le C(\bar\rho, M)A_1(T)C_0^{1/4} + C(\bar\rho)C_0 \le C(\bar\rho, M)C_0^{3/4}
\end{aligned} \tag{2.3.44}$$

provided $C_0 \le \epsilon_0$. By using of (2.3.20), (2.3.43) and (2.3.44), we obtain

$$\begin{aligned}
&\int_0^T\int \sigma^3|u|^4 dxdt \le C(\bar\rho)\int_0^T \sigma^3[C_0^{1/2}\|\nabla u\|_{L^2}^3 + C_0^{1/3}\|\nabla u\|_{L^2}^4]dt \\
&\le C(\bar\rho)C_0,
\end{aligned} \tag{2.3.45}$$

$$\begin{aligned}
&\int_0^T\int \sigma(|\nabla u||u|^2 + |\nabla u|^2|u|)dx \le \int_0^T\int |\nabla u|^2 dxdt + \int_0^T\int \sigma^2|u|^4 dxdt \\
&+ \int_0^T \sigma\|\nabla u\|_{L^3}\|\nabla u\|_{L^2}^2 dt \\
&\le \int_0^T\int |\nabla u|^2 dxdt + \int_0^T\int \sigma^2|u|^4 dxdt + C\int_0^T \sigma\|\nabla u\|_{L^3}^3 dt + \int_0^T \sigma\|\nabla u\|_{L^2}^3 dt \\
&\le C(\bar\rho, M)C_0^{3/4},
\end{aligned} \tag{2.3.46}$$

and

$$\begin{aligned}
&\int_0^T\int \sigma^3[|u||\nabla u||\dot{u}|+|\nabla u|^2|\dot{u}|]dxdt \le C\int_0^T \sigma^3\|u\|_{L^3}\|\nabla u\|_{L^2}\|\nabla \dot{u}\|_{L^2}dt\\
&+C\int_0^T \sigma^3\|\nabla u\|_{L^3}\|\nabla u\|_{L^2}\|\nabla \dot{u}\|_{L^2}dt\\
&\le C(\bar{\rho})\int_0^T \sigma^3[C_0^{1/4}\|\nabla u\|_{L^2}^{1/2}+C_0^{1/6}\|\nabla u\|_{L^2}]\|\nabla u\|_{L^2}\|\nabla \dot{u}\|_{L^2}dt\\
&+C\int_0^T \sigma^3\|\nabla u\|_{L^3}\|\nabla u\|_{L^2}\|\nabla \dot{u}\|_{L^2}dt \le C(\bar{\rho})C_0^{1/4}\int_0^T \sigma^3\|\nabla u\|_{L^2}^{3/2}\|\nabla \dot{u}\|_{L^2}dt\\
&+C(\bar{\rho})C_0^{1/6}\int_0^T \sigma^3\|\nabla u\|_{L^2}^2\|\nabla \dot{u}\|_{L^2}dt+C\int_0^T \sigma^3\|\nabla u\|_{L^3}\|\nabla u\|_{L^2}\|\nabla \dot{u}\|_{L^2}dt\\
&\le C(\bar{\rho},M)C_0^{3/4},
\end{aligned} \tag{2.3.47}$$

where we have used the following simply fact:

$$\begin{aligned}
&\int_0^T \sigma^3\|\nabla u\|_{L^3}\|\nabla u\|_{L^2}\|\nabla \dot{u}\|_{L^2}dt\\
&\le C\int_0^T \sigma^3\|\nabla u\|_{L^3}^3ds+C\int_0^T \sigma^3\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^{1/2}\|\nabla \dot{u}\|_{L^2}^{3/2}ds\\
&\le C(\bar{\rho},M)C_0^{3/4}+CA_1(T)^{1/2}(\int_0^T \sigma\|\nabla u\|_{L^2}^2ds)^{1/4}(\int_0^T \sigma^3\|\nabla \dot{u}\|_{L^2}^2ds)^{3/4}\\
&\le C(\bar{\rho},M)C_0^{3/4}.
\end{aligned}$$

Thus, it follows from (2.3.36) and (2.3.42) – (2.3.47) that the left hand side of (2.3.35) is bounded by

$$C(\bar{\rho},M)C_0^{3/4}\le C_0^{1/2}$$

provided

$$C_0\le \epsilon_1 \triangleq \min\{\epsilon_0,(C(\bar{\rho},M))^{-4}\}.$$

□

Lemma 2.3.6 *There exists a positive constant C depending only on μ, λ, $\tilde{\rho}$, A, γ, $\bar{\rho}$ and M such that the following estimates hold for a smooth solution (ρ,u) of*

(2.1.1) – (2.1.6) :

$$\sup_{t\in[0,T]} \|\nabla u\|_{L^2}^2 + \int_0^T \int \rho|\dot{u}|^2 dxdt \le C(\bar{\rho}, M), \tag{2.3.48}$$

$$\sup_{t\in(0,T]} \int \sigma\rho|\dot{u}|^2 dx + \int_0^T \int \sigma|\nabla\dot{u}|^2 dxdt \le C(\bar{\rho}, M), \tag{2.3.49}$$

provided $C_0 \le \epsilon_1$.

Proof: (2.3.48) is a direct consequence of (2.3.33) and (2.3.35). Hence we only need to show (2.3.49). Integrating (2.3.18) over $(0,T)$ and choosing $m = 1$, by (2.3.24) (2.3.42) and (2.3.48) we get

$$\begin{aligned}
&\sup_{t\in(0,T]} \int \sigma\rho|\dot{u}|^2 dx + \int_0^T \sigma\|\nabla\dot{u}\|_{L^2}^2 dt \\
&\le \int_0^{\sigma(T)} \int \rho|\dot{u}|^2 dxdt + C\int_0^T \sigma\|\nabla u\|_{L^4}^4 dt + C(\bar{\rho})C_0 \\
&+ \int_0^T \int \sigma|u|^4 dxds + C(\bar{\rho})\int_0^T \int \sigma[|u||\nabla u||\dot{u}| + |\nabla u|^2|\dot{u}|]dxdt \le C(\bar{\rho}, M) \\
&+ C\int_{\sigma(T)}^T \sigma^3\|\nabla u\|_{L^4}^4 dt + C\int_0^{\sigma(T)} \sigma\|\nabla u\|_{L^4}^4 dt \\
&+ C(\bar{\rho})\int_0^T \int \sigma[|u||\nabla u||\dot{u}| + |\nabla u|^2|\dot{u}|]dxdt \\
&\le C(\bar{\rho}, M) + C\int_0^{\sigma(T)} \sigma\|\nabla u\|_{L^2}(\|\rho\dot{u}\|_{L^2}^3 + \|P - P(\tilde{\rho})\|_{L^6}^3 + \|\nabla u\|_{L^2}^3 + \|P - P(\tilde{\rho})\|_{L^2}^3) \\
&+ C(\bar{\rho})\int_0^T \int \sigma[|u||\nabla u||\dot{u}| + |\nabla u|^2|\dot{u}|]dxdt \\
&\le C(\bar{\rho}, M) + C\sup_{t\in(0,\sigma(T))} [(\sigma^{1/2}\|\nabla u\|_{L^2})(\sigma^{1/2}\|\rho\dot{u}\|_{L^2})] \int_0^{\sigma(T)} \|\rho\dot{u}\|_{L^2}^2 dt \\
&+ \sup_{t\in(0,T]} (\sigma\|\nabla u\|_{L^2}^2) \int_0^{\sigma(T)} \|\nabla u\|_{L^2}^2 dt + \frac{1}{2}\int_0^T \sigma\|\nabla\dot{u}\|_{L^2}^2 dt \\
&\le C(\bar{\rho}, M) + C(\bar{\rho}, M)\sup_{t\in(0,T]} \sigma^{1/2}\|\rho\dot{u}\|_{L^2} + \frac{1}{2}\int_0^T \sigma\|\nabla\dot{u}\|_{L^2}^2 dt.
\end{aligned} \tag{2.3.50}$$

Then (2.3.49) follows from (2.3.50) and Young's inequality. □

Now we can derive a uniform (in time) upper bound for the density which turns out to be the key to obtain the global classical solution. In order to obtain this result, we need to use Lemma 1.0.11.

Lemma 2.3.7 *There exists a positive constants* $\epsilon = \epsilon(\bar{\rho}, M)$ *as described in Theorem 2.1.1 such that if* (ρ, u) *is a smooth solution of* (2.1.1) – (2.1.6) *as in Lemma 2.3.5 then*

$$\sup_{0\le t\le T} \|\rho(t)\|_{L^\infty} \le \frac{7\bar{\rho}}{4},$$

provided $C_0 \le \epsilon$.

Proof: Rewrite the equation of mass conservation (2.1.1) as

$$D_t\rho = g(\rho) + b'(t),$$

where

$$D_t\rho \triangleq \rho_t + u\cdot\nabla\rho, \quad g(\rho) = -\frac{A\rho}{2\mu+\lambda}(\rho^\gamma - \tilde{\rho}^\gamma), \quad b(t) \triangleq -\frac{1}{2\mu+\lambda}\int_0^t \rho F dt.$$

For $t \in [0, \sigma(T)]$, one can deduce from Gagliardo-Nirenberg inequality (1.0.3), (2.3.20) and (2.3.49) that for all $0 \le t_1 \le t_2 \le \sigma(T)$,

$$
\begin{aligned}
&|b(t_2) - b(t_1)| \le C \int_0^{\sigma(T)} \|\rho F(\cdot, t)\|_{L^\infty} dt \\
&\le C(\bar{\rho}) \int_0^{\sigma(T)} \|F\|_{L^2}^{1/4} \|\nabla F\|_{L^6}^{3/4} dt \\
&\le C(\bar{\rho}) \int_0^{\sigma(T)} (\|\nabla u\|_{L^2}^{1/4} + \|P - P(\tilde{\rho})\|_{L^2}^{1/4}) \\
&(\|\nabla \dot{u}\|_{L^2}^{3/4} + \|\rho \dot{u}\|_{L^2}^{3/4} + \|\nabla u\|_{L^2}^{3/4} + \|P - P(\tilde{\rho})\|_{L^2}^{3/4} + \|P - P(\tilde{\rho})\|_{L^6}^{3/4}) dt \\
&\le C(\bar{\rho}) \int_0^{\sigma(T)} (\sigma^{-1/2} (\sigma^{1/2} \|\nabla u\|_{L^2})^{1/4} + C_0^{1/8} \sigma^{-3/8}) [(\sigma \|\nabla \dot{u}\|_{L^2}^2)^{3/8} + (\sigma \|\rho \dot{u}\|_{L^2}^2)^{3/8}] dt \\
&+ C(\bar{\rho}) \int_0^{\sigma(T)} (\sigma^{1/2} \|\nabla u\|_{L^2})^{1/4} \sigma^{-1/8} (\|P - P(\tilde{\rho})\|_{L^2}^2)^{3/8} dt + C(\bar{\rho}) C_0^{1/2} \\
&\le C(\bar{\rho}) C_0^{1/16} \int_0^{\sigma(T)} (\sigma^{-1/2} + 1) [(\sigma \|\nabla \dot{u}\|_{L^2}^2)^{3/8} + (\sigma \|\rho \dot{u}\|_{L^2}^2)^{3/8}] dt \\
&+ C(\bar{\rho}) C_0^{1/16} \int_0^{\sigma(T)} \sigma^{-1/8} (\|P - P(\tilde{\rho})\|_{L^2}^2)^{3/8} dt + C(\bar{\rho}) C_0^{1/2} \\
&\le C(\bar{\rho}) C_0^{1/16} (1 + \int_0^1 \sigma^{-4/5} dt)^{5/8} [(\int_0^{\sigma(T)} \sigma \|\nabla \dot{u}\|_{L^2}^2 dt)^{3/8} + (\int_0^{\sigma(T)} \sigma \|\rho \dot{u}\|_{L^2}^2 dt)^{3/8}] \\
&+ C(\bar{\rho}) C_0^{1/16} (\int_0^{\sigma(T)} \sigma^{-1/5} dt)^{5/8} (\int_0^{\sigma(T)} \|P - P(\tilde{\rho})\|_{L^2}^2 dt)^{3/8} + C(\bar{\rho}) C_0^{1/2} \\
&\le C(\bar{\rho}, M) C_0^{1/16},
\end{aligned}
$$

provided $C_0 \le \epsilon_1$. Therefore, for $t \in [0, \sigma(T)]$, one can choose N_0 and N_1 in (1.0.11) as follows:

$$N_1 = 0, \quad N_0 = C(\bar{\rho}, M) C_0^{1/16},$$

and $\bar{\xi} = \tilde{\rho}$ in (1.0.12). Then

$$g(\xi) = -\frac{A\xi}{2\mu + \lambda}(\xi^\gamma - \tilde{\rho}^\gamma) \le -N_1 = 0, \quad \text{for} \quad \text{all} \quad \xi \ge \bar{\xi} = \tilde{\rho}.$$

From Lemma 1.0.11, we have

$$\sup_{t \in [0, \sigma(T)]} \|\rho\|_{L^\infty} \le \max\{\bar{\rho}, \tilde{\rho}\} + N_0 \le \bar{\rho} + C(\bar{\rho}, M) C_0^{1/16} \le \frac{3}{2}\bar{\rho}, \tag{2.3.51}$$

provided

$$C_0 \le \min\{\epsilon_1, \epsilon_2\}, \quad \text{for} \quad \epsilon_2 \triangleq (\frac{\bar{\rho}}{2C(\bar{\rho}, M)})^{16}.$$

On the other hand, for $t \in [\sigma(T), T]$, one can derive from Gagliardo-Nirenberg inequality (1.0.3) , (2.3.3), (2.3.20), (2.3.35) and (2.3.41) that for all $\sigma(T) \le t_1 \le t_2 \le T$,

$$
\begin{aligned}
&|b(t_2) - b(t_1)| \le C(\bar\rho) \int_{t_1}^{t_2} \|F\|_{L^\infty} dt \le \frac{A}{2\mu+\lambda}(t_2 - t_1) + C(\bar\rho) \int_{t_1}^{t_2} \|F\|_{L^\infty}^{8/3} dt \\
&\le \frac{A}{2\mu+\lambda}(t_2 - t_1) + C(\bar\rho) \int_{t_1}^{t_2} \|F\|_{L^2}^{2/3} \|\nabla F\|_{L^6}^2 dt \le \frac{A}{2\mu+\lambda}(t_2 - t_1) \\
&+ C(\bar\rho) \int_{t_1}^{t_2} (\|\nabla u\|_{L^2}^{2/3} + \|P - P(\tilde\rho)\|_{L^2}^{2/3})(\|\rho\dot u\|_{L^6}^2 + \|\nabla u\|_{L^2}^2 + \|P - P(\tilde\rho)\|_{L^6}^2 \\
&+ \|\rho\dot u\|_{L^2}^2 + \|P - P(\tilde\rho)\|_{L^2}^2) dt \\
&\le \frac{A}{2\mu+\lambda}(t_2 - t_1) + C(\bar\rho) C_0^{1/6} \int_{\sigma(T)}^{T} \|\nabla \dot u\|_{L^2}^2 dt + C(\bar\rho) C_0^{1/6} + C(\bar\rho) C_0^{1/2}(t_2 - t_1) \\
&+ C(\bar\rho) C_0^{1/6} \int_{\sigma(T)}^{T} \|\rho\dot u\|_{L^2}^2 \\
&\le (\frac{A}{2\mu+\lambda} + C(\bar\rho) C_0^{1/2})(t_2 - t_1) + C(\bar\rho) C_0^{2/3},
\end{aligned}
$$

provided $C_0 \le \epsilon_1$. Choosing

$$C(\bar\rho) C_0^{1/2} \le \frac{1}{2}(\frac{A}{2\mu+\lambda}),$$

that is

$$C_0 \le (\frac{A}{2(2\mu+\lambda)C(\bar\rho)})^2 \triangleq \epsilon_3,$$

and

$$N_1 = \frac{3}{2}\frac{A}{2\mu+\lambda}, \quad N_0 = C(\bar\rho) C_0^{2/3}.$$

Note that

$$g(\xi) = -\frac{A\xi}{2\mu+\lambda}(\xi^\gamma - \tilde\rho^\gamma) \le -N_1 = -\frac{3}{2}\frac{A}{2\mu+\lambda}, \quad \text{for} \quad \text{all} \quad \xi \ge (\tilde\rho + 1)\frac{3}{2} > \tilde\rho + 1.$$

So one can set $\bar\xi = \frac{3}{2}(\tilde\rho + 1)$ in (1.0.11). Lemma 2.3.7 and (2.3.51) thus yield that

$$\sup_{t \in [\sigma(T), T]} \|\rho\|_{L^\infty} \le \max\{\frac{3}{2}(\tilde\rho + 1), \frac{3}{2}\bar\rho\} + N_0 \le \frac{3}{2}\bar\rho + C(\bar\rho) C_0^{2/3} \le \frac{7\bar\rho}{4} \qquad (2.3.52)$$

provided

$$C_0 \le \epsilon \triangleq \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}, \quad \text{for} \quad \epsilon_4 \triangleq (\frac{\bar\rho}{4C(\bar\rho)})^{3/2}. \tag{2.3.53}$$

□

Lemma 2.3.8 *The following estimates hold*

$$\sup_{0\le t\le T}\int \rho|\dot u|^2dx + \int_0^T\int |\nabla \dot u|^2 dxdt \le C, \tag{2.3.54}$$

$$\sup_{0\le t\le T}(\|\nabla\rho\|_{L^2\cap L^6} + \|\nabla u\|_{H^1}) + \int_0^T \|\nabla u\|_{L^\infty}dt \le C. \tag{2.3.55}$$

Proof: Taking $m = 0$ in (2.3.18), one can deduce from Gagliardo-Nirenberg inequality (1.0.3), (2.3.20) and (2.3.48) that

$$\begin{aligned}
&(\int \rho|\dot u|^2dx)_t + \mu\int|\nabla\dot u|^2dx + (\mu+\lambda)\int|\mathrm{div}\dot u|^2dx\\
&\le C(\|\nabla u\|_{L^4}^4 + \|u\|_{L^4}^4) + C(\bar\rho)\|\nabla u\|_{L^2}^2\\
&+ C\int(|u||\nabla u||\dot u| + |\nabla u|^2|\dot u|)dx\\
&\le C\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}^3 + C\|\nabla u\|_{L^2}^3 + C\|\nabla u\|_{L^2}^4\\
&+ C\int(|u||\nabla u||\dot u| + |\nabla u|^2|\dot u|)dx + C\\
&\le C(\|F\|_{L^6}^3 + \|\omega\|_{L^6}^3 + \|P-P(\tilde\rho)\|_{L^6}^3) + \delta\|\nabla\dot u\|_{L^2}^2 + C\|\nabla u\|_{L^3}^3 + C\\
&\le C(\|\nabla F\|_{L^2}^3 + \|\nabla\omega\|_{L^2}^3) + \delta\|\nabla\dot u\|_{L^2}^2 + C\|\nabla u\|_{L^3}^3 + C\\
&\le C\|\rho\dot u\|_{L^2}^3 + \delta\|\nabla\dot u\|_{L^2}^2 + C\|\nabla u\|_{L^3}^3 + C\\
&\le C\|\rho^{1/2}\dot u\|_{L^2}^4 + \delta\|\nabla\dot u\|_{L^2}^2 + C\|\nabla u\|_{L^3}^3 + C.
\end{aligned} \tag{2.3.56}$$

Note that

$$\begin{aligned}
&\int_0^T\|\nabla u\|_{L^3}^3ds \le \int_0^{\sigma(T)}\|\nabla u\|_{L^3}^3ds + \int_{\sigma(T)}^T\sigma^3\|\nabla u\|_{L^3}^3ds\\
&\le C + \int_0^{\sigma(T)}\|\nabla u\|_{L^2}^{3/2}(\|\rho\dot u\|_{L^2}^{3/2} + C_0^{1/4} + C_0^{3/4} + \|\nabla u\|_{L^2}^{3/2})\\
&\le C.
\end{aligned} \tag{2.3.57}$$

Bearing in mind that the compatibility condition (2.1.10), we can define

$$\sqrt{\rho}\dot{u}(x,t=0)=\sqrt{\rho_0}g. \tag{2.3.58}$$

Choosing δ small enough, then (2.3.54) follows from Gronwall's inequality, (2.3.48), (2.3.56), (2.3.57) and (2.3.58).

Next, we will prove (2.3.55). For $2\le p\le 6$, $|\nabla\rho|^p$ satisfies

$$\begin{aligned}&(|\nabla\rho|^p)_t+\mathrm{div}(|\nabla\rho|^p u)+(p-1)|\nabla\rho|^p\mathrm{div}u\\&+p|\nabla\rho|^{p-2}(\nabla\rho)^t\nabla u(\nabla\rho)+p\rho|\nabla\rho|^{p-2}\nabla\rho\nabla\mathrm{div}u=0.\end{aligned}$$

Hence,

$$\begin{aligned}\partial_t\|\nabla\rho\|_{L^p}&\le C(1+\|\nabla u\|_{L^\infty})(\|\nabla\rho\|)_{L^p}+C\|\nabla^2u\|_{L^p}\\&\le C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^p}+C(\|\rho\dot{u}\|_{L^p}+\|\nabla P\|_{L^p}),\end{aligned} \tag{2.3.59}$$

due to

$$\|\nabla^2u\|_{L^p}\le C(\|\rho\dot{u}\|_{L^p}+\|\nabla P\|_{L^p}), \tag{2.3.60}$$

which follows from the standard L^p estimate for the elliptic system

$$-\mu\triangle u-(\mu+\lambda)\nabla\mathrm{div}u=-\rho\dot{u}-\nabla P,\quad \text{in}\quad \Omega,$$

$$(u_1,u_2,u_3)=\beta(u^1_{x_3},u^2_{x_3},0),\quad \text{in}\quad \partial\Omega.$$

Hence we need to estimate $\|\nabla u\|_{L^\infty}$.

Let $w=u-v$, where v satisfies

$$\begin{cases}-\mu\triangle v-(\mu+\lambda)\nabla\mathrm{div}v=-\nabla(P(\rho)-P(\tilde{\rho})),\quad \text{in}\quad\Omega\\ (v^1(x),v^2(x),v^3(x))=\beta(v^1_{x_3}(x),v^2_{x_3}(x),0),\quad\text{on}\quad\partial\Omega\end{cases} \tag{2.3.61}$$

then by the standard regularity estimate for elliptic systems, we have

$$\|\nabla v\|_{L^q}\le C\|P(\rho)-P(\tilde{\rho})\|_{L^q},\quad \|\nabla^2v\|_{L^q}\le C\|\nabla(P-P(\tilde{\rho}))\|_{L^q},\quad\text{for}\quad q\in[2,\infty) \tag{2.3.62}$$

and w satisfies

$$\begin{cases} -\mu\triangle w-(\mu+\lambda)\nabla \mathrm{div} w=\rho\dot{u}, \quad \text{in} \quad \Omega \\ (w^1(x),w^2(x),w^3(x))=\beta(w^1_{x_3}(x),w^2_{x_3}(x),0), \quad \text{on} \quad \partial\Omega \end{cases} \tag{2.3.63}$$

then using the standard regularity estimate for elliptic systems again, we have

$$\|\nabla^2 w\|_{L^q}\le C\|\rho\dot{u}\|_{L^q}, \quad \text{for} \quad q\in(1,\infty). \tag{2.3.64}$$

From the Sobolev's embedding theorem, we get

$$\|\nabla w\|_{L^\infty}\le C(\|\rho\dot{u}\|_{L^2}+\|\rho\dot{u}\|_{L^6}). \tag{2.3.65}$$

Now, we give the estimate for $\|\nabla v\|_{L^\infty}$ which is crucial to obtain the estimate of $\|\nabla\rho\|_{L^q}$. We have the similar results for half-space problem as in [69].

Lemma 2.3.9 *Let $\Omega=\{x\in\mathbb{R}^3: x_3>0\}$ and $\nabla v\in W^{1,q}(\Omega)$ with $q\in(3,\infty)$. There exists a constant C depending only on q such that*

$$\|\nabla v\|_{L^\infty}\le C(1+\ln(e+\|\nabla^2 v\|_{L^q})\|\nabla v\|_{BMO}), \quad \text{with} \quad q\in(3,\infty), \tag{2.3.66}$$

here

$$\|\nabla v\|_{BMO}=\|\nabla v\|_{L^2}+[\nabla v]_{BMO},$$
$$[\nabla v]_{BMO}=\sup_{r>0,x\in\Omega}\frac{1}{\Omega_r(x)}\int_{\Omega_r(x)}|\nabla v(y)-\nabla v_r(x)|dy,$$
$$\nabla v_r(x)=\frac{1}{|\Omega_r(x)|}\int_{\Omega_r(x)}\nabla v(y)dy,$$

where $\Omega_r(x)=\Omega\cap B_r(x)$, $B_r(x)$ is the ball with center x and radius r.

Proof: We know that there exists constant $A\ge 1$ such that for any $r>0$ and $x\in\Omega$,

$$|\Omega_r(x)|\le|B_r(x)|\le A|\Omega_r(x)|.$$

First, for $r\ge\frac{1}{2}r_0$, where $r_0\ge 1$, we have

$$|\nabla v_{r(x)}|\le\frac{1}{|\Omega_r(x)|}\int_{\Omega_r(x)}|\nabla v(y)|dy\le C\|\nabla v\|_{L^2}.$$

If $r \le \frac{1}{2} r_0$, then there exists some integer $k \ge 1$ such that

$$\frac{r_0}{2^{k+1}} \le r \le \frac{r_0}{2^k}, \quad k \le C(1 + |\ln r|).$$

Denoting $\Omega_j = \Omega_{2^j r}(x)$ for $j = 0, 1, \cdots, k$, we have

$$\begin{aligned}
|\nabla v_{r(x)}| &\le \sum_{j=1}^{k} |\nabla v_{j-1} - \nabla v_j| + |\nabla v_k| \\
&\le \sum_{j=1}^{k} \oint_{\Omega_{j-1}} |\nabla v(y) - \nabla v_j| dy + C\|\nabla v\|_{L^2} \\
&\le 2^N A \sum_{j=1}^{k} \oint_{\Omega_j} |\nabla v(y) - \nabla v_j| dy + C\|\nabla v\|_{L^2} \\
&\le Ck[\nabla v]_{BMO(\Omega)} + C\|\nabla v\|_{L^2} \le C(1 + |\ln r|)\|\nabla v\|_{BMO(\Omega)}.
\end{aligned}$$

From the Sobolev's embedding theorem, we have for small enough $\epsilon > 0$:

$$|\nabla v| \le |\nabla v(x) - \nabla v_{\epsilon(x)}| + |\nabla v_{\epsilon(x)}| \le C(\epsilon^{1-\frac{N}{q}} \|\nabla^2 v\|_{L^q} + (1 + |\ln \epsilon|)\|\nabla v\|_{BMO(\Omega)}).$$

By choosing suitable ϵ yields (2.3.66). □

By using of the classical theory for elliptic systems, we have

$$\|\nabla v\|_{BMO} \le C(\|\rho\|_{L^\infty(\Omega)} + \|\rho - \tilde{\rho}\|_{L^2(\Omega)}) \le C(\bar{\rho}). \tag{2.3.67}$$

Combining (2.3.66) and (2.3.67), yields

$$\|\nabla v\|_{L^\infty} \le C(1 + \ln(e + \|\nabla^2 v\|_{L^q})). \tag{2.3.68}$$

From (2.3.54) and (2.3.59), we have

$$\begin{aligned}
&\partial_t \|\nabla \rho\|_{L^q} \le C(1 + \|\nabla u\|_{L^\infty})\|\nabla \rho\|_{L^q} + C\|\nabla^2 u\|_{L^q} \\
&\le C(1 + \|\nabla \omega\|_{L^\infty} + \|\nabla v\|_{L^\infty})\|\nabla \rho\|_{L^q} + C(\|\rho \dot{u}\|_{L^q} + \|\nabla(P - P(\tilde{\rho})\|_{L^q}) \\
&\le C(\bar{\rho})(1 + \|\nabla \omega\|_{L^\infty} + \ln(e + \|\nabla^2 v\|_{L^q}))\|\nabla \rho\|_{L^q} + C(\|\rho \dot{u}\|_{L^q} + \|\nabla(P - P(\tilde{\rho})\|_{L^q}) \\
&\le C(\bar{\rho})(1 + \|\rho \dot{u}\|_{L^6} + \ln(e + \|\nabla \rho\|_{L^q}))\|\nabla \rho\|_{L^q} + C\|\rho \dot{u}\|_{L^q}.
\end{aligned} \tag{2.3.69}$$

Taking $q = 6$, and setting

$$f(t) \triangleq e + \|\nabla\rho\|_{L^6}, \quad g(t) = 1 + \|\rho\dot{u}\|_{L^6},$$

one gets

$$f'(t) \le Cf(t)g(t) + Cg(t)\ln f(t)f(t) + Cg(t),$$

which yields

$$(\ln f(t))' \le Cg(t) + Cg(t)\ln f(t), \tag{2.3.70}$$

due to $f(t) > 1$. Making use of (2.3.54), we have

$$\int_0^T g(t)dt \le C\int_0^T (1 + \|\rho\dot{u}\|_{L^6})dt \le C\int_0^T (1 + \|\nabla\dot{u}\|_{L^2})dt \le C, \tag{2.3.71}$$

which together with (2.3.70) and Gronwall's inequality, we obtain that

$$\sup_{0\le t\le T} f(t) \le C,$$

that is

$$\sup_{0\le t\le T} \|\nabla\rho\|_{L^6} \le C. \tag{2.3.72}$$

As a consequence of (2.3.68), (2.3.69), (2.3.71) and (2.3.72), one obtains

$$\int_0^T \|\nabla u\|_{L^\infty} dt \le C. \tag{2.3.73}$$

Next, taking $p = 2$ in (2.3.59), and using (2.3.48), (2.3.73) and Gronwall's inequality, one gets

$$\sup_{0\le t\le T} \|\nabla\rho\|_{L^2} \le C,$$

which, together with (2.3.48), (2.3.54), (2.3.60), (2.3.72) and (2.3.73), gives (2.3.55). We finish the proof of Lemma 2.3.9. □

In the following Lemmas 2.3.10-2.3.13, we will obtain the high order estimates of solutions which are needed to guarantee the extension of the local classical solution to be a global one.

Lemma 2.3.10 *The following estimates hold*

$$\sup_{0\le t\le T}\int \rho|u_t|^2dx + \int_0^T\int |\nabla u_t|^2dxdt \le C, \tag{2.3.74}$$

$$\sup_{0\le t\le T}(\|\rho-\tilde\rho\|_{H^2} + \|P(\rho)-P(\tilde\rho)\|_{H^2}) \le C. \tag{2.3.75}$$

Proof: Since

$$\begin{aligned}\int \rho|u_t|^2dx &\le \int \rho|\dot u|^2dx + \int \rho|u\cdot\nabla u|^2dx\\ &\le C + C\|\rho^{1/2}u\|_{L^2}\|u\|_{L^6}\|\nabla u\|_{L^6}^2\\ &\le C,\end{aligned}$$

and

$$\begin{aligned}\|\nabla u_t\|_{L^2}^2 &\le \|\nabla\dot u\|_{L^2}^2 + \|\nabla(u\cdot\nabla u)\|_{L^2}^2\\ &\le \|\nabla\dot u\|_{L^2}^2 + C\|u\|_{L^\infty}^2\|\nabla^2u\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4\\ &\le \|\nabla\dot u\|_{L^2}^2 + C,\end{aligned}$$

due to Lemma 2.3.9.

Now we proof (2.3.75). Note that P satisfies

$$P_t + u\cdot\nabla P + \gamma P\text{div}u = 0, \tag{2.3.76}$$

using (2.1.1), we have

$$\begin{aligned}&\frac{d}{dt}(\|\nabla^2P\|_{L^2}^2 + \|\nabla^2\rho\|_{L^2}^2)\\ &\le C(1+\|\nabla u\|_{L^\infty})(\|\nabla^2P\|_{L^2}^2 + \|\nabla^2\rho\|_{L^2}^2) + C\|\nabla u\|_{H^2}^2 + C.\end{aligned} \tag{2.3.77}$$

From (2.3.27), (2.3.28) and the standard elliptic regularity estimates, it holds

$$\|\nabla u\|_{H^2} \le C(\|F\|_{H^2} + \|\omega\|_{H^2} + \|P-P(\tilde\rho)\|_{H^2}).$$

So we need to estimate $\|F\|_{H^2}$ and $\|\omega\|_{H^2}$. Using the same idea as in (2.3.29) –

(2.3.32), we will have

$$\begin{aligned}
&\|F\|_{H^2} + \|\omega\|_{H^2} + \|P - P(\tilde{\rho})\|_{H^2} \\
&\leq C(\|\rho\dot{u}\|_{H^1} + \|F\|_{H^1} + \|\omega\|_{H^1} + \|P - P(\tilde{\rho})\|_{H^1} + \|\nabla^2 P\|_{L^2}) \\
&\leq C(1 + \|\rho\dot{u}\|_{L^2} + \|\nabla(\rho\dot{u})\|_{L^2} + \|\nabla^2 P\|_{L^2}) \\
&\leq C(1 + \|\nabla\rho\|_{L^3}\|\dot{u}\|_{L^6} + \|\nabla\dot{u}\|_{L^2} + \|\nabla^2 P\|_{L^2}),
\end{aligned}$$

which, together with (2.3.77), Lemma 2.3.9, and Gronwall's inequality, yields

$$\sup_{0\leq t\leq T}(\|\nabla^2 P\|_{L^2} + \|\nabla^2\rho\|_{L^2}) \leq C.$$

Hence, we finished the proof of Lemma 2.3.10. □

Lemma 2.3.11 *The following estimates hold*

$$\sup_{0\leq t\leq T}(\|\rho_t\|_{H^1} + \|P_t\|_{H^1}) + \int_0^T (\|\rho_{tt}\|_{L^2} + \|P_{tt}\|_{L^2})dt \leq C, \tag{2.3.78}$$

$$\sup_{0\leq t\leq T}\int |\nabla u_t|^2 dx + \int_0^T\int \rho u_{tt}^2 dxdt \leq C. \tag{2.3.79}$$

Proof: From (2.3.55) and (2.3.76), we have

$$\|P_t\|_{L^2} \leq C\|u\|_{L^\infty}\|\nabla P\|_{L^2} + C\|\nabla u\|_{L^2} \leq C. \tag{2.3.80}$$

Differentiating (2.3.76), we obtain

$$\nabla P_t + u\cdot\nabla\nabla P + \nabla u\cdot\nabla P + \gamma\nabla P \mathrm{div} u + \gamma P\nabla\mathrm{div} u = 0.$$

Hence, by (2.3.55) and (2.3.75), one gets

$$\|\nabla P_t\|_{L^2} \leq C\|u\|_{L^\infty}\|\nabla^2 P\|_{L^2} + \|\nabla u\|_{L^3}\|\nabla P\|_{L^6} + C\|\nabla^2 u\|_{L^2} \leq C, \tag{2.3.81}$$

then, (2.3.80) and (2.3.81) imply that

$$\sup_{0\leq t\leq T}\|P_t\|_{H^1} \leq C. \tag{2.3.82}$$

Differentiating (2.3.76) with respect to t, we have

$$P_{tt} + \gamma P_t \mathrm{div} u + \gamma P \mathrm{div} u_t + u_t \cdot \nabla P_t = 0. \qquad (2.3.83)$$

By using of (2.3.55), (2.3.74), (2.3.82) and (2.3.83), it holds that

$$\begin{aligned}
&\int_0^T \|P_{tt}\|_{L^2}^2 dt \\
&\le C \int_0^T (\|P_t\|_{L^6}\|\nabla u\|_{L^3} + \|\nabla u_t\|_{L^2} + \|u_t\|_{L^6}\|\nabla P\|_{L^3} + \|\nabla P_t\|_{L^2})^2 dt \\
&\le C.
\end{aligned}$$

Using the same method, we can obtain the similar estimate of ρ_t and ρ_{tt}.

Next, we prove (2.3.79). Differentiating (2.1.2) with respect to t, multiplying the resulting equation by u_{tt}, one gets after integrating by parts

$$\begin{cases}
\frac{d}{dt}\int (\frac{\mu}{2}|\nabla u_t|^2 + \frac{\lambda+\mu}{2}|\mathrm{div} u_t|^2)dx + \int \rho |u_{tt}|^2 dx + \frac{d}{dt}\int_{\partial\Omega} \beta^{-1}|u_t|^2 dS_x \\
= \frac{d}{dt}(-\frac{1}{2}\int \rho_t |u_t|^2 dx - \int \rho_t u \cdot \nabla u \cdot u_t dx + \int P_t \mathrm{div} u_t dx) \\
+ \frac{1}{2}\int \rho_{tt}|u_{tt}|^2 dx + \int (\rho u \cdot \nabla u)_t u_t dx - \int \rho u_t \cdot \nabla u \cdot u_{tt} dx \\
- \int \rho u \cdot \nabla u_t \cdot u_{tt} dx - \int P_{tt} \mathrm{div} u_t dx \\
\triangleq \frac{d}{dt} J_0 + \sum_{i=1}^{5} J_i.
\end{cases} \qquad (2.3.84)$$

It follows from (2.1.1), (2.3.55), (2.3.74) and (2.3.78), that

$$\begin{aligned}
|J_0| &= |-\frac{1}{2}\int \rho_t |u_t|^2 dx - \int \rho_t u \cdot \nabla u u_t dx + \int P_t \mathrm{div} u_t dx| \\
&\le |\int \mathrm{div}(\rho u)|u_t|^2 dx| + C\|\rho_t\|_{L^3}\|u \cdot \nabla u\|_{L^2}\|u_t\|_{L^6} + C\|P_t\|_{L^2}\|\nabla u_t\|_{L^2} \\
&\le C\int \rho |u||u_t||\nabla u_t| dx + C\|\nabla u_t\|_{L^2} \\
&\le C\|u\|_{L^6}\|\rho^{1/2}u_t\|_{L^2}^{1/2}\|u_t\|_{L^6}^{1/2}\|\nabla u_t\|_{L^2} + C\|\nabla u_t\|_{L^2} \\
&\le \delta\|\nabla u_t\|_{L^2}^2 + C_\delta,
\end{aligned} \qquad (2.3.85)$$

$$
\begin{aligned}
2|J_1| &= |\int \rho_{tt}|u_t|^2 dx| \\
&= |\int (\rho_t u + \rho u_t)\cdot \nabla(|u_t|^2) dx| \\
&\le C(\|\rho_t\|_{L^3}\|u\|_{L^\infty} + \|\rho^{1/2}u_t\|_{L^2}^{1/2}\|u_t\|_{L^6}^{1/2})\|u_t\|_{L^6}\|\nabla u_t\|_{L^2} \\
&\le C\|\nabla u_t\|_{L^2}^4 + C,
\end{aligned}
\tag{2.3.86}
$$

and

$$
\begin{aligned}
|J_2| &= |\int (\rho_t u\cdot\nabla u)_t \cdot u_t dx| \\
&= |\int (\rho_{tt} u\cdot\nabla u\cdot u_t + \rho_t u_t\cdot\nabla u\cdot u_t + \rho_t u\cdot\nabla u_t\cdot u_t) dx| \\
&\le \|\rho_{tt}\|_{L^2}\|u\cdot\nabla u\|_{L^3}\|u_t\|_{L^6} + \|\rho_t\|_{L^2}\||u_t|^2\|_{L^3}\|\nabla u\|_{L^6} \\
&+ \|\rho_t\|_{L^3}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\|u_t\|_{L^6} \\
&\le C\|\rho_{tt}\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2.
\end{aligned}
\tag{2.3.87}
$$

$$
\begin{aligned}
|J_3| + |J_4| &= |\int \rho u_t\cdot\nabla u\cdot u_{tt} dx| + |\int \rho u\cdot\nabla u_t\cdot u_{tt} dx| \\
&\le C\|\rho^{1/2}u_{tt}\|_{L^2}(\|u_t\|_{L^6}\|\nabla u\|_{L^3} + \|u\|_{L^\infty}\|\nabla u_t\|_{L^2}^2) \\
&\le \delta\|\rho^{1/2}u_{tt}\|_{L^2}^2 + C_\delta\|\nabla u_t\|_{L^2}^2,
\end{aligned}
\tag{2.3.88}
$$

and

$$
\begin{aligned}
|J_5| &= |\int P_{tt}\mathrm{div} u_t dx| \\
&\le \|P_{tt}\|_{L^2}\cdot\|\mathrm{div} u_t\|_{L^2} \\
&\le C\|P_{tt}\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2.
\end{aligned}
\tag{2.3.89}
$$

By using of the similar argument as in (2.3.26), we have $\sup\limits_{[0,T^*]}\|u_t\|_{L^2} \le C$ from $\nabla u_t \in L^\infty(0,\infty;L^2)$.

Due to the regularity of the local solution, (2.2.1), $t\nabla u_t \in C([0,T^*];L^2)$. Thus

$$
\|\nabla u_t(\cdot, T^*/2)\|_{L^2} \le \frac{2}{T^*}\|t\nabla u_t\|_{L^\infty(0,T^*;L^2)} \le C,
\tag{2.3.90}
$$

where C may depends on $\|\nabla g\|_{L^2}$.

Combining all the estimates (2.3.85) – (2.3.90), one deduces from (2.3.74), (2.3.78), (2.3.84) and Gronwall's inequality that

$$\sup_{T^*/2\le t\le T}\|\nabla u_t\|_{L^2}+\int_{T^*/2}^{T}\int\rho|u_{tt}|^2dxdt\le C. \tag{2.3.91}$$

On the other hand, (2.2.1) gives the estimate

$$\sup_{0\le t\le T^*/2}\|\nabla u_t\|_{L^2}+\int_{0}^{T^*/2}\int\rho|u_{tt}|^2dxdt\le C. \tag{2.3.92}$$

Now, we complete the proof of Lemma 2.3.11. □

Lemma 2.3.12 *We can obtain the following estimates*

$$\sup_{0\le t\le T}(\|\rho-\tilde\rho\|_{H^3}+\|P-P(\tilde\rho)\|_{H^3})\le C, \tag{2.3.93}$$

$$\sup_{0\le t\le T}(\|\nabla u_t\|_{L^2}+\|\nabla u\|_{H^2})+\int_0^T(\|\nabla u\|_{H^3}^2+\|\nabla u_t\|_{H^1}^2)dt\le C. \tag{2.3.94}$$

Proof: It follows from (2.3.55) and (2.3.79) that

$$\begin{aligned}\|\nabla(\rho\dot u)\|_{L^2}&\le\||\nabla\rho||u_t|\|_{L^2}+\||\nabla\rho||u||\nabla u|\|_{L^2}+\|\rho\nabla u_t\|_{L^2}\\&\quad+\|\rho|\nabla u|^2\|_{L^2}+\|\rho|u||\nabla^2u|\|_{L^2}\\&\le\|\nabla\rho\|_{L^3}\|u_t\|_{L^6}+C\|\nabla\rho\|_{L^3}\|u\|_{L^\infty}\|\nabla u\|_{L^6}+C\|\nabla u_t\|_{L^2}\\&\quad+C\|\nabla u\|_{L^3}\|\nabla u\|_{L^6}+C\|u\|_{L^\infty}\|\nabla^2u\|_{L^2}\le C,\end{aligned}$$

which together with (2.3.54) yields

$$\sup_{0\le t\le T}\|\rho\dot u\|_{H^1}\le C. \tag{2.3.95}$$

On the other hand, we have

$$\begin{aligned}\|\nabla^2u\|_{H^1}&\le C\|\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u\|_{H^1}\\&=C\|\rho\dot u+\nabla P\|_{H^1}\\&\le C(\|\rho\dot u\|_{H^1}+\|\nabla P\|_{H^1})\le C,\end{aligned} \tag{2.3.96}$$

due to (2.1.2), (2.3.75) and (2.3.95). Combining (2.3.55) and (2.3.96), one has

$$\sup_{0\le t\le T}\|\nabla u\|_{H^2}\le C. \tag{2.3.97}$$

Therefore, by the standard L^2-estimate for elliptic system, (2.3.55), and Lemma 2.3.11, we have

$$\begin{aligned}\|\nabla^2 u_t\| &\le C\|\mu\triangle u_t+(\mu+\lambda)\nabla \mathrm{div} u_t\|_{L^2}\\ &\le C\|\rho u_{tt}+\rho_t u_t+\rho_t u\cdot\nabla u+\rho u_t\cdot\nabla u+\rho u\cdot\nabla u_t+\nabla P_t\|_{L^2}\\ &\le C(\|\rho u_{tt}\|_{L^2}+\|\rho_t\|_{L^3}\|u_t\|_{L^6}+\|\rho_t\|_{L^3}\|u\|_{L^\infty}\|\nabla u\|_{L^6})\\ &+C(\|u_t\|_{L^6}\|\nabla u\|_{L^3}+\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}+\|\nabla P_t\|_{L^2})\\ &\le C\|\rho u_{tt}\|_{L^2}+C,\end{aligned} \tag{2.3.98}$$

from (2.3.79), we obtain

$$\int_0^T\|\nabla u_t\|_{H^1}^2dt\le C. \tag{2.3.99}$$

Applying the standard H^2-estimate for elliptic system again leads to

$$\begin{aligned}\|\nabla^2 u\|_{H^2}&\le C\|\mu\triangle u+(\mu+\lambda)\nabla\mathrm{div}u\|_{H^2}\\ &\le C\|\rho\dot{u}\|_{H^2}+C\|\nabla P\|_{H^2}\\ &\le C+C\|\nabla u_t\|_{H^1}+C\|\nabla^3 P\|_{L^2},\end{aligned} \tag{2.3.100}$$

where one has used (2.3.96) and the following estimates:

$$\begin{aligned}\|\nabla^2(\rho u_t)\|_{L^2}&\le C(\||\nabla^2\rho||u_t|\|_{L^2}+\||\nabla\rho||\nabla u_t|\|_{L^2}+\|\nabla^2 u_t\|_{L^2})\\ &\le C(\|\nabla^2\rho\|_{L^2}\|\nabla u_t\|_{H^1}+\|\nabla\rho\|_{L^3}\|\nabla u_t\|_{L^6}+\|\nabla^2 u_t\|_{L^2})\\ &\le C+C\|\nabla u_t\|_{H^1},\end{aligned}$$

and

$$\begin{aligned}\|\nabla^2(\rho u\cdot\nabla u)\|_{L^2}&\le C(\||\nabla^2(\rho u)||\nabla u|\|_{L^2}+\||\nabla(\rho u)||\nabla^2 u|\|_{L^2}+\|\nabla^3 u\|_{L^2})\\ &\le C(1+\|\nabla^2(\rho u)\|_{L^2}\|\nabla u\|_{H^2}+\|\nabla(\rho u)\|_{L^3}\|\nabla^2 u\|_{L^6})\\ &\le C(1+\|\nabla^2\rho\|_{L^2}\|u\|_{L^\infty}+\|\nabla\rho\|_{L^6}\|\nabla u\|_{L^3}+\|\nabla^2 u\|_{L^2})\\ &\le C,\end{aligned}$$

where we have used (2.3.75) and (2.3.97). By using (2.3.75), (2.3.97) and (2.3.100), it holds that

$$\begin{aligned}(\|\nabla^3 P\|_{L^2}^2)_t &\le C(\||\nabla^3 u||\nabla P|\|_{L^2} + \||\nabla^2 u||\nabla^2 P|\|_{L^2} \\ &\quad + \||\nabla u||\nabla^3 P|\|_{L^2} + \|\nabla^4 u\|_{L^2})\|\nabla^3 P\|_{L^2} \\ &\le C(\|\nabla^3 u\|_{L^2}\|\nabla P\|_{H^2} + \|\nabla^2 u\|_{L^3}\|\nabla^2 P\|_{L^6} + \|\nabla u\|_{L^\infty}\|\nabla^3 P\|_{L^2})\|\nabla^3 P\|_{L^2} \\ &\quad + C(1 + \|\nabla^2 u_t\|_{L^2} + \|\nabla^3 P\|_{L^2})\|\nabla^3 P\|_{L^2} \\ &\le C + C\|\nabla u_t\|_{H^1}^2 + C\|\nabla^3 P\|_{L^2}^2,\end{aligned}$$

by Gronwall's inequality and (2.3.99), one gets

$$\sup_{0\le t\le T} \|\nabla^3 P\|_{L^2} \le C. \tag{2.3.101}$$

Combining all the estimates (2.3.99) – (2.3.101) and (2.3.75), one have

$$\sup_{0\le t\le T} \|P - P(\tilde\rho)\|_{H^3} + \int_0^T \|\nabla u\|_{H^3}^2 dt \le C. \tag{2.3.102}$$

Using the similar argument for $\rho - \tilde\rho$, we can also obtain

$$\sup_{0\le t\le T} \|\rho - \tilde\rho\|_{H^3} \le C. \tag{2.3.103}$$

Hence, (2.3.94) follows from (2.3.79), (2.3.97), (2.3.99) and (2.3.102). □

Lemma 2.3.13 *For any $\tau \in (0, T)$, there exists some positive constant $C(\tau)$ such that*

$$\sup_{\tau\le t\le T} (\|\nabla u_t\|_{H^1} + \|\nabla^4 u\|_{L^2}) + \int_\tau^T \int |\nabla u_{tt}|^2 dx dt \le C(\tau). \tag{2.3.104}$$

Proof: Differentiating (2.1.2) with respect to t, we get

$$\begin{aligned}&\rho u_{ttt} + \rho u \cdot \nabla u_{tt} - \mu\Delta u_{tt} - (\mu+\lambda)\nabla \mathrm{div} u_{tt} \\ &= 2\mathrm{div}(\rho u)u_{tt} + \mathrm{div}(\rho u)_t u_t - 2(\rho u)_t \cdot \nabla u_t - (\rho_{tt} u + 2\rho_t u_t)\cdot \nabla u \\ &- \rho u_{tt} \cdot \nabla u - \nabla P_{tt}.\end{aligned} \tag{2.3.105}$$

Multiplying (2.3.105) by u_{tt} and then integrating the resulting equation over Ω, one gets

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|u_{tt}|^2dx + \int (\mu|\nabla u_{tt}|^2 + (\mu+\lambda)(\text{div}u_{tt})^2)dx + \int_{\partial\Omega} \beta^{-1}|u_{tt}|^2 dS_x \\
&= -4\int u^i_{tt}\rho u\cdot\nabla u^i_{tt}dx - \int (\rho u)_t[\nabla(u_t\cdot u_{tt}) + 2\nabla u_t\cdot u_{tt}]dx \\
&-\int (\rho_{tt}u + 2\rho_t u_t)\cdot\nabla u\cdot u_{tt}dx - \int \rho u_{tt}\cdot\nabla u\cdot u_{tt}dx + \int P_{tt}\text{div}u_{tt}dx \triangleq \sum_{i=5}^{5} I_i.
\end{aligned} \tag{2.3.106}$$

We estimate each I_i $(i=1,\cdots,6)$ as follows:

$$\begin{aligned}
|I_1| &\le C\|\rho^{1/2}u_{tt}\|_{L^2}\|\nabla u_{tt}\|_{L^2}\|u\|_{L^\infty} \\
&\le \delta\|\nabla u_{tt}\|^2_{L^2} + C_\delta\|\rho^{1/2}u_{tt}\|^2_{L^2}.
\end{aligned} \tag{2.3.107}$$

It follows from (2.3.74), (2.3.78), (2.3.79) and (2.3.55) that

$$\begin{aligned}
&|I_2| \le C(\|\rho u_t\|_{L^3} + \|\rho_t u\|_{L^3})(\|u_{tt}\|_{L^6}\|\nabla u_t\|_{L^3} + \|\nabla u_{tt}\|_{L^2}\|u_t\|_{L^6}) \\
&\le C(\|\rho^{1/2}u_t\|^{1/2}_{L^2}\|u_t\|^{1/2}_{L^6} + \|\rho_t\|_{L^6}\|u\|_{L^6})\|\nabla u_{tt}\|_{L^2} \\
&\le \delta\|\nabla u_{tt}\|^2_{L^2} + C_\delta.
\end{aligned} \tag{2.3.108}$$

$$\begin{aligned}
|I_3| &\le C(\|\rho_{tt}\|_{L^2}\|u\|_{L^\infty}\|\nabla u\|_{L^3} + \|\rho_t\|_{L^6}\|u_t\|_{L^6}\|\nabla u\|_{L^2})\|u_{tt}\|_{L^6} \\
&\le \delta\|\nabla u_{tt}\|^2_{L^2} + C_\delta\|\rho_{tt}\|^2_{L^2},
\end{aligned} \tag{2.3.109}$$

and

$$\begin{aligned}
|I_4| + |I_5| &\le C(\|\rho u_{tt}\|_{L^2}\|\nabla u\|_{L^3}\|u_{tt}\|_{L^6} + C\|P_{tt}\|_{L^2}\|\nabla u_{tt}\|_{L^2}) \\
&\le \delta\|\nabla u_{tt}\|^2_{L^2} + C_\delta\|\rho^{1/2}u_{tt}\|^2_{L^2} + C_\delta\|P_{tt}\|^2_{L^2}.
\end{aligned} \tag{2.3.110}$$

For any $\tau \in (0,T^*)$, since $t^{1/2}\sqrt{\rho}u_{tt} \in L^\infty(0,T^*;L^2)$ by (2.2.1), there exists some $t_0 \in (\tau/2,\tau)$ such that

$$\int \rho|u_{tt}|^2dx(t_0) \le \frac{1}{t_0}\|t^{1/2}\sqrt{\rho}u_{tt}\|^2_{L^\infty(0,T^*;L^2)} \le C(\tau). \tag{2.3.111}$$

Substituting (2.3.107) – (2.3.110) into (2.3.106), choosing δ suitably small, one obtains by using (2.3.78), (2.3.111) and Gronwall's inequality that

$$\sup_{t_0\le t\le T}\int \rho|u_{tt}|^2dx + \int_{t_0}^{T}\int |\nabla u_{tt}|^2dxdt \le C(\tau),$$

which, together with (2.3.98) and (2.3.79), leads to

$$\sup_{\tau \le t \le T} \|\nabla u_t\|_{H^1} + \int_\tau^T \int |\nabla u_{tt}|^2 dx dt \le C(\tau), \tag{2.3.112}$$

due to $t_0 \le \tau$. Now, (2.3.104) follows from (2.3.100), (2.3.112) and (2.3.93). We finish the proof of Lemma 2.3.13. □

2.4 Proof of Theorem 2.1.1

With all the a priori estimates in section 3, we are now in a position to proof the main result of this paper.

Proof of Theorem 2.1.1. From Lemma 2.2.1, there exists a T^* such that (2.1.1) – (2.1.6) has a unique classical solution (ρ, u) on $(0, T^*]$. We now want to extend the local solution to a global one by using the previous estimates.

First, from

$$A_1(0) + A_2(0) = 0, \quad A_3(0) \le M, \quad \rho(0) \le \bar{\rho},$$

we know that there exists a $T_1 \in (0, T^*]$ such that (2.3.1) holds for $T = T_1$.
Set

$$\tilde{T} = \sup\{T | (2.3.1) \quad \text{holds}\}. \tag{2.4.1}$$

Then $\tilde{T} \ge T_1 > 0$. Hence for $0 \le \tau \le T \le \tilde{T}$ with T finite, it follows from Lemma 2.3.12 and Lemma 2.3.13 that

$$\nabla u_t, \nabla^3 u \in C([\tau, T]; L^2 \cap L^4), \quad \nabla u, \nabla^2 u \in C([\tau, T]; L^2 \cap C(\bar{\Omega})), \tag{2.4.2}$$

where we have used the standard embedding

$$L^\infty(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \hookrightarrow C([\tau, T]; L^q), \quad \text{for} \quad \text{any} \quad q \in [2, 6).$$

By using of (2.3.74), (2.3.79) and (2.3.104), we have

$$\begin{aligned}
&\int_\tau^T \|(\rho|u_t|^2)_t\|_{L^1}dt \le \int_\tau^T (\|\rho_t|u_t|^2\|_{L^1} + 2\|\rho u_t u_{tt}\|_{L^1})dt \\
&\le C\int_\tau^T (\|\rho|\text{div}u||u_t|^2\|_{L^1} + \||u||\nabla\rho||u_t|^2\|_{L^1} + \|\rho^{1/2}u_t\|_{L^2}\|\rho^{1/2}u_{tt}\|_{L^2})dt \\
&\le C\int_\tau^T (\|\rho|u_t|^2\|_{L^1}\|\nabla u\|_{L^\infty} + \|u\|_{L^6}\|\nabla\rho\|_{L^2}\|u_t\|_{L^6}^2 + \|\rho^{1/2}u_{tt}\|_{L^2})dt \\
&\le C,
\end{aligned}$$

which yields

$$\rho^{1/2}u_t \in C([\tau,T];L^2).$$

This, together with (2.4.2), gives

$$\rho^{1/2}\dot{u}, \quad \nabla\dot{u} \in C([\tau,T];L^2).$$

We claim that

$$\tilde{T} = \infty. \tag{2.4.3}$$

Otherwise, $\tilde{T} < \infty$. Then by proposition 2.3.1, (2.3.2) holds for $T = \tilde{T}$. It follows from Lemma 2.3.12, Lemma 2.3.13 and (2.4.2) that $\rho(x,\tilde{T})$, $u(x,\tilde{T})$ satisfies (2.1.9) and (2.1.10) with $g(x) = \dot{u}(x,\tilde{T})$. Then Lemma 2.2.1 implies that there exists $\tilde{T}' > \tilde{T}$ such (2.3.1) holds for $T = \tilde{T}'$, which contradicts (2.4.1). Hence (2.4.3) holds. Lemma 2.2.1, Lemma 2.3.12, Lemma 2.3.13 and (2.4.1) show that (ρ, u) is in fact the unique classical solution defined on $(0,T]$ for any $0 < T < \tilde{T} = \infty$. Finally, in order to finish the proof of Theorem 2.1.1, we need to show (2.1.14).

Multiplying (2.3.39) by $4(P - P(\tilde{\rho}))^3$ and integrating over Ω, one gets

$$\begin{aligned}
&(\|P - P(\tilde{\rho})\|_{L^4}^4)'(t) \\
&= (4\gamma - 1)\int (P - P(\tilde{\rho}))^4 \text{div}u dx - \gamma\int P(\tilde{\rho})(P - P(\tilde{\rho}))^3 \text{div}u dx,
\end{aligned} \tag{2.4.4}$$

integrating the above equality over $(1,\infty)$, we obtain

$$\int_1^\infty |(\|P - P(\tilde{\rho})\|_{L^4}^4)'(t)|dt \le C\int_1^\infty (\|P - P(\tilde{\rho})\|_{L^4}^4 + \|\nabla u\|_{L^4}^4)dt \le C, \tag{2.4.5}$$

due to (2.3.42). Combining (2.3.42) with (2.4.5) leads to

$$\lim_{t\longrightarrow\infty}\|P-P(\tilde\rho)\|_{L^4}=0,$$

which together with (2.3.3) implies

$$\lim_{t\longrightarrow\infty}\int|\rho-\tilde\rho|^q dx=0,$$

for all q satisfying (2.1.14). From (2.3.3), we have

$$\int\rho^{1/2}|u|^4dx\le(\int\rho|u|^2dx)^{1/2}\|u\|_{L^6}^3\le C\|\nabla u\|_{L^2}^3.$$

Thus (2.1.14) follows provided that

$$\lim_{t\longrightarrow\infty}\|\nabla u\|_{L^2}=0. \tag{2.4.6}$$

Setting

$$I(t)\triangleq\frac{\mu}{2}\|\nabla u\|_{L^2}^2+\frac{\mu+\lambda}{2}\|\mathrm{div}u\|_{L^2}^2,$$

choosing $m=0$ in (2.3.6), and using (2.3.8) and (2.3.11), one gets

$$|I'(t)|\le C\int\rho|\dot u|^2dx+C\|\nabla u\|_{L^3}^3+CC_0^{1/2}\|\nabla\dot u\|_{L^2}, \tag{2.4.7}$$

where one has used

$$|\int\dot u\cdot\nabla Pdx|=|\int(P-P(\tilde\rho))\mathrm{div}\dot udx|\le CC_0^{1/2}\|\nabla\dot u\|_{L^2}.$$

We thus deduce from (2.4.7), (2.3.35) and (2.3.42) that

$$\int_1^\infty|I'(t)|^2dt\le C\int_1^\infty(\|\rho^{1/2}\dot u\|_{L^2}^4+\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^4}^4+\|\nabla\dot u\|_{L^2}^2)dt\le C,$$

which together with

$$\int_1^\infty|I(t)|^2dt\le C\int_1^\infty\|\nabla u\|_{L^2}^2dt\le C,$$

implies (2.4.6). The proof of Theorem 2.1.1 is finished. □

Proof of Theorem 2.1.2. Suppose the conclusion is false. Then there exist some constant $N_0 > 0$ and a subsequence $\{t_{n_j}\}_{j=1}^{\infty}$, $t_{n_j} \longrightarrow \infty$ such that $\|\nabla\rho(\cdot, t_{n_j})\|_{L^r} \leq N_0$. From the Gagliardo-Nirenberg inequality, it holds that

$$\|\rho(x, t_{n_j}) - \tilde{\rho}\|_{C(\bar{\Omega})} \leq C\|\nabla\rho(x, t_{n_j})\|_{L^r}^{\theta}\|\rho(x, t_{n_j}) - \tilde{\rho}\|_{L^3}^{1-\theta} \leq CN_0^{\theta}\|\rho(x, t_{n_j}) - \tilde{\rho}\|_{L^3}^{1-\theta}, \tag{2.4.8}$$

where $\theta = \dfrac{r}{2r-3} \in (0, 1)$.
By using of (2.1.14), we know that

$$\|\rho(x, t_{n_j}) - \tilde{\rho}\|_{C(\bar{\Omega})} \longrightarrow 0 \quad \text{as} \quad t_{n_j} \longrightarrow \infty. \tag{2.4.9}$$

On the other hand, since (ρ, u) is a classical solution, thus there exists a unique particle path $x_0(t)$ with $x_0(0) = x_0$ such that

$$\rho(x_0(t), t) \equiv 0 \quad \text{for} \quad \text{all} \quad t \geq 0. \tag{2.4.10}$$

So, we obtain from (2.4.10) that

$$\|\rho(x, t_{n_j}) - \tilde{\rho}\|_{C(\bar{\Omega})} \geq |\rho(x_0(t_{n_j}), t_{n_j}) - \tilde{\rho}| \equiv \tilde{\rho} > 0,$$

which contradicts (2.4.9). This completes the proof of Theorem 2.1.2. □

Chapter 3

Global behavior of spherically symmetric compressible Navier-Stokes system with degenerate viscosity coefficients

In this chapter, we study a free boundary value problem for spherically symmetric compressible Navier-Stokes system with degenerate viscosity coefficients, which include, in particular, a shallow water model. We obtain global existence, uniqueness and large time behavior of weak solution under some assumptions imposed on initial data. The results show that such a system is stable under small perturbations.

3.1 Introduction

In this chapter, we investigate the dynamical behavior for the spherically symmetric Navier-Stokes system with density-dependent viscosity coefficients in $\mathbb{R}^3$, which can be written in Eulerian coordinates as

$$\begin{cases} (r^2\rho)_\tau + (r^2\rho u)_r = 0, \\ \rho u_\tau + \rho u u_r + \partial_r P = \partial_r((\mu(\rho) + \lambda(\rho))(u_r + \frac{2}{r}u)) - 2\frac{u}{r}\partial_r \mu - \rho f_\infty, \end{cases} \tag{3.1.1}$$

for $(r, \tau) \in \Omega_\tau$, with

$$\Omega_\tau = \{(r, \tau) | 0 \le r \le a(\tau),\ \ 0 \le \tau \le \infty\}, \tag{3.1.2}$$

where $P = A\rho^2$, $A > 0$ is a constant and $f_\infty = \dfrac{G}{r^2}\displaystyle\int_0^r \rho s^2 ds$, G is a gravitational constant. Without loss of generality, we assume $P = \rho^2$, $\mu(\rho) = \rho$ and $\lambda(\rho) = 0$. The initial data are

$$(\rho, \rho u)(r, 0) = (\rho_0, m_0)(r) =: (\rho_0, \rho_0 u_0)(r),\ \ r \in (0, a). \tag{3.1.3}$$

The boundary conditions are

$$u|_{r=0} = 0, \quad \rho|_{r=a(\tau)} = 0, \tag{3.1.4}$$

where the free boundary $a(\tau)$ satisfies $a(0) = a$ and $a'(\tau) = u(a(\tau), \tau)$, $\tau > 0$.

Now, we consider the stationary problem, namely

$$(P(\rho_\infty))_r = -\rho_\infty G \frac{\int_0^r \rho_\infty(s) s^2 ds}{r^2} \tag{3.1.5}$$

in an interval $r \in (0, l_\infty)$, with the end l_∞ satisfying

$$\rho_\infty(l_\infty) = 0, \quad \int_0^{l_\infty} \rho_\infty r^2 dr = M := \int_0^a \rho_0 r^2 dr. \tag{3.1.6}$$

The unknown quantities are the stationary density $\rho_\infty \ge 0$ and free boundary $l_\infty > 0$. It is well-known that if $\gamma > \dfrac{2n-2}{n}$, where n denotes the dimension, then there exists a unique solution (ρ_∞, l_∞) to the stationary system (3.1.5) – (3.1.6), satisfying $\rho_\infty \sim (l_\infty^n - r^n)^{\frac{1}{\gamma-1}}$, $(\rho_\infty)_r < 0$, $0 < r < l_\infty$ with $l_\infty < +\infty$.

It is convenient to deal with the free boundary problem (3.1.1) and (3.1.3) – (3.1.4) in Lagrangian coordinates. Define the Lagrangian coordinates transformation

$$x = \int_0^r \rho y^2 dy,\ \ \tau = t, \tag{3.1.7}$$

then the fixed boundary $r = 0$ and free boundary $r = a(\tau)$ become

$$x = 0, \;\; x = \int_0^{a(\tau)} y^2 \rho dy = \int_0^a y^2 \rho_0 dy = M, \tag{3.1.8}$$

where M is the initially total mass. Moreover, the region $\{(r,\tau)|0 \le r \le a(\tau), \tau \ge 0\}$ is transformed into $\{(x,t)|0 \le x \le M, t \ge 0\}$. Under this Lagrangian coordinate, the equation (3.1.1) and (3.1.3) – (3.1.4) are changed to

$$\begin{cases} \rho_t + \rho^2 (r^2 u)_x = 0, \\ u_t + r^2 (\rho^2)_x = r^2 (\rho^2 (r^2 u)_x)_x - 2ur\rho_x - G\dfrac{x}{r^2}, \\ r^3 = 3\displaystyle\int_0^x \rho^{-1}(y,t)dy, \end{cases} \tag{3.1.9}$$

for $(x,t) \in (0,M) \times (0,\infty)$, with the following initial data and boundary conditions

$$(\rho, u)|_{t=0} = (\rho_0, u_0)(x), \;\; r|_{t=0} = r_0(x) = (3\int_0^x \rho_0^{-1}(y)dy)^{\frac{1}{3}}, \tag{3.1.10}$$

$$u|_{x=0} = 0, \;\; \rho|_{x=M} = 0, \;\; t > 0. \tag{3.1.11}$$

It is standard that if we can solve the problem (3.1.9) – (3.1.11), then the free boundary problem (3.1.1) and (3.1.3) – (3.1.4) have a solution.

From (3.1.5) – (3.1.6), it is easy to see that $\rho_\infty(x)$ is the solution to the stationary system

$$\begin{cases} r_\infty^2 (\rho_\infty^2)_x = -G\dfrac{x}{r_\infty^2}, \quad r_\infty^3(x) = 3\displaystyle\int_0^x \rho_\infty^{-1}(y)dy, \quad x \in (0,M), \\ \rho_\infty(M) = 0. \end{cases} \tag{3.1.12}$$

In this chapter, we can show that such a system is stable under small perturbations, it does not develop vacuum states or concentration states for all times, and the free boundary $a(\tau)$ propagates with finite speed.

The assumptions can be stated as follows:

(A_1). $C_1(M-x)^{\frac{1}{2}} \le \rho_0 \le C_2(M-x)^{\frac{1}{2}}$, where C_1 and C_2 are constants.

(A_2). $B_1[\rho_0, r_0] < \infty$, where

$$B_1[\rho, r] = \int_0^M r^{2+\alpha}(M-x)^{\frac{1}{2}}(\rho-\rho_\infty)_x^2 dx, \qquad (3.1.13)$$

and $\alpha \in (-1, 1)$.

(A_3). $B_2[\rho_0, u_0, r_0] < \infty$, $u_0(0) = 0$, where

$$B_2[\rho, u, r] = \int_0^M (\rho^2 r^4 u_x^2 + \frac{u^2}{r^2})dx + \int_0^M r^{\alpha+2}|(\rho^2(r^2u)_x)_x - 2\rho_x \frac{u}{r}|^2 dx. \quad (3.1.14)$$

Under above assumptions $(A_1) - (A_3)$, we will prove the existence of global weak solutions to the initial-boundary value problem (3.1.9) – (3.1.11) in the sense of the following definition.

Definition 3.1.1 *A pair of functions $(\rho, u, r)(x,t)$ is called a global weak solution to the initial-boundary value problem* (3.1.9) – (3.1.11) *if, for any $T > 0$,*

$$\rho, u \in L^\infty([0,M]\times[0,T]) \cap C^1([0,T]; L^2([0,M])),$$

$$r \in C^1([0,T]; L^\infty([0,M])),$$

$$\rho^{-1}, (ru)_x, (r^2)_x \in L^\infty([0,T]; L^1([0,M])),$$

and

$$\rho^2(r^2u)_x \in L^\infty([0,M]\times[0,T]) \cap C^{\frac{1}{2}}([0,T]; L^2([0,M])).$$

Furthermore, the following equations hold:

$$\rho_t + \rho^2(r^2u)_x = 0, \quad \rho(x,0) = \rho_0(x), \quad \textit{almost everywhere},$$

$$r_t = u, r(x,0) = r_0(x), r^3(x,t) = 3\int_0^x \rho^{-1}(y,t)dy, \quad \textit{almost everywhere},$$

$$\int_0^\infty \int_0^M [u\psi_t + (P - \rho^2(r^2u)_x)(r^2\psi)_x + 2\rho(ru\psi)_x - G\frac{x}{r^2}\psi]dxdt + \int_0^M u_0(x)\psi(x,0)dx = 0$$

for any test function $\psi(x,t) \in C_0^\infty(\Omega)$ with $\Omega = \{(x,t)|0 < x \le M, t \ge 0\}$. In what follows, we will use $C(C_i)$ to denote a generic positive constant depending only on the initial data, independent of the given time T.

3.2 Main result

We now state the main theorem in this section.

Theorem 3.2.1 *Under the conditions* $(A_1)-(A_3)$, *there exists a constant* $\epsilon_0 > 0$ *such that if*

$$\|g_0 - g_\infty\|_{L^\infty}^2 + \|\frac{u_0}{r_0}\|_{L^\infty}^2 + B_1[\rho_0, r_0] + B_2[\rho_0, u_0, r_0] \le \epsilon_0^2, \tag{3.2.1}$$

where $g_0(x) = (M-x)^{-\frac{1}{2}}\rho_0(x)$, $g_\infty(x) = (M-x)^{-\frac{1}{2}}\rho_\infty(x)$. *Then the system* $(3.1.9)-(3.1.11)$ *has a unique global weak solution* (ρ, u, r) *satisfying*

$$C^{-1}(M-x)^{\frac{1}{2}} \le \rho(x,t) \le C(M-x)^{\frac{1}{2}}, \tag{3.2.2}$$

$$C^{-1}x \le r^3(x,t) \le Cx, \tag{3.2.3}$$

$$B_1[\rho, r] + B_2[\rho, u, r] \le C, \tag{3.2.4}$$

and

$$\|\frac{u}{r}(\cdot,t)\|_{L^\infty} + \|\rho(r^2u)_x(\cdot,t)\|_{L^\infty} \le C \tag{3.2.5}$$

for all $t \ge 0$ *and* $x \in [0, M]$. *Furthermore*

$$\int_0^M [u^2 + (M-x)^{\frac{1}{2}}(g-g_\infty)^2 + x^{-2}(r^3 - a^3 - r_\infty^3)^2]dx \le C\epsilon_3^2(1+t)^{-1}, \tag{3.2.6}$$

$$\int_0^t\int_0^M (1+s)(\rho^2r^4u_x^2 + \frac{u^2}{r^2})dxds \le C\epsilon_3^2. \tag{3.2.7}$$

Remark 3.2.2 *The uniqueness of the solution in Theorem 3.2.1 means that if* (ρ_1, u_1, r_1) *and* (ρ_2, u_2, r_2) *are two solutions to the system* $(3.1.9)-(3.1.11)$ *with the same initial data* (ρ_0, u_0, r_0) *and satisfy regularity conditions in the theorem, then we have* $(\rho_1, u_1, r_1) = (\rho_2, u_2, r_2)$.

Theorem 3.2.3 *(Continuous Dependence) For each* $i = 1, 2$, *let* (ρ_i, u_i, r_i) *be the solution to the system* $(3.1.9)-(3.1.11)$ *with the initial data* $(\rho_{0i}, u_{0i}, r_{0i})$, *which* *satisfies regularity conditions in Theorem 2.1. Then, we have*

$$\int_0^M [(u_1 - u_2)^2 + (M-x)^{-1}(\rho_1 - \rho_2)^2$$
$$+ x^{-\frac{2}{3}}(r_1 - r_2)^2]dx$$
$$\le Ce^{Ct}\int_0^M [(u_{01} - u_{02})^2 + (M-x)^{-1}(\rho_{01} - \rho_{02})^2$$
$$+ x^{-\frac{2}{3}}(r_{01} - r_{02})^2]dx$$

for all $t \geq 0$.

Remark 3.2.4 *When the space dimension* $n = 2$, *similar results hold for shallow water model.*

3.3 The stationary problem

When $\gamma > \dfrac{2n-2}{n}$, we know that the stationary problem has a unique solution in [79]. Hence we will only give results.

Proposition 3.3.1 *If* $\gamma > \dfrac{2n-2}{n}$, *then the Lagrangian stationary problem* (3.1.12) *has a nonnegative solution* $\rho_\infty \in W^{1,\beta}([0,M])$ *satisfying* $C^{-1}(M-x)^{\frac{1}{\gamma}} \le \rho_\infty(x) \le C(M-x)^{\frac{1}{\gamma}}$, *where* $\beta \in (1, \min\{\frac{n}{n-2}, \frac{\gamma}{\gamma-1}\})$ *is a constant.*

Similar to [82], we say a stationary solution (ρ_∞, r_∞) is a statically stable if

$$\begin{aligned} J[W] := &\int_0^M (\gamma\rho_\infty^{1+\gamma}W_x^2 - (2n-2)Gxr_\infty^{2-3n}W^2dx) \\ \ge\ &\delta_1 \int_0^M ((M-x)^{\frac{1+\gamma}{\gamma}}W_x^2 + x^{-2}W^2)dx \end{aligned} \tag{3.3.1}$$

for some $\delta_1 > 0$ and all $W \in K_1 = \{f \in C([0,M]) | f(0) = 0, \frac{1}{f'} \in K\}$, where $K = \{f \in C([0,M]) | f \ge 0, \|\frac{(M-x)^{\frac{1}{\gamma}}}{f(x)}\|_{L^\infty} < \infty, \|\frac{f(x)}{(M-x)^{\frac{1}{\gamma}}}\|_{L^\infty} < \infty\}$.
Now, the static potential energy takes the following form:

$$S[V] = \int_0^M [\frac{1}{\gamma-1}(V_x)^{1-\gamma} + \int_1^V Gx(nh)^{\frac{2-2n}{n}}dh]dx. \tag{3.3.2}$$

We call $V \in K_1$ is a point of local quadratic minimum of S if

$$S[V+W]-S[V] \geq \delta_2 \int_0^M [(M-x)^{\frac{1+\gamma}{\gamma}} W_x^2 + x^{-2}W^2]dx, \qquad (3.3.3)$$

for all $V \in K_1$ and $\|(M-x)^{\frac{1}{\gamma}} W_x\|_{L^\infty([0,M])} + \|x^{-1}W\|_{L^\infty} \leq \delta_3$, for some $\delta_3 > 0$.

Proposition 3.3.2 *If $\gamma > \dfrac{2n-2}{n}$ and ρ_∞ is a solution of the problem* (2.1.13) *satisfying $\rho_\infty \in W^{1,\beta}([0,M])$ and $C^{-1}(M-x)^{\frac{1}{\gamma}} \leq \rho_\infty(x) \leq C(M-x)^{\frac{1}{\gamma}}$, then we have that* (3.3.2) *and* (3.3.3) *hold with $V = V_\infty = \dfrac{r_\infty^n}{n}$.*

Proposition 3.3.3 *Let ρ_∞ be a solution obtained in Proposition 3.3.1, and ρ_2 be another solution of the problem* (3.1.12) *satisfying $\rho_2 \in W^{1,\beta}([0,M])$ and $C^{-1}(M-x)^{\frac{1}{\gamma}} \leq \rho_2(x) \leq C(M-x)^{\frac{1}{\gamma}}$. If $\gamma > \dfrac{2n-2}{n}$ and $\|(M-x)^{-\frac{1}{\gamma}}(\rho_\infty - \rho_2)(x)\|_{L^\infty} \leq \delta_4$ with a small enough positive constant δ_4, then we have $\rho_2(x) = \rho_\infty(x)$, almost everywhere $x \in [0,M]$.*

Proposition 3.3.4 *If $\gamma > \dfrac{2n-2}{n}$, the Lagrangian stationary problem* (3.1.12) *has a unique solution $\rho_\infty \in K$.*

3.4 Approximate system

In this section, we will construct a sequence approximate solution. First, we can choose a sequence of suitable smooth functions $\{(\rho_{a0}, u_{a0}, r_{a0})\}$ satisfying

$$r_{0a}(x) = (a^3 + 3\int_0^x \rho_{a0}^{-1}(y)dy)^{\frac{1}{n}}, \quad u_{a0}(0) = 0,$$

$$(\rho_{a0}, u_{a0}, r_{a0}) \longrightarrow (\rho_0, u_0, r_0), \quad \text{in } C([0,M]),$$

$$\|g_{a0} - g_\infty\|_{L^\infty} \longrightarrow \|g_0 - g_\infty\|_{L^\infty},$$

$$\|\frac{u_{a0}}{r_{a0}}\|_{L^\infty} \longrightarrow \|\frac{u_0}{r_0}\|_{L^\infty},$$

$$B_1[\rho_{a0}, r_{a0}] \longrightarrow B_1[\rho_0, r_0],$$

$$B_2[\rho_{a0}, u_{a0}, r_{a0}] \longrightarrow B_1[\rho_0, u_0, r_0],$$

as $a \longrightarrow 0$, where $g_{a0} = (M-x)^{-\frac{1}{2}}\rho_{a0}$. Furthermore, we assume that $(\rho_{a0}, u_{a0}, r_{a0})$ satisfies

$$\|g_{a0} - g_\infty\|_{L^\infty}^2 + \|\frac{u_{a0}}{r_{a0}}\|_{L^\infty}^2 + B_1[\rho_{a0}, r_{a0}] + B_2[\rho_{a0}, u_{a0}, r_{a0}] \le 2\epsilon_0^2. \tag{3.4.1}$$

Then, we consider the following system with solid core.

$$\begin{cases} \rho_t = -\rho^2\partial_x(r^2u), \\ u_t = r^2\{\partial_x[\rho^2(r^2u)_x - P] - 2\partial_x\rho\frac{u}{r}\} - G\frac{x}{r^2}, \\ r^3(x,t) = a^3 + 3\int_0^x \rho^{-1}(y,t)dy, \end{cases} \tag{3.4.2}$$

where $(x,t) \in (0,M)\times(0,\infty)$, with the initial data

$$(\rho, u)|_{t=0} = (\rho_{a0}, u_{a0})(x), \quad r|_{t=0} = r_{a0}(x), \tag{3.4.3}$$

and the boundary conditions

$$u|_{x=0} = 0, \quad \rho|_{x=M} = 0, \quad t > 0. \tag{3.4.4}$$

Using the similar arguments as that in [11], we can obtain the following local existence and uniqueness result. We omit the proof.

Theorem 3.4.1 *(Local Result) Under the assumptions in Theorem 3.2.1 and* (3.4.1)*, there is a positive constant $T_1 > 0$ such that the free boundary problem* (3.4.2) – (3.4.4) *admits a unique weak solution $(\rho_a, u_a, r_a)(x,t)$ on $[0,M]\times[0,T_1]$ in the sense that*

$$\rho_a(x,t), u_a(x,t), r_a(x,t) \in L^\infty([0,M]\times[0,T_1]) \cap C^1([0,T_1]; L^2([0,M])),$$

$$\rho_a^2\partial_x(r_a^2u_a) \in L^\infty([0,M]\times[0,T_1]) \cap C^{\frac{1}{2}}([0,T_1]; L^2([0,M])),$$

$$\partial_x(r_a^2), \partial_x(ru) \in L^\infty([0,T_1], L^1([0,M])),$$

and following equations hold:

$$\partial_t\rho_a = -\rho_a^2\partial_x(r_a^2u_a), \quad \rho_a(x,0) = \rho_{a0},$$

$$\partial_t r_a(x,t) = u_a(x,t), \quad r_a^3 = a^3 + 3\int_0^x \rho_a^{-1}(y,t)dy, \tag{3.4.5}$$

$$\rho_a^2(r^2 u_a)_x = \rho_a^2 + 2\rho_a \frac{u_a}{r_a} + \int_x^M \{-\frac{(u_a)_t}{r_a^2} + 2\rho_a(\frac{u_a}{r_a})_x - Gyr_a^{-4}\}dy \tag{3.4.6}$$

for almost all $x \in [0, M]$, any $t \in [0, T_1]$,

$$\int_0^\infty \int_0^M [u_a \psi_t + (\rho_a^2 - \rho_a^2(r_a^2 u_a)_x(r_a^2\psi)_x$$
$$+ 2\rho_a(r_a u\psi)_x - Gxr_x^{-2}\psi]dxdt + \int_0^M u_{a0}(x)\psi(x,0)dx = 0$$

for any text function $\psi(x,t) \in C_0^\infty([0, M] \times [0, T_1))$. Furthermore, we have

$$N_1(M-x)^{\frac{1}{2}} \le \rho_a(x,t) \le N_2(M-x)^{\frac{1}{2}}, \quad (x,t) \in [0,M] \times [0,T_1], \tag{3.4.7}$$

$$(M-x)^{-\frac{1}{2}}\rho_a \in C([0,T_1]; L^\infty([0,M])). \tag{3.4.8}$$

$$(M-x)^{\frac{1}{4}}(\rho_a)_x, \quad (\rho_a)_t, \quad (u_a)_t \in L^\infty([0,T_1]; L^2[0,M]), \tag{3.4.9}$$

$$\rho_a(u_a)_{xt} \in L^2([0,M] \times [0,T_1]), \quad \rho_a \partial_x u_a \in L^\infty([0,M] \times [0,T_1]), \tag{3.4.10}$$

where N_1 and N_2 are positive constants.

Assume the maximum existence time of the weak solution of Theorem 3.4.1 is T^*. We can extend the existence interval by obtaining the following a priori estimates under suitable assumptions. In the following, we may assume that $(\rho_a, u_a, r_a)(x,t)$ is suitably smooth. Since all the argument used here can be applied to the weak solution in terms of the Friedrich's regularizing approximation. Throughout this chapter, $C(C_i)$ denotes a generic positive constant independent of the given time T and a. For simplicity, we omit the subscripts a in (ρ_a, u_a, r_a) and $(\rho_{a0}, u_{a0}, r_{a0})$ from now on.

3.5 A priori estimates

From Proposition 3.3.1 and (3.1.12), we can obtain the following Lemma easily.

Lemma 3.5.1 *Under the assumptions of Theorem 3.2.1, we have*

$$\rho_\infty^2(x) = \int_x^M G\frac{y}{r_\infty^4}dy \tag{3.5.1}$$

$$C^{-1}(M-x)^{\frac{1}{2}} \le \rho_\infty(x) \le C(M-x)^{\frac{1}{2}},\ C^{-1}x \le r_\infty^3 \le Cx, \tag{3.5.2}$$

$$\frac{d}{dx}(\rho_\infty^2(x)) = -G\frac{x}{r_\infty^4},\ C^{-1} \le (M-x)^{\frac{1}{2}}\frac{d}{dx}\rho_\infty(x) \le C, \tag{3.5.3}$$

for all $x \in [0, M]$.

Lemma 3.5.2 *Under the assumptions of Theorem 3.2.1, we have*

$$\frac{d}{dt}\int_0^M (\frac{1}{2}u^2 + \rho + \int_1^r \frac{Gx}{s^2}ds)dx + \int_0^M \rho^2 r^4 u_x^2 dx + \int_0^M \frac{u^2}{r^2}dx = 0. \tag{3.5.4}$$

Proof: Multiply $(3.1.9)_2$ by u, integrating over $[0, M]$ and using the boundary condition (3.1.11), we can obtain (3.5.4) easily.

□

Now, we define

$$I(t) = \|g - g_\infty\|_{L_x^\infty} + \|\frac{u}{r}\|_{L_x^\infty} + (1+t)^{\frac{1}{16}}\|x^{\frac{2+\alpha}{24}}(\rho^{\frac{3}{8}} - \rho_\infty^{\frac{3}{8}})\|_{L_x^\infty}, \tag{3.5.5}$$

where $g(x,t) = (M-x)^{-\frac{1}{2}}\rho(x,t)$ and $g_\infty(x) = (M-x)^{-\frac{1}{2}}\rho_\infty(x)$.
From the previous results, we know that $I(t) \in C([0, T^*))$. From (3.2.1) we have that $I(0) \le C_0\epsilon_0$. Now by using of the classical continuation method, we can obtain the estimate of $I(t)$, $t \in [0, T^*) \cap [0, T]$, for any $T > 0$.

Claim 1. Under the assumptions of Theorem 3.2.1, there is a small positive constant $\epsilon_1 > C_0\epsilon_0$, such that, for any $T_2 \in (0, T^*) \cap (0, T]$, if

$$I(t) \le 2\epsilon_1, \tag{3.5.6}$$

for all $t \in [0, T_2]$, then

$$I(t) \le \epsilon_1, \tag{3.5.7}$$

for all $t \in [0, T_2]$.

Using the results in Lemmas 3.5.3-3.5.14, we can give the definition of ϵ_1 and finish the proof of **Claim** 1.

Lemma 3.5.3 *Under the assumptions of Theorem 3.2.1 and* (3.5.6), *if* ϵ_1 *is small enough, we can obtain*

$$C_1^{-1}(M-x)^{\frac{1}{2}} \leq \rho(x,t) \leq C_1(M-x)^{\frac{1}{2}} \tag{3.5.8}$$

$$a^3 + C_1^{-1}x \leq r^3(x,t) \leq a^3 + C_1 x \tag{3.5.9}$$

for all $t \in [0, T_2]$ *and* $x \in [0, M]$.

Proof: From $(3.4.2)_3$, (3.5.5) and Lemma 3.5.1, we can easily obtain the estimate (3.5.8) – (3.5.9) when $4\epsilon_1 < \min\limits_{x\in[0,M]} g_\infty := \underline{g}$. □

Lemma 3.5.4 *Under the assumptions of Theorem 3.2.1, if* ϵ_1 *and* a *are small enough, it holds that*

$$\int_0^M \left(u^2 + (M-x)^{\frac{1}{2}}(g-g_\infty)^2 + \frac{(r^3-a^3-r_\infty^3)^2}{x^2}\right)dx \leq C_2\epsilon_0^2, \tag{3.5.10}$$

$$\int_0^t \int_0^M \left(\rho^2 r^4 u_x^2 + \frac{u^2}{r^2}\right)(x,s)\,dxds \leq C_2\epsilon_0^2, \tag{3.5.11}$$

for all $t \in [0, T_2]$.

Proof: From (3.3.2), (3.5.1) and (3.5.4) we have

$$\frac{d}{dt}\left(\int_0^M \frac{1}{2}u^2 dx + S[V] - S[V_\infty]\right) + \int_0^M \rho^2 r^4 u_x^2 dx + \int_0^M \frac{u^2}{r^2}dx = 0, \tag{3.5.12}$$

where $V_\infty = \frac{r_\infty^3}{3}$ and $V = \frac{r^3}{3}$. From (3.3.3), (3.5.2), (3.5.8) – (3.5.9) and Proposition 3.3.2, we can obtain

$$\begin{aligned} &C^{-1}\int_0^M (M-x)^{\frac{1}{2}}(g-g_\infty)^2 + x^{-2}(r^3-a^3-r_\infty^3)^2 dx \\ &\leq S[V-\frac{a^3}{3}] - S[V_\infty] \\ &\leq C\int_0^M (M-x)^{\frac{1}{2}}(g-g_\infty)^2 + x^{-2}(r^3-a^3-r_\infty^3)^2 dx, \end{aligned} \tag{3.5.13}$$

when $\|(M-x)^{\frac{1}{2}}(\rho^{-1}-\rho_\infty^{-1})\|_{L_x^\infty} + \|(3x)^{-1}(r^3-a^3-r_\infty^3)\|_{L_x^\infty} \leq C_3\epsilon_1 \leq \delta_3$ and

$$|S[V] - S[V-\frac{a^3}{3}]| = \int_0^M \int_{V-\frac{a^3}{3}}^V Gx(3h)^{-\frac{4}{3}}dhdx \leq Ca^3 \leq C\epsilon_0^2, \tag{3.5.14}$$

when $a^3 \le \epsilon_0^2$. Hence we can obtain that

$$\begin{aligned}&\int_0^M [u^2 + (M-x)^{\frac{1}{2}}(g-g_\infty)^2 + x^{-2}(r^3 - a^3 - r_\infty^3)^2]dx \\ &+ \int_0^t \int_0^M \rho^2 r^4 u_x^2 dxds + \int_0^t \int_0^M \frac{u^2}{r^2} dxds \\ &\le C\epsilon_0^2.\end{aligned} \tag{3.5.15}$$

□

Let $\alpha_1 \in (0, \frac{1}{16})$ be a constant. Define $\{\beta_i\}$ and $\{\theta_j\}$ by $\beta_{j+1} = \theta_j + 1$, where $\theta_j = \min\{\frac{\beta_j}{2} - \frac{7}{16} - \frac{\alpha_1}{4}, 0\}$ and $\beta_0 = 0$, $j = 0, 1, \ldots$. Let N be an integer satisfying $\beta_N = 1$ and $\theta_N = 0$. Then by induction we can obtain the following Lemma.

Lemma 3.5.5 *Under the assumptions of Theorem 3.2.1, if ϵ_1 and a are small enough, it holds that*

$$\int_0^M \left(u^2 + (M-x)^{\frac{1}{2}}(g-g_\infty)^2 + x^{-2}(r^3 - a^3 - r_\infty^3)^2\right)dx \le C_4\epsilon_3^2(1+t)^{-1}, \tag{3.5.16}$$

$$\int_0^t \int_0^M (1+s)(\rho^2 r^4 u_x^2 + \frac{u^2}{r^2})dxds \le C_4\epsilon_3^2, \tag{3.5.17}$$

$$\int_0^M (g-g_\infty)^2 dx + \int_0^t \int_0^M \rho_\infty(g-g_\infty)^2 + x^{-2}(r^3 - a^3 - r_\infty^3)^2 dxds \le C_4\epsilon_3^2, \tag{3.5.18}$$

for all $t \in [0, T_2]$, where $\epsilon_3 = \epsilon_0^{2^{-N}}$.

Proof: We can prove the following estimates by induction:

$$\int_0^M \left(u^2 + (M-x)^{\frac{1}{2}}(g-g_\infty)^2 + x^{-2}(r^3 - a^3 - r_\infty^3)^2\right)dx \le C\epsilon_0^{2^{1-j}}(1+t)^{-\beta_j}, \tag{3.5.19}$$

$$\int_0^t \int_0^M (1+s)^{\beta_j}(\rho^2 r^4 u_x^2 + \frac{u^2}{r^2})dxds \le C\epsilon_0^{2^{1-j}}, \tag{3.5.20}$$

$$\begin{aligned}&(1+t)^{\theta_j} \int_0^M (g-g_\infty)^2 dx \\ &+ \int_0^t \int_0^M (1+s)^{\theta_j}\left(\rho_\infty(g-g_\infty)^2 + x^{-2}(r^3 - a^3 - r_\infty^3)^2\right)dxds \le C\epsilon_0^{2^{-j}},\end{aligned} \tag{3.5.21}$$

for all $t \in [0, T_2]$, $j = 0, 1, \ldots, N$.

From (3.5.10) – (3.5.11), we have (3.5.19) – (3.5.20) hold with $j = 0$.

Suppose (3.5.19), (3.5.20) hold with $j = k \geq 0$, we want to prove (3.5.21) holds for $j = k$.

Multiplying $(3.4.2)_2$ by $(1+t)^{\theta_k} r^{-2}(\frac{r^3 - a^3 - r_\infty^3}{3})$, integrating over $[0, M] \times [0, t]$, using integration by parts and boundary conditions (3.4.4), we have

$$
\begin{aligned}
&\int_0^t \int_0^M (1+s)^{\theta_k}[(\rho_\infty^2 - \rho^2)(\rho^{-1} - \rho_\infty^{-1}) + Gx(r^{-4} - r_\infty^{-4})(\frac{r^3 - a^3 - r_\infty^3}{3})]dxds \\
&= -\int_0^t \int_0^M (1+s)^{\theta_k} \frac{u_t}{r^2}(\frac{r^3 - a^3 - r_\infty^3}{3})dxds + \int_0^t \int_0^M (1+s)^{\theta_k} \rho^2 \partial_x (r^2 u)(\rho_\infty^{-1} - \rho^{-1})dxds \\
&+ \int_0^t \int_0^M 2(1+s)^{\theta_k} \rho [\frac{u}{r}(\frac{r^3 - a^3 - r_\infty^3}{3})]_x dxds := \sum_{i=1}^3 E_i.
\end{aligned}
\tag{3.5.22}
$$

Rewriting the left hand side of (3.5.22), we obtain

$$
\begin{aligned}
&\int_0^t \int_0^M (1+s)^{\theta_k}[(\rho_\infty^2 - \rho^2)(\rho^{-1} - \rho_\infty^{-1}) + Gx(r^{-4} - r_\infty^{-4})(\frac{r^3 - a^3 - r_\infty^3}{3})]dxds \\
&= \int_0^t \int_0^M 2(1+s)^{\theta_k}[(2 + O(\epsilon_1))\rho_\infty^3(\rho^{-1} - \rho_\infty^{-1})^2 - (4 + O(\epsilon_1))Gxr_\infty^{-7}(\frac{r^3 - a^3 - r_\infty^3}{3})^2]dxds \\
&+ \int_0^t \int_0^M 2(1+s)^{\theta_k} \frac{Gx}{3}(r^3 - a^3 - r_\infty^3)(r^{-4} - (r^3 - a^3)^{-\frac{4}{3}})dxds.
\end{aligned}
$$

Similar to (3.3.1), we have

$$
\begin{aligned}
&\text{left hand side of (3.5.22)} + Ca^3(1+t) \\
&\geq C^{-1} \int_0^t (1+s)^{\theta_k} \int_0^M [\rho_\infty (g - g_\infty)^2 + \frac{(r^3 - a^3 - r_\infty^3)^2}{x^2}]dxds.
\end{aligned}
\tag{3.5.23}
$$

From (3.5.6), (3.5.8) – (3.5.11), we have

$$
\int_0^t \int_0^M u^2 dxds \leq C \int_0^t \int_0^M \frac{u^2}{r^2} dxds \leq C\epsilon_0^2, \tag{3.5.24}
$$

$$
\begin{aligned}
E_1 = & -\int_0^M (1+s)^{\theta_k} \frac{u}{r^2} (\frac{r^3 - a^3 - r_\infty^3}{3}) dx|_0^t \\
& + \theta_k \int_0^t \int_0^M (1+s)^{\theta_k - 1} \frac{u}{r^2} (\frac{r^3 - a^3 - r_\infty^3}{3}) dxds \\
& - \frac{2}{3} \int_0^t \int_0^M (1+s)^{\theta_k} \frac{u^2}{r^3} (r^3 - a^3 - r_\infty^3) + \int_0^t \int_0^M (1+s)^{\theta_k} u^2 dxds \\
& \le C \int_0^M [u^2 + x^{-2}(r^3 - a^3 - r_\infty^3)^2] dx + C\epsilon_0^2 + C \int_0^t \int_0^M u^2 dxds \\
& + C (\int_0^t \int_0^M u^2 dxds)^{\frac{1}{2}} (\int_0^t (1+s)^{2\theta_k - 2} ds)^{\frac{1}{2}} \le C\epsilon_0.
\end{aligned} \tag{3.5.25}
$$

$$
\begin{aligned}
E_2 = & -\int_0^t \int_0^M \rho_t (1+s)^{\theta_k} (\frac{1}{\rho_\infty} - \frac{1}{\rho}) dxds = -\int_0^t \int_0^M h(\rho, \rho_\infty)_t (1+s)^{\theta_k} dxds \\
= & -\int_0^M h(\rho, \rho_\infty)(1+s)^{\theta_k} dx|_0^t + \int_0^t \int_0^M h(\rho, \rho_\infty) \theta_k (1+s)^{\theta_k - 1} dxds \\
\le & -C(1+t)^{\theta_k} \int_0^M (g - g_\infty)^2 dx + C\epsilon_0^2,
\end{aligned} \tag{3.5.26}
$$

where $h(\rho, \rho_\infty) = \int_{\rho_\infty}^\rho (\frac{1}{\rho_\infty} - \frac{1}{s}) ds \sim (g - g_\infty)^2$, and

$$
\begin{aligned}
|r^3 - a^3 - r_\infty^3| & \le C \int_0^x |\rho^{-1} - \rho_\infty^{-1}| dy \\
& \le C \|x^{\frac{2+\alpha}{24}} (\rho^{\frac{3}{8}} - \rho_\infty^{\frac{3}{8}})\|_{L_x^\infty} \times \int_0^x y^{-\frac{2+\alpha}{24}} (M - y)^{-\frac{11}{16}} dy \\
& \le C(1+t)^{-\frac{1}{16}} x^{1 - \frac{2+\alpha}{24}},
\end{aligned} \tag{3.5.27}
$$

$$
\begin{aligned}
E_3 \le & C \int_0^t \int_0^M (1+s)^{\theta_k} \{ r^{-3} |r^3 - a^3 - r_\infty^3| (|\rho r^2 u_x| + |r^{-1} u|) + \rho |\frac{u}{r} (\frac{1}{\rho} - \frac{1}{\rho_\infty})| \} dxds \\
\le & C (\int_0^t \int_0^M (1+s)^{\beta_k} (\rho^2 r^4 u_x^2 + \frac{u^2}{r^2}) dxds)^{\frac{1}{2}} (\int_0^t (1+s)^{2\theta_k - \beta_k} \int_0^M |r^3 - a^3 - r_\infty^3|^2 x^{-2} dxds \\
+ & \int_0^t (1+s)^{2\theta_k - \beta_k} \|x^{\frac{2+\alpha}{24}} (\rho^{\frac{3}{8}} - \rho_\infty^{\frac{3}{8}})\|_{L^\infty}^2 \times \int_0^M x^{-\frac{2+\alpha}{12}} (M - x)^{-\frac{3}{8}} dxds)^{\frac{1}{2}} \\
\le & C (\int_0^t \int_0^M (1+s)^{\beta_k} (\rho^2 r^4 u_x^2 + \frac{u^2}{r^2}) dxds)^{\frac{1}{2}} (\int_0^t (1+s)^{-1 - \frac{\alpha_1}{2}} ds)^{\frac{1}{2}} \le C\epsilon_0^{2^{-k}},
\end{aligned} \tag{3.5.28}
$$

when $a^3(1+T) \le \epsilon_0$. From (3.5.22) – (3.5.28), we get (3.5.21) with $j = k$.

Now, suppose (3.5.19) – (3.5.21) hold with $j = k$, we have

$$\frac{d}{dt}\{(1+t)^{\beta_{k+1}}(\int_0^M \frac{1}{2}u^2dx + S[V] - S[V_\infty])\} + (1+t)^{\beta_{k+1}}\int_0^M (\rho^2 r^4 u_x^2 + \frac{u^2}{r^2})dx$$
$$= \beta_{k+1}(1+t)^{\theta_k}(\int_0^M \frac{1}{2}u^2(x,t)dx + S[V] - S[V_\infty]),$$

where $V = \frac{r_\infty^3}{3}$ and $V_\infty = \frac{r^3}{3}$. Integrating the above equality over $[0,t]$, from (3.4.1), (3.5.8) – (3.5.11), (3.5.13) – (3.5.14), (3.5.21) with $j = k$, (3.5.24), and $\theta_k \leq 0$, we obtain

$$(1+t)^{\beta_{k+1}}\int_0^M \{u^2(x,t) + (M-x)^{\frac{1}{2}}(g-g_\infty)^2 + \frac{(r^3 - a^3 - r_\infty^3)^2}{x^2}\}dx$$
$$+ \int_0^t\int_0^M (1+s)^{\beta_{k+1}}(\rho^2 r^4 u_x^2 + \frac{u^2}{r^2})dxds \leq C\epsilon_0^{2^{-k}},$$

when $a^3(1+T) \leq \epsilon_0$. Hence we have (3.5.19) – (3.5.20) with $j = k+1$. Then we obtain (3.5.16) – (3.5.18) immediately.

□

Now we can estimate $\|g(\cdot,t) - g_\infty(\cdot)\|_{L^\infty([\frac{M}{3},M])}$ by using of $r|_{x\in[\frac{M}{3},M]} \geq C^{-1} > 0$.

Lemma 3.5.6 *Under the assumptions of Theorem 3.2.1, if a is small enough, we have*

$$|g(x,t) - g_\infty(x)| \leq C_5\epsilon_4, \tag{3.5.29}$$

for all $x \in [\frac{M}{3}, M]$ *and* $t \in [0, T_2]$, *where* $\epsilon_4 = \epsilon_3^{\frac{1}{4}}$.

Proof: For any fixed $x \in [\frac{M}{3}, M]$, we have

$$I_1(x,t) + \int_0^t r^2(\rho^2(x,t) - \rho_\infty^2(x))ds = r_0^2\rho_0(x) + I_2(x,t), \ \ t \in [0,T_2], \tag{3.5.30}$$

where

$$I_1(x,t) = r_\infty^2(x)\rho(x,t) - (r_\infty^2(x) - r^2(x,t))\rho(x,t) + \int_x^M (\frac{2}{r_0} - \frac{2}{r})dy$$
$$- \int_x^M (u(y,t) - u_0(y))dy,$$

and

$$I_2(x,t) = -2\int_0^t\int_x^M r^{-1}\frac{\rho^2-\rho_\infty^2}{\rho}dyds + \int_0^t\int_x^M r^2Gy(r^{-4}-r_\infty^{-4})dyds.$$

From $(3.1.9)_3$, for any $x \in [\frac{M}{3}, M]$, we have

$$\begin{aligned} |r^3 - a^3 - r_\infty^3| &\le C\int_0^x |\frac{1}{\rho} - \frac{1}{\rho_\infty}|dy \\ &\le C\int_0^x |g-g_\infty|(M-y)^{-\frac{1}{2}}dy \\ &\le C[\int_0^M (g-g_\infty)^2dy]^{\frac{1}{4}} \le C\epsilon_3^{\frac{1}{2}}, \end{aligned} \tag{3.5.31}$$

$$\int_x^M (\frac{1}{r_0} - \frac{1}{r})dy = \int_x^M \frac{r-r_0}{rr_0}dy \le C\int_x^M |r-r_0|dy \le C\epsilon_3^{\frac{1}{2}}(M-x)^{\frac{1}{2}}, \tag{3.5.32}$$

and

$$|\int_x^M (u-u_0)dy| \le C(M-x)^{\frac{1}{2}}(\|u\|_{L^2} + \|u_0\|_{L^2}) \le C(M-x)^{\frac{1}{2}}\epsilon_3^{\frac{1}{2}}, \tag{3.5.33}$$

thus, if a is small enough, from (3.5.31) – (3.5.33), we have

$$|I_1(x,t) - r_\infty^2\rho| \le A_1(M-x)^{\frac{1}{2}}\epsilon_3^{\frac{1}{2}}, \tag{3.5.34}$$

and

$$|I_2(x,t_1) - I_2(x,t_2)| \le A_2\epsilon_3^{\frac{1}{2}}|t_2-t_1|(M-x), \;\; x \in [\frac{M}{3}, M]. \tag{3.5.35}$$

□

Claim 2. For any fixed $x \in [\frac{M}{3}, M]$, we have $I_1(x,t) \ge A_{1,1}$ for all $t \in [0,T_2]$, where

$$A_{1,1} = \min\{I_1(x,0), r_\infty^2[\rho_\infty^2 - \frac{A_2}{A_3}\epsilon_3^{\frac{1}{2}}(M-x)]^{\frac{1}{2}} - A_1\epsilon_3^{\frac{1}{2}}(M-x)^{\frac{1}{2}}\},$$

where $A_3 > 0$, satisfying $A_3 \le r^2$ in $(x,t) \in [\frac{M}{3}, M] \times [0,T_2]$.

Proof of **Claim 2**. If not, there exists $t_{1,1}$ such that $I_1(x,t) < A_{1,1}$, then we can find $t_{1,2} \in (0,t_{1,1})$ such that $I_1(x,t_{1,2}) = A_{1,1}$ and $I_1(x,t) < A_{1,1}$ for

$t \in (t_{1,2}, t_{1,1})$. From (3.5.30) and (3.5.35), we can get

$$\begin{aligned} & I_1(x, t_{1,1}) - I_1(x, t_{1,2}) + \int_{t_{1,2}}^{t_{1,1}} r^2(\rho^2 - \rho_\infty^2) ds \\ & \geq -A_2 \epsilon_3^{\frac{1}{2}} (M - x)(t_{1,1} - t_{1,2}). \end{aligned}$$

From (3.5.34), it holds that

$$\begin{aligned} \rho(x,t) &= r_\infty^{-2}[I_1(x,t) - (I_1(x,t) - r_\infty^2 \rho)] \\ &\leq r_\infty^{-2}(A_{1,1} + A_1(M-x)^{\frac{1}{2}} \epsilon_3^{\frac{1}{2}}) \leq (\rho_\infty^2 - \frac{A_2}{A_3}(M-x)\epsilon_3^{\frac{1}{2}})^{\frac{1}{2}}, \end{aligned}$$

and

$$\rho^2 \leq \rho_\infty^2 - \frac{A_2}{A_3}(M-x)\epsilon_3^{\frac{1}{2}},$$

then $I_1(x, t_{1,1}) \geq I_1(x, t_{1,2})$. Contradiction. Thus, **Claim 2** holds.

□

Similarly, we can obtain the following claim.

Claim 3. For any fixed $x \in [\frac{M}{3}, M]$, we have $I_1(x,t) \leq A_{1,2}$ for all $t \in [0, T_2]$, where

$$A_{1,2} = \max\{I_1(x,0), r_\infty^2[\rho_\infty^2 + \frac{A_2}{A_4}\epsilon_3^{\frac{1}{2}}(M-x)]^{\frac{1}{2}} + A_1 \epsilon_3^{\frac{1}{2}}(M-x)^{\frac{1}{2}}\},$$

where $A_4 > 0$, satisfying $A_4 \geq r^2$ in $(x,t) \in [\frac{M}{3}, M] \times [0, T_2]$.

From **Claim 2-3**, we have

$$|g(x,t) - g_\infty(x)| \leq C_5 \epsilon_4,$$

where $x \in [\frac{M}{3}, M]$, and $t \in [0, T_2]$.

Let $\phi \in C^\infty[0, M]$ satisfying $\phi \geq 0$, $\phi|_{x \in [0, \frac{M}{2}]} = 1$ and $\phi|_{x \in [\frac{3}{4}M, M]} = 0$. We have the following Lemma

Lemma 3.5.7 *Under the assumptions of Theorem 3.2.1, if ϵ_1 and a are small enough, we obtain that*

$$\int_0^M \phi^2 r^4 (\rho - \rho_\infty)_x^2 dx + \int_0^t \int_0^M \phi^2 r^4 (\rho - \rho_\infty)_x^2 dx ds \leq C_6 \epsilon_3, \tag{3.5.36}$$

for all $t \in [0, T_2]$.

Proof: From (3.4.2), we have

$$[u + r^2(\rho - \rho_\infty)_x]_t + 2r^2\rho\rho_x = -G\frac{x}{r^2} - 2ru\rho_{\infty x}. \tag{3.5.37}$$

Multiplying (3.5.37) by $\phi^2[u + r^2(\rho - \rho_\infty)_x]$, integrating over $[0, M]$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^M \phi^2|u + r^2(\rho - \rho_\infty)_x|^2 dx + 2\int_0^M \phi^2 r^2 \rho(\rho - \rho_\infty)_x[u + r^2(\rho - \rho_\infty)_x]dx \\
&+ 2\int_0^M \phi^2 r^2 \rho\rho_{\infty x}[u + r^2(\rho - \rho_\infty)_x]dx = -\int_0^M \phi^2 G\frac{x}{r^2}[u + r^2(\rho - \rho_\infty)_x]dx \\
&- 2\int_0^M \phi^2 ur\rho_{\infty x}[u + r^2(\rho - \rho_\infty)_x]dx,
\end{aligned}$$

by using the Cauchy-Schwarz inequality, we can obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_0^M \phi^2|u + r^2(\rho - \rho_\infty)_x|^2 dx + C_7\int_0^M \phi^2 r^4(\rho - \rho_\infty)_x^2 dx \\
&\le C\int_0^M \phi^2 u^2 dx + \int_0^M \phi^2|G\frac{x}{r^2} + r^2\rho\rho_{\infty x}||u + r^2(\rho - \rho_\infty)_x|dx \\
&+ C\int_0^M \phi^2|ur\rho_{\infty x}[u + r^2(\rho - \rho_\infty)_x]|dx \\
&\le C\int_0^M \phi^2 u^2 dx + \int_0^M \phi^2|G\frac{x}{r^2} + r^2\rho\rho_{\infty x}||u + r^2(\rho - \rho_\infty)_x|dx + \\
&C\int_0^M \phi^2 ru^2|\rho_{\infty x}|dx + C\int_0^M \phi^2 r^3|u\rho_{\infty x}(\rho - \rho_\infty)_x|dx \\
&\le \int_0^M \phi^2|G\frac{x}{r^2} + r^2\rho\rho_{\infty x}|^2 dx + \frac{1}{4}C_7\int_0^M \phi^2 r^4(\rho - \rho_\infty)_x^2 dx + C\int_0^M \phi^2 u^2 dx.
\end{aligned} \tag{3.5.38}$$

From (3.5.3) and (3.5.8) – (3.5.9), we have

$$\begin{aligned}
&\int_0^M \phi^2|G\frac{x}{r^2} + r^2\rho\rho_{\infty x}|^2 dx = \int_0^M \phi^2|G\frac{x}{r^2} - G\frac{xr^2\rho}{\rho_\infty r_\infty^4}|^2 dx \\
&\le C\int_0^M \phi^2[x^{-2}(r^3 - a^3 - r_\infty^3)^2 + \rho_\infty(g - g_\infty)^2]dx + Ca^3.
\end{aligned} \tag{3.5.39}$$

From (3.5.38) – (3.5.39), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^M \phi^2|u + r^2(\rho - \rho_\infty)_x|^2 dx + \frac{C_7}{2}\int_0^M \phi^2 r^4(\rho - \rho_\infty)_x^2 dx \\
&\le C\|u(\cdot, t)\|_{L^2}^2 + C\int_0^M [x^{-2}(r^3 - a^3 - r_\infty^3)^2 + \rho_\infty(g - g_\infty)^2] + Ca^3.
\end{aligned}$$

Combining (3.5.8) – (3.5.11), (3.5.18) and (3.5.24), we obtain (3.5.36) immediately. □

Lemma 3.5.8 *Under the assumptions of Theorem 3.2.1, if* ϵ_1 *and* a *are small enough, it holds that*

$$\int_0^t \int_0^M r^{\alpha-2}(\rho-\rho_\infty)^2 \phi dx ds \le C_8 \epsilon_3, \tag{3.5.40}$$

$$\int_0^M r^{\alpha-2} u^2 \phi^2 dx + \int_0^t \int_0^M \phi^2 r^{\alpha-2}(r^4 u_x^2 + \frac{u^2}{r^2}) dx ds \le C_8 \epsilon_3, \tag{3.5.41}$$

$$\int_0^M \phi^2 r^{\alpha+2}(\rho-\rho_\infty)_x^2 dx + \int_0^t \int_0^M \phi^2 r^{\alpha+2}(\rho-\rho_\infty)_x^2 dx ds \le C_8 \epsilon_3, \tag{3.5.42}$$

$$\|g(\cdot,t) - g_\infty(\cdot)\|_{L^\infty([0,M])} \le C_8 \epsilon_4 (1+t)^{-\epsilon_5}, \tag{3.5.43}$$

$$|r^3(x,t) - a^3 - r_\infty^3(x)| \le C_8 \epsilon_4 x, \quad x \in [0,M], \tag{3.5.44}$$

where $\epsilon_5 = \dfrac{p-1}{3p-2}$ *and* $p \in (1, \dfrac{6}{5+\alpha})$ *for all* $t \in [0, T_2]$.

Proof: We can prove following estimates by induction:

$$\int_0^t \int_0^M r^{\alpha_m}(\rho-\rho_\infty)^2 \phi dx ds \le C \epsilon_3, \tag{3.5.45}$$

$$\int_0^M r^{\alpha_m} u^2 \phi^2 dx + \int_0^t \int_0^M \phi^2 r^{\alpha_m}(r^4 u_x^2 + \frac{u^2}{r^2}) dx ds \le C \epsilon_3, \tag{3.5.46}$$

$$\int_0^M \phi^2 r^{\alpha_m+4}(\rho-\rho_\infty)_x^2 dx + \int_0^t \int_0^M \phi^2 r^{\alpha_m+4}(\rho-\rho_\infty)_x^2 dx ds \le C \epsilon_3, \tag{3.5.47}$$

for all $t \in [0, T_2]$ and $\alpha_{m+1} = \max\{\alpha-2, \alpha_m - 1\}$ with $\alpha_0 = 0$, $m = 0, 1, \ldots$.

From (3.5.8) – (3.5.11), (3.5.18) and (3.5.36), we know that (3.5.45) – (3.5.47) hold with $m = 0$.

Suppose that (3.5.45) – (3.5.47) hold with $m \le k-1$, then we prove the estimates (3.5.45) – (3.5.47) hold with $m = k$ as follows.

Since $\alpha \in (-1,1)$, we have

$$\begin{aligned}
&\int_0^M r^{\alpha_k}(\rho-\rho_\infty)^2\phi^2 dx = \int_0^M r^{\alpha_k}(\int_x^M \partial_x[(\rho-\rho_\infty)\phi]dy)^2 dx \\
&\le C\int_0^M \rho_\infty(g-g_\infty)^2 dx + C\int_0^M r^{\alpha_k}\int_x^M r^{4+\alpha_{k-1}}(\rho-\rho_\infty)_x^2\phi^2 dy\int_x^M r^{-4-\alpha_{k-1}}dydx \\
&\le C\int_0^M \rho_\infty(g-g_\infty)^2 dx + C\int_0^M r^{4+\alpha_{k-1}}(\rho-\rho_\infty)_x^2\phi^2 dx.
\end{aligned}$$

Combining (3.5.18) and (3.5.47) ($m=k-1$), we can obtain (3.5.45) $m=k$.

Multiplying $(3.4.2)_2$ by $ur^{\alpha_k}\phi^2$, integrating over $[0,M]$, using (3.4.4) and integration by parts, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^M \frac{1}{2}r^{\alpha_k}u^2\phi^2 dx - \int_0^M \frac{\alpha_k}{2}r^{\alpha_k-1}u^3\phi^2 dx \\
&= -\int_0^M \phi^2[\rho^2(r^2u)_x(r^{2+\alpha_k}u)_x - 2\rho(r^{\alpha_k+1}u^2)_x]dx \\
&+\int_0^M \phi^2(\rho^2-\rho_\infty^2)(r^{2+\alpha_k}u)_x dx + \int_0^M \phi^2 r^{2+\alpha_k}u(\frac{Gx}{r_\infty^4}-\frac{Gx}{r^4})dx \\
&+\int_0^M (\phi^2)_x[r^{2+\alpha_k}u(\rho^2-\rho_\infty^2-\rho^2(r^2u)_x+2\rho r^{\alpha_k+1}u^2]dx = \sum_{i=1}^4 F_i.
\end{aligned} \tag{3.5.48}$$

Now we estimate F_i as follows.

$$-F_1 \ge C_9\int_0^M \phi^2(r^{\alpha_k+4}u_x^2 + r^{\alpha_k-2}u^2)dx. \tag{3.5.49}$$

From (3.5.8) – (3.5.9) and Cauchy-Schwarz inequality, we obtain

$$F_2 \le \frac{C_9}{8}\int_0^M \phi^2(r^{4+\alpha_k}u_x^2 + r^{\alpha_k-2}u^2)dx + C\int_0^M \phi^2 r^{\alpha_k}(\rho-\rho_\infty)^2 dx, \tag{3.5.50}$$

$$\begin{aligned}
F_3 &\le \frac{C_9}{8}\int_0^M \phi^2 r^{\alpha_k-2}u^2 dx + C\int_0^M \phi^2 x^2 r^{\alpha_k+6}(r_\infty^{-4}-r^{-4})^2 dx \\
&\le \frac{C_9}{8}\int_0^M \phi^2 r^{\alpha_k-2}u^2 dx + C\int_0^M x^{-2}(r^3-a^3-r_\infty^3)^2 dx + Ca^2,
\end{aligned} \tag{3.5.51}$$

since $r-(r^3-a^3)^{\frac{1}{3}} \le Ca$, and

$$F_4 \le C\int_0^M [\rho^2 r^4 u_x^2 + \frac{u^2}{r^2} + (g-g_\infty)^2]dx. \tag{3.5.52}$$

From (3.5.6), (3.5.8) – (3.5.9) and (3.5.48) – (3.5.52) we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^M r^{\alpha_k}u^2\phi^2dx + C_{10}^{-1}\int_0^M r^{\alpha_k}(r^4u_x^2+\frac{u^2}{r^2})\phi^2dx\\
&\le C_{10}\epsilon_1\int_0^M r^{\alpha_k}\frac{u^2}{r^2}\phi^2dx + C\int_0^M r^{\alpha_k}\phi^2(\rho-\rho_\infty)^2dx\\
&+C\int_0^M[\rho^2r^4u_x^2+\frac{u^2}{r^2}+(g-g_\infty)^2]dx + C\int_0^M x^{-2}(r^3-a^3-r_\infty^3)^2dx + Ca^2.
\end{aligned}$$

If $\alpha\in(-1,1)$, we have that $\int_0^M r_0^{\alpha-2}u_0^2dx \le \|\frac{u_0^2}{r_0^2}\|_{L^\infty}\int_0^M r_0^\alpha dx \le C\epsilon_0^2$. When $C_{10}^2\epsilon_1\le 1$, using the estimate (3.5.11), (3.5.18) and (3.5.45)$(m=k)$, we can prove (3.5.46)$(m=k)$ holds.

From (3.4.2), we have

$$\begin{aligned}
&(r^{\frac{\alpha_k}{2}}u)_t + [r^{\frac{\alpha_k}{2}+2}(\rho-\rho_\infty)_x]_t = -2r^{\frac{\alpha_k}{2}+2}\rho\rho_x - Gr^{\frac{\alpha_k}{2}}\frac{x}{r^2} - 2r^{\frac{\alpha_k}{2}+1}u\rho_{\infty x}\\
&+\frac{\alpha_k}{2}r^{\frac{\alpha_k}{2}-1}u^2 + \frac{\alpha_k}{2}r^{\frac{\alpha_k}{2}-1}ur^2(\rho-\rho_\infty)_x.
\end{aligned} \tag{3.5.53}$$

Multiplying (3.5.53) with $\phi^2[r^{\frac{\alpha_k}{2}}u + r^{\frac{\alpha_k}{2}+2}(\rho-\rho_\infty)_x]$, integrating over $[0,M]$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^M \phi^2[r^{\frac{\alpha_k}{2}}u + r^{\frac{\alpha_k}{2}+2}(\rho-\rho_\infty)_x]^2dx\\
&= -2\int_0^M \phi^2 r^{\frac{\alpha_k}{2}+2}\rho\rho_x[r^{\frac{\alpha_k}{2}}u + r^{\frac{\alpha_k}{2}+2}(\rho-\rho_\infty)_x]dx\\
&-\int_0^M G\phi^2 r^{\frac{\alpha_k}{2}-2}x[r^{\frac{\alpha_k}{2}}u + r^{\frac{\alpha_k}{2}+2}(\rho-\rho_\infty)_x]dx\\
&-2\int_0^M \phi^2 r^{\frac{\alpha_k}{2}+1}u\rho_{\infty x}[r^{\frac{\alpha_k}{2}}u + r^{\frac{\alpha_k}{2}+2}(\rho-\rho_\infty)_x]\\
&+\frac{\alpha_k}{2}\int_0^M \phi^2 r^{\frac{\alpha_k}{2}-1}u^2[r^{\frac{\alpha_k}{2}}u + r^{\frac{\alpha_k}{2}+2}(\rho-\rho_\infty)_x]\\
&+\frac{\alpha_k}{2}\int_0^M \phi^2 r^{\frac{\alpha_k}{2}+1}u(\rho-\rho_\infty)_x[r^{\frac{\alpha_k}{2}}u + r^{\frac{\alpha_k}{2}+2}(\rho-\rho_\infty)_x]dx,
\end{aligned}$$

using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^M \phi^2(r^{\frac{\alpha_k}{2}}u + r^{\frac{\alpha_k}{2}+2}(\rho-\rho_\infty)_x)^2dx + C_{11}\int_0^M \phi^2 r^{\alpha_k+4}(\rho-\rho_\infty)_x^2dx\\
&\leq C\int_0^M \phi^2 r^{\alpha_k}u^2dx + \int_0^M \phi^2 r^{\frac{\alpha_k}{2}}|G\frac{x}{r^2}+\rho r^2\rho_{\infty x}||r^{\frac{\alpha_k}{2}}u + r^{\frac{\alpha_k}{2}+2}(\rho-\rho_\infty)_x|dx\\
&+(\frac{C_{12}}{4}+C_{12}\epsilon_1)\int_0^M \phi^2 r^{\alpha_k+4}(\rho-\rho_\infty)_x^2dx\\
&\leq C\int_0^M \phi^2 r^{\alpha_k}u^2dx + (\frac{C_{11}}{4}+C_{12}\epsilon_1)\int_0^M \phi^2 r^{\alpha_k+4}(\rho-\rho_\infty)_x^2dx\\
&+C\int_0^M \phi^2 r^{\alpha_k}|G\frac{x}{r^2}+\rho r^2\rho_{\infty x}|^2dx.
\end{aligned}$$

If $C_{12}\epsilon_1 \leq \dfrac{C_{11}}{4}$, using the estimates (3.5.6) and (3.5.45) – (3.5.46), we have

$$\begin{aligned}
&\frac{d}{dt}\int_0^M \phi^2[r^{\frac{\alpha_k}{2}}u + r^{\frac{\alpha_k}{2}+2}(\rho-\rho_\infty)_x]^2dx + \frac{C_{11}}{2}\int_0^M \phi^2 r^{\alpha_k+4}(\rho-\rho_\infty)_x^2dx\\
&\leq C\int_0^M \phi^2[r^{\alpha_k}u^2 + r^{\alpha_k}(\rho-\rho_\infty)^2 + x^{-2}(r^3-a^3-r_\infty^3)^2]dx + Ca^2,
\end{aligned}$$

from (3.5.18), (3.5.45) – (3.5.46)$(m=k)$ and $a^2(1+T)\leq\epsilon_0$, we have

$$\int_0^M \phi^2[r^{\frac{\alpha_k}{2}}u + r^{\frac{\alpha_k}{2}+2}(\rho-\rho_\infty)_x]^2dx + \int_0^t\int_0^M \phi^2 r^{\alpha_k+4}(\rho-\rho_\infty)_x^2dx \leq C\epsilon_3.$$

Using Galiardo-Nirenberg inequality, we have

$$\|(\rho-\rho_\infty)\phi\|_{L^\infty} \leq C\|(\rho-\rho_\infty)\phi\|_{L^2}^{\frac{2(p-1)}{3p-2}}\|\partial_x[(\rho-\rho_\infty)\phi]\|_{L^p}^{\frac{p}{3p-2}},$$

so we only need to estimate

$$\|\partial_x[(\rho-\rho_\infty)\phi]\|_{L^p}^{\frac{p}{3p-2}}.$$

If $\alpha\in(-1,1)$, we can choose $p\in(1,\dfrac{6}{5+\alpha})\subset(1,2)$. From (3.5.8) – (3.5.9) and (3.5.42), by using of Hölder's inequality, we have

$$\begin{aligned}
(\int_0^M |\partial_x(\rho-\rho_\infty)\phi|^pdx)^{\frac{1}{p}} &\leq (\int_0^M |\partial_x(\rho-\rho_\infty)|^2\phi^2 r^{2+\alpha}dx)^{\frac{1}{2}}(\int_0^M r^{-(2+\alpha)\frac{p}{2-p}}dx)^{\frac{2-p}{p}}\\
&\leq C\epsilon_3^{\frac{1}{2}}.
\end{aligned}$$

From above result and (3.5.16), we have

$$\|(\rho-\rho_\infty)\phi\|_{L^\infty}\le C\epsilon_3^{\frac{1}{2}}(1+t)^{-\epsilon_5},\ \ \epsilon_5=\frac{p-1}{3p-2},\ \ p\in(1,\frac{6}{5+\alpha}).$$

Now, we have proved the Lemma 3.5.8. □

Define $\psi\in C^\infty([0,M])$, $\psi\ge 0$, $\psi|_{[0,\frac{M}{4}]}=0$, and $\psi|_{[\frac{M}{2},M]}=1$, hence we have $\psi\cdot r^{-1}\le C$.

Lemma 3.5.9 *Under the assumptions of Theorem 3.2.1, if ϵ_1 is small enough, we have*

$$\|\psi^2u^2(M-x)^{\frac{1}{2}}\|_{L^\infty_x}\le C(\int_0^M\rho^2r^4u_y^2dy+\int_0^M\frac{u^2}{r^2}dy). \tag{3.5.54}$$

and

$$\begin{aligned}&\int_0^t(1+s)\|\psi^2u^2(M-x)^{\frac{1}{2}}\|_{L^\infty_x}ds\\&\le C\int_0^t(1+s)\int_0^M\rho^2r^4u_y^2dyds+C\int_0^t(1+s)\int_0^M\frac{u^2}{r^2}dyds\\&\le C\epsilon_3^2.\end{aligned} \tag{3.5.55}$$

Proof: First we have

$$\psi^2u^2(M-x)^{\frac{1}{2}}=(M-x)^{\frac{1}{2}}(\int_0^x\partial_y(\psi u)dy)^2.$$

Choosing $\alpha\in(0,\frac{1}{4})$, by (3.5.8) we can show that,

$$\begin{aligned}&\psi^2u^2(M-x)^{\frac{1}{2}}=[\int_0^x(M-x)^{\frac{1}{4}}\psi u_ydy+\int_0^x(M-x)^{\frac{1}{4}}\psi_yudy]^2\\&\le C[(\int_0^x(M-x)^{\frac{1}{4}}|\psi u_y|dy)^2+(\int_0^x(M-x)^{\frac{1}{4}}|\psi_yu|dy)^2]\\&=C[(\int_0^x(M-x)^{\frac{1}{4}-\alpha}(M-x)^{\alpha}\psi|u_y|dy)^2+(\int_0^x(M-x)^{\frac{1}{4}}|\psi_yu|dy)^2]\\&\le C(M-x)^{\frac{1}{2}-2\alpha}[\int_0^x(M-y)^{\alpha}\psi|u_y|dy]^2+C(M-x)^{\frac{1}{2}}(\int_0^x|\psi_yu|dy)^2\\&\le C(M-x)^{\frac{1}{2}-2\alpha}(\int_0^x\rho^2r^4u_y^2)(\int_0^x\rho^{-2}(M-y)^{2\alpha}dy)+C(M-x)^{\frac{1}{2}}(\int_0^M\frac{u^2}{r^2}dy)\\&\le C(\int_0^M\rho^2r^4u_y^2+\int_0^M\frac{u^2}{r^2}dy),\end{aligned}$$

hence we obtain (3.5.54). Once we have (3.5.54), the (3.5.55) holds by (3.5.17).

Lemma 3.5.10 *Under the assumptions of Theorem 3.2.1, if ϵ_1 and a are small enough, it holds that*

$$\int_0^t \int_0^M (g - g_\infty)^2 \psi^2 dxds \le C_{12}\epsilon_4^2, \tag{3.5.56}$$

where $t \in [0, T_2)$.

Proof: From (3.4.2), we have

$$(\rho^2 - \rho_\infty^2) = \int_x^M \frac{u_t}{r^2} dy + \int_x^M Gy(r^{-4} - r_\infty^{-4}) dy + \rho^2 (r^2 u)_x - 2\rho \frac{u}{r} - 2\int_x^M \rho (\frac{u}{r})_y dy.$$

Multiplying the above equality by $\psi (M-x)^{-2}(\rho^2 - \rho_\infty^2)$, integrating over $[0, M] \times [0, t]$ we obtain

$$\begin{aligned}
&\int_0^t \int_0^M \psi^2 (\rho^2 - \rho_\infty^2)^2 (M - x)^{-2} dxds \\
&= \int_0^t \int_0^M \psi^2 (M - x)^{-2} (\rho^2 - \rho_\infty^2) \int_x^M \frac{u_t}{r^2} dydxds \\
&+ \int_0^t \int_0^M \psi^2 (M - x)^{-2} (\rho^2 - \rho_\infty^2) \int_x^M Gy(r^{-4} - r_\infty^{-4}) dydxds \\
&+ \int_0^t \int_0^M \psi^2 (M - x)^{-2} (\rho^2 - \rho_\infty^2) \rho^2 (r^2 u)_x dxds \\
&- 2\int_0^t \int_0^M \psi^2 (M - x)^{-2} (\rho^2 - \rho_\infty^2) \rho \frac{u}{r} dxds \\
&- 2\int_0^t \int_0^M \psi^2 (M - x)^{-2} (\rho^2 - \rho_\infty^2) \int_x^M \rho (\frac{u}{r})_y dydxds = \sum_{i=1}^5 J_i.
\end{aligned} \tag{3.5.57}$$

Using (3.4.2), (3.5.8) – (3.5.9), (3.5.16) – (3.5.18), (3.5.24), (3.5.29), (3.5.43) – (3.5.44), (3.5.54) – (3.5.55), integrating by parts and the Cauchy-Schwarz inequality, we can estimate J_i as follows

$$J_1 = \{\int_0^M \psi^2(M-x)^{-2}(\rho^2-\rho_\infty^2)\int_x^M \frac{u}{r^2}dydx\}|_0^t + 2\int_0^t\int_0^M \psi^2(M-x)^{-2}\rho^3(r^2u)_x$$
$$\int_x^M \frac{u}{r^2}dydxds + 2\int_0^t\int_0^M \psi^2(M-x)^{-2}(\rho^2-\rho_\infty^2)\int_x^M \frac{u^2}{r^3}dydxds$$
$$\leq C\|g-g_\infty\|_{L^\infty}(\int_0^M u^2dx)^{\frac{1}{2}}\int_0^M (M-x)^{-\frac{1}{2}}dx + C\epsilon_0^2$$
$$+ C\int_0^t\int_0^M \psi^2(M-x)^{-\frac{1}{4}}|\rho(r^2u)_x|\|u(M-y)^{\frac{1}{4}}\|_{L^\infty[\frac{M}{4},M]}dxds$$
$$+ C\int_0^t\int_0^M \psi^2|g-g_\infty|\|\frac{u}{r}\|_{L_x^\infty}\|u(M-y)^{\frac{1}{4}}\|_{L^\infty[\frac{M}{4},M]}(M-x)^{-\frac{1}{4}}dxds$$
$$\leq C\|g-g_\infty\|_{L^\infty}(\int_0^M u^2dx)^{\frac{1}{2}} + C\epsilon_0^2 + C\int_0^t \|u(M-x)^{\frac{1}{4}}\|_{L^\infty[\frac{M}{4},M]}[\int_0^M \rho^2(r^2u)_x^2dxds]^{\frac{1}{2}}$$
$$+ \frac{1}{4}\int_0^t\int_0^M \psi^2(M-x)^{-2}(\rho^2-\rho_\infty^2)^2dxds + C\int_0^t \|u(M-x)^{\frac{1}{4}}\|^2_{L^\infty[\frac{M}{4},M]}ds$$
$$\leq \frac{1}{4}\int_0^t\int_0^M \psi^2(M-x)^{-2}(\rho^2-\rho_\infty^2)^2dxds + C\epsilon_4^2. \tag{3.5.58}$$

$$J_2 \leq C\int_0^t\int_0^M \psi^2(M-x)^{-2}|\rho^2-\rho_\infty^2|[\int_0^M \rho_\infty(g-g_\infty)^2dx]^{\frac{1}{2}}\int_x^M (M-y)^{-\frac{1}{4}}dydxds$$
$$+ Ca^3(1+t) \leq \frac{1}{4}\int_0^t\int_0^M \psi^2(M-x)^{-2}(\rho^2-\rho_\infty^2)^2dxds$$
$$+ C\int_0^t\int_0^M \rho_\infty(g-g_\infty)^2dxds\int_0^M (M-z)^{-\frac{1}{2}}dz + C\epsilon_0^2$$
$$\leq \frac{1}{4}\int_0^t\int_0^M \psi^2(M-x)^{-2}(\rho^2-\rho_\infty^2)^2dxds + C\epsilon_4^2. \tag{3.5.59}$$

$$J_3 = -\int_0^t\int_0^M \psi^2(M-x)^{-2}(\rho^2-\rho_\infty^2)\rho_t dxds \leq C\|g-g_\infty\|_{L^\infty}\int_0^M (M-x)^{-\frac{1}{2}}dx$$
$$+ C\epsilon_4^2 \leq C\epsilon_4^2. \tag{3.5.60}$$

$$
\begin{aligned}
J_4 &\le C\int_0^t\int_0^M \psi^2(M-x)^{-\frac{5}{8}}(M-x)^{-\frac{1}{4}}|u(M-x)^{\frac{1}{4}}|\|\rho^{\frac{1}{4}}-\rho_\infty^{\frac{1}{4}}\|_{L^\infty([\frac{M}{4},M])}dxds\\
&\le C\sup_{t\in[0,T_2]}[(1+s)^{\frac{1}{16}}\|x^{\frac{\alpha+2}{24}}(\rho^{\frac{3}{8}}-\rho_\infty^{\frac{3}{8}})\|_{L_x^\infty}]^{\frac{2}{3}}\\
&\int_0^t(1+s)^{-\frac{1}{24}}(1+s)^{-\frac{1}{2}}(1+s)^{\frac{1}{2}}\|u(M-x)^{\frac{1}{4}}\|_{L^\infty([\frac{M}{4},M])}ds\\
&\le C\sup_{t\in[0,T_2]}[(1+s)^{\frac{1}{16}}\|x^{\frac{\alpha+2}{24}}(\rho^{\frac{3}{8}}-\rho_\infty^{\frac{3}{8}})\|_{L_x^\infty}]^{\frac{2}{3}}\\
&(\int_0^t(1+s)^{-1-\frac{1}{12}}ds)^{\frac{1}{2}}(\int_0^t(1+s)\|u(M-x)^{\frac{1}{4}}\|^2_{L^\infty([\frac{M}{4},M])}ds)^{\frac{1}{2}}\\
&\le C\epsilon_4^2.
\end{aligned}
\tag{3.5.61}
$$

$$
\begin{aligned}
J_5 &\le C\int_0^t\int_0^M \psi^2(M-x)^{-1}|g-g_\infty|\int_x^M(|\rho u_x|+|u|)dydxds\\
&\le C\int_0^t\int_0^M \psi^2(M-x)^{-\frac{9}{8}}\|\rho^{\frac{1}{4}}-\rho_\infty^{\frac{1}{4}}\|_{L^\infty([\frac{M}{4},M])}[\int_x^M(\rho^2u_x^2+u^2)dy]^{\frac{1}{2}}(M-x)^{\frac{1}{2}}dxds\\
&\le C\int_0^t\int_0^M \psi^2(1+s)^{-\frac{1}{24}}(M-x)^{-\frac{5}{8}}[\int_x^M(\rho^2u_x^2+u^2)dy]^{\frac{1}{2}}dxds\\
&\le C[\int_0^t\int_0^M(1+s)(\rho^2r^4u_x^2+u^2)dxds]^{\frac{1}{2}}[\int_0^t(1+s)^{-1-\frac{1}{12}}ds]^{\frac{1}{2}}\le C\epsilon_4^2.
\end{aligned}
\tag{3.5.62}
$$

From (3.5.57) – (3.5.62), we get (3.5.56) immediately. □

Lemma 3.5.11 *Under the assumptions of Theorem 3.2.1, if a is small enough, it holds that*

$$
\int_0^M \psi^2(M-x)^{\frac{1}{2}}(\rho-\rho_\infty)_x^2dx+\int_0^t\int_0^M \psi^2(M-x)(\rho-\rho_\infty)_x^2dxds\le C_{13}\epsilon_4^2, \tag{3.5.63}
$$

for all $t\in[0,T_2]$.

Proof: From $(3.4.2)_2$, we have

$$
[u+r^2\rho_x-r_\infty^2(\rho_\infty)_x]_t+2\rho[r^2\rho_x-r_\infty^2(\rho_\infty)_x]+r^2G\frac{x}{r^4}+2r_\infty^2\rho(\rho_\infty)_x=0. \tag{3.5.64}
$$

Multiplying (3.5.64) by $\psi^2(M-x)^{\frac{1}{2}}[u+r^2\rho_x-r_\infty^2(\rho_\infty)_x]$, integrating over $[0,M]$, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_0^M(M-x)^{\frac{1}{2}}\psi^2|u+r^2\rho_x-r_\infty^2(\rho_\infty)_x|^2dx\\&+\int_0^M 2\psi^2(M-x)^{\frac{1}{2}}\rho(r^2\rho_x-r_\infty^2\rho_{\infty x})[u+r^2\rho_x-r_\infty^2(\rho_\infty)_x]dx\\&+\int_0^M\psi^2(M-x)^{\frac{1}{2}}[u+r^2\rho_x-r_\infty^2(\rho_\infty)_x][r^2G\frac{x}{r^4}+r_\infty^2\frac{\rho}{\rho_\infty}(\rho_\infty^2)_x]dx=0,\end{aligned}\tag{3.5.65}$$

using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_0^M(M-x)^{\frac{1}{2}}\psi^2|u+r^2\rho_x-r_\infty^2(\rho_\infty)_x|^2dx+C_{15}\int_0^M\psi^2(r^2\rho_x-r_\infty^2\rho_x)^2dx\\&\le C\int_0^M(M-x)\psi^2u^2dx+\int_0^M\psi^2|r^2G\frac{x}{r^4}+r_\infty^2\frac{\rho}{\rho_\infty}(\rho_\infty^2)_x|^2dx,\end{aligned}\tag{3.5.66}$$

making use of (3.5.8) – (3.5.9), we get

$$\begin{aligned}&\int_0^M\psi^2|r^2G\frac{x}{r^4}+r_\infty^2\frac{\rho}{\rho_\infty}(\rho_\infty^2)_x|^2dx=\int_0^M\psi^2|r^2\frac{Gx}{r^4}-\frac{Gx\rho r_\infty^2}{\rho_\infty r_\infty^4}|^2dx\\&\le C\int_0^M\psi^2[(r-r_\infty)^2+(g-g_\infty)^2]dx,\end{aligned}$$

then we have

$$\begin{aligned}&\frac{d}{dt}\int_0^M(M-x)^{\frac{1}{2}}\psi^2|u+r^2\rho_x-r_\infty^2(\rho_\infty)_x|^2dx+C_{23}\int_0^M\psi^2(r^2\rho_x-r_\infty^2\rho_x)^2dx\\&\le C\int_0^M\psi^2u^2dx+\int_0^M[(\frac{r^2-a^3-r_\infty^3}{x})^2+\psi^2(g-g_\infty)^2]dx+Ca^2,\end{aligned}\tag{3.5.67}$$

if $a^2(1+T)\le\epsilon_0$, by using of (3.5.18), (3.5.24), (3.5.57) and (3.5.66), we obtain (3.5.63) immediately. □

Lemma 3.5.12 *Under the assumptions of Theorem 3.2.1, it holds that*

$$B_1[\rho,r]\le C\epsilon_4^2,\tag{3.5.68}$$

$$(1+t)^{\frac{1}{4}}\|x^{\frac{\alpha+2}{6}}(\rho^{\frac{3}{2}}-\rho_{\infty}^{\frac{3}{2}})\|_{L_x^\infty}\le C_{14}\epsilon_4, \tag{3.5.69}$$

$$(1+t)^{\frac{1}{16}}\|x^{\frac{\alpha+2}{24}}(\rho^{\frac{3}{8}}-\rho_{\infty}^{\frac{3}{8}})\|_{L_x^\infty}\le C_{14}\epsilon_4^{\nu}, \tag{3.5.70}$$

for all $t\in[0,T_2]$, *where* $\nu=\dfrac{1}{4}$.

Proof: From (3.5.8) – (3.5.9), (3.5.42), (3.5.56) and (3.5.63), we have (3.5.68) and

$$\int_0^M r^{\alpha+2}(\rho^{\frac{3}{2}}-\rho_{\infty}^{\frac{3}{2}})_x^2dx\le C\epsilon_4^2,\ \ t\in[0,T_2], \tag{3.5.71}$$

using Galiardo-Nirenberg inequality $\|\phi\|_{L^\infty}\le\|\phi\|_{L^2}^{\frac{1}{2}}\|\phi'\|_{L^2}^{\frac{1}{2}}$ and (3.5.16), we have

$$(1+t)^{\frac{1}{4}}\|x^{\frac{\alpha+2}{6}}(\rho^{\frac{3}{2}}-\rho_{\infty}^{\frac{3}{2}})\|_{L_x^\infty}\le C\epsilon_4.$$

From (3.5.8) – (3.5.9) and (3.5.69), we have

$$\begin{aligned}&\|x^{\frac{\alpha+2}{24}}(\rho^{\frac{3}{8}}-\rho_{\infty}^{\frac{3}{8}})\|_{L^\infty([0,M])}\\&\le C\|x^{\frac{\alpha+2}{6}}(\rho^{\frac{3}{2}}-\rho_{\infty}^{\frac{3}{2}})\|_{L^\infty([0,M])}^{\frac{1}{4}}\le C(1+t)^{-\frac{1}{16}}\epsilon_4^{\frac{1}{4}}.\end{aligned}$$

□

Lemma 3.5.13 *Under the assumptions of Theorem 3.2.1, if a is small enough, it holds that*

$$(1+t)\int_0^M(\frac{u^2}{r^2}+\rho^2r^4u_x^2)dx+\int_0^t\int_0^M(1+s)u_t^2dxds\le C_{15}\epsilon_3^2[1+\|\rho(r^2u)_x\|_{L_{t,x}^\infty}], \tag{3.5.72}$$

for all $t\in[0,T_2]$.

Proof: Multiply $(3.4.2)_2$ by $(1+t)u_t$, integrating over $[0,M]\times[0,t]$, using integrating by parts and boundary conditions, we get

$$\begin{aligned}&\int_0^t\int_0^M(1+s)u_t^2dxds=\int_0^t\int_0^M(1+s)\rho^2(r^2u_t)_xdxds\\&-\int_0^t\int_0^M(1+s)\rho^2(r^2u)_x(r^2u_t)_xdxds+\int_0^t\int_0^M2(1+s)\rho(ruu_s)_xdxds\\&-\int_0^t\int_0^M(1+s)G\frac{x}{r^2}u_sdxds=\sum_{i=1}^4H_i.\end{aligned} \tag{3.5.73}$$

Using Cauchy-Schwarz inequality, A_3, (3.5.8)−(3.5.9), (3.5.16)−(3.5.17), (3.5.24), we obtain

$$H_2 + H_3 = \{(1+s)\int_0^M [-\frac{1}{2}\rho^2(r^2u)_x^2 + \rho(ru^2)_x]dx\}|_0^t$$
$$+ \int_0^t\int_0^M [\frac{1}{2}\rho^2(r^2u)_x^2 - \rho(ru^2)_x]dxds$$
$$+ \int_0^t\int_0^M (1+s)\{2\rho^2(r^2u)_x(ru^2)_x - \rho^3(r^2u)_x^3 + 2\rho^2(r^2u)_x^2\frac{u}{r} - 6\rho\frac{u^2}{r^2}(r^2u)_x + 6\frac{u^3}{r^3}\}dxds$$
$$\leq C\epsilon_3^2(1 + \|\rho(r^2u)_x\|_{L_x^\infty} - C_{18}(1+t)\int_0^M [\rho^2(r^2u)_x^2 + \frac{u^2}{r^2}]dx, \tag{3.5.74}$$

$$H_1 = (1+s)\int_0^M \rho^2(r^2u)_x dx|_0^t + \int_0^t\int_0^M 2(1+s)\rho^3(r^2u)_x^2 dxds$$
$$- \int_0^t\int_0^M 4(1+s)\rho^2\frac{u}{r}(r^2u)_x dxds + \int_0^t\int_0^M 6(1+s)\rho\frac{u^2}{r^2}dxds - \int_0^t\int_0^M \rho^2(r^2u)_x dxds$$
$$\leq (1+t)\int_0^M \rho^2(r^2u)_x dx + C\epsilon_3^2 - \int_0^t\int_0^M \rho^2(r^2u)_x dxdx \tag{3.5.75}$$

$$H_4 = -(1+s)\int_0^M G\frac{ux}{r^2}dx|_0^t - 2\int_0^t\int_0^M (1+s)Gxr^{-3}u^2 dxds + \int_0^t\int_0^M G\frac{ux}{r^2}dxds$$
$$\leq -(1+t)\int_0^M G\frac{ux}{r^2}dx + C\epsilon_3^2 + \int_0^t\int_0^M G\frac{ux}{r^2}dxds. \tag{3.5.76}$$

Using (3.5.8) − (3.5.9), (3.5.14), (3.5.16) − (3.5.18), (3.5.24), we obtain

$$(1+t)\int_0^M \rho^2(r^2u)_x dx - (1+t)\int_0^M G\frac{ux}{r^2}dx$$
$$= (1+t)\int_0^M [(\rho^2 - \rho_\infty^2)(r^2u)_x - Gr^2ux(r^{-4} - r_\infty^{-4})]dx$$
$$\leq \frac{C_{16}}{2}(1+t)\int_0^M [\rho^2(r^2u)_x^2 + \frac{u^2}{r^2}]dx \tag{3.5.77}$$
$$+ C\int_0^M (1+t)[\rho_\infty(g - g_\infty)^2 + \frac{(r^3 - a^3 - r_\infty^3)^2}{x^2}]dx$$
$$+ Ca^3(1+t) \leq \frac{C_{16}}{2}(1+t)\int_0^M [\rho^2(r^2u)_x^2 + \frac{u^2}{r^2}]dx + C\epsilon_3^2,$$

and

$$\begin{aligned}&\int_0^t\int_0^M[G\frac{ux}{r^2}-\rho^2(r^2u)_x]dxds\\&\le C\int_0^t\int_0^M(\rho^2(r^2u)_x^2+\frac{u^2}{r^2})dxds+C\int_0^t\int_0^M[\rho_\infty(g-g_\infty)^2\\&+\frac{(r^2-a^3-r_\infty^3)^2}{x^2}]dxds+Ca^3(1+t)\le C\epsilon_3^2.\end{aligned}\tag{3.5.78}$$

From (3.5.73) – (3.5.78), we obtain (3.5.72) immediately. □

Lemma 3.5.14 *Under the assumptions of Theorem 3.2.1, if ϵ_1 is small enough, it holds that*

$$\int_0^M r^{\alpha-2}u_t^2(x,t)dx+\int_0^t\int_0^M(\rho^2r^{2+\alpha}u_{xt}^2+r^{\alpha-4}u_t^2)dxds\le C_{17}\epsilon_3,\tag{3.5.79}$$

$$\|\frac{u}{r}(\cdot,t)\|_{L^\infty}+\|\rho(r^2u)_x(\cdot,t)\|_{L^\infty}\le C_{17}\epsilon_4,\tag{3.5.80}$$

$$\int_0^M(\frac{u^2}{r^2}+\rho^2r^4u_x^2)(x,t)dx+\int_0^t\int_0^M u_t^2(x,s)dxds\le C_{17}\epsilon_3,\tag{3.5.81}$$

for all $t\in[0,T_2]$.

Proof: Differentiate the equation $(3.4.2)_2$ with respect to t, multiply it by $r^{\alpha-2}u_t$, and integrating it over $[0,M]$, using the boundary conditions, we have

$$\begin{aligned}&\frac{d}{dt}\int_0^M\frac{1}{2}r^{\alpha-2}u_t^2dx-\frac{\alpha-2}{2}\int_0^M r^{\alpha-3}uu_t^2dx\\&=-\int_0^M[\rho^2(r^2u)_x-\rho^2-2\rho\frac{u}{r}](2r^{\alpha-1}uu_t)_xdx\\&-\int_0^M\partial_t[\rho^2(r^2u)_x-2\rho\frac{u}{r}-\rho^2+\rho_\infty^2](r^\alpha u_t)_xdx+\int_0^M 2[\rho r^2(\frac{u}{r})_x]_tr^{\alpha-2}u_tdx\\&-\int_0^M(G\frac{x}{r^2})_tr^{\alpha-2}u_tdx=\sum_{i=1}^4F_i.\end{aligned}\tag{3.5.82}$$

From (3.5.8) – (3.5.9) and (3.5.24), we have

$$\begin{aligned}
F_2+F_3 = &-\int_0^M [\rho^2(r^2u_t)_x(r^\alpha u_t)_x - 2\rho(r^{\alpha-1}u_t^2)_x]dx + \int_0^M \{2\rho^3(r^2u)_x^2 - 2\rho^2(ru^2)_x \\
&- 2\rho^3(r^2u)_x - 2\rho^2(r^2u)_x\frac{u}{r} - \rho\frac{u^2}{r^2}\}[\alpha\frac{r^{\alpha-3}u_t}{\rho} + r^\alpha u_{tx}]dx \\
&+ 2\int_0^M \{2r^{\alpha-1}u\rho(\frac{u}{r})_x u_t - r^\alpha\rho^2(r^2u)_x(\frac{u}{r})_x u_t - r^\alpha\rho(\frac{u^2}{r^2})_x u_t\}dx \\
&\le -C_{18}\int_0^M (\rho^2r^{\alpha+2}u_{tx}^2 + r^{\alpha-4}u_t^2)dx \\
&+ C(\|\rho(r^2u)_x\|_{L_x^\infty}^2 + 1)\int_0^M [\rho^2r^{2+\alpha}u_x^2 + r^{\alpha-4}u^2]dx,
\end{aligned} \tag{3.5.83}$$

$$F_1 \le \frac{C_{18}}{4}\int_0^M (\rho^2r^{\alpha+2}u_{tx}^2 + r^{\alpha-4}u_t^2)dx + C(\|\rho(r^2u)_x\|_{L_x^\infty}^2 + 1)\int_0^M [\rho^2r^{2+\alpha}u_x^2 + r^{\alpha-4}u^2]dx, \tag{3.5.84}$$

and

$$F_4 \le \frac{C_{18}}{4}\int_0^M r^{\alpha-4}u_t^2dx + C\int_0^M u^2r^\alpha dx \le \frac{C_{18}}{4}\int_0^M r^{\alpha-4}u_t^2dx + C\int_0^M r^{\alpha-4}u^2dx. \tag{3.5.85}$$

From (3.5.82) – (3.5.85), we have

$$\begin{aligned}
&\frac{d}{dt}\int_0^M r^{\alpha-2}u_t^2dx + \frac{C_{18}}{2}\int_0^M (\rho^2r^{\alpha+2}u_{tx}^2 + r^{\alpha-4}u_t^2)dx \\
&\le C(\|\rho(r^2u)_x\|_{L_x^\infty}^2 + 1)\int_0^M [\rho^2r^{2+\alpha}u_x^2 + r^{\alpha-4}u^2]dx + C\int_0^M u^2dx \\
&+ C_{19}\epsilon_1\int_0^M r^{\alpha-4}u_t^2dx,
\end{aligned}$$

if $C_{19}\epsilon_1 \le \frac{C_{18}}{4}$, that is, $4C_{19}\epsilon_1 \le C_{18}$ and (3.5.41), we have

$$\int_0^M r^{\alpha-2}u_t^2dx + \int_0^t\int_0^M (\rho^2r^{\alpha+2}u_{tx}^2 + r^{\alpha-4}u_t^2)dxds \le C\epsilon_3(\|\rho(r^2u)_x\|_{L_x^\infty}^2 + 1). \tag{3.5.86}$$

From $(3.4.2)_2$, we have

$$\frac{u_t}{r^2} + (\rho^2)_x = (\rho^2(r^2u)_x)_x - 2\frac{u}{r}\rho_x - G\frac{x}{r^4}.$$

Integrating the above equation over $[0, M]$, using the boundary conditions and integrating by parts, we obtain

$$\rho^2(r^2u)_x = \rho^2(x,t) - \int_x^M \frac{u_t}{r^2}dy + 2\int_x^M \rho(\frac{u}{r})_x dy + 2\rho\frac{u}{r} - \int_x^M Gy(r^{-4} - r_\infty^{-4})dy - \rho_\infty^2(x).$$

Hence we have

$$\rho(r^2u)_x = \frac{\rho^2(x,t) - \rho_\infty^2(x)}{\rho} - \frac{1}{\rho}\int_x^M \frac{u_t}{r^2}dy + \frac{2}{\rho}\int_x^M \rho(\frac{u}{r})_x dy + 2\frac{u}{r} - \frac{1}{\rho}\int_x^M Gy(r^{-4} - r_\infty^{-4})dy. \tag{3.5.87}$$

Using (3.5.8) – (3.5.9), (3.5.72) and (3.5.86), we conclude that

$$\begin{aligned} \|\rho(r^2u)_x\|_{L^\infty([\frac{M}{3},M])} &\le C\epsilon_4 + 2\epsilon_1 + C(\int_0^M [\rho^2 r^4 u_x^2 + \frac{u^2}{r^2} + r^{\alpha-2}u_t^2]dx)^{\frac{1}{2}} \\ &\le C\epsilon_4 + 2\epsilon_1 + C\epsilon_3^{\frac{1}{2}}\|\rho(r^2u)_x\|_{L^\infty_{tx}}, \ \ t \in [0, T_2], \end{aligned} \tag{3.5.88}$$

on the other hand, from the equation $(3.4.2)_2$, we have

$$r^2[\rho^2(r^2u)_x]_x = u_t + r^2(\rho^2)_x + 2ru\rho_x + G\frac{x}{r^2},$$

using the estimate (3.5.9) – (3.5.10), (3.5.40) – (3.5.42) and (3.5.86), we conclude that

$$\int_0^M r^{\alpha+2}|(\rho^2(r^2u)_x)_x|^2\phi^2 dx \le C\epsilon_3(1 + \|\rho(r^2u)_x\|^2_{L^\infty_{tx}}),$$

and

$$\int_0^{\frac{M}{2}} |\partial_x(\rho^2(r^2u)_x)|dx \le C\epsilon_3^{\frac{1}{2}}(1 + \|\rho(r^2u)_x\|_{L^\infty_{tx}}), \tag{3.5.89}$$

using (3.5.8), (3.5.72), (3.5.89) and Sobolev's embedding theorem $W^{1,1} \hookrightarrow L^\infty$, we have

$$\|\rho(r^2u)_x\|_{L^\infty([0,\frac{M}{2}])} \le C\epsilon_3^{\frac{1}{2}}(1 + \|\rho(r^2u)_x\|_{L^\infty_{tx}}),$$

and

$$\|\rho(r^2u)_x\|_{L^\infty_{tx}} \le C_{20}\epsilon_4(1 + \|\rho(r^2u)_x\|_{L^\infty_{tx}}) + C\epsilon_1, \tag{3.5.90}$$

if $C_{20}\epsilon_4 \le \frac{1}{2}$, then we can obtain that

$$\|\rho(r^2u)_x\|_{L^\infty_{tx}} \le C\epsilon_1 + C\epsilon_4.$$

Hence we can get (3.5.79) and (3.5.81) immediately. From (4.2.8), for $x \in [\frac{M}{3}, M]$, we have

$$\begin{aligned}
&\|\rho r^2 u_x\|_{L^\infty([\frac{M}{3},M])} \\
&\le C(\int_0^M r^{\alpha-2}u_t^2 dx)^{\frac{1}{2}} + C\|g-g_\infty\|_{L^\infty([\frac{M}{3},M])} + C(\int_0^M \rho^2 r^4 u_x^2 dx)^{\frac{1}{2}} \\
&+ C(\int_0^M u^2 dx)^{\frac{1}{2}} + C[\int_0^M x^{-2}(r^3 - a^3 - r_\infty^3)^2 dx]^{\frac{1}{2}} + Ca^3 \\
&\le C\epsilon_4.
\end{aligned}$$

For $x \in [\frac{M}{3}, M]$, we have $r \ge C > 0$, by $W^{1,1} \hookrightarrow L^\infty$, we have

$$\begin{aligned}
\frac{u}{r} &\le \frac{C}{r}(\int_{\frac{M}{3}}^M |u|dx + \int_{\frac{M}{3}}^M |u_x|dx) \\
&\le C(\int_0^M u^2 dx)^{\frac{1}{2}} + C\|\rho r^2 u_x\|_{L^\infty([\frac{M}{3},M])},
\end{aligned}$$

thus, we get

$$\|\frac{u}{r}\|_{L^\infty([\frac{M}{3},M])} \le C\epsilon_4.$$

For $x \in [0, \frac{M}{3}]$, we have

$$\frac{u}{r} = \frac{1}{r^3}\int_0^x (r^2 u)_y dy.$$

From the previous estimate, it is easy to obtain

$$\|\rho(r^2 u)_x\|_{L^\infty([0,\frac{M}{2}])} \le C\epsilon_3^{\frac{1}{2}},$$

hence we have

$$|\frac{u}{r}|_{L^\infty([0,M])} + \|\rho(r^2 u)_x\|_{L^\infty([0,M])} \le C_{19}\epsilon_4.$$

We finish the proof of the Lemma 3.5.10. □

Now we can choose

$$\epsilon_1 = C_0\epsilon_0 + (C_5 + C_8 + C_{17})\epsilon_4 + C_{14}\epsilon_4^\nu, \tag{3.5.91}$$

if

$$(\frac{4}{\underline{g}} + \frac{C_3}{\delta_3} + C_{10}^2 + 4\frac{C_{12}}{C_{11}} + 4\frac{C_{19}}{C_{18}})\epsilon_1 + (2C_{20} + \delta_5^{-1})\epsilon_4 \le 1. \tag{3.5.92}$$

Using the obtained Lemmas 3.5.3-3.5.14, we can proof the Claim 1. Finally, we can get $T^* > T$ and the following Lemma by using of the classical continuation method.

Lemma 3.5.15 *Under the assumptions in Theorem 3.2.1 and* $a(1+T) \le \epsilon_0$*, the solution* (ρ, u) *satisfies the estimates* (3.5.8)−(3.5.11)*,* (3.5.16)−(3.5.18)*,* (3.5.29)*,* (3.5.40) − (3.5.44)*,* (3.5.56)*,* (3.5.63)*,* (3.5.68) − (3.5.70) *and* (3.5.79) − (3.5.81) *for all* $t \in [0, T]$.

Remark 3.5.16 *For* $\lambda(\rho) \neq 0$*, we cannot obtain* $\|\rho r^2 u_x\|_{L^\infty([\frac{M}{3}, M])}$ *directly from the equation* $(3.1.9)_2$*. However, we can derive the uniform bounds of* $\|\rho^{1+\theta} r^2 u_x\|_{L^\infty([\frac{M}{3}, M])}$ *by using of* (3.5.54)*, where* $0 < \theta < 1$*. Once we obtain this result, we can use the similar argument to obtain* $\|\frac{u}{r}\|_{L^\infty([\frac{M}{3}, M])}$.

3.6 Global existence

From Lemmas 3.5.3-3.5.15, we obtain that the solution (ρ_a, u_a, r_a) exists on $[0, M] \times [0, T]$ and satisfies

$$C^{-1}(M-x)^{\frac{1}{2}} \le \rho_a(x,t) \le C(M-x)^{\frac{1}{2}}, \tag{3.6.1}$$

$$a^3 + C^{-1}x \le r_a^3(x,t) \le a^3 + Cx, \tag{3.6.2}$$

$$\int_0^T \int_0^M r_a^{\alpha-2}[\rho_a^2 r_a^4 (\partial_x u_a)^2 + \frac{u_a^2}{r_a^2}] dx ds \le C, \tag{3.6.3}$$

$$\sup_{t\in[0,T]} \int_0^M x^{\frac{2+\alpha}{3}} (M-x)^{\frac{1}{2}} (\rho_a - \rho_\infty)_x^2 dx + \int_0^T \int_0^M x^{\frac{2+\alpha}{3}} (M-x)(\rho_a - \rho_\infty)_x^2 dx ds \le C, \tag{3.6.4}$$

$$\sup_{t\in[0,T]}\int_0^M r_a^{2+\alpha}(\partial_t u_a)^2dx+\int_0^T\int_0^M[\rho_a^2 r_a^{2+\alpha}(u_a)_{xt}^2+r^{\alpha-4}(u_a)_t^2]dxds\le C, \quad (3.6.5)$$

$$\|\frac{u_a}{r_a}(\cdot,t)\|_{L^\infty}+\|\rho_a(r_a^2u_a)_x(\cdot,t)\|_{L^\infty}\le C, \quad (3.6.6)$$

$$\sup_{t\in[0,T]}\int_0^M[\frac{u_a^2}{r_a^2}+\rho_a^2r_a^4(u_a)_x^2]dx+\int_0^T\int_0^M(u_a)_t^2dxds\le C, \quad (3.6.7)$$

$$\sup_{t\in[0,T]}\int_0^M(\rho_a)_t^2dx\le C, \quad (3.6.8)$$

where $x\in[0,M]$ and $t\in[0,T]$.

Let $a\longrightarrow 0$, we have

$$(\rho_a,u_a,r_a)\longrightarrow(\rho,u,r)\quad in\quad C([0,M]\times[0,T]),$$

$$x^{\frac{2+\alpha}{6}}(M-x)^{\frac{1}{4}}(\rho_a)_x\stackrel{*}{\rightharpoonup}x^{\frac{2+\alpha}{6}}(M-x)^{\frac{1}{4}}\rho_x,$$

in $L^\infty([0,T];L^2[0,M])$,

$$\rho_a(r_a^2u_a)_x\stackrel{*}{\rightharpoonup}\rho(r^2u)_x\quad in\quad L^\infty([0,M]\times[0,T]),$$

$$r_a^{\frac{\alpha-2}{2}}\partial_tu_a\stackrel{*}{\rightharpoonup}r^{\frac{\alpha-2}{2}}\partial_tu\quad in\quad L^\infty([0,T];L^2[0,M]),$$

$$\rho_ar_a^{\frac{\alpha+2}{2}}(u_a)_{xt}\rightharpoonup\rho r^{\frac{\alpha+2}{2}}u_{xt}\quad in\quad L^2([0,M]\times[0,T]),$$

$$\partial_t\rho_a\stackrel{*}{\rightharpoonup}\partial_t\rho\quad in\quad L^\infty([0,T];L^2[0,M]),$$

where (ρ,u,r) is a weak solution to the system (3.1.9) – (3.1.11) on $[0,M]\times[0,T]$ and satisfies the regularity estimates (3.6.1) – (3.6.8). Since the constant

C is independent of T, we can extend the existence interval $[0,T]$ of the solution to $[0,\infty)$, and obtain the global existence of the solution to the system (3.1.9) – (3.1.11) .

3.7 Uniqueness

We can use the energy method to prove the uniqueness of the solution in Theorem 3.2.1. Let $(\rho_1,u_1,r_1)(x,t)$ and $(\rho_2,u_2,r_2)(x,t)$ be solutions in Theorem 3.2.1, then we have

$$C^{-1}(M-x)^{\frac{1}{2}} \le \rho_i(x,t) \le C(M-x)^{\frac{1}{2}},\ \ C^{-1}x^{\frac{1}{3}} \le r_i \le Cx^{\frac{1}{3}}, \tag{3.7.1}$$

$$|x^{-\frac{1}{3}}u_i(x,t)| + |(M-x)^{\frac{1}{2}}x^{\frac{2}{3}}\partial_x u_i(x,t)| \le C,\ \ i=1,2. \tag{3.7.2}$$

For simplicity, we may assume that $(\rho_1,u_1,r_1)(x,t)$ and $(\rho_2,u_2,r_2)(x,t)$ are suitably smooth. Otherwise we can use the Friedrichs mollifier to regularities solutions.

Let

$$\varrho = \rho_1-\rho_2,\ \ \omega = u_1-u_2,\ \ R = r_1-r_2.$$

Since $\partial_t r_i = u_i$, we have

$$\begin{aligned}\frac{d}{dt}\int_0^M x^{-\frac{2}{3}}R^2dx &= \int_0^M 2x^{-\frac{2}{3}}Rwdx\\ &\le \epsilon\int_0^M x^{-\frac{2}{3}}w^2dx + C_\epsilon\int_0^M x^{-\frac{2}{3}}R^2dx.\end{aligned} \tag{3.7.3}$$

From $(3.1.9)_1$ and (3.7.1) – (3.7.2), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_0^M (M-x)^{-1}\rho^2dx = 2\int_0^M (M-x)^{-1}\rho\partial_t(\rho_1-\rho_2)dx\\ &= 2\int_0^M (M-x)^{-1}\rho(-\rho_1^2r_1^2u_{1x} + \rho_2^2r_2^2u_{2x} - 2\rho_1\frac{u_1}{r_1} + 2\rho_2\frac{u_2}{r_2})dx\\ &\le \epsilon\int_0^M ((M-x)x^{\frac{4}{3}}w_x^2 + x^{-\frac{2}{3}}w^2)dx + C_\epsilon\int_0^M ((M-x)^{-1}\varrho^2 + x^{-\frac{2}{3}}R^2)dx.\end{aligned} \tag{3.7.4}$$

From the equation $(3.1.9)_2$ and the boundary conditions, we can get

$$\begin{aligned}
&\frac{d}{dt}\int_0^M \frac{1}{2}w^2(x,t)dx + \int_0^M \rho_1^2 r_1^4 w_x^2 dx + \int_0^M 2\frac{w^2}{r_1^2}dx \\
&= -\int_0^M \partial_x(r_1^2 w)(\rho_1^2-\rho_2^2)(r_1^2 u_2)_x dx - \int_0^M (r_1^2 w)_x \rho_2^2[(r_1^2-r_2^2)u_2]_x dx \\
&- \int_0^M [(r_1^2-r_2^2)w]_x(\rho_2^2(r_2^2 u_2)_x)dx + \int_0^M 2[r_1^2 w u_2(\frac{1}{r_1}-\frac{1}{r_2})]_x \rho_1 dx \\
&+ \int_0^M \partial_x(r_1^2 w)(\rho_1^2-\rho_2^2)dx + \int_0^M [(r_1^2-r_2^2)w]_x \rho_2^2 dx + \int_0^M 2(wr_1^2\frac{u_2}{r_2})(\rho_1-\rho_2)dx \\
&+ 2\int_0^M [w\frac{u_2}{r_2}(r_1^2-r_2^2)]_x \rho_2 dx + \int_0^M wGx(r_2^{-2}-r_1^{-2})dx.
\end{aligned} \tag{3.7.5}$$

From (3.7.1) – (3.7.2) and (3.7.5), we have

$$\begin{aligned}
&\frac{d}{dt}\int_0^M \frac{1}{2}w^2 dx + C\int_0^M \{(M-x)x^{\frac{4}{3}}w_x^2 + x^{-\frac{2}{3}}w^2\}dx \\
&\le C_\epsilon \int_0^M (x^{-\frac{2}{3}}R^2 + (M-x)^{-1}\varrho^2)dx + \epsilon\int_0^M x^{\frac{2}{3}}w^2 dx.
\end{aligned} \tag{3.7.6}$$

Since

$$\int_0^M x^{\frac{2}{3}}w^2 dx \le C\int_0^M w^2 dx,$$

we have

$$\begin{aligned}
&\frac{d}{dt}\int_0^M (w^2 + (M-x)^{-1}\varrho^2 + x^{-\frac{2}{3}}R^2)dx \\
&\le C\int_0^M (w^2 + (M-x)^{-1}\varrho^2 + x^{-\frac{2}{3}}R^2)dx.
\end{aligned}$$

Using Gronwall's inequality, we have for any $t \in [0,T]$,

$$\int_0^M (w^2 + (M-x)^{-1}\varrho^2 + x^{-\frac{2}{3}}R^2)dx = 0.$$

Thus, we prove the uniqueness of solution and finish the proof of Theorem 3.2.1. □

Chapter 4

Local well-posedness of Navier-Stokes-Poisson equations

4.1 Introduction

The motion of self-gravitating viscous gaseous stars can be described by the compressible Navier-Stokes-Poisson system:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \dfrac{\partial (\rho \mathbf{u})}{\partial t} + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P = -\rho \nabla \Phi + \operatorname{div}(\mu(\rho) \nabla \mathbf{u}), \\ \triangle \Phi = 4\pi \rho, \end{cases} \tag{4.1.1}$$

where $t \geq 0$, $x \in \mathbb{R}^3$, $\rho \geq 0$ is the density, $u \in \mathbb{R}^3$ the velocity, P the pressure, Φ the potential function of the self-gravitational force, and $\mu(\rho) \geq 0$ the viscosity coefficient. The equation of state is given by

$$P(\rho) = A\rho^\gamma,$$

where A is an entropy constant and $\gamma > 1$ is an adiabatic exponent. In this paper, we consider $\mu(\rho) = c\rho$, where $c > 0$ is a constant. Without loss of generality, we suppose $A = 1$, $c = 1$.

For the spherically symmetric motion, i.e. $\rho(\mathbf{x}, t) = \rho(r, t)$ and $\mathbf{u}(\mathbf{x}, t) = u(r, t)\frac{\mathbf{x}}{r}$, where u is a scalar function and $r = |\mathbf{x}|$, (4.1.1) can be written as follows:

$$\begin{cases} \rho_t + \frac{1}{r^2}(r^2\rho u)_r = 0, \\ \rho u_t + \rho u u_r + P_r + \frac{4\pi\rho}{r^2}\int_0^r \rho s^2 ds = \big(\rho(u_r + \frac{2}{r}u)\big)_r - 2\frac{u}{r}\rho_r. \end{cases} \tag{4.1.2}$$

We consider a free boundary value problem to (4.1.2) under the following vacuum boundary condition

$$\rho(R(t), t) = 0 \quad \text{and} \quad \rho(r, t) > 0 \quad \text{for} \quad r < R(t), \tag{4.1.3}$$

and the dynamic boundary condition

$$(\mu(\rho)u_r - P)(R(t), t) = 0. \tag{4.1.4}$$

Since this is a free boundary value problem, so we introduce the Lagrangian (mass) transformation to convert this free boundary value problem into a fixed boundary problem. We may also assume that the total mass is 4π. Let

$$x = \int_0^r \rho s^2 ds. \tag{4.1.5}$$

Then the domain of x is $[0, 1]$. Denoting the Lagrangian derivatives by D_t, D_x, then we have the following simple facts:

$$D_t r = u, \quad D_x r = \frac{1}{\rho r^2}. \tag{4.1.6}$$

The seconde relation formally leads to

$$r = (3\int_0^x \frac{1}{\rho} dy)^{1/3}. \tag{4.1.7}$$

By change of variables, the (4.1.2) convert to

$$\begin{cases} D_t\rho + \rho^2 D_x(r^2 u) = 0, \\ D_t u + r^2 D_x(\rho^\gamma) + \frac{4\pi x}{r^2} = r^2 D_x(\rho^2 D_x(r^2 u)) - 2ur D_x\rho, \\ \text{or} \quad D_t u + r^2 D_x(\rho^\gamma) + \frac{4\pi x}{r^2} + 2\frac{u}{r^2} = D_x(\rho^2 r^4 D_x u), \end{cases} \tag{4.1.8}$$

with the following boundary conditions

$$u(0,t)=0,\quad (\rho^2 r^2 D_x u-\rho^\gamma)(1,t)=0,\quad \text{and}\quad \rho(1,t)=0, \tag{4.1.9}$$

and initial value conditions

$$\rho(x,0)=\rho_0(x),\quad u(x,0)=u_0(x), \tag{4.1.10}$$

and

$$\rho_0(x)\sim(1-x)^\alpha,\quad \text{as}\quad x\sim 1,\quad 0\le\alpha\le\frac{1}{2}. \tag{4.1.11}$$

In order to obtain a priori estimates, we introduce the following cut-off functions χ and ζ. Consider $0<r_0<r_1<r_2<R$ such that

$$0<2d<r_0,\quad 0<3d<r_2-r_1,\quad \frac{1}{r_0-d}\le 1, \tag{4.1.12}$$

for small fixed constant d. Now let x_i be the initial position in Lagrangian coordinates corresponding to r_i, where $i=0,1,2$:

$$x_i=\int_0^{r_i}\rho_0(s)s^2ds. \tag{4.1.13}$$

Then by the positivity of ρ_0, we get $0<x_0<x_1<x_2<1$. Denoting the particle path emanating from r_i by $r_i(t)$, $r_i(t)$ characterizes x_i:

$$\frac{d}{dt}\int_0^{r_i(t)}\rho(s,t)s^2ds=0,\quad \text{i.e.}\quad \int_0^{r_i(t)}\rho(s,t)s^2ds=x_i. \tag{4.1.14}$$

The above equality follows from the conservation of mass and can be verified by using the continuity equation and

$$\frac{d}{dt}r(t)=u(r(t),t).$$

If we assume $|u(x,t)|\le K$ for all $0\le r\le R(t)$ and $0\le t\le T$, where T is sufficiently small. In particular, d in (4.1.12) will be chosen so that $KT\le d$. Then the smallness assumption on the time interval T prevents a dramatic change of r in time.

Now let $\chi \in C^{\infty}[0,1]$ be a smooth function of x such that

$$\begin{cases} 0 \le \chi \le 1 \quad \text{and} \quad \text{supp}(\chi) \subset [x_0, 1], \\ \chi(x) = 1 \quad \text{if} \quad x_1 \le x \le 1, \\ |\chi'| \le \dfrac{C}{x_1 - x_0} \quad \text{and} \quad |\chi''| < \infty. \end{cases}$$

Note that

$$|r_i(t) - r_i| \le d \quad \text{for} \quad 0 \le t \le T,$$

since

$$r_i(t) = r_i + \int_0^t u(r(\tau), \tau) d\tau, \quad \text{by} \quad (2.1.7).$$

Then we can deduce that for $0 \le t \le T$,

$$\chi(r,t) = 0 \quad \text{if} \quad r \le r_0 - d \quad \text{and} \quad \chi(x,t) = 1 \quad \text{if} \quad r \ge r_1 + d.$$

Similarly, we can construct smooth function ζ satisfied

$$\begin{cases} 0 \le \zeta \le 1 \quad \text{and} \quad \text{supp}(\zeta) \subset [0, r_2 - d], \\ \zeta(r) = 1 \quad \text{if} \quad 0 \le r \le r_1 + d, \\ |\zeta'| \le \dfrac{C}{r_2 - r_1 - 2d} \quad \text{and} \quad |\zeta''| < \infty. \end{cases}$$

ζ as a function of x, t satisfies for $0 \le t \le T$:

$$\zeta(x,t) = 1 \quad \text{if} \quad x \le x_1, \quad \text{and} \quad \zeta(x,t) = 0 \quad \text{if} \quad x \ge x_2.$$

We will view χ and ζ as functions of x, r and t without confusion. Define the energy functional

$$\xi(t) = \xi_E(t) + \xi_L(t),$$

$\xi_E(t)$ is the Eulerian energy and $\xi_L(t)$ is the Lagrangian energy, and

$$\xi_E(t) = \frac{1}{2} \sum_{|\alpha| \le 3} \{\gamma \int \zeta \rho^{\gamma-2} |\partial^\alpha \rho|^2 d\mathbf{x} + \int \zeta \rho |\partial^\alpha \mathbf{u}|^2 d\mathbf{x}\}; \tag{4.1.15}$$

$$\begin{aligned}\xi_L(t) &= \frac{1}{2}\int_{x_0}^1 \chi u^2 dx + \frac{1}{\gamma-1}\int_{x_0}^1 \chi\rho^{\gamma-1}dx + \frac{1}{2}\sum_{j=1}^{3}\{\int_{x_0}^1 \chi|D_t^j u|^2 dx\}\\ &+ \sum_{j=0}^{2}\int_{x_0}^1 \chi\{\rho^2 r^4|D_t^j D_x u|^2 + \frac{2|D_t^j u|^2}{r^2}\}dx + \sum_{j=0}^{1}\int_{x_0}^1 \chi\rho^{2\gamma-2}r^4|D_t^j D_x\rho|^2 dx\\ &+ \sum_{j=0}^{1}\frac{1}{2}\int_{x_0}^1 \chi\rho^{4\gamma}r^8|D_t^j D_x^2\rho|^2 dx + \frac{1}{2}\int_{x_0}^1 \chi\rho^{8\gamma}r^{12}|D_x^3\rho|^2 dx\\ &+ \frac{1}{2}\int_{x_0}^1 \chi\rho^{4\gamma-1}r^8|D_x^2\rho|^2 dx + \frac{1}{2}\int_{x_0}^1 \chi\rho^{2\gamma-3}r^4|D_x\rho|^2 dx.\end{aligned} \tag{4.1.16}$$

We define the dissipation

$$\begin{aligned}D(t) = D_E(t) + D_L(t) &= \sum_{|\alpha\leq 3|}\int \zeta\rho|\nabla\partial^\alpha \mathbf{u}|^2 d\mathbf{x} + \sum_{i=1}^{3}\int_{x_0}^1 \chi|D_t^i u|^2 dx\\ &+ \sum_{i=1}^{3}\int_{x_0}^1 \chi\{\rho^2 r^4|D_t^i D_x u|^2 + \frac{2D_t^i u}{r^2}\}dx.\end{aligned}$$

We suppose the following assumption (K):

$$\begin{aligned}&\sup_{x_0\leq x\leq 1}\{\rho, \frac{u}{r}, |\rho r^2 D_x u + \frac{2u}{r}| = |\frac{D_t\rho}{\rho}|, |\rho r^2 D_x u|, |\rho r^2 D_t D_x u|, |\frac{D_t u}{r}|, |\rho^{2\gamma-1}r^2 D_x\rho|\} \leq K,\\ &\sup_{0\leq r\leq r_2-d}\{\rho, |u|, |\partial_r u|, |\frac{\partial_t\rho}{\rho}|, |\frac{\partial_r\rho}{\rho}|, |\partial_t u|\} \leq K,\end{aligned} \tag{4.1.17}$$

which indicated what regularity strong solutions should enjoy. It is shown in Lemma 4.4.1 that K is closed by $\xi(t)$. Now we state the main result.

Theorem 4.1.1 *Suppose ρ, u are smooth solutions to the free boundary problem of the Navier-Stokes-Poisson system (4.1.2) with (4.1.3), (4.1.4), or (4.1.8) – (4.1.9) for given initial data such that $\xi(0)$ is bounded and satisfied (4.1.11). Then there exists $C_1 = C_1(K)$, $C_2 = C_2(K) > 0$ and $C_3 > 0$ such that for $0 \leq t < \frac{d}{K}$,*

$$\frac{d}{dt}\xi(t) + \frac{1}{2}D(t) \leq C_1\xi(t)^{\frac{1}{2}} + C_2\xi(t) + C_3\xi(t)^2, \tag{4.1.18}$$

moreover, there exists $T > 0$ and $A = A(T, C_1, C_2, C_3, \xi(0)) > 0$ such that

$$\sup_{0\leq t\leq T}\xi(t) \leq A. \tag{4.1.19}$$

In the following theorem, we establish the local in time well-posedness of strong solutions to the Navier-Stokes-Poisson system.

Theorem 4.1.2 *Let the initial data ρ_0, u_0 be given such that $\xi(0) < \infty$ and satisfied* (4.1.11)*. There exists $T^* > 0$ such that there exists a unique solution $R(t)$, $\rho(r,t)$, $u(r,t)$ to the Navier-Stokes-Poisson system* (4.1.2) *with* (4.1.3) – (4.1.4) *in $[0, T^*] \times [0, R(t)]$ such that*

$$\sup_{0 \le t \le T^*} \xi(t) \le 2\xi(0). \tag{4.1.20}$$

Moreover, $\rho(x,t)$, $u(x,t)$, $r(x,t)$ serve a unique solution to (4.1.8) – (4.1.9) *in $[0, T^*] \times [0, 1]$.*

4.2 Boundary estimates in Lagrangian coordinates

From the continuity equation in (4.1.8), we have

$$\rho(x,t) = \rho_0(x) \exp\{-\int_0^t (\rho r^2 D_x u + 2\frac{u}{r})(x,\tau) d\tau\}. \tag{4.2.1}$$

Hence we can see that if $|\rho r^2 D_x u + 2\frac{u}{r}|$ is bounded, then ρ is bounded too. In Lemma 4.4.1, we will show that

$$\sup_{0<x<1} |\rho r^2 D_x u + 2\frac{u}{r}| \le C_{in}(\xi(t) + \xi(t)^{1/2} + C)$$

where C_{in} depends only on the initial density ρ_0 and $C > 0$ is a constant. Moreover, if we let

$$M = \sup_{0<x<1} |\rho r^2 D_x u + 2\frac{u}{r}|, \tag{4.2.2}$$

then we have

$$\rho_0 e^{-MT} \le \rho(x,t) \le \rho_0 e^{MT}. \tag{4.2.3}$$

Firstly, we will establish the estimate on $\xi_L(t)$.

Lemma 4.2.1 *There exits $C_K > 0$ such that*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{i=0}^{3}\int_{x_0}^{1}\chi|D_t^i u|^2dx+\frac{3}{4}\sum_{i=0}^{3}\int_{x_0}^{1}\chi\{\rho^2r^4|D_t^iD_xu|^2+\frac{2|D_t^iu|^2}{r^2}\}dx\\&\le C_K(\xi_L+\xi_L^{1/2})+OL_1,\end{aligned}\tag{4.2.4}$$

where $OL_1 \le \tilde{C}_K\xi_E$ for some constant $\tilde{C}_K$.

Proof: Let $i=0$, multiply the momentum equation by χu, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{x_0}^{1}\chi|u|^2dx+\int_{x_0}^{1}\chi ur^2D_x(\rho^\gamma)dx-\int_{x_0}^{1}\chi ur^2D_x(\rho^2D_x(r^2u))dx\\&+\int_{x_0}^{1}2\chi u^2rD_x\rho dx+\int_{x_0}^{1}4\pi\chi u\frac{x}{r^2}dx=0,\end{aligned}\tag{4.2.5}$$

First we estimate the second, third and fourth terms in (4.2.5): integrating by parts and using the boundary condition (4.1.9), it holds that

$$\begin{aligned}&\int_{x_0}^{1}\chi ur^2D_x(\rho^\gamma)dx-\int_{x_0}^{1}\chi ur^2D_x(\rho^2D_x(r^2u))dx+\int_{x_0}^{1}2\chi u^2rD_x\rho dx\\&=-\int_{x_0}^{1}D_x(\chi ur^2)\rho^\gamma dx+\int_{x_0}^{1}D_x(\chi r^2u)\rho^2D_x(r^2u)dx-2\int_{x_0}^{1}D_x(\chi ru^2)\rho dx\\&=-\int_{x_0}^{1}\chi'r^2u\rho^\gamma dx-\int_{x_0}^{1}\chi D_x(r^2u)\rho^\gamma dx+\int_{x_0}^{1}\chi'r^2u\rho^2D_x(r^2u)dx\\&+\int_{x_0}^{1}\chi\rho^2r^4|D_xu|^2dx-2\int_{x_0}^{1}\chi'ru^2\rho dx+2\int_{x_0}^{1}\chi\frac{u^2}{r^2}dx\\&=-\int_{x_0}^{1}\chi'r^2u\rho^\gamma dx+\frac{1}{\gamma-1}\frac{d}{dt}\int_{x_0}^{1}\chi\rho^{\gamma-1}dx+\int_{x_0}^{1}\chi'r^2u\rho^2D_x(r^2u)dx\\&+\int_{x_0}^{1}\chi\rho^2r^4|D_xu|^2dx-2\int_{x_0}^{1}\chi'ru^2\rho dx+2\int_{x_0}^{1}\chi\frac{u^2}{r^2}dx,\end{aligned}$$

and

$$
\begin{aligned}
&-\int_{x_0}^{1}\chi' r^2 u\rho^\gamma dx+\int_{x_0}^{1}\chi' r^2 u\rho^2 D_x(r^2u)dx-2\int_{x_0}^{1}\chi' r u^2\rho dx\\
&=-\int_{x_0}^{x_1}\chi' r^2 u\rho^\gamma dx+\int_{x_0}^{x_1}\chi'\rho^2 r^4 D_x u u dx\\
&\le \frac{C}{x_1-x_0}\{\int_{r_0(t)}^{r_1(t)}\rho^{\gamma+1}|u|r^4 dr+\int_{r_0(t)}^{r_1(t)}\rho^2 r^4|\partial_r u u|dr\}\\
&\le \frac{C(r_1+d)^2}{x_1-x_0}\{\sup_{r\le r_1+d}\rho^{\frac{\gamma+1}{2}}(\int_{r_0(t)}^{r_1(t)}\rho^\gamma r^2 dr+\int_{r_0(t)}^{r_1(t)}\rho r^2u^2dr)\\
&+\sup_{r\le r_1+d}\rho(\int_{r_0(t)}^{r_1(t)}\rho r^2|\partial_r u|^2dr+\int_{r_0(t)}^{r_1(t)}\rho r^2u^2dr)\}\\
&\le \tilde{C}_k\xi_E.
\end{aligned}
$$

For the fifth term in (4.2.5), we can apply the Cauchy-Schwarz inequality to yield

$$
\int_{x_0}^{1}\chi u\frac{4\pi x}{r^2}dx\le\frac{C}{(r_0-d)^2}(\int_{x_0}^{1}\chi u^2dx)^{1/2}\le C\xi_L(t)^{1/2}.
$$

Then we get

$$
\frac{d}{dt}\int_{x_0}^{1}\chi\{\frac{1}{2}|u|^2+\frac{1}{\gamma-1}\rho^{\gamma-1}\}dx+\int_{x_0}^{1}\chi\{\rho^2r^4|D_xu|^2+\frac{2|u|^2}{r^2}\}dx\le C_K\xi_L^{1/2}+\tilde{C}_K\xi_E.
$$

From the continuity equation, we get

$$
\int_{x_0}^{1}\chi\rho^{-2}|D_t\rho|^2dx\le 3\int_{x_0}^{1}\chi\{\rho^2r^4|D_xu|^2+\frac{2u^2}{r^2}\}dx\le C_K\xi_L. \tag{4.2.6}
$$

Now let $i=1$. Take D_t of the momentum equation to get

$$
\begin{aligned}
&D_t^2u+r^2D_tD_x(\rho^\gamma)-D_x(\rho^2r^4D_xD_tu)+2\frac{D_tu}{r^2}=D_x(D_t(\rho^2r^4)D_xu)\\
&-D_tr^2D_x(\rho^\gamma)-D_t(\frac{2}{r^2})u-D_t(\frac{4\pi x}{r^2}),
\end{aligned}
\tag{4.2.7}
$$

Multiplying the above equation by χD_tu and integrating in x lead to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{x_0}^{1}\chi|D_tu|^2dx+\int_{x_0}^{1}\chi D_tur^2D_xD_t(\rho^\gamma)dx\\
&-\int_{x_0}^{1}\chi D_tuD_x(\rho^2r^4D_tD_xu)dx+\int_{x_0}^{1}\chi\frac{2|D_tu|^2}{r^2}dx\\
&=\int_{x_0}^{1}\chi D_tu\{D_x(D_t(\rho^2r^4)D_xu)-D_tr^2D_x(\rho^\gamma)-D_t(\frac{2}{r^2})u-D_t(\frac{4\pi x}{r^2})\}dx,
\end{aligned}
$$

we estimates above terms one by one as follows:

$$\begin{aligned}
&\int_{x_0}^1 \chi D_t u r^2 D_x D_t(\rho^\gamma)dx - \int_{x_0}^1 \chi D_t u D_x(\rho^2 r^4 D_t D_x u)dx \\
&= -\int_{x_0}^1 \gamma\chi' D_t u r^2 \rho^{\gamma-1} D_t\rho dx - \gamma\int_{x_0}^1 \chi D_x(D_t u r^2)\rho^{\gamma-1}D_t\rho dx \\
&+ \int_{x_0}^1 \chi' D_t u(\rho^2 r^4 D_t D_x u)dx + \int_{x_0}^1 \chi\rho^2 r^4 |D_t D_x u|^2 dx \\
&= \sum_{i=1}^3 I_i + \int_{x_0}^1 \chi\rho^2 r^4 |D_t D_x u|^2 dx
\end{aligned}$$

For I_1 term, we can use the assumption in (4.1.17) and change coordinate variables to obtain

$$|I_1| = |-\int_{x_0}^1 \gamma\chi' D_t u r^2 \rho^{\gamma-1} D_t\rho dx| \le \tilde{C}_k \xi_E.$$

For I_2 term, we have

$$\begin{aligned}
&|I_2| = |-\gamma\int_{x_0}^1 \chi(D_x D_t u r^2 + \frac{2D_t u}{\rho r})\rho^{\gamma-1}D_t\rho dx| \\
&\le \frac{1}{8}\int_{x_0}^1 \chi\rho^2 r^4 |D_t D_x u|^2 dx + \frac{1}{4}\int_{x_0}^1 \chi\frac{|D_t u|^2}{r^2}dx + C\int_{x_0}^1 \chi\rho^{2\gamma-4}|D_t\rho|^2 dx \\
&\le \frac{1}{8}\int_{x_0}^1 \chi\rho^2 r^4 |D_t D_x u|^2 dx + \frac{1}{4}\int_{x_0}^1 \chi\frac{|D_t u|^2}{r^2}dx + C\sup_{x_0\le x\le 1}|\rho^{2\gamma-2}|\int_{x_0}^1 \chi\rho^{-2}|D_t\rho|^2 dx.
\end{aligned} \tag{4.2.8}$$

Integrating by parts and using the boundary condition, we can estimate the last term I_3 by changing coordinate variables to derive

$$\begin{aligned}
|I_3| &= |\int_{x_0}^1 \chi' D_t u(\rho^2 r^4 D_t D_x u)dx| \\
&= |-\frac{1}{2}[\int_{x_0}^{x_1} \chi''|D_t u|^2 \rho^2 r^4 dx + \int_{x_0}^{x_1} \chi'|D_t u|^2 D_x(\rho^2 r^4)dx]| \\
&\le \tilde{C}_K \xi_E.
\end{aligned}$$

Now we only need to estimate

$$\int_{x_0}^1 \chi D_t u\{D_x(D_t(\rho^2 r^4)D_x u) - D_t r^2 D_x(\rho^\gamma) - D_t(\frac{2}{r^2})u - D_t(\frac{4\pi x}{r^2})\}dx. \tag{4.2.9}$$

The first term of (4.2.9) can be estimated as

$$\int_{x_0}^{1} \chi D_t u D_x(D_t(\rho^2 r^4) D_x u)$$
$$= -\int_{x_0}^{1} \chi' D_t(\rho^2 r^4) D_x u D_t u - \int_{x_0}^{1} \chi D_t D_x u D_t(\rho^2 r^4) D_x u dx$$
$$\leq \tilde{C}_K \xi_E + |\int_{x_0}^{1} \chi [4\rho^2 r^3 u + 2\rho D_t \rho r^4] D_x u D_x D_t u dx|$$
$$\leq \frac{1}{8}\int_{x_0}^{1} \chi \rho^2 r^4 |D_x D_t u|^2 dx + C \sup_{x_0 \leq x \leq 1} |\frac{4u}{r} + \frac{2D_t\rho}{\rho}|^2 \int_{x_0}^{1} \chi \rho^2 r^4 |D_x u|^2 dx + \tilde{C}_K \xi_E,$$

for the second term of (4.2.9), we have

$$-\int_{x_0}^{1} \chi D_t r^2 D_x(\rho^\gamma) D_t u dx = -2\gamma \int_{x_0}^{1} \chi r u \rho^{\gamma-1} D_x \rho D_t u dx$$
$$= -2\gamma \int_{x_0}^{1} \chi \frac{u}{r} r^2 \rho^{\gamma-1} D_x \rho D_t u dx$$
$$\leq 2\gamma \sup_{x_0 \leq x \leq 1} |\frac{u}{r}| \{\int_{x_0}^{1} \chi \rho^{2\gamma-2} r^4 |D_x \rho|^2 dx + \int_{x_0}^{1} \chi |D_t u|^2 dx\},$$

the third term of (4.2.9) can be estimated as follows

$$-\int_{x_0}^{1} \chi 2 D_t u D_t(\frac{1}{r^2}) u dx = 4\int_{x_0}^{1} \chi \frac{u}{r}\frac{D_t u}{r}\frac{u}{r} dx$$
$$\leq 4 \sup_{x_0 \leq x \leq 1} |\frac{u}{r}| \{\int_{x_0}^{1} \chi |\frac{u}{r}||\frac{D_t u}{r}| dx \leq \frac{1}{8}\int_{x_0}^{1} \chi \frac{u^2}{r^2} dx + C \sup_{x_0 \leq x \leq 1} |\frac{u}{r}|^2 \int_{x_0}^{1} \chi \frac{|D_t u|^2}{r^2} dx,$$

and for the last term, it holds that

$$-\int_{x_0}^{1} \chi D_t u D_t(\frac{4\pi x}{r^2}) dx = 8\pi \int_{x_0}^{1} \chi \frac{x}{r^3} u D_t u dx$$
$$\leq \frac{1}{8}\int_{x_0}^{1} \chi \frac{|D_t u|^2}{r^2} dx + \frac{C}{(r_0 - d)^2}\int_{x_0}^{1} \chi \frac{u^2}{r^2} dx.$$

Hence after absorbing the viscosity term into the left hand side and using the assumption of (4.1.17), we prove the result for $i = 1$.

Now let $i = 2, 3$. Take D_t^i of the momentum equation to get

$$D_t^{i+1} u + r^2 D_t^i D_x(\rho^\gamma) - D_x(\rho^2 r^4 D_x D_t^i u) + 2\frac{D_t^i u}{r^2}$$
$$= \sum_{j=0}^{i-1} \{D_x[D_t^{i-j}(\rho^2 r^4) D_t^j D_x u] - D_t^{i-j} r^2 D_x D_t^j(\rho^\gamma) - D_t^{i-j}(\frac{2}{r^2}) D_t^j u\} - D_t^i(\frac{4\pi x}{r^2})$$
$$(4.2.10)$$

Multiplying the above equality by $\chi D_t^i u$ and integrating in x, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{x_0}^1 \chi |D_t^i u|^2 dx + \int_{x_0}^1 \chi D_t^i u r^2 D_x D_t^i(\rho^\gamma) dx \\
&- \int_{x_0}^1 \chi D_t^i u D_x(\rho^2 r^4 D_x D_t^i u) dx + \int_{x_0}^1 \chi \frac{2|D_t^i u|^2}{r^2} dx \\
&= \sum_{j=0}^{i-1} \int_{x_0}^1 \chi D_t^i u \{ D_x[D_t^{i-j}(\rho^2 r^4) D_t^j D_x u] - D_t^{i-j} r^2 D_x D_t^j(\rho^\gamma) - D_t^{i-j}(\frac{2}{r^2}) D_t^j u \} dx \\
&- \int_{x_0}^1 \chi D_t^i u D_t^i(\frac{4\pi x}{r^2}) dx.
\end{aligned} \tag{4.2.11}$$

Integrating by parts and using the boundary condition (4.1.9), the second term and third term of (4.2.11) can be estimated as follows:

$$\begin{aligned}
&- \int_{x_0}^1 \chi' D_t^i u r^2 D_t^i(\rho^\gamma) dx + \int_{x_0}^1 \chi' D_t^i u D_x D_t^i u \rho^2 r^4 dx \\
&- \int_{x_0}^1 \chi D_x(r^2 D_t^i u) D_t^i(\rho^\gamma) dx + \int_{x_0}^1 \chi |D_x D_t^i u|^2 \rho^2 r^4 dx.
\end{aligned} \tag{4.2.12}$$

The first two terms can be bounded by $\tilde{C}_K \xi_E$ by using the change of variable. So we only need to estimate the third term of (4.2.12). From the continuity equation, we have for $0 \leq i \leq 2$,

$$D_t^{i+1}\rho = -\rho(\rho r^2 D_x D_t^i u + \frac{2D_t^i u}{r}) - \sum_{0\leq j<i, 0\leq k\leq i} D_t^{i-j-k}\rho^2 D_x(D_x^k r^2 D_t^j u), \tag{4.2.13}$$

hence

$$\int_{x_0}^1 \chi \rho^{-2} |D_t^j \rho|^2 dx \leq C_K \xi_L, \quad \text{for} \quad 1 \leq j \leq 3. \tag{4.2.14}$$

Then using the same idea as in (4.2.8) and (4.2.13), we can estimate the third term of (4.2.12). Note that each term in the RHS of (4.2.11) involves only lower order derivatives. Hence, by summing over i, we get the following

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{i=1}^3 \int_{x_0}^1 \chi |D_t^i u|^2 dx + \frac{3}{4}\sum_{i=1}^3 \int_{x_0}^1 \chi(\rho^2 r^4 |D_x D_t^i u|^2 + \frac{2|D_t^i u|^2}{r^2}) dx \\
&\leq \tilde{C}_k \xi_E + C_k(\xi_L + \xi_L^{1/2}).
\end{aligned}$$

□

Next, we estimate mixed derivatives with only one spatial derivative.

Lemma 4.2.2 *There exists $C_K > 0$, such that*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{i=0}^{2}\int_{x_0}^{1}\chi\{\rho^2 r^4|D_x D_t^i u|^2 + \frac{2|D_t^i u|^2}{r^2}\}dx + \frac{1}{2}\sum_{i=0}^{2}\int_{x_0}^{1}\chi|D_t^{i+1}u|^2 dx \\
&\le \frac{1}{4}\sum_{i=1}^{3}\int_{x_0}^{1}\chi\{\rho^2 r^4|D_x D_t^i u|^2 + \frac{2|D_t^i u|^2}{r^2}\}dx + C_K\xi_L + OL_2,
\end{aligned} \tag{4.2.15}$$

where $OL_2 \le \tilde{C}_K\xi_E$.

Proof: For $i = 0$, multiplying the momentum equation by $\chi D_t u$ and integrating, we have

$$\begin{aligned}
&\int_{x_0}^{1}\chi|D_t u|^2 dx = -\int_{x_0}^{1}\chi r^2 D_x(\rho^\gamma)D_t u dx + \int_{x_0}^{1}\chi D_x(\rho^2 r^4 D_x u)D_t u dx \\
&-\int_{x_0}^{1}\chi 2u\frac{D_t u}{r^2}dx - \int_{x_0}^{1}\chi\frac{4\pi x}{r^2}D_t u dx = \sum_{i=1}^{4}J_i.
\end{aligned} \tag{4.2.16}$$

Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
J_1 &= -\gamma\int_{x_0}^{1}\chi\rho^{\gamma-1}r^2 D_x\rho D_t u dx \\
&\le \frac{1}{2}\int_{x_0}^{1}\chi|D_t u|^2 dx + \frac{1}{2}\gamma^2\int_{x_0}^{1}\chi\rho^{2\gamma-2}r^4|D_x\rho|^2 dx,
\end{aligned} \tag{4.2.17}$$

For J_2, integrating by parts, using the boundary condition and by changing variables, we have

$$\begin{aligned}
J_2 &= -\int_{x_0}^{1}\chi'\rho^2 r^4 D_x u D_t u dx - \int_{x_0}^{1}\chi\rho^2 r^4 D_x u D_x D_t u dx \\
&= -\int_{x_0}^{1}\chi'\rho^2 r^4 D_x u D_t u dx - \frac{1}{2}\frac{d}{dt}\int_{x_0}^{1}\chi\rho^2 r^4|D_x u|^2 dx \\
&+ \frac{1}{2}\int_{x_0}^{1}\chi D_t\rho^2 r^4|D_x u|^2 dx + 2\int_{x_0}^{1}\chi\rho^2 r^3 u|D_x u|^2 dx \\
&\le -\frac{1}{2}\frac{d}{dt}\int_{x_0}^{1}\chi\rho^2 r^4|D_x u|^2 dx + \frac{5}{2}K\int_{x_0}^{1}\chi\rho^2 r^4|D_x u|^2 dx + \tilde{C}_K\xi_E,
\end{aligned} \tag{4.2.18}$$

where we have used $\sup_{x_0\le x\le 1}|\frac{D_t\rho}{\rho}| \le K$, and $\sup_{x_0\le x\le 1}|\frac{u}{r}| \le K$.

For J_3, we have

$$\begin{aligned}
J_3 &= -2\int_{x_0}^{1}\chi\frac{D_t u}{r^2}u dx = -\frac{1}{2}\frac{d}{dt}\int_{x_0}^{1}2\chi\frac{u^2}{r^2}dx - 2\int_{x_0}^{1}\chi\frac{u^3}{r^3}dx \\
&\le -\frac{1}{2}\frac{d}{dt}\int_{x_0}^{1}2\chi\frac{u^2}{r^2}dx + 2K\int_{x_0}^{1}\chi\frac{u^2}{r^2}dx
\end{aligned} \tag{4.2.19}$$

and

$$J_4 = -\int_{x_0}^1 \chi \frac{4\pi x}{r^2} D_t u dx \le \frac{C}{r_0 - d} (\int_{x_0}^1 \frac{|D_t u|^2}{r^2} dx)^{1/2}. \quad (4.2.20)$$

Then from (4.2.16) – (4.2.20), we complete the proof for $i = 0$. For $i = 1, 2$, multiply (4.2.10) by $\chi D_t^{i+1} u$ and integrate to get

$$\int_{x_0}^1 \chi |D_t^{i+1} u|^2 dx + \int_{x_0}^1 \chi r^2 D_x D_t^i(\rho^\gamma) D_t^{i+1} u dx - \int_{x_0}^1 \chi D_x(\rho^2 r^4 D_x D_t^i u) D_t^{i+1} u dx$$
$$+ \int_{x_0}^1 \chi \frac{2 D_t^i u D_t^{i+1} u}{r^2} dx = -\int_{x_0}^1 \chi D_t^i (\frac{4\pi x}{r^2}) D_t^{i+1} u dx$$
$$+ \sum_{j=0}^{i-1} \int_{x_0}^1 \chi D_t^{i+1} u \{ D_x(D_t^{i-j}(\rho^2 r^4) D_t^j D_x u) - D_t^{i-j} r^2 D_x D_t^j(\rho^\gamma) - D_t^{i-j}(\frac{2}{r^2}) D_t^j u \} dx.$$

Then all the estimates are similar to the case $i = 0$. Hence we obtain the desired results. □

Lemma 4.2.3 *There exists $C_K > 0$ such that*

$$\frac{1}{2}\frac{d}{dt} \sum_{i=0}^{1} \int_{x_0}^1 \chi \rho^{2\gamma - 2} r^4 |D_t^i D_x \rho|^2 dx \le C_k \xi_L. \quad (4.2.21)$$

Proof: Integrating the momentum equation from 1 to x for $x \ge x_0$, and using the boundary conditions (4.1.9), it holds that

$$-\rho^2 D_x(r^2 u) = -\rho^\gamma(x,t) - 2\frac{u}{r}\rho - \int_1^x \{ \frac{D_t u}{r^2} + \frac{4\pi y}{r^4} + \frac{2u}{r^4} - \frac{2 D_x u \rho}{r} \} dy. \quad (4.2.22)$$

Multiply the (4.2.22) by $\gamma \rho^{\gamma-1}$ and using the continuity equation, we have

$$D_t \rho^\gamma = -\gamma \rho^{2\gamma - 1} - \gamma \frac{2u}{r} \rho^\gamma - \gamma \rho^{\gamma-1} \int_1^x \{ \frac{D_t u}{r^2} + \frac{4\pi y}{r^4} + \frac{2u}{r^4} - \frac{2 D_x u \rho}{r} \} dy, \quad (4.2.23)$$

differentiating (4.2.23) with respect to x, it holds that

$$D_t D_x(\rho^\gamma) = -\gamma D_x(\rho^{2\gamma-1}) - \gamma D_x(\rho^\gamma) \frac{2u}{r} - \gamma \rho^\gamma D_x(\frac{2u}{r})$$
$$- \gamma D_x(\rho^{\gamma-1}) \int_1^x \{ \frac{D_t u}{r^2} + \frac{4\pi y}{r^4} + \frac{2u}{r^4} - \frac{2 D_x u \rho}{r} \} dy -$$
$$\gamma \rho^{\gamma-1} (\frac{D_t u}{r^2} + \frac{4\pi x}{r^4} + \frac{2u}{r^4} - \frac{2 D_x u \rho}{r}),$$

since $D_x(\frac{u}{r}) = \frac{D_x u}{r} - \frac{u}{\rho r^4}$, hence we have

$$\begin{aligned} &D_t D_x(\rho^\gamma) = -\gamma D_x(\rho^\gamma)\rho^{\gamma-1} - \gamma D_x(\rho^{\gamma-1})(\frac{2\rho u}{r} + \rho^2 r^2 D_x u) \\ &- \gamma D_x\rho\rho^{\gamma-1}\frac{2u}{r} - \gamma\rho^{\gamma-1}(\frac{D_t u}{r^2} + \frac{4\pi x}{r^4}). \end{aligned} \tag{4.2.24}$$

Multiply (4.2.24) by $\chi r^4 D_x(\rho^\gamma)$ and integrate to get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{x_0}^1 \chi r^4 |D_x(\rho^\gamma)|^2 dx = 2\int_{x_0}^1 \chi r^3 u |D_x(\rho^\gamma)|^2 dx - \gamma\int_{x_0}^1 \chi |D_x(\rho^\gamma)|^2 \rho^{\gamma-1} r^4 dx \\ &- \gamma\int_{x_0}^1 \chi r^4 D_x(\rho^{\gamma-1}) D_x(\rho^\gamma)(2\frac{\rho u}{r} + \rho^2 r^2 D_x u) dx - 2\int_{x_0}^1 \chi |D_x\rho^\gamma|^2 r^4 \frac{u}{r} dx \\ &- \gamma\int_{x_0}^1 \chi r^4 D_x(\rho^\gamma)\rho^{\gamma-1}(\frac{D_t u}{r^2} + \frac{4\pi x}{r^4}) dx = \sum_{i=1}^5 E_i. \end{aligned} \tag{4.2.25}$$

We estimate all the E_i as follows:

$$\begin{aligned} E_1 &= 2\int_{x_0}^1 \chi r^3 u |D_x(\rho^\gamma)|^2 dx = 2\int_{x_0}^1 \chi r^3 u \gamma^2 \rho^{2\gamma-2} |D_x\rho|^2 dx \\ &\le 2\gamma^2 \int_{x_0}^1 \chi |\frac{u}{r}| r^4 \rho^{2\gamma-2} |D_x\rho|^2 dx \le 2K\gamma^2 \int_{x_0}^1 \chi r^4 \rho^{2\gamma-2} |D_x\rho|^2 dx. \end{aligned} \tag{4.2.26}$$

$$E_2 = -\gamma^3 \int_{x_0}^1 \chi \rho^{2\gamma-2} |D_x\rho|^2 \rho^{\gamma-1} r^4 dx \le C_K \int_{x_0}^1 \chi\rho^{2\gamma-2}|D_x\rho|^2 r^4 dx. \tag{4.2.27}$$

$$\begin{aligned} E_3 &= -\gamma^2(\gamma-1)\int_{x_0}^1 \chi r^4 \rho^{2\gamma-2}|D_x\rho|^2 (2\frac{u}{r} + \rho r^2 D_x u) dx \\ &\le C_K \int_{x_0}^1 \chi\rho^{2\gamma-2}|D_x\rho|^2 r^4 dx. \end{aligned} \tag{4.2.28}$$

$$E_4 = -2\int_{x_0}^1 \chi |D_x\rho^\gamma|^2 r^4 \frac{u}{r} dx \le C_K\gamma^2 \int_{x_0}^1 \chi r^4 \rho^{2\gamma-2}|D_x\rho|^2 dx, \tag{4.2.29}$$

and

$$\begin{aligned} E_5 &= -\gamma\int_{x_0}^1 \chi r^4 D_x(\rho^\gamma)\rho^{\gamma-1}(\frac{D_t u}{r^2} + \frac{4\pi x}{r^4}) dx \\ &= -\gamma^2 \int_{x_0}^1 \chi r^4 \rho^{2\gamma-2} D_x\rho(\frac{D_t u}{r^2} + \frac{4\pi x}{r^4}) dx \\ &\le \gamma^2 |\int_{x_0}^1 \chi\rho^{2\gamma-2} r^2 D_x\rho(D_t u + \frac{4\pi x}{r^2}) dx| \le C_K \int_{x_0}^1 \chi r^4 \rho^{2\gamma-2}|D_x\rho|^2 dx \\ &+ C_K \int_{x_0}^1 \chi |D_t u|^2 dx + \frac{C_K}{(r_0-d)^4}\int_{x_0}^1 \chi\rho^{\gamma-1} dx. \end{aligned} \tag{4.2.30}$$

Combining (4.2.25) with (4.2.26) – (4.2.30), we obtain the result for $i = 0$. By using the previous result for t derivatives of u, we can get the similar estimate for $i = 1$. □

Corollary 4.2.4 *There exists a constant $C_K > 0$ such that*

$$\int_{x_0}^{1} \chi\rho^2 r^4 |D_x(\rho^2 r^2 D_t^i D_x u)|^2 dx \le C_K \xi_L + \tilde{C_K}\xi_E, \tag{4.2.31}$$

$$\int_{x_0}^{1} \chi\rho^2 r^4 D_x(\rho^2 r^2 D_t^i D_x u + 2\rho \frac{D_t^i u}{r})^2 dx \le C_K \xi_L + \tilde{C_K}\xi_E, \tag{4.2.32}$$

for $i = 0, 1$.

Proof: From the equality

$$r^2 D_x(\rho^2 r^2 D_x u) = D_t u + r^2 D_x(\rho^\gamma) + \frac{4\pi x}{r^2} - 2\rho r D_x u - 2\frac{u}{r^2},$$

and the assumption (4.1.17), we can easily obtain the (4.2.31) for $i = 0$.
Rewriting the above equality, we have

$$r^2 D_x(\rho^2 r^2 D_x u + 2\frac{\rho u}{r}) = D_t u + r^2 D_x(\rho^\gamma) + \frac{4\pi x}{r^2} + 2D_x\rho r u,$$

then we can get (4.2.32) for $i = 0$ by integrating in x.
Similarly, we can derive the estimate for $i = 1$. □
Now, we present the weighted estimate of $D_x^2\rho$.

Lemma 4.2.5 *There exists $C_K > 0$ such that*

$$\frac{1}{2}\frac{d}{dt}\sum_{i=0}^{1}\int_{x_0}^{1} \chi\rho^{4\gamma} r^8 |D_t^i D_x^2 \rho|^2 dx \le C_K \xi_L. \tag{4.2.33}$$

Proof: Differentiating the continuity equation with respect to x and using the momentum equation, we have

$$D_t D_x^2 \rho = -D_x(\frac{D_t u}{r^2} + \frac{4\pi x}{r^4} + \frac{2D_x\rho u}{r}) - \gamma\rho^{\gamma-1}D_x^2\rho - \gamma(\gamma-1)\rho^{\gamma-2}|D_x\rho|^2. \tag{4.2.34}$$

Multiplying (4.2.34) by $\chi\rho^{4\gamma}r^8D_x^2\rho$ and integrating in x, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{x_0}^1\chi\rho^{4\gamma}r^8|D_x^2\rho|^2dx=\frac{1}{2}\int_{x_0}^1\chi D_t(\rho^{4\gamma}r^8)|D_x^2\rho|^2dx\\&-\int_{x_0}^1\chi\gamma(\gamma-1)\rho^{5\gamma-2}r^8D_x^2\rho|D_x\rho|^2dx-\int_{x_0}^1\chi\gamma\rho^{5\gamma-1}r^8|D_x^2\rho|^2dx\\&-\int_{x_0}^1\chi\rho^{4\gamma}r^8D_x^2\rho D_x(\frac{D_tu}{r^2}+\frac{4\pi x}{r^4}+\frac{2D_x\rho u}{r})dx=\sum_{i=1}^4F_i.\end{aligned}\tag{4.2.35}$$

For F_1 term, we have

$$\begin{aligned}&F_1=2\gamma\int_{x_0}^1\chi\rho^{4\gamma-1}D_t\rho r^8|D_x^2\rho|^2dx+4\int_{x_0}^1\chi\rho^{4\gamma}r^7u|D_x^2\rho|^2dx\\&\le 2\gamma\sup_{x_0\le x\le 1}|\frac{D_t\rho}{\rho}|\int_{x_0}^1\chi\int_{x_0}^1\chi\rho^{4\gamma}r^8|D_x^2\rho|^2dx+4\sup_{x_0\le x\le 1}|\frac{u}{r}|\int_{x_0}^1\rho^{4\gamma}r^8|D_x^2\rho|^2dx.\end{aligned}\tag{4.2.36}$$

Using the same idea with F_1, we can estimate F_2 and F_3.

$$\begin{aligned}&F_2=-\int_{x_0}^1\chi\gamma(\gamma-1)\rho^{5\gamma-2}r^8D_x^2\rho|D_x\rho|^2dx\\&\le\gamma(\gamma-1)\sup_{x_0\le x\le 1}|r^2\rho^{2\gamma-1}D_x\rho|\int_{x_0}^1\chi\rho^{3\gamma-1}|D_x\rho||D_x^2\rho|r^6dx\\&\le\gamma(\gamma-1)\sup_{x_0\le x\le 1}|r^2\rho^{2\gamma-1}D_x\rho|(\int_{x_0}^1\chi\rho^{2\gamma-2}r^4|D_x\rho|^2dx+\int_{x_0}^1\chi\rho^{4\gamma}r^8|D_x^2\rho|^2dx).\end{aligned}\tag{4.2.37}$$

$$\begin{aligned}F_3&=-\gamma\int_{x_0}^1\chi\rho^{5\gamma-1}r^8|D_x^2\rho|^2dx\\&\le\gamma\sup_{x_0\le x\le 1}|\rho^{\gamma-1}|\int_{x_0}^1\chi\rho^{4\gamma}r^8|D_x^2\rho|^2dx,\end{aligned}\tag{4.2.38}$$

for the last term F_4, by a simple computation, we have

$$\begin{aligned}F_4&=-\int_{x_0}^1\chi\rho^{4\gamma}r^8D_x^2\rho D_x(\frac{D_tu}{r^2}+\frac{4\pi x}{r^4}+\frac{2D_x\rho u}{r})dx\\&=-\int_{x_0}^1\chi\rho^{4\gamma}r^8D_x^2\rho(\frac{D_tD_xu}{r^2}+\frac{4\pi}{r^4}-\frac{2D_tu}{\rho r^5}-\frac{16\pi x}{\rho r^7})dx\\&-2\int_{x_0}^1\chi\rho^{4\gamma}r^8D_x^2\rho(\frac{D_x^2\rho u}{r}+\frac{2D_x\rho D_xu}{r})dx\\&+4\int_{x_0}^1\chi\rho^{4\gamma}r^8D_x^2\rho\frac{D_x\rho u}{\rho r^4}dx=\sum_{i=1}^3H_i.\end{aligned}\tag{4.2.39}$$

Hence, we need to estimate H_i, $i = 1, 2, 3$.

For H_1, we have

$$\begin{aligned}
H_1 &= -\int_{x_0}^1 \chi\rho^{4\gamma}r^8 D_x^2\rho(\frac{D_tD_xu}{r^2} + \frac{4\pi}{r^4} - \frac{2D_tu}{\rho r^5} - \frac{16\pi x}{\rho r^7})dx \\
&\le \sup_{x_0\le x\le 1}|\rho^{2\gamma-1}|(\int_{x_0}^1 \chi\rho^2r^4|D_tD_xu|^2dx + \int_{x_0}^1 \chi\rho^{4\gamma}r^8|D_x^2\rho|^2dx) \\
&+ \sup_{x_0\le x\le 1}|\rho^{2\gamma-1}|(\int_{x_0}^1 \chi\frac{|D_tu|^2}{r^2}dx + \int_{x_0}^1 \chi\rho^{4\gamma}r^8|D_x^2\rho|^2dx) \\
&+ 2\pi\sup_{x_0\le x\le 1}|\rho^{\frac{3\gamma+1}{2}}|(\int_{x_0}^1 \chi\rho^{\gamma-1}dx + \int_{x_0}^1 \chi\rho^{4\gamma}r^8|D_x^2\rho|^2dx) \\
&+ 8\pi\sup_{x_0\le x\le 1}|\rho^{\frac{3\gamma-1}{2}}|(\frac{1}{(r_0-d)^6}\int_{x_0}^1 \chi\rho^{\gamma-1}dx + \int_{x_0}^1 \chi\rho^{4\gamma}r^8|D_x^2\rho|^2dx),
\end{aligned} \tag{4.2.40}$$

also, for H_2, we have

$$\begin{aligned}
H_2 \le\ & 2\sup_{x_0\le x\le 1}|\frac{u}{r}|\int_{x_0}^1 \chi\rho^{4\gamma}r^8|D_x^2\rho|^2dx + 4\sup_{x_0\le x\le 1}|\rho^{2\gamma-1}r^2D_x\rho| \\
&(\int_{x_0}^1 \chi\rho^{4\gamma}r^8|D_x^2\rho|^2dx + \frac{1}{(r_0-d)^2}\int_{x_0}^1 \chi\rho^2r^4|D_xu|^2dx),
\end{aligned} \tag{4.2.41}$$

and

$$H_3 \le 4\sup_{x_0\le x\le 1}|\rho^\gamma|(\int_{x_0}^1 \chi\rho^{4\gamma}r^8|D_x^2\rho|^2dx + \frac{1}{(r_0-d)^2}\int_{x_0}^1 \chi\rho^{2\gamma-2}r^4|D_x\rho|^2dx). \tag{4.2.42}$$

Combining (4.2.35) with (4.2.36) – (4.2.42), we finish the proof for $i = 0$. Using the similar argument, we can derive the result for $i = 1$. □

We can also obtain the mixed derivatives of u with three spatial derivatives.

Corollary 4.2.6 *There exists $C_K > 0$ such that*

$$\int_{x_0}^1 \chi|\rho r^2D_x(\rho r^2D_x(\rho^2r^2D_xu))|^2dx \le C_K\xi_L, \tag{4.2.43}$$

$$\int_{x_0}^1 \chi|\rho r^2D_x(\rho r^2D_x(\rho^2r^2D_tD_xu))|^2dx \le C_K\xi_L. \tag{4.2.44}$$

Proof: From the momentum equation, we have

$$D_x(\rho r^2 D_x(\rho^2 r^2 D_x u)) = D_x\big(\rho D_t u + \rho r^2 D_x(\rho^\gamma) + \frac{4\pi x\rho}{r^2} - 2\rho^2 r D_x u - \frac{2\rho u}{r^2}\big),$$

by a simple computation and integration in x, we can obtain the result for (4.2.43). Similarly, we can get (4.2.44).

The following Lemma gives the estimate to pure spatial derivative terms $D_x^3\rho$.

Lemma 4.2.7 *There exists $C_K > 0$ such that*

$$\frac{1}{2}\frac{d}{dt}\int_{x_0}^1 \chi\rho^{8\gamma} r^{12}|D_x^3\rho|^2 dx \le C_K\xi_L. \tag{4.2.45}$$

Proof: Taking D_x in (4.2.34), we get

$$\begin{aligned} &D_t D_x^3\rho + D_x^2\big(\frac{D_t u}{r^2} + \frac{4\pi x}{r^4} + \frac{2D_x\rho u}{r}\big) + \gamma(\gamma-1)\rho^{\gamma-2}D_x\rho D_x^2\rho + \gamma\rho^{\gamma-1}D_x^3\rho \\ &+ \gamma(\gamma-1)(\gamma-2)\rho^{\gamma-3}(D_x\rho)^3 + 2\gamma(\gamma-1)\rho^{\gamma-2}D_x\rho D_x^2\rho = 0. \end{aligned} \tag{4.2.46}$$

Multiplying (4.2.46) by $\chi\rho^{8\gamma}r^{12}D_x^3\rho$ and integrating over $[x_0, 1]$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{x_0}^1 \chi\rho^{8\gamma}r^{12}|D_x^3\rho|^2 dx = \frac{1}{2}\int_{x_0}^1 \chi D_t(\rho^{8\gamma}r^{12})|D_x^3\rho|^2 dx \\ &- \int_{x_0}^1 \chi\rho^{8\gamma}r^{12}D_x^3\rho D_x^2\big(\frac{D_t u}{r^2} + \frac{4\pi x}{r^4} + \frac{2D_x\rho u}{r}\big)dx \\ &- 3\gamma(\gamma-1)\int_{x_0}^1 \chi\rho^{8\gamma}r^{12}\rho^{\gamma-2}D_x\rho D_x^2\rho D_x^3\rho dx - \gamma\int_{x_0}^1 \chi\rho^{8\gamma}r^{12}\rho^{\gamma-1}|D_x^3\rho|^2 dx \\ &- \gamma(\gamma-1)(\gamma-2)\int_{x_0}^1 \chi\rho^{8\gamma}r^{12}D_x^3\rho\rho^{\gamma-3}(D_x\rho)^3 dx = \sum_{i=1}^5 A_i. \end{aligned}$$

For A_1, we have

$$\begin{aligned} A_1 &= 4\gamma\int_{x_0}^1 \chi\rho^{8\gamma-1}r^{12}D_t\rho|D_x^3\rho|^2 dx + 6\int_{x_0}^1 \chi\rho^{8\gamma}r^{11}u|D_x^3\rho|^2 dx \\ &\le 4\gamma\sup_{x_0\le x\le 1}|\frac{D_t\rho}{\rho}|\int_{x_0}^1 \rho^{8\gamma}r^{12}|D_x^3\rho|^2 dx + 6\sup_{x_0\le x\le 1}|\frac{u}{r}|\int_{x_0}^1 \rho^{8\gamma}r^{12}|D_x^3\rho|^2 dx. \end{aligned} \tag{4.2.47}$$

$$
\begin{aligned}
A_2 &= -\int_{x_0}^{1} \chi\rho^{8\gamma} r^{12} D_x^3\rho D_x^2\Big(\frac{D_t u}{r^2} + \frac{4\pi x}{r^4} + \frac{2D_x\rho u}{r}\Big)dx \\
&= -\int_{x_0}^{1} \chi\rho^{8\gamma} r^{12} D_x^3\rho D_x^2\Big(\frac{D_t u}{r^2} + \frac{4\pi x}{r^4}\Big)dx + \int_{x_0}^{1} \chi\rho^{8\gamma} r^{12} D_x^3\rho D_x^2\Big(\frac{2D_x\rho u}{r}\Big)dx \\
&= \int_{x_0}^{1} \chi\rho^{8\gamma} r^{12} D_x^3\rho D_x\Big(\frac{D_x^2\rho u}{r} + 2\frac{D_x\rho D_x u}{r}\Big)dx - 4\int_{x_0}^{1} \chi\rho^{8\gamma} r^{12} D_x^3\rho D_x\Big(\frac{D_x\rho u}{\rho r^4}\Big)dx \\
&+ \int_{x_0}^{1} \chi\rho^{8\gamma} r^{12} D_x^3\rho D_x\Big(\frac{D_t D_x u}{r^2} + \frac{4\pi}{r^4} - 2\frac{D_t u}{\rho r^5} - \frac{16\pi x}{\rho r^7}\Big)dx \\
&\le C_K \xi_L.
\end{aligned}
\tag{4.2.48}
$$

$$
\begin{aligned}
A_3 &= -3\gamma(\gamma-1)\int_{x_0}^{1} \chi\rho^{8\gamma} r^{12}\rho^{\gamma-2} D_x\rho D_x^2\rho D_x^3\rho dx \\
&\le 3\gamma(\gamma-1)\sup_{x_0\le x\le 1}(|\rho^{\gamma-1}||\rho^{2\gamma-1} r^2 D_x\rho|)\Big(\int_{x_0}^{1}\rho^{8\gamma} r^{12}|D_x^3\rho|^2 dx + \int_{x_0}^{1}\rho^{4\gamma} r^8 |D_x^2\rho|^2 dx\Big),
\end{aligned}
\tag{4.2.49}
$$

and

$$
A_4 \le \sup_{x_0\le x\le 1}|\rho^{\gamma-1}|\int_{x_0}^{1}\rho^{8\gamma} r^{12}|D_x^3\rho|^2 dx, \tag{4.2.50}
$$

$$
\begin{aligned}
A_5 &= -\gamma(\gamma-1)(\gamma-2)\int_{x_0}^{1}\chi\rho^{8\gamma} r^{12} D_x^3\rho\rho^{\gamma-3}(D_x\rho)^3 dx \\
&\le |\gamma(\gamma-1)(\gamma-2)|\sup_{x_0\le x\le 1}(|\rho^{2\gamma-1} r^2 D_x\rho|^2)\Big(\int_{x_0}^{1}\rho^{2\gamma-2} r^4 |D_x\rho|^2 dx + \int_{x_0}^{1}\rho^{8\gamma} r^{12}|D_x^3\rho|^2 dx\Big).
\end{aligned}
\tag{4.2.51}
$$

Hence, collecting (4.2.47) – (4.2.51), we obtain the (4.2.45). □

4.3 Interior estimates in Eulerian coordinates

In this section, we will obtain some interior estimates of $\xi_E(t)$. Away from the vacuum boundary, ρ is expected to be strictly positive and classical results of the Navier-Stokes theory can be applied. Recall the full Navier-Stokes-Poisson system (4.1.1):

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \\ \dfrac{\partial(\rho\mathbf{u})}{\partial t} + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P = -\rho\nabla\Phi + \operatorname{div}(\mu(\rho)\nabla\mathbf{u}), \\ \triangle\Phi = 4\pi\rho, \end{cases}$$

where $\mathbf{u}(\mathbf{x}) = u(r)\frac{\mathbf{x}}{r}$, $\mathbf{x} = (x_1, x_2, x_3)$, $r = |\mathbf{x}|$.

Lemma 4.3.1 *There exist constants C_K, $C > 0$ such that*

$$\frac{1}{2}\frac{d}{dt}\xi_E + \frac{1}{2}D_E \le C_K\xi_E + C(\xi_E)^2 + OL_3, \tag{4.3.1}$$

where $OL_3 \le \tilde{C}_K\xi_L$ for some $\tilde{C}_K$.

Proof: Multiply $(4.1.1)_2$ by ζu , use $(4.1.1)_1$, and integrate to get

$$\begin{aligned} &\frac{d}{dt}\{\frac{1}{2}\int\zeta\rho|\mathbf{u}|^2d\mathbf{x} + \frac{1}{\gamma-1}\int\zeta\rho^\gamma d\mathbf{x}\} + \int\zeta\rho|\nabla\mathbf{u}|^2d\mathbf{x} \\ &= \frac{1}{4\pi}\int\zeta\nabla\Phi\nabla\partial_t\Phi d\mathbf{x} + \frac{1}{2}\int\zeta\partial_t\rho|\mathbf{u}|^2d\mathbf{x} - \int\nabla\zeta\rho\nabla\mathbf{u}\mathbf{u}d\mathbf{x} \\ &- \int\zeta\rho(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}d\mathbf{x} + \frac{\gamma}{\gamma-1}\int\nabla\zeta\rho^\gamma\mathbf{u}d\mathbf{x} + \int\nabla\zeta\Phi\rho\mathbf{u}d\mathbf{x} \\ &+ \frac{1}{4\pi}\int\nabla\zeta\Phi\nabla\Phi_t d\mathbf{x}. \end{aligned}$$

By symmetry, we have

$$\begin{aligned} &|\frac{1}{4\pi}\int\zeta\nabla\Phi\nabla\partial_t\Phi d\mathbf{x}| = |\int_0^{r_2-d}\zeta(\frac{1}{r^2}\int_0^r\rho s^2ds)(\frac{1}{r^2}\int_0^r\partial_t\rho s^2ds)r^2dr| \\ &= |-\int_0^{r_2-d}\zeta(\frac{1}{r^3}\int_0^r\rho s^2ds)(\frac{\rho u}{r})r^4dr| \\ &\le C\int_0^{r_2-d}\zeta(\frac{1}{r^3}\int_0^r\rho s^2ds)^2\rho r^2dr + C\int\zeta\rho|\frac{u}{r}|^2r^6dr \\ &\le C\sup_{0\le r\le r_2-d}|\rho^{4-\gamma}|\int\zeta\rho^\gamma r^2dr + C\int\zeta\rho|\frac{u}{r}|^2r^6dr, \end{aligned}$$

where we have used the Cauchy-Schwarz inequality and for $r \le r_2 - d$,

$$\frac{1}{r^3}\int_0^r\rho s^2ds \le \frac{1}{3}\sup_{0\le r\le r_2-d}|\rho|,$$

and

$$\sup_{0\le x\le x_2}|\frac{1}{\rho}|\le \sup_{0\le x\le x_2}|\frac{1}{\rho_0}|e^{MT},$$

Other terms can be estimated as follows:

$$|\frac{1}{2}\int\zeta\partial_t\rho|\mathbf{u}|^2d\mathbf{x}|=\frac{1}{2}|\int\zeta\frac{\partial_t\rho}{\rho}\rho|\mathbf{u}|^2d\mathbf{x}|\le\frac{1}{2}\sup_{0\le r\le r_2-d}|\frac{\partial_t\rho}{\rho}|\int\zeta\rho|\mathbf{u}|^2d\mathbf{x},$$

$$\begin{aligned}|\int\nabla\zeta\rho\nabla\mathbf{u}\mathbf{u}d\mathbf{x}|&\le\frac{C}{r_2-r_1-2d}\int_{x_1}^{x_2}|\rho r^2D_xuu|dx\\&\le\frac{C}{r_2-r_1-2d}(\int_{x_0}^{1}\chi\rho^2r^4|D_xu|^2dx+\int_{x_0}^{1}\chi|u|^2dx)\\&\le C\xi_L(t).\end{aligned}$$

$$|\int\zeta\rho(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}d\mathbf{x}|\le\sup_{0\le r\le r_2-d}|u|(\int\zeta\rho|\nabla\mathbf{u}|^2d\mathbf{x}+\int\zeta\rho|\mathbf{u}|^2d\mathbf{x})$$

$$\begin{aligned}|\int\nabla\zeta\rho^\gamma\mathbf{u}d\mathbf{x}|&\le\frac{C}{r_2-r_1-2d}\int_{x_1}^{x_2}\rho^{\gamma-1}|u|dx\\&\le\frac{C}{r_2-r_1-2d}\sup_{x_1\le x\le x_2}|\rho^{\frac{\gamma-1}{2}}|\{\int_{x_1}^{x_2}\rho^{\gamma-1}dx+\int_{x_1}^{x_2}|u|^2dx\}.\end{aligned}$$

Hence we get the following zeroth-order estimate:

$$\frac{1}{2}\frac{d}{dt}\{\int\zeta\rho|\mathbf{u}|^2d\mathbf{x}+\frac{1}{\gamma-1}\int\zeta\rho^\gamma d\mathbf{x}\}+\frac{3}{4}\int\zeta\rho|\nabla\mathbf{u}|^2d\mathbf{x}\le C_K\xi_E+OL,\quad(4.3.2)$$

where $OL\le C_{K,in}\xi_L$, and $C_{K,in}$ depends also on the initial density ρ_0. The higher derivatives (up to third) can be estimated in the similar way. Let ∂ be any Eulerian derivative, we have

$$\begin{aligned}&\partial_t\partial\rho+\nabla\partial\rho\cdot\mathbf{u}+\nabla\partial\rho\cdot\partial\mathbf{u}+\partial\rho\nabla\mathbf{u}+\rho\nabla\cdot\partial\mathbf{u}=0\\&\rho\partial_t(\partial\mathbf{u})+\partial\rho\partial_t\mathbf{u}+\partial[\rho(\mathbf{u}\cdot\nabla)\mathbf{u}]+\nabla\partial\rho^\gamma+\rho\nabla\partial\Phi+\partial\rho\nabla\phi=\mathrm{div}(\partial(\rho\nabla\mathbf{u})).\end{aligned}\quad(4.3.3)$$

Multiplying $(4.3.3)_2$ by $\zeta\partial\mathbf{u}$, using $(4.3.2)_1$, integrating to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\zeta\rho|\partial\mathbf{u}|^2d\mathbf{x}+\gamma\int\zeta\rho^{\gamma-2}(\partial\rho)^2d\mathbf{x}\}+\int\zeta\rho|\nabla\partial\mathbf{u}|^2d\mathbf{x}\\
&=-\int\nabla\zeta\rho\nabla\partial\mathbf{u}\partial\mathbf{u}d\mathbf{x}-\int\zeta\nabla\rho\nabla\partial\mathbf{u}\partial\mathbf{u}d\mathbf{x}+\frac{\gamma(\gamma-2)}{2}\int\zeta\rho^{\gamma-3}\partial_t\rho(\partial\rho)^2d\mathbf{x}\\
&+\frac{1}{2}\int\zeta\partial_t\rho|\partial\mathbf{u}|^2d\mathbf{x}-\gamma\int\zeta\rho^{\gamma-2}\partial\rho\nabla\partial\rho\mathbf{u}d\mathbf{x}-\gamma\int\zeta\rho^{\gamma-2}\partial\rho\nabla\rho\partial\mathbf{u}d\mathbf{x}\\
&-\gamma\int\zeta\rho^{\gamma-2}(\partial\rho)^2\nabla\mathbf{u}d\mathbf{x}-\gamma\int\zeta\rho^{\gamma-1}\partial\rho\nabla\partial\mathbf{u}d\mathbf{x}-\int\zeta\partial\rho\partial_t\mathbf{u}\partial\mathbf{u}d\mathbf{x}\\
&-\int\zeta\partial[\rho(\mathbf{u}\cdot\nabla)\mathbf{u}]\partial\mathbf{u}d\mathbf{x}-\int\zeta\nabla\partial\rho^\gamma\partial\mathbf{u}d\mathbf{x}-\int\zeta\rho\nabla\partial\Phi\partial\mathbf{u}d\mathbf{x}\\
&-\int\zeta\partial\rho\nabla\Phi\partial\mathbf{u}d\mathbf{x}-\int\zeta\partial\rho\nabla\mathbf{u}\nabla\partial\mathbf{u}d\mathbf{x}-\int\nabla\zeta\partial\rho\partial\mathbf{u}\nabla\mathbf{u}d\mathbf{x}.
\end{aligned}$$

Note that

$$\begin{aligned}
-\gamma\int\zeta\rho^{\gamma-1}\partial\rho\nabla\partial\mathbf{u}d\mathbf{x}-\int\zeta\nabla\partial\rho^\gamma\partial\mathbf{u}d\mathbf{x}&=-\int\zeta\partial\rho^\gamma\nabla\partial\mathbf{u}d\mathbf{x}-\int\zeta\nabla\partial\rho^\gamma\partial\mathbf{u}d\mathbf{x}\\
&=\int\nabla\zeta\partial\rho^\gamma\cdot\partial\mathbf{u}d\mathbf{x}.
\end{aligned}$$

For another second derivative term $-\gamma\int\zeta\rho^{\gamma-2}\partial\rho\nabla\partial\rho\mathbf{u}d\mathbf{x}$, we integrate it by parts:

$$\frac{\gamma}{2}\int\nabla\zeta\rho^{\gamma-2}(\partial\rho)^2\mathbf{u}d\mathbf{x}+\frac{\gamma(\gamma-2)}{2}\int\zeta\rho^{\gamma-3}\nabla\rho(\partial\rho)^2\mathbf{u}d\mathbf{x}+\frac{\gamma}{2}\int\zeta\rho^{\gamma-2}(\partial\rho)^2\nabla\cdot\mathbf{u}d\mathbf{x}.$$

Potential term, in principle, lower order and the L^2 estimate $\|\partial^2\Phi\|_{L^2}\le C\|\rho\|_{L^2}$ is useful. Hence we get the following first order estimate:

$$\frac{1}{2}\frac{d}{dt}\{\gamma\int\zeta\rho^{\gamma-2}(\partial\rho)^2d\mathbf{x}+\int\zeta\rho|\partial\mathbf{u}|^2d\mathbf{x}\}+\int\zeta\rho|\nabla\partial\mathbf{u}|^2d\mathbf{x}\le C_K\xi_E+OL. \tag{4.3.4}$$

Now we estimate 3rd derivatives. Take one more derivative of the equation (4.3.3), we obtain

$$\begin{aligned}
&\partial_t(\partial^2\rho)+\nabla\partial^2\rho\cdot\mathbf{u}+2\nabla\partial\rho\cdot\partial\mathbf{u}+\nabla\rho\cdot\partial^2\mathbf{u}\\
&+\partial^2\rho\nabla\cdot\mathbf{u}+2\partial\rho\nabla\cdot\partial\mathbf{u}+\rho\nabla\cdot\partial^2\mathbf{u}=0\\
&\rho\partial_t(\partial^2\mathbf{u})+2\partial\rho\partial_t(\partial\mathbf{u})+\partial^2\rho\partial_t\mathbf{u}+\partial^2(\rho(\mathbf{u}\cdot\nabla)\mathbf{u})+\nabla\partial^2P+2\partial\rho\nabla\partial\Phi\\
&+\rho\nabla\partial^2\Phi+\partial^2\rho\nabla\Phi=\partial^2\mathrm{div}(\rho\nabla\mathbf{u}).
\end{aligned}\tag{4.3.5}$$

Multiplying (4.3.5) by $\zeta\partial^2\mathbf{u}$, using $(4.3.1)_1$ and integrating, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\int \gamma\zeta\rho^{\gamma-2}(\partial^2\rho)^2 d\mathbf{x} + \int \zeta\rho|\partial^2\mathbf{u}|^2 d\mathbf{x}\} + \int \zeta\rho|\nabla\partial^2\mathbf{u}|^2 d\mathbf{x} - \int \nabla\zeta\rho\nabla\partial^2\mathbf{u}\partial^2\mathbf{u}d\mathbf{x} \\
&= \frac{\gamma(\gamma-2)}{2}\int \zeta\rho^{\gamma-3}\partial_t\rho(\partial^2\rho)^2 d\mathbf{x} + \frac{1}{2}\int \zeta\partial_t\rho|\partial^2\mathbf{u}|^2 d\mathbf{x} - \gamma\int \zeta\rho^{\gamma-2}\partial^2\rho\nabla\partial^2\rho\mathbf{u}d\mathbf{x} \\
&- 2\gamma\int \zeta\rho^{\gamma-2}\partial^2\rho\nabla\partial\rho\partial\mathbf{u}d\mathbf{x} - \gamma\int \zeta\rho^{\gamma-2}\partial^2\rho\nabla\rho\partial^2\mathbf{u}d\mathbf{x} - \gamma\int \zeta\rho^{\gamma-2}(\partial^2\rho)^2\nabla\mathbf{u}d\mathbf{x} \\
&- 2\gamma\int \zeta\rho^{\gamma-2}\partial^2\rho\partial\rho\nabla\partial\mathbf{u}d\mathbf{x} - \gamma\int \zeta\rho^{\gamma-1}\partial^2\rho\nabla\partial^2\mathbf{u}d\mathbf{x} - 2\int \zeta\partial\rho\partial_t\partial\mathbf{u}\partial^2\mathbf{u}d\mathbf{x} \\
&- \int \zeta\partial^2\rho\partial_t\mathbf{u}\partial^2\mathbf{u}d\mathbf{x} - \int \zeta\partial^2(\rho(\mathbf{u}\cdot\nabla)\mathbf{u})\partial^2\mathbf{u}d\mathbf{x} - \int \zeta\nabla\partial^2 P\partial^2\mathbf{u}d\mathbf{x} \\
&- 2\int \zeta\partial\rho\nabla\partial\Phi\partial^2\mathbf{u}d\mathbf{x} - \int \zeta\rho\nabla\partial^2\Phi\partial^2\mathbf{u}d\mathbf{x} - \int \zeta\partial^2\rho\nabla\Phi\partial^2\mathbf{u}d\mathbf{x} \\
&- \int \zeta\nabla\partial^2\mathbf{u}\partial^2\rho\nabla\mathbf{u}d\mathbf{x} - 2\int \zeta\nabla\partial^2\mathbf{u}\partial\rho\nabla\partial\mathbf{u}d\mathbf{x} - \int \nabla\zeta\partial^2\mathbf{u}\partial^2\rho\nabla\mathbf{u}d\mathbf{x} - 2\int \nabla\zeta\partial^2\mathbf{u}\partial\rho\nabla\partial\mathbf{u}d\mathbf{x}.
\end{aligned}
$$

As in the first order estimates, for higher order estimates, either we use the integration by parts or they cancel each other. Eventually, we obtain the following :

$$
\frac{1}{2}\frac{d}{dt}\{\gamma\int \zeta\rho^{\gamma-2}(\partial^2\rho)^2 d\mathbf{x} + \int \zeta\rho|\partial^2\mathbf{u}|^2 d\mathbf{x}\} + \frac{3}{4}\int \zeta\rho|\nabla\partial^2\mathbf{u}|^2 d\mathbf{x} \le C_K\xi_E + OL. \tag{4.3.6}
$$

Taking one more derivative of the equation (4.3.5), it is routine to have the following high energy inequality:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\gamma\int \zeta\rho^{\gamma-2}(\partial^3\rho)^2 d\mathbf{x} + \int \zeta\rho|\partial^3\mathbf{u}|^2 d\mathbf{x}\} + \frac{1}{2}\int \zeta\rho|\nabla\partial^3\mathbf{u}|^2 d\mathbf{x} \\
&\le C_K\xi_E + C(\xi_E)^2 + OL,
\end{aligned} \tag{4.3.7}
$$

where $C(\xi_E)^2$ comes from the Gagliardo-Nirenberg inequality:

$$
\|f\|_{L^4} \le C\|\nabla f\|_{L^2}^{\frac{3}{4}}\|f\|_{L^2}^{\frac{1}{4}}
$$

to treat the nonlinear term such as $\int \partial^2\rho\partial_t\partial\mathbf{u}\partial^3\mathbf{u}d\mathbf{x}$. Now we finish the proof of the Lemma.

4.4 Weaving the estimates

In this section, we will verify the assumption (4.1.17) to show that energy estimates can be closed for $0 \le t \le T$ where T is sufficiently small.

Lemma 4.4.1 *Suppose (ρ, u, r) is a smooth solution to the Navier-Stokes-Poisson system. Then there exist $T > 0$, $C_{in} = C(\rho_0) > 0$ such that $K \le C_{in}(\xi(t)^{\frac{1}{2}} + \xi(t) + C)$ for $0 \le t \le T$, where $C > 0$ is a positive constant.*

Proof: In the view of (4.2.3), we have

$$\rho_0 e^{-MT} \le \rho(t,x) \le \rho_0 e^{MT}.$$

With the bounds of ρ, and

$$\rho_0(x) \sim (1-x)^{\alpha}, \quad \text{as} \quad x \sim 1, \quad \text{where} \quad 0 \le \alpha \le \frac{1}{2},$$

we can estimate $|\rho r^2 D_x u|$. Note that

$$\sup_{0<x<1} |\rho r^2 D_x u| = \sup\{\sup_{0\le r\le r_1+d} |\partial_r u|, \sup_{x_1\le x\le 1} |\rho r^2 D_x u|\}.$$

Applying the Sobolev imbedding theorem, we have

$$\begin{aligned}
&\sup_{0\le r\le r_1+d} |\partial_r u| \le \sum_{i=0}^{2} (\int_{B_{r_1+d}} |\nabla \partial_{\mathbf{x}}^i \mathbf{u}|^2 d\mathbf{x})^{\frac{1}{2}} \\
&\le \sup_{0\le r\le r_1+d} |\frac{1}{\sqrt{\rho}}| \sum_{i=0}^{2} (\int_{B_{r_1+d}} \rho |\nabla \partial_{\mathbf{x}}^i \mathbf{u}|^2 d\mathbf{x})^{\frac{1}{2}} \\
&\le C_{in} e^{MT/2} \xi_E(t)^{\frac{1}{2}}.
\end{aligned} \tag{4.4.1}$$

From the equation $(4.1.8)_2$, we have

$$\frac{D_t u}{r^2} + D_x(\rho^\gamma) = D_x(\rho^2 D_x(r^2 u)) - 2\frac{u}{r} D_x \rho - \frac{4\pi x}{r^4}.$$

Integrating the above equality over $[x, 1]$ for $x \in [x_1, 1)$, we have

$$\rho^2 D_x(r^2 u) = -\gamma \int_x^1 \rho^{\gamma-1} D_y \rho dy - \int_x^1 \frac{D_t u}{r^2} dy + 2\rho \frac{u}{r} + 2\int_x^1 D_y(\frac{u}{r}) \rho dy - \frac{4\pi y}{r^4} dy,$$

hence we get

$$\rho r^2 D_x u = -\frac{\gamma}{\rho}\int_x^1 \rho^{\gamma-1} D_y \rho dy - \frac{1}{\rho}\int_x^1 \frac{D_t u}{r^2} dy + \frac{2}{\rho}\int_x^1 D_y(\frac{u}{r})\rho dy - \frac{1}{\rho}\int_x^1 \frac{4\pi y}{r^4} dy,$$

and

$$\sup_{x_1 \le x \le 1} |\rho r^2 D_x u| \le C_{in} e^{MT} (\xi_L(t)^{1/2} + C). \tag{4.4.2}$$

Combining (4.4.1) with (4.4.2), we have

$$\sup_{0<x<1} |\rho r^2 D_x u| \le C_{in} e^{MT} (\xi(t)^{\frac{1}{2}} + C).$$

Once we obtain the estimate of $\sup\limits_{0<x<1} |\rho r^2 D_x u|$, we can now get the bound of $\sup\limits_{0<x<1} |\frac{u}{r}|$.

$$\sup_{0<x<1} |\frac{u}{r}| = \sup\{\sup_{0\le r \le r_1+d} |\frac{u}{r}|, \sup_{x_1 \le x \le 1} |\frac{u}{r}|\}.$$

Using the same argument as in (4.4.1), we can get

$$\sup_{0 \le r \le r_1+d} |\frac{u}{r}| \le C_{in} e^{MT/2} (\xi_E(t))^{\frac{1}{2}}. \tag{4.4.3}$$

On the other hand,

$$\begin{aligned} \sup_{x_1 \le x \le 1} |\frac{u}{r}| &\le C(\int_{x_1}^1 |u| dx + \int_{x_1}^1 |D_x u| dx) \\ &\le C_{in} e^{2MT} (\xi_L(t)^{\frac{1}{2}} + C). \end{aligned} \tag{4.4.4}$$

Collecting (4.4.1) – (4.4.4), using (4.2.2), we get

$$M \le C_{in} e^{2MT} (\xi(t)^{\frac{1}{2}} + C).$$

By Taylor expansion,

$$M - C_{in}(\xi(t)^{\frac{1}{2}} + C) \sum_{k=1}^{\infty} \frac{(2MT)^k}{k!} \le C_{in}(\xi(t)^{\frac{1}{2}} + C),$$

for sufficiently small T, we can obtain that

$$C_{in}(\xi(t)^{\frac{1}{2}} + C) \sum_{k=1}^{\infty} \frac{(2MT)^k}{k!} \le \frac{M}{2},$$

hence, we have

$$M \le C_{in}(\xi(t)^{\frac{1}{2}} + C) \quad \text{for} \quad 0 \le t \le T. \tag{4.4.5}$$

Because of (4.4.5) and Taylor expansion, we can derive for sufficiently small T,

$$\sup_{0<x<1} |\rho r^2 D_x u| \le C_{in}(\xi(t)^{\frac{1}{2}} + C), \tag{4.4.6}$$

and

$$\sup_{0<x<1} |\frac{u}{r}| \le C_{in}(\xi(t)^{\frac{1}{2}} + C). \tag{4.4.7}$$

Integrating (4.2.7) over $[x, 1]$, where $x \in [x_1, 1)$, we have

$$\begin{aligned}
\rho r^2 D_t D_x u = & -\frac{1}{\rho r^2}\int_x^1 D_t^2 u dy - \frac{1}{\rho r^2}\int_x^1 r^2 D_y D_t(\rho^\gamma) dy - \frac{2}{\rho r^2}\int_x^1 \frac{D_t u}{r^2} dy \\
& - \frac{1}{\rho r^2} D_t(\rho^2 r^4) D_x u - \frac{1}{\rho r^2}\int_x^1 2ur D_y(\rho^\gamma) dy - \frac{1}{\rho r^2}\int_x^1 D_t(\frac{2}{r^2}) u dy \\
& - \frac{1}{\rho r^2}\int_x^1 4\pi D_t(\frac{x}{r^4}) dy,
\end{aligned}$$

then we have

$$\sup_{x_1 \le x \le 1} |\rho r^2 D_t D_x u| \le C e^{2MT}(\xi(t)^{\frac{1}{2}} + \xi(t) + C). \tag{4.4.8}$$

Using the same idea as in the estimate of $\sup\limits_{0<x<1} |\rho r^2 D_x u|$, we have

$$\sup_{0 \le x \le 1} |\rho r^2 D_t D_x u| \le C e^{MT}(\xi(t)^{\frac{1}{2}} + \xi(t) + C). \tag{4.4.9}$$

Once we obtain (4.4.9), we can easily obtain

$$\sup_{0 \le x \le 1} |\frac{D_t u}{r}| \le C e^{2MT}(\xi(t)^{\frac{1}{2}} + \xi(t) + C). \tag{4.4.10}$$

Next, we estimate $|\rho^{2\gamma-1} r^2 D_x \rho|$ in $x_0 \le x \le 1$. Because the cutoff function χ values 1 only for $x_1 \le x \le 1$, $|\rho^{2\gamma-1} r^2 D_x \rho|$ for $x_0 \le x \le x_1$ should be estimated in Eulerian coordinates. Note that $r_0 - d \le r \le r_1 + d$ covers $x_0 \le x \le x_1$, we have

$$\sup_{0 \le x \le 1} |\rho^{2\gamma-1} r^2 D_x \rho| \le \sup\{\sup_{r_0 - d \le r \le r_1 + d} |\rho^{2\gamma-2} \partial_r \rho|, \sup_{x_1 \le x \le 1} |\rho^{2\gamma-1} r^2 D_x \rho|\}.$$

Applying the Sobolev embedding theorem $W^{1,1} \hookrightarrow L^\infty$ and Hölder inequality, one gets

$$\begin{aligned}
&\sup_{r_0-d\le r\le r_1+d}|\rho^{2\gamma-2}\partial_r\rho| \le \int_{r_0-d}^{r_1+d}|\rho^{2\gamma-2}\partial_r\rho|dr + \int_{r_0-d}^{r_1+d}|\partial_r(\rho^{2\gamma-2}\partial_r\rho)|dr \\
&\le (\int_{r_0-d}^{r_1+d}\frac{\rho^{3\gamma-2}}{r^2}dr)^{\frac{1}{2}}\sum_{i=1}^{2}(\int_{r_0-d}^{r_1+d}\rho^{\gamma-2}|\partial_r^i\rho|^2r^2dr)^{\frac{1}{2}} + \int_{r_0-d}^{r_1+d}\rho^{2\gamma-3}|\partial_r\rho|^2dr \\
&\le C_{in}e^{(\gamma-1)MT}\xi_E(t).
\end{aligned} \tag{4.4.11}$$

In order to estimate $\sup_{x_1\le x\le 1}|\rho^{2\gamma-1}r^2D_x\rho|$, we need following two Lemmas.

Lemma 4.4.2 *There exists $C_K>0$, such that*

$$\frac{1}{2}\frac{d}{dt}\int_{x_0}^{1}\chi\rho^{2\gamma-3}r^4|D_x\rho|^2dx \le C_K(\xi_L(t)+\xi_L(t)^{\frac{1}{2}}). \tag{4.4.12}$$

Proof: From the equation (4.1.8), we have

$$D_tD_x\rho = -D_x(\rho^\gamma) - 2D_x\rho\frac{u}{r} - \frac{D_tu}{r^2} - \frac{4\pi x}{r^4},$$

Multiplying the above equation by $\chi D_x\rho\rho^{2\gamma-3}r^4$ and integrating over $[x_0,1]$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{x_0}^{1}\chi\rho^{2\gamma-3}r^4|D_x\rho|^2dx = \frac{1}{2}\int_{x_0}^{1}\chi D_t(\rho^{2\gamma-3}r^4)|D_x\rho|^2dx \\
&-\int_{x_0}^{1}\chi D_x(\rho^\gamma)D_x\rho\rho^{2\gamma-3}r^4dx - 2\int_{x_0}^{1}\chi D_x\rho\frac{u}{r}D_x\rho\rho^{2\gamma-3}r^4dx \\
&-\int_{x_0}^{1}\chi\frac{D_tu}{r^2}D_x\rho\rho^{2\gamma-3}r^4dx - \int_{x_0}^{1}\chi\frac{4\pi x}{r^4}D_x\rho\rho^{2\gamma-3}r^4dx = \sum_{i=1}^{5}I_i.
\end{aligned}$$

For I_1, we have

$$\begin{aligned}
|I_1| &= |-2\int_{x_0}^{1}\chi\rho^{2\gamma-3}r^3u|D_x\rho|^2dx - \frac{1}{2}\int_{x_0}^{1}\chi(2\gamma-3)\rho^{2\gamma-4}D_t\rho r^4|D_x\rho|^2dx| \\
&\le C\sup_{x_0\le x\le 1}|\frac{u}{r}|\int_{x_0}^{1}\chi\rho^{2\gamma-3}r^4|D_x\rho|^2dx + C\sup_{x_0\le x\le 1}|\frac{D_t\rho}{\rho}|\int_{x_0}^{1}\chi\rho^{2\gamma-3}r^4|D_x\rho|^2dx,
\end{aligned} \tag{4.4.13}$$

For I_2, one can get

$$|I_2| \le C\int_{x_0}^{1} \chi\rho^{\gamma-1}|D_x\rho|^2\rho^{2\gamma-3}r^4dx \le \sup_{x_0\le x\le 1}|\rho^{\gamma-1}|\int_{x_0}^{1}\chi\rho^{2\gamma-3}r^4|D_x\rho|^2dx, \tag{4.4.14}$$

and

$$|I_3| \le C\sup_{x_0\le x\le 1}|\frac{u}{r}|\int_{x_0}^{1}\chi\rho^{2\gamma-3}r^4|D_x\rho|^2dx. \tag{4.4.15}$$

Making use of

$$\rho\sim(1-x)^{\alpha}, \quad \text{as} \quad x\sim 1 \quad \text{for} \quad 0\le\alpha\le\frac{1}{2},$$

we can estimate I_4 as follows:

$$\begin{aligned}|I_4| &\le C\sup_{x_0\le x\le 1}|\frac{D_tu}{r}|\int_{x_0}^{1}\chi\rho^{2\gamma-3}|D_x\rho|r^3dx\\ &\le C\sup_{x_0\le x\le 1}|\frac{D_tu}{r}|(\int_{x_0}^{1}\chi\rho^{2\gamma-3}r^4|D_x\rho|^2dx)^{\frac{1}{2}}(\int_{x_0}^{1}\chi\rho^{2\gamma-3}r^2dx)^{\frac{1}{2}}\\ &\le C_K\xi_L(t)^{\frac{1}{2}}.\end{aligned} \tag{4.4.16}$$

For I_5, we can easily obtain that

$$|I_5| \le C_K\xi_L(t)^{\frac{1}{2}}. \tag{4.4.17}$$

Combining all the estimate from (4.4.13) to (4.4.17), we finish the proof of this Lemma. □

Lemma 4.4.3 *There exists $C_K > 0$, such that*

$$\frac{1}{2}\frac{d}{dt}\int_{x_0}^{1}\chi\rho^{4\gamma-1}r^8|D_x^2\rho|^2dx \le C_K\xi_L(t). \tag{4.4.18}$$

Proof: From the equation, we have

$$D_tD_x^2\rho+\gamma(\gamma-1)\rho^{\gamma-2}|D_x\rho|^2+\gamma\rho^{\gamma-1}D_x^2\rho+D_x(\frac{D_tu}{r^2}+\frac{4\pi x}{r^4}+\frac{2D_x\rho u}{r})=0.$$

Multiplying the above equality by $\chi\rho^{4\gamma-1}r^8D_x^2\rho$ and integrating over $[x_0,1]$, one can get

$$\frac{1}{2}\frac{d}{dt}\int_{x_0}^1\chi\rho^{4\gamma-1}r^8|D_x^2\rho|^2dx=\frac{1}{2}\int_{x_0}^1\chi D_t(\rho^{4\gamma-1}r^8)|D_x^2\rho|^2dx$$
$$-\gamma(\gamma-1)\int_{x_0}^1\chi\rho^{5\gamma-3}|D_x\rho|^2r^8D_x^2\rho dx-\gamma\int_{x_0}^1\chi\rho^{5\gamma-2}|D_x^2\rho|^2r^8dx$$
$$-\int_{x_0}^1\chi D_x(\frac{D_tu}{r^2}+\frac{4\pi x}{r^4}+\frac{2D_x\rho u}{r})\rho^{4\gamma-1}D_x^2\rho r^8dx=\sum_{i=1}^4J_i.$$

For J_1, we have

$$|J_1|=|4\int_{x_0}^1\chi\rho^{4\gamma-1}ur^7D_x^2\rho dx+\frac{1}{2}\int_{x_0}^1\chi(4\gamma-1)\rho^{4\gamma-2}D_t\rho r^8|D_x^2\rho|^2dx|$$
$$\leq C\sup_{x_0\leq x\leq 1}|\frac{u}{r}|\int_{x_0}^1\chi\rho^{4\gamma-1}r^8|D_x^2\rho|^2dx+C\sup_{x_0\leq x\leq 1}|\frac{D_t\rho}{\rho}|\int_{x_0}^1\chi\rho^{4\gamma-1}r^8|D_x^2\rho|^2dx. \tag{4.4.19}$$

For J_2, one can get

$$|J_2|\leq C\sup_{x_0\leq x\leq 1}|\rho^{2\gamma-1}r^2D_x\rho|\int_{x_0}^1\chi\rho^{3\gamma-2}|D_x\rho||D_x^2\rho|r^6dx$$
$$\leq C\sup_{x_0\leq x\leq 1}|\rho^{2\gamma-1}r^2D_x\rho|(\int_{x_0}^1\chi\rho^{4\gamma-1}r^8|D_x^2\rho|^2dx+\int_{x_0}^1\chi\rho^{2\gamma-3}r^4|D_x\rho|^2dx), \tag{4.4.20}$$

and

$$|J_3|\leq C\sup_{x_0\leq x\leq 1}|\rho^{\gamma-1}|\int_{x_0}^1\chi\rho^{4\gamma-1}r^8|D_x^2\rho|^2dx. \tag{4.4.21}$$

For J_4, we have

$$J_4=\int_{x_0}^1\chi\rho^{4\gamma-1}r^8D_x^2\rho(\frac{D_tD_xu}{r^2}+\frac{4\pi}{r^4}-\frac{2D_tu}{\rho r^5}-\frac{16\pi x}{\rho r^7})dx$$
$$+2\int_{x_0}^1\chi\rho^{4\gamma-1}D_x^2\rho r^8(\frac{D_x^2\rho u}{r}+\frac{2D_x\rho D_xu}{r})dx+4\int_{x_0}^1\chi\rho^{4\gamma-1}r^8D_x^2\rho\frac{D_x\rho u}{\rho r^4}dx$$
$$=\sum_{i=1}^3E_i. \tag{4.4.22}$$

For each E_i, we can estimate as follows:

$$\begin{aligned} E_1 \le & C \sup_{x_0\le x\le 1} |\rho^{2\gamma-\frac{3}{2}}| (\int_{x_0}^1 \chi\rho^{4\gamma-1} r^8 |D_x^2\rho|^2 dx + \int_{x_0}^1 \chi\rho^2 r^4 |D_t D_x u|^2 dx) \\ & + C \sup_{x_0\le x\le 1} |\rho^{\frac{3}{2}\gamma}| (\int_{x_0}^1 \chi\rho^{\gamma-1} dx + \int_{x_0}^1 \chi\rho^{4\gamma-1} r^8 |D_x^2\rho|^2 dx) \\ & + C \sup_{x_0\le x\le 1} |\rho^{2\gamma-\frac{3}{2}}| (\int_{x_0}^1 \chi\rho^{4\gamma-1} r^8 |D_x^2\rho|^2 dx + \int_{x_0}^1 \chi \frac{|D_t u|^2}{r^2} dx) \\ & + C \sup_{x_0\le x\le 1} |\rho^{\frac{3}{2}\gamma-1}| (\int_{x_0}^1 \chi\rho^{4\gamma-1} r^8 |D_x^2\rho|^2 dx + \frac{1}{(r_0-d)^6}\int_{x_0}^1 \chi\rho^{\gamma-1} dx). \end{aligned} \tag{4.4.23}$$

$$\begin{aligned} E_2 = & 2\int_{x_0}^1 \chi\rho^{4\gamma-1} r^7 |D_x^2\rho|^2 u dx + 4\int_{x_0}^1 \chi\rho^{4\gamma-1} r^7 D_x^2\rho D_x u D_x\rho dx \\ \le & C \sup_{x_0\le x\le 1} |\frac{u}{r}| \int_{x_0}^1 \chi\rho^{4\gamma-1} r^8 |D_x^2\rho|^2 dx + \sup_{x_0\le x\le 1} |\rho^\gamma| \sup_{x_0\le x\le 1} |\rho r^2 D_x u| \\ & (\int_{x_0}^1 \chi\rho^{4\gamma-1} r^8 |D_x^2\rho|^2 dx + \frac{1}{(r_0-d)^2}\int_{x_0}^1 \chi\rho^{2\gamma-3} |D_x\rho|^2 r^4 dx), \end{aligned} \tag{4.4.24}$$

and

$$\begin{aligned} E_3 \le & C \sup_{x_0\le x\le 1} |\frac{u}{r}| \sup_{x_0\le x\le 1} |\rho^\gamma| \\ & (\int_{x_0}^1 \chi\rho^{4\gamma-1} r^8 |D_x^2\rho|^2 dx + \frac{1}{(r_0-d)^2}\int_{x_0}^1 \chi\rho^{2\gamma-3} |D_x\rho|^2 r^4 dx). \end{aligned} \tag{4.4.25}$$

Hence from (4.4.19) to (4.4.25), we obtain (4.4.18). □

Now we are in a position to prove $\sup_{x_1\le x\le 1} |\rho^{2\gamma-1} r^2 D_x\rho|$. By using of Sobolev embedding theorem, we have

$$\begin{aligned} & \sup_{x_1\le x\le 1} |\rho^{2\gamma-1} r^2 D_x\rho| \le C(\int_{x_1}^1 |\rho^{2\gamma-1} r^2 D_x\rho| dx + \int_{x_1}^1 |D_x(\rho^{2\gamma-1} r^2 D_x\rho)| dx) \\ & \le C \sup_{x_1\le x\le 1} |\rho^\gamma| (\int_{x_1}^1 \rho^{2\gamma-2} r^4 |D_x\rho|^2 dx)^{\frac{1}{2}} + C\int_{x_1}^1 |\rho^{2\gamma-1} r^2 D_x^2\rho| dx \\ & + C\int_{x_1}^1 \rho^{2\gamma-2} r^2 |D_x\rho|^2 dx + C\int_{x_1}^1 |\rho^{2\gamma-2} r^{-1} D_x\rho| dx \\ & \le C \sup_{x_1\le x\le 1} |\rho^\gamma| (\int_{x_1}^1 \rho^{2\gamma-2} r^4 |D_x\rho|^2 dx)^{\frac{1}{2}} + C(\int_{x_1}^1 \rho^{4\gamma-1} r^8 |D_x^2\rho|^2 dx)^{\frac{1}{2}} (\int_{x_1}^1 \rho^{-1} r^{-4} dx)^{\frac{1}{2}} \\ & + C\int_{x_1}^1 \rho^{2\gamma-2} r^2 |D_x\rho|^2 dx + C \sup_{x_1\le x\le 1} |\rho^{\gamma-1}| (\int_{x_1}^1 \rho^{2\gamma-2} r^2 |D_x\rho|^2 dx)^{\frac{1}{2}} (\int_{x_1}^1 r^{-6} dx)^{\frac{1}{2}} \\ & \le Ce^{\gamma MT}(\xi_L(t) + \xi_L^{\frac{1}{2}}(t)). \end{aligned} \tag{4.4.26}$$

Combining (4.4.11) with (4.4.26), we conclude that

$$\sup_{x_0\le x\le 1}|\rho r^2 D_x\rho|\le C_{in}(\xi(t)+\xi(t)^{\frac12}+C)\quad\text{for}\quad 0\le t\le T,\tag{4.4.27}$$

for small enough T. Now we have finished the verification of all the Lagrangian terms in (4.1.17). For Eulerian terms, we only give the estimate for $\sup_{0\le r\le r_2-d}|\frac{\partial_r\rho}{\rho}|$. Other terms such as $\frac{\partial_t\rho}{\rho}$ and $\partial_t u$ can be estimated in the same way by using the change of variable: $\partial_t=D_t-\rho r^2 D_x$ in the overlapping region to estimate them in Lagrangian interval $x_1\le x\le x_2$. First, we observe that

$$\sup_{0\le r\le r_2+d}|\frac{1}{\rho}|\le C_{in}e^{MT},$$

so it is sufficient to compute $\sup_{0\le r\le r_2-d}|\partial_r\rho|$. We know that

$$\sup_{0\le r\le r_2-d}|\partial_r\rho|=\sup\{\sup_{0\le r\le r_1+d}|\partial_r\rho|,\sup_{r_1+d\le r\le r_2-d}|\partial_r\rho|\}.$$

Since

$$\begin{aligned}\sup_{0\le r\le r_1+d}|\partial_r\rho|&\le\sum_{|\alpha|\le 2}(\int_{B_{r_1+d}}|\partial_{\mathbf{x}}^\alpha\partial_{\mathbf{x}}\rho|^2d\mathbf{x})^{\frac12}\\&\le\sup_{0\le r\le r_1+d}|\rho^{\frac{2-\gamma}{2}}|\sum_{|\alpha|\le 2}(\int_{B_{r_1+d}}\rho^{\gamma-2}|\partial_{\mathbf{x}}^\alpha\partial_{\mathbf{x}}\rho|^2d\mathbf{x})^{\frac12}\\&\le C_{in}e^{\frac{2-\gamma}{2}MT}\xi_E(t)^{\frac12}.\end{aligned}\tag{4.4.28}$$

Involving that $D_x(\rho r^2D_x\rho)=\frac{2D_x\rho}{r}+r^2|D_x\rho|^2+\rho r^2D_x^2\rho$, we have

$$\begin{aligned}&\sup_{r_1+d\le r\le r_2-d}|\partial_r\rho|\le\sup_{x_1\le x\le x_2}|\rho r^2D_x\rho|\le C(\int_{x_1}^{x_2}|\rho r^2D_x\rho|dx+\int_{x_1}^{x_2}|D_x(\rho r^2D_x\rho)|dx)\\&\le C(\int_{x_1}^{x_2}(\rho^{4-2\gamma}+\frac{1}{\rho^{2\gamma-2}r^6})dx)^{\frac12}(\int_{x_1}^{x_2}\rho^{2\gamma-2}r^4|D_x\rho|^2dx)^{\frac12}\\&+C\sup_{x_1\le x\le x_2}|\frac{1}{\rho^{2\gamma-2}}|\int_{x_1}^{x_2}\rho^{2\gamma-2}r^4|D_x\rho|^2dx\\&+C(\int_{x_1}^{x_2}\frac{1}{\rho^{4\gamma-2}r^4}dx)^{\frac12}(\int_{x_1}^{x_2}\rho^{4\gamma}r^8|D_x^2\rho|^2dx)^{\frac12}.\end{aligned}\tag{4.4.29}$$

This conclude the proof of the Lemma 4.4.1. Thus the a priori estimates can be closed at this point.

4.5 Approximate scheme and local well-posedness of strong solutions

In this section, we will construct the approximate scheme of solutions and obtain the existence, uniqueness of strong solutions. The approximate velocity is obtained by solving a linear parabolic PDE in Lagrangian coordinates and the approximate density is defined by the flow generated by the approximate velocity. Due to the singularity at the origin in Lagrangian formulation, the corresponding Eulerian formulation is invoked and both Lagrangian and Eulerian estimates are obtained.

Let initial data ρ_0, u_0 be given and satisfy

$$\begin{aligned} &\xi|_{(\rho_0,u_0)} \leq A \quad \text{for} \quad \text{some} \quad A > 0; \\ &\rho_0(R) = 0, \rho_0(r) > 0 \quad \text{for} \quad 0 \leq r < R, \int_0^R \rho_0(r) r^2 dr = 1. \end{aligned} \tag{4.5.1}$$

Introduce a Lagrangian variable x as follows:

$$x = \int_0^r \rho_0 s^2 ds, \quad 0 \leq x \leq 1.$$

Define

$$r_0 = (3\int_0^x \frac{1}{\rho_0(y)} dy)^{\frac{1}{3}}.$$

We would like to define the sequence $\{\rho^n, u^n, r^n\}$ inductively for $n \geq 0$. Suppose that ρ^n, r^n and u^n are known functions. Consider the following linear parabolic equation for u^{n+1}:

$$D_t u^{n+1} - D_x\big((\rho^n)^2 (r^n)^4 D_x u^{n+1}\big) + 2\frac{u^{n+1}}{(r^n)^2} = -(r^n)^2 D_x P^n - \frac{4\pi x}{(r^n)^2}, \tag{4.5.2}$$

with the initial data

$$u^{n+1}(x, 0) = u_0(x),$$

and boundary conditons

$$u^{n+1}(0, t) = 0, \quad ((\rho^n)^2 (r^n)^2 D_x u^{n+1} - P^n)(1, t) = 0.$$

By the change of variables

$$D_x = \frac{1}{\rho^n (r^n)^2}\partial_{r^n}, \quad \text{and} \quad D_t = \partial_t + (D_t r^n)\partial_{r^n}, \tag{4.5.3}$$

(4.5.2) can be written in Eulerian coordinates (r^n, t) as follows:

$$\partial_t u^{n+1} + D_t r^n \partial_{r^n} u^{n+1} - \frac{1}{\rho^n (r^n)^2}\partial_{r^n}(\rho^n (r^n)^2 \partial_{r^n} u^{n+1}) + \frac{2u^{n+1}}{(r^n)^2} = -\frac{1}{\rho^n}\partial_{r^n} P^n - \frac{4\pi x}{(r^n)^2}, \tag{4.5.4}$$

where $D_t r^n = -\frac{1}{(r^n)^2}\int_0^x \frac{D_t \rho^n}{(\rho^n)^2} dy$ and $\partial_t x = -\rho^n (r^n)^2 D_t r^n$; $\partial_{r^n} x = \rho^n (r^n)^2$. Next, define ρ^{n+1} by

$$\rho^{n+1}(x,t) = \rho_0(x)\exp\{-\int_0^t (\rho^n (r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n}) d\tau\}. \tag{4.5.5}$$

We can check easily that ρ^{n+1} satisfies the following equation

$$D_t \rho^{n+1} + \{\rho^n (r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n}\}\rho^{n+1} = 0. \tag{4.5.6}$$

In Eulerian coordinates, we can write (4.5.6) as

$$\partial_t \rho^{n+1} + D_t r^n \partial_{r^n} \rho^{n+1} + \rho^{n+1}(\partial_{r^n} u^{n+1} + \frac{2u^{n+1}}{r^n}) = 0. \tag{4.5.7}$$

Lastly, we define r^{n+1} by

$$r^{n+1} = (3\int_0^x \frac{1}{\rho^{n+1}} dy)^{\frac{1}{3}}. \tag{4.5.8}$$

We need to show that (4.5.2) is solvable and (4.5.5), (4.5.8) make sense in an appropriate sense. First, we investigate (4.5.2) in a weak formulation in Lagrangian coordinates, and establish the regularity of weak solution. Interior regularity is standard since (4.5.2) is a linear parabolic bounded away from the boundary, while boundary regularity is obtained with weights in the form of integrals. Once we have the regularity for u^{n+1}, we can check ρ^{n+1}, r^{n+1} are well-defined. From the equivalent in the interior of (4.5.2) and (4.5.4), we can easily obtain the Eulerian regularity. We will study the existence of weak solution in the frame of Galerkin's method [41]. Without confusion, we will drop the index n from now on. We assume that we have as much regularity of ρ and r as needed.

Firstly, we define the notion of the weak solution, as in [34]. Introduce a Hilbert space H.

$$H = Cl\{u \in C_0^\infty(0,1) : \int_0^1 (\rho^2 r^4 |D_x u|^2 + \frac{2u^2}{r^2}) dx < \infty, \quad u(0) = 0\}.$$

We can easily see that $H \subset L^2(0,1)$.

Definition 4.5.1 *We say $u \in L^2(0,T;H)$ with $u' \in L^2(0,T;H^*)$ is a weak solution of* (4.5.2) *provided*

$$\int_0^1 u' v dx + \int_0^1 \rho^2 r^4 D_x u D_x v dx + \int_0^1 \frac{2uv}{r^2} dx = \int_0^1 r^2 P D_x v dx + \int_0^1 (\frac{2P}{\rho r} - \frac{4\pi x}{r^2}) v dx$$

for each $v \in H$ a.e. time $0 \le t \le T$, and $u(0) = u_0(x)$. H^* is the dual space of H and $' = D_t$.

Lemma 4.5.2 *Assume that $u_0 \in L^2$, $\rho^{-1}P \in L^2(0,T;L^2)$ and $r^{-1}x \in L^2(0,T;L^2)$. There exists a unique weak solution $u \in L^2(0,T;H)$ with $u' \in L^2(0,T;H^*)$ to* (4.5.2)*. Furthermore, there exists a constant $C > 0$ such that*

$$\begin{aligned} &\max_{0 \le t \le T} \|u(t)\|_{L^2} + \|u\|_{L^2(0,T;H)} + \|u'\|_{L^2(0,T;H^*)} \\ &\le C(\|u_0\|_{L^2} + \|\frac{P}{\rho}\|_{L^2(0,T;L^2)} + \|\frac{4\pi x}{r}\|_{L^2(0,T;L^2)}). \end{aligned} \tag{4.5.9}$$

Proof: Let $\omega_k = \omega_k(x) (k = 1, 2, \cdots)$ be an orthogonal basis of H and orthonormal in L^2 when $t = 0$, i.e, $\rho(0) = \rho_0$, $r(0) = r_0$. Then $\{\omega_k\}$ forms a basis of H for $0 \le t \le T$, where T is sufficiently small, due to the smoothness of ρ, r. Fix a positive integer m. We seek a function $u_m : [0,T] \longrightarrow H$ of the form

$$u_m(t) = \sum_{k=1}^m d_m^k(t) \omega_k, \tag{4.5.10}$$

where

$$d_m^k(0) = \int_0^1 u_0(x) \omega_k(x) dx, \quad (k = 1, 2 \cdots, m) \tag{4.5.11}$$

and for each $k = 1, 2, \cdots, m$, $0 \le t \le T$,

$$\begin{aligned} &\int_0^1 u_m' \omega_k dx + \int_0^1 \rho^2 r^4 D_x u_m D_x \omega_k dx + \int_0^1 \frac{2u_m \omega_k}{r^2} dx \\ &= \int_0^1 r^2 P D_x \omega_k dx + \int_0^1 (\frac{2P}{\rho r} - \frac{4\pi x}{r^2}) \omega_k dx. \end{aligned} \tag{4.5.12}$$

Note that

$$\int_0^1 u_m'\omega_k dx = {d_m^k}'(t), \quad \int_0^1 \rho^2 r^4 D_x u_m D_x\omega_k dx + \int_0^1 \frac{2u_m\omega_k}{r^2}dx = \sum_{l=1}^m e^{kl}(t)d_m^l(t),$$

where

$$e^{kl} = \int_0^1 \rho^2 r^4 D_x\omega_l D_x\omega_k dx + \int_0^1 \frac{2\omega_l\omega_k}{r^2}dx.$$

Define

$$f^k(t) = \int_0^1 r^2 P D_x\omega_k dx + \int_0^1 (\frac{2P}{\rho r} - \frac{4\pi x}{r^2})\omega_k dx,$$

then (4.5.12) becomes the linear ODE systems

$${d_m^k}'(t) + \sum_{l=1}^m e^{kl}(t)d_m^l(t) = f^k(t), \tag{4.5.13}$$

with the initial condition (4.5.11). By the classical theory of ODEs, there exists a unique function $d_m^k(t)$ satisfying (4.5.11) and (4.5.13) for a.e $0 \le t \le T$. In addition, u_m defined by (4.5.10) solves (4.5.12).

Multiplying (4.5.12) by d_k^m and summing up, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 |u_m|^2 dx + \int_0^1 (\rho^2 r^4 |D_x u_m|^2 + \frac{2u_m^2}{r^2})dx \\ &= \int_0^1 r^2 P D_x u_m dx + \int_0^1 (\frac{2P}{\rho r} - \frac{4\pi x}{r^2})u_m dx \\ &\le \frac{1}{2}\int_0^1 (\rho^2 r^4 |D_x u_m|^2 + \frac{2u_m^2}{r^2})dx + \int_0^1 \rho^{2\gamma-2}dx + \int_0^1 \frac{16\pi^2 x^2}{r^2}dx. \end{aligned} \tag{4.5.14}$$

Integrating (4.5.14) over $[0,T]$, we have

$$\begin{aligned} \max_{0\le t\le T} \|u_m(t)\|_{L^2} + \int_0^T\int_0^1 (\rho^2 r^4 |D_x u_m|^2 + \frac{2u_m^2}{r^2})dxds \le \\ \|u_0\|_{L^2}^2 + 4(\|\frac{P}{\rho}\|_{L^2(0,T;L^2)}^2 + \|\frac{4\pi x}{r}\|_{L^2(0,T;L^2)}^2). \end{aligned} \tag{4.5.15}$$

Fix $v \in H$ with $\|v\|_H \le 1$. Write $v = v_1 + v_2$, $v_1 \in span\{\omega_k\}_{k=1}^m$, and $\int_0^1 v_2\omega_k dx = 0$, $k = 1, 2\cdots, m$. Since $\{\omega_k\}$ is orthogonal, $\|v_1\|_H \le \|v\|_H \le 1$. From (4.5.12), we have

$$\begin{aligned} \int_0^1 u_m' v dx = \int_0^1 u_m' v_1 dx = -\int_0^1 \rho^2 r^4 D_x u_m D_x v_1 dx - \int_0^1 \frac{2u_m v_1}{r^2}dx + \int_0^1 r^2 P D_x v_1 dx \\ + \int_0^1 (\frac{2P}{\rho r} - \frac{4\pi x}{r^2})v_1 dx \le C(\|\frac{P}{\rho}\|_{L^2} + \|\frac{4\pi x}{r}\|_{L^2} + \|u_m\|_H). \end{aligned}$$

Hence,

$$\|u_m'\|_{H^*} \le C(\|\frac{P}{\rho}\|_{L^2} + \|\frac{4\pi x}{r}\|_{L^2} + \|u_m\|_H),$$

and

$$\int_0^T \|u_m'\|_{H^*} dt \le C(\|u_0\|_{L^2} + \|\frac{P}{\rho}\|_{L^2(0,T;L^2)} + \|\frac{4\pi x}{r}\|_{L^2(0,T;L^2)}).$$

Now, we can pass to limit as $m \longrightarrow \infty$. Then we obtain the existence of weak solutions. The uniqueness easily follows from energy estimates.

In the following Lemmas, we provide the regularity of the weak solution. First, we establish Lagrangian regularity. Eulerian regularity is obtained by the change of variable (4.5.3) in the integral form. We skip the details of here. In the next Lemma, we will give the regularity in time.

Lemma 4.5.3 *Assume* $\sup_{0<x<1} |\frac{\rho'}{\rho}| \le C_1$, $\sup_{0<x<1} |\frac{r'}{r}| \le C_2$ *for* $0 \le t \le T$. *In addition, assume* $u_0 \in H$ *and* $r^2 D_x P + \frac{4\pi x}{r^2} \in L^2(0,T;L^2)$. *Then* $u \in L^\infty(0,T;H)$, $u' \in L^2(0,T;L^2)$ *with the estimate*

$$\begin{aligned}\int_0^T \|u'\|_{L^2}^2 dt + \sup_{0\le t\le T} \|u(t)\|_H^2 \le C(\|u_0\|_H^2 + \|u_0\|_{L^2}^2 \\ + \|r^2 D_x P + \frac{4\pi x}{r^2}\|_{L^2(0,T;L^2)}^2 + \|\frac{P}{\rho}\|_{L^2(0,T;L^2)}^2 + \|\frac{4\pi x}{r}\|_{L^2(0,T;L^2)}^2).\end{aligned} \tag{4.5.16}$$

Proof: Multiplying (4.5.12) by ${d_m^k}'$ and summing over k, it holds that

$$\int_0^1 u_m' u_m' dx + \int_0^1 (\rho^2 r^4 D_x u_m D_x u_m' + \frac{2 u_m u_m'}{r^2}) dx = -\int_0^1 (r^2 D_x P + \frac{4\pi x}{r^2}) u_m' dx,$$

hence we have

$$\begin{aligned}\|u_m'\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\int_0^1 (\rho^2 r^4 |D_x u_m|^2 + \frac{2|u_m|^2}{r^2}) dx \le \frac{1}{2}\|u_m'\|_{L^2}^2 + \frac{1}{2}\|r^2 D_x P + \frac{4\pi x}{r^2}\|_{L^2}^2 \\ + \frac{1}{2}\int_0^1 \big((2\rho\rho' r^4 + 4\rho^2 r^3 r')|D_x u_m|^2 - 4\frac{r'}{r^3}|u_m|^2\big) dx,\end{aligned} \tag{4.5.17}$$

since $\sup_{0<x<1} |\frac{\rho'}{\rho}| \le C_1$, $\sup_{0<x<1} |\frac{r'}{r}| \le C_2$, so

$$\frac{1}{2}\int_0^1 \big((2\rho\rho' r^4 + 4\rho^2 r^3 r')|D_x u_m|^2 - 4\frac{r'}{r^3}|u_m|^2\big) dx \le C\|u_m\|_H^2.$$

Integrating (4.5.17) over $[0,T]$, we have

$$\int_0^T \|u_m'\|_{L^2}^2 dt + \sup_{0<x<1} \int_0^1 (\rho^2 r^4 |D_x u_m|^2 + \frac{2|u_m|^2}{r^2}) dx$$
$$\leq \int_0^1 (\rho_0^2 r_0^4 |D_x u_m(0)|^2 + \frac{2|u_m(0)|^2}{r^2}) dx + C \int_0^T \|u_m\|_H^2 dt + \int_0^T \|r^2 D_x P + \frac{4\pi x}{r^2}\|_{L^2}^2 dt.$$

Pass to limit $m \longrightarrow \infty$. (4.5.16) holds and the Lemma follows.

Now we want to establish regularity in x variable. Bearing in mind that (4.5.2) is one-dimensional linear parabolic equation as long as x is bounded away from the boundary and hence interior regularity can be easily shown by using the standard differential quotients method, i.e. $u \in H^2_{loc}(0,1)$. Here H^2 represents the usual Sobolev space. Recall that

$$\begin{aligned} &\int_0^1 \rho^2 r^4 D_x u D_x v dx + \int_0^1 \frac{2uv}{r^2} dx \\ &= \int_0^1 r^2 P D_x v dx + \int_0^1 (\frac{2P}{\rho r} - \frac{4\pi x}{r^2} - u') v dx, \quad \forall v \in H. \end{aligned} \tag{4.5.18}$$

We can now integrate by parts in (4.5.18) by approximating H with $v \in C_c^\infty(0,1) \subset H$:

$$\begin{aligned} &-\int_0^1 D_x(\rho^2 r^4 D_x u) v dx + \int_0^1 \frac{2uv}{r^2} dx \\ &= -\int_0^1 (r^2 D_x P + \frac{4\pi x}{r^2} + u') v dx, \quad \forall v \in C_c^\infty(0,1). \end{aligned}$$

therefore, u actually solves the PDE a.e. , and the following estimate can be obtained from the equation:

$$\begin{aligned} \int_0^1 r^2 |r^2 D_x(\rho^2 r^2 D_x u)|^2 dx \leq &\int_0^1 r^2 |u'|^2 dx + \int_0^1 (\frac{4u^2}{r^2} + 4\rho^2 r^4 |D_x u|^2) dx \\ &+ \int_0^1 r^2 |r^2 D_x P + \frac{4\pi x}{r^2}|^2 dx. \end{aligned} \tag{4.5.19}$$

Note that $D_x(\rho^2 r^2 D_x u)$ is in $L^1(r^2 dx)$, so by the trace theorem, $\rho^2 r^2 D_x u$ at $x=1$ is well-defined. Thus in (4.5.18), we can integrate by parts up to the boundary for the first term to get:

$$\begin{aligned} &\rho^2 r^4 D_x u v(1,t) - \int_0^1 D_x(\rho^2 r^4 D_x u) v dx + \int_0^1 \frac{2uv}{r^2} dx \\ &= -\int_0^1 (r^2 D_x P + \frac{4\pi x}{r^2} + u') v dx, \quad \forall v \in H. \end{aligned}$$

We know that u solves the PDE a.e. and hence, we obtain that $\rho^2 r^2 D_x u(1,t) = 0$, the desired boundary condition. Also, we have proven the spatial regularity of u^{n+1}.

Lemma 4.5.4 *The weak solution u solves* (4.5.2) *and $u \in H^2_{loc}(0,1)$ and weak boundary regularity is given by* (4.5.19)*. Moreover, u satisfies the boundary condition.*

Using the same idea as the previous argument, we can build the high order regularity.

Lemma 4.5.5 *The weak solution $u \in H^4_{loc}(0,1)$ as long as initial data u_0 as well as coefficients ρ^n, r^n are regular. Moreover, weak boundary regularity is available in the integral form.*

Remark 4.5.6 *The boundary regularity is weak in a sense that ρ, r as weight functions vanish at $x = 1, 0$ respectively.*

Remark 4.5.7 *Since the equivalent of* (4.5.2) *and* (4.5.4) *away from the boundary, and the interior regularity in Lagrangian coordinates, we see u also solve* (4.5.4) *a.e. Corresponding Eulerian regularity can be obtained by the change of variable* (4.5.3)*.*

Now we are in a position to prove the local-wellposedness of strong solution. By using the similar argument as in [34] and the previous a prior estimate $\xi(t)$, we can obtain the uniform estimates of u^n, ρ^n and r^n on n for sufficiently small $T > 0$, which assure the existence of limit functions u, ρ and r. So we only need to show the uniqueness of strong solutions. Let (ρ_1, u_1, r_1) and (ρ_2, u_2, r_2) be two strong solutions to (4.1.13) satisfying the same initial condition. Considering momentum equations for u_1 and u_2 in Lagrangian coordinates:

$$D_t u_1 - D_x(\rho_1^2 r_1^4 D_x u_1) + \frac{2u_1}{r_1^2} = -r_1^2 D_x P_1 - \frac{4\pi x}{r_1^2}, \qquad (4.5.20)$$

$$D_t u_2 - D_x(\rho_2^2 r_2^4 D_x u_2) + \frac{2u_2}{r_2^2} = -r_2^2 D_x P_2 - \frac{4\pi x}{r_2^2}. \tag{4.5.21}$$

Subtracting (4.5.20) from (4.5.21), we obtain

$$\begin{aligned} &D_t(u_1 - u_2) - D_x(\rho_1^2 r_1^4 D_x(u_1 - u_2)) + \frac{2(u_1 - u_2)}{r_1^2} \\ &= -r_1^2 D_x P_1 - \frac{4\pi x}{r_1^2} + r_2^2 D_x P_2 + \frac{4\pi x}{r_2^2} \\ &+ D_x((\rho_1^2 r_1^4 - \rho_2^2 r_2^4) D_x u_2) - 2u_2(\frac{1}{r_1^2} - \frac{1}{r_2^2}). \end{aligned} \tag{4.5.22}$$

Multiplying (4.5.22) by $u_1 - u_2$ and integrating over $[0, 1]$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 |u_1 - u_2|^2 dx + \int_0^1 (\rho_1^2 r_1^4 |D_x(u_1 - u_2)|^2 + \frac{2|u_1 - u_2|^2}{r_1^2}) dx \\ &= \int_0^1 (r_1^2 P_1 - r_2^2 P_2) D_x(u_1 - u_2) dx + \int_0^1 (\frac{2P_1}{\rho_1 r_1} - \frac{2P_2}{\rho_2 r_2})(u_1 - u_2) dx \\ &- 4\pi \int_0^1 (\frac{x}{r_1^2} - \frac{x}{r_2^2})(u_1 - u_2) dx \\ &- \int_0^1 (\rho_1^2 r_1^4 - \rho_2^2 r_2^4) D_x u_2 D_x(u_1 - u_2) dx - 2\int_0^1 u_2(\frac{1}{r_1^2} - \frac{1}{r_2^2})(u_1 - u_2) dx \\ &\le (\int_0^1 \frac{1}{\rho_1^2 r_1^4} |r_1^2 P_1 - r_2^2 P_2|^2 dx)^{\frac{1}{2}} (\int_0^1 \rho_1^2 r_1^4 |D_x(u_1 - u_2)|^2 dx)^{\frac{1}{2}} \\ &+ \{(2\int_0^1 r_1^2 |\frac{P_1}{\rho_1 r_1} - \frac{P_2}{\rho_2 r_2}|^2 dx)^{\frac{1}{2}} + (8\pi^2 \int_0^1 r_1^2 x^2 |\frac{1}{r_1^2} - \frac{1}{r_2^2}|^2 dx)^{\frac{1}{2}}\} (\int_0^1 \frac{2(u_1 - u_2)^2}{r_1^2} dx)^{\frac{1}{2}} \\ &+ (\int_0^1 \frac{1}{\rho_1^2 r_1^4} |(\rho_1 r_1^4 - \rho_2 r_2^4) D_x u_2|^2 dx)^{\frac{1}{2}} (\int_0^1 \rho_1^2 r_1^4 |D_x(u_1 - u_2)|^2 dx)^{\frac{1}{2}} \\ &+ (2\int_0^1 r_1^2 u_2^2 |\frac{1}{r_1^2} - \frac{1}{r_2^2}|^2 dx)^{\frac{1}{2}} (\int_0^1 \frac{2(u_1 - u_2)^2}{r_1^2} dx)^{\frac{1}{2}}. \end{aligned} \tag{4.5.23}$$

Now we estimate the right hand side of (4.5.23) term by term. Bearing in mind that

$$\rho_1(x,t) = \rho_0(x) e^{-\int_0^t (\rho_1 r_1^2 D_x u_1 + \frac{2u_1}{r_1}) d\tau}; \quad \rho_2(x,t) = \rho_0(x) e^{-\int_0^t (\rho_2 r_2^2 D_x u_2 + \frac{2u_2}{r_2}) d\tau}.$$

Here we only provide the detail for $\int_0^1 \frac{1}{\rho_1^2 r_1^4} |(\rho_1 r_1^4 - \rho_2 r_2^4) D_x u_2|^2 dx$ since other terms can be estimated in the same way.

$$\begin{aligned}&\int_0^1 \frac{1}{\rho_1^2 r_1^4}|(\rho_1 r_1^4 - \rho_2 r_2^4)D_x u_2|^2 dx \le M^2 \int_0^1 \frac{1}{\rho_1^2 r_1^4}|\frac{\rho_1^2 r_1^4 - \rho_2^2 r_2^4}{\rho_2 r_2^2}|^2 dx \\ &= M^2 \int_0^1 |\frac{\rho_1 r_1^2}{\rho_2 r_2^2} - \frac{\rho_2 r_2^2}{\rho_1 r_1^2}|^2 dx,\end{aligned} \tag{4.5.24}$$

where M is the bound of $\rho_2 r_2^2 D_x u_2$. Note that

$$\begin{aligned}&|\frac{\rho_1 r_1^2}{\rho_2 r_2^2} - \frac{\rho_2 r_2^2}{\rho_1 r_1^2}|^2 = |\exp(\int_0^t (\rho_2 r_2^2 D_x u_2 - \rho_1 r_1^2 D_x u_1)ds) \\ &- \exp(-\int_0^t (\rho_2 r_2^2 D_x u_2 - \rho_1 r_1^2 D_x u_1)ds)|^2 \\ &\le C|\int_0^t (\rho_2 r_2^2 D_x u_2 - \rho_1 r_1^2 D_x u_1)ds|^2 \\ &\le C|\int_0^t \rho_1 r_1^2 |D_x(u_1 - u_2)ds|^2 + C|\int_0^t (\rho_1 r_1^2 - \rho_2 r_2^2) D_x u_2 ds|^2 \\ &\le Ct(\int_0^t \rho_1^2 r_1^4 |D_x(u_1 - u_2)|^2 ds) + CM^2|\int_0^t (\frac{\rho_1 r_1^2}{\rho_2 r_2^2} - 1)ds|^2.\end{aligned}$$

Since

$$CM^2|\int_0^t (\frac{\rho_1 r_1^2}{\rho_2 r_2^2} - 1)ds|^2 \le CM^2 t \int_0^t |\frac{\rho_1 r_1^2}{\rho_2 r_2^2} - \frac{r_2^2}{r_1^2}|^2 ds + CM^2 t \int_0^t |\frac{r_2^2}{r_1^2} - 1|^2 ds,$$

so we need to estimate $\int_0^t |\frac{\rho_1 r_1^2}{\rho_2 r_2^2} - \frac{r_2^2}{r_1^2}|^2 ds$. Using the same idea as (4.5.24), we have

$$\begin{aligned}&|\frac{\rho_1 r_1^2}{\rho_2 r_2^2} - \frac{r_2^2}{r_1^2}|^2 \le C|\int_0^t (\rho_2 r_2^2 D_x u_2 - \rho_1 r_1^2 D_x u_1 - 2(\frac{u_2}{r_2} - \frac{u_1}{r_1})ds|^2 \\ &\le Ct(\int_0^t \rho_1^2 r_1^4 |D_x(u_1 - u_2)^2 ds) + CM^2|\int_0^t (\frac{\rho_1 r_1^2}{\rho_2 r_2^2} - 1)ds|^2 \\ &+ Ct\int_0^t \frac{4(u_1 - u_2)^2}{r_1^2} ds + CM^2|\int_0^t (\frac{r_2}{r_1} - 1)ds|^2.\end{aligned}$$

Since we have

$$|\frac{r_2}{r_1} + 1| \le 1 + e^{\frac{2}{3}MT}, \quad \text{and} \quad CM^2|\int_0^t (\frac{r_2}{r_1} - 1)ds|^2 \le CM^2 t \int_0^t |\frac{r_2}{r_1} - 1|^2 ds,$$

we get, for $0 \le t \le T$ where T is sufficiently small to be fixed,

$$\begin{aligned}&|\frac{\rho_1 r_1^2}{\rho_2 r_2^2} - \frac{r_2^2}{r_1^2}|^2 \le CM^2 T \int_0^t |\frac{\rho_1 r_1^2}{\rho_2 r_2^2} - \frac{r_2^2}{r_1^2}|^2 ds + CM^2 T(1 + e^{\frac{4}{3}MT}) \int_0^t |\frac{r_2}{r_1} - 1|^2 ds \\ &+ Ct\int_0^t (\rho_1^2 r_1^4 |D_x(u_1 - u_2)^2 + \frac{4(u_1 - u_2)^2}{r_1^2})ds.\end{aligned}$$

On the other hand, in the similar argument, we can obtain

$$|\frac{r_2}{r_1}-1|^2 \le C|\int_0^t(\frac{u_2}{r_2}-\frac{u_1}{r_2})ds|^2 \le CM^2T\int_0^t|\frac{r_2}{r_1}-1|^2ds + Ct\int_0^t\frac{|u_1-u_2|^2}{r_1^2}ds,$$

by the Gronwall's inequality, we get

$$|\frac{\rho_1 r_1^2}{\rho_2 r_2^2}-\frac{r_2^2}{r_1^2}|^2+|\frac{r_2}{r_1}-1|^2 \le CT\int_0^t(\rho_1^2r_1^4|D_x(u_1-u_2)^2+\frac{2(u_1-u_2)^2}{r_1^2})ds$$
$$\cdot(1+CM^2T(1+e^{\frac{4}{3}MT})te^{CM^2T(1+e^{\frac{4}{3}MT})t}).$$

Taking this into account (4.5.24), we have

$$\int_0^1\frac{1}{\rho_1^2r_1^4}|(\rho_1r_1^4-\rho_2r_2^4)D_xu_2|^2dx$$
$$\le C_{M,T}\int_0^1\int_0^t(\rho_1^2r_1^4|D_x(u_1-u_2)|^2+\frac{2(u_1-u_2)^2}{r_1^2})dsdx.$$

Using the same argument, we can obtain that

$$\frac{1}{2}\frac{d}{dt}\int_0^1|u_1-u_2|^2dx+\int_0^1(\rho_1^2r_1^4|D_x(u_1-u_2)|^2+\frac{2|u_1-u_2|^2}{r_1^2})dx$$
$$\le C_{M,T}\int_0^t\int_0^1(\rho_1^2r_1^4|D_x(u_1-u_2)|^2+\frac{2(u_1-u_2)^2}{r_1^2})dxds.$$

Integrating the above inequality over $[0,t]$, it holds that for sufficient small T,

$$\frac{1}{2}\int_0^1|u_1-u_2|^2dx(t) \le \frac{1}{2}\int_0^1|u_1-u_2|^2dx(0),$$

that is $u_1=u_2$, a.e. Once we deriver this result, we can obtain that $r_1=r_2$ from $D_tr=u$, and $\rho_1=\rho_2$ from $r^3=3\int_0^x\frac{1}{\rho}dy$. Hence we get the uniqueness and finish the proof of theorem 4.1.2.

Chapter 5

Discuss on future work

In this section, we mainly list some related problems on the compressible Navier-Stokes equations.

In chapter 2, we establish a global well-posedness of classical solutions for compressible Navier-Stokes systems in a half-space under Navier boundary condition. But we do not know yet whether similar results hold for Dirichlet boundary condition. Since in that case, we cannot use the same argument as in chapter 2 to improve the regularity of F and ω, where F is the effective viscous flux and ω is the vorticity, due to the less boundary information. It seems that we need a new technology to deal with this kind of problem. This is a future work.

In chapter 3 and 4, concerning the multi-dimensional compressible Navier-Stokes equations with viscosity coefficient depending on density, can we obtain similar results? For spherically symmetric case, we can transfer this problem to be a one-dimensional model. By using of the method established for 1-d, we can obtain the existence, uniqueness and large time behavior of solutions. However, for the higher-dimensional case, it is a quite different story and would be more complicated.

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