Some New Results on Hyperbolic Gauss Curvature Flows

WO, Weifeng

A Thesis Submitted in Partial Fulfillment
of the Requirements for the Degree of
Doctor of Philosophy
in
Mathematics

The Chinese University of Hong Kong
February 2011

Thesis/Assessment Committee

Professor AU Kwok Keung Thomas (Chair)

Professor CHOU Kai Seng (Thesis Supervisor)

Professor XIN Zhou Ping (Committee Member)

Professor CHEUNG Wing Sum (External Examiner)

Abstract

In this thesis we study the hyperbolic curvature flows. Qusilinear hyperbolic equations are derived and studied for the motion of hypersurfaces under the hyperbolic mean curvature flows. As contrast to this, a new hyperbolic curvature flow (Gauss curvature flow) is proposed for convex hypersurfaces. The equations satisfied by the graphs of the hypersurfaces under these flows give rise to a new class of Euclidean invariant fully nonlinear hyperbolic equations. Based on this, we investigate the local solvability, finite time blow-up and asymptotic behavior for these flows. Group invariant solutions of the flows are also concerned.

In Chapter 2, we present a leisure study on the reducibility of a geometric motion to a differential equation for its graph for plane curves. It serves as a motivation for the introduction of normal and normal preserving flows. We show that any Euclidean invariant quasilinear equation arises as the associated equation of some normal flow and all fully nonlinear Euclidean invariant equations arise from normal preserving flows. We further study Affine type hyperbolic motion. Finally, some properties of these flows are presented.

In Chapter 3, the symmetry groups of the hyperbolic flows are determined and the corresponding group invariant solutions are discussed.

In Chapter 4, the motions for hypersurfaces are studied. Besides the equations satisfied by the graphs, we shall derive the equations for the support function of convex hypersurface. Based on this, we establish the local solvability of the

hyperbolic curvature flow. A preliminary discussion on topics such as finite time blow-up and asymptotic behavior will be given.

In the final part of this thesis, motion of free elastic curves is discussed. Conservation laws are derived by using the Noether's Theorem. We also consider group invariant solutions of this flow.

摘要

本論文研究一類雙曲高斯曲率流問題，類似於從平均曲率流倒出擬線性雙曲方程，我們研究了从新的曲率流得到的一類完全非線性雙曲方程。我們進一步討論了幾何流的局部可解性，有限時間爆破，長時間行為。對於平面曲線，我們考慮了曲率流的群不變解。

在論文的第一部分，我們首先給出如何把幾何流約化成單個圖像方程。接著引入了法方向流和法方向不變流，並建立了與擬線性和完全非線性雙曲方程之間的關係。最後我們給出了仿射幾何雙曲幾何流和各種雙曲幾何流的一些幾何性質。

第二部分，我們討論了第一部分平面曲率流的對稱群和相應的群不變解。

第三部分，我們考慮曲面運動。首先得到了圖像方程和支撐函數方程。基於支撐函數方程，我們給出了曲率流的局部可解性，討論了有限時間爆破。

最後一部分，我們討論了一類彈性曲線運動。利用 Noether 定理，我們得到了一系列守恆律. 最後還考慮了曲率流的群不變解。

Acknowledgement

I would like to express my deepest and sincere gratitude to my supervisor, Prof. Kai-Seng Chou, who accepted me as his Ph.D. student and offered me so much advice. His constant encouragement, understanding, and personal guidance have provided as good basis for the present thesis.

I also wish to express my sincere gratitude to Prof. Chang-zheng Qu, for his careful guidance during my studying in Northwest University and continuous encouragement.

I am also grateful to Dr. Leung-Fu Cheung for his kind help and useful discussions.

I should devote my special thanks to my family, for their continued support and selfless help. Their love and understanding have enabled me to complete this Ph.D. project.

Many thanks are also due to my friends and colleagues, Ms, W.W. Ao, Mr. S.Z. Du, Mr. C.Y. Ng, Ms. L.P. Wang, Mr. X.L. Wang, Mr. C.J. Xie, Mr. W. Yao, Mr. J. Yang, and Mr. Z.Y. Zhang.

Contents

Chapter 1

Introduction

In the mean curvature flow one studies the motion of a hypersurface whose velocity is equal to its mean curvature along its normal direction in the Euclidean space. Many results have been obtained over the years and one may consult the survey Huisken-Polden [HP] and the books Ecker [E], Giga [Gi] and Zhu [Z] for detailed discussions. From the point of view of differential equations, the mean curvature flow is a quasilinear parabolic equation which is invariant under the Euclidean motion.

In view of the intimate relation between the heat and the wave equations, it is natural to consider the hyperbolic version of the mean curvature flow. However, as we know, there is few results on hyperbolic version of curvature flow. Gurtin and Podio-Guidugli[GP] modeled a phenomenon which found in Melting crystals of helium : appearance of interfacial oscillations. They developed a hyperbolic theory for the evolution of the plane curves. In Yau [Y], it is proposed to study the motion of a hypersurface whose acceleration, instead of the velocity, is equal to its mean curvature along the normal direction. The hyperbolic version of curvature flow is important in both mathematics and applications, and has attracted many mathematicians. In He-Kong-Liu [HKL], local solvability of this problem is established and properties such as formation of singularities in finite time and asymptotic behavior of the flow are examined. However, this most direct analog of the mean curvature flow differs from its parabolic counterpart by not reducible to a Euclidean invariant hyperbolic equation. In LeFloch-Smoczyk [LS], the following motion law

$$\frac{\partial^2 X}{\partial t^2} = F\mathbf{n} - g^{ij}\Big\langle \frac{\partial X}{\partial t}, \frac{\partial^2 X}{\partial p_j \partial t}\Big\rangle \frac{\partial X}{\partial p_i}, \tag{1.1}$$

is studied. Here F is the driving force and g^{ij} is the inverse of the induced metric on the hypersurface $X(p,t)$ in $\mathbb{R}^{n+1}$. These authors call (1.1) the hyperbolic mean curvature flow for the specified choice of F,

$$F = \frac{1}{2}\Big(\Big|\frac{\partial X}{\partial t}\Big|^2 + n\Big)H,$$

where H is the mean curvature of X. This flow has the advantage of being derived from a Hamiltonian principle, and hence possesses some conservation laws. Besides, when the initial velocity is along the normal direction, the velocity of the hypersurface keeps pointing in the normal direction afterward. A flow with such property is called a normal flow. For a normal flow, the graph of the hypersurface satisfies a quasilinear Euclidean invariant hyperbolic equation. Subsequently, the hyperbolic curve shortening problem, that is, taking $n = 1$ and F to be the curvature of a plane curve in (1), is studied in Kong-Wang [KW] where several criteria on finite time blow-up for graphs are obtained. In Kong-Liu-Wang [KLW] they further study the problem for closed convex curves.

In chapter 2 of this thesis, we present a geometric view of the hyperbolic mean curvature flows in the plane. We show that every Euclidean invariant quasilinear equation arises as the associated equation of some normal flow.

Aside from the mean curvature flow, there are other curvature flows for convex hypersurfaces, notable ones including the Firey's model on worn stones [F] and the motion by the affine normal [A1] and [ST] which applies to image analysis. They depend on the Gauss curvature other than the mean curvature. The reader may look up [HP] and [Gi] for more information. The differential equations derived from these flows are no longer quasilinear. Usually, they are fully nonlinear. For flows involving the Gauss curvature, they are parabolic Monge-Ampère equations.

In chapter 4 of this thesis we propose a hyperbolic version of these fully nonlinear curvature flows. This is the main body of this thesis. For any driving force F, consider

$$\frac{\partial^2 X}{\partial t^2} = F\mathbf{n} - b^{ij}\frac{\partial F}{\partial p_i}\frac{\partial X}{\partial p_j}, \tag{1.2}$$

where b^{ij} is the inverse of the second fundamental form on the uniformly convex hypersurface. We say a flow is normal if the velocity is normal to the hypersurface, i.e. if

$$\left\langle \frac{dX}{dt}, \frac{\partial X}{\partial p_k} \right\rangle = 0.$$

A flow is called normal preserving if the normal of the hypersurface is independent of time $d\mathbf{n}/dt = 0$, i.e.

$$\left\langle \frac{\partial X_t}{\partial p_k}, \mathbf{n} \right\rangle = 0,$$

for each $k = 1, \cdots, n$. It can be shown that if this condition is fulfilled initially, then it holds for all time under (1.2). For any normal preserving flow, its graph satisfies a fully nonlinear Euclidean invariant hyperbolic equation. For instance, taking F to be the negative reciprocal of the Gauss curvature, we obtain the hyperbolic Monge-Ampère equation,

$$\det D^2_{x,t} u = -(1 + |\nabla u|^2)^{\frac{n+1}{2}},$$

and taking it to be the Gauss curvature, we have

$$\det D^2_{x,t} u = \frac{(\det D^2_x u)^2}{(1 + |\nabla u|^2)^{\frac{n+1}{2}}}.$$

It is interesting to observe that this new equation relates the Monge-Ampère operator in space-time to the Monge-Ampère operator in space. It is hyperbolic, and yet the solution is convex in (x,t). Different choices of F produces many new fully nonlinear hyperbolic equations.

Now we summarize the main results of the thesis in the following proposition and theorems.

Proposition 1.1. *Any Euclidean invariant quasilinear hyperbolic equation* $u_{tt} = au_{xx} + bu_{xt} + c$ *is the associated equation of the normal flow in the plane*

$$\frac{\partial^2\gamma}{\partial t^2} = F\boldsymbol{n} - \langle\gamma_t, \gamma_{ts}\rangle\boldsymbol{t}, \tag{1.3}$$

where F *is of the form* $F_1 + F_2k + F_3\langle\gamma_t, \gamma_{ts}\rangle$, *and* F_i, $i = 1, 2, 3$, *depend on* $\langle\gamma_t, \boldsymbol{n}\rangle$ *only.*

Proposition 1.2. *Any Euclidean invariant fully nonlinear hyperbolic equation* $u_{tt} = f(x, u, u_x, u_t, u_{xx}, u_{xt})$ *is the associated equation of a normal preserving flow*

$$\gamma_{tt} = F\boldsymbol{n} - \frac{1}{k}F_s\boldsymbol{t}, \tag{1.4}$$

where F *depends on* $\langle\gamma_t, \boldsymbol{n}\rangle, k,$ *and* $\langle\gamma_t, \gamma_{ts}\rangle$.

Proposition 1.3. *Let* $X(\cdot, t)$ *be a family of uniformly convex hypersurfaces satisfying* (4.1.1). *It is normal preserving if and only if it is given by* (1.2) *and*

$$\left\langle\frac{\partial X_t}{\partial p_j}, \boldsymbol{n}\right\rangle = 0, \qquad j = 1, \cdots, n,$$

at $t = 0$.

Theorem 1.1. *Consider*

$$\begin{cases} \dfrac{\partial^2 X}{\partial t^2} = F\boldsymbol{n} - b^{ij}\dfrac{\partial F}{\partial z_i}\dfrac{\partial X}{\partial z_j}, \\ X(0) \text{ and } X_t(0) \quad \text{are given.} \end{cases} \tag{1.5}$$

under

$$f \text{ is homogeneous of degree one on } \Gamma^+, \tag{1.6}$$
$$\frac{\partial f}{\partial R_j}(R_1, \cdots, R_n) < 0, \quad j = 1, \cdots, n, \quad (R_1, \cdots, R_n) \in \Gamma^+.$$

where $X(0)$ *is a uniformly convex hypersurface in* $\mathbb{R}^{n+1}$ *and* $X_t(0)$ *satisfies* $\langle \boldsymbol{n}, \partial X_t(0)/\partial z_j \rangle = 0$, $j = 1, \cdots, n$. *Suppose* $X(0) \in H^k(S^n)$ *and* $X_t(0) \in H^{k-1}(S^n), k > n/2 + 2$. *Let* $f \in C^\infty(\Gamma^+)$ *be a symmetric, positive function on the positive cone satisfying* (1.6). *There exists a positive* $T \leq \infty$ *such that* (1.5) *has a unique solution* X *in*

$$C([0,T), H^k(S^n)) \bigcap C^1([0,T), H^{k-1}(S^n))$$

which is uniformly convex at each t. *It is smooth provided* $X(0)$ *and* $X_t(0)$ *are smooth. Moreover, it is maximal in the sense that if* T *is finite, either the minimum of the principal curvatures of* $X(t)$ *tends to zero or*

$$||X(t)||_{C^2(S^n)} \to \infty,$$

as t *approaches* T.

This thesis is organized as follows. In Chapter 2 we study motion of plane curves. First we investigate the reducibility of a geometric motion to a differential equation for its graph for plane curves are presented. It serves as a motivation for the introduction of normal and normal preserving flows. Furthermore, we consider affine hyperbolic motion and obtain some properties for these flows. In chapter 3, we present a systematic study on group invariant solutions for the flows in chapter 2. Group invariant solutions such as traveling waves, rotating waves, expanding and contracting self-similar solutions play important roles in the study of parabolic flows. We apply Lie's theory of symmetries to determine the symmetry groups of these flows and examine some of the corresponding group invariant solutions. In chapter 4 the motions for hypersurfaces are discussed. We shall show that, when expressed in terms of the support function H for the convex hypersurface, the equation for (1.2) becomes

$$\frac{\partial^2 H}{\partial t^2} = -F.$$

It is the exact analog of

$$\frac{\partial H}{\partial t} = -F,$$

which is the corresponding equation arising from

$$\frac{\partial X}{\partial t} = F\mathbf{n}.$$

In this part, we first show the local solvability for hyperbolic flow of plane curves. Next we establish the local solvability of (1.2) for a large class of F based on the Caffarelli-Nirenberg-Spruck [CNS] theory of fully nonlinear elliptic equations. Finally, a preliminary discussion on topics such as finite time blow-up and asymptotic behavior will be given.

In the final part of this thesis, a new kind flow is established. The flow is derived from Hamilton principle based on a geometrically natural action, consisting of a kinetic term and elastic energy term. Conservation laws are derived by Noether's Theorem. We also consider group invariant solutions of this flow.

Chapter 2

Plane Curves

2.1 Euclidean invariant motions

We start by reviewing the reduction of the curve shortening problem to a quasi-linear parabolic equation. Consider the curve shortening problem or the more general problem where a family of plane curves $\gamma(p,t)$ is driven by the motion law

$$\frac{\partial\gamma}{\partial t} = F\mathbf{n} + G\mathbf{t}, \tag{2.1.1}$$

where $\mathbf{n}$ and $\mathbf{t}$ are respectively the unit normal and tangent vectors of the curve $\gamma(\cdot,t)$, and F and G are functions depending on γ and its derivatives with respect to p. The normal $\mathbf{n}$ is the inner one when the curve is closed. Suppose for $p \in (a,b)$ and $t \in (t_0,t_1)$, the curve $\gamma(p,t)$ can be expressed in the form of a graph $(x, u(x,t))$, $x = x(p,t)$, we have

$$\gamma_t = x_t(1,u_x) + (0,u_t). \tag{2.1.2}$$

Taking inner product with $\mathbf{n} = (-u_x,1)/\sqrt{1+u_x^2}$ and $\mathbf{t} = (1,u_x)/\sqrt{1+u_x^2}$, we see that (2.1.1) is split into two equations, namely,

$$u_t = \sqrt{1+u_x^2}\, F, \tag{2.1.3}$$

and

$$x_t = \frac{\sqrt{1+u_x^2}\, G - u_t u_x}{1+u_x^2}. \tag{2.1.4}$$

In the special case where F depends only on k, the curvature of γ, the formula $k = u_{xx}/(1+u_x^2)^{\frac{3}{2}}$ tells us that (2.1.3) is an evolution equation for u. In principle, one can solve (2.1.1) by first solving (2.1.3) for u and then determine x from (2.1.4). For instance, in the curve shortening problem $F(k) = k$ and $G \equiv 0$, so (2.1.3) and (2.1.4) become

$$u_t = \frac{u_{xx}}{1+u_x^2}, \tag{2.1.5}$$

and

$$x_t = \frac{-u_t u_x}{1+u_x^2}(x,t), \tag{2.1.6}$$

respectively. In case a solution u has been found for (2.1.5), x can be readily solved as the solution of the ODE (2.1.6). It is routine to verify that then $(x, u(x,t))$ constitutes a solution for the curve shortening problem.

Before proceeding further, we point out that for motions which only depend on the geometry of the curves, one should require the motion law to be a "geometric" one. Specifically, it means that solutions of (2.1.1) are preserved under any reparametrization as well as Euclidean motions. It turns out that the flow (2.1.1) is geometric when F and G depend only on the curvature and its derivatives with respect to the arc-length. For any geometric flow (2.1.1), the corresponding equation (2.1.1) is Euclidean invariant in the following sense. In case under a Euclidean motion R, $(y,v) = R(x,u)$, the graphs $(x, u(x,t))$ go over to graphs $(y, v(y,t))$, then v satisfies the same equation (2.1.3) with x and u replaced by y and v respectively. The reader is referred to Olver [O] for discussion on group invariant differential equations.

Now, consider the motion of curves where the velocity is replaced by the acceleration

$$\frac{\partial^2 \gamma}{\partial t^2} = F\mathbf{n} + G\mathbf{t}. \tag{2.1.7}$$

As the highest order of time derivative involved is two, the functions F and G are allowed to depend on γ, γ_t and their derivatives with respect to p. Typical geometric flows are formed from those F and G depending on $\langle \gamma_t, \mathbf{n} \rangle$, $\langle \gamma_t, \mathbf{t} \rangle$, $\langle \gamma_t, \gamma_{ts} \rangle$, k, etc, and their derivatives with respect to the arc-length. All these are invariants under reparametrizations and Euclidean motions.

When the curves are expressed as graphs $\gamma = (x, u(x,t))$, we have

$$\gamma_{tt} = x_{tt}(1, u_x) + (0, u_{xx}x_t^2 + 2u_{xt}x_t + u_{tt}).$$

Taking inner product with $\mathbf{n}$ and $\mathbf{t}$ respectively yields

$$u_{tt} + 2x_t u_{xt} + x_t^2 u_{xx} = \sqrt{1+u_x^2}\, F, \tag{2.1.8}$$

and

$$x_{tt} = \frac{G - u_x F}{\sqrt{1+u_x^2}}. \tag{2.1.9}$$

The situation is different from (2.1.2). In general, (2.1.8) not only depends on u and its derivatives, but also on x_t. (2.1.8) and (2.1.9) are coupled.

Is there some choice of x_t so that (2.1.8) reduces to an equation for u only? To examine this possibility, we note that from (2.1.8)

$$x_t = \frac{-u_{xt} \pm \sqrt{u_{xt}^2 - u_{xx}u_{tt} - u_{xx}\sqrt{1+u_x^2}\, F}}{u_{xx}}.$$

When (2.1.8) is reducible to an equation of the form $u_{tt} = \Psi(u_x, u_t, u_{xx}, u_{xt})$ for some function Ψ, plugging this equation into the above expression, one sees that x_t must be equal to $\Phi(u_x, u_t, u_{xx}, u_{xt})$ for some function Φ, assuming that F contains first and second derivatives of u only. Motivated by this, we introduce the following definitions. A flow (2.1.7) is called *reducible (to an equation)* if there exists a function $\Phi(z_1, z_2, z_3, z_4)$ such that whenever the flow is expressed as a graph $(x, u(x,t))$,

$$x_t = \Phi(u_x, u_t, u_{xx}, u_{xt})$$

must hold. For any reducible flow, the equation obtained by substituting $x_t = \Phi$ into (2.1.8) is called the *associated equation of the flow*. We may assume the variables of the function F can be expressed in terms of u and its derivatives.

Two remarks are in order. First, flows which are not reducible exist. In the end of this section we will show that for $F = k$ and $G \equiv 0$, that is, the most direct hyperbolic analog of the curve shortening problem, is not reducible. Second, when one is concerned with the initial value problem for (2.1.7), it is natural to wonder the flow is reducible for any initial values $\gamma(0)$ and $\gamma_t(0)$. The answer is no. Let us assume locally $\gamma(0) = (f_1(p), f_2(p))$ and $\gamma_t(0) = (g_1(p), g_2(p))$. As we have freedom in choosing the parameter, we may assume $x = p$, that is, f_1 is the identity map. Then the relation $x_t = \Phi$ at $t = 0$ gives the compatibility condition $g_1 = \Phi(f_2', g_2 - f_2'g_1, f_2'', (g_2 - f_2'g_1)')$. When the initial curve is fixed, that is, f_2 is given, this condition sets up a constraint between g_1 and g_2.

For a given function F, we will find two classes of "constrained" flows, namely, the normal and normal preserving flows, and the corresponding functions G so that the flows are reducible. Our approach is based on the observation that any associated equation of a reducible flow must be Euclidean invariant, so we start by classifying all Euclidean invariant equations. Of course, this is of interest in itself. After obtaining these equations, we may compare them with (2.1.8) to guess what the constraint Φ should be.

We examine the quasilinear case first. Consider

$$u_{tt} = au_{xx} + bu_{xt} + c, \tag{2.1.10}$$

where the coefficients a, b, and c depend on x, u, u_x, and u_t.

Proposition 2.1. *Any Euclidean invariant equation* (2.1.10) *is of the form*

$$\begin{aligned} b &= \frac{2u_x u_t}{1+u_x^2} + \frac{\varphi(z)}{\sqrt{1+u_x^2}} \\ a &= \frac{1}{1+u_x^2} - \frac{b^2}{4} + \frac{\chi(z)}{1+u_x^2} \\ c &= \sqrt{1+u_x^2}\,\psi(z), \qquad z = \frac{u_t}{\sqrt{1+u_x^2}}, \end{aligned}$$

where φ, χ *and* ψ *are arbitrary functions.*

Proof. The Euclidean group acts linearly on (x, u) and trivially on t. Its Lie algebra of infinitesimal symmetries is spanned by

$$\{\partial_x,\ \partial_u,\ -u\partial_x + x\partial_u\}.$$

According to Lie's theory of symmetries, (2.1.10) is Euclidean invariant if and only if

$$pr^{(2)}\mathbf{v}(u_{tt} - au_{xx} - bu_{xt} - c) = 0,$$

on $u_{tt} = au_{xx} + bu_{xt} + c$, where $\mathbf{v}$ is any infinitesimal symmetry and $pr^{(2)}\mathbf{v}$ is the second order prolongation of $\mathbf{v}$. By the prolongation formula [O], $pr^{(2)}\partial_x = \partial_x$, so

$$pr^{(2)}\partial_x(u_{tt} - au_{xx} - bu_{xt} - c) = -a_x u_{xx} - b_x u_{xt} - c_x = 0,$$

which implies that a, b, c are independent of x. Similarly, they are also independent of u. Now, for the rotation $\mathbf{r} \equiv -u\partial_x + x\partial_u$, the second prolongation is given by

$$\begin{aligned} pr^{(2)}\mathbf{r} &= -u\partial_x + x\partial_u + (1+u_x^2)\partial_{u_x} + u_x u_t \partial_{u_t} + 3u_x u_{xx}\partial_{u_{xx}} \\ &\quad + (2u_x u_{xt} + u_t u_{xx})\partial_{u_{xt}} + (u_{tt}u_x + 2u_t u_{xt})\partial_{u_{tt}}. \end{aligned}$$

Its action on (2.1.10) gives

$$u_{xx}\big(a_{u_x}(1+u_x^2) + a_{u_t}u_x u_t\big) + 3au_x u_{xx} + u_{xt}\big(b_{u_x}(1+u_x^2) + b_{u_t}u_x u_t\big) +$$

$$b(2u_x u_{xt} + u_t u_{xx}) + c_{u_x}(1+u_x^2) + c_{u_t}u_x u_t = u_{tt}u_x + 2u_t u_{xt},$$

on $u_{tt} = au_{xx} + bu_{xt} + c$. We eliminate u_{tt} in this equation using (2.1.10). Then the variables u_{xx}, u_{xt}, u_x, and u_t become free. By setting the coefficients of u_{xx} and u_{xt} to zero, we obtain

$$a_{u_x}(1+u_x^2) + a_{u_t}u_xu_t + 2au_x + bu_t = 0,$$

and

$$b_{u_x}(1+u_x^2) + b_{u_t}u_xu_t + bu_x - 2u_t = 0,$$

while the lower order terms give

$$c_{u_x}(1+u_x^2) + c_{u_t}u_xu_t - cu_x = 0.$$

These are first order linear PDE's for the coefficients. The second and third equations are readily solved to yield

$$b = \frac{2u_xu_t}{1+u_x^2} + \frac{1}{u_t}\varphi_1\Big(\frac{u_t}{\sqrt{1+u_x^2}}\Big),$$

and

$$c = \sqrt{1+u_x^2}\,\psi\Big(\frac{u_t}{\sqrt{1+u_x^2}}\Big).$$

Plugging b into the first equation gives

$$a = \frac{1}{1+u_x^2} - \frac{b^2}{4} + \frac{1}{u_t^2}\chi_1\Big(\frac{u_t}{\sqrt{1+u_x^2}}\Big).$$

Here φ_1, χ_1 and ψ are arbitrary functions. Clearly the proposition holds. □

Taking $\varphi = \chi = \psi = 0$, we obtain the simplest Euclidean invariant equation

$$u_{tt} - 2\frac{u_xu_t}{1+u_x^2}u_{xt} + \frac{u_x^2u_t^2}{(1+u_x^2)^2}u_{xx} = \frac{u_{xx}}{1+u_x^2}. \tag{2.1.11}$$

Comparing this equation with (2.1.8) where $F = k$, we see that $\Phi = -u_xu_t/(1+u_x^2)$. The meaning of this constraint becomes clear after using (2.1.2); it means that $\langle\gamma_t, \gamma_p\rangle = 0$ for all time. A flow with this property is called a *normal flow*. With this constraint at hand, G could be determined from (2.1.9), but here we use a different reasoning which is based on the fact that (2.1.7) must preserve this constraint. In other words, if $\langle \gamma_t, \gamma_p \rangle = 0$ at $t = 0$, then it holds for all time. Keeping this in mind, we compute

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\langle \gamma_t, \gamma_p \rangle \\ &= \langle \gamma_{tt}, \gamma_p \rangle + \langle \gamma_t, \gamma_{pt} \rangle \\ &= G|\gamma_p| + \langle \gamma_t, \gamma_{ts} \rangle |\gamma_p|, \end{aligned}$$

from which we deduce $G = -\langle \gamma_t, \gamma_{ts} \rangle$. Note that it is independent of F. Later we will see that G depends on F for a normal preserving flow. It is routine to check that for any given F and G in (2.1.7), starting from an initial velocity satisfying $\langle \gamma_t(0), \gamma_p(0) \rangle = 0$, the flow (whenever exists) is normal if and only if $G = -\langle \gamma_t, \gamma_{ts} \rangle$. In the following we show that any quasilinear Euclidean invariant equation (2.1.10) arises as the associated equation of some normal flow.

Proposition 2.2. *Any Euclidean invariant equation* (2.1.10) *is the associated equation of the normal flow*

$$\frac{\partial^2 \gamma}{\partial t^2} = F\boldsymbol{n} - \langle \gamma_t, \gamma_{ts} \rangle \boldsymbol{t}, \tag{2.1.12}$$

where F is of the form $F_1 + F_2 k + F_3 \langle \gamma_t, \gamma_{ts} \rangle$, and F_i, $i = 1, 2, 3$, depend on $\langle \gamma_t, \boldsymbol{n} \rangle$ only.

Proof. First, note that

$$\langle \gamma_t, \mathbf{n} \rangle = \frac{u_t}{\sqrt{1 + u_x^2}},$$

and $\langle \gamma_t, \mathbf{t} \rangle = 0$. We also claim

$$\langle \gamma_t, \gamma_{ts} \rangle = \frac{u_t u_{xt}}{(1 + u_x^2)^{3/2}} - \frac{u_x u_t^2 u_{xx}}{(1 + u_x^2)^{5/2}}.$$

To see this, we first use orthogonality to get $\gamma_t = \langle \gamma_t, \mathbf{n} \rangle \mathbf{n}$. It follows that

$$\gamma_{ts} = \langle \gamma_t, \mathbf{n}_s \rangle + (\langle \gamma_{ts}, \mathbf{n} \rangle + \langle \gamma_t, \mathbf{n}_s \rangle)\mathbf{n},$$

so

$$\langle \gamma_t, \gamma_{ts} \rangle = \langle \gamma_t, \mathbf{n} \rangle \langle \gamma_{ts}, \mathbf{n} \rangle,$$

after using Frenet's formula. Now, $\gamma_{ts} = x_{ts}(1, u_x) + x_t(0, u_{xx})x_s + (0, u_{tx})x_s$ where $x_s = 1/(1+u_x^2)$, hence

$$\langle \gamma_{ts}, \mathbf{n} \rangle = \frac{x_t u_{xx}}{1+u_x^2} + \frac{u_{tx}}{1+u_x^2},$$

and the claim follows.

Putting these into (2.1.8), we obtain

$$\begin{aligned} u_{tt} &= \sqrt{1+u_x^2}\Big[F_1 + F_2\frac{u_{xx}}{(1+u_x^2)^{3/2}} + F_3\Big(\frac{u_t u_{xt}}{(1+u_x^2)^{3/2}} - \frac{u_x u_t^2 u_{xx}}{(1+u_x^2)^{5/2}}\Big)\Big] \\ &\quad + \frac{2u_x u_t}{1+u_x^2}u_{xt} - \frac{u_x^2 u_t^2}{(1+u_x^2)^2}u_{xx}. \end{aligned}$$

Comparing with Proposition 2.1, we simply take $F_3(z) = \varphi/z$, $F_2(z) = 1 - \varphi^2(z)/4 + \chi(z)$, and $F_1(z) = \psi(z)$, then the proposition follows. □

Next we consider the fully nonlinear equation

$$u_{tt} = f(x, u, u_x, u_t, u_{xx}, u_{xt}). \tag{2.1.13}$$

Parallel to Proposition 2.1, we have

Proposition 2.3. *Any Euclidean invariant equation* (2.1.13) *is of the form*

$$u_{tt} = \frac{u_{xt}^2}{u_{xx}} + \sqrt{1+u_x^2}\;\Phi(z_1, z_2, z_3), \tag{2.1.14}$$

where $\Phi(z_1, z_2, z_3)$ *is an arbitrary function and* $z_1 = u_t/\sqrt{1+u_x^2}$, $z_2 = u_{xx}/(1+u_x^2)^{3/2}$, *and*

$$z_3 = \frac{u_{xt}}{1+u_x^2} - \frac{u_x u_t u_{xx}}{(1+u_x^2)^2}.$$

Proof. As in the proof of Proposition 2.1, f is independent of x and u by Euclidean invariance. From the action of the infinitesimal rotation, the prolongation formula gives

$$(1+u_x^2)f_{u_x} + u_x u_t f_{u_t} + 3u_x u_{xx} f_{u_{xx}} + (2u_x u_{xt} + u_t u_{xx})f_{xt} = u_x f + 2u_t u_{xt},$$

which implies

$$f = -\frac{u_x^2 u_t^2}{(1+u_x^2)^2} u_{xx} + \frac{2u_x u_t}{(1+u_x^2)} u_{xt} + (1+u_x^2)^{\frac{1}{2}} \Phi_1(z_1, z_2, z_3),$$

for some function Φ_1. The proposition now follows from letting $\Phi(z_1, z_2, z_3) = z_3^2/z_2 + \Phi_1(z_1, z_2, z_3)$. □

By comparing (2.1.14) with (2.1.8), we see that they are identical if we choose

$$x_t = -\frac{u_{xt}}{u_{xx}}. \tag{2.1.15}$$

This condition is readily checked to be equivalent to

$$\langle \gamma_{ts}, \mathbf{n} \rangle = 0. \tag{2.1.16}$$

A flow (2.1.7) is called a *normal preserving flow* if (2.1.16) holds for all time. To understand this definition, recall that the angle between the curve and the x−axis, α, is related to u_x by $\tan \alpha = u_x$. From

$$\sec^2 \alpha \frac{\partial \alpha}{\partial t} = u_{xx} x_t + u_{xt} = 0,$$

we see that α is independent of time during the flow. As the normal angle of the curve is equal to $\alpha + \pi/2$, it is also constant in time. In other words, $\mathbf{n}(p, t)$ is equal to $\mathbf{n}(p, 0)$, justifying the terminology.

Same as in the quasilinear case, we can determine G for a normal preserving flow. In fact,

$$\begin{aligned} \frac{\partial}{\partial t} \langle \gamma_{tp}, \mathbf{n} \rangle &= \langle \gamma_{ttp}, \mathbf{n} \rangle + \langle \gamma_{tp}, \mathbf{n}_t \rangle \\ &= \frac{\partial F}{\partial p} + |\gamma_p| k G + \langle \gamma_{tp}, \mathbf{t} \rangle \langle \mathbf{n}, \mathbf{t}_t \rangle \\ &= \frac{\partial F}{\partial p} + |\gamma_p| k G + \langle \gamma_{ts}, \mathbf{t} \rangle \langle \gamma_{tp}, \mathbf{n} \rangle. \end{aligned}$$

This is an ODE of the form $dy/dt = a + by$. Clearly, (2.1.7) preserves normal preserving flows if and only if $G = -k^{-1} F_s$. In fact, all fully nonlinear Euclidean invariant equations arise from this way.

Proposition 2.4. *Any Euclidean invariant equation* (2.1.13) *is the associated equation of a normal preserving flow*

$$\gamma_{tt} = F\boldsymbol{n} - \frac{1}{k}F_s\boldsymbol{t}, \tag{2.1.17}$$

where F depends on $\langle \gamma_t, \boldsymbol{n}\rangle, k$, and $\langle \gamma_t, \gamma_{ts}\rangle$.

Proof. Plug (2.1.15) into (2.1.8) and then use Proposition 2.3. □

Hyperbolic versions of the curve shortening problem can be found by choosing different F and G in (2.1.7). In LeFloch-Smoczyk [LS]

$$F = \frac{1}{2}(1 + |\gamma_t|^2)k, \quad G = -\langle \gamma_t, \gamma_{ts}\rangle,$$

is chosen. From the above discussion any normal flow is reducible with associated equation given by

$$u_{tt} = \frac{1 + u_x^2 + u_t^2 - 2u_x^2u_t^2}{2(1+u_x^2)^2}u_{xx} + 2\frac{u_xu_t}{1+u_x^2}u_{xt}. \tag{2.1.18}$$

In Kong-Wang [KW],

$$F = k, \qquad G = -\langle \gamma_t, \gamma_{ts}\rangle,$$

is chosen. Again, any normal flow is reducible and its associated equation is simply given by (2.1.11). Both equations are quasilinear hyperbolic. Now we may take

$$F = k, \quad G = -k^{-1}k_s$$

in (2.1.7). Any normal preserving flow is reducible, and its associated equation is

$$u_{tt}u_{xx} - u_{xt}^2 = \frac{u_{xx}^2}{1+u_x^2}, \tag{2.1.19}$$

This is a fully nonlinear, hyperbolic equation as long as the curve is uniformly convex.

Very often, in the study of the motions of convex curves, it is useful to express the flow in terms of the support function rather than the graph, Chou-Zhu [CZ]. Recall that the normal angle $\theta \in [0, 2\pi)$ of a curve satisfies

$$\mathbf{n} = -(\cos\theta,\ \sin\theta), \quad \mathbf{t} = (-\sin\theta,\ \cos\theta),$$

and the support function is a function of the normal angle given by

$$h(\theta, t) = \langle \gamma(p,t),\ -\mathbf{n} \rangle.$$

where $\gamma(p, t)$ is the point on the curve whose normal angle is equal to θ. Any closed convex curve can be determined from its support function. In fact, for $\gamma = (x, u(x,t))$, we have

$$x = h\cos\theta - h_\theta \sin\theta$$
$$u = h\sin\theta + h_\theta\cos\theta.$$

Differentiating the first of these relations in x and t, we have

$$1 = -(h + h_{\theta\theta})\theta_x \sin\theta$$
$$0 = h_t\cos\theta - h_{\theta t}\sin\theta - (h + h_{\theta\theta})\theta_t \sin\theta.$$

Therefore,

$$\theta_x = -\frac{k}{\sin\theta},$$

and

$$\theta_t = k\Big(\frac{h_t\cos\theta - h_{\theta t}\sin\theta}{\sin\theta}\Big),$$

after using the formula

$$k = \theta_s = \frac{1}{h_{\theta\theta} + h}.$$

By differentiating the second relation, we obtain,

$$
\begin{aligned}
u_x &= \frac{1}{k}\theta_x \cos\theta = -\cot\theta, \\
u_{xx} &= \frac{1}{\sin^2\theta}\theta_x = -\frac{k}{\sin^3\theta}, \\
u_{xt} &= \frac{1}{\sin^2\theta}\theta_t = -\frac{k}{\sin^3\theta}(h_t \sin\theta - h_{\theta t}\sin\theta), \\
u_t &= h_t \sin\theta + (h\cos\theta + h_{\theta\theta}\cos\theta)\theta_t + h_{\theta t}\cos\theta = \frac{h_t}{\sin\theta}, \\
u_{tt} &= \frac{h_{tt}}{\sin\theta} + \Big(\frac{h_{t\theta}}{\sin\theta} - \frac{h_t\cos\theta}{\sin^2\theta}\Big)\theta_t = \frac{h_{tt}}{\sin\theta} - \frac{k}{\sin^3\theta}(h_{t\theta}\sin\theta - h_t\cos\theta)^2.
\end{aligned}
$$

Using these formulas, we can express equations (2.1.18), (2.1.11) and (2.1.19) in terms of the support function. For (2.1.9) and (2.1.11), the equations are

$$h_{tt} = \frac{h_{t\theta}^2}{h_{\theta\theta} + h} - \frac{1 + h_t^2}{2},$$

and

$$h_{tt} = \frac{h_{t\theta}^2 - 1}{h_{\theta\theta} + h},$$

respectively. As for (2.1.19), the equation is

$$h_{tt} = -\frac{1}{h_{\theta\theta} + h},$$

which is the exact analog of the curve shortening problem when expressed in terms of the support function

$$h_t = -\frac{1}{h_{\theta\theta} + h}.$$

In concluding this section, let us show that the flow (2.1.7) is not reducible when $F(k) = k$ and $G \equiv 0$. To formulate the result, put the constraint $x_t = \Phi(u_x, u_t, u_{xx}, u_{xt})$ into (2.1.8) to get

$$u_{tt} + 2\Phi u_{xt} + \Phi^2 u_{xx} = \frac{u_{xx}}{1 + u_x^2}, \qquad (2.1.20)$$

(2.1.4) now reads as

$$x_{tt} = \frac{-u_x u_{xx}}{(1+u_x^2)^2}. \tag{2.1.21}$$

Proposition 2.5. *There is no such smooth function $\Phi(z_1, z_2, z_3, z_4)$ satisfying (i) (2.1.20) is solvable locally in space and time for arbitrary smooth initial data $u(0)$ and $u_t(0)$ and (ii) the constraint $x_t = \Phi(u_x, u_t, u_{xx}, u_{xt})$ fulfils (2.1.21).*

Proof. From the constraint we have

$$x_{tt} = \Phi_{z_1}(u_{xx}\Phi + u_{xt}) + \Phi_{z_2}(u_{xt}\Phi + u_{tt}) + \Phi_{z_3}(u_{xxx}\Phi + u_{xxt}) + \Phi_{z_4}(u_{xxt}\Phi + u_{xtt}). \tag{2.1.22}$$

On the other hand, from (2.1.20) we have

$$\begin{aligned} u_{xtt} &= \frac{u_{xxx}}{1+u_x^2} - \frac{2u_x u_{xx}^2}{(1+u_x^2)^2} - 2u_{xxt}\Phi - u_{xxx}\Phi^2 \\ &\quad -(2u_{xt} + 2u_{xx}\Phi)(\Phi_{z_1}u_{xx} + \Phi_{z_2}u_{xt} + \Phi_{z_3}u_{xxx} + \Phi_{z_4}u_{xxt}). \end{aligned}$$

Eliminating the term u_{xtt} in (2.1.22) by this equation and then identifying it with (2.1.21), we obtain a relation of the form $Au_{xxt} + Bu_{xxx} + C = 0$, between $u_x, u_t, u_{xx}, u_{xt}, u_{xxt}$ and u_{xxx}. By our assumption (i), all these variables are free. It follows that $A = B = 0$, that is,

$$\Phi_{z_3} + \Phi_{z_4}[-\Phi - (2z_4 + 2z_3\Phi)\Phi_{z_4}] = 0, \tag{2.1.23}$$

and

$$\Phi_{z_3}\Phi + \Phi_{z_4}\left[-\Phi^2 - (2z_4 + 2z_3\Phi)\Phi_{z_3} + \frac{1}{1+z_1^2}\right] = 0, \tag{2.1.24}$$

for all (z_1, z_2, z_3, z_4). The lower order term C also vanishes, but we do not need it.

We solve for Φ_{z_3} from (2.1.23) and plug it into (2.1.24) to get

$$\Phi_{z_4}\left[\left(1 - (2z_4 + 2z_3\Phi)\Phi_{z_4}\right)\left(\Phi + (2z_4 + 2z_3\Phi)\Phi_{z_4}\right) - \left(-\Phi^2 + \frac{1}{1+z_1^2}\right)\right] = 0.$$

Thus, either $\Phi_{z_4} = 0$ or

$$\Phi_{z_4} = \pm\frac{1}{(2z_4 + 2z_3\Phi)\sqrt{1+z_1^2}}.$$

If $\Phi_{z_4} = 0$, then $\Phi_{z_3} = 0$ and $x_t = \Phi(u_x, u_t)$. It is easy to see that this is impossible. We take

$$\Phi_{z_4} = \frac{1}{(2z_4 + 2z_2\Phi)\sqrt{1+z_1^2}}. \tag{2.1.25}$$

(The other case can be treated similarly.) From (1.24) we have

$$\Phi_{z_3} = \frac{\Phi\sqrt{1+z_1^2}+1}{(2z_4 + 2z_3\Phi)(1+z_1^2)}. \tag{2.1.26}$$

Differentiate (2.1.25) in z_3

$$\begin{aligned}\Phi_{z_4 z_3} &= -\frac{2\Phi + 2z_3\Phi_{z_3}}{(2z_4+2z_3\Phi)^2\sqrt{1+z_1^2}} \\ &= -\frac{2\Phi}{(2z_4+2z_3\Phi)^2\sqrt{1+z_1^2}} \\ &\quad -\frac{2z_3}{(2z_4+2z_3\Phi)^2\sqrt{1+z_1^2}}\frac{\Phi\sqrt{1+z_1^2}+1}{(2z_4+2z_3\Phi)(1+z_1^2)}.\end{aligned}$$

and differentiate (2.1.26) in z_4 to get

$$\begin{aligned}\Phi_{z_3 z_4} &= \frac{\Phi_{z_4}}{(2z_4+2z_3\Phi)\sqrt{1+z_1^2}} - \frac{(\Phi\sqrt{1+z_1^2}+1)(2+2z_3\Phi_{z_4})}{(2z_4+2z_3\Phi)^2(1+z_1^2)} \\ &= \frac{1}{(2z_4+2z_3\Phi)(1+z_1^2)} - \frac{2\Phi}{(2z_4+2z_3\Phi)^2\sqrt{1+z_1^2}} \\ &\quad -\frac{2}{(2z_4+2z_3\Phi)(1+z_1^2)} - \frac{2z_3(\Phi\sqrt{1+z_1^2}+1)}{(2z_4+2z_3\Phi)^3(1+z_1^2)}.\end{aligned}$$

We find

$$\Phi_{z_4 z_3} - \Phi_{z_3 z_4} = \frac{1}{(2z_4+2z_3\Phi)(1+z_1^2)} \neq 0,$$

contradiction holds, so the flow (2.1.7) ($F = k$ and $G \equiv 0$) is not reducible. □

2.2 Affine invariant motions

The affine curve shortening problem refers to $F = k^{1/3}$ and $G = -k^{-5/3}k_s/3$ in (2.1.1) has been studied in connection with image processing. Being called the fundamental equation of image processing in [AGLM], it is studied in [A1] and

[ST]. We may consider its hyperbolic analogs. Recall that the affine group is a subgroup of the Euclidean group whose infinitesimal symmetries are spanned by

$$\{\partial_x, \partial_u, u\partial_x, x\partial_u, x\partial_x - u\partial_u\}.$$

We call (2.1.1) is affine motion if the flow is invariant under affine group. Assume the flow can be reduced to a hyperbolic equation of form (2.1.13), then we have

Proposition 2.6. *Any affine invariant equation* (2.1.13) *is of the form*

$$u_{tt} = \frac{u_{xt}^2}{u_{xx}} + u_t \Phi\left(\frac{u_{xx}}{u_t^3}\right), \tag{2.2.1}$$

for some function Φ.

Proof. According to Lie's theory of symmetries, (2.1.13) is affine invariant if and only if

$$pr^{(2)}\mathbf{v}(u_{tt} - f) = 0,$$

on $u_{tt} = f$, where $\mathbf{v}$ is any infinitesimal symmetry in $\{\partial_x, \partial_u, u\partial_x, x\partial_u, x\partial_x - u\partial_u\}$ and $pr^{(2)}\mathbf{v}$ is the second order prolongation of $\mathbf{v}$. By the prolongation formula [O], $pr^{(2)}\partial_x = \partial_x$, so

$$pr^{(2)}\partial_x(u_{tt} - au_{xx} - bu_{xt} - c) = -a_x u_{xx} - b_x u_{xt} - c_x = 0,$$

which implies that a, b, c are independent of x. Similarly, they are also independent of u. Now, for the rotation $\mathbf{r} \equiv -u\partial_x + x\partial_u$, the second prolongation is given by

$$\begin{aligned} pr^{(2)}\mathbf{r} &= -u\partial_x + x\partial_u + (1 + u_x^2)\partial_{u_x} + u_x u_t \partial_{u_t} + 3u_x u_{xx}\partial_{u_{xx}} \\ &\quad + (2u_x u_{xt} + u_t u_{xx})\partial_{u_{xt}} + (u_{tt}u_x + 2u_t u_{xt})\partial_{u_{tt}}. \end{aligned}$$

Next, for $u\partial_x$ and $x\partial_u$, we have

$$\begin{aligned} pr^{(2)}u\partial_x &= u\partial_x - u_x\partial_{u_x} - 2u_{xx}\partial_{u_{xx}}, \\ pr^{(2)}x\partial_u &= x\partial_u + \partial_{u_x}. \end{aligned}$$

From these equations, we obtain

$$f = \frac{u_{xt}^2}{u_{xx}} + u_t \Phi\Big(\frac{u_{xx}}{u_t^3}\Big),$$

where Φ is an arbitrary function. □

Since the affine tangent and normal are related to $\mathbf{n}$ and $\mathbf{t}$ via

$$\mathcal{T} = k^{-\frac{1}{3}}\mathbf{n}, \quad \mathcal{N} = k^{\frac{1}{3}}\mathbf{n} - \frac{1}{3}k^{-\frac{5}{3}}k_s\mathbf{t},$$

the hyperbolic version of affine curve shortening problem reads as

$$\frac{\partial^2\gamma}{\partial t^2} = \mathcal{N} = k^{\frac{1}{3}}\mathbf{n} - \frac{1}{3}k^{-\frac{5}{3}}k_s\mathbf{t}, \tag{2.2.2}$$

or, in graph case,

$$\begin{cases} x_{tt} = -\frac{1}{3}u_{xx}^{-\frac{5}{3}}u_{xxx} \\ u_{tt} + 2u_{xt}x_t + u_{xx}x_t^2 = u_{xx}^{\frac{1}{3}}. \end{cases} \tag{2.2.3}$$

Letting

$$x_t = -\frac{u_{xt}}{u_{xx}}, \tag{2.2.4}$$

the second equation of (2.2.3) is reduced to

$$u_{tt} = \frac{u_{xt}^2}{u_{xx}} + (u_{xx})^{\frac{1}{3}}, \tag{2.2.5}$$

which corresponds to taking $\Phi(z) = z^{1/3}$ in Proposition 2.6. In fact, once we have a solution $u(x,t)$ for a solution $u(x,t)$ for (2.2.5), by differential (2.2.4), we get

$$\begin{aligned} x_{tt} &= -\frac{u_{xtt}}{u_{xx}} + \frac{2u_{xxt}u_{xt}}{u_{xx}^2} - \frac{u_{xxx}u_{xt}^2}{u_{xx}^2} \\ &= -\big(\frac{u_{xt}^2}{u_{xx}} + (u_{xx})^{\frac{1}{3}}\big)_x/u_{xx} + \frac{2u_{xxt}u_{xt}}{u_{xx}^2} - \frac{u_{xxx}u_{xt}^2}{u_{xx}^2} \\ &= -\frac{1}{3}u_{xx}^{-\frac{5}{3}}u_{xxx}, \end{aligned}$$

that is the first equation of (2.2.3) is satisfied. By the above calculation, we obtain the equivalence between the affine flow (2.2.2) and (2.2.5) under (2.2.3), that is, when the motion is normal preserving. We observe that by setting

$$p = u_{xx}, \quad q = -\frac{u_{xt}}{u_{xx}},$$

the equation (2.2.5) can be written as a conservation law

$$\begin{cases} p_t + (pq)_x = 0 \\ (pq)_t + (pq^2 + p^{\frac{1}{3}})_x = 0. \end{cases}$$

To conclude this section, we record the evolution of the curvature under this affine invariant flow. We have

$$\begin{aligned} k_{tt} &= \frac{u_{xxtt}}{(1+u_x^2)^{\frac{3}{2}}} - 2\frac{u_{xxxt}u_{xt}}{(1+u_x^2)^{\frac{3}{2}}u_{xx}} + \frac{u_{xxxx}u_{xt}^2}{(1+u_x^2)^{\frac{3}{2}}u_{xx}^2} \\ &- \frac{u_{xxx}u_{xtt}}{(1+u_x^2)^{\frac{3}{2}}u_{xx}} + 2\frac{u_{xxx}u_{xxt}u_{xt}}{(1+u_x^2)^{\frac{3}{2}}u_{xx}^2} - \frac{u_{xxx}^2u_{xt}^2}{(1+u_x^2)^{\frac{3}{2}}u_{xx}^3}. \end{aligned}$$

Since

$$\begin{aligned} u_{xtt} &= (\frac{u_{xt}^2}{u_{xx}} + (u_{xx})^{\frac{1}{3}})_x, \\ u_{xxtt} &= (\frac{u_{xt}^2}{u_{xx}} + (u_{xx})^{\frac{1}{3}})_{xx}, \end{aligned}$$

thus

$$k_{tt} = \frac{1}{3}k^{-\frac{2}{3}}k_{ss} - \frac{5}{9}k^{-\frac{5}{3}}k_s^2 + 2k_t^2 - 3k^2, \qquad (2.2.6)$$

after some computation.

2.3 Some direct consequences

For the normal flow (2.1.12), we assume $<\gamma_t, \mathbf{t}>= 0$, and $<\gamma_t, \mathbf{n}>= f$, then we have

$$\begin{aligned} \frac{\partial s}{\partial t} &= -fks, \\ \frac{\partial \mathbf{t}}{\partial t} &= f_s\mathbf{n}, \\ \frac{\partial \mathbf{n}}{\partial t} &= -f_s\mathbf{t}, \\ \frac{\partial f}{\partial t} &= k, \\ \frac{\partial k}{\partial t} &= fk^2 + f_{ss}. \end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial^2 k}{\partial t^2} &= (fk^2 + f_{ss})_t = f_t k^2 + 2fkk_t + f_{sst} \\ &= f_t k^2 + 2fkk_t + (f_{ts} - f_s f k)_s - f_{ss} f k \\ &= f_t k^2 + 2fkk_t + (f_t)_{ss} - 2f_{ss} f k - f_s^2 k \\ &= k^3 + 2fkk_t + k_{ss} - 2(k_t - fk^2) fk - f_s^2 k \\ &= k_{ss} + 2f^2 k^3 + k^3 - f_s^2 k.\end{aligned}$$

Thus

$$\frac{\partial^2 k}{\partial t^2} = k_{ss} + 2f^2 k^3 + k^3 - f_s^2 k, \tag{2.3.1}$$

where $f = < \gamma_t, \mathbf{n} >$.

Proposition 2.7. *The perimeter $\mathcal{L}(t)$ of the closed, convex curve $\gamma(\cdot, t)$ of* (2.1.12) *satisfies*

$$\frac{d\mathcal{L}(t)}{dt} = -\int_\gamma fkds$$

and

$$\frac{d^2\mathcal{L}(t)}{dt^2} = -\int_\gamma k^2 ds.$$

The area $\mathcal{A}(t)$ enclosed by the closed curve $\gamma(t)$ of (2.1.12) *satisfies*

$$\frac{d\mathcal{A}(t)}{dt} = -\int_\gamma fds,$$

and

$$\frac{d^2\mathcal{A}(t)}{dt^2} = \int_\gamma k(f^2 - 1)ds.$$

Proof. The length of γ satisfies

$$\frac{d\mathcal{L}(t)}{dt} = \int_I \frac{\partial s}{\partial t} dp = -\int_\gamma fkds,$$

and then

$$\begin{aligned}\frac{d^2\mathcal{L}(t)}{dt^2} &= -\int_\gamma f_t k + f k_t ds + \int_\gamma f^2 k^2 ds \\ &= -\int_\gamma [k^2 + f(fk^2 + f_{ss}) - f^2k^2] ds \\ &= -\int_\gamma k^2 + f_{ss} ds \\ &= -\int_\gamma k^2 ds.\end{aligned}$$

For an embedded closed solution we can use the formula for the enclosed area

$$\mathcal{A} = -\frac{1}{2}\int_\gamma < \gamma, \mathbf{n} > ds$$

to compute

$$\begin{aligned}\frac{d\mathcal{A}(t)}{dt} &= -\frac{1}{2}\int_\gamma f ds + \frac{1}{2}\int_\gamma f_s < \gamma, \mathbf{t} > ds \\ &+ \frac{1}{2}\int_\gamma fk < \gamma, \mathbf{n} > ds,\end{aligned}$$

which, after integration by parts, gives

$$\frac{d\mathcal{A}(t)}{dt} = -\int_\gamma f ds,$$

and then

$$\begin{aligned}\frac{d^2\mathcal{A}(t)}{dt^2} &= -\int_\gamma f_t ds + \int_\gamma f^2 k ds \\ &= \int_\gamma k(f^2 - 1) ds.\end{aligned}$$

□

When the normal preserving flow (2.1.17) is closed and convex for each time instant, its support function satisfies the equation

$$h_{tt} = -F. \qquad (2.3.2)$$

Hence, θ and t are independent variables and $k = 1/(h_{\theta\theta} + h)$. By a direct calculation,

$$k_t = -\frac{h_{\theta\theta t} + h_t}{(h_{\theta\theta} + h)^2},$$

and the curvature evolves according to the equation

$$k_{tt} = 2\frac{(h_{\theta\theta t}+h_t)^2}{(h_{\theta\theta}+h)^3} - \frac{h_{\theta\theta tt}+h_{tt}}{(h_{\theta\theta}+h)^2}.$$

Consider a special case where $F = k$ in (2.3.2). This equation takes a simpler form

$$k_{tt} = 2k^{-1}k_t^2 + k^2(k_{\theta\theta}+k). \tag{2.3.3}$$

Proposition 2.8. *(preserving convexity) Let k_0 be the curvature of initial curve γ_0, and set $\delta = \min_{\theta\in[0,2\pi]}\{k_0(\theta)\} > 0$. The initial velocity satisfies $k_t(0) \geq 0$. Then for a C^2-solution k of (2.3.3), one has*

$$k(\theta,t) \geq \delta,$$

for $t \in [0,T)$, where T is the maximal existence time for the solution γ of (2.1.17) for $F = k$.

Proof. Since the curvature satisfies (2.3.3), we define the operator L as follows

$$L[k] \equiv ak_{\theta\theta} + 2bk_{\theta t} + ck_{tt} + dk_\theta + ek_t, \tag{2.3.4}$$

where $a = k^2$, $b = 0$, $c = -1$, $d = 0$, $e = 2k^{-1}k_t$. a, b, c are twice continuously differentiable and d, e are continuously differentiable. By the direct computation,

$$b^2 - ac = k^2 > 0,$$

hence the operator L is defined by (2.3.5) is hyperbolic in the domain $S^1 \times [0,T)$.

We find k satisfies

$$\begin{cases} L[k-\delta] \equiv 0, & \text{in} S^1 \times [0,T_0), \\ k(\theta,0) - \delta \geq 0 & t = 0 \\ \dfrac{\partial(k-\delta)}{\partial\nu} = k_t(0) \geq 0 & t = 0. \end{cases} \tag{2.3.5}$$

Then we apply maximum principle for hyperbolic equation [PW] and conclude that

$$\delta \le k(\theta, t) \text{ in } S^1 \times [0, T_0)$$

with $T_0 \le T$. □

Proposition 2.9. *The perimeter $\mathcal{L}(t)$ of the closed, convex curve $\gamma(\cdot, t)$ of (2.1.17) satisfies*

$$\frac{d\mathcal{L}(t)}{dt} = \int_0^{2\pi} h_t d\theta$$

and

$$\frac{d^2\mathcal{L}(t)}{dt^2} = -\int_0^{2\pi} F d\theta.$$

The area $\mathcal{A}(t)$ enclosed by the closed curve $\gamma(t)$ satisfies

$$\frac{d\mathcal{A}(t)}{dt} = \int_0^{2\pi} h_t k^{-1} d\theta$$

and

$$\frac{d^2\mathcal{A}(t)}{dt^2} = \int_0^{2\pi} (-Fk^{-1} + h_t^2 - h_{\theta t}^2) d\theta.$$

Proof. By the definition of perimeter

$$\mathcal{L}(t) = \int_0^{2\pi} |\gamma_\theta| d\theta = \int_0^{2\pi} k^{-1} d\theta$$

By a direct calculation

$$\frac{d\mathcal{L}(t)}{dt} = \int_0^{2\pi} (-k^{-2}k_t) d\theta = \int_0^{2\pi} (h_{\theta\theta t} + h_t) d\theta = \int_0^{2\pi} h_t d\theta,$$

and then

$$\frac{d^2\mathcal{L}(t)}{dt^2} = \int_0^{2\pi} h_{tt} d\theta = -\int_0^{2\pi} F d\theta.$$

By the definition of area

$$\mathcal{A}(t) = -\frac{1}{2}\int_0^{2\pi} \langle \gamma, \mathbf{n} \rangle ds = \frac{1}{2}\int_0^{2\pi} hk^{-1} d\theta.$$

By a direct calculation

$$\begin{aligned}\frac{d\mathcal{A}(t)}{dt} &= \frac{1}{2}\int_0^{2\pi}(h_t k^{-1} - hk^{-2}k_t)d\theta = \frac{1}{2}\int_0^{2\pi}(h_t k^{-1} - h(h_{\theta\theta t} + h_t))d\theta \\ &= \frac{1}{2}\int_0^{2\pi}(h_t k^{-1} - h_t(h_{\theta\theta} + h))d\theta = \int_0^{2\pi} h_t k^{-1} d\theta,\end{aligned}$$

and then

$$\begin{aligned}\frac{d^2\mathcal{A}(t)}{dt^2} &= \int_0^{2\pi} h_{tt}k^{-1} - h_t k^{-2}k_t d\theta \\ &= \int_0^{2\pi}[-Fk^{-1} + h_t(h_{\theta\theta t} + h_t)]d\theta = \int_0^{2\pi}[-Fk^{-1} + h_t^2 - h_{\theta t}^2]d\theta.\end{aligned}$$

□

Proposition 2.10. *When $F = k$ in the normal preserving flow* (2.1.17), *any solution of this flow will blow up in finite time.*

Proof. When $F = k$ in the normal preserving flow (2.1.17), we have

$$\frac{d^2\mathcal{L}(t)}{dt^2} = -\int_0^{2\pi} k d\theta = -\int k^2 ds.$$

Using Cauchy-Schwarz inequality,

$$\int k^2 ds \int ds \geq (\int kds)^2 = 4\pi^2,$$

$$\frac{d^2\mathcal{L}}{dt^2} \leq -\frac{4\pi^2}{\mathcal{L}}.$$

If at some instant t_0, $\frac{d}{dt}\mathcal{L}(t_0) \leq 0$, then $d\mathcal{L}/dt$ and $\mathcal{L}$ will be decease for all $t \geq t_0$. It follows that $\frac{d^2\mathcal{L}}{dt^2} \leq -\frac{4\pi^2}{\mathcal{L}(t_0)}$ and $\mathcal{L}$ becomes zero in finite time. On the other hand, when $d\mathcal{L}/dt(t) > 0$ for all t, we have

$$\frac{1}{2}\frac{d}{dt}(\frac{d\mathcal{L}}{dt})^2 \leq -4\pi^2\frac{d}{dt}(\log \mathcal{L}),$$

as so,

$$(\frac{dL}{dt})^2(t) \leq (\frac{d\mathcal{L}}{dt})^2(t_0) - 8\pi^2 \log\frac{\mathcal{L}(t)}{\mathcal{L}(t_0)}.$$

It shows that $\mathcal{L}(t)$ cannot expand to infinity, and $\frac{d^2\mathcal{L}(t)}{dt^2} \leq -\frac{4\pi^2}{c_0}$, for some $c_0 > 0$. But then

$$\frac{d\mathcal{L}}{dt}(t) \leq \frac{d\mathcal{L}}{dt}(t_0) - \frac{4\pi^2}{c_0}(t - t_0),$$

contradiction holds as $t \to \infty$. □

Chapter 3

Group Invariant Solutions

In this chapter, we present a systematic study on the group invariant solutions for the following flows:

The normal hyperbolic flow:

$$\frac{\partial^2 \gamma}{\partial t^2} = k\mathbf{n} - \langle \gamma_t, \gamma_{ts} \rangle \mathbf{t}; \tag{3.1}$$

The normal preserving hyperbolic flow:

$$\frac{\partial^2 \gamma}{\partial t^2} = k\mathbf{n} - \frac{1}{k} k_s \mathbf{t}; \tag{3.2}$$

The affine hyperbolic flow:

$$\frac{\partial^2 \gamma}{\partial t^2} = \mathcal{N}, \tag{3.3}$$

where $\mathcal{N} = k^{\frac{1}{3}}\mathbf{n} - \frac{1}{3}k^{-\frac{5}{3}}k_s\mathbf{t}$ is the affine normal, $\mathbf{n}$, $\mathbf{t}$ are the Euclidean unit normal and tangent, and s is the Euclidean arc-length parameter.

When the plane curve $\gamma(\cdot, t)$ is given locally as a graph of the form $(x, u(x,t))$, each (3.1), (3.2), (3.3) are converted to

$$u_{tt} = \frac{1 + u_x^2 - u_x^2 u_t^2}{(1 + u_x^2)^2} u_{xx} + 2\frac{u_x u_t}{1 + u_x^2} u_{xt}, \tag{3.4}$$

$$u_{tt} u_{xx} - u_{xt}^2 = \frac{u_{xx}^2}{1 + u_x^2}, \tag{3.5}$$

and

$$u_{tt}u_{xx} - u_{xt}^2 = u_{xx}^{\frac{4}{3}}, \tag{3.6}$$

respectively.

A study for these hyperbolic equations have begun. We shall first determine the groups of symmetries of these equations. As these are, Euclidean invariant, the Euclidean group forms a invariant group in it. Furthermore, being not dependent on the time explicitly shows that the translation in time is also a one-parameter group. We shall determine the group of symmetries by determining the Lie algebra of the symmetries. The prolongation formula will be needed in the latter.

3.1 Normal hyperbolic flow

In this section, we attempt to investigate on group invariant solution of normal hyperbolic flow (3.1), with the graph equation (3.4). In Lie's theory of symmetry groups for differential equations a one-parameter group of symmetries is a family of local diffeomorphisms

$$\tilde{x} = \sum_{\epsilon}(x, u),$$
$$\tilde{u} = \Phi_\epsilon(x, u), \ \epsilon \ small,$$

satisfying $x = \sum_0(x, u)$ and $u = \Phi_0(x, u)$ which preserve solutions of (3.4). The vector field

$$\mathbf{v} = \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial u},$$

where

$$\xi(x, u) = \frac{\partial}{\partial \epsilon}|_{\epsilon=0}\sum_{\epsilon}(x, u),$$
$$\phi(x, u) = \frac{\partial}{\partial \epsilon}|_{\epsilon=0}\Phi_\epsilon(x, u),$$

is the infinitesimal symmetry for the 1-parameter group. In order to obtain all group invariant solutions of (3.4), we first determine the Lie algebra of all infinitesimal symmetries. Since the equation is geometric, it must admit the Euclidean motions (translations in x and u, rotation in $x-u$ but not the reflection because it is discrete) as its symmetries. Furthermore, being not dependent on t explicitly means that it admits translation in t. Finally, the special form of the equation suggests that it admits a certain scaling invariance.

Theorem 3.1. *The Lie algebra of all infinitesimal symmetries of* (3.4) *is spanned by*

$$\{\partial_x,\ \partial_u,\ \partial_t,\ -u\partial_x + x\partial_u,\ x\partial_x + u\partial_u + t\partial_t\}.$$

Proof. Let

$$\mathbf{v} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u} \tag{3.7}$$

be a vector field on the space (x,t,u). We wish to determine all possible coefficient functions ξ, τ and ϕ so that the corresponding one-parameter group $exp(\epsilon\mathbf{v})$ is a symmetry group of the equation. According to Theorem 2.31 in [O], $\mathbf{v}$ generates a symmetry of the equation (3.4) if and only if

$$\mathrm{pr}^2\mathbf{v}(u_{tt} - F) = 0 \ \textit{on}\ u_{tt} - F = 0,$$

where $\mathrm{pr}^{(2)}\mathbf{v}$ is the second prolongation of $\mathbf{v}$,

$$\mathrm{pr}^{(2)}\mathbf{v} = v + \phi^x\frac{\partial}{\partial u_x} + \phi^t\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xt}} + \phi^{xx}\frac{\partial}{\partial u_{xt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}},$$

and

$$
\begin{aligned}
\phi^x &= D_x(\phi - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{xt} \\
&= \phi_x + (\phi_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t, \\
\phi^t &= D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{tt} \\
&= \phi_t - \xi_t u_x + (\phi_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2, \\
\phi^{xx} &= D_x^2(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt} \\
&= \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\phi_{uu} - 2\xi_{xu})u_x^2 \\
&- 2\tau_{xu}u_x u_t - \xi_{uu}u_x^3 - \tau_{uu}u_x^3 u_t \\
&+ (\phi_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}, \qquad (3.8) \\
\phi^{tt} &= D_t^2(\phi - \xi u_x - \tau u_t) + \xi u_{xtt} + \tau_u u_{ttt} \\
&= \phi_{tt} + (2\phi_{tu} - \tau_{tt})u_t - \xi_{tt}u_x + (\phi_{uu} - 2\tau_{tu})u_t^2 \\
&- 2\xi_{tu}u_x u_t - \tau_{uu}u_t^3 - \xi_{uu}u_x u_t^3 \\
&+ (\phi_u - 2\tau_t)u_{tt} - 2\xi_t u_{xt} - 3\tau_u u_t u_{tt} - \xi_u u_x u_{tt} - 2\xi_u u_t u_{xt}, \\
\phi^{xt} &= D_{xt}(\phi - \xi u_x - \tau u_t) + \xi u_{xxt} + \tau u_{xtt} \\
&= \phi_{xt} + (\phi_{tu} - \xi_{xt})u_x + (\phi_{xu} - \tau_{xt})u_t \\
&- \xi_{tu}u_x^2 + (\phi_{uu} - \xi_{xu} - \tau_{tu})u_x u_t - \tau_{xu}u_t^2 - \xi_{uu}u_x^2 u_t \\
&- \tau_{uu}u_x u_t^2 + \phi_u u_{xt} - (\xi_u + 2\xi_u u_x)u_{xt} \\
&- (\tau_u + 2\tau_u u_t)u_{xt} - (\tau_x + \tau_u u_x)u_{tt} - (\tau_t + 2\xi_u u_t)u_{xx}
\end{aligned}
$$

For simplify, we set

$$
\begin{aligned}
\phi^{tt} &= A^1 u_{tt} + B^1 u_{xt} + C^1, \\
\phi^{xx} &= A^2 u_{xx} + B^2 u_{xt} + C^2, \\
\phi^{xt} &= A^3 u_{tt} + B^3 u_{xx} + C^3 u_{xt} + D, \\
u_{tt} &= f(u_x, u_t)u_{xx} + g(u_x, u_t)u_{xt}, \qquad (3.9)
\end{aligned}
$$

where A^i, B^i, C^i, and D depend on lower order terms.

Applying the second prolongation $\mathrm{pr}^2\mathbf{v}$ to (3.4), we find that ξ, τ, ϕ must satisfy the symmetry conditions.

$$\begin{aligned}\phi^{tt} &= f(u_x,u_t)\phi^{xx} + g(u_x,u_t)\phi^{xt} \\ &+ [f_{u_x}u_{xx} + g_{u_x}u_{xt}]\phi^x + (f_{u_t}u_{xx} + g_{u_t}u_{xt})\phi^t, \end{aligned} \tag{3.10}$$

which must be satisfied whenever $u_{tt} = fu_{xx}+gu_{xt}$. Substituting (3.8) into (3.10), replacing u_{tt} by $fu_{xx} + gu_{xt}$, we have

$$\begin{aligned}A^1(fu_{xx} &+ gu_{xt}) + B^1u_{xt} + C^1 = f(A^2u_{xx} + B^2u_{xt} + C^2) \\ &+ g[A^3(fu_{xx} + gu_{xt}) + B^3u_{xx} + C^3u_{xt} + D] \\ &+ [f_{u_x}u_{xx} + g_{u_x}u_{xt}]\phi^x + (f_{u_t}u_{xx} + g_{u_t}u_{xt})\phi^t. \end{aligned} \tag{3.11}$$

To solve (3.11), we look at the terms involving the mixed second order partial derivatives of u, namely u_{xx}, u_{xt}, each of which occurs linearly on the left-hand side. We find the defining equations for the symmetry group to be the following:

Monomial u_{xx}, coefficient

$$A^1f = fA^2 + g(A^3f + B^3) + f_{u_x}\phi^x + f_{u_t}\phi^t.$$

Monomial u_{xt}, coefficient

$$A^1g + B^1 = fB^2 + g(A^3g + C^3) + g_{u_x}\phi^x + g_{u_t}\phi^t.$$

Monomial 1, coefficient

$$C^1 = fC^2 + gD.$$

Multiplying these equations by $(1+u_x^2)^2$, and then setting the coefficients of $u_x^iu_t^j$ to zero in the above equations gives us certain determining systems for ξ, τ, ϕ. Therefore, after some calculation, we obtain

$$\tau_x = \tau_u = 0,$$

$$\xi_t = 0,$$

$$\phi_x = -\xi_u,$$

$$\phi_u = \xi_x$$

Thus we get

$$\xi = ax - bu + c,$$
$$\tau = at + d,$$
$$\phi = au + bx + e,$$

where a, b, c, d, e are constants. Finally we conclude that the Lie algebra of infinitesimal symmetries of the equation is spanned by the following five vector fields

$$\mathbf{v}_1 = \partial_x,$$
$$\mathbf{v}_2 = \partial_u,$$
$$\mathbf{v}_3 = \partial_t,$$
$$\mathbf{v}_4 = -u\partial_x + x\partial_u,$$
$$\mathbf{v}_5 = x\partial_x + u\partial_u + t\partial_t.$$

The proof is completed. □

In the study of group invariant solution it is more convenient to use the support function and normal angel to describe the flow sometimes. We have expressed the flow (3.1) as the equation of support function in Chapter 1, that is

$$h_{tt} = \frac{h_{t\theta}^2 - 1}{h_{\theta\theta} + h}. \tag{3.12}$$

When the curve γ is represented as a graph and described by the support function simultaneously, the following relations hold.

$$h = \frac{u - xu_x}{(1 + u_x^2)^{1/2}},$$
$$\tan\theta = -\frac{1}{u_x},$$

and

$$h_\theta = \frac{-x - uu_x}{(1+u_x^2)^{1/2}}.$$

Therefore,

$$\partial_x = -\frac{u_x}{(1+u_x^2)^{1/2}}\partial_h - \frac{1}{(1+u_x^2)^{1/2}}\partial_{h_\theta},$$
$$\partial_u = \frac{1}{(1+u_x^2)^{1/2}}\partial_h - \frac{u_x}{(1+u_x^2)^{1/2}}\partial_{h_\theta},$$
$$\partial_{u_x} = \frac{1}{1+u_x^2}\partial_\theta + \frac{-x-uu_x}{(1+u_x^2)^{\frac{3}{2}}}\partial_h + \frac{-u+xu_x}{(1+u_x^2)^{\frac{3}{2}}}\partial_{h_\theta}.$$

Now we can convert vector fields on the jet space (x, t, u, u_x) to the jet space (θ, t, h, h_θ) using these formulas. The following table shows the conversion. Of course, one can also compile it by applying the infinitesimal criterion to (3.12).

Table 3.1: Infinitesimal symmetries in (x, u, t) and (θ, h, t)

$u_{tt} = \frac{1+u_x^2-u_x^2u_t^2}{(1+u_x^2)^2}u_{xx} + 2\frac{u_xu_t}{1+u_x^2}u_{xt}$	$h_{tt} = \frac{h_{t\theta}^2 - 1}{h_{\theta\theta}+h}$
∂_x	$\cos\theta\partial_h$
∂_u	$\sin\theta\partial_h$
$-u\partial_x + x\partial_u$	∂_θ
∂_t	∂_t
$x\partial_x + u\partial_u + t\partial_t$	$t\partial_t + h\partial_h$

For each one-parameter subgroup of the full symmetry group there will be a corresponding class of group -invariant solutions which will be determined from a reduced ordinary differential equation. Given a group action G, there exist functionally independent invariants of form $y = \zeta^1(x, t, u)$, $v = \zeta^2(x, t, u)$. If we treat v as a function of y, we can compute formulae for the derivatives of u with respect to x and t in terms of y, v and the derivatives of v with respect to y. Once the relevant formulae relating derivatives of u with respect to x to those of v with respect to y have been determined, the reduced system of differential equations for the G-invariant solutions to the equation is determined by substituting these expressions into the equation whereever they occur. Upon substituting, we find the reduced ODE. After solving the reduced equation, for each solution $v = \phi(y)$ of the reduced equation there corresponds a G- invariant solution $u = f(x,t)$ of the original equation, which is given implicitly by the relation

$$\zeta^2(x,t,u) = \phi[\zeta^1(x,t,u)].$$

Now, let's determine the group invariant solutions of (3.1) for some 1-parameter subgroups of symmetries.

(a) *Self-Similar solutions.* First we take $\mathbf{v} = x\partial_x + t\partial_t + u\partial_u$, or $t\partial_t + h\partial_h$, in terms of support function. Two invariants are h/t and θ. Hence, invariant solution is of the form $h(\theta,t)/t = \phi(\theta)$, or $h(\theta,t) = t\cdot\phi(\theta)$. This is the self-similar solutions. Under the flow, the shape of the curve remains unchanged but it magnitude enlarges or shrinks. Putting this into equation (3.12) yields

$$0 = \phi'(\theta)^2 - 1.$$

It can be solved and

$$\phi(\theta) = \pm\theta + C,$$

where C is a constant. As a typical curve we take $\phi(\theta)$ in the following discussion. The self-similar solution, as a curve in (x,u)-plane, is given by

$$\begin{cases} x = \theta\cos\theta - \sin\theta, \\ u = \theta\sin\theta + \cos\theta. \end{cases}$$

This is a semi-infinite curve, which, at $\theta = 0$, passes $(0,1)$ with a vertical tangent and curls to infinity as θ tends to ∞, see Figure 3.1. One may connect this curve with the one starting at $(0,1)$ to form a complete curve. This C^1 curve satisfies the equation for the flow everywhere except at $(0,\pm1)$ and is a candidate for a weak solution, see Figure 3.2.

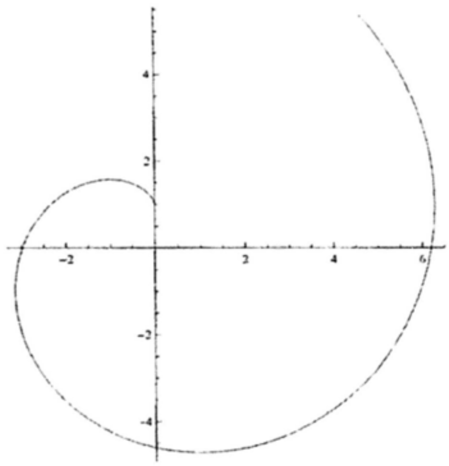

Figure 3.1: Self-similar solution

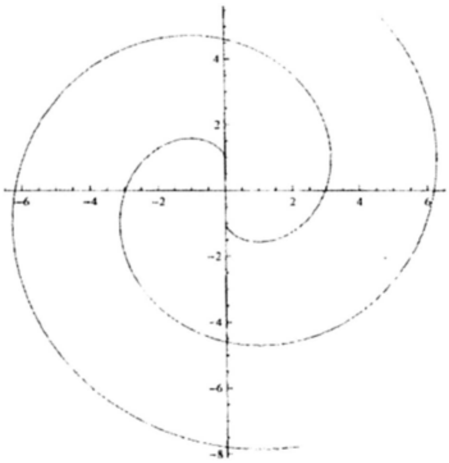

Figure 3.2: Self-similar solution

(b) *Circles.* Next, we take $\mathbf{v} = -u\partial_x + x\partial_u$. The resulting solutions are circles. Let γ_0 be a circle of radius r_0. If γ is a solution of (3.1) with the initial value $\gamma(x,0) = \gamma_0$, and $\frac{d}{dt}\gamma(x,0) = -r_1\mathbf{n}$. Then the curve is a circle with radius r, and the flow reduces to the ordinary differential equation

$$\begin{cases} r'' = -\dfrac{1}{r}, \\ r(0) = r_0 > 0, \ r_t(0) = r_1. \end{cases} \tag{3.13}$$

More details about (3.13) are discussed in Chapter 4. When $r_1 \leq 0$, the circles contract to a point. When $r_1 > 0$, the circles expand first, then contract to a point, see Figure 3.3, 3.4.

(c) *Spiral.* Finally, we consider $\mathbf{v} = -u\partial_x + x\partial_u - \partial_t = \partial_\theta - \partial_t$. Two invariants are h and $\theta - t$. Hence, the invariant solution is of the form $h(\theta, t) = \phi(\theta - t)$. Invariant solutions with respect to this group are called spirals. The resulting solution is a curve rotating around the origin with speed 1. A direct computation yields

$$h_{tt} = \phi'', \quad h_{\theta\theta} = \phi'', \quad h_{\theta t} = -\phi'',$$

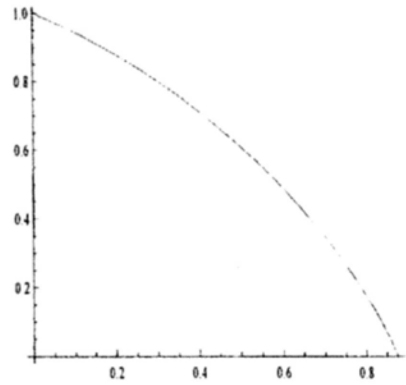

Figure 3.3: $r_1 \leq 0$

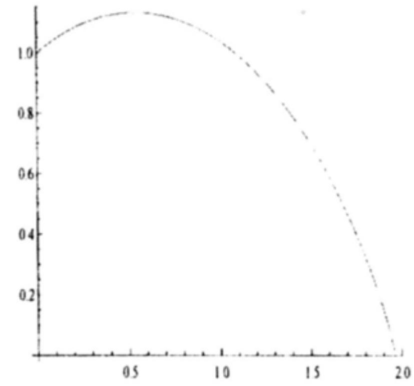

Figure 3.4: $r_1 > 0$

hence

$$\phi'' = -1/\phi. \tag{3.14}$$

We multiple ϕ' on the both sides of (3.14), then

$$\frac{\phi'^2}{2} = C - \ln|\phi|,$$

i.e.

$$\phi_\theta = \pm\sqrt{2}\sqrt{C - \ln|\phi|}.$$

As $h(\theta, t) = \phi(\theta - t)$, we may assume $\phi > 0$ at some θ. Take

$$\phi_\theta = \sqrt{2}\sqrt{C - \ln\phi},$$

then $C - \ln\phi$ means ϕ is increasing as θ increases, until $C = \ln\phi(\theta_1)$, $\phi_\theta(\theta_1) = 0$, $\phi(\theta) = e^C$. Consider, $\phi_\theta = -\sqrt{2}\sqrt{C - \ln\phi}$, then ϕ decreases as t increases. We extend ϕ to a smooth function on $[\theta_0, \theta_2]$, where $\phi'(\theta_1) = 0$, ϕ' blows up at θ_0, θ_2. We can also compute $\theta_2 - \theta_1$, since

$$d\theta = \frac{d\phi}{\sqrt{2C - \ln\phi^2}},$$

so

$$\begin{aligned} \theta_2 - \theta_1 &= 2\int_0^{e^C} \frac{d\phi}{\sqrt{2C - \ln\phi^2}} \\ &= 2\int_0^1 \frac{e^C d\psi}{\sqrt{-\ln(\psi)}} = e^C\sqrt{2\pi}. \end{aligned}$$

We have the figure of $\phi(\theta)$, see Figure 3.5.

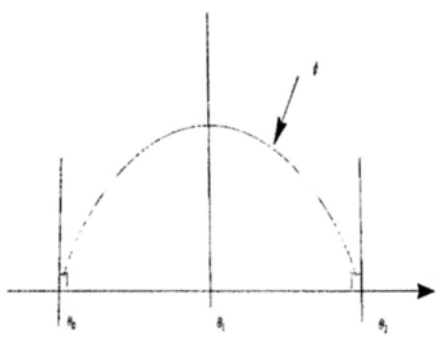

Figure 3.5: ϕ

The curve reads as (x, y) using support function ϕ and normal angle θ, $x = \phi\cos\theta - \phi'\sin\theta$, $y = \phi\sin\theta + \phi'\cos\theta$. We know that $(x(\theta), y(\theta))$ as a smooth plane curve needs $x'(\theta), y'(\theta) \neq 0$, at least, $x' = -(\phi''+\phi)\sin\theta$, $y' = (\phi''+\phi)\cos\theta$, when $\phi'' + \phi = -\frac{1}{\phi} + \phi = 0$, i.e. $\phi = \pm 1$, $x' = y' = 0$, we get a singularity.

Set $\theta_2 - \theta_1 = \Theta$, then $\max\limits_{\theta\in[\theta_0,\theta_2]} \phi(\theta) = \phi(\theta_1) = \Theta/\sqrt{2\pi}$. When $\Theta/\sqrt{2\pi} < 1$, there is no singularity, the graph of γ is hyperbole in a cone, which is showed in Figure 3.6.

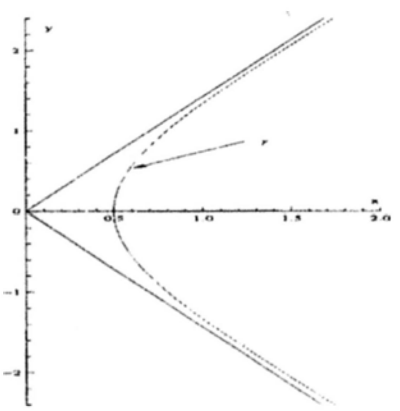

Figure 3.6: Spiral-1

When $\Theta/\sqrt{2\pi} \geq 1$, there are two singularities. We choose two examples and the show the correspoinding graphs of the curves in Figure 3.7, 3.8.

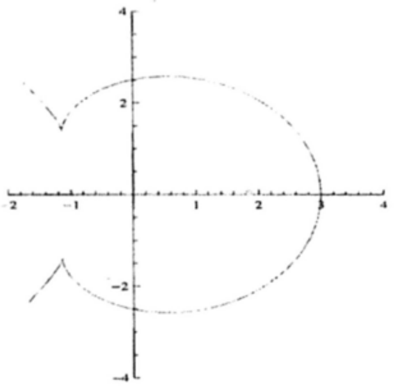

Figure 3.7: Spiral-2

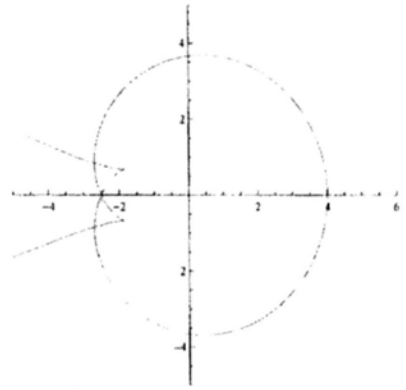

Figure 3.8: Spiral-3

3.2 Normal preserving hyperbolic flow

In this section, we attempt to investigate on group invariant solution of normal hyperbolic flow (3.2), with the graph equation (3.5).

Theorem 3.2. *The Lie algebra of all infinitesimal symmetries of* (3.5) *is spanned by*

$$\{\partial_x,\ \partial_u,\ \partial_t,\ -u\partial_x + x\partial_u, x\partial_x + u\partial_u + t\partial_t\}.$$

Proof. A typical vector field on the space of independent and dependent variables takes the form

$$\mathbf{v} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u},$$

where ξ, τ, ϕ depend on x, t, u. The coefficients of the second prolongation of $\mathbf{v}$, ϕ^{xx}, ϕ^{tt}, ϕ^{xt} are determined as the same in Theorem 3.1. A vector field $\mathbf{v}$ generates a one-parameter symmetry group if

$$\phi^{tt} = \frac{2u_{xt}}{u_{xx}}\phi^{xt} + (\frac{1}{1+u_x^2} - \frac{u_{xt}^2}{u_{xx}^2})\phi^{xx} - \frac{2u_x u_{xx}}{(1+u_x^2)^2}\phi^x, \qquad (3.15)$$

whenever u satisfies (3.5).

For simplify, we set

$$\phi^{tt} = A^1 u_{tt} + B^1 u_{xt} + C^1,$$
$$\phi^{xx} = A^2 u_{xx} + B^2 u_{xt} + C^2,$$
$$\phi^{xt} = A^3 u_{tt} + B^3 u_{xx} + C^3 u_{xt} + D,$$

where A^i, B^i, C^i, and D depend on lower order terms. (3.15) turn to be

$$\begin{aligned} u_{xx}^2(A^1 u_{tt} &+ B^1 u_{xt} + C^1) = 2u_{xt}u_{xx}(A^3 u_{tt} + B^3 u_{xx} + C^3 u_{xt} + D) \\ &+ (\frac{u_{xx}^2}{1+u_x^2} - u_{xt}^2)(A^2 u_{xx} + B^2 u_{xt} + C^2) - \frac{2u_x u_{xx}^3}{(1+u_x^2)^2}\phi^x. \end{aligned}$$

Replacing u_{tt} by $u_{xt}^2/u_{xx} + u_{xx}/(1+u_x^2)$ whenever it occurs, we have

$$\begin{aligned} u_{xx}^2[A^1(u_{xt}^2 &+ \frac{u_{xx}^2}{1+u_x^2}) + B^1 u_{xx}u_{xt} + C^1 u_{xx}] = 2u_{xt}u_{xx}[A^3(u_{xt}^2 + \frac{u_{xx}^2}{1+u_x^2}) \\ &+ B^3 u_{xx}^2 + C^3 u_{xt}u_{xx} + Du_{xx}] + (\frac{u_{xx}^2}{1+u_x^2} - u_{xt}^2)[A^2 u_{xx}^2 \\ &+ B^2 u_{xt}u_{xx} + C^2 u_{xx}] - \frac{2u_x u_{xx}^4}{(1+u_x^2)^2}\phi^x. \end{aligned} \tag{3.16}$$

To solve (3.16), we look first at the terms involving the various monomials in the second order derivatives of u, namely u_{xx}, u_{xt}, each of which occurs linearly on the left-hand side.

The coefficient of u_{xx}^4 is

$$\frac{A^1}{1+u_x^2} = \frac{A^2}{1+u_x^2} - \frac{2u_x\phi^x}{(1+u_x^2)^2}. \tag{3.17}$$

The coefficient of $u_{xx}^3 u_{xt}$ is

$$B^1 = 2\frac{A^3}{1+u_x^2} + 2B^3 + \frac{B^2}{1+u_x^2}. \tag{3.18}$$

The coefficient of $u_{xx}^2 u_{xt}^2$ is

$$A^1 = 2C^3 - A^2. \tag{3.19}$$

The coefficient of $u_{xx}u_{xt}^3$ is

$$0 = 2A^3 - B^2. \tag{3.20}$$

The coefficient of u_{xx}^3 is

$$C^1 = \frac{C^2}{1+u_x^2}. \tag{3.21}$$

The coefficient of $u_{xx}^2 u_{xt}$ is

$$0 = 2D. \tag{3.22}$$

The coefficient of $u_{xx}u_{xt}^2$ is

$$0 = -C^2. \tag{3.23}$$

(3.17) can be reduced to

$$\begin{aligned}(2\xi_x - 2\tau_t + 2u_x\xi_u - 2\tau_u u_t)(1+u_x^2) &= -2u_x[\phi_x + (\phi_u u - \xi_x)u_x \\ &- \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t],\end{aligned}$$

which implies

$$\begin{aligned}\xi_u &= -\phi_x, \\ \xi_x &= \tau_t, \\ \tau_u &= 0, \\ \phi_u &= \xi_x = \tau_t.\end{aligned}$$

(3.18) implies

$$\begin{aligned}\tau_x &= 0, \\ \tau_u &= 0.\end{aligned}$$

(3.21) implies

$$\begin{aligned}\phi_{tt} &= \phi_{xx}, \\ -\xi_{tt} &= 2(\phi_{xu} - \xi_{xx}), \\ \xi_{uu} &= \xi_{tt}.\end{aligned}$$

Finally, we have

$$\xi = ax - bu + c,$$
$$\tau = at + d,$$
$$\phi = au + bx + e,$$

where a, b, c, d, e are constants. Therefore, we conclude that the Lie algebra of infinitesimal symmetries of the equation is spanned by the following five vector fields

$$\mathbf{v}_1 = \partial_x,$$
$$\mathbf{v}_2 = \partial_u,$$
$$\mathbf{v}_3 = \partial_t,$$
$$\mathbf{v}_4 = -u\partial_x + x\partial_u,$$
$$\mathbf{v}_5 = x\partial_x + u\partial_u + t\partial_t.$$

□

The corresponding solutions of group $\{x\partial_x + t\partial_t + u\partial_u\}$ are straight lines. There is no support function for straight line. While if we expressed the flow as the equation of support function

$$h_{tt} = -\frac{1}{h_{\theta\theta} + h}.$$

We find that it is analog as the curve shortening problem(CSP) $h_t = -1/(h_{\theta\theta}+h)$. Thus we seek self-similar solutions which are form of $\gamma = \lambda(t)\hat{\gamma}(\cdot)$ as CSP. Then

$$\lambda''\lambda\gamma \cdot \mathbf{n} = k.$$

When this curve is not flat, similar as in Section 2.4 [CZ], $\lambda''\lambda$ must be a non-zero constant. After a rescaling, we may simply assume the constant is 1 (expanding self-similar solution) and -1 (contracting self-similar solution). When it is 1, $\lambda(t)$ expands for $\lambda_t(0) \geq 0$, and contracts first then expand for $\lambda_t(0) < 0$. When it is -1, $\lambda(t)$ contracts for $\lambda_t(0) \leq 0$, and expands first then contracts for $\lambda_t(0) > 0$. The reduced equation are the same as the CSP. The only difference is the deforming velocity. We refer [CZ] for more information.

Figure 3.9: Expanding self-similar curve

Figure 3.10: Contracting self-similar 3-petal curve

Since the reduced equation of Rotation Group of (3.4) and (3.5) are the same, then the corresponding group invariant solution are the same too.

3.3 Affine hyperbolic flow

In this section, we attempt to give an investigation on group invariant solutions of affine hyperbolic flow (3.3) with the graph equation (3.6). By the standard method we list in the first section, we obtain the following theorem.

Theorem 3.3. *The Lie algebra of all infinitesimal symmetries of* (3.6) *is spanned by*

$$\{\partial_x,\ \partial_t,\ \partial_u, u\partial_x,\ x\partial_u,\ x\partial_x - u\partial_u,\ 3x\partial_x + 2t\partial_t + 3u\partial_u\}.$$

Proof. The symmetry group of (3.6) will again be generated by vector fields of the form (3.7). Applying the second prolongation $pr^{(2)}\mathbf{v}$ to (3.6), we find that ξ, τ, ϕ must satisfy the symmetry conditions

$$u_{xx}\phi^{tt} + u_{tt}\phi^{xx} = 2u_{xt}\phi^{xt} + \frac{4}{3}u_{xx}^{1/3}\phi^{xx}, \tag{3.24}$$

where the coefficients ϕ^{xx}, ϕ^{xt}, ϕ^{tt} were determined in Theorem 3.1. Substituting u_{tt} by $u_{xt}^2/u_{xx} + u_{xx}^{1/3}$ whenever it occurs, we have

$$\begin{aligned} u_{xx}[A^1(u_{xt}^2 + u_{xx}^{4/3}) &+ B^1 u_{xt}u_{xx} + C^1 u_{xx}] + (u_{xt}^2 + u_{xx}^{4/3})(A^2 u_{xx} + B^2 u_{xt} + C^2) \\ &= 2u_{xt}[A^3(u_{xt}^2 + u_{xx}^{4/3}) + B^3 u_{xx}^2 + C^3 u_{xt}u_{xx} + Du_{xx}] \\ &+ \frac{4}{3}u_{xx}^{4/3}(A^2 u_{xx} + B^2 u_{xt} + C^2), \end{aligned} \tag{3.25}$$

The coefficient of $u_{xx}u_{xt}^2$ is

$$A^1 + A^2 = 2C^3. \tag{3.26}$$

The coefficient of $u_{xx}^{7/3}$ is

$$A^1 + A^2 = 4A^2/3. \tag{3.27}$$

The coefficient of $u_{xx}^2 u_{xt}$ is

$$B^1 = 2B^3. \tag{3.28}$$

The coefficient of u_{xx}^2 is

$$C^1 = 0. \tag{3.29}$$

The coefficient of u_{xt}^3 is

$$B^2 = 2A^3. \tag{3.30}$$

The coefficient of u_{xt}^2 is

$$C^2 = 0. \tag{3.31}$$

The coefficient of $u_{xx}^{4/3}u_{xt}$ is

$$B^2 = 2A^3. \tag{3.32}$$

The coefficient of $u_{xx}^{4/3}$ is

$$C^2 = 4C^2/3. \tag{3.33}$$

The coefficient of $u_{xx}u_{xt}$ is

$$0 = 2D. \tag{3.34}$$

(3.26), (3.28), (3.30), (3.32) are trivial. (3.27) implies

$$\phi_u = 3\tau_t - \xi_x,$$
$$\tau_u = 0.$$

(3.29) implies

$$\phi_{tt} = 0,$$
$$2\phi_{tu} - \tau_{tt} = 0,$$
$$-\xi_{tt} = 0,$$
$$\phi_{uu} - 2\tau_{tu} = 0$$
$$-2\xi_{tu} = 0,$$
$$-\tau_{uu} = 0,$$
$$-\xi_{uu} = 0.$$

(3.31) implies

$$\phi_{xx} = 0,$$
$$2\phi_{xu} - \xi_{xx} = 0,$$
$$-\tau_{xx} = 0,$$
$$\phi_{uu} - 2\xi_{xu} = 0$$
$$-2\tau_{xu} = 0,$$
$$-\xi_{uu} = 0,$$
$$-\tau_{uu} = 0,$$

(3.34) implies

$$\begin{aligned}
&\phi_{xt} = 0, \\
&\phi_{tu} - \xi_{xt} = 0, \\
&\phi_{xu} - \tau_{xt} = 0, \\
&-\xi_{tu} = 0, \\
&\phi_{uu} - \xi_{xu} - \tau_{tu} = 0, \\
&-\tau_{xu} = 0, \\
&-\xi_{uu} = 0, \\
&-\tau_{uu} = 0
\end{aligned}$$

Therefore

$$\begin{aligned}
\xi &= ax + bu + c, \\
\phi &= dx + eu + p, \\
\tau &= \frac{a+e}{3}t + q,
\end{aligned}$$

where a, b, c, d, e, p, q are constants. Now it is routine to check Theorem 3.3 holds. □

For the convenience, we use the support function and normal angel to describe the flow if the curve is convex. We have expressed the flow as the equation of support function in Chapter 1, it reads as

$$h_{tt} = -(\frac{1}{h_{\theta\theta} + h})^{\frac{1}{3}}. \tag{3.35}$$

We can also convert vector fields on the jet space (x, t, u, u_x) to the jet space (θ, t, h, h_θ). The following table shows the conversion for the simply ones.

Next, we shall list the reduced equations for these groups in the above table.

Table 3.2: Infinitesimal symmetries in (x, u, t) and (θ, h, t)

$u_{tt} = \frac{u_{xt}^2}{u_{xx}} + (u_{xx})^{\frac{1}{3}}$	$h_{tt} = -(\frac{1}{h_{\theta\theta} + h})^{\frac{1}{3}}$
∂_x	$\cos\theta\partial_h$
∂_u	$\sin\theta\partial_h$
$-u\partial_x + x\partial_u$	∂_θ
∂_t	∂_t
$3x\partial_x + 3u\partial_u + 2t\partial_t$	$2t\partial_t + 3h\partial_h$

(a) *Self-Similar solution.* We seek solutions of this flow whose shapes change homothetically during the evolution: $\hat{\gamma} = \lambda(t)\gamma(\cdot)$. $\hat{\gamma}$ is a self-similar solution if and only if

$$\lambda(t)''\lambda(t)^{\frac{1}{3}}\gamma \cdot \mathbf{n} = k^{\frac{1}{3}}.$$

When this curve is not flat, $\lambda(t)''\lambda(t)^{\frac{1}{3}}$ must be a non-zero constant. After a rescaling, we may assume the constant is $3/4$, and $-3/4$. On the other hand, the corresponding group of self-similar solutions is $\{3x\partial_x + 3u\partial_u + 2t\partial_t\}$. From Table 3.2, we get the corresponding group action of $\{2t\partial_t + 3h\partial_h\}$ is $(\theta, t, h) \to (\theta, e^{2\epsilon}t, e^{3\epsilon}h)$. Hence a self-similar solution could be taken to be $t^{3/2}h(\theta)$. That is $\lambda(t)''\lambda(t)^{\frac{1}{3}} = 3/4$. Plugging this into equation yields

$$\frac{3}{4}h = -(\frac{1}{h_{\theta\theta} + h})^{1/3}. \tag{3.36}$$

When $\lambda(t)''\lambda(t)^{\frac{1}{3}} = -3/4$, the support function satisfies

$$\frac{3}{4}h = (\frac{1}{h_{\theta\theta} + h})^{1/3}, \tag{3.37}$$

We call the former an expanding self-similar solution (λ expands to infinite) and the latter a contracting self-similar solution (λ tends to 0 at finite time). We shall study (3.36) and (3.37) separately.

First, a typical solution subject the initial conditions $h(0) = -\alpha$, $\alpha > 0$ and $h_\theta(0)$ of (3.36) is an even, convex function which is strictly increasing in $(0, \theta_0)$ where θ_0 is the zero of h and h_θ blows up as $\theta \uparrow \theta_0$. Then the invariant solution determined by h is a convex, complete noncompact curve lying inside the wedge $\{(x, y) : y < |x| \tan \theta_0\}$ (See Figure 3.11). Denote such curve by $\gamma_0(\theta)$. The invariant solution is the expanding self-similar solution $\gamma(\theta, t) = t^{3/2}\gamma_0(\theta)$.

Next, let's denote the solution of (3.37) subject to the initial conditions $h(0) = \alpha \geq 1$, $h_\theta(0) = 0$, by $h(\theta, \alpha)$. $h(\theta) \equiv \alpha$ which satisfies (3.37) defines the shrinking circle. The solutions of (3.37) are positive periodic functions and can be solved explicitly. They are support functions of ellipses(See Figure 12). For λ which $\lambda(t)''\lambda(t)^{\frac{1}{3}} = -3/4$, contracts to 0 if the initial velocity is nonpositive, and expands first then contract to 0 at finite time when the initial velocity positive.

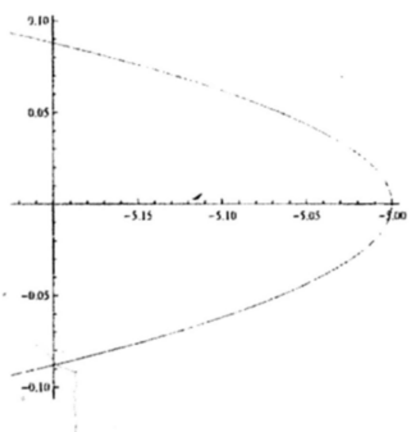

Figure 3.11: Expanding self-Similar solution

Figure 3.12: Contracting self-similar curves-ellipses

(b) *Circles.* We take $\{u\partial_x - x\partial_u\}$. Two group invariants are $v = u^2 + x^2$, $y = t$.

We look for invariant solutions as $v = v(y)$. Then we compute

$$
\begin{aligned}
&uu_x + x = 0 \\
&2uu_t = v_y \\
&u_x^2 + uu_{xx} = -1 \\
&u_x u_t + uu_{xt} = 0 \\
&2u_t^2 + 2uu_{tt} = v_{yy},
\end{aligned}
$$

so

$$
\begin{aligned}
u_x &= -\frac{x}{u} \\
u_t &= \frac{v_y}{2u} \\
u_{xx} &= \frac{-1 - u_x^2}{u} \\
u_{xt} &= -\frac{u_x u_t}{u} \\
u_{tt} &= \frac{v_{yy} - 2u_t^2}{2u}.
\end{aligned}
$$

Thus equation

$$u_{tt}u_{xx} - u_{xt}^2 = (u_{xx})^{\frac{4}{3}}$$

turn to be

$$
\begin{aligned}
\frac{v_{yy} - 2u_t^2}{2u}\frac{(-1 - u_x^2)}{u} - (\frac{u_x u_t}{u})^2 &= (\frac{(-1 - u_x^2)}{u})^{\frac{4}{3}} \\
-\frac{v}{2}(v_{yy} - 2u_t^2) - x^2u_t^2 &= (\frac{-v}{u^3})^{\frac{4}{3}},
\end{aligned}
$$

i.e.

$$2vv_{yy} - v_y^2 + 4v^{\frac{4}{3}} = 0, \tag{3.38}$$

If we choose $r^2 = v$, then (3.38) reduced to be

$$r^{\frac{1}{3}}r'' + 1 = 0,$$

The group invariant solution is a family of circles with radius r. We see $r'' < 0$, r' is a decreasing function. When the initial value $r'(0) > 0$, r increases first, then

decreases to 0 at finite time. The circle expands first, then contracts to a point in finite time.
When the initial value $r'(0) < 0$, r decreases to 0 at finite time. The circle contracts to a point in finite time.

We show the graph of the radius $r(t)$ in the following figure.

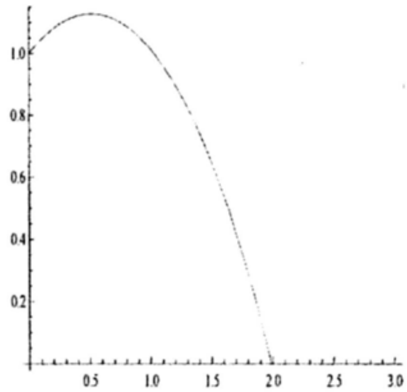

Figure 3.13: $r_1 > 0$.

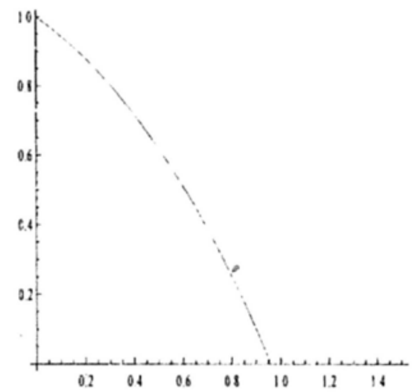

Figure 3.14: $r_1 \leq 0$.

(c)$\{u\partial_x + x\partial_u\}$-*hyperbolic rotation invariant solution*
Group Invariants: $v = u^2 - x^2$, $y = t$.
Group action: $(x, u, t) \to (x\cosh\epsilon + u\sinh\epsilon, u\cosh\epsilon + x\sinh\epsilon, t)$.
Invariant solution: $v = v(y)$.
We compute

$$\begin{aligned}
&uu_x - x = 0\\
&2uu_t = v_y\\
&u_x x^2 + uu_{xx} = 1\\
&u_x u_t + uu_{xt} = 0\\
&2u_t^2 + 2uu_{tt} = v_{yy}.
\end{aligned}$$

So

$$\begin{aligned} u_x &= \frac{x}{u} \\ u_t &= \frac{v_y}{2u} \\ u_{xx} &= \frac{1-u_x^2}{u} \\ u_{xt} &= -\frac{u_x u_t}{u} \\ u_{tt} &= \frac{v_{yy}-2u_t^2}{2u}. \end{aligned}$$

Thus equation

$$u_{tt}u_{xx} - u_{xt}^2 = u_{xx}^{4/3}$$

turn to be

$$\frac{v_{yy}-2u_t^2}{2u}\frac{(1-u_x^2)}{u} - (\frac{u_x u_t}{u})^2 = (\frac{(1-u_x^2)}{u})^{\frac{4}{3}}$$

$$\frac{v}{2}(v_{yy}-2u_t^2) - x^2u_t^2 = (v)^{\frac{4}{3}},$$

i.e. $v(y)$ satisfies

$$2vv_{yy} - v_y^2 - 4v^{\frac{4}{3}} = 0,$$

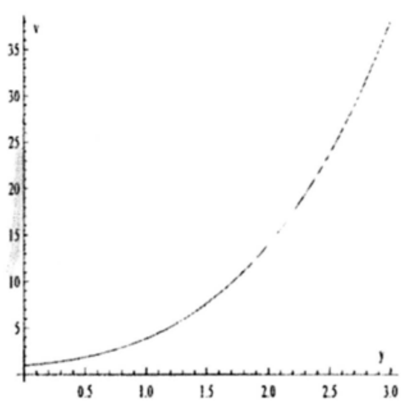

Figure 3.15: $v(y)$, $v'(0) \geq 0$.

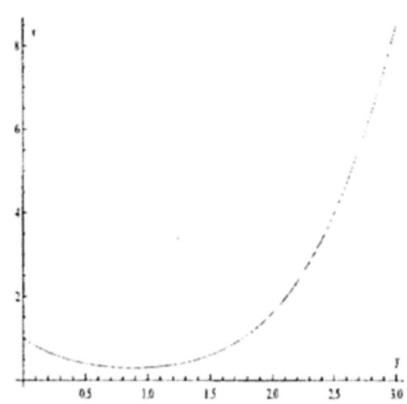

Figure 3.16: $v(y)$, $v'(0) < 0$.

Hence the solutions are translating hyperbolas.

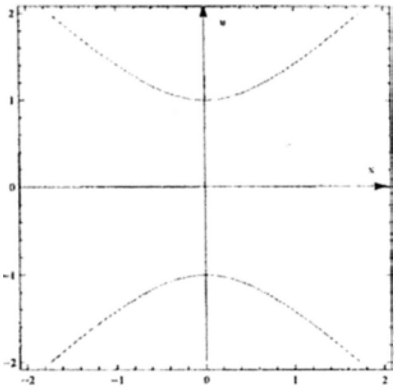

Figure 3.17: Hyperbolas

(d)$\{x\partial_x - u\partial_u\}$-*hyperbolas.* Two invariants are $xu = v$, $t = y$. We look at invariant solution of form $u = v(y)/x$. First, we compute

$$u + xu_x = 0$$
$$xu_t = v_y$$
$$2u_x + xu_{xx} = 0$$
$$u_t + xu_{xt} = 0$$
$$xu_{tt} = v_{yy}$$

and

$$u_x = -\frac{u}{x}$$
$$u_t = \frac{v_y}{x}$$
$$u_{xx} = \frac{2u}{x^2}$$
$$u_{xt} = \frac{v_y}{x^2}$$
$$u_{tt} = \frac{v_{yy}}{x},$$

then the equation

$$u_{tt}u_{xx} - u_{xt}^2 = u_t^4$$

is reduced to

$$\frac{v_{yy}}{x}\frac{2u}{x^2} - (\frac{v_y}{x^2})^2 = (\frac{v_y}{x})^4,$$

i.e.

$$2vv_{yy} - (v_y)^2 = (v_y)^4.$$

We know the invariant solutions are hyperbolas.

(1).When $v(0) > 0, v'(0) > 0$, $v(y)$ increases as y increase and blows up at finite y. So, the hyperbolas expands and blows up at finite time.

(2).If $v(0) > 0, v'(0) < 0$, $v(y)$ decreases as y increase and $v(y)$ tends to 0 at finite y. So, the hyperbolas contracts.

(3).When $v(0) < 0, v'(0) > 0$, $v(y)$ increases and tends to 0 at finite y. So, the hyperbolas contracts.

(4).If $v(0) > 0, v'(0) < 0$, $v(y)$ decreases as y increase and $v(y)$ blows up at finite y. So, the hyperbolas expands and blows up at finite time.

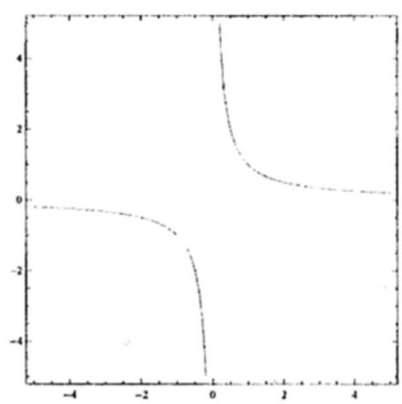

Figure 3.18: hyperbolas, $v(0) > 0$

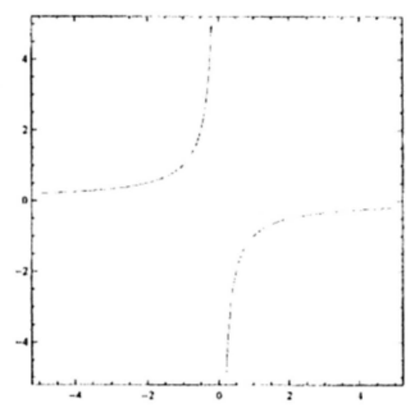

Figure 3.19: hyperbolas, $v(0) < 0$

Chapter 4

Hypersurfaces

4.1 The evolution equation

In this chapter, we first study the geometric motion of hypersurfaces given by

$$\frac{\partial^2 X}{\partial t^2} = F\mathbf{n} + G^j \frac{\partial X}{\partial p_j}, \tag{4.1.1}$$

where $X(\cdot, t)$ is a hypersurface in $\mathbb{R}^{n+1}$ at each t. The notion of a normal flow extends trivially to all dimensions, namely, $X(p,t)$ is a *normal flow* if $X_t(p,t)$ is orthogonal to the hypersurface at $X(p,t)$ for each t.

Proposition 4.1. *The flow* (4.1.1) *is normal if and only if it is given by* (1.1) *and*

$$\left\langle X_t, \frac{\partial X}{\partial p_j} \right\rangle = 0, \qquad j = 1, \cdots, n,$$

at $t = 0$.

Proof. From (4.1.1) we have

$$\frac{\partial}{\partial t}\langle X_t, X_k \rangle = G^j g_{jk} + \langle X_t, X_{tk} \rangle, \tag{4.1.2}$$

where $X_k \equiv \partial X / \partial p_k$. From this, it is readily seen that the proposition holds. □

Now, we write down the equation for the graph of the flow. Let $X(p,t) = (x, u(x,t))$, where $x = (x^1, \cdots, x^n)$ depends on (p,t). We have

$$\frac{\partial X}{\partial t} = \Big(\frac{\partial x}{\partial t}, \frac{\partial u}{\partial x_i}\frac{\partial x^i}{\partial t} + \frac{\partial u}{\partial t}\Big),$$

and

$$\frac{\partial^2 X}{\partial t^2} = \Big(\frac{\partial^2 x}{\partial t^2}, \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial t} + \frac{\partial u}{\partial x_i}\frac{\partial^2 x^i}{\partial t^2} + 2\frac{\partial^2 u}{\partial x_i \partial t}\frac{\partial x^i}{\partial t} + \frac{\partial^2 u}{\partial t^2}\Big).$$

Taking inner product of the last expression with $\mathbf{n}$ yields

$$u_{tt} + 2u_{jt}x_t^j + u_{ij}x_t^i x_t^j = F\sqrt{1 + |\nabla u|^2}. \tag{4.1.3}$$

To determine x_t we use the orthogonality condition $\langle X_t, X_k \rangle = 0$ to get

$$x_t^i g_{ki} + u_t u_k = 0,$$

for each k. Using $g_{ki} = \delta_{ki} + u_k u_i$, and $g^{ki} = \delta_{ki} - u_k u_i/(1 + |\nabla u|^2)$, we have

$$x_t^k = -g^{ki}u_i u_t = -\Big(\delta_{ki} - \frac{u_k u_i}{1 + |\nabla u|^2}\Big)u_i u_t.$$

So, the associated equation is

$$u_{tt} - \frac{2u_t u_i}{1 + |\nabla u|^2}u_{it} + \frac{u_t^2 u_i u_j}{(1 + |\nabla u|^2)^2}u_{ij} = F\sqrt{1 + |\nabla u|^2}. \tag{4.1.4}$$

When $F = A + BH$, where H is the mean curvature of $X(\cdot, t)$ and A, B depend on X up to its first order derivatives, (4.1.4) is hyperbolic if and only if B is positive.

When it comes to the fully nonlinear case, we consider uniformly convex hypersurfaces only. A family of (uniformly convex) hypersurfaces $X(\cdot, t)$ is called *normal preserving* if its normal at $X(p,t)$ is equal to its normal at $X(p,0)$, or equivalently, $\partial \mathbf{n}/\partial t = 0$.

Proposition 4.2. *Let $X(\cdot, t)$ be a family of uniformly convex hypersurfaces satisfying* (4.1.1). *It is normal preserving if and only if it is given by* (1.2) *and*

$$\Big\langle \frac{\partial X_t}{\partial p_j}, \boldsymbol{n} \Big\rangle = 0, \qquad j = 1, \cdots, n,$$

at $t = 0$.

Proof. As $\partial \mathbf{n}/\partial t$ is always orthogonal to $\mathbf{n}$, the flow is normal preserving if and only if

$$\Big\langle \frac{\partial \mathbf{n}}{\partial t}, \frac{\partial X}{\partial p_k} \Big\rangle = 0, \quad k = 1, \cdots, n.$$

We compute

$$\begin{aligned} \frac{\partial}{\partial t}\langle \mathbf{n}_t, X_k\rangle &= -\frac{\partial}{\partial t}\langle \mathbf{n}, X_{kt}\rangle \\ &= -\langle \mathbf{n}_t, X_{kt}\rangle - \langle \mathbf{n}, X_{ktt}\rangle \\ &= -\langle \mathbf{n}_t, X_{kt}\rangle - F_k - G^i\langle X_{ki}, \mathbf{n}\rangle. \end{aligned}$$

Using

$$\frac{\partial \mathbf{n}}{\partial t} = g^{ij}\Big\langle \frac{\partial \mathbf{n}}{\partial t}, \frac{\partial X}{\partial p_j}\Big\rangle \frac{\partial X}{\partial p_i},$$

we have

$$\frac{\partial}{\partial t}\Big\langle \frac{\partial \mathbf{n}}{\partial t}, \frac{\partial X}{\partial p_k}\Big\rangle = -g^{ij}\Big\langle \frac{\partial X}{\partial p_i}, \frac{\partial^2 X}{\partial p_k \partial t}\Big\rangle \Big\langle \frac{\partial \mathbf{n}}{\partial t}, \frac{\partial X}{\partial p_j}\Big\rangle - \frac{\partial F}{\partial p_k} - G^i\Big\langle \frac{\partial^2 X}{\partial p_k \partial p_i}, \mathbf{n}\Big\rangle.$$

This is a system of ODE of the form

$$\frac{\partial}{\partial t} y = By + a,$$

where $y = (y^1, \cdots, y^n)$, $y^k = \langle \mathbf{n}_t, X_k\rangle$, and $a^k = -F_k - G^i\langle X_{ki}, \mathbf{n}\rangle$. Now it is clear that the flow is normal preserving if and only if $a^k \equiv 0$ for all k. The proposition follows from the Weigarten equation

$$b_{ij} = \Big\langle \mathbf{n}, \frac{\partial^2 X}{\partial x_i \partial x_j}\Big\rangle.$$

□

To obtain the equation for the graph of a normal preserving flow, we use the normal preserving condition to obtain $u_{ij}x_t^j + u_{it} = 0$ for each i. It follows that

$$x_t^i = -u^{ij}u_{jt}, \; i = 1, \cdots, n.$$

Plugging this into (4.1.3) yields

$$u_{tt} - u^{ij}u_{it}u_{jt} = \sqrt{1 + |\nabla u|^2}\, F.$$

We claim that this equation can be rewritten as

$$\det D^2_{x,t}u = \det D^2_x u\sqrt{1+|\nabla u|^2}\ F, \tag{4.1.5}$$

For, first of all, using $u^{ij} = c_{ij}\det D^2_x u$, where c_{ij} is the $(i,j)-$ cofactor of $D^2_x u$, it suffices to show

$$\det D^2_{x,t}u = u_{tt}\det D^2_x u - c_{ij}u_{it}u_{jt}.$$

Denoting $x_0 = t$, we compute the determinant of the Hessian matrix $D^2_{x,t}u$ by expanding it along the first column

$$\det D^2_{x,t}u = \sum_{j=0}^{n}(-1)^j u_{j0}m_{j0}$$

where m_{j0} is the $(j,0)-$ minor of $D^2_{x,t}u$. By expanding along the first row $(u_{01}, u_{02}, \cdots, u_{0n})$ of the $n\times n$-matrix obtained from $D^2_{x,t}u$ by deleting its $0-$th column and $j-$th row, we have

$$m_{j0} = (-1)^{j+1}u_{0i}c_{ij}.$$

It follows that

$$\begin{aligned}\det D^2_{x,t}u &= \sum_{j=0}^{n}(-1)^j u_{j0}m_{j0}\\ &= u_{00}\det D^2_x u + \sum_{1}^{n}(-1)^j u_{j0}(-1)^{j+1}u_{0i}c_{ij}\\ &= u_{00}\det D^2_x u - c_{ij}u_{i0}u_{j0},\end{aligned}$$

and the claim holds.

The equation for the support function of a normal preserving flow assumes a simple form.

Recall that for any convex hypersurface X in $\mathbb{R}^{n+1}$, its support function H is a function of homogeneous one defined in $\mathbb{R}^{n+1}/\{0\}$ satisfying

$$H(z) = \langle z, X(p)\rangle, \quad |z| = 1,$$

where $X(p)$ is any point on the hypersurface whose unit outer normal is z. It is well-known that any uniformly convex hypersurface can be recovered by its support function via the formula

$$X^i(z) = \frac{\partial H}{\partial z_i}(z), \quad i = 1, \cdots, n,$$

where the unit outer normal z is used to parametrize the hypersurface.

Consider now $X(\cdot, t)$ a family of uniformly convex, closed hypersurfaces which is normal preserving. We may parametrize the initial hypersurface by its unit outer normal z. By the normal preserving property, z is always the unit outer normal for $X(\cdot, t)$ for all t. In particular, we have $\mathbf{n} = -z$. By taking inner product of (4.1.1) with z, we have

$$\begin{aligned} F &= \left\langle \frac{\partial^2 X}{\partial t^2}, -z \right\rangle \\ &= -\sum_1^{n+1} z_j \frac{\partial}{\partial z_j} \frac{\partial^2 H}{\partial t^2} \\ &= -\frac{\partial^2 H}{\partial t^2}, \end{aligned}$$

after using Euler's identity for homogeneous functions. We have the following equation for the support function of a normal preserving flow

$$\frac{\partial^2 H}{\partial t^2} = -F. \tag{4.1.6}$$

4.2 Local solvability

We start by considering the local solvability of hyperbolic flow for plane curves. Assume the plane curves γ can be written as the graphs $(x, u(x,t))$. By the standard theory of hyperbolic system, we have the following result.

Consider the quasi-linear system of $u(x,t) = (u^1, \cdots, u^N)$, $(x,t) \in \mathbb{R}^2$.

$$A_0 \frac{\partial}{\partial t} \begin{bmatrix} u^1 \\ \cdot \\ \cdot \\ \cdot \\ u^N \end{bmatrix} + A_1 \frac{\partial}{\partial x} \begin{bmatrix} u^1 \\ \cdot \\ \cdot \\ \cdot \\ u^N \end{bmatrix} + B = 0, \tag{4.2.1}$$

where A_0, A_1, B are $N \times N$-matrix depending on (x,t,u). It is called a symmetric hyperbolic system if

1. A_0, A_1 *are symmetric matrices*
2. A_0 *is positive definite.*

We recall the standard result[Ta].

Theorem 4.1. *Consider a symmetric hyperbolic system of type* (4.2.1) *with initial values*

$$u(x,0) = u_0. \tag{4.2.2}$$

For each $u_0 \in H^k$, $k \geq N/2 + 1$, *where* H^k *is the k-th order Sobolev space in* $\mathbb{R}^N$ *there exists a unique local solution* $u \in C(I, H^k)$ *for* (4.2.1) *and* (4.2.2), *for small time* $t_0 > 0$.

Now we apply this theorem to two cases.

First, consider the quasi-linear equation which is derived from a normal flow,

$$u_{tt} = f u_{xx} + g u_{xt} + h, \tag{4.2.3}$$

where f, g, h are given by

$$\begin{aligned} g &= \frac{2u_x u_t}{1+u_x^2} + \frac{1}{u_t}\varphi\left(\frac{u_t^2}{1+u_x^2}\right) \\ f &= \frac{1}{1+u_x^2} - \frac{g^2}{4} + \frac{1}{u_t^2}\chi\left(\frac{u_t^2}{1+u_x^2}\right) \\ h &= u_t \psi\left(\frac{u_t^2}{1+u_x^2}\right), \end{aligned}$$

and ϕ, φ, ψ are arbitrary functions.

Proposition 4.1. *Consider the Cauchy problem of (4.2.3) and (4.2.2). Assume that $\phi > 0$ on $\mathbb{R}$. Then for each $(u_0, u_1) \in H^{k+1} \times H^k$, $k \geq 2$, the Cauchy problem has a unique solution for t in $[0,T)$, $T > 0$, where T depends on (u_0, u_1).*

Proof. We rewrite (4.2.3) into a symmetric hyperbolic system.

Since

$$\begin{cases} \dfrac{\partial u}{\partial t} = u_t \\ \dfrac{\partial u_t}{\partial t} = f(u_x)_x + g(u_t)_x + h \\ \dfrac{\partial u_x}{\partial t} = (u_t)_x, \end{cases}$$

we have

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ u_t \\ u_x \end{bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & g & f \\ 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} u \\ u_t \\ u_x \end{bmatrix}_x + \begin{pmatrix} u_t \\ h \\ 0 \end{pmatrix}$$

Let $v = \begin{pmatrix} v^0 \\ v^1 \\ v^2 \end{pmatrix}$ and consider the system

$$\frac{\partial}{\partial t}\begin{bmatrix} v^0 \\ v^1 \\ v^2 \end{bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & g & f \\ 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} v^0 \\ v^1 \\ v^2 \end{bmatrix}_x + \begin{pmatrix} v^1 \\ h \\ 0 \end{pmatrix}, \tag{4.2.4}$$

where now f, g, h are functions of $\mathbf{v}$.

Let

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{g}{2} \\ 0 & -\frac{g}{2} & f + \frac{g^2}{2} \end{pmatrix}$$

Multiply both sides of (4.2.4) by R:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{g}{2} \\ 0 & -\frac{g}{2} & f + \frac{g^2}{2} \end{pmatrix} \mathbf{v}_t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{g}{2} & f \\ 0 & f & -\frac{fg}{2} \end{pmatrix} \mathbf{v}_x + \begin{pmatrix} v^1 \\ h \\ -\frac{g}{2} \end{pmatrix}. \tag{4.2.5}$$

Clearly, (4.2.5) is a symmetric system. It is also hyperbolic, for, we have

$$< \xi, R\xi >= \xi_1^2 + \xi_2^2 - g\xi_2\xi_3 + (f + \frac{g^2}{2})\xi_3^2.$$

By our assumption, $\phi > 0$, so

$$\begin{aligned}\Delta &= g^2 - 4(f + \frac{g^2}{2}) = -4f - g^2 \\ &= -4(\frac{1}{1+(v^2)^2} - \frac{g^2}{4} + \frac{1}{(v^1)^2}\phi) - g^2 \\ &= -\frac{4}{1+(v^2)^2} - \frac{4}{(v^1)^2}\phi < 0,\end{aligned}$$

so R is positive definite.

By theorem 4.1, there exists a unique local solution $\mathbf{v}$ for (4.2.5) satisying

$$\begin{aligned}v^0(x,0) &= u_0(x) \\ v^1(x,0) &= u_{0t}(x) \\ v^2(x,0) &= u_{0x}(x).\end{aligned}$$

From the first and the third equations of (4.2.4), we see that

$$\frac{\partial v^0}{\partial t} = v^1, \ \frac{\partial v^0}{\partial x} = v^2.$$

So if we set $u = v^0$, then the second equation of (4.2.4) shows that u solves (4.2.3). □

Next, we turn to the fully nonlinear hyperbolic equation

$$u_{tt} = \frac{u_{xt}^2}{u_{xx}} + \sqrt{1+u_x^2}\ \Phi(z_1, z_2, z_3),$$

where

$$z_1 = \frac{u_t^2}{1+u_x^2}, \ z_2 = k, \ z_3 = \frac{u_{xt}}{1+u_x^2} - \frac{u_{xx}u_xu_t}{(1+u_x^2)^2}.$$

We rewrite it into a quasi-linear equation

$$\begin{aligned}u_{xtt} &= \frac{2u_{xt}}{u_{xx}}u_{xxt} - \frac{u_{xt}^2}{u_{xx}}u_{xxx} + \sqrt{1+u_x^2}\Phi_{z_2}\frac{u_{xxx}}{(1+u_x^2)^{\frac{3}{2}}} \\ &+ \sqrt{1+u_x^2}\Phi_{z_3}(\frac{u_{xxt}}{1+u_x^2} - \frac{u_{xxx}u_xu_t}{(1+u_x^2)^2}) + l.o.t.\end{aligned}$$

where $l.o.t.$ stands for lower order terms.

Setting

$$v^0 = u,\ v^1 = u_x,$$

we get

$$\left\{\begin{array}{ll} v^0_{tt} & = \dfrac{(v^1_t)^2}{(v^1)_x} + \sqrt{1+(v^1)^2}\Phi(z_1, z_2, z_3) \\ & \\ v^1_{tt} & = \dfrac{2v^1_t}{v^1_x}v^1_{xt} - \dfrac{(v^1_t)^2}{(v^1_x)^2}v^1_{xx} + \Phi_{z_2}\dfrac{v^1_{xx}}{1+(v^1)^2} \\ & +\Phi_{z_3}(\dfrac{v^1_{xt}}{1+(v^1)^2} - \dfrac{v^1_{xx}v^1v^0_t}{(1+(v^1)^2)^2}) + l.o.t.. \end{array}\right. \tag{4.2.6}$$

This is a second order quasi-linear system for (v^0, v^1). We can also write it into a first order system as following

$$\begin{pmatrix} v^0 \\ v^1 \end{pmatrix} \to \begin{pmatrix} v^0_t \\ v^0_x \\ v^1_t \\ v^1_x \end{pmatrix} \equiv \begin{pmatrix} w^0 \\ w^1 \\ w^2 \\ w^3 \end{pmatrix}$$

(4.2.6) turns into

$$\left\{\begin{array}{ll} w^0_t = & F(w^0, w^1, w^2, w^3), \\ w^1_t = & w^2 \\ w^2_t = & \dfrac{2w^2}{w^3}w^2_x - \dfrac{(w^2)^2}{(w^3)^2}(w^3)_x + \Phi_{z_2}\dfrac{(w^3)_x}{1+(w^1)^2} \\ & +\Phi_{z_3}(\dfrac{(w^2)_x}{1+(w^1)^2} - \dfrac{w^0w^1(w^3)_x}{(1+(w^1)^2)^2}) + \chi \\ w^3_t = & w^2_x, \end{array}\right. \tag{4.2.7}$$

where the first equation is rewritten out completely by the first equation in (4.2.6) and the last equation is the computability condition

$$v^1_{xt} = v^1_{tx}.$$

Therefore (4.2.7) can be rewritten in the form

$$\begin{bmatrix} w^0 \\ w^1 \\ w^2 \\ w^3 \end{bmatrix}_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} w^0 \\ w^1 \\ w^2 \\ w^3 \end{bmatrix}_x + l \begin{pmatrix} w^0 \\ w^1 \\ w^2 \\ w^3 \end{pmatrix}, \tag{4.2.8}$$

where l is a vector function of $\mathbf{w}$ and

$$a = \frac{2w^2}{w^3} + \frac{\Phi_{z_3}}{1+(w^1)^2},$$
$$b = -\frac{(w^2)^2}{(w^3)^2} + \frac{\Phi_{z_2}}{1+(w^1)^2} - \frac{\Phi_{z_3} w^0 w^1}{(1+(w^1)^2)^2}.$$

Set

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Assume $\Phi_{z_3} \equiv 0$, and let

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{a}{2} \\ 0 & 0 & -\frac{a}{2} & b + \frac{a^2}{2} \end{pmatrix}.$$

Then

$$A' = RA = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{a}{2} & b \\ 0 & 0 & b & -\frac{ab}{2} \end{pmatrix}.$$

So A' is symmetric, and the system becomes

$$R\mathbf{w}_t = A'\mathbf{w}_x + l(\mathbf{w}), \tag{4.2.9}$$

where $l(\mathbf{w})$ is a vector function which independs of the derivative of $\mathbf{w}$, only depends on $\mathbf{w}$. We can see that R is positive definite. For

$$<\xi, R\xi>=\xi_1^2+\xi_2^2+\xi_3^2-2\frac{w^2}{w^3}\xi_2\xi_4+(\frac{(w^2)^2}{(w^3)^2}+\frac{\Phi_{z_2}}{1+(w^1)^2})\xi_4^2.$$

As

$$\begin{aligned}\Delta &= 4(\frac{w^2}{w^3})^2-4(\frac{(w^2)^2}{(w^3)^2}+\frac{\Phi_{z_2}}{1+(w^1)^2})\\ &= -4\frac{\Phi_{z_2}}{1+(w^1)^2}<0,\end{aligned}$$

provied $\Phi_{z_2}<0$. Then (4.2.8) is symmetric hyperbolic, and the initial value for (4.2.8) is given by $(u^1, u_x^0, u_x^1, u_{xx}^0)$, where $u^0 \doteq u(x,0)$, $u^1=u_t(x,0)$, by theorem 4.1 $\mathbf{w}$ is locally solvable. Claim that $w_x^0=w_t^1$, i.e. $(u_t)_x=(u_x)_t$. Since by (4.2.8) we have $(w^0)_{tx}=(w^1)_{tt}$, and $w_x^0|_{t=0}=w_t^1|_{t=0}=u_x^1$, thus $(u_t)_x=(u_x)_t$. Therefore u is locally solvable. We have proved the following proposition.

Proposition 4.2. *Consider the fully nonlinear equation*

$$u_{tt}=\frac{u_{xt}^2}{u_{xx}}+\sqrt{1+u_x^2}\,\Phi(z_1,z_2),$$

with initial values (4.2.2). *Assume that* $\Phi_{z_2}<0$. *Then the problem is locally solvable*

$$\forall(u_0(x), u_{0t}(x))\in H^{k+1}\times H^k,\ k\geq 2.$$

Now let's consider higher order cases. We will establish the local solvability for the normal preserving flow (1.2) where F is a function depending on the principal curvatures of the hypersurface. This will be achieved by reducing it to a fully nonlinear hyperbolic equation. Since the local solvability for normal flows which is related to quasilinear hyperbolic equations is similar to the plane curve case, we will not discuss it here.

Consider the initial value problem for the flow (1.2), that is,

$$\begin{cases} \dfrac{\partial^2 X}{\partial t^2} = F\mathbf{n} - b^{ij}\dfrac{\partial F}{\partial z_i}\dfrac{\partial X}{\partial z_j}, \\ X(0) \text{ and } X_t(0) \quad \text{are given.} \end{cases} \tag{4.2.10}$$

for a normal preserving flow. Due to the definition of a normal preserving flow, we may always take the independent variable z to be the unit outer normal of $X(\cdot, t)$. Here F is a curvature function. Following the formulation in Urbas [U1], which is based on the Caffarelli-Nirenberg-Spruck theory of fully nonlinear elliptic equations [CNS], we take it to be a function $f = f(R_1, \cdots, R_n)$, where $R_1, \cdots, R_n$ are the principal radii of curvature for a uniformly convex hypersurface in $\mathbb{R}^{n+1}$. The smooth function f is defined and symmetric in the positive cone $\Gamma^+ = \{R = (R_1, \cdots, R_n) : R_i > 0,\ i = 1, \cdots, n\}$. Moreover, it is assumed to satisfy the following conditions:

$$f \text{ is homogeneous of degree one on } \Gamma^+, \tag{4.2.11}$$
$$\frac{\partial f}{\partial R_j}(R_1, \cdots, R_n) < 0, \quad j = 1, \cdots, n, \quad (R_1, \cdots, R_n) \in \Gamma^+.$$

Theorem 4.2. *Consider* (4.2.10) *under* (4.2.11) *where* $X(0)$ *is a uniformly convex hypersurface in* $\mathbb{R}^{n+1}$ *and* $X_t(0)$ *satisfies* $\langle \mathbf{n}, \partial X_t(0)/\partial z_j\rangle = 0,\ j = 1, \cdots, n$. *Suppose* $X(0) \in H^k(S^n)$ *and* $X_t(0) \in H^{k-1}(S^n), k > n/2 + 2$. *Let* $f \in C^\infty(\Gamma^+)$ *be a symmetric, positive function on the positive cone satisfying* (4.2.11). *There exists a positive* $T \leq \infty$ *such that* (4.2.10) *has a unique solution* X *in*

$$C([0,T), H^k(S^n)) \bigcap C^1([0,T), H^{k-1}(S^n))$$

which is uniformly convex at each t. *It is smooth provided* $X(0)$ *and* $X_t(0)$ *are smooth. Moreover, it is maximal in the sense that if* T *is finite, either the minimum of the principal curvatures of* $X(t)$ *tends to zero or*

$$||X(t)||_{C^2(S^n)} \to \infty,$$

as t approaches T.

To prove this theorem, we look at the initial value problem for the associated equation satisfied by the support functions $H(z,t)$ of the hypersurfaces. By (4.1.6),

$$\begin{cases} \dfrac{\partial^2 H}{\partial t^2} = -f(R_1, \cdots, R_n), & (z,t) \in S^n \times [0,T), \\ H(0) \text{ and } H_t(0) \text{ are given }, \end{cases} \tag{4.2.12}$$

where $H(0)$ is the support function of a uniformly convex hypersurface and $H_t(0)$ is of homogeneous degree one. Our first job is to express the right hand side of the equation in (4.2.12) in terms of the support function and its derivatives.

Before proceeding further, we recall some basic facts concerning a convex hypersurface and its support function.

Let X be a convex hypersurface in $\mathbb{R}^{n+1}$. Its support function H is defined as a function of its unit outer normal by

$$H(x) = \sup \langle x, p \rangle, \quad x \in S^n,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^{n+1}$. We extend H to be homogenous function on $\mathbb{R}^{n+1}$ of degree one. Evidently we also have

$$H(x) = \sup \langle x, p \rangle, \quad \textit{for all } x \in \mathbb{R}^{n+1},$$

If X is in C^2, then so is H. When attains its maximum. The point $p = p(x)$ is given by

$$p^i = \frac{\partial H}{\partial x^i}, \quad i = 1, \cdots, n+1,$$

where H is differentiable. Notice that its normal is given by x.

The eigenvalues of the Hessian $(\partial^2 H/\partial x_i \partial x_j)(x)$, $i, j = 1, \cdots, n+1$, consists of zero (due to homogeneity of degree one) and the principle radii of curvature at $p(x)$. We will compute the metric and the second fundamental form of X in terms of the support function in the following.

Since H is of homogeneous degree one, it is uniquely determined by its restriction to $x^i = \pm 1$, $i = 1, \cdots, n+1$. As a typical case, we consider its restriction to $x^{n+1} = -1$. Let

$$u(x_1, \cdots, x_n) = H(x_1, \cdots, x_n, -1) = \sqrt{1+|x|^2} H\Big(\frac{(x_1, \cdots, x_n, -1)}{\sqrt{1+|x|^2}}\Big).$$

for $x \in \mathbb{R}^n$. The mapping

$$(x_1, \cdots, x_n) \to \frac{(x_1, \cdots, x_n, -1)}{\sqrt{1+|x|^2}}$$

maps $\mathbb{R}^n$ onto S^n_-. In this coordinate system the metric e_{ij} on S^n_- is given by

$$e_{ij} = (1+|x|^2)^{-1}(\delta_{ij} - \frac{x_i x_j}{1+|x|^2})$$

The second fundamental form of the hypersurface at the point $X(z)$ is given by

$$b_{ij}(x) = \frac{u_{ij}(x)}{\sqrt{1+|x|^2}}, \qquad z = \frac{(x, -1)}{\sqrt{1+|x|^2}}.$$

The radii of principal curvatures are the eigenvalues of the induced metric of X with respect to the second fundamental form, i.e., $\det\{g_{ij} - Rb_{ij}\} = 0$. It turns out they are the eigenvalues of the matrix (s_{ij}) given by

$$s_{ij} = (1+|x|^2)^{\frac{1}{2}}(\delta_{ik} + x_i x_k) u_{jk},$$

see [U1]. This matrix is not symmetric. However, observing that the symmetric matrix given by

$$\hat{s}_{ij} = \Big(\delta_{ik} + \frac{x_i x_k}{1+(1+|x|^2)^{\frac{1}{2}}}\Big)\Big(\delta_{jl} + \frac{x_j x_l}{1+(1+|x|^2)^{\frac{1}{2}}}\Big) u_{kl}, \tag{4.2.13}$$

shares the same eigenvalues with (b_{ij}) [CNS], we know there exists a smooth function F such that

$$F(\hat{s}_{ij}) = f(R_1, \cdots, R_n)$$

by our assumptions on f. The eigenvalues of the matrix $(\partial F/\partial z_{ij})$ are given precisely by $\partial f/\partial R_1, \cdots, \partial f/\partial R_n$, [CNS], so (4.2.11) is equivalent to

$$\frac{\partial F}{\partial z_{ij}}(A) < 0, \tag{4.2.14}$$

on all positive definite matrices A.

Restricting on the hyperplane $x_{n+1} = -1$, (4.2.12) becomes

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = -(1 + |x|^2) F(\hat{s}_{ij}), \\ u(0) \text{ and } u_t(0) \text{ are given }, \end{cases} \tag{4.2.15}$$

where $(\hat{s}_{ij})$ is in (4.2.13).

The above discussion leads us to the general fully nonlinear hyperbolic equation

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} = \phi(x, D_x^2 v), \quad (x,t) \in \mathbb{R}^{n+1} \times [0, T), \\ v(0) = f, \ \dfrac{\partial v}{\partial t}(0) = g, \end{cases} \tag{4.2.16}$$

where the smooth function $\phi(x, z^{ij})$ satisfies $\phi(x, z^{ij}) = \phi(x, z^{ji})$ and the ellipticity condition: There exists a symmetric matrix Z_0 such that for any symmetric matrix Z satisfying $Z_0 + Z$ is positive definite,

$$\frac{\partial \phi}{\partial z_{ij}}(x, Z) > 0. \tag{4.2.17}$$

Clearly, this condition is satisfied for (4.2.15) under (4.2.14) for v being $u - u(0)$ and Z_0 the Hessian of $u(0)$.

We would like to solve (4.2.16) locally in time. To do this we first reduce it to a quasilinear system of second order equations. In fact, for each $k = 1, \cdots, n$, $v_k = \partial v/\partial x_k$ satisfies

$$\begin{cases} \dfrac{\partial^2 v_k}{\partial t^2} = a^{ij} \dfrac{\partial^2 v_k}{\partial x_i \partial x_j} + b^k, \\ v_k(0) = \dfrac{\partial f}{\partial x_k}, \ \dfrac{\partial v_k}{\partial t}(0) = \dfrac{\partial g}{\partial x_k}, \end{cases}$$

where $a^{ij} = \phi_{z_{ij}}(x, D_x^2 v)$ and $b^k = \phi_{x_k}(x, D_x^2 v)$. Let us consider a second order system for $\mathbf{v} = (v^1, \cdots, v^n)$,

$$\begin{cases} \dfrac{\partial^2 v^k}{\partial t^2} = a^{ij}\dfrac{\partial^2 v^k}{\partial x_i \partial x_j} + b^k, \\ \\ \mathbf{v}(0) \text{ and } \dfrac{\partial \mathbf{v}}{\partial t}(0) \text{ are given,} \end{cases} \tag{4.2.18}$$

where $a^{ij} = a^{ij}(x, D_x\mathbf{v})$, $b^k = b^k(x, D_x\mathbf{v})$ and $D_x\mathbf{v} = (\nabla v^1, \cdots, \nabla v^n)$. Clearly, $\mathbf{v} = (\partial v/\partial x_1, \cdots, \partial v/\partial x_n)$ solves (4.2.18) whenever v is a solution of (4.2.16). On the other hand, we assert that if $\mathbf{v}$ solves (4.2.18) with $\mathbf{v}(0) = \nabla f$ and $\mathbf{v}_t(0) = \nabla g$, then a solution to (4.2.16) can be found.

For, we differentiate (4.2.18) in x_l to obtain

$$v^k_{ltt} = \phi_{z_{ij}} v^k_{lij} + \phi_{z_{ij}z_{mn}} v^m_{ln} v^k_{ij} + \phi_{z_{ij}x_l} v^k_{ij} + \phi_{x_k z_{ij}} v^i_{lj} + \phi_{x_k x_l}.$$

It follows that

$$\begin{aligned} (v^k_l - v^l_k)_{tt} &= \phi_{z_{ij}}(v^k_l - v^l_k)_{ij} + \phi_{z_{ij}z_{mn}}(v^m_{ln} v^k_{ij} - v^m_{kn} v^l_{ij}) \\ &\quad + \phi_{z_{ij}x_l} v^k_{ij} - \phi_{z_{ij}x_k} v^l_{ij} + \phi_{z_k z_{ij}} v^i_{lj} - \phi_{x_l z_{ij}} v^i_{kj} \\ &= \phi_{z_{ij}}(v^k_l - v^l_k)_{ij} + \phi_{z_{ij}z_{mn}}[v^i_{lj}(v^k_m - v^m_k)_n + v^m_{kn}(v^i_l - v^l_i)_j] \\ &\quad + \phi_{z_{ij}x_l}(v^k_i - v^i_k)_j + \phi_{z_{ij}x_k}(v^i_l - v^l_i)_j, \end{aligned}$$

after using

$$\phi_{z_{ij}z_{mn}} v^m_{ln} v^k_{ij} = \phi_{z_{ij}z_{mn}} v^i_{lj} v^k_{mn}.$$

Thus $\omega^{kl} \equiv v^k_l - v^l_k$ satisfies

$$\begin{cases} \dfrac{\partial^2 \omega^{kl}}{\partial t^2} = a^{ij}\dfrac{\partial^2 \omega^{kl}}{\partial x_i \partial x_j} + c^{klm}_{ij}\dfrac{\partial \omega^{ij}}{\partial x_m} \\ \\ \omega^{kl}(0) = 0, \ \dfrac{\partial \omega^{kl}}{\partial t}(0) = 0, \end{cases} \tag{4.2.19}$$

for some functions c^{klm}_{ij}.

(4.2.19) is a linear second order system. One can show that the solution to this system only admits the trivial solution. First by introducing a new variable $W = (w^{kl}, w_t^{kl}, w_j^{kl})$, we can make (4.2.19) into a first order linear system for W with zero initial data, namely

$$\begin{cases} \dfrac{\partial W}{\partial t} = A_j \partial_j W + B \\ \\ W(0) \ \text{given} . \end{cases} \tag{4.2.20}$$

By multiplying this system with the matrix which is the n^2 copies direct sum of $\mathcal{R}$ given below, we can turn it into a linear first symmetry hyperbolic system.

$$\mathcal{R}\frac{\partial W}{\partial t} = A^j \frac{\partial W}{\partial x_j} + \mathcal{R}B,$$

We apply to each side W and integrate the system

$$\mathcal{R}\frac{d}{dt}(\frac{1}{2}||W||_{L^2}^2) = (A_j \partial_j W, W) + (B, W).$$

By the energy estimate, we deduce the Grownall's inequality $d||W||_{L^2}^2/dt \leq C_1||W||_{L^2} + C_2$. It follows that W vanishes identically. Letting $v_l^k = v_k^l$ for each k, l and there exists a potential function $\tilde{v}$ such that $\partial\tilde{v}/\partial x_k = v^k$. Consequently,

$$\frac{\partial^2 \tilde{v}}{\partial t^2} = \phi(x, D_x^2 \tilde{v}) + c(t),$$

holds for some function $c(t)$. At $t = 0$,

$$\tilde{v}(x, 0) = f(x) + c_1 \quad \text{and} \quad \tilde{v}_t(x, 0) = g(x) + c_2,$$

for some constants c_1 and c_2. A solution for (4.2.16) is found by taking $v(x, t) = \tilde{v}(x, t) + \chi(t)$ where χ solves $\chi'' = -c(t)$, $\chi(0) = -c_1$ and $\chi'(0) = -c_2$.

We have reduced the solvability of (4.2.16) to that of (4.2.18). A further step is to reduce (4.2.18) to a first order system of quasilinear equations.

Consider the following system for an $\mathbb{R}^{(n+2)n}$-valued function $\mathbf{w}$

$$\begin{cases} \dfrac{\partial w^k}{\partial t} = w^{k0} \\ \dfrac{\partial w^{k0}}{\partial t} = a^{ij}\dfrac{\partial w^{ki}}{\partial x_j} + b^k \\ \dfrac{\partial w^{kj}}{\partial t} = \dfrac{\partial w^{k0}}{\partial x_j} \\ \mathbf{w}(0) \ \text{given}\ , \end{cases} \tag{4.2.21}$$

where $\mathbf{w} = (w^1, w^{10}, w^{11}, \cdots, w^{1n}, \cdots\cdots, w^n, w^{n0}, w^{n1}, \cdots, w^{nn})$ and the coefficients a^{ij} and b^k are evaluated at $(x, w^{11}, \cdots, w^{1n}, \cdots\cdots, w^{n1}, \cdots, w^{nn})$. It is clear that when

$$\mathbf{w}(0) = (v^1(0), v^1_t(0), v^1_1(0), \cdots, v^n_1(0), \cdots\cdots, v^n(0), v^n_t(0), v^n_1(0), \cdots, v^n_n(0))$$

where $\mathbf{v}(0)$ and $\mathbf{v}_t(0)$ are given in (4.2.18),

$$\mathbf{w} = (v^1, v^1_t, v^1_1, \cdots, v^1_n, \cdots\cdots, v^n, v^n_t, v^n_1, \cdots, v^n_n)$$

solves (4.2.21). Conversely, let $\mathbf{w}$ be a solution of (4.2.21) satisfying these special initial values. Then, for $k, l = 1, \cdots, n$,

$$\frac{\partial}{\partial t}\left(w^{lk} - \frac{\partial w^l}{\partial x_k}\right) = \frac{\partial w^{l0}}{\partial x_k} - \frac{\partial}{\partial x_k}\frac{\partial w^l}{\partial t} = 0,$$

whence $\mathbf{v} \equiv (w^1, \cdots, w^n)$ solves (4.2.18).

To solve (4.2.21), we note that for each k, $\mathbf{w}^k \equiv (w^k, w^{k0}, \cdots, w^{kn})$ satisfies

$$\frac{\partial \mathbf{w}^k}{\partial t} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a^{11} & \cdots & a^{n1} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \frac{\partial \mathbf{w}^k}{\partial x_1} + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a^{12} & \cdots & a^{n2} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \frac{\partial \mathbf{w}^k}{\partial x_2} +$$

$$\cdots + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a^{1n} & \cdots & a^{nn} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \frac{\partial \mathbf{w}^k}{\partial x_n} + \mathbf{b},$$

where $\mathbf{b} = \mathbf{b}(x, \mathbf{w}^k)$. By multiplying this system with the matrix $\mathcal{R}$ which is the the n-copies direct sum of the $(n+2) \times (n+2)$ matrix

$$R = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \cdot & \cdot & & a^{ij} & \\ 0 & 0 & & & \end{bmatrix},$$

we obtain

$$\mathcal{R} \frac{\partial \mathbf{w}}{\partial t} = A^j \frac{\partial \mathbf{w}}{\partial x_j} + \mathcal{R}\mathbf{b}, \tag{4.2.22}$$

where A^j's are now symmetric. When (4.2.22) is derived from (4.2.18), $\mathcal{R}$ is positive definite under (4.2.17), so this is a quasilinear symmetric hyperbolic system.

The theory of quasilinear symmetric hyperbolic systems is well-known. Consider a general system (4.2.22) where $\mathcal{R}, A^j$, and $\mathbf{c}$ are smooth functions of

$(x, \mathbf{w}) \in \mathbb{R}^n \times \mathcal{U}$, $\mathcal{U}$ an open set in $\mathbb{R}^N$ for some N. Moreoever, $\mathcal{R}$ and A^j's are symmetric $N \times N$-matrices, and all eigenvalues of $\mathcal{R}$ are positive in $\mathbb{R}^n \times \mathcal{U}$. The following facts can be found or derived easily from Taylor [T].

Lemma 4.1. *For any $\boldsymbol{w}(0) \in H^k(\mathbb{R}^n)$, $k > N/2+1$, with $\boldsymbol{w} \in \mathcal{V}$ where $\mathcal{V}$ is an open set compactly contained in $\mathcal{U}$, (4.2.22) has a unique classical solution $\boldsymbol{w}$ defined on some interval $[0,T)$, $T > 0, \boldsymbol{w}(t) \in \mathcal{V}$, which belongs to $C([0,T), H^k(\mathbb{R}^n)) \bigcap C^1([0,T), H^{k-1}(\mathbb{R}^n))$.*

Proof. The results of existence, uniqueness, and regularity for solutions to a system of the form

$$A_0(x,t,u)\frac{\partial u}{\partial t} = \sum_j A_j(t,x,u)u + g(t,x,u), \ u(0) = f,$$

where, all A_j are symmetric, and furthermore $A_0(x,t,u) \geq cI > 0$, are proved in [Ta]. So in our case, $\mathbf{w}$ satisfies

$$\mathcal{R}\frac{\partial \mathbf{w}}{\partial t} = A^j\frac{\partial \mathbf{w}}{\partial x_j} + \mathcal{R}\mathbf{b},$$

where $\mathcal{R} > 0$, and $\mathcal{R}$ and A^j are symmetric matrixes. Suppose $\mathbf{w} \in \mathcal{V}$ where $\mathcal{V}$ is an open set compactly contained in $\mathcal{U}$, then $\mathcal{R} > cI > 0$. Therefore there exist local solution on some interval $[0,T)$, $T > 0, \mathbf{w}(t) \in \mathcal{V}$, and $\mathbf{w} \in C([0,T), H^k(\mathbb{R}^n))$. By the equations we also have $\mathbf{w} \in C^1([0,T), H^{k-1}(\mathbb{R}^n))$. □

Lemma 4.2. *Suppose $||\boldsymbol{w}(t)||_{C^1}$ is uniformly bounded for $t \in [0,T)$. Then there exists $T_1 > T$ such that the solution extends to $C([0,T_1), H^k(\mathbb{R}^n))$ with $\boldsymbol{w}(t)$ in $\mathcal{U}$.*

Proof. We obtain the energy inequality

$$\frac{d||\mathbf{w}(t)||_{H^k}}{dt} \leq C(||\mathbf{w}||_C^1)||\mathbf{w}(t)||_{H^k}$$

Now Gronwall's inequality implies that $\mathbf{w}(t)||_{H^k}$ cannot blow up as $t \to T$ unless $||\mathbf{w}||_C^1$ does, so we can use the above lemma to obtain the conclusion. □

Lemma 4.3. *$\mathbf{w}(x,t)$ is smooth in $\mathbb{R}^n \times [0,T)$ if $\mathbf{w}(0)$ is smooth at $t=0$.*

Proof. Pick $k > n/2+1$, and apply Lemma (4.1) to get a solution $\mathbf{w} \in C([0,T), H^k(\mathbb{R}^n))$. We can also apply these results with $f \in H^l(\mathbb{R}^n)$, for l arbitrarily large, together with uniqueness, to get $u \in C([0,T_1), H^k(\mathbb{R}^n))$, for some interval $[0,T_1]$ for $T_1 \leq T$. But we can use Lemma (4.2) to obtain $\mathbf{w} \in C([0,T), H^k(\mathbb{R}^n))$. For arbitrarily large l and fixed T, this holds. By Soblev Embedding theorem, it follows that $\mathbf{w} \in C^\infty([0,T) \times H^k(\mathbb{R}^n))$. □

Suppose now $\mathcal{R}(x,Z)$ is positive definite at Z whenever $Z+Z_0$ is positive definite. From these facts one deduces that there exists a unique solution to (4.2.22) on a maximal interval $[0,T_{max}), T_{max} \leq \infty$, in the sense that when T_{max} is finite, either the lowest eigenvalue of $(w^{ij}(t))+Z_0$, $\lambda(t)$, satisfies

$$\inf_t \lambda(t) \to 0,$$

or

$$\sup_t ||\mathbf{w}(t)||_{C^1} \to \infty,$$

as $t \uparrow T_{max}$.

Proof of Theorem 4.2

Set $v = u - u(0)$ in (4.2.15) and consider the problem

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} = -(1+|x|^2)F(\hat{s}_{ij}) \\ \\ v(0) = 0 \text{ and } v_t(0) \text{ is given }, \end{cases} \tag{4.2.23}$$

where

$$\hat{s}_{ij} = \left(\delta_{ik} + \frac{x_i x_k}{1+(1+|x|^2)^{\frac{1}{2}}}\right)\left(\delta_{jl} + \frac{x_j x_l}{1+(1+|x|^2)^{\frac{1}{2}}}\right)(u(0)_{ij} + v_{ij})$$

and $v_t(0)$ is a function compactly supported in $\mathbb{R}^n$ which equals to $u_t(0)$ in the unit-ball $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$. From our discussion for $u(0) \in H^{k+2}(\mathbb{R}^n)$

and $v_t(0) \in H^{k+1}(\mathbb{R}^n)$, $k > n/2 + 1$,(4.2.23) has a solution $v(t)$, $t \in [0, T)$. Then $u = v + u(0)$ solves the equation in (4.2.15). By the finite speed of propagation of solutions (Mizohata [M]) for hyperbolic equations, there exists a time $T > 0$ such that the values of $v(x,t), (x,t) \in B_{1/2} \times [0,T)$ depend only on the initial values in B_1. Hence, u solves (4.2.15) in $B_{1/2} \times [0,T)$. Passing through the tangent space of each point z on the unit sphere, we can obtain a similar solution in $[0, T(z))$. The balls $\tilde{B}(z)$ obtained by projecting all $B_{1/2}$ on these tangent spaces to the sphere form an open cover of the sphere. We can choose finitely many balls to cover the sphere. Letting $T = \min\{T(z_1), \cdots, T(z_N)\}$ where $z_j, j = 1, \cdots, N$, are centers of these balls, it is clear that one can construct $H(z,t)$ on $[0,T)$ by putting these solutions u together. We have shown that (4.2.12) is locally solvable.

Letting $X^k(z,t) = \partial H/\partial z_k$, we have

$$\left\langle \frac{\partial \mathbf{n}}{\partial t}, \frac{\partial X}{\partial z_k} \right\rangle = -\left\langle \mathbf{n}, \frac{\partial^2 X}{\partial z_k \partial t} \right\rangle = -\sum_{j=1}^{n} z^j \frac{\partial}{\partial z_j} \frac{\partial H_t}{\partial z_k} = 0,$$

by Euler's identity for homogeneous functions. It follows that X satisfies the normal preserving condition. By Proposition 2.2 it solves (4.2.15) on $[0,T)$.

The assertion on smoothness of X follows from Fact (c) above. Finally, from the expression relating X and H we see that the C^3-norm of H is controlled by the C^2-norm of X. The proof of Theorem 4.2 is completed.

4.3 Finite time blow-up

After establishing the local solvability for some general normal preserving flows driven by curvatures, we turn to other properties of the flows such as the formation of finite time singularities and long time behavior. In the literature numerous results concerning these topics are available for fully nonlinear parabolic flows. As a preliminary study, we shall focus on the Gauss curvature flow. We take $F = K^\alpha, \alpha > 0$, where K is the Gauss curvature of the hypersurface in (4.2.10) and call the resulting flow the contracting Gauss curvature flow. Its parabolic counterpart has been studied by several authors including [F], [T], [C], [A3] and [A4]. A common feature is, for any closed uniformly convex hypersurface $X(0)$, $X(t)$ contracts to a point in finite time, and its ultimate shape is largely known when α is less than or equal to $1/n$. To examine the same question for the hyperbolic case, we first consider a special case, namely, the initial hypersurface is a sphere and its initial velocity is given by $R_1\mathbf{n}$, for some real number R_1. Under these assumptions, this flow reduces to an ODE for $R(t)$, the radius of the sphere at time t,

$$\begin{cases} R'' = -\dfrac{1}{R^{n\alpha}}, \\ R(0) = R_0 > 0, \ R'(0) = R_1. \end{cases} \tag{4.3.1}$$

The following proposition can be proved by elementary means.

Proposition 4.1. *Let* $c = R_1^2/2 - R_0^{1-n\alpha}/(n\alpha - 1)$. *For* $\alpha \in (1/n, \infty)$,

(a) when $R_1 \leq 0$ *and* $c \in \mathbb{R}$, *the sphere contracts to a point in finite time,*

(b) when $R_1 > 0$ *and* $c < 0$, *the sphere expands first and then contracts to a point in finite time; when* $c \geq 0$, *it expands to* ∞ *and*

$$R(t) = \begin{cases} O(t), & c > 0 \\ O(t^{\frac{2}{\alpha+1}}), & c = 0, \end{cases}$$

as $t \to \infty$.

For $\alpha \in (0, 1/n]$, *c is always positive,*

(c) when $R_1 > 0$, *the sphere expands first and then contracts to a point in finite time.*

(d) when $R_1 \leq 0$, *the sphere contracts to a point in finite time.*

Proof.

(a), (d). The initial velocity is nonpositive, i.e. $R_1 \leq 0$.

We prove it by contradiction. Assume that $R(t) > 0$ for all time $t > 0$. Then $R'' = -1/R^{n\alpha} < 0$ and $R'(t) < R'(0) = R_1 \leq 0$ for $t > 0$. Hence there exist a time t_0 such that $R(t_0) = 0$. This is a contradiction.

(b). The initial velocity is positive, $R_1 > 0$ and $c < 0$. $\alpha > 1/n$.

By (4.3.1), we obtain

$$\frac{R'^2}{2} = \frac{R^{1-n\alpha}}{n\alpha - 1} + c.$$

$R'(0) > 0$, then $\exists$ short time t_1, s.t. $R(t_1) > 0$, so

$$\frac{R'}{\sqrt{2}} = \sqrt{\frac{R^{1-n\alpha}}{n\alpha - 1} + c}.$$

If $c \geq 0$, $R'(0) > 0$, we know $\frac{R^{1-n\alpha}}{n\alpha-1} > 0$, then R increase and R' increase. R expands to ∞ and

$$R(t) = \begin{cases} O(t), & c > 0 \\ O(t^{\frac{2}{\alpha+1}}), & c = 0, \end{cases}$$

as $t \longrightarrow \infty$.

If $c < 0$, $R'(0) > 0$, then R increase first and R' attain to 0 in finite time. By the conclusion of the (a), the sphere contract to a point.

(c)Similar as (b), we omit the proof.

□

Thus, unlike the parabolic case, inward acceleration does not necessarily mean contraction for the hypersurface. The initial velocity plays a role. Nevertheless, for $\alpha \in (0, 1/n]$, although the sphere may expand for a while, it eventually contracts to its center in finite time. In general, we have

Proposition 4.2. *Any solution of the contracting Gauss curvature flow blows up in finite time for* $\alpha \in (0, 1/n]$.

Proof. Let $H(\cdot,t)$ be the support function of this flow. By (4.1.6), it satisfies

$$\begin{cases} \dfrac{\partial^2 H}{\partial t^2} = -K^\alpha \\ H(0) \text{ and } H_t(0) \text{ are given.} \end{cases}$$

Let us assume it exists for all time and draw a contradiction. First of all, we have

$$\begin{aligned} \sigma_n &= \int_X K ds \\ &\le \Big(\int_X K^{\alpha+1} ds\Big)^{\frac{1}{\alpha+1}} \Big(\int_X ds\Big)^{\frac{\alpha}{\alpha+1}} \\ &= \Big(\int_{S^n} K^\alpha dz\Big)^{\frac{1}{\alpha+1}} A(t)^{\frac{\alpha}{\alpha+1}}, \end{aligned}$$

where $\sigma_n = |S^n|$ and $\mathcal{A}(t)$ is the surface area of $X(t)$. On the other hand, from

$$H(z,t) = H(z,0) + \int_0^t \frac{\partial H}{\partial t}(z,s)ds \le H(z,0) + \sup_z \frac{\partial H}{\partial t}(z,0)t.$$

we see that the growth of the support function is at most linear. Therefore, the surface area satisfies

$$\mathcal{A}(t) \le C(1+t^n),$$

for some constant C. It follows that

$$\begin{aligned} \int H_{tt} &= -\int K^\alpha \\ &\le -\frac{\sigma_n^{\alpha+1}}{A(t)^\alpha} \\ &\le -\frac{\sigma_n^{\alpha+1}}{C(1+t^n)} \\ &\le C_1 - C_2 t^{1-\alpha n}, \end{aligned}$$

for some constants C_1 and C_2. When $n\alpha = 1$, the term $C_2 t^{1-n\alpha}$ should be replaced by $C_2 \log t$. Therefore,

$$\begin{aligned} \int H(z,t)dz &= \int H(z,0)dz + \int_0^t \int H_t(z,s)dzds \\ &\le \int H(z,0)dz + C_1 t - \frac{C_2}{2-n\alpha} t^{2-n\alpha}, \end{aligned}$$

becomes negative for large time. The same conclusion holds when $n\alpha = 1$. However, the integral of $H(t)$ is the mean width of the convex body enclosed by $X(t)$ and it cannot be negative. To see this we note that when the origin is contained inside the convex body, the support function is nonnegative everywhere, so this integral is nonnegative. When one uses different coordinates to represent the support functions, they differ from each other only by a linear function, hence the integrals are the same. Thus we have arrived at a contradiction. We conclude that the solution of (4.2.10) cannot exist for all time when $n\alpha$ is less than or equal to 1. □

A natural question is: Could the hypersurface develop a singularity before it contracts to a point under this contracting flow? We believe this is possible, although an example is out of our hand. Nevertheless, we present a noncompact example where an isolated singularity develops in finite time for α in $(0, 1/n]$.

Let Γ be a convex cone based at the origin in $\mathbb{R}^{n+1}$ whose cross section is bounded by a closed, uniformly convex hypersurface. According to Urbas [U1], there exists a uniformly convex hypersurface X^* sitting inside C and asymptotic to its boundary at ∞ satisfying

$$\langle X^*, \mathbf{n} \rangle = K^\alpha.$$

Consider the ODE for $\alpha \in (0, 1/n)$,

$$\lambda'' = \frac{1}{\lambda^{n\alpha}}, \qquad \lambda(0) = 1,\ \lambda'(0) = \lambda_1 < 0.$$

When λ_1 satisfies $\lambda_1^2 > 2/(1 - n\alpha)$, it is easy to see that it has a solution in $[0, T)$ and $\lambda(t) \to 0$, as $t \uparrow T$. Letting $X(t) = \lambda(t)X^*$, it is readily verified that $X(t)$ solves the contracting Gauss curvature flow with $X(0) = X^*$ and geometrically it collapses to the boundary of Γ as t approaches T. We see that the curvature blows up only at the origin. Away from the origin, the hypersurface remains smooth, but its Gauss curvature vanishes.

Next we present a necessary condition for the existence of global normal preserving flows (4.2.10) when F is positive. It leads to a criterion for finite time blow-up for special initial velocity.

Proposition 4.3. *Let X be a normal preserving flow solving (4.2.10) in $S^n \times [0,\infty)$ where $F > 0$. Then its support function $H(z,t)$ must satisfy*

$$H_t(z,0) + H_t(-z,0) \geq 0, \quad \textit{for all } z. \tag{4.3.2}$$

Proof. Let X be a global normal preserving solution of (4.2.10). Then $\tilde{X} \equiv (t, X(t))$ is a hypersurface in $[0,\infty) \times \mathbb{R}^{n+1}$. When expressed locally as a graph of some function, the Gauss curvature of $\tilde{X}$ is of the same sign as the determinant of the Hessian matrix of this function, which is positive by (4.1.5) when F is positive. Therefore, $\tilde{X}$ is a uniformly convex hypersurface in $[0,\infty) \times \mathbb{R}^{n+1}$. In a coordinate system, $\tilde{X}$ is expressed as the union of the graphs of two uniformly convex functions $u(x,t)$ and $v(x,t)$ defined in the closure of some convex domain Ω satisfying $v < u$ in Ω. Given a point $X(z_0, 0)$ on the initial hypersurface, we may choose a coordinate system such that this point is $(x_0, u(x_0,0))$ and its unit outer normal is $(0,\cdots,0,1)$, that is, $\nabla u(x_0,0) = (0,\cdots,0)$ holds. Let $(y_0, v(y_0,0))$ be the unique point on $X(0)$ satisfying $\nabla v(y_0,0) = (0,\cdots,0)$. Its unit outer normal is given by $(0,\cdots,0,-1)$. So the tangent hyperplanes at $(x_0, u(x_0,0))$ and $(y_0, v(y_0,0))$ are parallel in $\mathbb{R}^{n+1}$.

The tangent hyperplanes of $\tilde{X}$ at $(0, x_0, u(x_0))$ and $(0, y_0, v(y_0))$ are given respectively by

$$P_1 = \{(t,x,u) : \quad u_t(x_0,0)t = u - u(x_0,0)\},$$

and

$$P_2 = \{(t, x, u) : \quad v_t(y_0, 0)t = u - v(y_0, 0)\}.$$

When $\tilde{X}$ is global, P_1 always sits above P_2, so they never intersect. It means that these two hyperplanes either do not intersect or they intersect at negative time. When the latter happens, the intersection time is given

$$T = \frac{v(y_0, 0) - u(x_0, 0)}{u_t(x_0, 0) - v_t(y_0, 0)} < 0.$$

It follows that

$$u_t(x_0, 0) \geq v_t(y_0, 0),$$

must hold.

We express (4.3.2) in terms of the support function. By differentiating the relation $X(0) = (x, u(x, 0))$, we have $X_t = (x_t, u_t + u_j x_t^j)$. As the outer normal of $X(0)$ at z_0 is $(0, \cdots, 0, 1)$, $u_t(x_0, 0) = \langle X_t(z_0, 0), z_0 \rangle = H_t(z_0, 0)$. Similarly, we have $H_t(-z_0, 0) = -v_t(y_0, 0)$, hence $H_t(z_0, 0) + H_t(-z_0, t) \geq 0$ from (4.1).

□

Condition (4.3.2) can be rewritten as $\langle X_t(z, 0), z \rangle + \langle X_t(-z, 0), -z \rangle \geq 0$ for all outer normal z. Noting that $\langle X_t(z, 0), z \rangle$ is the "outer normal speed" along z, the sum of the inner normal speed along z and $-z$ may be called the "net outer normal speed" along z. This condition implies the following criterion for finite time blow-up: The solution cannot exist for all time when the "net outer normal speed" is negative for some z. In fact, an upper bound on its life span is given by

$$\inf \left\{ \frac{-\text{the width along } z}{\text{the net outer normal speed along } z} : \ z \in P \right\},$$

where P is the subset of the upper hemisphere consisting of all z along which the net inner normal speed are negative. Note that the width along z is given by $H(z, 0) + H(-z, 0)$ and it is equal to $u(x_0, 0) - v(y_0, 0)$ in the above proof.

Finally, we consider the expanding Gauss curvature flow by taking $F = -K^{-\beta}, \beta > 0$, in (4.2.19). Results on parabolic expanding Gauss curvature flows can be found in, for instance, Urabs [U1], [U3], and Chow-Tsai [CT]. The hypersurface expands to infinity in infinite time, and becomes round when β is less than or equal to $1/n$. When $\beta = 1$ and $n = 2$, it is known that the surface expands to infinity like a sphere in finite time by Schnürer [S]. In the hyperbolic case, we examine the motion of a sphere first. Indeed, when $X(0)$ is a sphere of radius R_0 and $X_t(0)$ has constant normal speed R_1, we have

Proposition 4.4. *Let* $c = R_1^2/2 - R_0^{1+n\beta}/(n\beta + 1)$. *For* $\beta > 0$,
(a) when $R_1 > 0$ *and* $c \in \mathbb{R}$, *the sphere expands to infinity as* $t \uparrow T$, *where* T *is finite when* $\beta \in (1/n, \infty)$ *and is infinite when* $\beta \in (0, 1/n]$. *In fact,*

$$R(t) = \begin{cases} O(t^{\frac{2}{1-n\beta}}), & \beta < \frac{1}{n}, \\ \\ O(e^t), & \beta = \frac{1}{n}. \end{cases}$$

(b) when $R_1 < 0$ *and* $c < 0$, *the sphere first contracts and then expands to infinity behaving like in (a); when* $R_1 < 0$ *and* $c \geq 0$, *the sphere contracts to a point in finite time.*

Proof. The normal flow reduces to an ODE for $R(t)$, the radius of the sphere at time t,

$$\begin{cases} R'' = R^{n\beta}, \\ \\ R(0) = R_0 > 0, \; R'(0) = R_1. \end{cases} \tag{4.3.3}$$

We multiply R' on both sides of the above equation to yield

$$\frac{R'^2}{2} = \frac{R^{1+n\beta}}{1+n\beta} + c,$$

where $c = R_1^2/2 - R_0^{1+n\beta}/(n\beta + 1)$. For the different choice of initial value, we obtain the proposition. □

There is a special case, namely, $\beta = 1$ and $n = 1$, where a rather complete analysis is possible. In this case the expanding flow becomes, in terms of its support function, a linear problem

$$\begin{cases} h_{tt} = h_{\theta\theta} + h, \\ h(0) = f, \ h_t(0) = g. \end{cases} \tag{4.3.4}$$

The solution can be represented by the cosine series, namely,

$$\begin{aligned} h(\theta, t) &= \frac{a_0 - a_0'}{2}e^{-t} + \frac{a_0 + a_0'}{2}e^t + (a_1 + a_1't)\cos\theta \\ &\quad + \sum_2^\infty \left(a_j \cos\sqrt{j^2-1}\ t + \frac{a_j'}{\sqrt{j^2-1}}\sin\sqrt{j^2-1}\ t\right)\cos j\theta, \end{aligned}$$

provided

$$f(\theta) = a_0 + \sum_1^\infty (a_j \cos j\theta + b_j \sin j\theta),$$

and

$$g(\theta) = a_0' + \sum_1^\infty (a_j' \cos j\theta + b_j' \sin j\theta).$$

For some choice of f and g, we show that a uniformly convex initial curve may develop an isolated singularity in finite time. For this purpose, let us take $f(\theta) = a_0(0) + a_1(0)\cos\theta + a_2(0)\cos 2\theta$, and $g(\theta) = a_0'(0)$, then

$$h(\theta, t) = \frac{a_0(0) - a_0'(0)}{2}e^{-t} + \frac{a_0(0) + a_0'(0)}{2}e^t + a_2(0)\cos\sqrt{3}t\cos 2\theta,$$

and

$$(h_{\theta\theta} + h)(\theta, t) = \frac{a_0(0) - a_0'(0)}{2}e^{-t} + \frac{a_0(0) + a_0'(0)}{2}e^t - 3a_2(0)\cos\sqrt{3}t\cos 2\theta.$$

At $t = 0$, $k = 1/(h_{\theta\theta} + h)$ is given by

$$k(0) = \frac{1}{a_0 - 3a_2\cos 2\theta}.$$

If we choose $a_0 = 2$ and $a_2 = -1/3$, then $k(0) > 0$ and the initial curve is uniformly convex. Moreover, we take $a_0' = -4$,

$$(h_{\theta\theta} + h)(\theta, t) = 3e^{-t} - e^t + \cos\sqrt{3}t\cos 2\theta.$$

which is positive at $t = 0$. However, there exists a time $T \sim 0.3$ such that $h_{\theta\theta} + h$ is positive on $[0, T)$ but it vanishes at $(\pm\pi/2, T)$. In other words, the flow is regular in $[0, T)$ and develops two isolated singularities at T. See Figure 4.1.

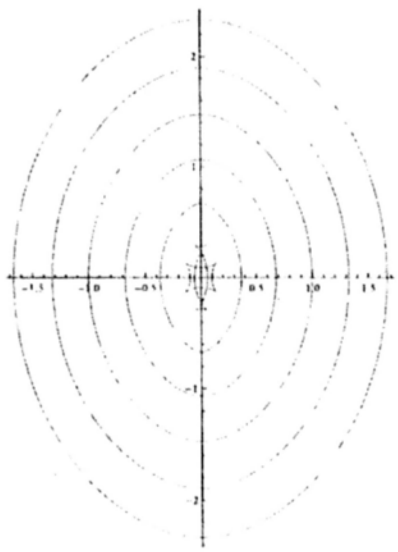

Figure 4.1: Singularity for problem (4.3.4)

On the other hand, for a class of initial values the flow behaves nicely.

Proposition 4.5. *Consider* (4.3.4) *where the initial values are smooth and satisfy* $f_{\theta\theta} + f,\ g_{\theta\theta} + g > 0$. *Then the flow remains smooth and expands to infinity like a circle.*

Proof. It suffices to show that $h_{\theta\theta} + h$ is positive for all t. We note that $\varphi = h_{\theta\theta} + h$ satisfies the one dimensional wave equation with a zero order term $\varphi_{tt} = \varphi_{\theta\theta} + \varphi$, and the initial values $\varphi(0) = f_{\theta\theta} + f,\ \varphi_t(0) = g_{\theta\theta} + g$ are positive. Therefore, we may apply the maximum principle for one-dimensional wave equation, see section 2 in chapter 4 of Protter-Weinberger [PW], to obtain the desired conclusion. The asymptotic behavior of the flow can be read off from the formula of the support function.

□

Chapter 5

Elastic Curves

In [LS], the authors study a mean curvature flow which stems from a geometrically natural action containing kinetic and internal energy terms. The equation under consideration models the nonlinear motion of an elastic membrane, driven by its surface tension only. It is natural to ask what will happen if we replace the internal energy by the elastic energy. We know that solutions of the corresponding Euler-Lagrange equation of elastic energy of a closed curve

$$\mathcal{E}(\gamma) = \int k^2 ds,$$

where k is the curvature of the curves, are called elastic curves. There are several types of equation of motion of elastic curves.In the previous works ([K], [BT]), the authors considered the motion of a fixed length elastic curves, governed by the elastic energy.

In this chapter, we consider the evolution of free (length unconstrained) elastic plane curves, which is derived from an Hamiltonian principle based on a geometrically natural action, consisting of a kinetic term and elastic energy term. The derived evolution equation turns out to be a coupled system of semi-linear 1-dimensional plane equation, where derivatives up to fourth order are involved. Koiso [K] used a perturbation to a composition of parabolic operators to prove the existence of a unique short-time solution for fixed elastic curve motion. We assume the flow is normal to get a single hyperbolic equation and investigate the conservation laws by Noether's Theorem.

5.1 The evolution equation

By Hamilton's principle, the equation of motion is given as critical points of the variational problem defined by the functional

$$\mathcal{L} = \int \mathcal{K}(t) - \mathcal{E}(t)dt, \tag{5.1}$$

where the kinetic energy at t is

$$\mathcal{K}(t) = \frac{1}{2}\int |\gamma_t|^2 ds,$$

and the elastic energy,

$$\mathcal{E}(t) = \frac{1}{2}\int k^2 ds.$$

According to the Hamilton principle, the flow we consider is the stationary for $\mathcal{L}$, i.e.

$$\frac{d}{d\epsilon}\mathcal{L}(\gamma + \epsilon\Phi)|_{\epsilon=0} = 0.$$

Proposition 5.1. *The stationary solutions of* (5.1) *satisfy the equation of motion*

$$\gamma_{tt} + \langle \mathbf{t}, \gamma_{ts}\rangle\gamma_t = -(k_{ss} + \frac{1}{2}k^3 + \frac{1}{2}|\gamma_t|^2 k)\mathbf{n} - \langle \gamma_t, \gamma_{ts}\rangle \mathbf{t}. \tag{5.2}$$

Proof. By copulation,

$$\begin{aligned}\frac{d}{d\epsilon}\mathcal{L}(\gamma + \epsilon\Phi)|_{\epsilon=0} &= \delta\mathcal{K} - \delta\mathcal{E} = \iint \delta(\frac{1}{2}|\gamma_t|^2)dsdt + \iint \frac{1}{2}|\gamma_t|^2(\delta ds)dt \\ &- \iint \delta(\frac{1}{2}k^2)dsdt - \iint \frac{1}{2}k^2(\delta ds)dt. \end{aligned}\tag{5.3}$$

We assume that

$$\frac{d\gamma}{dt} = \langle \gamma_t, \mathbf{t}\rangle\mathbf{t} + \langle \gamma_t, \mathbf{n}\rangle\mathbf{n},$$

and

$$\frac{d^2\gamma}{dt^2} = \langle \gamma_{tt}, \mathbf{t} \rangle \mathbf{t} + \langle \gamma_{tt}, \mathbf{n} \rangle \mathbf{n},$$

The first term of the right hand of (5.3) reads

$$\begin{aligned} \iint \delta(\frac{1}{2}|\gamma_t|^2) dsdt &= \iint \langle \gamma_t, \Phi_t \rangle dsdt \\ &= -\iint \langle \gamma_{tt}, \Phi \rangle dsdt - \iint \langle \gamma_t, \Phi \rangle (\frac{\partial}{\partial t}|\gamma_p|) dpdt \\ &= -\iint \langle \gamma_{tt}, \Phi \rangle dsdt - \iint \langle \gamma_t, \Phi \rangle \langle \gamma_{ts}, \mathbf{t} \rangle dsdt. \end{aligned}$$

The second term reads

$$\iint \frac{1}{2}|\gamma_t|^2 (\delta ds) dt = -\iint \langle \gamma_t, \gamma_{ts} \rangle \langle \mathbf{t}, \Phi \rangle - \iint \frac{1}{2}|\gamma_t|^2 k \langle \mathbf{n}, \Phi \rangle dsdt.$$

The third term turns to be

$$-\iint \delta(\frac{1}{2}k^2) dsdt = -\iint k\delta(k) dsdt = -\langle kk_s \mathbf{t} + (k_{ss} + k^3)\mathbf{n}, \Phi \rangle.$$

The fourth term is

$$-\iint \frac{1}{2}k^2 (\delta ds) dt = \iint kk_s \langle \mathbf{t}, \Phi \rangle + \iint \frac{1}{2}k^3 \langle \mathbf{n}, \Phi \rangle dsdt.$$

Combining the above identities together, we obtain the proposition. □

Since the tangential variations do not alter the shape of the curves, for simplify, we take the initial velocity is normal to the curve, i.e. its tangential part vanishes

$$\langle \gamma_t, \mathbf{t} \rangle|_{t=0} = 0.$$

We can prove this property holds for all times if they vanish initially. Under this assumption, we can reduce the flow to a single equation.

For the normal flow

$$-\langle \mathbf{t}, \gamma_{ts} \rangle \gamma_t = \langle \mathbf{t}_s, \gamma_t \rangle \gamma_t = k \langle \mathbf{n}, \gamma_t \rangle \langle \mathbf{n}, \gamma_t \rangle \mathbf{n} = k|\gamma_t|^2 \mathbf{n},$$

then the flow (5.2) yields

$$\gamma_{tt} = -(k_{ss} + \frac{1}{2}k^3 - \frac{1}{2}|\gamma_t|^2 k)\mathbf{n} - \langle \gamma_t, \gamma_{ts} \rangle \mathbf{t}. \tag{5.4}$$

It is convenient to begin our investigation with the case of graphs. We assume the curve can be locally written as an entire graph $\gamma = (x, u(x))$. Therefore we get a single equation for the graph

$$\begin{aligned} u_{tt} &- \frac{2u_x u_t}{(1+u_x^2)} u_{xt} + \frac{u_x^2 u_t^2}{(1+u_x^2)^2} u_{xx} \\ &= \sqrt{1+u_x^2}(-k_{ss} - \frac{1}{2}k^3 + \frac{1}{2}\frac{u_t^2 k}{1+u_x^2}) \\ &= -\frac{u_{xxxx}}{(1+u_x^2)^2} + \frac{10 u_x u_{xx} u_{xxx}}{(1+u_x^2)^3} - \frac{18 u_x^2 u_{xx}^3}{(1+u_x^2)^4} \\ &+ \frac{3u_{xx}^3}{(1+u_x^2)^3} + \frac{u_{xx}^3}{2(1+u_x^2)^4} + \frac{u_t^2 u_{xx}}{2(1+u_x^2)^2}. \end{aligned} \tag{5.5}$$

The equation is the Euler-Lagrange equation of variation

$$\begin{aligned} \mathcal{L} &= \iint \frac{1}{2}|\gamma_t|^2 - \frac{1}{2}k^2 ds dt \\ &= \iint (\frac{u_t^2}{2(1+u_x^2)} - \frac{u_{xx}^2}{2(1+u_x^2)^3})\sqrt{1+u_x^2} dx dt \\ &= \iint L \, dx dt, \end{aligned}$$

where

$$L = \frac{u_t^2}{2(1+u_x^2)^{\frac{1}{2}}} - \frac{u_{xx}^2}{2(1+u_x^2)^{\frac{5}{2}}}.$$

In order to state local well-posedness for the flow of graphs, we would construct various conservation laws satisfied by solution of the flow via Noether's theorem using a Lagrangian. In order to derive precise results, we will present some notation and preliminaries that will be used.

5.2 Notation and preliminaries

Let $x = (x^1, \cdots, x^n)$ be the independent variable with coordinate x^i, and $u = (u^1, \cdots, u^m)$ be the dependent variable with coordinates u^α. The derivatives of u with respect to x are

$$u_i^\alpha = D_i(u^\alpha),\ u_{ij}^\alpha = D_j D_i(u^\alpha),\ \cdots,$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, i = 1, \cdots, n$$

is the operator of total differentiation.

We say an current $P = (P^1, \cdots, P^n)$ is conserved if it satisfies

$$D_i P^i = 0$$

along the solution.

Theorem 5.1. *(Noether[O]) Suppose G is a (local) one-parameter group of symmetries of the variational problem $\mathcal{L} = \int L(x, u, u_{(1)}, \cdots, u_{(n)})\, dx$. Let*

$$\nu = \xi^i \frac{\partial}{\partial x^i} + \phi^\alpha \frac{\partial}{\partial u^\alpha}$$

be the infinitesimal generator of G, and

$$Q_\alpha(x, u) = \phi_\alpha - \xi^j u_j^\alpha,\ u_j^\alpha \partial u^\alpha / \partial x^j,$$

the corresponding characteristic of v. Then $Q = (Q_1, \cdots, Q_q)$ is also the characteristic of a conservation law for the Euler-Lagrange equations $E(L) = 0$; in other words, there is a p-tuple $P(x, u^{(m)}) = (P_1, \cdots, P_p)$ such that

$$Div P = Q \cdot E(L) = Q_i E_i(L)$$

is a conservation law in characteristic form for the Euler-Lagrange equations $E(L) = 0$.

The statement of Noether's theorem remains the same if we replace variational symmetry by divergence symmetry i.e. there exists a p-tuple $B(x, u^{(m)}) = (B_1, \cdots, B_p)$ of functions of x, u and derivatives of u such that

$$pr^n\nu(L) + L\ Div\xi = Div\ B$$

The Noether operator associated with a Lie-Bäcklund operator ν is defined by

$$N^i = \xi^i + Q^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1}\cdots D_{i_s}(Q^\alpha)\frac{\delta}{\delta u^\alpha_{ii_1\cdots i_s}},\ i = 1, \cdots, n,$$

where the Euler-Lagrange operator is defined by

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1}(-1)^s D_{j_1}\cdots D_{j_s}(Q^\alpha)\frac{\partial}{\partial u^\alpha_{ij_1\cdots j_s}}, \alpha = 1, \cdots, m.$$

L is referred to as a Lagrangian and the associated functional

$$\mathcal{L} = \int L(x, u, u_{(1)}, \cdots, u_{(r)})\ dx.$$

Corresponding to each ν, a conserved flow is obtained via Noether' theorem. A conserve vector is a tuple $P = (P^1, \cdots, P^n)$, where

$$P^i = N^i L - B_i,\ i = 1, \cdots, n.$$

such that

$$D_i(P^i) = 0.$$

5.3 Symmetries and conservation laws

In our work, we consider the scalar case in two dimensions, namely, $(x^1, x^2) = (x, t)$. Suppose $\nu = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \phi(x, t, u)\partial_u$ is a Noether point symmetry generator with gauge (f, g). Then the conserved flow (X, P) is

$$\begin{aligned} X &= L\xi + Q\frac{\delta L}{\delta u_x} + D_t(Q)\frac{\delta}{\delta u_{xt}} + D_x(Q)\frac{\delta}{\delta u_{xx}} + \cdots - f, \\ P &= L\tau + Q\frac{\delta L}{\delta u_t} + D_t(Q)\frac{\delta}{\delta u_{tt}} + D_x(Q)\frac{\delta}{\delta u_{tx}} + \cdots - g. \end{aligned}$$

Since the flow we consider is geometric, it is invariant under Euclidean group. It is easy to check that the generators

$$\{\partial_t, \partial_x, \partial_u, -u\partial_x + x\partial_u\}$$

are strict Noether symmetries of the Lagrangian

$$L = \frac{u_t^2}{2(1+u_x^2)^{\frac{1}{2}}} - \frac{u_{xx}^2}{2(1+u_x^2)^{\frac{5}{2}}}.$$

and are also symmetry group of the corresponding Euler-Lagrange equation

$$\begin{aligned} E(L) &= u_{tt} - \frac{2u_x u_t}{(1+u_x^2)} u_{xt} + \frac{u_x^2 u_t^2}{(1+u_x^2)^2} u_{xx} + \frac{u_{xxxx}}{(1+u_x^2)^2} - \frac{10 u_x u_{xx} u_{xxx}}{(1+u_x^2)^3} \\ &+ \frac{18 u_x^2 u_{xx}^3}{(1+u_x^2)^4} - \frac{3u_{xx}^3}{(1+u_x^2)^3} - \frac{u_{xx}^3}{2(1+u_x^2)^4} - \frac{u_t^2 u_{xx}}{2(1+u_x^2)^2} = 0. \end{aligned}$$

We found that the Euler-Lagrange equation admits other symmetry group (scaling group $x\partial_x + 2t\partial_t + u\partial_u$), but it is not a variational symmetry group of the original variational problem.

We now list the corresponding conserved vectors for these Noether symmetries and conserved Density.

(i) $-\partial_t$, $(Q_1 = u_t)$.

$$\begin{aligned} T_1 &= -\frac{u_t^2}{2(1+u_x^2)^{\frac{1}{2}}} + \frac{u_{xx}^2}{2(1+u_x^2)^{\frac{5}{2}}} + u_t \frac{u_t}{(1+u_x^2)^{\frac{1}{2}}} \\ &= \frac{u_{xx}^2}{2(1+u_x^2)^{\frac{5}{2}}} + \frac{u_t^2}{2(1+u_x^2)^{\frac{1}{2}}}, \\ X_1 &= u_t\Big(-\frac{u_x u_t^2}{2(1+u_x^2)^{\frac{3}{2}}} + \frac{5u_x u_{xx}^2}{2(1+u_x^2)^{\frac{7}{2}}} \\ &- D_x\big(-\frac{u_{xx}}{(1+u_x^2)^{\frac{5}{2}}}\big)\Big) + u_{xt}\big(-\frac{u_{xx}}{(1+u_x^2)^{\frac{5}{2}}}\big) \\ &= -\frac{u_x u_t^3}{2(1+u_x^2)^{\frac{3}{2}}} - \frac{5u_x u_t u_{xx}^2}{2(1+u_x^2)^{\frac{7}{2}}} + \frac{u_t u_{xxx}}{(1+u_x^2)^{\frac{5}{2}}} - \frac{u_{xx} u_{xt}}{(1+u_x^2)^{\frac{5}{2}}}. \end{aligned}$$

Thus,

$$D_t T_1 + D_x X_1 = u_t E(L) = Q_1 E(L) = 0.$$

Consider the functional

$$E = \int T_1 dx = \int \frac{u_{xx}^2}{2(1+u_x^2)^{\frac{5}{2}}} + \frac{u_t^2}{2(1+u_x^2)^{\frac{1}{2}}} dx$$

then

$$E(t) - E(0) = 0,$$

which means $E = \mathcal{K} + \mathcal{E}$ (the total Energy) is conserved.

(*ii*) $-\partial_x$, $(Q_2 = u_x)$.

$$\begin{aligned} T_2 &= u_x \frac{u_t}{(1+u_x^2)^{\frac{1}{2}}} = \frac{u_x u_t}{(1+u_x^2)^{\frac{1}{2}}}, \\ X_2 &= -\Big(\frac{u_t^2}{2(1+u_x^2)^{\frac{1}{2}}} + \frac{u_{xx}^2}{2(1+u_x^2)^{\frac{5}{2}}} + \frac{u_x^2 u_t^2}{2(1+u_x^2)^{\frac{3}{2}}} \\ &\quad - \frac{u_x u_{xxx}}{(1+u_x^2)^{\frac{5}{2}}} + \frac{5u_x^2 u_{xx}^2}{2(1+u_x^2)^{\frac{7}{2}}}\Big). \end{aligned}$$

Thus,

$$D_t T_2 + D_x X_2 = u_x E(L) = Q_2 E(L) = 0.$$

Consider the functional

$$\mathcal{M} = \int T_2 dx = \int \frac{u_x u_t}{(1+u_x^2)^{\frac{1}{2}}} dx.$$

then

$$\mathcal{M}(t) - \mathcal{M}(0) = 0,$$

which means $\mathcal{M}$ (the linear momenta) is conserved.

(*iii*) ∂_u, $(Q_3 = 1)$.

$$\begin{aligned} T_3 &= \frac{u_t}{(1+u_x^2)^{\frac{1}{2}}}, \\ X_3 &= -\frac{u_x u_t^2}{2(1+u_x^2)^{\frac{3}{2}}} - \frac{5u_x u_{xx}^2}{2(1+u_x^2)^{\frac{7}{2}}} + \frac{u_{xxx}}{(1+u_x^2)^{\frac{5}{2}}}. \end{aligned}$$

Thus,

$$D_t T_3 + D_x X_3 = E(L) = 0.$$

Consider the functional

$$\mathcal{P} = \int T dx = \int \frac{u_t}{(1+u_x^2)^{\frac{1}{2}}} dx$$

then

$$\mathcal{P}(t) - \mathcal{P}(0) = 0$$

which means $\mathcal{P}$ (the momenta) is conserved.

(*iv*) $-u\partial_x + x\partial_u$, ($Q_4 = x + uu_x$).

$$\begin{aligned} T_4 &= (x+uu_x)\frac{u_t}{(1+u_x^2)^{\frac{1}{2}}} = \frac{u_t(x+uu_x)}{(1+u_x^2)^{\frac{1}{2}}}, \\ X_4 &= -\frac{uu_t^2}{2(1+u_x^2)^{\frac{1}{2}}} + \frac{uu_{xx}^2}{2(1+u_x^2)^{\frac{5}{2}}} - \frac{u_x u_t^2(x+uu_x)}{2(1+u_x^2)^{\frac{3}{2}}} \\ &+ \frac{u_{xxx}(x+uu_x)}{(1+u_x^2)^{\frac{5}{2}}} + \frac{5u_x^2 u_{xx}^2(x+uu_x)}{2(1+u_x^2)^{\frac{7}{2}}} - \frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}} - \frac{uu_{xx}^2}{(1+u_x^2)^{\frac{5}{2}}}. \end{aligned}$$

Thus,

$$D_t T_4 + D_x X_4 = (x+uu_x)E(L) = 0.$$

Consider the functional

$$\mathcal{R} = \int T dx = \int \frac{u_t(x+uu_x)}{(1+u_x^2)^{\frac{1}{2}}} dx = \int \langle \gamma_t, \mathbf{n} \rangle \langle \gamma, \mathbf{t} \rangle ds$$

then

$$\mathcal{R}(t) - \mathcal{R}(0) = 0,$$

which means $\mathcal{R}$ is conserved.

Finally, we list all these quantities in the following table.

Symmetry	Characteristic	Conserved Density
$-\partial_x$	u_x	$E = \frac{u_x u_t}{2(1+u_x^2)^{\frac{1}{2}}}$
∂_u	1	$\mathcal{M} = \frac{u_t}{(1+u_x^2)^{\frac{1}{2}}}$
$-\partial_t$	u_t	$\mathcal{P} = \frac{u_{xx}^2}{2(1+u_x^2)^{\frac{5}{2}}} + \frac{u_t^2}{2(1+u_x^2)^{\frac{1}{2}}}$
$-u\partial_x + x\partial_u$	$(x + uu_x)$	$\mathcal{R} = \frac{u_t(x+uu_x)}{(1+u_x^2)^{\frac{1}{2}}}$

5.4 Group invariant solution

(a). $\{x\partial_x + 2t\partial_t + u\partial_u\}$

Group invariants: $y = \frac{x}{\sqrt{t}}$, $v = \frac{u}{\sqrt{t}}$.

Plug $u = \sqrt{|t|}v(\frac{x}{\sqrt{t}})$ into the equation.

Since

$$
\begin{aligned}
u_t &= \frac{1}{2\sqrt{t}}(v - yv') \\
u_{tt} &= -\frac{1}{4t^{\frac{3}{2}}}(v - yv' - y^2v'') \\
u_x &= v' \\
u_{xt} &= -v''\frac{y}{2t} \\
u_{xx} &= v''\frac{1}{\sqrt{t}} \\
u_{xxx} &= v'''\frac{1}{t} \\
u_{xxxx} &= v''''\frac{1}{t^{\frac{3}{2}}},
\end{aligned}
$$

the equation (5.5) turn to be

$$\begin{aligned}&-\frac{1}{4}(v - yv' - y^2v'') + \frac{v'(v - v'y)v''y}{2(1+v'^2)} + \frac{v'^2(v - v'y)^2v''}{4(1+v'^2)^2}\\&+\frac{v''''}{(1+v'^2)^2} - \frac{10v'v''v'''}{(1+v'^2)^3} + \frac{18v'^2v''^3}{(1+v'^2)^4} - \frac{3v''^3}{(1+v'^2)^3}\\&-\frac{v''^3}{2(1+v'^2)^4} - \frac{(v - v'y)^2v''}{8(1+v'^2)^2} = 0.\end{aligned}$$

It is a fourth order nonlinear ODE.

If we express the flow as the equation of support function, (5.5) turn to be

$$h_{tt} = \frac{h_{t\theta}^2}{h_{\theta\theta} + h} + 3\frac{(h_{\theta\theta\theta} + h_\theta)^2}{(h_{\theta\theta} + h)^5} - \frac{h_{\theta\theta\theta\theta} + h_{\theta\theta}}{(h_{\theta\theta} + h)^4} + \frac{2}{(h_{\theta\theta} + h)^3} - \frac{1}{2}\frac{h_t^2}{h_{\theta\theta} + h}. \tag{5.6}$$

The dilatation group $\{x\partial_x + 2t\partial_t + u\partial_u\}$ reads as $\{h\partial_h + 2t\partial_t\}$. Self-similar solution can be written as $h(t,\theta) = \sqrt{|t|}\phi(\theta)$, where ϕ satisfies

$$-\frac{\phi}{4} = \frac{\phi_\theta^2}{4(\phi_{\theta\theta} + \phi)} + 3\frac{(\phi_{\theta\theta\theta} + \phi_\theta)^2}{(\phi_{\theta\theta} + \phi)^5} - \frac{\phi_{\theta\theta\theta\theta} + \phi_{\theta\theta}}{(\phi_{\theta\theta} + \phi)^4} + \frac{2}{(\phi_{\theta\theta} + \phi)^3} - \frac{1}{8}\frac{\phi^2}{\phi_{\theta\theta} + \phi}.$$

Given suitable initial value, we can plot the graph for the curve.

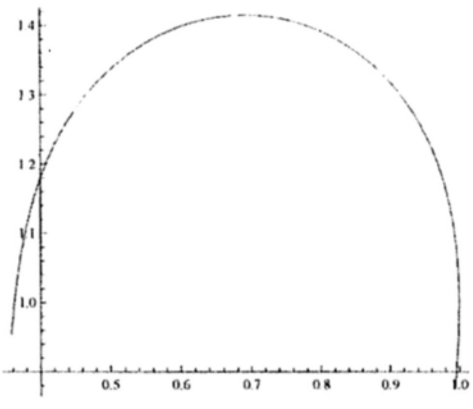

Figure 5.1: Self-similar circle

(b) *Circles.* Let γ_0 be a circle of radius r_0. If γ is a solution of (2) with the initial value $\gamma(x,0) = \gamma_0 > 0$, and $\frac{d}{dt}\gamma(x,0) = r_1\mathbf{n}$. Then the normal flow reduces to

$$r'' = \frac{1}{2r^3} - \frac{r'^2}{2r}.$$

1. When $r_1 \geq 0$, r increases and blows at finite time.

2. When $r_1 < 0$ and, $r_1 > f(r_0)$, f can be determined from the equation, r decreases first, then increases and blows at finite time.

3. When $r_1 < 0$ and $r_1 \leq f(r_0)$, r decreases and tend to 0 at finite time.

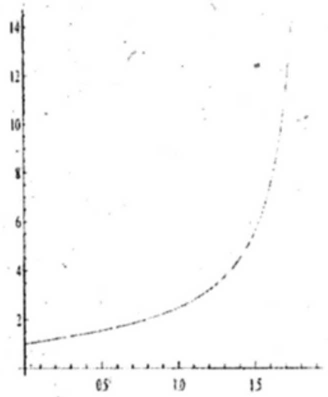

Figure 5.2: $r_1 \geq 0$.

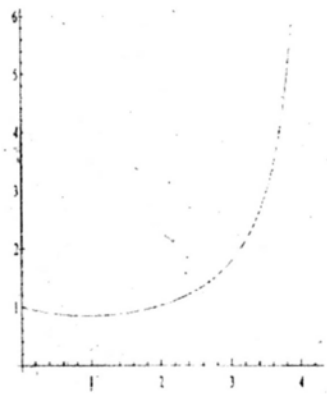

Figure 5.3: $r_0 = 1$, $r_1 = -1/3$.

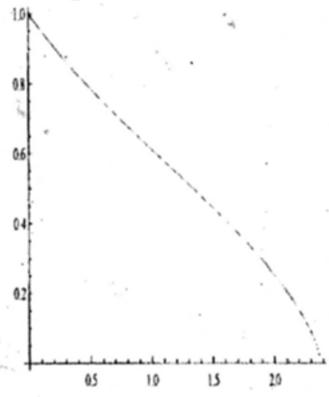

Figure 5.4: $r_0 = 1$, $r_1 = -1/2$.

(c). Traveling wave $\{\partial_x + \partial_t\}$

Invariants: $y = x - t$, $u = v(y)$.

$$\begin{aligned} u_x &= v' \\ u_t &= -v' \\ u_{xx} &= v'' \\ u_{xt} &= -v'' \\ u_{xxx} &= v''' \\ u_{xxxx} &= v'''' \end{aligned}$$

So the equation is reduced to

$$\begin{aligned} v'' - \frac{v'^2 v''}{(1+v'^2)} + \frac{v'^4 v''}{(1+v'^2)^2} + \frac{v''''}{(1+v'^2)^2} - \frac{10 v' v'' v'''}{(1+v'^2)^3} \\ + \frac{18 v'^2 v''^3}{(1+v'^2)^4} - \frac{3 v''^3}{(1+v'^2)^3} - \frac{v''^3}{2(1+v'^2)^4} - \frac{v'^2 v''}{2(1+v'^2)^2} = 0. \end{aligned}$$

It is a fourth order nonlinear ODE.

When $t = 0$, $u = v(y)$, the curve is showed in the following figure.

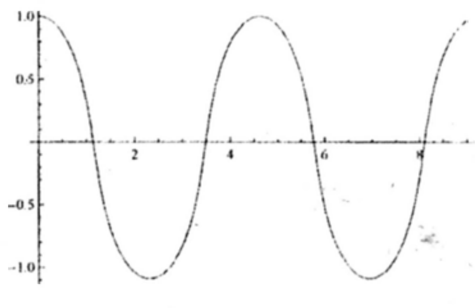

Figure 5.5: Traveling wave solution

Bibliography

[AGLM] L. Alvarez, F. Guichard, P. L. Lions and J. M. Morel, Axioms and fundamental equations of image processing, Arch. Rational Mech. Anal. **123** (1993), 199-257.

[A1] B. Andrews, Contraction of convex hypersurfaces in Euclidean space, Calc. Var. Partial Differential Equations **2** (1994), 151-171.

[A2] B. Andrews, Contraction of convex hypersurfaces by their affine normal, J. Differential Geom. **43** (1996), 207-230.

[A3] B. Andrews, Gauss curvature flow: the fate of the rolling stones, Invent. Math. **138** (1999), 151-161.

[A4] B. Andrews, Motion of hypersurfaces by Gauss curvature, Pacific J. Math. **195** (2000), 1-34.

[BT] A. Burchard, L.E. Thomas, On the Cauchy problem for a dynamical Euler's elastica, Comm. Partial Differential Equations **28** (2003), 271-300.

[C] B. Chow, Deforming convex hypersurfaces by the nth root of the Gaussian curvature, J. Differential Geom. **22** (1985), 117-138.

[CH] R. Courant and D. Hilbert, Methods of Mathematical Physics, vol. I, Interscience Publishers, Inc., New York, 1953.

[CNS] L. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian, Acta Math. **155** (1985), 261-301.

[CT] B. Chow and D. H. Tsai, Nonhomogeneous Gauss curvature flows, Indiana Univ. Math. J. **47** (1998), 965-994.

[CZ] K.-S. Chou and X.-P. Zhu, The Curve Shortening Problem. Chapman & Hall/CRC, Boca Raton 2001.

[E] K. Ecker, Regularity Theory for Mean Curvature Flow. Birkhuser Boston, Inc., Boston, MA, 2004.

[F] W. Firey, Shape of worn stones, Mathematika **21** (1974), 1-11.

[GH] M. Gage and R. S. Hamilton, The heat equation shrinking convex plane curves, J. Differential Geom. **23** (1986), 69-96.

[Gi] Y. Giga, Surface Evolution Equations. A level set approach. Birkhuser Verlag, Basel, 2006.

[G] M. A. Grayson, The heat equation shrinks embedded plane curves to round points, J. Differential Geom. **26** (1987), 285-314.

[GP] M. E. Gurtin, P. Podio-Guidugli, A hyperbolic theory for the evolution of plane curves, SIAM J. Math. Anal. **22** (1991), 575-586.

[HKL] C.-L. He, D.-X. Kong and K.-F. Liu, Hyperbolic mean curvature flow. J. Differential Equations **246** (2009), 373-390.

[HP] G. Huisken and A. Polden, Geometric evolution equations for hypersurfaces, 45-84, Lecture Notes in Math. **1713**, Springer, Berlin, 1999.

[KW] D.-X. Kong and Z.-G. Wang, Formation of singularities in the motion of plane curves under hyperbolic mean curvature flow, J. Differential Equations **247** (2009) 1694-1719.

[KLW] D.-X. Kong, K.-F. Liu and Z.-G. Wang, Hyperbolic mean curvature flow: Evolution of plane curves, Acta Math. Sci. **29** (2009) 493-514.

[K] N. Koiso, On motion of an elastic wire and singular perturbation of a 1-dimensional plate equation, Osaka J. Math **37** (2000) 905-924.

[LS] P. G. LeFloch and K. Smoczyk, The hyperbolic mean curvature flow, J. Math. Pures Appl. **90** (2008) 591-614.

[M] S. Mizohata, The Theory of Partial Differential Equations. Cambridge University Press, New York 1973.

[O] P. J. Olver, Applications of Lie Groups to Differential Equations, 2nd ed. Springer-Verlag, New York 1993.

[Ov] L. V. Ovsiannikov, Group Analysis of Differential Equations, Academic Press, New York 1982.

[PW] M. P. Protter and H. F. Weinberger, Maximum Principles in Differential Equations. Prentice-Hall, New Jersey 1967.

[S] O. Schnürer, Surfaces expanding by the inverse Gauss curvature flow, J. Reine Angew. Math. **600** (2006), 117-134.

[ST] G. Sapiro and A. Tannenbaum, On affine plane curve evolution, J. Funct. Anal. **119**(1994), 79-120.

[T] K. Tso, Deforming a hypersurface by its Gauss-Kronecker curvature, Comm. Pure Appl. Math. **38** (1985), 867-882.

[Ta] M. E. Taylor, Partial Differential Equations III: Nonlinear Equations. Springer-Verlag, New York 1996.

[U1] J. Urbas, An expansion of convex hypersurfaces. J. Differential Geom. **33** (1991), 91-125 and **35** (1992), 763-765.

[U2] J. Urbas, Complete noncompact self-similar solutions of Gauss curvature flows, I. Positive powers, Math. Ann. **311** (1998), 251-274.

[U3] J. Urbas, Complete noncompact self-similar solutions of Gauss curvature flows, II. Negative powers, Adv. Differential Equations **4** (1999), 323-346.

[Y] S.-T. Yau, Review of geometry and analysis, Asian J. Math. **4** (2000), 235-278.

[Z] X.-P. Zhu, Lectures on Mean Curvature Flows. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2002.