Concentration of Laplace eigenfunctions

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ABSTRACT

In this thesis, we give a review of known results concerning the concentration of Laplace eigenfunctions in the high-energy limit. We review asymptotic bounds on L^p norms of eigenfunctions, and possible quantum limits, under a variety of hypotheses on the manifold. We prove a new result that states that if eigenfunctions converge weakly to a quantum limit on the n-torus, they must also do so on a "rescaled" n-torus.

ABRÉGÉ

Dans ce mémoire, nous résumons les résultats connus concernant la concentration des fonctions propres du laplacien dans la limite de haute énergie. Nous révisons les bornes asymptotiques sur les normes L^p des fonctions propres, et les limites quantiques admissibles, sous une variété d'hypothèses sur la variété. Nous prouvons un résultat nouveau qui stipule que si les fonctions propres convergent faiblement vers une limite quantique sur le n-tore, ils doivent aussi le faire sur le n-tore rééchelonné.

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CHAPTER 1
Introduction

Let (M, g) be a compact n-dimensional Riemannian manifold. The Laplace operator Δ is given in local coordinates by

$$\Delta = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right),$$

where $\sqrt{g} = \sqrt{\det(g_{ij})}$. It is well-known that Δ is self-adjoint, and that $L^2(M)$ admits an orthonormal basis $\{\phi_j\}$ of Laplace eigenfunctions, where $\Delta\phi_j + \lambda_j\phi_j = 0$ and $\lambda_j \to \infty$ (see e.g. [40]).

In this thesis we give a survey of results about the concentration of Laplace eigenfunctions as their eigenvalue goes to infinity. This is known as the *high-energy limit*. By concentration we mean the phenomenon of an eigenfunction taking much larger values (in modulus) on some parts of the manifold than others.

One way to measure the concentration of an eigenfunction ϕ is by its L^p norms, which are given by

$$||\phi||_p = \left(\int_M |\phi|^p \mathrm{dV} \right)^{\frac{1}{p}},$$

where $2 \leq p < \infty$ and dV is the Riemannian volume form on M. The L^p norms are homogeneous, so any eigenfunction can be scaled to give a new eigenfunction with arbitrarily large L^p norms. Thus, we consider L^2 normalized eigenfunctions, i.e. those with $||\phi||_2 = 1$. The L^p norms are more sensitive to large values than small ones, so a large L^p norm indicates a non-uniformly distributed eigenfunction.

In the first part of the thesis, we review results about the asymptotics of L^p norms of eigenfunctions as $\lambda \to \infty$. In chapter 2 we give bounds on the L^p norms of eigenfunctions that are valid on any compact manifold, including the bounds of Avakumovič-Levitan-Hörmander and Sogge. In chapter 3 we show that Sogge's bound is sharp for the standard sphere S^n. Chapters 4 and 5 give results characterizing manifolds with eigenfunction growth that is respectively as small or as large as possible.

In chapters 6 - 9 we examine the concentration of eigenfunctions on a variety of classes of manifolds with ergodic geodesic flow. In chapter 7 we describe some consequences of a heuristic that states that eigenfunctions on such manifolds should behave like Gaussian random waves. We show that the general L^p bounds can be improved somewhat, and state conjectures about their true rate of growth. We also discuss another tool that measures concentration: quantum limits of eigenfunctions. We state the quantum ergodicity theorem and the related quantum unique ergodicity conjecture.

In chapter 10 we describe L^p bounds on tori, a special class of manifolds where exact formulas for eigenfunctions are known.

In chapter 11 we review results that show a symmetry in the L^p norms of the positive and negative parts of an eigenfunction.

In chapter 12 we give Burq-Gérard-Tzvetkov's bound for the L^p norms of the restrictions of eigenfunctions to submanifolds. We also apply a result of Khovanskii to give a bound on the number of points in the intersection of a collection of algebraic hypersurfaces and nodal sets.

In chapter 13 we prove a new result concerning eigenfunctions on "rescaled" tori. We show that if a sequence of toral eigenfunctions converges weakly to some quantum limit, the same weak convergence must also hold on a "rescaled copy" of the torus.

CHAPTER 2
General L^p bounds

Weyl's Law

Let (M, g) be a compact n-dimensional Riemannian manifold. *Weyl's law* gives an asymptotic formula for the number of eigenvalues of the Laplacian less than λ. It states:

$$N(\lambda) := \#\{j : \lambda_j \leq \lambda\} = \frac{|B_n|}{(2\pi)^n}\mathrm{Vol}(M, g)\lambda^{\frac{n}{2}} + O(\lambda^{\frac{n-1}{2}}),$$

where $|B_n|$ is the Euclidean volume of the unit ball in $\mathbb{R}^n$, and $\mathrm{Vol}(M, g)$ is the volume of M with respect to the metric g.

Given an orthonormal basis $\{\phi_j\}$ of $L^2(M)$ of Laplace eigenfunctions, the spectral function of the Laplacian is given by the formula:

$$e(x, y, \lambda) = \sum_{\lambda_j \leq \lambda} \phi_j(x)\overline{\phi_j(y)}.$$

Avakumovič [10] and Levitan [30] showed that

$$\sum_{\lambda_j \leq \lambda} |\phi_j(x)|^2 = e(x, x, \lambda) = \frac{1}{(2\pi)^n}|B^n|\lambda^{\frac{n}{2}} + R(\lambda, x), \tag{2.1}$$

where $R(\lambda, x) = O(\lambda^{\frac{n-1}{2}})$ uniformly in x, a result called *local Weyl's law*. It was shown for general elliptic differential operators by Hörmander in [22]. Integrating local Weyl's law over the whole manifold yields Weyl's law.

L^p bounds

Local Weyl's law can be applied to bound the L^∞ norms of eigenfunctions, as follows:

Theorem 1. *Let (M, g) be a compact n-dimensional C^∞ Riemannian manifold.*

Then if ϕ_λ, with $\Delta\phi_\lambda + \lambda\phi_\lambda = 0$, is an eigenfunction of the Laplacian,

$$\frac{||\phi_\lambda||_\infty}{||\phi_\lambda||_2} \leq c_1 \lambda^{\frac{n-1}{4}}. \tag{2.2}$$

Proof. In equation 2.1, the jump at λ of the left hand side is $\sum_{\lambda_j=\lambda} |\phi_{\lambda_j}|^2(x)$. The first term of the right hand side is continuous in λ, so the jump of the right hand side is just the jump in $R(\lambda, x)$, i.e. $\lim_{\epsilon\to 0} R(\lambda+\epsilon, x) - R(\lambda-\epsilon, x)$. Notice then:

$$\begin{aligned} R(\lambda+\epsilon, x) - R(\lambda-\epsilon, x) &\leq |R(\lambda+\epsilon, x)| + |R(\lambda-\epsilon, x)| \\ &\leq c(\lambda+\epsilon)^{\frac{n-1}{2}} + c(\lambda-\epsilon)^{\frac{n-1}{2}} \\ &\leq 2c(\lambda+\epsilon)^{\frac{n-1}{2}}. \end{aligned}$$

Taking $\epsilon \to 0$ gives that the jump is $O(\lambda^{\frac{n-1}{2}})$. Thus,

$$|\phi_\lambda(x)|^2 \leq \sum_{\lambda_j=\lambda} |\phi_{\lambda_j}(x)|^2 = O\left(\lambda^{\frac{n-1}{2}}\right).$$

Taking square-roots gives the result. □

Because this bound follows from local Weyl's law, proved by Avakumovič, Levitan, and Hörmander, it is referred to as the Avakumovič-Levitan-Hörmander L^∞ bound. In chapter 3, we show that it is sharp for the unit sphere S^n, where it is attained by the zonal spherical harmonics.

The Avakumovič-Levitan-Hörmander bound (2.2) leads easily to a bound on the multiplicity of eigenvalues. Let λ be an eigenvalue of Δ, with corresponding eigenspace E_λ. Set $m_\lambda = \dim E_\lambda$. Then we have

Lemma.

$$m_\lambda \leq c_2 \lambda^{(n-1)/2}. \tag{2.3}$$

Proof. Let ϕ_i be an orthonormal basis for E_λ. Let $K(x,y) = \sum_{i=1}^{m_\lambda} \phi_i(x)\phi_i(y)$, the Bergman kernel for orthogonal projection onto the eigenspace E_λ. Then $\int_M K(x,x)\mathrm{dV} = \sum_{i=1}^{m_\lambda} \int_M \phi_i^2(x)\mathrm{dV} = \sum_{i=1}^{m_\lambda} 1 = m_\lambda$. It follows that there must be a point $x_0 \in M$ such that $m_\lambda \leq K(x_0, x_0)\mathrm{Vol}(M)$. We choose a special orthonormal basis for E_λ as follows. Let $r_0 : L^2(M) \to \mathbb{C}$ denote the evaluation map at x_0, such that $r_0(f) = f(x_0)$ for any function $f \in L^2(M)$. Now pick each $\phi_i, i \geq 2$ to lie in $\ker(r_0)$. Then $K(x_0, x_0) = \phi_1^2(x_0, x_0) + \sum_{i=2}^{m_\lambda} 0 \leq c\lambda^{(n-1)/2}$ by (2.2), whence $m_\lambda \leq c\mathrm{Vol}(M)c\lambda^{(n-1)/2}$. □

In [16], Donnelly investigated the geometric dependence of the constant c_1 appearing in the Avakumovič-Levitan-Hörmander's bound (2.2). He showed:

Theorem 2. *Let (M, g) be a compact Riemannian manifold of dimension n. Suppose the injectivity radius of M is bounded below by c_3 and its sectional curvature is bounded above (in absolute value) by c_4. Then if ϕ, with $\Delta\phi + \lambda\phi = 0$, is an eigenfunction of the Laplacian, we have $||\phi||_\infty \leq c_1\lambda^{\frac{n-1}{4}}$, where c_1 depends only on c_3, c_4, and n. Furthermore, writing m_λ for the multiplicity of the eigenspace corresponding to λ, we have $m_\lambda \leq c_2\lambda^{\frac{n-1}{2}}$, where c_2 depends only on c_1 and an upper bound for the volume of M.*

Sogge proved general L^p bounds for arbitrary manifolds:

Theorem 3 (Sogge [45])**.** *Let (M, g) be a compact n-dimensional C^∞ Riemannian manifold. Then*

$$\frac{||\phi_\lambda||_p}{||\phi_\lambda||_2} = O(\lambda^{\delta(p)}), \quad 2 \leq p \leq \infty \tag{2.4}$$

where

$$\delta(p) = \begin{cases} \frac{n}{2}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{4}, & \frac{2(n+1)}{n-1} \leq p \leq \infty \\ \frac{n-1}{4}\left(\frac{1}{2} - \frac{1}{p}\right), & 2 \leq p \leq \frac{2(n+1)}{n-1}. \end{cases} \tag{2.5}$$

Notice that the case $p = \infty$, Sogge's bound reduces to the Avakumovič-Levitan-Hörmander bound.

CHAPTER 3
Estimates sharp on S^n

In this section, we show that Sogge's bound is sharp on the sphere for all $2 \leq p \leq \infty$. We recall that the k-th degree spherical harmonics on the sphere S^n are obtained by restricting k-th degree homogeneous harmonic polynomials of $\mathbb{R}^{n+1}$ to S^n. Further, they have eigenvalue $k(k+n-1)$. One class of spherical harmonics saturates Sogge's bound for $2 \leq p \leq \frac{2(n+1)}{n-1)}$, while another must be taken for $p > \frac{2(n+1)}{n-1}$.

Sharpness of Sogge's bound for $2 \leq p \leq \frac{2(n+1)}{n-1}$

We give a calculation for the case $n = 2$. The same example also gives sharpness for higher n.

In $\mathbb{R}^3$, consider the k-th degree homogeneous polynomial $p(x, y) = (x+iy)^k$. Then $\Delta_{\mathbb{R}^3} p(x, y) = k(k-1)(x+iy)^k - k(k-1)(x+iy)^k = 0$, so it is harmonic, and its restriction to S^2 is a spherical harmonic of degree k, with eigenvalue $k(k+1)$. This spherical harmonic attains its greatest modulus on the equator $z = 0$, where it is constantly 1. It is zero at the north and south poles. Furthermore, as k increases, its mass concentrates around the equator. We show that it saturates Sogge's L^p bound for $2 \leq p \leq 6$. We can find its L^2

norm using Gaussian integrals. First, notice,

$$\begin{aligned}
&\int_{\mathbb{R}^3} (x^2+y^2)^k e^{-(x^2+y^2+z^2)} \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&\qquad = \int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \left((r\sin\theta\cos\phi)^2+(r\sin\theta\sin\phi)^2\right)^k e^{-r^2} r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi \\
&\qquad = \int_{r=0}^{\infty} r^{2k} e^{-r^2} r^2 \int_{S^2} \left(x^2+y^2\right)^k \mathrm{dV}_{S^2}\mathrm{d}r \\
&\qquad = ||(x+iy)^k||_{L^2(S^2)}^2 \int_{r=0}^{\infty} r^{2k} e^{-r^2} r^2 \mathrm{d}r \\
&\qquad = ||(x+iy)^k||_{L^2(S^2)}^2 \int_{r=0}^{\infty} t^{k+\frac{1}{2}} e^{-t} \mathrm{d}t \\
&\qquad = ||(x+iy)^k||_{L^2(S^2)}^2 \Gamma\left(k+\frac{3}{2}\right).
\end{aligned}$$

We can also evaluate the first integral using cylindrical coordinates:

$$\begin{aligned}
\int_{\mathbb{R}^3} (x^2+y^2)^k e^{-(x^2+y^2+z^2)} \mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_{z=-\infty}^{\infty}\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} r^{2k} e^{-(r^2+z^2)} r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta \\
&= 2\pi \int_{-\infty}^{\infty} e^{-z^2}\mathrm{d}z \int_{r=0}^{\infty} r^{2k} e^{-r^2} r\mathrm{d}r \\
&= \pi\sqrt{\pi} \int_{r=0}^{\infty} t^k e^{-t}\mathrm{d}t \\
&= \pi\sqrt{\pi}\Gamma(k+1).
\end{aligned}$$

It follows that

$$||(x+iy)^k||_{L^2(S^2)}^2 = \pi\sqrt{\pi}\frac{\Gamma(k+1)}{\Gamma(k+\frac{3}{2})}. \tag{3.1}$$

Stirling's formula says that for $x>0$, $\Gamma(x) \sim \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x}$. It follows that

$$\begin{aligned}
||(x+iy)^k||_{L^2(S^2)}^2 &\sim \pi\sqrt{\pi}\frac{(k+1)^{k+1-\frac{1}{2}}e^{-(k+1)}}{(k+\frac{3}{2})^{k+\frac{3}{2}-\frac{1}{2}}e^{-(k+\frac{3}{2})}} && (3.2) \\
&\sim Ck^{-\frac{1}{2}}, && (3.3)
\end{aligned}$$

for some constant C. Thus, after L^2 normalizing, our function becomes (up to multiplication by a constant) $k^{1/4}(x+iy)^k$. Notice that $|k^{1/4}(x+iy)^k| = k^{1/4}(x^2+y^2)^{k/2} \leq k^{1/4}$, so its L^∞ norm is at most $k^{1/4}$, a value which is attained on the equator $z = 0$. We claim this function saturates Sogge's L^p bounds for $2 \leq p \leq 6$.

We may calculate the L^p norm of $k^{1/4}(x+iy)^k$ in the same manner, by considering instead the Gaussian integral

$$\int_{\mathbb{R}^3} (x^2+y^2)^{kp/2} e^{-(x^2+y^2+z^2)} \mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{3.4}$$

We find

$$||(x+iy)^k||^p_{L^p(S^2)} = \pi\sqrt{\pi}\frac{\Gamma\left(\frac{kp+2}{2}\right)}{\Gamma\left(\frac{kp+3}{2}\right)} \sim Ck^{-1/2}. \tag{3.5}$$

Thus, after L^2-normalizing, we have

$$||k^{1/4}(x+iy)^k||_{L^p(S^2)} = k^{1/4}k^{-1/2p} = k^{\frac{1}{4}-\frac{1}{2p}} \sim \lambda_k^{\frac{1}{8}-\frac{1}{4p}}, \tag{3.6}$$

where we have used that $\lambda_k \sim k^2$. The exponent $\frac{1}{8} - \frac{1}{4p}$ is precisely Sogge's exponent for $2 \leq p \leq \frac{2(n+1)}{n-1} = 6$.

Sharpness of Sogge's bound for $\frac{2(n+1)}{n-1} \leq p \leq \infty$

The k-th *zonal harmonic* on S^n is a spherical harmonic of degree k. Its value at each point $x \in S^n$ depends only on its inclination angle θ, and is given by the formula

$$Z_k(\theta) = C(k,n)P_k^{\left(\frac{n-2}{2},\frac{n-2}{2}\right)}(\cos\theta) \tag{3.7}$$

$$C(k,n) = \frac{2k+n-1}{n-1}\frac{\Gamma(n/2)\Gamma(k+n-1)}{\Gamma(n-1)\Gamma(k+n/2)}, \tag{3.8}$$

where $P_k^{(\alpha,\beta)}$ is the usual Jacobi polynomial.

The zonal harmonics take their maximum value at exactly one point, the north pole. We show that they saturate Sogge's bound for $\frac{2(n+1)}{n-1} \leq p \leq \infty$. A classical result in orthogonal polynomials (see [50]) states that

$$\int_0^1 (1-x)^\mu |P_k^{(\alpha,\beta)}(x)|^p \mathrm{d}x \sim \begin{cases} k^{\alpha p - 2\mu - 2}, & 2\mu < \alpha p - 2 + \frac{p}{2} \\ k^{-\frac{p}{2}} \log k, & 2\mu = \alpha p - 2 + \frac{p}{2} \\ k^{-\frac{p}{2}}, & 2\mu > \alpha p - 2 + \frac{p}{2}. \end{cases} \tag{3.9}$$

In particular, we have

$$\int_0^1 (1-x)^{\frac{n-2}{2}} |P_k^{\left(\frac{n-2}{2},\frac{n-2}{2}\right)}(x)|^p \mathrm{d}x \sim \begin{cases} k^{\frac{(n-2)p}{2} - n}, & n < \frac{(n-1)p}{2} \\ k^{-\frac{p}{2}}, & n > \frac{(n-1)p}{2}. \end{cases} \tag{3.10}$$

Then

$$\int_{S^n} |Z_k(\phi_1)|^p \mathrm{dV}_{S^n} \tag{3.11}$$

$$= \int_{\phi_1=0}^{\pi} \cdots \int_{\phi_{n-1}=0}^{\pi} \int_{\phi_n=0}^{2\pi} |Z_k(\phi_1)|^p \sin^{n-1}(\phi_1) \sin^{n-2}(\phi_2) \cdots \sin(\phi_{n-1}) \mathrm{d}\phi_1 \cdots \mathrm{d}\phi_n \tag{3.12}$$

$$= Vol_{S^{n-1}} \int_{\phi_1=0}^{\pi} \left| C(k,n) P_k^{\left(\frac{n-2}{2},\frac{n-2}{2}\right)}(\cos\phi_1) \right|^p \sin^{n-2}(\phi_1) \sin(\phi_1) \mathrm{d}\phi_1 \tag{3.13}$$

$$= C(k,n) Vol_{S^{n-1}} \int_{-1}^{1} \left| P_k^{\left(\frac{n-2}{2},\frac{n-2}{2}\right)}(x) \right|^p (1-x^2)^{\frac{n-2}{2}} \mathrm{d}x \tag{3.14}$$

$$= 2C(k,n) Vol_{S^{n-1}} \int_{0}^{1} \left| P_k^{\left(\frac{n-2}{2},\frac{n-2}{2}\right)}(x) \right|^p (1-x^2)^{\frac{n-2}{2}} \mathrm{d}x, \tag{3.15}$$

where we have used the fact that $|P_k^{(\alpha,\alpha)}(-x)| = |P_k^{(\alpha,\alpha)}(x)|$. On the interval $[0,1]$, we have the inequality $1 - x \leq 1 - x^2$, which gives

$$||Z_k||_{L^p(S^n)}^p \geq K \int_0^1 \left| P_k^{\left(\frac{n-2}{2},\frac{n-2}{2}\right)}(x) \right|^p (1-x)^{\frac{n-2}{2}} \mathrm{d}x. \tag{3.16}$$

If $p \geq \frac{2(n+1)}{n-1}$, then $n < \frac{(n-1)p}{2}$, so by equation 3.10 this last expression is asymptotic to $k^{\frac{(n-2)p}{2} - n}$. Taking p-th roots gives finally that

$$||Z_k||_{L^p(S^n)} \geq K' k^{\frac{n-2}{2} - \frac{n}{p}} \tag{3.17}$$

for some constant K' and k large enough.

Now we investigate the case $p = 2$. On the interval $[0, 1]$, we have the inequality $(1 - x^2) \leq 2(1 - x)$. Then equation 3.15 implies that

$$||Z_k||^2_{L^2(S^n)} \leq 4C(k,n)Vol_{S^{n-1}} \int_0^1 \left| P_k^{\left(\frac{n-2}{2},\frac{n-2}{2}\right)}(x) \right|^p (1-x)^{\frac{n-2}{2}} \mathrm{d}x. \tag{3.18}$$

When $p = 2$, $n > \frac{(n-1)p}{2}$, so by equation 3.10 this last expression is asymptotic (up to a constant) to $k^{-\frac{1}{2}}$. Taking square roots then implies that

$$||Z_k||_{L^2(S^n)} \leq K'' k^{-\frac{1}{2}} \tag{3.19}$$

for some constant K'' and k large enough.

Combining equations 3.17 and 3.19 gives

$$\frac{||Z_k||_{L^p(S^n)}}{||Z_k||_{L^2(S^n)}} \geq \frac{K'}{K''} k^{\frac{n-2}{2} - \frac{n}{p} + \frac{1}{2}} \tag{3.20}$$

$$= \frac{K'}{K''} k^{n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}} \tag{3.21}$$

$$\sim \frac{K'}{K''} \lambda^{\frac{n}{2}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{4}}, \tag{3.22}$$

which is Sogge's exponent for $\frac{2(n+1)}{n-1} \leq p \leq \infty$.

CHAPTER 4
Manifolds with completely integrable geodesic flow

We saw that the Avakumovič-Levitan-Hörmander bound is sharp for the zonal spherical harmonics. At the other extreme, there exist manifolds for which the L^∞ norms of L^2-normalized eigenfunctions are uniformly bounded with respect to their eigenvalues. Indeed, consider an irrational flat torus, i.e., $T^n = R^n/(2\pi\mathbb{Z})^n$, with metric given after diagonalization by $ds^2 = g_{11}\mathrm{d}x_1^2 + \cdots + g_{nn}\mathrm{d}x_n^2$, where the $g_{ii}, 1 \leq i \leq n$ are positive real numbers that are linearly independent over $\mathbb{Q}$. Each $\lambda = (\lambda_1, \cdots, \lambda_n), \lambda_i \in \mathbb{Z}, \lambda_i \geq 0, 1 \leq i \leq n$ is associated to a different eigenspace, spanned by the (at most) 2^n eigenfunctions $e^{i(\pm\lambda_1 x_1 \pm \cdots \pm \lambda_n x_n)}$. As we will see in chapter 10, the L^∞ norm of an L^2-normalized toral eigenfunction is bounded by the square root of the multiplicity of its eigenspace, so we have that for any eigenfunction ϕ of an irrational flat torus, $||\phi||_\infty \leq 2^{n/2}$, giving the claim.

It is natural to ask if we can characterize those manifolds with uniformly bounded eigenfunctions. To make this more precise, let us introduce some notation. Given a Riemannian manifold (M, g) with Laplacian Δ, define:

$$L^\infty(\lambda, g) := \sup_{\Delta\phi + \lambda\phi = 0, ||\phi||_2 = 1} ||\phi||_\infty$$
$$\ell^\infty(\lambda, g) := \inf_{\{\phi_j\} ONB} \left(\sup_j ||\phi_j||_\infty \right).$$

Then $\ell^\infty(\lambda, g) = O(1)$ if and only if M has an orthonormal basis of eigenfunctions with uniformly bounded L^∞ norms, and $L^\infty(\lambda, g) = O(1)$ if and only if *every* orthonormal basis of eigenfunctions of M has uniformly bounded L^∞ norms.

Given an n-dimensional Riemannian manifold (M, g), the metric canonically identifies its tangent bundle TM with its cotangent bundle T^*M as follows: for each $x \in M$, $v \in T_xM$ is identified with $\langle \cdot, v \rangle_{g(x)} \in T_x^*M$. We use this identification in what follows.

The geodesic flow on the cotangent bundle T^*M of M is defined by $g_t(x, \xi) = (x_t, \xi_t)$ where (x_t, ξ_t) is the terminal tangent vector at time t of the geodesic starting at x with direction ξ. Define the Hamiltonian function $q : T^*M \to \mathbb{R}$ by $q(x, \xi) = \left(\sum_{i,j} g^{ij}\xi_i\xi_j\right)^{1/2} = |\xi|_g$. Then the geodesic flow on T^*M coincides with the Hamiltonian flow of q.

In the language of quantum mechanics, the Laplacian Δ is a quantization of the Hamiltonian q. It is a traditional question to investigate the relationship between the behaviour of eigenfunctions of large eigenvalue (or high energy), and the dynamics of the geodesic flow on T^*M. This is known as the study of the semi-classical limit.

One case where we can say something about high energy eigenfunctions is when the geodesic flow is completely integrable. In the following, we use $T^*M \setminus 0$ to denote covectors with non-zero length. We say that the geodesic flow on T^*M is *completely integrable* if there exists a smooth function $\mathcal{P} = (p_1, \cdots, p_n) : T^*M \setminus 0 \to \mathbb{R}^n \setminus 0$, positive homogeneous of degree 1, called the *moment map*, and a conical open dense subset Ω of $T^*M \setminus 0$, such that:

1. $\mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_n \neq 0$ on Ω.
2. $p_1(x, \xi) = q(x, \xi) = |\xi|_g$.
3. the Poisson bracket $\{p_i, p_j\}$ is 0 $\qquad \forall\ 1 \leq i, j \leq n, i \neq j$.

Equivalently, we require that the functions $p_1, \ldots, p_n$ form a *completely integrable system*. When two functions p_i, p_j satisfy $\{p_i, p_j\}$, they are said to be in *involution*. The functions $p_2, \ldots, p_n$ are invariant under the geodesic flow, and are called *first integrals*. Each function p_j induces a Hamiltonian vector field

Ξ_{p_j} on the cotangent bundle. The composition of the associated Hamiltonian flows $\exp t_j \Xi_{p_j}$ gives an associated action Φ_t of $\mathbb{R}^n$ on T^*M:

$$\Phi_t := \exp t_1 \Xi_{p_1} \circ \cdots \circ \exp t_n \Xi_{p_n}.$$

By the Arnold-Liouville theorem, the orbits $\mathbb{R}^n \cdot (x, \xi)$ of this flow are diffeomorphic to $\mathbb{R}^k \times \mathbb{T}^m$, for some $k + m \leq n$. Those orbits with $k + m < n$ are called *singular orbits*.

Classical examples of manifolds with integrable geodesic flow include:

1. $\mathbb{R}^n$ or $\mathbb{T}^n$ with flat metrics
2. surfaces of revolution, i.e. S^2 with a rotationally invariant metric
3. $\mathbb{R}^2/\mathbb{Z}^2$ with a Liouville metric, i.e. where $\mathrm{d}s^2 = (f_1(x)+f_2(y))(\mathrm{d}x^2+\mathrm{d}y^2)$.
4. ellipsoids in $\mathbb{R}^n$ all of whose axes have different lengths.
5. $SO(3)$ with a left-invariant metric. This corresponds to a freely rotating rigid body in classical mechanics.

An even stronger condition than the geodesic flow on T^*M being completely integrable is the Laplacian being quantum completely integrable. We say that Δ is *quantum completely integrable* if there exist n pseudo-differential operators $P_1, \cdots, P_n \in \Psi^1(M)$, with principal symbols $p_1, \cdots, p_n$, such that

1. $P_1 = \sqrt{\Delta} \quad (p_1 = |\xi|_g)$
2. $[P_i, P_j] = 0$
3. $\mathrm{d}p_1 \wedge \cdots \mathrm{d}p_n \neq 0$ on a dense open set Ω of T^*M of finite complexity, defined below.

The symbol of $[P_i, P_j]$ is the poisson bracket $\{p_i, p_j\}$, so the conditions $[P_i, P_j] = 0$ and $\mathrm{d}p_1 \wedge \cdots \mathrm{d}p_n \neq 0$ on a dense open set mean that the functions $p_1, \cdots, p_n$ form a completely integrable system, with moment map $\mathcal{P} := (p_1, \cdots, p_n)$, and associated $\mathbb{R}^n$ action Φ_t. For each $b \in \mathcal{P}(T^*M \setminus 0)$, write $m_{\mathrm{cl}}(b)$ for the number of $\mathbb{R}^n$ orbits of the joint flow Φ_t on the level set $\mathcal{P}^{-1}(b)$. We say that Ω

is of *finite complexity* if there exists $M > 0$ such that for any $b \in \mathcal{P}(T^*M \setminus 0)$, $m_{\mathrm{cl}}(b) < M$.

We can now state a theorem that characterizes those manifolds with uniformly bounded eigenfunctions, assuming quantum integrability:

Theorem 4 (Toth-Zelditch [52]). *Suppose (M, g) is a compact Riemannian manifold with quantum completely integrable Laplacian Δ. Then:*

1. *If $L^\infty(\lambda, g) = O(1)$, (M, g) is flat;*
2. *If $\ell^\infty(\lambda, g) = O(1)$ and there exists M' such that*

$$\dim\{\phi; \Delta\phi + \lambda\phi = 0\} \leq M' \tag{4.1}$$

   for each eigenvalue λ, (M, g) is flat.

Stated another way, this means that if a compact Riemannian manifold with quantum completely integrable Laplacian is not flat, it must have a sequence of eigenfunctions whose L^∞ norms blow-up. In a later paper, Toth and Zelditch give a quantitative lower bound on the growth of this blowup, and show that the L^p norms must blow up also.

Theorem 5 (TZ [53]). *Suppose (M, g) is a compact Riemannian manifold with quantum completely integrable Laplacian, and suppose that the associated Hamiltonian $\mathbb{R}^n$ action is non-degenerate in the sense of Eliasson. Then, unless (M, g) is a flat torus, the $\mathbb{R}^n$ action has singular orbits of dimension less than n. If the minimum dimension of the singular orbits is ℓ, then for every $\epsilon > 0$ there exists a sequence $\{\phi_k\}$ of eigenfunctions satisfying:*

$$||\phi_k||_\infty \geq C(\epsilon)\lambda_k^{\frac{n-\ell}{4}-\epsilon} \tag{4.2}$$

$$||\phi_k||_p \geq C(\epsilon)\lambda_k^{\frac{(n-\ell)(p-2)}{4p}-\epsilon}, \quad 2 < p < \infty \tag{4.3}$$

If (M, g) is a convex surface of revolution, the equatorial geodesics (those invariant under the S^1 action) are singular orbits of degree 1, in which case the growth rate $||\phi_k||_\infty \geq C(\epsilon)\lambda_k^{\frac{n-1}{4}-\epsilon}$ saturates the Avakumovič-Levitan-Hörmander bound.

CHAPTER 5
Manifolds with maximal eigenfunction growth

We saw in the previous chapter that not only is the the Avakumovič-Levitan-Hörmander bound sharp for the sphere, but also for any convex surface of revolution. It is natural to ask if there are any other such manifolds. If we write $L^\infty(\lambda, g) = \sup_{\Delta\phi+\lambda\phi=0, ||\phi||_2=1} ||\phi||_\infty$, we can phrase the question as follows: for which (M, g) is $L^\infty(\lambda, g) = \Omega\left(\lambda^{\frac{n-1}{4}}\right)$? Here, $L^\infty(\lambda, g) = \Omega\left(\lambda^{\frac{n-1}{4}}\right)$ means it is $O\left(\lambda^{\frac{n-1}{4}}\right)$ but not $o\left(\lambda^{\frac{n-1}{4}}\right)$, i.e. for some *but not all* $C > 0$, $L^\infty(\lambda, g) \leq C\lambda^{\frac{n-1}{4}}$ for λ large enough. We say that such manifolds have *maximal eigenfunction growth.*

It can be shown that maximal eigenfunction growth requires the presence of "loops" in the set of geodesics of the manifold, i.e., geodesics which return to the point where they started. Given a manifold (M, g), and a point $x \in M$, we define the set of loops at x, $\mathcal{L}_x$, as follows:

$$\mathcal{L}_x := \{\xi \in S^*_x M : \exists T, \exp_x T\xi = x\}.$$

The theorem says that for a manifold to have maximal eigenfunction growth, it must have a point where the hypersurface measure of $\mathcal{L}_x$, denoted by $|\mathcal{L}_x|$, is nonzero. Such points are called *partial blow-down points.* A point where *all* geodesics are loops is called a *blow-down point.* As an example, given any surface of revolution, $|\mathcal{L}_x| = 2\pi$ at its north or south pole, i.e. they are blow-down points. Of course, $|\mathcal{L}_x| = 2\pi$ for *any* $x \in (S^2, \text{can})$.

Notice that loops are not required to close smoothly; they do not need to be closed geodesics. As an example, consider the triaxial ellipsoid,

$$E_{a_1,a_2,a_3} = \left\{ \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1; a_1 < a_2 < a_3 \right\}.$$

It has 4 umbilic points, located in the plane $x_2 = 0$, where all geodesics return at the same time. But only two of them close smoothly, namely the middle geodesic where x_2 is constantly 0, traversed in both directions.

Theorem 6 (Sogge-Zelditch [46]). *Let (M, g) be a compact Riemannian manifold of dimension n. Let $x \in M$. Suppose $|\mathcal{L}_x| = 0$. Then for all $\epsilon > 0$, there exists a neighbourhood $N(\epsilon)$ of x and a constant $\Lambda(\epsilon) > 0$, such that*

$$\sup_{\Delta\phi+\lambda\phi=0} \frac{||\phi||_{L^\infty(N)}}{||\phi||_{L^2(M)}} \leq \epsilon\lambda^{\frac{n-1}{4}},$$

for all $\lambda > \Lambda(\epsilon)$.

Furthermore, if $|\mathcal{L}_x| = 0$ for all $x \in M$, then for all $\epsilon > 0$ and $p > \frac{2(n+1)}{n-1}$, there exists $\Lambda(\epsilon, p)$, such that

$$\sup_{\Delta\phi+\lambda\phi=0} \frac{||\phi||_{L^p(M)}}{||\phi||_{L^2(M)}} \leq \epsilon\lambda^{\delta(p)},$$

for all $\lambda > \Lambda(\epsilon, p)$, where $\delta(p) = \frac{n}{2}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{4}$ is Sogge's exponent.

The naive converse to Theorem 6 is false. Sogge and Zelditch [46] exhibit a torus of revolution $(\mathbb{T}^2, g)$ with a point $x \in \mathbb{T}^2$ such that $|\mathcal{L}_x| > 0$, but $L^\infty(\lambda, g) = o\left(\lambda^{\frac{1}{4}}\right)$. The weaker converse of blow-up points implying maximal eigenfunction growth is also false. The triaxial ellipsoid mentioned before has blow-up points, but it will be seen below that it does not have maximal eigenfunction growth.

Adding the hypothesis of real analyticity gives an even stronger necessary condition for a manifold to have maximum eigenfunction growth:

Theorem 7 (S-Z [46])**.** *Suppose* (M, g) *is a compact real analytic Riemannian manifold such that* $L^\infty(\lambda, g) = \Omega\left(\lambda^{\frac{n-1}{4}}\right)$.

Then M *must be a* Y_l^m *manifold, i.e., there exist* $m \in M$ *and* $\ell > 0$ *such that all unit speed geodesics starting at* m *return there at time* ℓ*. If* $\dim M = 2$*, this implies that* M *must be topologically* S^2*.*

Sogge, Toth, and Zelditch strengthened the necessary condition of Theorem 6 by considering the growth of sequences of quasimodes, which are "approximate" eigenfunctions. A sequence $\{\psi_{\lambda_k}\}$ of functions on M with $||\psi_{\lambda_k}||_2 = 1$ is an *admissible sequence of quasimodes* if

$$||(\Delta + \lambda)\psi_{\lambda_k}||_2 + ||S^\perp_{4\lambda_k}\psi_{\lambda_k}||_\infty = o\left(\sqrt{\lambda}\right),$$

where $S^\perp_{4\lambda_k}$ is the projection onto the $[4\lambda_k, \infty)$ part of the spectrum of $-\Delta$. They also give a converse, by considering the dynamics of the "first return map". The motivation is the following: surfaces of revolution that are topologically S^2 have maximal eigenfunction growth (see [46]), and have the property that all geodesics leaving a pole return there and close smoothly. The triaxial ellipsoid does not have maximal eigenfunction growth, and at each of its umbilic points only two geodesics close smoothly. Given a manifold M and blow-down point $z \in M$, define the *first return map* Φ_z to be the natural action of the geodesic flow g^t on the sphere bundle:

$$\Phi_z = g^T_z : S^*_z M \to S^*_z M.$$

Thus, $\Phi_z = \mathrm{id}$ at the poles of a spherical surface of revolution, while at the umbilic points of a triaxial ellipsoid it has only two fixed points. We can now state the theorem.

Theorem 8 (Sogge-Toth-Zelditch [11])**.** *Let* (M, g) *be a compact Riemannian manifold of dimension* n*.*

1. *Suppose M admits an admissible sequence of quasimodes $\{\psi_{\lambda_k}\}$ such that $||\psi_{\lambda_k}||_\infty = \Omega\left(\lambda_k^{\frac{n-1}{4}}\right)$. Then M admits a partial blow-down point. If (M,g) is real analytic, it admits a blow-down point.*
2. *Conversely, suppose that M admits a blow-down point z, and that there exists an open set $\Gamma \subseteq S_z^*M$ such that $\Phi_z|_\Gamma = id$. Then there exists a quasimode sequence $\{\psi_k, r_k\}_{k\in\mathbb{Z}_+}$ with $||\psi_k||_2 = 1$ and $||(\Delta + r_k)\psi_k||_2 = O(1)$ such that $||\psi_k||_\infty = \Omega\left(\lambda_k^{\frac{n-1}{4}}\right)$.*

Sogge, Toth, and Zelditch give an even stronger requirement for maximal eigenfunction growth: the existence of a special type of partial blow-down point, called a *recurrent point*. Suppose $x \in M$ is a partial blow down point, that is, that $|\mathcal{L}_x| > 0$. The first return time function to x, $T_x : S_x^*M \to \mathbb{R}_+ \cup \{\infty\}$, is given by

$$T_x(\xi) = \begin{cases} \inf\{t > 0 : \exp_x t\xi = x\}, & \text{if } \xi \in \mathcal{L}_x; \\ \infty & \text{if no such } t \text{ exists.} \end{cases}$$

Then $\Phi_x := g_x^{T_x} : \mathcal{L}_x \to S_x^*M$, is the natural definition of the first return map at x.

$\mathcal{L}_x$ is not necessarily invariant under Φ_x. If we set

$$\mathcal{L}_x^\infty = \bigcap_{k\in\mathbb{Z}} (\Phi_x)^k \mathcal{L}_x,$$

then $\mathcal{L}_x^\infty$ is forward and backward invariant by construction, and $(\mathcal{L}_x^\infty, \Phi_x)$ defines a dynamical system. Define the set of *recurrent loop directions* at x, $\mathcal{R}_x$, by

$$\mathcal{R}_x = \{\xi \in \mathcal{L}_x^\infty : \xi \in \omega(\xi)\},$$

where $\xi \in \omega(\xi)$ means that for any neighbourhood $\Gamma \subseteq S_x^*M$ of ξ, we have $(\Phi_x)^k\xi \in \Gamma$ for infinitely many $k \in \mathbb{Z}$. Then x is called a *recurrent point* for the geodesic flow if $|\mathcal{R}_x| > 0$. They prove:

Theorem 9 (STZ [11]). *Let (M, g) be a compact Riemannian manifold of dimension n. Suppose M admits an admissible sequence of quasimodes $\{\psi_{\lambda_k}\}$ such that $||\psi_{\lambda_k}||_\infty = \Omega\left(\lambda_k^{\frac{n-1}{4}}\right)$.*

Then there exists $x \in M$ such that $|\mathcal{R}_x| > 0$.

Toth proved previously (see [54]) that Liouville metrics on spheres (which include the triaxial ellipsoid), satisfy $|\mathcal{R}_x| = 0$ at every point. It follows from Theorem 9 that the triaxial ellipsoid cannot have maximal eigenfunction growth.

Sogge, Toth, and Zelditch also prove:

Theorem 10 (STZ [11]). *A real analytic Riemannian surface with maximal eigenfunction growth does not have ergodic geodesic flow.*

By Theorem 7, we know that a real analytic Riemannian surface with maximal eigenfunction growth must be topologically S^2. Theorem 10 does add new information, as Burns-Donnay [9] and Donnay-Pugh [14], [15] have exhibited real analytic metrics on S^2 with ergodic geodesic flow.

CHAPTER 6
Manifolds with ergodic geodesic flow

In chapter 4 we considered eigenfunctions of manifolds whose geodesic flow was completely integrable. At the other extreme are manifolds whose geodesic flow is ergodic.

Given a Riemannian manifold (M, g), the Riemannian metric induces a measure $\mathrm{d}\mu_L$ on the bundle of unit cotangent vectors S^*M, called the Liouville measure. The geodesic flow g^t induces a unitary operator V_t on $L^2(S^*M, \mathrm{d}\mu_L)$, called the Koopman operator, given by $V_t(f) = f \circ g^t$. We say that the geodesic flow on S^*M is *ergodic* if and only if the only functions invariant under V_t (i.e. with $V_t f = f$) are the constant functions.

By well-known results of G. Hedlund and E. Hopf (see e.g. [2]), compact Riemannian manifolds with everywhere negative sectional curvature have ergodic geodesic flow. They are described in chapter 8. Arithmetic hyperbolic manifolds, described in chapter 9, form a special class of such manifolds.

Let $\{\phi_k\}_k$, $\triangle\phi_k + \lambda_k\phi_k = 0$, be a sequence of eigenfunctions of the Laplacian on a Riemannian manifold M such that $\lambda_k \to \infty$ as $k \to \infty$. Write $\mathrm{d}V$ for the Riemannian volume form on M. Then to each eigenfunction ϕ_k is associated a probability measure $\mathrm{d}\mu_k := |\phi_k|^2 \mathrm{d}V$, traditionally interpreted in quantum mechanics as the probability density of a particle in the state ϕ_k.

If there exists a measure $\mathrm{d}\mu$ on M such that for any test function $f \in L^2(M)$,

$$\int_M f|\phi_k|^2 \ \mathrm{d}V \to \int_M f \ \mathrm{d}\mu,$$

we say that the measures $\mathrm{d}\mu_k = |\phi_k|^2 \, \mathrm{d}V$ *converge weakly* to $\mathrm{d}\mu$, and write $\mathrm{d}\mu_k \rightharpoonup \mathrm{d}\mu$. We refer to such limit measures as *quantum limits.* A classical question is to determine which quantum limits are possible, in particular, whether eigenfunctions can concentrate on a subset of the manifold, or if they become equidistributed as $\lambda_k \to \infty$, in the sense that $|\phi_k|^2 \mathrm{d}V \rightharpoonup \frac{\mathrm{d}V}{\mathrm{Vol} M}$.

The famous Quantum Ergodicity theorem states that if a manifold has ergodic geodesic flow, then almost all of its eigenfunctions become equidistributed as $\lambda \to \infty$:

Theorem 11 (Shnirelman [44], Zelditch [55], Colin de Verdiere [13])**.** *Suppose (M, g) is a compact Riemannian manifold whose geodesic flow is ergodic. Let $\{\phi_k; \Delta\phi_k + \lambda_k\phi_k = 0, 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ be an orthonormal basis of $L^2(M)$.*

Then there is a subsequence k_j of k of full density such that

$$\mathrm{d}\mu_{k_j} := |\phi_{k_j}|^2 \mathrm{d}V \rightharpoonup \frac{\mathrm{d}V}{Vol(M)},$$

where full density means that

$$\lim_{\lambda \to \infty} \frac{\#\{\lambda_{k_j} \leq \lambda\}}{\#\{\lambda_j \leq \lambda\}} = 1.$$

Notice however, that the quantum ergodicity theorem does not rule out the possibility of non-trivial quantum limits; it is possible that an exceptional subsequence of eigenfunctions could localize on a subset of the manifold. Nevertheless, Rudnick and Sarnak have conjectured that in the case of compact manifolds of negative curvature, all sequences of eigenfunctions become equidistributed. This is called *Quantum Unique Ergodicity* [41].

CHAPTER 7
Gaussian random waves

A heuristic due to Berry [4] says that eigenfunctions on Riemannian manifolds with ergodic geodesic flow should behave like Gaussian random waves. Random waves are easier to study than individual eigenfunctions, so they provide one way to test conjectures.

We first define random waves on the n-sphere. Fix $N \geq 0$, and let $\mathcal{H}_N$ denote the real vector space of Nth degree spherical harmonics on $\mathbb{S}^n$. Fix an orthonormal basis $\{\phi_{N_j}\}_{j=1}^{d_N}$ of spherical harmonics, where $d_N = \dim \mathcal{H}_N$. Then any function $\psi \in \mathcal{H}_N$ can be written as $\psi = \sum_{j=1}^{d_N} c_j \phi_{N_j}$ for some real constants c_j. We endow $\mathcal{H}_N$ with a Gaussian measure $\mathrm{d}\gamma_N$, given by

$$\mathrm{d}\gamma_N = \left(\frac{d_N}{\pi}\right)^{\frac{d_N}{2}} e^{-d_N|c|^2} \mathrm{d}c, \tag{7.1}$$

where $\mathrm{d}c$ is the d_N-dimensional Lebesgue measure on the space of Fourier coefficients. The normalization is chosen so that $\mathbb{E}_{\gamma_N} \langle \psi, \psi \rangle = 1$. The preceding construction is equivalent to letting the $c_1, \cdots, c_{d_N}$ be independent identically distributed random variables, with mean 0 and variance $\frac{1}{2d_N}$, i.e.

$$\mathbb{E}_{\gamma_N} c_j = 0 \quad \text{and} \quad \mathbb{E}_{\gamma_N} c_j c_k = \frac{1}{2d_N} \delta_{jk}.$$

Nazarov and Sodin investigated the number of nodal domains of random spherical harmonics on $\mathbb{S}^2$. For $f \in \mathcal{H}_N$, let $Z(f) = \{x \in \mathbb{S}^2 : f(x) = 0\}$. The nodal domains of f are the connected components of $\mathbb{S}^2 \setminus Z(f)$. Let $N(f)$ denote the number of nodal domains. They showed that $\mathbb{E}N(f)/n^2$ concentrates exponentially around some value a as $n \to \infty$.

Theorem 12 (Nazarov-Sodin [38]). *For each $N \geq 1$, let the vector space $\mathcal{H}_N$ of N-th degree spherical harmonics be endowed with the Gaussian probability measure $\mathrm{d}\gamma_N$, as above. Then there exists a constant $a > 0$ such that, for every $\epsilon > 0$,*

$$\mathbb{P}\left\{\left|\frac{N(f)}{n^2} - a\right| > \epsilon\right\} \leq C(\epsilon)e^{-c(\epsilon)n},$$

where $c(\epsilon)$ and $C(\epsilon)$ are positive constants depending on ϵ only.

On a general Riemannian manifold (M, g) we partition the spectrum of $\sqrt{\Delta_g}$ into intervals I_N of length one. If the set of closed geodesics of M has measure zero the choice of I_N is arbitrary, and we put $I_N = [N, N+1]$ for convenience. The choice must be made more carefully on Zoll manifolds, all of whose geodesics are closed of equal length. Let d_N denote the number of eigenvalues in I_N. Let $\mathcal{H}_N$ denote the Hilbert space spanned by linear combinations of eigenfunctions $\Delta\phi + \lambda\phi = 0$ with $\sqrt{\lambda} \in I_N$. Then any function $\psi \in \mathcal{H}_N$ can be written as $\psi = \sum_{j=1}^{d_N} c_j \phi_{N_j}$ for some real constants c_j. We endow each $\mathcal{H}_N$ with the Gaussian measure $\mathrm{d}\gamma_N$ given by equation (7.1). The product space $\mathcal{H}_\infty := \prod_{N=1}^{\infty} \mathcal{H}_N$ is naturally endowed with the product probability measure $\mathrm{d}\gamma := \prod_{N=1}^{\infty} \mathrm{d}\gamma_N$.

In this setting, Shiffman and Zelditch showed that the L^∞ norms of Gaussian random waves should grow almost surely like $O(\sqrt{\log \lambda})$, and that their L^p norms are almost surely bounded.

Theorem 13 (Shiffman-Zelditch [43]). *Let (M, g) be a Riemannian manifold and $\{\mathcal{H}_N\}_{N=1}^{\infty}$ be a family of probability spaces of eigenfunctions with frequencies in the intervals $\{I_N\}_{N=1}^{\infty}$, as described above. Let $\{s_N\} \in \mathcal{H}_\infty := \prod_{N=1}^{\infty} \mathcal{H}_N$ be a sequence of eigenfunctions.*

Then with respect to the probability measure $\mathrm{d}\gamma := \prod_{N=1}^{\infty} \mathrm{d}\gamma_N$ we have:

1. $||s_N||_\infty / ||s_N||_2 = O(\sqrt{\log N})$ *almost surely;*
2. $||s_N||_p / ||s_N||_2 = O(1)$ *almost surely, for* $2 \leq p < \infty$.

CHAPTER 8
Negatively curved manifolds

In the case that (M, g) is a Riemannian manifold with negative sectional curvature, Bérard showed that the Avakumovič-Levitan-Hörmander estimate for the remainder in local Weyl's law can be improved by a factor of $\log \lambda$, leading to a corresponding L^∞ bound for eigenfunctions:

Theorem 14 (Bérard [3]). *Let (M, g) be a compact n-dimensional Riemannian manifold. Suppose that either $n = 2$ and M does not have conjugate points, or that M has nonpositive sectional curvature.*

Then there exists a constant $C > 0$ such that for any Laplace eigenfunction ϕ_λ with eigenvalue λ,

$$\frac{||\phi_\lambda||_\infty}{||\phi_\lambda||_2} \leq C \frac{\lambda^{\frac{n-1}{4}}}{\log \lambda}. \tag{8.1}$$

Hejhal and Rackner [21] examined the value distribution of eigenfunctions of the modular surface $M = \mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ numerically. The showed that as $\lambda \to \infty$, the eigenfunctions appear to have Gaussian value distributions. This would mean that for any sequences $\Delta\phi_j + \lambda_j \phi_j = 0, \lambda_j \to \infty$ of eigenfunctions, their moments should converge to the moments of the Gaussian distribution; in particular, the odd moments $\int_M \phi_j^{2k+1} \mathrm{d}A$ should go to zero as $j \to \infty$, and the even moments $\int_M \phi_j^{2k} \mathrm{d}A$ should remain bounded. A result of Iwaniec [23] on integrals of triple products of eigenfunctions has confirmed that the third moment $\int_{\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}} \phi_j^3 \, \mathrm{d}A$ does go to 0 as $j \to \infty$.

It is a conjecture of Sarnak that for hyperbolic manifolds of dimension 2 the bound in (8.1) can be improved to any λ^ϵ and any p:

Conjecture 15 (Sarnak [24])**.** *Let M be a Riemannian manifold of dimension 2 with constant Gaussian curvature -1. Let $2 < p \leq \infty$.*

Then for all $\epsilon > 0$ there exists a constant C_ϵ such that for any Laplace eigenfunction ϕ_λ of eigenvalue λ,

$$\frac{||\phi_\lambda||_p}{||\phi_\lambda||_2} \leq C_\epsilon \lambda^\epsilon.$$

The conjecture quantifies that eigenfunctions do not become localized as $\lambda \to \infty$. Notice that the conjecture is consistent with the random wave model (cf. Theorem 13), which predicts a rate of growth of $\sqrt{\log \lambda}$ for the L^∞ norms. Sarnak and Watson [42] have proved conjecture 15 in the case $p = 4$ for the modular surface $M = \mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$, a special example of the arithmetic hyperbolic surfaces considered in chapter 9.

CHAPTER 9
Arithmetic hyperbolic manifolds

Even more can be said about the eigenfunctions of arithmetic surfaces, which are a special case of manifolds of negative curvature. An *arithmetic surface* is a quotient $\Gamma \setminus \mathbb{H}^2$ of the upper half plane $\mathbb{H}^2$ by an arithmetic Fuchsian group Γ.

We give a characterization of arithmetic Fuchsian groups that is due to Takeuchi [51]. First, a *Fuchsian group of the first kind* is a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ such that the quotient $\Gamma \setminus \mathbb{H}^2$ has finite volume. A Fuchsian group Γ of the first kind is said to be an *arithmetic Fuchsian group* if and only if it satisfies the following two conditions:

1. If $k_1 = \mathbb{Q}(\mathrm{tr}(\gamma)|\gamma \in \Gamma)$ is the field generated by $\mathrm{tr}(\Gamma)$ over $\mathbb{Q}$, then k_1 is an algebraic number field of finite degree, and $\mathrm{tr}(\Gamma) \subseteq \mathcal{O}_{k_1}$, the ring of integers of k_1.
2. If $k_2 = \mathbb{Q}(\mathrm{tr}(\gamma)^2|\gamma \in \Gamma)$ is the field generated by $(\mathrm{tr}(\Gamma))^2$ over $\mathbb{Q}$, and ϕ is any embedding of k_1 into $\mathbb{C}$ such that $\phi|_{k_2}$ is not the identity, then $\phi(\mathrm{tr}(\Gamma))$ is bounded in $\mathbb{C}$.

The most important example of an arithmetic Fuchsian group is the modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Other examples are the Hecke groups Γ_4 and Γ_6, where Γ_k is the group generated by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2\cos\frac{\pi}{k} \\ 0 & 1 \end{bmatrix}$.

Given a quaternion division algebra $A = \left(\frac{a,b}{\mathbb{Q}}\right)$ over $\mathbb{Q}$, and a maximal order R of A, we can derive (see [28]) an arithmetic Fuchsian groups as follows. We have that A is linearly generated by 1, ω, Ω, $\omega\Omega$ over $\mathbb{Q}$, with $\omega^2 = a, \Omega^2 = b$, and $\omega\Omega + \Omega\omega = 0$. The norm is defined as usual: $N(\alpha) := \alpha\bar{\alpha}$. We fix an embedding ϕ of A into $\mathrm{M}_2(\mathbb{Q}(\sqrt{a}))$ by

$$x_0 + x_1\omega + x_2\Omega + x_3\omega\Omega \overset{\phi}{\mapsto} \begin{bmatrix} x_0 + x_1\sqrt{a} & x_2 + x_3\sqrt{a} \\ b(x_2 - x_3\sqrt{a}) & x_0 - x_1\sqrt{a} \end{bmatrix}.$$

For $m \geq 1$, define $R(m) := \{\alpha \in R | N(\alpha) = m\}$. Finally, set $\Gamma = \phi(R(1)) \subseteq M_2(\mathbb{R})$. It can be shown that Γ is an arithmetic Fuchsian group, and that the quotient $\Gamma \setminus \mathbb{H}$ is compact.

An essential tool in the study of eigenfunctions of arithmetic manifolds are Hecke operators, which arise from correspondences on the manifold. Given a manifold M, a correspondence C of order $r \geq 1$ on M is multivalued function

$$z \overset{C}{\mapsto} \{z_1, \cdots, z_r\},$$

such that the z_i are locally isometries.

We are interested in correspondences which arise in the following fashion: Let $X = \Gamma \setminus S$, where S is a Riemannian manifold and Γ is a subgroup of the group G of isometries of S. Define the commensurator of Γ, $\mathrm{COM}(\Gamma)$, by

$$\mathrm{COM}(\Gamma) := \{\delta \in G \quad | \quad \delta^{-1}\Gamma\delta \cap \Gamma =: B \text{ is finite index in both } \Gamma \text{ and } \delta^{-1}\Gamma\delta\}.$$

In other words, $\mathrm{COM}(\Gamma)$ is the set of all $\delta \in \Gamma$ such that $\delta^{-1}\Gamma\delta$ and Γ are *commensurable*. Each $\delta \in \mathrm{COM}(\Gamma)$ gives rise to a correspondence C_δ on $\Gamma \setminus S$, given by

$$\Gamma x \overset{C_\delta}{\mapsto} \{\Gamma\delta\alpha_1 x, \cdots, \Gamma\delta\alpha_r x\},$$

where $\Gamma = \bigcup_{j=1}^{r} B\alpha_j$ is a coset decomposition.

To each such correspondence C_δ is associated a *Hecke operator* $T_\delta : L^2(X) \to L^2(X)$, given by

$$T_\delta f(x) = \sum_{j=1}^{r} f(\delta\alpha_j x).$$

As $\delta\alpha_j \in G$, each T_δ commutes with the Laplacian Δ.

In the case $S = \mathbb{H}^2$, we have $G = SL_2(\mathbb{R})$. It is a result of Margulis [32] that $\mathrm{COM}(\Gamma)$ is dense in $\mathrm{SL}_2(\mathbb{R})$ if and only if $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is arithmetic. Thus arithmetic surfaces have an infinite family of correspondences and Hecke operators.

We say that Γ is a *congruence subgroup* if it is a subgroup of the unit group of a quaternion algebra, and there exists some q such that Γ contains $\Gamma(q) := \{\alpha \in \Gamma | \alpha \equiv 1 (\mod q)\}$. In this case, the Hecke operators form a self-adjoint commutative algebra. As they commute with the Laplacian, they can all be simultaneously diagonalized by a basis of joint eigenfunctions ϕ_j. The arithmetic groups derived from quaternion algebras mentioned before are congruence subgroups.

Iwaniec and Sarnak showed that in the arithmetic surface case, the Avakumovič-Levitan-Hörmander L^∞ bound can be improved (for joint Laplace-Hecke eigenbases) to $\lambda^{5/24+\epsilon}$, and that the L^∞ norms are not uniformly bounded, a result that is not obvious.

Theorem 16 (Iwaniec-Sarnak [24])**.** *Let G be an arithmetic group derived from a quaternion algebra as above, and $M = G \setminus \mathbb{H}^2$ its associated hyperbolic surface. Let $\{\phi_j\}$ be a joint orthonormal basis of eigenfunctions of the Laplacian and the Hecke operators. Then*

1. *For all j, and $\epsilon > 0$,*
$$||\phi_j||_\infty \leq C_\epsilon \lambda_j^{5/24+\epsilon},$$
where C_ϵ is a constant depending on ϵ.

2. *There exists a positive constant C such that*
$$||\phi_j||_\infty \geq C\sqrt{\log\log \lambda_j},$$
for infinitely many j.

Note that numerical evidence (see [21] and [48]) strongly suggests that the eigenvalues of arithmetic surfaces are of uniformly bounded multiplicity, so the restriction to a joint Laplace-Hecke eigenbasis is probably unnecessary.

Milićević later showed that L^∞ norms of some eigenfunctions must grow even faster than what was shown by Iwaniec and Sarnak:

Theorem 17 (Milićević [34])**.** *Let G be an arithmetic group derived from a quaternion algebra as above, and $M = G\backslash\mathbb{H}^2$ its associated hyperbolic surface. Let $\{\phi_j\}$ be a joint orthonormal basis of eigenfunctions of the Laplacian and the Hecke operators.*

Then for every fixed CM-point $z \in \mathbb{H}^2$, we have, as $j \to \infty$,

$$|\phi_j(z)| = \Omega\left(\exp\left(\sqrt{\frac{\log\lambda_j}{\log\log\lambda_j}}\left(1 + O\left(\frac{\log\log\log\lambda_j}{\log\log\lambda_j}\right)\right)\right)\right),$$

giving

$$||\phi_j||_\infty = \Omega\left(e^{(1+o(1))\sqrt{\frac{\log\lambda_j}{\log\log\lambda_j}}}\right).$$

This is a much stronger rate of growth than predicted by the random wave conjecture, which says that $||\phi_j||$ should grow like $\sqrt{\log\lambda_j}$ (cf. Theorem 13).

Lindenstrauss proved that "arithmetic" quantum unique ergodicity (QUE) holds for compact arithmetic hyperbolic surfaces, that is, that QUE holds for sequences of joint Laplace-Hecke eigenfunctions. Soundararajan extended the result to the non-compact case.

Theorem 18 (Lindenstrauss [31], Soundararajan [47])**.** *Let $M = \Gamma\setminus\mathbb{H}^2$ for some congruence subgroup Γ. Let $\{\phi_j\}$ be a joint orthonormal basis of eigenfunctions of the Laplacian and the Hecke operators. Then*

$$\mathrm{d}\mu_j := |\phi_j|^2\mathrm{d}V \rightharpoonup \frac{\mathrm{d}V}{Vol(M)},$$

where $\mathrm{d}V$ is the volume form on M.

As before, it is believed that arithmetic surfaces have eigenvalues of uniformly bounded multiplicity, so the restriction to a joint basis of eigenfunctions is a mild assumption.

The construction of 3-dimensional arithmetic hyperbolic manifolds is analogous to that of arithmetic surfaces. An *arithmetic hyperbolic 3-manifold* is a quotient $\Gamma \setminus \mathbb{H}^3$ of the upper half space $\mathbb{H}^3$ by an arithmetic Kleinian group $\Gamma < \mathrm{PSL}_2(\mathbb{C})$. As before, there exist Hecke operators that can be simultaneously diagonalized with the Laplacian, giving a joint eigenbasis.

Rudnick and Sarnak [41] exhibited an arithmetic hyperbolic 3-manifold X with a sequence of eigenfunctions ϕ_{j_k} such that for some constant D,

$$\frac{||\phi_{j_k}||_\infty}{||\phi_{j_k}||_2} \geq D\lambda^{\frac{1}{4}}.$$

Notice that this is much greater than the rate $\sqrt{\log \lambda_j}$ suggested by the random wave model. Strong growth is proved at special Heegner points where "almost all" eigenfunctions are known to vanish. It follows from local Weyl's law (2.1) that those that don't vanish must be very large. Similarly, Milićević [33] has demonstrated a class of arithmetic 3-manifolds for which $\frac{||\phi_j||_\infty}{||\phi_j||} = \Omega\left(\lambda_j^{1/4+o(1)}\right)$.

Donnelly [17] has extended the method of Rudnick and Sarnak to show that in all dimensions ≥ 5 there exist manifolds of constant negative curvature such that for some sequence ϕ_j of eigenfunctions and some constant C , one has $\frac{||\phi_j||_\infty}{||\phi_j||_2} \geq C\lambda^{\frac{n-4}{4}}$.

CHAPTER 10
Tori

L^∞ bounds

Given an eigenfunction ϕ_λ of the Laplacian on $\mathbb{T}^n := \mathbb{R}^n/(2\pi\mathbb{Z})^n$ with eigenvalue λ, we can write

$$\phi_\lambda(x) = \sum_{\xi\in\mathbb{Z}^n;|\xi|^2=\lambda} a_\xi e^{i(x,\xi)}, \tag{10.1}$$

with $a_\xi \in \mathbb{C}$. Suppose that the eigenfunction is L^2-normalized, so that $||\phi_\lambda||_2 = 1$. This means that

$$\sum_{\xi\in\mathbb{Z}^n;|\xi|^2=\lambda} |a_\xi|^2 = 1. \tag{10.2}$$

Taking the absolute value of (10.1) and applying the triangle inequality gives $|\phi_\lambda(x)| \le \sum_{\xi\in\mathbb{Z}^n;|\xi|^2=\lambda} |a_\xi|$, whence

$$||\phi_\lambda||_\infty \le \sum_{\xi\in\mathbb{Z}^n;|\xi|^2=\lambda} |a_\xi|. \tag{10.3}$$

Applying the method of Langrange multipliers gives that under the constraint (10.2), the right hand side of inequality (10.3) is maximized when all the $|a_\xi|$ are equal. The number of terms on the right hand side of (10.3) is the number of $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{Z}^n$ such that $\lambda = |\xi|^2 = \xi_1^2 + \cdots + \xi_n^2$, i.e., the number of ways to write λ as a sum of n integers. This number is traditionally denoted by $r_n(\lambda)$. Then by the argument made previously, the right hand side of (10.3) is maximized when each $|a_\xi| = \frac{1}{\sqrt{r_n(\lambda)}}$, giving the bound

$$||\phi_\lambda||_\infty \le \sum_{\xi\in\mathbb{Z}^n;|\xi|^2=\lambda} \frac{1}{\sqrt{r_n(\lambda)}} = \frac{r_n(\lambda)}{\sqrt{r_n(\lambda)}} = \sqrt{r_n(\lambda)}. \tag{10.4}$$

It is clear that the bound is attained at $x = 0$ if we set each $a_\xi = \frac{1}{\sqrt{r_n(\lambda)}}$.

The asymptotic behaviour of $r_n(\lambda)$ is a traditional problem in number theory (see [20]) . For $n = 2, 3, 4$, $\forall \epsilon$, $\exists C_\epsilon > 0$ such that

$$r_n(\lambda) < C_\epsilon \lambda^{\frac{n}{2}-1+\epsilon}.$$

For $n \geq 5$, the epsilon may be dropped and there is also a lower bound; there exist constants C_n, C'_n such that

$$C_n \lambda^{\frac{n}{2}-1} < r_n(\lambda) < C'_n \lambda^{\frac{n}{2}-1}. \tag{10.5}$$

In the case $n = 2$, there is an exact formula for $r_2(\lambda)$, given the arithmetic nature of λ. If $\lambda = 2^j \prod_{pi \equiv 1 \mod 4} p_1^{k_1} \cdots p_T^{k_T} \prod_{q_i \equiv 3 \mod 4} q_1^{l_1} \cdots q_S^{l_S}$ is the prime factor decomposition of λ, then $r_2(\lambda) = 4(k_1 + 1) \cdots (k_T + 1)$ if all the l_i are even, and $r_2(\lambda) = 0$ otherwise.

L^p bounds

In the case $n = 2$, G. Mockenhaupt showed that for each $p \geq 2$, eigenfunctions on the torus whose eigenvalues have a fixed number of distinct prime factors congruent to 1 modulo 3 have bounded L^p norms:

Theorem 19 (Mockenhaupt [35])**.** *Let $p \geq 2$. Suppose*

$$\lambda = 2^j \prod_{pi \equiv 1 \mod 4} p_1^{k_1} \cdots p_T^{k_T} \prod_{q_i \equiv 3 \mod 4} q_1^{l_1} \cdots q_S^{l_S},$$

where all the l_i are even, so that $r_2(\lambda) = 4(k_1 + 1) \cdots (k_T + 1)$. Then there exists $C(T, p)$, depending only on T and p, such that for any eigenfunction ϕ_λ of $\mathbb{T}^2$ with eigenvalue λ,

$$\frac{||\phi_\lambda||_p}{||\phi_\lambda||_2} < C(T, p) < \infty. \tag{10.6}$$

The next theorem shows that on $\mathbb{T}^n, n \geq 5$, and for p greater than the critical exponent $p_n = \frac{2n}{n-2}$, the L^p norms of eigenfunctions are not uniformly bounded.

Theorem 20. *Let $n \geq 5$. If $p > p_n = \frac{2n}{n-2}$, then there exists a sequence $\{\phi_k\}$, $\Delta\phi_k + \lambda_k\phi_k = 0$ of eigenfunctions of the Laplacian on $\mathbb{T}^n$ such that*

$$\frac{||\phi_k||_p}{||\phi_k||_2} \longrightarrow \infty \tag{10.7}$$

as $\lambda_k \longrightarrow \infty$.

Proof. We consider the same eigenfunctions as those that maximized the L^∞ norm in section 10.1. Set

$$\phi_k(x) = \frac{1}{\sqrt{r_n(\lambda_k)}} \sum_{|\xi|^2=\lambda_k} e^{i(x,\xi)} = \frac{1}{\sqrt{r_n(\lambda_k)}} \sum_{|\xi|^2=\lambda_k} 2\cos((x,\xi)). \tag{10.8}$$

As before, we have $\phi_k(0) = \sqrt{r_n(\lambda_k)}$.

Notice that if $|x| \leq \frac{\pi}{4\sqrt{\lambda_k}}$, it follows by Cauchy-Schwarz that

$$(x,\xi) \leq |x||\xi| = |x|\sqrt{\lambda_k} \leq \frac{\pi}{4}. \tag{10.9}$$

It follows that for such x, have $\cos((x,\xi)) \geq \frac{\sqrt{2}}{2}$, whence $\phi_k(x) \geq \sqrt{2r_n(\lambda_k)}$. Hence

$$\int_{\mathbb{T}^n} |\phi_k|^p \geq \mathrm{Vol}\left(B_0\left(\frac{\pi}{4\sqrt{\lambda_k}}\right)\right)\left(\sqrt{2r_n(\lambda)}\right)^p \tag{10.10}$$

$$= \frac{\mathrm{const}}{\lambda_k^{n/2}} r_n(\lambda_k)^{p/2} \tag{10.11}$$

$$\sim \lambda_k^{\frac{p}{2}\left(\frac{n}{2}-1\right)-\frac{n}{2}}, \tag{10.12}$$

where we have used that for $n \geq 5$ we have $r_n(\lambda) \sim \lambda^{\frac{n}{4}-\frac{1}{2}}$ (see 10.5). It follows that if $\frac{p}{2}\left(\frac{n}{2}-1\right) - \frac{n}{2} > 0$, then $||\phi_k||_p \to \infty$ as $\lambda_k \to \infty$. Rearranging this inequality gives the condition $p > p_n = \frac{2n}{n-2}$. $\square$

In the case $n = 2$, however, R. Cooke and A. Zygmund showed that the L^4 norms *are* uniformly bounded:

Theorem 21 (Cooke [12], Zygmund [56])**.** *For any eigenfunction ϕ of $\mathbb{T}^2$,*

$$\frac{||\phi||_4}{||\phi||_2} \leq 5^{1/4}. \tag{10.13}$$

Proof. Without loss of generality, we can take $||\phi||_2 = 1$. (Otherwise, consider the function $\frac{\phi}{||\phi||_2}$ and the conclusion follows by the homogeneity of the L^4 norm.) Suppose ϕ is an eigenfunction with eigenvalue λ. Then ϕ is a trigonometric polynomial, all of whose frequencies lie on a circle of radius $\sqrt{\lambda}$:

$$\phi(x) = \sum_{|\xi|=\sqrt{\lambda}} a_\xi e^{i(x,\xi)}. \tag{10.14}$$

We are interested in the quantity $||\phi||_4^4 = \int \left(|\phi|^2\right)^2$. With this in mind, write

$$|\phi(x)|^2 = \sum_{\eta=\sqrt{\lambda}=\xi} a_\xi \bar{a_\eta} e^{i(x,\xi-\eta)} = \sum_{\tau\in\mathbb{Z}^2} b_\tau e^{i(x,\tau)}, \tag{10.15}$$

where

$$b_\tau = \sum_{\xi-\eta=\tau} a_\xi \bar{a_\eta}. \tag{10.16}$$

By Parseval's formula,

$$\int \left(|\phi|^2\right)^2 = \sum_{\tau\in\mathbb{Z}^2} |b_\tau|^2 = |b_0|^2 + \sum_{\tau\neq 0} |b_\tau|^2. \tag{10.17}$$

We start by considering the part of the sum corresponding to $\tau \neq 0$. It is a simple geometric fact that for a fixed vector τ, there are at most two chords lying on a given circle which are congruent to that vector. This means that for each τ, the sum on the right hand side of equation 10.16 has at most two terms. Thus, it follows from the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, that for each

$\tau \neq 0$,

$$|b_\tau|^2 = \left| \sum_{\xi - \eta = \tau} a_\xi \bar{a}_\eta \right|^2 \leq \left(\sum_{\xi - \eta = \tau} |a_\xi| \, |a_\eta| \right)^2 \leq 2 \sum_{\xi - \eta = \tau} |a_\xi|^2 \, |a_\eta|^2 . \tag{10.18}$$

Summing over $\tau \neq 0$ gives

$$\sum_{\tau \neq 0} |b_\tau|^2 \leq 2 \sum_{\tau \neq 0} \sum_{\xi - \eta = \tau} |a_\xi|^2 \, |a_\eta|^2 . \tag{10.19}$$

The part of the right hand side of 10.19 containing any fixed ξ is exactly

$$2 \sum_{\tau \neq 0} |a_\xi|^2 \sum_{\xi - \eta = \pm\tau} |a_\eta|^2 = 4|a_\xi|^2 \sum_{\eta \neq \xi} |a_\eta|^2 = 4|a_\xi|^2(1 - |a_\xi|^2) \leq 4|a_\xi|^2, \tag{10.20}$$

Summing over ξ gives finally

$$\sum_{\tau \neq 0} |b_\tau|^2 \leq \sum_{\xi} 4|a_\xi|^2 = 4. \tag{10.21}$$

By Parseval's formula, $|b_0| = \sum_{\xi \in \mathbb{Z}^2} a_\xi^2 = ||\phi||_2^2 = 1$, so by equations 10.21 and 10.17, we have $||\phi||_4^4 \leq 5$, which is the desired result. □

The most general bounds are the following:

Theorem 22 (Bourgain [6])**.** *Let $n \geq 4$, $p \geq \frac{2(n+1)}{(n-3)}$, and $\epsilon > 0$. Then for each $\epsilon > 0$ there exists C_ϵ such that for any eigenfunction ϕ_λ on $\mathbb{T}^n$ with eigenvalue λ,*

$$\frac{||\phi_\lambda||_p}{||\phi_\lambda||_2} \leq C_\epsilon \lambda^{(n-2)/4 - n/2p + \epsilon}.$$

Theorem 23 (Bourgain [5])**.** *Let $n \geq 2$, $p \leq \frac{2n}{n-1}$. Then for each $\epsilon > 0$ there exists D_ϵ such that for any eigenfunction ϕ_λ on $\mathbb{T}^n$ with eigenvalue λ,*

$$\frac{||\phi_\lambda||_p}{||\phi_\lambda||_2} \leq D_\epsilon \lambda^{\epsilon}.$$

Bourgain [6] conjectured that in both cases the epsilon may be dropped, and that the lower bound on p in theorem 22 can be improved to the critical exponent $\frac{2n}{n-1}$.

Quantum Limits on the Torus

We recall the definition of quantum limits, given in chapter 6. If there exists a measure $\mathrm{d}\mu$ on $\mathbb{T}^n$ such that for any test function $f \in L^2(\mathbb{T}^n)$,

$$\int_{\mathbb{T}^n} f|\phi_k|^2 \ \mathrm{d}V \to \int_{\mathbb{T}^n} f \ \mathrm{d}\mu,$$

we say that the measures $\mathrm{d}\mu_k = |\phi_k|^2 \ \mathrm{d}V$ *converge weakly* to $\mathrm{d}\mu$, we write $\mathrm{d}\mu_k \rightharpoonup \mathrm{d}\mu$, and we say that $\mathrm{d}\mu$ is a *quantum limit*. It is natural to ask if we can say anything about such quantum limits. It is a result of Bourgain [25] that on the torus $\mathbb{T}^n$, any quantum limit $\mathrm{d}\mu$ is absolutely continuous with respect to the Lebesgue measure. By Radon-Nikodym, it follows that there exists a non-negative function $h \in L^1(\mathbb{T}^n)$ such that $\mathrm{d}\mu = h \ \mathrm{d}x$. As h is in $L^1(\mathbb{T}^n)$, it has a Fourier series, say

$$h(x) \sim \sum_{\tau \in \mathbb{Z}^n} c_\tau e^{i(x,\tau)}.$$

Jakobson [25] showed that any quantum limit on $\mathbb{T}^2$ must be a trigonometric polynomial whose frequencies τ have at most two different magnitudes. In the same paper, Jakobson also showed that estimates for quantum limits on $\mathbb{T}^{n+2}$ can be obtained from estimates for eigenfunctions on $\mathbb{T}^n$. With this in mind, we notice that by Parseval's theorem, we can interpret Cooke and Zygmund's theorem 21 as saying that on $\mathbb{T}^2$, the ℓ^2 norms of the Fourier coefficients of densities $|\phi|^2$ arising from Laplace eigenfunctions ϕ are uniformly bounded, independently of the eigenvalue. Jakobson [25] proved the corresponding statement for ℓ^3 norms on $\mathbb{T}^3$, and Jakobson, Nadirashvili, and Toth did the same for ℓ^4 norms on $\mathbb{T}^4$. In [25], Jakobson conjectured that the ℓ^n norms of the Fourier coefficients of $|\phi_j|^2$ should be uniformly bounded for each $\mathbb{T}^n$. Aissiou proved the conjecture for the remaining $n \geq 5$.

Theorem 24 (Cooke [12] , Zygmund [56] , Jakobson [25], Jakobson-Nadirashvili-Toth [27], Aissiou [1])**.** *Let $n \geq 2$. Then there exists a constant $C(n) < \infty$ such that for any Laplace eigenfunction $\phi, \Delta\phi + \lambda\phi = 0$ on $\mathbb{T}^n$,*

$$\left|\left|\widehat{|\phi|^2}\right|\right|_{\ell^n} \leq C(n)||\phi||_2^2.$$

Furthermore, $C(n) \to 2$ as $n \to \infty$.

Combining theorem 24 with the arguments of Jakobson [25] relating quantum limits on $\mathbb{T}^{n+2}$ to eigenfunctions on $\mathbb{T}^n$ gives the following bound on quantum limits:

Theorem 25 (Jakobson [25], Jakobson-Nadirashvili-Toth [27], Aissiou [1])**.** *Let $n \geq 1$. Then there exists a constant $C(n) < \infty$, such that for any quantum limit $\mathrm{d}\mu$ on $\mathbb{T}^{n+2}$, with Fourier coefficients c_τ,*

$$\left(\sum_{\tau \in \mathbb{Z}^{n+2}} |c_\tau|^n\right)^{1/n} \leq C(n) < \infty.$$

Furthermore, when $n \geq 2$, $C(n)$ is the same constant that appears in Theorem 24.

Notice that when $n = 1$, Theorem 25 implies that quantum limits on $\mathbb{T}^3$ must have absolutely convergent Fourier series, and when $n = 2$, it implies with Parseval's theorem that quantum limits on $\mathbb{T}^4$ must lie in $L^2(\mathbb{T}^4)$.

CHAPTER 11
Symmetry between ϕ^+ and ϕ^-

Given an eigenfunction ϕ of the Laplacian on a closed manifold M, define its positive part ϕ^+ and its negative part ϕ^- as follows:

$$\phi^+ := \phi \cdot \chi_{\{\phi \geq 0\}} \tag{11.1}$$

$$\phi^- := -\phi \cdot \chi_{\{\phi \leq 0\}}. \tag{11.2}$$

Then $\phi = \phi^+ - \phi^-$. If ϕ is non-constant, by Green's theorem we have

$$\int_M \phi \; \mathrm{d}V = \int_M \frac{\Delta\phi}{\lambda} \mathrm{d}V = \int_M \phi \; \Delta\left(\frac{1}{\lambda}\right) \mathrm{d}V = \int_M \phi \cdot 0 \; \mathrm{d}V = 0. \tag{11.3}$$

It follows that

$$||\phi^+||_1 = ||\phi^-||_1. \tag{11.4}$$

Nadirashvili showed that this symmetry partially extends to the L^∞ norms of the positive and negative parts, in the following sense:

Theorem 26 (Nadirashvili [37])**.** *Given an n-dimensional compact smooth manifold M, with eigenfunctions $\Delta\phi_i + \lambda_i\phi_i = 0$, $\lambda_1 \leq \lambda_2 \leq \cdots$, there exists $C > 0$, depending only on n, and a positive integer N, depending only on M, such that for every $i > N$,*

$$\frac{1}{C} < \frac{||\phi_i^+||_\infty}{||\phi_i^-||_\infty} < C. \tag{11.5}$$

Jakobson and Nadirashvili showed that this symmetry also holds for intermediate L^p norms:

Theorem 27 (Jakobson-Nadirashvili [26])**.** *Given a smooth compact manifold M, for each $p \geq 1$ there exists a constant C, depending only on p, such that* *for any eigenfunction ϕ of the Laplacian,*

$$\frac{1}{C} \leq \frac{||\phi^{+}||_p}{||\phi^{-}||_p} \leq C. \tag{11.6}$$

Quasi-symmetry between the positive and negative parts of eigenfunctions have been investigated in other ways. Donnelly and Fefferman demonstrated a lower bound on the volume of the set where an eigenfunction takes a constant sign. Nadirashvili later gave a different proof for the two-dimensional case.

Theorem 28 (Donnelly-Fefferman [18], Nadirashvili [37]). *Given an n-dimensional real analytic Riemannian manifold (M, g) with analytic metric, there exists a constant C, depending only on (M, g), such that for any eigenfunction ϕ of the Laplacian,*

$$vol\{\phi > 0\} > C. \tag{11.7}$$

The theorem is deduced as a corollary of the following "local" quasi-symmetry theorem:

Theorem 29 (D-F [18]). *Let (M, g) be an n-dimensional real analytic Riemannian manifold with analytic metric. Let $D \subseteq M$ be a fixed metric ball.*

Then there exists Λ depending on g and the radius of D such that for any Laplace eigenfunction ϕ_λ with eigenvalue $\lambda > \Lambda$,

$$\frac{vol(\{\phi_\lambda > 0\} \cap D)}{vol(D)} \geq a, \tag{11.8}$$

where the constant $a > 0$ depends only on the metric g.

In the case of surfaces with smooth metrics, Nazarov, Polterovich, and Sodin proved a related result for balls of any radius that satisfy a "deepness assumption":

Theorem 30 (Nazarov-Polterovich-Sodin [19]). *Let M be a closed connected surface with smooth metric. Let ϕ_λ be a Laplace eigenfunction with eigenvalue $\lambda \geq 3$. Suppose that there exists some metric disk $D \subseteq M$ such that $\{\phi_\lambda >$*

$0\} \cap \frac{1}{2}D \neq \emptyset$, where $\frac{1}{2}D$ is the disk with the same centre as D, but half its radius.

Then,

$$\frac{vol(\{\phi_\lambda > 0\} \cap D)}{vol(D)} \geq \frac{a}{\log \lambda \sqrt{\log \log \lambda}}, \tag{11.9}$$

where a is a constant depending only on the metric g.

They also showed that this theorem is sharp, up to the factor $\sqrt{\log \log \lambda}$, on the standard sphere $\mathbb{S}^2$:

Theorem 31 (N-P-S [19])**.** *Consider the 2-sphere $\mathbb{S}^2$ endowed with the standard metric. Then there exists a constant $C > 0$, a sequence of Laplace eigenfunctions ϕ_i, with eigenvalues $\lambda_i \to \infty$, and a sequence of disks $D_i \subseteq \mathbb{S}^2$, such that each ϕ_i vanishes at the centre of D_i, such that for all $i \in \mathbb{N}$,*

$$\frac{vol(\{\phi_i > 0\} \cap D_i)}{vol(D_i)} \leq \frac{C}{\log \lambda_i}. \tag{11.10}$$

CHAPTER 12
Restriction of eigenfunctions to submanifolds

Another approach to studying the concentration of eigenfunctions is to examine the growth of the L^p norms of their restrictions to submanifolds. Let (M, g) be a compact smooth Riemannian manifold, and Σ a smooth embedded submanifold. The metric endows M and Σ with canonical measures, so the spaces $L^p(M)$ and $L^p(\Sigma)$ are well-defined.

Burq, Gérard, and Tzvetkov proved a bound on the $L^p(\Sigma)$ norms of eigenfunctions in the case that Σ is a smooth curve.

Theorem 32 (Burq-Gérard-Tzvetkov, [36]). *Let (M, g) be a compact smooth Riemannian manifold, and let Σ be a smooth curve $\gamma : [a, b] \to M$, parameterized by arc length.*

Then there exists a constant C such that for every ϕ_λ, with $\Delta\phi_\lambda + \lambda\phi_\lambda = 0$,

$$||\phi_\lambda||_{L^p(\gamma)} \leq C(1 + \lambda)^{\delta(p)}||\phi_\lambda||_{L^2(M)}, \tag{12.1}$$

where

$$\delta(p) = \begin{cases} \frac{1}{4} - \frac{1}{2p} & 4 \leq p \leq \infty \\ \frac{1}{8} & 2 \leq p \leq 4 \end{cases}.$$

This bound is sharp if M is the standard sphere S^2 and γ is:

(i) any curve ($4 \leq p \leq \infty$) ;

(ii) a geodesic ($2 \leq p < 4$).

In the case $2 \leq p < 4$, they improve this bound for curves with non-vanishing geodesic curvature. Let $\frac{D}{dt}$ denote the covariant derivative along γ.

Theorem 33 (BGT, [36]). *Let γ be such that*

$$g\left(\frac{D}{\mathrm{d}t}\gamma', \frac{D}{\mathrm{d}t}\gamma'\right) \neq 0.$$

Then there exists a constant C such that for every ϕ_λ, with $\Delta\phi_\lambda + \lambda\phi_\lambda = 0$, and $2 \leq p \leq 4$,

$$||\phi_\Lambda||_{L^p(\gamma)} \leq C(1+\lambda)^{\tilde{\delta}(p)}||\phi_\lambda||_{L^2(M)}, \tag{12.2}$$

where

$$\tilde{\delta}(p) = \frac{1}{6} - \frac{1}{6p}.$$

This bound is sharp if M is the standard sphere S^2 and γ is any curve with non-vanishing geodesic curvature.

Burq, Gérard and Tzvetkov show that their techniques extend to give bounds for submanifolds of higher dimension.

Theorem 34 (BGT [36]). *Let (M, g) be a compact smooth Riemannian manifold of dimension d and Σ be a smooth submanifold of dimension k.*

Then there exists a constant $C > 0$ such that for any ϕ_λ, $\Delta\phi_\lambda + \lambda\phi_\lambda = 0$, we have

$$||\phi_\lambda||_{L^p(\Sigma)} \leq C(1+\lambda)^{\rho(k,d)}||\phi_\lambda||_{L^2(M)} \tag{12.3}$$

where

$$\rho(d-1,d) = \begin{cases} \frac{d-1}{4} - \frac{d-1}{2p} & p_0 = \frac{2d}{d-1} < p \leq \infty \\ \frac{d-1}{8} - \frac{d-2}{4p} & 2 \leq p < p_0 = \frac{2d}{d-1} \end{cases}$$
$$\rho(d-2,d) = \frac{d-1}{4} - \frac{d-2}{2p} \qquad 2 < p \leq \infty$$
$$\rho(k,d) = \frac{d-1}{4} - \frac{k}{2p} \qquad 1 \leq k \leq d-3.$$

If $p = p_0 = \frac{2d}{d-1}$ and $k = d-1$, we have

$$||\phi_\lambda||_{L^p(\Sigma)} \leq C(1+\lambda)^{\frac{d-1}{2d}} \log^{1/2}(\lambda)||\phi_\lambda||_{L^2(M)}$$

and if $p = 2$ and $k = d - 2$, we have

$$||\phi_\lambda||_{L^p(\Sigma)} \le C(1+\lambda)^{\frac{1}{2}} \log^{1/2}(\lambda)||\phi_\lambda||_{L^2(M)}.$$

All estimates are sharp, except for the log loss, if

1. *$k \le d-2$, M is the standard sphere S^d and Σ is any submanifold of dimension k.*
2. *$k = d-1$ and $\frac{2d}{d-1} \le p \le \infty$, M is the standard sphere S^d and Σ is any hypersurface.*
3. *$k = d-1$ and $2 \le p \le \frac{2d}{d-1}$, M is the standard sphere S^d and Σ is any hypersurface containing a piece of geodesic.*

In the special case where M is a compact hyperbolic surface and $p = 2$, Reznikov [39] has given proofs of the results of theorems 32 and 33 using the representation theory of $\mathrm{PGL}_2\mathbb{R}$.

The bound in equation 12.3 can be made uniform in λ in cases of standard flat d-dimensional tori $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$, with codimension one hypersurfaces Σ, and $p = 2$:

Theorem 35 (Bourgain-Rudnick [7]). *Let $d = 2, 3$ and let $\Sigma \subset \mathbb{T}^d$ be a smooth hypersurface with non-zero curvature. There are constants $0 < C_1 < C_2 < \infty$ and $\Lambda > 0$, all depending on Σ, so that all eigenfunctions ϕ_λ of the Laplacian on $\mathbb{T}^d$ with $\lambda > \Lambda$ satisfy*

$$C_1||\phi_\lambda||_2 \le ||\phi_\lambda||_{L^2(\Sigma)} \le C_2||\phi_\lambda||_2 \tag{12.4}$$

A weaker version holds in higher dimensions:

Theorem 36 (BR [7]). *For all $d \ge 4$ there is $\rho(d) < \frac{1}{6}$ so that if ϕ_λ is an eigenfunction of the Laplacian on $\mathbb{T}^d$, and $\Sigma \subset \mathbb{T}^d$ is a smooth compact* *hypersurface with positive curvature, then*

$$\frac{||\phi_\lambda||_{L^2(\Sigma)}}{||\phi_\lambda||_2} = O\left(\lambda^{\rho(d)}\right). \tag{12.5}$$

We give the proof of theorem 35 in the case $d = 2$, where it is relatively straightforward:

Proof. We suppose that the surface measure $\mathrm{d}\sigma$ is normalized, i.e., that

$$\int_\Sigma \mathrm{d}\sigma = 1. \tag{12.6}$$

The Fourier transform of the surface measure has the usual definition:

$$\hat{\sigma}(\xi) = \int_\Sigma e^{-ix\cdot\xi}\mathrm{d}\sigma(x). \tag{12.7}$$

It is a classical result in harmonic analysis (see e.g. [49]) that it decays as follows:

$$|\hat{\sigma}(\xi)| \leq C|\xi|^{-1/2}, \tag{12.8}$$

for some constant C depending on Σ. Notice that

$$|\hat{\sigma}(\xi)| \leq \int_\Sigma \left|e^{-ix\cdot\xi}\right| \mathrm{d}\sigma(x) = 1, \tag{12.9}$$

with equality only when $e^{-ix\cdot\xi}$ is of constant argument, which happens only when $\xi = 0$. It follows from (12.8) that there exists δ, depending on Σ, such that

$$|\hat{\sigma}(\xi)| \leq 1 - \delta, \quad \forall \xi \neq 0. \tag{12.10}$$

Suppose ϕ is an eigenfunction of the Laplacian on $\mathbb{T}^2$ with eigenvalue λ. Then

$$\phi(x) = \sum_{|n|=\lambda} c_n e^{in\cdot x}, \tag{12.11}$$

for some $c_n \in \mathbb{C}$.

The frequencies n lie in $\mathcal{E} := \mathbb{Z}^2 \cap \lambda S^1$, the set of lattice points on the circle of radius λ. It is a classical result that for c small enough, any arc of length $c\lambda^{1/3}$ on λS^1 contains at most two lattice points. Indeed, suppose there were such an arc with three lattice points, A, B, and C. Then λ is the circumradius of the triangle ΔABC, and we know from Euclidean geometry that

$$\text{Area}(\Delta ABC) = \frac{|AB||AC||BC|}{4\lambda} \leq \frac{(c\lambda^{1/3})^3}{4\lambda} = \frac{c^3}{4}. \tag{12.12}$$

Notice, however, that the vertices of ΔABC have integer coordinates, so its area is at least $\frac{1}{2}$. Taking $c < 2^{1/3}$ gives a contradiction.

It follows that we may partition $\mathcal{E}$ into disjoint subsets $\mathcal{E}_\alpha$, where $\#\mathcal{E}_\alpha \leq 2$ and $\text{dist}(\mathcal{E}_\alpha, \mathcal{E}_\beta) > c\lambda^{1/3}$. Then we can write

$$\phi = \sum_\alpha \phi^\alpha, \quad \phi^\alpha(x) = \sum_{n \in \mathcal{E}_\alpha} c_n e^{in\cdot x}. \tag{12.13}$$

It follows that

$$\int_\Sigma |\phi|^2 \mathrm{d}\sigma = \int_\Sigma \sum_\alpha \phi^\alpha \overline{\sum_\beta \phi^\beta} \mathrm{d}\sigma = \sum_\alpha \sum_\beta \int_\Sigma \phi^\alpha \overline{\phi^\beta} \mathrm{d}\sigma. \tag{12.14}$$

Look at this last integral for the nondiagonal terms, $\alpha \neq \beta$. We can write

$$\left| \int_\Sigma \phi^\alpha \overline{\phi^\beta} \mathrm{d}\sigma \right| = \left| \int_\Sigma \sum_{n_\alpha \in \mathcal{E}_\alpha, n_\beta \in \mathcal{E}_\beta} c_{n_\alpha} \overline{c_{n_\beta}} e^{i(n_\alpha - n_\beta)\cdot x} \mathrm{d}\sigma(x) \right| \tag{12.15}$$

$$\leq \sum_{n_\alpha \in \mathcal{E}_\alpha, n_\beta \in \mathcal{E}_\beta} \left| \int_\Sigma c_{n_\alpha} \overline{c_{n_\beta}} e^{i(n_\alpha - n_\beta)\cdot x} \mathrm{d}\sigma(x) \right| \tag{12.16}$$

$$= \sum_{n_\alpha \in \mathcal{E}_\alpha, n_\beta \in \mathcal{E}_\beta} |c_{n_\alpha}| \left|c_{n_\beta}\right| |\hat{\sigma}(n_\beta - n_\alpha)| \tag{12.17}$$

As $\#\mathcal{E}_\alpha, \mathcal{E}_\beta \leq 2$, this last sum has at most 4 terms. Let $k = \max_{n \in \mathcal{E}} |c_n|$. Then in each term, $|c_{n_\alpha}||c_{n_\beta}| \leq k^2 \leq \sum_{n \in \mathcal{E}} |c_n|^2 = ||\phi||_2^2$. Also, as $\text{dist}(\mathcal{E}_\alpha, \mathcal{E}_\beta) > c\lambda^{1/3}$, in each term we have $|n_\beta - n_\alpha| > c\lambda^{1/3}$, so by (12.8), we get $|\hat{\sigma}(n_\beta - n_\alpha)| \leq C\left(c\lambda^{1/3}\right)^{-1/2} = C\lambda^{-1/6}$, after combining constants. Putting it all together gives

$$\left|\int_\Sigma \phi^\alpha \overline{\phi^\beta} \mathrm{d}\sigma\right| \leq 4C||\phi||_2^2 \lambda^{-1/6}. \tag{12.18}$$

The multiplicity of λ is $\#\mathcal{E}$. It is well known (see Chapter 10) that

$$\#\mathcal{E} \leq D_\epsilon \lambda^\epsilon, \tag{12.19}$$

for all $\epsilon > 0$, and some D_ϵ depending on ϵ. The number of different α, β is each at most $\#\mathcal{E}$, so it follows that

$$\sum_\alpha \sum_\beta \left|\int_\Sigma \phi^\alpha \overline{\phi^\beta} \mathrm{d}\sigma\right| \leq (D_\epsilon \lambda^\epsilon)^2 4C||\phi||_2^2 \lambda^{-1/6} = 4CD_\epsilon^2 ||\phi||_2^2 \lambda^{-1/6+2\epsilon}. \tag{12.20}$$

It follows that by taking λ large enough, we can make the contribution of the nondiagonal terms to the integral (12.14) as small as we want.

Now let us look at the diagonal terms. We claim that

$$\delta ||\phi^\alpha||_2^2 \leq \int_\Sigma |\phi^\alpha|^2 \mathrm{d}\sigma \leq 2||\phi^\alpha||_2^2. \tag{12.21}$$

If $\mathcal{E}_\alpha = \{n\}$, then $||\phi^\alpha||_2^2 = |c_n|^2$ and $|\phi^\alpha| = |c_n|$, so the result is immediate. If $\mathcal{E}_\alpha = \{n, m\}$,

$$\begin{aligned}
\int_\Sigma |\phi^\alpha|^2 \mathrm{d}\sigma &= \int_\Sigma \left(c_n e^{in\cdot x} + c_m e^{im\cdot x}\right) \overline{(c_n e^{in\cdot x} + c_m e^{im\cdot x})} \mathrm{d}\sigma(x) && (12.22)\\
&= \int_\Sigma |c_n|^2 + |c_m|^2 + c_m \overline{c_n} e^{i(m-n)\cdot x} + \overline{c_m \overline{c_n} e^{i(m-n)\cdot x}} \mathrm{d}\sigma(x) && (12.23)\\
&= |c_n|^2 + |c_m|^2 + c_m \overline{c_n} \hat{\sigma}(n-m) + \overline{c_m \overline{c_n} \hat{\sigma}(n-m)} && (12.24)\\
&= |c_n|^2 + |c_m|^2 + 2\Re c_m \overline{c_n} \hat{\sigma}(n-m). && (12.25)
\end{aligned}$$

Then,

$$|2\Re c_m \overline{c_n} \hat{\sigma}(n-m)| \leq 2|c_m||c_n||\hat{\sigma}(n-m)| \tag{12.26}$$

$$\leq \left(|c_m|^2 + |c_n|^2\right) |\hat{\sigma}(n-m)| \tag{12.27}$$

$$= ||\phi^\alpha||_2^2 |\hat{\sigma}(n-m)| \tag{12.28}$$

$$\leq (1-\delta)||\phi^\alpha||_2^2, \tag{12.29}$$

where the last inequality comes from equation (12.10) ($n \neq m$!). Combining equations (12.25) and (12.29) gives equation (12.21).

Summing equation (12.21) over α gives

$$\delta||\phi||_2^2 \leq \sum_\alpha \int_\Sigma |\phi^\alpha|^2 \mathrm{d}\sigma \leq 2||\phi||_2^2. \tag{12.30}$$

Take λ large enough that the contribution of the nondiagonal terms in (12.20) is less than $\delta||\phi||_2^2$. Then adding contribution of the nondiagonal terms to (12.30) gives

$$C_1||\phi||_2^2 \leq \int_\Sigma |\phi|^2 \mathrm{d}\sigma \leq C_2||\phi||_2^2, \tag{12.31}$$

for some $C_1 < \delta, C_2 > 2$, which is what we wanted to show. □

Theorem 35 can be used to estimate the Hausdorff measure of the intersection of the nodal set of a toral eigenfunction on $\mathbb{T}^2, \mathbb{T}^3$ with a codimension one hypersurface. Other methods extend the result to all $\mathbb{T}^d$:

Theorem 37 (BR [8]). *Suppose Σ is a real-analytic codimension one hypersurface of $\mathbb{T}^d$ with non-vanishing Gauss curvature. Then there exists $\Lambda_\Sigma > 0$, and C_Σ such that for any eigenfunction ϕ_λ with eigenvalue $\lambda > \Lambda_\Sigma$ and nodal set N,*

$$h_{d-2}(N \cap \Sigma) \leq C_\Sigma \sqrt{\lambda},$$

where h_{d-2} is the $d-2$-dimensional Hausdorff measure.

In the case $d = 2$, the 0-dimensional Hausdorff measure is simply the counting measure, so the theorem says that the intersection $N \cap \Sigma$ is finite. If Σ is algebraic and it intersects transversally with N, we can show this another way: $N \cap \Sigma$ is the locus of points where a polynomial and a trigonometric polynomial both vanish. According to a Bezout-like result of Khovanskii, there can only be finitely many such points. The resulting bound is not as good as the one obtained by Bourgain-Rudnick, but the method generalizes to give other interesting results, so we present it below.

First, we give a bound on the number of intersections of the nodal set of an eigenfunction on $\mathbb{T}^2$ with an algebraic curve of degree m, in terms of its eigenvalue. Recall that the eigenspace corresponding to a given eigenvalue λ has as a basis $\{\sin(mx+ny), \cos(mx+ly) | \lambda = m^2 + l^2, \quad m, l \in \mathbb{Z}, m \geq 0\}$.

An algebraic curve of degree p is given as the locus of points where a polynomial $P_1(x, y)$ of degree p vanishes. Finding its points of intersection with a nodal set means solving the following system of two equations in two variables:

$$P_1(x, y) = 0 \tag{12.32}$$

$$P_2 = \sum_{m^2+l^2=\lambda, m\geq 0} a_{m,l} \cos(mx+ly) + b_{m,l} \sin(mx+ly) = 0 \tag{12.33}$$

We can use the sine and cosine sum identities to rewrite equation 12.33 as follows:

$$\begin{aligned} P_2 = \sum_{m^2+l^2=\lambda, m\geq 0} & a_{m,l} \left(\cos(mx)\cos(ly) - \sin(mx)\sin(ly)\right) \\ & + b_{m,l} \left(\sin(mx)\cos(ly) + \cos(mx)\sin(ly)\right) = 0. \end{aligned} \tag{12.34}$$

The cosine sum identity allows us to write $\cos(nx)$ as a polynomial in $\cos(x)$ for any positive integer n. Indeed,

$$\cos((n+1)x) = \cos x \cos nx - \sin x \sin nx \tag{12.35}$$
$$= 2\cos x \cos nx - (\cos x \cos nx + \sin x \sin nx) \tag{12.36}$$
$$= 2\cos x \cos nx - \cos(n-1)x, \tag{12.37}$$

and the result follows by induction on n. The nth Chebyshev polynomial of the first kind, T_n, is defined by $\cos(nx) = T_n(\cos(x))$. It follows from formula 12.37 that T_n is of degree n.

Similarly, notice that $\frac{\sin 2x}{\sin x} = 2\cos x$ and by the sine sum formula

$$\frac{\sin(n+1)x}{\sin x} = \frac{\sin nx \cos x}{\sin x} + \cos(nx). \tag{12.38}$$

By induction, it follows that $\frac{\sin(n+1)x}{\sin x}$ is also an nth degree polynomial in $\cos x$. It is called the nth Chebyshev polynomial of the second kind, and is denoted U_n. In particular, we have that $\sin nx$ is an nth degree polynomial in $\sin x$ and $\cos x$.

It follows from above that for each $m^2 + l^2 = \lambda$,

$$a_{m,l}\left(\cos(mx)\cos(ly) - \sin(mx)\sin(ly)\right) + b_{m,l}\left(\sin(mx)\cos(ly) + \cos(mx)\sin(ly)\right) = 0$$

is a polynomial in $\sin x$ and $\cos x$ of degree $m + l$. Thus, P_2 is a polynomial in $\sin x$ and $\cos x$ of degree at most

$$\max_{m^2+l^2=\lambda} m + l.$$

Using Lagrange multipliers, we can show this last expression is at most $\sqrt{2\lambda}$.

We now apply Khovanskii's result on the number of solutions of a system of trigonometric quasipolynomials. We quote the theorem in its full generality:

Theorem 38 (Khovanskii [29]). *Suppose* $P_1 = \cdots = P_d = 0$ *is a system of equations in* d *real variables* $x = x_1, \cdots, x_d$*, where* P_i *is a polynomial of degree* p_i *in the* $d + 2\rho$ *variables* $x, u_1, \cdots u_\rho, v_1, \cdots, v_\rho$*, where* $u_q = \sin(b_q, x)$, $\quad v_q = \cos(b_q, x)$, $\quad q = 1, \cdots, \rho$.

Then the number of nonsingular solutions of this system in the region bounded by the inequalities $|(b_q, x)| < \pi/2, q = 1, \cdots, \rho$ *is finite and less than*

$$p_1 \cdots p_d \left(\sum p_i + \rho + 1 \right)^\rho 2^{\rho + [\rho(\rho-1)/2]}.$$

In our case, we have $d = 2$, $x_1 = x$, $x_2 = y$, $p_1 = 2$, $p_2 \le \sqrt{2\lambda}$, and $\rho = 2$. It follows that in the region bounded by the inequalities $|x| < \pi/2, |y| < \pi/2$, the number of non-singular solutions to the system of equations 12.32, 12.33 is less than

$$8d\sqrt{2\lambda} \left(\sqrt{2\lambda} + d + 3 \right)^2.$$

Notice, however, that this region is only one quarter of the whole torus. We can cover the rest of the torus using 3 more copies of this region: moving it by $+\pi$ in the x direction, by $+\pi$ in the y direction, and by $+\pi$ in both directions. In each such region, by making a change of coordinates we can use the same method as above to bound the number of intersections of the nodal set with the curve, so that the final bound is 4 times the one obtained above. Thus we have:

Theorem 39. *Suppose* Γ *is an algebraic curve of degree* d *on the torus. Suppose* ϕ *is an eigenfunction of the Laplacian with eigenvalue* λ*. Then the number of non-singular intersections of the nodal set of* ϕ *with* Γ *is at most*

$$32d\sqrt{2\lambda} \left(\sqrt{2\lambda} + d + 3 \right)^2.$$

It is natural to ask if this result can be generalized to higher dimensions, say to $\mathbb{T}^d, d > 2$. Notice that the hypotheses of Theorem 38 require a system of d equations, so we cannot investigate intersection of a nodal set with only one hypersurface. One possibility, however, is to look at its intersection with $d-1$ hypersurfaces: Suppose $\Sigma_1, \cdots, \Sigma_{d-1}$ are algebraic hypersurfaces of $\mathbb{T}^d$ of degree $p_1, \cdots, p_{d-1}$. Suppose we have an eigenfunction ϕ of degree λ with nodal set N. The eigenspace corresponding to λ is spanned by

$$\left\{ \prod_{j=1}^{d} \binom{\sin}{\cos}(a_j x_j); \lambda = \sum_{j=1}^{d} a_j^2, a_j \geq 0 \right\},$$

which we know from before are trigonometric polynomials of degree $\sum_{j=1}^{n} a_j$. So ϕ is a trigonometric polynomial in the $2\rho = 2d$ variables $u_j = \sin(x_j), v_j = \cos(x_j)$, of degree at most

$$\max_{\sum_{j=1}^{d} a_j^2 = \lambda} \sum_{j=1}^{d} a_j.$$

Using Lagrange multipliers, we can show this last expression is at most $\sqrt{d\lambda}$. We conclude by Theorem 38 that in the region bounded by the inequalities $|x_j| < \pi/2, j = 1, \cdots, d$, the number of non-singular intersection points of $\Sigma_1, \cdots, \Sigma_{d-1}$ and N is at most

$$p_1 \cdots p_{d-1} \sqrt{d\lambda} \left(\sum_{i=1}^{d-1} p_i + \sqrt{d\lambda} + d + 1 \right)^d 2^{d + [d(d-1)/2]}.$$

Notice, however, that this region is only a fraction $(1/2)^d$ of the whole torus. We can cover the rest of the torus using $2^d - 1$ more copies of this region, each obtained by translating it by $+\pi$ in a different combination of the $\{x_i\}$ directions. In each such region, by making a change of coordinates we can use the same method as above to bound the number of intersections, so that the final bound is 2^d times the one obtained above. Thus we have:

Theorem 40. *Suppose $\Sigma_1, \cdots, \Sigma_{d-1}$ are algebraic hypersurfaces of $\mathbb{T}^d$ of degree $p_1, \cdots, p_{d-1}$. Suppose ϕ is an eigenfunction of the Laplacian with eigenvalue λ. Then the number of non-singular intersections of $\Sigma_1, \cdots, \Sigma_{d-1}$ and* *the nodal set of ϕ is at most*

$$2^d p_1 \cdots p_{d-1} \sqrt{d\lambda} \left(\sum_{i=1}^{d-1} p_i + \sqrt{d\lambda} + d + 1 \right)^d 2^{d+[d(d-1)/2]}.$$

Of course, instead of taking $d-1$ algebraic hypersurfaces above, we could have taken fewer, but considered their intersection with the nodal sets of more than one eigenfunction, as long as the total number of sets being intersected was d. In that case, we obtain the general theorem:

Theorem 41. *Suppose $\Sigma_1, \cdots, \Sigma_{d-k}$ are algebraic hypersurfaces of $\mathbb{T}^d$ of degree $p_1, \cdots, p_{d-k}$. Suppose $\phi_1, \cdots, \phi_k$ are eigenfunctions of the Laplacian with eigenvalues $\lambda_1, \cdots, \lambda_k$ and nodal sets $N_1, \cdots, N_k$. Then the number of non-singular intersections of $\Sigma_1, \cdots, \Sigma_{d-k}$ and $N_1, \cdots, N_k$ is at most*

$$2^d d^{k/2} p_1 \cdots p_{d-k} \sqrt{\lambda_1} \cdots \sqrt{\lambda_k} \left(\sum_{i=1}^{d-k} p_i + \sum_{i=1}^{k} \sqrt{d\lambda_i} + d + 1 \right)^d 2^{d+[d(d-1)/2]}.$$

CHAPTER 13
Limits of rescaled eigenfunctions on the torus

Let $\{\phi_k\}_k, \triangle\phi_k + \lambda_k\phi_k = 0$, be a sequence of eigenfunctions of the Laplacian on the flat torus $\mathbb{T}^n$ such that $\lambda_k \to \infty$ as $k \to \infty$. Write $\mathrm{d}x$ for the Lebesgue measure on $\mathbb{T}^n$. Given a measure $\mathrm{d}\mu$ on $\mathbb{T}^n$, if for any test function $f \in L^2(M)$,

$$\int_M f|\phi_k|^2 \mathrm{d}x \to \int_M f \ \mathrm{d}\mu,$$

we say that the measures $|\phi_k|^2 \ \mathrm{d}x$ *converge weakly* to $\mathrm{d}\mu$, and write $|\phi_k|^2 \ \mathrm{d}x \rightharpoonup \mathrm{d}\mu$. As before, we refer to such limit measures as *quantum limits*.

It is a result of Bourgain [25] that any quantum limit $\mathrm{d}\mu$ is absolutely continuous with respect to the Lebesgue measure. By Radon-Nikodym, it follows that there exists a non-negative function $h \in L^1(\mathbb{T}^n)$ such that $\mathrm{d}\mu = h \ \mathrm{d}x$.

Given positive real numbers $c_i, 1 \leq i \leq n$, we construct a *rescaled copy* N of $\mathbb{T}^n$ by setting

$$N = T^n \times \prod_{1 \leq i \leq n} [-c_i, c_i],$$

and defining projection maps $\Phi_k : \mathbb{T}^n \times \prod_{1 \leq i \leq n}[-c_i, c_i] \to \mathbb{T}^n$ by

$$\Phi_k(x, y) = x + \frac{y}{\sqrt{\lambda_k}},$$

for $x \in \mathbb{T}^n$ and $y \in \prod_{1 \leq i \leq n}[-c_i, c_i]$. For each k, we can imagine N as the manifold constructed by magnifying $\mathbb{T}^n$ *at each point* by a factor $\sqrt{\lambda_k}$, the approximate wavelength of ϕ_k, and then gluing the resulting spaces together.

We have the following new result:

Theorem 42. *Let $\{\phi_k\}_k, \triangle\phi_k + \lambda_k\phi_k = 0,$ be a sequence of eigenfunctions of the Laplacian on $\mathbb{T}^n$ such that $\lambda_k \to \infty$ as $k \to \infty$. Suppose that*

$$|\phi_k|^2 \,\mathrm{d}x \rightharpoonup h \,\mathrm{d}x,$$

for some $h \in L^1(\mathbb{T}^n)$.

Then $|\phi_k \circ \Phi_k|^2 \mathrm{d}x\mathrm{d}y \rightharpoonup H \mathrm{d}x\mathrm{d}y$ on any rescaled copy of $\mathbb{T}^n$, where $H(x,y) := h(x)$.

First we prove a special case when $n = 1$. Set $M = \mathbb{T}^1 = S^1$. Consider the eigenfunctions $\phi_k(x) = \sin(kx)$, with $\lambda_k = k^2$. Let $f \in L^2(S^1)$ be a test function on S^1. Then

$$\begin{aligned} \int_0^{2\pi} f(x)|\phi_k(x)|^2 \mathrm{d}x &= \int_0^{2\pi} f(x)\sin^2(kx)\mathrm{d}x \\ &= \int_0^{2\pi} \frac{f(x)}{2}\left(1 - \cos(2kx)\right)\mathrm{d}x \\ &\xrightarrow{k} \int_0^{2\pi} \frac{f(x)}{2}\mathrm{d}x \end{aligned}$$

by the Riemann-Lebesgue lemma. This means $|\phi_k|^2 \mathrm{d}x \rightharpoonup \frac{1}{2}\mathrm{d}x$ on S^1.

A rescaled copy of S^1 is $N = S^1 \times [-c, c]$, where $c > 0$. The projection map $\Phi_k : N \to S^1$ is given by $\Phi_k(x,y) = x + \frac{y}{k}$. We want to show $|\phi_k \circ \Phi_k|^2 \mathrm{d}x\mathrm{d}y \rightharpoonup \frac{1}{2}\mathrm{d}x\mathrm{d}y$ on N. Define

$$\begin{aligned} I_k(f) &:= \int_{x=0}^{2\pi} \int_{y=-c}^{c} f(x,y)\phi_k^2\left(\Phi_k(x,y)\right)\mathrm{d}y\mathrm{d}x \\ &= \int_{x=0}^{2\pi} \int_{y=-c}^{c} f(x,y)\phi_k^2\left(x + \frac{y}{k}\right)\mathrm{d}y\mathrm{d}x. \end{aligned}$$

Then $|\phi_k \circ \Phi_k|^2 \mathrm{d}x\mathrm{d}y \rightharpoonup \frac{1}{2}\mathrm{d}x\mathrm{d}y$ on N *if and only if* for any test function $f \in L^2(N)$,

$$I_k(f) \to \frac{1}{2}\int_N f \mathrm{d}x\mathrm{d}y.$$

Using trigonometric identities, we have:

$$\begin{aligned}
\phi_k^2\left(x+\frac{y}{k}\right) &= \sin^2\left(k\left(x+\frac{y}{k}\right)\right) \\
&= \sin^2\left(kx+y\right) \\
&= \frac{1}{2}\left(1-\cos\left(2kx+2y\right)\right) \\
&= \frac{1}{2}\left(1+\sin(2kx)\sin\left(2y\right)-\cos(2kx)\cos\left(2y\right)\right).
\end{aligned}$$

Let $f(x,y) \in L^2(N)$. Then

$$I_k\left(f(x,y)\right) = \frac{1}{2}\left(I_k^1+I_k^2+I_k^3\right),$$

where

$$\begin{aligned}
I_k^1 &:= \int_{x=0}^{2\pi}\int_{y=-c}^{c} f(x,y)\mathrm{d}y\mathrm{d}x = \int_N f\mathrm{d}x\mathrm{d}y; \\
I_k^2 &:= \int_{x=0}^{2\pi}\int_{y=-c}^{c} f(x,y)\sin(2kx)\sin\left(2y\right)\mathrm{d}y\mathrm{d}x; \\
I_k^3 &:= -\int_{x=0}^{2\pi}\int_{y=-c}^{c} f(x,y)\cos(2kx)\cos\left(2y\right)\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

The Riemann-Lebesgue lemma implies that as $k \to \infty$,

$$\int_{x=0}^{2\pi} f(x,y)\sin(2kx)\mathrm{d}x \to 0 \quad \text{and} \quad \int_{x=0}^{2\pi} f(x,y)\cos(2kx)\mathrm{d}x \to 0,$$

whence $I_k^2 \xrightarrow{k} 0$ and $I_k^3 \xrightarrow{k} 0$, which gives $I_k(f) \xrightarrow{k} \frac{1}{2}\int_N f\mathrm{d}x\mathrm{d}y$.

We conclude that $|\phi_k \circ \Phi_k|^2\mathrm{d}x\mathrm{d}y \rightharpoonup \frac{1}{2}\mathrm{d}x\mathrm{d}y$ on N, which is what we wanted to show.

Now we prove the general case.

Proof. Let $\{\phi_k\}_k, \triangle\phi_k + \lambda_k\phi_k = 0$, be a sequence of eigenfunctions of the laplacian on $\mathbb{T}^n$ such that $\lambda_k \to \infty$ as $k \to \infty$. Suppose that $|\phi_k|^2\mathrm{d}x \rightharpoonup h\mathrm{d}x$

for some $h \in L^1(\mathbb{T}^n)$. As h is integrable, it has a Fourier series, say

$$h(x) \sim \sum_{\tau \in \mathbb{Z}^n} b_\tau e^{i(x,\tau)}.$$

Given an eigenfunction ϕ_k of the Laplacian on $\mathbb{T}^n$ with eigenvalue λ_k, we can write

$$\phi_k(x) = \sum_{\xi \in \mathbb{Z}^n; |\xi|^2 = \lambda_k} a_{\xi,k} e^{i(x,\xi)},$$

with $a_{\xi,k} \in \mathbb{C}$. Then

$$|\phi_k(x)|^2 = \sum_{\tau \in \mathbb{Z}^n} b_{\tau,k} e^{i(x,\tau)}, \quad \text{with} \quad b_{\tau,k} := \sum_{\xi - \eta = \tau, |\xi|^2 = \lambda_k = |\eta|^2} a_{\xi,k} \bar{a}_{\eta,k}.$$

Taking the test function $e^{i(x,-\tau)}$ in the definition of the weak convergence $|\phi_k|^2 \mathrm{d}x \rightharpoonup h \mathrm{d}x$ gives that $b_{\tau,k} \xrightarrow{k} b_\tau$.

Let $N := \mathbb{T}^n \times \Pi_i[-c_i, c_i]$ be any rescaled copy of $\mathbb{T}^n$, with projection maps $\Phi_k : N \to \mathbb{T}^n$ defined as before. We want to show that for any test function $F \in L^2(N)$,

$$\int_N F |\phi_k \circ \Phi_k|^2 \mathrm{d}x \mathrm{d}y \to \int_N F H \mathrm{d}x \mathrm{d}y.$$

We claim that it is enough to show this for each test function $F_\mu(x, y) = e^{i(x,\mu)} f_\mu(y)$ with $\mu \in \mathbb{Z}^n$, $f_\mu(y) \in L^1 \left(\Pi_i[-c_i, c_i]\right)$. Why? As $C^\infty(N)$ is dense in $L^2(N)$, it is enough to only consider $F \in C^\infty(N)$. For such F, for each fixed $y \in \Pi_i[-c_i, c_i]$, we have $F(\cdot, y) \in L^2(\mathbb{T}^n)$, so we can expand it in Fourier series:

$$F(\cdot, y) = \sum_{\mu \in \mathbb{Z}^n} f_\mu(y) e^{i(\cdot,\mu)}, \quad \text{where} \quad f_\mu(y) = \int_{\mathbb{T}^n} F(x, y) e^{-i(x,\mu)} \mathrm{d}x.$$

It follows that

$$F(x, y) = \sum_{\mu \in \mathbb{Z}^n} f_\mu(y) e^{i(x,\mu)}.$$

Notice that $\int_{\Pi_i[-c_i,c_i]} |f_\mu(y)|\mathrm{d}y \leq \int_{\Pi_i[-c_i,c_i]} \int_{\mathbb{T}^n} |F(x,y)|\mathrm{d}x\mathrm{d}y < \infty$ ($F(x,y)$ is smooth!), so $f_\mu(y) \in L^1\left((\Pi_i[-c_i,c_i]\right)$. If we define

$$F_n(x,y) = \sum_{|\mu|\leq n} f_\mu(y)e^{i(x,\mu)},$$

then

$$||F - F_n||_2^2 = \int_{\Pi_i[-c_i,c_i]} \int_{\mathbb{T}^n} \left| \sum_{|\mu|>n} f_\mu(y)e^{i(x,\mu)} \right|^2 \mathrm{d}y\mathrm{d}x = \int_{\Pi_i[-c_i,c_i]} \sum_{|\mu|>n} f_\mu(y)^2 \mathrm{d}y$$

by Parseval's theorem. For fixed $y \in \Pi_i[-c_i,c_i]$, Parseval gives that $\sum_{\mu\in\mathbb{Z}^n} f_\mu(y)^2 = \int_{\mathbb{T}^n} |F(x,y)|^2\mathrm{d}x < \infty$ since $F(x,y)$ is smooth. It follows that for fixed y, $\lim_{n\to\infty} \sum_{|\mu|>n} f_\mu(y)^2 = 0$. Smoothness of $F(x,y)$ implies $\int_{\mathbb{T}} |F(x,y)|^2\mathrm{d}x$ attains a maximum for some y, which can be used as the dominating function in the dominated convergence theorem to conclude that

$$\lim_{n\to\infty} ||F - F_n||_2^2 = \lim_{n\to\infty} \int_{\Pi_i[-c_i,c_i]} \sum_{|\mu|>n} f_\mu(y)^2 \mathrm{d}y = \int_{\Pi_i[-c_i,c_i]} 0 \cdot \mathrm{d}y = 0.$$

Thus, that finite linear combinations of functions of the form $f_\mu(y)e^{i(x,\mu)}$ with $f_\mu(y) \in L^1\left(\Pi_i[-c_i,c_i]\right)$ are dense in $C^\infty(N)$, giving the claim.

Fix an $F_\mu(x,y) = f_\mu(y)e^{i(x,\mu)}$ and define

$$\begin{aligned} I_k &:= \int_N F_\mu |\phi_k \circ \Phi_k|^2 \mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{T}^n} \int_{\Pi_i[-c_i,c_i]} e^{i(x,\mu)} f_\mu(y) \left| \phi_k \left(x + \frac{y}{\sqrt{\lambda_k}} \right) \right|^2 \mathrm{d}y\mathrm{d}x \\ &= \int_{\mathbb{T}^n} \int_{\Pi_i[-c_i,c_i]} e^{i(x,\mu)} f_\mu(y) \sum_\tau b_{\tau,k} e^{i\left(x + \frac{y}{\sqrt{\lambda_n}}, \tau\right)} \mathrm{d}y\mathrm{d}x \end{aligned}$$

and

$$I := \int_N F_\mu H \mathrm{d}x\mathrm{d}y = \int_{\mathbb{T}^n} \int_{\Pi_i[-c_i,c_i]} e^{i(x,\mu)} f_\mu(y) h(x) \mathrm{d}y\mathrm{d}x.$$

We wish to show that $I_k \to I$ as $k \to \infty$. We have:

$$\begin{aligned}
I - I_k &= \int_{\mathbb{T}^n} \int_{\Pi_i[-c_i,c_i]} e^{i(x,\mu)} f_\mu(y) \left(h(x) - \sum_\tau b_{\tau,k} e^{i\left(x+\frac{y}{\sqrt{\lambda_k}},\tau\right)} \right) \mathrm{d}y\mathrm{d}x \\
&= \int_{\Pi_i[-c_i,c_i]} f_\mu(y) \left(\int_{\mathbb{T}^n} e^{i(x,\mu)} \left(h(x) - \sum_\tau b_{\tau,k} e^{i\left(x+\frac{y}{\sqrt{\lambda_k}},\tau\right)} \right) \mathrm{d}x \right) \mathrm{d}y \\
&= \int_{\Pi_i[-c_i,c_i]} f_\mu(y) \left(\int_{\mathbb{T}^n} e^{i(x,\mu)} \left(h(x) - \sum_\tau b_{\tau,k} e^{i\left(\frac{y}{\sqrt{\lambda_k}},\tau\right)} e^{i(x,\tau)} \right) \mathrm{d}x \right) \mathrm{d}y \\
&= \int_{\Pi_i[-c_i,c_i]} f_\mu(y) (2\pi)^n \left(b_{-\mu} - b_{-\mu,k} e^{i\left(\frac{y}{\sqrt{\lambda_k}},-\mu\right)} \right) \mathrm{d}y
\end{aligned}$$

As $k \to \infty$, $\sqrt{\lambda_k} \to \infty$ and $b_{-\mu,k} \to b_{-\mu}$, so for fixed y,

$$f_\mu(y) \left(b_{-\mu} - b_{-\mu,k} e^{i\left(\frac{y}{\sqrt{\lambda_k}},-\mu\right)} \right) \xrightarrow{k} 0.$$

By Lebesgue's dominated convergence theorem (with dominating function $(2\pi)^n \left|f_\mu(y)\right| (3|b_{-\mu}|)$), we conclude that $I_k \to I$ as $k \to \infty$. $\square$

CHAPTER 14
Conclusion

We have investigated the concentration of Laplace eigenfunctions on manifolds via a survey of L^p norms bounds and permissible quantum limits. We have investigated a variety of cases, including: general manifolds, manifolds with completely integrable geodesic flow, manifolds with ergodic geodesic flow, hyperbolic manifolds, arithmetic hyperbolic manifolds, tori, and spheres. We have also examined the symmetry between the positive and negative parts of an eigenfunction.

We have proven a new result that states that if a sequence of toral eigenfunctions converges weakly to some quantum limit, the same weak convergence must also hold on a "rescaled copy" of the torus.

References

[1] T. Aissiou. Semiclassical limits of eigenfunctions on flat n-dimensional tori. *Canadian Mathematical Bulletin*, first online 2011-08-03.

[2] V.I. Arnold and A. Avez. *Ergodic Problems of Classical Mechanics*. W.A. Benjamin, 1968.

[3] P. Berard. On the wave equation on a compact riemannian manifold without conjugate points. *Math. Z.*, 155:249–276, 1977.

[4] M. Berry. Regular and irregular semiclassical wavefunctions. *J. Phys. A*, 10(12):2083–2091, 1977.

[5] J. Bourgain. Moment inequalities for trigonometric polynomials with spectrum in curved hypersurfaces. *preprint*.

[6] J. Bourgain. Eigenfunction bounds for the laplacian on the $n-$torus. *International Mathematics Research Notices*, 3:61–66, 1993.

[7] J. Bourgain and Z. Rudnick. Restriction of toral eigenfunctions to hypersurfaces. *Comptes Rendus Math.*, 347:1249–1253, November 2009.

[8] J. Bourgain and Z. Rudnick. Restriction of toral eigenfunctions to hypersurfaces and nodal sets. *preprint 2011, arXiv:1105.0018v1*, 2011.

[9] K. Burns and V.J. Donnay. Embedded surfaces with ergodic geodesic flows. *Int. J. Bifurc. Chaos Appl. Sci. Eng.*, 7(7):1509–1572, 1997.

[10] V. G. Avakumovič. Uber die eigenfunktionen auf gschlossenen riemannschen manigfaltigkeiten. *Math. Z.*, 65:327–394, 1956.

[11] J. Toth C. D. Sogge and S. Zelditch. About the blowup of quasimodes on riemannian manifolds. *J. Geom. Anal.*, 21:150–173, 2011.

[12] R. Cooke. A cantor-lebesgue theorem in two dimensions. *Proc. Amer. Math. Soc.*, 30(3):547–550, 1971.

[13] Y. Colin de Verdiere. Ergodicite et fonctions propres du laplacien. *Comm. Math. Phys.*, 102:497–502, 1985.

[14] V.J. Donnay and C.C. Pugh. Anosov geodesic flows for embedded surfaces. *Geometric Methods in Dynamics*, 287:61–69, 2003.

[15] V.J. Donnay and C.C. Pugh. Finite horizon riemann structures and ergodicity. *Ergod. Theory Dyn. Syst.*, 24(1):89–106, 2004.

[16] H. Donnelly. Bounds for eigenfunctions of the laplacian on compact riemannian manifold. *J. Func. Anal.*, 187:247–261, 2001.

[17] H. Donnelly. Exceptional sequences of eigenfunctions for hyperbolic manifolds. *Proc. Am. Math. Soc.*, 135(5):1551–1555, 2007.

[18] H. Donnelly and C. Fefferman. Nodal sets of eigenfunctions on riemannian manifolds. *Inventiones mathematicae*, 93:161–183, 1988.

[19] L. Polterovich F. Nazarov and M. Sodin. Sign and area in nodal geometry of laplace eigenfunctions. *American J. Math.*, 127(4):879–910, 2005.

[20] E. Grosswald. *Representations of Integers As Sums of Squares*. Springer-Verlag, 1985.

[21] D. Hejhal and B. Rackner. On the topography of maass waveforms for psl(2,z). *Experimental Mathematics*, 1(4):275–305, 1992.

[22] L. Hormander. The spectral function of an elliptic operator. *Acta Math.*, 121:193–218, 1968.

[23] H. Iwaniec. The spectral growth of automorphic l-functions. *J. reine angew Math.*, 428:139–159, 1992.

[24] H. Iwaniec and P. Sarnak. l^∞ norms of eigenfunctions of arithmetic surfaces. *Annals of Mathematics*, 141(2):301–320, 1995.

[25] D. Jakobson. Quantum limits on flat tori. *Annals of Mathematics*, 145:235–266, 1997.

[26] D. Jakobson and N. Nadirashvili. Quasi-symmetry of l^p norms of eigenfunctions. *Communications in Analysis and Geometry*, 10(2):397–408, 2002.

[27] D. Jakobson, N. Nadirashvili, and J. Toth. Geometric properties of eigenfunctions. *Russian Math Surveys*, 56(6):1085–1106, 2001.

[28] S. Katok. *Fuchsian Groups*. Univ. Chicago, 1992.

[29] A. G. Khovanskii. *Fewnomials*, volume 88 of *Trans. Math. Mono.* Amer. Math. Soc., 1991.

[30] B. M. Levitan. On the asymptotic behaviour of the spectral function of a self-adjoint differential equation of second order. *Izv. Akad. Nauk SSR ser. Mat.*, 16:325–352, 1952.

[31] E. Lindenstrauss. Invariant measures and arithmetic quantum unique ergodicity. *Ann. of Math., 2nd Ser.*, 163(1):165–219, 2006.

[32] G. A. Margulis. *Discrete Subgroups of Semi-Simple Lie Groups*. Springer-Verlag, 1991.

[33] D. Milicevic. Large values of eigenfunctions on arithmetic hyperbolic 3-manifolds. *preprint*, 2010.

[34] D. Milicevic. Large values of eigenfunctions on arithmetic hyperbolic surfaces. *Duke Math. J.*, 155(2):365–401, 2010.

[35] G. Mockenhaupt. Bounds in lebesgue spaces of oscillatory integral operators, 1996. Habilitationsschrift, Univ. Siegen.

[36] P. Gérard N. Burq and N. Tzvetkov. Restrictions of the laplace-beltrami eigenfunctions to submanifolds. *Duke Math. J.*, 138(3):445–486, 2007.

[37] N. Nadirashvili. Metric properties of eigenfunctions of the laplace operator on manifolds. *Annales de l'institut Fourier*, 41(1):259–265, 1991.

[38] F. Nazarov and M. Sodin. On the number of nodal domains of random spherical harmonics. *Am. J. Math.*, 131(5):1337–1357, 2009.

[39] A. Reznikov. Estimates of geodesic restrictions of eigenfunctions on hyperbolic surfaces and representation theory. *preprint*, 2010.

[40] S. Roseberg. *The Laplacian on a Riemannian Manifold*. Cambridge University, 1997.

[41] Z. Rudnick and P. Sarnak. The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Comm. Math. Phys.*, 161:195–213, 1994.

[42] P. Sarnak. Spectra of hyperbolic surfaces. *BAMS*, 40(4):441–478, 2003.

[43] B. Shiffman and S. Zelditch. Random polynomials of high degree and levy concentration of measure. *Asian J. Math.*, 7(4):627–646, 2003.

[44] A.I. Shnirelman. Ergodic properties of eigenfunctions. *Uspenski Math. Nauk*, 29(6):181–182, 1974.

[45] C. Sogge. Concerning the l^p norm of spectral clusters for second-order elliptic operators on compact manifolds. *Journal of Functional Analysis*, 77:123–138, 1988.

[46] C. D. Sogge and S. Zelditch. Riemannian manifolds with maximal eigenfunction growth. *Duke Math. J.*, 113(3):387–438, 2002.

[47] K. Soundararajan. Quantum unique ergodicity for $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$. *Annals of Mathematics*, 172:1529–1538, 2010.

[48] G. Steil. Uber die eigenwerte des laplace operators und der hecke operatoren fur $\mathrm{SL}(2,\mathbb{Z})$. *preprint*, 1993.

[49] E. Stein. *Harmonic Analysis*. Princeton Univ., 1993.

[50] G. Szeg o. *Orthogonal Polynomials*, volume 23 of *Amer. Math. Soc. Colloq. Publ.* Amer. Math. Soc., 4th ed. edition, 1975.

[51] K. Takeuchi. A characterization of arithmetic fuchsian groups. *J. Math. Soc. Japan*, 27(4):600–612, 1975.

[52] J. Toth and S. Zelditch. Riemannian manifolds with uniformly bounded eigenfunctions. *Duke Math. J.*, 111(1):97–132, 2002.

[53] J. Toth and S. Zelditch. l^p norms of eigenfunctions in the completely integrable case. *Ann. Henri Poincaré*, 4:343–368, 2003.

[54] J.A. Toth. Eigenfunction localization in the quantized rigid body. *J. Differ. Geom.*, 43(4):844–858, 1996.

[55] S. Zelditch. Uniform distribution of eigenfunctions on compact hyperbolic surfaces,.

[56] A. Zygmund. On fourier coefficients and transforms of functions of two variables. *Studia Mathematica*, 50:189–201, 1974.