Some New Results on Nonlinear Elliptic Equations and Systems

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Abstract

This thesis is devoted to the study of nonlinear elliptic equations and systems. It is divided into two parts. In the first part, we study the uniqueness problem, and in the second part, we are concerns with traveling wave solutions.

In Chapter 2 we study the uniqueness problem of sign-changing solutions for a nonlinear scalar equation. It is well-known that positive solution is radially symmetric and unique up to a translation. Recently, there are many works on the existence and multiplicity of sign-changing solutions. However much less is known for uniqueness, even in the radially symmetric class. In Chapter 2, we solve this problem for nearly critical nonlinearity by Lyaponov-Schmidt reduction. Moreover, we can also prove the non-degeneracy.

In Chapter 3 we are concerned with the uniqueness problem for coupled nonlinear Schrödinger equations. The problem is to classify all positive solutions. In Chapter 3, some sufficient conditions are given. In particular, we have a sufficient and necessary condition in one dimension. The proof is elementary because only the implicit function theorem, integration by parts, and the uniqueness for scalar equation are needed.

In Chapter 4 we go back to the nonlinear scalar equation and consider the traveling wave solutions. Using an infinite dimensional Lyaponov-Schmidt reduction, new examples of traveling wave solutions are constructed. Our approach explains the difference between two dimension and higher dimensions, and also explores a connection between moving fronts and the mean curvature flow. This is the first such traveling waves connecting the same states.

摘要

本論文研究的是非線性橢圓方程和方程組。它可以分成兩個部分，其中之一研究的是唯一性問題，另外一個研究的是行波解。

首先在第二章我們研究一類非線性橢圓方程的變號解的唯一性問題。由於變號解的多樣性，我們只考慮具有徑向對稱性的變號解。當方程中的非線性項的次數足夠接近臨界值時，我們可以證明徑向對稱解具有唯一性和非退化性。證明的工具是一種有限維的Lyaponov-Schmidt約化方法。

接著在第三章我們研究耦合的非線性Schrödinger方程組的正解的唯一性問題。我們給出了一些條件使得耦合的非線性Schrödinger方程組有唯一的正解。特別的是，這個條件在一維的情況下也是必要的條件。

最後在第四章我們研究了一類非線性橢圓方程的行波解。運用一種無窮維的Lyaponov-Schmidt約化方法，我們可以構造出一些新的行波解。有趣的是他們的結構與平均曲率流以及Jacobi—Toda系統有關。

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This work is dedicated to **my family.**

Contents

Chapter 1

Introduction

In this thesis, we study two kinds of problems. The first one is the uniqueness problem for elliptic equations and systems. The second one concerns the traveling wave solutions. We will consider three nonlinear elliptic equations: the nonlinear Schrödinger equation of the form

$$\Delta u - u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N); \tag{NLS}$$

the coupled nonlinear Schrödinger equations of the form

$$\begin{cases} \Delta u_1 - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0 \quad \text{in } \mathbb{R}^N, \\ \Delta u_2 - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0 \quad \text{in } \mathbb{R}^N, \\ u_1, u_2 \in H^1(\mathbb{R}^N); \end{cases} \tag{CNLS}$$

and the traveling wave equation of the form

$$\Delta v + c\frac{\partial v}{\partial x_{N+1}} - v + v^p = 0 \quad \text{in } \mathbb{R}^{N+1}. \tag{TWc}$$

Next we will consider these problems separately.

1.1 Uniqueness of sign-changing solutions in NLS

Consider the general semi-linear elliptic equation

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \tag{1.1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(0) = 0$. Equations of type (1.1.1) arise in various contexts of physics, such as, constructive field theory, false vacuum in cosmology, nonlinear optics, laser propagation, etc. They are also called nonlinear Euclidean scalar field equations, see [15, 16] and the references therein.

Such equations arise in particular in the search of standing waves in nonlinear equations of the Klein-Gordon or Schrödinger type. Indeed, consider the nonlinear Schrödinger type equation

$$i\frac{\partial \Psi}{\partial t} - \Delta\Psi = g(|\Psi|^2)\Psi,$$

where Ψ is the complex-valued wave function, g is a real function such that $g(0) = 0$. Considering the so-called standing wave solutions, i.e., $\Psi(t, x) = e^{-i\lambda t}u(x)$, one is led to the following equation

$$\Delta u - \lambda u + f(u) = 0,$$

where $f(u) = g(u^2)u$.

The Lagrangian $E(u)$ associated with (1.1.1), is defined by

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \int_{\mathbb{R}^N} F(u)dx, \tag{1.1.2}$$

where $F(s) = \int_0^s f(t)\, dt$. The functional $E(u)$ is also called the "action" associated with (1.1.1). Moreover, by analogy with nonlinear elliptic problems in bounded domains, $E(u)$ is sometimes called the energy associated with (1.1.1). Roughly speaking, a solution u to (1.1.1) such that $E(u) < +\infty$ is called *bound state* solution. If a bound state solution u_0 has the property of having the least action among all non-trivial bound state solutions, namely, $0 < E(u_0) \leq E(u)$, for any non zero bound state solution u of (1.1.1), then u_0 is called *ground state* solution. For the existence of ground and bound state solutions, see [25, 15, 16, 13, 26, 27] and references therein.

In Chapter 2, we consider (NLS), i.e.,

$$\Delta u - u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N). \tag{1.1.3}$$

It is well-known that, if

$$1 < p < 2^* - 1, \quad \text{where} \;\; 2^* = \begin{cases} +\infty, & \text{if } N = 1, 2, \\ \frac{2N}{N-2}, & \text{if } N \geq 3, \end{cases}$$

then (1.1.3) has a ground state solution, which is positive (cf.[25, 15]). For $p \geq \frac{N+2}{N-2}$, Pohožaev's identity implies that (1.1.3) has no non-trivial solution (cf. [81, 84]).

The structure of positive solutions to (1.1.3) is completely classified thanks to the moving planes method developed in [44] and the uniqueness result in [53]. More precisely, the space of positive solutions to (1.1.3) is a smooth N dimensional manifold parameterized by $w(\cdot - x)$, $x \in \mathbb{R}^N$, where w is the unique positive radially symmetric solution to (1.1.3).

Next, we move on to sign-changing solutions to (1.1.3). Unlike positive solutions, sign-changing solutions have more complicated qualitative properties, such as the number and shapes of nodal domains and the measure of nodal sets. For a deeper discussion of a more general case, we refer the reader to the recent survey article [64], where various methods for obtaining sign-changing solutions developed in the last three decades are revisited. Apart from the survey article [64], there is a quite interesting paper [76], where the authors construct infinitely many nonradial solutions in any dimension $N \geq 2$ and explore a connection between finite-energy sign-changing solutions of the semilinear elliptic PDE and constant mean curvature surfaces in three dimensional Euclidean space.

To study the structure of sign-changing solutions, it is reasonable to consider the structure in the class of radially symmetric functions first. Given any integer $k \geq 1$, it is known that there exists a pair of radial solutions to (1.1.3) having precisely k nodes (cf. [13, Theorem 2.1]). However much less is known for further qualitative properties, such as the locations of nodes, non-degeneracy and uniqueness.

In Chapter 2, we regard the exponent p as a parameter and apply the *finite dimensional Lyaponov-Schmidt reduction* to study the structure of radially symmetric sign-changing solutions of (1.1.3), especially on the uniqueness problem. The approach is motivated by [93], where the author studies the uniqueness and critical spectrum of single boundary spike solutions for a singularly perturbed problem. To carry out the approach, it is worth pointing out that the so-called Emden-Fowler transformation is used to deal with the blow-up as p goes to the critical exponent $\frac{N+2}{N-2}$. For more details we refer the reader to Chapter 2.

Our first result in the thesis concerns the uniqueness of sign-changing solutions of (1.1.3) in the class of radially symmetric functions when p approaches the critical exponent $\frac{N+2}{N-2}$.

Theorem 1.1 (Uniqueness). *For $N \geq 3$ and $k \geq 1$, there exists a constant $\varepsilon_0 = \varepsilon_0(N, k) > 0$ depending only on N and k such that: if*

$$\frac{N+2}{N-2} - \varepsilon_0 < p < \frac{N+2}{N-2}, \tag{1.1.4}$$

then (1.1.3) admits a unique radially symmetric sign-changing solution having exactly k nodes, up to one sign.

Let us denote by u_p the unique radially symmetric sign-changing solution in Theorem 1.1. Our second result concerns the non-degeneracy of u_p.

Theorem 1.2 (Non-degeneracy). *For $N \geq 3$ and $k \geq 1$, there exists a positive constant $\varepsilon_1 \leq \varepsilon_0$ such that: if*

$$\frac{N+2}{N-2} - \varepsilon_1 < p < \frac{N+2}{N-2}, \tag{1.1.5}$$

then u_p is non-degenerate. Namely, if ϕ satisfies

$$\Delta\phi - \phi + p|u_p|^{p-1}\phi = 0 \quad \text{in } \mathbb{R}^N, \quad \phi \in H^1(\mathbb{R}^N),$$

then

$$\phi \in \operatorname{span}\left\{\frac{\partial u_p}{\partial x_1}, \cdots, \frac{\partial u_p}{\partial x_N}\right\}.$$

1.2 Uniqueness of positive solutions in CNLS

In Chapter 3, we study systems of nonlinear scalar field equations, that is, the coupled nonlinear Schrödinger equations:

$$\begin{cases} \Delta u_1 - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0 \quad \text{in } \mathbb{R}^N, \\ \Delta u_2 - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0 \quad \text{in } \mathbb{R}^N, \\ u_1, u_2 \in H^1(\mathbb{R}^N), \end{cases} \tag{CNLS}$$

where u_j's are unknown functions, λ_j, μ_j and β are given constants. $\beta > 0$ is the attractive case, while $\beta < 0$ is called repulsive. This system arises in mathematical model for various phenomena in physics, such as nonlinear optics and Bose-Einstein condensation. We refer to the survey articles [51, 43] for backgrounds.

When the spatial dimension is one, i.e., $N = 1$, system is integrable, and there are many analytical and numerical results on solitary wave solutions of the general m-coupled nonlinear Schrödinger equations by physicists ([47, 50, 99]). But it is still very hard to classify all solutions. One part of the work here was intended as an attempt to study this problem, see Section 3.2 of Chapter 3 for more details.

For higher dimension, as far as we know, the first general mathematical theorems were obtained by T.-C. Lin and J. Wei in [55, 61], where they consider a more general m-coupled nonlinear Schrödinger equations of the form

$$\begin{cases} \Delta u_j - \lambda_j u_j + \sum\limits_{k=1}^{m} \beta_{jk} |u_k|^2 u_j = 0 \quad \text{in } \mathbb{R}^N, \quad N \le 3, \\ u_j \in H^1(\mathbb{R}^N), \quad j = 1, \dots, m. \end{cases} \tag{1.2.1}$$

Indeed, they considered the following minimization problem:

$$c := \inf_{u \in \mathcal{N}} E[\vec{u}], \tag{1.2.2}$$

where the associated energy functional is given by

$$E[\vec{u}] := \frac{1}{2} \sum_{j=1}^{m} \int_{\mathbb{R}^N} \left(|\nabla u_j|^2 + \lambda_j |u_j|^2 \right) - \frac{1}{4} \sum_{j,k=1}^{m} \beta_{jk} \int_{\mathbb{R}^N} |u_j|^2 |u_k|^2, \tag{1.2.3}$$

for

$$\vec{u} = (u_1, \ldots, u_m) \in (H^1(\mathbb{R}^N))^m, \tag{1.2.4}$$

and the so-called Nehari manifold is defined by

$$\begin{aligned} \mathcal{N} := \Big\{ &\vec{u} = (u_1, \ldots, u_m) \in (H^1(\mathbb{R}^N))^m \,\big|\, u_j \geq 0, u_j \not\equiv 0, \\ &\textstyle\int_{\mathbb{R}^N} |\nabla u_j|^2 + \lambda_j \int_{\mathbb{R}^N} |u_j|^2 = \sum\limits_{k=1}^{2} \beta_{jk} \int_{\mathbb{R}^N} u_k^2 u_j^2,\ j = 1, \ldots, m \Big\}. \end{aligned} \tag{1.2.5}$$

A minimizer, if it exists, is called a ground state solution in [55]. Therefore, a ground state solution is a positive solution such that its energy is minimal among all the positive solutions.

Partially because the ground state solution defined in [55] might have Morse index m due to the fact that the Nehari manifold $\mathcal{N}$ defined in (1.2.5) has codimension m, a different definition of ground state is used in [11, 5]. In the following, we say that $\vec{u}$ is a *bound state* solution if $\vec{u} \in (H^1(\mathbb{R}^N))^2$ is a solution and satisfies $E[\vec{u}] < +\infty$. A bound state $\vec{u}$ whose energy is minimal among all non-trivial bound states, namely,

$$E[\vec{u}] = \min \Big\{ E[\vec{v}] \,\big|\, \vec{v} \in (H^1(\mathbb{R}^N))^2 \setminus \{0\}, E'[\vec{v}] = 0 \Big\},$$

is called a *ground state* solution. We emphasize that all the ground state solutions may be semi-trivial, i.e., one of its components $u_j \equiv 0$. When all $\lambda_j, \beta_{jk} > 0$, using the Nehari manifold approach and symmetrization arguments T. Bartsch and Z.-Q. Wang [11] proved that (CNLS) has a semi-positive radially symmetric ground state solution. Moreover, it is of mountain pass type and has Morse index 1 considered as critical point of E on $(H^1(\mathbb{R}^N))^m$ and on $(H_r^1(\mathbb{R}^N))^m$. Here $H_r^1(\mathbb{R}^N)$ consists of all radially symmetric functions in $H^1(\mathbb{R}^N)$.

Compare the two different definitions of ground state, there raises a quite interesting problem: *under what conditions* (1.2.1) *has a positive ground state solution?* Recently some sufficient conditions have been obtained in [4, 11, 68, 12, 87, 5] for large coupling parameters by differential methods, such as minimax method and the method of invariant sets.

In Chapter 3 we study the uniqueness of positive solutions to (CNLS) in the attractive case, i.e.,

$$\beta > 0. \tag{1.2.6}$$

Under this assumption, using a classical "bootstrap" argument, all positive solutions are classical solutions and tend to zero as $|x| \to \infty$. Moreover, applying Moving Planes Method (cf. [17, Theorem 1]), all positive solutions are radially symmetric and strictly decreasing with respect to some point x_0. Without loss of generality we assume $x_0 = 0$. Without loss of generality we write $u_j(x) = u_j(r)$ for $r = |x|$ and $j = 1, 2$. Then (CNLS) becomes

$$\begin{cases} u_1'' + \frac{N-1}{r}u_1' - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0 \quad \text{in } (0, +\infty), \\ u_2'' + \frac{N-1}{r}u_2' - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0 \quad \text{in } (0, +\infty), \\ u_1'(0) = u_2'(0) = 0 \quad \text{and} \quad u_1(r),\, u_2(r) \to 0 \text{ as } r \to +\infty. \end{cases} \tag{1.2.7}$$

Denote by w the unique radial positive solution of

$$\Delta w - w + w^3 = 0, \quad w \in H^1(\mathbb{R}^N).$$

Our first result in Chapter 3 concerns the one-dimensional case.

Theorem 1.3. *Suppose $N = 1$ and $\lambda_1 = \lambda_2 = \lambda > 0$. Then*

(i) for

$$0 < \beta \notin \big[\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}\big], \tag{1.2.8}$$

(CNLS) *admits a unique positive solution (u_1^*, u_2^*) explicitly given by*

$$(u_1^*, u_2^*) = \left(\sqrt{\frac{\lambda(\beta - \mu_2)}{\beta^2 - \mu_1\mu_2}}\, w\big(\sqrt{\lambda}x\big), \sqrt{\frac{\lambda(\beta - \mu_1)}{\beta^2 - \mu_1\mu_2}}\, w\big(\sqrt{\lambda}x\big)\right), \tag{1.2.9}$$

up to a translation;

(ii) for $\beta \in \big[\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}\big]$ but $\mu_1 \neq \mu_2$, positive solution doesn't exist;

(iii) for $\mu_1 = \mu_2 = \beta > 0$, all positive solutions of (CNLS) *are of the form*

$$\left(\cos\theta\sqrt{\frac{\lambda}{\beta}}w(\sqrt{\lambda}x), \sin\theta\sqrt{\frac{\lambda}{\beta}}w(\sqrt{\lambda}x)\right), \quad \forall\theta \in \left(0, \frac{\pi}{2}\right), \tag{1.2.10}$$

up to a translation.

Unlike to one dimension case, the situation is more complicated for higher dimensions. In general, the question of uniqueness of positive solutions to nonlinear equations is difficult For scalar equation, the shooting method and Pohozaev's identity can give uniqueness (cf. [53, 20]). However for systems, there are very few results on uniqueness, and it seems very difficult to apply shooting method because there are at least two shooting parameters. We briefly discuss here two feasible ways. One way is based on the implicit function theorem. The restriction of this technique is that only local uniqueness can be obtained mostly. Another way based on the uniqueness for scalar equation is perhaps more efficient. However, it is not easy to reduce a problem of systems to that of single equation.

Our second result in Chapter 3 concerns the higher dimensions.

Theorem 1.4. *Suppose $N = 2, 3$.*

(i) There exists $\beta_0 > 0$ depending only on λ_j's, μ_j's and N such that if $0 < \beta < \beta_0$, then (CNLS) *admits a unique positive solution up to a translation;*

(ii) If assume further $\lambda_1 = \lambda_2$, then for $\beta > \max\{\mu_1, \mu_2\}$, (u_1^*, u_2^*) explicitly defined at (1.2.9) *is the unique positive solution to* (CNLS) *up to translation; for $\beta \in \big[\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}\big]$ but $\mu_1 \neq \mu_2$, positive solution doesn't exist;*

(iii) For $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2 = \beta > 0$, all positive solutions of (CNLS) *are of the form* (1.2.10) *up to translation.*

There are still many quite interesting and open problems regarding (CNLS). We will discuss them in Chapter 3.

1.3 Traveling wave solutions to TWc

In a totally different context, a solution of (1.1.1) can also be interpreted as stationary solution for a nonlinear heat equation

$$\frac{\partial \psi}{\partial t} = \Delta\psi + f(\psi), \quad \psi = \psi(t,x),\ t \geq 0,\ x \in \mathbb{R}^N. \tag{1.3.1}$$

Such problems arise in biology, especially in population dynamics theory, for a recent account of the theory we refer to the survey articles [41, 77, 78, 94].

In Chapter 4 of the thesis, we are interested with the traveling wave solutions to (1.3.1). It is well known that traveling wave solutions play an important role in nonlinear science. These solutions may well describe various phenomena in nature, such as vibrations, solitons and propagation with a finite speed, etc. In mathematics, they form a specially important class of time-global solutions of evolution equations. For a recent account of the theory we refer the reader to the survey article [89], especially on the stability theory.

To begin our study, we first introduce a generalization of traveling wave solution which is defined in [14] and stated as follows:

Definition 1.1 ([14]). *Let $k \geq 1$ be a given integer and let $u_1, \ldots, u_k$ be k time-global classical solutions of* (1.3.1). *A generalized transition wave (or traveling wave solution) between $u_1, \ldots, u_k$ is a time-global classical solution u of* (1.3.1) *such that $u \neq u_j$ for all $1 \leq j \leq k$, and there exist k families $(\Omega_t^j)_{t\in\mathbb{R}}$, $1 \leq j \leq k$ of open pairwise disjoint nonempty subsets of $\mathbb{R}^N$ and a family $(\Gamma_t)_{t\in\mathbb{R}}$ of nonempty subsets of $\mathbb{R}^N$, such that*

$$\begin{cases} \forall\, t \in \mathbb{R}, \quad \bigcup\limits_{1\leq j\leq k} \partial\Omega_t^j = \Gamma_t, \quad \Gamma_t \bigcup \bigcup\limits_{1\leq j\leq k} \Omega_t^j = \mathbb{R}^N, \\ \forall\, 1 \leq j \leq k, \quad \sup\left\{ d_\Omega(x,\Gamma_t) \,\middle|\, t \in \mathbb{R},\ x \in \Omega_t^j \right\} = +\infty, \end{cases} \tag{1.3.2}$$

and

$$\begin{aligned} u(t,x) - u_j(t,x) \to 0 \quad &\text{uniformly in } t \in \mathbb{R} \text{ and } x \in \overline{\Omega_t^j} \\ \text{as} \quad d_\Omega(x,\Gamma_t) \to +\infty, \quad &\text{for all } 1 \leq j \leq p. \end{aligned} \tag{1.3.3}$$

In the particular case where $k = 1$ and Γ_t is a singleton in Definition 4.1, u is called a *localized pulse*. Here the set Γ_t will be called *traveling front* or *front*.

In the following we study the traveling wave solutions of

$$\frac{\partial u}{\partial t} = \Delta u - u + |u|^{p-1}u, \quad x \in \mathbb{R}^{N+1}, \ t > 0.$$

As a first step, we look for traveling wave solutions in the following form:

$$u(t, x) = v(x', x_{N+1} - ct), \quad x = (x', x_{N+1}) \in \mathbb{R}^{N+1}, \tag{1.3.4}$$

which is called *curved travelling fronts* in [14]. Then the profile v satisfies

$$\Delta v + c\frac{\partial v}{\partial x_{N+1}} - v + |v|^{p-1}v = 0 \quad \text{in } \mathbb{R}^{N+1}. \tag{TWc}$$

For stationary wave solutions (sometimes also called standing wave solutions), i.e., $c = 0$, there is a surprisingly rich and very interesting structure. We refer the reader to [70, 34, 85], where a surprising link between the solutions of the (continuous or discrete) Toda type system and entire solutions of above semi-linear elliptic equation is provided.

The objective of Chapter 4 is to show that a similar construction can be obtained for the positive traveling wave solutions of (TWc). Roughly speaking, the approach in Chapter 4 explores a connection between traveling wave solutions of (TWc) and eternal solutions to the mean curvature flow. In particular, three new kinds of traveling wave solutions are constructed. The first one is that a traveling wave solution with one convex non planar front. The second one is that with one non convex front. The third one is that with two non planar fronts. It is worth pointing out that the approach used here is motivated by [36], where the authors construct traveling wave solutions to the parabolic Allen-Cahn equation with multiple and non convex fronts for $N \geq 2$. Their approach also explores a connection between traveling wave solutions of parabolic Allen-Cahn equation and eternal solutions to the mean curvature flow.

To explain the difference between the study of Allen-Cahn equation and that of (TWc), we consider the one dimensional case. It is known that the linearized operator of heteroclinic solution to the Allen-Cahn equation is stable and has only one bound element in the kernel. However, the linearized operator of one-dimensional bump to the nonlinear Schrödinger equation has a negative eigenvalue, where resonance phenomena may occurs. The main tool used in Chapter 4 is the infinite dimensional Lyaponov-Schmidt reduction, which have been well developed in the last three decades.

For the readers' convenience, Chapters 2, 3 and 4 are independent and can be viewed as individual papers. In fact,

Chapter 2 is based on the paper: *Juncheng Wei and Wei Yao, Uniqueness and non-degeneracy of sign-changing radial solutions of an almost critical problem, preprint.*

Chapter 3 is based on the paper: *Juncheng Wei and Wei Yao, Uniqueness of positive solutions for some coupled nonlinear Schrödinger equations, Communications on Pure and Applied Analysis, to appear.*

Chapter 4 is based on the paper: *Manuel del Pino, Juncheng Wei and Wei Yao, Traveling waves with one and two fronts for an autonomous parabolic Equation, preprint.*

Chapter 2

Uniqueness and non-degeneracy of sign-changing solutions

In this chapter we are concerned with the semi-linear elliptic equation

$$\Delta u - u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \tag{2.0.1}$$

where

$$N \geq 3 \quad \text{and} \quad 1 < p < \frac{N+2}{N-2}.$$

It is well-known that (2.0.1) admits a unique positive solution (called ground state solution), which is radially symmetric up to translations. Unlike positive solutions, sign-changing solutions have more complicated and interesting structure. For instance, infinitely many nonradial solutions with geometric characteristics are constructed in [76].

To study the structure of sign-changing solutions, we consider the problem in the class of radially symmetric functions first. Motivated by [93], we apply the *Lyaponov-Schmidt reduction* to study the uniqueness and non-degeneracy of radially symmetric sign-changing solutions. When the exponent p goes to the critical exponent $\frac{N+2}{N-2}$ from below, the uniqueness and non-degeneracy of sign-changing solutions will be proved in the desired class. To carry out the approach, the so-called Emden-Fowler transformation is used.

2.1 Introduction

Sign-changing solutions of nonlinear elliptic equations and systems have attracted much attention in the last three decades. One reason is that sign-changing solutions arise naturally from mathematical models in science. Another reason is that there are richer structures of sign-changing solutions than that of positive and negative solutions for generic linear and nonlinear elliptic problems. For a deeper discussion we refer the reader to the recent survey article [64], where various methods for obtaining sign-changing solutions developed in the last three decades are revisited, such as Nehari manifold technique, heat flow method, Morse theory and the method of invariant sets.

In this chapter we consider the semi-linear elliptic equation

$$\Delta u - u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \tag{2.1.1}$$

where

$$N \geq 3 \quad \text{and} \quad 1 < p < \frac{N+2}{N-2}. \tag{2.1.2}$$

It is well-known that (2.1.1) admits a unique positive radially symmetric solution, called ground state solution. But compared with positive solutions, sign-changing solutions have more complicated and interesting structure. For this it is worth to mention a quite interesting article [76], where the authors construct infinitely many nonradial solutions in any dimension $N \geq 2$ and explores a connection between finite-energy sign-chaing solutions of the semilinear elliptic PDE and constant mean curvature surfaces in three dimensional Euclidean space.

To study the structure of sign-changing solutions of (2.1.1), we consider the problem in the class of radially symmetric functions first. Applying the standard "bootstrap" argument, we are concerned with a boundary value problem of the nonlinear ordinary differential equation

$$\begin{cases} u'' + \frac{N-1}{r}u' - u + |u|^{p-1}u = 0, & r \in (0, \infty), \\ u'(0) = 0, \quad \lim\limits_{r \to \infty} u(r) = 0. \end{cases} \tag{2.1.3}$$

This raises a quite interesting and challenging problem:

Problem 2.1. *Classify all the solutions of* (2.1.3).

When $N = 1$, the situation is trivial by the Hamiltonian identity. In fact, only positive solution exists. But for $N \geq 2$, as far as we know, this problem is largely open.

To study Problem 2.1, it is reasonable to use the numbers of zeros of solutions to distinguish all solutions. Therefore, first we consider the set of positive and negative solutions. Initiated by Coffman [25] and finally by Kwong [53], it is proved that this set contains only two points. One is positive and another one is negative. Combining the symmetry result in [44], the uniqueness of positive solution to (2.1.1) follows. The main method used in [25] and [53] is the so-called *shooting method*. The main idea is to study the behavior of solution $u(r, \alpha)$ to the initial value problem:

$$\begin{cases} u'' + \frac{N-1}{r}u' - u + |u|^{p-1}u = 0, & r \in (0, \infty), \\ u(0) = \alpha, \quad u'(0) = 0. \end{cases} \tag{2.1.4}$$

Let us mention two important properties of (2.1.4). One is the existence and uniqueness of $u(r, \alpha)$. Namely, given any $\alpha \in \mathbb{R}$, there exists a unique solution to (2.1.4). The other one is the oddness of nonlinearity in (2.1.4). Therefore, without loss of generality we consider the case $\alpha > 0$. After a series of comparison results between two solutions to (2.1.4) with different initial values, the authors in [25, 53] proved that there exists a unique $\alpha_0 > 0$ such that $u(r, \alpha_0) > 0$ for all r and $u(r, \alpha_0) \to 0$ as r goes to infinity. One feature of their approach is that it can be extended to more general nonlinearities (cf. [73, 72]), balls and annulus (cf. [20, 88]), quasilinear operators (cf. [37, 86]) and fully nonlinear operators (cf. [40]). However, it seems very hard to apply the approach to sign-changing solutions if one don't understand the complicated intersection between two solutions to (2.1.4) in the second nodal domain.

Now we are interested in the sign-changing solutions. For the existence, it is known that given any integer $k \geq 1$ there exists a pair of solutions to (2.1.3) having precisely k nodes. For example, this was proved in [25] and [13] both by variational methods. However, much less is known for further qualitative properties, such as the locations of nodes, the uniqueness and stability problems.

This chapter is intended as an attempt to solve Problem 2.1. Our first result concerns the uniqueness of sign-changing solutions of (2.1.3).

Theorem 2.1. *For $N \geq 3$ and $k \geq 1$, there exists a positive constant $\varepsilon_0 = \varepsilon_0(N,k)$ depending only on N and k such that: if*

$$\frac{N+2}{N-2} - \varepsilon_0 < p < \frac{N+2}{N-2}, \tag{2.1.5}$$

then (2.1.3) *admits a unique sign-changing solution having precisely k nodes, up to a sign.*

As a corollary of Theorem 2.1, Theorem 1.1 in the introduction of this thesis is proved. For the convenience of the reader we repeat it as follows.

Corollary 2.1 (Theorem 1.1). *Under the same hypotheses and conditions of Theorem 2.1,* (2.1.1) *admits a unique radially symmetric sign-changing solution having exactly k nodes, up to a sign.*

To prove Theorem 2.1, we regard exponent p as a parameter and apply the *Lyaponov-Schmidt reduction*, which is a powerful tool for obtaining solutions of nonlinear problems. The idea of applying the Lyaponov-Schmidt reduction to a uniqueness problem, is motivated by [93], where the author studied the uniqueness and critical spectrum of single boundary spike solutions for the singularly perturbed problem

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Here $\varepsilon > 0$ is a small parameter, Ω is a smooth bounded domain in $\mathbb{R}^N$ and ν is the unit outer normal on $\partial\Omega$. Roughly speaking, the approach carried out in [93] is to establish a one-to-one connection between the singularly perturbed problem and a finite-dimensional problem. Then the uniqueness problem is reduced to count the number of critical points for a finite-dimensonal problem, where the degree theory can be easily applied. A similar approach has been used in [46].

For the proof of Theorem 2.1, let us mention one more important point. Here we do not apply the Lyaponov-Schmidt reduction to (2.1.3) since the single point blow-up phenomenon occurs for (2.1.3) as p goes to the critical exponent. To overcome this difficulty, we take the so-called *Emden-Fowler transformation*

$$v(t) = r^{\frac{2}{p-1}} u(r), \quad r = e^t,\ t \in (-\infty, \infty). \tag{2.1.6}$$

Then (2.1.3) becomes

$$v'' - \beta v' - (\gamma + e^{2t})v + |v|^{p-1}v = 0, \quad t \in (-\infty, \infty), \tag{2.1.7}$$

where

$$p = \frac{N+2}{N-2} - \varepsilon, \quad \beta = \frac{(N-2)^2\varepsilon}{4-(N-2)\varepsilon}, \quad \gamma = \frac{(N-2)^2}{4} - \frac{\beta^2}{4}. \tag{2.1.8}$$

Note that if $p \to \frac{N+2}{N-2}$ then $\varepsilon \to 0$. Now the Lyaponov-Schmidt reduction can be applied to (2.1.7) since v is uniformly bounded by (2.1.6). As far as we know, the Emden-Fowler transformation has been always used in the study of the Lane-Emden equation

$$-\Delta u = |u|^{p-1}u.$$

It seems to be the first time to use the Emden-Fowler transformation in the study of (2.1.1).

Let us denote by u_p the unique radially symmetric sign-changing solution to (2.1.1) in Theorem 1.1 or Corollary 2.1. An invertibility theory for the linearized operator associated to u_p is very important for the construction of new solutions with u_p. We consider the linear problem

$$\Delta\phi - \phi + p|u_p|^{p-1}\phi = 0 \quad \text{in } \mathbb{R}^N, \quad \phi \in H^1(\mathbb{R}^N). \tag{2.1.9}$$

Clearly $\frac{\partial u_p}{\partial x_j}$ satisfies (2.1.9) for all $1 \leq j \leq N$.

Our second result shows that the converse is also true, which proves the non-degeneracy of u_p, i.e., Theorem 1.2 in the introduction of this thesis. For the convenience of the reader we repeat it as follows.

Theorem 2.2 (Theorem 1.2). *For $N \geq 3$ and $k \geq 1$, there exists a positive constant $\varepsilon_1 \leq \varepsilon_0$ depending only on N and k such that: if*

$$\frac{N+2}{N-2} - \varepsilon_1 < p < \frac{N+2}{N-2}, \tag{2.1.10}$$

then u_p is non-degenerate. Namely, if ϕ satisfies (2.1.9), *then*

$$\phi \in \operatorname{span}\left\{\frac{\partial u_p}{\partial x_1}, \cdots, \frac{\partial u_p}{\partial x_N}\right\}.$$

To prove Theorem 2.2, we first expand ϕ into spherical harmonics as

$$\phi(x) = \sum_{m=0}^{\infty} \phi_m(r) e_m(\theta), \quad r > 0,\ \theta \in S^{N-1} \tag{2.1.11}$$

where e_m's are the eigenfunctions of the Laplace-Beltrami operator $-\Delta_{S^{N-1}}$ on the sphere S^{N-1}, normalized so that they constitute an orthonormal system in $L^2(S^{N-1})$. Then ϕ_m's satisfy the following differential equations

$$\phi_m'' + \frac{N-1}{r}\phi_m' - \phi_m + p|u_\varepsilon|^{p-1}\phi_m + \frac{(-\lambda_m)}{r^2}\phi_m = 0 \quad \text{in } (0,\infty)$$
$$\text{and } \lim_{r\to\infty} \phi_m(r) = 0,$$

where $\lambda_m = m(N-2+m)$ denotes the eigenvalue associated to e_m.

Using the Emden-Fowler transformation, above eigenvalue problem becomes

$$L_\varepsilon[\psi] := \psi'' - \beta\psi' - (\gamma + e^{2t})\psi + p|v_\varepsilon|^{p-1}\psi = \lambda_m \psi \text{ in } (-\infty,\infty)$$
$$\text{and } \lim_{|t|\to\infty} \psi(t) = 0.$$

For $m = 0$, i.e., $\lambda_m = 0$, we study the small eigenvalue μ_ε of L_ε and prove that $\frac{\mu_\varepsilon}{\varepsilon}$ converges to a nonzero constant. Thus $\psi \equiv 0$ for the mode $m = 0$. For $m = 1$ the situation is clear since we have an explicit solution u_ε to the equation of ϕ_m.

Finally consider $m \geq 2$, we study the first eigenvalue $-\nu_1(p)$ of L_ε, i.e., $-\nu_1(p)$ is the largest eigenvalue. If one can show that $-\nu_1(p) < 2N$, then Theorem 2.2 follows. By the variational characterization of eigenvalues and the properties of v_ε we have obtained, it can be shown that $-\nu_1(p) \to N-1$ as $p \to \frac{N+2}{N-2}$. Since $N-1 < 2N$ we prove Theorem 2.2.

The organization of the chapter is the following. In Section 2.2 we give some preliminary analysis. In Section 2.3 a finite dimensional reduction procedure is given. In Section 2.4 we show the existence and uniqueness. Finally in Section 2.5 the small eigenvalue estimate and the proof of theorem 2.2 are given.

2.2 Preliminary analysis

In this section, some preliminary analysis are given. In oder to make the argument more transparent, we will consider the special case of one, i.e., $k = 1$. The corresponding results for general case will be given in the remarks. Furthermore, without loss of generality we can assume that $u(0) > 0$ due to the oddness of nonlinearity in (2.1.3).

We consider the equation

$$v'' - \beta v' - (\gamma + e^{2t})v + |v|^{p-1}v = 0, \quad t \in (-\infty, \infty), \tag{2.2.1}$$

where

$$p = \frac{N+2}{N-2} - \varepsilon, \quad \beta = \frac{(N-2)^2\varepsilon}{4-(N-2)\varepsilon}, \quad \gamma = \frac{(N-2)^2}{4} - \frac{\beta^2}{4}. \tag{2.2.2}$$

Recall that the corresponding energy functional of (2.1.3) is given by

$$\widetilde{E}_\varepsilon(u) = \frac{1}{2}\int_0^\infty (|u'|^2 + |u|^2)r^{N-1}\,dr - \frac{1}{p+1}\int_0^\infty |u|^{p+1}r^{N-1}\,dr, \tag{2.2.3}$$

and by the Emden-Fowler transformation,

$$\int_0^\infty |u'|^2 r^{N-1}\,dr = \int_{-\infty}^\infty \Big[|v'|^2 + \gamma|v|^2\Big] e^{-\beta t}\,dt;$$

$$\int_0^\infty |u|^2 r^{N-1}\,dr = \int_{-\infty}^\infty e^{2t}|v|^2 e^{-\beta t}\,dt;$$

$$\int_0^\infty |u|^{p+1} r^{N-1}\,dr = \int_{-\infty}^\infty |v|^{p+1} e^{-\beta t}\,dt.$$

Thus the corresponding energy functional of (2.2.1) is

$$E_\varepsilon(v) = \frac{1}{2}\int_{-\infty}^\infty \Big[|v'|^2 + (\gamma + e^{2t})|v|^2\Big] e^{-\beta t}\,dt - \frac{1}{p+1}\int_{-\infty}^\infty |v|^{p+1} e^{-\beta t}\,dt. \quad (2.2.4)$$

Moreover, $u(r) \in H^1(\mathbb{R}^N)$ if and only if $v(t) \in \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space defined by

$$\mathcal{H} = \Big\{ v \in H^1(\mathbb{R}) \mid \int_{-\infty}^\infty \Big[|v'|^2 + (\gamma + e^{2t})|v|^2\Big] e^{-\beta t}\,dt < \infty \Big\}$$

with the inner product

$$(v, w)_\varepsilon = \int_{-\infty}^\infty \Big[v'w' + (\gamma + e^{2t})vw\Big] e^{-\beta t}\,dt.$$

Similarly, we define the weighted L^2-product as follows:

$$\langle v, w\rangle_\varepsilon = \int_{-\infty}^\infty vw e^{-\beta t}\,dt.$$

To obtain the asymptotic behavior of the solutions, by the standard blow-up analysis, we get the a priori estimate.

Lemma 2.1. *Let v_ε satisfies (2.2.1). Then there exists a positive constant C depending only on N such that*

$$\|v_\varepsilon\|_\infty \le C. \quad (2.2.5)$$

Since the solution u of (2.1.1) is unique in fixed ball and annulus, so is v. Then the a priori estimate of energy of v_ε can be proved.

Lemma 2.2. *Let v_ε satisfies (2.2.1). Then there exists a small positive constant δ such that*

$$E(v_\varepsilon) < 2E(w_0) + \delta < 3E(w_0), \quad (2.2.6)$$

where w_0 *is the unique positive solution of*

$$\begin{cases} w'' - \frac{(N-2)^2}{4}w + w^{2^*-1} = 0, w > 0 \quad \text{in } \mathbb{R}; \\ w(0) = \max\limits_{t\in\mathbb{R}} w(t), \quad w(t) \to 0, \text{ as } |t| \to \infty. \end{cases} \tag{2.2.7}$$

Proof. First we show that the local maximum point must go to $-\infty$. Suppose not, there exists a sequence of local maximum points t_ε of v_ε such that $t_\varepsilon \to t_0$. By the estimate of energy of v_ε, we get $v_\varepsilon(t + t_\varepsilon) \to v_0$ in C^2_{loc}, where v_0 satisfies

$$v'' - (\gamma_0 + \delta e^{2t})v + v^{2^*-1} = 0, \ v \geq 0 \text{ in } \mathbb{R}, \tag{2.2.8}$$

$$v(t) \to 0, \text{ as } |t| \to \infty. \tag{2.2.9}$$

But by Pohozave's identity, $v_0 \equiv 0$. This contradicts with $v_0(0) \geq \gamma_0^{1/(2^*-1)} > 0$.

Using the similar argument for local minimum point, we get that w_0 is an approximation of v_ε near the maximum or minimum point. Note that in each nodal domain, v_ε has one sign, thus it is one and the least energy solution by the uniqueness of positive solutions. Compare the energies between v_ε and a proper cut-off of w_0, the conclusion follows. □

Remark 2.1. For the general case $k \geq 1$, using the similar argument, we can get

$$E(v_\varepsilon) < (k+1)E(w_0) + \delta < (k+2)E(w_0). \tag{2.2.10}$$

Using the a priori estimate (2.2.6), one may follow the argument of [79] to prove the following asymptotic behavior of v_ε.

Lemma 2.3. *Suppose* v_ε *is a sign-changing once solution of (2.2.1), then* v_ε *has exactly one local maximum point* t_1 *and one local minimum point* t_2 *in* $(-\infty, \infty)$*, provided that* ε *is sufficiently small. Moreover,*

$$v_\varepsilon(t) = w_0(t - t_1) - w_0(t - t_2) + o(1) \tag{2.2.11}$$

and

$$t_1 < t_2, \quad t_1 \to -\infty, \quad t_2 \to -\infty, \quad |t_2 - t_1| \to \infty, \tag{2.2.12}$$

where w_0 *is the unique positive solution to equation (2.2.7) and* $o(1) \to 0$ *as* $\varepsilon \to 0$.

Proof. As shown in the proof of Lemma 2.2, the local maximum and minimum points must go to $-\infty$. Next we prove that the distance between local maximum point and zero point of v_ε approaches to ∞. Suppose not, using the same notation above, there exists $d \in \mathbb{R}$ such that, $v_\varepsilon(t + t_\varepsilon) \to v_0$ in $C^2_{loc}((-\infty, d))$, where v_0 satisfies

$$v'' - (\gamma_0 + \delta e^{2t})v + v^{2^*-1} = 0, \ v \geq 0 \text{ in } (-\infty, d),$$
$$v(d) = 0, \quad v(t) \to 0, \text{ as } |t| \to \infty.$$

This is also a contradiction to the Pohozave's identity.

Now we show that there only exists one local maximum point. Suppose not, there are at least two local maximum points t_1 and t_2. We first show that $|t_1 - t_2| \to \infty$. Suppose not, $|t_1 - t_2|$ is bounded. Then using the same notations, $v_\varepsilon(t + t_1) \to v_0$ in $C^2_{loc}(\mathbb{R})$, where v_0 satisfies (2.2.8). Moreover since $v'_\varepsilon(0) = 0$, $v'_0(0) = 0$, then applying Lemma 4.2 in [79] and the paragraph right after the proof of Lemma 4.2, we get a contradiction. Thus $|t_1 - t_2| \to \infty$. Now we estimate the energy from below to get $E(v_\varepsilon) > 2E(w_0) + C_1 > 2E(w_0) + \delta$, a contradiction follows.

For the negative part, we can get the similar result and complete the proof. □

Remark 2.2. Similarly, for general case $k \geq 1$ we have

$$v_\varepsilon(t) = \sum_{j=1}^{k+1} (-1)^{j+1} w_0(t - t_j) + o(1),$$

where t_j's are the local maximum and minimum points satisfying

$$t_j < t_{j+1}, \quad t_j \to -\infty, \quad |t_j - t_{j+1}| \to \infty.$$

Now we set

$$S_\varepsilon[v] = v'' - \beta v' - (\gamma + e^{2t})v + |v|^{p-1}v. \tag{2.2.13}$$

To get more accurate information on asymptotic behavior, we introduce the function w to be the unique positive solution of

$$\begin{cases} w'' - \frac{(N-2)^2}{4} w + w^p = 0 \quad \text{in } \mathbb{R}; \\ w(0) = \max_{t\in\mathbb{R}} w(t), \quad w(t) \to 0, \text{ as } |t| \to \infty. \end{cases} \tag{2.2.14}$$

It is standard to see that

$$\begin{cases} w(t) = A_{\varepsilon,N} e^{-(N-2)t/2} + O\big(e^{-p(N-2)t/2}\big), \quad t \geq 0; \\ w'(t) = -\frac{N-2}{2} A_{\varepsilon,N} e^{-(N-2)t/2} + O\big(e^{-p(N-2)t/2}\big), \quad t \geq 0, \end{cases} \tag{2.2.15}$$

where $A_{\varepsilon,N} > 0$ is a constant depending only on ε and N. Actually the function $w(t)$ can be written explicitly and has the following form

$$w(t) = \gamma_0^{\frac{1}{p-1}} \Big(\frac{p+1}{2}\Big)^{\frac{1}{p-1}} \Big[\cosh\Big(\frac{p-1}{2}\gamma_0^{1/2} t\Big)\Big]^{-\frac{2}{p-1}}, \tag{2.2.16}$$

where $\gamma_0 = (N-2)^2/4$. Testing (2.2.15) with w and w' and integrating by parts, one arrives at the following identity:

$$\int_{\mathbb{R}} |w'|^2 \, dt = \Big(\frac{1}{2} - \frac{1}{p+1}\Big) \int_{\mathbb{R}} w^{p+1} \, dt = \gamma_0 \Big(\frac{p-1}{p+3}\Big) \int_{\mathbb{R}} w^2 \, dt. \tag{2.2.17}$$

Note that $w \notin H$ when $N = 3, 4$. Next we introduce an important definition. For each $t_j \in \mathbb{R}$, we set w_{j,t_j} to be the unique solution of

$$v'' - (\gamma_0 + e^{2s}) v + w_{t_j}^p = 0, \quad \text{where } w_{t_j}(s) = w(s - t_j), \tag{2.2.18}$$

in the Hilbert space $\mathcal{H}$. The existence and uniqueness of w_{j,t_j} are derived from the Riesz's representation theorem.

Using the ODE analysis, we can obtain the asymptotic expansion of w_{j,t_j}, $j = 1, 2$.

Lemma 2.4. *For ε sufficient small,*

$$w_{j,t_j} = w_{t_j} + \phi_{j,t_j} + O(e^{2t_j}).$$

Due to the different asymptotic behavior of w in different dimension spaces, we have the following cases:

(i) for $N = 3$,

$$\phi_{j,t_j}(s) = -e^{t_j/2} A_{\varepsilon,3}\, e^{-s/2}\big(1 - e^{-e^s}\big); \tag{2.2.19}$$

(ii) for $N = 4$,

$$\phi_{j,t_j}(s) = -e^{t_j} A_{\varepsilon,4}\, e^{-s}\big[1 - \rho_0(\frac{1}{4}e^{2s})\big], \tag{2.2.20}$$

and

$$\rho_0(r) = 2\sqrt{r}K_1(2\sqrt{r}),$$

where $K_1(z)$ *is the modified Bessel function of second kind and satisfies*

$$z^2K_1''(z) + zK_1'(z) - (z^2+1)K_1(z) = 0,$$

see for example [67];

(iii) for $N = 5$,

$$\phi_{j,t_j}(s) = -e^{3t_j/2} A_{\varepsilon,5}\, e^{-3s/2}\Big[1 - (1+e^s)e^{-e^s}\Big]; \tag{2.2.21}$$

(iv) for $N = 6$,

$$\phi_{j,t_j} = -e^{2t_j} A_{\varepsilon,6} e^{-2s}\big[1 - u_0(\frac{1}{16^2}e^{4s})\big],$$

where

$$u_0(r) = 8\sqrt{r}K_2(4r^{1/4}),$$

where $K_2(z)$ *is the modified Bessel function of second kind and satisfies*

$$z^2K_2''(z) + zK_2'(z) - (z^2+4)K_2(z) = 0;$$

(v) for $N \geq 7$, $\phi_{j,t_j} = 0$.

Proof. For the convenience of the reader we postpone the details in the Appendix A. □

Remark 2.3. To obtain more asymptotic expansion of w_{j,t_j}'s, some remarks are given below.

(i) By the maximum principle, we get the following useful estimates:

$$0 < w_{j,t_j} < w_{t_j}, \quad -w_{t_j} < \phi_{j,t_j} < 0, \tag{2.2.22}$$

and

$$|w'_{t_j}| \le c_1 w_{t_j} \le c_2, \ |w'_{j,t_j}| \le c_1 w_{j,t_j} \le c_2. \tag{2.2.23}$$

(ii) In the case of $N = 3$, the contribution of ϕ_j in the integration estimate is e^{t_j}; in the case of $N = 4$, the contribution of ϕ_j in the integration estimate is $O(\delta_j |\ln \delta_j|)$; in the case of $N \ge 5$, the contribution of ϕ_j in the integration estimate is $O(\delta_j)$ as we will see later in Appendix.

From the above lemma and (2.2.12), we see that $w_{j,t_j} = w_{t_j} + o(1) = w_{0,t_j} + o(1)$ in all the cases for $j = 1, 2$. Thus by (2.2.11),

$$v_\varepsilon(t) = w_{\varepsilon,\mathbf{t}} + o(1),$$

where

$$w_{\varepsilon,\mathbf{t}}(t) = w_{1,t_1}(t) - w_{2,t_2}(t). \tag{2.2.24}$$

Before studying the properties of $w_{\varepsilon,\mathbf{t}}$, we need some preliminary lemmas. The first one is a useful inequality.

Lemma 2.5. *For $x \ge 0$, $y \ge 0$,*

$$|x^p - y^p| \le \begin{cases} |x-y|^p, & \text{if } 0 < p < 1, \\ p|x-y|(x^{p-1} + y^{p-1}), & \text{if } 1 \le p < \infty. \end{cases} \tag{2.2.25}$$

The second is about the interactions of two w's.

Lemma 2.6. *For $|r - s| \gg 1$ and $\eta > \theta > 0$, there hold*

$$w^\eta(t-r) w^\theta(t-s) = O(w^\theta(|r-s|)); \tag{2.2.26}$$

$$\int_{-\infty}^{\infty} w^\eta(t-r) w^\theta(t-s)\, dt = \big(1 + o(1)\big) w^\theta(|r-s|) \int_{-\infty}^{\infty} w^\eta(t) e^{-\theta\sqrt{\gamma_0} t}\, dt, \tag{2.2.27}$$

where $o(1) \to 0$ as $|t - s| \to \infty$.

Proof. The conclusion follows from (2.2.15) and Lebesgue's Dominated Convergence Theorem. □

Now we can prove the error estimates.

Lemma 2.7. *For ε sufficiently small and t_1, t_2 satisfy (2.2.12), there is a constant C independent of ε, t_1 and t_2 such that*

$$\|S_\varepsilon[w_{\varepsilon,t}]\|_\infty + \int_{-\infty}^{\infty} \Big|S_\varepsilon[w_{\varepsilon,t}]\Big| e^{-\beta t}\, dt \le C\big[\beta + e^{\tau t_2} + e^{-\tau|t_1-t_2|/2}\big] \quad \text{for } N=3;$$

$$\|S_\varepsilon[w_{\varepsilon,t}]\|_\infty + \int_{-\infty}^{\infty} \Big|S_\varepsilon[w_{\varepsilon,t}]\Big| e^{-\beta t}\, dt \le C\big[\beta + t_2^\tau e^{2\tau t_2} + e^{-\tau|t_1-t_2|}\big] \quad \text{for } N=4;$$

$$\|S_\varepsilon[w_{\varepsilon,t}]\|_\infty + \int_{-\infty}^{\infty} \Big|S_\varepsilon[w_{\varepsilon,t}]\Big| e^{-\beta t}\, dt \le C\big[\beta + e^{2\tau t_2} + e^{-\tau(N-2)|t_1-t_2|/2}\big] \quad \text{for } N\ge 5,$$

where τ satisfies $\frac{1}{2} < \tau < \frac{\min\{p,2\}}{2}$.

Proof. By the equation of w_{j,t_j}, we have

$$S_\varepsilon[w_{\varepsilon,\mathbf{t}}] = -\beta w'_{\varepsilon,\mathbf{t}} - (\gamma-\gamma_0)w_{\varepsilon,\mathbf{t}} + |w_{\varepsilon,\mathbf{t}}|^{p-1}w_{\varepsilon,\mathbf{t}} - w_{t_1}^p + w_{t_2}^p. \tag{2.2.28}$$

From the exponential decay of w_j and (2.2.22), (2.2.23) we deduce that

$$|\beta w'_{\varepsilon,\mathbf{t}}| \le C\beta(w_{t_1} + w_{t_2}).$$

Using (2.2.22), (2.2.23), the exponential decay of w_j and the fact that $\gamma - \gamma_0 = -\beta^2/4$, we get

$$|(\gamma-\gamma_0)w_{\varepsilon,\mathbf{t}}| \le C\beta^2(w_{t_1} + w_{t_2}).$$

Next, we divide $(-\infty,\infty)$ into two intervals I_1, I_2 defined by

$$I_1 = (-\infty, \frac{t_1+t_2}{2}), \quad I_2 = [\frac{t_1+t_2}{2}, \infty).$$

Then on I_i, $i=1,2$, we have $w_{t_j} \le w_{t_i}$ and then $w_{j,t_j} \le w_{i,t_i}$ by the maximum principle. So on I_1 we use inequality (2.2.25) to get

$$\begin{aligned}\Big||w_{\varepsilon,\mathbf{t}}|^{p-1}w_{\varepsilon,\mathbf{t}} - w_{t_1}^p + w_{t_2}^p\Big| &\le Cw_{t_1}^{p-1}w_{t_2} + Cw_{t_1}^{p-1}\phi_{1,t_1} \\ &\le Cw_{t_1}^{p-\tau}w_{t_2}^\tau + Cw_{t_1}^{p-\tau}\phi_{1,t_1}^\tau,\end{aligned}$$

for any $\tau \in (0,1]$. Similarly on I_2 the following inequality holds,

$$\begin{aligned}\Big||w_{\varepsilon,\mathbf{t}}|^{p-1}w_{\varepsilon,\mathbf{t}} - w_{t_1}^p + w_{t_2}^p\Big| &\leq C w_{t_2}^{p-1} w_{t_1} + C w_{t_2}^{p-1}\phi_{2,t_2}\\ &\leq C w_{t_2}^{p-\tau} w_{t_1}^{\tau} + C w_{t_2}^{p-\tau}\phi_{2,t_2}^{\tau},\end{aligned}$$

for any $\tau \in (0,1]$.

By the above inequalities and using Lemma 2.6, the desired result follows. □

Remark 2.4. A similar result holds for the general case $k \geq 1$. Usually t_2 will be replaced by t_{k+1}, and $|t_1 - t_2|$ by $\sup_j |t_j - t_{j+1}|$ thanks to the exponential decay of w_j's and the one dimension space. This remark is also true when similar estimate appears.

In order to obtain the a priori estimate of t_1, t_2 and compute $E_\varepsilon[w_{\varepsilon,\mathbf{t}}]$, we give the estimates of

$$\|v_\varepsilon - w_{\varepsilon,\mathbf{t}}\|_\infty \quad \text{and } \|v_\varepsilon - w_{\varepsilon,\mathbf{t}}\|_H$$

in the following lemma.

Lemma 2.8. *For ε sufficiently small, there is a constant C independent of ε such that*

$$v_\varepsilon = w_{\varepsilon,\boldsymbol{t}} + \phi_\varepsilon,$$

where

$$\begin{cases} \|\phi_\varepsilon\|_\infty + \|\phi_\varepsilon\|_H \leq C\big[\beta + e^{\tau t_2} + e^{-\tau|t_1-t_2|/2}\big] & \text{for } N = 3;\\ \\ \|\phi_\varepsilon\|_\infty + \|\phi_\varepsilon\|_H \leq C\big[\beta + t_2^{\tau} e^{2\tau t_2} + e^{-\tau|t_1-t_2|}\big] & \text{for } N = 4;\\ \\ \|\phi_\varepsilon\|_\infty + \|\phi_\varepsilon\|_H \leq C\big[\beta + e^{2\tau t_2} + e^{-\tau(N-2)|t_1-t_2|/2}\big] & \text{for } N \geq 5,\end{cases}$$

where τ satisfies $\frac{1}{2} < \tau < \frac{\min\{p,2\}}{2}$.

Proof. We may follow the arguments given in the proof of Lemma 2.4 in [93]. First by the properties of w_{j,t_j}'s we can choose proper t_j's such that the maximum points r_ε and minimum points s_ε of v_ε are also the ones of $w_{\varepsilon,\mathbf{t}}$, respectively. Let $v_\varepsilon = w_{\varepsilon,\mathbf{t}} + \phi_\varepsilon$, then $\phi_\varepsilon \to 0$ and satisfies

$$\phi'' - \beta\phi' - (\gamma + e^{2t})\phi + p|w_{\varepsilon,\mathbf{t}}|^{p-1}\phi + S_\varepsilon[w_{\varepsilon,\mathbf{t}}] + N_\varepsilon[\phi] = 0,$$

where

$$N_\varepsilon[\phi] = |w_{\varepsilon,\mathbf{t}} + \phi|^{p-1}(w_{\varepsilon,\mathbf{t}} + \phi) - |w_{\varepsilon,\mathbf{t}}|^{p-1}w_{\varepsilon,\mathbf{t}} - p|w_{\varepsilon,\mathbf{t}}|^{p-1}\phi.$$

Now we prove the result by contradiction. Denote the right hand side order term by K_ε and suppose that

$$\|\phi_\varepsilon\|_\infty / K_\varepsilon \to \infty.$$

Let $\widetilde{\phi}_\varepsilon = \phi_\varepsilon / \|\phi_\varepsilon\|_\infty$. Then $\widetilde{\phi}_\varepsilon$ satisfies

$$\widetilde{\phi}'' - \beta\widetilde{\phi}' - (\gamma + e^{2t})\widetilde{\phi} + p\big|w_{\varepsilon,\mathbf{t}}\big|^{p-1}\widetilde{\phi} + \frac{S_\varepsilon[w_{\varepsilon,\mathbf{t}}]}{\|\phi_\varepsilon\|_\infty} + \frac{N_\varepsilon[\phi_\varepsilon]}{\|\phi_\varepsilon\|_\infty} = 0. \tag{2.2.29}$$

Note that

$$\Big|\frac{S_\varepsilon[w_{\varepsilon,\mathbf{t}}]}{\|\phi_\varepsilon\|_\infty}\Big| \le CK_\varepsilon / \|\phi_\varepsilon\|_\infty, \quad \Big|\frac{N_\varepsilon[\phi_\varepsilon]}{\|\phi_\varepsilon\|_\infty}\Big| \le C\|\phi_\varepsilon\|_\infty^{\min\{p-1,1\}}. \tag{2.2.30}$$

Let t_ε be such that $\widetilde{\phi}_\varepsilon(t_\varepsilon) = \|\widetilde{\phi}_\varepsilon\|_\infty = 1$ (the same proof applies if $\widetilde{\phi}_\varepsilon(t_\varepsilon) = -1$). Then by (2.2.29), (2.2.30) and the Maximum Principle, we have $|t_\varepsilon - t_1| \le C$ or $|t_\varepsilon - t_2| \le C$. Thus $|t_\varepsilon - r_\varepsilon| \le C$ or $|t_\varepsilon - s_\varepsilon| \le C$. WLOG we assume that $|t_\varepsilon - r_\varepsilon| \le C$. Then by the usual elliptic regular theory, we may take a subsequence $\widetilde{\phi}_\varepsilon(t + r_\varepsilon) \to \widetilde{\phi}_0(t)$ as $\varepsilon \to 0$ in $C^1_{\mathrm{loc}}(\mathbb{R})$ since $|r_\varepsilon - t_1| \to 0$, where $\widetilde{\phi}_0$ satisfies

$$\widetilde{\phi}_0'' - \frac{(N-2)^2}{4}\widetilde{\phi}_0 + \frac{N+2}{N-2}w_0^{4/(N-2)}\widetilde{\phi}_0 = 0, \text{ and } \widetilde{\phi}_0'(0) = 0,$$

which implies $\widetilde{\phi}_0 \equiv 0$. This contradicts to the fact that $1 = \widetilde{\phi}_\varepsilon(t_\varepsilon) \to \widetilde{\phi}_0(t_0)$ for some t_0. Therefore we complete the proof. □

The following is the basic technical estimate in this paper which gives the a priori estimates of t_1 and t_2.

Lemma 2.9. *For ε sufficient small we have for $N = 3$,*

$$\begin{cases} t_1 = \log a + 2\log b + 3\log\beta; \\ t_2 = \log a + \log\beta, \end{cases}$$

where a, b are constants and

$$a \longrightarrow a_{0,3}, \ \ b \longrightarrow b_{0,3}.$$

Here $a_{0,3}, b_{0,3}$ are positive constants.

For $N = 4$,

$$\begin{cases} t_1 - t_2 = \log b + \log\beta; \\ -2l_2 e^{2t_2} = a\beta, \end{cases}$$

where a, b are constants and

$$a \longrightarrow a_{0,4}, \ \ b \longrightarrow b_{0,4}.$$

Here $a_{0,4}, b_{0,4}$ are positive constants.

For $N \geq 5$,

$$\begin{cases} t_1 = \frac{1}{2}\log a + \frac{2}{N-2}\log b + \frac{N+2}{2(N-2)}\log\beta; \\ t_2 = \frac{1}{2}\log a + \frac{1}{2}\log\beta, \end{cases}$$

where a, b are constants and

$$a \longrightarrow a_{0,N}, \ \ b \longrightarrow b_{0,N}.$$

Here $a_{0,N}, b_{0,N}$ are positive constants.

Proof. From $S_\varepsilon[v_\varepsilon] = 0$ and $v_\varepsilon = w_{\varepsilon,\mathbf{t}} + \phi$ we deduce that

$$L_{\varepsilon,\mathbf{t}}[\phi] + S_\varepsilon[w_{\varepsilon,\mathbf{t}}] + N_\varepsilon[\phi] = 0, \tag{2.2.31}$$

where

$$L_{\varepsilon,\mathbf{t}}[\phi] = \phi'' - \beta\phi' - (\gamma + e^{2t})\phi + p|w_{\varepsilon,\mathbf{t}}|^{p-1}\phi,$$

and

$$N_\varepsilon[\phi] = |w_{\varepsilon,\mathbf{t}} + \phi|^{p-1}(w_{\varepsilon,\mathbf{t}} + \phi) - |w_{\varepsilon,\mathbf{t}}|^{p-1}w_{\varepsilon,\mathbf{t}} - p|w_{\varepsilon,\mathbf{t}}|^{p-1}\phi.$$

Multiplying (2.2.31) by w'_{1,t_1} and integrating over $\mathbb{R}$, we obtain

$$\int_{-\infty}^{\infty} L_{\varepsilon,\mathbf{t}}[\phi]w'_{1,t_1}\,dt + \int_{-\infty}^{\infty} S_\varepsilon[w_{\varepsilon,\mathbf{t}}]w'_{1,t_1} + \int_{-\infty}^{\infty} N_\varepsilon[\phi]w'_{1,t_1}\,dt = 0.$$

Integrating by parts and using Lemma 2.8 we have

$$\begin{aligned}
\int_{-\infty}^{\infty} L_{\varepsilon,\mathbf{t}}[\phi]w'_{1,t_1}\,dt &= \int_{-\infty}^{\infty} \Big[w'''_{1,t_1} + \beta w''_{1,t_1} - (\gamma + e^{2t})w'_{1,t_1} + p|w_{\varepsilon,\mathbf{t}}|^{p-1}w'_{1,t_1}\Big]\phi\,dt \\
&= \int_{-\infty}^{\infty} \beta\big[(\gamma_0 + e^{2t})w_{1,t_1} - w_{t_1}^p\big]\phi\,dt - (\gamma - \gamma_0)\int_{-\infty}^{\infty} w'_{1,t_1}\phi\,dt \\
&\quad + 2\int_{-\infty}^{\infty} e^{2t}w_{1,t_1}\phi\,dt + p\int_{-\infty}^{\infty}\Big[|w_{\varepsilon,\mathbf{t}}|^{p-1}w'_{1,t_1} - w_{t_1}^{p-1}w'_{t_1}\Big]\phi\,dt \\
&= o\big(\big[\beta + e^{t_1} + e^{t_2} + e^{-|t_1-t_2|/2}\big]\big).
\end{aligned}$$

Similarly we can obtain

$$\int_{-\infty}^{\infty} L_{\varepsilon,\mathbf{t}}[\phi]w'_{2,t_2}\,dt = o\big(\big[\beta + e^{t_1} + e^{t_2} + e^{-|t_1-t_2|/2}\big]\big).$$

For the nonlinearity term, using (2.2.25) we get

$$\Big|N_\varepsilon[\phi]\Big| = \Big||w_{\varepsilon,\mathbf{t}} + \phi|^{p-1}(w_{\varepsilon,\mathbf{t}} + \phi) - |w_{\varepsilon,\mathbf{t}}|^{p-1}w_{\varepsilon,\mathbf{t}} - p|w_{\varepsilon,\mathbf{t}}|^{p-1}\phi\Big| \le C|\phi|^{\min\{p,2\}},$$

so using the exponential decay of w and taking $\tau > \max\{\frac{1}{2}, \frac{1}{p}\}$ we deduce

$$\int_{-\infty}^{\infty} N_\varepsilon[\phi]w'_{1,t_1}\,dt = O(\|\phi\|_\infty^2) = o\big(\big[\beta + e^{t_1} + e^{t_2} + e^{-|t_1-t_2|/2}\big]\big).$$

Similarly we can obtain

$$\int_{-\infty}^{\infty} N_\varepsilon[\phi]w'_{2,t_2}\,dt = o\big(\big[\beta + e^{t_1} + e^{t_2} + e^{-|t_1-t_2|/2}\big]\big).$$

To estimate $\int_{-\infty}^{\infty} S_\varepsilon[w_{\varepsilon,\mathbf{t}}]w'_{1,t_1}\,dt$, we write

$$\begin{aligned}&\int_{-\infty}^{\infty} S_\varepsilon[w_{\varepsilon,\mathbf{t}}]w'_{1,t_1}\,dt\\ &=\int_{-\infty}^{\infty}\Big[-\beta w'_{\varepsilon,\mathbf{t}}-(\gamma-\gamma_0)w_{\varepsilon,\mathbf{t}}+|w_{\varepsilon,\mathbf{t}}|^{p-1}w_{\varepsilon,\mathbf{t}}-w_{t_1}^p+w_{t_2}^p\Big]w'_{1,t_1}\,dt\\ &=E_1+E_2+E_3,\end{aligned}$$

where

$$\begin{aligned}E_1&=-\beta\int_{-\infty}^{\infty} w'_{\varepsilon,\mathbf{t}}w'_{1,t_1}\,dt;\\ E_2&=-(\gamma-\gamma_0)\int_{-\infty}^{\infty} w_{\varepsilon,\mathbf{t}}w'_{1,t_1}\,dt;\\ E_3&=\int_{-\infty}^{\infty}\Big[|w_{\varepsilon,\mathbf{t}}|^{p-1}w_{\varepsilon,\mathbf{t}}-w_{t_1}^p+w_{t_2}^p\Big]w'_{1,t_1}\,dt.\end{aligned}$$

Using (2.2.23) and Lemma 2.6 we obtain

$$E_1=-\beta\int_{-\infty}^{\infty}|w'_{1,t_1}|^2\,dt+\beta\int_{-\infty}^{\infty}w'_{1,t_1}w'_{2,t_2}\,dt=-\beta\int_{-\infty}^{\infty}|w'|^2\,dt+o(\beta).$$

Note that $\gamma-\gamma_0=-\beta^2/4$ and using (2.2.23) we get

$$E_2=\frac{\beta^2}{4}\int_{-\infty}^{\infty}w_{\varepsilon,\mathbf{t}}w'_{1,t_1}\,dt=O(\beta^2).$$

To estimate E_3, following the argument in the proof of Lemma 2.7. We divide $(-\infty,\infty)$ into two intervals I_1, I_2 defined by

$$I_1=(-\infty,\frac{t_1+t_2}{2}),\quad I_2=[\frac{t_1+t_2}{2},\infty).$$

Then on I_i, $i=1,2$, we have $w_{t_j}\le w_{t_i}$ and then $w_{j,t_j}\le w_{i,t_i}$ by the maximum principle. So on I_1 the following equality holds:

$$\begin{aligned}&\big|w_{\varepsilon,\mathbf{t}}\big|^{p-1}w_{\varepsilon,\mathbf{t}}-w_{t_1}^p+w_{t_2}^p\\ =\;&\big[(w_{1,t_1}-w_{2,t_2})^p-w_{1,t_1}^p+pw_{1,t_1}^{p-1}w_{2,t_2}\big]-pw_{1,t_1}^{p-1}w_{2,t_2}\\ &+\big[(w_{t_1}+\phi_{1,t_1})^p-w_{t_1}^p-pw_{t_1}^{p-1}\phi_{1,t_1}\big]+pw_{t_1}^{p-1}\phi_{1,t_1}.\end{aligned}$$

We use inequality (2.2.25) to get

$$\left|(w_{1,t_1}-w_{2,t_2})^p-w_{1,t_1}^p+pw_{1,t_1}^{p-1}w_{2,t_2}\right|\le Cw_{t_1}^{p-\delta}w_{t_2}^{\delta},$$
$$\left|(w_{t_1}+\phi_{1,t_1})^p-w_{t_1}^p-pw_{t_1}^{p-1}\phi_{1,t_1}\right|\le Cw_{t_1}^{p-\delta}\phi_{1,t_1}^{\delta},$$

for any $1<\delta<2$. Then using Lemma 2.6 and integrating by parts, we get

$$\begin{aligned}&\int_{I_1}\left[\left|w_{\varepsilon,\mathbf{t}}\right|^{p-1}w_{\varepsilon,\mathbf{t}}-w_{t_1}^p+w_{t_2}^p\right]w_{1,t_1}'\,dt\\&=\int_{-\infty}^{\infty}w_{t_1}^pw_{t_2}'\,dt-\int_{-\infty}^{\infty}w_{t_1}^p\phi_{1,t_1}'\,dt+o(e^{-|t_1-t_2|/2})+o(e^{t_1}).\end{aligned}$$

On the other hand, on I_2, using $w_{1,t_1}\le w_{2,t_2}$, (2.2.23) and inequality (2.2.25) we get

$$\left|\left[\left|w_{\varepsilon,\mathbf{t}}\right|^{p-1}w_{\varepsilon,\mathbf{t}}-w_{t_1}^p+w_{t_2}^p\right]w_{1,t_1}'\right|\le Cw_{t_1}^{\delta}w_{t_2}^{p+1-\delta}+C\phi_{2,t_2}^{\delta}w_{t_2}^{p+1-\delta},$$

for any $1<\delta<2$. Using Lemma 2.6 we get

$$\int_{I_2}\left[\left|w_{\varepsilon,\mathbf{t}}\right|^{p-1}w_{\varepsilon,\mathbf{t}}-w_{t_1}^p+w_{t_2}^p\right]w_{t_1}'\,dt=o(e^{-|t_1-t_2|/2})+o(e^{t_2}).$$

Thus

$$\begin{aligned}E_3&=\frac{1}{2}e^{-|t_1-t_2|/2}A_{\varepsilon,3}\int_{-\infty}^{\infty}w^pe^{t/2}\,dt\\&\quad+\frac{1}{2}e^{t_1}A_{\varepsilon,3}\int_{-\infty}^{\infty}w^pe^{t/2}\,dt+o(e^{-|t_1-t_2|/2})+o(e^{t_2}),\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty}S_\varepsilon[w_{\varepsilon,\mathbf{t}}]w_{1,t_1}'\,dt&=-\beta\int_{-\infty}^{\infty}|w'|^2\,dt+\frac{1}{2}e^{-|t_1-t_2|/2}A_{\varepsilon,3}\int_{-\infty}^{\infty}w^pe^{t/2}\,dt\\&\quad+o(\beta)+o(e^{-|t_1-t_2|/2})+o(e^{t_2}).\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{-\infty}^{\infty}S_\varepsilon[w_{\varepsilon,\mathbf{t}}]w_{2,t_2}'\,dt&=\beta\int_{-\infty}^{\infty}|w'|^2\,dt+\frac{1}{2}\left[e^{-|t_1-t_2|/2}-e^{t_2}\right]A_{\varepsilon,3}\int_{-\infty}^{\infty}w^pe^{t/2}\,dt\\&\quad+o(\beta)+o(e^{-|t_1-t_2|/2})+o(e^{t_2}).\end{aligned}$$

Combining all the estimates above, β, e^{t_2} and $e^{-|t_1-t_2|/2}$ have the same order.

Therefore,

$$\begin{cases} -\beta\int_{-\infty}^{\infty}|w'|^2\,dt+\frac{1}{2}e^{-|t_1-t_2|/2}A_{\varepsilon,3}\int_{-\infty}^{\infty}w^pe^{t/2}\,dt=o(\beta); \\ \beta\int_{-\infty}^{\infty}|w'|^2\,dt+\frac{1}{2}\left[e^{-|t_1-t_2|/2}-e^{t_2}\right]A_{\varepsilon,3}\int_{-\infty}^{\infty}w^pe^{t/2}\,dt=o(\beta). \end{cases}$$

Let

$$e^{t_2}=a\beta,\quad e^{-|t_1-t_2|/2}=b\beta,$$

then

$$a=\frac{4\int_{-\infty}^{\infty}|w'|^2\,dt}{A_{\varepsilon,3}\int_{-\infty}^{\infty}w^pe^{t/2}\,dt}+o(1)\to a_{0,3}$$

and

$$b=\frac{2\int_{-\infty}^{\infty}|w'|^2\,dt}{A_{\varepsilon,3}\int_{-\infty}^{\infty}w^pe^{t/2}\,dt}+o(1)\to b_{0,3},$$

where $a_{0,3},b_{0,3}$ are positive constants. Thus

$$\begin{cases} t_1=\log a+2\log b+3\log\beta; \\ t_2=\log a+\log\beta, \end{cases}$$

where

$$a\longrightarrow a_{0,3},\ \ b\longrightarrow b_{0,3}.$$

□

Remark 2.5. The above estimates for the general case $k\geq 1$ are:

$$\begin{cases} (-1)^j\beta\int_{-\infty}^{\infty}|w'|^2\,dt+\frac{1}{2}\left[e^{-\frac{|t_{j-1}-t_j|}{2}}+e^{-\frac{|t_j-t_{j+1}|}{2}}\right]A_{\varepsilon,3}\int_{-\infty}^{\infty}w^pe^{t/2}\,dt=o(\beta),\ j=1,\dots,k, \\ \int_{-\infty}^{\infty}|w'|^2\,dt+\frac{1}{2}\left[e^{-|t_k-t_{k+1}|/2}-(-1)^{k+1}e^{t_{k+1}}\right]A_{\varepsilon,3}\int_{-\infty}^{\infty}w^pe^{t/2}\,dt=o(\beta). \end{cases} \tag{2.2.32}$$

By Lemma 2.9 we introduce the following set

$$\Lambda\equiv\begin{cases} \left\{\mathbf{t}=(t_1,t_2)|\frac{1}{2}a_{0,3}\beta<e^{t_2}<\frac{3}{2}a_{0,3}\beta,\frac{1}{2}b_{0,3}\beta<e^{(t_1-t_2)/2}<\frac{3}{2}b_{0,3}\beta\right\} \text{ for } N=3; \\ \left\{\mathbf{t}=(t_1,t_2)|\frac{1}{2}a_{0,4}\beta<-2t_2e^{2t_2}<\frac{3}{2}a_{0,4}\beta,\frac{1}{2}b_{0,4}\beta<e^{t_1-t_2}<\frac{3}{2}b_{0,4}\beta\right\} \text{ for } N=4; \\ \left\{\mathbf{t}=(t_1,t_2)|\frac{1}{2}a_{0,N}\beta<e^{2t_2}<\frac{3}{2}a_{0,N}\beta,\frac{1}{2}b_{0,N}\beta<e^{(N-2)(t_1-t_2)/2}<\frac{3}{2}b_{0,N}\beta\right\} \text{ for } N\geq 5. \end{cases}$$

Then by Lemma 2.9, for ε sufficient small, $\mathbf{t} = (t_1, t_2) \in \Lambda$ if v_ε is a sign-changing solution to equation (2.2.1). In the next section, the set Λ will be the configuration space in the Lyapunov-Schmidt reduction method.

2.3 The existence result

In this section we outline the main steps of the so-called Lyapunov-Schmidt reduction method or localized energy method, which reduces the infinite problem to finding a critical point for a functional on a finite dimensional space. A very important observation is the following Lemma 2.12. To achieve this, we first study the solvability of a linear problem and then apply some standard fixed point theorem for contraction mapping to solve the nonlinear problem. Since the procedure has been used in many papers, we will omit most of the details. We refer to [71] for further detailed proofs.

2.3.1 An auxiliary linear problem

In this subsection we study a linear theory which allows us to perform the finite-dimensional reduction procedure.

Fix $\mathbf{t} \in \Lambda$. Integrating by parts, one can show that orthogonality to $\frac{\partial w_{\varepsilon,\mathbf{t}}}{\partial t_j}$ in H, $j = 1, 2$, is equivalent to orthogonality to the following functions

$$Z_{\varepsilon,t_j} = -(\partial_{t_j} w_{\varepsilon,\mathbf{t}})'' + \beta(\partial_{t_j} w_{\varepsilon,\mathbf{t}})' + (\gamma + e^{2t})\partial_{t_j} w_{\varepsilon,\mathbf{t}}, \quad j = 1, 2, \tag{2.3.1}$$

in the weighted L^2-product $\langle\ ,\ \rangle_\varepsilon$. By (2.2.24) and elementary computations, we obtain for $j = 1, 2$,

$$\partial_{t_j} w_{\varepsilon,\mathbf{t}} = (-1)^{j+1}\partial_{t_j} w_{j,t_j} = (-1)^{j+1}\big(\partial_{t_j} w_{t_j} + \partial_{t_j}\phi_{t,t_j}\big) + O(e^{2t_j}),$$

and

$$Z_{\varepsilon,t_j} = (-1)^j\Big[p w_{t_j}^{p-1} w_{t_j}' - \beta(\partial_{t_j} w_{j,t_j})' - (\gamma - \gamma_0)\partial_{t_j} w_{j,t_j}\Big]. \tag{2.3.2}$$

In this section, we consider the following linear problem. Given $h \in L^\infty(\mathbb{R})$, find a function ϕ satisfying

$$\begin{cases} L_{\varepsilon,\mathbf{t}}[\phi] := \phi'' - \beta\phi' - (\gamma + e^{2t})\phi + p|w_{\varepsilon,\mathbf{t}}|^{p-1}\phi = h + \sum\limits_{j=1}^{2} c_j Z_{\varepsilon,t_j}; \\ \langle \phi, Z_{\varepsilon,t_j}\rangle_\varepsilon = 0, \ j = 1, 2, \end{cases} \tag{2.3.3}$$

for some constants c_j, $j = 1, 2$. For this purpose, define the norm

$$\|\phi\|_* := \|\phi\|_\infty. \tag{2.3.4}$$

Using contradiction argument, we have the following result. Since its proof is now standard, we omit the details here.

Proposition 2.1. *Let ϕ satisfy (2.3.3). Then for ε sufficiently small, we have*

$$\|\phi\|_* \le C\|h\|_*, \tag{2.3.5}$$

where C is a positive constant independent of ε and $\mathbf{t} \in \Lambda$.

Proof. The proof is now standard, we refer to [71]. □

Using Fredholm's alternative we can show the following existence result.

Proposition 2.2. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ the following property holds. Given $h \in L^\infty(\mathbb{R})$, there exists a unique pair (ϕ, c_1, c_2) such that*

$$\begin{cases} L_{\varepsilon,t}[\phi] = h + \sum\limits_{j=1}^{2} c_j Z_{\varepsilon,t_j}; \\ \langle \phi, Z_{\varepsilon,t_j}\rangle_\varepsilon = 0, \ j = 1, 2. \end{cases} \tag{2.3.6}$$

Moreover, we have

$$\|\phi\|_* + \sum_{j=1}^{2} |c_j| \le C\|h\|_* \tag{2.3.7}$$

for some positive constant C.

Proof. The result follows from Proposition 2.1 and the Fredholm's alternative theorem, see for example [71]. □

In the following , if ϕ is the unique solution given in Proposition 2.2, we set

$$\phi = A_\varepsilon(h). \tag{2.3.8}$$

Note that (2.3.7) implies

$$\|A_\varepsilon(h)\|_* \leq C\|h\|_*. \tag{2.3.9}$$

2.3.2 The nonlinear problem

In this subsection we reduce problem (2.2.1) to a finite-dimensional one. This amounts to finding a function $\phi_{\varepsilon,\mathbf{t}}$ such that for some constant c_j, $j = 1, 2$, the following equation holds true

$$\begin{cases} (w_{\varepsilon,\mathbf{t}} + \phi)'' - \beta(w_{\varepsilon,\mathbf{t}} + \phi)' - (\gamma + e^{2t})(w_{\varepsilon,\mathbf{t}} + \phi) + |w_{\varepsilon,\mathbf{t}} + \phi|^{p-1}(w_{\varepsilon,\mathbf{t}} + \phi) = \sum_{j=1}^{2} c_j Z_{\varepsilon,t_j}; \\ \langle \phi, Z_{\varepsilon,t_j}\rangle_\varepsilon = 0, \ j = 1, 2. \end{cases} \tag{2.3.10}$$

The first equation in (2.3.10) can be written as

$$\phi'' - \beta\phi' - (\gamma + e^{2t})\phi + p|w_{\varepsilon,\mathbf{t}}|^{p-1}\phi = -S_\varepsilon[w_{\varepsilon,\mathbf{t}}] - N_\varepsilon[\phi] + \sum_{j=1}^{2} c_j Z_{\varepsilon,t_j},$$

where

$$N_\varepsilon[\phi] = |w_{\varepsilon,\mathbf{t}} + \phi|^{p-1}(w_{\varepsilon,\mathbf{t}} + \phi) - |w_{\varepsilon,\mathbf{t}}|^{p-1}w_{\varepsilon,\mathbf{t}} - p|w_{\varepsilon,\mathbf{t}}|^{p-1}\phi_{\varepsilon,\mathbf{t}}. \tag{2.3.11}$$

Lemma 2.10. *For $\mathbf{t} \in \Lambda$ and ε sufficiently small, we have for $\|\phi\|_* + \|\phi_1\|_* + \|\phi_2\|_* \leq 1$,*

$$\|N_\varepsilon[\phi]\|_* \leq C\|\phi\|_*^{\min\{p,2\}}; \tag{2.3.12}$$

$$\|N_\varepsilon[\phi_1] - N_\varepsilon[\phi_2]\|_* \leq C(\|\phi_1\|_*^{\min\{p-1,1\}} + \|\phi_2\|_*^{\min\{p-1,1\}})\|\phi_1 - \phi_2\|_*. \tag{2.3.13}$$

Proof. These inequalities follows from the mean-value theorem and inequality (2.2.25). □

By the standard fixed point theorem for contraction mapping and Implicit Function Theorem, we have the following proposition.

Proposition 2.3. *For $\boldsymbol{t} \in \Lambda$ and ε sufficiently small, there exists a unique $\phi = \phi_{\varepsilon,\boldsymbol{t}}$ such that (2.3.10) holds. Moreover, $\boldsymbol{t} \to \phi_{\varepsilon,\boldsymbol{t}}$ is of class C^1 as a map into H, and we have*

$$\|\phi_{\varepsilon,\boldsymbol{t}}\|_* + \sum_{j=1}^{2} |c_j| \leq \begin{cases} C\big[\beta + e^{\tau t_2} + e^{-\tau|t_1-t_2|/2}\big] & \text{for } N=3; \\ C\big[\beta + t_2^{\tau} e^{2\tau t_2} + e^{-\tau|t_1-t_2|}\big] & \text{for } N=4; \\ C\big[\beta + e^{2\tau t_2} + e^{-\tau(N-2)|t_1-t_2|/2}\big] & \text{for } N\geq 5, \end{cases} \tag{2.3.14}$$

where τ satisfies $\frac{1}{2} < \tau < \frac{\min\{p,2\}}{2}$.

Proof. The result follows from the standard fixed point theorem and the implicit function theorem, see for example [71]. □

2.3.3 Expansion of the reduced energy functional

In this subsection we expand the quantity

$$K_\varepsilon(\mathbf{t}) = E_\varepsilon[w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}] : \Lambda \to \mathbb{R} \tag{2.3.15}$$

in ε and $\mathbf{t}$, where $\phi_{\varepsilon,\mathbf{t}}$ is given by Proposition 2.3.

Lemma 2.11. *For $\boldsymbol{t} \in \Lambda$ and ε sufficiently small, we have for $N=3$,*

$$\begin{aligned} K_\varepsilon(\boldsymbol{t}) &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big)(e^{-\beta t_1} + e^{-\beta t_2}) \int_{-\infty}^{\infty} w^{p+1}\, dt + \frac{1}{2} e^{t_2} A_{\varepsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2}\, dt \\ &\quad + e^{-|t_1-t_2|/2} A_{\varepsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2}\, dt + o(\beta) + o(e^{t_2}) + o(e^{-|t_1-t_2|/2}) \\ &= \widetilde{K}_\varepsilon(\boldsymbol{t}) + o(\beta) + o(e^{t_2}) + o(e^{-|t_1-t_2|/2}). \end{aligned}$$

For $N=4$,

$$K_\varepsilon(\boldsymbol{t}) = \Big(\frac{1}{2}-\frac{1}{p+1}\Big)(e^{-\beta t_1}+e^{-\beta t_2})\int_{-\infty}^{\infty} w^{p+1}\,dt - \frac{1}{4}t_2 e^{2t_2}A_{\varepsilon,4}\int_{-\infty}^{\infty} w^p e^t\,dt$$
$$+ e^{-|t_1-t_2|}A_{\varepsilon,4}\int_{-\infty}^{\infty} w^p e^t\,dt + o(\beta) + o(t_2e^{2t_2}) + o(e^{-|t_1-t_2|})$$
$$= \widetilde{K}_\varepsilon(\boldsymbol{t}) + o(\beta) + o(t_2e^{2t_2}) + o(e^{-|t_1-t_2|}).$$

For $N\ge 5$,

$$K_\varepsilon(\boldsymbol{t}) = \Big(\frac{1}{2}-\frac{1}{p+1}\Big)(e^{-\beta t_1}+e^{-\beta t_2})\int_{-\infty}^{\infty} w^{p+1}\,dt + \frac{1}{2}e^{2t_2}\int_{-\infty}^{\infty} w^2 e^{2t}\,dt$$
$$+ e^{-(N-2)|t_1-t_2|/2}A_{\varepsilon,N}\int_{-\infty}^{\infty} w^p e^{(N-2)t/2}\,dt + o(\beta) + o(e^{2t_2}) + o(e^{-(N-2)|t_1-t_2|/2})$$
$$= \widetilde{K}_\varepsilon(\boldsymbol{t}) + o(\beta) + o(e^{2t_2}) + o(e^{-(N-2)|t_1-t_2|/2}).$$

Proof. We write

$$K_\varepsilon(\mathbf{t}) = E_\varepsilon[w_{\varepsilon,\mathbf{t}}] + K_1 + K_2 - K_3, \tag{2.3.16}$$

where

$$K_1 = \int_{-\infty}^{\infty}\Big[w'_{\varepsilon,\mathbf{t}}\phi'_{\varepsilon,\mathbf{t}} + (\gamma+e^{2t})w_{\varepsilon,\mathbf{t}}\phi_{\varepsilon,\mathbf{t}}\Big]e^{-\beta t}\,dt - \int_{-\infty}^{\infty}|w_{\varepsilon,\mathbf{t}}|^{p-1}w_{\varepsilon,\mathbf{t}}\phi_{\varepsilon,\mathbf{t}}e^{-\beta t}\,dt;$$

$$K_2 = \frac{1}{2}\int_{-\infty}^{\infty}\Big[|\phi'_{\varepsilon,\mathbf{t}}|^2 + (\gamma+e^{2t})|\phi_{\varepsilon,\mathbf{t}}|^2 - p|w_{\varepsilon,\mathbf{t}}|^{p-1}|\phi_{\varepsilon,\mathbf{t}}|^2\Big]e^{-\beta t}\,dt;$$

$$K_3 = \frac{1}{p+1}\int_{-\infty}^{\infty}\Big[|w_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}}|^{p+1} - |w_{\varepsilon,\mathbf{t}}|^{p+1} - (p+1)|w_{\varepsilon,\mathbf{t}}|^{p-1}w_{\varepsilon,\mathbf{t}}\phi_{\varepsilon,\mathbf{t}}$$
$$- \frac{1}{2}p(p+1)|w_{\varepsilon,\mathbf{t}}|^{p-1}|\phi_{\varepsilon,\mathbf{t}}|^2\Big]e^{-\beta t}\,dt.$$

Integrating by parts and using Lemmas 2.7, 2.8, we have

$$|K_1| = \Big|-\int_{-\infty}^{\infty}S_\varepsilon[w_{\varepsilon,\mathbf{t}}]\phi_{\varepsilon,\mathbf{t}}e^{-\beta t}\,dt\Big| = o\Big(\big[\beta + e^{t_2} + e^{-|t_1-t_2|/2}\big]\Big). \tag{2.3.17}$$

To estimate K_2, we note that $\phi_{\varepsilon,\mathbf{t}}$ satisfies

$$\phi''_{\varepsilon,\mathbf{t}} - \beta\phi'_{\varepsilon,\mathbf{t}} - (\gamma+e^{2t})\phi_{\varepsilon,\mathbf{t}}$$
$$= -|w_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}}|^{p-1}(w_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}}) + |w_{\varepsilon,\mathbf{t}}|^{p-1}w_{\varepsilon,\mathbf{t}} - S_\varepsilon[w_{\varepsilon,\mathbf{t}}] + \sum_{j=1}^{2}c_jZ_{\varepsilon,t_j}. \tag{2.3.18}$$

Integrating by parts and using the orthogonality condition (2.3.10), we have

$$2K_2 = \int_{-\infty}^{\infty} \Big[|w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}|^{p-1}(w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - |w_{\varepsilon,\mathbf{t}}|^{p-1} w_{\varepsilon,\mathbf{t}} - p|w_{\varepsilon,\mathbf{t}}|^{p-1}\phi_{\varepsilon,\mathbf{t}} + S_\varepsilon[w_{\varepsilon,\mathbf{t}}] \Big] \phi_{\varepsilon,\mathbf{t}} e^{-\beta t}\, dt.$$

By the mean value theorem and inequality (2.2.25) we get

$$\Big| |w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}|^{p-1}(w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - |w_{\varepsilon,\mathbf{t}}|^{p-1} w_{\varepsilon,\mathbf{t}} - p|w_{\varepsilon,\mathbf{t}}|^{p-1}\phi_{\varepsilon,\mathbf{t}} \Big| \le C|\phi_{\varepsilon,\mathbf{t}}|^{\min\{p,2\}},$$

so using Lemmas 2.7 and 2.8 we deduce

$$K_2 = o\Big(\big[\beta + e^{t_2} + e^{-|t_1 - t_2|/2}\big] \Big). \tag{2.3.19}$$

For K_3, using the mean value theorem and inequality (2.2.25),

$$\Big| |w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}|^{p+1} - |w_{\varepsilon,\mathbf{t}}|^{p+1} - (p+1)|w_{\varepsilon,\mathbf{t}}|^{p-1} w_{\varepsilon,\mathbf{t}}\phi_{\varepsilon,\mathbf{t}} - \frac{1}{2}p(p+1)|w_{\varepsilon,\mathbf{t}}|^{p-1}|\phi_{\varepsilon,\mathbf{t}}|^2 \Big|$$
$$\le C|\phi_{\varepsilon,\mathbf{t}}|^{\min\{p+1,3\}},$$

so, again, using Lemmas 2.7 and 2.8 it follow that

$$K_3 = o\Big(\big[\beta + e^{t_2} + e^{-|t_1 - t_2|/2}\big] \Big). \tag{2.3.20}$$

Combing with (2.3.16), (2.3.17), (2.3.19), (2.3.20) and Lemma 2.3.16, we obtain the conclusion. □

A very important observation both for existence and uniqueness is the following fact.

Lemma 2.12. *$v_{\varepsilon,t} = w_{\varepsilon,t} + \phi_{\varepsilon,t}$ is a critical point of E_ε if and only if $\mathbf{t}$ is a critical point of K_ε in Λ.*

Proof. The proof follows from the proofs in [93]. For the sake of completeness, we include a proof here.

By Proposition 2.3, there exists an ε_0 such that, for $0 < \varepsilon < \varepsilon_0$, we have a C^1 map $\mathbf{t} \to \phi_{\varepsilon,\mathbf{t}}$ from Λ into H such that

$$S_\varepsilon[v_{\varepsilon,\mathbf{t}}] = \sum_{j=1}^{2} c_j(\mathbf{t}) Z_{\varepsilon,t_j}, \quad v_{\varepsilon,\mathbf{t}} = w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}, \tag{2.3.21}$$

for some constants c_j, which also are of class C^1 in $\mathbf{t}$.

First by the integration by parts we get

$$\begin{aligned}\partial_{t_j} K_\varepsilon(\mathbf{t}) &= \int_{-\infty}^{\infty} \Big[v'_{\varepsilon,\mathbf{t}} \big(\partial_{t_j} w_{\varepsilon,\mathbf{t}} + \partial_{t_j} \phi_{\varepsilon,\mathbf{t}}\big)' + (\gamma + e^{2t}) v_{\varepsilon,\mathbf{t}} \big(\partial_{t_j} w_{\varepsilon,\mathbf{t}} + \partial_{t_j} \phi_{\varepsilon,\mathbf{t}}\big)\Big] e^{-\beta t}\, dt \\ &\quad - \int_{-\infty}^{\infty} |v_{\varepsilon,\mathbf{t}}|^{p-1} v_{\varepsilon,\mathbf{t}} \big(\partial_{t_j} w_{\varepsilon,\mathbf{t}} + \partial_{t_j} \phi_{\varepsilon,\mathbf{t}}\big) e^{-\beta t}\, dt \qquad (2.3.22) \\ &= -\int_{-\infty}^{\infty} S_\varepsilon[v_{\varepsilon,\mathbf{t}}] \big(\partial_{t_j} w_{\varepsilon,\mathbf{t}} + \partial_{t_j} \phi_{\varepsilon,\mathbf{t}}\big) e^{-\beta t}\, dt.\end{aligned}$$

If $v_{\varepsilon,\mathbf{t}} = w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}$ is a critical point of E_ε, then $S_\varepsilon[v_{\varepsilon,\mathbf{t}}] = 0$. By (2.3.22) we get

$$\partial_{t_j} K_\varepsilon(\mathbf{t}) = -\int_{-\infty}^{\infty} S_\varepsilon[v_{\varepsilon,\mathbf{t}}] \big(\partial_{t_j} w_{\varepsilon,\mathbf{t}} + \partial_{t_j} \phi_{\varepsilon,\mathbf{t}}\big) e^{-\beta t}\, dt = 0,$$

which implies that $\mathbf{t}$ is a critical point of K_ε.

On the other hand, let $\mathbf{t}_\varepsilon \in \Lambda$ be a critical point of K_ε, that is $\partial_{t_j} K_\varepsilon(\mathbf{t}_\varepsilon) = 0$, $j = 1, 2$, by (2.3.22) we get

$$0 = \partial_{t_j} K_\varepsilon(\mathbf{t}_\varepsilon) = -\int_{-\infty}^{\infty} S_\varepsilon[v_{\varepsilon,\mathbf{t}_\varepsilon}] \big(\partial_{t_j} w_{\varepsilon,\mathbf{t}_\varepsilon} + \partial_{t_j} \phi_{\varepsilon,\mathbf{t}_\varepsilon}\big) e^{-\beta t}\, dt$$

for $j = 1, 2$. Hence by (2.3.21) we have

$$\sum_{i=1}^{2} c_i(\mathbf{t}_\varepsilon) \int_{-\infty}^{\infty} Z_{\varepsilon,t_{\varepsilon,i}} \big(\partial_{t_j} w_{\varepsilon,\mathbf{t}_\varepsilon} + \partial_{t_j} \phi_{\varepsilon,\mathbf{t}_\varepsilon}\big) e^{-\beta t}\, dt = 0.$$

By Proposition 2.3 and the fact $\langle \phi_{\varepsilon,\mathbf{t}_\varepsilon}, Z_{\varepsilon,t_{\varepsilon,i}} \rangle_\varepsilon = 0$,

$$\langle Z_{\varepsilon,t_{\varepsilon,i}}, \partial_{t_j} \phi_{\varepsilon,\mathbf{t}_\varepsilon} \rangle_\varepsilon = -\langle \phi_{\varepsilon,\mathbf{t}_\varepsilon}, \partial_{t_j} Z_{\varepsilon,t_{\varepsilon,i}} \rangle_\varepsilon = o(1). \qquad (2.3.23)$$

On the other hand,

$$\int_{-\infty}^{\infty} Z_{\varepsilon,t_{\varepsilon,i}} \partial_{t_j} w_{\varepsilon,\mathbf{t}_\varepsilon} e^{-\beta t}\, dt = \langle Z_{\varepsilon,t_{\varepsilon,i}}, \partial_{t_j} w_{\varepsilon,\mathbf{t}_\varepsilon} \rangle_\varepsilon = \delta_{ij} p \int_{-\infty}^{\infty} w^{p-1} |w'|^2\, dt + o(1). \qquad (2.3.24)$$

By (2.3.23) and (2.3.24), the matrix

$$\int_{-\infty}^{\infty} Z_{\varepsilon,t_{\varepsilon,i}} \big(\partial_{t_j} w_{\varepsilon,\mathbf{t}_\varepsilon} + \partial_{t_j} \phi_{\varepsilon,\mathbf{t}_\varepsilon}\big) e^{-\beta t}\, dt$$

is diagonally dominant and thus is non-singular, which implies $c_i(\mathbf{t}_\varepsilon) = 0$ for $i = 1, 2$. Hence $v_{\varepsilon,\mathbf{t}_\varepsilon} = w_{\varepsilon,\mathbf{t}_\varepsilon} + \phi_{\varepsilon,\mathbf{t}_\varepsilon}$ is a critical point of E_ε. □

Remark 2.6. Note that in the proof the theorem, we assume that the solution v_ε of equation (2.2.1) can be written as $v_\varepsilon = w_{\varepsilon,\mathbf{t}} + \phi_\varepsilon$ with ϕ_ε satisfying

$$\langle \phi_\varepsilon, Z_{\varepsilon,t_j} \rangle_\varepsilon = 0, \; j = 1, 2. \tag{2.3.25}$$

For generality, using (2.3.24) we can decompose

$$\phi_\varepsilon = \overline{\phi_\varepsilon} + \sum_{j=1}^{2} d_j \partial_{t_j} w_{\varepsilon,\mathbf{t}}, \tag{2.3.26}$$

where $\overline{\phi_\varepsilon}$ satisfies (2.3.25) and $d_j = O(\|\phi_\varepsilon\|_\infty)$. Thus we can write

$$v_\varepsilon = w_{\varepsilon,\mathbf{t}} + \sum_{j=1}^{2} d_j \partial_{t_j} w_{\varepsilon,\mathbf{t}} + \overline{\phi_\varepsilon} \tag{2.3.27}$$

and get the desired result using the same argument for $w_{\varepsilon,\mathbf{t}} + \sum_{j=1}^{2} d_j \partial_{t_j} w_{\varepsilon,\mathbf{t}}$.

2.4 The uniqueness result

By Lemma 2.12, the number of sign-changing once solutions of (2.2.1) equals the one of critical points of $K_\varepsilon(\mathbf{t})$. To count the number of critical points of $K_\varepsilon(\mathbf{t})$, we need to compute $\partial K_\varepsilon(\mathbf{t})$ and $\partial^2 K_\varepsilon(\mathbf{t})$.

Recall that $K_\varepsilon(\mathbf{t})$ and $\widetilde{K}_\varepsilon(\mathbf{t})$ are define in (2.3.15) and Lemma 2.11. The crucial estimate to prove uniqueness of v_ε and u_ε is the following proposition.

Proposition 2.4. *$K_\varepsilon(\boldsymbol{t})$ is of C^2 in Λ and for ε sufficiently small, we have*

(1) $K_\varepsilon(\boldsymbol{t}) - \widetilde{K}_\varepsilon(\boldsymbol{t}) = o(\beta)$;

(2) $\partial K_\varepsilon(\boldsymbol{t}) - \partial \widetilde{K}_\varepsilon(\boldsymbol{t}) = o(\beta)$ uniformly for $\mathbf{t} \in \Lambda$;

(3) if $\boldsymbol{t}_\varepsilon \in \Lambda$ is a critical point of K_ε, then

$$\partial^2 K_\varepsilon(\boldsymbol{t}_\varepsilon) - \partial^2 \widetilde{K}_\varepsilon(\boldsymbol{t}_\varepsilon) = o(\beta). \tag{2.4.1}$$

The proof of proposition 2.4 will be delayed until the end of this section. Let us now use it to prove the uniqueness of v_ε.

Proof of theorem 2.1. By Lemma 2.12, we just need to prove that $K_\varepsilon(\mathbf{t})$ has only one critical point in Λ. We prove it in the following steps as in [93].

On one hand, By (2) of proposition 2.4, both $K_\varepsilon(\mathbf{t})$ and $\widetilde{K}_\varepsilon(\mathbf{t})$ have no critical points on $\partial\Lambda$ and a continuous deformation argument shows that $\partial K_\varepsilon(\mathbf{t})$ has the same degree as $\partial\widetilde{K}_\varepsilon(\mathbf{t})$ on Λ. By the definition of $\widetilde{K}_\varepsilon(\mathbf{t})$, we have $\deg(\widetilde{K}_\varepsilon(\mathbf{t}),\Lambda,0) = (-1)^m$, where m is the number of negative eigenvalues of $(\partial_{t_i}\partial_{t_j}\widetilde{K}_\varepsilon(\mathbf{t}_\varepsilon))$. Therefore, $\deg(\partial K_\varepsilon(\mathbf{t}),\Lambda,0) = (-1)^m$. On the other hand, at each critical point $\mathbf{t}_\varepsilon$ of $K_\varepsilon(\mathbf{t})$, we have

$$\deg(\partial K_\varepsilon(\mathbf{t}),\Lambda\cap B_{\delta_\varepsilon}(\mathbf{t}_\varepsilon),0) = (-1)^m,$$

fro δ_ε is sufficiently small. This follows from (3) of proposition 2.4 and the fact that the eigenvalues of the matrix $\beta^{-1}(\partial_{t_i}\partial_{t_j}\widetilde{K}_\varepsilon(\mathbf{t}_\varepsilon))$ are away from 0 (cf. 2.4.14 and 2.4.15). Hence we deduce that $K_\varepsilon(\mathbf{t})$ has only a finite number of critical points in Λ, say, k_ε. By the properties of the degree, we have

$$\deg(\partial K_\varepsilon(\mathbf{t}),\Lambda\cap B_{\delta_\varepsilon}(\mathbf{t}_\varepsilon),0) = k_\varepsilon.$$

Therefore, $k_\varepsilon = 1$ and then theorem 2.1 is thus proved. □

In the rest of this section, we shall prove proposition 2.4.

Proof of proposition 2.4. The proof of part (1) is postponed in Appendix B.

We now prove (2) of proposition 2.4 as follows,

$$\begin{aligned}\partial_{t_j}K_\varepsilon(\mathbf{t}) &= \int_{-\infty}^{\infty}\Big[v'_{\varepsilon,\mathbf{t}}(\partial_{t_j}v_{\varepsilon,\mathbf{t}})' + (\gamma+e^{2t})v_{\varepsilon,\mathbf{t}}\partial_{t_j}v_{\varepsilon,\mathbf{t}}\Big]e^{-\beta t}\,dt - \int_{-\infty}^{\infty}|v_{\varepsilon,\mathbf{t}}|^{p-1}v_{\varepsilon,\mathbf{t}}\partial_{t_j}v_{\varepsilon,\mathbf{t}}e^{-\beta t}\,dt\\ &= -\int_{-\infty}^{\infty}S_\varepsilon[v_{\varepsilon,\mathbf{t}}]\partial_{t_j}v_{\varepsilon,\mathbf{t}}e^{-\beta t}\,dt\\ &= I_1 + I_2,\end{aligned} \tag{2.4.2}$$

where

$$I_1 \equiv -\int_{-\infty}^{\infty} S_\varepsilon[w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}]\partial_{t_j} w_{\varepsilon,\mathbf{t}} e^{-\beta t}\, dt, \tag{2.4.3}$$

and

$$I_2 \equiv -\int_{-\infty}^{\infty} S_\varepsilon[w_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}]\partial_{t_j} \phi_{\varepsilon,\mathbf{t}} e^{-\beta t}\, dt. \tag{2.4.4}$$

Using the similar argument in Lemma 2.9, in the case of $N = 3$, we can obtain

$$I_1 = \begin{cases} -\beta \int_{-\infty}^{\infty} |w'|^2\, dt + \frac{1}{2} e^{(t_1-t_2)/2} A_{\varepsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2}\, dt + o(\beta), & \text{for } j = 1; \\ \\ -\beta \int_{-\infty}^{\infty} |w'|^2\, dt - \frac{1}{2}\left[e^{(t_1-t_2)/2} - e^{t_2}\right] A_{\varepsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2}\, dt + o(\beta), & \text{for } j = 2. \end{cases} \tag{2.4.5}$$

By (2.3.10) and proposition 2.3,

$$\begin{aligned} I_2 &= -\sum_{i=1}^{2} c_i(\mathbf{t}) \int_{-\infty}^{\infty} Z_{\varepsilon,t_i} \partial_{t_j} \phi_{\varepsilon,\mathbf{t}} e^{-\beta t}\, dt \\ &= \sum_{i=1}^{2} c_i(\mathbf{t}) \int_{-\infty}^{\infty} \phi_{\varepsilon,\mathbf{t}} \partial_{t_j} Z_{\varepsilon,t_i} e^{-\beta t}\, dt = o(\beta). \end{aligned} \tag{2.4.6}$$

Combining the estimates (2.4.5) and (2.4.6), part (2) of proposition 2.4 is thus proved.

In the rest we shall prove part (3) of proposition 2.4. By definition and (2.4.2),

$$\begin{aligned} \partial_{t_i}\partial_{t_j} K_\varepsilon(\mathbf{t}) &= \partial_{t_i}\Big[-\int_{-\infty}^{\infty} S_\varepsilon[v_{\varepsilon,\mathbf{t}}]\partial_{t_j} v_{\varepsilon,\mathbf{t}} e^{-\beta t}\, dt\Big] \\ &= -\int_{-\infty}^{\infty} S_\varepsilon[v_{\varepsilon,\mathbf{t}}]\partial_{t_i}\partial_{t_j} v_{\varepsilon,\mathbf{t}} e^{-\beta t}\, dt - \int_{-\infty}^{\infty} \partial_{t_i} S_\varepsilon[v_{\varepsilon,\mathbf{t}}]\partial_{t_j} v_{\varepsilon,\mathbf{t}} e^{-\beta t}\, dt. \end{aligned} \tag{2.4.7}$$

By (2.3.21) we get

$$\partial_{t_i} S_\varepsilon[v_{\varepsilon,\mathbf{t}}] = \sum_{k=1}^{2} c_k(\mathbf{t})\partial_{t_i} Z_{\varepsilon,t_k} + \sum_{k=1}^{2} \partial_{t_i} c_k(\mathbf{t}) Z_{\varepsilon,t_k} \tag{2.4.8}$$

Let $\mathbf{t}_\varepsilon$ be a critical point of $K_\varepsilon(\mathbf{t})$ in Λ, then

$$S_\varepsilon[v_{\varepsilon,\mathbf{t}_\varepsilon}] = 0 \quad \text{and } c_k(\mathbf{t}_\varepsilon) = 0, \tag{2.4.9}$$

which implies

$$\partial_{t_i} S_\varepsilon[v_{\varepsilon,\mathbf{t}}]\Big|_{\mathbf{t}=\mathbf{t}_\varepsilon} = \sum_{k=1}^{2} \partial_{t_i} c_k(\mathbf{t}_\varepsilon) Z_{\varepsilon,t_\varepsilon,k}. \tag{2.4.10}$$

Note that

$$\partial_{t_i} S_\varepsilon[v_{\varepsilon,\mathbf{t}}] = L_\varepsilon[\partial_{t_i} v_{\varepsilon,\mathbf{t}}] + p\big[|v_{\varepsilon,\mathbf{t}}|^{p-1} - |w_{\varepsilon,\mathbf{t}}|^{p-1}\big]\partial_{t_i} v_{\varepsilon,\mathbf{t}} =: \overline{L}_\varepsilon[\partial_{t_i} v_{\varepsilon,\mathbf{t}}]. \tag{2.4.11}$$

As in Lemma 2.2, multiplying (2.4.11) by $\partial_{t_j} w_{j,t_j}$ and integrating by parts, we get $\partial_{t_i} c_k(\mathbf{t}_\varepsilon) = O(\beta)$. Hence

$$\begin{aligned}\int_{-\infty}^{\infty} \partial_{t_i} S_\varepsilon[v_{\varepsilon,\mathbf{t}}]\partial_{t_j}\phi_{\varepsilon,\mathbf{t}} e^{-\beta t}\,dt\Big|_{\mathbf{t}=\mathbf{t}_\varepsilon} &= \sum_{k=1}^{2} \partial_{t_i} c_k(\mathbf{t}_\varepsilon) \int_{-\infty}^{\infty} Z_{\varepsilon,t_\varepsilon,k}(\partial_{t_j}\phi_{\varepsilon,\mathbf{t}_\varepsilon}) e^{-\beta t}\,dt \\ &= -\sum_{k=1}^{2} \partial_{t_i} c_k(\mathbf{t}_\varepsilon) \int_{-\infty}^{\infty} (\partial_{t_j} Z_{\varepsilon,t_\varepsilon,k})\phi_{\varepsilon,\mathbf{t}_\varepsilon} e^{-\beta t}\,dt = o(\beta),\end{aligned} \tag{2.4.12}$$

and then

$$\begin{aligned}\partial_{t_i}\partial_{t_j} K_\varepsilon(\mathbf{t}_\varepsilon) &= -\int_{-\infty}^{\infty} \partial_{t_i} S_\varepsilon[v_{\varepsilon,\mathbf{t}}]\partial_{t_j} v_{\varepsilon,\mathbf{t}} e^{-\beta t}\,dt\Big|_{\mathbf{t}=\mathbf{t}_\varepsilon} \\ &= -\int_{-\infty}^{\infty} \overline{L}_\varepsilon[\partial_{t_i} w_{\varepsilon,\mathbf{t}} + \partial_{t_i}\phi_{\varepsilon,\mathbf{t}}]\partial_{t_j} w_{\varepsilon,\mathbf{t}} e^{-\beta t}\,dt\Big|_{\mathbf{t}=\mathbf{t}_\varepsilon} + o(\beta).\end{aligned} \tag{2.4.13}$$

Note that

$$\int_{-\infty}^{\infty} \overline{L}_\varepsilon[\partial_{t_i}\phi_{\varepsilon,\mathbf{t}}]\partial_{t_j} w_{\varepsilon,\mathbf{t}} e^{-\beta t}\,dt = \int_{-\infty}^{\infty} \partial_{t_i}\phi_{\varepsilon,\mathbf{t}} \overline{L}_\varepsilon[\partial_{t_j} w_{\varepsilon,\mathbf{t}}] e^{-\beta t}\,dt = o(\beta),$$

and

$$\overline{L}_\varepsilon[\partial_{t_j} w_{\varepsilon,\mathbf{t}}] = -Z_{\varepsilon,t_j} + p|v_{\varepsilon,\mathbf{t}}|^{p-1}\partial_{t_j} w_{\varepsilon,\mathbf{t}} = O(\beta^\tau).$$

Therefore, we have

$$\partial_{t_i}\partial_{t_j}K_\varepsilon(\mathbf{t}_\varepsilon) = -\int_{-\infty}^{\infty} \overline{L}_\varepsilon[\partial_{t_i}w_{\varepsilon,\mathbf{t}}]\partial_{t_j}w_{\varepsilon,\mathbf{t}}e^{-\beta t}\,dt\Big|_{\mathbf{t}=\mathbf{t}_\varepsilon} + o(\beta). \tag{2.4.14}$$

Using the following important estimate:

$$\begin{aligned}&\int_{-\infty}^{\infty} \overline{L}_\varepsilon[\partial_{t_i}w_{\varepsilon,\mathbf{t}}]\partial_{t_j}w_{\varepsilon,\mathbf{t}}e^{-\beta t}\,dt\\
&= \begin{cases} -\frac{1}{4}e^{(t_1-t_2)/2}A_{\varepsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt + o(\beta), & \text{for } i=j=1;\\[2ex] \frac{1}{4}e^{(t_1-t_2)/2}A_{\varepsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt + o(\beta), & \text{for } i\neq j;\\[2ex] -\left[\frac{1}{4}e^{(t_1-t_2)/2}+\frac{1}{2}e^{t_2}\right]A_{\varepsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt + o(\beta), & \text{for } i=j=2,\end{cases}\end{aligned} \tag{2.4.15}$$

which we have proved in Appendix C, we get the desired result. □

Remark 2.7. For the general case $k\geq 1$, we denote the Hessian matrix

$$\left(\partial_{t_i}\partial_{t_j}K_\varepsilon(\mathbf{t}_\varepsilon)\right)_{i,j=1,\cdots,k+1}$$

by $\mathcal{M}$. We also set for convenience $t_0=-\infty$ and $t_{k+2}=+\infty$. Using the argument above, we can get

$$\begin{aligned}&\mathcal{M}_{ii} = (-1)^{i+1}\tfrac{1}{4}\left[e^{\frac{t_{i-1}-t_i}{2}} - e^{\frac{t_i-t_{i+1}}{2}}\right] \quad \text{for } i=1,\dots,k.\\ &\mathcal{M}_{k+1,k+1} = (-1)^k\tfrac{1}{4}e^{\frac{t_k-t_{k+1}}{2}} - \tfrac{1}{2}e^{t_{k+1}},\end{aligned} \tag{2.4.16}$$

$$\begin{aligned}&\mathcal{M}_{i,i+1} = \mathcal{M}_{i+1,i} = (-1)^{i+1}\tfrac{1}{4}e^{\frac{t_i-t_{i+1}}{2}}, \quad i=1,\dots,k,\\ &\mathcal{M}_{ij}=0 \quad \text{for } |i-j|\geq 2.\end{aligned} \tag{2.4.17}$$

We show that $\mathcal{M}$ is invertible and has fixed number of negative and positive eigenvalue. In fact let $\eta=(\eta_1,\dots,\eta_{k+1})^T$ and we compute

$$\sum_{i,j=1,\cdots,k+1}\mathcal{M}_{ij}\eta_i\eta_j = \sum_{j=1}^{k}\left[(-1)^j\frac{1}{4}e^{\frac{t_j-t_{j+1}}{2}}(\eta_j-\eta_{j+1})^2\right] - \frac{1}{2}e^{t_{k+1}}\eta_{k+1}^2. \tag{2.4.18}$$

Recall that by (2.2.32), $e^{\frac{t_j-t_{j+1}}{2}}$'s and $e^{t_{k+1}}$ satisfies

$$\begin{aligned} &(-1)^j\big[e^{\frac{t_{j-1}-t_j}{2}}+e^{\frac{t_j-t_{j+1}}{2}}\big]=c_0\beta+o(\beta), \quad j=1,\dots,k \\ &(-1)^{k+1}e^{\frac{t_k-t_{k+1}}{2}}-e^{t_{k+1}}=c_0\beta+o(\beta), \end{aligned} \tag{2.4.19}$$

where $c_0<0$ is a constant independent of β. Therefore, $\beta^{-1}e^{\frac{t_j-t_{j+1}}{2}}$ and $\beta^{-1}e^{t_{k+1}}$ converge to non-zero positive constants, which proves the desired result.

2.5 The non-degenerate result

In this section we want to investigate under what conditions the following homogeneous problem admits only trivial solution:

$$\Delta\phi-\phi+p|u_\varepsilon|^{p-1}\phi=0 \quad \text{in } \mathbb{R}^N \quad \text{and} \quad \lim_{|x|\to\infty}\phi(x)=0. \tag{2.5.1}$$

First we expand ϕ into spherical harmonics as

$$\phi(x)=\sum_{m=0}^{\infty}\phi_m(r)e_m(\theta), \quad r>0,\ \theta\in S^{N-1} \tag{2.5.2}$$

where e_m, $m\ge 0$ are the eigenfunctions of the Laplace-Beltrami operator $-\Delta_{S^{N-1}}$ on the sphere S^{N-1}, normalized so that they constitute an orthonormal system in $L^2(S^{N-1})$. Let λ_m denotes the eigenvalue associated to e_m, we repeat eigenvalues according to their multiplicity and we arrange them in an non-decreasing sequence. We recall that the set of eigenvalues is given by $\{j(N-2+j)\mid j\ge 0\}$.

The components ϕ_m then satisfies the following differential equations

$$\begin{aligned} \phi_m''+\frac{N-1}{r}\phi_m'-\phi_m+p|u_\varepsilon|^{p-1}\phi_m+\frac{(-\lambda_m)}{r^2}\phi_m=0 \quad \text{in } (0,\infty) \\ \text{and } \lim_{r\to\infty}\phi_m(r)=0. \end{aligned} \tag{2.5.3}$$

To this end, let us consider the eigenvalues of the problem

$$\begin{aligned} \phi_m''+\frac{N-1}{r}\phi_m'-\phi_m+p|u_\varepsilon|^{p-1}\phi_m+\frac{\nu}{r^2}\phi_m=0 \quad \text{in } (0,\infty) \\ \text{and } \lim_{r\to\infty}\phi_m(r)=0. \end{aligned} \tag{2.5.4}$$

The l-th eigenvalue of (2.5.4) can be characterized variationally as

$$\nu_l(p) = \max_{\dim(V)<l} \inf_{\phi\in V^\perp} \frac{\int_0^\infty \left[|\phi'|^2 + |\phi|^2\right] r^{N-1}\,dr - p\int_0^\infty |u_\varepsilon|^{p-1}|\phi|^2 r^{N-1}\,dr}{\int_0^\infty |\phi|^2 r^{N-3}\,dr}, \quad (2.5.5)$$

where V runs through subspaces of $H_r^1(\mathbb{R}^N)$ and $V^\perp$ is the set of $\phi \in H_{0,r}^1(\mathbb{R}^N)$ satisfying $\int_0^\infty \phi u r^{N-3} = 0$ for all $u \in V$, and $H_r^1(\mathbb{R}^N)$ be the space of radial functions in $H^1(\mathbb{R}^N)$. Thanks to Hardy's inequality:

$$\frac{(N-2)^2}{4}\int_0^\infty |\phi|^2 r^{N-3}\,dr \le \int_0^\infty |\phi'|^2 r^{N-1}\,dr,$$

the eigenvalues $\nu_1(p) \le \nu_2(p) \le \cdots$ are well defined. Using Hardy's embedding and a simple compactness argument involving the fast decay of $|u_\varepsilon|^{p-1}$, there is an extremal for $\nu_l(p)$ which represents a solution to problem (2.5.4) for $\nu = \nu_l(p)$.

To prove Theorem 2.2 we need to know whether and when $\nu_l(p)$ equals $-\lambda_m$. To show this, more information about solutions is required. As before, we consider the corresponding problems for v_ε using the Emden-Fowler transformation. Then the eigenvalue problem (2.5.4) becomes

$$L_\varepsilon[\psi] := \psi'' - \beta\psi' - (\gamma + e^{2t})\psi + p|v_\varepsilon|^{p-1}\psi = -\nu\psi \text{ in } (-\infty,\infty) \quad (2.5.6)$$
$$\text{and } \lim_{|t|\to\infty}\psi(t) = 0.$$

For the proof of Theorem 2.2, let us consider first the radial mode $m = 0$, namely $\lambda_m = 0$. The following result, which contains elements of independent interest, gives the small eigenvalue estimates of L_ε and shows that $\psi_m = 0$ for the mode $m = 0$.

Proposition 2.5. *For ε small enough, the eigenvalue problem*

$$L_\varepsilon\phi_\varepsilon = \mu_\varepsilon\phi_\varepsilon \quad (2.5.7)$$

has exactly two small eigenvalues μ_ε^j, $j = 1, 2$, which satisfy

$$\frac{\mu_\varepsilon^j}{\varepsilon} \to -c_0\nu_j, \quad \text{up to a subsequence as } \varepsilon \to 0, \text{ for } j = 1, 2, \quad (2.5.8)$$

where ν_j's are the eigenvalues of the Hessian matrix $\nabla^2 \widetilde{K}_\varepsilon$ and c_0 is a positive constant. Furthermore, the corresponding eigenfunctions ϕ_ε^j's satisfy

$$\phi_\varepsilon^j = \sum_{i=1}^{2} \left[a_{ij} + o(1)\right] \partial_{t_i} w_{\varepsilon,\boldsymbol{t}} + O(\varepsilon), \quad j = 1, 2,$$

where $\boldsymbol{a}_j = (a_{1,j}, \ldots, a_{2,j})^T$ is the eigenvector associated with ν_j, namely,

$$\nabla^2 \widetilde{K}_\varepsilon \boldsymbol{a}_j = \nu_j \boldsymbol{a}_j.$$

Remark 2.8. By (2.5.8) we know that $\mu_\varepsilon \neq 0$ and then obtain the non-degeneracy of v_ε in the space of H^1-radial symmetric functions.

Proof of proposition 2.5. To prove this proposition, one may follow the arguments given in section 5 of [93] and use the following estimates

$$\int_{-\infty}^{\infty} \overline{L}_\varepsilon[\partial_{t_i} w_{\varepsilon,\mathbf{t}}] \partial_{t_j} w_{\varepsilon,\mathbf{t}} e^{-\beta t}\, dt$$
$$= \begin{cases} -\frac{1}{4} e^{(t_1-t_2)/2} A_{\varepsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2}\, dt + o(\beta), & \text{for } i = j = 1; \\ \\ \frac{1}{4} e^{(t_1-t_2)/2} A_{\varepsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2}\, dt + o(\beta), & \text{for } i \neq j; \\ \\ -\left[\frac{1}{4} e^{(t_1-t_2)/2} + \frac{1}{2} e^{t_2}\right] A_{\varepsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2}\, dt + o(\beta), & \text{for } i = j = 2, \end{cases} \tag{2.5.9}$$

which is given in Appendix C. □

Let us consider now mode 1 for (2.5.3), namely $m = 1, \ldots, N$, for which $\lambda_m = N - 1$. In this case we have an explicit solution $u_\varepsilon'(r)$. Now we show that $\phi_m = C_m u_\varepsilon'$ for some constants C_m for $m = 1, \ldots, N$. This is not trivial since $u_\varepsilon'(r)$ change sign once. Suppose that ϕ_m solve (2.5.3). We first multiply equation of ϕ_m by u_ε' and the equation of u_ε' by ϕ_m, and integrate over the ball B_r centered at the origin with radius r. Since they satisfy the same equation, we get

$$\phi_m'(r) u_\varepsilon'(r) - \phi_m(r) u_\varepsilon''(r) = 0,$$

from which we get $\phi_m = C_m u_\varepsilon'$ for some constants C_m.

Finally let us consider modes 2 and higher. Assume now that $m \geq N+1$ for which $\lambda_m \geq 2N$. Since $u_\varepsilon'(r)$ has exactly one zero in $(0,\infty)$ and $\lambda_m > \lambda_1$, by the standard Sturm-Liouville comparison theorem, ϕ_m does not change sign in $(0,\infty)$. On the other hand, by Sturm-Liouville theory, it is well known that the eigenfunctions corresponding to ν_l much change sign in $(0,\infty)$ at least $l-1$ times. Thus the only possibility for equation (2.5.3) to have a nontrivial solution for a given $m \geq N+1$ is that $\lambda_m = -\nu_1(p)$. In the next proposition we shall show that $-\nu_1(p) \to \lambda_1 = N-1$ as $p \to \frac{N+2}{N-2}$. Therefore we get $\lambda_m \neq -\nu_1(p)$ for $k \geq N+1$ and p is closed to $\frac{N+2}{N-2}$ and then complete the proof of Theorem 2.2.

Proposition 2.6. *As $p \uparrow \frac{N+2}{N-2}$, we have that $-\nu_l(p) \to \lambda_1 = N-1$ for $l \leq 2$.*

Proof of Proposition 2.6. One may follow the arguments given in section 3 of [30] to prove this kinds of proposition. Note that by the Emden-Fowler transformation, the eigenvalues has a variational characterization

$$\nu_l(p) = \max_{\dim(W)<l} \inf_{\psi \in W^\perp} \frac{\int_{-\infty}^{\infty}\left[|\psi'|^2+(\gamma+e^{2t})|\psi|^2\right]e^{-\beta t}\,dt - p\int_{-\infty}^{\infty}|v_\varepsilon|^{p-1}|\psi|^2 e^{-\beta t}\,dt}{\int_{-\infty}^{\infty}|\psi|^2 e^{-\beta t}\,dt}, \tag{2.5.10}$$

where W runs through the subspaces of H and $W^\perp$ is the set of $\psi \in W$ satisfying $\int_{-\infty}^{\infty}\psi v e^{-\beta t}\,dt = 0$ for all $v \in W$. Note that the term involving the weight is relatively compact and it follows from a previous argument that the eigenvalues exist.

We observe that the limiting eigenvalue problem

$$\psi'' - \frac{(N-2)^2}{4}\psi + \frac{N+2}{N-2}w_0^{\frac{4}{N-2}}\psi = \mu\psi, \quad \psi(\pm\infty) = 0, \tag{2.5.11}$$

admits eigenvalues

$$\mu_1 = N-1, \quad \mu_2 = 0, \quad \mu_3 < 0, \quad \cdots, \tag{2.5.12}$$

where the corresponding eigenfunction for the principal eigenvalue μ_1 is positive and denoted by Ψ_1. A simple computation shows that we can take $\Psi_1 = w_0^{\frac{N}{N-2}}$.

Now we take $\psi_j = w_{j,\mathbf{t}_{\varepsilon,j}}^{\frac{p+1}{2}}$, $j = 1, 2$. Let W be a given one-dimensional subspace. Then there exists c_1, c_2 (not all equal to 0) such that $\int_{-\infty}^{\infty} \big(\sum_{j=1}^{2} c_j\psi_j\big)ve^{-\beta t}\,dt = 0$ for all $v \in W$. We then compute that

$$\begin{aligned}&\int_{-\infty}^{\infty}\big[|\psi'|^2 + (\gamma + e^{2t})|\psi|^2\big]e^{-\beta t}\,dt - p\int_{-\infty}^{\infty}|v_\varepsilon|^{p-1}|\psi|^2e^{-\beta t}\,dt\\ &\qquad \le \sum_{j=1}^{2} c_j^2\big(-\mu_1 + o(1)\big)\int_{-\infty}^{\infty}|\psi|^2e^{-\beta t}\,dt,\end{aligned}$$

and hence by variational characterization of ν_2 we deduce that

$$\nu_l(p) \le \nu_2(p) \le -(N-1) + o(1), \quad l = 1, 2. \tag{2.5.13}$$

On the other hand, according to (2.5.12), $v_l(p) \to \mu_k \ge -(N-1)$ for some k. Thus we have $\nu_l(p) \to -(N-1)$ as $p \to \frac{N+2}{N-2}$ for $l \le 2$. □

Remark 2.9. Take $\psi_j = w_{j,\mathbf{t}_{\varepsilon,j}}^{\frac{p+1}{2}}$, $j = 1, \ldots, k$ for the general case $k \ge 1$, by a similar argument we get the desired result.

2.6 Conclusion and comment

To study the structure of sign-changing solutions to (2.1.1), we first consider the radially symmetric sign-changing solutions. Then we shall study a boundary value problem on an infinite interval:

$$\begin{cases} u'' + \frac{N-1}{r}u' - u + |u|^{p-1}u = 0, & r \in (0,\infty),\\ u'(0) = 0, \quad \lim_{r\to\infty} u(r) = 0. \end{cases}$$

In this work we regard exponent p as a parameter and prove the uniqueness and non-degeneracy of sign-changing solutions having exactly k nodes as p goes to $\frac{N+2}{N-2}$ from below. The method used here is the *Lyaponov-Schmidt reduction*, which has been well developed in the last three decades. We refer the reader to [80, 79, 93, 71, 76] and references therein for more details.

To our knowledge, there are only two work concerning the uniqueness of sign-changing solution. So let us compare our results with them.

The first article is [92], where the author studied the existence and uniqueness of positive solution and sign-changing **once** solution (so-called second bound state solution) to

$$\begin{cases} u'' + \frac{2}{r}u' - u + f(u) = 0, \quad r \in (0, \infty), \\ u'(0) = 0, \quad \lim_{r\to\infty} \big(u(r), u'(r)\big) = (0, 0), \end{cases}$$

for

$$f(u) = \begin{cases} u + 1, \quad u \leq -\frac{1}{2}, \\ -u, \quad -\frac{1}{2} \leq u \leq \frac{1}{2}, \\ u - 1, \quad u \geq \frac{1}{2}. \end{cases}$$

His approach follows from [25] and a careful analysis of the behavior of the first variation of $u(r, \alpha)$. The special form of $f(u)$ plays an important role in the proof as well as the number of nodes. As a comparison, our argument in this chapter can be applied to any number of nodes.

The second article is [28] where the authors study a more general nonlinearity. More precisely, they established the uniqueness of the second bound state solution (sign-changing **once** solution) of

$$\begin{cases} u'' + \frac{N-1}{r}u' + f(u) = 0, \quad r \in (0, \infty), \ N > 2, \\ u'(0) = 0, \quad \lim_{r\to\infty} u(r) = 0, \end{cases} \tag{2.6.1}$$

under some convexity and growth conditions of $f(u)$. If we consider the canonical example

$$f(u) = |u|^{p-1}u - |u|^{q-1}u,$$

then conditions of $f(u)$ in [28] are given by

$$p \geq 1, \quad 0 < q < p, \quad \text{and} \quad p + q \leq \frac{2}{N-2}.$$

Therefore, q can not equal to 1, i.e., there must be a sub linear term in the equation. The main idea in [28] goes back to [25, 53] and is carried out through a careful analysis of the intersection between two different solutions. So their approach depends on the number of nodes. But our method can deal with any number of nodes.

Therefore, there are two possible way to study the uniqueness of sign-changing solutions. One way is to apply the shooting method, initiated from [25]. It will be very interesting to study the uniqueness of sign-changing solutions to (2.1.1) by this method and to (2.6.1) under weaker assumptions on the function f and on the number of nodes. The other way is to combine the approach in this chapter and a bifurcation argument, which is suggested by Professor Wei. For an application, we refer the reader to [30].

2.7 Appendices

2.7.1 Appendix A

In this subsection we shall give the estimates of w_{j,t_j}, $j=1,2$. Recall that w_{j,t_j} is the unique solution to the following equation

$$v'' - (\gamma_0 + e^{2s})v + w_{t_j}^p = 0, \tag{2.7.1}$$

in the Hilbert space H, whose existence are given by the Riesz's representation theorem. Here w is the unique positive even solution of

$$w'' - \gamma_0 w + w^p = 0. \tag{2.7.2}$$

Actually the function $w(t)$ can be written explicitly and has the following form

$$w(t) = \gamma_0^{\frac{1}{p-1}} \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \left[\cosh\left(\frac{p-1}{2}\gamma_0^{1/2} t\right)\right]^{-\frac{2}{p-1}} = A_{\varepsilon,N}\left[e^{\frac{p-1}{2}\gamma_0^{1/2}t} + e^{-\frac{p-1}{2}\gamma_0^{1/2}t}\right]^{-\frac{2}{p-1}}.$$

Note that now w has the following expansion

$$\begin{cases} w(t) = A_{\varepsilon,N} e^{-\sqrt{\gamma_0}t} + O(e^{-p\sqrt{\gamma_0}t}), & t \geq 0; \\ w'(t) = -\sqrt{\gamma_0} A_{\varepsilon,N} e^{-\sqrt{\gamma_0}t} + O(e^{-p\sqrt{\gamma_0}t}), & t \geq 0, \end{cases}$$

where $A_{\varepsilon,N} > 0$ is a constant depending on ε and N.

To get the estimates of w_{j,t_j}, we write $w_{j,t_j} = w_{t_j} + \phi$, then by (2.7.1) and (2.7.2), ϕ satisfies

$$\phi'' - (\gamma_0 + e^{2s})\phi - e^{2s} w_{t_j} = 0. \tag{2.7.3}$$

Note that as $s \to \infty$,

$$e^{2s} w_{t_j}(s) = e^{\frac{N-2}{2}t_j} A_{\varepsilon,N} e^{-\frac{N-6}{2}s}. \tag{2.7.4}$$

Hence when $N > 6$, $\phi \in H$ and $\phi = O(e^{2t_j})$. Therefore,

$$w_{j,t_j} = w_{t_j} + O(e^{2t_j}) \quad \text{when } N > 6. \tag{2.7.5}$$

Next we consider $N \le 6$. Let ϕ_N be the unique solution of

$$\phi'' - (\gamma_0 + e^{2s})\phi - e^{-\frac{N-6}{2}s} = 0, \quad |\phi(s)| \to 0, \text{ as } |s| \to \infty. \tag{2.7.6}$$

Then when $N \le 6$, we have

$$w_{j,t_j} = w_{t_j} + e^{\frac{N-2}{2}t_j} A_{\varepsilon,N}\phi_N + O(e^{2t_j}) =: w_{t_j} + \phi_{j,t_j} + O(e^{2t_j}). \tag{2.7.7}$$

The rest of this subsection is to solve ϕ_N. The key point is that

$$\phi_0 = -e^{-\frac{N-2}{2}s} \tag{2.7.8}$$

is a special solution of (2.7.6). Thus we only need to solve the homogeneous equation

$$-\phi'' + (\gamma_0 + e^{2s})\phi = 0. \tag{2.7.9}$$

Note that $\gamma_0 = (N-2)^2/4$, let

$$\phi(s) = e^{-\frac{N-2}{2}s}\widetilde{\phi}\big(\lambda_N e^{(N-2)s}\big), \quad \text{where } \lambda_N = (N-2)^{-(N-2)}. \tag{2.7.10}$$

Then $\widetilde{\phi}$ satisfies

$$\widetilde{\phi}''(s) = s^{-\frac{2N-6}{N-2}}\widetilde{\phi}(s), \quad \widetilde{\phi}(0) = 1, \quad \widetilde{\phi}(\infty) = 0 \tag{2.7.11}$$

and thus

$$\phi_N = -e^{-\frac{N-2}{2}s}\Big[1 - \widetilde{\phi}\big(\lambda_N e^{(N-2)s}\big)\Big]. \tag{2.7.12}$$

In the case of $N = 3$, $\lambda_3 = 1$ and $\widetilde{\phi} = e^{-s}$. Then

$$\phi_3 = -e^{-s/2}\big(1 - e^{-e^s}\big).$$

In the case of $N = 4$, $\lambda_4 = 1/4$ and

$$\widetilde{\phi}(r) = 2\sqrt{r}K_1(2\sqrt{r}) =: \rho_0,$$

where $K_1(z)$ is the modified Bessel function of second kind and satisfies

$$z^2K_1''(z) + zK_1'(z) - (z^2+1)K_1(z) = 0,$$

see for example [67]. Then

$$\phi_4 = -e^{-s}\Big[1 - \rho_0(\frac{1}{4}e^{2s})\Big]. \tag{2.7.13}$$

For $N = 5$,

$$\phi_5 = -e^{-3s/2}\Big[1 - (1+e^s)e^{-e^s}\Big].$$

In the case of $N = 6$,

$$\phi_6 = -e^{-2s}\big[1 - u_0(\frac{1}{16^2}e^{4s})\big],$$

where u_0 satisfies

$$u''(r) = \frac{u(r)}{r^{3/2}},\ u(0) = 1, u(\infty) = 0.$$

Actually, we have

$$u_0(r) = 8\sqrt{r}K_2(4r^{1/4}),$$

where $K_2(z)$ is the modified Bessel function of second kind and satisfies

$$z^2K_2''(z) + zK_2'(z) - (z^2+4)K_2(z) = 0.$$

2.7.2 Appendix B

In this appendix we expand the quality $E_\varepsilon[w_{\varepsilon,\mathbf{t}}]$ as a function of ε and $\mathbf{t}$.

Lemma 2.13. *For $\mathbf{t} \in \Lambda$ and ε sufficiently small, we have for $N = 3$,*

$$E_\varepsilon[w_{\varepsilon,t}] = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)(e^{-\beta t_1} + e^{-\beta t_2}) \int_{-\infty}^{\infty} w^{p+1}\,dt + \frac{1}{2}e^{t_2} A_{\varepsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2}\,dt$$
$$+ e^{-|t_1-t_2|/2} A_{\varepsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2}\,dt + o(\beta) + o(e^{t_2}) + o(e^{-|t_1-t_2|/2}).$$

For $N = 4$,

$$E_\varepsilon[w_{\varepsilon,t}] = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)(e^{-\beta t_1} + e^{-\beta t_2}) \int_{-\infty}^{\infty} w^{p+1}\,dt - \frac{1}{4}t_2 e^{2t_2} A_{\varepsilon,4} \int_{-\infty}^{\infty} w^p e^{t}\,dt$$
$$+ e^{-|t_1-t_2|} A_{\varepsilon,4} \int_{-\infty}^{\infty} w^p e^{t}\,dt + o(\beta) + o(t_2 e^{2t_2}) + o(e^{-|t_1-t_2|}).$$

For $N \geq 5$,

$$E_\varepsilon[w_{\varepsilon,t}] = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)(e^{-\beta t_1} + e^{-\beta t_2}) \int_{-\infty}^{\infty} w^{p+1}\,dt + \frac{1}{2}e^{2t_2} \int_{-\infty}^{\infty} w^2 e^{2t}\,dt$$
$$+ e^{-(N-2)|t_1-t_2|/2} A_{\varepsilon,N} \int_{-\infty}^{\infty} w^p e^{(N-2)t/2}\,dt + o(\beta) + o(e^{2t_2}) + o(e^{-(N-2)|t_1-t_2|/2}).$$

Proof. Since the proof are similar for different cases, we give the details for $N = 3$ here. Integrating by parts we get

$$\begin{aligned} E_\varepsilon[w_{\varepsilon,\mathbf{t}}] &= \frac{1}{2}\int_{-\infty}^{\infty} \Big[- S_\varepsilon[w_{\varepsilon,\mathbf{t}}] + |w_{\varepsilon,\mathbf{t}}|^{p-1} w_{\varepsilon,\mathbf{t}} \Big] w_{\varepsilon,\mathbf{t}} e^{-\beta t}\,dt - \frac{1}{p+1}\int_{-\infty}^{\infty} |w_{\varepsilon,\mathbf{t}}|^{p+1} e^{-\beta t}\,dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \Big[\beta w'_{\varepsilon,\mathbf{t}} + (\gamma - \gamma_0) w_{\varepsilon,\mathbf{t}} + w_{t_1}^p - w_{t_2}^p \Big] w_{\varepsilon,\mathbf{t}} e^{-\beta t}\,dt - \frac{1}{p+1}\int_{-\infty}^{\infty} |w_{\varepsilon,\mathbf{t}}|^{p+1} e^{-\beta t}\,dt \\ &= E_1 + E_2 + E_3 - E_4 + E_5, \end{aligned}$$

where

$$E_1 = \frac{\beta}{2}\int_{-\infty}^{\infty} w'_{\varepsilon,\mathbf{t}} w_{\varepsilon,\mathbf{t}} e^{-\beta t}\,dt = \frac{\beta^2}{4}\int_{-\infty}^{\infty} w_{\varepsilon,\mathbf{t}}^2 e^{-\beta t}\,dt = O(\beta^2);$$

$$E_2 = \frac{(\gamma - \gamma_0)}{2}\int_{-\infty}^{\infty} w_{\varepsilon,\mathbf{t}}^2 e^{-\beta t}\,dt = -\frac{\beta^2}{8}\int_{-\infty}^{\infty} w_{\varepsilon,\mathbf{t}}^2 e^{-\beta t}\,dt = O(\beta^2);$$

$$E_3 = -\frac{1}{2}\int_{-\infty}^{\infty} w_{t_1}^p w_{2,t_2} e^{-\beta t}\,dt - \frac{1}{2}\int_{-\infty}^{\infty} w_{1,t_1} w_{t_2}^p e^{-\beta t}\,dt;$$

$$E_4 = \frac{1}{p+1}\int_{-\infty}^{\infty}\Big[|w_{1,t_1} - w_{2,t_2}|^{p+1} - w_{t_1}^p w_{1,t_1} - w_{t_2}^p w_{2,t_2}\Big] e^{-\beta t}\,dt;$$

$$E_5 = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\Big[\int_{-\infty}^{\infty} w_{t_1}^p w_{1,t_1} e^{-\beta t}\,dt + \int_{-\infty}^{\infty} w_{t_2}^p w_{2,t_2} e^{-\beta t}\,dt\Big].$$

First for E_3, by Lemma 2.6, we have

$$E_3 = -e^{-|t_1-t_2|/2} A_{\varepsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt + o(\beta) + o(e^{t_2}) + o(e^{-|t_1-t_2|/2}).$$

To estimate E_4, we divide $\mathbb{R}$ into two intervals I_1, I_2 defined by

$$I_1 = \Big(-\infty, \frac{t_1+t_2}{2}\Big), \quad I_2 = \Big[\frac{t_1+t_2}{2}, \infty\Big).$$

So on I_1 the following equality holds:

$$\begin{aligned}
& \frac{1}{p+1}\Big[|w_{1,t_1} - w_{2,t_2}|^{p+1} - w_{t_1}^p w_{1,t_1} - w_{t_2}^p w_{2,t_2}\Big] \\
= \; & \frac{1}{p+1}\big[(w_{1,t_1} - w_{2,t_2})^{p+1} - w_{1,t_1}^{p+1} + (p+1) w_{1,t_1}^p w_{2,t_2}\big] - w_{1,t_1}^p w_{2,t_2} \\
& + \frac{1}{p+1}\big[(w_{t_1} + \phi_{1,t_1})^p - w_{t_1}^p - p w_{t_1}^{p-1}\phi_{1,t_1}\big] w_{1,t_1} + \frac{p}{p+1} w_{t_1}^p \phi_{1,t_1} \\
& + \frac{p}{p+1} w_{t_1}^{p-1}\phi_{1,t_1}^2 - \frac{1}{p+1} w_{t_2}^p w_{2,t_2}.
\end{aligned}$$

As in the proof of Lemma 2.9, by the mean value theorem and inequality (2.2.25) we have

$$\begin{aligned}
& \Big|\frac{1}{p+1}\Big[|w_{1,t_1} - w_{2,t_2}|^{p+1} - w_{t_1}^p w_{1,t_1} - w_{t_2}^p w_{2,t_2}\Big] + w_{1,t_1}^p w_{2,t_2} - \frac{p}{p+1} w_{t_1}^p \phi_{1,t_1}\Big| \\
& \le C w_{t_1}^{p+1-\delta} w_{t_2}^{\delta},
\end{aligned}$$

for any $1 < \delta < 2$. Using Lemma 2.6 and integrating by parts, we get

$$\begin{aligned}
& \frac{1}{p+1}\int_{I_1}\Big[|w_{t_1} - w_{t_2}|^{p+1} - w_{t_1}^p w_{1,t_1} - w_{t_2}^p w_{2,t_2}\Big] e^{-\beta t}\,dt \\
= & -\frac{p}{p+1} e^{t_1} A_{\varepsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt - e^{-|t_1-t_2|/2} A_{\varepsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt + o(e^{-|t_1-t_2|/2}).
\end{aligned}$$

Similarly,

$$\frac{1}{p+1}\int_{I_2}\Big[|w_{t_1}-w_{t_2}|^{p+1}-w_{t_1}^p w_{1,t_1}-w_{t_2}^p w_{2,t_2}\Big]e^{-\beta t}\,dt$$
$$=-\frac{p}{p+1}e^{t_2}A_{\varepsilon,3}\int_{-\infty}^{\infty}w^p e^{t/2}\,dt-e^{-|t_1-t_2|/2}A_{\varepsilon,3}\int_{-\infty}^{\infty}w^p e^{t/2}\,dt+o(e^{-|t_1-t_2|/2}).$$

Hence

$$E_4=-\frac{p}{p+1}e^{t_2}A_{\varepsilon,3}\int_{-\infty}^{\infty}w^p e^{t/2}\,dt-2e^{-|t_1-t_2|/2}A_{\varepsilon,3}\int_{-\infty}^{\infty}w^p e^{t/2}\,dt+o(e^{-|t_1-t_2|/2}).$$

Regarding the term E_5, by Lemma 2.6 we have

$$\begin{aligned}E_5=&\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\Big[\int_{-\infty}^{\infty}w_{t_1}^{p+1}e^{-\beta t}\,dt+\int_{-\infty}^{\infty}w_{t_1}^p\phi_{1,t_1}e^{-\beta t}\,dt\\&\qquad+\int_{-\infty}^{\infty}w_{t_2}^{p+1}e^{-\beta t}\,dt+\int_{-\infty}^{\infty}w_{t_2}^p\phi_{2,t_2}e^{-\beta t}\,dt\Big]\\=&\Big(\frac{1}{2}-\frac{1}{p+1}\Big)(e^{-\beta t_1}+e^{-\beta t_2})\int_{-\infty}^{\infty}w^{p+1}\,dt-\Big(\frac{1}{2}-\frac{1}{p+1}\Big)e^{t_2}A_{\varepsilon,3}\int_{-\infty}^{\infty}w^p e^{t/2}\,dt+o(\beta).\end{aligned}$$

Combining the above estimates for E_1, E_2, E_3, E_4 and E_5, we obtain

$$\begin{aligned}E_\varepsilon[w_{\varepsilon,\mathbf{t}}]=&\Big(\frac{1}{2}-\frac{1}{p+1}\Big)(e^{-\beta t_1}+e^{-\beta t_2})\int_{-\infty}^{\infty}w^{p+1}\,dt+\frac{1}{2}e^{t_2}A_{\varepsilon,3}\int_{-\infty}^{\infty}w^p e^{t/2}\,dt\\&+e^{-|t_1-t_2|/2}A_{\varepsilon,3}\int_{-\infty}^{\infty}w^p e^{t/2}\,dt+o(\beta)+o(e^{t_2})+o(e^{-|t_1-t_2|/2}).\end{aligned}$$

□

2.7.3 Appendix C

In this section we want to prove (2.4.15) for $N=3$, that is,

$$\int_{-\infty}^{\infty}\overline{L}_\varepsilon[\partial_{t_i}w_{\varepsilon,\mathbf{t}}]\partial_{t_j}w_{\varepsilon,\mathbf{t}}e^{-\beta t}\,dt=\begin{cases}-\frac{1}{4}e^{(t_1-t_2)/2}A_{\varepsilon,3}\int_{-\infty}^{\infty}w^p e^{t/2}\,dt+o(\beta), & \text{for } i=j=1;\\ \frac{1}{4}e^{(t_1-t_2)/2}A_{\varepsilon,3}\int_{-\infty}^{\infty}w^p e^{t/2}\,dt+o(\beta), & \text{for } i\neq j;\\ -\Big[\frac{1}{4}e^{(t_1-t_2)/2}+\frac{1}{2}e^{t_2}\Big]A_{\varepsilon,3}\int_{-\infty}^{\infty}w^p e^{t/2}\,dt+o(\beta), & \text{for } i=j=2.\end{cases}$$

Proof. Note that by (2.3.1) and (2.3.2), we obtain

$$\begin{aligned}\overline{L}_\varepsilon[\partial_{t_j}w_{\varepsilon,\mathbf{t}}]=&-Z_{\varepsilon,t_j}+p|v_{\varepsilon,\mathbf{t}}|^{p-1}\partial_{t_j}w_{\varepsilon,\mathbf{t}}\\=&(-1)^j\Big[-pw_{t_j}^{p-1}w_{t_j}'+\beta(\partial_{t_j}w_{j,t_j})'+(\gamma-\gamma_0)\partial_{t_j}w_{j,t_j}-p|v_{\varepsilon,\mathbf{t}}|^{p-1}\partial_{t_j}w_{j,t_j}\Big].\end{aligned}\tag{2.7.14}$$

And by the definition of $w_{\varepsilon,\mathbf{t}}$,

$$\partial_{t_j} w_{\varepsilon,\mathbf{t}} = (-1)^{j+1}\partial_{t_j} w_{j,t_j} = (-1)^{j+1}\big(\partial_{t_j} w_{t_j} + \partial_{t_j}\phi_{j,t_j}\big) + O(e^{2t_j}). \qquad (2.7.15)$$

We divide $(-\infty,\infty)$ into two intervals I_1, I_2 defined by

$$I_1 = (-\infty, \frac{t_1+t_2}{2}), \quad I_2 = [\frac{t_1+t_2}{2}, \infty).$$

First we computer the case of $i \neq j$, by (2.7.14) and (2.7.15) we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} \overline{L}_\varepsilon[\partial_{t_1} w_{\varepsilon,\mathbf{t}}]\partial_{t_2} w_{\varepsilon,\mathbf{t}} e^{-\beta t}\, dt \\
= &\int_{-\infty}^{\infty} p w_{t_1}^{p-1} w'_{t_1} w'_{t_2} - \int_{I_1} p|v_{\varepsilon,\mathbf{t}}|^{p-1} w'_{t_1} w'_{t_2} - \int_{I_2} p|v_{\varepsilon,\mathbf{t}}|^{p-1} w'_{t_1} w'_{t_2} + o(\beta) \\
= &-\int_{I_2} p w_{t_2}^{p-1} w'_{t_1} w'_{t_2} + o(\beta) \\
= &\int_{-\infty}^{\infty} w_{t_2}^p w''_{t_1} + o(\beta) \\
= &\frac{1}{4} e^{(t_1-t_2)/2} A_{\varepsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2}\, dt + o(\beta).
\end{aligned}$$

For the case of $i = j$, on I_i, $i = 1, 2$, we have $w_{t_j} \leq w_{t_i}$ and then $w_{j,t_j} \leq w_{i,t_i}$ by the maximum principle. Recall that $v_{\varepsilon,\mathbf{t}} = w_{\varepsilon,\mathbf{t}} + \phi$, where $\phi = \phi_{\varepsilon,\mathbf{t}}$ is given by proposition 2.3. Then on I_1 (Here we give the details for $i = j = 1$, the other cases is similar),

$$\begin{aligned}
&p|v_{\varepsilon,\mathbf{t}}|^{p-1}(w'_{t_1})^2 - p|w_{t_1}|^{p-1}(w'_{t_1})^2 \\
= &-p(p-1)w_{t_1}^{p-2}(w'_{t_1})^2 w_{t_2} + p(p-1)w_{t_1}^{p-2}(w'_{t_1})^2\phi + o(\beta).
\end{aligned}$$

So by (2.7.14) and (2.7.15) we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \overline{L}_\varepsilon[\partial_{t_1} w_{\varepsilon,\mathbf{t}}]\partial_{t_1} w_{\varepsilon,\mathbf{t}} e^{-\beta t}\,dt \\
=&\ -\int_{I_1} p w_{t_1}^{p-1}(w_{t_1}')^2 e^{-\beta t}\,dt + \int_{I_1} p|v_{\varepsilon,\mathbf{t}}|^{p-1}(w_{t_1}')^2 e^{-\beta t}\,dt + o(\beta) \\
=&\ -\int_{I_1} p(p-1) w_{t_1}^{p-2}(w_{t_1}')^2 w_{t_2}\,dt + \int_{I_1} p(p-1) w_{t_1}^{p-2}(w_{t_1}')^2 \phi + o(\beta) \\
=&\ T_1 + T_2 + o(\beta). \qquad (2.7.16)
\end{aligned}
$$

Recall that w_{j,t_j} satisfies

$$w_{j,t_j}'' - (\gamma_0 + e^{2t})w_{j,t_j} + w_{t_j}^p = 0.$$

So $\partial_{t_j} w_{j,t_j}$ and $\partial_{t_j}^2 w_{j,t_j}$ satisfy

$$(\partial_{t_j} w_{j,t_j})'' - (\gamma_0 + e^{2t})(\partial_{t_j} w_{j,t_j}) + p w_{t_j}^{p-1}(\partial_{t_j} w_{t_j}) = 0,$$

and

$$(\partial_{t_j}^2 w_{j,t_j})'' - (\gamma_0 + e^{2t})(\partial_{t_j}^2 w_{j,t_j}) + p w_{t_j}^{p-1}(\partial_{t_j}^2 w_{t_j}) + p(p-1) w_{t_j}^{p-2}(\partial_{t_j} w_{t_j})^2 = 0,$$

which implies

$$p(p-1) w_{t_1}^{p-2}(w_{t_1}')^2 = -L_\varepsilon[\partial_{t_1}^2 w_{1,t_1}] + o(\beta).$$

Hence

$$T_2 = -\int_{I_1} \phi L_\varepsilon[\partial_{t_1}^2 w_{1,t_1}] + o(\beta).$$

By (2.2.28) and proposition 2.3, on I_1 we have

$$L_\varepsilon[\phi] = \beta w_{t_1}' + p w_{t_1}^{p-1} w_{t_2} + o(\beta).$$

Thus

$$
\begin{aligned}
T_2 &= -\int_{\mathbb{R}} w_{t_1}''\left[\beta w_{t_1}' + p w_{t_1}^{p-1} w_{t_2}\right] dt + o(\beta) &(2.7.17)\\
&= -\int_{\mathbb{R}} w_{t_1}'' p w_{t_1}^{p-1} w_{t_2}\,dt + o(\beta). &(2.7.18)
\end{aligned}
$$

On the other hand,

$$\begin{aligned} T_1 &= -\int_{\mathbb{R}} p(p-1)w_{t_1}^{p-2}(w_{t_1}')^2 w_{t_2}\,dt + o(\beta) \\ &\quad = \int_{\mathbb{R}} L_0[w_{t_1}'']w_{t_2}\,dt + o(\beta) = \int_{\mathbb{R}} w_{t_1}'' L_0[w_{t_2}]\,dt + o(\beta) \\ &= \int_{\mathbb{R}} w_{t_1}''\left[w_{t_2}^p + pw_{t_1}^{p-1}w_{t_2}\right]dt + o(\beta) \\ &= -\int_{\mathbb{R}} w_{t_1}''w_{t_2}^p\,dt + \int_{\mathbb{R}} w_{t_1}''pw_{t_1}^{p-1}w_{t_2}\,dt + o(\beta), \end{aligned} \tag{2.7.19}$$

where

$$L_0[\phi] := \phi'' - \gamma_0\phi + pw_{t_1}^{p-1}\phi.$$

Combining (2.7.16), (2.7.17) and (2.7.19), we get the desired result. □

Chapter 3

Uniqueness of positive solutions to some coupled nonlinear Schrödinger equations

In the chapter we study the uniqueness of positive solutions to the coupled nonlinear Schrödinger equations:

$$\begin{cases} \Delta u_1 - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0 & \text{in } \mathbb{R}^N, \\ \Delta u_2 - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0 & \text{in } \mathbb{R}^N, \\ u_1, u_2 \in H^1(\mathbb{R}^N), \end{cases} \tag{CNLS}$$

where $1 \le N \le 3$, $\lambda_1, \lambda_2, \mu_1, \mu_2$ are positive constants, and $\beta > 0$ is a coupling constant. In Section 3.1 we first introduce the background of (CNLS) and review some of the recent results. Later our main results on uniqueness are stated. Section 3.2 is devoted to the study of the uniqueness of positive solutions to (CNLS) in one dimension. In particular, we prove the uniqueness for $\beta \notin [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$ when $\lambda_1 = \lambda_2$. The higher dimensional cases are studied in Sections 3.3 and 3.4. More precisely, in Section 3.3, among other things, the uniqueness of positive solution is proved for both sufficiently small β and $\beta > \max\{\mu_1, \mu_2\}$; Section 3.4 establishes a Liouville type theorem for $\mu_1 = \mu_2 = \beta$. Finally we discuss some possible extensions and applications in Section 3.5. There are still many quite interesting and open problems regarding (CNLS). We will discuss them later.

3.1 Background and main results

We are interested in nonlinear elliptic Schrödinger system of the form:

$$-\Delta u_j + \lambda_j u_j = \sum_{k=1}^{m} \beta_{jk}|u_k|^2 u_j, \quad \text{in } \Omega \subset \mathbb{R}^N, \quad j = 1, \ldots, m, \tag{3.1.1}$$

where u_j's are unknown functions, λ_j, β_{jk} are given constants. Here Ω is an open subset of $\mathbb{R}^N$, N is the spatial dimension and m is the number of equations. This system arises in mathematical model for various phenomena in physics, such as nonlinear optics and Bose-Einstein condensation. We refer for this to the survey articles [51, 43].

Recall that system (3.1.1) is satisfied by the amplitudes of standing or solitary wave solutions of the form $\Psi_j(t,x) = e^{i\lambda_j t}u_j(x)$, for the time-dependent m-coupled Gross-Pitaevskii equations given by

$$\begin{cases} -i\frac{\partial \Psi_j}{\partial t} = \Delta \Psi_j + \sum_{k=1}^{m} \beta_{jk}|\Psi_k|^2 \Psi_j \quad j = 1, \ldots, m, \\ \Psi_j = \Psi_j(t,x) \in \mathbb{C}, \quad t > 0, \quad x \in \Omega \subset \mathbb{R}^N, \end{cases} \tag{GPE}$$

where $i = \sqrt{-1}$ is the imaginary unit and $\mathbb{C}$ is the set of all complex numbers.

Physically, when $\Omega = \mathbb{R}^N$, system (GPE) arises in the study of incoherent solitons in nonlinear optics. The j-th component Ψ_j of solution denotes the j-th component of the beam in Kerr-like photorefractive media [1]. Denote $\mu_j = \beta_{jj}$ and suppose $\mu_j > 0$. Then the positive constant μ_j is for self-focusing in the j-th component of the beam and the coupling constant β_{jk} $(j \neq k)$ is the interaction between the j-th and the k-th component of the beam. As $\beta_{jk} > 0$, the interaction is *attractive*, but the interaction is *repulsive* if $\beta_{jk} < 0$.

When Ω is a bounded domain and $m = 2$, system (GPE) also arises in the Hartree-Fock theory for a double condensate, i.e., a binary mixture of Bose-Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ ([38]). Now $|\Psi_1|$ and $|\Psi_2|$ are the corresponding condensate amplitudes. Constants $\mu_j := \beta_{jj}$ and $\beta := \beta_{12}$ are the intraspecies and interspecies scattering lengths respectively. The sign of the scattering length β determines whether the interactions of states $|1\rangle$ and $|2\rangle$ are *repulsive* or *attractive*. When $\beta < 0$, the interactions of states $|1\rangle$ and $|2\rangle$ are *repulsive* ([91]). In contrast, when $\beta > 0$, the interactions of $|1\rangle$ and $|2\rangle$ are *attractive*. For atoms of the single state $|j\rangle$, when $\mu_j > 0$, the interactions of the single state $|j\rangle$ are *attractive*.

Throughout the whole chapter due to physical considerations, the coupling constants β_{jk}'s satisfy $\beta_{jk} = \beta_{kj}$, and we always denote $\mu_j := \beta_{jj}$ for $j = 1, \ldots, m$. Unless otherwise stated we always assume that $\Omega = \mathbb{R}^N$ and consider the nonlinear Schrödinger system like the following:

$$\begin{cases} \Delta u_j - \lambda_j u_j + \sum\limits_{k=1}^{m} \beta_{jk}|u_k|^2 u_j = 0 \quad \text{in } \mathbb{R}^N, \quad N \leq 3 \\ u_j \in H^1(\mathbb{R}^N), \quad j = 1, \ldots, m. \end{cases} \tag{3.1.2}$$

When the spatial dimension is one, i.e., $N = 1$, system (GPE) is integrable, and there are many analytical and numerical results on solitary wave solutions of the general m-coupled nonlinear Schrödinger equations by physicists ([47, 50, 99]). But it is still very hard to classify all solutions. One part of the work here was intended as an attempt to study this problem, see Section 3.2 for more details.

For the high dimensional m-component solitons, from physical experiment ([74]), two dimensional photorefractive screening solitons and a two dimensional self-trapped beam were observed. It is natural to believe that there are two dimensional m-component ($m \geq 2$) solitons and self-trapped beams. As far as we know the first general mathematical theorems for m-component solitary wave solutions of system (GPE) in two and three spatial dimensions were obtained by T.-C. Lin and J. Wei in [55, 61]. They established some general theorems for the existence and non-existence of ground state solutions of steady-state m-coupled nonlinear Schrödinger equations (3.1.2) using a modified Nehari manifold approach and symmetrization arguments. Here a *ground state* solution is defined as a constrained minimum on the called Nehari manifold:

$$\mathcal{N} := \Big\{\vec{u} = (u_1, \dots, u_m) \in (H^1(\mathbb{R}^N))^m \,\Big|\, u_j \geq 0, u_j \not\equiv 0, \\ \textstyle\int_{\mathbb{R}^N} |\nabla u_j|^2 + \lambda_j \int_{\mathbb{R}^N} |u_j|^2 = \sum_{k=1}^{m} \beta_{jk} \int_{\mathbb{R}^N} u_k^2 u_j^2,\ j = 1, \dots, m\Big\}. \tag{3.1.3}$$

It is worth pointing out that by this definition it is also a positive solution of (3.1.2) such that its energy is minimal among all the positive solutions of (3.1.2). In fact, they considered the following minimization problem:

$$c := \inf_{u \in \mathcal{N}} E[\vec{u}], \tag{3.1.4}$$

where the associated energy functional is given by

$$E[\vec{u}] := \frac{1}{2}\sum_{j=1}^{m} \int_{\mathbb{R}^N} \left(|\nabla u_j|^2 + \lambda_j |u_j|^2\right) - \frac{1}{4}\sum_{j,k=1}^{m} \beta_{jk} \int_{\mathbb{R}^N} |u_j|^2 |u_k|^2. \tag{3.1.5}$$

for

$$\vec{u} = (u_1, \dots, u_m) \in (H^1(\mathbb{R}^N))^m. \tag{3.1.6}$$

Since we assume $N \leq 3$, the Sobolev embedding theorem implies that the energy functional E is well-defined and of class C^2.

The sign of coupling constants β_{jk}'s is crucial for the existence of ground state solutions. The first result in [55] concerns the all repulsive case:

Theorem 3.A ([55]). *Suppose $\lambda_j, \mu_j > 0$ for all $j = 1, \dots, m$. If $\beta_{jk} < 0$ for all $j \neq k$, then the ground state solution does not exist, i.e., c defined at* (3.1.4) *can not be attained.*

Remark 3.1. Some results for $\lambda_j \leq 0$ or $\mu_j \leq 0$ can be found in [59, 60]. It is related to the vortex solution of Ginzburg-Landau equation. The existence and properties of semiclassical state solutions have been studied in [56, 58, 75, 62, 31], where the following singularly perturbed nonlinear Schrödinger system with or without trapping potentials is studied:

$$\begin{cases} \varepsilon^2 \Delta u - V_1(x) u + \mu_1 u^3 + \beta u v^2 = 0 \quad \text{in } \Omega, \\ \varepsilon^2 \Delta v - V_2(x) v + \mu_2 v^3 + \beta u^2 v = 0 \quad \text{in } \Omega, \\ u, v > 0 \text{ in } \Omega, \text{ and } u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.1.7}$$

where $\Omega \subset \mathbb{R}^N$ $(N \leq 3)$ is a smooth and bounded domain, $\varepsilon > 0$ is a small parameter, $\mu_1, \mu_2 > 0$ are positive constants, V_1, V_2 are positive functions and may be constants and β is a coupling constant. For some more general nonlinearities, see [68, 6, 83] and references given there.

From now on, we will restrict the discussion to the case of $\lambda_j, \mu_j > 0$ for all $j = 1, \ldots, m$. The second result in [55, 61] concerns the all attractive case.

Theorem 3.B ([55, 61]). *Under hypotheses of Theorem 3.A. There exists $\beta_0 > 0$ depending on λ_j's, μ_j's, N and m such that if $0 < \beta_{jk} < \beta_0$ and the matrix $\Sigma := \left(|\beta_{jk}|\right)$ is positively definite, then there exists a ground state solution with all components are positive, radially symmetric and strictly decreasing.*

When attraction and repulsion coexist, things become very complicated. The third result in [55, 61] shows that if one state is repulsive to all the other states, then the ground state solution doesn't exist.

Theorem 3.C ([55, 61]). *Under hypotheses of Theorem 3.A. There exists $\beta_0 > 0$ depending on λ_j's, μ_j's, N and m such that if the matrix Σ is positively definite,*

$$\beta_{j_0,k} < 0, \ \forall k \neq j_0, \ \text{and } 0 < \beta_{jk} < \beta_0, \ \forall j \neq j_0, k \notin \{j, j_0\},$$

for some $j_0 \in \{1, \ldots, m\}$, then the ground state solution to (3.1.2) *doesn't exist.*

Remark 3.2. For $m = 3$ and $\lambda_j, \mu_j > 0$ for all $j = 1, \ldots, 3$, T.-C. Lin and J. Wei [55] constructed certain coefficient matrices $\left(\beta_{jk}\right)$ for which there is a non-radially symmetric bound state solution of (3.1.2) by using Lyapunov-Schmidt reduction and variational arguments. Here a *bound state* solution $\vec{u}$ is defined as a solution of (3.1.2) with finite energy, i.e., $E[\vec{u}] < +\infty$.

Remark 3.3. Note that β_0 in Theorem 3.B is a (unknown) small constant. Recently some explicit estimates of β_0 have been obtained, see for instance [87, 5] where they also gave some explicit ranges for large coupling parameters. The methods there are different. One [87] considered a minimization problem with m constraints and compared the energies, the other [5] used a minimax argument for the energy functional $E[\vec{u}]$ with one natural constraint and evaluated the Morse indices. As far as we know, it is still open, and quite interesting, to find out *what the optimal ranges for existence are.* Some progress may be found in [4, 11, 68, 12, 87, 5, 33, 19, 18].

It is worth pointing out a different definition of ground state is used in [11, 5]. One of the reason is the following [5]: In the case of a single nonlinear Schrödinger equation

$$\Delta u - \lambda u + \mu u^3 = 0 \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \tag{3.1.8}$$

a *ground state* solution is a solution u of (3.1.8) such that

$$I(u) = \min\Big\{I(v) \,|\, v \in H^1(\mathbb{R}^N) \setminus \{0\}, v \geq 0, I'(v)v = 0\Big\},$$

where

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + \lambda |v|^2\right) - \frac{1}{4}\mu \int_{\mathbb{R}^N} |v|^4.$$

It is well known that, for all $\lambda, \mu > 0$, (3.1.8) has a unique radial ground state solution $U_{\lambda,\mu}$ [53], which is positive and non-degenerate [80]. Furthermore, since $U_{\lambda,\mu}$ arises as a mountain-pass critical point of I [7], it has Morse index equal to 1 [3]. Hence it is natural to expect that similar properties are shared by a ground state solution of m-coupled nonlinear Schrödinger systems. In particular, a ground state solution should have Morse index 1. However, according to the definition of ground state solution in [55], it might have Morse index m due to the fact that the Nehari manifold $\mathcal{N}$ defined in (3.1.3) has codimension m. Therefore, in this chapter we will use the following definition of ground state solution as in [11, 5]:

Definition 3.1. *We say that $\vec{u}$ is a bound state solution of* (3.1.2) *if $\vec{u} \in (H^1(\mathbb{R}^N))^m$ is a solution of* (3.1.2) *satisfying $E[\vec{u}] < +\infty$. A bound state $\vec{u}$*

such that its energy is minimal among all non-trivial bound states, namely,

$$E[\vec{u}] = \min\Big\{E[\vec{v}] \,\Big|\, \vec{v} \in (H^1(\mathbb{R}^N))^m \setminus \{0\}, E'[\vec{v}]v = 0\Big\},$$

is called a ground state solution of (3.1.2).

Note that here we do not require all the components of ground state solution are strictly positive, which is slightly different from Definition 2.1 in [5].

Since we assume $N \leq 3$, the Sobolev embedding theorem implies that the energy functional E is well-defined on $(H^1(\mathbb{R}^N))^m$ and of class C^2. With the help of a classical "bootstrap" argument, solutions of (3.1.2) which are in $(H^1(\mathbb{R}^N))^m$ are also in $(C^2(\mathbb{R}^N))^m$ and tend to zero as $|x| \to +\infty$. Clearly, if there exists a ground state solution then there also exists a *semi-positive* one, which satisfies $u_j \geq 0$ for all j and $u_j \not\equiv 0$ for at least one j. In this case, note that u_j satisfies a linear equation

$$\Delta u_j - \lambda_j u_j + \Big(\sum_{k=1}^{m} \beta_{jk}|u_k|^2\Big)u_j = 0,$$

the Strong Maximum Principle implies that u_j is strictly positive or $u_j \equiv 0$. Moreover, if assume further $\beta_{jk} > 0$ for all $j, k = 1, \dots, m$ satisfying (H4) in [17], then all positive u_j's are radial symmetric and strictly decrease with respect to the same origin in $\mathbb{R}^N$ by Moving Planes method [17]. For $u_j \equiv 0$ for all j, the vector $0 := (0, \dots, 0)$ will be referred to as the *trivial* solution. For a solution $\vec{u} \not\equiv 0$, if one of its components $u_j \equiv 0$, then it will be called a *semi-trivial* solution; if all of its components are positive we will call it a *positive* solution.

About the ground state, it is natural to ask the following question:

Problem 3.1. *When does ground state solution exist? When is it positive? Is it unique? And what other properties does it have, like symmetry and non-degeneracy?*

If $m = 1$, the answer is complete, we refer to [25, 15] for the existence; [44] for the radial symmetry; [25, 53] for the uniqueness; and [79] for the non-degeneracy.

When $m \geq 2$, the problem above remains largely open. For the existence of ground state solution, T. Bartsch and Z.-Q. Wang [11] gave an answer as follows.

Theorem 3.D ([11]). *Suppose $\lambda_j, \beta_{jk} > 0$ for all $j, k = 1, \ldots, m$. Then system (3.1.2) has a semi-positive radially symmetric ground state solution. Moreover, it is of mountain pass type and has Morse index 1 considered as critical point of E on $(H^1(\mathbb{R}^N))^m$ and on $(H_r^1(\mathbb{R}^N))^m$. Here $H_r^1(\mathbb{R}^N)$ consists of all radially symmetric functions in $H^1(\mathbb{R}^N)$.*

Remark 3.4. To prove Theorem 3.D, the authors [11] used the Nehari manifold approach and symmetrization arguments similar to [55]. But the Nehari manifold is different. In [55] the Nehari manifold is defined at (3.1.3) with m constraints. T. Bartsch and Z.-Q. Wang [11] correspondingly considered the Nehari manifold with one constraint:

$$M := \left\{ \vec{u} \in (H^1(\mathbb{R}^N))^m \setminus \{0\} \mid E'[\vec{u}]\vec{u} = 0 \right\} \tag{3.1.9}$$

and the radial Nehari manifold $M_r := M \cap (H_r^1(\mathbb{R}^N))^m$. Similar idea was also used by A. Ambrosetti and E. Colorado [5].

Remark 3.5. When $\beta_{jk} < 0$ for some $j \neq k$, there maybe doesn't exist a ground state solution, see [55, 4, 87, 5] for more details. Moreover, in this situation, the structure of bound state solutions to (3.1.2) is more complicated. For examples,

(i) symmetry-breaking may occurs for positive bound state solution, see [55, 57, 48] for small interactions and [95] for large interactions;

(ii) system (3.1.2) admits infinitely many positive radial bound state solutions [97, 96, 31, 90, 32, 10], a relation between which and sign-changing radial solutions of (3.1.8) can be found in [97, 90], which provides a theoretical indication of phase separation into many nodal domains for the m-mixtures of Bose-Einstein condensates with strong repulsion;

(iii) The relation between a priori bounds and multiple existence of positive bound state solutions has been studied in [32, 10] by establishing some new Liouville type theorems. After that the local and global bifurcation structure of positive bound state solutions are investigated in [10] by using spectral analysis;

(iv) For sign-changing bound state solutions, the existence and multiplicity have been studied in [63] for both small and large interactions by different approaches, but there is no precise nodal property of the solutions and explicit estimates on small and large interactions. This raises some quite interesting questions: *what are the optimal ranges for existence? what's the precise nodal property? Does it determine the solution, similar to that of scalar equation* (3.1.8)*?*

We emphasize that all the ground state solutions may be semi-trivial. An example for such a situation is contained in the next result [11].

Theorem 3.E ([11]). *Assume that λ_j's are non-increasing and β_{jk}'s are non-decreasing in j and k. Then* (3.1.2) *admits no positive solution unless $\lambda_j = \lambda$ and $\beta_{jk} = \beta$ for some positive constants λ, β and all $j, k = 1, \ldots, m$.*

Remark 3.6. The above result still holds with both of the monotonicity conditions reversed for λ_j and β_{jk}. It also gives a non-existence result for positive solutions to (3.1.2).

Hence it raises a more complicated and interesting question: *under what conditions* (3.1.2) *has a positive ground state solution?* Recently some sufficient conditions for the existence of positive ground state solution have been obtained in [4, 11, 68, 12, 87, 5] for large coupling parameters. The methods there are different. One is the minimax method on a Nehari manifold [4, 11, 68, 87, 5] and the other is the method of critical point theory in the setting of invariant sets of the (negative) gradient flow [12]. Later on, some improved explicit estimates have been done in [33, 19, 18] by comparing the energies or Morse indices. For small coupling parameters, there are some results on the non-existence of positive ground state solution, see for instance [4, 11, 87, 5]. But there is no general result on *the optimal range for existence.* For $m = 2$ and $\lambda_1 = \lambda_2$, it is a simple matter to check that (3.1.2) admits a positive solution if and only if $\beta \notin \left[\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}\right]$ or $\mu_1 = \mu_2 = \beta$; and there exists a positive ground state solution if and only if $\beta > \max\{\mu_1, \mu_2\}$ or $\mu_1 = \mu_2 = \beta$ (cf. [11]). Here $\beta := \beta_{12}$.

Remark 3.7. Among other properties of ground and bound state solutions,

(i) the orbital stability of ground and bound state solutions to a more general nonlinear Schrödinger system:

$$\begin{cases} \Delta u_j - \lambda_j u_j + \sum\limits_{k=1}^{m} \beta_{jk} |u_k|^p |u_j|^{p-2} u_j = 0 \quad \text{in } \mathbb{R}^N, \quad N \leq 3 \\ u_j \in H^1(\mathbb{R}^N), \quad j = 1, \ldots, m, \end{cases} \tag{3.1.10}$$

has been studied in [62] for $p = 1 + \frac{2}{n}$ (critical case) and [69] for $p < 1 + \frac{2}{n}$ (subcritical case);

(ii) the blowup solutions of (GPS) have been investigated in [57, 39, 22], which may describe nonlinear wave collapse.

Remark 3.8. To get a positive ground state solution of (3.1.2), it is sufficient to show that there exist semi-positive ground state solutions different from all the semi-trivial solutions. So it is very important to know all the semi-trivial solutions. For $m = 2$, let $\vec{u} = (u_1, u_2)$ be a semi-trivial solution with non-negative components. Then either $u_1 \equiv 0$ or $u_2 \equiv 0$. In any case, the non-zero component satisfies

$$\Delta u_j - \lambda_j u_j + \mu_j u_j^3 = 0, \ u_j \geq 0 \text{ in } \mathbb{R}^N \quad \text{and} \quad u_j \in H^1(\mathbb{R}^N),$$

for $j = 1$ or 2, then $u_j = U_{\lambda_j,\mu_j}$ by the uniqueness of positive solution to (3.1.8) for all $\lambda, \mu > 0$ (cf. [53]). Therefore, all the semi-trivial non-negative solutions are given by

$$\vec{u}_1 = (U_{\lambda_1,\mu_1}, 0), \quad \vec{u}_2 = (0, U_{\lambda_2,\mu_2}).$$

Similarly, to get a positive ground state for $m \geq 3$, it is important to know all the non-negative solutions of (3.1.2) having one component equal to zero. Precisely, if the component u_1 is identically zero, then the remaining pair $(u_2, \ldots, u_m)$ solve the system

$$\begin{cases} \Delta u_j - \lambda_j u_j + \sum\limits_{k=2}^{m} \beta_{jk} |u_k|^2 u_j = 0 \quad \text{in } \mathbb{R}^N, \quad N \leq 3 \\ u_j \in H^1(\mathbb{R}^N), \quad j = 2, \ldots, m. \end{cases} \tag{3.1.11}$$

It is noting but a $(m-1)$-coupled nonlinear Schrödinger equations with β_{jk}, $j, k = 2, \ldots, m$. Therefore, to search for all the non-negative semi-trivial solutions, one need to know the uniqueness of positive solutions to (3.1.11).

Next as shown in [11] we give a simple condition which guarantees the existence and non-existence of a positive solution of (3.1.2) when $\lambda_1 = \cdots = \lambda_m$. In fact in this case there is an explicit solution.

Theorem 3.F (cf. [11]). *Assume $\lambda_1 = \cdots = \lambda_m = 1$. Then* (3.1.2) *has a positive solution of the form*

$$u_j(x) = \alpha_j\, w(x), \quad j = 1, \ldots, m, \tag{3.1.12}$$

where w is the unique radial positive solution of

$$\Delta w - w + w^3 = 0, \quad w \in H^1(\mathbb{R}^N), \tag{3.1.13}$$

if and only if the matrix $B = (\beta_{jk})$ can be written as $B = SD$, where S is a square matrix with each row summing to 1 *and D is a diagonal matrix with positive entries.*

Remark 3.9. The above result still holds for some $\beta_{jk} < 0$. Let $\vec{a} = (\alpha_1^2, \dots, \alpha_m^2)$, then the function $\vec{u} = (\alpha_1 w, \dots, \alpha_m w)$ is a solution of (3.1.2) if and only if $B\vec{a} = \vec{1}$ where $\vec{1} = (1, \dots, 1)$. Hence all possibility of the number of positive solutions of the form (3.1.12) are zero, one and infinity.

From Theorem 3.F we can get all positive solutions if one can prove that all positive solutions of (3.1.2) are of the form (3.1.12). This raises an important and quite interesting question:

Problem 3.2. *For $\lambda_1 = \cdots = \lambda_m = 1$, are all positive solutions of* (3.1.2) *of the form* (3.1.12)*?*

If the answer is affirmative, then non-existence, uniqueness and infinitely multiplicity of positive solutions will follow from that of $B\vec{a} = \vec{1}$. Moreover, when the uniqueness holds, this explicit solution (3.1.12) will be a positive ground state if the latter exists. As far as I know, the first attempt to study this problem is given in [87], where B. Sirakov conjectured that under the hypotheses of $m = 2$ and $0 < \beta \notin \big[\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}\big]$ the answer is affirmative and uniqueness follows.

In the remainder of this chapter we proceed with the study of Problem 3.2 and mostly focus on the case $m = 2$ of two equations, namely,

$$\begin{cases} \Delta u_1 - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0 \quad \text{in } \mathbb{R}^N, \\ \Delta u_2 - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0 \quad \text{in } \mathbb{R}^N, \\ u_1, u_2 \in H^1(\mathbb{R}^N). \end{cases} \tag{CNLS}$$

Some extensions to systems with more than two equations will be discussed and a few applications are also indicated. From now on, unless otherwise stated we assume that

$$\lambda_1, \lambda_2, \mu_1, \mu_2, \beta > 0. \tag{3.1.14}$$

Under this assumption, using a classical "bootstrap" argument, all positive solutions of (CNLS) are classical solutions and tend to zero as $x \to \infty$. Moreover, applying Moving Planes method (cf. [17, Theorem 1]), they are radial symmetric and strictly decrease with respect to some origin x_0. Without loss of generality we assume $x_0 = 0$. It will cause no confusion if we write $u_j(x) = u_j(r)$ for $r = |x|$ and $j = 1, 2$. Then (CNLS) becomes

$$\begin{cases} u_1'' + \frac{N-1}{r}u_1' - \lambda_1 u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0 & \text{in } (0, +\infty), \\ u_2'' + \frac{N-1}{r}u_2' - \lambda_2 u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0 & \text{in } (0, +\infty), \\ u_1'(0) = u_2'(0) = 0 \quad \text{and} \quad u_1(r),\ u_2(r) \to 0 \text{ as } r \to +\infty. \end{cases} \tag{3.1.15}$$

Our first result concerns the one-dimensional case.

Theorem 3.1. *Suppose $N = 1$ and $\lambda_1 = \lambda_2 = \lambda > 0$. Then the function (u_1^*, u_2^*) explicitly given by*

$$(u_1^*, u_2^*) = \left(\sqrt{\frac{\lambda(\beta - \mu_2)}{\beta^2 - \mu_1\mu_2}}\, w\big(\sqrt{\lambda}x\big), \sqrt{\frac{\lambda(\beta - \mu_1)}{\beta^2 - \mu_1\mu_2}}\, w\big(\sqrt{\lambda}x\big)\right) \tag{3.1.16}$$

is the unique positive solution to (CNLS) *up to a translation as long as*

$$0 < \beta \notin \big[\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}\big]. \tag{3.1.17}$$

Remark 3.10. The condition 3.1.17 is necessary. Indeed, if $\mu_1 = \mu_2 = \beta$ there are infinitely many positive solutions

$$\left(\cos\theta\sqrt{\frac{\lambda}{\beta}}\, w\big(\sqrt{\lambda}x\big), \sin\theta\sqrt{\frac{\lambda}{\beta}}\, w\big(\sqrt{\lambda}x\big)\right), \quad \forall\, \theta \in \big(0, \frac{\pi}{2}\big); \tag{3.1.18}$$

and if $\beta \in \big[\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}\big]$ but $\mu_1 \neq \mu_2$, positive solution doesn't exist by Theorem 3.E.

The above result gives an affirmative answer to Problem 3.2 under the condition (3.1.17) for $N = 1$. Our second theorem below deals with the case of $\mu_1 = \mu_2 = \beta$. Combining these results Problem 3.2 is completely solved in one dimension.

Theorem 3.2. *Under the hypotheses of Theorem 3.1, suppose further $\mu_1 = \mu_2 = \beta > 0$. Then all positive solutions of* (CNLS) *are of the form* (3.1.18) *up to a translation.*

For higher dimensions, the situation is more complicated. In general, the question of uniqueness of positive solutions to nonlinear equations is difficult. For scalar equation, the shooting method and Pohozaev's identity can give uniqueness (cf. [53, 20]). However for systems, there are very few results on uniqueness and it seems very difficult to apply shooting method because there are at least two free initial values. We briefly discuss here two feasible ways. One way is based on the implicit function theorem. To apply it one need the non-degeneracy and a compactness result. The restriction of applying this technique is that only local uniqueness can be obtained mostly. Another way based on the uniqueness for scalar equation is perhaps more efficient. But how to reduce a problem of systems to that of equations is a big problem. Some results have been obtained in [66, 54, 65, 49, 23], in which Hamiltonian and integral identities are very useful.

Our first uniqueness result in higher dimensions concerns small β. To get it, the implicit function theorem, a compactness result, and the uniqueness of positive solutions to (3.1.8) are needed.

Theorem 3.3. *Suppose $N = 2, 3$. There exists $\beta_0 > 0$ depending only on λ_j's, μ_j's and N such that if $0 < \beta < \beta_0$, then* (CNLS) *admits a unique positive solution up to a translation.*

Remark 3.11. We do not know how small β_0 is. It would be interesting to find an explicit estimate and we conjecture that $\beta_0 = \min\{\mu_1, \mu_2\}$ for $\lambda_1 = \lambda_2$.

But for large β, using a simple integral identity, we can obtain a result for higher dimensions similar to $N = 1$.

Theorem 3.4. *Under the hypotheses of Theorem 3.3, suppose further $\lambda_1 = \lambda_2$. Then (u_1^*, u_2^*) explicitly defined at* (3.1.16) *is the unique positive solution to* (CNLS) *up to translation when*

$$\beta > \max\{\mu_1, \mu_2\}. \tag{3.1.19}$$

To answer Problem 3.2 in higher dimensions for $\mu_1 = \mu_2 = \beta$, we have the following Liouville type theorem.

Theorem 3.5. *Under hypotheses of Theorem 3.4, assume further* $\mu_1 = \mu_2 = \beta > 0$. *Then all positive solutions of* (CNLS) *are of the form* (3.1.18) *up to translation.*

The organization of this chapter is as follows. In Section 3.2, we consider the one-dimensional case and prove Theorems 3.1, 3.2. Sections 3.3 and 3.4 are devoted to the proofs of Theorems 3.3, 3.4, 3.5. Section 3.5 presents some extensions and applications.

3.2 The one-dimensional case

In this section we consider the one-dimensional case of (3.1.15) under the condition $\lambda_1 = \lambda_2 = \lambda$, i.e., the following ODE system with two boundary conditions:

$$\begin{cases} u_1'' - \lambda u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0 \quad \text{in } (0, +\infty), \\ u_2'' - \lambda u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0 \quad \text{in } (0, +\infty), \\ u_1'(0) = u_2'(0) = 0 \quad \text{and} \quad u_1(r),\, u_2(r) \to 0 \text{ as } r \to +\infty. \end{cases} \tag{3.2.1}$$

Proof of Theorem 3.1. Let (u_1, u_2) be a positive solution of (3.2.1). The basic idea of the proof is to show that $u_1(r) = a^{-1}\, u_2(r)$ for all $r > 0$, where $a = \sqrt{\frac{\beta-\mu_1}{\beta-\mu_2}}$ well-defined by (3.1.17). Define $u(r) = u_1(r)$ and $v(r) = a^{-1}\, u_2(r)$ for $r > 0$. Then (u, v) satisfies

$$\begin{cases} u'' - \lambda u + \mu_1 u^3 + \beta a^2 u v^2 = 0 \quad \text{in } (0, +\infty), \\ v'' - \lambda v + \mu_2 a^2 v^3 + \beta u^2 v = 0 \quad \text{in } (0, +\infty), \\ u'(0) = v'(0) = 0 \quad \text{and} \quad u(r),\, v(r) \to 0 \text{ as } r \to +\infty. \end{cases} \tag{3.2.2}$$

The proof of $u \equiv v$ will be divided into four steps.

Step 1: We claim that

$$(u'v - uv')' + (\mu_1 - \beta)uv(u^2 - v^2) = 0 \quad \text{in } (0, +\infty). \tag{3.2.3}$$

Indeed, multiplying the first equation in (3.2.2) by v and second one by u yields

$$\begin{cases} (u'v)' - u'v' - \lambda uv + \mu_1 u^3 v + \beta a^2 uv^3 = 0, \\ (uv')' - u'v' - \lambda uv + \mu_2 a^2 uv^3 + \beta u^3 v = 0. \end{cases}$$

Subtracting the second equation above from the first one, our claim follows. Integrating (3.2.3) over $(0, +\infty)$ we get

$$(\mu_1 - \beta) \int_0^\infty uv(u^2 - v^2) = 0.$$

Hence $u \equiv v$ otherwise the function $u - v$ changes sign.

Step 2: Suppose that $u - v$ changes sign. We claim that there exists $r_1 > 0$ such that either

$$u(r) - v(r) > 0 \quad \text{for all } r > r_1 \quad \text{and} \quad u(r_1) - v(r_1) = 0, \tag{3.2.4}$$

or

$$u(r) - v(r) < 0 \quad \text{for all } r > r_1 \quad \text{and} \quad u(r_1) - v(r_1) = 0. \tag{3.2.5}$$

Indeed, by (3.2.2) $u - v$ satisfies

$$f'' - \lambda f + \left[\mu_1 u^2 + (\mu_1 - \beta)uv + \mu_2 a^2 v^2\right] f = 0 \quad \text{in } (0, +\infty).$$

Since $\lambda > 0$ and

$$\left[\mu_1 u^2 + (\mu_1 - \beta)uv + \mu_2 a^2 v^2\right](r) \to 0 \quad \text{as } r \to +\infty,$$

our claim is proved.

Step 3: Without loss of generality we assume that (3.2.4) holds, since otherwise we consider $v - u$. We claim that there exists $r_2 > r_1$ such that

$$(u'v - uv')(r_2) = 0. \tag{3.2.6}$$

Indeed, multiplying the first equation in (3.2.2) by u' and second one by v' we get

$$\begin{cases} \frac{1}{2}[(u')^2]' - \frac{1}{2}\lambda\,(u^2)' + \frac{1}{4}\mu_1 (u^4)' + \frac{1}{4}\beta a^2 (u^2 v^2)' + \frac{1}{2}\beta a^2 uv(u'v - uv') = 0, \\ \frac{1}{2}[(v')^2]' - \frac{1}{2}\lambda\,(v^2)' + \frac{1}{4}\mu_2 a^2 (v^4)' + \frac{1}{4}\beta (u^2 v^2)' + \frac{1}{2}\beta uv(uv' - u'v) = 0. \end{cases}$$

Subtracting the second equation above from the first one and integrating over (r_1, ∞) yields

$$-\frac{1}{2}\left[(u')^2 - (v')^2\right](r_1) + \frac{a^2+1}{2}\beta \int_{r_1}^{\infty} uv(u'v - uv') = 0, \tag{3.2.7}$$

It follows from (3.2.4) that

$$0 > u'(r_1) > v'(r_1) \quad \text{and} \quad (u'v - uv')(r_1) > 0. \tag{3.2.8}$$

Hence by (3.2.7) and $\beta > 0$,

$$\int_{r_1}^{\infty} uv(u'v - uv') < 0,$$

from which and (3.2.8) our claim follows.

Step 4: Integrating (3.2.3) over (r_2, ∞) yields

$$(\mu_1 - \beta) \int_{r_2}^{\infty} uv(u^2 - v^2) = 0,$$

which contradicts the fact that $u, v > 0$, $r_2 > r_1$ and (3.2.4). □

Next we prove Theorem 3.2 in a one-dimensional way. For other proof we refer to Section 3.4.

Proof of Theorem 3.2. Let (u_1, u_2) be a positive solution of (3.2.1) with $\mu_1 = \mu_2 = \beta$. Define the Hamiltonian functional $H(r)$ of (3.2.1) by

$$H(r) = \frac{1}{2}\left[(u_1')^2 + (u_2')^2\right] + \frac{\beta}{4}(u_1^4 + u_2^4) + \frac{\beta}{2}u_1^2 u_2^2 - \frac{\lambda}{2}(u_1^2 + u_2^2). \tag{3.2.9}$$

Thanks to $N = 1$, we always have the Hamiltonian identity

$$H(r) \equiv C \quad \text{in } (0, +\infty). \tag{3.2.10}$$

By the exponential decay of solutions to (3.2.1) (cf. [25]), we get $C = 0$. Hence

$$0 = H(0) = \frac{\beta}{4}\left[u_1^4(0) + u_2^4(0)\right] + \frac{\beta}{2}u_1^2(0)u_2^2(0) - \frac{\lambda}{2}\left[u_1^2(0) + u_2^2(0)\right], \tag{3.2.11}$$

which implies that

$$u_1^2(0) + u_2^2(0) = \frac{2\lambda}{\beta}.$$

Then there is a $\theta \in (0, \frac{\pi}{2})$ such that

$$u_1(0) = \frac{2\lambda}{\beta} \cos\theta \quad \text{and} \quad u_2(0) = \frac{2\lambda}{\beta} \sin\theta.$$

Note that $w(0) = \sqrt{2}$ for $N = 1$. Thus (u_1, u_2) has the same initial values with

$$\left(\cos\theta \sqrt{\frac{\lambda}{\beta}} w(\sqrt{\lambda}x), \sin\theta \sqrt{\frac{\lambda}{\beta}} w(\sqrt{\lambda}x) \right).$$

Therefore, applying the standard uniqueness theorem of initial value problem in ODE theory, our conclusion follows. □

Remark 3.12. The proof above gives more, namely all non-trivial solutions of (CNLS) are of the form (3.1.18) up to a translation except $\theta \in [0, 2\pi)$ under hypotheses of Theorem 3.2. But for $\lambda_1 \neq \lambda_2$, the solution structure is much more complicated, see [99] and references therein.

3.3 The higher dimensional case

In this section, we prove Theorems 3.3, 3.4 and discuss some possible extensions.

3.3.1 Uniqueness for small β

By the symmetry and regularity result mentioned above, we need only consider (3.1.15) and work on the space $C_{r,0}(\mathbb{R}^N) \times C_{r,0}(\mathbb{R}^N)$, where $C_{r,0}(\mathbb{R}^N)$ denotes the space of continuous radial functions vanishing at infinity.

To prove Theorem 3.3, we first establish the following more general lemma, applying which we get uniqueness from local uniqueness. Before we state and prove the lemma, let us define

$$a_0 = \inf_{\phi \in H^1} \frac{\int_{\mathbb{R}^N} (|\nabla \phi|^2 + \lambda_2 |\phi|^2)}{\int_{\mathbb{R}^N} U^2_{\lambda_1, \mu_1} |\phi|^2}, \quad b_0 = \inf_{\phi \in H^1} \frac{\int_{\mathbb{R}^N} (|\nabla \phi|^2 + \lambda_1 |\phi|^2)}{\int_{\mathbb{R}^N} U^2_{\lambda_2, \mu_2} |\phi|^2}. \tag{3.3.1}$$

Lemma 3.1. *Suppose that the set of nonnegative solutions of* (CNLS) *is compact in* $C_{r,0}(\mathbb{R}^N) \times C_{r,0}(\mathbb{R}^N)$ *for bounded* β. *Let* $\vec{u}_0$ *is a positive solution of* (CNLS) *for* $\beta = \beta_0$*, where* $\beta_0 \neq a_0, b_0$*. Assume further* $\vec{u}_0$ *is unique and non-degenerate in* $C_{r,0}(\mathbb{R}^N) \times C_{r,0}(\mathbb{R}^N)$*. Then there exists* $\epsilon_0 > 0$ *such that* (CNLS) *admits a unique positive solution in* $C_{r,0}(\mathbb{R}^N) \times C_{r,0}(\mathbb{R}^N)$ *for* $\beta \in (\beta_0 - \epsilon_0, \beta_0 + \epsilon_0)$*.*

Proof of Lemma 3.1. Denote $\Phi(\beta, \vec{u}) = E'[\vec{u}]$. Then $\Phi(\beta_0, \vec{u}_0) = 0$. Moreover, $\Phi_u(\beta_0, \vec{u}_0) = E''[\vec{u}_0]$ is invertible. By the implicit function theorem, there exists $c_0, r_0 > 0$ and function $\phi : (\beta_0 - c_0, \beta_0 + c_0) \mapsto B_{r_0}(\vec{u}_0)$ such that for any $\beta \in (\beta_0 - c_0, \beta_0 + c_0)$, the functional $\Phi(\beta, \vec{u})$ has a unique solution $\vec{u} = \phi(\beta)$ in $B_{r_0}(\vec{u}_0)$. Namely, (CNLS) admits a unique solution in $B_{r_0}(\vec{u}_0)$ for $\beta \in (\beta_0 - \epsilon_0, \beta_0 + \epsilon_0)$. To get the uniqueness result, it suffices to prove that positive solutions is contained in $B_{r_0}(\vec{u}_0)$ for $|\beta - \beta_0|$ sufficiently small. Suppose not, there is a sequence $\{\vec{u}_n\}$ of positive solutions to (CNLS) for $\beta = \beta_n$ such that $\vec{u}_n \notin B_{r_0}(\vec{u}_0)$ and β_n goes to β_0 as n goes to infinity. By the compactness condition, $\vec{u}_n$ converges to a nonnegative solution $\vec{u}_* \neq \vec{u}_0$ of (CNLS) for $\beta = \beta_0$. By the uniqueness of $\vec{u}_0$, $\vec{u}_*$ is a semi-trivial solution. That is either $\vec{u}_* = (U_{\lambda_1,\mu_1}, 0)$ or $\vec{u}_* = (0, U_{\lambda_2,\mu_2})$. By non-degeneracy of $U_{\lambda,\mu}$, as in [31], it is easily seen that $(U_{\lambda_1,\mu_1}, 0)$ is non-degenerate if $\beta \neq a_0$. Thus by the implicit function theorem again, (CNLS) admits a unique solution in a neighborhood of $(U_{\lambda_1,\mu_1}, 0)$ for $\beta \neq a_0$. Since $(U_{\lambda_1,\mu_1}, 0)$ is one such solution, it is the unique one. Hence $\vec{u}_* \neq (U_{\lambda_1,\mu_1}, 0)$ if $\beta_0 \neq a_0$. Similarly, we conclude that $\vec{u}_* \neq (0, U_{\lambda_2,\mu_2})$ if $\beta_0 \neq b_0$. This leads to a contradiction and completes the proof. □

The compactness condition in Lemma 3.1 is satisfied by the Lemma 2.4 of [31]. For the convenience of the reader we repeat it without proof, thus making our exposition self-contained.

Lemma 3.2 (cf.[31])**.** *Let* $\beta \geq 0$ *be bounded. Then the set of nonnegative solutions of* (CNLS) *is compact in* $C_{r,0}(\mathbb{R}^N) \times C_{r,0}(\mathbb{R}^N)$*.*

Proof of Theorem 3.3. Combining Lemma 3.2 and the uniqueness and non-degeneracy of $U_{\lambda,\mu}$ to (3.1.8) in [53, 79], our Theorem 3.3 follows from Lemma 3.1 for $\beta_0 = 0$. □

Remark 3.13. Let us mention that, N. Ikoma in [49] proved a result similar to Theorem 3.3 and also extended to radially symmetric solutions of (CNLS) with trapping potentials by perturbation argument. By Lemma 3.1 we have proved more, namely that there exists an optimal $\beta_0 > 0$ such that (CNLS) admits a unique positive solution for $0 < \beta < \beta_0$. But we don't know any explicit estimate of β_0. Recently, there are a few results on uniqueness for other systems, see for instance [66, 54, 23].

3.3.2 Uniqueness for large β

Now we consider the case of large β and prove Theorem 3.4.

Proof of Theorem 3.4. As in the proof of Theorem 3.1. The basic idea is to show that $u_1(x) = a^{-1} u_2(x)$ for all $x \in \mathbb{R}^N$, where $a = \sqrt{\frac{\beta-\mu_1}{\beta-\mu_2}}$.

Define $u(x) = u_1(x)$ and $v(x) = a^{-1} u_2(x)$ for $x \in \mathbb{R}^N$. Then (u, v) satisfies

$$\begin{cases} \Delta u - \lambda u + \mu_1 u^3 + \beta a^2 u v^2 = 0 \quad \text{in } \mathbb{R}^N, \\ \Delta v - \lambda v + \mu_2 a^2 v^3 + \beta u^2 v = 0 \quad \text{in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N). \end{cases} \tag{3.3.2}$$

Let $\Gamma_+ = \{x \in \mathbb{R}^N \,|\, u(x) > v(x)\}$ and $\Gamma_- = \{x \in \mathbb{R}^N \,|\, u(x) < v(x)\}$. We divide the proof of $u \equiv v$ into two steps.

Step 1: We claim that

$$\operatorname{div}(v\nabla u - u\nabla v) + (\mu_1 - \beta)uv(u^2 - v^2) = 0 \quad \text{in } \mathbb{R}^N. \tag{3.3.3}$$

Indeed, multiplying the first equation in (3.3.2) by v and second one by u yields

$$\begin{cases} \operatorname{div}(v\nabla u) - \nabla u \cdot \nabla v - \lambda uv + \mu_1 u^3 v + \beta a^2 u v^3 = 0, \\ \operatorname{div}(u\nabla v) - \nabla u \cdot \nabla v - \lambda uv + \mu_2 a^2 u v^3 + \beta u^3 v = 0. \end{cases}$$

Subtracting the second equation above from the first one, our claim follows.

Step 2: We claim that $\Gamma_+ = \emptyset$ and $\Gamma_- = \emptyset$. Then $u \equiv v$. Indeed, integrating (3.5.21) over Γ_+ yields

$$\int_{\partial\Gamma_+} \left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} \right) + (\mu_1 - \beta) \int_{\Gamma_+} uv(u^2 - v^2) = 0, \tag{3.3.4}$$

where ν denotes the unit outward normal to $\partial\Gamma_+$. Note that $u - v > 0$ in Γ_+ and $u = v > 0$ on $\partial\Gamma_+$, we get

$$\left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} \right) \le 0 \quad \text{on } \partial\Gamma_+.$$

Hence the first term of (3.5.22) is non-positive. Since $\beta > \max\{\mu_1, \mu_2\}$ the second term of (3.5.22) is also non-positive. Therefore,

$$\int_{\Gamma_+} uv(u^2 - v^2) = 0,$$

which implies $\Gamma_+ = \emptyset$. By a similar argument, we can prove that $\Gamma_- = \emptyset$. □

3.4 Liouville-type theorem

This section is devoted to the proof of Theorem 3.5. It also gives another proof of Theorem 3.2 with a PDE method.

To prove Theorem 3.5, we will use the following Liouville type result for the equation $\nabla \cdot (\varphi^2 \nabla \sigma) = 0$, where $\nabla\cdot$ denotes the divergence operator. This has previously been used by L. Ambrosio and X. Cabré to study a conjecture of De Giorgi in $\mathbb{R}^3$ (cf. [8, Proposition 2.1]).

Proposition 3.1 ([8]). *Let $\varphi \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ be a positive function. Suppose that $\sigma \in H^1_{\text{loc}}(\mathbb{R}^N)$ satisfies*

$$\sigma \nabla \cdot (\varphi^2 \nabla \sigma) \ge 0 \quad \text{in } \mathbb{R}^N$$

in the distributional sense. For every $R > 1$, let $B_R = \{|x| < R\}$ and assume that

$$\int_{B_R} (\varphi\sigma)^2 \le CR^2, \tag{3.4.1}$$

for some constant C independent of R. Then σ is constant.

Proof of Theorem 3.5. Let (u_1, u_2) be a positive solutions of (CNLS) with

$$\lambda_1 = \lambda_2 = \lambda, \quad \text{and} \quad \mu_1 = \mu_2 = \beta.$$

Direct computation yields

$$\nabla \cdot \left[u^2 \nabla \left(\frac{v}{u} \right) \right] = 0. \tag{3.4.2}$$

Since $v \in H^1(\mathbb{R}^N)$, applying Proposition 3.1 with $\varphi = u$ and $\sigma = v/u$, we get that v/u is a constant. That is, there exists $\theta \in (0, \pi/2)$ such that $u \equiv \tan\theta\, v$. Substituting this identity into (CNLS) yields Theorem 3.5. □

3.5 Extension and application

In this section we will discuss some possible extensions and applications.

3.5.1 General domain and trapping potentials

First we consider generalizations of Theorem 3.4 to the following coupled nonlinear Schrödinger equations with trapping potentials:

$$\begin{cases} \Delta u_1 - V_1(x) u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0 \quad \text{in } \Omega, \\ \Delta u_2 - V_2(x) u_2 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0 \quad \text{in } \Omega, \\ u_1, u_2 > 0 \text{ in } \Omega, \quad u_1 = u_2 = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.5.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth (bounded or unbounded) domain, $N \leq 3$, $V_1(x), V_2(x)$ are trapping potentials and μ_1, μ_2, β are positive constants.

Now we consider the case of large β and pose the following conditions on the trapping potentials and coupling constant:

$$V_1(x) = V_2(x) = V(x) > 0 \quad \text{in } \Omega, \tag{3.5.2}$$

$$\beta > \max\{\mu_1, \mu_2\}. \tag{3.5.3}$$

Using only integration by part, we obtain the following result.

Theorem 3.6. *Let (u_1, u_2) is a solution of system (3.5.1), then under the conditions (3.5.2) and (3.5.3), we obtain*

$$u_2(x) = au_1(x), \quad \text{where } a = \sqrt{\frac{\beta - \mu_1}{\beta - \mu_2}} \tag{3.5.4}$$

and u_1 satisfies the scalar equation:

$$\Delta u - V(x)u + \frac{\beta^2 - \mu_1\mu_2}{\beta - \mu_2} u^3 = 0. \tag{3.5.5}$$

Proof. Let $u(x) = u_1(x)$ and $v(x) = a^{-1}u_2(x)$ we have

$$\mathrm{div}(v\nabla u - u\nabla v) + (\mu_1 - \beta)uv(u^2 - v^2) = 0 \quad \text{in } \Omega. \tag{3.5.6}$$

Then conclusion follows from the argument in Step 2 of the proof for Theorem 3.4. □

Remark 3.14. The equality (3.5.6) is the basic of our argument.

(i) The conclusion is also true for homogeneous Neumann boundary condition;

(ii) It is worth pointing out that the proof above doesn't use any symmetric assumption compared to [49].

3.5.2 Non-degeneracy and existence

Let (u_1, u_2) be a solution of system (CNLS). We say that (u_1, u_2) is *non-degenerate* if the solution set of the linearized equation

$$\begin{cases} \Delta\phi_1 - \lambda_1\phi_1 + 3\mu_1 u_1^2\phi_1 + \beta u_2^2\phi_1 + 2\beta u_1 u_2 \phi_2 = 0, \\ \Delta\phi_2 - \lambda_2\phi_2 + 3\mu_2 u_2^2\phi_2 + \beta u_1^2\phi_2 + 2\beta u_1 u_2 \phi_1 = 0, \\ \phi_1, \phi_2 \in H^1(\mathbb{R}^N), \end{cases} \tag{3.5.7}$$

is exactly N-dimensional, namely,

$$\phi_j = \sum_{l=1}^{N} a_l \frac{\partial u_j}{\partial x_l}, \quad j = 1, 2,$$

for some constants a_l, $l = 1, \dots, N$.

Assume that $\lambda_1 = \lambda_2$. By Theorem 3.3 and Theorem 3.4, we conclude that $(u_1, u_2) = (u_1^*, u_2^*)$, where (u_1^*, u_2^*) is defined by (3.1.16), provided

$$\beta \notin \big[\beta_0(N), \max\{\mu_1, \mu_2\}\big], \tag{3.5.8}$$

where $\beta_0(N) \le \min\{\mu_1, \mu_2\}$ and $\beta_0(1) = \min\{\mu_1, \mu_2\}$. By Lemma 2.2 and Theorem 3.1 of [31], (u_1^*, u_2^*) is non-degenerate. We state it in the following corollary.

Corollary 3.1. *Assume that* $\lambda_1 = \lambda_2$ *and* (3.5.8) *holds. Then the positive solution* (u_1^*, u_2^*) *to* (CNLS) *is non-degenerate.*

In [58], the authors constructed ground states in coupled nonlinear Schrödinger equations with trapping potentials. Using the non-degeneracy result, we can consider bound states of the following system

$$\begin{cases} \varepsilon^2 \Delta u_1 - V_1(x) u_1 + \mu_1 u_1^3 + \beta u_1 u_2^2 = 0 & \text{in } \mathbb{R}^1, \\ \varepsilon^2 \Delta u_2 - V_2(x) u_1 + \mu_2 u_2^3 + \beta u_1^2 u_2 = 0 & \text{in } \mathbb{R}^1. \end{cases} \tag{3.5.9}$$

To simplify the technical difficulties, we assume that $0 < C_1 \le V_1, V_2 \le C_2$. We have the following two results.

Theorem 3.7. *Assume that* V_1 *and* V_2 *has a strictly local minimum at* x_0. *That is, there exists* $\delta > 0$ *such that* $V_1(x) > V_1(x_0)$, $V_2(x) > V_2(x_0)$ *for* $x_0 \neq x \in (x_0 - \delta, x_0 + \delta)$. *Furthermore we assume that*

$$V_1(x_0) = V_2(x_0), \quad \beta \notin \big[\max\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}\big]. \tag{3.5.10}$$

Then for ε *sufficiently small, problem* (3.5.9) *has a solution* $(u_{1,\varepsilon}, u_{2,\varepsilon})$ *with spikes near* x_0.

Theorem 3.8. *Assume that* V_1 *and* V_2 *has a strictly local maximum at* x_0. *That is, there exists* $\delta > 0$ *such that* $V_1(x) < V_1(x_0)$, $V_2(x) < V_2(x_0)$ *for* $x_0 \neq x \in (x_0 - \delta, x_0 + \delta)$. *Furthermore suppose* (3.5.10) *holds. Then for positive integer* $K \ge 2$ *and* ε *sufficiently small, problem* (3.5.9) *has a solution* $(u_{1,\varepsilon}, u_{2,\varepsilon})$ *with* K *spikes near* x_0.

Theorem 3.8 seems to be the first result on the existence of bound states with multiple spikes. Under the condition (3.5.10), we have uniqueness and non-degeneracy of the limiting equations. The proofs of both Theorem 3.7 and Theorem 3.8 follow form the same reduction procedure in [98] for single equations. We omit the details.

Another application of our uniqueness results is in the article [18]. The author consider the existence of positive radial ground states of 3-coupled nonlinear Schrödinger equations

$$\begin{cases} -\Delta u_1 + \lambda u_1 = \mu_1 u_1^3 + \beta_{12} u_2^2 u_1 + \beta_{13} u_3^2 u_1, & \text{in } \mathbb{R}^n, \\ -\Delta u_2 + \lambda u_2 = \beta_{12} u_1^2 u_2 + \mu_2 u_2^3 + \beta_{23} u_3^2 u_2, & \text{in } \mathbb{R}^n, \\ -\Delta u_3 + \lambda u_3 = \beta_{13} u_1^2 u_3 + \beta_{23} u_2^2 u_3 + \mu_3 u_3^3, & \text{in } \mathbb{R}^n, \\ u_1(x) \to 0, \ u_2(x) \to 0, \ u_3(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{3.5.11}$$

where $n = 1, 2, 3$, $\lambda > 0$, $\mu_j > 0$ and $\beta_{ij} > 0$ $(i < j)$ for $i, j = 1, 2, 3$. To conclude the existence of positive ground state, the author first use Theorem 3.D of T. Bartsch and Z.-Q. Wang. Secondly the author applies our unique result to obtain all the semitrivial solution of (3.5.11). Thirdly the author provides a sufficient condition to guarantee each Morse index of nontrivial and semitrivial solutions is at least 2 and then gets the following result:

Theorem 3.9 ([18]). *If β_{ij}'s satisfy the following conditions:*

$$\beta_{ij} > \max\{\mu_i, \mu_j\} \text{ and } \beta_{ij}^2 - \mu_i \mu_j < \beta_{ik}(\beta_{ij} - \mu_j) + \beta_{jk}(\beta_{ij} - \mu_i), \tag{3.5.12}$$

for $i, j, k = 1, 2, 3$, $i < j$ and $i \neq k, j \neq k$, then (3.5.11) has a positive radial ground state.

It seems that the condition ((3.5.12)) is only a sufficient condition, we can give some comments below.

(i) The first kind of condition: $\beta_{ij} > \max\{\mu_i, \mu_j\}$ in condition (3.5.12) is a sufficient condition on the uniqueness and existence, So we can firstly relax this kind of condition for any considered dimension by using the non-existence result for coupled equations. For example,

$$\begin{cases} \mu_3 < \mu_2 < \mu_1, \quad \beta_{12} < \beta_{13} < \beta_{23}, \\ \beta_{23} - \beta_{12} = \mu_1 - \mu_3, \quad \beta_{13} - \beta_{12} = \mu_2 - \mu_3, \\ \beta_{12} \le \mu_1 \text{ and } \mu_1 - \beta_{12} \text{ is small enough.} \end{cases} \tag{3.5.13}$$

Secondly we can relax this kind of condition at least in dimension one by Theorem 3.1

(ii) The second kind of condition:

$$\beta_{ij}^2 - \mu_i\mu_j < \beta_{ik}(\beta_{ij} - \mu_j) + \beta_{jk}(\beta_{ij} - \mu_i) \tag{3.5.14}$$

in condition (3.5.12) is one part of condition on the existence of positive solution of system (3.5.11) with $u_1 = c_1u_3$ and $u_2 = c_2u_3$ for some positive constant c_1 and c_2. It is quite interesting to see whether it is sufficient and necessary (exclude the case of $\beta_{12} = \beta_{13} = \beta_{23} = \mu_1 = \mu_2 = \mu_3$).

3.5.3 Systems with more than two equations

Now we consider Problem (3.2) for the general m-coupled nonlinear Schrödinger equations

$$\begin{cases} \Delta u_j - \lambda_j u_j + \sum\limits_{k=1}^{m} \beta_{jk}|u_k|^2 u_j = 0 \quad \text{in } \mathbb{R}^N, \quad N \le 3, \\ u_j \in H^1(\mathbb{R}^N), \quad j = 2, \dots, m. \end{cases} \tag{3.5.15}$$

Namely, for $\lambda_1 = \dots, \lambda_m = 1$, does any positive solution of (3.5.15) have the following form:

$$u_j(x) = \alpha_j\, w(x), \quad j = 1, \dots, m, \tag{3.5.16}$$

where w is the unique radial positive solution of

$$\Delta w - w + w^3 = 0, \quad w \in H^1(\mathbb{R}^N). \tag{3.5.17}$$

As we mentioned before, there is a sufficient and necessary condition of $B = (\beta_{jk})$ such that (3.5.15) has a positive solution of the form (3.5.16) (cf. Theorem 3.F). Let us denote the solvable domain of B by $\mathcal{B}$, i.e., $B \in \mathcal{B}$ is the sufficient and necessary condition. For simplicity of notation, we consider the case $m = 3$:

$$\begin{cases} -\Delta u_1 + \lambda u_1 = \mu_1 u_1^3 + \beta_{12} u_2^2 u_1 + \beta_{13} u_3^2 u_1, & \text{in } \mathbb{R}^N, \\ -\Delta u_2 + \lambda u_2 = \beta_{12} u_1^2 u_2 + \mu_2 u_2^3 + \beta_{23} u_3^2 u_2, & \text{in } \mathbb{R}^N, \\ -\Delta u_3 + \lambda u_3 = \beta_{13} u_1^2 u_3 + \beta_{23} u_2^2 u_3 + \mu_3 u_3^3, & \text{in } \mathbb{R}^N, \\ u_1, u_2, u_3 \in H^1(\mathbb{R}^N), \ N \le 3. \end{cases} \tag{3.5.18}$$

Follow the argument of the proof for Theorem 3.3, one can show that

Theorem 3.10. *Suppose $N = 1, 2, 3$. There exists $\beta_0 > 0$ depending only on λ_j's, μ_j's and N such that if $0 < \beta_{jk} < \beta_0$ for all j, k's, then (3.5.18) admits a unique positive solution up to a translation.*

For large β_{jk}'s we can get a result similar to Theorem 3.4.

Theorem 3.11. *For $N = 1, 2, 3$. Suppose $B \in \mathcal{B}$ and*

$$\beta_{13} = \beta_{23}, \quad \beta_{12} > \max\{\mu_1, \mu_2\} \quad \beta_{13} > \mu_3. \tag{3.5.19}$$

Then any positive solution of (3.5.18) has the form (3.5.16) and then is unique up to a translation.

Proof. First we can prove that $u_1 \equiv a^{-1} u_2$ where $a = \sqrt{\frac{\beta_{12}-\mu_1}{\beta_{12}-\mu_2}}$. Indeed, define $u(x) = u_1(x)$ and $v(x) = a^{-1} u_2(x)$ for $x \in \mathbb{R}^N$. Then (u, v, u_3) satisfies

$$\begin{cases} \Delta u - \lambda u + \mu_1 u^3 + \beta_{12} a^2 u v^2 + \beta_{13} u_3^2 u = 0 \quad \text{in } \mathbb{R}^N, \\ \Delta v - \lambda v + \mu_2 a^2 v^3 + \beta_{12} u^2 v + \beta_{23} u_3^2 v = 0 \quad \text{in } \mathbb{R}^N, \\ \Delta u_3 - \lambda u_3 + \mu_3 u_3^3 + \beta_{13} u^2 u_3 + \beta_{23} a^2 v^2 u_3 = 0 \quad \text{in } \mathbb{R}^N, \\ u, v, u_3 \in H^1(\mathbb{R}^N). \end{cases} \tag{3.5.20}$$

Let $\Gamma_+ = \{x \in \mathbb{R}^N \,|\, u(x) > v(x)\}$ and $\Gamma_- = \{x \in \mathbb{R}^N \,|\, u(x) < v(x)\}$. We divide the proof of $u \equiv v$ into two steps.

Step 1: We claim that

$$\operatorname{div}(v\nabla u - u\nabla v) + (\mu_1 - \beta_{12})uv(u^2 - v^2) = 0 \quad \text{in } \mathbb{R}^N. \tag{3.5.21}$$

Indeed, multiplying the first equation in (3.5.20) by v and second one by u yields

$$\begin{cases} \operatorname{div}(v\nabla u) - \nabla u \cdot \nabla v - \lambda uv + \mu_1 u^3 v + \beta_{12} a^2 uv^3 + \beta_{13} u_3^2 uv = 0, \\ \operatorname{div}(u\nabla v) - \nabla u \cdot \nabla v - \lambda uv + \mu_2 a^2 uv^3 + \beta_{12} u^3 v + \beta_{23} u_3^2 uv = 0. \end{cases}$$

Subtracting the second equation above from the first one, our claim follows.

Step 2: We claim that $\Gamma_+ = \emptyset$ and $\Gamma_- = \emptyset$. Then $u \equiv v$. Indeed, integrating (3.5.21) over Γ_+ yields

$$\int_{\partial\Gamma_+} \left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} \right) + (\mu_1 - \beta_{12}) \int_{\Gamma_+} uv(u^2 - v^2) = 0, \tag{3.5.22}$$

where ν denotes the unit outward normal to $\partial\Gamma_+$. Note that $u - v > 0$ in Γ_+ and $u = v > 0$ on $\partial\Gamma_+$, we get

$$\left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} \right) \leq 0 \quad \text{on } \partial\Gamma_+.$$

Hence the first term of (3.5.22) is non-positive. Since $\beta_{12} > \max\{\mu_1, \mu_2\}$ the second term of (3.5.22) is also non-positive. Therefore,

$$\int_{\Gamma_+} uv(u^2 - v^2) = 0,$$

which implies $\Gamma_+ = \emptyset$. By a similar argument, we can prove that $\Gamma_- = \emptyset$.

Step 3: We claim that $u_3 = bu$ for some positive constant b. Indeed, (u, u_3) satisfies

$$\begin{cases} \Delta u - \lambda u + (\mu_1 + \beta_{12} a^2) u^3 + \beta_{13} u_3^2 u = 0 \quad \text{in } \mathbb{R}^N, \\ \Delta u_3 - \lambda u_3 + \mu_3 u_3^3 + (\beta_{13} + \beta_{23} a^2) u^2 u_3 = 0 \quad \text{in } \mathbb{R}^N. \end{cases} \tag{3.5.23}$$

Define $w = b^{-1} u_3$, then (u, w) satisfies

$$\begin{cases} \Delta u - \lambda u + (\mu_1 + \beta_{12} a^2) u^3 + \beta_{13} b^2 w^2 u = 0 \quad \text{in } \mathbb{R}^N, \\ \Delta w - \lambda w + \mu_3 b^2 w^3 + (\beta_{13} + \beta_{23} a^2) u^2 w = 0 \quad \text{in } \mathbb{R}^N. \end{cases} \tag{3.5.24}$$

Multiplying the first equation above by w and second one by u, we get

$$\mathrm{div}(w\nabla u - u\nabla w) + (\mu_1 + \beta_{12}a^2 - \beta_{13} - \beta_{23}a^2)u^3 w - (\mu_3 - \beta_{13})b^2 u w^3 = 0. \quad (3.5.25)$$

Since $B \in \mathcal{B}$ and $\mu_{13} > \mu_3$, there exists $b > 0$ such that

$$\mu_1 + \beta_{12}a^2 - \beta_{13} - \beta_{23}a^2 = (\mu_3 - \beta_{13})b^2. \quad (3.5.26)$$

Therefore, we have

$$\mathrm{div}(w\nabla u - u\nabla w) + (\mu_3 - \beta_{13})b^2 uw(u^2 - w^2) = 0. \quad (3.5.27)$$

Repeat the argument in Step 1 and Step 2, we get the desired result. □

Remark 3.15. Similar argument can be applied to the case $\beta_{12} = \beta_{13}$ or $\beta_{12} = \beta_{23}$.

Chapter 4

Traveling wave solutions

In this chapter, we study the positive traveling wave solutions for the semi-linear parabolic equation

$$u_t = \Delta u - u + u^p \quad \text{in } (0, \infty) \times \mathbb{R}^{N+1}, \quad N \geq 1,$$

in the form

$$u(t, x) = v(x', x_{N+1} - ct), \quad (x', x_{N+1}) \in \mathbb{R}^{N+1}.$$

Some new examples are constructed. The first one is that of a traveling wave solution with one convex non planar front. The second one is that with one non convex front. The third one is that with two non planar fronts. Our approach explains the difference between two dimension and higher dimensions, and also explores a connection between moving fronts and the mean curvature flow. The main tool is the infinite dimensional Lyaponov-Schmidt reduction, which have been well developed in the last three decades.

4.1 Introduction

Traveling wave solutions play an important role in nonlinear science. These solutions may well describe various phenomena in nature, such as vibrations, solitons and propagation with a finite speed, etc. In mathematics, they form a specially important class of time-global solutions of evolution equations. For a recent account of the theory we refer the reader to the survey article [89], especially on the stability theory.

In this chapter we consider the traveling wave solutions in the homogeneous case, for the semi-linear parabolic equation

$$\frac{\partial u}{\partial t} = \Delta u + f(u), \quad x \in \mathbb{R}^{N+1},\ t > 0, \tag{4.1.1}$$

where u is a (unknown) real-valued function on $\mathbb{R}^{N+1}$ so that the level sets of u are N dimensional.

Let us start with the *planar traveling wave solution*, which propagates in a given unit direction $\vec{e}$ with a speed c. Two properties characterize such solutions: their level sets are parallel hyperplanes which are orthogonal to the direction $\vec{e}$ and the solution is invariant in the moving frame with speed c in the direction $\vec{e}$. Then it can be written as $u(t,x) = U(\vec{e}\cdot x - ct)$. The profile U satisfies the ordinary differential equation

$$U'' + cU + f(U) = 0 \quad \text{in } \mathbb{R}.$$

Existence and possible uniqueness of such solutions are well-known and depend upon the profile of the function f, see for instance, [9, 42, 52].

Recently, the non planar traveling waves have been well studied for the reaction diffusion equations. For a recent account of the theory, we refer the reader to [14], where a generalization of travelling wave solutions is introduced. More generally speaking, waves with multiple transitions can be defined as follows:

Definition 4.1 ([14]). *Let $k \geq 1$ be a given integer and let $u_1, \ldots, u_k$ be k time-global classical solutions of* (4.1.1). *A generalized transition wave (or traveling* *wave solution) between $u_1, \dots, u_k$ is a time-global classical solution u of* (4.1.1) *such that $u \neq u_j$ for all $1 \le j \le k$, and there exist k families $(\Omega_t^j)_{t\in\mathbb{R}}$, $1 \le j \le k$ of open pairwise disjoint nonempty subsets of Ω and a family $(\Gamma_t)_{t\in\mathbb{R}}$ of nonempty subsets of Ω, such that*

$$\begin{cases} \forall t \in \mathbb{R}, \quad \bigcup_{1\le j\le k} \left(\partial\Omega_t^j \cap \Omega\right) = \Gamma_t, \quad \Gamma_t \bigcup \bigcup_{1\le j\le k} \Omega_t^j = \Omega, \\ \forall 1 \le j \le k, \quad \sup\left\{d_\Omega(x,\Gamma_t) \,\middle|\, t\in\mathbb{R},\ x\in\Omega_t^j\right\} = +\infty, \end{cases} \tag{4.1.2}$$

and

$$\begin{aligned} u(t,x) - u_j(t,x) \to 0 \quad &\text{uniformly in } t\in\mathbb{R} \text{ and } x\in\overline{\Omega_t^j} \\ &\text{as} \quad d_\Omega(x,\Gamma_t)\to+\infty, \quad \text{for all } 1\le j\le p. \end{aligned} \tag{4.1.3}$$

In the particular case where $k = 1$ and Γ_t is a singleton in Definition 4.1, u is called a *localized pulse*. In the following the set Γ_t will be called *traveling front* or *front*.

In the following we are concerned with the case

$$f(u) = -u + |u|^{p-1}u,$$

which appears in various nonlinear equations, such as the nonlinear Schrödinger equation and the Gray-Scott or Gierer-Meinhardt systems in Turing's biological theory of pattern formation (cf. [94]). Namely, we study the traveling wave solutions of

$$\frac{\partial u}{\partial t} = \Delta u - u + |u|^{p-1}u, \quad x\in\mathbb{R}^{N+1},\ t>0.$$

As a first step, we look for traveling wave solutions in the following form:

$$u(t,x) = v(x', x_{N+1} - ct), \quad x = (x', x_{N+1}) \in \mathbb{R}^{N+1}, \tag{4.1.4}$$

which is called *curved travelling fronts* in [14]. Then the profile v satisfies

$$\Delta v + c\,\partial_{N+1}v - v + |v|^{p-1}v = 0 \quad \text{in } \mathbb{R}^{N+1}. \tag{TWc}$$

In particular, we say the traveling wave u is *stationary* if it does not depend on t, i.e., u satisfies

$$\Delta u - u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^{N+1}. \tag{SWc}$$

For stationary wave solutions (sometimes also called standing wave solutions), there is a surprisingly rich and very interesting structure. First, the solutions of (SWc) which are positive and decay to zero at infinity are well understood. Secondly, the solutions of (SWc) which decay to zero at infinity but change sign have a more complicated structure. Much less is known about solutions which are defined in the entire space and which do not decay to zero at infinity uniformly. Entire solutions of (SWc) are known to be bounded thanks to [82]. Observe that the solution of (SWc) can be trivially extended as a solution of (SWc) which is defined in $\mathbb{R}^{N+1}$ and which only depends on N variables. Starting from the unique positive radially symmetric H^1 solution in $\mathbb{R}^N$, a new class of entire, positive solutions has been discovered by N. Dancer [29] using a bifurcation argument. Later more positive entire solutions are constructed, see for instance [70, 34, 85]. These results provides a surprising link between the solutions of the (continuous or discrete) Toda type system and entire solutions of above semi-linear elliptic equation. In particular, in [34] the authors construct a new class of positive entire solutions of (SWc) in $\mathbb{R}^2$ when $p > 2$. These solutions are close to the function

$$\sum_{j=1}^{k} w\big(\mathrm{dist}(\cdot, \gamma_j)\big),$$

where w is the unique positive even solution of

$$w'' - w + w^p = 0, \quad w \in H^1(\mathbb{R}), \tag{4.1.5}$$

and $\gamma_j = \big\{(x, z) \in \mathbb{R}^2 \,\big|\, x = f_j(z)\big\}$ are embedded curves which are asymptotic to oriented half lines at infinity. Moreover, f_j's satisfies a Toda system:

$$c_p^2 f_j'' = e^{f_{j-1} - f_j} - e^{f_j - f_{j+1}}, \quad j = 1, \ldots, k. \tag{4.1.6}$$

where we agree that $f_0 = -\infty$, $f_{k+1} = +\infty$ and $c_p > 0$ is an explicit constant.

The objective of this chapter is to show that a similar construction can be obtained for the positive traveling wave solutions of (TWc). Rough speaking, three new kinds of traveling wave solutions are constructed. The first one is that a traveling wave solution with one convex non planar front. The second one is that with one non convex front. The third one is that with two non planar fronts.

To introduce our results in this chapter more precisely, we should mention some known facts about the relation between traveling wave solutions of semi-linear parabolic equation and the so-called self-translating solutions (also called eternal solutions) to the mean curvature flow. Consider the traveling wave solutions to the parabolic Allen-Cahn equation of the form (4.1.4), that is, the solutions of

$$\Delta v + c\partial_{N+1} v + v - v^3 = 0 \quad \text{in } \mathbb{R}^{N+1}. \tag{4.1.7}$$

In [21], cylindrically symmetric traveling waves with paraboloid like interfaces are constructed for $N \geq 2$ and that with hyperbolic cosine like interface is constructed for $N = 1$. It is also shown that the asymptotic shape of the interfaces (level sets) are related to mean curvature flow. Moreover, there is a monotonicity condition on v in [21], so the traveling fronts in all cases are connected, convex surfaces. Recently, in [36] the authors construct traveling wave solutions with multiple and non convex fronts for $N \geq 2$. Their approach explores a connection between traveling wave solutions of parabolic Allen-Cahn equation and eternal solutions to the mean curvature flow. More precisely, the first example of their construction is that of a traveling wave solution with two non planar fronts that move with the same speed. The second example in [36] is a traveling wave solution with a non convex moving front.

The objective of this chapter is to show that a similar construction can be obtained for the positive traveling wave solutions of (TWc). To explain the difference between the study of Allen-Cahn equation and that of (TWc), we consider the one dimensional case first, which are the basic models in both constructions. It is known that the heteroclinic solution to the Allen-Cahn equation is stable and has only one bound element in its kernel. However, the one dimensional bump to the nonlinear Schrödinger equation is unstable. It is a mountain-pass type solution and has Morse index one. Hence resonance phenomena may occurs for the nonlinear Schrödinger equation.

Now we review some known facts about the eternal solutions to the mean curvature flow. In general, we say that an evolving in time family of surfaces moves by mean curvature if the following is satisfied:

$$V = \mathbf{H},$$

where V is the normal velocity of the surface and $\mathbf{H}$ denotes the mean curvature vector. Self-translating solutions are represented by surfaces that do not change shape and are translated by the mean curvature (MC) flow in a fixed direction and with constant velocity. After a rigid motion and rescaling we may assume that a translating solution of the MC flow is represented by a family of surfaces $\{\Gamma + cte_{N+1}\}_{t\in\mathbb{R}}$, where Γ is a fixed surface and $c \in \mathbb{R}$ is a fixed number. From this Γ must satisfies

$$H = c\nu_{N+1}, \tag{4.1.8}$$

where H is the mean curvature and $\vec{\nu}$ is the unit normal vector of the (oriented) surface Γ (here $\mathbf{H} = H\vec{\nu}$).

Fix a surface Γ for which (4.1.8) holds and such that $c = 1$. Let us define its scaling Γ_ε by

$$y \in \Gamma_\varepsilon \Longleftrightarrow \varepsilon y \in \Gamma, \tag{4.1.9}$$

and denote the mean curvature of Γ_ε by H_{Γ_ε}. Then,

$$H_{\Gamma_\varepsilon} = \varepsilon\nu_{N+1}. \tag{4.1.10}$$

In this chapter we will consider ε to be a small parameter, or in other words, we will be interested in translating solutions of the MC flow moving with a small speed.

Several examples of translating solution to the MC equation are known, see for example [2, 24] and the references therein. Here we will discuss a special eternal solution of the mean curvature flow for which Γ is a graph of a smooth function $F : \mathbb{R}^N \to \mathbb{R}$, that is, $\Gamma = \{(x', F(x')), x' \in \mathbb{R}^N\}$. In this case (4.1.8) reduces to

$$\nabla\left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}}\right) = \frac{1}{\sqrt{1+|\nabla F|^2}}. \tag{4.1.11}$$

It is known from [2, 24] that there exists a unique rotationally symmetric solution F of (4.3.1) with the following asymptotic behavior:

$$F(r) = \frac{r^2}{2(N-1)} - \log r + 1 + O(r^{-1}), \quad r \gg 1.$$

In what follows we will denote the rotationally symmetric translating solution of the MC flow by Γ and the corresponding scaled surface by Γ_ε, i.e.,

$$\Gamma_\varepsilon = \{x_{N+1} = \varepsilon^{-1}F(\varepsilon r)\}.$$

The first result in this chapter concerns the existence of a positive traveling wave solution to (TWc) with one convex non planar front.

Theorem 4.1. *Suppose $N \geq 2$. Then for each sufficiently small ε, the traveling wave problem* (TWc) *has a solution v_ε moving with speed $c = \varepsilon$, and with only one front, which is a rotationally symmetric paraboloid-like hypersurface.*

Our second result is about existence of a traveling wave solution to (TWc) which has two traveling fronts, each of which is asymptotically a paraboloid-like surface in a neighborhood of the rotationally symmetric eternal solution to the mean curvature flow.

Theorem 4.2. *Suppose $N \geq 2$. Then for each sufficiently small ε,* (TWc) *has a traveling wave solution v_ε moving with speed $c = \varepsilon$, and with the following properties:*

(1) the fronts of v_ε consists of two disjoint, rotationally symmetric and convex hypersurfaces $\Gamma_\varepsilon^\pm$.

(2) For any $r > 0$, let C_r be the cylinder $C_r = \{(x', x_{N+1} \,|\, |x'| \le r\}$. Let $\Gamma_\varepsilon^{\pm}(r) = \Gamma_\varepsilon^{\pm} \setminus C_r$, and similarly $\Gamma_\varepsilon(r) = \Gamma_\varepsilon \setminus C_r$. Here Γ_ε is the eternal solution of mean curvature flow with $c = \varepsilon$. Then it holds:

$$d(\Gamma_\varepsilon^{\pm}, \Gamma_\varepsilon^{\pm}) = O\Big(\log\big(\frac{1+\varepsilon^2 r^2}{\varepsilon^2}\big)\Big), \tag{4.1.12}$$

where d is the Hausdorff distance between sets.

Our third result is denoted to prove the existence of traveling wave solutions whose traveling front are non convex surfaces.

Theorem 4.3. *Suppose $N \ge 2$. Then for each sufficiently small ε, the traveling wave problem* (TWc) *has a solution v_ε moving with speed $c = \varepsilon$, and with only one front, which is a rotationally symmetric non convex hypersurface.*

Our last result is denoted to the study of traveling wave solutions in two dimension.

Theorem 4.4. *Suppose $N = 1$. Then for each sufficiently small ε, the traveling wave problem* (TWc) *has a solution v_ε moving with speed $c = \varepsilon$, and with only one front, which is a rotationally symmetric hyperbolic cosine like hypersurface.*

The existence results in this chapter explain the complicated bifurcation structure of traveling wave solutions to (TWc). In the following we will focus on the second result since it is the most complicated one.

This chapter is organized as follows. In Section 4.2 we explore on the formal level the relation between the traveling wave solutions to (TWc) and the eternal solutions to the mean curvature flow and introduce the Jacobi-Toda system for the moving fronts. In Section 4.3 we review some known results on the eternal solutions to MC flow. In Section 4.4 we study of the Jacobi-Toda system and its linearization. Section 4.5, 4.6 and 4.7 is denoted to carry out the infinite dimensional Lyapunov-Schmidt reduction to prove Theorem 4.2.

4.2 Preliminaries and motivations

To understand the role played by the mean curvature flow and the Jacobi-Toda system in the existence of traveling wave solutions with multiple traveling fronts, we first introduce some important notations and tools from Riemannian geometry. Secondly, after describing a model for the traveling wave solutions with multi fronts, we will derive formally the Jacobi-Toda system. The notations and many calculations presented here will be used throughout the chapter.

4.2.1 Geometric background

In this section, we assume that $N \geq 1$ and that Γ is an oriented smooth hypersurface embedded in the $(N+1)$ dimensional Euclidean space $\mathbb{R}^{N+1}$, which separates $\mathbb{R}^{N+1}$ into two different connected components in the sense that Γ is the zero set of a smooth function for which 0 is a regular value.

The first important tool is the use of *Fermi coordinates* to parameterize a neighborhood of Γ in $\mathbb{R}^{N+1}$.

Denote by $\vec{\nu}$ the unit normal vector field on Γ which defines the orientation of Γ. We define

$$\mathfrak{X}(y,z) = y + z\,\vec{\nu}(y),$$

where $y \in \Gamma$ and $z \in \mathbb{R}$. The implicit function theorem implies that $\mathfrak{X}$ is a local diffeomorphism from a neighborhood of a point $(y,0) \in \Gamma\times\mathbb{R}$ onto a neighborhood of $y \in \mathbb{R}^{N+1}$.

Given $z \in \mathbb{R}$, we define Γ_z by

$$\Gamma_z = \left\{\mathfrak{X}(y,z) \in \mathbb{R}^{N+1} \,|\, y \in \Gamma\right\}.$$

Observe that for z small enough (depending on y), Γ_z restricted to a neighborhood of y is a smooth hypersurface which will be referred to as the *hypersurface parallel to* Γ *at height* z. The induced metric on Γ_z will be denoted by g_z.

The first result in this section is a consequence of Gauss's Lemma. It gives the expression of the metric g of $\mathbb{R}^{N+1}$ parameterized by $\mathfrak{X}$. For a treatment of a more general case we refer the reader to the book "Tubes" of Alfred Gray [45].

Lemma 4.1. *The metric g of $\mathbb{R}^{N+1}$ parameterized by $\mathfrak{X}$, in a tubular neighborhood, is*

$$\mathfrak{X}^* g = g_z + dz^2,$$

where

$$\begin{aligned} &g_z = g_0 - 2zA_\Gamma + z^2 A_\Gamma \otimes A_\Gamma, \\ &A_\Gamma(t_1, t_2) = -g_0(\nabla^g_{t_1}\vec{\nu}, t_2), \quad A_\Gamma \otimes A_\Gamma(t_1, t_2) = g_0(\nabla^g_{t_1}\vec{\nu}, \nabla^g_{t_2}\vec{\nu}). \end{aligned}$$

Here g_0, A_Γ are the induced metric and second fundamental form on Γ, respectively.

Moreover, the mean curvature H_z of Γ_z for z small, has the explicit formula

$$H_z = H_\Gamma + z|A_\Gamma|^2 + z^2 R_z, \tag{4.2.1}$$

where H_Γ is the mean curvature of Γ and

$$R_z = \sum_{m=2}^{\infty} \Big(\sum_{j=1}^{N} k_j^{m+1} \Big) z^{m-2}.$$

Here we denote by k_j's the principal curvatures of Γ.

Recall that the Laplace-Beltrami operator is given by

$$\Delta_g = \frac{1}{\sqrt{|g|}} \partial_{x_i} \Big(g^{ij} \sqrt{|g|}\, \partial_{x_j} \Big)$$

in local coordinates. Therefore, in a tubular neighborhood of Γ, the Euclidean Laplacian Δ in $\mathbb{R}^{N+1}$ can be expressed in Fermi coordinates by the well-known formula

$$\Delta = \partial_z^2 - H_z \partial_z + \Delta_{g_z}. \tag{4.2.2}$$

Denote π_{N+1} be the projection on the $N+1$-th coordinate, then the Euclidean $N+1$-th partial derivative ∂_{N+1} has the expression

$$\partial_{N+1} = \nu_{N+1} \partial_z + \nabla^{g_z} \pi_{N+1} \cdot \nabla^{g_z}, \tag{4.2.3}$$

in Fermi coordinates.

Finally, for future reference, consider the scaled version of Γ_ε and denote its parametrization and the unit normal by $q_\varepsilon, \vec{\nu}_\varepsilon$, respectively. It is easy to see that the following relations hold:

$$q_\varepsilon(y) = \varepsilon^{-1} q(\varepsilon y), \quad \vec{\nu}_\varepsilon(y) = \vec{\nu}(\varepsilon y). \tag{4.2.4}$$

Therefore,

$$g_{\Gamma_\varepsilon, z}(y) = g_{\Gamma, z}(\varepsilon y), \quad \nabla^{g_{\Gamma_\varepsilon, z}}(y) = \varepsilon \nabla^{g_{\Gamma, z}}(\varepsilon y), \quad \Delta_{g_{\Gamma_\varepsilon, z}}(y) = \varepsilon^2 \Delta_{g_{\Gamma, z}}(\varepsilon y),$$
$$H_{\Gamma_\varepsilon}(y) = \varepsilon H_\Gamma(\varepsilon y), \quad |A_{\Gamma_\varepsilon}|^2(y) = \varepsilon^2 |A_\Gamma|^2(\varepsilon y), \quad k_{\Gamma_\varepsilon, j}(y) = \varepsilon k_{\Gamma, j}(\varepsilon y).$$

4.2.2 A model for the traveling wave solutions with multi fronts

In this section we will first describe a model for the traveling wave solutions with multi fronts to (TWc), where $c = \varepsilon$ is considered to be a small parameter. Then using the Fermi coordinates and the expansions of operators in the previous section, we explore formally the relation between the traveling wave solutions to (TWc) and the Jacobi-Toda system defined on an eternal solutions to the MC flow.

Let w be the unique positive and even solution of

$$w'' - w + w^p = 0 \quad \text{in } \mathbb{R}. \tag{4.2.5}$$

For future reference let us recall that

$$w(t) = \big(\frac{p+1}{2}\big)^{\frac{1}{p-1}} \Big[\cosh\big(\frac{p-1}{2} t\big)\Big]^{-\frac{2}{p-1}}.$$

We agree that Γ is a smooth embedded hypersurface in $\mathbb{R}^{N+1}$ and let

$$\zeta_j : \Gamma \to \mathbb{R}, \quad j = 1, \ldots, k, \ k \geq 1,$$

be any (sufficiently small) smooth functions such that $\zeta_j < \zeta_{j+1}$. Now we assume that, in a tubular neighborhood of Γ_ε,

$$\bar{v} = \mathfrak{X}^* v(y, z) = v_\varepsilon(y, z) + \phi(y, z), \tag{4.2.6}$$

where

$$v_\varepsilon(y,z)=\sum_{j=1}^{k} w_{\varepsilon,j}(y,z), \quad w_{\varepsilon,j}(y,z)=w\big(z-\zeta_j(\varepsilon y)\big), \tag{4.2.7}$$

and ϕ is a small perturbation. Later on we will have to be more specific about the way the approximate solution is defined.

For simplicity of notation, we use the same $S_\varepsilon(v)$ for the nonlinear operator in (TWc) regardless of the coordinates. Using the expressions of operators in Fermi coordinates in the previous section, in a tubular neighborhood of Γ_ε,

$$\begin{aligned} S_\varepsilon(w_{\varepsilon,j})= &\ \big(\partial_z^2 w_{\varepsilon,j}-w_{\varepsilon,j}+w_{\varepsilon,j}^p\big)\\ &+\big(\varepsilon\nu_{\varepsilon,N+1}-H_{\Gamma_\varepsilon,z}\big)\partial_z w_{\varepsilon,j}\\ &+\Delta_{g_{\Gamma_\varepsilon,z}} w_{\varepsilon,j}+\varepsilon\nabla^{g_{\Gamma_\varepsilon,z}}\pi_{\varepsilon,N+1}\cdot\nabla^{g_{\Gamma_\varepsilon,z}} w_{\varepsilon,j}. \end{aligned}$$

Direct computation yields

$$\partial_z^2 w_{\varepsilon,j}-w_{\varepsilon,j}+w_{\varepsilon,j}^p=(w''-w+w^p)\big(z-\zeta_j(\varepsilon y)\big)=0,$$

$$\begin{aligned} \big(\varepsilon\nu_{\varepsilon,N+1}-H_{\Gamma_\varepsilon,z}\big)\partial_z w_{\varepsilon,j}= &\ \varepsilon(\nu_{N+1}-H_\Gamma)(\varepsilon y)w'\big(z-\zeta_j(\varepsilon y)\big)\\ &-\varepsilon^2\big(z|A_\Gamma|^2+O(\varepsilon)\big)(\varepsilon y)w'\big(z-\zeta_j(\varepsilon y)\big), \end{aligned}$$

$$\Delta_{g_{\Gamma_\varepsilon,z}} w_{\varepsilon,j}=-\varepsilon^2(\Delta_{g_z}\zeta_j)(\varepsilon y)w'\big(z-\zeta_j(\varepsilon y)\big)+\varepsilon^2|\nabla_{g_z}\zeta_j|^2(\varepsilon y)w''\big(z-\zeta_j(\varepsilon y)\big),$$

and

$$\varepsilon\nabla^{g_{\Gamma_\varepsilon,z}}\pi_{\varepsilon,N+1}\cdot\nabla^{g_{\Gamma_\varepsilon,z}} w_{\varepsilon,j}=-\varepsilon^2\,\nabla_{g_z}\pi_{N+1}\cdot\big(\nabla_{g_z}\zeta_j\big)(\varepsilon y)\,w'\big(z-\zeta_j(\varepsilon y)\big).$$

From the above computations, since we assume that $w_{\varepsilon,j}$ is a good approximation, it is reasonable to choose Γ such that $H_\Gamma=\nu_{N+1}$, i.e., Γ is an eternal solution of the mean curvature flow. In the following, we will agree this choice.

To obtain a traveling wave solution to (TWc) near our model v_ε, at least in a tubular neighborhood of Γ_ε, a standard way is to apply the method of Lyapunov-Schmidt reduction. To apply this method, it is needed to know the kernel of the linearized operator associated to v_ε. Since near each front w is a model of v_ε, we first review the known facts about the linearized operator associated to w.

The linearized operator of (4.2.5) about w is given by

$$L_0 = \partial_t^2 - 1 + pw^{p-1}. \tag{4.2.8}$$

It is known that L_0 has a unique principal eigenvalue $\lambda_1 > 0$ and $\lambda_2 = 0$ while the rest of the spectrum is strictly negative, see for example [34]. For future reference let us recall that

$$\lambda_1 = \frac{1}{4}(p-1)(p+3),$$

and the function

$$Z = \frac{w^{(p+1)/2}}{\sqrt{\int_{\mathbb{R}} w^{p+1}\, dx}}$$

is a positive eigenfunction associated to λ_1.

As already mentioned, the discussion to follow is based on the understanding of the kernel of the operator

$$L_* = \partial_t^2 + \Delta_{\mathbb{R}^N} - 1 + pw^{p-1}, \tag{4.2.9}$$

which is now acting on functions defined on the product space $\mathbb{R} \times \mathbb{R}^N$. It is easy to check that both the functions $w'(t)$ and $\Phi(y)Z(t)$ are in the kernel of L_*, where $\Phi(y)$ is a bounded radial solution of

$$\Delta_{\mathbb{R}^N}\Phi + \lambda_1\Phi = 0 \quad \text{in } \mathbb{R}^N.$$

Therefore, the situation is more complicated than that of Allen-Cahn equation.

To deal with this difficulty, we introduce another family of parameter functions as in [35, 34]. Let

$$\eta_j : \Gamma \to \mathbb{R}, \quad j = 1, \ldots, k, \ k \geq 1,$$

be any (sufficiently small) smooth functions with the property:

$$\|\eta_j\| \leq \varepsilon^{2+\kappa}, \tag{4.2.10}$$

where the norm and a small number κ will be chosen later on. Now we consider the approximate solution of (TWc) is a tubular neighborhood of Γ_ε,

$$\bar{v_\varepsilon} = \sum_{j=1}^{k} \Big[w_{\varepsilon,j}(y,z) + \eta_j(\varepsilon y) Z_{\varepsilon,j}(y,z) \Big], \quad Z_{\varepsilon,j}(y,z) = Z\big(z - \zeta_j(\varepsilon y)\big). \tag{4.2.11}$$

We will look for a solution of (TWc) in the form:

$$v = \bar{v}_\varepsilon + \phi. \tag{4.2.12}$$

Substituting into (TWc) with $c = \varepsilon$ we get for the function ϕ,

$$\Delta\phi + \varepsilon\partial_{N+1}\phi + f'(\bar{v}_\varepsilon)\phi = -S_\varepsilon(\bar{v}_\varepsilon) - N(\phi), \tag{4.2.13}$$

where $f(u) = -u + u^p$, and

$$S_\varepsilon(\bar{v}_\varepsilon) = \Delta\bar{v}_\varepsilon + \varepsilon\partial_{N+1}\bar{v}_\varepsilon + f(\bar{v}_\varepsilon),\ N(\phi) = f(\bar{v}_\varepsilon + \phi) - f(\bar{v}_\varepsilon) - f'(\bar{v}_\varepsilon)\phi.$$

For future references let us denote as well:

$$\mathbb{L}(\phi) = \Delta\phi + \varepsilon\partial_{N+1}\phi + f'(\bar{v}_\varepsilon)\phi. \tag{4.2.14}$$

To solving (4.2.13) for ϕ one would like to use a fixed point argument for the operator

$$\phi \longmapsto -\mathbb{L}^{-1}\left(S(\bar{v}_\varepsilon) + N(\phi)\right),$$

provided that $\mathbb{L}$ has a uniformly (in small ε) bounded inverse in a suitable function space. To explain the theory we will need let us observe that locally, that is near Γ_ε, for small ε the linear operator $\mathbb{L}$ resembles the following form:

$$L_\varepsilon\phi = \Delta_{\Gamma_\varepsilon}\phi + \varepsilon\nabla_{\Gamma_\varepsilon}\pi_{\varepsilon,N+1}\cdot\nabla_{\Gamma_\varepsilon}\phi + \partial_z^2\phi + f'(\bar{v}_\varepsilon)\phi,$$

where $\nabla_{\Gamma_\varepsilon}, \Delta_{\Gamma_\varepsilon}$ are the gradient vector field and Laplace-Betrami operator on Γ_ε respectively. Observe that

$$L_\varepsilon(w') = o(1), \quad L_\varepsilon(\Phi Z) = o(1),$$

and consequently we do not expect to find a uniformly bounded inverse of $\mathbb{L}$ without introducing some restriction on its range. In this chapter we deal with this difficulty using a version of infinite Lyapunov-Schmidt reduction (cf. [35, 34]).

The essence of this method is to introduce a function $c(y)$ and $d(y)$, $y \in \Sigma_\varepsilon$ and consider the following problem:

$$\begin{cases} L_\varepsilon\phi = -S_\varepsilon(\bar{v}_\varepsilon) - N(\phi) + c(y)w'(z) + d(y)Z(z), \quad \text{in } \Sigma_\varepsilon \times \mathbb{R}, \\ \int_{\mathbb{R}} \phi(y,z)w'(z)\,dz = 0 = \int_{\mathbb{R}} \phi(y,z)Z(z)\,dz, \quad \text{for all } y \in \Gamma_\varepsilon. \end{cases} \tag{4.2.15}$$

Recall that the ansatz $\bar{v}_\varepsilon$ depends on, still undetermined, functions ζ_j's and η_j's, $j = 1, \ldots, k$. Solving (4.2.15) for given ζ_j's and η_j's and then adjusting them in such a way that

$$c(y; \zeta_j, \eta_j) = 0 = d(y; \zeta_j, \eta_j), \quad \forall y \in \Sigma_\varepsilon, \tag{4.2.16}$$

we get a solution of (TWc). Actually, the following extra steps are needed to solve (TWc): (a) gluing the local (inner) solution of (4.2.15) and a suitable outer solution; (b) a fixed point argument to solve (4.2.15); (c) solve system (4.2.16), called here the reduced problem. It is a nonlocal PDE system for ζ_j's and η_j's and its solvability is a nontrivial step extensively in this chapter.

Next we explore formally the relation between the traveling wave solutions to (TWc) and the Jacobi-Toda system. Because of the L^2-orthogonality of w' and Z, the reduced problem (4.2.16) is equivalent to

$$\begin{cases} \int_{\mathbb{R}}(L_\varepsilon\phi)(y,z)w'(z)\,dz + \int_{\mathbb{R}} S_\varepsilon(\bar{v}_\varepsilon)(y,z)w'(z)\,dz + \int_{\mathbb{R}} N(\phi)(y,z)w'(z)\,dz = 0, \\ \int_{\mathbb{R}}(L_\varepsilon\phi)(y,z)Z(z)\,dz + \int_{\mathbb{R}} S_\varepsilon(\bar{v}_\varepsilon)(y,z)Z(z)\,dz + \int_{\mathbb{R}} N(\phi)(y,z)Z(z)\,dz = 0. \end{cases}$$

Neglecting formally terms involving $N(\phi)$ and $L_\varepsilon(\phi)$, which should be of lower order, this condition reads:

$$\int_{\mathbb{R}} S_\varepsilon(\bar{v}_\varepsilon)(y,z)Z_{\varepsilon,j}(y,z)\,dz \approx 0, \quad j = 1, \ldots, k,$$

and

$$\int_{\mathbb{R}} S_\varepsilon(\bar{v}_\varepsilon)(y,z)w'_{\varepsilon,j}(y,z)\,dz \approx 0, \quad j = 1, \ldots, k.$$

Using the expressions for Δ, ∂_{N+1}, and neglecting small terms (as in the previous section), we get:

$$\begin{aligned} S_\varepsilon(\bar{v}_\varepsilon) \sim\ & \partial_z^2\bar{v}_\varepsilon + f(\bar{v}_\varepsilon) \\ & +(\varepsilon\nu_{\varepsilon,N+1} - H_{\Gamma_\varepsilon})\partial_z\bar{v}_\varepsilon \\ & +\Big[(\Delta_{\Gamma_\varepsilon} - z|A_{\Gamma_\varepsilon}|^2\partial_z)\bar{v}_\varepsilon + \varepsilon\nabla_{\Gamma_\varepsilon}\bar{v}_\varepsilon\cdot\nabla_{\Gamma_\varepsilon}\pi_{\varepsilon,N+1}\Big]. \end{aligned}$$

Consecutive terms above are organized in such a way that the first term can be estimated by the definitions of $w_{\varepsilon,j}$ and $Z_{\varepsilon,j}$ as follows:

$$E_1 \sim \lambda_1\eta_j(\varepsilon\cdot)Z_{\varepsilon,j} + f(\bar{v}_\varepsilon) - f\big(w_{\varepsilon,j} + \eta_j(\varepsilon\cdot)Z_{\varepsilon,j}\big).$$

The second term is also 0 since Γ_ε is an eternal solution of the mean curvature flow translating with speed $c = \varepsilon$, and the third is of order $O(\varepsilon^2)$. In this term we will separate those parts that are parallel to $w'_{\varepsilon,j}$ and $Z_{\varepsilon,j}$ from the rest:

$$\begin{aligned} E_3 \sim\ & \varepsilon^2\left[(-\Delta_\Gamma\zeta_j - \nabla_\Gamma\pi_{N+1}\cdot\nabla_\Gamma\zeta_j - |A_\Gamma|^2\zeta_j)\,w'_{\varepsilon,j}\right] \\ & +\varepsilon^2\left(|\nabla_\Gamma\zeta_j|^2 w''_{\varepsilon,j}\right) - \varepsilon^2\left(|A_\Gamma|^2(z-\zeta_j)w'_{\varepsilon,j}\right) \\ & +\varepsilon^2\left[(\Delta_\Gamma\eta_j + \nabla_\Gamma\pi_{N+1}\cdot\nabla_\Gamma\eta_j)\,Z_{\varepsilon,j}\right] \\ & -\varepsilon^2\left(|A_\Gamma|^2(z-\zeta_j)Z'_{\varepsilon,j}\right). \end{aligned}$$

Here $w_{\varepsilon,j} = w(z-\zeta_j(\varepsilon\cdot)), w'_{\varepsilon,j} = w'(z-\zeta_j(\varepsilon\cdot))$ and $w''_{\varepsilon,j} = w''(z-\zeta_j(\varepsilon\cdot))$. Taking this formula into account, since

$$\int_{\mathbb{R}} w''w'\,dz = 0 = \int_{\mathbb{R}} z(w')^2\,dz,$$

it is not hard to show that

$$\int_{-\delta/\varepsilon}^{\delta/\varepsilon}\left(E_3 w'_{\varepsilon,j}\right)(y,z)\,dz \sim -\varepsilon^2 c_0\left(\Delta_\Gamma\zeta_j + |A_\Gamma|^2\zeta_j + \nabla_\Gamma\zeta_j\cdot\nabla_\Gamma\pi_{N+1}\right)(\varepsilon y),$$

where $c_0 = \int_{\mathbb{R}}(w')^2\,dt$.

Similarly we will separate the integrand in the second integral in E_1 into parts which are parallel to $w'_{\varepsilon,j}$ and the rest. After some elementary manipulations we find

$$f(\bar{v}_\varepsilon) - f\big(w_{\varepsilon,j} + \eta_j(\varepsilon\cdot)Z_{\varepsilon,j}\big) \sim p w_{\varepsilon,j}^{p-1}w_{\varepsilon,j-1} + p w_{\varepsilon,j}^{p-1}w_{\varepsilon,j+1}$$

since $f(u) = -u + u^p$ and the terms we have neglected turn out to have small contributions when projected onto $w'_{\varepsilon,j}$. To compute the projection let us recall the following asymptotic formula

$$w(x) = e^{-|x|} + O((\cosh x)^{-2})$$

and denoting:

$$c_1 = p \int_{\mathbb{R}} w^{p-1} w' e^t \, dt = -\int_{\mathbb{R}} w^p e^t \, dt < 0,$$

we get the following as the leading order term in the second integral in E_1:

$$\begin{aligned}
& \int_{-\delta/\varepsilon}^{\delta/\varepsilon} \Big(\big[f\big(\textstyle\sum_{i=0}^2 w_{\varepsilon,j+i-1} \big) - \sum_{i=1}^2 f\left(w_{\varepsilon,j+i-1} \right) \big] w'_{\varepsilon,j} \Big)(y,z) \, dz \\
\sim \ & \int_{-\delta/\varepsilon}^{\delta/\varepsilon} p w_{\varepsilon,j}^{p-1} w'_{\varepsilon,j} w_{\varepsilon,j-1} \, dz + \int_{-\delta/\varepsilon}^{\delta/\varepsilon} p w_{\varepsilon,j}^{p-1} w'_{\varepsilon,j} w_{\varepsilon,j+1} \, dz \\
\sim \ & \int_{-\delta/\varepsilon}^{\delta/\varepsilon} p w^{p-1}(z) w'(z) w(z + \zeta_j(\varepsilon y) - \zeta_{j-1}(\varepsilon y)) \, dz \\
& + \int_{-\delta/\varepsilon}^{\delta/\varepsilon} p w^{p-1}(z) w'(z) w(z + \zeta_j(\varepsilon y) - \zeta_{j+1}(\varepsilon y)) \, dz \\
\sim \ & c_1 \left[-e^{\zeta_{j-1} - \zeta_j} + e^{\zeta_j - \zeta_{j+1}} \right] (\varepsilon y).
\end{aligned}$$

Denoting

$$\alpha_0 = -\frac{c_0}{c_1} = \frac{\int (w')^2}{\int w^p e^t \, dt} > 0, \tag{4.2.17}$$

we find that to the leading order of $\int_{-\delta/\varepsilon}^{\delta/\varepsilon} \big(S_\varepsilon(\bar{v}_\varepsilon) w'_{\varepsilon,j} \big)(y,z) \, dz = 0$ is equivalent to:

$$\varepsilon^2 \alpha_0 \left(\Delta_\Gamma \zeta_j + \nabla_\Gamma \pi_{N+1} \cdot \nabla_\Gamma \zeta_j + |A_\Gamma|^2 \zeta_j \right) - \left[e^{\zeta_{j-1} - \zeta_j} - e^{\zeta_j - \zeta_{j+1}} \right] = \mathbf{P}(\zeta_j, \eta_j). \tag{4.2.18}$$

Similarly the leading order of $\int_{-\delta/\varepsilon}^{\delta/\varepsilon} \big(S_\varepsilon(\bar{v}_\varepsilon) Z_{\varepsilon,j} \big)(y,z) \, dz = 0$ is equivalent to:

$$\varepsilon^2 \left(\Delta_\Gamma \eta_j + \nabla_\Gamma \pi_{N+1} \cdot \nabla_\Gamma \eta_j \right) + \lambda_1 \eta_j = \mathbf{Q}(\zeta_j, \eta_j). \tag{4.2.19}$$

Remark 4.1. Using a similar argument, for $N = 1$, we can get the following type of Jacobi-Toda system:

$$\varepsilon^2 \alpha_0 (\zeta_j'' + \zeta_j') - \left[e^{\zeta_{j-1} - \zeta_j} - e^{\zeta_j - \zeta_{j+1}} \right] = 0. \tag{4.2.20}$$

4.3 Eternal solutions to the mean curvature flow

In this section, we review some results about the eternal solutions to the mean curvature flow.

First we consider the entire solutions to the mean curvature flow. Assuming that the surface Γ is given as a graph $\Gamma = \{x_N = F(x') \mid x' \in \mathbb{R}^N\}$, and that $c = 1$, we obtain that (4.1.8) is equivalent to:

$$\nabla\left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}}\right) = \frac{1}{\sqrt{1+|\nabla F|^2}} \quad \text{in } \mathbb{R}^N.$$

We will further assume that $F(x') = F(|x'|)$, i.e, Γ is rotationally symmetric. Denoting $|x'| = r$ we get:

$$\frac{F_{rr}}{1+F_r^2} + \frac{(N-1)}{r}F_r = 1 \quad \text{in } \mathbb{R}^N. \tag{4.3.1}$$

The following result is proven in [2] in the case $N = 2$ and in general in [24]:

Proposition 4.1 ([24]). *Suppose $N \geq 2$. Then there exists an entire, rotationally symmetric, and strictly convex, graphical eternal solution to the mean curvature flow (4.3.1). This solution is translating with speed* 1 *and has the following asymptotic expansion as $r \to \infty$:*

$$F(r) = \frac{r^2}{2(N-1)} - \log r + C_1 + O(r^{-1}), \tag{4.3.2}$$

where C_1 is a constant depending on $F(1)$.

Remark 4.2. The dynamically stability of these eternal solutions to the mean curvature flow have been proved in [24] and there is no decay rate imposed on initial value.

In the sequel by Γ we will denote the surface corresponding to the rotationally symmetric eternal solution described in Proposition 4.1.

Next we consider the complete non-convex translating solution to the mean curvature flow. We repeat here an existence result proven in [24] for the convenience of the reader.

Proposition 4.2 ([24]). *Suppose $N \geq 2$. For every $R > 0$, there exists rotationally symmetric, graphical solution to the mean curvature flow, W_R^+, W_R^- : $\mathbb{R}^N \setminus B_R \times [0, \infty) \to \mathbb{R}$, translating with speed 1. We have the following asymptotic expansion as r goes to infinity:*

$$W_R^{\pm}(r,t) = t + \frac{r^2}{2(N-1)} - \log r + C^{\pm} + O(r^{-1}). \tag{4.3.3}$$

Moreover, the union of these graphs forms a complete non-convex translating solution to the mean curvature flow.

In what follows by Σ we denote the non-convex surface obtained by taking the union of the graphs of $W_R^{\pm}$ and by Σ_ε we denote its scaled version.

4.4 The Jacobi-Toda system and its linearization

The general theory of solvability of the Jacobi-Toda system (4.2.18) is beyond the scope of this chapter. Here we only consider the Jacobi-Toda system on Γ (or Σ):

$$\begin{cases} \varepsilon^2 \alpha_0 \left(\Delta_\Gamma \zeta_j + \nabla_\Gamma F \cdot \nabla_\Gamma \zeta_j + |A_\Gamma|^2 \zeta_j \right) - e^{\zeta_{j-1} - \zeta_j} + e^{\zeta_j - \zeta_{j+1}} = 0 \\ j = 1, \ldots, k. \end{cases} \tag{JT}$$

Our theory of solvability of the Jacobi-Toda system will be valid for functions of the radial variable r only and so we need to express the Jacobi-Toda system on Γ in terms of the radial variable r first. For what follows it will be convenient to denote:

$$L[v] := \Delta_\Gamma v + \nabla_\Gamma F \cdot \nabla_\Gamma v + |A_\Gamma|^2 v. \tag{4.4.1}$$

Now we will find the expression of this operator when restricted to functions $v = v(r)$, i.e., functions depending on the radial variable only. The Laplace-Betrami operator for a surface $x_{N+1} = F(r)$ acting on $v = v(r)$ is

$$\Delta_\Gamma v = \frac{v_{rr}}{1 + F_r^2} + \left(\frac{N-1}{r} - \frac{F_r}{1 + F_r^2} \right) v_r.$$

The principal curvatures are given by

$$k_1 = \cdots = k_{N-1} = \frac{F_r}{r\sqrt{1+F_r^2}}, \quad k_N = \frac{F_{rr}}{(1+F_r^2)^{3/2}},$$

hence

$$|A_\Gamma|^2 = \sum_{j=1}^{N} k_j^2 = \frac{(N-1)F_r^2}{r^2(1+F_r^2)} + \frac{F_{rr}^2}{(1+F_r^2)^3}.$$

Finally we have

$$\nabla_\Gamma F \cdot \nabla_\Gamma v = \frac{F_r}{1+F_r^2} v_r.$$

Hence the expression for the operator L acting on radial functions, denoted by L_{rad}, is

$$L_{\text{rad}}[v] = \frac{v_{rr}}{1+F_r^2} + \frac{(N-1)v_r}{r} + \left[\frac{(N-1)F_r^2}{r^2(1+F_r^2)} + \frac{F_{rr}^2}{(1+F_r^2)^3}\right] v. \tag{4.4.2}$$

We will now proceed to define some weighted norms that we will use in the sequel. For radial functions the following relations hold:

$$\begin{aligned}
|\nabla_\Gamma h(r)| &\le \frac{C|\partial_r h(r)|}{\sqrt{1+|F_r(r)|^2}}, \\
|\partial_r h(r)| &\le C\sqrt{1+|F_r(r)|^2}|\nabla_\Gamma h(r)|, \\
|D_\Gamma^2 h(r)| &\le \frac{C(|\partial_r^2 h(r)| + r^{-1}|\partial_r h(r)|}{1+|F_r(r)|^2}, \\
|\partial_r^2 h(r)| &\le C(1+|F_r(r)|^2)(|D_\Gamma^2 h(r)| + |\nabla_\Gamma h(r)|),
\end{aligned}$$

where ∇_Γ is the gradient derivative vector and D_Γ^2 is the second derivative matrix on Γ.

We define the following weighted norms for $C^{2,\mu}$ function h on Γ:

$$\|h\|_{C_\beta^{0,\mu}(\Gamma)} := \sup_{y\in\Gamma}(2+|F_r(|y'|)|^2)^\beta \|h\|_{C^{0,\mu}(B(y,1)\cap\Gamma)}, \quad y = (y', y_N) \in \mathbb{R}^{N+1},$$

$$\|h\|_{C_\beta^{2,\mu}(\Gamma)} := \|h\|_{C_\beta^{0,\mu}(\Gamma)} + \|\nabla_\Gamma h\|_{C_\beta^{0,\mu}(\Gamma)} + \|D_\Gamma^2 h\|_{C_\beta^{0,\mu}(\Gamma)},$$

and for radial $C^{0,\mu}$ function g on Γ:

$$\|g\|_{C_{\beta,\nu}^{0,\mu}(\Gamma)} := \sup_{y\in\Gamma}\left\{(2+|F_r(|y'|)|^2)^\beta \left(\log\frac{2+|F_r(|y'|)|^2}{\varepsilon^2}\right)^\nu \|g\|_{C^{0,\mu}(B(y,1)\cap\Gamma)}\right\},$$

$$\|g\|_{C_{\beta,\nu}^{2,\mu}(\Gamma)} := \|h\|_{C_{\beta,\nu}^{0,\mu}(\Gamma)} + \|\nabla_\Gamma h\|_{C_{\beta,\nu}^{0,\mu}(\Gamma)} + \|D_\Gamma^2 h\|_{C_{\beta,\nu}^{0,\mu}(\Gamma)}.$$

With these weighted norms, we first consider the solvability of the Jacobi-Toda system in the case $k = 1$, i.e., consider the solvability of the linear equation on Γ:

$$L[\zeta_1] = \Delta_\Gamma \zeta_1 + \nabla_\Gamma F \cdot \nabla_\Gamma \zeta_1 + |A_\Gamma|^2 \zeta_1 = 0. \tag{4.4.3}$$

The key observation is that the equation $L[\phi] = 0$ has a decaying, positive solution

$$\phi_0 = \frac{1}{\sqrt{1+F_r^2}} \sim \frac{1}{r}, \quad r \gg 1. \tag{4.4.4}$$

from which we can solve (4.4.3) by a standard ODE method. We rewrite this observation in the following lemma, which has been proved in [36].

Lemma 4.2. *Function $\phi_0 = \frac{1}{\sqrt{1+F_r^2}}$ satisfies $L[\phi_0] = 0$, that is, ϕ_0 is a positive, decaying element in the kernel of L.*

Proof. Let us consider the nonlinear operator

$$\mathcal{H}(\Phi) = \frac{\Phi_{rr}}{1+\Phi_r^2} + (N-1)\frac{\Phi_r}{r}. \tag{4.4.5}$$

Let $\Phi_\sigma = F + \sigma\phi$, $\phi = \phi(r)$ we get

$$\frac{d}{d\sigma}\mathcal{H}(\Phi_\sigma)\big|_{\sigma=0} = \mathcal{H}'[\phi] = \frac{\phi_{rr}}{1+F_r^2} - \frac{2F_{rr}F_r\phi_r}{(1+F_r^2)^2} + \frac{(N-1)\phi_r}{r}.$$

In particular we have $\mathcal{H}'[1] = 0$. On the other hand, it is not hard to check that

$$L[\phi] = \mathcal{H}'[\phi\sqrt{1+F_r^2}].$$

From this the assertion of the lemma follows. □

The second lemma concerns the solvability of $L[v] = g$, which has also been proved in [36].

Lemma 4.3. *Let g be a $\mathcal{C}^{0,\mu}(\Gamma)$ radial function such that*

$$\|g\|_{\mathcal{C}^{0,\mu}_\beta(\Gamma)} < \infty, \quad \beta \geq 1.$$

There exists a unique, bounded solution to

$$L[v] = g, \tag{4.4.6}$$

such that:

$$\|v\|_{C^{2,\mu}_{\beta-1}(\Gamma)} \leq C\|g\|_{C^{0,\mu}_{\beta}(\Gamma)}. \tag{4.4.7}$$

Proof. This lemma is also proved in [36] by the reduction of order formula in the standard ODE theory. Here the uniqueness means we can define v in one unique form. More precisely,

$$v(r) = -\phi_0(r)\int_0^r \frac{\phi_1(\rho)\tilde{g}(\rho)}{W(\rho)} + \phi_1(r)\int_0^r \frac{\phi_0(\rho)\tilde{g}(\rho)}{W(\rho)},$$

where

$$\begin{aligned}
&\tilde{g}(r) = (1 + |F_r(r)|^2)g(r),\\
&\phi_1(r) = \phi_0(r)\textstyle\int_r^\infty (1 + |F_r(\rho)|^2)e^{-A(\rho)}\,d\rho,\\
&A(\rho) = \textstyle\int_1^\rho \frac{(N-1)(1+|F_r(\eta)|^2)}{\eta}\,d\eta.
\end{aligned}$$

□

Next we consider the Jacobi-Toda system in the case $k = 2$. Generally we consider the non-homogeneous problem:

$$\begin{cases} \varepsilon^2\alpha_0(\Delta_\Gamma\zeta_1 + \nabla_\Gamma F\cdot\nabla_\Gamma\zeta_1 + |A_\Gamma|^2\zeta_1) + e^{\zeta_1-\zeta_2} = \varepsilon^2 h_1,\\ \varepsilon^2\alpha_0(\Delta_\Gamma\zeta_2 + \nabla_\Gamma F\cdot\nabla_\Gamma\zeta_2 + |A_\Gamma|^2\zeta_2) - e^{\zeta_1-\zeta_2} = \varepsilon^2 h_2. \end{cases} \tag{4.4.8}$$

where $\zeta_j\colon \Gamma \to \mathbb{R}$. To describe the strategy let us denote

$$u = \zeta_2 - \zeta_1,\ v = \zeta_1 + \zeta_2,\ h = \frac{1}{\alpha_0}(h_2 - h_1),\ \text{and}\ g = \frac{1}{\alpha_0}(h_1 + h_2). \tag{4.4.9}$$

Then the decoupled system holds:

$$\begin{cases} \mathcal{S}_\varepsilon[u] := L[u] - \frac{2}{\varepsilon^2\alpha_0}e^{-u} = h,\\ L[v] = g. \end{cases} \tag{4.4.10}$$

The second equation has been solved by lemma 4.3. The solvability theory for the nonlinear equation in (4.4.10) is where the real difficulty lies.

We summarize here the main results in Section 3 of [36], which concerns the solvability theory of the Jacobi-Toda system.

Lemma 4.4. *The nonlinear homogeneous equation*

$$\mathcal{S}_\varepsilon[u] := L[u] - \frac{2}{\varepsilon^2\alpha_0}e^{-u} = 0, \tag{4.4.11}$$

has a bounded solution u_0 *satisfying*

$$u_0(r) = \log\frac{2}{\varepsilon^2\alpha_0|A_\Gamma(r)|^2} + \mathcal{O}(\log\log\frac{1}{\varepsilon^2|A_\Gamma(r)|^2}), \ \text{as } \varepsilon r \to 0^+, \text{or } \varepsilon r \gg 1, \tag{4.4.12}$$

where $|A_\Gamma(r)|^2$ *is the norm of the second fundamental form on* Γ*. Moreover, for the linearized operator* $\mathcal{L}_\varepsilon[\phi] := \mathcal{S}_\varepsilon'[\phi]$ *at* u_0*, suppose that* $\beta > 0$*,* $\nu > 0$*, then there exist a constant* $C > 0$ *and solution* ϕ *to* $\mathcal{L}_\varepsilon[\phi] = g$ *such that*

$$\|\phi\|_{C^{0,\mu}_{\beta,\nu}(\Gamma)} + \|\nabla_\Gamma\phi\|_{C^{0,\mu}_{\beta+1,\nu}(\Gamma)} + \|D^2_\Gamma\phi\|_{C^{0,\mu}_{\beta+1,\nu}(\Gamma)} \le C(\log\frac{1}{\varepsilon^2})^{4+2\beta}\|g\|_{C^{0,\mu}_{\beta+1,\nu+1}(\Gamma)}.$$

Remark 4.3. The asymptotic expansion is found by solving for u_0 the following equation:

$$|A_\Gamma|^2 u_0 = \frac{2}{\varepsilon^2\alpha_0}e^{-u_0}.$$

From this first approximation, the authors in [36] define a sequence of approximations by solving a sequence algebraic equations like u_0. Once an accurate enough approximation is found the nonlinear problem can be reduced to a fixed point theorem.

Using a fixed point argument as in [36] one can solve the following nonlinear and non-homogeneous problem:

$$L[u] - \frac{2}{\varepsilon^2\alpha_0}e^{-u} = h \tag{4.4.13}$$

in the following lemma:

Lemma 4.5. *Let* h *be a* $\mathcal{C}^{0,\mu}(\Gamma)$ *radial function such that*

$$\|h\|_{C^{0,\mu}_\beta(\Gamma)} \le C\varepsilon^\tau, \quad \tau > 0, \ \beta > 1.$$

Then there exists a bounded solution to (4.4.13) *satisfying* (4.4.12).

We will finish this section with a discussion of another important ODE, however not directly related to the Jacobi-Toda system considered above, plays as important role in the sequel. We consider the solvability theory of the linear equation:

$$\Delta_\Gamma \eta + \nabla_\Gamma F \cdot \nabla_\Gamma \eta + \frac{\lambda_1}{\varepsilon^2}\eta = g. \tag{4.4.14}$$

The key point is that it has a exponential decaying solution, from which we can solve (4.4.14) by a standard ODE method and have the following:

Lemma 4.6. *Let g be a $\mathcal{C}^{0,\mu}(\Gamma)$ radial function such that*

$$\|g\|_{\mathcal{C}^{0,\mu}_\beta(\Gamma)} < \infty, \quad \beta \geq 1.$$

There exist a constant $C > 0$ and bounded solution to (4.4.14) *such that:*

$$\|\eta\|_{\mathcal{C}^{0,\mu}_\beta(\Gamma)} + \varepsilon\|\nabla_\Gamma \eta\|_{\mathcal{C}^{0,\mu}_\beta(\Gamma)} + \varepsilon^2\|D^2_\Gamma \eta\|_{\mathcal{C}^{0,\mu}_\beta(\Gamma)} \leq C\|g\|_{\mathcal{C}^{0,\mu}_\beta(\Gamma)}. \tag{4.4.15}$$

Proof. The proof of this lemma follows arguments in section 3 of [36] by the reduction of order formula in the standard ODE theory. Recall that we define another parametrization of Γ, which is obtained by taking the arc length along the curve $(r, F(r))$. Thus we define

$$s = \int_0^r \sqrt{1+F_r^2}\, d\rho. \tag{4.4.16}$$

Using the asymptotic formula for F we get that

$$s \sim r, \quad r \ll 1, \quad s = \frac{r^2}{2(N-1)} + O(\log r), \quad r \gg 1. \tag{4.4.17}$$

By a straightforward computation we obtain the following expression for the operators but now with the arc-length variable s:

$$\Delta_\Gamma \eta + \nabla_\Gamma F \cdot \nabla_\Gamma \eta = \frac{\eta_{rr}}{1+F_r^2} + \frac{(N-1)\eta_r}{r} = \eta_{ss} + a(s)\eta_s. \tag{4.4.18}$$

where

$$a(s) = \frac{F_r(r(s)) + \frac{N-1}{r(s)}}{\sqrt{1+|F_r|^2(r(s))}}. \tag{4.4.19}$$

Note that

$$a(s) = \frac{N-1}{s}(1 + O(s^2)), \quad s \ll 1, \quad a(s) = 1 + O(s^{-1}), \quad s \gg 1. \tag{4.4.20}$$

Denote $L_1[\eta] = \Delta_\Gamma \eta + \nabla_\Gamma F \cdot \nabla_\Gamma \eta + \frac{\lambda_1}{\varepsilon^2}\eta$. Then given one solution ψ_0 of $L_1[\psi] = 0$ we find the second linear independent solution ψ_1 of $L_1[\psi] = 0$ by reduction of order formula:

$$\psi_1(s) = \psi_0(s) \int_s^\infty \frac{exp\left(-\int_1^\tau a(\eta)\, d\eta\right)}{\psi_0^2(\tau)}\, d\tau. \tag{4.4.21}$$

By $W(s)$ we will denote the Wronskian of ψ_0, ψ_1. By the Abel formula we have

$$W(s) = W(1) exp\left(-\int_1^s a(\tau)\, d\tau\right). \tag{4.4.22}$$

We make the following Liouville transformation

$$\hat{\eta}(s) = exp\left(\frac{1}{2}\int_1^s a(\tau)\, d\tau\right)\eta(s). \tag{4.4.23}$$

Then

$$\hat{\eta}(s) \sim s^{\frac{N-1}{2}}\eta(s), \quad \hat{\eta}(s) \sim e^{s/2}\eta(s). \tag{4.4.24}$$

Now η satisfies

$$\hat{\eta}'' + \left(\frac{\lambda_1}{\varepsilon^2} - \hat{a}(s)\right)\hat{\eta} = \hat{g}, \tag{4.4.25}$$

where

$$\hat{a} = \frac{1}{2}a' + \frac{1}{4}a^2, \quad \hat{g} = exp\left(\frac{1}{2}\int_1^s a(\tau)\, d\tau\right) g. \tag{4.4.26}$$

Let us denote

$$\hat{L}[\eta] = \eta'' + \hat{p}(s)\eta, \quad \hat{p} = \frac{\lambda_1}{\varepsilon^2} - \hat{a}(s). \tag{4.4.27}$$

When we consider the operator $\hat{L}$ for functions defined in the interval $I_1 = (0, s_1)$, for some $s_1 > 0$ then we refer to this problem as the inner problem. We speak of the out problem when we take $I_{s_\varepsilon} = (s_\varepsilon, \infty)$, $s_\varepsilon \geq s_1 > 0$ as the domain of the functions involved.

First we will describe the way we choose s_1 and s_ε. For $s \to 0$, we have

$$\hat{a}(s) = s^{-2}\left[\frac{(N-2)^2}{4} - \frac{1}{4}\right](1 + O(s^2)). \tag{4.4.28}$$

As a consequence we see that there exist an $M > 0$ and $s_1 > M\varepsilon > 0$ such that

$$\hat{p}(s) > 0, \quad M\varepsilon \leq s \leq s_1. \tag{4.4.29}$$

When $s \to \infty$ we have

$$\hat{a}(s) = \frac{1}{4} + O(s^{-1}), \tag{4.4.30}$$

with similar formula for the derivatives. Actually we have

$$\hat{a} = \left[\frac{(N-1)F_r}{r(1+F_r^2)} - \frac{1}{2}\right]^2 + \left[\frac{(N-2)^2}{4} - \frac{1}{4}\right]\frac{1}{r^2(1+F_r^2)} + \frac{1}{4(1+F_r^2)}. \tag{4.4.31}$$

Also we can use the asymptotic behavior of $\hat{a}(s)$ for s large to infer the existence of $s_2 \geq s_1$ such that for $s > s_2$ it holds

$$\hat{p}(s) \geq \frac{\lambda_1}{\varepsilon^2} - \frac{1}{4}, \quad \hat{p}'(s) \geq 0. \tag{4.4.32}$$

Observe that s_1 and s_2 in general do not coincide and we need to solve an intermediate problem to glue the inner solution and the solution for s between s_1 and s_2. Finally, we will assume that ε is chosen sufficiently small, so that

$$\hat{p}(s) > 0, \quad s_1 < s < s_2. \tag{4.4.33}$$

We first solve the inner problem:

$$\hat{L}[\eta_i] = g, \quad \text{in } I_1 = (0, s_1), \tag{4.4.34}$$

$$\eta_i(0) = 0, \quad \eta_i'(0) = 0. \tag{4.4.35}$$

Our goal is to show that there exists a unique solution η_i such that

$$\|\eta_i\|_{C^{0,\beta}} \leq C\|g\|_{C^{0,\beta}}. \tag{4.4.36}$$

For convenience we will denote $\lambda = \frac{\lambda_1}{\varepsilon^2}$. Taking into account the asymptotic behavior of $\hat{a}(s)$ when $s \to 0$ we see that the operator $\hat{L}$ can be written in the form:

$$\hat{L}[\hat{\eta}] = \hat{\eta}'' + \left[\lambda^2 - s^{-2}\left(\frac{(N-2)^2}{4} - \frac{1}{4}\right)\right](1 + O(s^2))\hat{\eta}. \tag{4.4.37}$$

It is convenient to make further change of variables setting:

$$\hat{\eta}(s) = \tilde{\eta}(\lambda s),\ \hat{g}(s) = \tilde{g}(\lambda s),\ \hat{p}(s) = \lambda^2\tilde{p}(\lambda s). \tag{4.4.38}$$

Then denoting by $\tilde{L}$ the scaled operator we have

$$\tilde{L}[\tilde{\eta}] = \tilde{\eta}'' + \left[1 - s^{-2}\left(\frac{(N-2)^2}{4} - \frac{1}{4}\right)\right](1 + O(\lambda^{-2}s^2))\tilde{\eta}, \tag{4.4.39}$$

and

$$\tilde{L}[\tilde{\eta}_i] = \lambda^{-2}\tilde{g}, \quad \text{in } I_\lambda = (0, \lambda s_1). \tag{4.4.40}$$

Formally $\tilde{L}[\tilde{\eta}] = 0$ resembles the modified Bessel equation and the operator $\tilde{L}$ should have an element of the the kernel $\tilde{\eta}_{i,1}$ such that

$$\tilde{\eta}_{i,1}(s) \sim s^{\frac{1}{2}} J_{\frac{N-2}{2}}(s), \tag{4.4.41}$$

where $J_{\frac{N-2}{2}}(s)$ is the Bessel function. The second, linearly independent element in the kernel is such that

$$\tilde{\eta}_{i,2}(s) \sim s^{\frac{1}{2}} J_{\frac{-N+2}{2}}(s), \tag{4.4.42}$$

when $\frac{N-2}{2}$ is not an integer and

$$\tilde{\eta}_{i,2}(s) \sim s^{\frac{1}{2}} Y_{\frac{N-2}{2}}(s), \tag{4.4.43}$$

when $\frac{N-2}{2}$ is an integer, where $Y_{\frac{N-2}{2}}$ is the modified Bessel function of the second kind.

We choose a solution given by

$$\tilde{\eta}_i(s) = -\lambda^{-2}\tilde{\eta}_{i,1}(s)\int_0^s \tilde{\eta}_{i,2}(\tau)\tilde{g}(\tau)\,d\tau + \lambda^{-2}\tilde{\eta}_{i,2}(s)\int_0^s \tilde{\eta}_{i,1}(\tau)\tilde{g}(\tau)\,d\tau. \tag{4.4.44}$$

Note that $\tilde{\eta}_i(0) = 0, \tilde{\eta}_i'(0) = 0$ since after the change of variables we have $\tilde{g}(s) = O(s^{\frac{N-1}{2}})$.

Now we will make a useful observation: let $\tilde{\eta}$ be a solution of $\tilde{L}[\tilde{\eta}] = 0$ in $(K, \lambda s_1)$ and consider the following expressions:

$$Q_1(\tilde{\eta}) = [\tilde{\eta}'(s)]^2 + \tilde{p}[\tilde{\eta}(s)]^2, \quad Q_2(\tilde{\eta}) = \frac{[\tilde{\eta}'(s)]^2}{\tilde{p}(s)} + [\tilde{\eta}(s)]^2. \tag{4.4.45}$$

It is easy to see that

$$\frac{d}{ds}Q_1(\tilde{\eta}) = \tilde{p}'[\tilde{\eta}]^2, \quad \frac{d}{ds}Q_2(\tilde{\eta}) = -\frac{\tilde{p}'}{[\tilde{p}]^2}[\tilde{\eta}']^2. \tag{4.4.46}$$

Now, the asymptotic formulas of $\tilde{\eta}_{i,1}$ and $\tilde{\eta}_{i,2}$ for s small and the uniform bound on $\tilde{\eta}_{i,j}$ together with the variation of parameters formula give the following bound:

$$\|s^{\frac{1-N}{2}}\tilde{\eta}_i\| \le \frac{C}{\lambda^2}\|s^{2+\frac{1-N}{2}}\tilde{g}\|. \tag{4.4.47}$$

On the other hand uniform bounds on $\tilde{\eta}_{i,j}$ yield:

$$\|s^{\frac{1-N}{2}}\tilde{\eta}_i\| \le \frac{C}{\lambda^2}\|s^{1+\frac{1-N}{2}}\tilde{g}\|. \tag{4.4.48}$$

Scaling back this estimates we get for the solution of inner problem estimates.

Since

$$0 < \frac{\lambda_1}{\varepsilon^2} - \frac{1}{4} - a \le \hat{p}(s) \le \frac{\lambda_1}{\varepsilon^2} - \frac{1}{4}, \tag{4.4.49}$$

let ξ_1, ξ_2 are two consecutive zeros of $\hat{\eta}$, then by the theory of Sturm-Liouville we have

$$\frac{\pi}{\sqrt{\frac{\lambda_1}{\varepsilon^2} - \frac{1}{4}}} \le \operatorname{dist}(\xi_1, \xi_2) \le \frac{\pi}{\sqrt{\frac{\lambda_1}{\varepsilon^2} - \frac{1}{4} - a}}. \tag{4.4.50}$$

By the similar argument above we have

$$|\hat{\eta}(s)| \le \frac{C}{\sqrt{\frac{\lambda_1}{\varepsilon^2} - \frac{1}{4}}}\int_{s_2}^{s} e^{\tau/2}\tau^{-\frac{3}{2}}\, d\tau \le \frac{C}{\sqrt{\frac{\lambda_1}{\varepsilon^2} - \frac{1}{4}}}e^{s/2}s^{-1/2}. \tag{4.4.51}$$

Therefore,

$$|\eta(s)| \le \frac{C}{\sqrt{\frac{\lambda_1}{\varepsilon^2} - \frac{1}{4}}}s^{-1/2}. \tag{4.4.52}$$

□

4.5 The infinite dimensional reduction

Let Γ be the eternal solution of the mean curvature flow with speed 1 and let Γ_ε be the corresponding surface translating with speed $c = \varepsilon \ll 1$. We will use the natural representation of Γ as a graph of the radial function $\Gamma = \{x_{N+1} = F(r)\}$. The scaled surface is given by $\Gamma_\varepsilon = \{x_{N+1} = F_\varepsilon(r) \,|\, F_\varepsilon(r) = \varepsilon^{-1}F(\varepsilon r)\}$. In general we will take advantage of the radially symmetry of the eternal solution and apply the infinite dimensional Lyapunov-Schmidt reduction, whose approach has been sketched in Section 4.2, to reduce the original PDE:

$$\Delta v + \varepsilon \partial_{N+1} v + f(v) = 0, \quad \text{in } \mathbb{R}^{N+1},$$

to a one dimensional system whose independent variable is the radial variable $r = |x'|$.

4.5.1 An infinite dimensional family of approximate solutions

We will now proceed to define an approximation of solution which depends on the radial variable $r = |x'|$ and the signed distance z to Γ_ε. We will use the notations introduced in Sections 4.2, with obvious modifications taking into account the fact that Γ_ε is radially symmetric and thus has a globally defined parameterization as follows:

$$\Gamma = \{(r\Theta, F(r)) \,|\, r > 0, \Theta \in S^{N-1}\}, \quad \Gamma_\varepsilon = \{(\varepsilon r\Theta, \varepsilon^{-1}F(\varepsilon r)) \,|\, r > 0, \Theta \in S^{N-1}\}.$$

We choose an orientation $\vec{\nu}(y)$ on Γ and take $z = z(x) = \text{dist}(x, \Gamma)$ compatible with this orientation. Let us introduce the following weight functions:

$$\omega(x) = 2 + |F_r(r)|^2, \ \omega_\varepsilon(x) = 2 + |F_r(\varepsilon r)|^2, \ x = (x', x_{N+1}), \ r = |x'|.$$

Recall that $F_r(r) \sim r$ as $r \gg 1$ for $N \geq 2$. It is not hard to show that there exists an $\eta_0 > 0$ such that for all points x such that $|z(x)| \leq \eta_0 \log \omega(r)$ the map

$$x \mapsto y + z\,\vec{\nu}(y), \quad y \in \Gamma$$

is a diffeomorphism, denoted by $\mathfrak{X}(x) = (y, z)$ and called Fermi coordinates of Γ. Similar claims are true when we consider Γ_ε and points x such that $|z(x)| \leq \frac{\eta_0}{\varepsilon}\log\omega_\varepsilon(r)$. Taking this into account we introduce the following neighborhood of Γ_ε:

$$U_{\varepsilon,M} := \{x \in \mathbb{R}^{N+1} \mid |z(x)| \leq M \log \frac{\omega_\varepsilon(r)}{\varepsilon^2}\}.$$

Clearly Fermi coordinates are well defined in $U_{\varepsilon,M}$ for all $M > 0$ large and $\varepsilon > 0$ small. If by $\mathfrak{X}_\varepsilon$ we denote the diffeomorphism in $U_{\varepsilon,M}$ defined by $\mathfrak{X}_\varepsilon(x) = (y, z)$ then for a function v defined in this neighborhood we set

$$(\mathfrak{X}_\varepsilon^* v)(y, z) = (v \circ \mathfrak{X}_\varepsilon^{-1})(y, z).$$

We will describe functions f_j representing the leading order for the location of the fronts of our traveling wave solution. In this section we consider the case of two fronts since other situation can be deal with in a similar approach (most may be more easily). Let f_j, $j = 1, 2$ to be solutions of the Jacobi-Toda system (JT). We get that f_j's satisfies

$$f_j(r) = \frac{(-1)^j}{2} \log \frac{2}{\varepsilon^2 \alpha_0 |A_\Gamma(r)|^2} + O(\log\log \frac{1}{\varepsilon^2 |A_\Gamma(r)|^2}). \tag{4.5.1}$$

In addition we have $f_1 = -f_2$. In the sequel we will use scaled versions of these functions, namely $f_{\varepsilon,j} : \Gamma_\varepsilon \to \mathbb{R}$, defined by

$$f_{\varepsilon,j}(r) = f_j(\varepsilon r), \quad r = r(y) = |y'|,\ y = (y', y_{N+1}) \in \Gamma_\varepsilon.$$

We recall here that $|A_{\Gamma_\varepsilon}(r)|^2 = \varepsilon^2 |A_\Gamma(\varepsilon r)|^2$. In the course of the Lyapunov-Schmidt scheme for our problem we further need two family of small functions, which will be for a moment unknown parameters. Thus we let h_j, e_j, $j = 1, 2$, be functions of the radial variable r on Γ such that for some $\beta, \tau \in (0, 1)$ we have

$$\|h_j\|_{C^{2,\mu}_\beta(\Gamma)} \leq \varepsilon^\tau, \quad \|e_j\|_{C^{2,\mu}_\beta(\Gamma)} \leq \varepsilon^\tau. \tag{4.5.2}$$

Then by the relation between the weighted norms on Γ and Γ_ε we get

$$\|h_{\varepsilon,j}\|_{C^{2,\mu}_\beta(\Gamma_\varepsilon)} \leq \varepsilon^\tau, \quad \|e_{\varepsilon,j}\|_{C^{2,\mu}_\beta(\Gamma_\varepsilon)} \leq \varepsilon^\tau. \tag{4.5.3}$$

Given the functions $f_{\varepsilon,j}, h_{\varepsilon,j}$ and $e_{\varepsilon,j}$ as described above we will denote

$$\mathbf{f}_\varepsilon = (f_{\varepsilon,1}, f_{\varepsilon,2}), \quad \mathbf{h}_\varepsilon = (h_{\varepsilon,1}, h_{\varepsilon,2}), \quad \mathbf{e}_\varepsilon = (e_{\varepsilon,1}, e_{\varepsilon,2}),$$

etc.

To define a proper initial approximation in the whole space. We will need various cutoff functions in our construction. Therefore, for $m = 1, \ldots, 6$, we define the cut-off function χ_m by

$$\chi_m(t) = \begin{cases} 1, & |t| < 1 - \frac{2m-1}{100}, \\ 0, & |t| > 1 - \frac{2m-2}{100}. \end{cases}$$

Now let $M > 0$ be a fixed large number and let

$$\chi_{\varepsilon,m,j}(x) = \chi_m\left(\frac{z_j(x)}{M \log \frac{\omega_\varepsilon(r)}{\varepsilon^2}}\right), \quad z_j(x) = \mathrm{dist}(x, \Gamma_{\varepsilon,j}),$$

where $\Gamma_{\varepsilon,j}$ is a normal, rotationally symmetric graph over Γ_ε defined by

$$\Gamma_{\varepsilon,j} = \left\{x = \left(r, \varepsilon^{-1}F(\varepsilon r)\right) + \zeta_{\varepsilon,j}\,\vec{\nu}_\varepsilon(r)\right\}, \quad \text{where} \quad \zeta_{\varepsilon,j} = f_{\varepsilon,j} + h_{\varepsilon,j}.$$

Based on the analysis we have done in the previous sections, taking M large and ε small, we define the initial approximation $\tilde{v}_\varepsilon$ of the solution by

$$\tilde{v}_\varepsilon(x) = \sum_{j=1}^{2} \chi_{\varepsilon,1,j}(x)\Big[w\left(z - \zeta_{\varepsilon,j}(r)\right) + \varepsilon e_{\varepsilon,j}(r) Z\left(z - \zeta_{\varepsilon,j}(r)\right)\Big], \tag{4.5.4}$$

where

$$\zeta_{\varepsilon,j}(r) = f_{\varepsilon,j}(r) + h_{\varepsilon,j}(r). \tag{4.5.5}$$

4.5.2 Reduction to the projected nonlinear problem

Now we look for a solution of

$$S_\varepsilon(v) = \Delta v + \varepsilon \partial_{N+1} v - v + v^p = 0,$$

as a perturbation of $\tilde{v}_\varepsilon$, and hence, we define

$$v = \tilde{v}_\varepsilon + \phi,$$

so that the equation we need to solve can be written as

$$S_\varepsilon(\tilde{v}_\varepsilon + \phi) = S_\varepsilon(\tilde{v}_\varepsilon) + \mathbb{L}(\phi) + N(\phi) = 0, \tag{4.5.6}$$

where

$$\begin{aligned} \mathbb{L}(\phi) &= \Delta\phi + \varepsilon\partial_{N+1}\phi - \phi + p\tilde{v}_\varepsilon^{p-1}\phi, \\ N(\phi) &= (\tilde{v}_\varepsilon + \phi)^p - \tilde{v}_\varepsilon^p - p\tilde{v}_\varepsilon^{p-1}\phi. \end{aligned} \tag{4.5.7}$$

To solve (4.5.6), we use a very nice trick which was already used in [35, 34, 36]. This trick amounts to decompose the function ϕ into three functions and instead of solving (4.5 6), solve a coupled system. At first glance this might look rather counterintuitive but, as we will see, this strategy allows one to work in a tubular neighborhood of Γ_ε. Therefore, we set

$$\phi = \sum_{j=1}^{2} \chi_{\varepsilon,4,j}\, \phi_j + \psi,$$

where the function ψ solves

$$\begin{aligned} &\Delta\psi + \varepsilon\partial_{N+1}\psi - \psi = -\Big(1 - \sum_{j=1}^{2}\chi_{\varepsilon,4,j}\Big)\Big[S_\varepsilon(\tilde{v}_\varepsilon) + N(\phi) + p\tilde{v}_\varepsilon^{p-1}\psi\Big] \\ &- \sum_{j=1}^{2}\big[\Delta(\chi_{\varepsilon,4,j}\phi_j) - \chi_{\varepsilon\, 4,j}\Delta\phi_j\big] - \varepsilon\sum_{j=1}^{2}\big[\partial_{N+1}(\chi_{\varepsilon,4,j}\phi_j) - \chi_{\varepsilon,4,j}\partial_{N+1}\phi_j\big] \end{aligned} \tag{4.5.8}$$

For short, the right hand side will be denoted by $N_\varepsilon(\phi_j, \psi, h_j. e_j)$ so that this equation reads

$$\Delta\psi + \varepsilon\partial_{N+1}\psi - \psi = N_\varepsilon(\phi_j, \psi, h_j, e_j). \tag{4.5.9}$$

Observe that the right hand side vanishes when $\chi_{\varepsilon\, 4,j} = 1$. Hence it can be written as the product of a function with $(1 - \sum_{j=1}^{2}\chi_{\varepsilon,5,j})$.

Taking the difference between the equation satisfied by ϕ and the equation by ψ, it is enough that ϕ_j's satisfies

$$\begin{aligned} &\chi_{\varepsilon,4,j}\left[\Delta\phi_j + \varepsilon\partial_{N+1}\phi_j - \phi_j + p\tilde{v}_\varepsilon^{p-1}\phi_j\right] \\ = \;& -\chi_{\varepsilon,4,j}\left[S_\varepsilon(\tilde{v}_\varepsilon) + N(\phi) + p\tilde{v}_\varepsilon^{p-1}\psi\right], \quad j = 1, 2. \end{aligned} \tag{4.5.10}$$

Since we only need this equation to be satisfied on the support of $\chi_{\varepsilon,4,j}$, we can as well solve the equation

$$\begin{aligned} L_{\varepsilon,j}\phi_j = & -\chi_{\varepsilon,3,j}\left[\Delta\phi_j + \varepsilon\partial_{N+1}\phi_j - \phi_j + p\tilde{v}_\varepsilon^{p-1}\phi_j - L_{\varepsilon,j}\phi_j\right] \\ & -\chi_{\varepsilon,3,j}\left[S_\varepsilon(\tilde{v}_\varepsilon) + N(\phi) + p\tilde{v}_\varepsilon^{p-1}\psi\right], \quad j = 1, 2, \end{aligned} \tag{4.5.11}$$

where the operator $L_{\varepsilon,j}$ is defined on functions whose domain is $\Gamma_\varepsilon \times \mathbb{R}$ by

$$L_{\varepsilon,j} = \Delta_{\Gamma_\varepsilon} + \partial_z^2 + f'(w_{\varepsilon,j}). \tag{4.5.12}$$

For short, the right hand side of (4.5.11) will be denoted by $M_{\varepsilon,j}(\phi_j, \psi, h_j, e_j)$ so that (4.5.11) reads

$$\Delta_{\Gamma_\varepsilon}\phi_j + \partial_z^2\phi_j + f'(w_{\varepsilon,j})\phi_j = M_{\varepsilon,j}(\phi_j, \psi, h_j, e_j). \tag{4.5.13}$$

It is convenient to rewrite this system in the following way. First we introduce the shifted Fermi coordinates:

$$t_j = z - f_{\varepsilon,j}, \quad j = 1, 2.$$

Then each of the operators has the following form in terms of these new coordinates:

$$\begin{aligned} \Delta_{\Gamma_\varepsilon} + \partial_z^2 + f'(w_{\varepsilon,j}) = & \ \Delta_{\Gamma_\varepsilon} + \partial_{t_j}^2 + f'(w(t_j)) \\ & -\Delta_{\Gamma_\varepsilon} f_{\varepsilon,j}\partial_{t_j} - \nabla_{\Gamma_\varepsilon} f_{\varepsilon,j} \cdot \nabla_{\Gamma_\varepsilon}\partial_{t_j} + |\nabla_{\Gamma_\varepsilon} f_{\varepsilon,j}|^2\partial_{t_j}^2. \end{aligned}$$

Usually the second line above is small in the sense that its norm can be controlled by the norm of solution times a small factor and thus can be absorbed on the right hand side of the corresponding equation. Note also that variables t_j are related through the formula:

$$t_1 - t_2 = f_{\varepsilon,2} - f_{\varepsilon,1}.$$

Letting

$$\tilde{M}_{\varepsilon,j} = M_{\varepsilon,j} + \chi_{\varepsilon,3,j}\left[\Delta_{\Gamma_\varepsilon} f_{\varepsilon,j}\partial_{t_j} + \nabla_{\Gamma_\varepsilon} f_{\varepsilon,j} \cdot \nabla_{\Gamma_\varepsilon}\partial_{t_j} - |\nabla_{\Gamma_\varepsilon} f_{\varepsilon,j}|^2\partial_{t_j}^2\right],$$

we obtain the following system

$$\Delta_{\Gamma_\varepsilon}\phi_j + \partial_{t_j}^2\phi_j + f'(w(t_j))\phi_j = \tilde{M}_{\varepsilon,j}(\phi_j, \psi, h_j, e_j), \tag{4.5.14}$$

where now, with some abuse of notation, $\phi_j = \phi_j = \phi_j(y, t_j)$. This system can be considered as a system for functions defined on $\Gamma_\varepsilon \times \mathbb{R}$, and it looks at first sight as being decoupled. However,

$$\tilde{M}_{\varepsilon,j} = \tilde{M}_{\varepsilon,j}(y, z; \phi_j, \psi, h_j, e_j).$$

Therefore, when considering the equation for ϕ_1 in the shifted variable t_1 we need to use the above relation between t_1 and t_2. As a result we will obtain a nonlinear and nonlocal system for ϕ_j, $j = 1, 2$. The advantage of making this transformation is that we always work in the same, basic linearized operator. Again it is worth to point out that all the functions involved depend on y through the radial variable.

4.6 The linear theory

Given a $C^{2,\mu}(\Gamma_\varepsilon \times \mathbb{R})$ function u we define its weighted norm by:

$$\|u\|_{C^{0,\mu}_{\beta,\eta}(\Gamma_\varepsilon\times\mathbb{R})} = \sup_{(y,z)\in\Gamma_\varepsilon\times\mathbb{R}} (\cosh z)^\eta \omega_\varepsilon^\beta(r(y)) \|u\|_{C^{0,\mu}(B(y,1)\cap\Gamma_\varepsilon\times(z-1,z+1))},$$
$$\|u\|_{C^{2,\mu}_{\beta,\eta}(\Gamma_\varepsilon\times\mathbb{R})} = \|u\|_{C^{0,\mu}_{\beta,\eta}(\Gamma_\varepsilon\times\mathbb{R})} + \|\nabla_{\Gamma_\varepsilon\times\mathbb{R}}u\|_{C^{0,\mu}_{\beta,\eta}(\Gamma_\varepsilon\times\mathbb{R})} + \|D^2_{\Gamma_\varepsilon\times\mathbb{R}}u\|_{C^{0,\mu}_{\beta,\eta}(\Gamma_\varepsilon\times\mathbb{R})}. \tag{4.6.1}$$

Above $\nabla_{\Gamma_\varepsilon\times\mathbb{R}}$ and $D^2_{\Gamma_\varepsilon\times\mathbb{R}}$ denote the gradient and second derivative on the manifold $\Gamma_\varepsilon\times\mathbb{R}$ equipped with a natural product metric and the associated Levi-Civita connection.

In this section we will consider the following basic linearized operator:

$$L_\varepsilon\phi := \Delta_{\Gamma_\varepsilon}\phi + \partial_t^2\phi + f'(w(t))\phi.$$

Recall that the linearized operator of (4.2.5) about w is given by

$$L_0 = \partial_t^2 - 1 + pw^{p-1}. \tag{4.6.2}$$

It is known that L_0 has a unique principal eigenvalue $\lambda_1 > 0$ and $\lambda_2 = 0$ while the rest of the spectrum is strictly negative, see for example [34]. For future reference let us recall that

$$\lambda_1 = \frac{1}{4}(p-1)(p+3),$$

and the function

$$Z = \frac{w^{(p+1)/2}}{\sqrt{\int_{\mathbb{R}} w^{p+1}\,dx}}$$

is a positive eigenfunction associated to λ_1. Hence there exists a positive constant γ_0 such that

$$\langle L_0(\phi), \phi\rangle \geq \gamma_0 \|\phi\|_2^2,$$

whenever

$$\langle \phi, w'\rangle = \langle \phi, Z\rangle = 0.$$

From the equation of w it also follows that there exists a $\gamma > 0$ such that:

$$\langle L_0(\phi), \phi\rangle \geq \gamma \|\phi\|_{H^1}^2.$$

As a consequence the problem

$$L_0(\phi) - \xi^2\phi = h,$$

is uniquely solvable whenever $\xi^2 \neq \lambda_1, 0$ for any $h \in L^2(\mathbb{R})$. Actually, rather standard argument, using comparison principle and the fact that L_0 is of the form

$$L_0(\phi) = \partial_t^2\phi - \phi + q(t)\phi, \quad |q(t)| \leq Ce^{-c|t|},$$

can be used to show that the solution is an exponentially decaying function whenever h is for instance a compactly supported function.

In general we will consider the following problem:

$$\begin{cases} \Delta_{\Gamma_\varepsilon}\phi + \partial_t^2\phi + f'(w(t))\phi = g, & \text{in } \Gamma_\varepsilon \times \mathbb{R}, \\ \int_{\mathbb{R}} \phi(y,t)w'(t)\,dt = 0 = \int_{\mathbb{R}} \phi(y,t)Z(t)\,dt, & \text{for all } y \in \Gamma_\varepsilon. \end{cases} \tag{4.6.3}$$

We will assume that

$$\|g\|_{C^{0,\mu}_{\beta,\eta}(\Gamma_\varepsilon\times\mathbb{R})} < +\infty,$$

with some $\beta, \eta > 0$.

4.6.1 The a priori estimates and an existence result

Most of what will be stated in this section follows the arguments of [34] and so we will only outline the main points.

The followng lemma is about the kernel of L_ε.

Lemma 4.7. *Let ϕ be a bounded radial solution of the problem*

$$\Delta_{\Gamma_\varepsilon}\phi + \partial_t^2\phi + f'(w(t))\phi = 0 \quad \text{in } \Gamma_\varepsilon \times \mathbb{R}. \tag{4.6.4}$$

Then $\phi(y,t)$ is a linear combination of the functions $w'(t)$ and $\Phi(y)Z(t)$, where $\Phi(y)$ satisfies

$$\Delta_{\Gamma_\varepsilon}\phi + \lambda_1\phi = 0 \quad \text{in } \Gamma_\varepsilon.$$

Proof. Let ϕ be a bounded solution of equation (4.6.4). First we claim that ϕ has exponential decay in t, uniformly in y. Secondly let

$$\widetilde{\phi}(y,t) = \phi(y,t) - \left(\int_{\mathbb{R}} w'(t)\phi(y,t)\,dt\right)\frac{w'(t)}{\int_{\mathbb{R}} w'^2\,dt} - \left(\int_{\mathbb{R}} Z(t)\phi(y,t)\,dt\right)\frac{Z(t)}{\int_{\mathbb{R}} Z^2\,dt},$$

then $L_\varepsilon\widetilde{\phi} = 0$. We claim that $\widetilde{\phi} \equiv 0$ and then we get the desired result. To prove the claim, we define

$$\varphi(y) := \int_{\mathbb{R}} \widetilde{\phi}^2(y,t)\,dt,$$

which is well defined by the first claim. In fact so are its first and second derivatives by elliptic regularity theory applied to ϕ, and differentiation under the integral sign is thus justified. Now observe that

$$\Delta_{\Gamma_\varepsilon}\varphi = 2\int_{\mathbb{R}} \Delta_{\Gamma_\varepsilon}\widetilde{\phi}\cdot\widetilde{\phi}\,dt + 2\int_{\mathbb{R}} |\nabla_{\Gamma_\varepsilon}\widetilde{\phi}|^2\,dt$$

and hence

$$\begin{aligned} 0 &= \int_{\mathbb{R}} L_\varepsilon(\widetilde{\phi}) \cdot \widetilde{\phi}\, dt \\ &= \tfrac{1}{2}\Delta_{\Gamma_\varepsilon}\varphi - \int_{\mathbb{R}} |\nabla_{\Gamma_\varepsilon}\widetilde{\phi}|^2\, dt + \int_{\mathbb{R}} L_0(\widetilde{\phi}) \cdot \widetilde{\phi}\, dt. \end{aligned}$$

It follows that

$$\Delta_{\Gamma_\varepsilon}\varphi - \gamma_0\varphi \geq 0.$$

Since $\varphi \geq 0$ is bounded, from maximum principle we find that φ must be identically equal to zero and then $\widetilde{\phi} \equiv 0$. This means that

$$\phi(y,t) = \phi_1(y)w'(t) + \phi_2(y)Z(t). \tag{4.6.5}$$

Substitute it into the equation (4.6.4) we get

$$\Delta_{\Gamma_\varepsilon}\phi_1(y)w'(t) + (\Delta_{\Gamma_\varepsilon}\phi_2 + \lambda_1\phi_2)(y)Z(t) = 0,$$

which implies

$$\Delta_{\Gamma_\varepsilon}\phi_1 = 0, \quad \Delta_{\Gamma_\varepsilon}\phi_2 + \lambda_1\phi_2 = 0.$$

Liouville's theorem implies that $\phi_1 = C_1$ and $\phi_2 = C_2\Phi(y)$ for some constants C_1, C_2 because ϕ_1 and ϕ_2 are bounded radial functions on Γ_ε. □

Follows the above lemma, we show get the a priori estimate:

Lemma 4.8. *Let ϕ be a solution of the problem* (4.6.3)*. There holds:*

$$\|\phi\|_{C^{0,\mu}_{\beta,\eta}(\Gamma_\varepsilon\times\mathbb{R})} \leq C\|g\|_{C^{0,\mu}_{\beta,\eta}(\Gamma_\varepsilon\times\mathbb{R})}. \tag{4.6.6}$$

By the a priori estimate in lemma 4.8 one may get the following existence result:

Lemma 4.9. *Given $g \in C^{0,\mu}_{\beta,\eta}(\Gamma_\varepsilon \times \mathbb{R})$ such that*

$$\int_{\mathbb{R}} \phi(y,z)w'(z)\, dz = 0 = \int_{\mathbb{R}} \phi(y,z)Z(z)\, dz, \quad \text{for all } y \in \Gamma_\varepsilon,$$

there exists a unique solution of (4.6.3).

4.6.2 Study of a strongly coercive operator

In this section we will consider the following problem:

$$\Delta\psi + \varepsilon\partial_{N+1}\psi - \psi = h. \tag{4.6.7}$$

Observe that if h depends on $r = |x'|$ and x_{N+1} only, so does ψ.

We will use the following weighted norms:

$$\|h\|_{C^{0,\mu}_\beta(\mathbb{R}^N\times\mathbb{R})} = \sup_{x'\in\mathbb{R}^N} \left(1+\varepsilon^2|x'|^2\right)^\beta \|h\|_{C^{0,\mu}(B(x',1)\times\mathbb{R})}, \quad \beta > 0. \tag{4.6.8}$$

The weighted Hölder norms $C^{2,\mu}_\beta(\mathbb{R}^N\times\mathbb{R})$ are defined similarly. Note that if

$$\|h\|_{C^{0,\mu}_\beta(\mathbb{R}^N\times\mathbb{R})} < +\infty,$$

then

$$\|h\|_{C^{0,\mu}(\mathbb{R}^N\times\mathbb{R})} < +\infty.$$

Therefore, by a standard argument, we obtain the existence of a solution $\psi \in C^{2,\mu}_\beta(\mathbb{R}^N\times\mathbb{R})$ to (4.6.7).

Now to show that in fact

$$\|\psi\|_{C^{2,\mu}_\beta(\mathbb{R}^N\times\mathbb{R})} \le C\|h\|_{C^{0,\mu}_\beta(\mathbb{R}^N\times\mathbb{R})}, \tag{4.6.9}$$

one may use a comparison argument based on the fact that the reciprocal of the weigh function $(1+\varepsilon^2|x'|^2)^\beta$ is a positive supersolution. Details are left to the reader.

4.7 Proof of Theorem 4.2

In this section we will prove Theorem 4.2.

4.7.1 Error estimates

Our first goal is to estimate functions $\tilde{M}_{\varepsilon,\jmath}$. Whenever convenient we will indicate the fact that these functions depend on their functional arguments by writing

$\tilde{M}_{\varepsilon,j} = \tilde{M}_{\varepsilon,j}(\phi_{\varepsilon,1}, \phi_{\varepsilon,2}, \psi_\varepsilon, \mathbf{h}_\varepsilon, \mathbf{e}_\varepsilon)$. In general, besides the assumptions on $\mathbf{h}_\varepsilon, \mathbf{e}_\varepsilon$ we made in (4.5.2) we will also assume that, for some $\sigma \in (0,1)$ and $K > 0$,

$$\|\phi_{\varepsilon,j}\|_{C^{2,\mu}_{1-\sigma,\sigma}(\Gamma_\varepsilon \times \mathbb{R})} \leq K\varepsilon^{2-2\sigma}. \tag{4.7.1}$$

About the function ψ_ε we assume that, with some $\kappa > 3$, we have

$$\|\psi_\varepsilon\|_{C^{2,\mu}_{\kappa}(\mathbb{R}^N \times \mathbb{R})} \leq K\varepsilon^3. \tag{4.7.2}$$

Lemma 4.10. *Under the preceding assumption there exists a $\sigma \in (0,1)$ such that the following estimate holds:*

$$\|\tilde{M}_{\varepsilon,j}\|_{C^{2,\mu}_{1-\sigma,\sigma}(\Gamma_\varepsilon \times \mathbb{R})} \leq C\Big[\varepsilon^{2-2\sigma} + o(1)\sum_{j=1}^{2} \|\phi_{\varepsilon,j}\|_{C^{2,\mu}_{1-\sigma,\sigma}(\Gamma_\varepsilon \times \mathbb{R})} + \|\psi_\varepsilon\|_{C^{2,\mu}_{\kappa}(\mathbb{R}^N \times \mathbb{R})}\Big]. \tag{4.7.3}$$

Function $\tilde{M}_{\varepsilon,j}$ is Lipschitz function of its arguments and we have:

$$\begin{aligned} &\|\tilde{M}_{\varepsilon,j}(\phi^{(1)}_{\varepsilon,1}, \phi^{(1)}_{\varepsilon,2}, \psi^{(1)}_\varepsilon, \boldsymbol{h}^{(1)}_\varepsilon, \boldsymbol{e}^{(1)}_\varepsilon) - \tilde{M}_{\varepsilon,j}(\phi^{(2)}_{\varepsilon,1}, \phi^{(2)}_{\varepsilon,2}, \psi^{(2)}_\varepsilon, \boldsymbol{h}^{(2)}_\varepsilon, \boldsymbol{e}^{(2)}_\varepsilon)\|_{C^{2,\mu}_{1-\sigma,\sigma}(\Gamma_\varepsilon \times \mathbb{R})} \\ &\leq C\Big[\varepsilon^{2-2\sigma}\|\boldsymbol{h}^{(1)}_\varepsilon - \boldsymbol{h}^{(2)}_\varepsilon\|_{C^{2,\mu}_{1-\sigma}} + \|\boldsymbol{e}^{(1)}_\varepsilon - \boldsymbol{e}^{(2)}_\varepsilon\|_{C^{2,\mu}_{1-\sigma}} \\ &\qquad + o(1)\sum_{j=1}^{2} \|\phi^{(1)}_{\varepsilon,j} - \phi^{(2)}_{\varepsilon,j}\|_{C^{2,\mu}_{1-\sigma,\sigma}(\Gamma_\varepsilon \times \mathbb{R})} + \|\psi^{(1)}_\varepsilon - \psi^{(2)}_\varepsilon\|_{C^{2,\mu}_{\kappa}(\mathbb{R}^N \times \mathbb{R})}\Big]. \end{aligned} \tag{4.7.4}$$

Next we will consider $N_\varepsilon(\phi_{\varepsilon,1}, \phi_{\varepsilon,2}, \psi_\varepsilon, \mathbf{h}_\varepsilon, \mathbf{e}_\varepsilon)$ defined in (4.5.9). We have:

Lemma 4.11. *Under the same hypothesis as in Lemma 4.10, and assuming that the constant M is large enough, there existes $\kappa > 3$ and $\gamma > 1$ such that we have*

$$\|N_\varepsilon\|_{C^{2,\mu}_{\kappa}(\mathbb{R}^N \times \mathbb{R})} \leq C\Big[\varepsilon^3 + \varepsilon^\gamma \sum_{j=1}^{2} \|\phi_{\varepsilon,j}\|_{C^{2,\mu}_{1-\sigma,\sigma}(\Gamma_\varepsilon \times \mathbb{R})} + o(1)\|\psi_\varepsilon\|_{C^{2,\mu}_{\kappa}(\mathbb{R}^N \times \mathbb{R})}\Big]. \tag{4.7.5}$$

Furthermore, considering N_ε as a function of $(\phi_{\varepsilon,1}, \phi_{\varepsilon,2}, \psi_\varepsilon, \boldsymbol{h}_\varepsilon, \boldsymbol{e}_\varepsilon)$, it is a Lipschitz function of its arguments and

$$\begin{aligned} &\|N_\varepsilon(\phi^{(1)}_{\varepsilon,1}, \phi^{(1)}_{\varepsilon,2}, \psi^{(1)}_\varepsilon, \boldsymbol{h}^{(1)}_\varepsilon, \boldsymbol{e}^{(1)}_\varepsilon) - N_\varepsilon(\phi^{(2)}_{\varepsilon,1}, \phi^{(2)}_{\varepsilon,2}, \psi^{(2)}_\varepsilon, \boldsymbol{h}^{(2)}_\varepsilon, \boldsymbol{e}^{(2)}_\varepsilon)\|_{C^{2,\mu}_{1-\sigma,\sigma}(\Gamma_\varepsilon \times \mathbb{R})} \\ &\leq C\Big[\varepsilon^3\|\boldsymbol{h}^{(1)}_\varepsilon - \boldsymbol{h}^{(2)}_\varepsilon\|_{C^{2,\mu}_{1-\sigma}} + \varepsilon^{1+\sigma}\|\boldsymbol{e}^{(1)}_\varepsilon - \boldsymbol{e}^{(2)}_\varepsilon\|_{C^{2,\mu}_{1-\sigma}} \\ &\qquad + \varepsilon^\gamma \textstyle\sum_{j=1}^{2} \|\phi^{(1)}_{\varepsilon,j} - \phi^{(2)}_{\varepsilon,j}\|_{C^{2,\mu}_{1-\sigma,\sigma}(\Gamma_\varepsilon \times \mathbb{R})} + o(1)\|\psi^{(1)}_\varepsilon - \psi^{(2)}_\varepsilon\|_{C^{2,\mu}_{\kappa}(\mathbb{R}^N \times \mathbb{R})}\Big]. \end{aligned} \tag{4.7.6}$$

The proof of this estimates is omitted, since similar results are proven in [34] and no essentially new elements are needed to carry out the argument in the present case.

4.7.2 Projected nonlinear problem

Our objective in this section is to solve (4.5.9)-(4.5.14), i.e.,

$$\begin{cases} \Delta\psi + \varepsilon\partial_{N+1}\psi - \psi = N_\varepsilon(\phi_j, \psi, h_j, e_j), \\ \Delta_{\Gamma_\varepsilon}\phi_j + \partial_{t_j}^2\phi_j + f'(w(t_j))\phi_j = \tilde{M}_{\varepsilon,j}(\phi_j, \psi. h_j, e_j). \end{cases} \tag{4.7.7}$$

Given the linear theory available and the results of the preceding section, we will achieve this by a simple fixed point argument.

Let functions $\widetilde{\phi}_{\varepsilon,j}$, $j = 1, 2$ and $\widetilde{\psi}_\varepsilon$ satsifying assumptions (4.7.1)-(4.7.2) be fixed. We will also choose $\mathbf{h}_\varepsilon, \mathbf{e}_\varepsilon$ to satisfy (4.5.2). We first use the linear theory to solve the following system:

$$\begin{cases} (\Delta + \varepsilon\partial_{N+1} - 1)\psi_\varepsilon = N_\varepsilon(x; \widetilde{\phi}_{\varepsilon,1}, \widetilde{\phi}_{\varepsilon,2}, \widetilde{\psi}_\varepsilon, \mathbf{h}_\varepsilon, \mathbf{e}_\varepsilon), \\ \left[\Delta_{\Gamma_\varepsilon} + \varepsilon\nabla_{\Gamma_\varepsilon}F_\varepsilon \cdot \nabla_{\Gamma_\varepsilon} + \partial_{t_j}^2 + f'(w(t_j))\right]\phi_{\varepsilon,j} \\ \qquad = \tilde{M}_{\varepsilon,j}(y, t_j; \widetilde{\phi}_{\varepsilon,1}, \widetilde{\phi}_{\varepsilon,2}, \widetilde{\psi}_\varepsilon, \mathbf{h}_\varepsilon, \mathbf{e}_\varepsilon) + c_{\varepsilon,j}w'(t_j) + d_{\varepsilon,j}Z(t_j), \quad j = 1, 2, \\ \int_{\mathbb{R}} \phi_{\varepsilon,j}(y, t_j)w'(t_j)\, dt_j = 0 = \int_{\mathbb{R}} \phi_{\varepsilon,j}(y, t_j)Z(t_j)\, dt_j. \end{cases} \tag{4.7.8}$$

Using Lemma 4.10 and Lemma 4.11 we obtain existence of such a fixed point satisfying (4.7.8) by the Banach fixed point theorem. Hence we have the following:

Lemma 4.12. *Under the above hypothesis there exists a unique solution* $(\phi_{\varepsilon,1}, \phi_{\varepsilon,2}, \psi_\varepsilon)$ *of* (4.7.8) *satisfying* (4.7.1) *and* (4.7.2).

Let us observe that the map

$$(\widetilde{\phi}_{\varepsilon,1}, \widetilde{\phi}_{\varepsilon,2}, \widetilde{\psi}_\varepsilon, \mathbf{h}_\varepsilon, \mathbf{e}_\varepsilon) \mapsto (\phi_{\varepsilon,1}, \phi_{\varepsilon,2}, \psi_\varepsilon)$$

is a uniform contraction with respect to $\mathbf{h}_\varepsilon, \mathbf{e}_\varepsilon$. It follows that $(\phi_{\varepsilon,1}, \phi_{\varepsilon,2}, \psi_\varepsilon)$ are Lipschitz functions of $\mathbf{h}_\varepsilon, \mathbf{e}_\varepsilon$ with a small Lipschitz constant.

4.7.3 Solution of the reduced problem

At this point we are left with the task of adjusting $\mathbf{h}_\varepsilon, \mathbf{e}_\varepsilon$ in such a way that $c_{\varepsilon,j} \equiv 0$ and $d_{\varepsilon,j} \equiv 0$. First we will find the exact conditions for $\mathbf{h}_\varepsilon, \mathbf{e}_\varepsilon$ which guarantee that $c_{\varepsilon,j} \equiv 0$ and $d_{\varepsilon,j} \equiv 0$. We will show that they result in a non-homogeneous and non-local ODE system since we have assumed initially that all the functions are functions of r, where $r = |x'|$, $(x', x_{N+1}) \in \mathbb{R}^{N+1}$, quite similar to the one already studied in Section 4.4. From the theory developed in this section the existence of $\mathbf{h}_\varepsilon$ and $\mathbf{e}_\varepsilon$ will follow immediately, thus completing the proof of Theorem 4.2. Our first task is then to justify rigorously formal calculations in section 4.2. In fact, with the notations as in the previous sections we need to adjust $\mathbf{h}_\varepsilon$ and $\mathbf{e}_\varepsilon$ so that

$$\int_{\mathbb{R}} \tilde{M}_{\varepsilon,j}(r,t_j) w'(t_j)\, dt_j = 0 = \int_{\mathbb{R}} \tilde{M}_{\varepsilon,j}(r,t_j) Z(t_j)\, dt_j, \quad j = 1, 2.$$

Let us recall that $\tilde{M}_{\varepsilon,j}$ depends non-locally on $\mathbf{h}_\varepsilon$ and $\mathbf{e}_\varepsilon$ and this dependence involves the first and second derivatives of $\mathbf{h}_\varepsilon$ and $\mathbf{e}_\varepsilon$. Thus its projection onto $w'(t_j)$ and $Z(t_j)$ will be a non-local, second order ODE system in terms of the radial variable r.

Let us write

$$\tilde{M}_{\varepsilon,j} = \chi_{\varepsilon,3,j} S_\varepsilon(\bar{v}_\varepsilon) + \hat{M}_{\varepsilon,j}, \quad \hat{M}_{\varepsilon,j} = \hat{q}_{\varepsilon,j}(\phi_{\varepsilon,1}, \phi_{\varepsilon,2}, \psi_\varepsilon, \mathbf{h}_\varepsilon, \mathbf{e}_\varepsilon).$$

It is easy to see that $c_{\varepsilon,j} \equiv 0$ when

$$\begin{aligned} \textstyle\int_{\mathbb{R}} \tilde{M}_{\varepsilon,j}(r,t_j) w'(t_j)\, dt_j &= \textstyle\int_{\mathbb{R}} \chi_{\varepsilon,3,j} S_\varepsilon(\bar{v}_\varepsilon)(r,t_j) w'(t_j)\, dt_j + \int_{\mathbb{R}} \hat{M}_{\varepsilon,j}(r,t_j) w'(t_j)\, dt_j \\ &= I_{\varepsilon,j} + \hat{I}_{\varepsilon,j} = 0. \end{aligned}$$

As we have argued in Section 4.2, the main order term in the above integral comes from $I_{\varepsilon,j}$ while the remaining part of the projection, denoted $\hat{I}_{\varepsilon,j}$ is a lower order term. Repeating calculations in Section 4.2, one can derive the following expression:

$$I_{\varepsilon,j} = \alpha_0 J_{\Gamma_\varepsilon}(f_{\varepsilon,j} + h_{\varepsilon,j}) + \mathcal{T}_j(\mathbf{f}_\varepsilon + \mathbf{h}_\varepsilon) + q_{\varepsilon,j}(\mathbf{f}_\varepsilon + \mathbf{h}_\varepsilon), \tag{4.7.9}$$

where, for a vector function $\mathbf{v} = (v_1, v_2)$, on Γ_ε we have denoted:

$$J_{\Gamma_\varepsilon}(v_j) = \Delta_{\Gamma_\varepsilon} v_j + |A_{\Gamma_\varepsilon}|^2 v_j + \varepsilon \nabla_{\Gamma_\varepsilon} F_\varepsilon \cdot \nabla_{\Gamma_\varepsilon} v_j, \quad \mathcal{T}_j(\mathbf{v}) = -e^{v_{j-1} - v_j} + e^{v_j - v_{j+1}}.$$

We observe that the main order term in $q_{\varepsilon,j}$ comes from

$$z^2 \zeta_{\varepsilon,j} R_{\Gamma_\varepsilon} \partial_z u_\varepsilon \approx (t_j - f_{\varepsilon,j})^2 \sum_{i=1}^{N} k^3_{\Gamma_\varepsilon,i} w'(t_j - h_{\varepsilon,j}),$$

where $k_{\Gamma_\varepsilon,i}$ are the principal curvatures of Γ_ε. Direct calculations show that

$$|k_{\Gamma_\varepsilon,i}| \approx \varepsilon^3 \omega_\varepsilon^{-3/2}.$$

Taking into account the assumptions we have made at the beginning on $\mathbf{h}_\varepsilon$, and $\mathbf{e}_\varepsilon$ in (4.5.2), we see that there exist $\beta > 0$ and $\rho > 0$ such that

$$\|q_{\varepsilon,j}\|_{C^{0,\mu}_{1+\beta}(\Gamma_\varepsilon)} \le C\varepsilon^{2+\rho}.$$

Identifying functions on Γ_ε and Γ by $v_\varepsilon(r) = v(\varepsilon r)$, so that $q_{\varepsilon,j}(r) = q_j(\varepsilon r)$ we get

$$\|q_j\|_{C^{0,\mu}_{1+\beta}(\Gamma)} \le C\varepsilon^{2+\rho-\mu}.$$

Function q_j now depends on the functions $\mathbf{h}_\varepsilon$ and $\mathbf{e}_\varepsilon$ defined on Γ. Similar statements hold for the remaining term in $\hat{M}_{\varepsilon,j}$, namely we have

$$\|\hat{I}_j\|_{C^{0,\mu}_{1+\beta}(\Gamma)} \le C\varepsilon^{2+\rho-\mu}.$$

We let $\mu > 0$ to be a small number such that $\rho - \mu > \tau$. Denoting by J_Γ the scaled operator for Γ, and setting $\hat{q}_j = q_j + \hat{I}_j$ we get

$$\alpha_0 \varepsilon^2 J_\Gamma(f_j + h_j) + \mathcal{T}_j(\mathbf{f} + \mathbf{h}) = \hat{q}_j. \qquad (4.7.10)$$

This is a Jacobi-Toda system, which can be solved using the theory we developed in Lemma 4.6. In fact $\hat{q}_j$ is a Lipschitz function of $\mathbf{h}$ and $\mathbf{e}$ since it follows from the Lipschitz character of $S(w_\varepsilon), \phi_{\varepsilon,j}, \psi_\varepsilon$ as functions of $\mathbf{h}$ and $\mathbf{e}$. Defining

$$\mathcal{T}_j(\mathbf{f} + \mathbf{h}) - \mathcal{T}_j(\mathbf{f}) - \mathcal{T}_j'(\mathbf{f})\mathbf{h} = \mathcal{N}_j(\mathbf{h}),$$

we also have

$$\|\mathcal{N}_j(\mathbf{h})\|_{C^{0,\mu}_{1+\beta}(\Gamma)} \le C\varepsilon^{2+\tau}\|\mathbf{h}\|_{C^{0,\mu}_{\beta}(\Gamma)}.$$

Similarly, $\mathcal{N}_j(\mathbf{h})$ is a Lipschitz function of $\mathbf{h}$. Since we have chosen $\mathbf{f}$ to be a solution of the homogeneous version of (JT) we are left with:

$$\alpha_0\varepsilon^2 J_\Gamma(h_j) + \mathcal{T}_j'(\mathbf{f})\mathbf{h} = \widetilde{q}_j, \quad \widetilde{q}_j = \hat{q}_j - \mathcal{N}_j. \tag{4.7.11}$$

Similarly, the condition such that $d_{\varepsilon,j} \equiv 0$ is

$$\alpha_0\varepsilon^2 J_\Gamma(e_j) + \lambda_1 e_j = \widetilde{p}_j. \tag{4.7.12}$$

The left hand side of this equation is the linearized Jacobi-Toda system, and now Lemma 4.4 can be employed directly to solve it using Banach fixed point theorem. As similar arguments can be found for instance in [34], we omit the details here. With this last step we complete our proof.

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