Applications of the h-Cobordism Theorem

A Thesis by

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The following faculty members have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirement for the degree of Master of Science in Mathematics.

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ABSTRACT

We provide an exposition of J. Milnor's proof of the h-Cobordism Theorem. This theorem states that a smooth, compact, simply connected n-dimensional manifold W with $n \geq 6$, whose boundary ∂W consists of a pair of closed simply connected $(n-1)$-dimensional manifolds M_0 and M_1 and whose relative integral homology groups $H_*(W, M_0)$ are all trivial, is diffeomorphic to the cylinder $M_0 \times [0,1]$. The proof makes heavy use of Morse Theory and in particular the cancellation of certain pairs of Morse critical points of a smooth function. We pay special attention to this cancellation and provide some explicit examples. An important application of this theorem concerns the generalized Poincaré conjecture, which states that a closed simply connected n-dimensional manifold with the integral homology of the n-dimensional sphere is homeomorphic to the sphere. We discuss the proof of this conjecture in dimension $n \geq 6$, which is a consequence of the h-Cobordism Theorem.

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1 Introduction

One fundamental problem in mathematics is the problem of classification. In order to understand the structures that we study, it helps to have theorems which give us a way to contruct examples or to find obstructions preventing two objects from being the same. Examples appear in many areas of mathematics. For instance, in Algebra an important problem that was solved recently was to classify all finite simple groups up to isomorphism. For topology in particular, classification plays a central role. Given two topological spaces, we want an easy way to tell if they are *homeomorphic*. If the spaces are smooth manifolds and therefore have a differental structure, then we might also ask if they are *diffeomorphic*. Both of these are quite difficult questions. We could instead loosen our criteria and ask if the spaces are merely *homotopy equivalent*. However even this presents a problem. Spaces that are homotopy equivalent have isomorphic fundamental groups π_1, so we must first be able to classify these groups. However, every finitely presented group G is the fundamental group for some manifold and it is known that classifying finitely presented groups is an unsolvable problem [1]. It may seem that there's no hope for a solution to the classification problem, but we can still obtain partial results.

In dimensions 1 and 2, classifying manifolds is not difficult. For dimension 1, the only closed connected manifold is the circle S^1. Dimension 2 is more interesting: we have the sphere S^2, the torus T^2, the projective plane $\mathbb{R}\mathrm{P}^2$ and others, yet a complete classification is known [2]. In fact every closed connected 2-manifold is the connected sum of a collection of copies of T^2 and $\mathbb{R}\mathrm{P}^2$, and the only one that is simply connected is the sphere S^2. Unfortunately for higher dimensions the problem is much more difficult. In 1904, in his paper "Cinquième complément à l'analysis situs" [3], Henri Poincaré put forward the question: is the only simply connected closed 3-manifold is the 3-sphere S^3? This remained an open question for about one hundred years and the assumption that the answer is 'yes' became known as the Poincaré Conjecture. Poincaré himself showed that the assumption of simple connectivity is necessary by constructing the manifold known as the Poincaré dodecahedral sphere [3]. This manifold is a closed 3-manifold with the same homology as S^3, yet has fundamental group π_1 of order 120. Hence this space is not even homotopy equivalent to S^3. Reaching further into higher dimensions, the Generalized Poincaré Conjecture (GPC) was formulated as follows:

Generalized Poincaré Conjecture: If a simply connected, closed n-manifold has the same homology groups as the n-sphere S^n, then it is homeomorphic to S^n.

Almost parodoxically, it turns out that working in higher dimensions made the problem easier, and in 1962, Stephen Smale [4] proved the GPC for smooth manifolds for dimensions $n \geq 5$, though this result was quickly improved to topological manifolds through the work of John Stallings [5] and Christopher Zeeman [6]. Smale's original proof involved expressing a manifold as a collection of handles, making what is called a *handlebody*. He proceeded to show that under certain conditions some of the handles could cancel each other out, eventually proving what is known as the h-Cobordism Theorem. In 1965, John Milnor translated his proof into the language of Morse functions and critical points [7]. This paper follows Milnor's ideas.

Other results have been discovered since Smale's work. In 1982, Michael Freedman [8] proved the GPC for manifolds of dimension 4, though he had to use very different methods since a 4-dimensional version of the h-Cobordism Theorem turns out to be false. Even more recently in 2002 and 2003, Grigori Perelman posted three papers on the arXiv [9] [10] [11], giving the final pieces of the proof of Poincaré's original conjecture. His proof used the Ricci flow developed by Richard Hamilton [12] and was general enough to prove the stronger Geometrization Conjecture of William Thurston [13], giving a geometric characterization of closed 3-manifolds.

In the early years of topology, it was generally assumed that homeomorphic smooth manifolds are in fact diffeomorphic since continuous maps can be approximated by smooth ones. For dimensions lower than 3 it is true [14] [15], however in dimensions 4 and above there are counterexamples. In fact in 1987 Clifford Taubes [16] showed that even $\mathbb{R}^4$ itself has uncountably many distinct smooth structures, though this is the only dimension in which this occurs. Therefore we can consider the stronger question of whether the GPC holds with homeomorphic replaced by diffeomorphic. Smale's work does in fact prove that for dimensions 5 and 6 the smooth GPC is true. However, even before the GPC was proven, in 1956 Milnor [17] discovered a 7-manifold Σ^7 that is homeomorphic to the 7-sphere S^7, but not diffeomorphic to S^7. Such a manifold is known as an exotic sphere, and they exist in most dimensions higher than 7. It is still an open question whether or not the smooth GPC holds in dimension 4.

Chapter 2

2 Background Material

Our primary objects of study are smooth manifolds. Topology allows us to generalize the concept of continuity. In order to generalize differentiation we consider spaces that locally look like Euclidean space in such a way that local neighborhoods match smoothly on overlaps.

Definition 2.1. An *n-dimensional manifold* (or simply a n-manifold) is a topological space M with a collection $\mathcal{S}$ of pairs (U, ϕ) of open sets $U \subseteq V$ and maps $\phi : U \to \mathbb{R}^n_+$ where $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$ such that

(1) for $(U, \phi) \in \mathcal{S}$, ϕ maps U homeomorphically onto an open subset of $\mathbb{R}^n_+$,

(2) the sets U such that $(U, \phi) \in \mathcal{S}$ form a cover of M,

(3) If $(U, \phi), (V, \psi) \in \mathcal{S}$ then the map $\phi\psi^{-1} : \psi(U \cap V) \to \mathbb{R}^n_+$ is smooth,

(4) The collection $\mathcal{S}$ is maximal under inclusion among collections satisfying the other conditions.

Each set U is called a *coordinate neighborhood* and ϕ is a *coordinate map* or *coordinate chart*. The collection $\mathcal{S}$ is called a *smooth structure* on M. As in the case of topological spaces where we rarely name the topology, the smooth structure is usually left unnamed. For the most part, we can also ignore condition (4) as a cover of coordinate neighborhoods will uniquely determine the whole smooth structure [18].

We frequently abbreviate the phrase "M is an n-manifold" by "M^n is a manifold". The set of points of M^n that do not have neighborhoods homeomorphic to $\mathbb{R}^n$ is called the *boundary* of M. The boundary is denoted ∂M and turns out to be an $(n-1)$-manifold.

Given two manifolds M^m and N^n and a map $f : M \to N$ we will discuss what it means for f to be differentiable. At a point $p \in M$, consider a coordinate neighborhood (U, ϕ) of p and a coordinate neighborhood (V, ψ) of $f(p)$. If the map $\psi f \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth at $\phi^{-1}(p)$ then f is *differentiable* at p. If f is differentiable at each $p \in M$, then f is simply called differentiable or *smooth.* A bijective smooth map $f : M \to N$ with a smooth inverse is a *diffeomorphism* and the manifolds M, N are said to be *diffeomorphic.* Two diffeomorphisms $f_0, f_1 : M \to N$ are *isotopic* if there is a smooth map $f : M \times I \to N$ such that $f_0(p) = f(p, 0)$ and $f_1(p) = f(p, 1)$ and for each $t \in I$ the map $f_t(p) = f(p, t)$ is a diffeomorphism.

Although many of the definitions and results hold in general, we will restrict our attention to compact manifolds. A compact manifold with empty boundary is called a *closed manifold.* Given two closed n-manifolds M_0 and M_1, a *cobordism* between them is an $(n+1)$-manifold W such that the boundary ∂W is the disjoint union $M_0 \sqcup M_1$. Figure 2.1 shows an example of a cobordism between two 2-manifolds. If such a cobordism exists we say that M_0 and M_1 are *cobordant.* As we will be using cobordisms frequently, we will call the triple (W, M_0, M_1) a *triad.* Two triads (W, M_0, M_1) and (W', M_0, M_1) are *equivalent* if there exists a diffeomorphism from W to W' that is the identity on M_0 and M_1.

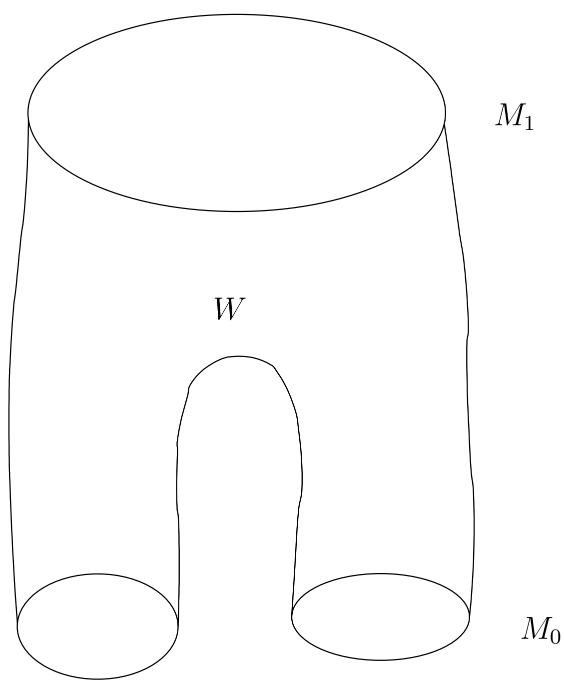

Figure 2.1: The "pair of pants" cobordism between two circles and one circle.

We mainly wish to study critical points of smooth real valued function on these triads. A point $p \in W$ is a *critical point* of a smooth function $f : W^n \to \mathbb{R}$ if there is a coordinate neighborhood (U, ϕ) of p where $\frac{\partial f \phi^{-1}}{\partial x_i}(p) = 0$ for $i = 1, \ldots, n$. As a result of the chain rule, this condition is independent of coordinate chart and we usually write $\frac{\partial f}{\partial x_i}(p)$ for $\frac{\partial f \phi^{-1}}{\partial x_i}(p)$. Such a critical point is *degenerate* if $\det\left(\frac{\partial^2 f}{\partial x_i \partial_j}(p)\right)_{n\times n} = 0$. Again this condition is independent of coordinate chart. Near non-degenerate critical points, a smooth function is well behaved. In fact, Marston Morse [19] discovered in 1934 that near a non-degenerate critical point, a smooth function can take the form of a quadratic polynomial.

Theorem 2.1 (Morse Lemma)**.** If $f : M \to \mathbb{R}$ is a smooth function and p a non-degenerate critical point of f, then there is a coordinate neighborhood U of p and a nonnegative integer λ such that for points with coordinates $(x_1, \ldots, x_n)$ in U, the function f is given by

$$f(x_1, \ldots, x_n) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$$

The λ given in the lemma is independent of the choice of coordinate neighborhood of p for which f has a quadratic form. We call λ the *Morse index* of p and denote it index(p). If all critical points of a smooth function f are non-degenerate, f is called a *Morse function*. For a triad (W, V_0, V_1), we restrict our attention to Morse functions $f : W \to [0, 1]$ such that $f^{-1}(0) = V_0$, $f^{-1}(1) = V_1$ and no critical point of f lies on either M_0 or M_1. The form of f in a quadratic chart of p prevents any other point in the neighborhood from being a critical point, and so a non-degenerate critical point is isolated. This also means that a Morse function on a compact manifold can only have finitely many critical points.

Given a closed manifold M, we denote by $C^\infty(M)$ the set of all smooth real valued functions on M. We can define a topology on $C^\infty(M)$ by considering a finite cover $\{(U_\alpha, \phi_\alpha)\}$ of coordinate charts on M with a closed cover $\{C_\alpha\}$ where $C_\alpha \subseteq U_\alpha$ for each α. For brevity, if $f \in C^\infty(M)$ we denote $f_\alpha = f h_\alpha^{-1}$. For a neighborhood base of f, we take the sets

$$N(f, \delta) = \Big\{ g \in C^\infty(M) : \forall \alpha, 1 \le i, j \le n, |f_\alpha - g_\alpha| < \delta, \\ \left|\frac{\partial f_\alpha}{\partial x_i} - \frac{\partial g_\alpha}{\partial x_i}\right| < \delta, \left|\frac{\partial^2 f_\alpha}{\partial x_i \partial x_j} - \frac{\partial^2 g_\alpha}{\partial x_i \partial x_j}\right| < \delta \Big\}$$

This will form a topology on $C^\infty(M)$ called the C^2 topology. Its importance here is that the subset Morse(M) of all Morse functions M is open and dense in $C^\infty(M)$ under the C^2 topology [7]. Using this fact, we can show that every triad (W, M_0, M_1) has a Morse function. We define the *Morse number* of a triad to be the minimum number of critical points over all Morse functions on the triad, denoted $\mu(W, M_0, M_1)$.

The smooth structure on a manifolds allows us to define tangent vectors at a point, and therefore a vector field which associates each point in the manifold with a vector based at that point. A *gradient-like vector field* for a Morse function f on a triad (W, M_0, M_1) is a vector field ξ on W such that the directional derivative at any non-critical point is $\xi(f) > 0$, and at any critical point p of f, there is a coordinate chart about p such that f has the form $f(x_1, \ldots, x_n) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$ and ξ has coordinates $(-x_1, \ldots, -x_\lambda, x_{\lambda+1}, \ldots, x_n)$. Once we have a gradient-like vector field, we can use techniques from differential equations to get curves $\phi : [a, b] \to W$ such that $\frac{d}{dt}(f \circ \phi) = \xi(f)$ which we call *integral curves*. If the Morse number of a triad (W, M_0, M_1) is zero, then every point of W lies on a unique integral curve starting in M_0. Hence we can construct a diffeomorphism $W \cong M_0 \times I$ so that the triad is equivalent to the product $(M_0 \times I, M_0 \times \{1\}, M_0 \times \{0\})$. We call such a triad a *product cobordism*. From this result, we get a couple of useful tools in the following two lemmas.

Lemma 2.2 (Collar Neighborhood)**.** Let W be a compact manifold with boundary. There is a "collar" neighborhood of ∂W diffeomorphic to $\partial W \times [0, 1)$.

Lemma 2.3 (Bicollar Neighborhood)**.** Let W be a compact manifold and $M \subseteq W \setminus \partial W$ a closed submanifold such that every component of M separates W (if C is a component of M then $W \setminus C$ is disconnected). Then there is a "bicollar" neighborhood of M in W diffeomorphic to $M \times (-1, 1)$ such that M corresponds to $M \times 0$.

Using these lemmas, if we have two triads (W, M_0, M_1) and (W', M_1', M_2') with a diffeomorphism $h : M_1 \to M_1'$, we can glue them together to get a triad $(W \cup_h W', M_0, M_2')$ as in Figure 2.2. This also implies that cobordism is an equivalence relation on closed manifolds.

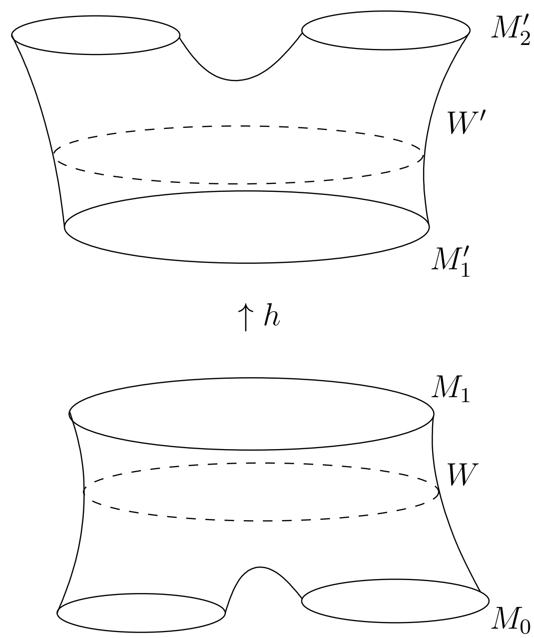

Figure 2.2: Gluing cobordisms W and W' with collar neighborhoods.

With some more work with integral curves, we can also strengthen this result so that if we have Morse functions f and f' on W and W' respectively and gradient-like vector fields ξ and ξ' associated with them, we can piece f and f' together into a Morse function g on $W \cup_h W'$ and similarly piece together ξ and ξ' to get a gradient-like vector field for g. A triad that has a Morse function with only one critical point is called an *elementary cobordism*, and by the previous statement, any triad can be decomposed into elementary cobordisms that glue back together into the original triad. The index of the critical point in an elementary cobordism is in fact independent of the Morse function, and so we can define the index of the elementary cobordism to be the index of the critical point.

Now consider an elementary cobordism (W^n, M_0, M_1), a Morse function $f : W \to [0, 1]$ with one critical point p with index λ, and ξ a gradient-like vector field for f. Suppose $f(p) = c$. For small ϵ we set $V_\epsilon = f^{-1}(c+\epsilon)$. There is an $\epsilon > 0$ and a coordinate neighborhood (U, g) such that $g : D^n_{2\epsilon} \to U$ and $(f \circ g)(\mathbf{x}, \mathbf{y}) = c - |\mathbf{x}|^2 + |\mathbf{y}|^2$. Setting $\phi : S^{\lambda-1} \times D^{n-\lambda} \to V_{-\epsilon}$ as $\phi(u, \theta v) = g(\epsilon u \cosh\theta, \epsilon v \sinh\theta)$ for $u \in S^{\lambda-1}$, $v \in S^{n-\lambda-1}$, and $0 \leq \theta \leq 1$, we define the characteristic embedding $\phi_L : S^{\lambda-1} \times D^{n-\lambda} \to V_0$ by following integral curves of ξ back to V_0. The lower sphere $S_L(p)$ is the image $\phi_L(S^{\lambda-1} \times 0)$ and the lower disk $D_L(p)$ is the union of the integral curves from $S_L(p)$ to p. The characteristic embedding $\phi_U : D^\lambda \times S^{n-\lambda-1} \to V_1$, upper sphere $S_U(p)$, and upper disk $D_U(p)$ are defined similarly. The embedding ϕ also

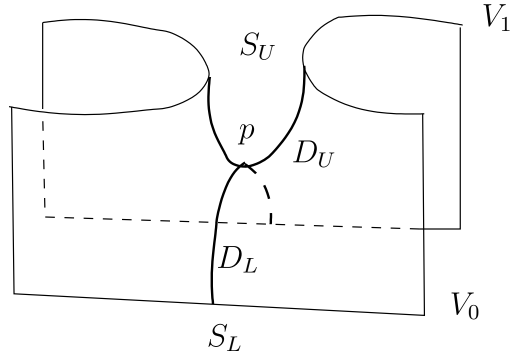

Figure 2.3: A critical point p with upper and lower spheres.

allows us to prove the following lemma which will help us make the connection between the critical points of a Morse function and the homology of W

Lemma 2.4. Let (W, V_0, V_1) be a triad and f a Morse function on W with one critical point p of index λ. If D_L is the lower disk of p then $V \cup D_L$ is a deformation retract of W. Consequently the homology $H_k(W, V_0)$ is only nonzero for $k = \lambda$ where it is $\mathbb{Z}$ with a generator represented by D_L.

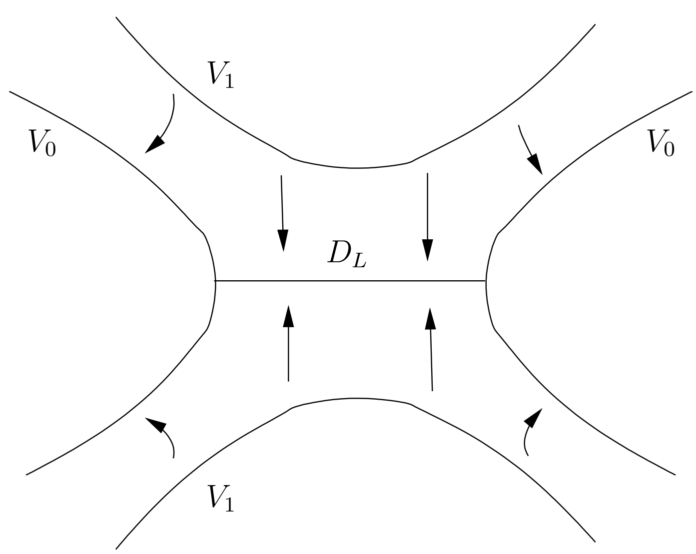

Figure 2.4: Retraction from W to $V_0 \cup D_L$

This lemma is easily seen by considering the embedding ϕ shown in Figure 2.4 and defining the retraction by following the integral curves backward until either we reach V_0 or a small neighborhood of D_L after which we move to D_L.

Elementary cobordisms also have a strong connection to surgery. Given a manifold M^{n-1} and an embedding $\phi : S^{\lambda-1} \times D^{n-\lambda} \to M$ we denote by $\chi(M, \phi)$ the manifold obtained from the disjoint union $(M - \phi(S^{\lambda-1} \times 0)) \cup (D^{\lambda} \times S^{n-\lambda-1})$ by identifying $\phi(u, \theta v)$ with $(\theta u, v)$ for $u \in S^{\lambda-1}$, $v \in S^{n-\lambda+1}$, and $\theta \in (0, 1)$. Any manifold diffeomorphic to $\chi(M, \phi)$ is said to be obtained from M by a surgery of type $(\lambda, n - \lambda)$. In essence, we removed an embedded sphere of dimension $\lambda - 1$ and replaced it with an embedded sphere of dimension $n - \lambda - 1$. If M' is obtained from M by surgery of type $(\lambda, n - \lambda)$, then there is an elementary cobordism (W, M, M') with index λ.

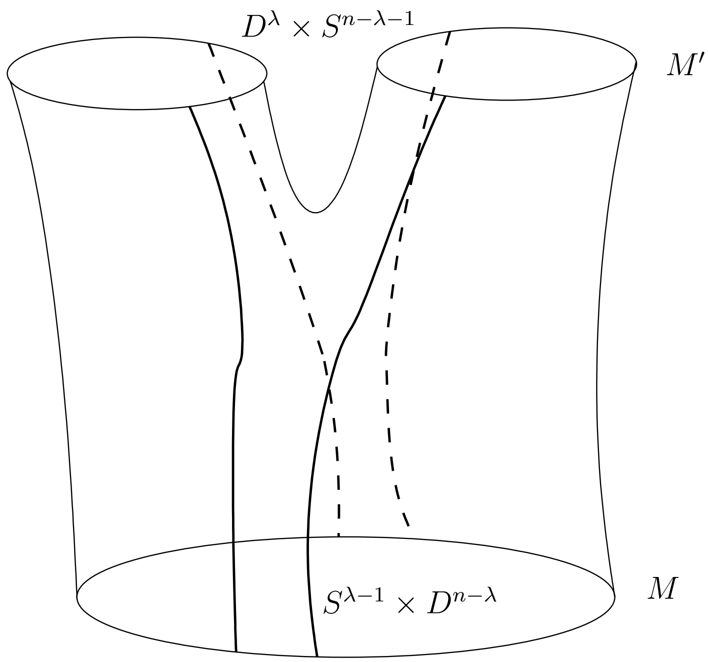

Figure 2.5: Surgery on M using cobordism.

We will also need to make use of transversality in later sections. Two submanifolds M, N of a manifold V intersect *transversely* if at every point $p \in M \cap N$ the tangent space T_pV is spanned by the tangent spaces T_pM and T_pN. Note that if the sum of the dimensions of M and N is less than the dimension of V, this condition is impossible, so in this case M and N intersect transversely if and only if they are disjoint.

Chapter 3

3 Rearranging Critical Points

In order to simplify things, it would be helpful to be able to reorder the critical points of a Morse function. In other words, if we have a Morse function f on a cobordism (W, V_0, V_1) with critical points p, q and $f(p) < f(q)$, is it possible to deform f into a new Morse function f' for which $f'(q) > f'(p)$? It turns out that it depends on the relative indices of the critical points. Given a gradient-like vector field, K_p denotes the set of all points on integral curves leading to or from p, and K_q is the corresponding set for q.

Theorem 3.1. Let (W, V_0, V_1) be a triad with a Morse function $f : W \to [0, 1]$. Suppose that f has exactly two critical points p, q and $K_p \cap K_q = \emptyset$ for some gradient-like vector field ξ. Then for any choice of $a, b \in (0, 1)$ there exists a Morse function g such that

a) the vector field ξ is a gradient-like vector field for g,

b) the critical points of g are also p, q, yet $g(p) = a$ and $g(q) = b$, and

c) near $V_0 \cup V_1$ the functions g and f agree, yet $g = f + c_p$ near p and $g = f + c_q$ near q for constants c_p, c_q.

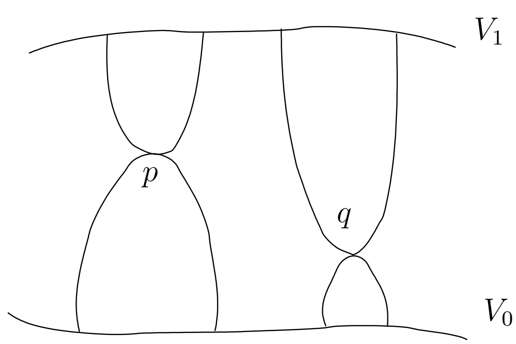

Figure 3.6: The sets K_p and K_q.

Proof Set $K = K_p \cup K_q$. Any integral curve through points not in K cannot pass through p or q. Since there are no other critical points, such an integral curve must go from V_0 to V_1. We can define a function $\pi : W \setminus K \to V_0$ by assigning to each point in $W \setminus K$ the unique point of V_0 we get by traveling back along integral curves. This function π will be smooth and $\pi(w)$ will be near K whenever w is near K. If $\mu : V_0 \to [0,1]$ is any smooth function that is zero in a neighborhood of $K_p \cap V_0$ and one in a neighborhood of $K_q \cap V_0$, then we can extend to a smooth function $\bar{\mu} : W \to [0,1]$ by assigning $\bar{\mu}(w) = \mu(\pi(w))$ for $w \notin K$, $\bar{\mu}(w) = 0$ for $w \in K_p$, and $\bar{\mu}(w) = 1$ for $w \in K_q$. From real analysis there exists a smooth function $G : [0,1]^2 \to [0,1]$ with the properties:

(1) $\frac{\partial G}{\partial x}(x,y) > 0$ for all x, y,

(2) $G(x,y)$ increases from 0 to 1 as x increases,

(3) $G(f(p),0) = a$ and $G(f(q),1) = b$,

(4) $G(x,y) = x$ for x near 0 and 1, and

(5) $\frac{\partial G}{\partial x}(x,0) = 1$ for x near $f(p)$ and $\frac{\partial G}{\partial x}(x,1) = 1$ for x near $f(q)$

Defining $g : W \to [0,1]$ by $g(w) = G(f(w), \bar{\mu}(w))$ will give us the desired Morse function.

Q.E.D.

This result shifts the question to what conditions guarantee that K_p and K_q are disjoint. The following lemma gives us the tool we need to make this happen.

Lemma 3.2. Let M, N be submanifolds of a manifold V. If M has a product neighborhood in V, then there is a diffeomorphism $h : V \to V$ isotopic to the identity such that $h(M)$ has transverse intersection with N. Specifically, if $\dim(M) + \dim(N) < \dim(V)$, then $h(M)$ and N are disjoint.

Proof Set $m = \dim(M)$, $n = \dim(N)$, and $v = \dim(V)$. Let U be a product neighborhood of M as in figure 3.7 and $k : M \times \mathbb{R}^{v-m} \to U$ be a diffeomorphism with $k(M \times \{0\}) = M$. Set $N_0 = U \cap N$ and consider the map $g = \pi \circ k^{-1}|_{N_0}$ where $\pi : M \times \mathbb{R}^{v-m} \to \mathbb{R}^{v-m}$ is the projection. A manifold $k(M \times \{x\})$ for any $x \in \mathbb{R}^{v-m}$ will not have transverse intersection with N if and only if $x = g(q)$ for a critcal point $q \in N_0$ at which g does not have maximal rank. By Sard's Theorem [20], the image $g(C)$ of the set C of critical points of g in N_0 has measure zero in $\mathbb{R}^{v-m}$. In particular there exists a point $u \in \mathbb{R}^{v-m} \setminus g(C)$. Hence the submanifold $k(M \times \{u\})$ has transverse intersection with N.

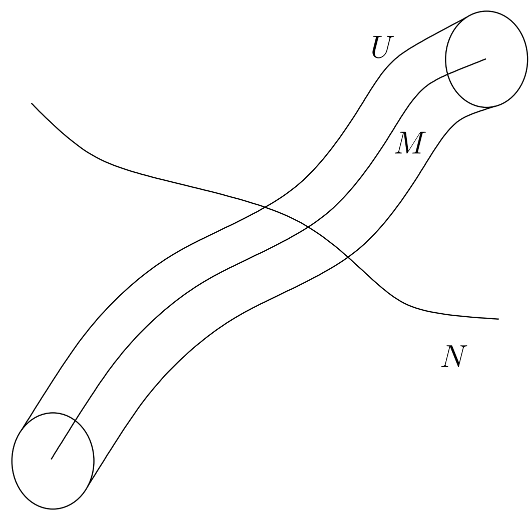

Figure 3.7: 1-dimensional submanifolds M and N in 3-dimensional V.

We have a smooth vector field ξ on $\mathbb{R}^{v-m}$ such that $\xi(x) = u$ for $|x| \leq |u|$ and 0 for $|x| \geq 2|u|$. Since ξ has compact support and $\mathbb{R}^{v-m}$ has no boundary, the integral curves $\psi(t, x)$ are defined for all real t. Then $\psi(0, x)$ is the identity on $\mathbb{R}^{v-m}$, $\psi(1, x)$ is a diffeomorphism mapping 0 to u, and $\psi(t, x)$ is a smooth isotopy from $\psi(0, x)$ to $\psi(1, x)$. Setting $h_t(w) = k(q, \psi(t, x))$ for $w = k(q, x) \in U$ and $h_t(w) = w$ otherwise, $h = h_1$ is a diffeomorphism $V \to V$ with $h(M) = k(M \times u)$ having transverse intersection with N and isotopic to $h_0 = 1_V$ as was desired.

Q.E.D.

Lemma 3.3. Let (W, V_0, V_1) be a triad with a Morse function f, gradient-like vector field ξ, a non-critical level $V = f^{-1}(b)$, and a diffeomorphism $h : V \to V$ isotopic to the identity. If $f^{-1}[a, b]$ contains no critical points, then there is a new gradient-like vector field $\bar{\xi}$ for f such that $\bar{\xi} = \xi$ outside $f^{-1}(a, b)$ and $\bar{\phi} = h \circ \phi$ where ϕ and $\bar{\phi}$ are the diffeomorphisms $f^{-1}(a) \to V$ determined by following the trajectories of ξ and $\bar{\xi}$.

Proof Normalize ξ by setting $\hat{\xi} = \xi/\xi(f)$. By assumption, h is isotopic to the identity, so let $h_t : V \to V$, $t \in [a, b]$, be a smooth isotopy such that h_t is the identity for t near a and $h_t = h$ for t near b. Define the diffeomorphism H from $[a, b] \times V$ to itself by $H(t, q) = (t, h_t(q))$. Then $\xi' = (\phi \circ H \circ \phi^{-1})_* \hat{\xi}$ is a smooth vector field on $f^{-1}[a, b]$ that coincides with $\hat{\xi}$ near

$f^{-1}(a)$ and $f^{-1}(b) = V$, yet $\xi'(f) \equiv 1$. Then the vector field $\bar{\xi}$ equal to $\xi(f)\xi'$ on $f^{-1}[a,b]$ and equal to ξ elsewhere is a smooth gradient-like vector field for f that satisfies the required conditions.

Q.E.D.

Theorem 3.4. If p, q are critical points of f with $f(p) < f(q)$, then on a given neighborhood N of $V = f^{-1}(c)$ with $f(p) < c < f(q)$, containing no critical points, we can alter the gradient-like vector field so that the upper sphere of p and the lower sphere of q have transverse intersection.

Proof Let S_U be the upper sphere of p and S_L the lower sphere of q in V. Since S_U has a product neighborhood in V, then by Lemma 3.2, there is a diffeomorphism $h : V \to V$ isotopic to the identity such that $h(S_U)$ and S_L have transverse intersection. Choose a number a such that $f^{-1}[a,c]$ lies in the neighborhood N. Then as $f^{-1}[a,c]$ contains no critical points, by Lemma 3.3, we can deform ξ into a new vector field $\bar{\xi}$. Outside $f^{-1}(a,c)$, the fields $\bar{\xi}$ and ξ agree and if ϕ and $\bar{\phi}$ are the diffeomorphisms $f^{-1}(a) \to V$ determined by the integral curves of ξ and $\bar{\xi}$ respectively, then $\bar{\phi} = h \circ \phi$. Hence the upper sphere S_U of p under ξ is carried into $h(S_U)$ while the lower sphere S_L of q is unaffected. Since $h(S_U) \cap S_L = \emptyset$, then $\bar{\xi}$ is the required new gradient-like vector field.

Q.E.D.

Putting the previous theorem together with Theorem 3.1, we immediately get the following.

Corollary 3.5. Let (W, V_0, V_1) be a triad with a Morse function $f : W \to [0,1]$ and gradient-like vector field ξ. If f has two critical points p, q with $f(p) < f(q)$ and $\text{index}(p) \geq \text{index}(q)$, then there is a new Morse function g on W with the same critical points as f still having ξ as a gradient-like vector field but with $g(q) < g(p)$.

In fact, since Theorem 3.1 allows us to choose the critical values, then if our Morse function f has critical points $p_1, \ldots, p_n$, we can alter f to a new Morse function g without changing the gradient-like vector field so that $g(V_0) = -\frac{1}{2}$, $g(V_1) = n + \frac{1}{2}$, and $g(p_i) = \text{index}(p_i)$. Since having the critical points ordered in this way is very convenient, we will frequently assume that this alteration has already been done and will call such a Morse function self-indexing or nice. Figure 3.8 shows an example of a self-indexing Morse function f with critical points p_1 and p_2 such that $\text{index}(p_1) = 0$ and $\text{index}(p_2) = 1$.

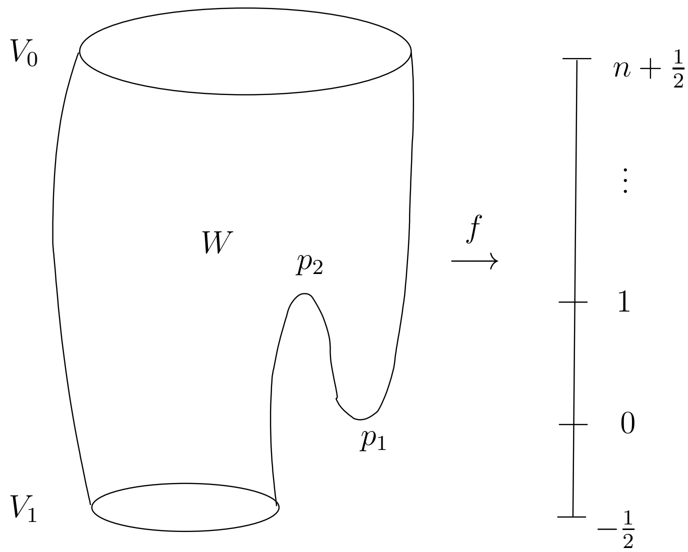

Figure 3.8: A self-indexing Morse function f.

Chapter 4

4 Cancelling Critical Points

We now know that we can reorder the critical points of a Morse function in order of their index. However, is it possible to alter a Morse function to eliminate a critical point altogether? In figure 4.9 we see an example of a cobordism W where a Morse function given by height would have two critical points, p with index 0 and q with index 1. However, W is equivalent to the product cobordism $V_0 \times [0,1]$ which eliminates both critical points. We want to find conditions for which this happens.

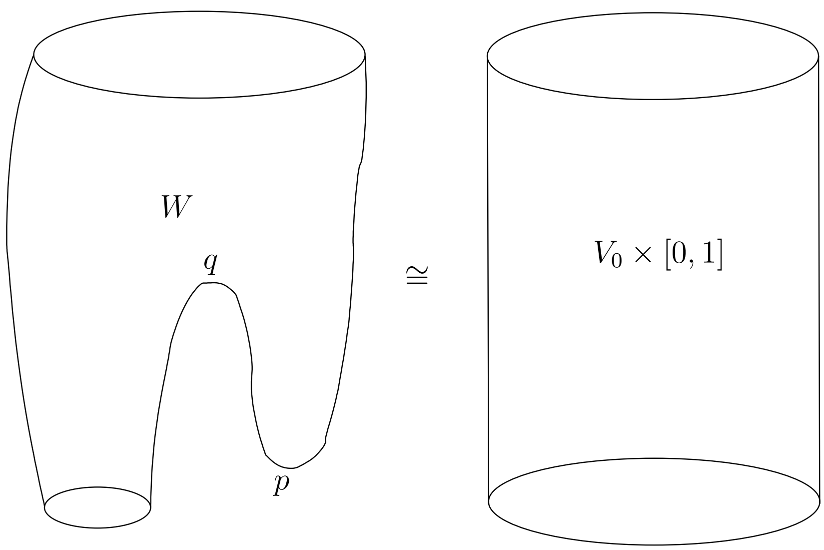

Figure 4.9: Equivalent cobordisms.

Let (W, V_0, V_1) be a triad and f a Morse function on W that has two critical points p and q with consecutive indices λ and $\lambda+1$. At a non-critical level V between p and q, we know that the lower sphere $S_L(q)$ of q has dimension λ and the upper sphere $S_U(p)$ of p has dimension $n-\lambda-1$. Assuming the two spheres have transverse intersection, then the intersection of them, if nonempty, must have dimension

$$\dim S_L(q) + \dim S_U(p) - \dim V = \lambda + (n-\lambda-1) - (n-1) = 0.$$

Hence the spheres intersect in a number of isolated points. The simplest case is when there is only one point of intersection.

Theorem 4.1. Let (W, V_0, V_1) be a triad, f a self-indexed Morse function on W and ξ a gradient-like vector field for f. Assume f has only the critical points p, q with indices λ and $\lambda + 1$ and such that the intersection of the right hand sphere $S_U(p)$ of p intersects the left hand sphere $S_L(q)$ of q transversely at a single point. Then it is possible to alter ξ on an arbitrarily small neighborhood of the single trajectory T from p to q to get a nowhere zero vector field ξ' that is a gradient-like vector field for a Morse function f' with no critical points and agreeing with f near $V_0 \cup V_1$.

Proof We know by the definition of the gradient-like vector field ξ there are charts $g_1 : U_1 \to \mathbb{R}^n$ and $g_2 : U_2 \to \mathbb{R}^n$ about p and q with

$$(f \circ g_1)(x) = x_1^2 - x_2^2 - \cdots - x_{\lambda+1}^2 + x_{\lambda+2}^2 + \cdots + x_n^2$$
$$(f \circ g_2)(x) = -x_1^2 - x_2^2 - \cdots - x_{\lambda+1}^2 + x_{\lambda+2}^2 + \cdots + x_n^2$$

and on which ξ has coordinates $(\pm x_1, -x_2, \ldots, -x_{\lambda+1}, x_{\lambda+2}, \ldots, x_n)$. We would like to have a chart $g : U_T \to \mathbb{R}_n$ around the trajectory T that essentially extends g_1 and g_2 so that p and q correspond to $(0, \ldots, 0)$ and $(1, 0, \ldots, 0)$. Under this chart ξ will have coordinates $(\nu(x_1), -x_2, \ldots, -x_\lambda, -x_{\lambda+1}, x_{\lambda+2}, \ldots, x_n)$ where ν is a smooth function as in Figure 4.10 that is positive on $(0, 1)$, $\nu(0) = \nu(1) = 0$, negative elsewhere, and $\left|\frac{\partial \nu}{\partial x_1}(x_1)\right| = 1$ near 0 and 1. Doing this requires careful patching together of coordinate charts along T and some analysis to make sure the charts match up smoothly near p and q. Full details are in Milnor [7].

Let U be a neighborhood of T whose closure lies in U_T. We want to show that there is a smaller neighborhood U' of T contained in U such that no trajectory leads from U' to outside U and back into U'. Assume this is not true. Then there is a sequence of trajectories $T_1, \ldots, T_k, \ldots$ where T_k goes from $r_k \in U'$ through $s_k \in W \setminus U$ and back to $t_k \in U'$ where

$\{r_k\}$ and $\{t_k\}$ approach T. By compactness of $W \setminus U$, we may assume that $\{s_k\}$ is convergent to a point $s \in W \setminus U$. Since s is not on T, the integral curve through s must meet either V_0 or V_1 and without loss of generality, we'll assume it meets V_0. By continuity of integral curves in their starting points, there is a neighborhood N of s such that there are trajectories T_x through points $x \in N$ from V_0. Since T and all T_x are compact, under some metric on W, we have that the least distance $d(x)$ from T_x to T exists and depends continuously on x. Since $d(s) > 0$, for some $\epsilon > 0$ there is a neighborhood $N' \subseteq N$ of s such that $d(x) \leq \epsilon$ for $x \in N'$. By construction, $r_k \in T_{s_k}$ for each k and r_k approaches T as $k \to \infty$. Therefore $d(s_k) \to 0$. However, by continuity, $s_k \to s$ implies $d(s_k) \to d(s)$, contradicting $d(s) \neq 0$. Therefore such a neighborhood U' must exist.

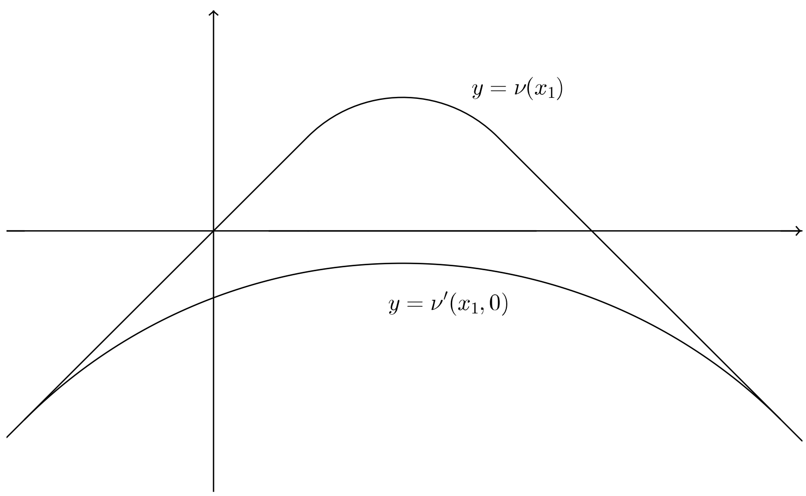

Figure 4.10: The functions ν and ν'.

Let $\nu' : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function like in Figure 4.10 such that $\nu'(x_1, 0)$ is negative and $\nu'(x_1, \rho) = \nu(x_1)$ where $\rho = \sqrt{x_2^2 + \cdots + x_n^2}$ outside a compact neighborhood of $g(T)$ contained in $g(U')$. Consider the smooth vector field

$$\eta'(x) = (\nu'(x_1, \rho), -x_2, \ldots, -x_\lambda, -x_{\lambda+1}, x_{\lambda+2}, \ldots, x_n)$$

on U. Then $\eta'(x)$ extends to a nowhere zero vector field ξ' on W. Integral curves of ξ' satisfy $\frac{dx_1}{dt} = \nu'(x_1, 0)$, $\frac{dx_i}{dt} = -x_i$ for $2 \leq i \leq \lambda + 1$, and $\frac{dx_i}{dt} = x_i$ for $\lambda + 2 \leq i \leq n$

on U_T. Consider an integral curve $x(t) = (x_1(t), \ldots, x_n(t))$ with intial value $(x_1^0, \ldots, x_n^0)$. Assume that one of the numbers x_i^0 is nonzero for $i \geq \lambda + 2$. Then we can solve the differential equation $\frac{dx_i}{dt} = x_i$ to get $x_i(t) = x_i^0 e^t$. Since this grows exponentially, $x(t)$ must eventually leave the bounded set $g(U)$. Otherwise, we have $x_{\lambda+2} = \cdots = x_n = 0$. The solutions to the differential equations $\frac{dx_i}{dt} = -x_i$ give us that $x_i(t) = x_i e^{-t}$ for $2 \leq i \leq \lambda + 1$. Therefore $\rho(x(t)) = e^{-t}\sqrt{(x_2^0)^2 + \cdots + (x_{\lambda+1}^0)^2}$. If $x(t)$ stays in $g(U)$, then as $\nu'(x_1, \rho(x))$ is negative on the x_1 axis, there is a $\delta > 0$ such that $\nu'(x_1, \rho(x))$ is negative on the compact set $K_\delta = \{x \in g(U) : \rho(x) \leq \delta\}$. There is an $\alpha > 0$ such that $-\alpha$ bounds $\nu'(x_1, \rho(x))$ above on K_δ. As ρ is decreasing exponentially with t, eventually $\rho \leq \delta$ and hence $\frac{dx_1}{dt} \leq -\alpha$. Thus $x(t)$ eventually leaves $g(U)$ after all. Similarly, $x(t)$ will leave $g(U)$ as t decreases as well.

If an integral curve of ξ' is never in U' then it has to go from V_0 to V_1. If it is every in U', then by the previous paragraph it will eventually leave U and never reenter U' so that it will hit V_1. Simiarly, it also must have come from V_0. Therefore every trajectory of ξ' goes from V_0 to V_1.

Let $\psi(t, x)$ be the integral curve of ξ' through the point $x \in W$. Since ξ' is never tangent to ∂W, then by the Implicit Function Theorem, the function $\tau_1(x)$ that assigns to x the t at which $\psi(t, x)$ reaches V_1 is smooth. Similarly, the function $\tau_0(x)$ that assigns to x the number $-t$ at which $\psi(t, x)$ reaches V_0 is also smooth. The smooth vector field $\tau_1(\psi(-\tau_0(x), x))\xi'(x)$ has integral curves that go from V_0 to V_1 as t goes from 0 to 1. We can assume that ξ' had this initially. Defining $\phi : V_0 \times [0, 1] \to W$ by $(x, t) \mapsto \psi(t, x)$ we see that ϕ has inverse $x \mapsto (\psi(-\tau_0(x), x), \tau_0(x))$ so that ϕ is a diffeomorphism between $V_0 \times [0, 1]$ and W with $V_0 \times 0$ and $V_0 \times 1$ corresponding to V_0 and V_1.

Set $f' = f \circ \phi$. There exists a δ such that for either $t < \delta$ or $t > 1 - \delta$ we have $\frac{\partial f'}{\partial t}(x, t) > 0$ for all $x \in V_0$. Let $\lambda : [0, 1] \to [0, 1]$ be a smooth function that is zero on $[\delta, 1 - \delta]$ and one near 0 and 1. Choosing δ sufficiently small so that

$$k(x) = \frac{1 - \displaystyle\int_0^1 \lambda(\tau)\frac{\partial f'}{\partial t}(\tau, x)\, d\tau}{\displaystyle\int_0^1 (1 - \lambda(\tau))\, d\tau} > 0$$

for all $x \in V_0$, then the function

$$g(x,t) = \int_0^t \left[\lambda(\tau)\frac{\partial f'}{dt}(\tau, x) + (1 - \lambda(\tau))k(x)\right] d\tau$$

is a Morse function on $V_0 \times [0,1]$ that agrees with f' near $(V_0 \cup V_1) \times [0,1]$ and $\frac{\partial g}{\partial t} > 0$ everywhere. Hence $g \circ \phi^{-1}$ is the required Morse function on W.

Q.E.D.

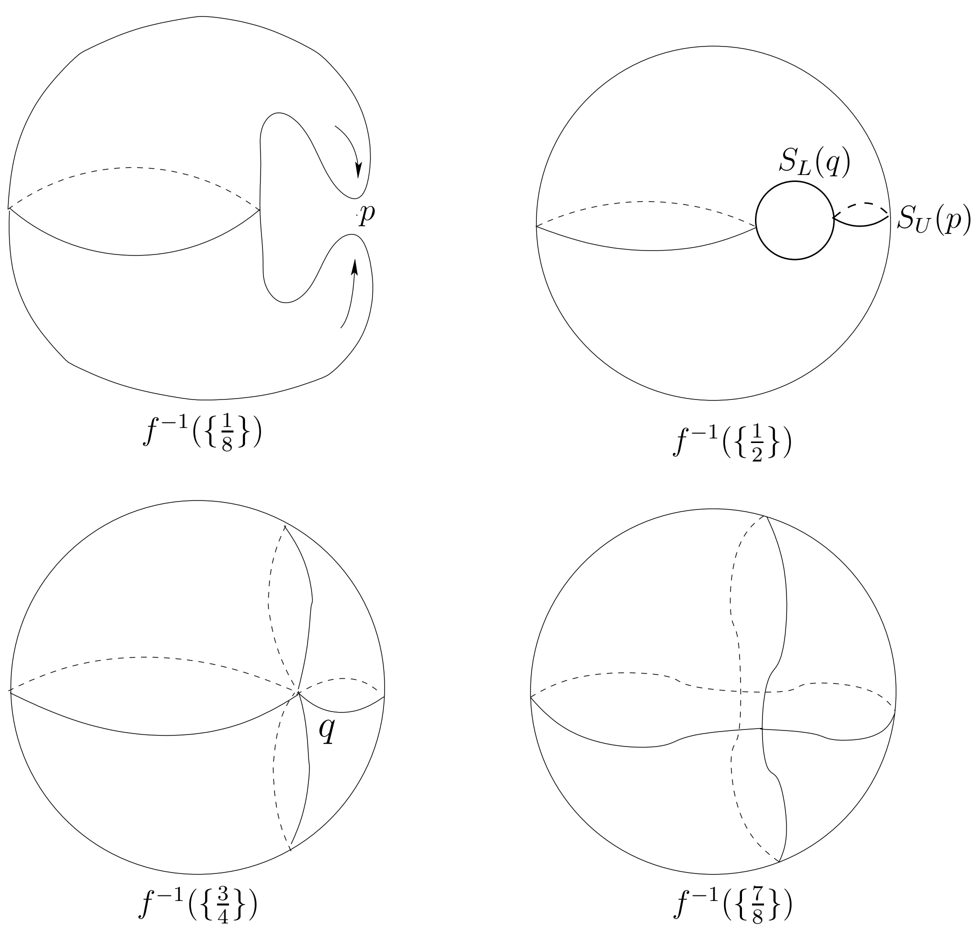

Figure 4.11: Level sets of the cobordism W.

To demonstrate the importance of the transverse intersection, let's look at an example of a 3-dimensional cobordism W between the sphere S^2 and itself. Our Morse function $f : W \to [0,1]$ will have critical points p and q with $f(p) = \frac{1}{4}$ and $f(q) = \frac{3}{4}$. For clarity, we will focus on the level sets and describe how they change as we pass through the critical values. Some of these level sets are shown in Figure 4.11. We start with the sphere S^2 which then grows horns which come to meet at a critical point p of index 1 when we get to $t = \frac{1}{4}$. The level set for $\frac{1}{4} < t < \frac{3}{4}$ is a torus. As $t \to \frac{3}{4}$, the hole of the torus collapses to a critical point q with index 2. After this the space bulges out in the middle and eventually smooths back out into a sphere. Overall the cobordism W is diffeomorphic to $S^2 \times [0,1]$. Examining the noncritical level $V = f^{-1}(\{\frac{1}{2}\})$ we can clearly see the spheres $S_U(p)$ and $S_L(q)$ as two circles in three dimensional space transversely intersecting in one point allowing the cancellation to occur.

More generally, the left and right-spheres will intersect in multiple points. Assume we have two oriented submanifolds M^m and N^n of an oriented manifold W^{m+n}, that intersect transversely in distinct points $p_1, \ldots, p_k$. At each point p_i we can assign the number $+1$ if positively oriented frames in M and N combine to make a positively oriented frame in W. Otherwise we will assign the number -1 to p_i. The *intersection number* $I(M, N)$ is the sum of these numbers. Note that if M and N only have one intersection point, then $I(M, N) = \pm 1$. As the intersection number of two submanifolds does not change when they are smoothly deformed, then it is natural to ask if the condition $I(M, N) = \pm 1$ implies that M and N can be untangled until they have exactly one intersection. With a few more conditions, this turns out to be the case.

Theorem 4.2. Using the notation from the previous theorem, suppose W, V_0, and V_1 are simply connected and $3 \leq \lambda \leq n - 3$. If on a noncritical level V between p and q we have $I(S_U(p), S_L(q)) = \pm 1$, then ξ can be altered near V so that the right and lower spheres intersect instead in a single point. The result of Theorem 4.1 will then apply.

Proof Note first that the condition on λ implies that the spheres $S_U(p)$ and $S_L(q)$ must be simply connected. Simple connectivity of these spheres, W, V_0, and V_1 imply that the manifolds are orientable and hence it is possible to define the intersection number $I(S_U(p), S_L(q))$. Also, using the fact that $\lambda \geq 2$ and $n - \lambda \geq 3$ we have that $\pi_1(V) \cong \pi_1(D_U(p) \cup V \cup D_L(q))$ from Van Kampen's Theorem and by Lemma 2.4, the space $D_U(p) \cup V \cup D_L(q)$ is a deformation retract of the simply connected W. Hence $\pi_1(V) = 1$ so that V is simply connected.

We will assume that we have already made sure that $S_U(p)$ and $S_L(q)$ intersect transversely. If $S_U(p) \cap S_L(q)$ is a single point, there is nothing to do, so assume there are multiple points of intersection. Since $I(S_U(p), S_L(q)) = \pm 1$ there must be a pair x, y of intersection points that have opposite intersection numbers. Let C_U be a smooth path in $S_U(p)$ from x to y that does not pass through any other intersection points and similarly for a smooth path C_L in $S_L(q)$ from y to x. Then C_U and C_L form a loop L in V. Note that as V is simply connected, this loop is contractible to a point and this will allow us to alter one of the spheres in such a way to remove the intersections at x and y.

We construct a model in $\mathbb{R}^2$ as in Figure 4.12. Take two open arcs C_0 and C_1 in $\mathbb{R}^2$ intersecting transversely at a and b and enclosing a disk D with corners. Let U be an open neighborhood of D. There is a smooth embedding $\phi : U \times \mathbb{R}^{n-3} \to V$ such that $\phi^{-1}(S_U(p)) = (U \cap C_0) \times \mathbb{R}^{\lambda-1} \times 0$ and $\phi^{-1}(S_L(q)) = (U \cap C_1) \times 0 \times \mathbb{R}^{n-\lambda-1}$. We can choose

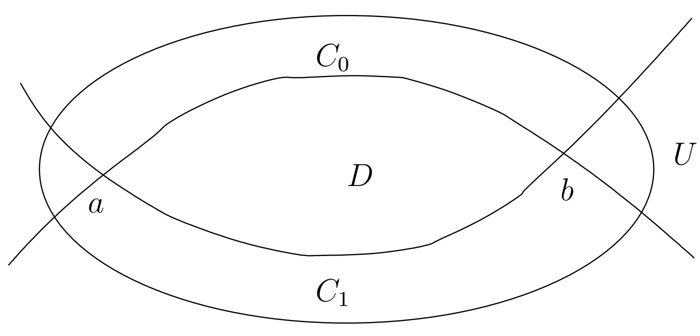

Figure 4.12: The standard model.

an isotopy $G_t : U \to U$ such that G_0 is the identity, G_t is the identity in a neighborhood of ∂U for each t, and $G_1(C_0)$ does not intersect C_1. Such an isotopy is shown in Figure 4.13. Next, let $\rho : \mathbb{R}^{n-2} \to [0, 1]$ be a smooth function such that $\rho(r) = 1$ for $|r| \leq 1$ and 0 for

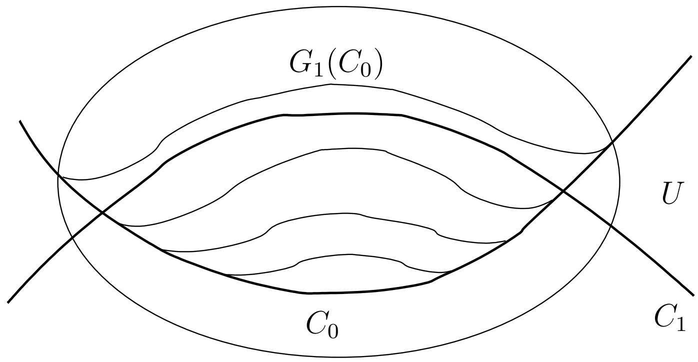

Figure 4.13: The isotopy G_t in U

$|r| \geq 2$. Define $H_t(u,r) = (G_{t\rho(r)}(u), r)$ for $r \in \mathbb{R}^{n-3}$ and $u \in U$ and define $F_t : V \to V$ by $F_t(v) = (\phi \circ H_t \circ \phi^{-1})(v)$ for $v \in \operatorname{im} \phi$ and $F_t(v) = v$ otherwise. Then F_t is an isotopy on V such that $F_1(C_U)$ is disjoint from C_L. Hence the isotopy removes the intersection points x and y while not affecting the other intersections of $S_U(p)$ and $S_L(q)$. Continuing to cancel pairs of intersection points, we eventually get to the point where there is exactly one point of intersection.

Q.E.D.

Note that the conditions of the theorem imply that $n \geq 6$. Also, replacing the Morse function f with its negative $-f$ and similarly with the gradient-like vector field, shows that the theorem is also valid for $\lambda = 2$ so that it holds for $2 \leq \lambda \leq n - 3$.

Chapter 5

5 The Main Result

Consider the oriented triads (W, V_0, V) and (W', V, V_1) and a Morse function f on $W \cup W'$ with critical points $p_1, p_2, \ldots, p_\ell$ all of index λ on one level and critical points $q_1, \ldots, q_m$ of index $\lambda + 1$ on another with V a noncritical level between them. Choose orientations for the left and righ-hand disks so that $I(D_L(p_i), D_U(p_i)) = +1$ and $I(D_L(q_j), D_U(q_j)) = +1$ for all i and j. By Lemma 2.4, the homology groups $H_\lambda(W, V_0)$ and $H_{\lambda+1}(W \cup W', W) \cong H_{\lambda+1}(W', V)$ are free abelian on generators $[D_L(p_i)]$ and $[D_L(q_j)]$ respectively. With respect to these bases, the boundary map $\partial : H_{\lambda+1}(W \cup W', W) \to H_\lambda(W, V_0)$ in the long exact sequence of the triple $(W \cup W', W, V_0)$ can be represented by the matrix of intersection numbers $\big[I(S_U(p_i), S_L(q_j))\big]$.

Now given a triad (W, V_0, V_1) with a self-indexing Morse function f, we can decompose W into a sequence of cobordisms W_λ such that $V = W_{-1} \subseteq W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n = W$ and W_λ contains only the critical points up to index λ (see Figure 5.14). For each λ, define $C_\lambda = H_\lambda(W_\lambda, W_{\lambda+1})$. If the map $\partial_\lambda : C_\lambda \to C_{\lambda+1}$ is the boundary morphism from the exact

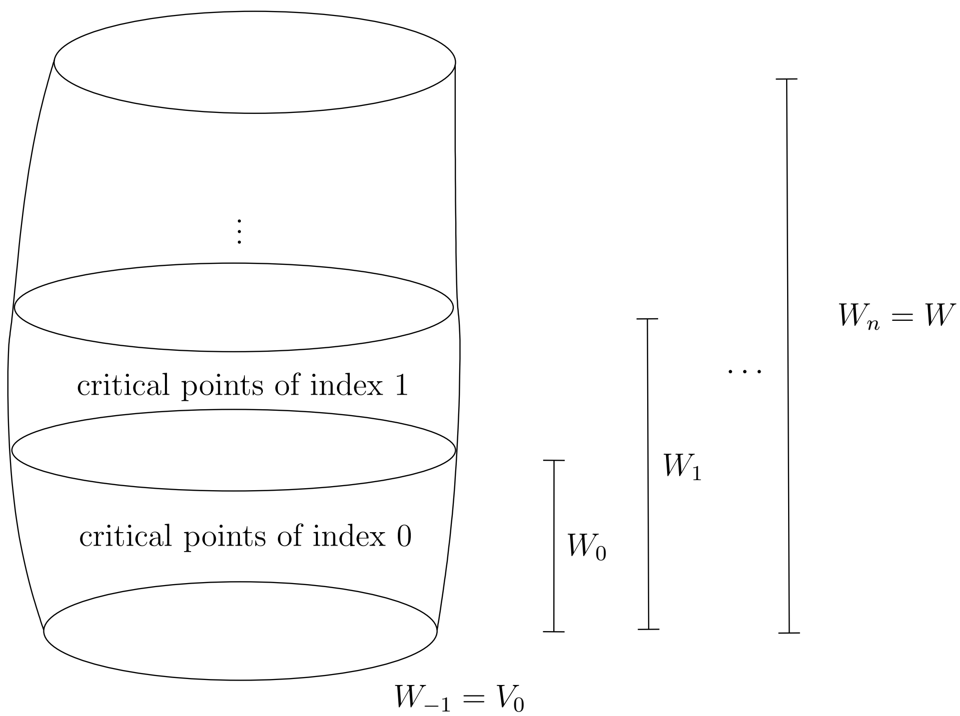

Figure 5.14: Cobordism W decomposed.

sequence of the triple $(W_\lambda, W_{\lambda-1}, W_{\lambda-2})$, then $\{C_\lambda, \partial_\lambda\}$ becomes a chain complex C_*. Taking homology, we can show that $H_\lambda(C_*) \cong H_\lambda(W, V_0)$ for all λ.

Theorem 5.1 (Basis Theorem)**.** Suppose (W^n, V_0, V_1) is a connected triad, f a Morse function on W having all critical points of index λ all on the same level where $n \geq 6$ and $2 \leq \lambda \leq n-2$, and ξ a gradient-like vector field for f. Given any basis for $H_\lambda(W, V_0)$, there exist a Morse function f' and a gradient-like vector field ξ' for f' agreeing wih f and ξ is a neighborhood of ∂W such that f' has the same critical points of f and the lower disks of those critical points under ξ' determine the given basis.

Proof Let $p_1, \ldots, p_k$ be the critical points of f and denote $b_i = [D_L(p_i)]$ so that $b_1, \ldots, b_k$ is a basis of $H_\lambda(W, V_0)$. By our convention on orientations, the matrix of intersection numbers $I(D_U(p_i), D_L(p_j))$ is the identity $k \times k$ matrix. Consider an oriented λ-disk D embedded in W with $\partial D \subseteq V_0$. Then $[D] \in H_\lambda(W, V_0)$ and therefore there are integers α_i such that $[D] = \alpha_1 b_1 + \cdots + \alpha_k b_k$. Taking intersection numbers,

$$\begin{aligned} I(D_U, D) &= I\big(D_U(p_j), \alpha_1 D_L(p_1) + \cdots + \alpha_k D_L(p_k)\big) \\ &= \alpha_1 I(D_U(p_j), D_L(p_1)) + \cdots + \alpha_k I(D_U(p_j), D_L(p_k)) = \alpha_j. \end{aligned}$$

Thus $[D] = I(D_U(p_1), D)b_1 + \cdots + I(D_U(p_k), D)b_k$.

Using rearrangement we can alter f only on a neighborhood of p_1 to a Morse function g such that $g(p_1) > f(p)$ and the critical points of g are the same as f. Choose a noncritical level V under g between p_1 and the rest of the critical points. Consider the lower sphere S_L of p_1 in V and the upper spheres $S_U(p_i)$ of p_i in V. These spheres are pairwise disjoint. Choose points $a \in S_L$ and $b \in S_U(p_2)$ and let $\phi_0 : (0,3) \to V_0$ be a path in V that intersects S_L transversely at $\phi_0(1) = a$, intersects $S_U(p_2)$ transversely at $\phi_0(2) = b$, and does not intersect any other $S_U(p_i)$. We can extend this to an embedding $\phi : (0,3) \times \mathbb{R}^{n-2} \to V$ where $\phi(s, 0) = \phi_0(s)$, $\phi^{-1}(S_L) = \{1\} \times \mathbb{R}^{\lambda-1} \times \{0\}$, $\phi^{-1}(S_U(p_2)) = \{2\} \times \{0\} \times \mathbb{R}^{n-\lambda-1}$, and ϕ still misses the other $S_U(p_i)$. In addition, we can choose ϕ such that it maps $\{1\} \times \mathbb{R}^{\lambda-1} \times \{0\}$

into S_L with positive orientation and $\phi((0,3) \times \mathbb{R}^{\lambda-1} \times \{0\})$ intersects $S_U(p_2)$ at b with intersection number $+1$. Fix $\delta > 0$ and let $\alpha : \mathbb{R} \to [1, \frac{5}{2}]$ be a smooth function such that $\alpha(x) = 1$ for $x \geq 2\delta$ and $\alpha(x) > 2$ for $u \leq \delta$. There is an isotopy H_t of $(0,3) \times \mathbb{R}^{n-2}$ such that $H_t(1, x, 0) = (t\alpha|x|^2 + 1 - t, x, 0)$ for $x \in \mathbb{R}^{\lambda-1}$ and H_t is the identity outside a compact subset of $(0,3) \times \mathbb{R}^{n-2}$. The functions $F_t : V \to V$ defined by $F_t(v) = (\phi \circ H_t \circ \phi^{-1})(v)$ for $v \in \operatorname{im} \phi$ and $F_t(v) = v$ otherwise forms an isotopy on V. Using F_t, alter ξ to ξ' in a neighborhood of V containing no critical points. Under ξ' the upper spheres $S_U(p_2), \ldots, S_U(p_k)$ are the same while the lower sphere of p_1 is now $S'_L = F_0(S_L)$. Again using rearrangment, let f' be a Morse function putting the critical points back on the same level.

Under ξ' the only lower disk that has changed is $D'_L(p_1)$ which now intersects $D_U(p_2)$ at a single point with intersection number $+1$ and does not intersect any of $D_U(p_3), \ldots, D_U(p_k)$. Hence the basis for $H_\lambda(W, V_0)$ represented by these lower disks is $b_1 + b_2, b_2, \ldots, b_k$. Any changes of basis in the free abelian $H_\lambda(W, V_0)$ can be made up of a combination of such sums or negations of basis elements. Since negations can be obtained by just changing orientation on the corresponding disk, we can obtain any desired basis.

Q.E.D.

Putting together the Basis Theorem and the comments preceding it, we can prove the h-Cobordism Theorem. The proof will precede by first eliminating critical points of indices 0 and 1 and the dual critical points of indices $n-1$ and n. After that, we will use the Basis Theorem to show that the rest cancel as well.

Theorem 5.2 (h-Cobordism Theorem)**.** Let (W^n, V, V') be a triad such that W, V, V' are simply connected, $n \geq 6$, and $H_*(W, V) = 0$. Then W is diffeomorphic to $V \times [0,1]$.

Proof Take a self-indexing Morse function f on W. Using the notation for the chain complex in the comments before the Basis Theorem and working in $\mathbb{Z}_2$ coefficients, $H_0(W, V) = 0$ implies that the map $\partial : H_1(W_1, W_0) \to H_0(W_0, V)$ is surjective. However, ∂ is given by the matrix of intersection numbers modulo 2 of the upper spheres and lower spheres for the index 0 and 1 critical points. Hence for each upper sphere S_U there is at least one lower sphere

S_L such that $I(S_U, S_L) \neq 0 \pmod 2$. Hence $S_L \cap S_U$ contains an odd number of points, and since $S_L \cong S^0$ consists of only two points to begin with, then S_L intersects S_U in exactly one point. Hence the corresponding critical points will cancel, and continuing this way all critical points of index 0 cancel.

Now for each critical point p of index 1 we want to contruct new critical points of indices 2 and 3 so that the index 2 point will cancel with p so that in essence we promote p to index 3. Since f is self-indexing, let V_1 be a noncritical level between the index 1 and index 2 critical points and V_2 a noncritical level between the index 2 and index 3 critical points. Fix a critical point of index 1 and let S_U be the upper sphere of p in V_1. Take a small embedded 1-disk D in V_1 that intersects S_U transversely at a point and not hitting any other upper sphere. We can move the endpoints back along integral curves to V, and since V is simply connected with dimension more than 2, there is a smooth path between them in V that avoids the lower spheres of the index 1 critical points. Lift this path to V_1 so that we have a smooth path between the endpoints of D that avoids all upper spheres. We can smooth these out to get an embedding of S^1 into $S \subseteq V_1$ that intersects S_U transversely in a single point and does not intersect any other upper spheres. In addition, by Lemma 3.2 we can assume that S does not intersect any of the lower spheres of the index 2 critical points. In a small collar neighborhood of V_2 we can take coordinate functions $x_1, \ldots, x_n$ so that f acts as x_n, and by contructing a smooth function $\mathbb{R}^n \to \mathbb{R}$ with nondegenerate critical points of index 2 and 3 we can add critical points q, r to f with $f(q) < f(r)$ such that q has index 2 and r has index 3. Since V_2 is simply connected, we can adjust ξ so that the lower sphere of q coincides with S. Hence p and q can cancel leaving r, the critical point of index 3. We continue to do this until f no longer has any critial points of index 0 or 1.

Replacing f and ξ with $-f$ and $-\xi$ we exchange critical points of index λ with those of index $n - \lambda$. Since $H_*(W, V) = 0$ will also imply that $H_*(W, V') = 0$, then the previous two paragraphs allow us to remove the index 0 and 1 critical points from $-f$. In other words, we can eliminate the critial points of index n and $n - 1$ from f. At this point f only has critical points of index λ with $2 \leq \lambda \leq n-2$.

Recall the chain complex

$$C_{n-2} \xrightarrow{\partial_{n-2}} C_{n-3} \to \cdots \to C_3 \xrightarrow{\partial_3} C_2$$

Since $H_*(W,V) = 0$ and we've shown that $H_\lambda(C_*) \cong H_\lambda(W,V) = 0$, then the sequence is exact. Now for each λ, choose a basis $\{z_1^{\lambda+1}, \cdots, z_{k(\lambda+1)}^{\lambda+1}\}$ for the kernel of the morphism $\partial_{\lambda+1} : C_{\lambda+1} \to C_\lambda$. By exactness, there are $b_i \in C_{\lambda+1}$ such that $\partial_{\lambda+1}(b_i^{\lambda+1}) = z_i^\lambda$ for i from 1 to $k(\lambda)$. Hence $\{z_1^{\lambda+1}, \cdots, z_{k(\lambda+1)}^{\lambda+1}, b_1^{\lambda+1}, \cdots, b_{k(\lambda)}^{\lambda+1}\}$ is a basis for $C_{\lambda+1}$. By the Basis Theorem, since $2 \leq \lambda \leq \lambda+1 \leq n-2$, we can alter the Morse function f and gradient-like vector field ξ so that for each λ the lower disks of the critical points of index λ represent the above basis for C_λ. However, the fact that $\partial_{\lambda+1}(b_i^{\lambda+1}) = z_i^\lambda$ implies that the right and lower spheres of the corresponding critical points of index λ and $\lambda+1$ have intersection number ± 1 and hence we can cancel them. As each critical points of f is in one of these pairs, then by repeating the process we can eliminate all the critical points of f. Therefore we get a Morse function on W with no critical points, so that $W \cong V \times [0,1]$.

Q.E.D.

The h-Cobordism Theorem immediately has important consequences for the classification problem. Almost immediately we can characterize the disk D^n up to diffeomorphism in dimensions at least 6.

Corollary 5.3. Suppose W^n is a compact simply connected manifold with $n \geq 6$. If ∂W is simply connected and W has the homology of a point, then W is diffeomorphic to the n-disk D^n.

Proof Let D_0 be an n-disk in $\text{Int}(W)$. By excision, $H_*(W \setminus \text{Int}(D_0), \partial D_0) \cong H_*(W, D_0) = 0$ and so the pair $(W \setminus \text{Int}(D_0), \partial D_0)$ satisfies the conditions of the h-Cobordism Theorem. Hence $W \setminus \text{Int}(D_0) \cong \partial D_0 \times [0,1] \cong S^{n-1} \times [0,1]$ and therefore $W = (W \setminus \text{Int}(D_0)) \cup D_0 \cong D^n$.

Q.E.D.

With a similar technique as Corollary 5.3 and a little more work, we can finally prove the Generalized Poincaré Conjecture in dimensions 6 and above, characterizing the sphere S^n.

Corollary 5.4 (Generalized Poincaré Conjecture)**.** Let M^n be a closed simply connected smooth manifold with dimension $n \geq 6$ and having the homology of the n-sphere S^n. Then M is homeomorphic to S^n.

Proof Let D_0 be an n-disk embedded in M. By Poincaré duality, excision, and the exact sequence of the pair (M, D_0), for each i we have that

$$H_i(M \setminus \mathrm{Int}(D_0)) \cong H^{n-i}(M \setminus \mathrm{Int}(D_0), \partial D_0) \cong H^{n-i}(M, D_0) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases}$$

Hence by the previous theorem, $M \setminus \mathrm{Int}(D_0)$ is diffeomorphic to D^n and hence M is the union of two n-disks D_1 and D_2 with boundaries identified through some diffeomorphism $h : \partial D_1 \to \partial D_2$. Let $g_0 : D_1 \to S^n$ be an embedding into the southern hemisphere. Each point of D_2 can be written as tv for $v \in \partial D_2$ and $0 \leq t \leq 1$. Define $g : M \to S^n$ by $g(u) = g_0(u)$ for $u \in D_1$ and $g(tv) = \sin \frac{\pi t}{2} g_0(h^{-1}(v)) + \cos \frac{\pi t}{2}(0, \ldots, 0, 1)$ for $tv \in D_2$. Then g is a homeomorphism between M and S^n.

Q.E.D.

Unlike Corollary 5.3, the Generalized Poincaré Conjecture only guarantees a homeomorphism. In dimension 7, Milnor [17] constructed an example of a manifold Σ^7 that satisfies the conditions of the GPC, yet is not diffeomorphic to S^7. Such manifolds are called *exotic spheres* and there are exotic spheres in most dimensions 7 and above. Examining the proof of the GPC above, we can see that an exotic sphere in dimension n is at least diffeomorphic to two copies of the n-disk D^n attached along their boundaries by a diffeomorphism. In dimensions 5 and 6, a theorem by Kervaire and Milnor [21] [22] states that that a manifold M^n satisfying the conditions of GPC for $n = 5$ or 6 must be the boundary of a compact, contractible smooth manifold. Hence by Corollary 5.3, M is in fact the boundary of the disk D^{n+1} and therefore diffeomorphic to S^n.

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