SYZ Mirror Symmetry for Toric Calabi-Yau Manifolds

LAU, Siu Cheong

A Thesis Submitted in Partial Fulfilment
of the Requirements for the Degree of
Doctor of Philosophy
in
Mathematics

The Chinese University of Hong Kong

June 2011

Thesis/Assessment Committee

Professor Y.H. Wan (Chair)

Professor N.C. Leung (Thesis Supervisor)

Professor K.K. Au (Committee Member)

Professor W.P. Li (External Examiner)

Abstract

This thesis gives a procedure to carry out SYZ construction of mirrors with quantum corrections by Fourier transform of open Gromov-Witten invariants. Applying to toric Calabi-Yau manifolds, one obtains the Hori-Iqbal-Vafa mirror together with a map from the Kähler moduli to the complex moduli of the mirror, called the SYZ map.

It is conjectured that the SYZ map equals to the inverse mirror map. In dimension two this conjecture is proved, and in dimension three supporting evidences of the equality are studied in various examples. Since the SYZ map is expressed in terms of open Gromov-Witten invariants, this conjectural equality established an enumerative meaning of the inverse mirror map.

Moreover a computational method of open Gromov-Witten invariants for toric Calabi-Yau manifolds is invented. As an application, the Landau-Ginzburg mirrors of compact semi-Fano toric surfaces are computed explicitly.

摘要

本論文通過 open GW 不變量 (open Gromov-Witten invariant) 的傅里葉變換，得出 SYZ 鏡構造的量子校正 (quantum correction)。應用到環狀卡拉比丘空間 (toric Calabi-Yau manifold) 上，即可得出其 Hori-Iqbel-Vafa 鏡流形，以及一個由其 Kähler 模空間到複結構鏡模空間 (mirror moduli) 的映射，稱為 SYZ 映射。

本文提出以下猜想，並證明其在二維情形下成立：SYZ 映射與鏡映射 (mirror map) 的逆相等。在三維情況下，本文列舉了一些支持這個猜想的經典例子。因為 SYZ 映射是通過 open GW 不變量給出的，由此猜想可得出逆鏡映射的幾何計數意義。

另外，本文提出計算環狀卡拉比丘空間 open GW 不變量的新方法。此計算還可應用到半 Fano (semi-Fano) 的環狀流形 (toric manifold) 上，得出其超勢能 (superpotential) 的具體表達式。

ACKNOWLEDGMENTS

I express my deepest gratitude to my supervisor *Naichung Conan Leung*. He brought me to this interesting topic of using T-duality and Fourier transform to understand mirror symmetry, and spent a lot of time and patience to explain basic stuffs to me during my course of research. Without his guidance and encourangement, this research program would have not even started.

I am very grateful to my two collaborators, *Kwokwai Chan* and *Baosen Wu*. Kwokwai Chan is my former academic brother, who taught me a lot of stuffs in SYZ mirror symmetry during the tea time. Baosen Wu helped me a lot in studying Gromov-Witten invariants when he was a postdoctorate in The Chinese University of Hong Kong. They generously shared many ideas with me and contributed a lot to this piece of work.

During my course of research, I have had helpful academic discussions with brilliant mathematicians, and I would like to thank them here: *Changzheng Li*, who is my former academic brother, invited me to KIAS in January 2010 and we had joyful discussions. I visited *Mark Gross* at UCSD in February 2010 and learnt a lot about tropical geometry and mirror symmetry from him. During the same visit *Ricardo Castano Bernard* and *Diego Matessi* told me about their interesting work on Lagrangian fibrations on Calabi-Yau manifolds. In June 2010, *Cheol-Hyun Cho* invited me to Seoul National University and he told me a lot about Lagrangian Floer theory during hiking. Afterwards *Bumsig Kim* invited me to KIAS and generously offered me a chance to talk about this piece of work in the Summer School in Mirror Symmetry. In the autumn semester of 2010, I visited *Andrei Caldararu* at University of Wisconsin and had joyful discussions with him and his student *Junwu Tu*. Moreover during this visit I learnt more about mirror symmetry of toric manifolds from *Yong-Geun Oh*. In Yamabe Memorial Symposium 2010, I learnt more about open Gromov-Witten invaraints and coherent-constructible correspondence from *Chiu-Chu Liu*. In October 2010, I visited *Shing-Tung Yau* and he generously shared his ideas on SYZ mirror symmetry for K3 surfaces with me. In the same month I visited *Eric Zaslow* and he generously told me his ideas on SYZ mirror symmetry of universal ellptic curve. In February 2011, *Tsian-hua Tseng* invited me to Ohio State University and taught me a lot about Givental's mirror map. Moreover I am very grateful to his generous help in my flight delay due to snowstorm. *David Treumann* explained his work on constructible plumbing model in details during his visit to The Chinese University of Hong Kong in March 2011. I would also like to express my gratitude to *Denis Auroux, Kenji Fukaya, Abouzaid Mohammed, Rongbin Ruan, Ed Segal, Yan Soibelman* and *Ilia Zharkov* for answering my academic questions patiently.

Let me also thank my fellow colleagues in the Institute of Mathematical Science and Mathematics Department of The Chinese University of Hong Kong, including *Kai-Leung Chan, Yunxia Chen, Kim-Hong Chiu, Sin-Tsun Fan, Yong-Dong Huang, Kwok-Kun Kwong, Changzheng Li, Chit Ma, Ding Ma, Kit-Ho Mak, Sheng Mao, Tse-Leung So, Dan Wu, Matthew Wong, Le Yin, Jiajing Zhang, Yi Zhang*. We have had a lot of joyful discussions. Finally, I express my gratitude to *Yi Zhang* helping me with the Chinese tranlation of the abstract.

Finally, I would like to give special thanks to my dear parents, who always give me unlimited support and care anywhere, anytime.

Contents

Chapter 1

An overview

Mirror symmetry states that for a Calabi-Yau manifold X, there exists another Calabi-Yau manifold $\check{X}$ (which is called the mirror of X), such that the *complex geometry* of $\check{X}$ reflects the *symplectic geometry* of X, and vice versa. Since its discovery by the physicists [7], it astonished many mathematicians again and again by its powerful predictions, especially on enumerative geometry.

A lot of efforts have been put to lay a rigorous mathematical foundation for mirror symmetry (for example, [33, 34]), and its developments have opened a new era of geometry and physics. Important concepts such as Gromov-Witten invariants [5, 15, 41] and Fukaya category [20, 21] were developed, and they help mathematicians and physicists to obtain a deeper understanding of the subject. Moreover, mirror symmetry beyond the Calabi-Yau setting has also been extensively studied (for instance, [12, 22, 18, 10, 14, 47, 13, 31]).

Yet mirror symmetry is far from being fully understood. While new computational techniques of Gromov-Witten invariants were developed to verify the predictions from mirror symmetry, these computations reveal little on how and why they work. This thesis attempt to answer this question in the context of *toric Calabi-Yau manifolds* from the *Strominger-Yau-Zaslow* perspective [48].

1.1 SYZ mirror symmetry

For a pair of mirror Calabi-Yau manifolds X and $\check{X}$, the Strominger-Yau-Zaslow (SYZ) conjecture [48] asserts that there exist special Lagrangian torus fibrations $\mu : X \to B$ and $\mu : \check{X} \to B$ which are fiberwise-dual to each other. In particular, this suggests an intrinsic construction of the mirror $\check{X}$ by fiberwise dualizing a special Lagrangian torus fibration on X. This process is called *T-duality*.

The SYZ program has been carried out successfully in the *semi-flat* case [35, 39, 38], where the discriminant loci of Lagrangian torus fibrations are empty (that is, all fibers are regular). On the other hand, mirror symmetry has been extended to non-Calabi-Yau settings, and the SYZ construction has been shown to work in the *toric* case [10, 14], where the discriminant locus appears as the boundary of the base B.

In general, by fiberwise dualizing a Lagrangian torus fibration $\mu : X \to B$ *away* from the discriminant locus, one obtains a manifold $\check{X}_0$ equipped with a complex structure J_0, the so-called *semi-flat complex structure*. In both semi-flat and compact toric cases, $(\check{X}_0, J_0)$ already serves as the mirror manifold of X.[1] However, when the discriminant locus Γ appears in the interior of B, (X_0, J_0) does not give the mirror that physicists suggest, and it loses the geometric information of the discriminant locus. It is expected that the complex structure J on X can be obtained from J_0 by *quantum corrections*, which capture symplectic enumerative information on X (see Fukaya [16], Kontsevich-Soibelman [36], Gross-Siebert [27]). This is one manifestation of the mirror principle that the *complex geometry of the mirror X encodes symplectic enumerative data of X*.

The SYZ mirror construction with quantum corrections for general compact Calabi-Yau manifolds is still unclear to mathematicians. A good starting point is to work with local models of Lagrangian torus fibrations on Calabi-Yau mani-

[1] In the toric case, there is an additional data in the mirror called the superpotential, which is a holomorphic function on X_0.

folds, which are realized by *non-toric Lagrangian fibrations* [2] *on toric Calabi-Yau manifolds* constructed by Gross [26] and Goldstein [24] independently. Interior discriminant loci are present in these fibrations, leading to the *wall-crossing phenomenon of open GW invariants* (which is roughly speaking the counting of holomorphic disks) and nontrivial quantum corrections of the mirror complex structure. Understanding SYZ mirror symmetry with corrections for such local models would be an essential first step to study mirror symmetry for compact Calabi-Yau manifolds.

1.2 The main results

In this thesis we study SYZ mirror symmetry with corrections for toric Calabi-Yau manifolds. The first three chapters aim at building up terminologies and notations. Section 4.3 gives the procedures to construct the SYZ mirror, and the main results are contained in Chapter 5, 6 and 7. What follows is a brief introduction to these results.

1.2.1 SYZ mirror construction for toric Calabi-Yaus

An essential step of the SYZ construction discussed in the last section is quantum correction, which is still unclear to mathematicians. In Section 4.3 we propose a way to handle quantum correction using *Fourier transform of open Gromov-Witten invariants*, and in Chapter 5 we show that it works for all toric Calabi-Yau manifolds. The following is a simplified version of Theorem 5.5.1:

Theorem 1.2.1 (Restatement of Theorem 5.5.1). *Let (X, ω) be a toric Calabi-Yau n-fold equipped with a toric Kähler form. The SYZ mirror $\mathcal{F}_{\mathrm{SYZ}}(X, \omega)$ is*

[2] Here, "non-toric" means the fibrations are not those provided by moment maps of Hamiltonian torus actions on toric varieties.

$(\check{X}, \check{\Omega})$, where

$$\check{X} = \{(u, v, z) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} : uv = g(z)\}$$

is a complex manifold defined by an explicit Laurent polynomial g, and $\check{\Omega}$ is a holomorphic volume form on $\check{X}$ transformed from ω.

While this result agrees with Hori-Iqbal-Vafa's physical prediction [28], *the SYZ approach gives more than the Hori-Iqbal-Vafa recipe*: The Laurent polynomial g in the above theorem is expressed explicitly in terms of Kähler parameters and open Gromov-Witten invariants. Thus Theorem 1.2.1 gives the complex manifold mirror to (X, ω), rather than a mirror family given by the Hori-Iqbal-Vafa recipe. This construction produces the SYZ map $\mathcal{F}_{\mathrm{SYZ}}$ from the Kähler moduli of X to the complex moduli of its mirror.

There are two main ingredients in the construction. First, to carry out torus duality, one needs a Lagrangian fibration on X, which is written down by M. Gross [26] and E. Goldstein independently. We give a review on such Lagrangian fibrations in Section 5.1. Figure 1.1 depicts how this fibration looks like in the two-dimensional case.

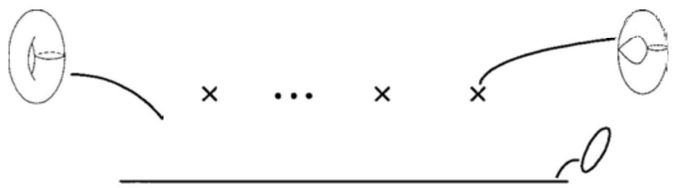

Figure 1.1: Gross' Lagrangian fibrations on toric Calabi-Yau manifolds.

The second ingredient has to do with quantum corrections, which encode the one-pointed genus-zero open Gromov-Witten invariants of a fiber of the above fibration. It is observed by D. Auroux [3, 4] that open GW invariants admit wall-crossing in various examples such as $\mathbb{P}^2$ and the Hirzebruch surface $\mathbb{F}_2$. In Section 5.4 we study wall-crossing phenomenon for all toric Calabi-Yau manifolds in a uniform way (see Figure 1.2 for an example). Then by taking Fourier transform of these open invariants, one obtains the corrected mirror complex coordinates (Section 5.5).

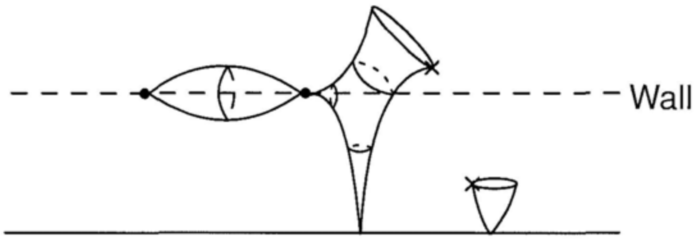

Figure 1.2: Wall-crossing phenomenon for $X = K_{\mathbb{P}^1}$. There is only one class of holomorphic disks bounded by fibers below the wall. Crossing the wall, other classes of holomorphic disks come up, such as the one drawn in the left of this figure.

1.2.2 Mirror map = SYZ map

The (inverse) mirror map $\mathcal{F}_{\text{mirror}} : \mathcal{M}_K(X) \to \mathcal{M}_C(\check{X})$ provides a canonical local isomorphism between the Kähler moduli $\mathcal{M}_K(X)$ and the mirror complex moduli $\mathcal{M}_C(\check{X})$ near the large complex structure limit, and it plays a key role in the study of mirror symmetry. For instance, the success of mirror symmetry on counting of rational curves in the quintic threefold relies in an essential way on identifying the mirror map.

Yet geometric meanings of $\mathcal{F}_{\text{mirror}}$ remain unclear to mathematicians. Integrality of coefficients of certain series expansion of the mirror map has been observed and studied [45, 50, 37], and it is expected that these coefficients contain enumerative meanings.

Since $\mathcal{F}_{\text{SYZ}}$ is also defined in a canonical way, it is natural to expect that the two maps $\mathcal{F}_{\text{mirror}}$ and $\mathcal{F}_{\text{SYZ}}$ are equal. The precise formulation of this conjecture is contained in Section 6 2 Moreover observe that $\mathcal{F}_{\mathrm{SYZ}}$ is defined in terms of enumerative invariants of X Establishing this conjectural equality will give an enumerative meaning to the mirror map

In Section 7 3, it is shown to be true for all toric Calabi-Yau manifolds in dimension two

Theorem 1.2.2 (Restatement of Theorem 7 3 6).

$$\mathcal{F}_{\mathrm{mirror}} = \mathcal{F}_{\mathrm{SYZ}}$$

for every toric Calabi-Yau surface

In dimension three, various typical examples such as $X = K_{\mathbb{P}^2}$, $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ are studied, and strong evidences of the conjecture have been found See Section 7 3 2 for the details

Remark 1.2.3. *We follow the existing literatures to call* $\mathcal{F}_{\mathrm{mirror}}$ $\mathcal{M}_K(X) \to \mathcal{M}_C(\check{X})$ *the inverse mirror map In Chapter 6 and 7 we will denote the mirror map by* ϕ $\mathcal{M}_C(\check{X}) \to \mathcal{M}_K(X)$

1.2.3 Computation of open GW invariants

A key step to prove $\mathcal{F}_{\mathrm{mirror}} = \mathcal{F}_{\mathrm{SYZ}}$ is a computation of one-pointed open Gromov-Witten invariants of Lagrangian toric fibers in toric Calabi-Yau manifolds Cho-Oh [12] gave the answer when the toric manifold is Fano, in which case the corresponding moduli problem has no obstruction Little is known about these invariants beyond the Fano case due to the presence of obstructions When X is semi-Fano (meaning that its anti-canonical line bundle is ample), the only previous known result is the Hirzebruch surface $\mathbb{F}_2$, which is computed by Auroux [4] using wall-crossing, and independently by FOOO [19] using their big machinery (Using wall-crossing Auroux also computed the invariants for $\mathbb{F}_3$)

In Section 7.2, the open Gromov-Witten invariants of Lagrangian toric fibers in toric Calabi-Yau manifolds are computed, where the obstructions in the corresponding moduli problems are non-trivial. The strategy is to first equate them with some closed Gromov-Witten invariants, whose computational techniques are better developed. Such relation was proven by Chan [9] when the toric Calabi-Yau is the total space of the canonical line bundle K_S of a toric manifold S. It can be generalized to other toric Calabi-Yau manifolds as well.

Figure 1.3 gives an illustration to this strategy. The statement is as follows. (The readers are referred to Section 7.2 for more detailed explanations of the terminologies involved.)

Theorem 1.2.4 (Restatement of Theorem 7.2.4). *Let X be a toric Calabi-Yau manifold, $\mathbf{T} \subset X$ be a regular toric fiber and $\beta \in \pi_2(X, \mathbf{T})$ be a disk class bounded by $\mathbf{T}$. Then the one-pointed genus zero open Gromov-Witten invariant $n_\beta \neq 0$ only when β is a basic disk class, in which case $n_\beta = 1$, or $\beta = b + \alpha$ where b is a basic disk class whose corresponding toric divisor is compact, and $\alpha \in H_2(X, \mathbb{Z})$ is represented by a rational curve.*

Let $\bar{X}$ be the compactification of X along the direction of b, and $h \in H_2(\bar{X})$ denote the fiber class. Then

$$n_{b+\alpha} = \langle [\mathrm{pt}] \rangle^{\bar{X}}_{0,1,h+\alpha}$$

provided that every complex curve in $\bar{X}$ representing $\alpha \in H_2(X) \subset H_2(\bar{X})$ is contained in $X \subset \bar{X}$. Here $\langle [\mathrm{pt}] \rangle^{\bar{X}}_{0,1,h+\alpha}$ denotes the one-pointed genus zero closed GW invariant of the class $h + \alpha$.

1.2.4 Open invariants of semi-Fano toric manifolds

This method of relating open Gromov-Witten invariants with closed invariants can be applied to other toric manifolds as well. Then one obtains an explicit

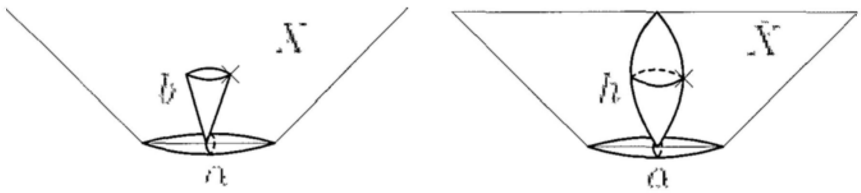

Figure 1.3: Equating open invariants with some closed invariants. Counting disks shown in the left hand side equals to counting rational curves shown in the right hand side.

expression of their Landau-Ginzburg mirrors, which are written in terms of the open GW invariants. This application will be discussed in Section 7.4.

Three dimensional examples include fiberwise compactifications of $K_{\mathbb{P}^2}$ and $K_{\mathbb{P}^1\times\mathbb{P}^1}$. Restricting to the surface case, one has the following result (which is a simplified version of Theorem 7.4.5:

Theorem 1.2.5 (Restatement of Theorem 7.4.5). *Let X be a compact semi-Fano toric surface, and $\beta \in \pi_2(X, \mathbf{T})$ be a disk class bounded by a Lagrangian torus fiber $\mathbf{T} \subset X$. Then $n_\beta = 1$ if β is an admissible disk class, and $n_\beta = 0$ otherwise. Thus its Landau-Ginzburg mirror superpotential*

$$W : (\mathbb{C}^\times)^2 \to \mathbb{C},$$

is written as

$$W = \sum_{\beta \text{ admissible}} Z_\beta$$

where Z_β are monomials whose coefficients are explicitly in terms of Kähler parameters of X.

See Theorem 7.4.5 for the meaning of admissibility. Figure 1.4 shows an example for illustration.

Remark 1.2.6. *The 'semi-Fano' condition is needed to ensure that the open GW invariants are well-defined (that is, independent of auxiliary choices such as*

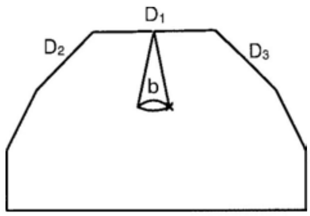

Figure 1.4: An example of a semi-Fano toric surface. The class $b+2D_1+D_2+D_3$ is admissible, while $b+2D_1+D_2$ is not.

complex structures). Moreover in the surface case, it implies the self-intersection number of every curve is at least (-2), so that the related closed GW invariants can be computed using the result of Bryan-Leung [6].

A recent result of FOOO [17] showed that

$$QH^*(X) \cong \mathrm{Jac}(W)$$

for all compact toric manifolds X. Now since we give an explicit expression to the right-hand-side, one obtains the following corollary:

Corollary 1.2.7. *When X is a semi-Fano toric surface, the quantum cohomology ring $QH^*(X)$ has an explicit presentation.*

1.3 Key examples

Let's illustrate the main ideas in this thesis by examples and figures. First we start with the simplest example $\mathbf{T}^2$ to explain T-duality. Then we consider $\mathbb{P}^1$ and its Landau-Ginzburg mirror. Wall-crossing phenomenon appears when one consider the Gross fibration on $\mathbb{C}^2$, which was studied by Auroux [3, 4]. Counting disks becomes more complicated when one considers $K_{\mathbb{P}^1}$, in which a bubbled disk occurs. It becomes even more complicated for $K_{\mathbb{P}^2}$, in which infinitely many classes of bubbled disks appear. Finally, we illustrate an application to semi-Fano toric manifold by considering the example $\bar{K}_{\mathbb{P}^2}$. The description in this section is expository, and the readers are referred to the later chapters for the precise definitions and statements.

1.3.1 Two-tori

Given a vector space V, one may take its dual V^* by collecting all the linear functionals on V. A family version of this duality is the duality between the tangent bundle TB and the cotangent bundle T^*B of a base manifold B.

Yet the total space TB is non-compact. To make it compact, one consider a lattice bundle $\Lambda \subset TB$ over B. Then one obtains a torus bundle TB/Λ. In the dual side one has T^*B/Λ^*, the *dual torus bundle.*

This is called *T-duality.* It works perfectly to explain mirror symmetry, without any corrections, when the base B is a torus. Without losing much, we may simply take $B = \mathbf{S}^1$. We refer to [39, 38] for the details.

$X = T^*B/\Lambda^*$ has a canonical symplectic structure $\mathrm{d}r \wedge \mathrm{d}\theta$, where r is the base coordinate and θ is the fiber coordinate. On the other side, $\check{X} = TB/\Lambda$ has a canonical complex coordinate $z = \exp(-r + \mathbf{i}\,\check{\theta})$.[3] This gives an illustration that symplectic structure is 'mirror' to complex structure. Moreover, homological mirror symmetry has been verified in this case [46], in which line bundles on $\check{X}$ corresponds to Lagrangian section of $X \to B$.

1.3.2 The complex projective line

However, compact torus bundles are certainly too restrictive. Most proper Lagrangian fibrations have singularities. Consider the sphere X in $\mathbb{R}^3$ with radius R centered at 0. It has a symplectic structure ω, which is simply its volume form. Then the projection to the x-axis gives a Lagrangian fibration $X \to B$ where

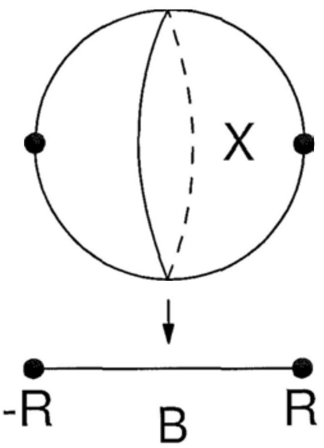

Figure 1.5: The moment map fibration on $\mathbb{P}^1$. The endpoints of the interval $B = [-R, R]$ are critical values, whose fibers degenerate as points.

$B = [-R, R] \subset \mathbb{R}$. The discriminant locus is $\partial B = \{-R, R\}$. See Figure 1.5.

Away from the singular fibers, one may take the dual torus bundle and obtain $B \times \mathbf{S}^1$, which has a complex coordinate $z = \exp(-r + \mathbf{i}\,\check{\theta})$, where r is the coordinate on $B = [-R, R]$ and $\check{\theta}$ is the coordinate on $\mathbf{S}^1$. z embeds $B \times \mathbf{S}^1$ into $\mathbb{C}^\times$ as an open subset. The physicist takes $\check{X} = \mathbb{C}^\times$ as the mirror manifold, which equals to B at the large volume limit $R = +\infty$.

The above duality does not capture the geometry of discriminant locus. To do this, one has to consider the Landau-Ginzburg mirror $(\mathbb{C}^\times, W)$, where W is a holomorphic function on $\mathbb{C}^\times$ given by

$$W = \mathrm{e}^{-R} z + \frac{\mathrm{e}^{-R}}{z}.$$

The two monomials in the above expression correspond to the two disks bounded by a fiber of $X \to B$ (see Figure 1.6). Moreover, their coefficients are e^{-R}, the Kähler parameter of X recording the area of the sphere. In general for compact toric manifolds, W is always written in terms of disk counting and

[3] In this thesis it is more convenient to take this convention, instead of setting $z = \exp(r + \mathbf{i}\,\check{\theta})$.

Kähler parameters, and the behavior of its critical points reflects the symplectic enumerative geometry of X [17].

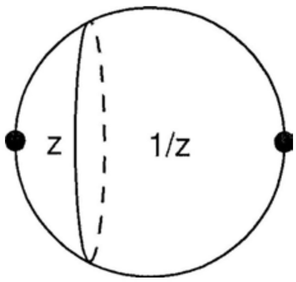

Figure 1.6: The two disks bounded by a toric fiber. One corresponds to the term z, and the other corresponds to $1/z$ in the expression of the superpotential.

For T-duality for compact toric manifolds, the readers are referred to [10].

1.3.3 The complex plane

For $X = \mathbb{C}^2$, the situation looks similar and it is tempting to write its mirror as $((\mathbb{C}^\times)^2, W = z_1 + z_2)$ (see Figure 1.7). However, such W does not possess any critical point, and it is not helpful to understand the symplectic geometry of X.

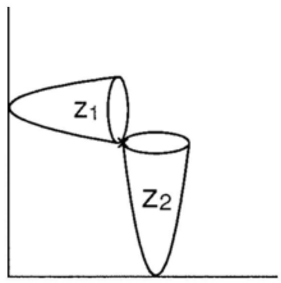

Figure 1.7: The two disks bounded by a toric fiber in $\mathbb{C}^2$, which should contribute to two terms in the superpotential.

Instead, the physicists [28] wrote down the mirror of X as

$$\{(u, v, z) \in \mathbb{C}^2 \times \mathbb{C}^\times : uv = 1 + z\} \tag{1.3.1}$$

and it is interesting to see how this comes up from the SYZ perspective. We will give the answer to this in Chapter 5. One needs to consider the fibration $\mu : X \to B$, $\mu(z_1, z_2) = (|z_1|^2 - |z_2|^2, |z_1 z_2 - 1| - 1)$ where B is the upper half plane. Figure 1.8 depicts the fibers of this fibration.

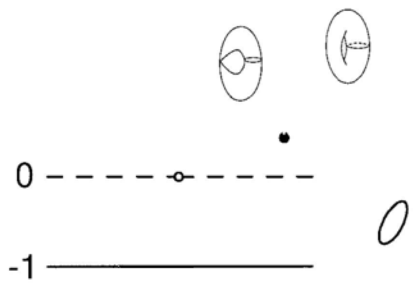

Figure 1.8: The fibers of μ on $\mathbb{C}^2$. Generic fibers are tori. The dotted line is the 'wall'.

The key ingredient is the wall-crossing phenomenon of disk counting studied extensively by Auroux [3, 4]. Roughly speaking, the base B is divided into chambers by 'walls', so that disk counting changes drastically as one moves from one chamber to another chamber. In this example, the wall is $\mathbb{R} \times \{0\}$. Below the wall, torus fibers only bound one disk; Above the wall, torus fibers bound two (see Figure 1.9). Such wall-crossing phenomenon forces one to correct the complex structure coming from T-duality. In Section 4.3 we introduce such a correction procedure, such that after correction one obtains the mirror (1.3.1).

Notice that there are two disks bounded by toric fiber, and there are two terms (1 and z) in the right hand side of (1.3.1). We will see in Chapter 5 that this is not a coincidence.

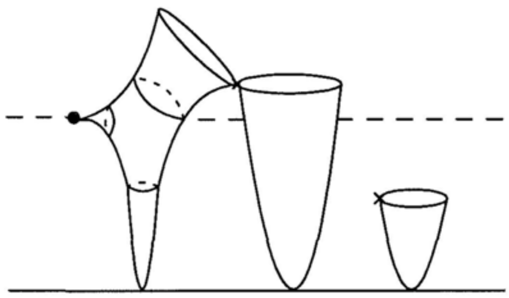

Figure 1.9: Wall-crossing phenomenon in $\mathbb{C}^2$. Fibers at a point below the wall bound one holomorphic disk, while those above the wall bound two holomorphic disk.

1.3.4 Canonical line bundle of the projective line

Disk counting in the toric Calabi-Yau manifold $K_{\mathbb{P}^1}$ is more complicated than in $\mathbb{C}^2$, since there is a holomorphic sphere in $K_{\mathbb{P}^1}$, leading to sphere bubbling of holomorphic disks (see Figure 1.10). In the presence of sphere bubbling, the moduli problem involves obstructions and one needs to consider the theory of virtual fundamental class, and so the task of disk counting becomes highly non-trivial.

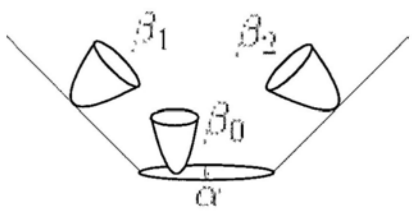

Figure 1.10: Sphere bubbling of holomorphic disks in $K_{\mathbb{P}^1}$. The moduli space of disks contains a holomorphic disk union with a sphere as shown in the figure.

In Chapter 7, we will count the disks in the presence of such obstruction. For this example, the result is that a toric fiber bounds four disks, which are in the classes $\beta_1, \beta_2, \beta_0$ and $\beta_0 + \alpha$ respectively (see Figure 1.10). The mirror of $K_{\mathbb{P}^1}$ is then

$$\{(u, v, z) \in \mathbb{C}^2 \times \mathbb{C}^\times : uv = 1 + \mathrm{e}^{-A} + z + \mathrm{e}^{-A} z^{-1}\}$$

where A is the symplectic area of the zero-section $\mathbb{P}^1 \subset K_{\mathbb{P}^1}$. Again the four terms corresponds to the four disks bounded by toric fibers.

1.3.5 Canonical line bundle of the projective plane

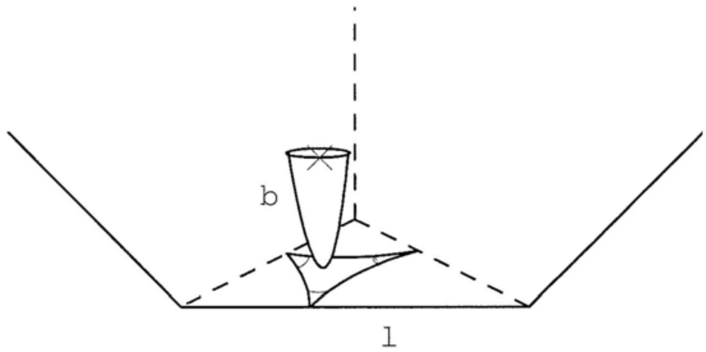

Figure 1.11: Sphere bubbling of holomorphic disks in $K_{\mathbb{P}^2}$. There are infinitely many classes of bubbled disks.

Similarly, sphere bubbling occurs in $K_{\mathbb{P}^2}$, and here the situation is even more complicated since a toric fiber bounds infinitely many disks. The mirror of $K_{\mathbb{P}^2}$ is The mirror of $K_{\mathbb{P}^2}$ is then

$$\{(u, v, z_1, z_2) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^2 : uv = 1 + \sum_{k=1}^{\infty} n_k \mathrm{e}^{-kA} + z_1 + z_2 + \mathrm{e}^{-A} z_1^{-1} z_2^{-1}\}$$

where A is the symplectic area of the line class $l \in H_2(X)$ of $\mathbb{P}^2$, and for each k, n_k is the number of disks in the class $b + kl$, b is the basic disk class corresponding to the zero section $\mathbb{P}^2 \subset K_{\mathbb{P}^2}$ (see Figure 1.11). In Chapter 7.3, we will see that n_k equal to the local invariants of the Hirzebruch surface $\mathbb{F}_1$, which can be computed via local mirror symmetry [11]. The first few terms are

$$\begin{aligned} n_1 &= -2, \\ n_2 &= 5, \\ n_3 &= -32, \\ n_4 &= 286, \\ n_5 &= -3038, \\ &\vdots \end{aligned}$$

1.3.6 The fiberwise compactification

The same computational method can be used to count the disks in the fiberwise compactification $X = \bar{K}_{\mathbb{P}^2}$ of $K_{\mathbb{P}^2}$. This application is discussed in Section 7.4. The Landau-Ginzburg mirror of X is $W : (\mathbb{C}^\times)^3 \to \mathbb{C}$,

$$W = \left(1 + \sum_{k=1}^{\infty} n_k \mathrm{e}^{-kA}\right) z_3 + z_1 z_3 + z_2 z_3 + \mathrm{e}^{-A} z_1^{-1} z_2^{-1} z_3 + z_3^{-1}$$

where A is the symplectic area of the line class $l \in H_2(X)$ of $\mathbb{P}^2$ and n_k is the same number for each k as that in the last example.

Chapter 2

Symplectic aspects of toric manifolds

This chapter aims at quickly building up basic terminologies and notations in symplectic geometry and toric geometry we need throughout the thesis.

2.1 Basic notions in symplectic geometry

This section introduces some very basic notions in symplectic geometry, aiming at setting up the necessary notations. We refer to the excellent book [8] by Cannas da Silva for more details. X is a smooth manifold throughout this section.

Definition 2.1.1. *A symplectic structure on a smooth manifold X is a closed non-degenerate two-form ω on X, that is, $\mathrm{d}\omega = 0$ and $\omega(v, \cdot) : T_pX \to T_p^*X$ is an isomorphism for all $v \in T_pX$. The pair (X, ω) is called a symplectic manifold.*

By the non-degeneracy condition on ω, $\dim X = 2n$ for some $n \in \mathbb{N}$. The following Darboux theorem tells us that (X, ω) is locally trivial, and so there is no local symplectic invariant (in contrast to Riemannian structures):

Theorem 2.1.2 (Darboux). *Let (X, ω) be a symplectic manifold. For each $p \in X$, there exists a coordinate chart (U, ρ), where $U \subset X$ is an open set containing*

p and $\rho : U \to V \subset \mathbb{C}^n$ is a diffeomorphism, such that $\rho^*(\omega_0|_V) = \omega|_U$. Here ω_0 is the standard symplectic form on $\mathbb{C}^n$, which is written as $\omega_0 = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n$ in terms of the standard coordinates $z_i = x_i + \mathbf{i}\, y_i$ on $\mathbb{C}^n$.

The objects of interest are Lagrangian submanifolds $L \subset X$:

Definition 2.1.3. *Let (X, ω) be a symplectic manifold.*

1. *A submanifold $L \subset X$ is said to be Lagrangian if L is of dimension n and $\omega|_L = 0$.*
2. *A pair (L, ∇) is called a Lagrangian brane if L is a Lagrangian submanifold of X and ∇ is a flat $U(1)$ connection on L.*

Remark 2.1.4. *From the perspective of mirror symmetry, a reason to study Lagrangian branes (instead of merely Lagrangian submanifolds) is the following.*

For a complex n-fold Y, a point $p \in Y$ has real $2n$-dimensional freedom of deformation. By mirror symmetry, in the symplectic side we should study objects which also has $2n$-dimensional freedom of movement. For a Lagrangian torus L in a symplectic $2n$-fold, the space of infinitesimal deformation of L is given by $H^1(L)$ which is of n-dimension only. To 'complexify' its deformation space one equips L with a flat $U(1)$ connection ∇, which adds n-dimension of freedom to the deformation since the space of flat $U(1)$ connections on L is also given by $H^1(L)$.

One of the key notions in SYZ mirror symmetry is a Lagrangian fibration on X defined as follows:

Definition 2.1.5. *Let (X, ω) be a symplectic manifold and B be a smooth manifold. A surjective map $\mu : X \to B$ is called a Lagrangian fibration if for every regular point $p \in X$ of μ, $\mathrm{Ker}(\mathrm{d}\mu(p)) \subset T_pX$ is a Lagrangian subspace.*

In this thesis we always assume that μ is proper (that is, inverse images of every compact set is compact) and all its fibers are connected. In such a setting

Arnold-Liouville theorem can be applied to infer that every regular fiber of μ is an n-torus:

Theorem 2.1.6 (Arnold-Liouville [2]). *Let $\mu : X \to B$ be a proper Lagrangian fibration with connected fibers. For every regular value $r_0 \in B$ of μ, there exists an open set $U \subset B$ containing r_0 such that one has a diffeomorphism $(\mu^{-1}(U), \omega, \mu) \cong (T^*U/\mathbb{Z}^n, \omega_0, \pi)$, where $\mathbb{Z}^n$ acts freely on each cotangent fiber T_r^*U by translations, ω_0 is the canonical symplectic form on T^*U and it descends to $T^*U/\mathbb{Z}^n$, and $\pi : T^*U \to U$ is the canonical bundle map. In particular, every regular fiber $\mu^{-1}\{r\}$ is diffeomorphic to a torus $T_r^*U/\mathbb{Z}^n$.*

As a consequence, if μ is a submersion (that is, all its values are regular), then μ is a Lagrangian torus bundle defined as follows:

Definition 2.1.7. *Let (X, ω) be a symplectic manifold and B be a smooth manifold. A fiber bundle $\mu : X \to B$ is called a Lagrangian torus bundle if each of its fibers is diffeomorphic to a torus and is a Lagrangian submanifold in (X, ω).*

2.2 Toric geometry

An important class of Lagrangian fibrations consists of moment maps on toric Kahler manifolds. Let $N \cong \mathbb{Z}^n$ be a lattice, and for simplicity we shall always use the notation $N_R := N \otimes R$ for a $\mathbb{Z}$-module R. In this thesis Σ always denotes a simplicial convex fan supported in $N_{\mathbb{R}}$, and X_Σ is the associated toric manifold admitting an action from the complex torus $N_{\mathbb{C}}/N \cong (\mathbb{C}^\times)^n$, which accounts for its name 'toric manifold'. X is compact if Σ is complete.

We now give a short explanation of some terminologies appearing in the previous paragraph and the construction of toric manifolds as GIT quotients. We begin with the definition of a fan:

Definition 2.2.1. *A fan Σ is a collection of (closed) cones sitting in $N_{\mathbb{R}}$, which satisfies the following compatibility conditions:*

1. *If $\sigma \in \Sigma$, then its faces (which are again cones) are also elements of Σ.*

2. *Any two cones in Σ intersect along their faces.*

The support of Σ is denoted by $|\Sigma| \subset N_{\mathbb{R}}$, which is the union of all cones contained in Σ. A ray of Σ is a one-dimensional cone in Σ.

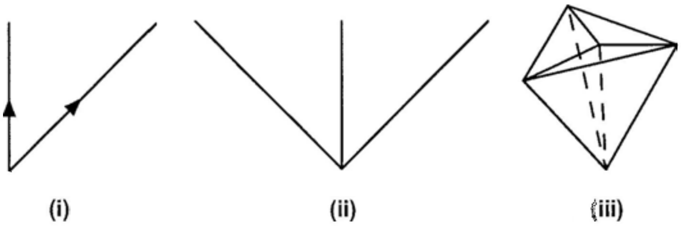

Figure 2.1: Fan pictures of some toric Calabi-Yau manifolds. (i.) $\mathbb{C}^2$. (ii.) $K_{\mathbb{P}^1}$. (iii.) $K_{\mathbb{P}^2}$.

Σ is said to be convex if its support $|\Sigma|$ is a convex subset of $N_{\mathbb{R}}$, and it is said to be strongly convex if in addition $|\Sigma|$ does not contain any whole line through the origin $0 \in N_{\mathbb{R}}$. It is said to be complete if $|\Sigma| = N_{\mathbb{R}}$. A convex fan Σ is said to be simplicial if every n-dimensional cone contained in Σ can be written as $\mathbb{R}_{\geq 0}\langle v_1, \ldots, v_n\rangle$ for a basis $\{v_i\}_{i=1}^n$ of N [1].

See Figure 2.1 for some examples. From now on Σ always denote a simplicial convex fan. Let $v_j \in N$ for $j = 0, \ldots, m-1$ be all the primitive generators of rays of Σ.

Then one defines a linear map

$$\mathbb{Z}^m \to N$$

[1] Usually a simplicial cone is defined as a cone in $N_{\mathbb{R}}$ spanned by n independent elements in N. For our purpose we require in addition that those elements form a basis.

by sending the standard basic vectors $e_i \in \mathbb{Z}^m$ to $v_i \in N$. Let $K \subset \mathbb{Z}^m$ be the kernel. $\mathbb{C}^\times(K) := K_\mathbb{C}/K$ acts on $\mathbb{C}^m$ by

$$[(a_1, \ldots, a_m)] \cdot (z_1, \ldots, z_m) = (\exp(2\pi \mathbf{i}\, a_1)z_1, \ldots, (\exp(2\pi \mathbf{i}\, a_m)z_m).$$

X_Σ is defined as a suitable quotient (the 'GIT' quotient) of $\mathbb{C}^m$ by $\mathbb{C}^\times(K)$. To do so we need to remove a 'bad subset' Z_Σ in $\mathbb{C}^m$ defined as follows.

A subset $S = \{v_{i_1}, \ldots, v_{i_k}\}$ of generators is called a primitive collection if $\mathbb{R}_{\geq 0}\langle S \rangle$ is not a cone of Σ, but every proper subset of S generates a cone of Σ. Z_Σ is defined as

$$Z_\Sigma := \bigcup_S \{(z_1, \ldots, z_m) \in \mathbb{C}^m : z_{i_1} = \ldots = z_{i_k} = 0\}$$

where the union is taken over primitive collections $S = \{v_{i_1}, \ldots, v_{i_k}\}$. Then X_Σ is defined as

$$X_\Sigma := (\mathbb{C}^m - Z_\Sigma)/\mathbb{C}^\times(K).$$

X_Σ admits a residual action by $N_\mathbb{C}/N$. There is an open subset in X_Σ on which $N_\mathbb{C}/N$ acts freely, which is $(\mathbb{C}^\times)^m/\mathbb{C}^\times(K) \subset X_\Sigma$. By abuse of notation we'll also denote this orbit by $N_\mathbb{C}/N \subset X_\Sigma$.

We denote by M the dual lattice of N. Every lattice point $\nu \in M$ gives a nowhere-zero holomorphic function $\exp 2\pi\mathbf{i}\ (\nu\, , \cdot) : N_\mathbb{C}/N \to \mathbb{C}$ which extends as a meromorphic function on X_Σ. Its zeroes and poles give a toric divisor which is linearly equivalent to 0. (A divisor $\mathcal{D}$ in X_Σ is toric if $\mathcal{D}$ is invariant under the action of $N_\mathbb{C}/N$ on X_Σ.)

We may further equip X_Σ with a Kähler form ω which is invariant under the torus action by $\mathbf{T} := N_\mathbb{R}/N$. It induces a moment map

$$\mu_0 : \mathbb{P}_\Sigma \to M_\mathbb{R}$$

whose image is a polyhedral set $P \subset M_\mathbb{R}$ defined by a system of inequalities

$$(v_j\, , \cdot) \geq c_j$$

where $c_j \in \mathbb{R}$ are some suitable constants. μ_0 provides a Lagrangian fibration on X_Σ. Every interior point r of P is regular, at which the fiber is $\mu_0^{-1}\{r\} \cong \mathbf{T}^n$. We will also call μ_0 as a toric fibration on X_Σ (to distinguish with the Gross fibration which will be introduced in Section 5.1).

P admits a natural stratification by its faces. Each codimension-one face $T_j \subset P$ which is normal to $v_j \in N$ corresponds to an irreducible toric divisor $\mathcal{D}_j = \mu_0^{-1}(T_j) \subset X_\Sigma$ for $j = 0, \ldots, m$, and all other toric divisors are generated by $\{\mathcal{D}_j\}_{j=0}^{m-1}$. For example, the anti-canonical divisor K_X^{-1} is $\sum_{j=0}^{m-1} \mathcal{D}_j$.

In this thesis we mainly focus on toric Calabi-Yau manifolds, whose definition is given as follows:

Definition 2.2.2. *A toric manifold $X = X_\Sigma$ is Calabi-Yau if there exists a toric linear equivalence between its canonical divisor K_X and the zero divisor. In other words, there exists a dual lattice point $\underline{\nu} \in M$ such that*

$$(\underline{\nu}, v_i) = 1$$

for all $i = 0, \ldots, m-1$.

As an illustration Figure 2.1 gives the fan picture of some familiar toric Calabi-Yau manifolds. An important subclass of toric Calabi-Yau manifolds is given by total spaces of canonical line bundles of compact toric manifolds:

Example 2.2.3. *Let N' be a lattice of rank $n-1$, and $Y = Y_{\Sigma'}$ be a toric $(n-1)$-fold associated to a simplicial convex fan Σ' supported in $N_\mathbb{R}$, whose rays are generated by primitive generators $v'_1, \ldots, v'_{m-1} \in N'$.*

Define $N := N' \times \mathbb{Z}$, $v_0 := (0,1) \in N$, and $v_i := (v'_i, 1) \in N$ for all $i = 1, \ldots, m-1$. Every cone $\mathbb{R}_{\geq 0}\langle v'_{i_1}, \ldots, v'_{i_{k-1}}\rangle$ of Σ' induces a cone

$$\mathbb{R}_{\geq 0}\langle v_0, v_{i_1}, \ldots, v_{i_{k-1}}\rangle \subset N_\mathbb{R}.$$

Σ is defined as the collection of all such cones together with the trivial cone

$\{0\} \subset N_{\mathbb{R}}$, *and this gives a simplicial convex fan supported in* $N_{\mathbb{R}}$. *Define*

$$\underline{\nu} := (0,1) \in M = M' \times \mathbb{Z}$$

and by definition

$$(\underline{\nu}\,,\, v_i) = 1$$

for all $i = 0, \ldots, m-1$. *Thus the corresponding toric manifold* $X = X_\Sigma$ *is a toric Calabi-Yau manifold. Indeed* X *is the total space of the canonical line bundle of* Y.

2.3 Holomorphic disks bounded by toric fibers

Lagrangian fibrations are essential in SYZ mirror symmetry, since by taking their duals one obtains an approximation of the mirror, which is the so-called semi-flat mirror (see Section 4.3). Another key ingredient of SYZ is the counting of holomorphic (or more rigorously, stable) disks bounded by Lagrangian fibers, which is the data to correct the semi-flat mirror. More precisely:

Definition 2.3.1. *Let* (X, ω) *be a symplectic manifold and* $L \subset X$ *be a Lagrangian submanifold. Equip* X *with an almost complex structure* J *compatible with* ω.

A pseudoholomorphic disk bounded by L *is a smooth map* $u : (\Delta, \partial\Delta) \to (X, L)$,*where* $\Delta \subset \mathbb{C}$ *is the closed unit disk (and the notation means that* u *is a map from* Δ *to* X *and* $u(\partial\Delta) \subset L$*), such that* u *is holomorphic with respect to the almost complex structure* J, *that is,*

$$\mathrm{d}u \circ j = J \circ \mathrm{d}u,$$

where j *is the standard complex structure on the disk* $\Delta \subset \mathbb{C}$. *For simplicity we'll abbreviate 'pseudoholomorphic disk' to 'holomorphic disk'.*

In this thesis, L is taken to be a regular fiber of a Lagrangian fibration. The moduli of holomorphic disks and the definition of their counting will be discussed in Chapter 3.

For toric fibers in toric manifolds $X_\Sigma = (\mathbb{C}^m - Z_\Sigma)/\mathbb{C}^\times(K)$ (Section 2.2), holomorphic disks are easy to describe: Every holomorphic disk can be lifted to $(\mathbb{C}^m, L)$ where $L = \{|z_i| = r_i \text{ for } i = 1, \ldots, m\} \subset \mathbb{C}^m$ for some $r_i > 0$, and these can be written down explicitly. This is the work of Cho-Oh [12] who classified holomorphic disks bounded by Lagrangian toric fibers:

Theorem 2.3.2 (Theorem 5.3 of [12]). *Let $X = X_\Sigma$ be a toric manifold and $\mathbf{T} \subset X$ be a Lagrangian toric fiber. A holomorphic map $u : (\Delta, \partial\Delta) \to (X, \mathbf{T})$ can be lifted to*

$$\tilde{u} : (\Delta, \partial\Delta) \to (\mathbb{C}^m - Z_\Sigma, \pi^{-1}(\mathbf{T}))$$

where $\pi : \mathbb{C}^m - Z_\Sigma \to X$ is the natural quotient map. Moreover $\tilde{u} = (w_1, \ldots, w_m)$, where

$$w_j(z) = c_j \prod_{k=1}^{\mu_j} \frac{z - \alpha_{j,k}}{1 - \bar{\alpha}_{j,k} z}$$

for $c_j \in \mathbb{C}^\times$ and $\mu_j \in \mathbb{Z}_{\geq 0}$ for each $j = 1, \ldots, m$.

The theorem will be applied to compute the open Gromov-Witten invariants for toric Calabi-Yau manifolds in Section 5.4 and Section 7.2. The next chapter gives a brief introduction to the FOOO's formulation of open Gromov-Witten invariants [20, 21].

Chapter 3

Open Gromov-Witten invariants

In this chapter we introduce the concept of open Gromov-Witten invariant, which is the main ingredient for the quantum correction procedure to be discussed in Section 4.3. The idea of open Gromov-Witten invariant is along the same line as its closed counterpart, in which one has to define corresponding moduli spaces and investigate the compactness and transversality issues. Yet it is more technical and complicated since the moduli spaces involved to define open GW invariants usually have boundaries, which make cobordism arguments fail. Nevertheless, under some assumptions which are satisfied in the context of this thesis, this issue can be avoided (Proposition 3.2.7). We follow the approach of FOOO [20, 21], and the readers are referred to there for details.

Throughout this chapter, (X, ω) is a symplectic manifold equipped with an almost complex structure J compatible with ω (that is, $g(v, w) := \omega(Jv, w)$ gives a Hermitian metric on X with respect to J).

3.1 Classical invariants of disks

In this section we introduce two classical symplectic invariants: symplectic area and Maslov index. These invariants are associated to homotopy classes of disks.

Definition 3.1.1. *1. For a submanifold $L \subset X$, $\pi_2(X, L)$ is the group of homotopy classes of maps*

$$u : (\Delta, \partial\Delta) \to (X, L),$$

where $\Delta := \{z \in \mathbb{C} : |z| \leq 1\}$ denotes the closed unit disk in $\mathbb{C}$. We have a natural homomorphism

$$\partial : \pi_2(X, L) \to \pi_1(L)$$

defined by $\partial[u] := [u|_{\partial\Delta}]$.

2. For two submanifolds $L_0, L_1 \subset X$, $\pi_2(X, L_0, L_1)$ is the set of homotopy classes of maps

$$u : ([0,1] \times \mathbf{S}^1, \{0\} \times \mathbf{S}^1, \{1\} \times \mathbf{S}^1) \to (X, L_0, L_1).$$

Similarly we have the natural boundary maps $\partial_+ : \pi_2(X, L_0, L_1) \to \pi_1(L_1)$ and $\partial_- : \pi_2(X, L_0, L_1) \to \pi_1(L_0)$.

Given a disk class $\beta \in \pi_2(X, L)$, one may measure its symplectic area

$$\int_\beta \omega := \int_\Delta u^*\omega$$

where $u : (\Delta, \partial\Delta) \to (X, L)$ is a map representing β, and by Stokes' theorem the above integration is independent of the choice of representatives. Another important topological invariant for β is its Maslov index:

Definition 3.1.2. *Let L be a Lagrangian submanifold and $\beta \in \pi_2(X, L)$. Let $u : (\Delta, \partial\Delta) \to (X, L)$ be a representative of β. Since Δ is simply connected, one has the trivialization*

$$u^*TX \cong \Delta \times V,$$

where V is a symplectic vector space. Thus the subbundle $(\partial u)^*TL \subset (\partial u)^*TX|_L \cong \partial\Delta \times V$ over $\partial\Delta$ induces the Gauss map

$$\partial\Delta \to U(n)/O(n) \to U(1)/O(1) \cong \mathbf{S}^1,$$

where $U(n)/O(n)$ parameterizes all Lagrangian subspaces in V. The Maslov index $\mu(\beta) \in \mathbb{Z}$ of β is defined as the degree of this map [1]*, which is independent of the choice of representative of β and trivialization of TX over Δ.*

From the perspective of open Gromov-Witten theory, $\mu(\beta)$ is important for open Gromov-Witten theory because it determines the expected dimension of the moduli space of holomorphic disks (see Equation (3.2.1)).

In the next section, we'll see that to make the open Gromov-Witten theory well-behaved (see Proposition 3.2.7), one may impose the condition that L is an compact oriented spin Lagrangian submanifold with minimal Maslov index at least two (see Definition 3.2.6 and Proposition 3.2.7).

3.2 Moduli spaces of stable disks

To define genus-zero open Gromov-Witten invariants of (X, L), which are roughly speaking countings of pseudoholomorphic disks in X bounded by L, we need to define the moduli of pseudoholomorphic disks.

Recall that we have defined the notion of pseudoholomorphic disk in Definition 2 3 1. Their moduli is defined as follows:

Definition 3.2.1. *Let (X, ω) be a symplectic manifold equipped with a compatible almost complex structure J, $L \subset X$ be a Lagrangian submanifold, and $\beta \in \pi_2(X, L)$ be a disk class bounded by L.*

1. *The moduli space $\mathcal{M}_k^\circ(L, \beta)$ of pseudoholomorphic disks representing $\beta \in \pi_2(X, L)$ with k ordered boundary marked points is defined as the quotient by $\mathrm{Aut}(\Delta)$ of the set of all pairs $(u, (p_i)_{i=0}^{k-1})$, where*

$$u : (\Delta, \partial\Delta) \to (X, L)$$

[1] It should be clear from the context whether μ refers to a Lagrangian fibration or the Maslov index

is a pseudoholomorphic disk bounded by L with homotopy class $[u] = \beta$, and $(p_i \in \partial\Delta : i = 0, \ldots, k-1)$ is a sequence of boundary points labelled in the counterclockwise fashion. For convenience the notation $(u, (p_i)_{i=0}^{k-1})$ is usually abbreviated as u.

2. *The evaluation map* $\mathrm{ev}_i : \mathcal{M}_k^{\circ}(L,\beta) \to L$ *for $i = 0, \ldots, k-1$ is defined as*

$$\mathrm{ev}_i([u, (p_i)_{i=0}^{k-1}]) := u(p_i).$$

By index theory of the elliptic operator $\bar{\partial}$, one has

Proposition 3.2.2 ([20]). *$\mathcal{M}_k^{\circ}(L,\beta)$ has expected dimension*

$$\dim_{virt}(\mathcal{M}_k^{\circ}(L,\beta)) = \dim L + \mu(\beta) + k - 3. \tag{3.2.1}$$

The shorthand notation 'virt' stands for the word 'virtual', which refers to 'virtual fundamental chain' discussed below.

To define countings of holomorphic disks, one requires an intersection theory on $\mathcal{M}_k^{\circ}(L,\beta)$. This involves various issues:

1. *Compactification of moduli.*

$\mathcal{M}_k^{\circ}(L,\beta)$ is non-compact in general, and one needs to compactify the moduli. Analogous to closed Gromov-Witten theory, this involves the concept of stable disks. A *stable disk* bounded by a Lagrangian L with k ordered boundary marked points [2] is a pair $(u, (p_i)_{i=0}^{k-1})$, where

$$u : (\Sigma, \partial\Sigma) \to (X, L)$$

is a pseudoholomorphic map whose domain Σ is a 'semi-stable' Riemann surface of genus-zero with a non-empty connected boundary $\partial\Sigma$ [3], and $(p_i \in \partial\Sigma)$ is

[2] More generally one considers stable disks with both boundary and interior marked points. For our purpose we consider boundary marked points only.

[3] Roughly speaking, this means Σ consists of disk and sphere components, and each singular point is a 'normal crossing', that is, the neighborhood around every singular point is locally isomorphic to $\{(x, y \in \mathbb{C}) : xy = 0\}$.

a sequence of boundary points labelled in the counterclockwise fashion (with respect to the orientation of each disk components), such that it satisfies the stability condition: If a sphere component C of Σ is contracted under u (that is, $u|_C$ is constant), then C contains at least three special points of Σ If a disk component Δ is contracted under u, then Δ either contains at least two interior special points, or three boundary special points, or one interior and one boundary special points A point in Σ is said to be *special* if it is a singular or marked point. (See Figure 3.1 for an illustration.)

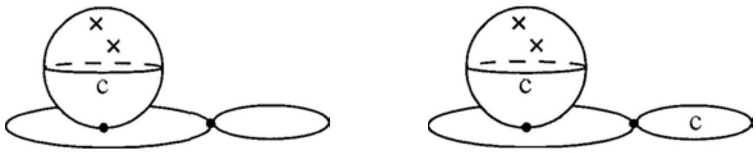

Figure 3.1: Examples of stable and unstable disks. The left one is stable while the right one is unstable. The components labelled by 'c' are contracted, and all others are not contracted The points represented by crosses are marked points, and those represented by dots are singular.

A compactification of $\mathcal{M}_k^\circ(L,\beta)$ is then given by the moduli space of stable disks:

Definition 3.2.3 (Definition 2.27 of [20]). *Let L be a compact Lagrangian submanifold in X and $\beta \in \pi_2(X, L)$. Then $\mathcal{M}_k(L,\beta)$ is defined to be the set of isomorphism classes of stable disks representing β with k ordered boundary marked points. Two stable disks $(u,(p_i))$ and $(u',(p'_i))$ are isomorphic if the maps u and u' have the same domain Σ and there exists $\phi \in \mathrm{Aut}(\Sigma)$ such that $u' = u \circ \phi$ and $\phi(p'_i) = p_i$.*

Remark 3.2.4. *In the above definition we require that the ordering of marked points respects the cyclic order of $\partial\Sigma$. In the terminologies and notations of*

[20], the above moduli is called the main component and is denoted by $\mathcal{M}_k^{main}(\beta)$ instead.

2. *Orientation.*

According to Chapter 9 of [21], $\mathcal{M}_k(L,\beta)$ is canonically oriented by fixing a relative spin structure on L. Thus the issue of orientation can be avoided by assuming that the Lagrangian L is relatively spin, which we shall always do from now on. Indeed, in this paper, L is always a torus, and so this assumption is satisfied.

3. *Transversality.*

An essential difficulty in Gromov-Witten theory is that in general, the moduli space $\mathcal{M}_k(L,\beta)$ is not of the expected dimension, which indicates the issue of non-transversality. To remedy the situation, one needs to do a generic perturbation and construct the virtual fundamental chain instead of working directly with $\mathcal{M}_k(L,\beta)$. This is done by Fukaya-Oh-Ohta-Ono [20, 21] which uses *Kuranishi structure* on $\mathcal{M}_k(L,\beta)$. We briefly recall its construction in the following. See Appendix A1 of the book [21] for more details.

Definition 3.2.5 (Definitions A1.1, A1.3, A1.5 in [21]). *Let $\mathcal{M}$ be a compact metrizable space. A Kuranishi structure on $\mathcal{M}$ of (real) virtual dimension d consists of the following data:*

(1) For each point $\sigma \in \mathcal{M}$,

(1.1) A smooth manifold V_σ (with boundary or corners) and a finite group Γ_σ acting smoothly and effectively on V_σ.

(1.2) A real vector space E_σ on which Γ_σ has a linear representation and such that $\dim V_\sigma - \dim E_\sigma = d$.

(1.3) A Γ_σ-equivariant smooth map $s_\sigma : V_\sigma \to E_\sigma$.

(1.4) A homeomorphism ψ_σ from $s_\sigma^{-1}(0)/\Gamma_\sigma$ onto a neighborhood of σ in $\mathcal{M}$.

(2) For each $\sigma \in \mathcal{M}$ and for each $\tau \in Im\ \psi_\sigma$,

(2.1) A Γ_τ-invariant open subset $V_{\sigma\tau} \subset V_\tau$ containing $\psi_\tau^{-1}(\tau)$.[4]

(2.2) A homomorphism $h_{\sigma\tau} : \Gamma_\tau \to \Gamma_\sigma$.

(2.3) An $h_{\sigma\tau}$-equivariant embedding $\varphi_{\sigma\tau} : V_{\sigma\tau} \to V_\sigma$ and an injective $h_{\sigma\tau}$-equivariant bundle map $\hat{\varphi}_{\sigma\tau} : E_\tau \times V_{\sigma\tau} \to E_\sigma \times V_\sigma$ covering $\varphi_{\sigma\tau}$.

Moreover, these data should satisfy the following conditions:

(i) $\hat{\varphi}_{\sigma\tau} \circ s_\tau = s_\sigma \circ \varphi_{\sigma\tau}$.[5]

(ii) $\psi_\tau = \psi_\sigma \circ \varphi_{\sigma\tau}$.

(iii) If $\xi \in \psi_\tau(s_\tau^{-1}(0) \cap V_{\sigma\tau}/\Gamma_\tau)$, then in a sufficiently small neighborhood of ξ,

$$\varphi_{\sigma\tau} \circ \varphi_{\tau\xi} = \varphi_{\sigma\xi},\ \hat{\varphi}_{\sigma\tau} \circ \hat{\varphi}_{\tau\xi} = \hat{\varphi}_{\sigma\xi}.$$

The spaces E_σ are called obstruction spaces (or obstruction bundles), the maps $\{s_\sigma : V_\sigma \to E_\sigma\}$ are called Kuranishi maps, and $(V_\sigma, E_\sigma, \Gamma_\sigma, s_\sigma, \psi_\sigma)$ is called a Kuranishi neighborhood of $\sigma \in \mathcal{M}$.

The Kuranishi structure on $\mathcal{M}_k(L, \beta)$ can be described as follows. Let $(u, (p_i)_{i=0}^{k-1})$ be representing a point $\sigma \in \mathcal{M}_k(L, \beta)$. Let $W^{1,p}(\Sigma; u^*(TX); L)$ be the space of sections v of $u^*(TX)$ of $W^{1,p}$ class such that the restriction of v to $\partial\Sigma$ lies in $u^*(TL)$, and $W^{0,p}(\Sigma; u^*(TX) \otimes \Lambda^{0,1})$ be the space of $u^*(TX)$-valued $(0,1)$-forms of $W^{0,p}$ class. Then consider the linearization of the Cauchy-Riemann operator $\bar{\partial}$

$$D_u\bar{\partial} : W^{1,p}(\Sigma; u^*(TX); L) \to W^{0,p}(\Sigma; u^*(TX) \otimes \Lambda^{0,1}).$$

[4]Here and in C2 below, we regard ψ_τ as a map from $s_\tau^{-1}(0)$ to $\mathcal{M}$ by composing with the quotient map $V_\tau \to V_\tau/\Gamma_\tau$

[5]Here and after, we also regard s_σ as a section $s_\sigma : V_\sigma \to E_\sigma \times V_\sigma$.

This map is not always surjective (i.e. u may not be regular), and this is exactly why we need to introduce the notion of Kuranishi structures. Define the obstruction space E_σ to be the cokernel of $D_u\bar{\partial}$. We also define Γ_σ to be the automorphism group of $(u, (p_i)_{i=0}^{k-1})$.

To construct V_σ, first let $V'_{\mathrm{map},\sigma}$ be the space of solutions of the equation

$$D_u\bar{\partial}v = 0 \bmod E_\sigma.$$

Now, the Lie algebra $\mathrm{Lie}(\mathrm{Aut}(\Sigma, (p_i)_{i=0}^{k-1}))$ of the automorphism group of $(\Sigma, (p_i)_{i=0}^{k-1})$ can naturally be embedded in $V'_{\mathrm{map},\sigma}$. Take its complement and let $V_{\mathrm{map},\sigma}$ be a neighborhood of its origin. On the other hand, let $V_{\mathrm{domain},\sigma}$ be a neighborhood of the origin in the space of first order deformations of the domain curve $(\Sigma, (p_i)_{i=0}^{k-1})$. Now, V_σ is given by $V_{\mathrm{map},\sigma} \times V_{\mathrm{domain},\sigma}$.

Next. one needs to prove that there exist a Γ_σ-equivariant smooth map $s_\sigma : V_\sigma \to E_\sigma$ and a family of smooth maps $u_{v,\zeta} : (\Sigma_\zeta, \partial\Sigma_\zeta) \to (X, L)$ for $(v, \zeta) \in V_\sigma$ such that $\bar{\partial}u_{v,\zeta} = s_\sigma(v, \zeta)$, and there is a map ψ_σ mapping $s_\sigma^{-1}(0)/\Gamma_\sigma$ onto a neighborhood of $\sigma \in \mathcal{M}_k(L, \beta)$. The proofs of these are very technical and thus omitted.

After introducing Kuranishi structure on $\mathcal{M}_k(L, \beta)$, one perturb the moduli by *Kuranishi multi-sections*. We will not give the precise definition of multi-sections here. See Definitions A1.19, A1.21 in [21] for details. Roughly speaking, a multi-section $\mathfrak{s}$ is a system of multi-valued perturbations $\{s'_\sigma : V_\sigma \to E_\sigma\}$ of the Kuranishi maps $\{s_\sigma : V_\sigma \to E_\sigma\}$ satisfying certain compatibility conditions. For a Kuranishi space with certain extra structures (this is the case for $\mathcal{M} = \mathcal{M}_k(L, \beta)$), there exist multi-sections $\mathfrak{s}$ which are transversal to 0. Furthermore, suppose that $\mathcal{M}$ is oriented. Let $ev : \mathcal{M} \to Y$ be a strongly smooth map to a smooth manifold Y (in this case $Y = L^k$), i.e. a family of Γ_σ-invariant smooth maps $\{ev_\sigma : V_\sigma \to Y\}$ such that $ev_\sigma \circ \varphi_{\sigma\tau} = ev_\tau$ on $V_{\sigma\tau}$. Then, using these transversal multisections, one can define the virtual fundamental chain $[\mathcal{M}]^{\mathrm{vir}}$ as a $\mathbb{Q}$-singular chain in Y (Definition A1.28 in [21]).

4. *Boundary strata of the moduli space.*

Another difficulty in the theory is that in general $\mathcal{M}_k(L,\beta)$ has codimension-one boundary strata, which consist of stable disks whose domain Σ has more than one disk components. Then intersection theory on $\mathcal{M}_k(L,\beta)$ is still not well-defined (because it depends on the choice of perturbation). A way to get around this problem is by imposing the condition that L has minimal Maslov index at least two, so that the moduli $\mathcal{M}_k(L,\beta)$ has no codimension-one boundary stratum:

Definition 3.2.6. *The minimal Maslov index of a Lagrangian submanifold L is defined as*

$$\min\{\mu(\beta) \in \mathbb{Z} : \beta \neq 0 \text{ and } \mathcal{M}_0(L,\beta) \text{ is non-empty}\}.$$

Proposition 3.2.7. *Let $L \subset X$ be a compact Lagrangian submanifold which has minimal Maslov index at least two, that is, L does not bound any non-constant stable disks of Maslov index less than two. Also let $\beta \in \pi_2(X,L)$ be a class with $\mu(\beta) = 2$. Then $\mathcal{M}_k(L,\beta)$ has no codimension-one boundary stratum.*

Proof. Let $u \in \beta$ be a stable disk belonging to a codimension-one boundary stratum of $\mathcal{M}_k(L,\beta)$. Then, by the results of [20, 21], u is a union of two non-constant stable disks u_1 and u_2. By forgetting the marked points (and contracting the unstable components in the domain if necessary), for each $i = 1,2$ u_i corresponds to an element $\tilde{u}_i \in \mathcal{M}_0(L,\beta)$, and $\mu(u_i) = \mu(\tilde{u}_i)$. By assumption $\mu(\tilde{u}_i) \geq 2$. Then $2 = \mu([u]) = \mu([u_1]) + \mu([u_2]) \geq 4$, which is false. □

When $\mathcal{M}_k(L,\beta)$ is compact oriented without codimension-one boundary strata, the virtual fundamental chain is a *cycle*. Hence, we have the virtual fundamental cycle $ev_*[\mathcal{M}_k(L,\beta)] \in H_d(L^k,\mathbb{Q})$, where $d = \dim_{\mathrm{virt}} \mathcal{M}_k(L,\beta)$. While one cannot do intersection theory on the moduli due to non-transversality, by introducing the virtual fundamental cycles, one may do intersection theory on L^k instead.

3.3 Generating functions of open GW invariants

With the above preparations, one-pointed genus-zero open Gromov-Witten invariants can be defined as follows.

Definition 3.3.1. *Let $L \subset X$ be a compact relatively spin Lagrangian submanifold which has minimal Maslov index at least two. For a class $\beta \in \pi_2(X, L)$ with $\mu(\beta) = 2$, we define*

$$n_\beta := \text{P.D.}(\mathcal{M}_1(L, \beta)) \cup \text{P.D.}([\text{pt}]) \in \mathbb{Q},$$

where $[\text{pt}] \in H_0(L, \mathbb{Q})$ is the point class in L, P.D. denotes the Poincaré dual, and $\cup$ is the cup product on $H^*(L, \mathbb{Q})$.

The number n_β is invariant under deformation of complex structure and under Lagrangian isotopy in which all Lagrangian submanifolds in the isotropy have minimal Maslov index at least two (see Remark 3.7 of [3]). This justifies that n_β is said to be an 'invariant'. Also, notice that the virtual dimension of $\mathcal{M}_1(L, \beta)$ equals $n + \mu(\beta) - 2 \geq n$ and it is equal to $n = \dim L$ only when $\mu(\beta) = 2$. Thus

Proposition 3.3.2. *Assume the setting as in Definition 3.3.1. Then $n_\beta = 0$ if $\mu(\beta) \neq 2$.*

As in closed Gromov-Witten theory, a good way to pack the data of open Gromov-Witten invariants is to form a generating function:

Definition 3.3.3. *Let $L \subset X$ be a compact relatively spin Lagrangian submanifold with minimal Maslov index at least two. For each $\lambda \in \pi_1(L)$, we have the generating function*

$$\mathcal{F}_X(L, \lambda) := \sum_{\beta \in \pi_2(X, L)_\lambda} n_\beta \exp\left(-\int_\beta \omega\right) \tag{3.3.1}$$

where

$$\pi_2(X, L)_\lambda := \{\beta \in \pi_2(X, L) : \partial\beta = \lambda\}. \tag{3.3.2}$$

For simplicity we'll abbreviate $\mathcal{F}_X$ to $\mathcal{F}$ when the background symplectic manifold X is clear from the context.

Intuitively $\mathcal{F}(L, \lambda)$ is a weighted count of stable disks bounded by the loop λ which passes through a generic point in L. In general, the above expression for $\mathcal{F}(L, \lambda)$ can be an infinite series, and one has to either take care of convergence issues or bypass the issues by considering the Novikov ring $\Lambda_0(\mathbb{Q})$, as done by Fukaya-Oh-Ohta-Ono in their works.

Definition 3.3.4. *The Novikov ring $\Lambda_0(\mathbb{Q})$ is the set of all formal series*

$$\sum_{i=0}^{\infty} a_i T^{\lambda_i}$$

where T is a formal variable, $a_i \in \mathbb{Q}$ and $\lambda_i \in \mathbb{R}_{\geq 0}$ such that $\lim_{i\to\infty} \lambda_i = \infty$.

Then $\mathcal{F}(L, \lambda) \in \Lambda_0(\mathbb{Q})$ is defined by

$$\mathcal{F}(L, \lambda) = \sum_{\beta \in \pi_2(X,L)_\lambda} n_\beta T^{\int_\beta \omega}.$$

The evaluation $T = \mathbf{e}^{-1}$ recovers Equation (3.3.1), if the corresponding series converges. In the rest of this paper, Equation (3.3.1) will be used, while we keep in mind that we can bypass the convergence issues by invoking the Novikov ring $\Lambda_0(\mathbb{Q})$.

Chapter 4

T-duality and Fourier transform

This chapter discusses how to use Fourier transform of Gromov-Witten invariants to obtain the quantum corrections in SYZ mirror symmetry. Section 4.1 is a quick review on Fourier series for tori, and Section 4.2 gives a family version for torus bundles. Section 4.3 is the main section, in which we propose a procedure to carry out SYZ construction with corrections. It will be used to construct the mirrors of toric Calabi-Yau manifolds in Chapter 5.

4.1 Fourier series

Let $\underline{\Lambda}$ be a lattice (that is, a free Abelian group with finite rank), and $V := \underline{\Lambda} \otimes \mathbb{R}$ be the corresponding real vector space. Then $\mathbf{T} := V/\underline{\Lambda}$ is an n-dimensional torus. We use V^*, $\underline{\Lambda}^*$ and $\mathbf{T}^*$ to denote the dual of V, $\underline{\Lambda}$ and $\mathbf{T}$ respectively. There exists a unique $\mathbf{T}$-invariant volume form dVol on $\mathbf{T}$ such that $\int_{\mathbf{T}} \mathrm{dVol} = 1$. One has the following well-known Fourier transform for complex-valued functions:

$$\begin{aligned} l^2(\underline{\Lambda}^*) &\cong L^2(\mathbf{T}) \\ f &\leftrightarrow \check{f} \end{aligned}$$

where for each $\breve{\theta} \in \mathbf{T}$,

$$\breve{f}(\breve{\theta}) = \sum_{\lambda \in \underline{\Lambda}^*} f(\lambda) \mathbf{e}^{2\pi \mathbf{i} (\lambda, \breve{\theta})} \tag{4.1.1}$$

and for each $\lambda \in \underline{\Lambda}^*$,

$$f(\lambda) = \int_{\mathbf{T}} \breve{f}(\breve{\theta}) \mathbf{e}^{-2\pi \mathbf{i} (\lambda, \breve{\theta})} \mathrm{dVol}(\breve{\theta}). \tag{4.1.2}$$

The above familiar expressions have the following natural interpretation. $\underline{\Lambda}^* = \mathrm{Hom}(\mathbf{T}, U(1))$ parametrizes all characters of the Abelian group $\mathbf{T}$, and conversely $\mathbf{T} = \mathrm{Hom}(\underline{\Lambda}^*, U(1))$ parametrizes all characters of $\underline{\Lambda}^*$. Consider the following diagram:

$$\begin{array}{ccc} & \underline{\Lambda}^* \times \mathbf{T} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \underline{\Lambda}^* & & \mathbf{T} \end{array}$$

$\underline{\Lambda}^* \times \mathbf{T}$ admits the universal character function $\chi : \underline{\Lambda}^* \times \mathbf{T} \to U(1)$ defined by

$$\chi(\lambda, \breve{\theta}) := \mathbf{e}^{2\pi \mathbf{i} (\lambda, \breve{\theta})}$$

which has the property that $\chi|_{\{\lambda\} \times \mathbf{T}}$ is exactly the character function on $\mathbf{T}$ corresponding to λ, and $\chi|_{\underline{\Lambda}^* \times \{\breve{\theta}\}}$ is the character function on $\underline{\Lambda}^*$ corresponding to $\breve{\theta}$. For a function $f : \underline{\Lambda}^* \to \mathbb{C}$, we have the following natural transformation

$$\breve{f} := (\pi_2)_* \big((\pi_1^* f) \cdot \chi \big)$$

where $(\pi_2)_*$ denotes integration along fibers using the counting measure of $\underline{\Lambda}^*$. This gives equation (4.1.1). Conversely, given a function $\breve{f} : \mathbf{T} \to \mathbb{C}$, we have the inverse transform

$$f := (\pi_1)_* \big((\pi_2^* \breve{f}) \cdot \chi^{-1} \big)$$

where $(\pi_1)_*$ denotes integration along fibers using the volume form dVol of $\mathbf{T}$. This gives equation (4.1.2).

We will mainly focus on the subspace $C^\infty(\mathbf{T})$ of smooth functions on $\mathbf{T}$. Then Fourier transform restricted on this subspace gives

$$C^{\text{r.d.}}(\underline{\Lambda}^*) \cong C^\infty(\mathbf{T})$$

where $C^{\text{r.d.}}(\underline{\Lambda}^*)$ consists of rapid-decay functions f on $\underline{\Lambda}^*$. f decays rapidly means that for all $k \in \mathbb{N}$,

$$||\lambda||^k f(\lambda) \to 0$$

as $\lambda \to \infty$. Here we have chosen a linear metric on V and

$$||\lambda|| := \sup_{|v|=1} |\,(\lambda\,,\, v)\,|.$$

The notion of rapid decay is independent of the choice of a linear metric on V.

4.2 Fiberwise Fourier transform

Now consider a Lagrangian torus bundle $\mu : X \to B$ (see Definition 2.1.7). We want to introduce Fourier transform in this setting. First we need a family version of the notion 'dual torus'.

In the previous section, for a torus $\mathbf{T} = V/\underline{\Lambda}$, the dual torus is defined as $\mathbf{T}^* := V^*/\underline{\Lambda}^*$. Note that there is a canonical identification

$$\pi_1(\mathbf{T}) \cong \underline{\Lambda}$$

and so

$$\begin{aligned} &V^*/\underline{\Lambda}^* = \mathrm{Hom}(\pi_1(\mathbf{T}), U(1)) \\ &\cong \{(L, \nabla) : L \text{ is a } \mathbb{C}\text{-line bundle; } \nabla \text{ is a flat } U(1) \text{ connection on } L\}/\text{gauge changes} \end{aligned}$$

where the last isomorphism is by recording the holomony of each flat $U(1)$ connection (L, ∇). ((L_1, ∇_1) and (L_2, ∇_2) differ by a gauge change if there is a bundle isomorphism $\Phi : L_1 \xrightarrow{\cong} L_2$ and $\nabla_1 = \Phi^* \nabla_2$.) Thus we may define the dual to a smooth torus $\mathbf{T}$ as the moduli of all flat $U(1)$ connections (L, ∇) on $\mathbf{T}$. This definition has the advantage that it is intrinsic, that is, it does not require choosing a diffeomorphism from the smooth torus $\mathbf{T}$ to $(\pi_1(\mathbf{T}))_{\mathbb{R}}/\pi_1(\mathbf{T})$.

With this intrinsic definition in hand, it is natural to define the dual torus bundle in the following way:

Definition 4.2.1. *For a Lagrangian torus bundle $\mu : X \to B$, $\check{X}$ is defined as the set of all $(F_r, \mathcal{L}, \nabla)$ up to gauge changes, where $r \in B$ and $F_r := \mu^{-1}(\{r\})$ is a fiber, $\mathcal{L}$ is a Hermitian line bundle over F_r and ∇ is a flat $U(1)$ connection on L. By forgetting (L, ∇) one has the map $\check{\mu} : \check{X} \to B$, which is called the dual torus bundle to μ.*

By taking local trivializations of μ it is a routine check that $\check{\mu}$ is a fiber bundle. Moreover since the space of flat $U(1)$ connections on F_r is the dual torus as we have explained, the fibers of $\check{\mu}$ are tori. Notice that $\check{\mu}$ always has the zero section by taking the trivial connection on the trivial complex line bundle over each fiber F_r (while μ may not possess a Lagrangian section in general), which is essential in the definition of Fourier transform.

One also has a family version of the lattice $\underline{\Lambda}^*$ as follows:

Definition 4.2.2. *For a torus bundle $\mu : X \to B$, define the lattice bundle*

$$\Lambda^* := \bigcup_{r \in B} \pi_1(F_r)$$

whose bundle map $\Lambda^* \to B$ is given by taking the base point $r \in B$.

We are ready to introduce the fiberwise Fourier transform. Analogous to Section 4.1, we have the following commutative diagram

$$
\begin{array}{ccccc}
 & & \Lambda^* \times_{B_0} \check{X}_0 & & \\
 & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\
\Lambda^* & & & & \check{X}_0 \\
 & \searrow & & \swarrow & \\
 & & B_0 & &
\end{array}
$$

Each fiber $\check{F}_r$ parametrizes the characters of Λ_r^* (which are homomorphisms $\Lambda_r^* \to U(1)$), and vice versa. $\Lambda^* \times_{B_0} \check{X}_0$ admits the universal character function $\chi : \Lambda^* \times_{B_0} \check{X}_0 \to U(1)$ defined as follows. For each $r \in B_0$, $\lambda \in \Lambda_r^*$ and $\nabla \in \check{F}_r$,

$$\chi(\lambda, \nabla) := \mathrm{Hol}_\nabla(\lambda)$$

which is the holonomy of the flat $U(1)$-connection ∇ over F_r around the loop λ. Thus we have the corresponding Fourier transform between functions on Λ^* and $\check{X}_0$ similar to Section 4.1:

$$C^{\mathrm{r.d.}}(\Lambda^*) \cong C^\infty(\check{X}_0)$$

where $C^{\mathrm{r.d.}}(\Lambda^*)$ consists of smooth functions f on Λ^* such that for each $r \in B_0$, $f|_{\Lambda_r^*}$ is a rapid-decay function. Explicitly, $f \in C^{\mathrm{r.d.}}(\Lambda^*)$ is transformed to

$$
\begin{aligned}
\check{f} : \check{X}_0 &\to \mathbb{C}, \\
\check{f}(F_r, \nabla) &= \sum_{\lambda \in \Lambda_r^*} f(\lambda)\mathrm{Hol}_\nabla(\lambda).
\end{aligned}
$$

In the next section, fiberwise Fourier transform will be used to construct the corrected SYZ mirror.

4.3 SYZ construction with corrections

In this section we introduce a construction procedure of mirrors which employs the SYZ program. This involves two steps. First one constructs the semi-flat mirrors of Lagrangian fibrations, which have been studied extensively in [38, 39].

Then we carry out the quantum correction procedure using Fourier transform of open Gromov-Witten invariants, with the assumption that the base of the Lagrangian fibration is a polytope. The result of the construction would be a 'Landau-Ginzburg mirror', which consists of a complex manifold $\check{X}$ together with a holomorphic function W on $\check{X}$.

4.3.1 The semi-flat mirror

Let (X, ω) be a symplectic manifold, B be a smooth manifold and $\mu : X \to B$ be a proper Lagrangian fibration with connected fibers (See Section 2.1). We first introduce the following notations:

Definition 4.3.1. *For a Lagrangian fibration $\mu : X \to B$, let*

$$\Gamma := \{r \in B : r \text{ is a critical value of } \mu\} \subset B$$

which is called the discriminant locus of μ. Then

$$B_0 := B - \Gamma$$

consists of the regular values of μ, and μ restricted to

$$X_0 := \mu^{-1}(B_0) \tag{4.3.1}$$

is a Lagrangian submersion onto B_0.

By Arnold-Liouville Theorem (Theorem 2.1.6), $\mu|_{X_0}$ is a Lagrangian torus bundle, and one may take its dual to obtain $\check{\mu} : \check{X}_0 \to B_0$ according to Definition 4.2.1. $\check{X}_0$ consists of all flat $U(1)$ connections on regular torus fibers of μ.

Moreover, $\check{X}_0$ admits a natural complex structure defined as follows:

Definition 4.3.2. *Let $\mu : X_0 \to B_0$ be a Lagrangian torus bundle and $\check{\mu} : \check{X}_0 \to B_0$ be its dual torus bundle. Fix any $r_0 \in B_0$, let $U \subset B_0$ be a contractible* *neighborhood of* r_0 *and trivialize* $\mu : \mu^{-1}(U) \to U$ *as* $U \times \mathbf{T} \to U$. *Let* $\{\lambda_i\}_{i=1}^n$ *be the standard basis of* $\pi_1(\mathbf{T}) \cong \mathbb{Z}^n$.

For each $r \in U$, *choose a path* $\gamma : [0,1] \to U$ *with* $\gamma(0) = r_0$ *and* $\gamma(1) = r$, *and by abuse of notation denote a map* $\mathbf{S}^1 \to \mathbf{T}$ *representing* $\lambda_i \in \pi_1(\mathbf{T})$ *also by* λ_i. *Let* $h_i(r)$ *denote the cylinder*

$$h_i(r) := \gamma \times \lambda_i : [0,1] \times \mathbf{S}^1 \to U \times \mathbf{T} \cong \mu^{-1}(U).$$

By Stokes' theorem, its symplectic area $\int_{h_i(r)}$ *is independent of choices of* γ *and a representative of* λ_i. *Moreover fixing a flat* $U(1)$*-connection* ∇ *over* F_r, *denote its holonomy around* $\lambda_i \in \pi_1(\mathbf{T}) \cong \pi_1(F_r)$ *by* $\mathrm{Hol}_\nabla(\lambda_i)$. *Then for* $i = 1, \ldots, n$, $z_i : \check{\mu}^{-1}(U) \to \mathbb{C}$ *is defined by*

$$z_i(F_r, \nabla) := \exp\left(-\int_{h_i(r)} \omega\right) \mathrm{Hol}_\nabla(\lambda_i). \tag{4.3.2}$$

It is a routine check that $\{z_i\}_{i=1}^n$ *defines local complex coordinates on* $\check{X}_0$. *They are called the semi-flat complex coordinates which define the corresponding semi-flat complex structure on* $\check{X}_0$.

Notations in the above definition are depicted in Figure 4.1. $\check{X}_0$ endowed with this semi-flat complex structure is called the *semi-flat mirror of* X [39, 38].

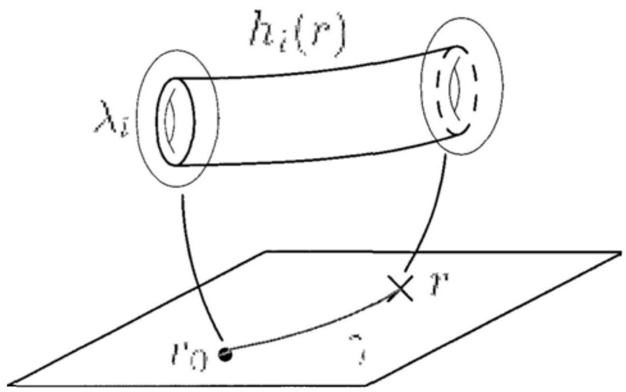

Figure 4.1: The cylinders h_i.

Furthermore, if the monodromy of the torus bundle $X_0 \to B_0$ (which is a homomorphism $\pi_1(B_0) \to GL(n, \mathbb{Z})$) has image inside $SL(n, \mathbb{Z})$, then one has a nowhere vanishing holomorphic n-form on $\check{X}_0$ which is

$$\mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n$$

when written in terms of the semi-flat complex coordinates. We call this the semi-flat holomorphic volume form. It was shown in [10] that this holomorphic volume form can be obtained by taking Fourier transform of $\exp(-\omega)$. Thus it encodes the symplectic geometric information of X_0.

While the definition of semi-flat mirror is simple and canonical, it does not capture the symplectic geometry of the singular fibers of μ. Moreover, as pointed out by Gross-Siebert [27], one has to 'correct' its semi-flat complex structure in order to compactify it to produce the mirror of X. This procedure is introduced in the next section.

4.3.2 Quantum corrections

We now define a procedure to construct the SYZ mirror out of the dual torus bundle and symplectic enumerative information of X. This employs open Gromov-Witten invariants introduced in Chapter 3 and Fourier transform introduced in the last section.

The setting is as follows.

Assumption 4.3.3. *Let X be a Kähler manifold of complex dimension n, and $\mu : X \to B$ be a proper Lagrangian fibration with connected fibers. We'll make the following assumptions:*

1. *$B \subset \mathbb{R}^n$ is a polytope.*

2. *Denote the facets of B by Ψ_j, $j = 0, \ldots, m-1$. The preimages*

$$D_j := \mu^{-1}(\Psi_j) \subset X$$

are Weil divisors of X *for all* $j = 0, \ldots, m-1$. *They are called the* boundary divisors. *Moreover* $-K_X = \sum_{j=0}^{m-1} D_j$.

3. *For every regular fiber* F_r *and* $\beta \in \pi_2(X, F_r)$,

$$\mu(\beta) = 2\,\beta \cdot \sum_{j=0}^{m-1} D_j.$$

4. *Let* B_0 *be the set of regular values of* μ. *For generic* $r \in B_0$ *(that is, away from a proper closed subset of* B_0*), the torus fiber* F_r *has minimal Maslov index at least two.*

Figure 4.2 gives a simple illustration of the notations in the above setting for the moment map on $\mathbb{P}^1$.

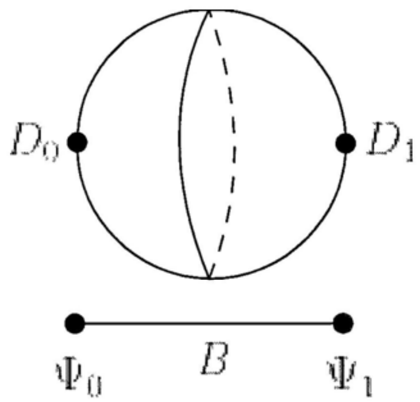

Figure 4.2: The moment map fibration over $\mathbb{P}^1$.

Example 4.3.4. *Let* X *be a compact toric manifold endowed with a toric Kähler form. Let* μ *be the moment map induced by the torus action, which gives a Lagrangian fibration on* X. *The image* B *of* μ *is a polytope. The inverse images* $D_j = \mu^{-1}(\Psi_j) \subset X$ *of the facets* Ψ_j *are toric divisors in* X, *and* $\sum_{j=0}^{m-1} D_j$ *is an anti-canonical divisor of* X. *Thus Assumptions (1), (2), (3) are satisfied (Formula (3) for the Maslov index is proven by Cho-Oh [12]).*

Furthermore, if X is Fano, then $-K_X \cdot C > 0$ for every non-constant holomorphic curve C. Suppose $u \in \mathcal{M}_0(\mathbf{T}, \beta)$ is a stable disk with Maslov index less than two bounded by a Lagrangian toric fiber $\mathbf{T} \subset X$. By Cho-Oh's classification of holomorphic disks [12] (Theorem 2.3.2), each non-constant disk component of u intersect $\sum_{j=0}^{m-1} D_j$ at least once, and so has Maslov index at least two. Also by the Fano condition each non-constant sphere component has Maslov index at least two. This forces u to be a constant map. Hence Assumption (4) is also satisfied.

In Chapter 5 we will consider non-toric Lagrangian fibrations which satisfy these assumptions. By the previous section, by considering μ away from the discriminant locus $\Gamma \subset B$ one constructs the semi-flat mirror $\check{X}_0$. As a consequence to the above assumptions:

Proposition 4.3.5. *Under the setting of Assumption 4.3.3, ∂B is a subset of the discriminant locus Γ.*

Proof. Assume $r \in \partial B$ is a regular point of μ. Since regularity is an open condition, there exists a neighborhood $U \subset B$ of r such that μ is regular over U. Since B is a polytope, U has non-empty intersection with the interior of a facet Ψ_j for some j. Then $D_j = \mu^{-1}(\Psi_j)$ should have real dimension $2n-1$, contradicting that it is a Weil divisor in X. □

Now consider quantum corrections of the semi-flat mirror by open Gromov-Witten invariants, which are well-defined for Lagrangian fibers with minimal Maslov index at least two (see Section 3.3.1). This motivates the following definition:

Definition 4.3.6. *The wall $H \subset B_0$ is defined to be the set of all $r \in B_0$ such that F_r has minimal Maslov index less than two.*

The open Gromov-Witten invariants $n_\beta^{F_r}$ for $\beta \in \pi_2(X, F_r)$ are locally constant with respect to $r \in B_0 - H$. However, $n_\beta^{F_r}$ may change drastically as r vary from one connected component to another component of $B_0 - H$, and this is called the *wall-crossing phenomenon* which is studied extensively by Auroux [3, 4]. It will be studied in Section 5.4 for Gross fibrations on toric Calabi-Yau manifolds.

Then for a Lagrangian torus fiber F_r at $r \in B_0 - H$, a loop class $\lambda \in \pi_1(F_r)$ and a boundary divisor D_i, we may consider the weighted count of stable disks bounded by λ and passing through D_i:

$$\sum_{\beta \in \pi_2(X, F_r)_\lambda} (\beta \cdot D_i)\, n_\beta \exp\left(-\int_\beta \omega\right).$$

$(\beta \cdot D_i)$ is the intersection number in the above expression.

Recall that Λ^* is the lattice bundle over B_0 whose fiber Λ_r^* is $\pi_1(F_r)$. By taking a family version of the above expression, One has the following generating function of open Gromov-Witten invariants (which is similar to the one given in Definition 3.3.1):

Definition 4.3.7. *For each $i = 0, \ldots, m-1$, The generating function $\mathcal{I}_{D_i}$: $\Lambda^*|_{B_0 - H} \to \mathbb{R}$ of open Gromov-Witten invariants is defined by*

$$\mathcal{I}_{D_i}(\lambda) := \sum_{\beta \in \pi_2(X, F_r)_\lambda} (\beta \cdot D_i)\, n_\beta \exp\left(-\int_\beta \omega\right) \tag{4.3.3}$$

where $r \in B_0 - H$ is the image of λ under the bundle map $\Lambda^* \to B_0$; $\pi_2(X, F_r)_\lambda \subset \pi_2(X, F_r)$ consists of elements $\beta \in \pi_2(X, F_r)$ with $\partial\beta = \lambda$; $(\beta \cdot D_i)$ is the intersection number between β and D_i, which is well-defined because $D_i \cap F_r = \emptyset$. We may abbreviate $\mathcal{I}_{D_i}$ as $\mathcal{I}_i$.

Now apply the family version of Fourier transform introduced in the previous section on $\mathcal{I}_i$:

Definition 4.3.8. *Let $\check{X}_0 \to B_0$ be the semi-flat mirror. Define $\tilde{z}_i : \check{\mu}^{-1}(B_0 -$

$H) \to \mathbb{C}$ for $i = 0, \ldots, m-1$ as the Fourier transform of $\mathcal{I}_i$, which is

$$\begin{aligned}\tilde{z}_i(F_r, \nabla) &= \sum_{\lambda \in \pi_1(F_r)} \mathcal{I}_i(\lambda)\mathrm{Hol}_\nabla(\lambda) \\ &= \sum_{\beta \in \pi_2(X, F_r)} (\beta \cdot D_i)\, n_\beta \exp\left(-\int_\beta \omega\right) \mathrm{Hol}_\nabla(\partial\beta).\end{aligned}$$

$\tilde{z}_i$ are referred as the corrected holomorphic functions.

As we have mentioned above, n_β may vary drastically when r moves from one component to another component of $B_0 - H$. Thus in general $\tilde{z}_i$ cannot be extended to be a holomorphic function on the whole $\check{X}_0$. This indicates that the complex structure of $\check{X}_0$ requires a correction, and $\tilde{z}_i$ serves as the holomorphic functions with respect to the corrected complex structure. In view of this, define

Definition 4.3.9. *Let R be the subring of holomorphic functions on $(\check{\mu})^{-1}(B_0 - H) \subset \check{X}_0$ generated by the corrected holomorphic functions $\{\tilde{z}_i\}_{i=0}^{m-1}$. Define $\check{X} :=$* Spec$R$.

The above procedure applied on toric fibrations on toric Fano manifolds would be trivial: There is no wall (that is, $H = \emptyset$) and no quantum correction, and thus $\check{X} = (\mathbb{C}^\times)^n$. But one recalls that the mirror of a toric Fano manifold consists of not just the space $(\mathbb{C}^\times)^n$, but also a holomormphic function $W : (\mathbb{C}^\times)^n \to \mathbb{C}$ known as the Landau-Ginzburg superpotential. In order to obtain the superpotential, we consider the generating function $\mathcal{F}(F_r)$ given in Definition 3.3.1 for every $r \in B_0 - H$, and then apply Fourier transform:

Definition 4.3.10. *Define the superpotential $W : \check{\mu}^{-1}(B_0 - H) \to \mathbb{C}$ to be the Fourier transform of the generating function $\mathcal{F}(F_r)$ of open Gromov-Witten invariants, that is,*

$$\begin{aligned}W(F_r, \nabla) &= \sum_{\lambda \in \pi_1(F_r)} \mathcal{F}(F_r, \lambda)\mathrm{Hol}_\nabla(\lambda) \\ &= \sum_{\beta \in \pi_2(X, F_r)} n_\beta \exp\left(-\int_\beta \omega\right) \mathrm{Hol}_\nabla(\partial\beta).\end{aligned}$$

The pair $(\check{X}, W)$ is defined to be the *SYZ mirror* of X. In good situations (which is the case when we consider Gross fibrations on toric Calabi-Yau manifolds in Chapter 5), the superpotential W takes a particularly simple form:

Proposition 4.3.11. *Assume that for every $r \in B_0 - H$ and $\beta \in \pi_2(X, F_r)$ such that $n_\beta \neq 0$, there exists some $i \in \{0, \ldots, m-1\}$ such that $\beta \cdot D_i = 1$ and $\beta \cdot D_j = 0$ for all $j \neq i$. Then*

$$W = \tilde{z}_0 + \ldots + \tilde{z}_{m-1} \in R.$$

Proof. By definition,

$$\mathcal{F}(\lambda) = \sum_{\beta \in \pi_2(X,\lambda)} n_\beta \exp\left(-\int_\beta \omega\right).$$

The sum is over all β with $\mu(\beta) = 2$. By the above assumption, each β appearing in the above sum intersect exactly one of the boundary divisors D_i once. Thus

$$\mathcal{F}(\lambda) = \sum_{i=0}^{m-1} \mathcal{I}_i(\lambda)$$

and so its Fourier transform W is $\sum_{i=0}^{m-1} \tilde{z}_i$. □

In Section 5.5, the above procedure will be carried out in details for toric Calabi-Yau manifolds, which will reproduce the mirrors written down by Hori-Iqbal-Vafa [28] from the physical perspective. More than that, it produces the SYZ map which is the central object to study in this thesis.

Chapter 5

SYZ mirrors of toric Calabi-Yaus

Throughout this chapter, we will always take X to be a toric Calabi-Yau manifold. Gross [26] and Goldstein [24] have independently written down a non-toric proper Lagrangian fibration $\mu : X \to B$, and we give a brief review of them in Section 5.1. These Lagrangian fibrations have interior discriminant loci of codimension two, leading to the wall-crossing of genus-zero open Gromov-Witten invariants which will be discussed in Section 5.4. In Section 5.5 we apply the procedure given in Section 4.3 to construct the SYZ mirror $\check{X}$.

5.1 Gross fibrations on toric Calabi-Yau manifolds

First recall some notations for toric geometry introduced in Section 2.2. N is a lattice of rank n and Σ is a simplicial fan supported in $N_{\mathbb{R}} := N \otimes \mathbb{R}$. $X = X_\Sigma$ denotes the toric manifold associated to a simplicial convex fan Σ. The primitive generators of rays of Σ are denoted by v_i for $i = 0, \ldots, m-1$, where m is the number of these generators. Each v_i corresponds to an irreducible toric divisor $\mathcal{D}_i$.

In this chapter we work with toric Calabi-Yau manifolds (see Definition 2.2.2), and without loss of generality we always assume that $m \geq n$ and Σ is strongly convex. Recall that by the definition of toric CY, there is a distinguished lattice point $\underline{\nu} \in M$. The following collects some basic facts in this setting:

Proposition 5.1.1 ([26]). *The meromorphic function w corresponding to $\underline{\nu} \in M$ is holomorphic. The corresponding divisor (w) is $-K_X = \sum_{i=0}^{m-1} \mathcal{D}_i$.*

Proof. For each cone C in Σ, let $v_{i_1}, \ldots, v_{i_n}$ be its primitive generators, which form a basis of N because C is simplicial by smoothness of X_Σ. Let $\{\nu_j \in M\}_{j=1}^n$ be the dual basis, which corresponds to coordinate functions $\{\zeta_j\}_{j=1}^n$ on the affine piece U_C corresponding to the cone C. We have

$$\underline{\nu} = \sum_{j=1}^n \nu_j$$

because $(\underline{\nu}, v_{i_j}) = 1$ for all $j = 1, \ldots, n$. Then

$$w|_{U_C} = \prod_{j=1}^n \zeta_j$$

which is a holomorphic function whose zero divisor is exactly the sum of irreducible toric divisors of U_C. □

Proposition 5.1.2 ([26]). *Let $\{\nu_j\}_{j=0}^{n-1} \subset M$ be the dual basis of $\{v_0, \ldots, v_{n-1}\}$, and ζ_j be the meromorphic functions corresponding to ν_j for $j = 0, \ldots, n-1$. Then*

$$\mathrm{d}\zeta_0 \wedge \ldots \wedge \mathrm{d}\zeta_{n-1}$$

extends to a nowhere-zero holomorphic n-form Ω on X.

Proof. $\mathrm{d}\zeta_0 \wedge \ldots \wedge \mathrm{d}\zeta_{n-1}$ defines a nowhere-zero holomorphic n-form on the affine piece corresponding to the cone $\mathbb{R}_{\geq 0}\langle v_0, \ldots, v_{n-1}\rangle$. Let C be an n-dimensional cone in Σ, $\{\nu'_j\}_{j=0}^{n-1} \subset M$ be a basis of M which generates the dual cone of C, and let $\zeta'_0, \ldots, \zeta'_{n-1}$ be the corresponding coordinate functions on the affine piece U_C

corresponding to C. Then

$$\begin{aligned}
\mathrm{d}\zeta_0 \wedge \ldots \wedge \mathrm{d}\zeta_{n-1} &= \zeta_0 \ldots \zeta_{n-1} \mathrm{d}\log\zeta_0 \wedge \ldots \mathrm{d}\log\zeta_{n-1} \\
&= w\,\mathrm{d}\log\zeta_0 \wedge \ldots \mathrm{d}\log\zeta_{n-1} \\
&= (\det A) w\,\mathrm{d}\log\zeta_0' \wedge \ldots \mathrm{d}\log\zeta_{n-1}' \\
&= (\det A)\mathrm{d}\zeta_0' \wedge \ldots \wedge \mathrm{d}\zeta_{n-1}'
\end{aligned}$$

where A is the matrix such that $\nu_i = \sum_j A_{ij}\nu_j'$. Since the fan Σ is simplicial, $A \in \mathrm{GL}(n,\mathbb{Z})$ and hence $\det A = \pm 1$. Thus $\mathrm{d}\zeta_0 \wedge \ldots \wedge \mathrm{d}\zeta_{n-1}$ extends to a nowhere-zero holomorphic n-form on U_C. This proves the proposition because X is covered by affine pieces. □

Remark 5.1.3. *In Proposition 5.1.2 we have chosen the basis* $\{v_0, \ldots, v_{n-1}\} \subset N$. *If we take another basis* $\{u_0, \ldots, u_{n-1}\} \subset N$ *which spans some cone of* Σ, *then the same construction gives*

$$\mathrm{d}\zeta_0' \wedge \ldots \wedge \mathrm{d}\zeta_{n-1}' = \pm \mathrm{d}\zeta_0 \wedge \ldots \wedge \mathrm{d}\zeta_{n-1}$$

where ζ_j'*'s are coordinate functions corresponding to the dual basis of* $\{u_i\}$. *The reason is that both* $\{v_i\}$ *and* $\{u_i\}$ *are basis of* N, *and thus the basis change belongs to* $\mathrm{GL}(n,\mathbb{Z})$, *and its determinant is* ± 1. *Thus the holomorphic volume form, up to a sign, is independent of the choice of the cone and its basis.*

Let ω be a toric Kähler form on $\mathbb{P}_\Sigma$ and $\mu_0 : \mathbb{P}_\Sigma \to P$ be the corresponding moment map, where P is a polyhedral set defined by the system of inequalities

$$(v_j\,,\,\cdot) \geq c_j \tag{5.1.1}$$

for $j = 1, \ldots, m$ and constants $c_j \in \mathbb{R}$ as shown in Figure 5.1.

The moment map corresponding to the action of the subtorus

$$\mathbf{T}^{\perp\underline{\nu}} := N_{\mathbb{R}}^{\perp\underline{\nu}} / N^{\perp\underline{\nu}} \subset N_{\mathbb{R}}/N$$

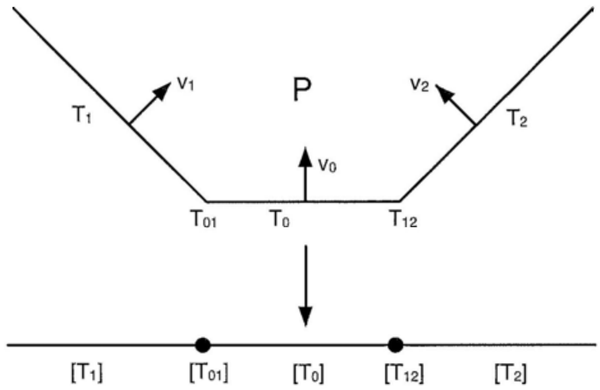

Figure 5.1: The toric moment map image of $K_{\mathbb{P}^1}$.

on X_Σ is

$$[\mu_0] : X_\Sigma \to M_\mathbb{R}/\mathbb{R}\langle \underline{\nu} \rangle$$

which is the composition of μ_0 with the natural quotient map $M_\mathbb{R} \to M_\mathbb{R}/\mathbb{R}\langle \underline{\nu} \rangle$.

Definition 5.1.4. *Fixing $K > 0$, the Gross fibration corresponding to K is*

$$\begin{array}{rcl} \mu : X & \to & M_\mathbb{R}/\mathbb{R}\langle \underline{\nu} \rangle \quad \times \quad \mathbb{R}_{\geq -K^2} \\ x & \mapsto & \big([\mu_0(x)] \quad , \quad |w(x) - K|^2 - K^2 \big). \end{array}$$

The base $(M_\mathbb{R}/\mathbb{R}\langle \underline{\nu} \rangle) \times \mathbb{R}_{\geq -K^2}$ is denoted by B.

Since another constant K_1 will appear in the next section, we will denote K by K_2 from now on.

One has to justify the term 'fibration' in the above definition, that is, $\mu : X \to B$ is surjective:

Proposition 5.1.5. *Under the natural quotient $M_\mathbb{R} \to M_\mathbb{R}/\mathbb{R}\langle \underline{\nu} \rangle$, ∂P is homeomorphic to $M_\mathbb{R}/\mathbb{R}\langle \underline{\nu} \rangle$. Thus μ maps X onto B.*

Proof. For any $\xi \in M_\mathbb{R}$, since $(v_j \, , \, \underline{\nu}) = 1$ for all $j = 1, \ldots, m$, we may take $t \in \mathbb{R}$ sufficiently large such that $\xi + t\underline{\nu}$ satisfies the above system of inequalities

$$(v_j \, , \, \xi + t\underline{\nu}) \geq c_j$$

and hence $\xi + t\underline{\nu} \in P$. Let t_0 be the infimum among all such t. Then $\xi + t_0\underline{\nu}$ still satisfies all the above inequalities, and at least one of them becomes equality. Hence $\xi + t_0\underline{\nu} \in \partial P$, and such t_0 is unique. Thus the quotient map gives a bijection between ∂P and $M_{\mathbb{R}}/\mathbb{R}\langle\underline{\nu}\rangle$. Moreover, the quotient map is continuous and maps open sets in ∂P to open sets in $M_{\mathbb{R}}/\mathbb{R}\langle\underline{\nu}\rangle$, and hence it is indeed a homeomorphism. □

It is proved by Gross that the above fibration is special Lagrangian using techniques of symplectic reduction:

Proposition 5.1.6 ([26]). *With respect to the symplectic form ω and the holomorphic volume form $\Omega/(w-K)$ defined on $\mu^{-1}(B^{\mathrm{int}}) \subset X$, μ is a special Lagrangian fibration, that is, there exists $\theta_0 \in \mathbb{R}/2\pi\mathbb{Z}$ such that for every regular fiber F of μ, $\omega|_F = 0$ and*

$$\mathrm{Re}\left(\frac{\mathbf{e}^{2\pi \mathbf{i}\theta_0}\,\Omega}{w-K}\right)\bigg|_F = 0.$$

This gives a proper Lagrangian fibration $\mu : X \to B$ where the base B is the upper half space. The inverse image $\mu^{-1}(\partial B) = \{w = K\}$ is referred as a 'boundary divisor' since it is a divisor whose image is the boundary of B.

5.2 Toric modification

To construct the mirror $\check{X}$ as a complex manifold, the idea is to construct coordinate functions of $\check{X}$ by counting holomorphic disks emanating from boundary divisors of $\mu : X \to B$ (Section 4.3). The problem is that in our situation, the base B has only one codimension-one boundary, while we need to construct at least n holomorphic functions. To resolve this issue, we consider a one-parameter family of Lagrangian fibrations $X^{(t)} \to B^{(t)}$, where $B^{(t)}$ is a polytope, such that $X \to B$ appears as the limit of this family. Then one may count holomorphic disks in $X^{(t)}$ to construct holomorphic functions of the mirror.

Choose a basis of N which generate a cone in Σ, say, the one given by $v_0, \ldots, v_{n-1}$. Since this is simplicial, $\{v_j\}_{j=0}^{n-1}$ forms a basis of N. We denote its dual basis by $\{\nu_j\}_{j=0}^{n-1} \subset M$ as before.

Remark 5.2.1. *While all the constructions from now on depend on the choice of this basis, we will see in Proposition 5.5.9 that the mirrors resulted from different choices of basis differ simply by a coordinate change.*

We define the following modification to $X = X_\Sigma$:

Definition 5.2.2. *Fix $K_1 > 0$.*

1. *Let*

$$P^{(K_1)} := \left\{\xi \in P : (-v_0\,,\, \xi) \geq -K_1; -K_1 \leq \left(v'_j\,,\, \xi\right) \leq K_1 \text{ for all } j = 1, \ldots, n-1\right\}$$

where P is the moment map image of the toric Calabi-Yau manifold X and $v'_j := v_j - v_0$ for $j = 1, \ldots, n-1$. We'll write $v_\infty := -v_0$ and $v'_{-j} := -v'_j$. More explicitly, the defining inequalities of $P^{(K_1)}$ are

$$\begin{cases} (v_i\,,\, \xi) & \geq c_i \quad \text{for } i = 0, \ldots, m-1; \\ (v_\infty\,,\, \xi) & \geq -K_1; \\ \left(v'_j\,,\, \xi\right) & \geq -K_1 \quad \text{for } j = 1, \ldots, n-1; \\ \left(v'_{-j}\,,\, \xi\right) & \geq K_1 \quad \text{for } j = 1, \ldots, n-1 \end{cases} \tag{5.2.1}$$

where c_i are the constants appearing in (5.1.1). *K_1 is assumed to be sufficiently large such that none of the defining inequalities of $P^{(K_1)}$ is redundant.*

2. *Let $\Sigma^{(K_1)}$ be the inward normal fan to $P^{(K_1)}$, whose rays are generated by $v_0, \ldots, v_{m-1}, v'_1, \ldots, v'_{n-1}, v'_{-1}, \ldots, v'_{-(n-1)}, v_\infty$.*

3. *Let $X^{(K_1)}$ be the toric Kähler manifold corresponding to $P^{(K_1)}$ and denote by*

$$\mu_0^{(K_1)} : X^{(K_1)} \to P^{(K_1)}$$

the corresponding moment map.

Since a toric CY can never be compact, $X^{(K_1)}$ is no longer a toric Calabi-Yau manifold. For notation simplicity, we will suppress the dependency on K_1 and write Σ' in place of $\Sigma^{(K_1)}$ and $\mu_0' : X' \to P'$ in place of $\mu_0^{(K_1)} : X^{(K_1)} \to P^{(K_1)}$ in the rest of this paper. The fan Σ' and toric moment map image P' of X' are demonstrated in Figure 5.2.

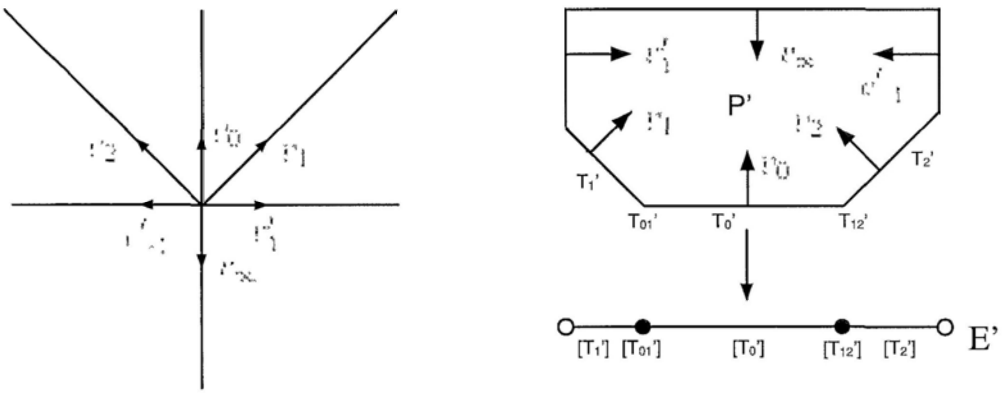

Figure 5.2: The fan and moment map polytope of X' when $X = K_{\mathbb{P}^1}$.

Analogously, one has a special Lagrangian fibration on X'. The definitions and propositions below are similar to that of Section 5.1, so they are written in a brief way. The proofs are similar and thus omitted.

Proposition 5.2.3. *Let w' be the meromorphic function on X' corresponding to $\underline{\nu}$. The corresponding divisor is*

$$(w') = \sum_{i=0}^{m-1} \mathcal{D}_i - \mathcal{D}_\infty$$

where we denote each irreducible toric divisor corresponding to v_i by $\mathcal{D}_i$ for $i = 0, \infty, \pm 1, \ldots, \pm(m-1)$. (Notice that w' is non-zero holomorphic on $\mathcal{D}_j'$'s, and so $\mathcal{D}_j'$'s do not appear in the above expression of (w').)

Proposition 5.2.4. *Let ζ_j be the meromorphic functions corresponding to ν_j for $j = 0, \ldots, n-1$. (Recall that $\{\nu_j\}_{j=0}^{n-1} \subset M$ is the dual basis to $\{v_j\}_{j=0}^{n-1} \subset N$.)*

Then

$$\Omega' := \mathrm{d}\zeta_0 \wedge \ldots \wedge \mathrm{d}\zeta_{n-1}$$

extends to a meromorphic n-form on X' with

$$(\Omega') = -\sum_{j=1}^{n-1} \mathcal{D}'_j - \sum_{j=1}^{n-1} \mathcal{D}'_{-j} - 2\mathcal{D}_\infty.$$

Definition 5.2.5. *Let*

$$E^{(K_1)} := \left\{ q \in M_\mathbb{R}/\mathbb{R}\langle \underline{\nu} \rangle : -K_1 \leq \left(v'_j\,,\, q \right) \leq K_1 \text{ for all } j = 1, \ldots, n-1 \right\}$$

and

$$B^{(K_1)} := E^{(K_1)} \times [-1, 1].$$

We have the fibration

$$\begin{array}{rcl} \mu^{(K_1)} : X^{(K_1)} & \to & B^{(K_1)} \\ x & \mapsto & \left([\mu_0^{(K_1)}(x)], f(|w'(x) - K_2|^2) \right). \end{array}$$

where $f : [0, +\infty] \to [-1, 1]$,

$$f(x) := -\frac{2K_2^2}{x + K_2^2} + 1$$

which is a smooth increasing function with $f(0) = -1$, $f(K_2^2) = 0$ and $f(+\infty) = 1$.

Again we'll suppress the dependency on K_1 for notation simplicity and use the notations E and $\mu' : X' \to B'$ instead.

Figure 5.2 shows an example of E', and Figure 5.3 depicts an example of the fibration μ'.

Proposition 5.2.6. *Under the natural quotient $M_\mathbb{R} \to M_\mathbb{R}/\mathbb{R}\langle \underline{\nu} \rangle$, the image of P' is E. As a consequence, $\mu' : X' \to B'$ is onto.*

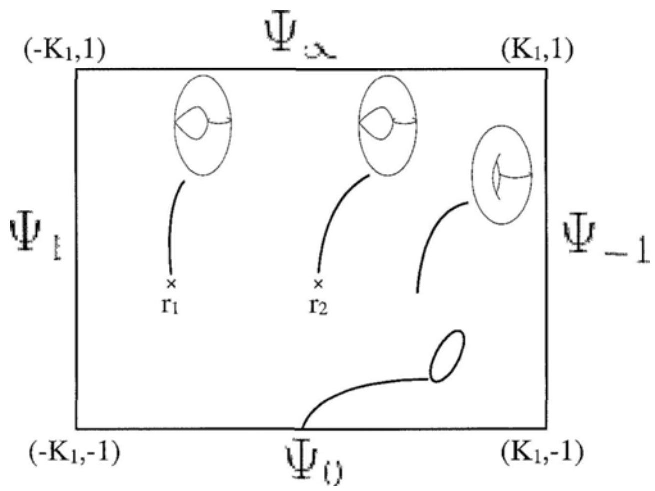

Figure 5.3: The base B' is a rectangle when $X = K_{\mathbb{P}^1}$. The discriminant locus is $\{r_1, r_2\} \cup \partial B'$.

B' is a polygon whose facets of B' are denoted as

$$\Psi_j := \{(q_1, q_2) \in B' : \left(v'_j\,,\, q_1\right) = -K_1\}$$

and

$$\Psi_{-j} := \{(q_1, q_2) \in B' : \left(-v'_j\,,\, q_1\right) = K_1\}$$

for $j = 1, \ldots, n-1$;

$$\Psi_0 := \{(q_1, q_2) \in B' : q_2 = -1\}$$

and

$$\Psi_\infty := \{(q_1, q_2) \in B' : q_2 = 1\}.$$

Their preimages under μ' are denoted as D_j, D_{-j}, D_0 and D_∞ respectively $D_{\pm j}$ are the toric divisors $\mathfrak{D}_{\pm j}$, D_∞ is the toric divisor $\mathfrak{D}_\infty$, and $D_0 = \{w' = K_2\}$ which is NOT a toric divisor. They are called the boundary divisors.

Proposition 5.2.7. *The polar divisor of the meromorphic volume form* $\Omega'/(w' -$

K_2) is

$$\left(\frac{\Omega'}{w'-K_2}\right) = -D_0 - D_\infty - \sum_{j=1}^{n-1} D_j - \sum_{j=1}^{n-1} D_{-j}.$$

$\mu' : X' \to B'$ is a special Lagrangian fibration with respect to the toric Kähler form and $\Omega'/(w'-K_2)$.

See Figure 5.3 for an illustration of the above notations. As the parameter $K_1 \to +\infty$, the boundary divisors $D_{\pm j}$ for $j = 1, \ldots, n-1$ and D_∞ move to infinity and so the Lagrangian fibration μ is recovered. Its mirror will be constructed in Section 5.5.

5.3 Topological properties of Gross fibrations

In this section we compute the discriminant locus of μ and fix a choice of generators of $\pi_2(X, F)$, where $F \subset X$ is a regular fiber of μ. We do the same things for the modified fibration μ'.

5.3.1 The Gross fibration

The discriminant locus

First we fix some notations:

Definition 5.3.1. *For each index set $\emptyset \neq I \subset \{0, \ldots, m-1\}$ such that $\{v_i : i \in I\}$ generates some cone C in Σ, let*

$$T_I := \left\{\xi \in P : (v_i\,, \xi) = c_i \text{ for all } i \in I\right\} \tag{5.3.1}$$

which is a codimension-$(|I|-1)$ face of ∂P.

Via the homeomorphism given in Proposition 5.1.5, $[T_I]$ gives a stratification of $M_{\mathbb{R}}/\mathbb{R}\langle \underline{\nu} \rangle$. This is demonstrated in Figure 5.1.

We are now ready to describe the discriminant locus Γ of μ:

Proposition 5.3.2. *Let μ be the Gross' fibration given in Definition 5.1.4. The discriminant locus of μ is*

$$\Gamma = \partial B \cup \left(\left(\bigcup_{|I|=2} [T_I] \right) \times \{0\} \right).$$

Proof. The critical points of $\mu = ([\mu_0], |w - K_2|^2 - K_2^2)$ are where the differential of $[\mu_0]$ or that of $|w - K_2|^2 - K_2^2$ is not surjective. The first case happens at the codimension-two toric strata of X, and the second case happens at the divisor defined by $w = K_2$. The images under μ of these sets are $\left(\bigcup_{|I|=2}[T_I]\right) \times \{0\}$ and ∂B respectively. □

An illustration of the discriminant locus is given by Figure 5.4.

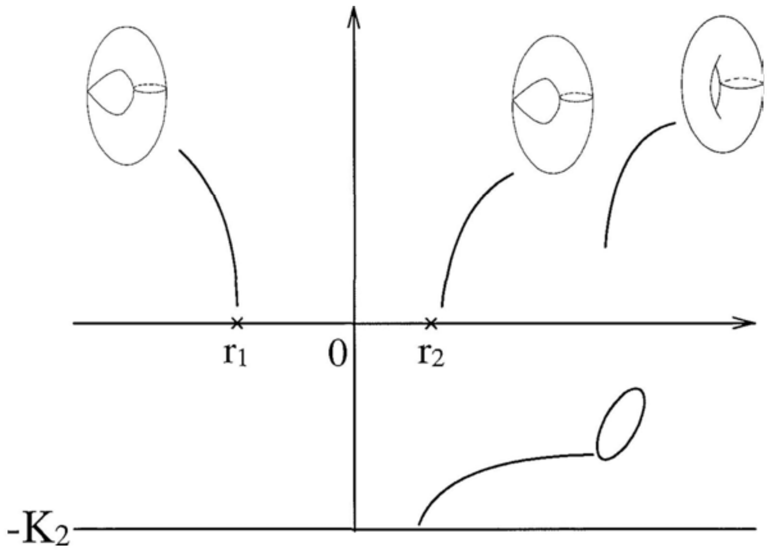

Figure 5.4: The base of the fibration $\mu : X \to B$ when $X = K_{\mathbb{P}^1}$. In this example, $\Gamma = \{r_1, r_2\} \cup \mathbb{R} \times \{-K_2\}$.

Local trivialization

By removing the singular fibers, we obtain a torus bundle $\mu : X_0 \to B_0$ (see Equation (4.3.1) for the notations). We now write down explicit local trivializations of this torus bundle, which will be used to make an explicit choice of generators of generators of $\pi_1(F)$ and $\pi_2(X, F)$. Let

$$U_i := B_0 - \bigcup_{k \neq i} ([T_k] \times \{0\})$$

for $i = 0, \ldots, m-1$, which are contractible open sets covering B_0, and hence $\mu^{-1}(U_i)$ can be trivialized. Without loss of generality, we will always stick to the open set

$$U := U_0 = B_0 - \bigcup_{k \neq 0} ([T_k] \times \{0\}) = \{(q_1, q_2) \in B_0 : q_2 \neq 0 \text{ or } q_1 \in [T_0]\}.$$

Proposition 5.3.3.

$$[T_0] = \left\{q \in M_{\mathbb{R}}/\mathbb{R}\langle \underline{\nu} \rangle : \left(v'_j\, , \, q\right) \geq c_j - c_0 \textit{ for all } j = 1, \ldots, m-1\right\}$$

where

$$v'_j := v_j - v_0 \tag{5.3.2}$$

defines linear functions on $M_{\mathbb{R}}/\mathbb{R}\langle \underline{\nu} \rangle$ *for* $j = 1, \ldots, m-1$.

Proof. T_0 consists of all $\xi \in M_{\mathbb{R}}$ satisfying

$$\begin{cases} (v_j\, , \, \xi) & \geq & c_j \text{ for all } j = 1, \ldots, m-1; \\ (v_0\, , \, \xi) & = & c_0. \end{cases}$$

which implies $\left(v'_j\, , \, q\right) \geq c_j - c_0$ for all $j = 1, \ldots, m-1$.

Conversely, if $q = [\xi] \in M_{\mathbb{R}}/\mathbb{R}\langle \underline{\nu} \rangle$ satisfies $\left(v'_j\, , \, q\right) \geq c_j - c_0$ for all $j = 1, \ldots, m-1$, then since $(\underline{\nu}\, , \, v_0) = 1$, there exists $t \in \mathbb{R}$ such that $(v_0\, , \, \xi + t\underline{\nu}) = c_0$. And we still have $\left(v'_j\, , \, \xi + t\underline{\nu}\right) \geq c_j - c_0$ for all $j = 1, \ldots, m-1$ because $\left(v'_j\, , \, \underline{\nu}\right) = 0$. Then $(v_j\, , \, \xi) \geq c_j$ for all $j = 1, \ldots, m-1$. Hence the preimage of q contains $\xi + t\underline{\nu} \in T_0$. □

Using the above proposition, the open set $U = U_0$ can be written as

$$\left\{(q_1, q_2) \in B^{\text{int}} : q_2 \neq 0 \text{ or } \left(v'_j\,,\, q_1\right) > c_j - c_0 \text{ for all } j = 1, \ldots, m-1\right\}$$

where v'_j is defined by Equation 5.3.2. Now we are ready to right down an explicit coordinate system on $\mu^{-1}(U)$.

Definition 5.3.4. *Let*

$$\mathbf{T}_N/\mathbf{T}\langle v_0\rangle := \frac{N_{\mathbb{R}}/\mathbb{R}\langle v_0\rangle}{N/\mathbb{Z}\langle v_0\rangle}.$$

We have the trivialization

$$\mu^{-1}(U) \xrightarrow{\sim} U \times (\mathbf{T}_N/\mathbf{T}\langle v_0\rangle) \times (\mathbb{R}/2\pi\mathbb{Z})$$

given as follows. The first coordinate function is simply given by μ.

To define the second coordinate function, let $\{\nu_0, \ldots, \nu_{n-1}\} \subset M$ be the dual basis to $\{v_0, \ldots, v_{n-1}\} \subset N$. Let ζ_j be the meromorphic functions corresponding to ν_j for $j = 1, \ldots, n-1$. Then the second coordinate function is given by

$$\left(\frac{\arg\zeta_1}{2\pi}, \ldots, \frac{\arg\zeta_{n-1}}{2\pi}\right) : \mu^{-1}(U) \to (\mathbb{R}/2\pi\mathbb{Z})^{n-1} \cong (\mathbf{T}_N/\mathbf{T}\langle v_0\rangle)$$

which is well-defined because for each $j = 1, \ldots, n-1$, $\nu_j \in M^{\perp v_0}$, implying ζ_j is a nowhere-zero holomorphic function on $\mu^{-1}(U)$.

The third coordinate is given by $\arg(w - K_2)$, which is well-defined because $w \neq K_2$ on $\mu^{-1}(U)$.

Explicit generators of $\pi_1(F_r)$ and $\pi_2(X, F_r)$

Now we define explicit generators of $\pi_1(F_r)$ and $\pi_2(X, F_r)$ for $r \in U$ in terms of the above coordinates. For $r \in U$, one has

$$F_r \cong (\mathbf{T}_N/\mathbf{T}\langle v_0\rangle) \times (\mathbb{R}/2\pi\mathbb{Z})$$

and hence

$$\pi_1(F_r) \cong (N/\mathbb{Z}\langle v_0\rangle) \times \mathbb{Z}$$

which has generators $\{\lambda_i\}_{i=0}^{n-1}$, where $\lambda_0 = (0,1)$ and $\lambda_i = ([v_i], 0)$ for $i = 1, \ldots, n-1$. This gives a basis of $\pi_1(F_r)$.

We take explicit generators of $\pi_2(X, F_r)$ in the following way. First we write down the generators for $\pi_2(X, \mathbf{T})$, which are well-known in toric geometry. Then we fix $r_0 = (q_1, q_2) \in U$ with $q_2 > 0$, and identify $\pi_2(X, \mathbf{T})$ with $\pi_2(X, F_{r_0})$ by choosing a Lagrangian isotopy between F_{r_0} and $\mathbf{T}$. (The choice $q_2 > 0$ seems arbitrary at this moment, but it will be convenient for the purpose of describing holomorphic disks in Section 5.4.) Finally $\pi_2(X, F_r)$ for every $r \in B_0$ is identified with $\pi_2(X, F_{r_0})$ by using the trivialization of $\mu^{-1}(U) \cong U \times F_{r_0}$. In this way we have fixed an identification $\pi_2(X, F_r) \cong \pi_2(X, \mathbf{T})$. The details are given below.

1. *Generators for* $\pi_2(X, \mathbf{T})$. Let $\mathbf{T} \subset X$ be a Lagrangian toric fiber, which can be identified with the torus $\mathbf{T}_N$. By [12], $\pi_2(X, \mathbf{T})$ is generated by the basic disk classes $\beta_j^{\mathbf{T}}$ corresponding to primitive generators v_j of a ray in Σ for $j = 0, \ldots, m-1$. One has

$$\partial \beta_j^{\mathbf{T}} = v_j \in N \cong \pi_1(\mathbf{T}_N).$$

These basic disk classes $\beta_i^{\mathbf{T}}$ can be expressed more explicitly in the following way. We take the affine chart $U_C \cong \mathbb{C}^n$ corresponding to the cone $C = \langle v_0, \ldots, v_{n-1} \rangle$ in Σ. Let

$$\mathbf{T}_\rho := \{(\zeta_0, \ldots, \zeta_{n-1}) \in \mathbb{C}^n : |\zeta_j| = \mathbf{e}^{\rho_j} \text{ for } j = 0, \ldots, n-1\} \subset X$$

be a toric fiber at $\rho = (\rho_0, \ldots, \rho_{n-1}) \in \mathbb{R}^n$. For $i = 0, \ldots, n-1$, $\beta \mathbf{T}_i$ is represented by the holomorphic disk $u : (\Delta, \partial\Delta) \to (U_C, \mathbf{T}_\rho)$,

$$u(\zeta) = (\mathbf{e}^{\rho_0}, \ldots, \mathbf{e}^{\rho_{i-1}}, \mathbf{e}^{\rho_i}\zeta, \mathbf{e}^{\rho_{i+1}}, \ldots, \mathbf{e}^{\rho_{n-1}}).$$

By taking other affine charts, other disk classes can be expressed in a similar way. Figure 5.5 gives a drawing for $\beta_i^{\mathbf{T}}$ when $X = K_{\mathbb{P}^1}$. Since every disk class $\beta_i^{\mathbf{T}}$ intersects the anti-canonical divisor $\sum_{i=0}^{m-1} \mathcal{D}_i$ exactly once, it has Maslov index two (Maslov index is twice the intersection number [12]).

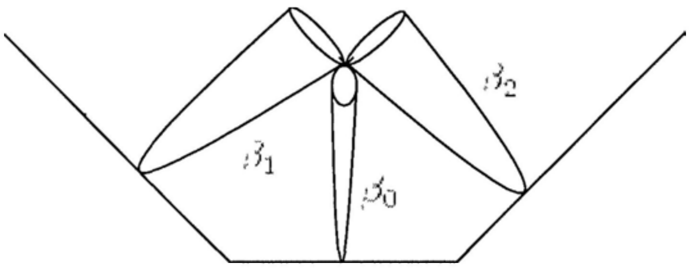

Figure 5.5: The basic disk classes in $\pi_2(X, \mathbf{T})$ for a toric fiber $\mathbf{T}_\rho$ of $X = K_{\mathbb{P}^1}$.

2. *Lagrangian isotopy between F_{r_0} and* $\mathbf{T}$.

Fix $r_0 = (q_1, q_2) \in B_0$ with $q_2 > 0$. We have the following Lagrangian isotopy relating fibers of μ and Lagrangian toric fibers:

$$L_t := \{x \in X : [\mu_0(x)] = q_1; |w(x) - t|^2 = K_2^2 + q_2\} \tag{5.3.3}$$

where $t \in [0, K_2]$. L_0 is a Lagrangian toric fiber, and $L_{K_2} = F_{r_0}$. (This is also true for $q_2 < 0$. We fix $q_2 > 0$ for later purpose.)

The isotopy gives an identification between $\pi_2(X, F_{r_0})$ and $\pi_2(X, \mathbf{T})$. Thus we may identify $\{\beta_j^{\mathbf{T}}\}_{j=0}^{m-1} \subset \pi_2(X, \mathbf{T})$ as a generating set of $\pi_2(X, F_{r_0})$, and we denote the corresponding disk classes by $\beta_j \in \pi_2(X, F_{r_0})$. They are depicted in Figure 5.6.

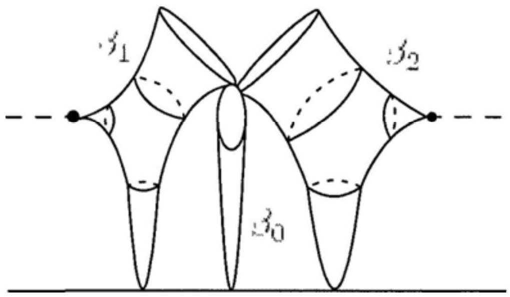

Figure 5.6: Disks generating $\pi_2(X, \mathbf{F_r})$ when $X = K_{\mathbb{P}^1}$.

Finally by the trivialization of $\mu^{-1}(U)$, every fiber F_r at $r \in U$ is identified with F_{r_0}, and thus $\{\beta_j\}_{j=0}^{m-1}$ may be identified as a generating set of $\pi_2(X, F_r)$.

Notice that since Maslov index is invariant under Lagrangian isotopy, each $\beta_j \in \pi_2(X, F_r)$ remains to have Maslov index two. We will need the following description for the boundary classes of β_j:

Proposition 5.3.5.

$$\partial\beta_j = \lambda_0 + \sum_{i=1}^{n-1} (\nu_i\,,\, v_j)\,\lambda_i \in (N/\mathbb{Z}\langle v_0\rangle) \times \mathbb{Z} \cong \pi_1(F_r)$$

for all $j = 0, \ldots, m-1$, where $\{\nu_i\}_{i=0}^{n-1} \subset M$ is the dual basis of $\{v_i\}_{i=0}^{n-1} \subset N$.

Proof. Under the identification

$$\mathbf{T}_N \xrightarrow{\cong} (\mathbf{T}_N/\mathbf{T}\langle v_0\rangle) \times (\mathbb{R}/2\pi\mathbb{Z})$$

where the last coordinate is given by $(\underline{\nu}\,,\,\cdot)$, $\partial\beta_j^{\mathbf{T}} = v_j \in \pi_1(\mathbf{T}_N)$ is identified with

$$([v_j], 1) = \left(\sum_{i=1}^{n-1} (\nu_i\,,\, v_j)\,[v_i], 1\right) = \sum_{i=1}^{n-1} (\nu_i\,,\, v_j)\,\lambda_i + \lambda_0 \in \pi_1\left((\mathbf{T}_N/\mathbf{T}\langle v_0\rangle) \times (\mathbb{R}/2\pi\mathbb{Z})\right)$$

because $(\underline{\nu}\,,\, v_j) = 1$ for all $j = 0, \ldots, m-1$. Under the isotopy given in Equation (5.3.3), this relation is preserved. □

The following proposition gives the intersection numbers of the disk classes with various divisors:

Proposition 5.3.6. *Let $r = (q_1, q_2) \in U$ with $q_2 \neq 0$, and $\beta_i \in \pi_2(X, F_r)$ be the disk classes defined above. Then*

$$\beta_i \cdot D_0 = 1$$

for all $i = 0, \ldots, m-1$, where

$$D_0 := \{x \in X : w(x) = K_2\} \tag{5.3.4}$$

is the boundary divisor whose image under μ is ∂B.

Proof. We need to use the following topological fact: Let $\{L_t : t \in [0,1]\}$ be an isotopy between L_0 and L_1, and $\{S_t : t \in [0,1]\}$ be an isotopy between the cycles S_0 and S_1. Suppose that for all $t \in [0,1]$, $L_t \cap S_t = \emptyset$. Then for $\beta \in \pi_2(X, L_0)$, one has the following equality of intersection numbers:

$$\beta \cdot S_0 = \beta' \cdot S_1$$

where $\beta' \in \pi_2(X, L_1)$ corresponds to β under the isotopy L_t.

First consider the case that $r = r_0$. From the isotopy L_t given by Equation (5.3.3) and the equalities

$$\beta_0^{\mathbf{T}} \cdot \mathcal{D}_j = 0$$

for $j = 1, \ldots, m-1$ and

$$\beta_i^{\mathbf{T}} \cdot \mathcal{D}_j = \delta_{ij}$$

for $i = 1, \ldots, m-1$, $j = 1, \ldots, m-1$, one has

$$\beta_0 \cdot \mathcal{D}_j = 0$$

for all $j = 1, \ldots, m-1$ and

$$\beta_i \cdot \mathcal{D}_j = \delta_{ij}$$

for all $i = 1, \ldots, m-1$, $j = 1, \ldots, m-1$.

We also have the isotopy

$$S_t = \{x \in X : w(x) = t\}$$

for $t = [0, K_2]$ between the anti-canonical divisor $-K_X = \sum_{l=0}^{m-1} \mathcal{D}_l$ and D_0. One has $S_t \cap L_t = \emptyset$ for all t, and so

$$\beta_i \cdot D_0 = \beta_i^{\mathbf{T}} \cdot \sum_{l=0}^{m-1} \mathcal{D}_l = 1$$

for all $i = 0, \ldots, m-1$.

For general $r \in U$, since $U \cap \mathcal{D}_j = \emptyset$ for all $j = 1, \ldots, m-1$ and $U \cap D_0 = \emptyset$, the isotopy between F_r and F_{r_0} never intersect D_0 and $\mathcal{D}_j$ for all $j = 1, \ldots, m-1$. Thus the above equalities of intersection numbers are preserved. □

5.3.2 The modified fibration

We write down the discriminant locus of μ' and generators of $\pi_2(X', F)$, where F is a fiber of μ', in this section. This is almost the same as the discussion for X in the previous subsection, except that we have more disk classes due to the additional toric divisors. The proofs are similar and thus omitted.

The discriminant locus of μ'

Definition 5.3.7. *For each $\emptyset \neq I \subset \{0, \ldots, m-1\}$ such that $\{v_i : i \in I\}$ generates some cone in Σ', we define*

$$T'_I := T_I \cap \left\{\xi \in P' : -K_1 < \left(v'_j\,,\, \xi\right) < K_1 \text{ for all } j = 1, \ldots, n-1\right\}$$

where T_I is a face of P given in Definition 5.3.1. T'_I is a codimension-$(|I|-1)$ face of

$$\left\{\xi \in \partial P' : -K_1 < \left(v'_j\,,\, \xi\right) < K_1 \text{ for all } j = 1, \ldots, n-1\right\}.$$

Proposition 5.3.8. *The discriminant locus of μ' is*

$$\Gamma' := \left(\left(\bigcup_{|I|=2} [T'_I]\right) \times \{0\}\right) \cup \partial B'.$$

Figure 5.3 gives an example for the base and discriminant locus of μ'.

By removing the singular fibers of μ', we get a Lagrangian torus bundle $\mu' : X'_0 \to B'_0$, where

$$\begin{aligned} B'_0 &:= B' - \Gamma'; \\ X'_0 &:= (\mu')^{-1}(B'_0). \end{aligned}$$

Local trivialization

We define

$$U'_i := B'_0 - \bigcup_{k \neq i} ([T'_k] \times \{0\})$$

which is a contractible set for $i = 0, \ldots, m-1$, so that $(\mu')^{-1}(U_i')$ is trivialized. Without loss of generality we stick to the trivialization over the open set

$$U' := U_0' = B_0' - \bigcup_{k \neq 0} ([T_k'] \times \{0\}) = \{(q_1, q_2) \in B_0' : q_2 \neq 0 \text{ or } q_1 \in [T_0']\}. \quad (5.3.5)$$

Similar to Proposition 5.3.3, one has

$$[T_0'] = \{q \in E^{\text{int}} : (v_j', q) \geq c_j - c_0 \text{ for all } j = 1, \ldots, m-1\}.$$

Thus the open set $U' = U_0'$ can be written as

$$\{(q_1, q_2) \in E^{\text{int}} \times (-1, 1) : q_2 \neq 0 \text{ or } (v_j', q_1) > c_j - c_0 \text{ for all } j = 1, \ldots, m-1\}.$$

Then the trivialization is explicitly written as

$$(\mu')^{-1}(U') \stackrel{\cong}{\to} U' \times (\mathbf{T}_N / \mathbf{T}\langle v_0 \rangle) \times (\mathbb{R}/2\pi\mathbb{Z})$$

which is given in the same way as in Definition 5.3.4.

Explicit generators of $\pi_1(F_r)$ and $\pi_2(X, F_r)$

For $r \in U'$, every F_r is identified with the torus $(\mathbf{T}_N / \mathbf{T}\langle v_0 \rangle) \times (\mathbb{R}/2\pi\mathbb{Z})$ via the above trivialization. Then a basis of $\pi_1(F_r)$ is given by $\{\lambda_i\}_{i=0}^{n-1}$, where $\lambda_0 = (0, 1) \in N/\mathbb{Z}\langle v_0 \rangle \times \mathbb{Z}$ and $\lambda_i = ([v_i], 0) \in N/\mathbb{Z}\langle v_0 \rangle \times \mathbb{Z}$ for $i = 1, \ldots, n-1$.

We use the same procedure as that given in Section 5.3.1 to write down explicit generators of $\pi_2(X', F_r)$ for $r \in B_0'$. First of all, $\pi_2(X', \mathbf{T})$ is generated by β_i, β_∞ and $\beta_{\pm j}'$ corresponding to v_i, v_∞ and $v_{\pm j}'$ respectively, where $i = 0, \ldots, m-1$ and $j = 1, \ldots, n-1$. They are depicted in Figure 5.7.

Then fixing a based point $r_0 = (q_1, q_2) \in U'$ with $0 < q_2 < 1$, the isotopy

$$L_t := \{x \in X : [\mu_0'(x)] = q_1; f(|w'(x) - t|^2) = q_2\}$$

between F_{r_0} and a toric fiber $\mathbf{T}$ gives an identification $\pi_2(X', F_{r_0}) \cong \pi_2(X', \mathbf{T})$. Finally the trivialization of $\mu^{-1}(U')$ gives an identification between F_r and F_{r_0}

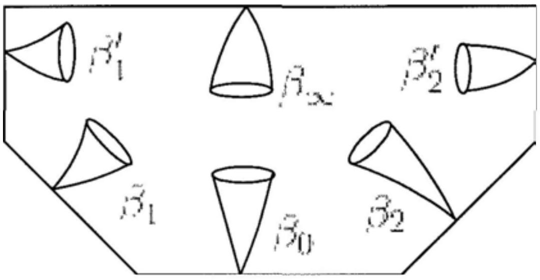

Figure 5.7: Disks generating $\pi_2(X', T)$ for a regular moment-map fiber T when $X = K_{\mathbb{P}^1}$.

for any $r \in U'$. Thus $\{\beta_i\}_{i=0}^{m-1} \cup \{\beta_\infty\} \cup \{\beta'_j\}_{j=1}^{n-1} \cup \{\beta'_{-j}\}_{j=1}^{n-1}$ can be regarded as a generating set of $\pi_2(X', F_r)$. See Figure 5.8 to get a feeling of what they look like topologically.

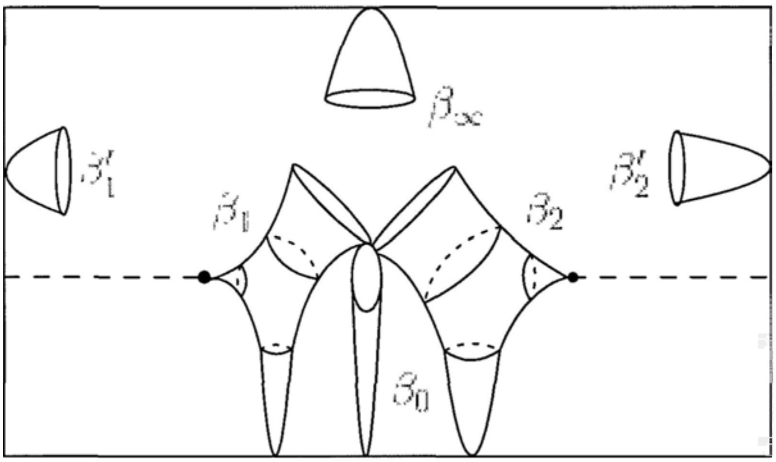

Figure 5.8: Disk generating $\pi_2(X', F_r)$.

Proposition 5.3.9.

$$\partial\beta_j = \lambda_0 + \sum_{i=1}^{n-1} (\nu_i \,,\, v_j)\, \lambda_i \in (N/\mathbb{Z}\langle v_0\rangle) \times \mathbb{Z} \cong \pi_1(F_r)$$

for all $j = 0, \ldots, m-1$,

$$\partial\beta_\infty = -\lambda_0$$

and

$$\partial\beta'_k = \lambda_k; \partial\beta'_{-k} = -\lambda_k$$

for $k = 1, \ldots, n-1$.

We will need to know the following intersection numbers when we compute the generation functions of open Gromov-Witten invariants:

Proposition 5.3.10. *Let* $r = (q_1, q_2) \in U'$ *with* $q_2 \neq 0$, *and* $\beta_i \in \pi_2(X', F_r)$ *be the disk classes defined above. For all* $i = 0, \ldots, m-1$ *and* $l, k = \pm 1, \ldots, \pm n-1$,

$$\beta_i \cdot D_0 = 1;\ \beta_\infty \cdot D_0 = 0;\ \beta'_l \cdot D_0 = 0;$$
$$\beta_i \cdot D_k = 0;\ \beta_\infty \cdot D_k = 0;\ \beta'_l \cdot D_k = \delta_{lk};$$
$$\beta_i \cdot D_\infty = 0;\ \beta_\infty \cdot D_\infty = 1;\ \beta'_l \cdot D_\infty = 0.$$

5.4 Wall crossing phenomenon

In this section we show that when X is a toric Calabi-Yau manifold and F_r is a fiber of the Gross fibration, the open Gromov-Witten invariants n_β for a disk class $\beta \in \pi_2(X, F_r)$ bounded by F_r (Definition 3.3.1) exhibit a phenomenon called wall-crossing. This is an application of the ideas and techniques introduced by Auroux [3, 4] to the case of toric Calabi-Yau manifolds. The main results are Proposition 5.4.7 and 5.4.9. In Chapter 7 we will give a method to compute these open Gromov-Witten invariants.

Let's begin with the Maslov index of disks (Definition 3.1.2), which is important because it determines the expected dimension of the corresponding moduli (Equation 3.2.1). The following lemma which appeared in [3] gives a formula for computing the Maslov index, which can be regarded as a generalization of the corresponding result by Cho-Oh [12] for moment-map fibers of toric manifolds.

Lemma 5.4.1 (Lemma 3.1 of [3]). *Let Y be a Kähler manifold of dimension n, σ be a nowhere-zero meromorphic n-form on Y, and let D denote its pole divisor (and so D is the anti-canonical divisor). If $L \subset Y - D$ is a compact oriented special Lagrangian submanifold with respect to σ, then for each $\beta \in \pi_2(Y, L)$,*

$$\mu(\beta) = 2\beta \cdot D.$$

Using the above lemma:

Corollary 5.4.2. *For a regular fiber F of the Gross fibration μ, the Maslov index of $\beta \in \pi_2(X, F)$ is*

$$\mu(\beta) = 2\beta \cdot D_0.$$

For a regular fiber F of the modified fibration μ', the Maslov index of $\beta \in \pi_2(X', F_r)$ is

$$\mu(\beta) = 2\beta \cdot \left(D_0 + D_\infty + \sum_{j=1}^{n-1} D_j + \sum_{j=1}^{n-1} D_{-j} \right). \tag{5.4.1}$$

Proof. Recall that the regular fibers F_r of $\mu : X \to B$ are special Lagrangian with respect to $\Omega/(w - K_2)$ whose pole divisor is D_0 (see Equation (5.3.4) for the definition of D_0). Using the above lemma, the Maslov index of $\beta \in \pi_2(X, F_r)$ is

$$\mu(\beta) = 2\beta \cdot D_0.$$

Similarly $\mu' : X' \to B'$ are special Lagrangian with respect to $\Omega'/(w' - K_2)$ whose pole divisor is $D_0 + D_\infty + \sum_{j=1}^{n-1} D_j + \sum_{j=1}^{n-1} D_{-j}$. Thus the Maslov index of $\beta \in \pi_2(X', F_r)$ is

$$\mu(\beta) = 2\beta \cdot \left(D_0 + D_\infty + \sum_{j=1}^{n-1} D_j + \sum_{j=1}^{n-1} D_{-j} \right).$$

□

Corollary 5.4.3. *Let F be a regular fiber of the Gross fibration μ. For every $\beta \in \pi_2(X, F)$, if $\mathcal{M}_0(F, \beta) \neq \emptyset$, then $\mu(\beta) \geq 0$.*

Proof. From the above formula, it follows that the Maslov index of any holomorphic disks in $\beta \in \pi_2(X, F_r)$ or $\beta \in \pi_2(X', F_r)$ is non-negative.

Every stable disk consists of holomorphic disk components and holomorphic sphere components, and its Maslov index is the sum of Maslov indices of its disk components and two times Chern numbers of its sphere components. The disk components have non-negative Maslov index as mentioned above. Since X is Calabi-Yau, every holomorphic sphere in X has Chern number zero. Thus the sum is non-negative. □

5.4.1 Stable disks in a toric CY manifold

First consider a toric Calabi-Yau manifold X. The lemma below gives an expression of the wall (see Definition 4.3.6).

Lemma 5.4.4. *For $r = (q_1, q_2) \in B_0$, $\mathcal{M}_0(F_r, \beta) \neq \emptyset$ for some $\beta \in \pi_2(X, F_r)$ with $\mu(\beta) = 0$ if and only if $q_2 = 0$.*

Proof. Since X is Calabi-Yau, sphere bubbles in a stable disk have Chern number zero and hence do not affect the Maslov index. We can restrict our attention to a holomorphic disk $u : (\Delta, \partial\Delta) \to (X, F_r)$ whose Maslov index is zero. By Lemma 5.4.1, u has intersection number zero with the boundary divisor $D_0 = \{w = K_2\}$. But since u is holomorphic and D_0 is a complex submanifold, the multiplicity for each intersection point between them is positive. This implies

$$\mathrm{Im}(u) \subset \mu^{-1}(B^{\mathrm{int}}).$$

Then $w \circ u - K_2$ is a nowhere-zero holomorphic function on the disk. Moreover, $|w \circ u - K_2|$ is constant on $\partial\Delta$. By applying maximum principle on $|w \circ u - K_2|$ and $|w \circ u - K_2|^{-1}$, $w \circ u$ must be constant with value z_0 in the circle

$$\left\{|z - K_2|^2 = K_2^2 + q_2\right\} \subset \mathbb{C}.$$

Unless $z_0 = 0$, $w^{-1}(z_0)$ is topologically $\mathbb{R}^{n-1} \times \mathbf{T}^{n-1}$, which contains no non-constant holomorphic disks whose boundary lies in $F_r \cap w^{-1}(z_0) \cong T^{n-1} \subset \mathbb{R}^{n-1} \times \mathbf{T}^{n-1}$. Hence $z_0 = 0$, which implies $q_2 = 0$. Conversely, if $q_2 = 0$, F_r intersects a toric divisor along a (degenerate) moment map fiber, and hence bounds holomorphic disks which are part of the toric divisor. They have Maslov index zero because they never intersect D_0. □

Combining the above lemma with Corollary 5.4.3, one has

Corollary 5.4.5. *For $r = (q_1, q_2) \in B_0$ with $q_2 \neq 0$, F_r has minimal Maslov index at least two.*

Using the terminology introduced in Definition 4.3.6, the wall is

$$H = M_{\mathbb{R}}/\mathbb{R}\langle \underline{\nu} \rangle \times \{0\}.$$

$B_0 - H$ consists of two connected components

$$B_+ := M_{\mathbb{R}}/\mathbb{R}\langle \underline{\nu} \rangle \times (0, +\infty) \tag{5.4.2}$$

and

$$B_- := M_{\mathbb{R}}/\mathbb{R}\langle \underline{\nu} \rangle \times (-K_2, 0). \tag{5.4.3}$$

For $r \in B_0 - H$, the fiber F_r has minimal Maslov index two, and thus n_β is well-defined for $\beta \in \pi_2(X, F_r)$. There are two cases: $r \in B_+$ and $r \in B_-$.

1. $r \in B_+$.

One has the following lemma relating a Gross fiber F_r to a Lagrangian toric fiber $\mathbf{T}$:

Lemma 5.4.6. *For $r \in B_+$, the Gross fiber F_r is Lagrangian-isotopic to a Lagrangian toric fiber $\mathbf{T}$, and all the Lagrangians in this isotopy do not bound non-constant stable disks of Maslov index zero.*

Proof. Let $r = (q_1, q_2)$ with $q_2 > 0$. The Lagrangian isotopy has already been given in Equation (5.3.3), which is

$$L_t := \{x \in X : [\mu_0(x)] = q_1; |w(x) - t|^2 = K_2^2 + q_2\}$$

where $t \in [0, K_2]$. Since $q_2 > 0$, for each $t \in [0, K_2]$, w is never zero on L_t. By Lemma 5.4.4, L_t does not bound non-constant stable disks of Maslov index zero. □

Using the above lemma, one shows that the open Gromov-Witten invariants of F_r when $r \in B_+$ are the same as that of $\mathbf{T}$:

Proposition 5.4.7. *For $r \in B_+$ and $\beta \in \pi_2(X, F_r)$, let $\beta^{\mathbf{T}} \in \pi_2(X, \mathbf{T}) \cong \pi_2(X, F_r)$ be the corresponding class under the isotopy given in Lemma 5.4.6. Then*

$$n_\beta = n_{\beta^{\mathbf{T}}}.$$

$n_\beta \neq 0$ only when

$$\beta = \beta_j + \alpha$$

where $\alpha \in H_2(X)$ is represented by rational curves, and $\beta_j \in \pi_2(X, F_r)$ is a basic disk class given in Section 5.3.1. Moreover, $n_{\beta_j} = 1$ for all $j = 0, \ldots, m-1$.

Proof. It suffices to consider those $\beta \in \pi_2(X, F_r)$ with $\mu(\beta) = 2$, or otherwise $n_\beta = 0$ due to dimension reason.

The Lagrangian isotopy given in Lemma 5.4.6 gives an identification between $\pi_2(X, F_r)$ and $\pi_2(X, \mathbf{T})$, where $\mathbf{T}$ is a regular fiber of μ_0. Moreover, since every Lagrangian in the isotopy has minimal Maslov index two, the isotopy gives a cobordism between $\mathcal{M}_1(F_r, \beta)$ and $\mathcal{M}_1(\mathbf{T}, \beta^{\mathbf{T}})$, where $\beta^{\mathbf{T}} \in \pi_2(X, \mathbf{T})$ is the disk class corresponding to $\beta \in \pi_2(X, F_r)$ under the isotopy. Hence n_β keeps constant along this isotopy, which implies

$$n_\beta = n_{\beta^{\mathbf{T}}}.$$

By dimension counting of the moduli space, $n_{\beta^{\mathbf{T}}}$ is non-zero only when $\beta^{\mathbf{T}}$ is of Maslov index two (see Equation 3.2.1 and the explanation below Definition 3.3.1).

Using Theorem 11.1 of [22], $\mathcal{M}_1(\mathbf{T}, \beta^{\mathbf{T}})$ is non-empty only when $\beta^{\mathbf{T}} = \beta_j + \alpha$, where $\alpha \in H_2(X)$ is represented by rational curves, and $\beta_j \in \pi_2(X, \mathbf{T}) \cong \pi_2(X, F_r)$ are the basic disk classes given in Section 5.3.1. For completeness we also give the reasoning here. Let $u \in \mathcal{M}_1(\mathbf{T}, \beta^{\mathbf{T}})$ be a stable disk of Maslov index two. u is composed of holomorphic disk components and sphere components. Since every holomorphic disk bounded by a toric fiber $\mathbf{T} \subset X$ must intersect some toric divisors, which implies that it has Maslov index at least two, u can have only one disk component. Moreover a holomorphic disk of Maslov index two must belong to a basic disk class β_j [12]. Thus $\beta = [u]$ is of the form $\beta_j + \alpha$.

Moreover, by Cho-Oh's result [12], $n_{\beta_j} = 1$ for all $j = 0, \ldots, m-1$. □

2. $r \in B_-$.

When $r \in B_-$, the open Gromov-Witten invariants behave differently compared to the case $r \in B_+$ (see Equation 5.4.3 for the definition of B_-). For $X = \mathbb{C}^n$, n_β has been studied by Auroux [3, 4] (indeed he considered the cases $n = 2, 3$, but there is no essential difference for general n). We give the detailed proof here for readers' convenience:

Lemma 5.4.8 ([3]). *When the toric Calabi-Yau manifold is $X = \mathbb{C}^n$ and $F_r \subset X$ is a Gross fiber at $r \in B_-$, we have*

$$n_\beta = \begin{cases} 1 & \text{when } \beta = \beta_0; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $(\zeta_0, \ldots, \zeta_{n-1})$ be the standard complex coordinates of $\mathbb{C}^n$. In these coordinates the Gross fibration is written as

$$\mu = (|\zeta_0|^2 - |\zeta_1|^2, \ldots, |\zeta_{n-2}|^2 - |\zeta_{n-1}|^2, |\zeta_0 \ldots \zeta_{n-1} - K_2|^2 - K_2^2).$$

Due to dimension reason, $n_\beta = 0$ whenever $\mu(\beta) \neq 2$. Thus it suffices to consider the case $\mu(\beta) = 2$. Write $\beta = \sum_{i=0}^{n-1} k_i \beta_i$, where $\beta_i \in \pi_2(X, F_r)$ are the basic disk classes defined in Section 5.3.1. We claim that $k_0 = 1$ and $k_i = 0$ for all $i = 1, \ldots, n-1$ if the moduli space $\mathcal{M}_1(F_r, \beta)$ is non-empty.

Let u be a stable disk in $\mathbb{C}^n$ representing β with $\mu(\beta) = 2$. Since $\mathbb{C}^n$ supports no non-constant holomorphic sphere, u has no sphere component. Also by Corollary 5.4.5, F_r has minimal Maslov index two, and so u consists of only one disk component (see Proposition 3.2.7). Thus u is indeed a holomorphic map $\Delta \to \mathbb{C}^n$.

Since $q_2 < 0$, one has $|(\zeta_0 \ldots \zeta_{n-1}) \circ u - K_2| < K_2$ on $\partial\Delta$. By maximum principle this inequality holds on the whole disk Δ. In particular, $\zeta_0 \ldots \zeta_{n-1}$ is never zero on Δ, and so u never hits the toric divisors $\mathcal{D}_i = \{\zeta_i = 0\}$ for $i = 0, \ldots, n-1$. Thus $\beta \cdot \mathcal{D}_i = 0$ for all $i = 0, \ldots, n-1$. By Proposition 5.3.6, $(\beta_0 \,, \mathcal{D}_j) = 0$ for all $j = 1, \ldots, n-1$, and $(\beta_i \,, \mathcal{D}_j) = \delta_{ij}$ for $i = 1, \ldots, n-1$ and $j = 0, \ldots, n-1$. Thus

$$(\beta \,, \mathcal{D}_j) = k_j = 0$$

for $j = 1, \ldots, n-1$. Thus $\beta = k_0 \beta_0$. But $\mu(\beta) = k_0 \mu(\beta_0) = 2$ and $\mu(\beta_0) = 2$, and so $k_0 = 1$.

This proves that $n_\beta \neq 0$ only when $\beta = \beta_0$. Now we prove that $n_{\beta_0} = 1$. Since every fiber F_r is Lagrangian isotopic to each other for $r \in B_-$ and the Lagrangian fibers have minimal Maslov index 2, n_{β_0} keeps constant as $r \in B_-$ varies. Hence it suffices to consider $r = (0, q_2)$ for $q_2 < 0$, which means that $|\zeta_0| = |\zeta_1| = \ldots = |\zeta_{n-1}|$ for every $(\zeta_0, \ldots, \zeta_{n-1}) \in F_r$.

In the following we prove that for every $p \in F_r \subset (\mathbb{C}^\times)^n$, the preimage of p under the evaluation map $\mathrm{ev}_0 : \mathcal{M}_1(F_r, \beta_0) \to F_r$ is a singleton, and so $n_{\beta_0} = 1$.

Write $p = (p_0, \ldots, p_{n-1}) \in (\mathbb{C}^\times)^n$. $p \in F_r$ implies that $|p_0| = |p_1| = \ldots = |p_{n-1}|$. Consider the line

$$l := \{(\zeta p_0, \zeta p_1, \ldots, \zeta p_{n-1}) \in (\mathbb{C}^\times)^n : \zeta \in \mathbb{C}^\times\}$$

spanned by p. Then $w = \zeta_0 \ldots \zeta_{n-1}$ gives an n-to-one covering $l \to \mathbb{C}^\times$. The disk

$$\Delta_{K_2} := \{\zeta \in \mathbb{C} : |\zeta - K_2| \leq (K_2^2 + q_2)^{1/2}\}$$

never intersects the negative real axis $\{\mathrm{Re}(\zeta) \leq 0\}$, and hence we may choose a branch to obtain a holomorphic map $\tilde{u} : \Delta_{K_2} \to l$ (There are n such choices). Moreover there is a unique choice such that $\tilde{u}(p_0 \ldots p_{n-1}) = (p_0, \ldots, p_{n-1})$. The image of $\partial \Delta_{K_2}$ under $\tilde{u}$ lies in F_r: Let $\zeta \in \partial \Delta_{K_2}$ and $z = \tilde{u}(\zeta)$. Then $w(z) = \zeta$ satisfies $|w(z) - K_2|^2 = K_2^2 + q_2$. Moreover $z \in l$, and so $|z_0| = |z_1| = \ldots = |z_{n-1}|$. $\tilde{u}$ represents β_0 because it never intersects the toric divisors $\mathcal{D}_j$ for $j = 0, \ldots, n-1$ and it intersect with $D_0 = \{w = 0\}$ once.

The above proves that there exists a holomorphic disk representing β_0 such that its boundary passes through p. In the following we prove that indeed this is unique.

Let $u \in \mathcal{M}_1(F_r, \beta_0)$ such that $\mathrm{ev}_0(u) = p$. By the above consideration u is a holomorphic disk. Since $\beta_0 \cdot \mathcal{D}_i = 0$, u never hits the toric divisors $\{\zeta_i = 0\}$ for $i = 0, \ldots, n-1$, and hence $\zeta_i \circ u : \Delta \to \mathbb{C}$ are nowhere-zero holomorphic functions. By applying maximum principle on $|\zeta_i/\zeta_1 \circ u|$ and $|\zeta_1/\zeta_i \circ u|^{-1}$ for each $i = 2, \ldots, n$, which has value 1 on $\partial\Delta$, we infer that u must lie on the complex line

$$\{(\zeta, c_1\zeta, \ldots, c_{n-1}\zeta) \in (\mathbb{C}^\times)^n : \zeta \in \mathbb{C}^\times\}$$

where $|c_i| = 1$ are some constants for $i = 1, \ldots, n-1$. Moreover, The line passes through p, and so this is the line l defined above.

Consider the holomorphic map $w \circ u : \Delta \to \mathbb{C}^\times$. Since u has Maslov index two, it has intersection number one with the divisor $\{w - K_2 = 0\}$, implying that $w \circ u|_{\partial\Delta}$ winds around K_2 only once. Hence $w \circ u$ gives a biholomorphism $\Delta \overset{\cong}{\to} \Delta_{K_2}$ defined above. One has $\tilde{u} \circ (w \circ u) = (\tilde{u} \circ w) \circ u = u$, where $\tilde{u}$ is the one-side inverse of w defined above. This means u is the same as $\tilde{u}$ up to the biholomorphism $w \circ u$. Thus $\tilde{u}$ is unique. □

Indeed the same statement holds for all toric Calabi-Yau manifolds:

Proposition 5.4.9. *For $r \in B_-$ and $\beta \in \pi_2(X, F_r)$,*

$$n_\beta = \begin{cases} 1 & \text{when } \beta = \beta_0; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Due to dimension reason, $n_\beta = 0$ if $\mu(\beta) \neq 2$, and so it suffices to assume $\mu(\beta) = 2$. Let $r = (q_1, q_2)$ with $q_2 < 0$.

First of all, one observes that when $r \in B_-$, every holomorphic disk $u : (\Delta, \partial\Delta) \to (X, F_r)$ has image

$$\mathrm{Im}(u) \subset S_- := \mu^{-1}(\{(q_1, q_2) \in B : q_2 < 0\}).$$

This is because $(w - K_2) \circ u$ defines a holomorphic function on Δ. Since $r \in B_-$, $|w - K_2|$ is constant with value less than K_2 on F_r. By maximum principle, $|w - K_2| \circ u < K_2$. This proves the observation.

Notice that (S_-, F_r) is homeomorphic to $((\mathbb{C}^\times)^{n-1} \times \mathbb{C}, T)$, where

$$T = \{(\zeta_1, \ldots, \zeta_n) \in (\mathbb{C}^\times)^{n-1} \times \mathbb{C} : |\zeta_1| = \ldots = |\zeta_n| = c\}$$

for $c > 0$. In particular, $\pi_2(S_-) = 0$ which implies that S_- supports no non-constant holomorphic sphere. Moreover, every non-constant holomorphic disk bounded by F_r with image lying in S_- must intersect D_0, and thus it has Maslov index at least two.

Now let $v \in \mathcal{M}_1(F_r, \beta)$ be a stable disk of Maslov index two, where $r \in B_-$. By the above observation, each disk component of v has Maslov index at least two, and so v has only one disk component.

Moreover, the image of a non-constant holomorphic sphere $h : \mathbb{CP}^1 \to X$ does not intersect S_-: Consider $w \circ h$, which is a holomorphic function on $\mathbb{CP}^1$ and hence must be constant. Thus image of h lies in $w^{-1}(c)$ for some c. But for $c \neq 0$, $w^{-1}(c)$ is $(\mathbb{C}^\times)^{n-1}$ which supports no non-constant holomorphic sphere. Thus $c = 0$. But w is never zero on S_-, implying that $w^{-1}(0) \cap S_- = \emptyset$.

Thus v does not have any sphere component, because any non-constant holomorphic sphere in X never intersect its disk component. This proves for all $\beta \in \pi_2(X, F_r)$, $\mathcal{M}_1(\beta, F_r)$ consists of holomorphic maps $u : (\Delta, \partial\Delta) \to (X, F_r)$, that is, neither disk nor sphere bubbling never occurs.

In particular, all elements in $\mathcal{M}_1(\beta, F_r)$ have images in S_- and never intersect the toric divisors. Writing $\beta = \sum_{i=0}^{m-1} k_i \beta_i$, one has

$$(\beta\,,\,\mathcal{D}_j) = k_j = 0$$

(see Proposition 5.3.6). Moreover, $\mu(\beta) = 2$ forces $k_0 = 1$. Thus $\mathcal{M}_1(\beta, F_r)$, where β has Maslov index two, is non-empty only when $\beta = \beta_0$. Thus $n_\beta = 0$ whenever $\beta \neq \beta_0$.

Let $V = \mathbb{C}^n \hookrightarrow X$ be the complex coordinate chart corresponding to the cone $\langle v_0, \ldots, v_{n-1} \rangle$. We have $F_r \subset S_0 \subset V$, and since $\beta_0 \cdot \mathcal{D} = 0$ for every toric divisor $\mathcal{D} \subset X$, any holomorphic disk representing β_0 in X is indeed contained in V. Thus

$$\mathcal{M}_1^X(\beta_0, F_r) \cong \mathcal{M}_1^V(\beta_0, F_r).$$

Then $n^X_{\beta_0} = n^V_{\beta_0}$, where the later has been proven to be 1 in Lemma 5.4.8. □

From the above propositions, one sees that n_β for $\beta \in \pi_2(X, F_r)$ changes dramatically as r crosses the wall H, and this is the so-called wall-crossing phenomenon.

5.4.2 Stable disks in the modified fibration

Now we consider open Gromov-Witten invariants of X'. The statements are very similar, except that there are more disk classes due to the additional toric divisors. The proofs are also very similar and thus omitted.

Lemma 5.4.10. *For $r = (q_1, q_2) \in B_0'$, $\mathcal{M}_0(F_r, \beta) \neq \emptyset$ for some $\beta \in \pi_2(X', F_r)$ with $\mu(\beta) = 0$ if and only if $q_2 = 0$.*

As a consequence,

Corollary 5.4.11. *The fibration $\mu' : X' \to B'$ satisfies Assumption 4.3.3.*

Proof. By construction μ' is a Lagrangian torus fibration whose image B' is a polytope. As discussed in Section 5.2, the inverse images of the facets of B' are divisors in X'. Corollary 5.4.2 gives us the formula for Maslov index, which is the statement of Assumption (3). By the above lemma, for every $r = (q_1, q_2) \in B_0$ with $q_2 \neq 0$, F_r has minimal Maslov index two. Thus Assumption (4) is satisfied. □

The wall (see Definition 4.3.6) is

$$H' = E^{\text{int}} \times \{0\}.$$

The two connected components of $B'_0 - H'$ are denoted by

$$B'_+ := E^{\text{int}} \times (0, 1)$$

and

$$B'_- := E^{\text{int}} \times (-1, 0)$$

respectively. Again we have two cases to consider:

1. $r \in B'_+$.

Lemma 5.4.12. *For $r \in B'_+$, the fiber F_r is Lagrangian-isotopic to a Lagrangian toric fiber of X', and all the Lagrangians in this isotopy do not bound non-constant stable disks of Maslov index zero.*

Proposition 5.4.13. *For $r \in B'_+$ and $\beta \in \pi_2(X', F_r)$, $n_\beta \neq 0$ only when $\beta = \beta_\infty$, $\beta = \beta'_k$ for $k = \pm 1, \ldots, \pm(n-1)$ or*

$$\beta = \beta_j + \alpha \text{ for } j = 0, \ldots, m-1$$

where $\alpha \in H_2(X)$ is represented by rational curves of Chern number zero. Moreover, $n_\beta = 1$ when $\beta = \beta_0, \ldots, \beta_{m-1}, \beta_\infty$ or $\beta'_1, \ldots, \beta'_{n-1}$ or $\beta'_{-1}, \ldots, \beta'_{-(n-1)}$.

2. $r \in B'_-$.

Proposition 5.4.14. *For $r \in B'_-$ and $\beta \in \pi_2(X', F_r)$, $n_\beta = 1$ when*

$$\beta = \beta_0, \beta'_j \text{ or } \beta_\infty + (\beta_i - \beta_0) \text{ for } i = 0, \ldots, m-1; j = \pm 1, \ldots, \pm(n-1)$$

and zero otherwise.

These invariants contribute to the quantum corrections of the complex structure of the mirror, as we shall discuss in the next section.

5.5 SYZ construction of mirrors

We have seen that the fibration $\mu' : X' \to B'$ satisfies Assumption 4.3.3 (Corollary 5.4.11). Now we are ready to follow the procedure given in Section 4.3 to construct the mirror of a toric Calabi-Yau. The following is the main theorem:

Theorem 5.5.1. *Let $\mu : X \to B$ be the Gross fibration over a toric Calabi-Yau n-fold $X = X_\Sigma$, and $\mu' : X' \to B'$ be the modified fibration given by Definition 5.2.2.*

1. *Applying SYZ construction with quantum corrections described in Section 4.3 on the Lagrangian fibration $\mu' : X' \to B'$, one obtains a complex manifold*

$$\check{X} = \left\{ (u, v, z_1, \ldots, z_{n-1}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} : uv = G(z_1, \ldots, z_{n-1}) \right\} \quad (5.5.1)$$

where G is a polynomial given by

$$G(z_1, \ldots, z_{n-1}) = (1 + \delta_0) + \sum_{j=1}^{n-1} (1 + \delta_j) z_j + \sum_{i=n}^{m-1} (1 + \delta_i) q_{i-n+1} z^{v_i}. \quad (5.5.2)$$

The notations δ_j, q_a and z^v appeared above are explained in the end of this theorem.

2. *Let $H \subset B_0$ be the wall (Definition 4.3.6). There exists a canonical map*

$$\rho : \breve{\mu}^{-1}(B_0 - H) \to \check{X}$$

such that the holomorphic n-form

$$\check{\Omega} := \operatorname{Res}\left(\frac{1}{uv - G(z_1, \ldots, z_{n-1})} \mathrm{d}\log z_1 \wedge \ldots \wedge \mathrm{d}\log z_{n-1} \wedge \mathrm{d}u \wedge \mathrm{d}v\right) \tag{5.5.3}$$

defined on $\check{X} \subset \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1}$ is pulled back to the semi-flat holomorphic volume form on $\breve{\mu}^{-1}(B_0 - H)$ under ρ. In this sense the semi-flat holomorphic volume form on $\check{X}_0$ extends to $\check{X}$.

3. *Let $\mathcal{F}_X$ be the generating function of open Gromov-Witten invariants (Definition 3.3.3). The Fourier transform of $\mathcal{F}_X$ is given by $\rho^*(C_0 u)$, where C_0 is some constant (defined by Equation (5.5.9)).*

$(\check{X}, \check{\Omega}, W = C_0 u)$ is called the SYZ mirror of the toric CY manifold X, where $\check{\Omega}$ is the holomorphic volume form and W is the superpotential. The definitions of δ_i, q_a and z^v appeared above are as follows:

- *δ_i's are constants defined by*

$$\delta_i := \sum_{\alpha \neq 0} n_{\beta_i + \alpha} \exp\left(-\int_\alpha \omega\right) \tag{5.5.4}$$

for $i = 0, \ldots, m-1$, in which the summation is over all $\alpha \in H_2(X, \mathbb{Z}) - \{0\}$ represented by rational curves. (Recall that $\beta_i \in \pi_2(X, \mathbf{T})$ are the basic disk classes bounded by a Lagrangian toric fiber $\mathbf{T}$.)

- *z^{v_i} denotes the monomial*

$$\prod_{j=1}^{n-1} z_j^{(\nu_j, v_i)}$$

where $\{\nu_j\}_{j=0}^{n-1} \subset M$ is the dual basis of $\{v_j\}_{j=0}^{n-1} \subset N$ which spans a cone of the fan Σ.

- *For* $a = 1, \ldots, m-n$, q_a *are Kähler parameters defined as follows. Let* $S_a \in H_2(X, \mathbb{Z})$ *be the classes defined by*

$$S_a := \beta_{a+n-1} - \sum_{j=1}^{n-1} (\nu_j\,,\, v_{a+n-1})\, \beta_j \tag{5.5.5}$$

Then $q_a := \exp(-\int_{S_a} \omega)$.

Notice that the Laurent polynomial (5.5.2) is independent of the parameters K_1, K_2.

We need to check that the above expression (5.5.5) of S_a does define classes in $H_2(X, \mathbb{Z})$:

Proposition 5.5.2. $\{S_a\}_{a=1}^{m-n}$ *is a basis of* $H_2(X, \mathbb{Z})$.

Proof. One has the short exact sequence

$$0 \to H_2(X) \to \pi_2(X, \mathbf{T}) \to \pi_1(\mathbf{T}) \to 0$$

where $\mathbf{T}$ is a Lagrangian toric fiber, and the second last arrow is given by the boundary map ∂. For $i = n, \ldots, m-1$,

$$\begin{aligned}\partial\left(\beta_i^{\mathbf{T}} - \sum_{j=1}^{n-1} (\nu_j\,,\, v_i)\, \beta_j^{\mathbf{T}}\right) &= \partial\beta_i^{\mathbf{T}} - \sum_{j=1}^{n-1} (\nu_j\,,\, v_i)\, \partial\beta_j^{\mathbf{T}} \\ &= v_i - \sum_{j=1}^{n-1} (\nu_j\,,\, v_i)\, v_j \\ &= v_i - v_i = 0\end{aligned}$$

where $\beta_i^{\mathbf{T}}$'s are the basic disk classes given in Section 5.3.1. Thus

$$\beta_i^{\mathbf{T}} - \sum_{j=1}^{n-1} (\nu_j\,,\, v_i)\, \beta_j^{\mathbf{T}} \in H_2(X, \mathbb{Z}).$$

Moreover, they are linearly independent for $i = n, \ldots, m-1$, because $\beta_i^{\mathbf{T}}$'s are linearly independent. But $H_2(X, \mathbb{Z}) \cong \mathbb{Z}^{m-n}$, and so they form a basis of $H_2(X, \mathbb{Z})$.

β_i's are identified with $\beta_i^{\mathbf{T}}$'s under the Lagrangian isotopy between F_r and $\mathbf{T}$ given in Section 5.3.1. Thus $\{S_a\}_{a=1}^{m-n}$ is a generating subset of $H_2(X, \mathbb{Z})$. □

By the above proposition, δ_i, and so $\check{X}$, can be expressed in terms of Kähler parameters q_a and open GW invariants n_β.

While throughout the construction we have fixed a choice of ordered basis $\{v_i\}_{i=0}^{n-1}$ of N which generates a cone of Σ, in Proposition 5.5.9 we will see that another choice of the basis amounts to a coordinate change of the mirror. In this sense the mirror $\check{X}$ is independent of choice of this ordered basis.

We now apply the construction procedure given in Section 4.3 and prove Theorem 5.5.1.

5.5.1 Semi-flat complex structure

First let's write down the semi-flat complex coordinates on the chart $(\check{\mu}')^{-1}(U') \subset \check{X}_0$, where $U' \subset B'$ is given in Equation (5.3.5), and $\check{\mu}' : \check{X}_0' \to B_0'$ is the dual torus bundle to $\mu' : X_0' \to B_0'$.

Fix a base point $r_0 \in U'$. For each $r \in U'$, let $\lambda_i \subset \pi_1(F_r)$ be the loop classes given in Section 5.3.2. Define the cylinder classes $[h_i(r)] \in \pi_2((\mu')^{-1}(U'), F_{r_0}, F_r)$ as follows. Recall that we have the trivialization

$$(\mu')^{-1}(U') \cong U' \times (\mathbf{T}_N/\mathbf{T}\langle v_0\rangle) \times (\mathbb{R}/2\pi\mathbb{Z})$$

given in Section 5.3.2. Let $\gamma : [0,1] \to U'$ be a path with $\gamma(0) = r_0$ and $\gamma(1) = r$. For $j = 1, \ldots, n-1$,

$$h_j : [0,1] \times \mathbb{R}/\mathbb{Z} \to U' \times (\mathbf{T}_N/\mathbf{T}\langle v_0\rangle) \times (\mathbb{R}/2\pi\mathbb{Z})$$

is defined by

$$h_j(R,\Theta) := \left(\gamma(R), \frac{\Theta}{2\pi}[v_k], 0\right)$$

and

$$h_0(R,\Theta) := (\gamma(R), 0, 2\pi\Theta).$$

The classes $[h_i(r)]$ is independent of the choice of γ.

Then the semi-flat complex coordinates z_i on $(\breve{\mu}')^{-1}(U')$ for $i = 0, \ldots, n-1$ are defined as

$$z_i(F_r, \nabla) := \exp(\rho_i + 2\pi \mathbf{i}\, \breve{\theta}_i) \tag{5.5.6}$$

where $\mathbf{e}^{2\pi \mathbf{i} \breve{\theta}_i} := \mathrm{Hol}_\nabla(\lambda_i(r))$ and $\rho_i := -\int_{[h_i(r)]} \omega$.

$\mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_{n-1} \wedge \mathrm{d}z_0$ defines the semi-flat holomorphic volume form on $(\breve{\mu}')^{-1}(U')$.

5.5.2 Fourier transform of generating functions

Next we correct the semi-flat complex structure by open Gromov-Witten invariants. The corrected coordinate functions $\tilde{z}_i$ are expressed in terms of Fourier series whose coefficients are FOOO's disk-counting invariants of X. The leading terms of these Fourier series give the original semi-flat complex coordinates. In this sense the semi-flat complex structure is an approximation to the corrected complex structure. The corrected functions have the following expressions:

Proposition 5.5.3. *Let $\mathcal{I}_i = \mathcal{I}_{D_i}$ for $i = 0, \infty, \pm 1, \ldots, \pm(n-1)$ be the generating functions defined by Equation (4.3.3). The Fourier transforms of $\mathcal{I}_i$'s are holomorphic functions $\tilde{z}_i$ on $(\breve{\mu}')^{-1}(B_0' - H')$ respectively.*

1. *For $i = 1, \ldots, n-1$,*

$$\tilde{z}_i = C_i' z_i$$

where C_i' are constants defined by

$$C_i' = \exp\left(-\int_{\beta_i'(r_0)} \omega\right) > 0. \tag{5.5.7}$$

(Recall that r_0 is the based point chosen to define the semi-flat complex coordinates $z_0, \ldots, z_{n-1}$ in the previous subsection.)

2. *For $i = 1, \ldots, n-1$,*

$$\tilde{z}_{-i} = \mathbf{e}^{-2K_1} C_i' z_i^{-1}.$$

3.

$$\tilde{z}_0 := \begin{cases} C_0 z_0 & \text{on } (\check{\mu}')^{-1}(B'_-) \\ z_0 g(z_1, \ldots, z_{n-1}) & \text{on } (\check{\mu}')^{-1}(B'_+) \end{cases}$$

where $g(z_1, \ldots, z_{n-1})$ is the Laurent polynomial

$$g(z_1, \ldots, z_{n-1}) := \sum_{i=0}^{m-1} C_i(1 + \delta_i) \prod_{j=1}^{n-1} z_j^{(\nu_j, v_i)}, \tag{5.5.8}$$

C_i are constants defined by

$$C_i := \exp\left(-\int_{\beta_i(r_0)} \omega\right) > 0 \tag{5.5.9}$$

for $i = 0, \ldots, m-1$, and δ_i are constants previously defined by Equation (5.5.4).

4.

$$\tilde{z}_\infty := \begin{cases} \mathrm{e}^{-K_1 - c_0} z_0^{-1} \left(C_0^{-1} g(z_1, \ldots, z_{n-1})\right) & \text{on } (\check{\mu}')^{-1}(B'_-) \\ \mathrm{e}^{-K_1 - c_0} z_0^{-1} & \text{on } (\check{\mu}')^{-1}(B'_+). \end{cases}$$

(Recall that K_1 and c_0 are constants appearing in the defining equations (5.2.1) *of the polytope P'.)*

Proof. The Fourier transform of each $\mathcal{I}_i$ is a complex-valued function $\tilde{z}_i$ on $(\check{\mu}')^{-1}(B'_0 - H')$ given by

$$\tilde{z}_i = \sum_{\lambda \in \pi_1(X', F_r)} \mathcal{I}_i(\lambda) \mathrm{Hol}_\nabla(\lambda) = \sum_{\beta \in \pi_2(X', F_r)} (\beta \cdot D_i) n_\beta \exp\left(-\int_\beta \omega\right) \mathrm{Hol}_\nabla(\partial\beta).$$

By Proposition 5.4.13 and 5.4.14, $n_\beta = 0$ unless $\beta = \beta'_j, \beta_k + \alpha$ or $\beta_\infty + (\beta_k - \beta_0)$ for $j = \pm 1, \ldots, \pm n-1$ and $k = 0, \ldots, m-1$, where $\alpha \in H_2(X')$ is a class represented by rational curves with Chern number zero, which implies that $\alpha \in H_2(X) \subset H_2(X')$.

1. By Proposition 5.3.10, among the above classes $\beta \cdot D_i \neq 0$ only when $\beta = \beta'_i$, in which case $\beta'_i \cdot D_i = 1$. Thus

$$\tilde{z}_i = \exp\left(-\int_{\beta'_i(r)} \omega\right) \mathrm{Hol}_\nabla(\partial\beta'_i) = \exp\left(-\int_{\beta'_i(r_0)} \omega - \int_{[h_i(r)]} \omega\right) \mathrm{Hol}_\nabla(\lambda_i)$$
$$= C'_i z_i.$$

2. By the same argument
$$\tilde{z}_{-i} = \mathrm{e}^{-2K_1} C'_i z_i^{-1}$$
for $i = 1, \ldots, n-1$.

3. Among the above classes such that $n_\beta \neq 0$, $\beta \cdot D_0 \neq 0$ only when $\beta = \beta_k + \alpha$, in which case $(\beta_k + \alpha) \cdot D_0 = 1$. There are two cases: When $r \in B'_-$,
$$n_{\beta_k+\alpha} = \begin{cases} 1 & \text{for } k = 0 \text{ and } \alpha = 0; \\ 0 & \text{otherwise.} \end{cases}$$
In this case
$$\begin{aligned}\tilde{z}_0 &= \exp\left(-\int_{\beta_0(r)} \omega\right) \mathrm{Hol}_\nabla(\partial\beta_0) = \exp\left(-\int_{\beta_0(r_0)} \omega - \int_{[h_0(r)]} \omega\right) \mathrm{Hol}_\nabla(\lambda_0) \\ &= C_0 z_0.\end{aligned}$$
When $r \in B'_+$,
$$\begin{aligned}&\tilde{z}_0 \\ =&\sum_{j=0}^{m-1} \sum_\alpha n_{\beta_j(r)+\alpha} \exp\left(-\int_{\beta_j(r)+\alpha} \omega\right) \mathrm{Hol}_\nabla(\partial\beta_j(r)) \\ =&\sum_{j=0}^{m-1} \left(\sum_\alpha n_{\beta_j(r)+\alpha} \exp\left(-\int_\alpha \omega\right)\right) \exp\left(-\int_{\beta_j(r_0)} - \int_{[h_0(r)]} - \sum_{i=1}^{n-1} (\nu_i\,,\, v_j) \int_{[h_i(r)]}\right) \omega \\ &\cdot \mathrm{Hol}_\nabla\left(\lambda_0 + \sum_{i=1}^{n-1} (\nu_i\,,\, v_j)\, \lambda_i\right) \\ =&\sum_{j=0}^{m-1} C_j \left(\sum_\alpha n_{\beta_j(r)+\alpha} \exp\left(-\int_\alpha \omega\right)\right) z_0 \prod_{i=1}^{n-1} z_i^{(\nu_i\,,\, v_j)} \\ =&z_0 \sum_{j=0}^{m-1} C_j (1+\delta_j) \prod_{i=1}^{n-1} z_i^{(\nu_i\,,\, v_j)}.\end{aligned}$$

4. Among the above classes such that $n_\beta \neq 0$, $\beta \cdot D_\infty \neq 0$ only when $\beta_\infty + (\beta_k - \beta_0)$ for $k = 0, \ldots, m-1$, in which case the intersection number is 1. The same argument as above shows that
$$\tilde{z}_\infty := \begin{cases} \mathrm{e}^{-K_1-c_0} z_0^{-1} \left(C_0^{-1} g(z_1, \ldots, z_{n-1})\right) & \text{on } (\check{\mu}')^{-1}(B'_-) \\ \mathrm{e}^{-K_1-c_0} z_0^{-1} & \text{on } (\check{\mu}')^{-1}(B'_+). \end{cases}$$

□

Remark 5.5.4. *Let $r_0 \in U'$ be chosen such that C_0 equals to a specific constant, say, $1/2$. One may also choose the toric Kähler form such that the symplectic sizes of the disks β_i are very large for $i = 1, \ldots, m-1$, and so $C_i \ll 1$ (under this choice every non-zero two-cycle in X' has large symplectic area, so this Kähler structure is said to be near the large Kähler limit). According to the above expression of $\tilde{z}_0$, g is approximately C_0, and so $C_0 z_0 = z_0/2$ approximates $\tilde{z}_0$. Similarly $\mathrm{e}^{-K_1 - c_0} z_0^{-1}$ gives an approximation to $\tilde{z}_\infty$. Thus the semi-flat complex coordinates of $\check{X}_0$ are approximations to the corrected complex coordinates, and the correction terms encode the enumerative data of X'.*

5.5.3 The mirror manifold

Now take R to be the subring of functions on $\mu^{-1}(X' - H')$ generated by $\{\tilde{z}_i : i = 0, \infty, \pm 1, \ldots, \pm(n-1)\}$. From the expression of $\tilde{z}_i$ given in Proposition 5.5.3, one may write R as follows:

Proposition 5.5.5.

$$R \cong R_- \times_{R_0} R_+$$

where $R_- = R_+ := \mathbb{C}[z_0^{\pm 1}, \ldots, z_{n-1}^{\pm 1}]$ and R_0 is the localization of $\mathbb{C}[z_0^{\pm 1}, \ldots, z_{n-1}^{\pm 1}]$ at $g = \sum_{i=0}^{m-1} C_i (1 + \delta_i) z^{v_i}$ (see Equation (5.5.8)). The gluing homomorphisms are given by $[Id] : R_- \to R_0$ and

$$\begin{array}{rcl} R_+ & \to & R_0, \\ z_k & \mapsto & [z_k] \text{ for } k = 1, \ldots, n-1 \\ z_0 & \mapsto & \left[g^{-1} z_0\right]. \end{array}$$

The isomorphism is given by

$$\begin{cases} \tilde{z}_0 & \mapsto & (C_0 z_0, \quad z_0 g); \\ \tilde{z}_\infty & \mapsto & (\mathrm{e}^{-K_1 - c_0} C_0^{-1} z_0^{-1} g, \quad \mathrm{e}^{-K_1 - c_0} z_0^{-1}); \\ \tilde{z}_j & \mapsto & (C_j' z_j, \quad C_j' z_j); \\ \tilde{z_{-j}} & \mapsto & (\mathrm{e}^{-2K_1} C_j' z_j^{-1}, \quad \mathrm{e}^{-2K_1} C_j' z_j^{-1}) \end{cases} \tag{5.5.10}$$

for $j = 1, \ldots, n-1$.

Setting $u = (C_0 z_0, z_0 g) \in R$ and $v = (C_0^{-1} z_0^{-1} g, z_0^{-1})$, one has

$$R \cong \frac{\mathbb{C}[u, v, z_1^{\pm 1}, \ldots, z_{n-1}^{\pm 1}]}{\langle uv - g \rangle}.$$

Thus $\check{X} := \mathrm{Spec}(R)$ is geometrically realized as

$$\check{X} = \left\{ (u, v, z_1, \ldots, z_{n-1}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} : uv = g(z_1, \ldots, z_{n-1}) \right\}.$$

One has the canonical map

$$\rho_0 : \check{\mu}^{-1}(B_0 - H) \to \check{X} \tag{5.5.11}$$

by setting

$$u := \begin{cases} C_0 z_0 & \text{on } (\check{\mu}')^{-1}(B_-); \\ z_0 g & \text{on } (\check{\mu}')^{-1}(B_+). \end{cases}$$

and

$$v := \begin{cases} C_0^{-1} z_0^{-1} g & \text{on } (\check{\mu}')^{-1}(B_-); \\ z_0^{-1} & \text{on } (\check{\mu}')^{-1}(B_+). \end{cases}$$

By a change of coordinates, the defining equation of $\check{X}$ can be transformed to the form appeared in Theorem 5.5.1:

Proposition 5.5.6. *By a coordinates change on* $\mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1}$, *the defining equation*

$$uv = \sum_{i=0}^{m-1} C_i (1 + \delta_i) z^{v_i}$$

can be transformed to

$$uv = (1+\delta_0) + \sum_{j=1}^{n-1}(1+\delta_j)z_j + \sum_{i=n}^{m-1}(1+\delta_i)q_{i-n+1}z^{v_i}$$

where C_i's are the constants defined by Equation (5.5.9).

Proof. Consider the coordinates change

$$\hat{z}_j = \frac{C_i}{C_0}z_j$$

for $j = 0, \ldots, n-1$ on $(\mathbb{C}^\times)^{n-1}$. Then

$$C_0 z^{v_0} = C_0 \hat{z}^{v_0}$$

and for $i = 1, \ldots, m-1$,

$$\begin{aligned} C_i z^{v_i} &= C_i \hat{z}^{v_i} \prod_{j=1}^{n-1} \left(\frac{C_0}{C_j}\right)^{(\nu_j\,,\,v_i)} \\ &= C_0 C_i \hat{z}^{v_i} \left(\prod_{j=1}^{n-1} C_j^{(\nu_j\,,\,v_i)}\right)^{-1}. \end{aligned}$$

The last equality in the above follows from the equality

$$\sum_{j=1}^{n-1}(\nu_j\,,\,v_i) = \sum_{j=0}^{n-1}(\nu_j\,,\,v_i) = (\underline{\nu}\,,\,v_i) = 1$$

for $i = 1, \ldots, m-1$.

Thus for $i = 0, \ldots, n-1$,

$$C_i z^{v_i} = C_0 \hat{z}^{v_i} = C_0 \hat{z}^{v_i}.$$

For i $= n, \ldots, m-1$,

$$C_i \left(\prod_{j=1}^{n-1} C_j^{(\nu_j\,,\,v_i)}\right)^{-1}$$

is $\exp(-A_{i-n+1})$, where A_{i-n+1} is the symplectic area of

$$S_{i-n+1} = \beta_{i-n+1} - \sum_{j=1}^{n-1}(\nu_j\,,\,v_{i-n+1})\,\beta_j.$$

Thus it equals to q_{i-n+1}.

Now set $\hat{u} = u/C_0$, the equation

$$uv = \sum_{i=0}^{m-1} C_i(1+\delta_i)z^{v_i}$$

is transformed to

$$\hat{u}v = (1+\delta_0) + \sum_{j=1}^{n-1}(1+\delta_j)\hat{z}_j + \sum_{i=n}^{m-1}(1+\delta_i)q_{i-n+1}\hat{z}^{v_i}.$$

□

This proves part (1) of Theorem 5.5.1 that the construction procedure given in Section 4.3 produces the mirror as stated.

Notice that the defining equation of $\check{X}$ is independent of the parameter K_1 used to define the modification X' in Section 5.2, while the toric Calabi-Yau manifold X appears as the limit of X' as $K_1 \to \infty$. Thus the mirror manifold of X is also taken to be $\check{X}$.

Remark 5.5.7. *Hori-Iqbal-Vafa [28] has written down the mirror of a toric Calabi-Yau manifold X as*

$$uv = 1 + \sum_{j=1}^{n-1} z_j + \sum_{i=n}^{m-1} q_{i-n+1}z^{v_i}$$

by physical considerations. They realize that the above equation needs to be 'quantum corrected', but they did not write down the correction in terms of the symplectic geometry of X. From the SYZ construction, now we see that the corrections (which are the factors $(1+\delta_i)$) can be expressed in terms of open Gromov-Witten invariants of X.

Composing the canonical map ρ_0 (5.5.11) with the coordinate changes given above, one obtains a map

$$\rho : \check{\mu}^{-1}(B_0 - H) \to \check{X} \tag{5.5.12}$$

where

$$u := \begin{cases} z_0 & \text{on } (\check{\mu}')^{-1}(B_-); \\ z_0 G(z_1, \ldots, z_{n-1}) & \text{on } (\check{\mu}')^{-1}(B_+). \end{cases}$$

and

$$v := \begin{cases} z_0^{-1} G(z_1, \ldots, z_{n-1}) & \text{on } (\check{\mu}')^{-1}(B_-); \\ z_0^{-1} & \text{on } (\check{\mu}')^{-1}(B_+). \end{cases}$$

Recall that G is the Laurent polynomial defined by Equation (5.5.2).

In the following we consider part (2) of Theorem 5.5.1.

5.5.4 Holomorphic volume form.

Recall that one has the semi-flat holomorphic volume form on $\check{X}_0$, which is written as $\mathrm{d}\log z_1 \wedge \ldots \wedge \mathrm{d}\log z_{n-1} \wedge \mathrm{d}\log z_0$ in Section 5.5.1. Under the natural map ρ (see Equation (5.5.12)) this semi-flat holomorphic volume form extends to a holomorphic volume form $\check{\Omega}$ on $\check{X}$ which is exactly the one appearing in previous literatures (for example, see P.3 of [32]):

Proposition 5.5.8. *There exists a holomorphic volume form $\check{\Omega}$ on $\check{X}$ which has the property that $\rho^*\check{\Omega} = \mathrm{d}\log z_0 \wedge \ldots \wedge \mathrm{d}\log z_{n-1}$ on $\check{\mu}^{-1}(B_0 - H)$, where $\rho : \check{\mu}^{-1}(B_0 - H) \to \check{X}$ is the canonical map given by* (5.5.12). *Indeed in terms of the coordinates of $\mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1}$,*

$$\check{\Omega} = \mathrm{Res}\left(\frac{1}{uv - G(z_1, \ldots, z_{n-1})} \mathrm{d}\log z_1 \wedge \ldots \wedge \mathrm{d}\log z_{n-1} \wedge \mathrm{d}u \wedge \mathrm{d}v\right)$$

where G is the polynomial defined by Equation (5.5.2).

Proof. Let $F = uv - G(z_1, \ldots, z_{n-1})$ be the defining function of $\check{X}$. On $\check{X} \cap (\mathbb{C}^\times)^{n+1}$, we have the nowhere-zero holomorphic n-form

$$\mathrm{d}\log z_1 \wedge \ldots \wedge \mathrm{d}\log z_{n-1} \wedge \mathrm{d}\log u$$

whose pull-back by ρ is $\mathrm{d}\log z_1 \wedge \ldots \wedge \mathrm{d}\log z_{n-1} \wedge \mathrm{d}\log z_0$. It suffices to prove that this form extends to $\check{X} = \{(u, v, z_1, \ldots, z_{n-1}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} : F = 0\}$. It is clear that the form extends to the open subset of $\check{X}$ where $u \neq 0$. By writing the form as

$$-\mathrm{d}\log z_1 \wedge \ldots \wedge \mathrm{d}\log z_{n-1} \wedge \mathrm{d}\log v$$

we see that it also extends to the open subset where $v \neq 0$. Since

$$u\mathrm{d}v + v\mathrm{d}u = \sum_{i=1}^{m-1} Q_i(1+\delta_i) \prod_{j=1}^{n-1} z_j^{(\nu_j\,,\,v_i)} \left(\sum_{k=1}^{n-1} (\nu_k\,,\,v_i)\,\mathrm{d}\log z_k \right)$$

where $Q_i := 1$ for $i = 0, \ldots, n-1$ and $Q_i = q_i$ for $i = n, \ldots, m-1$, the above n-form can also be written as

$$\begin{aligned}
&\frac{u\mathrm{d}v + v\mathrm{d}u}{\sum_{i=1}^{m-1} Q_i(1+\delta_i) \prod_{j=1}^{n-1} z_j^{(\nu_j\,,\,v_i)} (\nu_1\,,\,v_i)} \wedge \mathrm{d}\log z_2 \wedge \ldots \wedge \mathrm{d}\log z_{n-1} \wedge \mathrm{d}\log u \\
= &\left(\frac{\partial F}{\partial z_1} \right)^{-1} \mathrm{d}v \wedge \mathrm{d}\log z_2 \wedge \ldots \wedge \mathrm{d}\log z_{n-1} \wedge \mathrm{d}u
\end{aligned}$$

which is holomorphic when $\frac{\partial F}{\partial z_1} \neq 0$. By similar change of variables, we see that the form is holomorphic whenever $\mathrm{d}F \neq 0$, which is always the case because $\check{X}$ is smooth.

For $u \neq 0$,

$$\begin{aligned}
&\frac{1}{F}\mathrm{d}\log z_1 \wedge \ldots \wedge \mathrm{d}\log z_{n-1} \wedge \mathrm{d}u \wedge \mathrm{d}v \\
=&\mathrm{d}\log z_1 \wedge \ldots \wedge \mathrm{d}\log z_{n-1} \wedge \mathrm{d}\log u \wedge \frac{u\mathrm{d}v}{F} \\
=&\mathrm{d}\log z_1 \wedge \ldots \wedge \mathrm{d}\log z_{n-1} \wedge \mathrm{d}\log u \wedge \frac{\mathrm{d}F}{F}
\end{aligned}$$

whose residue is $\mathrm{d}\log z_1 \wedge \ldots \wedge \mathrm{d}\log z_{n-1} \wedge \mathrm{d}\log u$. □

This proves part (2) of Theorem 5.5.1.

5.5.5 Independence of choices of basis

If in the beginning we have chosen another ordered basis which generates a cone of Σ to construct the mirror, the complex manifold given in Theorem 5.5.1 differs the original one by a biholomorphism which preserves the holomorphic volume form:

Proposition 5.5.9. *Let $\{u_0, \ldots, u_{n-1}\} \subset N$ and $\{v_0, \ldots, v_{n-1}\} \subset N$ be two ordered basis, each generating a cone of Σ. Let $(\widetilde{X}, \tilde{\Omega})$ and $(\check{X}, \check{\Omega})$ be the two mirror complex manifolds constructed from these two choices respectively. Then there exists a biholomorphism $\phi : \widetilde{X} \to \check{X}$ with the property that $\phi^*\check{\Omega} = \pm\tilde{\Omega}$.*

Proof. The mirror complex manifolds constructed from the two choices are

$$\check{X} = \left\{(u, v, z_1, \ldots, z_{n-1}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} : uv = \sum_{i=0}^{m-1} Q_i(1+\delta_i) \prod_{j=1}^{n-1} z_j^{(\nu_j, v_i)}\right\}$$

and

$$\widetilde{X} = \left\{(u, v, \zeta_1, \ldots, \zeta_{n-1}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} : uv = \sum_{i=0}^{m-1} Q_i(1+\delta_i) \prod_{j=1}^{n-1} \zeta_j^{(\mu_j, v_i)}\right\}$$

respectively, where $Q_i = 1$ for $i = 0, \ldots, n-1$ and $Q_i = q_i$ for $i = n, \ldots, m-1$; $\{\nu_0, \ldots, \nu_{n-1}\}$ is the dual basis of $\{v_0, \ldots, v_{n-1}\}$, and $\{\mu_0, \ldots, \mu_{n-1}\}$ is the dual basis of $\{u_0, \ldots, u_{n-1}\}$. Notice that the constants Q_i and δ_i are independent of choice of the basis (see Equation (5.5.4)). Let $\mu_j = \sum_k a_{jk}\nu_k$ be the change of basis, where a_{jk} form a matrix $A \in \mathrm{GL}(n, \mathbb{Z})$. Then

$$\begin{aligned}
\sum_{i=0}^{m-1} Q_i(1+\delta_i) \prod_{j=1}^{n-1} \zeta_j^{(\mu_j, v_i)} &= \sum_{i=0}^{m-1} Q_i(1+\delta_i) \prod_{j=1}^{n-1}\prod_{k=1}^{n-1} \zeta_j^{a_{jk}(\nu_k, v_i)} \\
&= \sum_{i=0}^{m-1} Q_i(1+\delta_i) \prod_{k=1}^{n-1} \left(\prod_{j=1}^{n-1} \zeta_j^{a_{jk}}\right)^{(\nu_k, v_i)}.
\end{aligned}$$

Thus the coordinates change

$$z_k = \prod_{j=1}^{n-1} \zeta_j^{a_{jk}}$$

gives the desired biholomorphism. Moreover under this coordinates change

$$\begin{aligned}
\check{\Omega} &= \mathrm{d}\log z_1 \wedge \ldots \wedge \mathrm{d}\log z_{n-1} \wedge \mathrm{d}\log u \\
&= (\det A)\, \mathrm{d}\log \zeta_1 \wedge \ldots \wedge \mathrm{d}\log \zeta_{n-1} \wedge \mathrm{d}\log u \\
&= \pm\tilde{\Omega}.
\end{aligned}$$

7.3.2 Three dimensional cases

Let $X = X_\Sigma$ be a toric Calabi-Yau and $\beta \in \pi_2(X, \mathbf{T})$ be a disk class bounded by a Lagrangian toric fiber $\mathbf{T} \subset X$. By Theorem 7.2.4, the one-pointed open Gromov-Witten invariant n_β is non-zero only when $\beta = \beta_k$ for $k = 0, \ldots, m-1$, in which case $n_{\beta_k} = 1$, or β is of the form $\beta_i + \alpha$ for some α represented by rational curves in X. For the later case

$$n_{\beta_i+\alpha} = \langle [\mathrm{pt}] \rangle_{0,1,h+\alpha}^{\bar{X}}$$

provided that rational curves in $\bar{X}$ representing α are contained in X. (Recall that $\bar{X}$ is a toric compactification of X along the v_i direction, and $h \in H_2(\bar{X})$ denote the fiber class.)

When $\dim X = 3$, we may apply a flop to relate the above expression to some known local invariants as follows. Without loss of generality let $i = 0$ in the above expression. Write $Y = \bar{X}$ and let $\mathcal{D}_\infty$ be the toric divisor corresponding to $v_\infty = -v_0$. Let $y \in Y$ be one of the torus-fixed points contained in $\mathcal{D}_\infty$, which corresponds to a 3-cone $\langle v_\infty, u_1, u_2 \rangle_{\mathbb{R}}$ of Σ_Y. First we blow up y to get Y_1, whose fan Σ_1 is obtained by adding the ray generated by $w = v_\infty + u_1 + u_2$ to Σ_Y. By using the blow-up formula of Hu-Gathmann [30, 23] (Theorem 7.1.1), one has

$$\langle [\mathrm{pt}] \rangle_{0,1,h+\alpha}^{Y} = \langle 1 \rangle_{0,0,\pi^!(h)+\alpha-e_1}^{Y_1}$$

where $\pi : Y_1 \to Y$ is the blow-up described above, $e_1 \in H_2(Y_1)$ is the corresponding exceptional class, and $\pi^! := \mathrm{PD} \circ \pi^* \circ \mathrm{PD} : H_2(Y) \to H_2(Y_1)$. By abuse of notations if $\alpha \in H_2(X) \subset H_2(Y)$, then $\pi^!(\alpha)$ is still denoted as α.

Write $h \in H_2(Y, \mathbb{Z})$ as $h' + \delta$, where $h' \in H_2(Y, \mathbb{Z})$ is the class corresponding to the 2-cone $\langle u_1, u_2 \rangle_{\mathbb{R}}$ of Σ_Y and $\delta \in H_2(X, \mathbb{Z})$. Then

$$\pi^!(h) + \alpha - e_1 = \pi^!(h') - e_1 + \delta + \alpha = h'' + \delta + \alpha$$

where $h'' \in H_2(Y_1, \mathbb{Z})$ is the class corresponding to the 2-cone $\langle u_1, u_2 \rangle_{\mathbb{R}}$ of Σ_1.

Thus the above expression can be written as

$$\langle 1\rangle^{Y_1}_{0,0,\pi'(h)+\alpha-e_1} = \langle 1\rangle^{Y_1}_{0,0,h''+\delta+\alpha}.$$

Now we take a flop of Y_1 along h'' as follows. There exists a unique primitive vector $u_0 \neq w$ such that $\{u_0, u_1, u_2\}$ generates a simplicial cone in Σ_1: If $\{v_0, u_1, u_2\}$ spans a cone of Σ_1, then take $u_0 = v_0$; otherwise since Σ_1 is simplicial, there exists a primitive vector $u_0 \subset \mathbb{R}\langle v_0, u_1, u_2\rangle$ with the required property. Now $\langle u_1, u_2, w\rangle_{\mathbb{R}}$ and $\langle u_1, u_2, u_0\rangle_{\mathbb{R}}$ form two adjacent simplicial cones in Σ_1, and we may employ a flop to obtain a new toric variety W, whose fan Σ_W contains the adjacent cones $\langle w, u_0, u_1\rangle_{\mathbb{R}}$ and $\langle w, u_0, u_2\rangle_{\mathbb{R}}$. (See Figure 7.5).

Figure 7.5: A flop.

The flop induces a natural isomorphism $H_2(Y_1) \cong H_2(W)$, which sends $h'' \in H_2(Y_1)$ (which corresponds to the cone $\langle u_1, u_2\rangle_{\mathbb{R}}$) to $-e \in H_2(W)$, where $e \in H_2(W)$ is the exceptional class corresponding to the cone $\langle u_0, w\rangle_{\mathbb{R}}$. A class $a \neq h'' \in H_2(Y_1)$ corresponding to a 2-cone C of Σ_1 is sent to a class in $H_2(W)$ corresponding to C as a cone of Σ_W, and by abuse of notations it is still denoted as $a \in H_2(W)$. By Li-Ruan's flop formula (Theorem 7.1.2),

$$\langle 1\rangle^{Y_1}_{0,0,h''+\delta+\alpha} = \langle 1\rangle^{W}_{0,0,-e+\delta+\alpha}.$$

W is the compactification of another toric Calabi-Yau W_0 whose fan is as follows: First we add the ray generated by w to Σ_X, and then we apply a flop to the adjacent cones $\langle w, u_1, u_2\rangle$ and $\langle u_0, u_1, u_2\rangle$. W_0 is Calabi-Yau because

$$(\underline{\nu}, w) = 1$$

and a flop preserves this Calabi-Yau condition. Moreover, $-e, \delta, \alpha \in H_2(W)$ are indeed classes in $H_2(W_0) \subset H_2(W)$. Assuming that curves in W representing $\alpha + \delta - e$ are contained in W_0, then

$$\langle 1 \rangle^{W}_{0,0,-e+\delta+\alpha} = \langle 1 \rangle^{W_0}_{0,0,\alpha+\delta-e}$$

and the later is a local invariant which can be computed by localization or the mirror principle [11, 42, 43, 44]. For any curve class $\alpha \in H_2(X)$, $\alpha' = \alpha + \delta - e \in H_2(W_0)$ is a curve class called the strict transform of α. To conclude,

Proposition 7.3.7. *Let X be a toric Calabi-Yau manifold and $\beta = \beta_i + \alpha \in \pi_2(X, \mathbf{T})$, where $\mathcal{D}_i$ is compact and $\alpha \in H_2(X)$ is represented by some rational curves. Let $\bar{X}$, W and W_0 be the toric varieties constructed as above. Denote the strict transform of α (defined above) by α'. Then*

$$n_\beta = \langle 1 \rangle^{W_0}_{0,0,\alpha'}$$

provided that curves in $\bar{X}$ representing α are contained in X, and curves in W representing α' are contained in W_0.

The above formula provides a way to compute the open Gromov-Witten invariants of a toric CY threefold. Yet at present we don't have a proof of the Mirror Conjecture 6.2.1 which works uniformly for all toric CY threefolds. Instead, we apply the above computational method to some familiar threefolds to give evidences of Conjecture 6.2.1.

1. $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

For $X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, the generators of the rays of the defining fan Σ are $v_0 = (0, 0, 1), v_1 = (1, 0, 1), v_2 = (0, 1, 1)$ and $v_3 = (1, -1, 1)$. We equip X with a toric Kähler structure ω so that the associated moment polytope P is given by

$$P = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \geq 0, x_1 + x_3 \geq 0, x_2 + x_3 \geq 0, x_1 - x_2 + x_3 \geq -s_1\},$$

where $s_1 = \int_l \omega > 0$ and $l \in H_2(X, \mathbb{Z})$ is the class of the embedded $\mathbb{P}^1 \subset X$. To complexify the Kähler class, we set $\omega^{\mathbb{C}} = \omega + 2\pi\sqrt{-1}B$, for some real two-form B (the B-field). We let $s = \int_l \omega^{\mathbb{C}} \in \mathbb{C}$.

Since there is no compact toric prime divisors in X (see Figure 7.6 below), by Theorem 7.2.4, $n_\beta \neq 0$ only when $\beta = \beta_i$ is a basic disk class. Thus the SYZ mirror written in Theorem 5.5.1 is given by

$$\check{X} = \{(u, v, z_1, z_2) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^2 : uv = 1 + z_1 + z_2 + qz_1z_2^{-1}\},$$

where $q = \exp(-s)$.

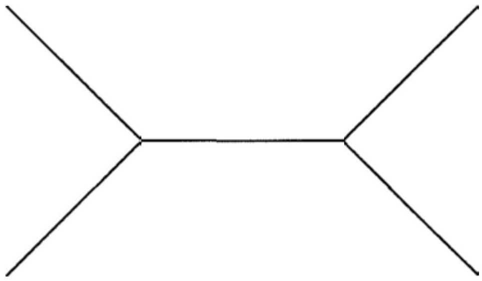

Figure 7.6: $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

Both $\mathcal{M}_C(\check{X})$ and $\mathcal{M}_K(X)$ can be identified with the punctured unit disk Δ^* and the SYZ map $\mathcal{F}_{\mathrm{SYZ}} : \Delta^* \to \Delta^*$ given by (6.2.1) is the identity map.

On the other hand, $\Phi(\check{q}) = -\log \check{q}$ is the unique (up to multiplication by constants) solution with a single logarithm of the Picard-Fuchs equation

$$((1 - \check{q})\theta_{\check{q}}^2)\Phi(\check{q}) = 0,$$

where $\theta_{\check{q}}$ denotes $\check{q}\frac{\partial}{\partial \check{q}}$, which implies that that the mirror map ψ is the identity. Hence, $\psi \circ \mathcal{F}_{\mathrm{SYZ}} = \mathrm{Id}$, and so Conjecture 6.2.1 holds true for this example. □

2. $K_{\mathbb{P}^2}$.

The primitive generators of the rays of the fan Σ defining $X = K_{\mathbb{P}^2}$ can be chosen to be $v_0 = (0, 0, 1), v_1 = (1, 0, 1), v_2 = (0, 1, 1)$ and $v_3 = (-1, -1, 1)$. We equip X with a toric Kähler structure ω associated to the moment polytope

$$P = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \geq 0, x_1 + x_3 \geq 0, x_2 + x_3 \geq 0, -x_1 - x_2 + x_3 \geq -s_1\},$$

where $s_1 = \int_l \omega > 0$ and $l \in H_2(X, \mathbb{Z}) = H_2(\mathbb{P}^2, \mathbb{Z})$ is the class of a line in $\mathbb{P}^2 \subset X$. To complexify the Kähler class, set $\omega^{\mathbb{C}} = \omega + 2\pi\sqrt{-1}B$, where B is a real two-form (the B-field), and let $s = \int_l \omega^{\mathbb{C}} \in \mathbb{C}$.

There is only one compact toric prime divisor $\mathcal{D}_0$ which is the zero section $\mathbb{P}^2 \hookrightarrow K_{\mathbb{P}^2}$ and it corresponds to v_0. By Theorem 7.2.4, $n_\beta \neq 0$ only when $\beta = \beta_i$ is a basic disk class or $\beta = \beta_0 + kl$ for $k \geq 0$. The SYZ mirror $\check{X}$ in Theorem 5.5.1 is given by

$$\check{X} = \left\{(u, v, z_1, z_2) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^2 : uv = \left(1 + \sum_{k=1}^{\infty} n_{\beta_0+kl} q^k\right) + z_1 + z_2 + \frac{q}{z_1 z_2}\right\},$$

where $q = \exp(-s)$.

Now we apply Proposition 7.3.7 to compute n_{β_0+kl}. In the construction of $\bar{X}$, W and W_0, $v_\infty = (0, 0, -1)$, $u_1 = (1, 0, 1)$, $u_2 = (0, 1, 1)$, $w = (1, 1, 1)$ and $u_0 = (0, 0, 1)$. Thus $W_0 = K_{\mathbb{F}_1}$ and W is the fiberwise compactification of W_0, where $\mathbb{F}_1$ is the blowup of $\mathbb{P}^2$ at a point. Moreover, $\delta = 0$ and e is the (-1)-curve in $\mathbb{F}_1 \subset W_0$. Denote the fiber class of $\mathbb{F}_1$ by f, one has $kl - e = kf + (k-1)e$. See Figure 7.7 below.

Since $\mathbb{P}^2$ and $\mathbb{F}_1$ are Fano, curves in $\bar{X}$ representing kl are contained in X, and curves in W representing $kf + (k-1)e$ are contained in W_0. Thus we have

$$n_{\beta_0+kl} = \langle 1 \rangle^{K_{\mathbb{F}_1}}_{0,0,kf+(k-1)e}.$$

The local BPS invariants $\langle 1 \rangle^{K_{\mathbb{F}_1}}_{0,0,kf+(k-1)e}$ have been computed by Chiang-

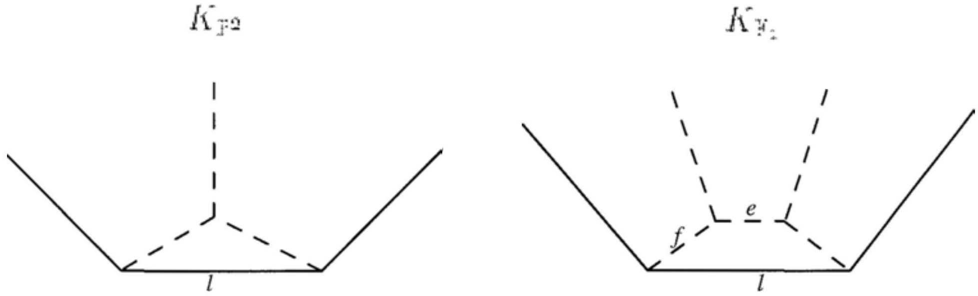

Figure 7.7: Polytope picture for $K_{\mathbb{P}^2}$ and $K_{\mathbb{F}_1}$.

Klemm-Yau-Zaslow and the results can be found in Table 10 in [11]. Thus

$$\begin{aligned} n_{\beta_0+l} &= -2, \\ n_{\beta_0+2l} &= 5, \\ n_{\beta_0+3l} &= -32, \\ n_{\beta_0+4l} &= 286, \\ n_{\beta_0+5l} &= -3038, \\ n_{\beta_0+6l} &= 35870, \\ &\vdots \end{aligned}$$

Using these results, we can write down the SYZ mirror explicitly as

$$\check{X} = \left\{ (u, v, z_1, z_2) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^2 : uv = 1 + \delta_0(q) + z_1 + z_2 + \frac{q}{z_1 z_2} \right\},$$

where

$$\delta_0(q) = -2q + 5q^2 - 32q^3 + 286q^4 - 3038q^5 + \dots.$$

Now, both $\mathcal{M}_C(\check{X})$ and $\mathcal{M}_K(X)$ can be identified with the punctured unit disk Δ^*. Our map $\mathcal{F}_{\text{SYZ}} : \Delta^* \to \Delta^*$ (6.2.1) is given by

$$\mathcal{F}_{\text{SYZ}}(q) = q(1 - 2q + 5q^2 - 32q^3 + 286q^4 - 3038q^5 + \dots)^{-3}.$$

On the other hand, the mirror map and its inverse have been computed by Graber-Zaslow in [25]. First of all, the Picard-Fuchs equation associated to $K_{\mathbb{P}^2}$

is

$$[\theta_{\check{q}}^3 + 3\check{q}\theta_{\check{q}}(3\theta_{\check{q}}+1)(3\theta_{\check{q}}+2)]\Phi(\check{q}) = 0,$$

where $\theta_{\check{q}}$ denotes $\check{q}\frac{\partial}{\partial \check{q}}$, the solution of which with a single logarithm is given by

$$\Phi(\check{q}) = -\log \check{q} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{(3k)!}{(k!)^3} \check{q}^k.$$

Hence, the mirror map $\psi : \Delta^* \to \Delta^*$ can be written explicitly as

$$\check{q} \mapsto q(\check{q}) = \exp(-\Phi(\check{q})) = \check{q} \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{(3k)!}{(k!)^3} \check{q}^k \right).$$

The inverse mirror map is then computed by inverting the above formal power series and is given by

$$\begin{aligned} q &\mapsto q + 6q^2 + 9q^3 + 56q^4 + 300q^5 + 3942q^6 + \ldots \\ &= q(1 - 2q + 5q^2 - 32q^3 + 286q^4 - 3038q^5 + \ldots)^{-3} \end{aligned}$$

which agrees with the above expression for $\mathcal{F}_{\mathrm{SYZ}}$.

3. $K_{\mathbb{P}^1 \times \mathbb{P}^1}$.

For $X = K_{\mathbb{P}^1 \times \mathbb{P}^1}$, the primitive generators of rays of the defining fan Σ can be taken to be $v_0 = (0,0,1), v_1 = (1,0,1), v_2 = (0,1,1), v_3 = (-1,0,1)$ and $v_4 = (0,-1,1)$. We equip X with a toric Kähler structure ω so that the associated moment polytope P is defined by the following inequalities

$$x_3 \geq 0, x_1 + x_3 \geq 0, x_2 + x_3 \geq 0, -x_1 + x_3 \geq -s_1', -x_2 + x_3 \geq -s_2'.$$

Here, $s_1' = \int_{l_1} \omega, s_2' = \int_{l_2} \omega > 0$ and $l_1, l_2 \in H_2(X, \mathbb{Z}) = H_2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})$ are the classes of the $\mathbb{P}^1$-factors in $\mathbb{P}^1 \times \mathbb{P}^1$. To complexify the Kähler class, we set $\omega^{\mathbb{C}} = \omega + 2\pi\sqrt{-1}B$, where B is a real two-form (the B-field), and let $s_1 = \int_{l_1} \omega^{\mathbb{C}}, s_2 = \int_{l_2} \omega^{\mathbb{C}} \in \mathbb{C}$.

There is only one compact toric prime divisor $\mathcal{D}_0$ which is the zero section $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow K_{\mathbb{P}^1 \times \mathbb{P}^1}$. By Theorem 7.2.4, $n_\beta \neq 0$ only when $\beta = \beta_i$ is a basic disk class or $\beta = \beta_0 + k_1 l_1 + k_2 l_2$ for $k_1, k_2 \geq 0$. The SYZ mirror $\check{X}$ in Theorem 5.5.1 is given by

$$\check{X} = \left\{ (u, v, z_1, z_2) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^2 : uv = 1 + \delta_0(q_1, q_2) + z_1 + z_2 + \frac{q_1}{z_1} + \frac{q_2}{z_2} \right\},$$

where $q_a = \exp(-s_a)$ $(a = 1, 2)$ and

$$1 + \delta_0(q_1, q_2) = \sum_{k_1, k_2 \geq 0} n_{\beta_0 + k_1 l_1 + k_2 l_2} q_1^{k_1} q_2^{k_2}.$$

For simplicity, denote $n_{\beta_0 + k_1 l_1 + k_2 l_2}$ by n_{k_1, k_2}. Let's apply Proposition 7.3.7 to compute n_{k_1, k_2}. In the construction of $\bar{X}$, W and W_0, $v_\infty = (0, 0, -1)$, $u_1 = (-1, 0, 1)$, $u_2 = (0, -1, 1)$, $w = (-1, -1, 1)$ and $u_0 = (0, 0, 1)$. Thus $W_0 = K_{dP_2}$ and W is the fiberwise compactification of W_0, dP_2 is the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point. Moreover, $\delta = 0$ and e is the exceptional curve in $dP_2 \subset W_0$ under the blow-up map $dP_2 \to \mathbb{P}^1 \times \mathbb{P}^1$. See Figure 7.7 below.

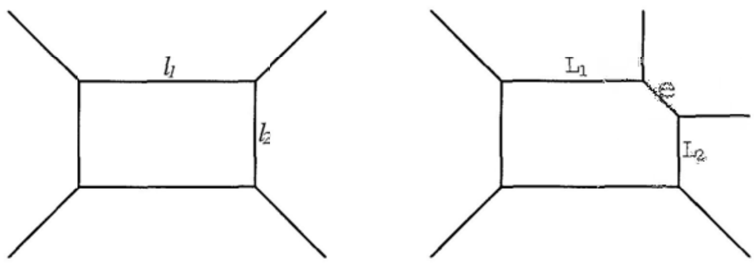

Figure 7.8: Polytope picture for $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ and K_{dP_2}.

Since $\mathbb{P}^2$ and $\mathbb{F}_1$ are Fano, curves in $\bar{X}$ representing kl are contained in X, and curves in W representing $k_1 L_1 + k_2 L_2 + (k_1 + k_2 - 1)e$ are contained in W_0. Thus we have

$$n_{\beta_0 + kl} = \langle 1 \rangle^{K_{dP_2}}_{0,0,k_1 L_1 + k_2 L_2 + (k_1 + k_2 - 1)e}$$

and the local invariants appeared in the right hand side above have been computed by Chiang-Klemm-Yau-Zaslow and the results can be read off from Table 3 on p.

□

5.5.6 The superpotential.

Recall that we have defined the generating function $\mathcal{F}_X$ of open Gromov-Witten invariants (Definition 3.3.3). By taking Fourier transform, we obtain the superpotential, which is a holomorphic function on $(\check{\mu})^{-1}(B_0 - H)$, and it extends to be a holomorphic function on $\check{X}$:

Proposition 5.5.10. *Let $\tilde{z}_i$ be the holomorphic functions on $(\check{\mu}')^{-1}(B_0'-H)$ given in Proposition 5.5.3 for $i = 0, \infty, \pm 1, \ldots, \pm(n-1)$.*

1. *The Fourier transform of $\mathcal{F}_{X'}$ is the function*
$$W' = \tilde{z}_0 + \tilde{z}_\infty + \sum_{i=1}^{n-1} \tilde{z}_i$$
on $(\check{\mu}')^{-1}(B_0' - H)$.

2. *The Fourier transform of $\mathcal{F}_X$ is the function*
$$W = \tilde{z}_0$$
on $(\check{\mu}')^{-1}(B_0' - H)$, which equals to $\rho^*(C_0 u)$. (C_0 is a constant defined by Equation (5.5.9).)

Proof. The first part is just Proposition 4.3.11.

For the second part, the Fourier transform of $\mathcal{F}_X$ is
$$W = \sum_{\lambda \in \pi_1(X, F_r)} \mathcal{F}_X(\lambda) \mathrm{Hol}_\nabla(\lambda) = \sum_{\beta \in \pi_2(X, F_r)} n_\beta \exp\left(-\int_\beta \omega\right) \mathrm{Hol}_\nabla(\partial\beta).$$

For $r \in B_+$, by Proposition 5.4.7, $n_\beta = 0$ unless $\beta = \beta_k + \alpha$ for $k = 0, \ldots, m-1$ and $\alpha \in H_2(X)$ represented by rational curves. Moreover, $n_{\beta_k} = 1$. Thus
$$\begin{aligned} W &= \sum_{j=0}^{m-1} \sum_\alpha n_{\beta_j(r)+\alpha} \exp\left(-\int_{\beta_j(r)+\alpha} \omega\right) \mathrm{Hol}_\nabla(\partial\beta_j(r)) \\ &= \tilde{z}_0. \end{aligned}$$

For $r \in B_-$, by Proposition 5.4.9, $n_\beta = 0$ unless $\beta = \beta_0$, and $n_{\beta_0} = 1$. Thus

$$\begin{aligned} W &= \exp\left(-\int_{\beta_0(r)} \omega\right) \mathrm{Hol}_\nabla(\partial\beta_0) \\ &= \tilde{z}_0. \end{aligned}$$

By Equation (5.5.12), $\tilde{z}_0 = \rho^*(C_0 u)$. □

This ends the proof of Theorem 5.5.1. In the next Chapter, we will state a conjecture on the relation between the SYZ mirror and the mirror map, which plays a fundamental role in the study of mirror symmetry and its applications.

Chapter 6

Mirror maps and SYZ maps

For a pair $(X, \check{X})$ of mirror Calabi-Yau manifolds, mirror symmetry asserts that there is a local isomorphism between the moduli space $\mathcal{M}_C(\check{X})$ of complex structures of $\check{X}$ and the complexified Kähler moduli space $\mathcal{M}_K(X)$ of X near the large complex structure limit and large volume limit respectively, such that the Frobenius structures over the two moduli spaces get identified. This is called the *mirror map*. It induces canonical flat coordinates on $\mathcal{M}_C(\check{X})$ from the natural flat structure on $\mathcal{M}_K(X)$. A remarkable feature of the SYZ mirror is that for typical toric Calabi-Yau manifolds, it is inherently written in these canonical flat coordinates. In this Chapter we shall formulate this feature as a conjecture. The conjecture will be proved for toric CY surfaces and some toric CY threefolds in the next Chapter.

6.1 A quick review on the mirror map

Let $X = X_\Sigma$ be a toric Calabi-Yau n-fold. We adopt the notation used in Chapter 5: $v_i \in N, i = 0, \ldots, m-1$ are primitive generators of rays in the fan Σ, and $\{\nu_j\}_{i=0}^{n-1} \subset M$ is the dual basis of $\{v_j\}_{j=0}^{n-1} \subset N$. Moreover, $H_2(X, \mathbb{Z})$ is of rank $l = m - n$ generated by $\{S_a\}_{i=1}^{m-n}$ (see Equation (5.5.5) and Proposition 5.5.2).

1. *The complexified Kähler moduli.*

Let $\mathcal{K}(X)$ be the Kähler cone of X, i.e. $\mathcal{K}(X) \subset H^2(X,\mathbb{R})$ is the space of Kähler classes on X. Then let

$$\mathcal{M}_K(X) = \mathcal{K}(X) + 2\pi\sqrt{-1}H^2(X,\mathbb{R})/H^2(X,\mathbb{Z}).$$

This is the complexified Kähler moduli space of X. An element in $\mathcal{M}_K(X)$ is represented by a complexified Kähler class $\omega^{\mathbb{C}} = \omega + 2\pi\sqrt{-1}B$, where $\omega \in \mathcal{K}(X)$ and $B \in H^2(X,\mathbb{R})$. B is usually called the B-field. We have the map $\mathcal{M}_K(X) \to (\Delta^*)^l$ defined by

$$q_i = \exp\left(-\int_{S_{n+i-1}} \omega^{\mathbb{C}}\right)$$

for $i = 1,\ldots,l$. This map is a local biholomorphism from an open subset $U \subset \mathcal{M}_K(X)$ to $(\Delta^*)^l$, where $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ is the punctured unit disk. The inclusion $(\Delta^*)^l \hookrightarrow \Delta^l$, where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk, gives an obvious partial compactification, and the origin $0 \in \Delta^l$ is called a *large radius limit* point. From now on, by abuse of notation we will take $\mathcal{M}_K(X)$ to be this open neighborhood of large radius limit.

2. *The complex moduli.*

On the other hand, let $\mathcal{M}_C(\check{X}) = (\Delta^*)^l$. We have a family of noncompact Calabi-Yau manifolds $\{\check{X}_{\check{q}}\}$ parameterized by $\check{q} \in \mathcal{M}_C(\check{X})$ defined as follows. For $\check{q} = (\check{q}_1,\ldots,\check{q}_l) \in \mathcal{M}_C(\check{X})$,

$$\check{X}_{\check{q}} := \left\{(u,v,z_1,\ldots,z_{n-1}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} : uv = \sum_{i=0}^{m-1} t_i z^{v_i}\right\}, \qquad (6.1.1)$$

where $t_i \in \mathbb{C}$ are subject to the constraints

$$t_{n+a-1}\prod_{i=0}^{n-1} t_i^{-(\nu_i, v_{a+n-1})} = \check{q}_a,\ a = 1,\ldots,l. \qquad (6.1.2)$$

The origin $0 \in \Delta^l$ in the partial compactification $\mathcal{M}_C(\check{X}) \hookrightarrow \Delta^l$ is called a *large complex structure limit* point. Each $\{\check{X}_{\check{q}}\}$ is equipped with a holomorphic volume form

$$\check{\Omega}_{\check{q}} := \operatorname{Res}\left(\frac{1}{uv - \sum_{i=0}^{m-1} t_i z^{v_i}} \mathrm{d}\log z_1 \wedge \ldots \wedge \mathrm{d}\log z_{n-1} \wedge \mathrm{d}u \wedge \mathrm{d}v\right). \qquad (6.1.3)$$

3. *The mirror map.* The mirror map $\psi : \mathcal{M}_C(\check{X}) \to \mathcal{M}_K(X)$ is defined by periods:

$$\psi(\check{q}) := \left(\exp\left(-\int_{\gamma_1} \check{\Omega}_{\check{q}}\right), \ldots, \exp\left(-\int_{\gamma_l} \check{\Omega}_{\check{q}}\right)\right),$$

where $\{\gamma_1, \ldots, \gamma_l\}$ is a basis of $H_n(\check{X}, \mathbb{Z})$.

Mirror symmetry asserts that ψ is a local isomorphism (around the large complex structure limit), and this induces canonical flat coordinates on $\mathcal{M}_C(\check{X})$ by pulling back the flat coordinates on $\mathcal{M}_K(X)$ via ψ.

In practice, one computes the mirror map by solving a system of linear differential equations, namely the Picard-Fuchs equations, associated to the toric Calabi-Yau manifold X [1, 25]. For $i = 0, 1, \ldots, m-1$, denote by θ_i the differential operator $t_i \frac{\partial}{\partial t_i}$. For $j = 1, \ldots, n$, let

$$\mathcal{T}_j = \sum_{i=0}^{m-1} v_i^j \theta_i,$$

where $v_i^j = (\nu_j \, , \, v_i)$. For $a = 1, \ldots, l$, let

$$\Box_a = \prod_{i:Q_i^a > 0} \left(\frac{\partial}{\partial t_i}\right)^{Q_i^a} - \prod_{i:Q_i^a < 0} \left(\frac{\partial}{\partial t_i}\right)^{-Q_i^a}$$

where $Q_j^a = -(\nu_j \, , \, v_{a+n-1})$ for $j = 0, \ldots, n-1$, and $Q_i^a = \delta_{i,a+n-1}$ for $i = n, \ldots, m-1$. Then the A-hypergeometric system (also called GKZ system) of linear differential equations associated to X is given by

$$\mathcal{T}_j \Phi(t) = 0 \ (j = 1, \ldots, n), \quad \Box_a \Phi(t) = 0 \ (a = 1, \ldots, l).$$

If we denote by $\check{X}_t$ the noncompact Calabi-Yau manifold (6.1.1) parameterized by $t = (t_0, t_1, \ldots, t_{m-1}) \in \mathbb{C}^m$ and $\check{\Omega}_t$ the holomorphic volume form (6.1.3) on $\check{X}_t$, then, for any n-cycle $\gamma \in H_3(\check{X}, \mathbb{Z})$, the period

$$\Pi_\gamma(t) := \int_\gamma \check{\Omega}_t,$$

as a function of $t = (t_0, t_1, \dots, t_{m-1})$, satisfies the above A-hypergeometric system (see e.g. [29] and [32]).

By imposing the constraints (6.1.2), the A-hypergeometric system is reduced to a set of Picard-Fuchs equations, which are satisfied by the periods

$$\Pi_\gamma(\check{q}) = \int_\gamma \check{\Omega}_{\check{q}}, \ \gamma \in H_n(\check{X}, \mathbb{Z}),$$

as functions of $\check{q} \in \mathcal{M}_C(\check{X})$. Now, let $\Phi_1(\check{q}), \dots, \Phi_l(\check{q})$ be a basis of the solutions of this set of Picard-Fuchs equations with a single logarithm. Then there is a basis $\gamma_1, \dots, \gamma_l$ of $H_n(\check{X}, \mathbb{Z})$ such that

$$\Phi_a(\check{q}) = \int_{\gamma_a} \check{\Omega}_{\check{q}}$$

for $a = 1, \dots, l$, and the mirror map $\psi : \mathcal{M}_C(\check{X}) \to \mathcal{M}_K(X)$ is given by

$$\psi(\check{q}) = (\exp(-\Phi_1(\check{q})), \dots, \exp(-\Phi_l(\check{q}))).$$

6.2 A mirror conjecture

1. *The SYZ map.*

In the last Chapter, for every toric Calabi-Yau manifold we have constructed its SYZ mirror which is defined by (Theorem 5.5.1)

$$uv = (1 + \delta_0) + \sum_{j=1}^{n-1}(1 + \delta_j) z_j + \sum_{i=n}^{m-1}(1 + \delta_i) q_{i-n+1} z^{v_i}.$$

Recall that $\mathcal{M}_C(\check{X}) \cong (\Delta^*)^l$ with each point $\check{q} = (\check{q}_1, \dots, \check{q}_l) \in \mathcal{M}_C(\check{X})$ parametrizing the complex manifold defined by

$$uv = t_0 + \sum_{j=1}^{n-1} t_j z_j + \sum_{i=n}^{m-1} t_i z^{v_i}$$

subject to the constraints

$$\check{q}_a = t_{n+a-1} \prod_{i=0}^{n-1} t_i^{-(\nu_i, v_{a+n-1})}, \ a = 1, \dots, l.$$

Thus setting $t_0 = 1 + \delta_0$, $t_j = 1 + \delta_j$ for $j = 1, \ldots, n-1$ and $t_i = (1+\delta_i)q_{i-n+1}$ for $i = n, \ldots, m-1$, one has a map $\mathcal{F}_{\mathrm{SYZ}} : \mathcal{M}_K(X) \to \mathcal{M}_C(\check{X})$, $(\check{q}_1, \ldots, \check{q}_l) = \mathcal{F}_{\mathrm{SYZ}}(q_1, \ldots, q_l)$ defined by

$$\check{q}_a = q_a(1+\delta_{a+n-1})\prod_{j=0}^{n-1}(1+\delta_j)^{-(\nu_j, v_{a+n-1})}, \; a = 1, \ldots, l. \tag{6.2.1}$$

We will call $\mathcal{F}_{\mathrm{SYZ}}$ to be the SYZ map since it is defined by the SYZ mirror construction in Section 4.3 and Chapter 5.

2. *The conjecture.*

Notice that the SYZ map $\mathcal{F}_{\mathrm{SYZ}} : \mathcal{M}_K(X) \to \mathcal{M}_C(\check{X})$ is *canonical*, in the sense that it is constructed by T-duality and open Gromov-Witten invariants, which involve *intrinsic* structures of the Lagrangian fibration $X \to B$. On the other hand, mirror symmetry tells us that the *canonical* way to identify $\mathcal{M}_K(X)$ with $\mathcal{M}_C(\check{X})$ is via the mirror map ψ. This gives rise to a natural conjecture that these two canonical maps equal to each other:

Conjecture 6.2.1. *The map $\mathcal{F}_{\mathrm{SYZ}}$ is the inverse of the mirror map ψ, that is, $\psi \circ \mathcal{F}_{\mathrm{SYZ}} = \mathrm{Id}$. In other words, there exists a basis $\gamma_1, \ldots, \gamma_l$ of $H_n(\check{X}, \mathbb{Z})$ such that*

$$q_a = \exp\left(-\int_{\gamma_a} \check{\Omega}_{\check{q}}\right),$$

for $\check{q} = \mathcal{F}_{\mathrm{SYZ}}(q)$ and $a = 1, \ldots, l$.

In the literature, various integrality properties of mirror maps and their inverses (see e.g. [49]) have been established. This suggests that the coefficients in the Taylor expansions of these maps have enumerative meanings. This is a consequence of what the above conjecture says for the inverse mirror map, namely, it can be expressed in terms of the open Gromov-Witten invariants for X. In the next Chapter, we will see that the open GW invariants involved are integers (Theorem 7.2.4).

To verify Conjecture 6.2.1, we need to compute the open Gromov-Witten invariants $n_{\beta_i+\alpha}$ and then compare the map $\mathcal{F}_{\mathrm{SYZ}}$ define by (6.2.1) with the inverse mirror map. These are investigated in the next chapter.

Chapter 7

Computation of open Gromov-Witten invariants

In this chapter we develop a method to compute the open Gromov-Witten invariants of toric Calabi-Yau manifolds. The main difficulty is that obstruction is non-trivial in this situation, and there is no systematic tool to compute open GW invariants. On the other hand, many tools (such as localization and degeneration methods) have been developed for computation of closed GW invariants. Thus the main idea is to relate the open invariants that we want to compute to some closed invariants which are computable by current techniques. This idea has been used by Chan [9] to investigate the open Gromov-Witten invariants of the canonical line bundle K_S of a toric manifold S.

This method is applied to verify the Mirror Conjecture 6.2.1 for toric Calabi-Yau surfaces and some typical threefolds such as $K_{\mathbb{P}^2}$ and $K_{\mathbb{P}^1\times\mathbb{P}^1}$. As another application, it is applied to compute open GW invariants of semi-Fano toric manifolds in Section 7.4.

7.1 GW invariants under blowups and flops

First we fix the notations for closed Gromov-Witten invariants and review some previous results on their transformations under birational maps which will be used in this Chapter. The readers are referred to [41, 15] for the definition of Gromov-Witten invariants.

For a projective manifold X, let $\mathcal{M}_{g,k}(X,\alpha)$ be the moduli space of stable maps $f:(C;x_1,\cdots x_k)\to X$ with genus $g(C)=g$ and $[f(C)]=\alpha\in H_2(X,\mathbb{Z})$. Denote by $\mathrm{ev}_i:\mathcal{M}_{g,k}(X,\alpha)\to X$ the evaluation maps at marked points $f\mapsto f(x_i)$. The genus g k-pointed Gromov-Witten invariant for classes $\gamma_i\in H^*(X)$, $i=1,\ldots,k$, is defined as

$$\langle\gamma_1,\cdots,\gamma_k\rangle^X_{g,k,\alpha}=\int_{[\overline{M}_{g,k}(X,\alpha)]^{\mathrm{vir}}}\prod_{i=1}^{k}\mathrm{ev}_i^*(\gamma_i)$$

where $[\mathcal{M}_{g,k}(X,\alpha)]^{\mathrm{vir}}$ denotes the virtual fundamental class of $\mathcal{M}_{g,k}(X,\alpha)$. (This is very similar to the discussion in Section 3.2, in which Kuranishi structure is used. See [41, 15] for the details.)

When the expected dimension of $\mathcal{M}_{g,k}(X,\alpha)$ is zero, for instance when X is a Calabi-Yau threefold and $k=0$, we will be primarily interested in the invariant

$$\langle 1\rangle^X_{g,0,\alpha}=\int_{[\mathcal{M}_{g,0}(X,\alpha)]^{\mathrm{vir}}}1 \tag{7.1.1}$$

which equals to the degree of the 0-cycle $[\mathcal{M}_{g,0}(X,\alpha)]^{\mathrm{vir}}$ of $\mathcal{M}_{g,0}(X,\alpha)$.

Roughly speaking, the invariant $\langle\gamma_1,\cdots,\gamma_k\rangle^X_{g,k,\alpha}$ is a counting of genus g curves in the class α which intersect with generic representatives of the Poincaré dual $PD(\gamma_i)$ of γ_i. In particular, if we want to count curves in a homology class α passing through a generic point $x\in X$, we simply take some γ_i to be [pt], the Poincaré dual of a point. In the genus zero case, there is an alternative way to do this counting: Let $\pi:\tilde{X}\to X$ be the blow-up of X at one point x; we count curves in the homology class $\pi^!(\alpha)-e$, where $\pi^!(\alpha):=PD(\pi^*PD(\alpha))$ and e is the line class of the exceptional divisor (which is $\mathbb{CP}^{n-1}$). By the result of Hu [30] (or the result of Gathmann [23] using algebraic geometry), this gives the desired counting:

Theorem 7.1.1. *([23],[30]) Let $\pi : \tilde{X} \to X$ be the blow-up of X at one point, and e be the line class in the exceptional divisor. Let $\alpha \in H_2(X, \mathbb{Z})$ and $\gamma_1, \cdots, \gamma_k \in H^*(X)$. Then one has*

$$\langle \gamma_1, \cdots, \gamma_k, [\mathrm{pt}] \rangle^X_{0,k+1,\alpha} = \langle \pi^*\gamma_1, \cdots, \pi^*\gamma_k \rangle^{\tilde{X}}_{0,k,\pi^!(\alpha)-e}$$

where $\pi^!(\alpha) = PD(\pi^* PD(\alpha))$.

Another result that we will use is the transformation of Gromov-Witten invariants under a simple flop. Let $f : X \dashrightarrow X_f$ be a simple flop between two threefolds along a smooth $(-1,-1)$ rational curve. Denote by Γ the $(-1,-1)$ curve in X and Γ_f the corresponding exceptional curve in X_f. There is a natural isomorphism

$$\varphi : H_2(X, \mathbb{Z}) \longrightarrow H_2(X_f, \mathbb{Z})$$

and it has the property that

$$\varphi([\Gamma]) = -[\Gamma_f].$$

Moreover, φ determines a homomorphism $\varphi^* : H^{\mathrm{even}}(X_f, \mathbb{R}) \to H^{\mathrm{even}}(X, \mathbb{R})$ whose restriction on $H^2(X_f, \mathbb{R}) \to H^2(X, \mathbb{R})$ is the dual map and restriction on $H^4(X_f, \mathbb{R}) \to H^4(X, \mathbb{R})$ is via Poincaré duality.

The following theorem is proved by A.-M. Li and Y. Ruan.

Theorem 7.1.2. *([40]) Let $f : X \dashrightarrow X_f$ be a simple flop between threefolds along a smooth $(-1,-1)$ rational curve $\Gamma \subset X$ and $\varphi : H_2(X, \mathbb{Z}) \longrightarrow H_2(X_f, \mathbb{Z})$ be the natural isomorphism. If $\alpha \neq m[\Gamma] \in H_2(X, \mathbb{Z})$ for any $m \in \mathbb{Z}$, then*

$$\langle \varphi^*\gamma_1, \cdots, \varphi^*\gamma_k \rangle^X_{g,k,\alpha} = \langle \gamma_1, \cdots, \gamma_k \rangle^{X_f}_{g,k,\varphi(\alpha)}$$

for all $\gamma_1, \ldots, \gamma_k \in H^{\mathrm{even}}(X, \mathbb{R})$.

7.2 A relation between open and closed GW invariants

In this section we study open Gromov-Witten invariants of a toric Calabi-Yau manifold $X = X_\Sigma$ by relating them to some closed Gromov-Witten invariants (Theorem 7.2.4). The proof is based on the work of Chan [9]. Then we specialize to two and three dimensional cases and compute their open Gromov-Witten invariants.

Let $\mathbf{T} \subset X$ be a regular toric fiber and $\beta \in \pi_2(X, \mathbf{T})$ be a disk class bounded by $\mathbf{T}$. Recall that the genus zero one-pointed open Gromov-Witten invariant in the class β is denoted by n_β (see Chapter 3 for a quick review on its definition). The main difficulty of computing n_β is the failure of transversality due to sphere bubbling (see Section 3.2). Fortunately for toric Calabi-Yau manifolds, one can locate the holomorphic spheres using the following simple lemma:

Lemma 7.2.1. *Let Y be a toric manifold and assume the notations of Section 2.2 for toric geometry. Suppose there exists $\nu \in M$ such that ν defines a holomorphic function on Y whose zeros contain all toric divisors of Y. Then the image of any non-constant holomorphic map $u : \mathbf{P}^1 \to Y$ lies in the toric divisors of Y. In particular this holds for a toric Calabi-Yau manifold.*

Proof. Denote the holomorphic function corresponding to $\nu \in M$ by f. Then $f \circ u$ gives a holomorphic function on $\mathbf{P}^1$, which must be a constant by maximal principle. $f \circ u$ cannot be constantly non-zero, or otherwise the image of u lies in $(\mathbb{C}^\times)^n \subset Y$, forcing u to be constant. Thus $f \circ u \equiv 0$, implying the image of u lies in the toric divisors of Y.

For a toric Calabi-Yau variety X, $(\underline{\nu}, v_i) = 1 > 0$ for all $i = 0, \ldots, m-1$ implies that the meromorphic function corresponding to $\underline{\nu}$ indeed has no poles. □

As a consequence:

Proposition 7.2.2. *Let X be a toric Calabi-Yau manifold and $\mathbf{T}$ be a Lagrangian toric fiber. For a disk class $\beta \in \pi_2(X, \mathbf{T})$ which has Maslov index two, $\mathcal{M}_1(\mathbf{T}, \beta)$ is empty unless*

1. *$\beta = \beta_i$ for some i; or*
2. *$\beta = \beta_i + \alpha$, where the i-th toric divisor $\mathcal{D}_i$ is compact and $\alpha \in H_2(X, \mathbb{Z})$ is represented by a rational curve with non-empty intersection with $\mathcal{D}_i$.*

Proof. By Theorem 11.1 of [22], $\mathcal{M}_1(X, \beta)$ is empty unless $\beta = \sum_i k_i \beta_i + \sum_j \alpha_j$ where $k_i \in \mathbb{Z}_{\geq 0}$ and each $\alpha_j \in H_2(X, \mathbb{Z})$ is realized by a holomorphic sphere. Since X is Calabi-Yau, every α_j has Chern number zero. Thus

$$2 = \mu(\beta) = \sum_i k_i \mu(\beta_i) = \sum_i 2k_i$$

where $\mu(\beta)$ denotes the Maslov index of β. Thus $\beta = \beta_i + \alpha$ for some $i = 0, \ldots, m-1$ and $\alpha \in H_2(X, \mathbb{Z})$ is realized by some chains Q of non-constant holomorphic spheres in X.

Now suppose that $\alpha \neq 0$, and so Q is not a constant point. By Lemma 7.2.1, Q must lie inside $\bigcup_{i=0}^{m-1} \mathcal{D}_i$. Q must have non-empty intersection with the holomorphic disk representing $\beta_i \in \pi_2(X, L)$ for generic L, implying some components of Q lie inside $\mathcal{D}_i$ and have non-empty intersection with the torus orbit $(\mathbb{C}^\times)^2 \subset \mathcal{D}_i$. But if $\mathcal{D}_i$ is non-compact, then the fan of $\mathcal{D}_i$ (as a toric manifold) is simplicial convex incomplete, and so $\mathcal{D}_i$ is a toric manifold satisfying the condition of Lemma 7.2.1. Then Q has empty empty intersection with the open orbit $(\mathbb{C}^\times)^2 \subset \mathcal{D}_i$, which is a contradiction. □

By classifying all the holomorphic disks bounded by regular toric fibers, Cho-Oh [12] proved that $n_\beta = 1$ for the basic disc classes $\beta = \beta_i$. The remaining task is to compute n_β for $\beta = \beta_i + \alpha$ with nonzero $\alpha \in H_2(X)$. To do this we relate n_β to certain closed Gromov-Witten invariants by compactifying X.

First consider the special case that $X = K_S$ is the canonical bundle of a toric Fano manifold S. Then the only compact toric divisor in X is $\mathcal{D}_0 = S$, and so it only remains to compute $n_{\beta_0+\alpha}$, where $\alpha \in H_2(S)$ is represented by some rational curves in S. Now take the fiberwise compactification $\bar{X}$, whose fan $\bar{\Sigma}$ is the refinement of Σ by adding the ray generated by $v_\infty := -v_0$. Let h denote the fiber class of $\bar{X}$.

Let $p \in \mathbf{T} \subset X$, $M_{\text{op}}^p := \mathcal{M}_1(X, \beta_0 + \alpha) {}_{\text{ev}_0}\times_X \{p\}$ denote the moduli space of stable disks representing $\beta_0 + \alpha$ whose boundary passes through p, and $M_{\text{cl}}^p := \mathcal{M}_1(\bar{X}, h + \alpha) {}_{\text{ev}_0}\times_{\bar{X}} \{p\}$ denote the moduli space of rational curves representing $h + \alpha$ and passing through p. It was shown by Chan [9] that the Kuranishi structures on M_{op}^p and M_{cl}^p are the same (see Figure 7.1 for an illustration), and so the corresponding invariants equal to each other:

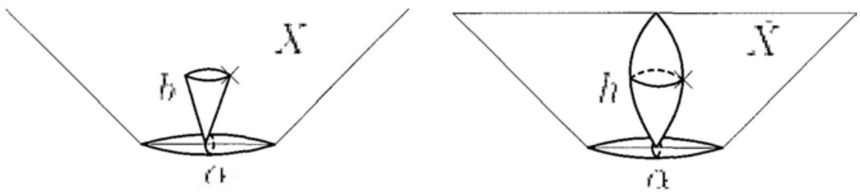

Figure 7.1: Equating open invariants with some closed invariants.

Theorem 7.2.3 (Chan [9]). *Let $X = K_S$ be the canonical line bundle of a toric Fano surface S and $\alpha \in H_2(S, \mathbb{Z})$, and denote by $\bar{X}$ its fiberwise compactification. Then*

$$n_{\beta_0+\alpha} = \langle [\text{pt}] \rangle_{0,1,h+\alpha}^{\bar{X}}.$$

Now consider a toric Calabi-Yau manifold X which may not be of the form K_S. To compute $n_{\beta_0+\alpha}$, consider a toric compactification $\bar{X}$ of X as follows. Let v_0 be the primitive generator corresponding to $\mathcal{D}_0$, and we take $\bar{\Sigma}$ to be the refinement of Σ by adding the ray generated by $v_\infty := -v_0$ (and then completing it into a convex fan). By further refinement is necessary, we may assume that

$\bar{\Sigma}$ is simplicial so that $\bar{X} = X_{\bar{\Sigma}}$ is smooth. Denote by $h \in H_2(\bar{X}, \mathbb{Z})$ to be the unique curve class such that $h \cdot \mathcal{D}_0 = h \cdot \mathcal{D}_\infty = 1$ and $h \cdot D = 0$ for all other irreducible toric divisors D. We also call h to be the 'fiber class'. Then we have the following result:

Theorem 7.2.4. *Let X be a toric Calabi-Yau manifold, $\mathbf{T} \subset X$ be a regular toric fiber and $\beta \in \pi_2(X, \mathbf{T})$ be a disk class bounded by $\mathbf{T}$. Then $n_\beta \neq 0$ only when $\beta = \beta_i$ is a basic disk class for $i = 0, \ldots, m-1$, in which case $n_{\beta_i} = 1$, or $\beta = \beta_i + \alpha$ where the i-th toric divisor $\mathcal{D}_i$ is compact and $\alpha \in H_2(X, \mathbb{Z})$ is represented by a rational curve with non-empty intersection with $\mathcal{D}_i$.*

By relabelling if necessary, set $i = 0$. Let $\bar{X}$ be the compactification constructed above and $h \in H_2(\bar{X})$ denote the fiber class. Assume that all rational curves in $\bar{X}$ representing α are contained in X. Then

$$n_{\beta_0+\alpha} = \langle [\mathrm{pt}] \rangle^{\bar{X}}_{0,1,h+\alpha}. \tag{7.2.1}$$

Proof. The first paragraph is just a repetition of Proposition 7.2.2 and the result of Cho-Oh [12]. The second paragraph is the main part.

For simplicity let $M_{\mathrm{op}} := \mathcal{M}_1(X, \beta_0 + \alpha)$ denote the moduli space of stable disks representing $\beta_0 + \alpha$ and $M_{\mathrm{cl}} := \mathcal{M}_1(\bar{X}, h + \alpha)$ denote the moduli space of rational curves representing $h + \alpha$. By evaluation at the marked point we have a $\mathbf{T}$-equivariant fibration

$$\mathrm{ev} : M_{\mathrm{op}} \to \mathbf{T}$$

whose fiber at $p \in \mathbf{T} \subset X$ is denoted as M^p_{op}. Similarly we have a $\mathbf{T}_{\mathbb{C}}$-equivariant fibration

$$\mathrm{ev} : M_{\mathrm{cl}} \to \bar{X}$$

whose fiber is M^p_{cl}. By the assumption that all rational curves in $\bar{X}$ representing α is contained in X, which in turn is contained in the toric divisors of X, one has

$$M^p_{\mathrm{op}} = M^p_{\mathrm{cl}}.$$

There is a Kuranishi structure on M^p_{cl} which is induced from that on M_{cl} (please refer to [15] and [20, 21] for the detailed definitions of Kuranishi structures). Transversal multisections of the Kuranishi structures give the virtual fundamental cycles $[M_{\mathrm{op}}] \in H_n(X, \mathbb{Q})$ and $[M^p_{\mathrm{op}}] \in H_0(\{p\}, \mathbb{Q})$. In the same way we obtain the virtual fundamental cycles $[M_{\mathrm{cl}}] \in H_{2n}(\bar{X}, \mathbb{Q})$ and $[M^p_{\mathrm{cl}}] \in H_0(\{p\}, \mathbb{Q})$. By taking the multisections to be $\mathbf{T}_{\mathbb{C}}$-($\mathbf{T}$-) equivariant so that their zero sets are $\mathbf{T}_{\mathbb{C}}$- ($\mathbf{T}$-) invariant,

$$\deg[\mathcal{M}^p_{\mathrm{cl/op}}] = \deg[\mathcal{M}_{\mathrm{cl/op}}]$$

which are the invariants $\langle[\mathrm{pt}]\rangle^X_{0,1,h+\alpha}$ and $n_{\beta_0+\alpha}$ respectively. It remains to prove that the Kuranishi structures on M^p_{cl} and M^p_{op} are the same, so that their virtual fundamental classes have the same degree.

Let $[u_{\mathrm{cl}}] \in M^p_{\mathrm{cl}}$, which corresponds to an element $[u_{\mathrm{op}}] \in M^p_{\mathrm{op}}$. $u_{\mathrm{cl}} : (\Sigma, q) \to \bar{X}$ is a stable holomorphic map with $u_{\mathrm{cl}}(q) = p$. Σ can be decomposed as $\Sigma_0 \cup \Sigma_1$, where $\Sigma_0 \cong \mathbf{P}^1$ such that $u_*[\Sigma_0]$ represents h, and $u_*[\Sigma_1]$ represents α. Similarly the domain of u_{op} can be decomposed as $\Delta \cup \Sigma_1$, where $\Delta \subset \mathbb{C}$ is the closed unit disk.

We have the Kuranishi chart $(V_{\mathrm{cl}}, E_{\mathrm{cl}}, \Gamma_{\mathrm{cl}}, \psi_{\mathrm{cl}}, s_{\mathrm{cl}})$ around $u_{\mathrm{cl}} \in M^p_{\mathrm{cl}}$, where we recall that $E_{\mathrm{cl}} \oplus \mathrm{Im}(D_{u_{\mathrm{cl}}}\bar{\partial}) = \Omega^{(0,1)}(\Sigma, u^*_{\mathrm{cl}}TX)$ and $V_{\mathrm{cl}} = \{\bar{\partial} f \in E; f(q) = p\}$. On the other hand let $(V_{\mathrm{op}}, E_{\mathrm{op}}, \Gamma_{\mathrm{op}}, \psi_{\mathrm{op}}, s_{\mathrm{op}})$ be the Kuranishi chart around $u_{\mathrm{op}} \in M^p_{\mathrm{op}}$.

Now comes the key: since the obstruction space for the deformation of $u_{\mathrm{cl}}|_{\Sigma_0}$ is 0, E_{cl} is of the form $0 \oplus E' \subset \Omega^{(0,1)}(\Sigma_0, u_{\mathrm{cl}}|^*_{\Sigma_0}TX) \times \Omega^{(0,1)}(\Sigma_1, u_{\mathrm{cl}}|^*_{\Sigma_1}TX)$. Similarly E_{op} is of the form $0 \oplus E'' \subset \Omega^{(0,1)}(\Delta, u_{\mathrm{op}}|^*_{\Delta}TX) \times \Omega^{(0,1)}(\Sigma_1, u_{\mathrm{op}}|^*_{\Sigma_1}TX)$. But since $D_{u_{\mathrm{cl}}|_{\Sigma_1}}\bar{\partial} = D_{u_{\mathrm{op}}|_{\Sigma_1}}\bar{\partial}$, E' and E'' can be taken as the same subspace. Thus the two Kuranishi charts agree: $(V_{\mathrm{cl}}, E_{\mathrm{cl}}, \Gamma_{\mathrm{cl}}, \psi_{\mathrm{cl}}, s_{\mathrm{cl}}) = (V_{\mathrm{op}}, E_{\mathrm{op}}, \Gamma_{\mathrm{op}}, \psi_{\mathrm{op}}, s_{\mathrm{op}})$.

To see that $\langle[\mathrm{pt}]\rangle^{\bar{X}}_{0,1,h+\alpha}$ is an integer, let $Y = \bar{X}$ and use Hu-Gathmann's blow-up formula (Theorem 7.1.1) to remove the point condition:

$$\langle[\mathrm{pt}]\rangle^Y_{0,1,h+\alpha} = \langle 1 \rangle^{\tilde{Y}}_{0,0,h'+\alpha}$$

where h' is represented by the strict transform of h.

□

In the next section, we apply the above formula to two and three dimensional cases. The Mirror Conjecture 6.2.1 is proven in the two-dimensional case, while strong evidences have been found in dimension three.

7.3 Computations in two and three dimensions

7.3.1 Two-dimensional cases

Let $X = X_\Sigma$ be a toric Calabi-Yau surface. First of all, toric Calabi-Yau surfaces are classified by $m \in \mathbb{Z}_{\geq 0}$:

Proposition 7.3.1. *For $m \in \mathbb{Z}_{\geq 0}$, let Σ_m be the convex fan supported in $\mathbb{R}^2$ whose rays are generated by $(i,1)$ for $i = 0, \ldots, m-1$. Then X_{Σ_m} is a toric Calabi-Yau surface. Conversely, if X_Σ is a toric Calabi-Yau manifold, then $X_\Sigma \cong X_{\Sigma_m}$ as toric manifolds for some $m \geq 0$.*

Proof. Taking $\underline{\nu} = (0,1) \in \mathbb{Z}^2$, one has $(\underline{\nu}, (i,1)) = 1$ for all $i = 0, \ldots, m-1$. Thus X_{Σ_m} is a toric Calabi-Yau.

Now suppose X_Σ is a toric Calabi-Yau surface whose fan Σ has rays generated by $v_i \in N$ for $i = 0, \ldots, m-1$. We may take $\{v_0, v_1\}$ as a basis of N and identify it with $\{(0,1), (1,1)\} \subset \mathbb{Z}^2$. Then $(\underline{\nu}, v_0) = (\underline{\nu}, v_1) = 1$ implies that $\underline{\nu}$ is identified with $(0,1)$. Moreover, since for each $i = 0, \ldots, m-1$, $(\underline{\nu}, v_i) = 1$, v_i must be identified with $(k_i, 1)$ for some $k_i \in \mathbb{Z}$. Without lose of generality we may assume that $v_0, \ldots, v_{m-1}$ are labelled in the clockwise fashion, so that $\{k_i\}$ is an increasing sequence. Inductively, using the fact that $\{v_{i-1}, v_i\}$ is simplicial, one can see that $k_i = i$ for all $i = 0, \ldots, m-1$. □

The following is a list of some familiar toric Calabi-Yau surfaces:

$m = 0$: $X = (\mathbb{C}^\times)^2$.

$m = 1$: $X = \mathbb{C}^\times \times \mathbb{C}$.

$m = 2$: $X = \mathbb{C}^2$.

$m = 3$: $X = K_{\mathbb{P}^1}$.

Remark 7.3.2. *Every toric Calabi-Yau surface X_{Σ_m} for $m \geq 3$ is the toric resolution of A_{m-2} singularity $\mathbb{C}^2/\mathbb{Z}_{m-1}$, whose fan is given by the cone $\mathbb{R}_{\geq 0}\langle (0,1),(m-1,1)\rangle \subset \mathbb{R}^2$. (See Figure 7.2.) $\{\mathcal{D}_i\}_{i=1}^{m-2}$ is the set of compact irreducible toric divisors, and it generates $H_2(X,\mathbb{Z})$. The Kähler moduli of X_{Σ_m} has canonical Kähler coordinates given by*

$$q_i := \exp\left(-\int_{\mathcal{D}_i} \omega\right)$$

for $i = 1, \ldots, m-2$.

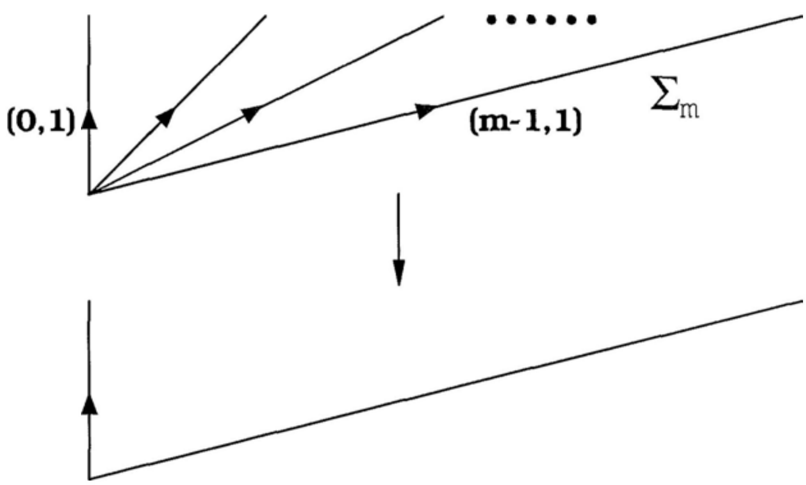

Figure 7.2: Toric resolution of $\mathbb{C}^2/\mathbb{Z}_{m-1}$.

We are ready to compute the open Gromov-Witten invariants of a Lagrangian toric fiber of a toric Calabi-Yau surface. To state the theorem it is more convenient to have the following definition:

Definition 7.3.3. *Let $\{s_k\}_{k=N_1}^{N_2}$ be a finite sequence of integers and $l \in \mathbb{Z}$ with $N_1 \leq l \leq N_2$. $\{s_k\}_{k=N_1}^{N_2}$ is said to be admissible with center l if*

1. $s_k \geq 0$ *for all* $k = N_1, \ldots, N_2$.
2. $s_i \leq s_{i+1} \leq s_i + 1$ *when* $i < l$;
3. $s_i \geq s_{i+1} \geq s_i - 1$ *when* $i \geq l$;
4. $s_{N_1}, s_{N_2} \leq 1$.

Theorem 7.3.4. *Let $X = X_{\Sigma_m}$ be a toric Calabi-Yau surface and $\mathbf{T}$ be a Lagrangian toric fiber. For every $\beta \in \pi_2(X, \mathbf{T})$, n_β is either 0 or 1, and $n_\beta = 1$ if and only if $\beta = \beta_0, \ldots, \beta_{m-1}$, or $\beta = \beta_l + \alpha \in \pi_2(X, \mathbf{T})$ for $l \in \{1, \ldots, m-2\}$, where*

$$\alpha = \sum_{k=1}^{m-2} s_k [\mathcal{D}_k],$$

$\{s_k\}_{k=1}^{m-2}$ is an admissible sequence of integers with center l and $\mathcal{D}_k$ are irreducible compact toric divisors of X corresponding to the rays generated by $(k, 1)$ respectively.

The proof involves using Formula (7.2.1) to relate the open invariants n_β to closed invariants, the blow-up formula (Theorem 7.1.1) by Hu-Gathmann, and Bryan-Leung's result [6] on local invariants of surfaces. As a consequence,

$$\delta_i = \sum_{\alpha \neq 0} n_{\beta_i + \alpha} \exp\left(-\int_\alpha \omega\right) \text{ for } i = 0, \ldots, m-1$$

are explicit and it is a direct computation to check that

Corollary 7.3.5. *Let $X = X_{\Sigma_m}$ be a toric Calabi-Yau surface. The defining equation of the SYZ mirror $\check{X}$ in Theorem 5.5.1 can be simply written as*

$$uv = (1 + z)(1 + q_1 z)(1 + q_1 q_2 z) \ldots (1 + q_1 \ldots q_{m-2} z)$$

where

$$q_j := \exp\left(-\int_{\mathcal{D}_j}\omega\right) \text{ for } j = 1,\ldots,m-2$$

are the Kähler parameters.

Proof of Theorem 7.3.4. By the first part of Theorem 7.2.4, $n_\beta \neq 0$ only when $\beta = \beta_i$ is a basic disk class for $i = 0,\ldots,m-1$, in which case $n_{\beta_i} = 1$, or $\beta = \beta_i + \alpha$ where $i = 1,\ldots,m-2$ so that $\mathcal{D}_i$ is compact, and $\alpha \in H_2(X,\mathbb{Z})$ is represented by a rational curve.

To compute $n_{\beta_l+\alpha}$, let $\bar{X}$ be the toric compactification along the v_l direction as in Section 7.2: The fan of $\bar{X}$ is convex consisting of rays generated by $v_i = (i,1)$ for $i = 0,\ldots,m-1$, $(1,0)$, $(-1,0)$ and $v_\infty = -v_l$. Since $\mathcal{D}_k$ are (-2)-curves, rational curves in $\bar{X}$ representing $\alpha = \sum_{k=1}^{m-2} s_k[\mathcal{D}_k]$ have images contained in $\cup_{k=1}^{m-2}\mathcal{D}_k \subset X$.

Recall that $h \in H_2(\bar{X})$ is called the fiber class with the properties that $h\cdot D_l = h\cdot\mathcal{D}_\infty = 1$ and $h\cdot D = 0$ for all other irreducible toric divisors D (see Figure 7.3). Then Formula (7.2.1) says that

$$n_{\beta_l+\alpha} = \langle[\mathrm{pt}]\rangle^{\bar{X}}_{0,1,h+\alpha}.$$

Thus it remains to compute $\mathrm{GW}^{Y,h+\alpha}_{0,1}([\mathrm{pt}])$.

Write $Y = \bar{X}$, now we may apply the result by Hu [30] and Gathmann [23] which removes the point condition by blow-up:

$$\langle[\mathrm{pt}]\rangle^{Y}_{0,1,h+\alpha} = \langle 1\rangle^{\tilde{Y}}_{0,0,\pi^!(h+\alpha)-e}$$

where $\pi : \tilde{Y} \to Y$ is the blow-up of Y at a point, $e \in H_2(\tilde{Y})$ is the corresponding exceptional class, and $\pi^!(b) := \mathrm{PD}(\pi^*\mathrm{PD}(b))$ for $b \in H_2(\bar{X})$.

Since $\alpha = \sum_{k=1}^{m-2} s_k[\mathcal{D}_k]$, one has

$$\pi^!(h+\alpha) - e = [C] + \sum_{k=1}^{m-2} s_k[\mathcal{D}_k]$$

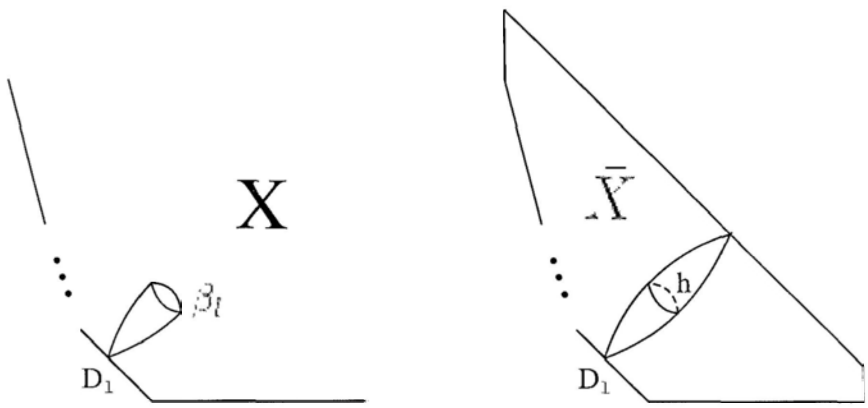

Figure 7.3: A sphere representing $h \in H_2(\bar{X})$.

where C is a (-1)-curve and $\mathcal{D}_k$ are (-2)-curves, and their intersection configuration is as shown in Figure 7.4. The Gromov-Witten invariant $\langle 1 \rangle_{0,0,[C]+\sum_{k=1}^{m-2} s_k[\mathcal{D}_k]}^{\tilde{Y}}$ has already been computed by Bryan-Leung [6], and the result is that the invariant is 1 when the sequence $\{s_k\}_{k=1}^{m-2}$ is admissible with center l, and 0 otherwise.

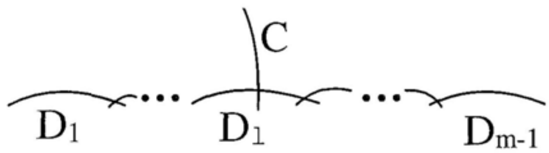

Figure 7.4: A chain of $\mathbb{P}^1$'s.

□

According to Corollary 7.3.5, the SYZ mirror $\check{X}$ is given by

$$uv = (1+z)(1+q_1 z)(1+q_1 q_2 z)\dots(1+q_1\dots q_{m-2}z)$$

where for $j = 1, \dots, m-2$,

$$q_j = \exp\left(-\int_{\mathcal{D}_j} \omega\right).$$

Having this explicit expression, it is then a direct calculation to prove the Mirror Conjecture 6.2.1 in surface case, which can be restated as follows:

Theorem 7.3.6 (Conjecture 6.2.1 in surface case). *For every toric Calabi-Yau surface, the inverse mirror map is the SYZ map. More precisely, let $X = X_{\Sigma_m}$ be a toric Calabi-Yau surface, and $(\check{X}, \check{\Omega})$ be its SYZ mirror. Then there exists a basis $\{\gamma_l\}_{l=1}^{m-2}$ of $H_2(\check{X}, \mathbb{Z})$ such that*

$$\int_{\mathcal{D}_l} \omega = \int_{\gamma_l} \check{\Omega} \tag{7.3.1}$$

for all $l = 1, \ldots, m-2$.

Proof. Let $\gamma_l := -[S_l] \in H_2(\check{X}, \mathbb{Z})$ for $l = 1, \ldots, m-2$, where

$$S_l := \left\{ (u, v, z) \in \check{X} : |u| = |v|; z \in \left[-\prod_{i=0}^{l} q_i^{-1}, -\prod_{i=0}^{l-1} q_i^{-1} \right] \right\}$$

are two-dimensional submanifolds of $\check{X}$. Fix $l \in \{1, \ldots, m-2\}$. Then $\{\gamma_l\}_{l=1}^{m-2}$ forms a basis of $H_2(\check{X}, \mathbb{Z})$.

Notice that $[S_l]$ can be written as a sum of 2-chains $C_1 + C_2 + C_3$, where

$$C_1 := \left\{ (u, v, z) \in \check{X} : |u| \leq 1; z = -\prod_{i=0}^{l} q_i^{-1} \right\};$$
$$C_2 := \left\{ (u, v, z) \in \check{X} : |u| = 1; z \in \left[-\prod_{i=0}^{l} q_i^{-1}, -\prod_{i=0}^{l-1} q_i^{-1} \right] \right\};$$
$$C_3 := \left\{ (u, v, z) \in \check{X} : |u| \leq 1; z = -\prod_{i=0}^{l-1} q_i^{-1} \right\}.$$

Since $\check{\Omega}|_{z=c} = 0$ for every $c \in \mathbb{C}^\times$, one has $\int_{C_1} \check{\Omega} = \int_{C_3} \check{\Omega} = 0$ and so

$$\int_{\gamma_l} \check{\Omega} = -\int_{C_2} \check{\Omega}.$$

Recall that locally $\check{\Omega}$ is written as $\mathrm{d}\log z \wedge \mathrm{d}\log u$. Thus

$$\int_{C_2} \check{\Omega} = \int_{-(q_1 \ldots q_l)^{-1}}^{-(q_1 \ldots q_{l-1})^{-1}} \mathrm{d}\log z = \log q_l = -\int_{\theta_l} \omega.$$

□

42 in [11]:

$$
\begin{aligned}
n_{0,0} &= 1,\\
n_{1,0} = n_{0,1} &= 1,\\
n_{2,0} = n_{0,2} = 0, n_{1,1} &= 3,\\
n_{3,0} = n_{0,3} = 0, n_{2,1} = n_{1,2} &= 5,\\
n_{4,0} = n_{0,4} = 0, n_{3,1} = n_{1,3} = 7, n_{2,2} &= 35,\\
n_{5,0} = n_{0,5} = 0, n_{4,1} = n_{1,4} = 9, n_{3,2} = n_{2,3} &= 135,\\
&\vdots
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\delta_0(q_1, q_2) \;&=\; q_1 + q_2 + 3q_1q_2 + 5q_1^2q_2 + 5q_1q_2^2 + 7q_1^3q_2 + 35q_1^2q_2^2 + 7q_1q_2^3\\
&\quad +9q_1^4q_2 + 135q_1^3q_2^2 + 135q_1^2q_2^3 + 9q_1q_2^4 + \dots.
\end{aligned}
$$

Then the SYZ map (6.2.1) $\mathcal{F}_{\mathrm{SYZ}}$ is given by

$$
\mathcal{F}_{\mathrm{SYZ}}(q_1, q_2) = (q_1(1 + \delta_0(q_1, q_2))^{-2}, q_2(1 + \delta_0(q_1, q_2))^{-2}).
$$

On the other hand, we can compute the mirror map and its inverse by solving the following Picard-Fuchs equations:

$$
\begin{aligned}
(\theta_1^2 - 2\check{q}_1(\theta_1 + \theta_2)(1 + 2\theta_1 + 2\theta_2))\Phi(\check{q}_1, \check{q}_2) = 0,\\
(\theta_2^2 - 2\check{q}_2(\theta_1 + \theta_2)(1 + 2\theta_1 + 2\theta_2))\Phi(\check{q}_1, \check{q}_2) = 0,
\end{aligned}
$$

where θ_a denotes $\check{q}_a \frac{\partial}{\partial \check{q}_a}$ for $a = 1, 2$. The two solutions to these equations with a single logarithm are given by

$$
\Phi_1(\check{q}_1, \check{q}_2) = -\log \check{q}_1 - f(\check{q}_1, \check{q}_2),\;\; \Phi_2(\check{q}_1, \check{q}_2) = -\log \check{q}_2 - f(\check{q}_1, \check{q}_2),
$$

where

$$\begin{aligned} f(\check{q}_1,\check{q}_2) &= 2\check{q}_1 + 2\check{q}_2 + 3\check{q}_1^2 + 12\check{q}_1\check{q}_2 + 3\check{q}_2^2 + \frac{20}{3}\check{q}_1^3 + 60\check{q}_1^2\check{q}_2 + 60\check{q}_1\check{q}_2^2 + \frac{20}{3}\check{q}_2^3 \\ &\quad + \frac{35}{2}\check{q}_1^4 + 280\check{q}_1^3\check{q}_2 + 630\check{q}_1^2\check{q}_2^2 + 280\check{q}_1\check{q}_2^3 + \frac{35}{2}\check{q}_2^4 \\ &\quad + \frac{252}{5}\check{q}_1^5 + 1260\check{q}_1^4\check{q}_2 + 5040\check{q}_1^3\check{q}_2^2 + 5040\check{q}_1^2\check{q}_2^3 + 1260\check{q}_1\check{q}_2^4 + \frac{252}{5}\check{q}_2^5 \\ &\quad + \ldots. \end{aligned}$$

This gives the mirror map $\psi : (\Delta^*)^2 \to (\Delta^*)^2$:

$$(\check{q}_1,\check{q}_2) \mapsto (\check{q}_1 \exp(f(\check{q}_1,\check{q}_2)), \check{q}_2 \exp(f(\check{q}_1,\check{q}_2))).$$

We can then invert this map and the result is given by

$$(q_1,q_2) \mapsto (q_1(1-F(q_1,q_2)), q_2(1-F(q_1,q_2)))$$

where

$$\begin{aligned} F(q_1,q_2) &= 2q_1 + 2q_2 - 3q_1^2 - 3q_2^2 + 4q_1^3 + 4q_1^2q_2 + 4q_1q_2^2 + 4q_2^3 \\ &\quad -5q_1^4 + 25q_1^2q_2^2 - 5q_2^4 + \ldots. \end{aligned}$$

Now, we compute

$$\begin{aligned} (1-F(q_1,q_2))^{-1/2} &= 1 + q_1 + q_2 + 3q_1q_2 + 5q_1^2q_2 + 5q_1q_2^2 \\ &\quad +7q_1^3q_2 + 35q_1^2q_2^2 + 7q_1q_2^3 \\ &\quad +9q_1^4q_2 + 135q_1^3q_2^2 + 135q_1^2q_2^3 + 9q_1q_2^4 + \ldots \end{aligned}$$

We see that this agrees with the expression of δ_0.

7.4 An application to semi-Fano toric manifolds

The same computational method by compactification may be applied to other situations as well. In this section we study compact semi-Fano toric manifolds. 'Semi-Fano' means the following:

Definition 7.4.1. *A complex manifold X is said to be semi-Fano if its anti-canonical divisor $-K_X$ is numerically effective* [1], *that is, for every complex curve $C \subset X$, $-K_X \cdot C \geq 0$.*

The Hirzebruch surface $\mathbb{F}_2$ provides the first non-trivial example of a semi-Fano toric manifold. It contains a (-2)-curve C such that $K_X \cdot C = 0$, so that it cannot be a Fano manifold.

In [17], Fukaya-Oh-Ono-Ohta wrote down the Landau-Ginzburg mirror of a compact toric manifold. Yet they did not compute its coefficients, which are the open Gromov-Witten invariants. In this section we compute the open Gromov-Witten invariants in surface case and some three-dimensional examples.

7.4.1 The SYZ mirror

Let X be a compact semi-Fano toric manifold (equipped with a toric Kähler form), and denote the corresponding moment map by $\mu : X \to P$, where $P \subset M_{\mathbb{R}}$ is a polytope (see Section 2.2). Let

$$X_0 = X - \bigcup_{i=0}^{m-1} \mathcal{D}_i$$

which is the inverse image of P^{int} under μ. One has

$$X_0 \cong P^{\text{int}} \times N_{\mathbb{R}}/N$$

as symplectic manifolds. In this case the dual torus bundle is simply

$$\check{X}_0 = P^{\text{int}} \times M_{\mathbb{R}}/M.$$

Fixing a choice of basis of N, say $v_0, \ldots, v_{n-1}$, the semi-flat complex coordinates are simply given by the natural pairings

$$z_i(r, \theta) := \exp\left(-\left(v_i \, , \, r + \mathbf{i}\,\theta\right)\right)$$

[1] A more common definition would be requiring $-K_X$ to be ample. But for our purpose numerical effectiveness is enough.

for $(r, \theta) \in \check{X}_0$.

Then one considers quantum correction by open Gromov-Witten invariants. First of all, we need to verify Assumption 4.3.3 in this situation:

Proposition 7.4.2. *Assumption 4.3.3 is satisfied for the moment map fibration $\mu : X \to B$ on a compact semi-Fano toric manifold X. Moreover, the wall H (see Definition 4.3.6) of the fibration μ is empty.*

Proof. The image B of the moment map fibration μ on a toric manifold is a polytope. The inverse images of the facets of B are toric divisors $\mathcal{D}_j$ in X, and $\sum_{j=0}^{m-1} \mathcal{D}_j$ is an anti-canonical divisor of X. Moreover by Cho-Oh [12], one has the following formula for the Maslov index $\mu(\beta)$ of a disk class $\beta \in \pi_2(X, \mathbf{T})$ bounded by a Lagrangian toric fiber $\mathbf{T}$:

$$\mu(\beta) = 2\,\beta \cdot \sum_{j=0}^{m-1} \mathcal{D}_j.$$

Thus Assumptions (1), (2), (3) are satisfied.

Now suppose β is a disk class such that the moduli space $\mathcal{M}_0(\mathbf{T}, \beta)$ is non-empty. By Theorem 11.1 of [22], β must be of the form $\sum_i k_i \beta_i + \sum_j \alpha_j$ where $k_i \in \mathbb{Z}_{\geq 0}$, $\{k_i\}$ cannot be all zero, and each $\alpha_j \in H_2(X, \mathbb{Z})$ is realized by a holomorphic sphere. Since X is semi-Fano, $-K_X \cdot \alpha_j \geq 0$. Thus

$$\mu(\beta) = \sum_i k_i \mu(\beta_i) + \sum_j (-K_X \cdot \alpha_j) = 2k_i + \sum_j (-K_X \cdot \alpha_j) \geq 2.$$

This proves that every Lagrangian toric fiber $\mathbf{T}$ has minimal Maslov index two, which implies Assumption (4). In this case the wall H, which consists of $r \in B_0$ such that F_r has minimal Maslov index less than two, is empty. □

Since there is no wall, the semi-flat complex structure receives no quantum correction, and so the mirror manifold is simply $\mathrm{Spec}(\mathbb{C}[z_0^{\pm 1}, \ldots, z_{n-1}^{\pm 1}]) = (\mathbb{C}^\times)^n$. The quantum corrections are recorded by the superpotential (Definition 4.3.10)

$$W(F_r, \nabla) = \sum_{\beta \in \pi_2(X, F_r)} n_\beta \exp\left(-\int_\beta \omega\right) \mathrm{Hol}_\nabla(\partial\beta).$$

By Theorem 11.1 of [22], $\mathcal{M}_1(\mathbf{T}, \beta)$ is non-empty only when $\beta = \beta_j + \alpha$ where $\alpha \in H_2(X)$ is represented by rational curves of Chern number zero, and $n_{\beta_j} = 1$. Thus

$$W = \sum_{i=0}^{m-1} (1 + \delta_i) Z_{\beta_i} \tag{7.4.1}$$

where

$$\delta_i = \sum_{\alpha \neq 0} n_{\beta_i + \alpha} \exp\left(- \int_\alpha \omega \right)$$

can be expressed in terms of Kähler parameters of X, and

$$Z_\beta := \exp\left(- \int_\beta \omega \right) \mathrm{Hol}_\nabla(\partial\beta)$$

can be expressed in terms of Kähler parameters of X and $\{z_j\}_{j=0}^{n-1}$.

The superpotential itself contains enumerative information of X, namely, its coefficients are expressed in terms of open Gromov-Witten invariants. Furthermore, recently Fukaya-Oh-Ono-Ohta [17] proved that for a compact toric manifold X, its quantum cohomology $QH^*(X)$ is isomorphic as a ring to

$$\mathrm{Jac}(W) := \frac{\mathbb{C}[z_0^{\pm 1}, \ldots, z_{n-1}^{\pm 1}]}{\langle \partial W/\partial z_0, \ldots, \partial W/\partial z_{n-1} \rangle}.$$

(When X is not semi-Fano, W may not be a Laurent polynomial, and the above expression of the Jacobian ring needs to be modified. Since we only consider semi-Fano cases in this thesis, we omit such complication and the readers are referred to [17] for details.) Thus, an explicit expression of the superpotential W gives an explicit presentation of the quantum cohomology ring.

7.4.2 Open GW invariants and the superpotential

In the expression (7.4.1) of the superpotential, the only non-explicit terms are $n_{\beta_i+\alpha}$ for $\alpha \neq 0$ represented by rational curves of Chern number zero. If X is Fano, then any rational curve has positive Chern number, so that the superpotential is simply $W = \sum_{i=0}^{m-1} Z_{\beta_i}$, which is already known by Cho-Oh [12]. But when X

is non-Fano, we do have such non-trivial terms and one has to compute them in order to write down W explicitly. We compute these open invariants using the method introduced in Section 7.2 for semi-Fano toric surfaces and some threefolds.

For $\dim X = 2$, let's label the irreducible toric divisors $\{\mathcal{D}_i\}_{i=0}^{m-1}$ in the anti-clockwise fashion. Figure 7.9 gives an illustration of such a labelling.

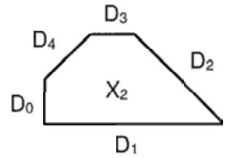

Figure 7.9: An example of semi-Fano toric manifold. Its irreducible toric divisors $\{\mathcal{D}_i\}_{i=0}^{4}$ are labelled in the anticlockwise fashion.

One has the following simple formula on the self-intersection number of toric divisors in a toric surface:

Proposition 7.4.3. *Let $X = X_\Sigma$ be a toric surface, and denote by $\{v_i\}$ the primitive generators of rays of the fan Σ, where the labelling $i \in \mathbb{Z}_m$ is in the counterclockwise fashion. Then the self-intersection of $\mathcal{D}_i$ is*

$$\mathcal{D}_i \cdot \mathcal{D}_i = v_{i+1} \wedge v_{i-1} \in \mathbb{Z}.$$

For example in Figure 7.9, $\mathcal{D}_3$ has self-intersection number -2. As a consequence to the above formula,

Proposition 7.4.4. *Let $X = X_\Sigma$ be a toric surface, $\{v_i\}_{i\in\mathbb{Z}_m}$ be the primitive generators of rays of the fan Σ labelled in the counterclockwise fashion, and $\{\mathcal{D}_i\}_{i\in\mathbb{Z}_m}$ be the corresponding irreducible toric divisors. Suppose that for some $N_1, N_2 \in \mathbb{Z}$ with $N_1 \leq N_2$, $\mathcal{D}_i$ has self-intersection (-2) for every $i = [N_1], [N_1+1], \ldots, [N_2]$.*

Then there exists $\underline{\nu} \in M$ such that $(\underline{\nu}, v_i) = 1$ for all $i = [N_1-1], [N_1], \ldots, [N_2+1]$. Thus $X = X_\Sigma$ is a compactification of the toric Calabi-Yau surface whose fan consists of the rays generated by v_i for $i = [N_1-1], [N_1], \ldots, [N_2+1]$.

Proof. Without loss of generality, let $N_1 = 1$. $\{v_0, v_1\}$ forms an oriented basis of N, which give $N \cong \mathbb{Z}^2$ by identifying v_0 with $(1,0)$ and v_1 with $(1,1)$. Let $v_2 = (a,b)$. Since $\mathcal{D}_1$ has self-intersection (-2), by Proposition 7.4.3 $v_0 \wedge v_2 = 2$, and so $b = 2$. Since X is smooth, $v_1 \wedge v_2 = 1$, forcing $a = 1$. Thus $v_2 = (1,2)$. Inductively $v_i = (1,i)$ for $i = [0], \ldots, [N_2+1]$. Thus by taking $\underline{\nu} = (1,0)$ result follows. □

Theorem 7.4.5. *Let X be a compact semi-Fano toric surface whose rays are labelled in a counterclockwise fashion, and $\beta \in \pi_2(X, \mathbf{T})$ be a disk class bounded by a Lagrangian toric fiber $\mathbf{T}$. Then $n_\beta = 1$ if $\beta = \beta_i$ is a basic disk class, or $\beta = \beta_i + \alpha$ for some $i = 0, \ldots, m-1$, where α is of the form*

$$\alpha = \sum_{k=N_1}^{N_2} s_k \mathcal{D}_k$$

for some $N_1 \leq N_2$, such that $\{s_k\}_{k=N_1}^{N_2}$ is admissible with center i in the sense of Definition 7.3.3, and for every $k = N_1, \ldots, N_2$, $\mathcal{D}_k$ has self-intersection (-2). Such β is said to be an admissible disk class. Otherwise $n_\beta = 0$.

In particular the mirror superpotential has the explicit expression

$$W = \sum_{\beta \text{ admissible}} Z_\beta.$$

Proof. By Theorem 11.1 of [22], n_β is non-zero only when $\beta = \beta_i + \alpha$ for some i and $\alpha \in H_2(X, \mathbb{Z})$ represented by rational curves with $c_1(\alpha) = 0$. Moreover by [12], $n_{\beta_i} = 1$, so it suffices to consider $\beta = \beta_i + \alpha$ with $\alpha \neq 0$.

Since $c_1(\alpha) = 0$ and X is semi-Fano, α must be of the form

$$\alpha = \sum_{k=N_1}^{N_2} s_k \mathcal{D}_k$$

where $0 \leq N_1 \leq i \leq N_2 \leq m-1$, each $\mathcal{D}_k$ has self-intersection -2 and $s_k \geq 0$. Then every rational curve representing α has its image contained in $\cup_{k=N_1}^{N_2} \mathcal{D}_k$. Moreover, by Proposition 7.4.4, one has the Calabi-Yau manifold $X_0 \subset X$, whose fan consists of the rays generated by $v_{i_{i=N_1-1}}^{N_2+1]}$. Since the moduli spaces $\mathcal{M}_1(X,\beta)$ and $\mathcal{M}_1(X_0,\beta)$ are the same, one has $n_\beta^X = n_\beta^{X_0}$. Now the result follows from Theorem 7.3.4 applying to the toric Calabi-Yau surface X_0. □

Appendix A gives a list of the superpotentials of all semi-Fano but non-Fano toric surfaces (as we have mentioned before, the Fano case is unobstructed and was known by [12]).

Similar to Section 7.3.2, we may also compute the superpotentials for some semi-Fano toric threefolds.

Example 7.4.6. $\bar{K}_{\mathbb{P}^2}$. *Let $X = \bar{K}_{\mathbb{P}^2}$ be the fiberwise compactification of $X = K_{\mathbb{P}^2}$. The primitive generators of the rays of the fan Σ defining X can be chosen to be $v_0 = (0,0,1), v_1 = (1,0,1), v_2 = (0,1,1), v_3 = (-1,-1,1)$ and $v_4 = (0,0,-1)$. We equip X with a toric Kähler structure ω associated to the moment polytope*

$$P = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : 0 \leq x_3 \leq s_2, x_1+x_3 \geq 0, x_2+x_3 \geq 0, -x_1-x_2+x_3 \geq -s_1\},$$

where $s_1 = \int_l \omega > 0$, $l \in H_2(X,\mathbb{Z})$ is the class of a line in $\mathbb{P}^2 \subset X$, and $s_2 = \int_h \omega > 0$, $h \in H_2(X,\mathbb{Z})$ is the fiber class of $\bar{K}_{\mathbb{P}^2} \to \mathbb{P}^2$. We may complexify the Kähler class and let $\omega^{\mathbb{C}} = \omega + 2\pi\sqrt{-1}B$, where B is a real two-form (the B-field).

All the curve class C satisfies $(-K_X)\cdot C \geq 0$, and the only curve class with $(-K_X)\cdot C = 0$ is $C = kl$ where $l \in H_2(X)$ is the line class of $\mathbb{P}^2 \subset X$. Now except for $i = 0$, the moduli space $\mathcal{M}_1(X,\beta_i + kl)$ is empty. Thus $n_\beta = 0$ unless $\beta = \beta_i + kl$ for some $k \geq 0$, and $n_{\beta_0} = 1$. The superpotential is

$$W = \left(1 + \sum_{k=1}^{\infty} n_k q_1^k\right) z_3 + z_1 z_3 + z_2 z_3 + q_1 z_1^{-1} z_2^{-1} z_3 + q_2 z_3^{-1}$$

where $q_\iota = \mathrm{e}^{-s_\iota}$ for $i = 1, 2$.

Every curves representing kl are contained in $\mathbb{P}^2 \subset X$. Thus $\mathcal{M}_1(X,\beta_0+kl) = \mathcal{M}_1(X_0,\beta_0+kl)$, where $X_0 = K_{\mathbb{P}^2}$. Thus $n_{\beta_\iota+kl}$ equals to those invariants of $K_{\mathbb{P}^2}$ which has been computed in Example 2 of Section 7.3.2.

Example 7.4.7. $\bar{K}_{\mathbb{P}^1\times\mathbb{P}^1}$. *Similar to the previous example, The mirror of* $X = \bar{K}_{\mathbb{P}^1\times\mathbb{P}^1}$ *is* $W : (\mathbb{C}^\times)^3 \to \mathbb{C}$,

$$W = \left(1 + \sum_{k=1}^{\infty} n_{k_1,k_2} q_1^{k_1} q_2^{k_2}\right) z_3 + z_1 z_3 + q_1 z_1^{-1} z_3 + z_2 z_3 + q_2 z_2^{-1} z_3 + q_3 z_3^{-1}$$

where n_{k_1,k_2} *is the invariants of* $K_{\mathbb{P}^1\times\mathbb{P}^1}$ *computed in Example 3 of Section 7.3.2.*

Appendix A

A list of superpotentials of all semi-Fano toric surfaces

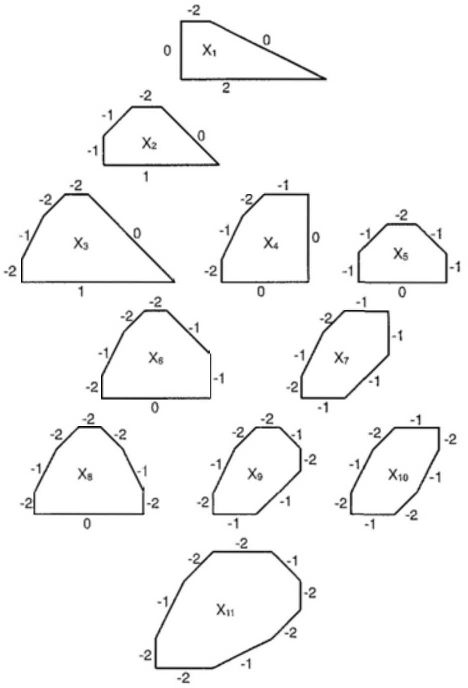

Figure A.1: Polytopes defining the semi-Fano but non-Fano toric surfaces. The numbers indicate the self-intersection numbers of the toric divisors.

	$\rho(\Sigma)$	polytope P	superpotential W
X_1	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,-2)$ $v_4=(0,-1)$	$x_1\geq 0$ $x_2\geq 0$ $2t_1+t_2-x_1-2x_2\geq 0$ $t_1-x_2\geq 0$	$z_1+z_2+\frac{q_1^2q_2}{z_1z_2^2}+(1+q_2)\frac{q_1}{z_2}$
X_2	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,-1)$ $v_4=(0,-1)$ $v_5=(1,-1)$	$x_1\geq 0$ $x_2\geq 0$ $t_1+t_2+2t_3-x_1-x_2\geq 0$ $t_1+t_3-x_2\geq 0$ $t_1+x_1-x_2\geq 0$	$z_1+z_2+\frac{q_1q_2q_3^2}{z_1z_2}+(1+q_2)\frac{q_1q_3}{z_2}+\frac{q_1z_1}{z_2}$
X_3	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,-1)$ $v_4=(0,-1)$ $v_5=(1,-1)$ $v_6=(2,-1)$	$x_1\geq 0$ $x_2\geq 0$ $t_1+t_2+2t_3+3t_4-x_1-$ $x_2\geq 0$ $t_1+t_3+2t_4-x_2\geq 0$ $t_1+t_4+x_1-x_2\geq 0$ $t_1+2x_1-x_2\geq 0$	$(1+q_1)z_1+z_2+\frac{q_1q_2q_3^2q_4^3}{z_1z_2}+(1+q_2+$ $q_2q_3)\frac{q_1q_3q_4^2}{z_2}+(1+q_3+q_2q_3)\frac{q_1q_4z_1}{z_2}+\frac{q_1z_1^2}{z_2}$
X_4	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,0)$ $v_4=(0,-1)$ $v_5=(1,-1)$ $v_6=(2,-1)$	$x_1\geq 0$ $x_2\geq 0$ $t_2+t_3+t_4-x_1\geq 0$ $t_1+t_3+2t_4-x_2\geq 0$ $t_1+t_4+x_1-x_2\geq 0$ $t_1+2x_1-x_2\geq 0$	$(1+q_1)z_1+z_2+\frac{q_2q_3q_4}{z_1}+\frac{q_1q_3q_4^2}{z_2}+(1+$ $q_3)\frac{q_1q_4z_1}{z_2}+\frac{q_1z_1^2}{z_2}$
X_5	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,0)$ $v_4=(-1,-1)$ $v_5=(0,-1)$ $v_6=(1,-1)$	$x_1\geq 0$ $x_2\geq 0$ $t_2+t_3+t_4-x_1\geq 0$ $t_1+t_3+2t_4-x_1-x_2\geq 0$ $t_1+t_4-x_2\geq 0$ $t_1+x_1-x_2\geq 0$	$z_1+z_2+\frac{q_2q_3q_4}{z_1}+\frac{q_1q_3q_4^2}{z_1z_2}+(1+$ $q_3)\frac{q_1q_4}{z_2}+\frac{q_1z_1}{z_2}$
X_6	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,0)$ $v_4=(-1,-1)$ $v_5=(0,-1)$ $v_6=(1,-1)$ $v_7=(2,-1)$	$x_1\geq 0$ $x_2\geq 0$ $t_2+t_3+t_4+t_5-x_1\geq 0$ $t_1+t_3+2t_4+3t_5-x_1-$ $x_2\geq 0$ $t_1+t_4+2t_5-x_2\geq 0$ $t_1+t_5+x_1-x_2\geq 0$ $t_1+2x_1-x_2\geq 0$	$(1+q_1)z_1+z_2+\frac{q_2q_3q_4q_5}{z_1}+\frac{q_1q_3q_4^2q_5^3}{z_1z_2}+$ $(1+q_3+q_3q_4)\frac{q_1q_4q_5^2}{z_2}+(1+q_4+$ $q_3q_4)\frac{q_1q_5z_1}{z_2}+\frac{q_1z_1^2}{z_2}$
X_7	$v_1=(1,0)$ $v_2=(0,1)$ $v_3=(-1,1)$ $v_4=(-1,0)$ $v_5=(0,-1)$ $v_6=(1,-1)$ $v_7=(2,-1)$	$x_1\geq 0$ $x_2\geq 0$ $t_2+t_3-t_1-t_5-x_1+x_2\geq 0$ $t_3+t_4+t_5-x_1\geq 0$ $t_1+t_4+2t_5-x_2\geq 0$ $t_1+t_5+x_1-x_2\geq 0$ $t_1+2x_1-x_2\geq 0$	$(1+q_1)z_1+z_2+\frac{q_2q_3z_2}{q_1q_5z_1}+\frac{q_3q_4q_5}{z_1}+$ $\frac{q_1q_4q_5^2}{z_2}+(1+q_4)\frac{q_1q_5z_1}{z_2}+\frac{q_1z_1^2}{z_2}$

	$\rho(\Sigma)$	polytope P	superpotential W
X_8	$v_1 = (1,0)$ $v_2 = (0,1)$ $v_3 = (-1,0)$ $v_4 = (-2,-1)$ $v_5 = (-1,-1)$ $v_6 = (0,-1)$ $v_7 = (1,-1)$ $v_8 = (2,-1)$	$x_1 \geq 0$ $x_2 \geq 0$ $t_2+t_3+t_4+t_5+t_6-x_1 \geq 0$ $t_1+t_3+2t_4+3t_5+4t_6-2x_1-x_2 \geq 0$ $t_1+t_4+2t_5+3t_6-x_1-x_2 \geq 0$ $t_1+t_5+2t_6-x_2 \geq 0$ $t_1+t_6+x_1-x_2 \geq 0$ $t_1+2x_1-x_2 \geq 0$	$(1+q_1)z_1 + z_2 + (1+\frac{q_1q_5q_6^2}{q_2^2q_3})\frac{q_2q_3q_4q_5q_6}{z_1} + \frac{q_1q_3q_4^2q_5^3q_6^4}{z_1^2z_2} + (1+q_3+q_3q_4+q_3q_4q_5)\frac{q_1q_4q_5^2q_6^3}{z_1z_2} + (1+q_4+q_3q_4+q_4q_5+q_3q_4q_5+q_3q_4^2q_5)\frac{q_1q_5q_6^2}{z_2} + (1+q_5+q_4q_5+q_3q_4q_5)\frac{q_1q_6z_1}{z_2} + \frac{q_1z_1^2}{z_2}$
X_9	$v_1 = (1,0)$ $v_2 = (0,1)$ $v_3 = (-1,1)$ $v_4 = (-1,0)$ $v_5 = (-1,-1)$ $v_6 = (0,-1)$ $v_7 = (1,-1)$ $v_8 = (2,-1)$	$x_1 \geq 0$ $x_2 \geq 0$ $t_2+2t_3+t_4-t_1-t_6-x_1+x_2 \geq 0$ $t_3+t_4+t_5+t_6-x_1 \geq 0$ $t_1+t_4+2t_5+3t_6-x_1-x_2 \geq 0$ $t_1+t_5+2t_6-x_2 \geq 0$ $t_1+t_6+x_1-x_2 \geq 0$ $t_1+2x_1-x_2 \geq 0$	$(1+q_1)z_1 + z_2 + \frac{q_2q_3^2q_4z_2}{q_1q_6z_1} + (1+q_2)\frac{q_3q_4q_5q_6}{z_1} + \frac{q_1q_4q_5^2q_6^3}{z_1z_2} + (1+q_4+q_4q_5)\frac{q_1q_5q_6^2}{z_2} + (1+q_5+q_4q_5)\frac{q_1q_6z_1}{z_2} + \frac{q_1z_1^2}{z_2}$
X_{10}	$v_1 = (1,0)$ $v_2 = (0,1)$ $v_3 = (-1,1)$ $v_4 = (-2,1)$ $v_5 = (-1,0)$ $v_6 = (0,-1)$ $v_7 = (1,-1)$ $v_8 = (2,-1)$	$x_1 \geq 0$ $x_2 \geq 0$ $t_2+t_3+t_4-t_1-t_6-x_1+x_2 \geq 0$ $2t_4+t_5-t_1-t_3-2x_1+x_2 \geq 0$ $t_4+t_5+t_6-x_1 \geq 0$ $t_1+t_5+2t_6-x_2 \geq 0$ $t_1+t_6+x_1-x_2 \geq 0$ $t_1+2x_1-x_2 \geq 0$	$(1+q_1)z_1 + z_2 + (1+\frac{q_1q_5q_6^2}{q_2^2q_3})\frac{q_2q_3q_4z_2}{q_1q_6z_1} + \frac{q_4^2q_5z_2}{q_1q_3z_1^2} + (1+q_3)\frac{q_4q_5q_6}{z_1} + \frac{q_1q_5q_6^2}{z_2} + (1+q_5)\frac{q_1q_6z_1}{z_2} + \frac{q_1z_1^2}{z_2}$
X_{11}	$v_1 = (1,0)$ $v_2 = (0,1)$ $v_3 = (-1,2)$ $v_4 = (-1,1)$ $v_5 = (-1,0)$ $v_6 = (-1,-1)$ $v_7 = (0,-1)$ $v_8 = (1,-1)$ $v_9 = (2,-1)$	$x_1 \geq 0$ $x_2 \geq 0$ $t_2+2t_3+3t_4+t_5-2t_1-t_6-3t_7-x_1+2x_2 \geq 0$ $t_3+2t_4+t_5-t_1-t_7-x_1+x_2 \geq 0$ $t_4+t_5+t_6+t_7-x_1 \geq 0$ $t_1+t_5+2t_6+3t_7-x_1-x_2 \geq 0$ $t_1+t_6+2t_7-x_2 \geq 0$ $t_1+t_7+x_1-x_2 \geq 0$ $t_1+2x_1-x_2 \geq 0$	$(1+q_1+\frac{q_2q_3^2q_4^3q_5}{q_1q_6q_7^3})z_1 + (1+\frac{q_2q_3^2q_4^3q_5}{q_1^2q_6q_7^3}+\frac{q_2q_3^2q_4^3q_5}{q_1q_6q_7^3})z_2 + \frac{q_2q_3^2q_4^3q_5z_2^2}{q_1^2q_6q_7z_1} + (1+q_2+q_2q_3)\frac{q_3q_4^2q_5z_2}{q_1q_7z_1} + (1+q_3+q_2q_3)\frac{q_4q_5q_6q_7}{z_1} + \frac{q_1q_5q_6^2q_7^3}{z_1z_2} + (1+q_5+q_5q_6)\frac{q_1q_6q_7^2}{z_2} + (1+q_6+q_5q_6)\frac{q_1q_7z_1}{z_2} + \frac{q_1z_1^2}{z_2}$

Bibliography

[1] M. Aganagic, A. Klemm, and C. Vafa, *Disk instantons, mirror symmetry and the duality web*, Z. Naturforsch. A **57** (2002), no. 1-2, 1–28.

[2] V. I. Arnold, *Mathematical methods of classical mechanics*, Graduate Texts in Mathematics, vol. 60, Springer-Verlag, New York, 1991.

[3] D. Auroux, *Mirror symmetry and T-duality in the complement of an anti-canonical divisor*, J. Gökova Geom. Topol. GGT **1** (2007), 51–91.

[4] ———, *Special Lagrangian fibrations, wall-crossing, and mirror symmetry*, Surv. Differ. Geom., vol. 13, Int. Press, Somerville, MA, 2009, pp. 1–47.

[5] K. Behrend and Yu. Manin, *Stacks of stable maps and Gromov-Witten invariants*, Duke Math. J. **85** (1996), no. 1, 1–60.

[6] J. Bryan and N.-C. Leung, *The enumerative geometry of $K3$ surfaces and modular forms*, J. Amer. Math. Soc. **13** (2000), no. 2, 371–410.

[7] P. Candelas, X. C. De la Ossa, P. S. Green, and L. Parkes, *An Exactly soluble superconformal theory from a mirror pair of Calabi-Yau manifolds*, Phys. Lett. **B258** (1991), 118–126.

[8] A. Cannas da Silva, *Lectures on symplectic geometry*, Lecture Notes in Mathematics, vol. 1764, Springer-Verlag, Berlin, 2001.

[9] K.-W. Chan, *A formula equating open and closed gromov-witten invariants and its applications to mirror symmetry*, preprint 2010, arXiv:1006.3827.

[10] K.-W. Chan and N.-C. Leung, *Mirror symmetry for toric Fano manifolds via SYZ transformations*, Adv. Math. **223** (2010), no. 3, 797–839.

[11] T.-M. Chiang, A. Klemm, S.-T. Yau, and E. Zaslow, *Local mirror symmetry: calculations and interpretations*, Adv. Theor. Math. Phys. **3** (1999), no. 3, 495–565.

[12] C.-H. Cho and Y.-G. Oh, *Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds*, Asian J. Math. **10** (2006), no. 4, 773–814.

[13] A.I. Efimov, *Homological mirror symmetry for curves of higher genus*, preprint 2009, arXiv:0907.3903.

[14] B. Fang, C.-C. Liu, D. Treumann, and E. Zaslow, *The coherent-constructible correspondence and homological mirror symmetry for toric varieties*, 2009, arXiv:0901.4276.

[15] K. Fukaya, *Arnold conjecture and Gromov-Witten invariant*, Topology **38** (1999), no. 5, 933–1048.

[16] K. Fukaya, *Multivalued Morse theory, asymptotic analysis and mirror symmetry*, Graphs and patterns in mathematics and theoretical physics, Proc. Sympos. Pure Math., vol. 73, Amer. Math. Soc., Providence, RI, 2005, pp. 205–278.

[17] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian Floer theory and mirror symmetry on compact toric manifolds*, preprint 2010, arXiv:1009.1648.

[18] ———, *Lagrangian Floer theory on compact toric manifolds II : Bulk deformations*, preprint 2008, arXiv:0810.5654.

[19] ———, *Toric degeneration and non-displaceable Lagrangian tori in $S^2 \times S^2$*, preprint 2010, arXiv:1002.1660.

[20] ———, *Lagrangian intersection Floer theory: anomaly and obstruction. Part I*, AMS/IP Studies in Advanced Mathematics, vol. 46, American Mathematical Society, Providence, RI, 2009.

[21] ———, *Lagrangian intersection Floer theory: anomaly and obstruction. Part II*, AMS/IP Studies in Advanced Mathematics, vol. 46, American Mathematical Society, Providence, RI, 2009.

[22] ———, *Lagrangian Floer theory on compact toric manifolds. I*, Duke Math. J. **151** (2010), no. 1, 23–174.

[23] A. Gathmann, *Gromov-Witten invariants of blow-ups*, J. Algebraic Geom. **10** (2001), no. 3, 399–432.

[24] E. Goldstein, *Calibrated fibrations on noncompact manifolds via group actions*, Duke Math. J. **110** (2001), no. 2, 309–343.

[25] T. Graber and E. Zaslow, *Open-string Gromov-Witten invariants: calculations and a mirror "theorem"*, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, pp. 107–121.

[26] M. Gross, *Examples of special Lagrangian fibrations*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 81–109.

[27] M. Gross and B. Siebert, *From real affine geometry to complex geometry*, preprint 2007, arXiv:math/0703822 .

[28] K. Hori, A. Iqbal, and C. Vafa, *D-branes and mirror symmetry*, preprint 2000, arXiv:hep-th/0005247.

[29] S. Hosono, *Central charges, symplectic forms, and hypergeometric series in local mirror symmetry*, Mirror symmetry. V, AMS/IP Stud. Adv. Math., vol. 38, Amer. Math. Soc., Providence, RI, 2006, pp. 405–439.

[30] J. Hu, *Gromov-Witten invariants of blow-ups along points and curves*, Math. Z. **233** (2000), no. 4, 709–739.

[31] A. Kapustin, L. Katzarkov, D. Orlov, and M. Yotov, *Homological mirror symmetry for manifolds of general type*, preprint 2010, arXiv:1004.0129.

[32] Y. Konishi and S. Minabe, *Local B-model and Mixed Hodge Structure*, preprint 2009, arXiv:0907.4108.

[33] M. Kontsevich, *Enumeration of rational curves via torus actions*, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 335–368.

[34] ______, *Homological algebra of mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 120–139.

[35] M. Kontsevich and Y. Soibelman, *Homological mirror symmetry and torus fibrations*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 203–263.

[36] ______, *Affine structures and non-Archimedean analytic spaces*, The unity of mathematics, Progr. Math., vol. 244, Birkhäuser Boston, Boston, MA, 2006, pp. 321–385.

[37] C. Krattenthaler and T. Rivoal, *On the integrality of the Taylor coefficients of mirror maps*, Duke Math. J. **151** (2010), no. 2, 175–218.

[38] N.-C. Leung, *Mirror symmetry without corrections*, Comm. Anal. Geom. **13** (2005), no. 2, 287–331.

[39] N.-C. Leung, S.-T. Yau, and E. Zaslow, *From special Lagrangian to Hermitian-Yang-Mills via Fourier-Mukai transform*, Adv. Theor. Math. Phys. **4** (2000), no. 6, 1319–1341.

[40] A.-M. Li and Y.-B. Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds*, Invent. Math. **145** (2001), no. 1, 151–218.

[41] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), First Int. Press Lect. Ser., I, Int. Press, Cambridge, MA, 1998, pp. 47–83.

[42] B. H. Lian, K.-F. Liu, and S.-T. Yau, *Mirror principle. I*, Asian J. Math. **1** (1997), no. 4, 729–763.

[43] ______, *Mirror principle. II*, Asian J. Math. **3** (1999), no. 1, 109–146, Sir Michael Atiyah: a great mathematician of the twentieth century.

[44] ______, *Mirror principle. III*, Asian J. Math. **3** (1999), no. 4, 771–800.

[45] B. H. Lian and S.-T. Yau, *Integrality of certain exponential series*, **2** (1998), 215–227.

[46] A. Polishchuk and E. Zaslow, *Categorical mirror symmetry: the elliptic curve*, Adv. Theor. Math. Phys. **2** (1998), no. 2, 443–470.

[47] P. Seidel, *Homological mirror symmetry for the genus two curve*, preprint 2008, arXiv:0812.1171.

[48] A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B **479** (1996), no. 1-2, 243–259.

[49] J. Zhou, *Some integrality properties in local mirror symmetry*, preprint 2010, arXiv:1005.3243.

[50] V. V. Zudilin, *On the integrality of power expansions related to hypergeometric series*, Mat. Zametki **71** (2002), no. 5, 662–676.