Three Dimensional Contact Topology

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A Thesis Submitted in Partial Fulfilment
of the Requirements for the Degree of
Master of Philosophy
in
Mathematics

The Chinese University of Hong Kong
August 2013

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Abstract

Contact geometry is the study of contact structures on odd dimensional manifolds. It is usually perceived as a twin of symplectic geometry. In dimension three, a contact structure on a 3-manifold is a maximally non-integrable tangent 2-plane field. Since 1980's, topological methods were introduced to contact geometry, and contact topology was born as a result. Its contributions to low dimension topology were then discovered one by one. For example, Cerf's theorem ($\Gamma_4 = 0$) and Property P for knots in $\mathbb{S}^3$. Recent development includes contact homology, studying J-holomorphic curves of which boundaries lie on contact manifolds.

In this survey, I shall introduce fundamental concepts in 3-dimensional contact topology. They include Bennequin's theorem, convex surface theory and the method of filling by holomorphic disks. In particular, the 3-manifold is closed (except for $\mathbb{R}^3$) and the contact structure is coorientable.

摘要

切觸幾何常被認作是奇數維版本的辛幾何，它主要的研究對象，爲切觸結構。在三維的流型上，切觸結構是切叢裡的一個完全不可積的二維平面場。起於二十世紀八十年代，拓樸方法被引進於切觸幾何，如是者，誕生出切觸拓樸。自此，它在低維拓樸中的應用，不斷被發現，例如 Cerf 定理($\Gamma_4 = 0$)，及在三維球面裡繩結的 Property P。切觸同調乃近年此領域發展方向之一，當中包括對其邊界處於一個切觸流型的全純曲線的研究。

在這份報告裡，我將介紹一些三維切觸拓樸中的基礎概念、理論和結果。其中有 Bennequin 定理、凸曲線理論與填充全純圓盤的論述。特別一提，我們將探討的流型全爲閉流型（除了歐氏三維空間），切觸結構皆爲一個切觸形式的核。

ACKNOWLEDGMENTS

I wish to express my gratitude to my supervisor Professor Au Kwok Keung for all his teaching and sharing since I was admitted to CUHK. Thank very much for all help and support from everyone in the Department of Mathematics.

Contents

Introduction

Contact geometry originates from notions of contact elements and contact transformations by Sophus Lie in studying systems of differential equations. A detailed description of its historical background is written by H. Geiges in [15]. In 1980's, topological viewpoint was introduced to this field and provided insights on low dimensional topology. *Contact Topology*, refers to modern development of contact geometry along this direction. For more historical background, readers may also look at [10] by Y. Eliashberg.

Early results clarify how research in contact topology should go on. In 1882, M. G. Darboux showed that there are no local invariants in contact geometry [7]. Later in 1959, J. W. Gray proved that all deformations of contact structures on closed manifolds are trivial. These theorems were originally argued by solving differential equations. Now, they can be understood by Moser trick and Lie differentiation theory.

Standard examples of contact structures on $\mathbb{R}^3$ and $\mathbb{S}^3$ were constructed long ago. Some early examples came from symplectic geometry. On the other hand, contact structures also provide examples of symplectic structures (symplectisation). However, it remained unclear how to classify them. In 1970, R. Lutz constructed a non-standard contact structure on $\mathbb{S}^3$. This technique helps construct a contact structure on an arbitrary closed 3-manifold, see Martinet's theorem. In 1982, D. Bennequin showed the existence of an exotic (overtwisted) contact structure on $\mathbb{R}^3$ [4]. His argument bases on knot theory. It suggests that contact structures should be divided into two groups: tight contact structures and overtwisted contact structures.

Classification of overtwisted contact structures is comparably straightforward. In 1989, Y. Eliashberg established 1-1 correspondence between isotopy classes of overtwisted contact structures and homotopy classes of cooriented tangent 2-plane fields on a closed 3-manifold [8].

Classification results of tight contact structures remain isolated. One important tool to deal with this problem is the method of convex surface theory. It is created by E. Giroux in 1991 and then developed by J. Etnyre, K. Honda and other mathematicians. By convex surface theory, classification of tight contact structures is complete on $\mathbb{R}^3$, $\mathbb{S}^3$, lens spaces or 3-torus [23].

Another approach is by holomorphic and symplectic filling. It was invented by Y. Eliashberg [9] and based on M. Gromov's work to compactify the moduli space of holomorphic curves [20]. Nowadays, holomorphic filling is more often heard as 'Stein filling'. Symplectic filling is a generalization of holomorphic filling.

In this survey, I try to deliver elementary concepts and results in contact 3-manifold. Some figures are copied from references for convenience. The first chapter serves as an introduction of contact structures. Examples, theorems are mostly extracted from [17] by H. Geiges and [11] by J. Etynre.

In Chapter 2, Martinet's construction of a contact structure on an arbitrary closed 3-manifold will be discussed. I follow [17] for main steps, but background on Dehn surgery is selected from [35]. Actually, Martinet's construction is not the single way to Martinet's theorem. Another method is open book decomposition by W. Thurston and H. Winkelnkemper [37].

In Chapter 3, I shall explain D. Bennequin's ideas in [3]. [27] by Xiao-Song Lin and [1] are taken as complementary sources. This paper is originally written in French. It was translated to Russian by M. B. Mishustin in 1989. Then its Russian version was translated to English, under the title "Linkings and Pfaff's equations ".

The second half of Chapter 3 is about elimination lemma and its consequence. This method of eliminating singularities provides some basic classification results. Materials are copied from [17]. I put them here because this technique is comparable to D. Bennequin's step in reduction of sacks. At the end, Y. Eliashberg's classification of overtwisted contact structures is stated.

Chapter 4 is a brief introduction of convex surface theory. I refer to J. Etynre's lecture notes on convex surfaces [12] in this part. Giroux criterion helps justify that the standard (tight) contact structure on $\mathbb{R}^3$ is unique up to isotopy. Classification results on other spaces involve the study of *bypass*. Readers can find details from Honda's paper [23] as well.

The final chapter is devoted to holomorphic filling. I shall first illustrate general results about holomorphic disks. [19] by H. Geiges and K. Zehmisch is taken as a guideline. Elaborations are selected from [1], [29] and [30]. Then, I shall outline an approach to fill a 2-sphere by holomorphic disks stated in [1]. Gromov compactness theorem [20] is shortly stated afterwards. A reference quoted in this part, [22] by H. Hofer, is indeed about *Weinstein conjecture* in 3-dimension:

> *"for every contact form α on a closed 3-manifold M with $H^1(M, \mathbb{R}) = 0$, the Reeb vector field of α has at least one periodic orbit."*

I apologize to readers that this introduction of contact topology in 3-dimension is not complete or up-to-date. For example, symplectic cobordism and contact surgery are not mentioned. (see [17]) It would also improve if contact homology is contained as a new chapter. Overall speaking, I hope that readers can at least know about some important techniques or viewpoints in contact topology.

Chapter 1

Background

Contact structures on a differentiable 3-manifold M are tangent 2-plane fields that are maximally non-integrable. The 3-manifold M is closed or $\mathbb{R}^3$ in most cases, and compact with boundary in rare situations. Initial materials in this chapter come mainly from [11], [16] and [17].

Standard examples of contact structures on $\mathbb{R}^3$ and $\mathbb{S}^3$ provide a basic idea about how contact structures look like. They are in fact important to the whole subject of contact topology. If there is a 1-form α on M globally which orients a contact structure ξ, we say that ξ is **cooriented**. In this article, all contact structures except the example in Section 1.2 are cooriented. For the rest of this introductory chapter, I shall discuss neighborhood models of some submanifolds in a contact 3-manifold (M, ξ). General results can be found in [17].

Definition 1.1. A *contact structure* ξ on a closed 3-manifold M is a smooth tangent 2-plane field on M such that whenever locally $\xi = \ker \alpha$ for a local 1-form α, the later satisfies the contact condition: $\alpha \wedge d\alpha \neq 0$. ■

In general, the local 1-form α exists within a coordinate chart U from a trivialization of the orthogonal complement of this 2-plane field inside TM, $\xi^{\perp}$. If we take a non-vanishing vector field $X \in \Gamma(U, \xi^{\perp})$, the local 1-form can be defined by: $\alpha|_U = g(X, \cdot)$, where g is a Riemannian metric on M.

The contact condition holds if and only if $d\alpha$ is non-degenerate on ξ. Using the fact that $d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$, it is also equivalent to the maximally non-integrable condition:

$$[X,Y] \notin \xi \quad \forall \text{ linearly indpendent } X,Y \in \xi$$

which is opposite to the Frobenius integrability condition.

Definition 1.2. A contact structure ξ is *cooriented* if a global 1-form α exists and $\xi = \ker \alpha$. In this case, α is called a contact form. ■

When α is a contact form, $(f\alpha) \wedge d(f\alpha) \neq 0$ for any non-vanishing smooth function f on M. Therefore, contact forms of a contact manifold (M,ξ) exists up to multiplying by non-vanishing functions. Coorientedness means exactly that the line bundle $\xi^\perp$ is orientable or trivial, due to the presence of a global non-vanishing section of $\xi^\perp$. A particular choice is the Reeb vector field associated with a contact form, whose existence comes from basic results in linear algebra:

Definition 1.3. For any contact form α on the contact manifold (M,ξ), there exists the *Reeb vector field* $R_\alpha \in TM$ assoicated with α such that $\alpha(R_\alpha) \equiv 1$ and $d\alpha(R_\alpha, \cdot) = 0$. ■

If a contact form α exists, $\alpha \wedge d\alpha$ is a volume form on M by contact condition. Hence, M must be orientable. Conversely, if M is an oriented 3-manifold, such an orthogonal section to ξ exists if and only if ξ is orientable. As I mention before, contact structures are all cooriented in in this article except one example. Furthermore, when $\boldsymbol{\xi = \ker \alpha}$ appears in this article, I refer to α being a positive contact form of the contact structure ξ, i.e., $\alpha \wedge d\alpha$ matches with the orientation of M.

1.1 Contact structures on $\mathbb{R}^3$ and $\mathbb{S}^3$

Example 1.1. Let (x, y, z) be the Cartesian coordinates on $\mathbb{R}^3$, the standard contact structure on $\mathbb{R}^3$ is defined by $\xi_{std} = \ker \eta$ with $\eta = dz - ydx$. Note that $\eta \wedge d\eta = dz \wedge dx \wedge dy > 0$ under natural orientation, and the contact plane is spanned by ∂_y and $\partial_x + y\partial_z$ at every point.

Equivalence between contact manifolds is called *contactomorphism*. Throughout this article, I reserve the congruence sign '$\cong$' for a contactomorphism or symplectomorphism. The equality sign '$=$' usually stands for diffeomorphism when it lies between manifolds.

Definition 1.4. Two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) are *contactomorphic* if there exists a diffeomorphism $\phi : M_1 \to M_2$ such that its differential $d\phi$ maps ξ_1 isomorphically to ξ_2 at every $p \in M_1$. If α_1 and α_2 are contact forms of (M_1, ξ_1) and (M_2, ξ_2), it is equivalent that $\phi^*\alpha_2 = g\alpha_1$ for some non-vanishing function g on M_1. ■

Example 1.2. Let $\eta_2 = dz + xdy$ be a contact form of the corresponding contact structure ξ_2 on $\mathbb{R}^3$. Considering the reflection map $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\phi(x, y, z) = (y, -x, z)$, we have $\phi^*\eta = \eta_2$. Hence, $(\mathbb{R}^3, \xi_2)$ is contactomorphic to $(\mathbb{R}^3, \xi_{std})$.

Example 1.3. The rotationally symmetric contact structure ξ_3 on $\mathbb{R}^3$ is defined by its contact form $\eta_3 = dz + xdy - ydx$, which equals to $dz + r^2 d\theta$ under cylindrical coordinates. $(\mathbb{R}^3, \xi_3)$ is contactomorphic to $(\mathbb{R}^3, \xi_{std})$ as well. It can be verified through a diffeomorphism $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ in the form of

$$(u, v, w) = \phi(x, y, z) = (\frac{x + y}{2}, \frac{x - y}{2}, f(x, y, z)).$$

Note that $\phi^*\eta_3 = \eta$ if $df = \frac{x}{2}dy + \frac{y}{2}dx - dz$. One choice of f is the function $\frac{xy}{2} - z$. From now on, we call both ξ_{std}, ξ_1 and ξ_2 the standard contact structure on $\mathbb{R}^3$.

The Hopf contact structure ξ_H can be defined by *complex tangencies* to $\mathbb{S}^3$ at every point induced by the natural almost complex structure J on $\mathbb{R}^4$. See [11]. In Chapter 5, we will see that an almost complex structure helps to define a contact structure.

Definition 1.5. An *almost complex structure* J on a real vector space V is a linear operator on V satisfying $J^2 = -id_V$. Note that V must be an even dimensional space. ■

Denote the Cartesian coordinates on $\mathbb{R}^4$ by (x, y, u, v). At every $p \in \mathbb{R}^4$ we take the almost complex structure $J : T_p\mathbb{R}^4 \to T_p\mathbb{R}^4$ as:

$$J(\partial_x) = \partial_y \ , \ J(\partial_y) = -\partial_x \ , \ J(\partial_u) = \partial_v \text{ and } J(\partial_v) = -\partial_u.$$

Since $\mathbb{S}^3$ is the level set of function $f : \mathbb{R}^4 \to \mathbb{R}$, $f(x, y, u, v) = x^2 + y^2 + u^2 + v^2$ at value 1, $T_p\mathbb{S}^3 = \ker(df_p)$ for every $p \in \mathbb{S}^3$. Also observe that $J(\ker df_p) = \ker(df_p \circ J)$ for $p \in \mathbb{R}^4$. Hence,

$$\xi_H = T_p\mathbb{S}^3 \cap J(T_p\mathbb{S}^3) = T_p\mathbb{S}^3 \cap \ker(df_p \circ J) = \ker(df_p \circ J)|_{T_p\mathbb{S}^3}.$$

Set 1-form $\alpha \triangleq df \circ J$. On solving, $\alpha = 2(ydx - xdy + vdu - udv)$ and so $\xi_H = \ker(\alpha|_{T_p\mathbb{S}^3}) = \ker(ydx - xdy + vdu - udv|_{T_p\mathbb{S}^3})$. One can show that the contact form α satisfies the contact condition, and the corresponding Reeb vector field is given by $R = \frac{1}{2}(y\partial_x - x\partial_y + v\partial_u - u\partial_v)$, where $\iota_R d\alpha = df \equiv 0$ on $T_p\mathbb{S}^3$.

By the Reeb vector field, we can see how the Hopf contact structure is related to the Hopf fibration: $\mathbb{R}^4 = \mathbb{C}^2 \supset \mathbb{S}^3 \to \mathbb{C}P^1 = \mathbb{S}^2$. It sends $(z, w) \in \mathbb{S}^3$ to $[z : w] \in \mathbb{C}P^1$ in homogeneous coordinates. In fact, the flow of R generates fibres of Hopf fibration. The later at a point (z_0, w_0) is a circle parametrized by

$$\gamma(\theta) = (e^{i\theta}z_0, e^{i\theta}w_0) \quad \text{for} \quad \theta \in [0, 2\pi]$$

$$\Longrightarrow \ d\gamma|_0(\partial_\theta) = x_0\partial_y - y_0\partial_x + u_0\partial_v - v_0\partial_u = -2R$$

for $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$.

Note that $\alpha \wedge d\alpha < 0$ with respect to induced orientation of $\mathbb{S}^3$ from $\mathbb{R}^4$. For later convenience, I shall give a minus sign to α and R to adjust orientation. Let $\alpha_H = \frac{1}{2}(xdy - ydx + udv - vdu)$ and $R_\alpha = 2(x\partial_y - y\partial_x + u\partial_v - v\partial_u)$. We denote the Hopf contact structure on $\mathbb{S}^3$ by $\xi_H = \ker \alpha_H$ cooriented by the contact form α_H.

Theorem 1.1. [16]. For any point $P \in \mathbb{S}^3$, the contact manifold $(\mathbb{S}^3 - \{P\}, \xi_H)$ is contactomorphic to $(\mathbb{R}^3, \xi_{std})$, the standard contact structure on $\mathbb{R}^3$.

Proof. Here we adapt the proof from [16]. Consider
$\mathbb{S}^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ and $\mathbb{R}^3 = \{(\zeta = x + iy, s) \mid \zeta \in \mathbb{C}\ ,\ s \in \mathbb{R}\}$.
So,

$$\eta = ds + xdy - ydx = ds + \frac{i}{2}(\zeta d\overline{\zeta} - \overline{\zeta} d\zeta);$$
$$\alpha_H = \frac{i}{2}(zd\overline{z} - \overline{z}dz + wd\overline{w} - \overline{w}dw).$$

Wlog, $P = (0, -1)$ in complex coordinates. Here we use a complex analogue of stereographic projection,

$$f(z,w) = (\frac{z}{1+w}\ ,\ \frac{-i(w-\overline{w})}{2|1+w|^2}) = (\frac{z}{1+w}\ ,\ \mathrm{Im}\frac{w}{1+w}),$$

which is a diffeomorphism from $\mathbb{S}^3 - \{P\}$ to $\mathbb{R}^3$. One can show that

$$f^*\eta = \frac{\alpha_H}{|1+w|^2}.$$

□

Remark. The usual stereographic projection $\sigma : \mathbb{S}^3 - \{P\} \to \mathbb{R}^3$ is given by

$$\sigma(x,y,u,v) = (\frac{x}{1+u}\ ,\ \frac{y}{1+u}\ ,\ \frac{v}{1+u}).$$

1.2 Coorientedness

We are going to construct a non-coorientable contact structure on $\mathbb{R}^2 \times \mathbb{R}P^1$ [17]. Let (x_1, x_2) be the Cartesian coordinates on $\mathbb{R}$ and $[y_1 : y_2]$ be the homogeneous coordinates on $\mathbb{R}P^1$, which owns a differentiable structure consisted of charts (U_1, ϕ_1) and (U_2, ϕ_2):

$$\phi_1 : U_1 \to \mathbb{R}P^1 \ , \ \phi_1(u_1) = [1 : u_1]$$

$$\phi_2 : U_2 \to \mathbb{R}P^1 \ , \ \phi_2(u_2) = [u_2 : 1]$$

Change of coordinates is then given by $\phi_2^{-1} \circ \phi_1(u_1) = \frac{1}{u_1}$ We can define 1-forms

$$\alpha_1 = dx_1 + u_1 dx_2 = dx_1 + \frac{y_2}{y_1} dx_2 \text{ on } \mathbb{R}^2 \times U_1;$$

$$\alpha_2 = dx_2 + u_2 dx_1 = dx_2 + \frac{y_1}{y_2} dx_1 \text{ on } \mathbb{R}^2 \times U_2.$$

such that the 2-plane field $\xi = \ker(\alpha_k)$ on U_k, $k = 1, 2$, is a contact structure on $\mathbb{R}^2 \times \mathbb{R}P^1$. To show that ξ is non-coorientable, we introduce the canonical line bundle γ^1 over $\mathbb{R}P^1$ and its pull back $E = \pi^* \gamma^1$ through projection map $\pi : \mathbb{R}^2 \times \mathbb{R}P^1 \to \mathbb{R}P^1$. Explicitly,

$$E = \left\{ (\vec{x}, [y_1 : y_2], v) \in \mathbb{R}^2 \times \mathbb{R}P^1 \times \mathbb{R}^2 \mid v = \lambda(y_1 \partial_{y_1} + y_2 \partial_{y_2}) \right\}.$$

The complementary bundle TM/ξ is isomorphic to E, given by

$$\alpha_1(\partial_{x_1} + \frac{y_2}{y_1}\partial_{x_2}) = (dx_1 + \frac{y_2}{y_1} dx_2)(\partial_{x_1} + \frac{y_2}{y_1}\partial_{x_2}) = 1 + \frac{y_2^2}{y_1^2} > 0 \text{ on } U_1;$$

$$\alpha_2(\partial_{x_2} + \frac{y_1}{y_2}\partial_{x_1}) = (dx_2 + \frac{y_1}{y_2} dx_1)(\partial_{x_2} + \frac{y_1}{y_2}\partial_{x_1}) = 1 + \frac{y_1^2}{y_2^2} > 0 \text{ on } U_2.$$

The bundle isomorphism maps ∂_{x_i} to ∂_{y_i} at every base point. Since γ^1 is non-trivial, E is non-trivial and the contact structure ξ is non-coorientable.

1.3 Neighborhood theorems

Darboux theorem states that contact manifolds look like the same locally near a point. There is a similar result in symplectic geometry which is also called Darboux theorem. I will show both the symplectic version and contact version.

The original proof of Darboux theorem can be found in [7]. Here I shall refer to proofs applying Lie differentiation theory. Background of Lie differentation can be found in [33] and [38].

Definition 1.6. A symplectic manifold (W, ω) is an even dimensional smooth manifold W equipped with a closed and non-degenerate 2-form $\omega \in \Omega^2(M)$, called the symplectic form. ■

Theorem 1.2 (Moser, see [36])**.** Let ω_0 and ω_1 be two symplectic forms on W^{2n} such that on closed submanifold N of W, $\omega_0|_N = \omega_1|_N$. Then there exist two neighborhoods U and V of N in W and a diffeomorphism $\phi : U \xrightarrow{\cong} V$ with

$$\phi^* \omega_1 = \omega_0 \quad \text{and} \quad \phi|_N = id_N.$$

Proof. We first define a family of closed 2-forms by $\omega_t = (1-t)\omega_0 + t\omega_1$. Note $\omega_t \equiv \omega_0$ over N. For every point $p \in N$, there exists a neighborhood U_p of p on which ω_t's are all non-degenerate. We call the union of U_p's by U_0. Hence, $\omega_t \neq 0$ on this neighborhood of N.

Next, we refer to the relative Poincaré lemma. Since $d(\omega_1 - \omega_0) = 0$ and $(\omega_1 - \omega_0)|_N = 0$, there exists a neighborhood $U_1 \subset U_0$ of N and a 1-form $\alpha \in \Omega^1(U_1)$ with $\alpha|_N = 0$ and $(\omega_1 - \omega_0) = d\alpha$ on U_1.

To achieve our goal, we find an isotopy (ϕ_t), $t \in [0,1]$, satisfying $\phi_0 = id$ and $\phi_t^* \omega_t = \omega_0$ by Moser trick:

$$\phi_t^* \omega_t = \omega_0 \implies \frac{d}{dt}(\phi_t^* \omega_t) = \phi_t^*(\mathcal{L}_{X_t}\omega_t + \frac{d\omega_t}{dt}) = 0,$$

where X_t is the family of generating vector fields associated with (ϕ_t) near N. Using Cartan formula and $\frac{d\omega_t}{dt} = d\alpha$, we have

$$\mathcal{L}_{X_t}\omega_t + \frac{d\omega_t}{dt} = 0 \iff d(\iota_{X_t}\omega_t) + d\alpha = 0.$$

Guaranteed by non-degeneracy of ω_t, X_t is uniquely defined by Moser equation: $\iota_{X_t}\omega_t + \alpha = 0$ on U_1. Hence, being the flow of (X_t), ϕ_t's are local diffeomorphism over a smaller neighborhood U of N.

$$\alpha|_N = 0 \implies X_t = 0 \text{ on N} \implies \phi_t|_N = id$$

Take $\phi = \phi_1$ and $V = \phi_1(U)$ to finish the proof. □

The following Darboux theorems are in particular dimensions so that unnecessary details can be avoided. However, it is straightforward to generalize these results.

Theorem 1.3 (Darboux- symplectic version, see [36])**.** Let (M^4, ω) be a symplectic manifold and consider a point p in W. Then, there exists a neighborhood U of p and local coordinates (x_1, y_1, x_2, y_2) on U with $\omega|_U = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

Proof. Take a basis $\{u_1, v_1, u_2, v_2\}$ for T_pW such that $\omega_p = du_1 \wedge dv_1 + du_2 \wedge dv_2$. We can define a 1-form $\omega_1 = du_1 \wedge dv_1 + du_2 \wedge dv_2$ over a coordinate chart U_0 containing p. Note that ω_1 is a symplectic form on U_0 and $\omega_1|_p = \omega|_p$. Using Moser theorem, there exist neighborhoods U and $V \subset U_0$ of p, and a diffeomorphism $\phi : U \xrightarrow{\cong} V$ with $\phi(p) = p$ and $\phi^*\omega_1 = \omega$. Regarding u_i and v_i as coordinate functions, we define change of coordinate functions by $x_i = u_i \circ \phi$ and $y_i = v_i \circ \phi$. Hence,

$$\omega = \phi^*\omega_1 = \phi^*(du_1 \wedge dv_1 + du_2 \wedge dv_2) = dx_1 \wedge dy_1 + dx_2 \wedge dy_2.$$

□

Theorem 1.4 (Darboux- contact version, see [17])**.** Let (M^3, ξ) be a contact 3-manifold and consider a point p in M. Then, there exists a neighborhood U of p and local coordinates (x, y, z) on U such that $\alpha|_U = dz + xdy$, where ξ is locally cooriented by the 1-form α near p.

Proof. It is much the same as the previous proof. Suppose that ξ is locally cooriented by 1-form α_0. We take a basis $\{u, v, w\}$, u, $v \in \xi$ and $w = R_\alpha|_p$ for T_pM such that $\alpha|_p = dw + udv$. Extend the RHS to local 1-form $\alpha_1 = dw + udv$

over a coordinate chart containing p, and so the proof will be finished if we can find a diffeomorphism ϕ between two neighborhoods of p pulling back α_1 to α.

Some modifications in Moser theorem are required. We let $\alpha_t = (1-t)\alpha_0 + t\alpha_1$ and so $\frac{d\alpha_t}{dt} = \alpha_1 - \alpha_0$. The Moser trick starts with $\phi_t^*\alpha_t = \alpha_0$, giving

$$\frac{d}{dt}(\phi_t^*\alpha_t) = \phi_t^*(\frac{d\alpha}{dt} + \mathcal{L}_{V_t}\alpha_t) = 0,$$

where V_t is a family of vector fields near p to be determined. Decompose V_t by $V_t = f_t R_t + Y_t$, where R_t's are the Reeb vector field of α_t, $Y_t \in \ker \alpha_t$ and f_t's are functions near the point p. So we have

$$\frac{d\alpha_t}{dt} + df_t + \iota_{Y_t} d\alpha_t = 0.$$

Solve for f_t by putting R_t inside, giving

$$\frac{d\alpha_t}{dt}(R_t) + df_t(R_t) = 0.$$

Requiring that f_t and df_t vanish at p, Y_t and V_t will be uniquely determined. The flow of V_t defines local diffeomorphisms near p and preserves p. □

Moser trick argument can be applied to show Gray stability theorem. It states that deformations of contact structures on a closed 3-manifold are all trivial in the sense of contactomorphism.

Theorem 1.5 (Gray stability, [17])**.** Suppose (ξ_t), $t \in [0, 1]$ is a smooth family of contact structures on a closed 3-manifold M. Then there exists an isotopy (ψ_t) of M such that $(d\psi_t)\xi_0 = \xi_t$. If $\xi_t = \ker \alpha_t$ for a smooth family of 1-forms α, the statement translates to

$$\psi_t^*\alpha_t = \lambda_t\alpha_0$$

for a smooth family of functions $\lambda_t : M \to \mathbb{R}^+$.

Proof. We start with

$$\frac{d}{dt}(\psi_t^*\alpha_t) = \psi_t^*(\frac{d\alpha_t}{dt} + \mathcal{L}_{X_t}\alpha_t) = \frac{d\lambda_t}{dt}\alpha_0 = \frac{d\lambda_t}{dt}(\frac{\psi_t^*\alpha_t}{\lambda_t}).$$

Let $\mu_t = \frac{d(\log \lambda_t)}{dt} \circ \psi_t^{-1}$ and suppose that $X_t \in \xi_t$ for all t. Then we have

$$\frac{d\alpha_t}{dt} + \iota_{X_t} d\alpha_t = \mu_t \alpha_t.$$

Put the Reeb vector fields R_t of α_t to this equation. It yields $\mu_t = \frac{d\alpha_t}{dt}(R_t)$. The horizontal vector field X_t is then uniquely determined. □

A transverse knot γ is an embedded circle in a contact manifold (M, ξ) always transverse to ξ. If $\xi = \ker \alpha$, we say γ is positively (or negatively) transverse to ξ if $\alpha(T) > 0$ (or < 0), where T is the positive unit tangent of γ.

An odd dimensional submanifold N in (M, ξ) is a contact submanifold, if $\xi' = \xi \cap TN$ is a contact structure on N. A transverse knot is then a 1-dimensional contact submanifold though the induced contact structure is trivial. Due to this property, it admits a nice standard model.

Theorem 1.6. [1]. Let γ be a transverse knot in a contact manifold (M, ξ). Then there exists a neighborhood N of γ on which $\xi = \ker(d\theta + xdy - ydx)$ in terms of local coordinates.

Proof. We assume that ξ is cooriented by α and γ is positively transverse to ξ. Notice that a sufficient small neighborhood N_0 of γ is diffeomorphic to $\mathbb{S}^1 \times D^2$, let say by a diffeomorphism g. As before, let $T = dg^{-1}(\partial_\theta)$ and then we can replace α by another contact form $f\alpha$ such that T coincides with R_α along γ:

$$d\alpha(T, \cdot) = 0 \quad \text{and} \quad \alpha(T) = 1.$$

Now, α and $\alpha_0 = d\theta + xdy - ydx$ define two contact structures ξ and ξ_0 on $\mathbb{S}^1 \times D^2$. Consider these two symplectic vector bundles $\xi \to \mathbb{S}^1$ and $\xi_0 \to \mathbb{S}^1$ and we can define a bundle map h with $h^* d\alpha_0 = d\alpha$ at every $(\theta, 0) \in \gamma$.

$$\begin{array}{ccc} \xi & \xrightarrow{h} & \xi_0 \\ \downarrow & & \downarrow \\ \mathbb{S}^1 & = & \mathbb{S}^1 \end{array}$$

Through compositing with exponential maps, this bundle map h defines a diffeomorphism h_1 between two neighborhood of $\gamma = \mathbb{S}^1$ in $\mathbb{S}^1 \times D^2$. Along this circle, $h_1 = id$ and $h_1^* d\alpha_0 = d\alpha$. After replacing g by $h_1 \circ g$ and rescaling the model, $g : N_1 \xrightarrow{\cong} \mathbb{S}^1 \times D^2$ for some neighborhood $N_1 \subset N_0$ and $g^* d\alpha_0 = d\alpha$ on γ. Since h_1 preserves γ, we still have $dg(T) = \partial_\theta$. On N_1, let α_0 denote $g^*\alpha_0$ and $\alpha_1 = \alpha$. So we have

$$\alpha_0 = \alpha_1 \quad \text{and} \quad d\alpha_0 = d\alpha_1 \quad \text{on } \gamma.$$

The proof will be completed by Moser trick argument. As a remark, we require f_t and df_t vanish along γ this time. □

Local neighborhoods of an embedded surface S inside a contact manifold (M, ξ) can be studied by the characteristic foliation induced from ξ on S. Giroux theorem states that some neighborhoods of two closed surfaces are contactomorphic to each other provided that the characteristic foliations on them are the same. It also comes from applying Moser trick argument, through a horizontal vector field X_t preserving leaves on the surface.

Definition 1.7. Let S be an embedded surface in a contact 3-manifold (M, ξ). The *characteristic foliation* S_ξ is the singular foliation on S from ξ, defined by $\xi_p \cap T_pS$ at every point $p \in S$. ■

Remark. If ξ is cooriented by 1-form α, globally or near S, and Ω is an area form on S, then S_ξ is generated by the vector field X defined by $\iota_X \Omega = \alpha|_{TS} \triangleq \beta$.

Theorem 1.7 (Giroux, see [17])**.** Let S be an oriented closed surface in a closed 3-manifold M on which ξ_0 and ξ_1 are two contact structures. Suppose that there is a diffeomorphism $\phi : S \to S$ with $\phi(S_{\xi_0}) = S_{\xi_1}$, i.e. sending leaves of S_{ξ_0} onto S_{ξ_1}, as oriented characteristic foliations. Then, there exists a contactomorphism

$$\psi : (N_0, \xi_0) \xrightarrow{\cong} (N_1, \xi_1)$$

for some neighborhoods N_0 and N_1 of S with $\psi(S) = S$ and $\psi|_S$ isotopic to ϕ via an isotopy preserving the characteristic foliation. ■

Example 1.4. Consider the unit sphere $\mathbb{S}^2$ in $(\mathbb{R}^3, \xi_{std})$ cooriented by $\eta = dz + xdy - ydx$. (Figure 1.1) A generating vector field X of the characteristic foliation $S_{\xi_{std}}$ can be solved by $\eta(X) = 0$ and $dr(X) = 0$. Explicitly,

$$X = (xz - y)\partial_x + (yz + x)\partial_y - (x^2 + y^2)\partial_z.$$

and always cuts the equator $\{z = 0\}$ of $\mathbb{S}^2$ by $\frac{\pi}{4}$ downwards. In fact, by considering the area form $\Omega = \iota_{\partial_r} dx \wedge dy \wedge dz = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$, one can show that $\iota_X \Omega = \eta$ on $\mathbb{S}^2$.

Example 1.5. Under the cylindrical coordinates (r, θ, z) on $\mathbb{R}^3$, we define another contact structure by the 1-form

$$\alpha_{ot} = \cos r dz + r \sin r d\theta.$$

Note $r \sin r d\theta$ smoothly extends to $r^2 d\theta = xdy - ydx$ and $\alpha_{ot} \wedge d\alpha_{ot} = 2dx \wedge dy \wedge dz$ at origin, so α_{ot} is a contact form. We call $\xi_{ot} = \ker \alpha_{ot}$ the standard overtwisted contact structure on $\mathbb{R}^3$.

Consider the disk $\Delta = \{z = 0, r \leq \pi\}$ in $\mathbb{R}^3$. The characteristic foliation $\Delta_{\xi_{ot}}$ consists of all boundary points and the origin as singular points. If we push up Δ to $\Delta^\epsilon = \{z = \epsilon r^2, r \leq \pi\}$, then $\Delta^\epsilon_{\xi_{ot}}$ is generated by

$$V = r \sin r\partial_r - 2\epsilon r \cos r\partial_\theta + 2\epsilon r^2 \sin r\partial_z.$$

It has a singular point only at the origin and a closed leaf on boundary. Projecting Δ^ϵ onto (x, y)-plane, integral curves are given by $\theta = -2\epsilon \ln \sin r + C$. (Figure 1.1) Both $\Delta_{\xi_{ot}}$ and $\Delta^\epsilon_{\xi_{ot}}$ are called the **standard overtwisted disk**. The shorthand notation '**OT**' appeared later always stands for 'overtwisted'.

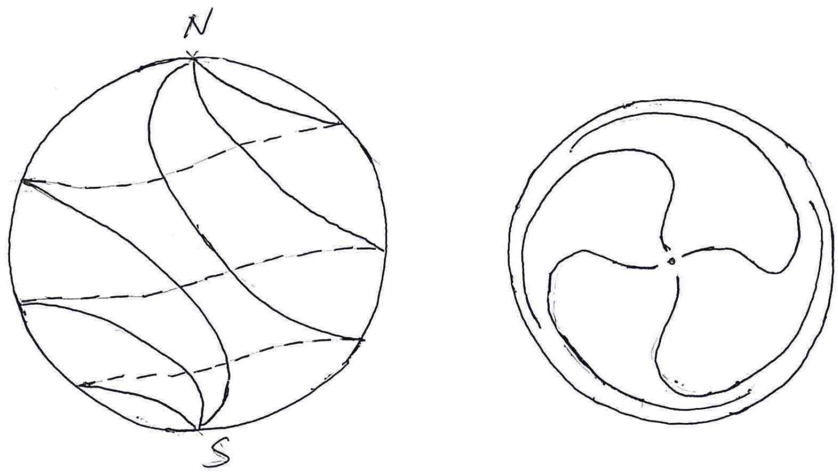

Figure 1.1: $\mathbb{S}^2_{\xi_{std}}$ (L) and $\Delta_{\xi_{ot}}$ (R)

Example 1.6. By the covering map $q : \mathbb{R}^3 \to \mathbb{S}^1 \times \mathbb{R}^2$, α_{ot} induces a contact form $\alpha = \cos r d\phi + r \sin r d\theta$ on $M = \mathbb{S}^1 \times \mathbb{R}^2$, where $\mathbb{S}^1$ is parametrized by ϕ and $\mathbb{R}^2$ by (r, θ). Denote the underlying contact structure on M by ξ. This time we look at the torus $T_a = \{r = a\}$ and its characteristic foliation $(T_a)_\xi$, given by

$$\frac{d\phi}{d\theta} = -a \tan a \quad \text{and} \quad dr = 0.$$

Hence the flow lines are given by the lines $\phi = -(a \tan a)\theta + C$.

Chapter 2

Construction of contact 3-manifolds

Contact structures exist on $\mathbb{R}^3$ and $\mathbb{S}^3$ as well as on other closed 3-manifolds. This result is called Martinet's theorem. Further contribution by R. Lutz leads to existence of a contact structure in every homotopy class of cooriented 2-plane fields on a closed 3-manifold. Martinent's theorem can be verified through two different ways. The first way, which I shall discuss in this chapter, makes use of the Hopf contact structure on $\mathbb{S}^3$ and bases on Lickorish-Wallace theorem [17, 35].

Theorem 2.1 (Lickorish-Wallace)**.** Any orientable closed 3-manifold can be obtained by cutting out finite number of solid tori from $\mathbb{S}^3$ and then pasting them back in by some homeomorphisms between their boundaries. It can be further assumed that all these solid tori are unknotted in $\mathbb{S}^3$. ■

Following this decomposition, we try to extend the Hopf contact structure from outer to inner solid tori across their boundaries. The other way is using an *open book decomposition* of M, constructed by Thurston and Winkelnkemper. Content and results of this method can be found in [17] and [37].

2.1 Martinet's construction

Take a non-separating curve γ on a torus T. Cut T into a cylinder along γ. Fix one boundary circle of this cylinder. We rotate the boundary circle at another end by 2π radian and then glue the points on these two circles back to their original positions. Denoting the resulting torus by T', the identification of points on T and T' after this procedure is a homeomorphism, called a Dehn twist of T along γ.

Theorem 2.2 (Dehn-Lickorish, [35])**.** The mapping class group (of orientation preserving homeomorphisms) of any closed surface is generated by Dehn twists along non-separating curves. ■

Furthermore, since the mapping class group of a torus is isomorphic to $SL(2, \mathbb{Z})$, it is generated by the two Dehn twists along α and β, generators of $H_1(T, \mathbb{Z})$.

A Dehn surgery along a knot K lying in a closed 3-manifold M is performed by removing a tubular neighborhood $N(K) = \mathbb{S}^1 \times D^2$ near K first. Then we glue back a solid tori by some homeomorphism along boundary tori to obtain a new 3-manifold M'. Isotopic homeomorphisms of torus onto itself, or representatives in the same equivalence class of the mapping class group, result in homeomorphic 3-manifolds after corresponding Dehn surgeries.

On any solid torus N, parametrized by $(\phi, r, \theta) \in \mathbb{S}^1 \times D^2$ respectively with $r \leq 1$, we call a circle μ on ∂N bounding a copy of D^2 by a meridian of N. A copy λ of S^1 circle on ∂N is called a longitude (or parallel). See Figure 2.1

Let $N(K)$ be the original solid torus near K in M, and $N_1 = \mathbb{S}^1 \times D^2$ the solid torus glued into $M - N(K)$ along the boundary. Under the above notations, let

$$\begin{array}{rclcrcl} \alpha & = & \text{meridian of } N_1 & ; & \beta & = & \text{longitude of } N_1; \\ \alpha' & = & \text{meridian of } N(K) & ; & \beta' & = & \text{longitude of } N(K). \end{array}$$

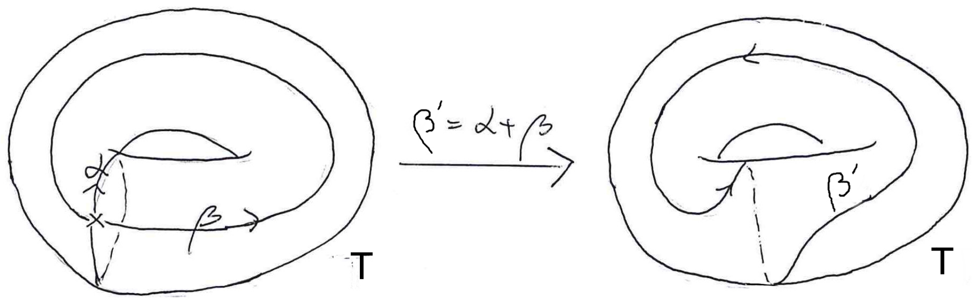

Figure 2.1: Dehn twist along α

The Dehn surgery, also called the rational surgery, can be described by maps between meridians and longitudes:

$$\alpha \mapsto p\alpha' + q\beta' \quad \text{and} \quad \beta \mapsto m\alpha' + n\beta'.$$

with $\begin{pmatrix} p & m \\ q & n \end{pmatrix} \in GL(2, \mathbb{Z})$. In fact, a Dehn surgery is uniquely determined by the surgery coefficient, also called framing index, $r \triangleq \frac{p}{q}$. Once we have the image of meridian α, one can attach a 2-disk along $p\alpha' + q\beta'$ in $M - N(K)$ and then recover M' through gluing in a 3-ball, which is unique up to isotopy. Explicitly,

$$\begin{pmatrix} p & m+tp \\ q & n+tq \end{pmatrix} = \begin{pmatrix} p & m \\ q & n \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

So it suffices to show that all Dehn surgeries preserving meridians are trivial.

Example 2.1. $\mathbb{S}^3$ has a genus one Heegaard splitting. Two boundary tori are glued by torus switch: a Dehn surgery with $r = 0$ switching meridians and longitudes.

Consider $\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ as a submanifold of $\mathbb{C}^2$. We can decompose $\mathbb{S}^3$ into the following tori:

$$M_1 = \{(z, w) \in \mathbb{S}^3 \mid |z| \leq |w|\} \quad \text{and} \quad M_2 = \{(z, w) \in \mathbb{S}^3 \mid |z| \geq |w|\}.$$

Under polar coordinates, any point in $\mathbb{S}^3$ is represented by $(ae^{i\alpha}, be^{i\beta})$ with $a^2 + b^2 = 1$. So M_1 is a torus parametrized by $(r = \frac{a}{b}, \alpha, \beta) \in D^2 \times \mathbb{S}^1$. Similarly,

M_2 can be parametrized by $(\alpha, r' = \frac{b}{a}, \beta) \in \mathbb{S}^1 \times D^2$. The gluing map across boundaries of M_1 and M_2 is then described by

$$f : \partial M_1 = D^2 \times \mathbb{S}^1 \to \mathbb{S}^1 \times D^2 = \partial M_2 \quad \text{and} \quad f(\alpha, \beta) = (\beta, \alpha).$$

Example 2.2. The Poincaré homology sphere can be obtained by a Dehn surgery on $\mathbb{S}^3$ along the right knot trefoil with surgery coefficient $r = 1$. By investigation on this surgery, one can show that the fundamental group of the Poincaré homology sphere is isomorphic to the binary icosahedral group I^*. See [35].

$$I^* = \{a, b \mid a^5 = b^3 = (ba)^2\}$$

Usually we construct the Poincaré homology sphere by the quotient space $SO(3)/I$, where I is the icosahedral rotation group, isomorphic to A_5. $SO(3)$ is diffeomorphic to $\mathbb{R}P^2$, which is doubly covered by $\mathbb{S}^3$. Hence two ways of construction give spaces of the same fundamental group: $I^* = I \times \mathbb{Z}^*$.

We are ready to show Martinet's theorem. For a closed 3-manifold M, by Lickorish-Wallace theorem, we can construct M by finite steps of Dehn surgeries. Inductively, it suffices to consider how to extend the Hopf contact structure $\xi_H = \ker \alpha_H$ on $\mathbb{S}^3 - N(K)$ to the glued-in solid torus. Since every knot in a contact 3-manifold can be C^0-approximated by a transverse knot, we can assume that the knot $K \pitchfork^+ \xi_H$. (This pitchfork sign $\pitchfork$ means 'transverse to'. When it is superscripted by a positive sign, it means 'positively transverse to'.)

By previous result, there is a neighborhood $N(K)$ of K diffeomorphic to a solid torus $\mathbb{S}^1 \times D^2_\delta$ on which K is identified with $\mathbb{S} \times \{0\}$ and $\alpha_H = d\bar{\theta} + \bar{r}^2 d\bar{\phi}$. Further assume that the Dehn surgery is carried out in a smaller solid torus $N_0(K) = \mathbb{S}^1 \times D^2_{\delta'}$ with $\delta' < \delta$. Let N_1 be the solid torus glued with $\mathbb{S}^3 - N_0(K)$. Let

$$\begin{aligned} \mu &= \text{meridian of } N_0(K) \quad ; \quad & \lambda &= \text{longitude of } N_0(K); \\ \mu_0 &= \text{meridian of } N_1 \quad ; \quad & \lambda_0 &= \text{longitude of } N_1. \end{aligned}$$

This surgery can be described by $\mu_0 \mapsto p\mu + q\lambda$ and $\lambda_0 \mapsto m\mu + n\lambda$. Parametrizing N_1 by $(\theta, r, \phi) \in \mathbb{S}^1 \times D^2$, it can be rephrased by

$$\bar{\theta} = n\theta + q\phi \quad \text{and} \quad \bar{\phi} = m\theta + p\phi.$$

It means that if we rotate by 2π in both meridian and longitude directions on $\partial N_0(K)$, $\bar{\theta}$ increases by $(p+m)2\pi$ and $\bar{\phi}$ increases by $(q+n)2\pi$ on ∂N_1 through the mapping. Hence after pulling back,

$$\alpha_H = d(n\theta + q\phi) + r^2 d(m\theta + p\phi) \quad \text{near } \partial N_1.$$

It remains to extend α_H inside N_1 in terms of local coordinates. Suppose $pn - qm = 1$, and then we let the extension be $\alpha = h_1(r)d\theta + h_2(r)d\phi$, $0 \le r \le 1$. It turns out that if $(h_1, h_2) = (1, r^2)$ near $r = 0$; $= (n + mr^2, q + pr^2)$ near the boundary torus $(r = 1)$ and it is never parallel to (h_1', h_2') when $r \neq 0$, then

$$\alpha = \begin{cases} (n + mr^2)d\theta + (q + pr^2)d\phi & \text{near } \partial N_1; \\ d\theta + r^2 d\phi & \text{near } r = 0. \end{cases}$$

Hence, α extends α_H.

2.2 Homotopy Class

Martinet's theorem will be extended in this section, following steps in [17]. Basic results from algebraic and differential topology are more involved in this section. For details, one can find them in [21] and [31].

Theorem 2.3 (Lutz-Martinet)**.** Let M be an oriented closed 3-mainfold. Every cooriented 2-plane field in TM is homotopic to a contact structure.

For a cooriented 2-plane field η on M, its properties are completely determined by its Gauss map $g_\eta : M \to \mathbb{S}^2$. The Gauss map can be defined only if we specify a trivialization of TM. Here we use the fact that every close and orientable 3-manifold is parallelizable. The later can be argued by $\mathbb{S}^3$ being a Lie group together with the following theorem.

Theorem 2.4 (Hilden-Montesinos)**.** For every closed 3-manifold M, there exists a 3-fold covering $p : M \to \mathbb{S}^3$ branching along a knot. ■

Given an arbitrary initial contact structure on M, we try to perform Lutz twist along suitable knots so that the resulting contact structure lies inside the prescribed homotopy class of cooriented 2-plane fields. Lutz twist is a trivial Dehn surgery followed by extending the contact structure to an inner solid torus generated by a knot. Concrete definition will be shown later.

For convenience, we introduce a CW structure on M and denote the 2-skeleton of M by $M^{(2)}$. Obstructions in homotopic equivalence are measured by d^2 and d^3 defined by

$$\begin{aligned} \text{On } M^{(2)} \quad &: \quad d^2(\xi_1, \xi_2) \triangleq PD_M[L_{\xi_1}] - PD_M[L_{\xi_2}] \in H^2(M, \mathbb{Z}); \\ \text{On } M \quad &: \quad d^3(\xi_1, \xi_2) \triangleq H(G_{\xi_1,\xi_2}) \in \mathbb{Z}. \end{aligned}$$

In the definition of d^2, L_ξ stands for the Pontryagin manifold associated with the Gauss map g_ξ. For further information about Pontryagin manifold, one can refer to [31]. $PD_M[L_\xi]$ denotes the Poincaré duality of L_ξ in M and measures degree of g_ξ. Explicitly, let $p \in \mathbb{S}^2$ be a regular value of g_ξ,

$$PD_M[L_\xi] = PD_M[g_\xi^{-1}(p)] = g_\xi^*(PD_{\mathbb{S}^2}[p]) = g_\xi^*\omega_0,$$

where $\omega_0 \in H^2(\mathbb{S}^2, \mathbb{Z})$ is a generator of this group. By the notion of framed cobordism, $d^2(\xi_1, \xi_2) = 0 \iff \xi_1 \simeq \xi_2$ over $M^{(2)}$.

We define $d^3(\xi_1, \xi_2)$ only when $d^2(\xi_1, \xi_2) = 0$. In this case we further assume that $\xi_1 = \xi_2$ outside a 3-disk D^3 in M. $\mathbb{S}^3$ can be identified with two 3-discs D_+, the upper hemisphere, and D_-, the lower hemisphere sticking together in opposite orientations. Consider stereographic projections corresponding to south and north poles respectively and write them as $\pi_+ : D_+ \to \mathbb{S}^3$ and $\pi_- : D_- \to \mathbb{S}^3$. We then define $G_{\xi_1,\xi_2} : \mathbb{S}^3 \to \mathbb{S}^2$ by

$$G_{\xi_1,\xi_2} = \begin{cases} g_{\xi_1} \circ \pi_+ & \text{on } D_+; \\ g_{\xi_2} \circ \pi_- & \text{on } D_-. \end{cases}$$

Since $\xi_1 = \xi_2$ on $M - \text{int}(D^3)$ is assumed, G_{ξ_1,ξ_2} is continuous across the boundary 2-sphere. In this way, we take the Hopf invariant (H) of G_{ξ_1,ξ_2} by perturbing it to a smooth function $G \simeq G_{\xi_1,\xi_2}$. Recall that $H : \pi_2(\mathbb{S}^3) \to \mathbb{Z}$ is an isomorphism, see [34]. For any smooth $f : \mathbb{S}^3 \to \mathbb{S}^2$,

$$H(f) = \int_{\mathbb{S}^3} \omega \wedge d\omega = link(f^{-1}(p), f^{-1}(q)).$$

$f^*\Omega = d\omega$ is given by an area form Ω of $\mathbb{S}^2$, and p, q are two different regular values of f. As a remark, any vector bundle of rank 2 is trivial over an one dimensional manifold so we don't need to consider the 1-skeleton $M^{(1)}$.

Definition 2.1. Let $(M, \xi = \ker \alpha)$ be a closed oriented contact 3-manifold. The *Lutz twist* along a knot K is a trivial Dehn surgery described by $\mu_0 \to \mu$ and $\lambda_0 \to \lambda$ using previous notations on meridians and longitudes. The positive contact form α is extended by $\alpha = h_1(r)d\theta + h_2(r)d\phi$ over the glued-in solid torus N_1, satisfying

(1) $(h_1, h_2) = (-1, -r^2)$ near $r = 0$;

(2) $(h_1, h_2) = (1, r^2)$ near ∂N_1; and

(3) $h_1 h_2' - h_1' h_2 > 0$ whenever $r \neq 0$. ■

Proof (Theorem 2.3, see [17])**.** The key point is to understand how a Lutz twist affects obstruction classes d^2 and d^3. A starting contact structure ξ_M comes from the Martinet's theorem. One can show that its Euler class $e(\xi_M) \in H^2(M, \mathbb{Z})$ is an even element, and let $e(\xi_M) = 2c$. First, we perform a Lutz twist along a transverse knot K_0 with $PD_M(c) = [K_0]$ so that the outcome ξ_0 has zero Euler class.

Regarding to the prescribed homtopy class η, we pick another transverse knot K_1 satisfying $PD_M[K_1] = d^2(\xi_0, \eta)$ on cohomology level. Denote the resulting contact structure by ξ_1 and note that

$$d^2(\eta, \xi_1) = d^2(\eta, \xi_0) + d^2(\xi_0, \xi_1) = d^2(\eta, \xi_0) + (0 - PD_M[-K_1]) = 0.$$

Since η and ξ_1 agree on $M^{(2)}$, we can further assume that they agree on M outside a small 3-disk D contained inside a Darboux chart. Also, it can be guaranteed that K_0 and K_1 lie outside D. The final step is to cancel d^3. We perform a Lutz twist again on components of a transverse link L in D, of which the self-linking number $sl(L) = d^3(\eta, \xi_1) \in \mathbb{Z}$. Since $e(\xi_1|_D) = 0$, so we can show that $sl(L) = H(G_{\xi,\xi_1})$, where ξ is the contact structure created by this Lutz twist. Therefore,

$$d^3(\xi, \eta) = d^3(\xi, \xi_1) + d^3(\xi_1, \eta) = sl(L) - sl(L) = 0. \quad \square$$

Remark. Self-linking number sl of transverse knots is defined in Chapter 3.

Chapter 3

Knots and Overtwistedness

Classification of (cooriented) contact structures on a closed 3-manifold starts with a crucial observation: the standard (tight) contact structure on $\mathbb{R}^3$, ξ_{std}, doesn't contain any standard overtwisted(OT) disks. One method to show this result is by Bennequin inequality on $(\mathbb{S}^3, \xi_H)$, the Hopf contact structure mentioned in Chapter 1. I shall first introduce knots and knot invariants in $(\mathbb{R}^3, \xi_{std})$, and then follow Bennequin [4] to verify it. Xiao-Song Lin also explains some details in [27].

A contact structure containing an embedded standard OT disks is called **overtwisted**. Otherwise, it is called **tight**. Classification of tight and overtwisted contact structures is treated separately. In later part, I shall introduce some basic classification results. Study of singularities on certain foliation plays an important role throughout this chapter.

3.1 Knot invariants

For basic ideas of knots and knot invariants, one can see [13], [17] and [35].

Definition 3.1. A *Legendrian knot* in a contact 3-manifold (M, ξ) is a closed curve $\gamma : \mathbb{S}^1 \to M$ of which tangent vector $\gamma'(t) \in \xi_{\gamma(t)}$. (We say that a tangent vector field X is *horizontal*, if $X \in \xi$ at every point we concern.) ■

Comparing to transverse knots, Legendrian knots also own standard neighborhoods in a contact 3-manifold.

Theorem 3.1. Let λ be a Legendrian knot in a contact 3-manifold (M, ξ). Then there is a neighborhood N of λ in M on which λ is mapped to $\mathbb{S}^1 \times \{0\}$, and $\xi = \ker(\cos\theta dx + \sin\theta dy)$ in local coordinates $(\theta, x, y) \in \mathbb{S}^1 \times D^2$. ■

We will focus on Legendrian knots in $(\mathbb{R}^3, \xi_{std})$. We consider the positive contact form $\eta = dz + xdy$. If γ is a Legendrian knot, then $\eta(\gamma') = 0$ or $z' + xy' = 0$ in Cartesian coordinates. Front and Lagrangian projections help us to visualize Legendrian knots in $(\mathbb{R}^3, \xi_{std})$.

Definition 3.2. Suppose $\gamma(s) = (x(s), y(s), z(s))$ is a parametrized curve in $(\mathbb{R}^3, \xi_{std})$. *Front projection* of γ, γ_F, is the projection of γ on yz-plane. *Lagrangian projection* of γ, γ_L, is the projection of γ on xy-plane. Explicitly, $\gamma_F = (y(s), z(s))$ and $\gamma_L = (x(s), y(s))$. ■

The Legendrian condition $z' + xy' = 0$ forces that γ_F has no vertical tangencies. Instead, the point on γ where $y' = 0$ ($\implies z' = 0$) projects to a cusp point on γ_F. A Legendrian curve is generic if cusp points are isolated and $y'' \neq 0$ there. By perturbation, any Legendrian curve γ will become generic, and cusp points of γ_F will behave like $\tilde{\gamma}(s)$ near $s = 0$:

$$\tilde{\gamma}(s) = (s + a, \lambda s^2 + b, -\lambda(\frac{2s^3}{3} + as^2) + c).$$

Away from cusp points, γ can be recovered from γ_F by $x = -\frac{dz}{dy}$. Finally, at any crossing, the strand of γ with smaller slope must lie in front of the strand with larger slope.

Conditions on the Lagrangian projection γ_L are fewer. If we specify a point $\gamma(s_0) = (x(s_0), y(s_0), z(s_0))$, then γ can be recovered by γ_L through

$$z(s) = z(s_0) - \int_{s_0}^{s} x(s) y'(s) ds.$$

This formula also shows that γ_L encloses a region of zero oriented area.

Front projection of a transverse knot τ, also denoted by τ_F, is also restricted by the contact condition. Let say $\tau \pitchfork^+ \xi_{std}$, so $z' + xy' > 0$. First, no downward vertical tangencies are allowed on τ_F. Secondly, whenever a downward crossing occurs, where $z' < 0$ locally, the strand with $y' > 0$ overlaps the other strand with $y' < 0$.

Any knot can be C^0-approximated by a Legendrian knot or a transverse knot of the same isotopy class.

Theorem 3.2. Suppose γ is a knot in a contact 3-manifold (M, ξ). Then γ can be C^0-approximated by a Legendrian knot isotopic to γ. If ξ is cooriented, γ can also be C^0-approximated by a positively (or negatively) transverse knot isotopic to γ. ■

An isotopy between Legendrian knots, i.e., all γ_t are Legendrian, is called a Legendrian isotopy. Two knots are isotopic if and only if their link diagrams are related by finite step of Reidemeister moves. (See [13]). We have a similar result in Legendrian isotopy.

Theorem 3.3. Two Legendrian knots are Legendrian isotopic if and only if front projections are related by a finite step of Legendrian Reidemeister moves. ■

There are three important classical knot invariants, (A) the Thurston-Bennequin invariant *tb*, (B) the rotation number *rot* and (C) the self-linking number *sl*. The former two are for Legendrian knots. The third is for transverse knots. In this part, let $(M, \xi = \ker \alpha)$ be a contact 3-manifold.

Definition 3.3. Let K be a nullhomologous knot in M. A connected, compact and oriented surface S with $\partial S = K$ is called a *Seifert surface* for K. ■

Very often K is considered as an oriented knot. In this sense, $-K$ represents the same knot but of opposite direction. We always assume that orientation of

S is compatible with its boundary orientation induced on K. In other words, if K is oriented, then its Seifert surface S is oriented by K. Actually, the choice of orientation on S doesn't affect either tb, rot or sl.

Definition 3.4. Let K_1 and K_2 be two disjoint nullhomologous oriented knots. Suppose D_2 is a Seifert surface for K_2 being transverse to K_1. Then, the *linking number* of K_1 and K_2 is defined by $link(K_1, K_2) \triangleq [K_1] \cdot [D_2]$, where RHS denotes their intersection product on homology level. ■

Note that this linking number is independent of the choice of Seifert surfaces. We also have $link(K_1, K_2) = link(K_2, K_1)$. When K_1 and K_2 are knots in $\mathbb{R}^3$, $link(K_1, K_2)$ equals the number of signed crossings where K_2 crosses under K_1 on a link diagram.

(A) Let λ be a homologically trivial Legendrian knot in (M, ξ), S a Seifert surface for λ. Pick a transverse vector field V along λ and then let $\tilde{\lambda}$ be a transverse push-off of λ along V. The Thurston-Bennequin invariant of λ is defined by

$$tb(\lambda) = link(\lambda, \tilde{\lambda}) = [\tilde{\lambda}] \cdot [S].$$

Note that $tb(\lambda)$ doesn't depend on the choice of Seifert surface S. Since ξ is cooriented, the complementary bundle TM/ξ is trivial. $tb(\lambda)$ can also be interpreted as the twisting of the contact framing, represented by the transverse vector field V, relative to the surface framing of λ, a trivialization of the normal bundle of λ in TS. See [17]. As a remark, $tb(\lambda)$ is independent of orientation of λ.

Example 3.1 (See [17])**.** This example illustrates how to compute tb in actual cases. Consider the Hopf contact structure $\alpha_H = xdy - ydx + zdt - tdz$ on $\mathbb{S}^3$. Parametrize the circle $\gamma = \{x^2 + z^2 = 1\}$ by $\gamma(\theta) = (\cos\theta, 0, \sin\theta, 0) \in \mathbb{S}^3$. Its unit tangent vector field $T = x\partial_z - z\partial_x$ is horizontal, so it is an oriented Legendrian knot.

We first state an orientation for $\mathbb{S}^3$ by the volume form

$$\Omega = xdy \wedge dz \wedge dt - ydz \wedge dt \wedge dx + zdt \wedge dx \wedge dy - tdx \wedge dy \wedge dz.$$

The 2-disk $D = \{x^2 + y^2 + z^2 = 1 \ , \ y \geq 0\}$ is a Seifert surface for γ. It is oriented positively by the basis $< T, N >$ along γ, where $N = \partial_y$, the inner unit normal. A positive orthonormal basis for $T\mathbb{S}^3$ along γ is then given by $< T, N, B = -\partial_t >$. Positivity can be verified by

$$\iota_N \iota_T \Omega = (-x^2 - z^2)dt - t(xdz + zdx) = -dt.$$

We are ready to find $tb(\gamma)$. Consider the transverse vector field
$R = x\partial_y - y\partial_x + z\partial_t - t\partial_z$. Transversal of R guarantees that its projection R' on $< N, B >$ along γ is never zero, so the required linking number is given by the winding number of R' relative to the positive basis $< N, B >$ along γ. Note

$$R'(\theta) = x(\theta)N - z(\theta)B = \cos\theta N - \sin\theta B$$

along γ. It is obvious that this winding number is -1 so $tb(\gamma) = -1$.

The Thurston-Bennequin invariant can also be obtained through the font projection γ_F and Lagrangian projection γ_L. For an oriented Legendrian knot γ in $(\mathbb{R}^3, \xi_{std})$, we have (The symbol '#' means 'the number of'.)

$$tb(\gamma) = \#writhe(\gamma_F) - \frac{1}{2}\#cusp(\gamma_F) = \#writhe(\gamma_L).$$

Example 3.2. Consider this front projection of a Legendrian knot K in $(\mathbb{R}^3, \xi_{std})$. See Figure 3.1. We have $tb(K) = -2$ and $rot(K) = 1$.

(B) Denote the pair of an oriented Legendrian knot and its Seifert surface by(λ, S). Note that ξ is oriented by $d\alpha$ globally. Since S retracts to its
1-skeleton, we can then take a trivialization of ξ over S, and so a non-vanishing horizontal vector field X near S in (M, ξ).

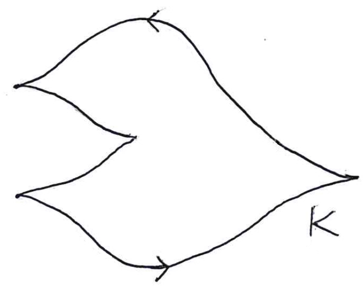

Figure 3.1: Front projection of knot K

Denote the unit tangent of λ by T. Let $\theta : \mathbb{S}^1 \to \mathbb{S}^1$ be the angle measured from X to T. (We can simply use the underlying Riemannian metric g on M.) The rotation number is defined by

$$rot(\lambda, S) = \deg \theta.$$

This integer rot depends on the choice of orientation on λ. Moreover, since the trivialization of ξ we took is local near S, rot depends on the choice of Seifert surface S as well. If ξ is globally trivial, rot will be independent of S. In this case, we replace $rot(\lambda, S)$ by $rot(\lambda)$. Explicitly, when we have two Seifert surfaces S and S', $rot(K, S) - rot(K, S') = e(\xi) \cdot ([S] - [S'])$.

Example 3.3. Refer to Example 3.1. This time we compute $rot(\gamma)$. Since $X = x\partial_z - z\partial_x + t\partial_y - y\partial_t$ is a non-vanishing horizontal vector field over $\mathbb{S}^3$, $e(\xi_H) = 0$. This implies that rot is independent of the choice of Seifert surfaces. Note that $X = T$ along γ, so $rot(\gamma) = 0$.

Similar to tb, when λ is Legendrian in $(\mathbb{R}^3, \xi_{st})$, $rot(\lambda)$ can be computed through its front projection λ_F. Let $L_\uparrow$ or $L_\downarrow$ be the number of upward or downward left cusps in λ_F, $R_\uparrow$ or $R_\downarrow$ the number of upward or downward right cusps in λ_F. We have

$$rot(\lambda) = L_\downarrow - R_\uparrow = R_\downarrow - L_\uparrow = \frac{1}{2}(L_\downarrow + R_\downarrow - L_\uparrow - R_\uparrow).$$

As a remark, both *tb* and *rot* are invariants under Legendrian isotopy. It can be verified by the Legendrian isotopy extension theorem [17]. Roughly speaking, it shows that an isotopy of Legendrian knots can be modified to a contact isotopy, i.e. an isotopy of contactomorphisms. On the other hand, the self-linking number *sl* is invariant under isotopy of transverse knots. This result comes from the isotopy extension theorem for contact submanifolds, since transverse knots are contact submanifolds.

(C) Let γ be a transverse knot in (M, ξ) and S a Seifert surface for γ. Take a non-vanishing horizontal vector field X over S. Denote $\tilde{\gamma}$ be a horizontal push-off of γ along X. The self-linking number of γ relative to S is defined by

$$sl(\gamma, S) = link(\gamma, \tilde{\gamma}) = [\tilde{\gamma}] \cdot [S].$$

sl depends on S in the same way as *rot*. Yet, the choice of orientation on γ doesn't matter. Explicitly, $sl(\gamma, S) - sl(\gamma, S') = \mp e(\xi) \cdot ([S] - [S'])$. The sign depends on whether γ is positively or negatively transverse to ξ.

Example 3.4. Refer to Example 3.1 and 3.3 once more. Consider the positively transverse knot $\gamma_0 = \{x^2 + y^2 = 1\}$ in $(\mathbb{S}^3, \xi_H = \ker \alpha_H)$. $T_0 = x\partial_y - y\partial_x$ is the unit tangent vector field along γ_0.

From Example 3.3, we know that ξ_H is trivial. Choose a Seifert surface $D_0 = \{x^2 + y^2 + z^2 = 1, z \geq 0\}$ for γ_0. The inner unit normal of γ_0 in TD_0 is denoted by $N_0 \triangleq \partial_z$, so $< T_0, N_0, B_0 = \partial_t >$ forms a positive frame of TM along γ_0. We can verify it by $\iota_{N_0}\iota_{T_0}\Omega = dt$ on γ_0.

Take the horizontal vector field $X = x\partial_z - z\partial_x + t\partial_y - y\partial_t$ as the direction of push-off. Along γ_0, $X = x\partial_z - y\partial_t = \cos\theta N_0 - \sin\theta B_0$. Therefore, $sl(\gamma_0) = -1$.

For a transverse knot γ in $(\mathbb{R}^3, \xi_{std})$, its font projection determines its self-linking number by the formulae $sl(\gamma) = \#writhe(\gamma_F)$. We end this part with a theorem applied in Chapter 2. See [17].

Theorem 3.4. Every integer can be realized as the self-linking number of some transverse link in $(\mathbb{R}^3, \xi_{std})$. ■

3.2 Bennequin inequality

The basic version of Bennequin inequality states that, for a Legendrian knot K and its Seifert surface Σ lying in $(\mathbb{R}^3, \xi_{std})$, we have $tb(K) + |rot(K)| \leq -\chi(\Sigma)$. It suggests that the standard overtwisted contact structure ξ_{ot} is not contactomorphic to the standard contact structure ξ_{std} on $\mathbb{R}^3$.

We are not going to show the Bennequin inequality on $(\mathbb{R}^3, \xi_{std})$ directly. Instead, we verify this inequality on $(\mathbb{S}^3, \xi_H)$. As a result, $(\mathbb{S}^3, \xi_H)$ contains no embedded standard OT disks, so as $(\mathbb{R}^3, \xi_{std})$ under Theorem 1.1. In fact, on any contact 3-manifolds (M, ξ), if Δ is a standard OT disk on M, then $tb(\partial\Delta) = 0$ but $-\chi(\Delta) = -1 < 0$. Conversely, if (M, ξ) is tight, i.e., contains no standard OT disks, then by later result, Bennequin inequality on (M, ξ) holds.

Before presenting Bennequin's ideas, I shall discuss relation between tb, rot and sl by horizontal push-off. Suppose that λ is an oriented Legendrian knot in a contact 3-manifold (M, ξ). Let T be its unit tangent vector, N a horizontal positive normal to T on ξ. Define λ^+ and λ^- by push-offs of λ along N and $-N$ respectively. In fact, $\lambda^+ \pitchfork^- \xi$ while $\lambda^- \pitchfork^+ \xi$.

Theorem 3.5. [1]. If λ is a homologically trivial Legendrian knot in (M, ξ), then

$$sl(\lambda^\pm) = tb(\lambda) \pm rot(\lambda).$$

Proof. Fix a Seifert surface S for λ. Extend N over S and denote the push-off of S along N by S^+. Take a non-vanishing transverse vector field X over S. Denote the push-off of λ along ϵX by $\lambda + \epsilon X$, for a small $\epsilon > 0$. Similar notations apply to other push-off curves. By definitions,

(1) $tb(\lambda) = link(\lambda, \lambda + \epsilon X) = link(\lambda^+, \lambda^+ + \epsilon X)$; and

(2) $sl(\lambda^+) = link(\lambda^+, \lambda^+ + \epsilon' N)$ for a small $\epsilon' > 0$.

Therefore, $sl(\lambda^+) - tb(\lambda)$ describes how N twists over X along λ^+. Since λ^+ is close to λ, it also equals to twisting of N over X along λ. Therefore,

$$sl(\lambda^+) - tb(\lambda) = rot(\lambda),$$

and similar result holds for λ^-. □

Using Theorem 3.5 and the transverse Bennequin inequality, $sl(\gamma) \leq -\chi(\Sigma)$, on $(\mathbb{S}^3, \xi_H)$, we get to the desired (Legendrian) Bennequin inequality.

3.2.1 Markov surface

Consider the Heegaard splitting $\mathbb{S}^3 = M \cup P$ adopted in Example 2.1.
$\mathbb{S}^3 = \{|z|^2 + |w|^2 = 1\}$,

$$M = \{|z| \geq |w|\} = \{|z| \geq \frac{1}{2}\} \quad \text{and} \quad P = \{|w| \geq \frac{1}{2}\}.$$

Call the boundary torus by $T = \partial M = \partial P$. Parametrize the generating circles $\Gamma_0 = \{|z| = 1\}$ in M and $\Gamma_1 = \{|w| = 1\}$ in P by $\theta, \phi \in \mathbb{S}^1$ respectively.

Reeb foliations on M and P induce an oriented foliation $\mathcal{R}$ on $\mathbb{S}^3$, also called as Reeb foliation. We choose a particular orientation of $\mathcal{R}$ such that Γ_0 and Γ_1 is positively transverse to $\mathcal{R}$. A general description of Reeb foliation on a solid torus can be found in [32]. Consider the solid torus $D \times \mathbb{S}^1$, where $D = \{|z| = 1\} \subset \mathbb{C}$. It can be covered by the quotient map $\pi : D \times \mathbb{R} \to D \times \mathbb{S}^1$. Let x be the coordinate on $\mathbb{R}$. The boundary torus is one leaf of Reeb foliation. Other leaves of the Reeb foliation are the images of graphs of

$$x = e^{\frac{1}{1-|z|^2}} + C \triangleq F(z) + C$$

on $D \times \mathbb{S}^1$ for different constants C.

Denote the unique fibre in M intersecting $\theta \in \Gamma_0$ by $\mathcal{R}_\theta$. T is the only compact fibre of $\mathcal{R}$. On T, circles parallel and homotopic to Γ_0 are called *parallels* and circles parallel and homotopic to Γ_1 are called *meridians*. We can consider any knot γ in $\mathbb{S}^3$ as a closed braid, winding around Γ_1 positively (like Γ_0), lying completely inside M and transverse to all fibres $\mathcal{R}_\theta$. A Seifert surface V for γ is said to be a *Markov surface*, if

(1) V is connected;

(2) $T \cap V$ is the union of finitely many parallels p_{ϕ_i}, and
$P \cap V$ are the union of parallel discs P_{ϕ_i}, bounded by p_{ϕ_i} in V; and

(3) singularities of $V_{\mathcal{R}}$ in M are all saddle points. (No closed loop in $(V \cap M)_{\mathcal{R}}$.)

All restrictions mentioned above are valid throughout this section. Genericity of Markov surfaces comes from the following theorem.

Theorem 3.6. Let V_0 be a Seifert surface for a closed braid γ with maximum Euler characteristic $\chi(\gamma)$. Then there is a Markov surface V isotopic to V_0 in $\mathbb{S}^3$ relative to γ. ■

The singular foliation $V_{\mathcal{R}}$ of the Markov surface V has *no saddle-saddle connection*. It will be assumed below.

Given a closed braid γ, suppose V is a Markov surface of maximum Euler characteristic for γ. Let n be the number of threads of γ (also called braid index) and c the algebraic length of γ. We try to see how n and c relate to hyperbolic points of $V_{\mathcal{R}}$ in M and parallel disks P_{ϕ_i}. Beforehand, we observe how $\mathcal{R}_\theta \cap V$ looks like on each fibre $\mathcal{R}_\theta$. Maximality of $\chi(V)$ ensures that no closed loops appear on $\mathcal{R}_\theta$. On the fibres, there are deleted points representing points in $\mathcal{R}_\theta \cap \gamma$. The boundary circle denotes points on boundary torus T. Arcs are called free if they go from deleted points to a parallel p_ϕ, and called connected if they join two parallels. (Figure 3.2)

Figure 3.2: R_θ (L) and V near a hyperbolic point (R) ([4])

Assume no two hyperbolic points take place on the same fibre in M. We say a hyperbolic point is positive (h_+) or negative (h_-) depending on whether orientations of $\mathcal{R}$ and V match at that point or not. Let A^+ be the number of h_+ and A^- the number of h_-. Two arcs are switched along fibres after passing through a hyperbolic point. (Figure 3.3)

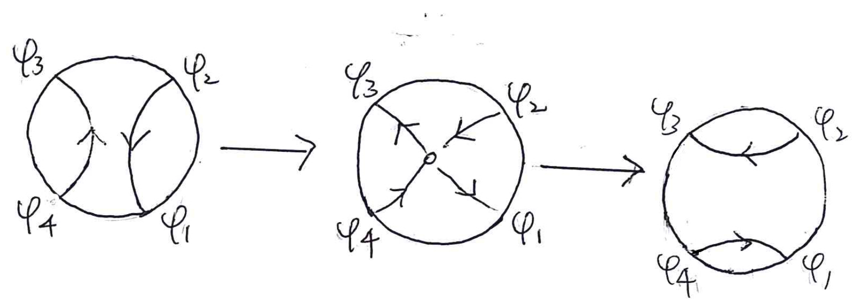

Figure 3.3: Change of R_θ near a h_- ([4])

Theorem 3.7. $c = A^+ - A^-$.

Proof. Let X_0 be a non-vanishing vector field over M such that $X_0 \in T\mathcal{R}_\theta$ on every $\mathcal{R}_\theta$. For example, we can choose a constant vector field on D and then map it to the graphs of $x = F(z) + C$. Let Y be a generating vector field of $V_\mathcal{R}$. Note that $Y \pitchfork \gamma$. So we can define a map $\alpha : \gamma \to \mathbb{S}^1$ where α measures the angle from Y to X_0 along γ. Observe $c = \deg \alpha$. On other components of

$\partial(V \cap M)$, parallels p_{ϕ_i} are pushed along X_0, but the push-offs don't intersect $V \cap M$. Therefore, passing from $\partial(V \cap M)$ to small circles near hyperbolic points, we have $c = A^+ - A^-$. □

Foliation on P_ϕ consists of an elliptic point and surrounding circles. the parallel p_ϕ will be an attracting or repulsing cycle depending on how it is oriented relative to P_ϕ. Let S^+ be the number of attracting cycles and S^- the number of repulsing cycles. Immediately we have $n = S^+ - S^-$ and $\chi(V) = (S^+ + S^-) - (A^+ + A^-)$.

3.2.2 $c - n \leq -\chi$

We consider again a closed braid γ and a Markov surface V for γ of the greatest possible Euler characteristic. (under certain assumptions and restrictions.) Our goal is to show that $c(\gamma) - n(\gamma) \leq -\chi(V)$ and $sl(\gamma) = c(\gamma) - n(\gamma)$. In this way, Bennequin inequality for Markov surfaces is established. For the first inequality, we introduce sacks.

Definition 3.5. A *sack* is a domain of a repulsing disc P_ϕ. An elementary sack is a sack whose closure contains only two attracting cycles. ■

An *elementary sack* contains two sinks and two hyperbolic points of opposite signs, together with surrounding flow lines all in attracting direction. If we can carry out a topological surgery to cancel all but one sink in this sack, this new surface will has the same Euler characteristic as V. Moreover, since $n = S^+ - S^-$ and $c = A^+ - A^-$, automatically we preserve $n(\gamma)$ and $c(\gamma)$ in this surgery. Here comes the core lemma.

Lemma. If a Markov surface V for a closed braid γ has an elementary sack, then there is another pair (γ', V') of a closed braid and its Markov surface such that $c(\gamma) = c(\gamma')$, $n(\gamma) = n(\gamma')$, $\chi(V) = \chi(V')$, and V' has one sack fewer than V has, i.e. $S^-(V') < S^-(V)$. ■

Theorem 3.8. $c(\gamma) - n(\gamma) \leq -\chi(V)$.

Proof. It will be done if we can show that $A^- \geq S^-$ for such a pair (γ', V') in previous lemma. When $S^- = 0$, we are done. If $S^- = 1$, a connected arc emitted from the repulsing cycle p_ϕ will always rotate anticlockwise if it passes through a positive hyperbolic point in the sack. Since this arc will come back to the original position on R_θ after a loop of Γ_0, there must be one negative hyperbolic point in the sack. So, $A^- \geq S^-$.

In general, if $S^- > A^-$, there is a sack in V with $\#h_- \leq 1$. It is because each h_- appears at most in two sacks, so

$$\#\{\text{sacks with at least two } h_-\} \leq A^- < S^-.$$

Once the Markov surface V has a sack of one h_-, we deform V into another Markov surface V' (for γ) where V' has an elementary sack. Applying the lemma, induction on S^- will finish our proof. The remaining task is to find out how this deformation looks like. Here I will consider phase diagrams, and denote an arc in the form of "start $\rightarrow$ end".

Let p_ψ be the repulsing parallel in this sack of singleton h_-. After passing through this h_-, as θ increases, the free arc $p_\psi \rightarrow p_{\phi_0}$, or $\psi \rightarrow \phi_0$ for simplicity, will rotate clockwise to $\psi \rightarrow \phi_1$. Then let say this arc shift to $\psi \rightarrow \phi_2$ and $\psi \rightarrow \phi_3$ after passing through two h_+'s at θ_1 and θ_2. In phase diagram, among ϕ_1, ϕ_2 and ϕ_3, ϕ_1 is the leftmost parallel, and ϕ_3 is the rightmost one. Denoting two h_+ by Q_1 and Q_2 respectively, we set a vertical membrane separating them on all fibres R_θ, $\theta \in [\theta_1 - \epsilon, \theta_2 + \epsilon]$. Using this membrane, we carry out a deformation to the solid cylinder bounded by $\theta_1 - \epsilon$ and $\theta_2 + \epsilon$ in M. The resulting sack on V will have one h_+ lesser, as p_ψ jumps from p_{ϕ_1} directly to p_{ϕ_3}. □

As a remark, once we destroy a repulsing disk on V, we need to destroy an attracting disk, a positive hyperbolic point and a negative one. It guarantees that c, n and χ keep constant in this surgery.

We turn back to the Hopf contact structure $\xi_H = \ker \alpha_H$. Recall that Γ_0 is positively transverse to ξ_H, so any closed braid γ sufficiently close to Γ_0 is positively transverse to ξ_H.

Theorem 3.9. If γ is a closed braid positively transverse to ξ_H, then

$$sl(\gamma) = c(\gamma) - n(\gamma).$$

Proof. Take the global non-vanishing horizontal vector field over $\mathbb{S}^3$ in Example 3.3. Rename it as Y_0 for convenience. Under notations in the proof of Theorem 1.5, $sl(\gamma) = link(\gamma, \gamma + \epsilon Y_0)$ for a small $\epsilon > 0$. Let X_0 is be a non-vanishing vector field of $\mathcal{R}$ over M. So we have $c(\gamma) = link(\gamma, \gamma + \epsilon' X_0)$, for a small $\epsilon' > 0$.

Therefore, $sl(\gamma) - c(\gamma)$ counts the total variation of Y_0 against X_0 along γ. Along every thread of γ, this total variation acts similar to the total variation of Y_0 against X_0 along Γ_0, which equals to $sl(\Gamma_0) = -1$. Thus, $sl - c = -n$. □

Denotes the number of $\pm$ elliptic points by $s^{\pm}$, and assume S intersects Reeb foliation $\mathcal{R}$ generically. It worths to mention that $A^- \geq s^-$ holds even for any such Seifert surface S for a transverse knot γ. One can obtain $A^- \geq s^-$ and $sl(\gamma) = -s^+ + s^- + A^+ - A^-$.

Overall speaking, for an arbitrary positively transverse knot γ in $(\mathbb{S}^3, \xi_H)$, we can assume that it lies in $(\mathbb{R}^3, \xi_{std})$ and transverse isotopic to another transverse knot $\tilde{\gamma}$ close enough to Γ_0. $\tilde{\gamma}$ is a closed braid in M so we have

$$sl(\gamma) = sl(\tilde{\gamma}) = c(\tilde{\gamma}) - n(\tilde{\gamma}) \leq -\max_{\partial V = \tilde{\gamma}} \chi(V) = -\max_{\partial \Sigma = \gamma} \chi(\Sigma).$$

It is the transverse Bennequin inequality on $(\mathbb{S}^3, \xi_H)$ mentioned after Theorem 3.5.

3.3 Elimination lemma

Recall the definitions of tightness and overtwistedness.

Definition 3.6. A contact structure ξ on a closed 3-manifold M is *overtwisted* if there is a contact embedding of the standard overtwisted disk (see Example 1.5), $\phi : (\Delta, \xi_{ot}) \hookrightarrow (M, \xi)$. A contact structure is *tight* if it is not overtwisted. ■

Equivalently, ξ is overtwisted if there is an embedded disk Δ in (M, ξ) such that $\partial\Delta$ is Legendrian, $tb(\partial\Delta) = 0$ and Δ_ξ contains a unique singular point on $\text{int}(\Delta)$. The presence of this singular point is unnecessary. Comparing to the standard overtwisted disk, if D is a 2-disk embedded in $(\mathbb{R}^3, \xi_{std})$ with Legendrian boundary, then $D_{\xi_{std}}$ contains singular points on its boundary.

We will see some primary results on classification in this section. We concern singularities on a nice kind of surfaces, surfaces of Morse-Smale type. For convenience, **MS** stands for 'Morse-Smale'. Elimination lemma helps reduce singularities on such a surface and it is applied in Theorem 3.13, 3.14 and 3.15.

Before anything, there is an extension of Lutz-Martinet theorem on OT contact structures.

Theorem 3.10. [17]. For any closed 3-manifold M, there is an OT contact structure in every homotopy class of cooriented 2-plane field. ■

Suppose S is a closed oriented surface in a contact 3-manifold (M, ξ), where $\xi = \ker \alpha$. (S can be compact sometimes. In this case, we require ∂S to be Legendrian.) Let Ω be a positive area form of S and X the generating vector field of S_ξ defined by $\iota_X \Omega = \alpha|_S$.

As a remark, a vector field X on S defines the characteristic foliation of some contact structure near S if and only if $\text{div}_\Omega(X) \neq 0$ at $p \in M$ whenever $X(p) = 0$. Recall that divergence is given by $\sum_{i=1}^{n} < \nabla_{e_i} X, e_i >$, where for $\{e_i\}$ is an orthonormal frame. We also have $\mathcal{L}_X \Omega = \text{div}_\Omega(X)\Omega$.

Notions about non-degenerate zeros, elliptic and hyperbolic points follow common notations. For example, from [31]. The divergence condition guarantees that elliptic points are never accumulated by circles. In addition to topological index, we define the sign of a singular point to be ± 1, depending on whether orientation of TS matches with that of ξ at this singular point or not. ($X \in TS \cap \xi$). In fact, sign is given by $\text{div}_\Omega X$.

Definition 3.7. A characteristic foliation S_ξ (oriented by X) on a closed oriented surface S is of *Morse-Smale type* (**MS type**) if all singularities and closed orbits in are non-degenerate, no saddle-saddle connection occurs, and the Poincaré-Bendixson property is satisfied. ■

Non-degeneracy of closed orbits means that all Poincaré return maps are non-degenerate. Therefore, singular points are finite and isolated, and there is no accumulation of circles towards a closed orbit. Usually, Poincaré-Bendixson property means that the $\alpha-$ and $\omega-$limit set of each flow line is either a singular point, a closed orbit or a union of zeros and connecting flow lines. However, since we don't allow saddle-saddle connection, any $\alpha-$ and $\omega-$limit set cannot be such a union of singular points and orbits.

A singular foliation on S induced by the gradient of a Morse function is in particular of MS type. Accumulation of circles toward a union of zeros and flow lines is not allowed since no hyperbolic points are connected by separatrices. So the number of closed orbits is finite. In general, we can perturb a surface so that its characteristic foliation is of MS type.

Theorem 3.11. Let S be a closed oriented surface in (M, ξ). Then there is a surface S' isotopic and C^∞-close to S such that S'_ξ is of MS type. ■

For a characteristic foliation S_ξ of MS type, we define $e_\pm$ to be the number of positive or negative (in sign) elliptic points, and $h_\pm$ to be the number of positive or negative hyperbolic points. Straightforwardly, we have
$\chi(S) = (e_+ - h_+) + (e_- - h_-)$ and $< e(\xi), [S] >= (e_+ - h_+) - (e_- - h_-)$. Here a diffeomorphism (or a homeomorphism) between two characteristic foliations is assumed to respect signs of hyperbolic points.

Example 3.5 (see [17])**.** We consider the torus $T = T^2 \times \{0\} \subset T^2 \times \mathbb{R}_z$. A fundamental domain of T in $\mathbb{R}^2$ is given by the square $Q = \{(x, y)| -1 \leq x \leq 1, -1 \leq y \leq 1\}$. (Figure 3.4) Two contact structures ξ_1 and ξ_2 on $T^2 \times \mathbb{R}^2$ are defined by the contact forms

$$\alpha_1 = \sin \pi y dx + 2 \sin \pi x dy + (2 \cos \pi x - \cos \pi y)dz;$$

$$\alpha_2 = \sin \pi y dx + (1 - \frac{1}{K} \cos \pi x) \sin \pi x dy + (\cos \pi x - \frac{1}{K} \cos 2\pi x \cos \pi y)dz,$$

respectively for $K \gg 1$. They induce the same topological foliation diagram on T^2 and Q, but note that the signs of hyperbolic point p differ in these two characteristic foliations. As a result, T_{ξ_1} is not diffeomorphic to T_{ξ_2}, or ξ_1 not contactomorphic to ξ_2 near T.

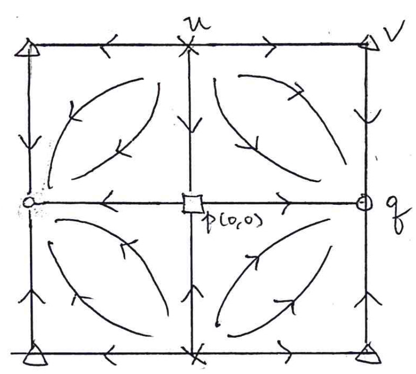

Figure 3.4: Fundamental domain of T ([17])

Suppose S is a closed oriented surface in a contact 3-manifold (M, ξ), and its characteristic foliation S_ξ of MS type. The elimination lemma, by suitable perturbation of S in M, helps cancel out singular points if possible. I shall focus on its implication rather than its proof.

Theorem 3.12 (Elimination lemma, [17])**.** If there exists an elliptic point x_e and a hyperbolic point x_h of same sign on S, connected by a separatrix γ of x_h, then there is an arbitrary C^0-small isotopy $\psi_t : S \to M$ and an arbitrary small neighborhood U of γ on S such that (i) ψ_0 is the inclusion map;(ii) ψ_t fixes γ and $S - U$ for every t; and (iii) $\psi_1(S)_\xi$ has no singularities on $\psi_1(U)$. ■

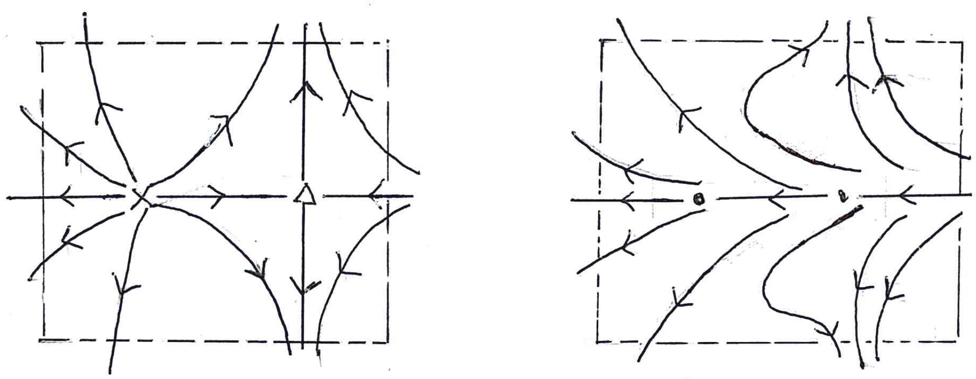

Figure 3.5: Elimination lemma ([17])

Apart from destroying singularities, introduction of a pair of singularities can be analogously constructed. In the elimination process, S remains of MS-type after perturbation.

Among several applications, we start with clarifying (standard) overtwisted disks as stated at the beginning of this section.

Theorem 3.13. [17]. Suppose Δ is a 2-disk in (M, ξ) with Legendrian boundary, whether of degenerate points or not, and $tb(\partial\Delta) = 0$. Then ξ is an OT contact structure.

Proof. Assume that Δ_ξ is of MS type, and $\partial\Delta$ is non-degenerate. By passing to smaller disk inside, assume also that no closed orbits exist in $\text{int}(\Delta)$. If there is virtually no hyperbolic point on Δ, $\chi(\Delta) = 1$ implies the existence of a unique elliptic point. So we are done. We then consider the problem in three other cases one by one.

Case 1. If there is a positive hyperbolic point x_{h+} on Δ, then its stable separatrix must come from a source x_{e+}. So x_{h+} and x_{e+} cancel out each other. It rules out all cases that $h_+ > 0$, so we can assume that $h_+ = 0$.

Case 2. If there is a negative hyperbolic point connected with a sink, they can be automatically destroyed.

Case 3. If there is a negative hyperbolic point x_{h-} not in Case 2, we create a sink x_{e-} attracting both unstable separatrices γ and γ' of x_{h-}. By the elimination process, orientation of γ changes, and it bounds a smaller disk Δ' of non-degenerate Legendrian boundary in Δ.

From Case 3, We can pass everything to Δ' and reconsider every possible cases. Note the number of negative hyperbolic points in Δ'_ξ is one less than that in Δ_ξ. □

The genus bound inequality is an achievement of this elimination process. It will be of great advantage to prove it under convex surface theory. Here, I shall draft the long proof for convenience.

Theorem 3.14. Let (M, ξ) be a tight contact structure and S a closed oriented surface in M. Then

$$| < e(\xi), [S] > | \leq \max\{0, -\chi(S)\}.$$

Proof. see [17]. For S_ξ of MS type, one has $< e(\xi), [S] > +\chi(S) = 2(e_+ - h_+)$. It is done if we can cancel sources one by one, or reach special cases midway. After breaking down all attracting cycles, we pick a source x_{e+} and consider the Legendrian polygon B centered at x_{e+}. This Legendrian polygon is exactly a sack, the closure of flow lines initiated from x_{e+}.

On ∂B, we call sinks by vertices (of B). Edges are corresponding hyperbolic points and their unstable separatrices. Note that $\text{int}(B) \hookrightarrow S$ but not necessarily ∂B. We investigate this Legendrian polygon B by three steps.

Step 1. If there is no saddle points, then $S = \mathbb{S}^2$, $e_\pm = 1$ and $h_\pm = 0$. We are done. If there is a positive hyperbolic point on ∂B, cancel out x_{e+} immediately. It will only create a repelling cycle. In this way we can assume that all saddle points on ∂B are negative. Note if $B \hookrightarrow S$, then B is an OT disk. We are also done.

Step 2. Separate any two vertices as long as none of their adjacent edges are identified. Suppose there is an edge of distinct (on S) vertices v_1 and v_2 in polygon B. If it is not identified with any other edge, we can cancel it and replace all representatives of v_2 on B by v_1. Else if it is identified with another edge, note that the later edge should not be adjacent to v_1 or v_2, so we can also cancel this pair of edges. After Step 2, all vertices of the polygon B are glued to a single point on S.

Step 3. If there is no edge on B, then $S = \mathbb{S}^2$, $e_\pm = 1$ and $h_\pm = 0$ from Step 1. If ∂B consists of one edge, $B \hookrightarrow S$ and so it is an OT disk.

We can merge two non-identified edges to one edge. Such a non-identified edge always bounds a region outside this polygon B on S. After repetition, we will arrive at a conclusion unless there is a pair of identified edges, with at most one non-identified edge. Since the loop on S represented by this pair is non-separating, a little perturbation of S helps us to separate the pair of edges and singularities on them.

As all processes above work for compact surfaces as well, we can repeat Steps 1, 2 and 3 on new polygon B'. Each time we reduce the number of edges, since at least two vertices of B' are not glued together.

Overall speaking, for every positive elliptic point x_{e+}, either (1) x_{e+} is connected with a positive hyperbolic point, or (2) $S = \mathbb{S}^2$, $e_\pm = 1$ and $h_\pm = 0$. Similar being for $- < e(\xi), [S] > + \chi(S) = 2(e_- - h_-)$. □

Remark. A Legendrian polygon with all negative singularities on boundary exists only in situation (2). This happens because one contractible component of the dividing curve lies inside this polygon B on S. We will see it in convex surface theory.

Theorem 3.15. [17]. On a closed oriented 3-manifold M, only finite elements of $H^2(M, \mathbb{Z})$ can be realized as the Euler class $e(\xi)$ of a tight contact structure ξ on M.

Proof. It follows from the Universal Coefficient theorem. See [28] for this theorem. For simplicity, write $H^*(M, \mathbb{Z}) = H^*(M)$, $\mathrm{Hom}_{\mathbb{Z}} = \mathrm{Hom}$ and $\mathrm{Ext}_{\mathbb{Z}} = \mathrm{Ext}$. The splitting short exact sequence is
Let us pick an element $e(\xi) \in H^2(M)$ for a tight contact structure ξ on M. Note $h(e(\xi))[S] =< e(\xi), [S] >$ for all $[S] \in H_2(M)$. Suppose $H_2(M)$ is generated by $[S_i]$, $i = 1, \cdots, n$, so the genus bound inequality tells that

$$\min(0, \chi(S_i)) \quad \leq \quad < e(\xi), [S_i] > \quad \leq \quad \max(0, \chi(S_i))$$

for all i. Therefore, the number of choices for $h(e(\xi))$ in $\mathrm{Hom}(H_2(M), \mathbb{Z})$ is finite. Recall that $\mathrm{Ext}(H_1(M), \mathbb{Z})$ is the torsion group of $H^1(M)$. It is a finite group. Hence the total number of elements of $H^2(M, \mathbb{Z})$ possibly realized as the Euler class of a tight contact structure must be finite. □

3.4 Classification of OT contact structures

At the end of this chapter, I would like to state Eliashberg's classification of overtwisted contact structures on a closed 3-manifold M. It says that two OT contact structures ξ_1 and ξ_2 are homotopic as cooriented tangent plane fields if and only if they are isotopic as contact structures [8].

Fixed an orientation for M. Write $\mathrm{Distr}(M)$ for the space of cooriented tangent 2-plane fields in C^∞-topology and $\Xi^{\mathrm{ot}}(M)$ for its subspace of positive cooriented OT contact structures on M. So, it means that the inclusion map

$$i : \Xi^{\mathrm{ot}}(M) \to \mathrm{Distr}(M)$$

induces a bijection on path component.

Fix an oriented 2-disk Δ on M with its center $\mathbf{0}$. Write $\Xi^{\text{ot}}(M, \Delta)$ for the space of contact structures $\xi \in \Xi^{\text{ot}}(M)$ that contains Δ as a standard OT disk with a unique positive elliptic point at $\mathbf{0}$. Also, denote $\text{Distr}(M, \Delta)$ by the subspace of $\text{Distr}(M)$ in which those 2-plane fields are (positively) tangent to Δ at $\mathbf{0}$. In this way, the previous result comes from the following theorem.

Theorem 3.16 (Eliashberg [8], see also [17])**.** The inclusion map

$$i_\Delta : \Xi^{\text{ot}}(M, \Delta) \to \text{Distr}(M, \Delta)$$

is a weak homotopy equivalence. ■

Since both $\Xi^{\text{ot}}(M, \Delta)$ and $\text{Distr}(M, \Delta)$ have the homotopy type of CW-complexes, so weak homotopy equivalence means homotopy equivalence here by the Whitehead theorem. A complete proof of Theorem 3.16 can be found in [8] or [17].

Chapter 4

Convex surface theory

Convex surface theory is introduced to study classification of tight contact structures. A convex surface S in a contact 3-manifold (M, ξ) enjoys a vertically invariant neighborhood. The flexibility theorem provides a base that dividing curves of this convex surface S are already enough to represent the characteristic foliation S_ξ. Immediate applications include the Giroux's criterion, and an alternative approach to previous theorems. Materials mainly come from [12] and [17].

Later in this chapter, I shall state some classification results on tight contact structures. For example, the standard tight contact structure ξ_{std} on $\mathbb{R}^3$ is unique [17]. Other results cover $T^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, the solid torus $\mathbb{S}^3 \times D^2$ and the lens spaces $L(p, q)$. They are given by Honda in [23].

Definition 4.1. For a contact 3-manifold (M, ξ), a *contact vector field* Y is a vector field whose flow ϕ_t preserves ξ. If ξ is cooriented by a 1-form α, then

$$\mathcal{L}_Y \alpha = g\alpha$$

for a smooth function $g : M \to \mathbb{R}$. ■

For example, the Reeb vector field R_α associated with α is a contact vector field. Note that a contact vector field is horizontal if and only if it is identically zero [12]. On the other hand, it is transverse to ξ if and only if it represents the Reeb vector field of a contact form α'.

Any locally defined contact vector fields can be globally extended, given by their Hamiltonian functions. Recall that if (X, ω) is a symplectic manifold, and $f : X \to \mathbb{R}$ is a smooth function on it, then the Hamiltonian vector field H_f of f is given by $\iota_{H_f}\omega = df$.

Theorem 4.1 ([12])**.** Given a contact 3-manifold $(M, \xi = \ker \alpha$, Y is a contact vector field if and only if there is a (Hamiltonian) function $h : M \to \mathbb{R}$ satisfying

$$\alpha(Y) = -h,$$

$$\iota_Y d\alpha = dh - dh(R_\alpha)\alpha,$$

where R_α is the Reeb vector field associated with α.

Proof. Given a contact vector field Y, define h by $\alpha(Y) = -h$. Note for a prescribed smooth function g on M,

$$g\alpha = \mathcal{L}_Y \alpha = d(\alpha(Y)) + \iota_Y d\alpha = -dh + \iota_Y d\alpha.$$

Put R_α into this equation, and define $g \triangleq -dh(R_\alpha)$. The converse holds as well. □

Definition 4.2. A *convex surface* S in a contact 3-manifold (M, ξ) is an oriented surface such that there exists a contact vector field Y defined near S and transverse to S. ■

Example 4.1. Consider the standard tight contact structure on $\mathbb{R}^3$ cooriented by $\eta = dz + xdy - ydx$. The unit sphere $\mathbb{S}^2$ is a convex surface since the contact vector field $Y = x\partial_x + y\partial_y + 2z\partial_z$ is transverse to it. Note $\mathcal{L}_Y \eta = 2\eta$.

If S is a convex surface (M, ξ), then it has a neighborhood $N_v(S)$ contactomorphic to $S \times \mathbb{R}$ equipped with the contact form $\alpha = \beta + udt$. The 1-form

$\beta = \alpha|_S$, u is a smooth function on S, and t is the coordinate on $\mathbb{R}$. Such a neighborhood $N_v(S)$ is called *vertically invariant* (i.e. $\mathbb{R}$-invariant), because the pull-back contact structure in $N_v(S)$ is vertically invariant.

Theorem 4.2. [12]. In a contact 3-manifold $(M, \xi = \ker\alpha)$, a closed surface S is convex if and only if there is an embedding $\Phi : S \times \mathbb{R} \hookrightarrow M$ such that $\Phi : S \times \{0\} \to M$ is the inclusion map, and the pull-back contact structure is vertically invariant on $S \times \mathbb{R}$.

Proof. Suppose Y is a transverse contact vector field defined near S, $\mathcal{L}_Y\alpha = g\alpha$ for a smooth function g on M. Denote the flow of Y by ϕ_t. Note we can extend Y by its Hamiltonian function $h = -\alpha(Y)$, compactly supported inside a small neighborhood. So the flow ϕ_t is well defined for all $t \in \mathbb{R}$. We try to multiply the contact form α by $\lambda : M \to \mathbb{R}^+$. Note $\mathcal{L}_Y(\lambda\alpha) = (d\lambda(Y) + \lambda g)\alpha$. Solve $d\lambda(Y) + \lambda g = 0$ by $\lambda(\phi_t(p)) = \exp(-\int_0^t g(\phi_s(p))ds)$ near the surface S. Therefore, $\mathcal{L}_Y(\lambda\alpha) = 0$ and $\Phi(p,t) \triangleq \phi_t(p)$ is the required embedding. □

In the previous theorem, S doesn't need to be closed or oriented. In the former case we apply this argument to any open and relative compact subset of S. In the remaining sections, convex surfaces will usually be closed and oriented.

4.1 Giroux's criterion

Let S be a closed oriented convex surface in $(M, \xi = \ker\alpha)$, and Y be the corresponding contact vector field. Define the set

$$\Gamma_S \triangleq \{p \in S \mid Y(p) \in \xi_p\} = \{\alpha(Y) = 0\}.$$

Consider the local model $(N_v(S), \alpha) \cong (S \times \mathbb{R}, \beta + udz)$ for a small neighborhood $N_v(S)$ of S. We have $\Gamma_S = \{p \in S \mid u = 0\}$. The contact condition $ud\beta + \beta \wedge du > 0$ guarantees that $du \neq 0$ on Γ_S. Hence Γ_S is exactly a non-empty union of disjoint circles on a closed surface S transverse to S_ξ. See [17].

The following definition of dividing curves characterizes the properties of Γ_S: divides the characteristic foliation S_ξ.

Definition 4.3. Let $\mathcal{F}$ be a singular 1-dimensional foliation on a closed oriented surface S. A collection Γ of disjoint oriented circles is said to *divide* $\mathcal{F}$ if

(i) Γ is transverse to $\mathcal{F}$; (so it doesn't meet singularities of $\mathcal{F}$.)

(ii) There exists an area form Ω on S, a vector field X generating $\mathcal{F}$ such that

$$\mathcal{L}_X\Omega \neq 0 \text{ on } S - \Gamma \quad \text{and} \quad S - \Gamma = S_- \cup S_+.$$

$$S_\pm \triangleq \{p \in S \mid \pm \operatorname{div}_\Omega X > 0\} \text{ and } X \text{ points from } S_+ \text{ to } S_- \text{ along } \Gamma. \qquad \blacksquare$$

Note that when both Γ_1 and Γ_2 divide $\mathcal{F}$, they are isotopic through dividing curves Γ_t (of $\mathcal{F}$) [12]. When S is a convex surface, $\Gamma_S = \{Y \in \xi\}$ exactly divides S_ξ. In this special case, we call Γ_S the dividing set of S.

Theorem 4.3 (Giroux, see [17])**.** Let S be a compact oriented surface in (M, ξ), closed or with Legendrian boundary. Then, S is a convex surface if and only if S_ξ has dividing curves. ■

Example 4.2. Back to the Example 4.1, the unit sphere $\mathbb{S}^2$ in $(\mathbb{R}^3, \xi_{std})$ is divided by the equator $\Gamma = \{z = 0\}$. A generating vector field can be taken as $X = (xz - y)\partial_x + (yz + x)\partial_y - (x^2 + y^2)\partial_z$ from that example. Recall that the area form is $\Omega = \iota_{\partial_r} dx \wedge dy \wedge dz$.Note

$$dr \wedge \mathcal{L}_X\Omega = 2z dx \wedge dy \wedge dz.$$

So $\operatorname{div}_\Omega X$ is positive or negative on upper or lower hemisphere respectively.

Example 4.3. Let M be the 3-manifold $\mathbb{R}^2 \times \mathbb{S}^1$ parametrized by (r, θ, ϕ). The standard contact structure on $\mathbb{R}^3$ induces a contact structure ξ on M cooriented by $\alpha = d\phi + r^2 d\theta$. Consider the torus $T = \{r = c\}$ in M, and note that leaves of T_ξ are linear in ϕ and θ.

If T is convex in (M, ξ), then a vertically neighborhood of T is given by $(T \times \mathbb{R}, \alpha = \beta + udt)$. $\beta = d\phi + c^2 d\theta$ implies that $d\beta = 0$ on T. For a generating vector field X of T_ξ, the contact condition $\beta \wedge du > 0$ shows that $du(X) < 0$. So u strictly decreases along the flow lines of X. However, it is impossible since flow lines are either periodic orbits or dense in M.

In the convex surface theory, we deal with surfaces of almost Morse-Smale type. By the definition below, surfaces of Morse-Smale type are automatically of almost Morse-Smale type.

Definition 4.4. A singular foliation $\mathcal{F}$ (or $S_\mathcal{F}$) on a closed oriented surface S is of *almost Morse-Smale type* if all singularities and closed orbits in $\mathcal{F}$ are non-degenerate, and no flow lines run from a negative saddle point to a positive saddle point. ∎

Here come two theorems relating surfaces of almost Morse-Smale type to convex surfaces by Giroux. See [12].

Theorem 4.4. Suppose that S is a closed oriented surface in a contact 3-manifold (M, ξ). If S_ξ is of almost MS type, then S is convex in (M, ξ). ∎

Theorem 4.5. Suppose that $\tilde{S}$ is a compact oriented surface in (M, ξ) with Legendrian boundary, and $\tilde{S}_\xi$ satisfies the Poincaré-Bendixson property. Then $\tilde{S}$ is convex if and only if all closed orbits are non-degenerate, and no flow lines run from a negative saddle point to a positive saddle point. ∎

Density of convex surfaces is carried from that of MS type surfaces: for a closed oriented surface S in a contact 3-manifold (M, ξ), there exists a convex surface S' isotopic and C^∞-close to S.

Theorem 4.6. Let S be a compact oriented surface in (M, ξ), with Legendrian boundary of knots λ_i. If $tb(\lambda_i) \leq 0$ for all i, then there exists a convex surface S' C^∞-close to S on the interior and C^0-close to S near the boundary. ∎

Giroux flexibility theorem states that on a convex surface S, the dividing set Γ_S has already determined the neighborhood model of S. In general cases, from Theorem 1.7, we need the whole characteristic foliation S_ξ.

Theorem 4.7 (Flexibility, [12])**.** Suppose S is a compact oriented convex surface in $(M, \xi = \ker \alpha)$, either closed or of Legendrian boundary. Let $\mathcal{F}$ be a singular 1-dimensional foliation on S divided by the dividing set Γ_S (with respect to S_ξ). Then, there exists an isotopy $\phi_t : S \to M$ such that

(1) $\phi_0 = id_S$, $\phi_t|_{\Gamma_S} = id_{\Gamma_S}$ for all t;

(2) $\phi_1(S)_\xi = \phi_1(S)_{\mathcal{F}}$; and

(3) $\phi_t(S)$ is convex with dividing set $\phi_t(\Gamma_S) = \Gamma_S$. ■

Giroux's criterion helps us to classify if vertically invariant neighborhoods of a convex surface is tight or overtwisted by the topology of its dividing set.

Theorem 4.8 (Giroux's criterion, [12])**.** Let S be a closed oriented convex surface in a contact 3-manifold (M, ξ). Let $N_v(S)$ be a vertically invariant neighborhood of S. Denote the dividing set of S_ξ by Γ_S. Then,

(1) if $S = \mathbb{S}^2$, $\quad (N_v(S), \xi)$ is tight $\iff$ Γ_S is connected;

(2) if $S \neq \mathbb{S}^2$, $\quad (N_v(S), \xi)$ is tight $\iff$ Γ_S has no contractible components.

Proof. Note that one vertically neighborhood is tight if and only if all are tight. Case (1) is obvious since $\mathbb{S}^2$ is simply connected. If Γ_S has more than two components, then a Legendrian knot lies between two distinct components and bounds an OT disk. If Γ_S is a circle, then Example 4.2 applies. By flexibility theorem and Giroux theorem, $(N_v(S), \xi)$ is contactomorphic to a neighborhood of $\mathbb{S}^2$ in $(\mathbb{R}^3, \xi_{std})$. Hence, $N_v(S)$ is tight. Now, we can suppose that $S \neq \mathbb{S}^2$.

($\Rightarrow$) Assume that γ is a component of Γ_S bounding a disk Δ on S. *Wlog*, γ is the only component bounding a disk with $\mathrm{int}(\Delta) \subset S_-$ under previous notations.

When Γ_S has another component β, we push γ in S_+ for two parallel copies γ_0 and γ_+, bounding disks Δ_0 and Δ_+ respectively.

We try to construct a singular foliation $\mathcal{F}$ on S also divided by Γ_S. Inside Δ_+, take γ_0 as a closed orbit of which surrounding flow lines flow into Δ_0 until hitting a sink inside, or flow out of Δ_+. Next on $S - \Delta_+$, $\mathcal{F}$ is defined by a vector field of positive divergence, flowing from γ_+ transversely to other components of Γ_S. By flexibility theorem, $S_{\mathcal{F}}$ can be realized in $N_v(S)$. Δ_0 is the required OT disk.

When Γ_S has no other component than γ (bounding Δ as before), we will deform S to a convex surface S' of which the dividing set $\Gamma_{S'}$ has an additional circle. It makes use a non-separating circle $\bar{\gamma}$ in S disjoint form γ. We define a singular 1-dimensional foliation $\mathcal{F}_0$ on S. $\mathcal{F}_0$ has $\bar{\gamma}$ as a repelling closed leaf inside an annulus A, flowing out from two components of ∂A to Δ.

Identify a neighborhood N of A in (M, ξ) with $N' = A' \times (-\delta, \delta)$ in $\mathbb{R}^3$, where $A' = \{1 - \epsilon \leq s \leq 1 + \epsilon\}$ on $\mathbb{R}^2$ for polar coordinates (s, θ). Particularly, we map A onto $A' = A' \times \{0\}$. By Giroux theorem, we can assume

$$(N, A, \xi) \cong (N', A', \ker(dz - (s-1)d\theta + ds))$$

in contactomorphism with $z \in (-\delta, \delta)$. Deform A' from $s \in (1 - \epsilon, 1 + \epsilon)$, $z = 0$ to $(s, z) = (s(t), z(t))$ described on below figure, and then put everything back to the annulus A on S. That is, S is deformed to a new surface S'.

Call the deformed singular foliation on S' by $\mathcal{F}'$. Note on the (perturbed) annulus A, $A_{\mathcal{F}'} = A_\xi$, while outside A, $S' = S$ and $\mathcal{F} = \mathcal{F}'$. Hence, $S'_{\mathcal{F}}$ is divided by $\Gamma_{S'}$, but the later consists of two additional components. This finishes the "only if" part.

($\Leftarrow$) Given that $S \neq \mathbb{S}^2$, let N_S be a vertically invariant neighborhood of S. We take $\tilde{S} = \mathbb{R}^2$ to be a universal cover of S and V_S to be a universal cover of N_S. Thus, V_S is a vertically invariant neighborhood of N_S. When Γ_S has no contractible components, it lifts to a union of lines and arcs, called by $\tilde{\Gamma_S}$.

Let $\phi : V_S \rightarrow N_S$ be the covering map. We claim that the contact structure ξ' induced by $\phi^*\alpha$ on V_S is tight. Pick G as a Legendrian graph on S that realizes $S^{(1)}$, the 1-skeleton, and name the lifting of G on $\tilde{S}$ by $\tilde{G}$. If there is an OT disk in V_S, we can assume that it lies over a disk D in $\tilde{S}$ as a union of regions in $\tilde{S} - \tilde{G}$. So ∂D is Legendrian, D is convex, and its dividing set Γ_D consists of arcs but not closed loops.

We then argue that the characteristic foliation $D_{\xi'}$ can be realized on a disc $D' \subset \mathbb{S}^2$ in $(\mathbb{R}^3, \xi_{std})$. (Figure 4.1) By flexibility theorem, an OT disk on D will then map to an OT disk in $(\mathbb{R}^3, \xi_{std})$, driving to a contradiction. Therefore, V_S is tight , so is N_S. □

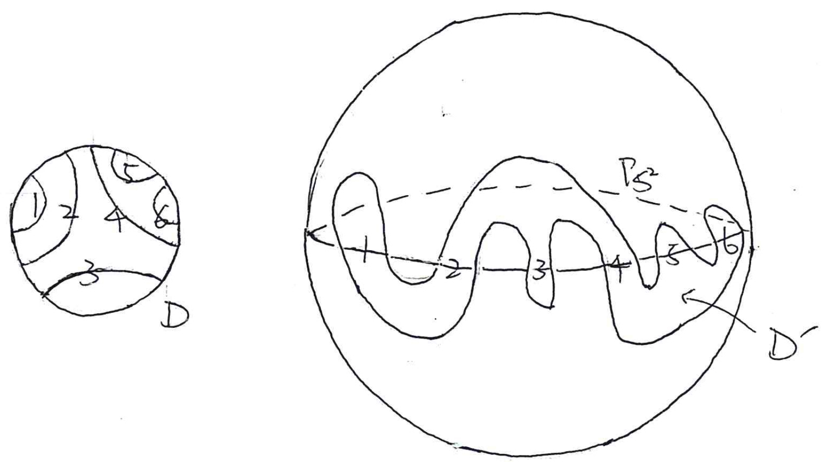

Figure 4.1: Realize $D_{\xi'}$ on $\mathbb{S}^2_{\xi_{std}}$ ([12])

It worths to mention how the genus bound inequality can be obtained through convex surface theory. Recall Theorem 3.14 states that for a closed oriented surface S on a tight contact 3-manifold (M, ξ),

$$| < e(\xi), [S] > | \leq -\chi(S)$$

when $S \neq \mathbb{S}^2$, and the LHS equals 0 when $S = \mathbb{S}^2$. Roughly speaking, we can assume that S is convex with its dividing set Γ_S. Since M is tight, Γ_S doesn't bound any disks on S and so both $\chi(S_+)$ and $\chi(S_-)$ are non-positive. Each component of S_+ or S_- is homeomorphic to a 2-sphere with at least two disks removed. In this way, $\chi(S) = \chi(S_+) + \chi(S_-)$and $< e(\xi), [S] >= \chi(S_+) - \chi(S_-)$, where the later is given by counting singularities. Therefore,

$$< e(\xi), [S] > \pm\chi(S) = \pm 2\chi(S_\pm) \geq 0.$$

In addition to the genus bound inequality, Giroux criterion helps to generalize Legendrian and transverse Bennequin inequality for tight contact 3-manifolds.

Theorem 4.9. Let (M, ξ) be a tight contact 3-manifold. For a Legendrian knot λ bounding a Seifert surface S, we have $tb(\lambda) + |rot(\lambda)| \leq -\chi(S)$. For a transverse knot γ bounding a Seifert surface S, we have $sl(\gamma) \leq -\chi(S)$. ■

4.2 Classification results

Here I shall state classification results of tight contact structures on different manifolds. That on $\mathbb{S}^3$, $\mathbb{R}^3$ and $\mathbb{S}^2 \times \mathbb{S}^1$ can be obtained by the use of *tomography.* For example, when $M = \mathbb{S}^2 \times [-1, 1]$, we study how characteristic foliations of $\mathbb{S}^2 \times \{t\}$ behave and vary. See [17].

Lemma. A tight contact structure on $\mathbb{S}^2 \times [-1, 1]$ is determined by characteristic foliations on boundary, $\mathbb{S}^2 \times \{\pm 1\}$, up to isotopy rel boundary. ■

Theorem 4.10. Each of $\mathbb{S}^3$, $\mathbb{R}^3$ and $\mathbb{S}^2 \times \mathbb{S}^1$ admits a unique tight contact structure up to isotopy. Tight contact structures on $\mathbb{D}^3$ inducting same characteristic foliation on $\partial\mathbb{D}^3 = \mathbb{S}^2$ are isotopic rel boundary. ■

Instead of using tomography, some classification results of tight contact structures are obtained by studying bypasses and twisting. We first start with the

3-manifold $T^3 = \mathbb{R}^3/\mathbb{Z}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. On T^3, we have the following tight contact structures in Cartesian coordinates on $\mathbb{R}^3$,

$$\xi_n = \ker(\cos(2\pi nz)dx + \sin(2\pi nz)dy)$$

Theorem 4.11 (Kanda, Giroux)**.** The contact structures ξ_n, for integers $n > 0$, are distinct. ■

Theorem 4.12. Any tight contact structure ξ on T^3 is contactomorphic to ξ_n for exactly one $n > 0$. ■

Next statement is about lens spaces. We assume that p and q are relative prime and $p > q > 0$. In this case the lens space $L(p, q) = \mathbb{S}^3/\mathbb{Z}_q$ obtained by identifying $(z, w) \sim (ze^{\frac{2\pi ir}{q}}, we^{\frac{2\pi ipr}{q}})$ in terms of coordinates on $\mathbb{C}^2$ for $r \in \mathbb{Z}_q$. For simplicity, we write $-\frac{p}{q}$ as the continued fraction

$$-\frac{p}{q} = r_0 - \frac{1}{r_1 - \frac{1}{r_2 - \cdots \frac{1}{r_k}}}$$

with all $r_i < -1$

Theorem 4.13. Under this convention, there exists exactly

$$|(r_0 + 1)(r_1 + 1) \cdots (r_k + 1)|$$

tight contact structures on the lens space $L(p, q)$. All these tight contact structures are holomorphically fillable. ■

Story on solid torus $\mathbb{S}^1 \times \mathbb{D}^2$ is also summarized. Similarly, r_0, ..., r_k are the coefficients of the continued fraction expansion of $-\frac{p}{q}$.

Theorem 4.14. Let Γ be a multi-curve on the boundary torus T of $\mathbb{S}^1 \times \mathbb{D}^2$, having exactly two components with slope $-\frac{p}{q}$, $p \geq q > 0$ and $(p, q) = 1$. Fix a characteristic foliation $\mathcal{F}$ adapted to Γ. Then, there exists exactly

$$|(r_0 + 1)(r_1 + 1) \cdots (r_{k-1} + 1)(r_k)|$$

tight contact structures on $\mathbb{S}^1 \times \mathbb{D}^2$ under a prescribed convex boundary $T_\xi = \mathcal{F}$, up to isotopy fixing T. ■

Recall the Poincaré sphere obtained by Dehn twist in Example 2.2. Sometimes it is expressed in terms of a Brieskorn 3-sphere, $\Sigma(2,3,5)$. Let

$$S(p,q,r) \triangleq \{(x,y,z) \in \mathbb{C}^3 \mid x^2 + y^3 + z^5 = 0\}.$$

Then $\Sigma(2,3,5) = S(2,3,5) \cap \mathbb{S}^5$. If any one of p, q or r is 1, then $\Sigma(p,q,r)$ is homeomorphic to $\mathbb{S}^3$.

Some closed 3-manifolds contain no tight contact structures. One example is given by Etnyre and Honda in [14].

Theorem 4.15. There exist no positive tight contact structures on $\Sigma(2,3,5)$ with reverse orientation. ■

Chapter 5

Holomorphic Filling

The method of filling by holomorphic disks first appears in [9] and [20]. It is a powerful technique to show that a contact 3-manifold (M, ξ) is tight. It means that when $M = \partial W$ for a 4-dimensional manifold W, we can fill a 2-sphere Σ of two elliptic points in Σ_ξ by boundary of holomorphic disks under certain conditions or after perturbation. In our discussion, we require that (M, ξ) is the J-convex boundary of a 4-dimensional almost complex manifold (W, J).

Usually, We say that a compact complex manifold W is a holomorphic filling of (M, ξ) if $M = \partial W$ is the J-convex boundary. Here, I refer 'holomorphic filling' to almost complex manifold in general, since my attention lies on 4-dimension manifolds.

One remarkable application of holomorphic filling is to show *Cerf's theorem*, $\Gamma_4 = 0$. It means that every diffeomorphism of $\mathbb{S}^3$ extends to a diffeomorphism of the unit 4-ball. For its proof, see [17] and [19].

Let (W, J) be a 4-dimensional complex manifold and (S, j) be a compact Riemann surface. A smooth map $u : S \to W$ is a J-holomorphic curve if its differential du is complex linear with respect to j and J: $J \circ du = du \circ j$. (In this chapter, I shall denote '(j, J)-holomorphic' by 'J-holomorphic'.) Here we will deal with the unit disk $\mathbb{D}$ on $\mathbb{C}$ under the standard complex structure. So the Cauchy-Riemann equation becomes:

$$\bar{\partial}_J u = \frac{1}{2}(\partial_s u + J(u)\partial_t u) = 0$$

for $\theta = s + it \in \mathbb{C}$.

We usually regard (W, J) as a symplectic manifold with the symplectic form ω. The almost complex structure J is not necessarily compatible with ω. We just assume that J is *ω-tamed*: $\omega(X, JX) > 0$ for all nonzero $X \in TW$. Over this ω-tamed almost complex manifold (W, J, ω), we can define a Hermitian metric

$$g_J(X, Y) = \frac{1}{2}(\omega(X, JY) + \omega(Y, JX)).$$

5.1 J-convexity

Suppose now that a closed oriented 3-manifold M lies inside the tamed triple (W, J, ω). First of all, we regard M as a *hypersurface* in W. A hypersurface of W is a 3-dimensional submanifold of W being the preimage of a regular value of a smooth function $f : W \to \mathbb{R}$, i.e., $M = \{f = 0\}$ and $df|_M \neq 0$. Recall the Hamiltonion vector field H_f to the function f is defined by $\iota_{H_f}\omega = df$, so $H_f \in TM$. For convenience, we require $df(N) > 0$ for a positive normal to M in TW.

On every tangent plane T_xM, $x \in M$, there is a unique complex line $\xi_x \subset T_xM$ satisfying $J(\xi_x) = \xi_x$. The complex tangency ξ can be characterized by the 1-form $\alpha = -J^*df$. Since ξ is oriented by J, a positive normal to ξ in TM is given by JN. Hence $\alpha(JN) > 0$. We say that a hypersurface M is *J-convex* if

$$d\alpha(T, JT) = -d(J^*df)(T, JT) > 0$$

for all nonzero $T \in \xi$. J^*df stands for $df \circ J$. In this way, $(M, \xi = \ker\alpha)$ is a positive contact 3-manifold.

'J-convex' means that M cannot be touched inside by a J-holomorphic curve. The idea of J-convexity originates from strictly pseudoconvexity on $\mathbb{C}^n$. Denote the standard complex structure on $\mathbb{C}^2$ by J_0. Consider a domain $D = \{\rho < 0\} \subset \mathbb{C}^2$ and a smooth function $\rho : \mathbb{C}^2 \to \mathbb{R}$ such that $d\rho|_{\partial D} \neq 0$. The differential form $\alpha = -J_0^* d\rho$ can be expressed in local coordinates $(x+iy, s+it) \in \mathbb{C}^2$.

$$\alpha = -i(\partial\rho - \bar{\partial}\rho) = (\partial_x \rho dy - \partial_y \rho dx) + (\partial_s \rho dt - \partial_t \rho ds)$$

A relevant concept to J-convexity is a hypersurface M of *contact type* inside a symplectic manifold (P^4, ω). A hypersurface (M^3, f) in a symplectic 4-manifold (P, ω) is of contact type if there exists a 1-form $\alpha \in \Omega^1(M)$ such that $d\alpha = \omega|_{TM}$ and $\alpha(H_f) \neq 0$.

Another definition of hypersurface of contact type is related to Liouville vector field: $Y \in TP$ such that $\mathcal{L}_Y \omega = \omega$.

Theorem 5.1 (Weinstein, see [1])**.** A hypersurface M in (P, ω) is of contact type if and only if there exists a Liouville vector field Y defined near M and transverse to M. ■

In particular, given such a Liouville vector field Y, the 1-form α is recovered by $\alpha = \iota_Y \omega$.

Theorem 5.2. [1] If M is a hypersurface of contact type in a symplectic 4-manifold (P, ω), then there exists an almost complex structure J on P tamed by ω such that $M \subset (P, J, \omega)$ is J-convex. ■

Combining J-convexity with tameness, we have the following definition.

Definition 5.1. A closed contact 3-manifold (M, ξ) is *holomorphically fillable* if it is the J-convex boundary (defined by a function f) of a compact symplectic 4-manifold (W, J, ω) such that the almost complex structure J is tamed by ω. ■

Briefly speaking, holomorphic filling refers to filling a surface S in W by J-holomorphic disks: $u_t : (\mathbb{D}, \partial\mathbb{D}) \to (W, S)$. The most important result about holomorphic filling is that all holomorphically fillable contact structures are tight. For example, $(\mathbb{S}^3, \alpha_H)$ is the J-convex boundary (let say by $f = r^2$) of the 4-disk $(\mathbb{D}^4, J_0, \omega_{std} = dx \wedge dy + ds \wedge dt)$. Hence the Hopf contact structure on $\mathbb{S}^3$ is tight.

5.2 Filling near elliptic points

Consider an oriented surface Σ embedded in a 4-dimensional almost complex manifold (W, J). Any point $p \in \Sigma$ with $T_p\Sigma = J(T_p\Sigma)$ is called a complex point. In generic cases, complex points are isolated. We say that a surface $\tilde{\Sigma}$ is completely real if $\tilde{\Sigma}$ contains no complex points, i.e., $T\tilde{\Sigma} \oplus J(T\tilde{\Sigma}) = TW$. From [5] by E. Bishop , we can generically describe a local neighborhood of the complex point $p \in S$ by holomorphic coordinates $(w = u + iv, z = x + iy) \in \mathbb{C}^2$.

Identify p with $(0, 0) \in \mathbb{C}^2$ and $T_p\Sigma$ with $< \partial_u, \partial_v >$ at $(0, 0)$. Σ is locally the graph of a function $z = z(w)$ with $z(0) = 0$ and $z'(0) = 0$. We can write

$$z = \alpha w^2 + \beta \bar{w}^2 + \gamma w \bar{w} + O(w^3).$$

Assume $\gamma \neq 0$. One can simplify it to a normal form

$$z = w\bar{w} + 2\beta_0 Re(w^2) + O(w^3) = (u^2 + v^2) + 2\beta_0(u^2 - v^2) + O(w^3)$$

with $\beta_0 \geq 0$. We call the complex point p elliptic if $\beta_0 \in [0, \frac{1}{2})$, hyperbolic if $\beta_0 \in (\frac{1}{2}, \infty)$, and parabolic if $\beta_0 = \frac{1}{2}$. Generically, parabolic case can be avoided. See [1] and [5].

Let us assume that Σ is a closed oriented surface in $\mathbb{C}^2$. In this case, the Gauss map associating the tangent plane T_qS at $q \in \Sigma$ to an element of the Grassmannian manifold $G_{2,2}$, consisting of oriented 2-dimensional (in $\mathbb{R}$) subspaces of $\mathbb{C}^2$, through the Plücker coordinates a_{ij} on $G_{2,2}$.

Since every element $P \in G_{2,2}$ is determined by an ordered orthonormal basis $< v_1, v_2 >$, we set $v_1 \wedge v_2 = \sum\limits_{i<j} a_{ij} e_i \wedge e_j$ for $e_1 = \partial_u$, $e_2 = \partial_v$, $e_3 = \partial_x$ and $e_4 = \partial_y$. Since $|v_1 \wedge v_2|^2 = 1$ and $(v_1 \wedge v_2) \wedge (v_1 \wedge v_2) = 0$, we have $\sum_{i<j} a_{ij}^2 = 1$ and

$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$. These properties help identify $G_{2,2}$ with $S_1 \times S_2$, a product of two 2-spheres, under a diffeomorphism $g : G_{2,2} \to S_1 \times S_2$. For $(\vec{s}, \vec{t}) = g(v_1 \wedge v_2)$ (in Cartesian coordinates on $\mathbb{R}^3 \times \mathbb{R}^3$), we have

$$\vec{s} = (a_{12} + a_{34}, a_{23} + a_{14}, -a_{13} + a_{24});$$

$$\vec{t} = (a_{12} - a_{34}, a_{23} - a_{14}, -a_{13} - a_{24}).$$

Particularly, for a complex point $p \in \Sigma$ where orientation of $T_p\Sigma$ agrees with the complex structure, regarding $T_p\Sigma \subset \mathbb{C}^2$, we have $\vec{s}(T_p\Sigma) = (1, 0, 0)$. In general, complex lines in $\mathbb{C}^2$ are mapped onto the subset $S_2^+ \sqcup S_2^-$, where
$S_2^{\pm} = \{(\pm 1, 0, 0)\} \times S_2$.

Filling holomorphic disks near an elliptic complex point p is guaranteed by Bishop's theorem. The original proof in [5] requires that the almost complex structure J is integrable near p. It uses the normal form described above as a start. Here, however, I shall just state another version of Bishop's theorem by R. Ye [40].

Theorem 5.3 (Bishop,Ye)**.** Let (W, J) be a 4-dimensional almost complex manifold and Σ an embedded surface in W. Suppose that $p \in \text{int}(\Sigma)$ is a complex point. Then, there is a unique smooth 1-parameter family of mutually disjoint, embedded J-holomorphic disks $(u_t) : (\mathbb{D}, \partial\mathbb{D}) \to (W, \Sigma)$ for $0 < t < 1$ and $\mathbb{D}$ being the unit disk in $\mathbb{C}$. Moreover, $u_t(\partial\mathbb{D}) \subset \Sigma - \{p\}$, its union fill a neighborhood of p on S and $\lim_{t \to 0} u_t = p$. ■

5.2.1 Maximum principle

We will focus on the case that Σ is diffeomorphic to $\mathbb{S}^2$ with only two elliptic points $p_{\pm}$, and it lies on the J-convex boundary M of a tamed almost complex 4-manifold (W, J, ω). Under this assumption, those properties displayed on the Bishop's theorem hold whenever (u_t) is a family of holomorphic disks filling some neighborhood of a complex point. Our goal is to fill this completely real surface

$\tilde{\Sigma} = \Sigma - \{p_\pm\}$ by a unique family of mutually disjoint, embedded J-holomorphic disks.

In the following, we always stick with the above condition that

$$\mathbb{S}^2 = \Sigma \subset (M = \partial W, f, \xi) \subset (W, J, \omega).$$

I shall discuss how one guarantees that such a family of holomorphic disks (u_t) filling near the complex point p is unique, mutually disjoint and embedded in this section.

For any holomorphic disk $u : (\mathbb{D}, \partial\mathbb{D}) \to (W, \tilde{\Sigma})$, if $u|_{\partial D}$ is an embedding, then u is simple. 'Simple' means that no two disjoint non-empty open sets U and V in $\mathbb{D}$ with $u(U) = u(V)$. It can be derived from the maximum principle below. See [19].

Theorem 5.4 (Maximum principle)**.** Consider a completely real surface $\tilde{\Sigma} \subset (Mf, \xi = \ker\alpha) \subset (W, J, \omega)$ as stated above and $\tilde{\Sigma}_\xi$ is non-singular. Let $u : (\mathbb{D}, \partial\mathbb{D}) \to (W, \tilde{\Sigma})$ be a non-constant holomorphic disk. Then u maps $\mathrm{int}(\mathbb{D})$ to the $\mathrm{int}(W)$ and $u|_{\partial D}$ is an immersion transverse to the characteristic foliation $\tilde{\Sigma}_\xi$.

Proof. Recall that the contact form is defined by $\alpha = -J^*df$. Compose f with a smooth function $g : \mathbb{R} \to \mathbb{R}$, $g'' \gg g' > 0$, so that $d\alpha(X, JX) > 0$ for all nonzero $X \in TW$. $(\alpha = -J^*d(g \circ f))$ Now we can consider the map $h = f \circ u : (\mathbb{D}, \partial\mathbb{D}) \to \mathbb{R}$. Denote the standard complex structure by j. Note that $d(j^*d\rho) = -\Delta\rho d\mathbf{x}$ for any $\rho \in C^\infty$. Therefore, $d(j^*dh) = -u^*d\alpha$ and thus

$$\Delta h = (u^*d\alpha)(\partial_x, \partial_y) = d\alpha(du(\partial_x), du(\partial_y)) > 0,$$

whenever $du \neq 0$. Back to the issue, f attains maximum at points on $\bar{S}$, so the maximum principle guarantees that u map $\mathrm{int}\mathbb{D}$ to $\mathrm{int}(W)$. Also, if $du_z(T\mathbb{D}) \subset \xi_u(z)$ at some point $z \in \partial\mathbb{D}$, then $dh_z = 0$, contradicting to the Hopf maximum principle. $\square$

5.2.2 Positivity of intersections

We now investigate how two distinct holomorphic disks $u_1, u_2 : (\mathbb{D}, \partial\mathbb{D}) \to (W, \tilde{\Sigma})$ intersect under this background. 'Distinct' means that $u_1(\mathbb{D}) \neq u_2(\mathbb{D})$. Using terminology in [19], let $S = S(u_1, u_2)$ be the set of intersection points between u_1 and u_2, i.e.,

$$S = \{(z_1, z_2) \in \mathbb{D} \times \mathbb{D} \mid u_1(z_1) = u_2(z_2)\}.$$

We can divide S to two subsets by $S = S_{int} \sqcup S_\partial$ where $S_{int} = S \cap (\mathrm{int}\mathbb{D} \times \mathrm{int}\mathbb{D})$ and $S_\partial = S \cap (\partial\mathbb{D} \times \partial\mathbb{D})$. In part (I), we know that any holomorphic disk u simply intersects itself. This time it turns out that any two distinct holomorphic disks u_1 and u_2 nicely intersect: $|S(u_1, u_2)| < +\infty$. Obviously it implies that $u_1(U) \neq u_2(V)$ for any open sets U, V in $\mathbb{D}$.

Under the fact that $|S(u_1, u_2)| < +\infty$, we can talk about the topological intersection number at a pair of intersection points $(z_1, z_2) \in S$, denoted by $\iota(z_1, z_2)$. The intersection number of a holomorphic pair (u_1, u_2) is then defined by

$$u_1 \cdot u_2 \triangleq 2 \sum_{(z_1,z_2) \in S_{int}} \iota(z_1, z_2) + \sum_{(z_1,z_2) \in S_\partial} \iota(z_1, z_2).$$

This topological intersection can also be understood through a nice local model given by M. J. Micallef and B. White. See [29] and [30].

Theorem 5.5 (Micallef-White)**.** Let u_1 and u_2 be two distinct holomorphic disks described in the above setting. Suppose $(z_1, z_2) \in S_{int} = S_{int}(u_1, u_2)$. Then we can find coordinates chart $\phi_i : (U_i, z_i) \to (\mathbb{C}, 0)$ at $z_i \in \mathbb{D}$ for $i = 1, 2$ and $\Phi : V \to \mathbb{C}^2$ for a neighborhood $V \subset W$ of $u_1(z_1) = u_2(z_2)$ such that there exist

$$\tilde{u_1} = \Phi \circ u_1 \circ \phi_1^{-1} : (\mathbb{C}, 0) \to (\mathbb{C}^2, \mathbf{0});$$

$$\tilde{u_2} = \Phi \circ u_2 \circ \phi_2^{-1} : (\mathbb{C}, 0) \to (\mathbb{C}^2, \mathbf{0})$$

given by $\tilde{u_1}(z) = (z^{k_1}, f_1(z))$ and $\tilde{u_2}(z) = (z^{k_2}, f_2(z))$ for integers k_i's and holomorphic functions f_i's with $ord_0 f_i \geq k_i$. Moreover, intersection number at (z_1, z_2)

is given by

$$\iota(z_1, z_2) = \frac{1}{k_1 k_2} \sum_{\lambda^k=1} ord_0(f_1(\lambda^{k_2} z^{k_2}) - f_2(z^{k_1})).$$

$\iota(z_1, z_2) \geq 1$ with equality holds if and only if $u_1 \pitchfork u_2$ at $u_1(z_1) = u_2(z_2)$. ■

To illustrate how $\iota(z_1, z_2)$ is defined, let us take a ball $B_\epsilon(0) \subset \mathbb{D}$ so that $\tilde{u}_1$ and $\tilde{u}_2$ intersect only at 0 over this ball. Perturb $\tilde{u}_1$ to the function

$$\tilde{u}_1^t(z) = \tilde{u}_1(z) + t\psi(|z|).$$

$\psi : \mathbb{R} \to \mathbb{R}$ is smooth and $B_\eta(0) \prec \psi \prec B_\epsilon(0)$ for $0 < \eta < \epsilon$. Let S_{int}^t be the set of intersection points between $\tilde{u}_1$ and $\tilde{u}_2$ over $B_\epsilon(0)$. Then, when $k_1 = k_2 = k$,

$$(\zeta, \tau) \in S_{int}^t \iff \zeta^k = \tau^k \quad \text{and} \quad f_1(\zeta) - t\psi = f_2(\tau).$$

Further restricted our attention to $B_\eta(0)$ by smaller range of t, the previous condition is equivalent to $\zeta = \lambda\tau$ for a k^{th}-root of unity λ and $f_1(\lambda\tau) - f_2(\tau) = t$. When t is generic, the later equation will have simple zeros near 0 as many as $ord_0(f_1(\lambda z) - f_2(z))$. As a result,

$$\iota(z_1, z_2) = \sum_{\lambda^k=1} ord_0(f_1(\lambda z) - f_2(z))$$

in this special case. The general case follows similarly by concerning $h_1 = \tilde{u}_1(z^{k_2})$ and $h_2 = \tilde{u}_2(z^{k_1})$.

Micallef-White theorem is indeed an important ingredient to show that $|S_{int}| < \infty$. For simplicity, let two holomorphic functions $u_1, u_2 : \mathbb{C} \to \mathbb{C}^2$ be given by $u_i(z) = (z^{k_i}, f_i(z))$ and have an isolated intersection at 0. If a non-trivial sequence (z_n, w_n) tends to $(0, 0)$ and $u_1(z_n) = u_2(w_n)$ for every n, then we can say that $u_1(z^{k_2}) \equiv u_2(\lambda^{k_1} z^{k_1})$ for some $(k_1 k_2)^{th}$-root of unity λ. Hence $u_1(U) = u_2(V)$ for some neighborhood U, V of 0, since holomorphic maps are open maps.

Intersection at boundary can be settled through the Schwartz reflection principle. For $(z_1, z_2) \in S_\partial = S_\partial(u_1, u_2)$, we can identify z_1 and z_2 with $0 \in \mathbb{H}$ and then write down $u_1, u_2 : (\mathbb{H}, \mathbb{R}, 0) \to (\mathbb{H} \times \mathbb{C}, \mathbb{R} \times \mathbb{R}, \mathbf{0})$ near 0 by $u_1(z) = (z, 0)$ and $u_2(z) = (z, h(z))$. Note $u_1|_{\mathbb{R}}$, $u_2|_{\mathbb{R}} \in \mathbb{R} \times \mathbb{R}$, we can extend h by $h(z) = \overline{h(\overline{z})}$ when $Im(z) < 0$. After this extension, we obtain $u_1 \cdot u_2$ by the same procedures over interior intersection. This method apples to both transverse intersections and tangential intersections on boundary.

A detailed proof for $|S_\partial| < \infty$ can be found in [19]. From Theorem 5.5, $\iota(z_1, z_2) \geq 1$ if $(z_1, z_2) \in S_{int}$. From the arguments in [30], if $(z_1, z_2) \in S_\partial$, $\iota(z_1, z_2) = 1$ for a transverse intersection, and ≥ 2 for a tangential one. Therefore we obtain positivity of intersections.

Theorem 5.6 (Positivity of intersections)**.** For two distinct holomorphic disks $u_1, u_2 : (\mathbb{D}, \partial\mathbb{D}) \to (W, \bar{S})$,

$$u_1 \cdot u_2 \geq 2|S_{int}(u_1, u_2)| + |S_\partial(u_1, u_2)|.$$

Equality holds if and only if all intersections are transverse. Particularly, $u_1(\mathbb{D})$ and $u_2(\mathbb{D})$ are disjoint if and only if $u_1 \cdot u_2 = 0$. ■

The following theorem from R. Ye [40] enhances the power of this intersection number. By Theorem 5.6 and 5.7, if two holomorphic disks, $[u_1], [u_2] \in \pi_2(W, \Sigma)$, have disjoint boundaries, they must be disjoint disks. It is because $u_1 \cdot u_2$ equals the linking number of $u_1(\partial\mathbb{D})$ and $u_2(\partial\mathbb{D})$ in M. Both of them lie on $\Sigma = \mathbb{S}^2$, so this linking number must be zero.

Theorem 5.7. The intersection number $u_1 \cdot u_2$ depends only on the homotopy classes $[u_1], [u_2] \in \pi_2(W, \Sigma)$. ■

5.2.3 Embedding

We have known that any holomorphic disk u we concern is simple. It guarantees that its set of self-intersection points

$$S'(u) = \{(w_1, w_2) \in \mathbb{D} \times \mathbb{D} \mid u(w_1) = u(w_2)\ ,\ w_1 \neq w_2\}$$

is a finite set. Decompose $S'(u) = S'_{int}(u) \sqcup S'_{\partial}(u)$ similarly. We can speak of the topological intersection number at a self-intersection pair $(w_1, w_2) \in S'(u)$, in the same manner as Section 5.2.2. Theorem 5.7 also provides us a well-defined version of the self-intersection number of u, $u \cdot u$. Unlike Section 5.2.2, the set of critical points of u, $Crit(u)$, is also considered. As $u|_{\partial\mathbb{D}}$ is an immersion, $Crit(u)$ lies inside $\text{int}(\mathbb{D})$. From Chapter 2 of [29], $Crit(u)$ is a discrete set and hence finite.

The embedding deficit $D : \pi_2(W, \Sigma) \to \mathbb{Z}$ is defined by

$$D(A) = A \cdot A - \mu(A) + 2$$

for $A \in \pi_2(W, \Sigma)$. For a holomorphic disk u, $\mu([u])$ stands for the Maslov index $\mu(u^*TW, u^*T\Sigma)$, where $(u^*TW, u^*T\Sigma)$ is a bundle pair over $(\mathbb{D}, \partial\mathbb{D})$. Since $\mathbb{D}$ is contractible, we take a global trivialization

$$\begin{array}{ccc} u^*TW & \xrightarrow{\Phi} & \mathbb{D} \times \mathbb{C}^2 \\ \downarrow & & \downarrow \\ \mathbb{D} & = & \mathbb{D} \end{array}$$

Denote the fiber of $\Phi(u^*TS)$ at $\theta \in \partial\mathbb{D} = \mathbb{S}^1$ by F_θ, so F_θ is a completely real subspace of $T_\theta\mathbb{C}^2$. Hence we get $\mu(u^*TW, u^*T\Sigma) = \mu(\mathbb{D} \times \mathbb{C}^2, F)$, where the later is defined by V. I. Arnold [2, 29].

Let $R(n) = GL(n, \mathbb{C})/GL(n, \mathbb{R})$ be the manifold of completely real subspaces of $\mathbb{C}^n$. $R(n)$ is retracted to its submanifold $L(n) = U(n)/O(n)$ of Lagrangian subspaces in $\mathbb{C}^n$, i.e. completely real subspaces of real dimension n. Define a map $\det^2 : L(n) \to \mathbb{S}^1$ by $\det^2(a \cdot O(n)) = \det^2(a)$ for $a \in U(n)$. Note that the function $\det^2$ induces an isomorphism on fundamental groups, so $\pi_1(L(n)) = \mathbb{Z}$.

Given a closed curve $\Lambda : \mathbb{S}^1 \to L(n)$, consider the composition

$$\mathbb{S}^1 \xrightarrow{\Lambda} L(n) \xrightarrow{\det^2} \mathbb{S}^1.$$

The Maslov index of Λ is defined by $\mu(\Lambda) = \deg(\det^2 \circ \Lambda)$. For any two loops $\Lambda_1, \Lambda_2 : \mathbb{S}^1 \to L(n)$, Λ_1 is homotopic to Λ_2 if and only if $\mu(\Lambda_1) = \mu(\Lambda_2)$ [2].

This definition of Maslov index immediately carries to the case that a bundle pair $(\mathbb{C}^2, F')$ defined over $(\mathbb{D}, \partial\mathbb{D})$ whenever F' is a completely real subbundle in $\mathbb{C}^2$ along $\partial\mathbb{D}$. In our case, $\mu(u^*TW, u^*T\Sigma) = \mu(\mathbb{D} \times \mathbb{C}^2, F) = \mu(\Lambda)$,where $\Lambda : \mathbb{S}^1 \to L(2)$ is defined by $\Lambda(\theta) = F_\theta \subset T_\theta\mathbb{C}^2 = \mathbb{C}^2$.

As a remark, if we handle the oriented Lagrangian subspaces in $\mathbb{C}^n$, called by $L^+(n)$, then the determinant map $\det : L^+(n) \to \mathbb{S}^1$ induces isomorphism on fundamental groups instead of $\det^2$. The resulting Maslov index will be different by a multiple of 2 accordingly. The method of Gauss map to Grassmannian $G_{2,2} = S_1 \times S_2$ mentioned before also helps us to determine the degree of $\det \circ \Lambda$.

The relative adjunction inequality [19] provides a lower bound for the embedding deficit $D([u])$ in terms of $S'(u)$ and $Crit(u)$.

Theorem 5.8 (Relative adjunction inequality)**.** Let $u : (\mathbb{D}, \partial\mathbb{D}) \to (W, \tilde{\Sigma})$ be a holomorphic disk under the prior condition. We have

$$D([u]) \geq 2|S'_{int}(u)| + |S'_\partial(u)| + 4|Crit(u)|.$$

When $Crit(u) = \emptyset$, equality holds if and only if all self-intersections are transverse. In particular, u is an embedding if and only if $D([u]) = 0$. ■

In summary, given a family of holomorphic disks $u_t : (\mathbb{D}, \partial\mathbb{D}) \to (W, \tilde{\Sigma})$ near an elliptic point $p \in (W, J)$ as stated in Theorem 5.3, if u_t's have mutually disjoint boundaries, then they are mutually disjoint as well as $u_t \cdot u_t = 0$ for all t. Therefore $D([u_t]) = 0$ if and only if $\mu([u_t]) = 2$. In this case, by the relative adjunction inequality, all holomorphic disks u_t are embedding. Uniqueness of such a family u_t filling around p follows as well. Given a holomorphic disk $v : (\mathbb{D}, \partial\mathbb{D}) \to (W, \tilde{\Sigma})$ distinct from all u_t which intersects some $u_{t_0} \in (u_t)$ on the boundary, $v \cdot u_t > 0$ by positivity of intersections. However, as there exists a disk $u_{t_1} \in (u_t)$ disjoint from v, $v \cdot u_t = v \cdot u_{t_1} = 0$. Contradiction arrives.

5.3 Global filling

Last section we clarify how a local family of holomorphic disks $u_t : (\mathbb{D}, \partial\mathbb{D}) \to (W, \tilde{\Sigma})$ near an elliptic point $p \in \Sigma = \mathbb{S}^2$ behaves, and how goals in Bishop theorem can be achieved. This section turns to extension of holomorphic disks. While extension in a small region doesn't face much difficulties, additional condition should be added in order to create a global filling over our 2-sphere Σ, i.e.

$$\Sigma - \{p_\pm\} = \cup u_t(\partial\mathbb{D})$$

for a smooth family of mutually disjoint and embedded holomorphic disks $u_t : (\mathbb{D}, \partial\mathbb{D}) \to (W, \tilde{\Sigma})$.

The problem of extending a holomorphic disk over a small open interval of t is settled by H. Hofer in [1, 22]. He mainly targeted at the case when the symplectic 4-manifold W is defined as the *symplectisation* of a contact 3-manifold $(M, \xi = \ker\alpha)$. That is, $W = M \times \mathbb{R}$ equipped with a symplectic form $\omega = d(e^t\alpha)$. Nevertheless, his proof holds for any almost complex 4-manifold and especially for our cases. Before H. Hofer, E. Bedford and B. Gaveau proved this result when $W = \mathbb{C}^2$ under the standard complex structure [3].

Theorem 5.9 (Hofer [1, 22])**.** Let $u_0 : (\mathbb{D}, \partial\mathbb{D}) \to (W, \tilde{\Sigma})$ be an embedded holomorphic disk with Maslov index $\mu(u_0) = 2$. Then there exists a smooth embedding $\Phi : (-\epsilon, \epsilon) \times \mathbb{D} \hookrightarrow W$ such that $\Phi(0, \cdot) = u_0$, $u_\tau = \Phi(\tau, \cdot) \in H^{1,p}(\mathbb{D}, W)$ whenever $p > 2$, $u_\tau(\partial\mathbb{D}) \subset \tilde{\Sigma}$, and $\bar{\partial}_J u_\tau = 0$ on $\mathbb{D}$ for all $\tau \in (-\epsilon, \epsilon)$. ■

In [39], the outcome that u_τ are distinct embedding results from differentiation theory. Under argument in previous section, we just need to show that u_τ's are distinct and $u_\tau(\partial\mathbb{D}) \subset \tilde{\Sigma}$ for small τ near 0 so that we will not obtain one-side or two-side trivial extension.

Any global filling of holomorphic disks on Σ is unique, so it remains to question if this extension of holomorphic disks is compact, in other words, if the limit of a sequence of holomorphic disks is also a holomorphic disk. This kind of difficulties can be overcome by Gromov's compactness theorem [20]. Beforehand, I shall introduce the notion of cusp-curves, or singular Riemann surfaces.

5.3.1 Compactness

Let us consider first a smooth map from a compact Riemann surface to an almost complex manifold, $f : (S, j, \nu) \to (W, J, \mu)$, with compatible metrics ν and μ on both side. The area of f is defined by $area(f) \triangleq \int_S \sigma_{f^*\mu}$, where $\sigma_{f^*\mu}$ denotes the top form on S with respect to $f^*\mu$. In case $S = \mathbb{D}$, we have $\sigma_{f^*\mu} = \sqrt{\det(\mu \circ df)}dxdy$ in local coordinates. When f is J-holomorphic, we also have

$$area(f) \triangleq \int_S \sigma_{f^*\mu} = \int_S ||df||^2 \sigma_\nu \triangleq E(f).$$

It is because $f^*\mu = ||df||^2\nu$. Note $||df||^2 = \frac{1}{2}tr((df)^*(df))$. If ω_J is a compatible symplectic form on W, then in general

$$area(f) \geq \int_S f^*\omega_J,$$

and equality holds if and only if f is J-holomorphic.

Let (S_k, j_k)'s, $k = 1, ..., N$, be disjoint compact Riemann surfaces. Write $\hat{S} = \sqcup S_k$. A Riemann surface with nodes, or singular Riemann surface, $\bar{S}$ is obtained by identifying finitely many points in $\hat{S}$, called nodes or singular points. This identification can be clarified by a map $\alpha : \hat{S} \to \bar{S}$, where $\alpha(p) = \alpha(q)$ if p and q are identified. If some S_k have boundaries, we require that boundary nodes glued with boundary nodes only. For a Riemann surface with nodes, let $sing(\bar{S})$ be the set of singular points. A continuous map $\bar{f} : \bar{S} \to (W, J)$ is called J-holomorphic if its composition with α, $\bar{f} \circ \alpha : \hat{S} \to W$ is J-holomorphic. In this case, we call $\bar{f}$ a *cusp-curve* with $area(\bar{f}) = area(\bar{f} \circ \alpha)$.

A Riemann surface with nodes $\bar{S}$ is usually deformed from a compact Riemann surface S. Take a finite family $\{\gamma_i\}$ of simple, and pairwise disjoint curves in S, either closed in $\text{int}(S)$ or with endpoints on ∂S. Then $\hat{S}$ is obtained from one-point compactification on $\text{int}(S) - \cup\gamma_i$, and the identification followed can be described by α. A continuous surjective map $\phi : S \to \bar{S}$ is naturally defined by α. We call it a deformation of S.

Definition 5.2. Let $\phi : \bar{S}' \to \bar{S}$ be a continuous surjective map between two Riemann surfaces with nodes. ϕ is a *node map* if (1) For every $x \in sing(\bar{S})$, $\phi^{-1}(x)$ is either a node, a simple closed curve in $\text{int}(\bar{S}') - sing(\bar{S}')$ or a simple arc disjoint from $sing(\bar{S}')$ with endpoints lying on $\partial\bar{S}'$; and
(2) $\phi : \bar{S}' - \phi^{-1}(sing\bar{S}) \to \bar{S} - sing(\bar{S})$ is a diffeomorphism. A node map ϕ is called a *deformation* if $\bar{S}'$ is a Riemann surface. ■

Following M. Gromov's paper [20], a sequence of J-holomorphic curves (f_n) weakly converge to a cusp-curve $\bar{f}$ in the following sense. We use μ for a Hermitian metric on W as well. Also see [1].

Definition 5.3 (Weak convergence)**.** Let $f_n : (S, j_n) \to (W, J, \mu)$ be a sequence of J-holomorphic curves, and $\bar{f} : (\bar{S}, \bar{j}) \to (W, J, \mu)$ be a cusp curve. Here $\bar{j}$ denotes the almost complex structure on $\bar{S}$ induced from $(\hat{S}, \alpha)$, valid outside $sing(\bar{S})$. We say that $(f_n)_{n\geq 1}$ *weakly converges* to $\bar{f}$ if the following four conditions are satisfied.

(1) There exists a deformation $\phi : S \to \bar{S}$ such that f_n uniformly converges to $f \circ \phi$ on S (i.e., in C^0-topology) ;
(2) j_n weakly C^∞-converges to $\phi^*\bar{j}$ on $S - \phi^{-1}(sing\bar{S})$;
(3) f_n weakly C^∞-converges to $\bar{f} \circ \phi$ on $S - \phi^{-1}(sing\bar{S})$; and
(4) under the hermitian metric μ on W, $area(f_n) \to area(\bar{f})$ as $n \to \infty$. ■

As a remark, weak C^∞-convergence means locally uniform C^k-convergence for every integer $k \geq 0$. Note that if f_n weakly converges to $\bar{f}$, then $[\bar{f}] = [f_n] \in H_2(W, \mathbb{Z})$ for sufficiently large n. According to [39], for closed cusp-curves and cusp-curves with boundary on completely real submanifolds, all C^k-topologies are equivalent. Therefore, the condition (3) can be dropped. The argument behind involves the generalized Weierstrass theorem [24].

Theorem 5.10. Let S be a compact connected surface equipped with a sequence of complex structures (j_n) which converges to another complex structure j on S in weak C^∞-topology. Suppose that $f_n : (S, j_n) \to (W, J, \mu)$ is a sequence of J-holomorphic maps which uniformly converges to a map $f : S \to W$ on S. Then, f is (j, J)-holomorphic, and f_n converges to f on S in weak C^∞-topology. ■

If the sequence (f_n) is reparametrized to $(f_n \circ \chi_n)$, then the later sequence will converge to $\bar{f}$ as well, after reparametrization. In this way a sequence of holomorphic curves (f_n) converges to the limit $\bar{f}$ even there is a sequence of deformations ϕ_n such that $f_n \circ \phi_n$ converges to $\bar{f} \circ \phi$ in the sense of Definition 5.3. See [1], p.155. Introduction of cusp-curves to the space of holomorphic curves makes the later compact, stated by Gromov's compactness theorem. See [20] and [39].

Theorem 5.11 (Gromov [20] - closed curves)**.** Let (W, J, μ) be a compact almost complex manifold with a hermitian metric, S a closed Riemann surface under a sequence of complex structure j_n. Suppose that $(f_n) : (S, j_n) \to (W, J, \mu)$ is a sequence of J-holomorphic curves with $area_\mu(f_n) \leq C$ for a constant C independent of n. Then, there is a subsequence (f_{n_k}) of (f_n) weakly converges to a cusp-curve $\bar{f} : (\bar{S}, \bar{j}) \to (W, J, \mu)$. ■

Theorem 5.12. [39]. Consider a compact, completely real submanifold N in the compact almost complex manifold (W, J, μ). Let S be a compact Riemann surface with boundaries under a sequence of complex structure j_n. Suppose that $(f_n) : (S, j_n) \to (W, J, \mu)$ is J-holomorphic, $f(\partial S) \subset N$, and $area_\mu(f_n) \leq C$ for a constant C. Then, there is a subsequence (f_{n_k}) weakly converges to a cusp-curve $\bar{f} : (\bar{S}, \partial\bar{S}, \bar{j}) \to (W, N, J, \mu)$. ■

As a remark, when (W, J) is tamed by a symplectic form ω, then this taming property helps to control the area. Therefore, if a sequence of J-holomorphic curves (f_n) lies in a fixed homology class in $H_2(W, \mathbb{Z})$, $area_\mu(f_n)$ is then bounded by a constant so the conclusion of Gromov's compactness theorem holds. Another important result is the following theorem. From now on we just take $j_n \equiv j$ as a complex structure on S.

Theorem 5.13. [1]. Let $(f_n) : (S, j) \to (W, J, \mu)$ be a sequence of J-holomorphic curves with $area_\mu(f_n) \leq C$ and weakly converges to a cusp-curve $\bar{f} : (\bar{S}, \bar{j}) \to (W, J, \mu)$. Then $(\hat{S}, \bar{j})$ is conformally equivalent to the disjoint union of S and finitely many 2-spheres. ■

We call these holomorphic 2-spheres by bubbles. According to [29], this phenomenon of bubbling off occurs only when the sequence $||df_n||_{L^\infty}$ is unbounded. More precisely, if a sequence $(f_n) : (S, j) \to (W, J, \mu)$ has uniformly bounded first derivative in local sense on S, i.e., $||df_n||_{L^\infty_{loc}} < C$ for some constant C, then (f_n) has a subsequence C^∞-converging to a J-holomorphic curve $f : S \to W$.

5.3.2 Main result

Recall our setting involves a compact almost complex 4-manifold (W, J) tamed by a symplectic form ω. Its boundary M is J-convex and so holomorphically fillable by our definition. Picking an embedded 2-sphere Σ of only two elliptic points $p_\pm$ from M, we try to fill the whole $\tilde{\Sigma} = \Sigma - \{p_\pm\}$ by a family of holomorphic disks

$$u_t : (\mathbb{D}, \partial\mathbb{D}) \to (W, \tilde{\Sigma}, J).$$

Starting with the elliptic points, there is a local family of holomorphic disks nearby. After clarifying its properties, Hofer's theorem ensures that we don't need further assumption to extend this family in open sense. In order to obtain compact extension, we refer to Gromov's compactness theorem. As long as bubbling off doesn't occur under further assumption, this family (u_t) can extend to a global filling, particulary of mutually disjoint boundaries.

Theorem 5.14 (Gromov [20]). Let (W, J, ω), $M = \partial W$ and $(\Sigma, p_\pm)$ be as stated. If either (i) $J$is integrable near Σ, and Σ is real analytic, i.e., transition functions are real analytic; or (ii) (W, J) is holomorphically aspherical, i.e., there are no holomorphic embedding $\mathbb{C}P^1 \hookrightarrow (W, J)$, then there exists a unique smooth family of mutually disjoint, embedded J-holomorphic disks

$$u_t : (\mathbb{D}, \partial\mathbb{D}) \hookrightarrow (W, \tilde{\Sigma}, J)$$

for $0 < t < 1$, $\lim_{t\to 0} u_t(x) = p_+$, and $\lim_{t\to 1} u_t(x) = p_-$. ■

Uniqueness here is up to reparametrization in t. For simplicity, we say Σ filled by holomorphic disks when the conclusion of Theorem 5.3 holds. Finally, we have prepared everything and get to the core result of this section.

Theorem 5.15 (Gromov [20], Eliashberg [9]). Let (M, ξ) be a holomorphically fillable contact 3-manifold in (W, J, ω), where ξ is the contact structure on M induced by J. Then, ξ is a tight contact structure.

Proof. In case that assumption (ii) in Theorem 5.14 is satisfied, we can follow the guideline in [1], p.214. Let Δ be an embedded overtwisted disk on M. By Giroux theorem, there is a neighborhood Σ of Δ diffeomorphic to a 2-sphere such that Σ_ξ has exactly two elliptic points of opposite sides $p_\pm$ and two limit cycles. See [8]. By Theorem 5.14, Σ can be filled by holomorphic disks, whose boundaries are all transverse to Σ_ξ on $\Sigma - p_\pm$. However, a limit cycle γ on Σ_ξ must hit one $\partial u_{t_0}(\mathbb{D})$ tangentially. Let say $t_0 = \max\{t \mid u_t(\mathbb{D}) \cap \gamma \neq \emptyset\}$. Contradiction arrives.

In general, filling every 2-sphere by holomorphic disks is not necessary. We refer to [9]. Any 2-sphere Σ can be C^2-approximated by a surface $\tilde{\Sigma}$ filled by holomorphic disks. Then, Y. Eliashberg argues that Bennequin inequality holds for (M, ξ). Therefore ξ is a tight contact structure. □

At the end of the thesis, I would like to state generalized results of Gromov-Eliashberg theorem. Comparing to 'holomorphic filling', it is more common to hear of 'symplectic filling' and 'Stein filling'.

Definition 5.4. A contact 3-manifold $(M, \xi = \ker \alpha)$ is *symplectically fillable* if there exists a compact symplectic 4-manifold (W, ω) such that $M = \partial W$ (compatible with boundary orientation) and $\omega|_\xi > 0$. ■

This definition is usually referred to '*weak-symplectically fillable*'. A Stein manifold is a complex manifold which is holomorphically convex and holomorphically separable. For example, if S is a Riemann surface, it is a Stein manifold if and only if it is non-compact. ■

Definition 5.5. Let (W, J, ω) be a Stein manifold with its complex structure J tamed by a symplectic form ω. We can assume that W lies properly in an ambient space W'. Suppose that $M = \partial W$ is the J-convex boundary, and a 2-plane field ξ is defined by $TM \cap J(TM)$. Then, (M, ξ) is said to be *Stein fillable.* ■

Different versions of filling are related by the final theorem.

Theorem 5.16. [18]. Stein fillable $\Longrightarrow$ Symplectically fillable $\Longrightarrow$ Tight. ■

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