Retrocession for Portfolio Optimization in Reinsurance

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Abstract

Reinsurance is the insurance protection of an insurance company. Retrocession is reinsurance for a portfolio of reinsurance contracts. Reinsurance portfolios can comprise several thousand contracts that may be contingent on the same events, which makes retrocession a complex decision. This thesis develops an optimization model for retrocession, where the aim is to maximize the expected result and satisfy contraints on risk. A review and development of risk measures that can be included in the model is performed. The optimization model is implemented and applied to a large portfolio of reinsurance contracts using mathematical programming algorithms. Results suggest that a benefit amounting to several percent of the annual expected result may be obtained by applying optimal retrocession to the reinsurance portfolio. The results depend on several assumptions that, if not fulfilled, may diminish the benefit.

Keywords: Reisurance, Retrocession, Portfolio Optimization, Risk Measures, Tail-Value-at-Risk.

Master's Thesis
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Retrocession för optimering av återförsäkringsportföljer

Sammanfattning

Återförsäkring är försäkringsskydd för försäkringsbolag. Retrocession är återförsäkring för en portfölj av återförsäkringskontrakt. Återförsäkringsportföljer kan bestå av flera tusen kontrakt som kan vara beroende av samma händelser, vilket gör beslutsfattande om retrocession komplext. Denna rapport utvecklar en optimeringsmodell för retrocession, med målet att maximera det förväntade resultatet samt uppfylla begränsningar på risk. En överblick och utveckling av riskmått som kan tillämpas i modellen genomförs. Optimeringsmodellen implementeras och tillämpas på en stor återförsäkringsportfölj genom att använda optimeringsalgoritmer. Utfallet av optimeringen indikerar att en förbättring på flera procent av det årliga förväntade resultatet skulle kunna uppnås genom att tillämpa optimal retrocession på återförsäkringsportföljen. Utfallet beror dock på att flera antaganden gäller. Om antagandena inte gäller så kan förbättringen utebli.

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Contents

Nomenclature

Cede	In the context of reinsurance: Transferring a share of insurance business to a reinsurer. Premiums are *ceded* to the reinsurer, who then stands to cover a share of losses, the *ceded* losses.
Cession	See under Cede.
Cover	Cover refers to the amount of protection against risks that is provided by an insurance contract.
Direct Insurer	Insurance company that has a direct contractual relationship with a client who is exposed to the original risk requiring insurance.
Excess of Loss Reinsurance	A form of non-proportional reinsurance.
Gross Portfolio	Portfolio of all underwritten reinsurance, taken before the impact of retrocession contracts.
Gross Exposure	Exposure to the gross portfolio, see under Gross Portfolio.
Gross Loss	Loss incurred before the impact of retrocession contracts.
Non-Proportional Reinsurance	A reinsurance contract that does not qualify as proportional. Ceded premiums, losses or both can be allocated according to a non-linear function.
Overriding Commission	A percentage commission that the retrocessionaire pays to a ceding reinsurer for retrocession.
Premium Principle	Function or principle specifying the relation between ceded exposure to losses and ceded premiums.
Profit Commission	A form of reinsurance commission that is paid to the cedent if the ceded portfolio is profitable
Proportional Reinsurance	Reinsurance protection where premiums and losses are allocated between cedent and reinsurer according to a predefined ratio.
Reinsurance	Transfer of insurance risk from an insurer to another party, the reinsurer.
Reinsurer	Company that accepts a share of an insurer's risk in return for payment of a premium [15].
Retained Share	The share of the exposure to losses that is retained after proportional retrocession
Retrocession	Passing on of risk by the reinsurer to other insurance companies [15].
Retrocessionaire	A reinsurer who insures another reinsurer.
Underwriting	The process of selling reinsurance protection.

1 Introduction

Insurance is an agreement whereby one party seeks protection against a risk by finding another party, the insurance company, that is willing to cover part of a loss pertaining to the risk, should it occur. In return for the cover the insurance company receives proper compensation from the first party. Reinsurance arises when there are some risks that the insurance company, in its turn, does not want to be exposed to. If the insurer deems some risks to be too large or is seeking to reduce overall exposure, then the insurer can seek insurance protection from a reinsurer.

Definition 1 (Cession)**.** The insurer is said to *cede* business to the reinsurer in the situation described above. Cession, in the context of insurance, is used to describe when one party allocates exposure to some losses and some premium revenues to another party.

In modern reinsurance it is also common for the reinsurer to seek cover from another reinsurer, which is called retrocession. Retrocession can have many different goals, such as improving profitability, reducing variability, reducing the risk of ruin or complying with regulatory standards. Reinsurance portfolios may be comprised of several thousand contracts that cover diverse risks, from earthquakes to liability [9]. Performing appropriate retrocession to protect such a portfolio can be complex, especially if the retrocession has several goals.

The purpose of this thesis is to find an optimal reinsurance portfolio given certain preferences regarding result and risk. Retrocession decisions will be modelled as an optimization problem for which optimal decisions can be determined by mathematical programming algorithms. Approaches to measuring risk from both the financial mathematics literature and the insurance literature are thoroughly reviewed and developed to underpin the model.

The thesis is structured as follows. The rest of this section gives an introduction to reinsurance. Section 2 is devoted to giving a theoretical background on optimization and the measurement of risk. Section 3 further develops risk measures found in the literature. Section 4 deals with the modelling of retrocession as an optimization problem. In Section 5 the results of optimizing a large reinsurance portfolio are presented. Finally, Section 6 concludes the thesis.

1.1 Reinsurance and Retrocession

Reinsurance is the insurance protection of an insurer. In order to distinguish between different forms of insurance, the party who has a contractual relationship with the client that is exposed to the original risk (for example an airplane) is called a *direct insurer* whereas an insurer who only has a contractual relationship to another insurer is called a *reinsurer*. The need for reinsurance can arise because a direct insurer has too much exposure to one individual risk or because protection is needed in the case of a catastrophe. In some cases the reinsurer may deem itself to have too large exposure to individual risks or there might be a reason to change the composition of the portfolio of reinsurance contracts. In such an instance the reinsurer can elect to retrocede risk, that is, to buy insurance cover from another reinsurer [15].

Several different types of reinsurance cover are available. This section will give definitions of different covers and highlight the options that are available in retrocession. Furthermore some other aspects of reinsurance contracts will be outlined insofar as they are important to take into account when modelling a reinsurance portfolio.

Reinsurance often takes the form of a contract signed between insurer and reinsurer but there are other forms as well, such as catastrophe bonds that are sold to investors [9]. The important factor in optimization of the portfolio is what protection is supplied by the market and the cost of that protection. Thus, the purpose of this section will be to highlight the available flexibility in risk transfer and the payment terms for different types of contracts.

The retrocession contracts considered in this report mainly apply to classes of reinsurance contracts. An example might be for the reinsurer to cede part of the exposure to earthquakes in Japan.

1.1.1 Proportional Reinsurance

The main principle in proportional reinsurance is that premiums and losses are divided between the reinsurer and the direct insurer at a predefined ratio. However, as the direct insurer incurs costs in the acquisition and administration of the individual contracts, a commission is usually allocated to the direct insurer. The commission is often contractually defined as a percentage of the original premiums [15].

Let γ be a class of insurance contracts for which the proportional reinsurance is bought and let D_γ denote the losses in that class of contracts. The losses retained by the direct insurer can then be expressed as

$$D_\gamma^{\text{Retained}} = (1 - x_\gamma) \cdot D_\gamma$$

where $x_\gamma \in (0, 1)$ is the share of the total exposure to losses that is ceded for the class γ. Furthermore, the premium retained by the direct insurer is given by

$$P_\gamma^{\text{Retained}} = (1 - x_\gamma(1 - c)) \cdot P_\gamma \tag{1.1}$$

where P_γ is the sum of the original premiums of all contracts in the class γ and where $c \in [0, 1]$ is the reinsurance commission.

When one reinsurer cedes risk to another reinsurer the situation might be different from the one above. Acquisition and administration costs are likely to be higher for a direct insurer and thus a higher commission would be motivated in the above case. Reinsurance expertise at Sirius International have stated that the ceding party in retrocession typically receives a commission and that the above premium principle is in use. In the case of retrocession, the above commission is called the *overriding commission* [9].

Sometimes a profit commission is also included, in addition to the overriding commission. The profit commission is an amount returned to the ceding reinsurer when the ceded portfolio is profitable. The profit commission will be introduced by showing how to calculate the ceding reinsurer's result on the class of contracts that is subject to cession. Let R_γ denote the result and let $c_1, c_2, c_3 \in [0, 1]$ be constants that are known and pertain to the contract for retrocession. The result with profit commission is then

$$R_\gamma = P_\gamma(1 - x_\gamma) - D_\gamma(1 - x_\gamma) + x_\gamma c_1 P_\gamma + c_2 x_\gamma \max(P_\gamma - D_\gamma - c_1 P_\gamma - c_3 P_\gamma, 0). \tag{1.2}$$

The result R_γ in (1.2) includes an overriding commission, which is represented by c_1. c_3 is a base profit for the reinsurer selling the cover and c_2 is the share of excess profits returned.

Note that although the profit commission is determined by the non-linear maximum function it is still linear in the retrocession share x_γ.

1.1.2 Non-proportional Reinsurance

Reinsurance that is not proportional can allocate losses, premiums or both according to a non-linear function. For a prospective non-proportional cover sold to a direct insurer, the reinsurer demands a share of premiums that it finds suitable [15]. The most common form of non-proportional reinsurance is excess of loss reinsurance [9]. In excess of loss reinsurance, the ceding insurer has to bear all losses related to an event if they are smaller than the *deductible* but losses above the deductible are fully covered by the reinsurer, up to a certain *limit* [15].

Definition 2 (Limit and Deductible)**.** The *deductible* of a reinsurance contract is a number specifying the smallest loss that can give rise to a transfer of losses to the reinsurer.

The *limit* of a reinsurance contract specifies the maximum amount of losses that the reinsurer can be allocated under the contract.

It will be assumed here that the excess of loss contract covers losses pertaining to a class of contracts, γ, for a certain time period. To express the contract in mathematical terms, let d_γ denote the deductible for an excess of loss cover on the class of contracts γ and let l_γ denote the corresponding cover limit. The losses incurred by the party buying the excess of loss cover, $D_\gamma^{\text{Retained}}$, can then be expressed as

$$D_\gamma^{\text{Retained}} = \min(D_\gamma, d_\gamma) + \max(D_\gamma - l_\gamma - d_\gamma, 0). \tag{1.3}$$

Excess of loss reinsurance can also be limitless. The rule for allocation of losses to the ceding party is then

$$D_\gamma^{\text{Retained}} = \min(D_\gamma, d_\gamma). \tag{1.4}$$

Note that the excess of loss cover with a limit above basically amounts to buying a limitless excess of loss cover at d_γ and selling a limitless excess of loss cover at $l_\gamma + d_\gamma$ because

$$D_\gamma^{\text{Retained}} = \min(D_\gamma, d_\gamma) + \max(D_\gamma - l_\gamma - d_\gamma, 0) = \min(D_\gamma, d_\gamma) + D_\gamma - \min(D_\gamma, l_\gamma + d_\gamma). \tag{1.5}$$

The above specification of excess of loss ignores some subtleties in how the excess of loss contract can be specified. For an overview of a few different specifications see [15]. To completely describe the different types of excess of loss contracts it could be necessary to introduce many reinsurance contract classes for one single excess of loss retrocession contract. However, henceforth it will be assumed that each non-proportional contract covers one class γ as above.

There is little literature covering the premium principles for non-proportional retrocession but it can be of interest to note that for non-proportional reinsurance in general there is extensive literature, see for example [3], [10] and [11].

1.1.3 Details on the Reinsurance Contract

The previous sections convey how risk is transferred by a reinsurance contract and how the cost of such a contract is specified. Cost and risk transfer are central to retrocession decision making but there are also other characteristics of a prospective contract that need to be considered. Outlined below are such characteristics and how they affect a decision on retrocession.

1. *Multiplicative contracts* is a term here used to describe reinsurance contracts that are layered on top of each other. This is illustrated by way of an example. Suppose that a contract covers 30% of all losses in Germany and another contract covers 10% of losses in Europe. If the contracts are layered, a loss in Germany will be reduced by 10% by the Europe-wide contract only after deducting the 30% cover of the German contract. Thus the total loss allocated to the Europe-wide contract will only be $0.7 \cdot 0.1 = 0.07 = 7\%$ of the total loss. Practically, this form of contract leads to complicated relationships between all the parties involved.

2. *Additive contracts* are here used to describe contracts that act in parallel, which is the alternative to layered contracts. Drawing on the same example, the total cover for any loss in Germany will be $10\% + 30\% = 40\%$. According to expertise at Sirius International, the usual form for reinsurance contracts is the additive one.

3. *Reinstatement premium.* Once a reinsurance contract has been allocated a loss it is usual to have a *reinstatement* of the contract. A reinstatement is the payment of a premium, the reinstatement premium, to continue the cover offered by the contract [9]. To the extent that reinstatement premiums are obligatory they need to be included when calculating the simulations based result of a contract.

4. *Minimum retention.* In reinsurance situations it may not be possible to cede all risk in a portfolio. In the case of proportional reinsurance this implies that there is a lower bound, such as 30%, for the share of losses that are retained.

2 Theoretical Background

2.1 Optimization

Optimization is the pursuit of the "best" solution to a problem. A typical optimization model is composed of a specific objective that depends on a set of decisions. Several constraints may exist on what the permissible decisions are. An optimization model is an attempt to express such a problem in mathematical terms. This section will very briefly present the theory of optimization models.

An optimization model includes a set of variables that represent decisions to be made, a mathematical function to be optimized (the objective function) and a set of constraints that define the allowable decisions. A linear optimization model (also known as a *linear program*) is an optimization model where the function to be optimized and the constraints are linear in the decision variables [8].

Suppose that there are n decision variables, denoted x_j, $j = 1, \dots, n$, and that for each decision variable there is an associated cost, c_j. Suppose further that there are m constraints that need to be satisfied. Then the model can be written as

$$\begin{aligned} \text{minimize} \quad & z = \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \;\; i = 1, \dots, m. \end{aligned} \tag{2.1}$$

By introducing matrix notation the model can be written more succinctly as

$$\begin{aligned} \text{minimize} \quad & z = c^T x \\ \text{subject to} \quad & \\ & Ax \leq b, \end{aligned} \tag{2.2}$$

where $c = (c_1, \dots, c_n)^T$, $x = (x_1, \dots, x_n)^T$ and $b = (b_1, \dots, b_m)^T$. A is an m-by-n matrix with the coefficients of the linear constraints, a_{ij}. A *feasible point* of (2.2) is a point in the space of the decision variables that satisfies $Ax \leq b$.

By using appropriate adjustments of the problem (2.2) it is possible to include a quite general linear program. Maximizing z is equivalent to minimizing $-z$. It is also possible to include equality constraints in the model.

The problem (2.2) either has a *finite optimal solution*, is *unbounded* or has *no feasible points*. Several algorithms are available to find a feasible point of (2.2), and if one exists, to determine whether the problem is bounded, and if it is, to find a feasible optimal solution. The most widely used algorithm to solve linear programming problems is the simplex algorithm. Extended versions of the algorithm exist that guarantee encountering an optimal feasible solution or the categorization of the problem as either unbounded or unfeasible [8].

Generally, optimization problems may be non-linear. The optimization model can then be written as

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & \\ & g_i(x) \leq 0, \text{ for } i = 1, 2, \ldots, m. \end{aligned} \tag{2.3}$$

In (2.3) $f(\cdot)$ is the objective function, which can be non-linear. The functions $\{g_i(\cdot)\}_{i=1}^m$ are constraint functions that are possibly non-linear.

When the problem is non-linear it is possible that there are multiple local minima. A local minimum is some x^* for which it holds that $f(x) \geq f(x^*)$ in some neighbourhood of x^*. For the case of linear programs it is possible to find a global minimizer but when the model is non-linear with multiple local minima it may not be possible to guarantee that a local minimizer of $f(\cdot)$ is a global minimizer of $f(\cdot)$ [8].

An important exception is when the problem (2.3) is a convex optimization problem.

Definition 3 (Convex Function)**.** A function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is convex if for every $x, y \in \mathbb{R}^n$ and every $\lambda \in (0, 1)$ it holds that

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \tag{2.4}$$

If strict inequality holds in (2.4) then the function is *strictly convex*.

The problem (2.3) is a convex optimization problem if the function $f(\cdot)$ is convex and the functions $g_i(\cdot)$ are convex for $i = 1, 2 \ldots, m$.

Using (2.4) it can be verified that the linear optimization (2.2) is in fact convex. In general, when a non-linear optimization problem is convex it holds that a local minimizer is also a global minimizer. Furthermore, the global minimizer is unique when the objective function is strictly convex .

2.2 Risk Measures

Activities that are undertaken to achieve any type of gain often have an associated uncertainty regarding the eventual outcome. In finance and insurance the variability of the outcome can be considerable. In such a circumstance it might be possible, and desirable, to obtain an approximate probability distribution for the outcomes, based on experience, modelling or other means. Often, risk measures are defined as a function of a probability distribution with the aim of distilling the distribution to a single number that represents the risk that arises from the variability. In this thesis it is assumed that risk is measured based on the portfolio outcome for one year.

Two broad categories of risk measures are introduced in this thesis. First, risk measures that depend on the quantile of a given outcome are presented. Second, risk measures that depend on the size of the outcome are presented and developed, in a separate section below. The development is carried out to allow comparison and identification of the merits of each approach.

To introduce risk measures mathematically, let X be a stochastic variable on a probability space Ω, $X : \Omega \to \mathbb{R}$, and let P be the associated probability measure. A risk measure will be denoted by $\rho(X)$ and ρ is a function $\rho : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ from the space of stochastic variables $\mathcal{X}$ to the extended real line.

Remark 1 (Sign Convention). Throughout this report the variable measured with a risk measure, X, is assumed to represent the negative of a portfolio result, or the portfolio loss function. This implies that gains are represented by negative values of X and losses by positive values. The risk measure is then of large absolute value and positive when there is a large risk. Considering liabilities as positive and assets as negative is referred to as an actuarial view in [12].

Artzner et al. [4] introduce the concept of a coherent risk measure as one that satisfies a number of properties. The properties are given below, with interpretations, in order to underpin the discussion of what an appropriate risk measure is. The presentation is similar to the one in [7].

Definition 4. A risk measure is defined to be *convex* if ρ is a convex function, that is, if given any $\lambda \in (0,1)$ and any $X, Y \in \mathcal{X}$, it holds that

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y). \tag{2.5}$$

Definition 5 (Monetary risk measure)**.** A risk measure ρ is called a *monetary risk measure* if $\rho(0)$ is finite and for all $X, Y \in \mathcal{X}$, ρ satisfies

- *Monotonicity*: If $X \leq Y$, then $\rho(X) \leq \rho(Y)$
- *Cash (translation) invariance*: If $m \in \mathbb{R}$, then $\rho(X - m) = \rho(X) - m$.

The definition of cash invariance is important because it allows the difference between a given risk tolerance and the risk measure to be interpreted as the minimum amount of capital that has to be added as a risk-free amount to put the risk level of the portfolio in accordance with the risk tolerance.

Definition 6 (Positive homogeneity)**.** A risk measure ρ is defined to be *positively homogeneous* if for $\lambda \geq 0$ it holds that $\rho(\lambda X) = \lambda\rho(X)$.

Definition 7 (Subadditivity)**.** The risk measure ρ is *subadditive* if for all $X, Y \in \mathcal{X}$

$$\rho(X + Y) \leq \rho(X) + \rho(Y). \tag{2.6}$$

Subadditivity implies that the combination of any two portfolios does not have a higher risk than the two portfolios on a stand-alone basis. Convexity is equivalent to subadditivity for a positively homogeneous risk measure.

2.2.1 Quantile Risk Measures

One widely used risk measure is Value-at-Risk (VaR), which measures the maximum loss at a specified confidence level, $\alpha > 0$ [18] . Value-at-Risk measures the quantile of α. For a formal definition, let X be a stochastic variable as above and let $F(x)$ be the distribution function of X, $F(x) = P(X \leq x)$.

Definition 8 (Value-at-Risk)**.** Value-at-Risk of a stochastic variable is defined as

$$\mathrm{VaR}_\alpha(X) = F^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}. \tag{2.7}$$

VaR is positively homogenous but it has some undesirable properties. First, it is not subadditive, which implies that combining two portfolios can increase overall risk. To see how risk is increased, consider the following example.

Example 1. Consider the sum of two independent and identically distributed stochastic variables, X_i, $i = 1, 2$, that have

$$P(X_i = 1) = 0.04$$

$$P(X_i = 0) = 0.96, \text{ for } i = 1, 2.$$

Then $\mathrm{VaR}_{0.95}(X_i) = 0$ for $i = 1, 2$ but $\mathrm{VaR}_{0.95}(X_1 + X_2) = 1$. $\mathrm{VaR}_\alpha(\cdot)$ is not convex as it lacks subadditivity, which implies that an optimization involving minimization of VaR can have multiple local extrema [18].

Second, VaR does not take the severity of improbable outcomes into account, which is illustrated by the following example.

Example 2. Suppose that there is an opportunity to sell protection against either earthquakes or windstorms. Let X_1 denote the loss variable if protecting against earthquakes and let X_2 denote the loss if protecting against windstorms. Suppose that

$$P(X_1 = 60) = 0.01 \text{ and } P(X_1 = 0) = 0.99$$

$$P(X_2 = 1) = 0.05 \text{ and } P(X_2 = 0) = 0.95.$$

Then $\mathrm{VaR}_{0.95}(X_1) = 0$ but $\mathrm{VaR}_{0.95}(X_2) = 1$. In this case VaR asserts that the risk is *higher* in the case of protecting against windstorms because the severity of the 1% outcome is not taken into account.

A measure of risk that does not share these shortcomings of VaR is Tail-Value-at-Risk (TVaR). Tail-Value-at-Risk measures the expected loss conditional on the outcome being in the $(1-\alpha)$-tail of the distribution. Loosely stated, Tail-Value-at-Risk measures the expected loss conditional on the loss being larger than or equal to Value-at-Risk. It is, however, necessary to make an adjustment for the case where $P(X = \mathrm{VaR}_\alpha(X)) > 0$. Two alternative definitions are given below. The first definition is adapted from the definition of Expected Shortfall in [2].

Definition 9 (Tail-Value-at-Risk)**.** The Tail-Value-at-Risk is defined by

$$\mathrm{TVaR}_\alpha(X) = \frac{1}{1-\alpha}\mathbb{E}\left[X\mathbf{1}^{(\alpha)}_{\{X \geq \mathrm{VaR}_\alpha(X)\}}\right]. \tag{2.8}$$

The function $\mathbf{1}^{(\alpha)}_{\{X \geq x\}}$ is defined by

$$\mathbf{1}^{(\alpha)}_{\{X \geq x\}} = \begin{cases} \mathbf{1}_{\{X \geq x\}}, & \text{if } P(X = x) = 0 \\ \mathbf{1}_{\{X \geq x\}} + \frac{1-\alpha-P(X \geq x)}{P(X=x)}\mathbf{1}_{\{X = x\}}, & \text{if } P(X = x) > 0 \end{cases} \tag{2.9}$$

where

$$\mathbf{1}_{\{X \geq x\}} = \begin{cases} 1 & \text{if } X \geq x \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

and analogously for the equality condition.

The adjusted indicator function $\mathbf{1}^{(\alpha)}_{\{X \geq x\}}$ is introduced to account for the case where $P(X = \mathrm{VaR}_\alpha(X)) > 0$. The adjusted indicator function has the following properties

$$\mathbf{1}^{(\alpha)}_{\{X \geq \mathrm{VaR}_\alpha(X)\}} \in [0, 1] \text{ and} \tag{2.11}$$

$$\mathbb{E}\left[\mathbf{1}^{(\alpha)}_{\{X\geq \mathrm{VaR}_\alpha(X)\}}\right] = 1-\alpha. \tag{2.12}$$

Proposition 10. *TVaR as defined by* (2.8) *is positively homogenous, monotonic, cash invariant, subadditive and convex.*

Proof. Monotonicity. Suppose $X, Y \in \mathcal{X}$ are stochastic variables on the probability space Ω and that $X \leq Y$. Then, taking the difference between the Tail-Value-at-Risk of Y and X,

$$\begin{aligned}(1-\alpha)\left(\mathrm{TVaR}_\alpha(Y)-\mathrm{TVaR}_\alpha(X)\right) &= \mathbb{E}\left[Y\mathbf{1}^{(\alpha)}_{\{Y\geq \mathrm{VaR}_\alpha(Y)\}} - X\mathbf{1}^{(\alpha)}_{\{X\geq \mathrm{VaR}_\alpha(X)\}}\right]\\ \geq \mathbb{E}\left[Y\mathbf{1}^{(\alpha)}_{\{Y\geq \mathrm{VaR}_\alpha(Y)\}} - Y\mathbf{1}^{(\alpha)}_{\{X\geq \mathrm{VaR}_\alpha(X)\}}\right]&\\ \geq \mathrm{VaR}_\alpha(Y)\mathbb{E}\left[\mathbf{1}^{(\alpha)}_{\{Y\geq \mathrm{VaR}_\alpha(Y)\}} - \mathbf{1}^{(\alpha)}_{\{X\geq \mathrm{VaR}_\alpha(X)\}}\right] &= \mathrm{VaR}_\alpha(Y)((1-\alpha)-(1-\alpha)) = 0.\end{aligned} \tag{2.13}$$

The first inequality follows directly from $X \leq Y$. The second inequality follows from the fact that

$$\begin{cases}\mathbf{1}^{(\alpha)}_{\{Y\geq \mathrm{VaR}_\alpha(Y)\}} - \mathbf{1}^{(\alpha)}_{\{X\geq \mathrm{VaR}_\alpha(X)\}} \geq 0 & \text{if } Y > \mathrm{VaR}_\alpha(Y)\\ \mathbf{1}^{(\alpha)}_{\{Y\geq \mathrm{VaR}_\alpha(Y)\}} - \mathbf{1}^{(\alpha)}_{\{X\geq \mathrm{VaR}_\alpha(X)\}} \leq 0 & \text{if } Y < \mathrm{VaR}_\alpha(Y)\end{cases} \tag{2.14}$$

and that equality holds when $Y = \mathrm{VaR}_\alpha(Y)$ due to linearity of the expectation.
Cash invariance can be shown using the linearity of expectation, (2.12) and a manipulation to demonstrate that the adjusted indicator function will be the same.
Positive homogeneity. VaR is positively homogeneous so linearity of the expectation directly implies positive homogeneity for $\mathrm{TVaR}_\alpha(\cdot)$.
Subadditivity. Introduce a new stochastic variable $Z = X + Y$. Then

$$\begin{aligned}&(1-\alpha)(\mathrm{TVaR}_\alpha(X)+\mathrm{TVaR}_\alpha(Y)-\mathrm{TVaR}_\alpha(Z))\\ &= \mathbb{E}\left[X\mathbf{1}^{(\alpha)}_{\{X\geq \mathrm{VaR}_\alpha(X)\}} + Y\mathbf{1}^{(\alpha)}_{\{Y\geq \mathrm{VaR}_\alpha(Y)\}} - Z\mathbf{1}^{(\alpha)}_{\{Z\geq \mathrm{VaR}_\alpha(Z)\}}\right]\\ &= \mathbb{E}\left[X\left(\mathbf{1}^{(\alpha)}_{\{X\geq \mathrm{VaR}_\alpha(X)\}} - \mathbf{1}^{(\alpha)}_{\{Z\geq \mathrm{VaR}_\alpha(Z)\}}\right) + Y\left(\mathbf{1}^{(\alpha)}_{\{Y\geq \mathrm{VaR}_\alpha(Y)\}} - \mathbf{1}^{(\alpha)}_{\{Z\geq \mathrm{VaR}_\alpha(Z)\}}\right)\right]\\ &\geq \mathrm{VaR}_\alpha(X)\mathbb{E}\left[\left(\mathbf{1}^{(\alpha)}_{\{X\geq \mathrm{VaR}_\alpha(X)\}} - \mathbf{1}^{(\alpha)}_{\{Z\geq \mathrm{VaR}_\alpha(Z)\}}\right)\right]\\ &\qquad + \mathrm{VaR}_\alpha(Y)\mathbb{E}\left[\left(\mathbf{1}^{(\alpha)}_{\{Y\geq \mathrm{VaR}_\alpha(Y)\}} - \mathbf{1}^{(\alpha)}_{\{Z\geq \mathrm{VaR}_\alpha(Z)\}}\right)\right]\\ &= \mathrm{VaR}_\alpha(X)((1-\alpha)-(1-\alpha)) + \mathrm{VaR}_\alpha(Y)((1-\alpha)-(1-\alpha)) = 0.\end{aligned} \tag{2.15}$$

The inequality above follows from exactly the same argument as the one made when proving monotonicity above but applied to each of the terms. The expectation of the adjusted indicator function is obtained from (2.12).

Convexity then follows from TVaR being subadditive and positively homogenous. □

An alternative definition of TVaR that lends itself to optimization is given in [16].

Definition 11 (Tail-Value-at-Risk alternative definition)**.**

$$\mathrm{TVaR}_\alpha(X) = \inf_{\phi\in\mathbb{R}}\left\{\phi + \frac{1}{1-\alpha}\mathbb{E}[X-\phi]^+\right\}. \tag{2.16}$$

The definitions (2.16) and (2.8) can be shown to be equivalent [18, 2]. When the solution to the optimization problem in (2.16) is unique, then $\phi_{\text{inf}} = \mathrm{VaR}_\alpha(X)$ [16].

The risk measures presented above all depend on a single confidence level. A question that might arise is whether it is possible to define a risk measure that depends on multiple confidence levels. For example, a user of risk measures might perceive expected losses at one significance level to be offset if losses are small at another significance level. A suitable risk measure might then be $\rho(X) = \mathrm{TVaR}_{\alpha_1}(X) + \mathrm{TVaR}_{\alpha_2}(X)$ where α_1 and α_2 are two different significance levels.

Acerbi [1] presents a generalization of the tail-value-at-risk which allows for consideration of several different confidence levels, or a continuous spectrum of confidence levels.[1] The first definition of a risk measure in [1] is

$$M_\mu(X) = \int_0^1 d\mu(\alpha)(1-\alpha)\mathrm{TVaR}_\alpha(X), \tag{2.17}$$

where $d\mu(\alpha)$ is a measure on $\alpha \in [0,1]$. The risk measure defined by (2.17) indeed makes it possible to consider different confidence levels when expressing risk preferences. In order for (2.17) to be cash invariant, monotonic and subadditive it is necessary that

$$\int_0^1 (1-\alpha)d\mu(\alpha) = 1. \tag{2.18}$$

By using the definition of TVaR and interchanging integrals it is shown that the risk measure defined by (2.17) has an alternative representation as the integral over a continuous risk spectra

$$\phi(p) = \int_0^p d\mu(\alpha). \tag{2.19}$$

Definition 12 (Spectral risk measure)**.** The spectral risk measure presented in [1] is defined as

$$M_\phi(X) = \int_0^1 \mathrm{VaR}_p(X)\phi(p)dp \tag{2.20}$$

where $\phi \in \mathcal{L}^1([0,1])$.

In [1] it is shown that the conditions that ϕ be positive, increasing and that $\|\phi\| = 1$ imply that $M_\phi(X)$ is convex, monotonic, positively homogeneous and cash invariant. Additionally, it is shown that the continuous measure (2.20) can be estimated by convex combinations of simulated outcomes,

$$M_\phi^{(N)}(X) = \sum_{i=1}^N X_{i:N}\phi_i \tag{2.21}$$

where $X_{i:N}$ are ordered realizations of the stochastic variable X and

$$\phi_i = \frac{\phi(\frac{i}{N})}{\sum_{k=1}^N \phi(\frac{k}{N})}, \qquad i \in \{1,\dots,N\}. \tag{2.22}$$

It is further shown that, under suitable conditions on $\phi(\cdot)$ and the stochastic variables being measured, the sequence of approximations converges to the continuous risk measure (2.20) almost surely.

The spectral risk measures presented in [1] are especially interesting to consider in relation to the following section, where weighting with respect to the size of losses is considered. The conditions that ϕ be positive and increasing correspond especially well to the cases presented below.

[1] The theory presented in [1] has here been adjusted to correspond to the conventions used in this thesis.

3 Development of Risk Measures

3.1 Loss Size Risk Measures

When deciding how much a given outcome should contribute to the total riskiness of a portfolio one alternative is to consider its relative position in the distribution of outcomes, as in the previous section. Another alternative that is suggested in [12] is to let the riskiness of an outcome depend on the size of that outcome.

The following definition introduces a risk measure that depends only on the size of outcomes or losses. The definition is a variation of the one in [12]. The reason for using a different definition is given in Appendix A.

Definition 13 (Weighted Outcome Risk)**.** If X is a stochastic variable for which the probability density function $f(x)$ exists. The weighted outcome risk is then defined as

$$\rho(X) = \mathrm{WR}_L(X) = \int_{-\infty}^{\infty} L(x)xf(x)dx. \tag{3.1}$$

The risk measure (3.1) can be considered in the context of the properties presented in section 2.2. To show convexity, it is sufficient to show that the stochastic variable being integrated over the probability space is convex, which is the content of the following proposition.

Proposition 14. *The risk measure defined by*

$$\rho(X) = \int_{-\infty}^{\infty} R(x)f(x)dx \tag{3.2}$$

is convex in the sense of (2.5) *if and only if the function* $R(x)$ *is convex.*

Proof. The convexity of the risk measure $\rho(X)$ is a direct consequence of the linearity of the integral.

$$\begin{aligned} \rho(\lambda X + (1-\lambda)Y) &= \int_{-\infty}^{\infty} R(\lambda x + (1-\lambda)y) f_{X,Y}(x,y)dxdy \leq \\ \leq \int_{-\infty}^{\infty} (\lambda R(x) + (1-\lambda)R(y)) f_{X,Y}(x,y)dxdy &= \\ = \lambda \int_{-\infty}^{\infty} R(x)f_X(x)dx + (1-\lambda)\int_{-\infty}^{\infty} R(y)f_Y(y)dy &= \lambda\rho(X) + (1-\lambda)\rho(Y) \end{aligned} \tag{3.3}$$

The necessity of $R(x)$ being convex is proved similarly. □

Corollary 15. *The risk measure defined by* (3.1) *is convex if any of the following are satisfied:*

(i) The function $L(\cdot)$ *is convex, non-decreasing and the stochastic variables in the set being measured,* $X \in \mathcal{X}$*, are non-negative.* $X \in \mathcal{X}$ *is non-negative if for all* $\omega \in \Omega$ *it holds that* $X(\omega) \geq 0$*.*

(ii) The function $L(\cdot)$ *is non-decreasing, convex on* $(0, \infty)$ *and concave on* $(-\infty, 0)$*.*

Proof. In order to prove that the risk measure defined by (3.1) is convex when *(i)* above holds consider the definition of convexity in (2.5).

$$\begin{aligned} \rho(\lambda X + (1-\lambda)Y) &= \mathbb{E}\left[L(\lambda X + (1-\lambda)Y)(\lambda X + (1-\lambda)Y)\right] \leq \\ &\leq \mathbb{E}[(\lambda L(X) + (1-\lambda)L(Y))(\lambda X + (1-\lambda)Y)] = \\ &= \mathbb{E}[\lambda^2 L(X)X + (1-\lambda)^2 L(Y)Y] + \mathbb{E}[\lambda L(X)(1-\lambda)Y + (1-\lambda)L(Y)\lambda X] \end{aligned} \tag{3.4}$$

where the inequality follows from the fact that $L(\cdot)$ is convex and that $\lambda X + (1-\lambda)Y \geq 0$ because X and Y do not take negative values. Now note that as $L(\cdot)$ is non-decreasing it holds that $(X-Y)(L(X)-L(Y)) \geq 0$, which implies that

$$\mathbb{E}[XL(Y) + YL(X)] \leq \mathbb{E}[XL(X) + YL(Y)]. \tag{3.5}$$

By applying (3.5) to the last member of (3.4) it follows that

$$\begin{aligned} \rho(\lambda X + (1-\lambda)Y) &\leq \mathbb{E}[\lambda^2 L(X)X + (1-\lambda)^2 L(Y)Y] \\ &\quad + \mathbb{E}[\lambda L(X)(1-\lambda)X + (1-\lambda)L(Y)\lambda Y] \\ &= \lambda \mathbb{E}[L(X)X] + (1-\lambda)\mathbb{E}[L(Y)Y] = \lambda\rho(X) + (1-\lambda)\rho(Y). \end{aligned} \tag{3.6}$$

(ii) above can be demonstrated in a similar manner but an alternative proof is available when the function $L(\cdot)$ is smooth. Then both *(i)* and *(ii)* as well as other criteria for convexity can be glanced by differentiating $R(x) = L(x)x$ twice.

$$(L(x)x)'' = L''(x)x + 2L'(x), \tag{3.7}$$

which has to be non-negative for $R(x)$ to be convex. If *(ii)* above holds then it is indeed non-negative. □

Section 2.2 presents subadditivity, positive homogeneity, monotonicity and cash invariance, in addition to convexity. The weighted outcome risk defined by (3.1) lacks positive homogeneity, cash invariance and subadditivity for many choices of the weighting function $L(\cdot)$. The reason is that the definitions of positive homogeneity, cash invariance and subadditivity all alter the size of outcomes, and thus the value of $L(\cdot)$. Taking a simple example, such as $L(X) = X$, makes it clear that the risk measure (3.1) lacks the above-mentioned properties. The weighted outcome risk is if the function $L(\cdot)$ is non-decreasing, which can be verified by applying the definition of monotonicity.

Condition *(ii)* in Proposition 15 is sufficient to show that some interesting special cases of (3.1) are convex.

(i) Let $Z = X - \mu_X$ and define $L(z) = z$ in (3.1). Then the measure of risk obtained is the *variance*, $\mathrm{VAR}(X) = \int_{-\infty}^{\infty}(x-\mu_X)^2 f(x)dx$.

(ii) If $L(x) = C \in \mathbb{R}$ for $x \in (-\infty, 0)$ and $L(x)$ is convex on $(0,\infty)$ then *(ii)* is satisfied. If $C > 0$ then portfolio gains will contribute to reduced risk but only at a constant rate whereas portfolio losses are weighted depending on the size of the portfolio loss. If $C = 0$ then the risk measure asserts that portfolio gains have no impact on risk whatsoever.

3.2 General Risk Measures

The goal of this section is to introduce risk measures that allow further flexibility by including a dependency on both the size and the quantile of outcomes. Some of the previously presented risk measures are included as special cases of these more general risk measures.

As a first development towards a more general risk measure, consider a modification of $\mathrm{TVaR}_\alpha(\cdot)$ as defined by (9). The modification is introduced as a preliminary but can be interesting in its own right.

Definition 16 (Weighted Tail Risk)**.** Assume that $L(X)$ is a function $L : \mathbb{R} \mapsto \mathbb{R}$. Then the weighted tail risk is defined as

$$\rho(X) = \frac{1}{1-\alpha}\mathbb{E}\left[L(X)X\mathbf{1}^{(\alpha)}_{\{L(X)X \geq \mathrm{VaR}_\alpha(L(X)X)\}}\right] = \mathrm{TVaR}_\alpha(L(X)X). \tag{3.8}$$

The following result gives conditions on $L(\cdot)$ that guarantee convexity of the risk measure defined by (3.8).

Proposition 17. *If $R(x) = L(x)x$ is convex then the weighted tail risk defined by* (3.8) *defines a convex risk measure.*

Proof. $\mathrm{TVaR}_\alpha(X)$ is convex and non-decreasing (monotonic) by Proposition 10. Therefore (3.8) is the composition of a convex non-decreasing risk measure with a convex function. For some stochastic variables, $X, Y \in \mathcal{X}$ it then holds that

$$\begin{aligned} &\mathrm{TVaR}_\alpha(R(\lambda X + (1-\lambda)Y)) \\ &\leq \mathrm{TVaR}_\alpha(\lambda R(X) + (1-\lambda)R(Y)) \\ &\leq \lambda \mathrm{TVaR}_\alpha(R(X)) + (1-\lambda)\mathrm{TVaR}_\alpha(R(Y)). \end{aligned} \tag{3.9}$$

The first inequality in (3.9) follows from $\mathrm{TVaR}_\alpha(\cdot)$ being non-decreasing and $R(\cdot)$ being convex. The second inequality follows from TVaR_α being convex. □

One question that arises when introducing a risk measure that depends on both the quantile and the absolute size of outcomes as in (3.8) is what difference there is between the two. In a practical setting, measuring the size of the outcome and the quantile might seem very similar. Indeed, when the portfolio is a priori known, it is possible to use the portfolio distribution function to convert between quantiles and outcomes. There are, however, fundamental differences that become apparent when considering changes that can greatly affect the portfolio distribution function. A difference that can be easily appreciated is that quantile risk measures do not allow for different treatment of losses and gains. The following example illustrates an expression of risk that is only possible with the generalized measure.

Example 3. Tail-Value-at-Risk allows that profitable outcomes in the tail reduce the tail expectation and thereby the measured risk. Suppose that it is of interest to measure Tail-Value-at-Risk without allowing profits to reduce risk. To express such a risk preference define $\rho(X) = \mathrm{TVaR}_\alpha(L(X)X)$ as in (3.8) and let

$$L(X) = \begin{cases} 1 & \text{when } X \geq 0 \\ 0 & \text{when } X < 0 \end{cases} \tag{3.10}$$

and remember that $X < 0$ for gains. $L(\cdot)$ defines a convex risk measure by (3.8) because

$$xL(x) = \begin{cases} x & \text{when } X \geq 0 \\ 0 & \text{when } X < 0. \end{cases} \tag{3.11}$$

The weighted tail risk associated with the weighting function $L(\cdot)$ defined by (3.10) is the Tail-Value-at-Risk when profits are not allowed to reduce risk.

The weighted tail risk allows risk to be measured by the tail of a stochastic variable other than the portfolio result. One possibly unattractive feature of (3.8) is, however, that the adjusted indicator function for the tail, $\mathbf{1}^{(\alpha)}_{\{L(X)X \geq \mathrm{VaR}_\alpha(L(X)X)\}}$, depends not only on the distribution of the portfolio loss function, X, but also on the outcome weighting function $L(\cdot)$. The question would be why $\mathbf{1}^{(\alpha)}_{\{L(X)X \geq \mathrm{VaR}_\alpha(L(X)X)\}}$ is used instead of $\mathbf{1}^{(\alpha)}_{\{X \geq \mathrm{VaR}_\alpha(X)\}}$. The following example shows that measuring the tail of $L(X)X$ can indeed be a very desirable feature when $L(X)X$ is interpreted as a stochastic variable representing the actual riskiness of the stochastic variable X.

Example 4. It can be of interest to measure the tail of the variance. To define such a risk measure, let Z be the centered portfolio, $Z = X - E[X]$, and let $L(Z) = Z$. Then $L(Z)Z$ is the variance of X and $L(\cdot)$ defines the risk measure

$$\rho(Z) = \frac{1}{1-\alpha}\mathbb{E}\left[Z^2 \mathbf{1}^{(\alpha)}_{\{Z^2 \geq \mathrm{VaR}_\alpha(Z^2)\}}\right] \tag{3.12}$$

by (3.8). This risk measure could be termed the $(1-\alpha)-$tail-variance of X and might be related to the existence of outliers that greatly contribute to variability.

In the above example the riskiness variable Z^2 changes the relative severity of outcomes. If it indeed represents the actual risk of the portfolio there would be little sense in considering the tail of the original stochastic variable X.

A way to formalize the question of whether it would be possible to consider only the tail of the original stochastic variable is to ask when the adjusted indicator functions are equivalent. Sufficient conditions for equivalence of the tails are given in the following proposition.

Proposition 18. *If $L(\cdot)$ is a function such that*

(i) $L(\cdot)$ is non-decreasing and

(ii) $L(x)$ is positive for all $x \in \mathbb{R}$

then

$$TVaR_\alpha(L(X)X) = \mathbb{E}\left[L(X)X\mathbf{1}^{(\alpha)}_{\{L(X)X \geq VaR_\alpha(L(X)X)\}}\right] = \mathbb{E}\left[L(X)X\mathbf{1}^{(\alpha)}_{\{X \geq VaR_\alpha(X)\}}\right]. \tag{3.13}$$

Proof. Intuitively, one has to prove that the tail of one variable is the same as the tail of the other. Below, this will be proved formally.

First note that the stochastic variables X and $L(X)X$ have the same probability atoms. As $L(\cdot)$ is positive it holds that if $P(X = x) = \delta > 0$ for some $x \in \mathbb{R}$ then $P(XL(X) = xL(x)) = \delta$. In order to prove that (3.13) holds it is then sufficient to show that

$$X \geq \mathrm{VaR}_\alpha(X) \iff XL(X) \geq \mathrm{VaR}_\alpha(XL(X)). \tag{3.14}$$

As $L(\cdot)$ is positive and non-decreasing it holds that

$$x \geq y \iff xL(x) \geq yL(y). \tag{3.15}$$

As a direct application of (3.15) it is possible to rewrite (3.14) as

$$XL(X) \geq \mathrm{VaR}_\alpha(X)L(VaR_\alpha(X)) \iff XL(X) \geq \mathrm{VaR}_\alpha(XL(X)). \tag{3.16}$$

It is clear that (3.16) holds if

$$\mathrm{VaR}_\alpha(XL(X)) = \mathrm{VaR}_\alpha(X)L(VaR_\alpha(X)). \tag{3.17}$$

To show that (3.17) is true, consider the definition of Value-at-Risk in (2.7),

$$\mathrm{VaR}_\alpha(X) = \inf \{x \in \mathbb{R} : P(X \leq x) \geq \alpha\} = \inf A. \tag{3.18}$$

The distribution function $F(x) = P(X \leq x)$ is continuous from the right implying that the set A in (3.18) is closed. Assume that $\alpha > 0$ as the case when $\alpha = 0$ is trivial because the adjusted indicator functions are then identically 1. When $\alpha > 0$ the set A is bounded from below.

Using once again (3.15) gives that

$$A = \{x \in \mathbb{R} : P(X \leq x) \geq \alpha\} = \{x \in \mathbb{R} : P(XL(X) \leq xL(x)) \geq \alpha\}. \tag{3.19}$$

Also define the set B as

$$B = \{z \in \mathbb{R} : P(XL(X) \leq z) \geq \alpha\}. \tag{3.20}$$

Then it holds that for all $z \in B$ there is an $x \in A$ such that $z = xL(x)$. A and B are both closed and bounded from below, implying that they contain their infima, that is

$$VaR_\alpha(X) \in A \text{ and } \mathrm{VaR}_\alpha(XL(X)) \in B.$$

Then, in particular, there is a $z^* \in B$ such that $z^* = \mathrm{VaR}_\alpha(X)L(VaR_\alpha(X))$. Finally, as

$$x \geq \mathrm{VaR}_\alpha(X) \; \forall x \in A,$$

(3.15) gives that

$$xL(x) \geq \mathrm{VaR}_\alpha(X)L(VaR_\alpha(X)) \; \forall x \in A,$$

which implies that $\mathrm{VaR}_\alpha(X)L(VaR_\alpha(X)) = z^* = \inf B = \mathrm{VaR}_\alpha(XL(X))$. Then (3.17) holds, completing the proof. □

The conditions of Proposition 18 are quite strict so for many choices of $L(\cdot)$ it will not be possible to change the stochastic variable in the adjusted indicator function. In fact, neither of the examples 3 and 4 fulfill the criteria of Proposition 18.

A quantile risk measure depending on several confidence levels is available by the spectral risk measure (2.20). It is then natural to consider whether an extension of the risk measure (3.8) to several confidence levels is also available.

Definition 19 (Weighted quantile risk measure)**.** Let X be a stochastic variable as above and let $\alpha = \{\alpha_i\}_{i=1}^{\infty}$ be a set of confidence levels. Furthermore, let $L = \{L_i\}_{i=1}^{\infty}$ be a set of functions, $L_i : \mathbb{R} \mapsto \mathbb{R}$. The weighted quantile risk measure is then defined as

$$\mathrm{WQR}_{\alpha,L}(X) = \sum_{i=1}^{\infty} \mathrm{TVaR}_{\alpha_i}(L_i(X)X). \tag{3.21}$$

Proposition 20. *The risk measure $WQR_{\alpha,L}(X)$ is convex if $l(x)x$ is convex for all functions $l \in L$.*

Proof. $\mathrm{WQR}_{\alpha,L}(X)$ is a sum of risk measures of the type (3.8). Each of the terms is a convex risk measure because the functions $L_i \in L$ satisfy the conditions of Proposition 17. It is easy to verify that a sum of convex risk measures is also a convex risk measure. Therefore it results that the weighted quantile risk measure is convex under the given conditions. □

If the sum in (3.21) converges and the functions $l \in L$ all satisfy the conditions of Proposition 18, then the terms in (3.21) only depend on the weighting functions $L_i(\cdot)$ and the α_i-tails of the original stochastic variable. Stated otherwise,

$$\mathrm{WQR}_{\alpha,L}(X) = \sum_{i=1}^{\infty} \mathrm{TVaR}_{\alpha_i}(L_i(X)X) = \mathbb{E}\left[\sum_{i=1}^{\infty} X L_i(X)\mathbf{1}^{(\alpha_i)}_{\{X \geq \mathrm{VaR}_{\alpha_i}(X)\}}\right]. \tag{3.22}$$

$\mathrm{WQR}_{\alpha,L}(X)$ makes it possible to express risk preference as a function of both the size of the portfolio outcome and the quantile of the distribution. However, it is important to note that for a given probability interval, for example $[\alpha_i, \alpha_{i+1}]$, the weight for an outcome x, is the sum of all weighting functions $L_i(x)$ such that $F_{L_i(X)X}(L(x)x) \geq \alpha_{i+1}$. As the functions in L are non-negative, the risk measure defined by (3.21) is non-decreasing in both X and in additional probability levels. An example of a risk measure defined by (3.21) is shown in Figure 3.1, assuming that the representation (3.22) is available. The example set α has just two members, α_1 and α_2. Note that although the weighting functions vary, the sum in the last graph shows that the total weight for the outcomes of X is non-decreasing.

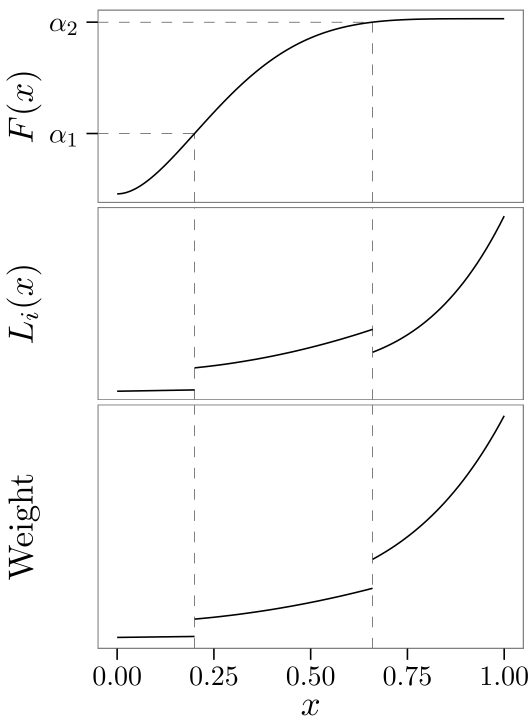

Figure 3.1: The figure shows how different additional weight functions $L_i(x)$ (middle) are assigned depending on the quantiles of the distribution of the stochastic variable X (top). The total weight for an outcome is obtained as $\sum_{i=1}^{\infty} L_i(x)\mathbf{1}^{(\alpha_i)}_{\{x \geq \mathrm{VaR}_{\alpha_i}(X)\}}$ (bottom).

It is possible to extend the risk measure (3.21) to a continuous interval of significance levels. The development is similar to that in [1] bar the consideration of absolute loss size in addition to the significance level.

Definition 21 (Composite quantile-loss risk measure)**.** Suppose that $h(x,\alpha)$ is a function, $h : \mathbb{R} \times [0,1] \mapsto \mathbb{R}$ and that X is a stochastic variable, $X \in \mathcal{X}$. The composite quantile-loss risk measure is then defined as

$$\rho(X) = \mathbb{E}\left[Xh(X, F(X))\right]. \tag{3.23}$$

Proposition 22. *Suppose that the stochastic variables $X \in \mathcal{X}$ all have continuous probability distribution functions, $F_X(\cdot)$. The composite quantile-loss risk measure is then a convex risk measure on $\mathcal{X}$ if the function $h(x,\alpha)$ is absolutely continuous in α for all x and satisfies*

(i) $xh(x,0)$ is convex in x,

(ii) $x\frac{\partial h(x,\alpha)}{\partial \alpha}$ *is convex in* x *for all* $\alpha \in [0,1]$,

(iii) $\frac{\partial h(x,\alpha)}{\partial \alpha}$ *is non-decreasing in* x *and*

(iv) $\frac{\partial h(x,\alpha)}{\partial \alpha}$ *is positive for all* $x \in \mathbb{R}$.

Proof. The risk measure defined by (3.23),

$$\rho(X) = \mathbb{E}\left[Xh(X, F(X))\right], \tag{3.24}$$

can be rewritten by dividing the weight-function $h(X, F(X))$ into two parts. Due to absolute continuity of $h(x, \cdot)$ it holds that

$$h(x, \alpha) = h(x, 0) + \int_0^{\alpha} \frac{\partial h(x, \alpha')}{\partial \alpha'} \mathrm{d}\alpha', \tag{3.25}$$

which makes it possible to write (3.24) as

$$\rho(X) = \mathbb{E}\left[Xh(X, 0)\right] + \mathbb{E}\left[X \int_0^{F(X)} \frac{\partial h(X, \alpha)}{\partial \alpha} \mathrm{d}\alpha\right]. \tag{3.26}$$

Using the indicator function defined by (2.10), the second term in (3.26) can be written as

$$\mathbb{E}\left[X \int_0^{1} \frac{\partial h(X, \alpha)}{\partial \alpha} \mathbf{1}_{\{F(X) \geq \alpha\}} \mathrm{d}\alpha\right]. \tag{3.27}$$

With the definition of $\mathrm{VaR}_\alpha(\cdot)$ in (2.7), it can be shown that

$$\mathbf{1}_{\{X \geq \mathrm{VaR}_\alpha(X)\}} = \mathbf{1}_{\{F(X) \geq \alpha\}}. \tag{3.28}$$

By assumption, the stochastic variables $X \in \mathcal{X}$ have continuous distribution functions, which implies that the adjusted indicator function defined by (2.11) is equal to the indicator function defined by (2.10). Applying that and (3.28) to (3.27) gives

$$\mathbb{E}\left[X \int_0^{1} \frac{\partial h(X, \alpha)}{\partial \alpha} \mathbf{1}^{(\alpha)}_{\{X \geq \mathrm{VaR}_\alpha(X)\}} \mathrm{d}\alpha\right]. \tag{3.29}$$

Under the condition that

$$\mathbb{E}\left[\int_0^{1} \left| X \frac{\partial h(X, \alpha)}{\partial \alpha} \mathbf{1}^{(\alpha)}_{\{X \geq \mathrm{VaR}_\alpha(X)\}} \right| \mathrm{d}\alpha\right] < \infty,$$

the Fubini-Tonelli theorem allows exchanging the order of taking the expectation and the integral, which gives

$$\int_0^{1} \mathbb{E}\left[X \frac{\partial h(X, \alpha)}{\partial \alpha} \mathbf{1}^{(\alpha)}_{\{X \geq \mathrm{VaR}_\alpha(X)\}}\right] \mathrm{d}\alpha. \tag{3.30}$$

Now note that, due to the conditions (iii) and (iv) above, the weighted tail risk obtained by taking $L(X) = X\frac{\partial h(X,\alpha)}{\partial \alpha}$ in (3.8) fulfills the criteria of Proposition 18. Therefore, it is possible to change the stochastic variable in the adjusted indicator function. Changing the stochastic variable of which the tail is measured to $X\frac{\partial h(X,\alpha)}{\partial\alpha}$ in (3.30) then gives

$$\begin{aligned}&\int_0^1 \mathbb{E}\left[X\frac{\partial h(X,\alpha)}{\partial\alpha}\mathbf{1}^{(\alpha)}_{\{X\frac{\partial h(X,\alpha)}{\partial\alpha}\geq \mathrm{VaR}_\alpha(X\frac{\partial h(X,\alpha)}{\partial\alpha})\}}\right]\mathrm{d}\alpha\\ =&\int_0^1(1-\alpha)\mathrm{TVaR}_\alpha\left(X\frac{\partial h(X,\alpha)}{\partial\alpha}\right)\mathrm{d}\alpha.\end{aligned} \tag{3.31}$$

The equality in (3.31) follows by definition from (2.8).

Bringing (3.31) and (3.26) together yields

$$\rho(X)=\mathbb{E}\left[Xh(X,0)\right]+\int_0^1(1-\alpha)\mathrm{TVaR}_\alpha\left(X\frac{\partial h(X,\alpha)}{\partial\alpha}\right)\mathrm{d}\alpha. \tag{3.32}$$

The first term of (3.32) is a weighted outcome risk measure with $L(X)=h(X,0)$, as defined by (3.1) and therefore convex, as (i) corresponds to the condition of Proposition 14.

The second term of (3.32) can be shown to be convex by directly applying the definition of convexity, (2.5).

Consider the risk measure $\rho_I(\cdot)$ defined by

$$\rho_I(X)=\mathrm{TVaR}_\alpha\left((1-\alpha)X\frac{\partial h(X,\alpha)}{\partial\alpha}\right). \tag{3.33}$$

$\rho_I(\cdot)$ is the composition of a convex and non-decreasing risk measure, $\mathrm{TVaR}_\alpha(\cdot)$, with a function $L(x,\alpha)=(1-\alpha)x\frac{\partial h(x,\alpha)}{\partial\alpha}$ that is convex due to (ii) above. The composition of a convex and non-decreasing risk measure with a convex function is a convex risk measure, as is shown in the proof of Proposition 17, and therefore $\rho_I(\cdot)$ is a convex risk measure.

To apply the definition of convexity (2.5) to the second term in (3.32), take two stochastic variables, $X,Y\in\mathcal{X}$, and a $\lambda\in(0,1)$. For the second term in (3.32) it then holds that

$$\begin{aligned}&\int_0^1(1-\alpha)\mathrm{TVaR}_\alpha\left((\lambda X+(1-\lambda)Y)\frac{\partial h(\lambda X+(1-\lambda)Y,\alpha)}{\partial\alpha}\right)\mathrm{d}\alpha\\ =&\int_0^1\rho_I(\lambda X+(1-\lambda)Y)\mathrm{d}\alpha\leq\int_0^1\lambda\rho_I(X)+(1-\lambda)\rho_I(Y)\mathrm{d}\alpha\\ =&\lambda\int_0^1(1-\alpha)\mathrm{TVaR}_\alpha\left(X\frac{\partial h(X,\alpha)}{\partial\alpha}\right)+(1-\lambda)\int_0^1(1-\alpha)\mathrm{TVaR}_\alpha\left(Y\frac{\partial h(Y,\alpha)}{\partial\alpha}\right)\mathrm{d}\alpha.\end{aligned} \tag{3.34}$$

The inequality in (3.34) follows from the fact that $\rho_I(\cdot)$ is a convex risk measure. The first inequality follows from the fact that $(1-\alpha)$ can be moved inside TVaR as is allowed due to positive homogeneity. The second equality follows from linearity of the integral and that $(1-\alpha)$ can be moved outside TVaR.

(3.34) shows that the second term of (3.32) defines a convex risk measure. Finally, the sum of convex risk measures is a convex risk measure, which implies that (3.32) is a convex risk measure. □

Proposition 22 assumes that the stochastic variables $X\in\mathcal{X}$ all have continuous distribution functions. The following proposition holds even if some stochastic variables in $\mathcal{X}$ have discontinuous distribution functions.

Proposition 23. *Let $L_0(x)$ be a function, $L_0:\mathbb{R}\mapsto\mathbb{R}$ and $L(x,\alpha)$ a function, $L:\mathbb{R}\times[0,1]\mapsto\mathbb{R}$.*

Suppose that $h(X, F(X))$ is a function of the form

$$h(X, F(X)) = L_0(X) + \int_0^1 L(X, \alpha)\mathbf{1}^{(\alpha)}_{\{X \geq VaR_\alpha(X)\}} d\alpha. \tag{3.35}$$

The composite quantile-loss risk measure defined using $h(X, F(X))$ is then a convex risk measure on $\mathcal{X}$ if

(i) $L_0(x)$ *is convex in* x,

(ii) $xL(x, \alpha)$ *is convex in* x *for all* $\alpha \in [0, 1]$,

(iii) $L(x, \alpha)$ *is non-decreasing in* x *and*

(iv) $L(x, \alpha)$ *is positive for all* $x \in \mathbb{R}$.

Proof. The proof of Proposition 23 follows the same idea as the proof of Proposition 22 but starting from (3.29). □

3.3 Comments on the Risk Measures

The sections above present several risk measures that differ in some respects. It is interesting to consider what implications the differences have if the risk measures are interpreted as a reflection of risk preferences.

First of all, Value-at-Risk has already been shown to have some undesirable properties that it does not share with Tail-Value-at-Risk. Another attractive feature of TVaR is that it leads to a linear optimization problem, which is not generally true for the loss size risk measures introduced in sections 3.1 and 3.2.

All of the measures bar VaR have been shown to be convex under suitable assumptions. Convexity appears to be a reasonable feature of a risk measure as it can be interpreted as diversification not leading to an increase in risk. However, in contrast to TVaR, the loss size measures are not generally positively homogeneous or subadditive. Positive homogeneity implies that doubling the size of a portfolio doubles the associated risk. Convexity and the lack of positive homogeneity for the loss size risk measures implies that the risk can more than double when the size of the portfolio doubles. The upshot is that if risk is viewed as something that can be mitigated just as easily regardless of the size of a portfolio then positive homogeneity is a desirable feature but if it becomes increasingly costly or hard to mitigate risk as the portfolio grows, then it might not be.

It is also interesting to consider what the difference might be when using the composite quantile-loss risk measure. The conditions imposed by Proposition 22 are quite strict. Following is an example that fulfills the criteria when the stochastic variables $X \in \mathcal{X}$ are non-negative, that is, $X \geq 0$ for all $X \in \mathcal{X}$.

Example 5. Consider the function $h(x, \alpha) = e^x\alpha$, which defines a risk measure

$$\mathbb{E}\left[Xe^X\alpha\right] \tag{3.36}$$

by (3.23). If $X \geq 0$ for all $X \in \mathcal{X}$ then the risk measure is convex as it satisfies the conditions of Proposition 22. The risk measure asserts that risk increases linearly in the significance level of an outcome but exponentially in the size of the loss, thus giving very big weight to big losses, irrespective of them being improbable.

The risk measure (3.36) is clearly not positively homogeneous as $\lambda X e^{\lambda X} \neq \lambda X e^{X}$, for general $\lambda \in \mathbb{R}^{+}, X \in \mathcal{X}$. For any stochastic variable $X \in \mathcal{X}$ that has a tail density that falls off slower than the exponential function, the associated risk as measured by (3.36) is infinite.

4 Optimization Modelling of Retrocession

A model for optimization of retrocession should contain the basic components of an optimization problem. That is, it is necessary to define a cost function that is to be minimized (or a utility function to be maximized) and which constraints are to be applied with regard to the decision variables. Central decisions made in retrocession are

(i) for which regions and perils should risk be retroceded,

(ii) what type of proportional or non-proportional contract should be used and

(iii) by how much should the current exposure to losses be reduced.

In this section the business considerations of retrocession are related to alternative mathematical models that can reflect the considerations in a way that can be used to find optimal answers to the questions above. The first model considered in this section is a simple linear model, where the risk ceded is proportional to the total risk and each contract can be subject to only one single retrocession contract. Subsequently, a more general model is developed which allows for multiple overlapping retrocession contracts and retrocession contracts with non-linear features. Approaches to solving the optimization problem are considered for each of the models.

4.1 Linear Model

Suppose that there are n_Γ disjoint classes of contracts and that there are n_γ contracts in the class denoted by $\gamma \in \{1, 2, ..., n_\Gamma\}$. The contracts in each class γ are assigned an index, $i \in \{1, 2, ..., n_\gamma\}$. Suppose further that each contract has an associated loss $D_{\gamma i}$, which is a stochastic variable. The loss from a contract is considered to be the losses from that contract during a certain period of time, such as one year. Then the loss pertaining to a class of contracts is $D_\gamma = \sum_{i=1}^{n_\gamma} D_{\gamma i}$.

For the purpose of this model, retrocession is assumed to be available for each class of contracts and the terms are that a proportional share, $x_\gamma \in [0, 1]$, of the risk can be ceded in return for a share $(1 - c)x_\gamma$ of the premiums of class γ, P_γ. c here reflects the overriding commission and, as noted in section 1.1, usually $c > 0$. Suppose that $\rho = (\rho_1, \rho_2, \ldots, \rho_{n_\rho})$ is a risk measure that measures n_ρ dimensions of risk. Each dimension of risk represents some risk measure that it is deemed necessary to constrain in order to reflect company risk preferences. One possible objective is to maximize the result contribution of the portfolio, which then gives the optimization problem

$$
\begin{aligned}
&\max_{(x_1, x_2, \ldots, x_{n_\Gamma})} \quad \mathbb{E}[R] \\
&\text{subject to} \\
&\qquad\qquad 0 \leq x_\gamma \leq 1 \text{ for } \gamma = 1, 2, \ldots, n_\Gamma \\
&\qquad\qquad \rho_i(-R) \leq f_i \text{ for } i = 1, 2, \ldots, n_\rho \\
&\qquad\qquad R = \sum_{\gamma=1}^{n_\Gamma} P_\gamma(1 - (1 - c)x_\gamma) - \sum_{\gamma=1}^{n_\Gamma} D_\gamma(1 - x_\gamma)
\end{aligned}
\tag{4.1}
$$

where $f = (f_1, f_2, ..., f_{n_\rho})$ is a vector of constants that define the maximum acceptable risk in each dimension. Here R is linear in x_γ and thus convex so the problem is a convex optimization problem if ρ is convex. Furhtermore it is possible to impose linear contraints on the decision variables, x_γ and still maintain a convex model. There can also be a class of contracts that is not available for retrocession, implying $x_\gamma = 0$ for that class. The reason for including such a class is that it affects portfolio outcomes and thereby also the risk measure.

The above model makes the important simplification that the reinsurance contracts are divided into several disjoint classes for which at most one retrocession contract is available for each class. Although this is not always the case it gives a model that can be solved using linear programming if there is such a technique for the risk measures in the set ρ. The model defined by (4.1) is also linear for premium principles that include a profit commission.

4.2 Model Generalizations

The linear model presented above does not allow for optimization of excess of loss contracts and makes the implausible assumption that all the prospective retrocession contracts cover disjoint classes of contracts. The purpose of this section is to consider these shortcomings in turn and suggest generalizations that overcome them. Each generalization will be investigated to see if it defines a problem that can be solved by linear programming, and if not, whether there are other methods available for finding the optimal solution.

4.2.1 Overlapping Contract Covers

To model overlapping contracts assume, as for the linear model above, that contracts in the portfolio have been divided into several disjoint classes, denoted by $\gamma \in \{1, 2, ..., n_\Gamma\} = \Gamma$, possibly encompassing only individual contracts. Now suppose, in contrast to the linear model, that n_Ψ proportional retrocession contracts are available, denoted by $\psi \in \{1, 2, \ldots, n_\Psi\} = \Psi$, and that each retrocession contract offers proportional cover for a subset of the disjoint classes. Let $\Gamma_\psi \subset \Gamma$ denote the subset of contracts covered by the retrocession contract, ψ. Under these assumptions it is allowed that any of the classes γ is covered by two or more of the retrocession contracts, which immediately raises the question of what should be meant by having ceded a share of the same risk to several different contracts. Section 1.1.3 presents two different answers to that question, multiplicative contracts and additive contracts. Only the additive case will be considered as it is the one used practically.

In the additive case, the total proportional cover for the class γ is given by the sum of the cover from all retrocession contracts that include γ. Form the set of all such retrocession contracts, $\Psi_\gamma = \{\psi \in \Psi : \gamma \in \Gamma_\psi\}$ and let x_ψ denote the proportional share ceded to contract ψ. Let $X = -R$ denote the loss function of the portfolio. The expression for the loss function of class γ, omitting for a moment the premium principle, is then identified as

$$X_\gamma = \left(1 - \sum_{\psi \in \Psi_\gamma} x_\psi\right) D_\gamma - \left(1 - \sum_{\psi \in \Psi_\gamma} x_\psi\right) P_\gamma. \tag{4.2}$$

The above expression is linear but a problem with it is that optimization based on X_γ would allow the cover to be more than 100% of the loss, which is not usual in the context of insurance. Cession of more than 100% of premiums is assumed to be possible. The problem can be solved by introducing a new variable for the retained share of each class, $r_\gamma = \left(1 - \sum_{\psi \in \Psi_\gamma} x_\psi\right)$ and requiring that no more than 100% can be ceded. The optimization obtained is then

$$\begin{aligned}\min_{(r_1,r_2,\ldots,r_{n_\Gamma},x_1,x_2,\ldots,x_{n_\Psi})} \quad & \mathbb{E}[X] \\ \text{subject to} \quad & \\ & 0 \leq x_\psi \leq 1 \text{ for } \psi = 1,2,\ldots,n_\Psi \\ & \rho_i(X) \leq f_i \text{ for } i = 1,2,\ldots,n_\rho \\ & r_\gamma = \left(1 - \sum_{\psi\in\Psi_\gamma} x_\psi\right) \text{ for } \gamma = 1,2,\ldots,n_\Gamma \\ & X = \sum_{\gamma=1}^{n_\Gamma} \max(r_\gamma, 0) D_\gamma - \sum_{\gamma=1}^{n_\Gamma} r_\gamma P_\gamma\end{aligned} \tag{4.3}$$

The optimization problem (4.3) is non-linear because the function $\max(\cdot, 0)$ is non-linear. If, however, linear programming can be applied to the risk measures $\rho_i(\cdot)$, for $i = 1, 2, \ldots n_\rho$, it is possible to rewrite the optimization problem (4.3) as a linear program. To do so, introduce auxiliary variables $z_\gamma = \max(r_\gamma, 0)$ for each class γ and impose the conditions $z_\gamma \geq 0$ and $z_\gamma \geq r_\gamma$. Then the optimization problem becomes

$$\begin{aligned}\min_{(r_1,r_2,\ldots,r_{n_\Gamma},x_1,x_2,\ldots,x_{n_\Psi},z_1,z_2,\ldots,z_{n_\Gamma})} \quad & \mathbb{E}[X] \\ \text{subject to} \quad & \\ & 0 \leq x_\psi \leq 1 \text{ for } \psi = 1,2,\ldots,n_\Psi \\ & \rho_i(X) \leq f_i \text{ for } i = 1,2,\ldots,n_\rho \\ & z_\gamma \geq 0 \text{ for } \gamma = 1,2,\ldots,n_\Gamma \\ & z_\gamma \geq r_\gamma \text{ for } \gamma = 1,2,\ldots,n_\Gamma \\ & r_\gamma = \left(1 - \sum_{\psi\in\Psi_\gamma} x_\psi\right) \text{ for } \gamma = 1,2,\ldots,n_\Gamma \\ & X = \sum_{\gamma=1}^{n_\Gamma} z_\gamma D_\gamma - \sum_{\gamma=1}^{n_\Gamma} r_\gamma P_\gamma.\end{aligned} \tag{4.4}$$

In the above problem the auxiliary variables z_γ will always equal $\max(r_\gamma, 0)$ if the conditions $\rho_i(X) \leq R_i$ are such that if they are satisfied for some z^*_γ then they are also satisfied for any $z < z^*_\gamma$. The reason is that as z_γ has a positive coefficient in the objective function, at least one of the only two conditions constraining z_γ from below must be active.

The condition on the constraints $\rho_i(X) \leq f_i$ is satisfied precisely when the coefficient of z_γ in $\rho_i(X)$ is positive. This gives a condition on the risk measures $\rho_i(\cdot)$ for (4.4) to define a linear program. TVaR is one risk measure that satisfies this condition.

Remark 2. In practice it is unnecessary to introduce auxiliary variables when $r_\gamma = (1 - x_\psi)$ for some ψ because then the maximum is of no concern as $0 \leq x_\psi \leq 1$ for all $\psi \in \Psi$.

4.2.2 Non-Proportional Retrocession

One of the shortcomings of the linear model is that it does not enable modelling excess of loss contracts. Unfortunately, non-proportional retrocession does not lead to an optimization problem that is nearly as computationally tractable as the models previously discussed. This section will give an example showing why that is the case and presents a simplistic model that may be useful in some cases.

Suppose that there is only a single limitless excess of loss contract available, which covers a single class of the portfolio. This will serve as an example but the case of limited excess of loss is similar. Denote the class that the prospective contract covers by γ. The limitless excess of loss principle of section 1.1.2 gives the retained losses as $D_\gamma^{\text{Retained}} = \min(D_\gamma, d_\gamma)$, where d_γ denotes the deductible of the cover.

An explicit function describing the relation between the cost of an excess of loss cover, the limit and the deductible may be harder to come by than in the case of proportional reinsurance contracts. For the model presented here it is thus assumed that the reinsurer cannot influence the contracts offered by the market and is thus left to decide between a finite number of prospective contracts.

The deductible d_1 of the available contract is assumed to be known, and so is the cost of buying the protection. Let s be a binary decision variable that takes the value 1 if the contract is signed. Further assume that c is the cost of the protection if the contract is signed.

The assumptions made gives the optimization problem

$$\begin{aligned}
&\min_{s} \quad \mathbb{E}[X] \\
&\text{subject to} \\
&\qquad \rho_i(X) \leq f_i \text{ for } i = 1, 2, \ldots, n_\rho \\
&\qquad s \in \{0, 1\} \\
&\qquad X = D_{\Gamma\setminus\gamma} - P_{\Gamma\setminus\gamma} + D_\gamma(1-s)s\min(D_\gamma, d_\gamma) - P_\gamma + cs,
\end{aligned} \tag{4.5}$$

where $D_{\Gamma\setminus\gamma}$ denotes the losses of all classes $g \in \Gamma$ except the class γ, and analogously for the premiums, $P_{\Gamma\setminus\gamma}$.

Similarly to the case with overlapping proportional contracts there is a non-linear objective function. In contrast, however, there is a minimization of the minimum of two numbers, as opposed to minimization of the maximum in the previous case. The present optimization problem does not lend itself to the technique of introducing an auxiliary variable to circumvent the non-linearity. Further complicating the objective function is the fact that the minimum is taken with respect to a stochastic variable, D_γ. There is another technique available that involves the introduction of binary variables. That technique will now be introduced but it should be noted that the binary variable introduced will be a stochastic variable, dependent on D_γ.

To proceed, let $D_\gamma^{\text{Retained}} = \min(D_\gamma, d_\gamma)$ denote the retained loss of class γ. Introduce a binary variable b,

$$b = \begin{cases} 1 & \text{if } D_\gamma \leq d_\gamma \\ 0 & \text{if } D_\gamma > d_\gamma \end{cases} \tag{4.6}$$

It is then possible to write the retained loss as $D_\gamma^{\text{Retained}} = bD_\gamma + (1-b)d_\gamma$. The variable b can be forced to take the desired values, which will be further explored after presenting the optimization problem. Let $\epsilon > 0$ be a small positive number and let $M \in \mathbb{R}$ be a positive number that is very large compared to the values taken by the other variables in the optimization problem. The optimization problem is

$$
\begin{aligned}
&\min_{s,b} \quad \mathbb{E}[X] + \epsilon b \\
&\text{subject to} \\
&\qquad s, b \in \{0, 1\} \\
&\qquad \rho_i(X) \leq f_i \text{ for } i = 1, 2, \ldots, n_\rho \\
&\qquad d_\gamma - D_\gamma - Mb \leq 0 \\
&\qquad X = D_{\Gamma \setminus \gamma} - P_{\Gamma \setminus \gamma} + D_\gamma(1 - s) + sbD_\gamma + s(1 - b)d_\gamma - P_\gamma + cs.
\end{aligned}
\tag{4.7}
$$

Note that in (4.7) the variable b has to take the value 1 when $d_1 > D_1$ in order to satisfy the last constraint. On the other hand, $D_1 \geq d_1$ implies that b will take the value 0 because the last constraint is satisfied irrespective of the value of b and the objective function is minimized by minimizing b.

The model (4.7) is simplistic and it mainly serves as an example of the issues related to modelling excess of loss. Indeed, if the expected ceded loss is smaller than the binary cost of the excess of loss cover, then there is no benefit of signing the contract if the risk constraints are satisfied. If the risk constraints are not satisfied then the contract has to be signed if that leads to a feasible solution, as signing the contract is the only way to reduce risk in the above model. In the following, several ways to extend the model and incorporate it in other optimization models are considered.

The combination of (4.7) with (4.4) yields a model where both non-proportional and proportional retrocession is available. It is allowable that there is overlap between the contracts covered by non-proportional cover and the contracts covered by proportional cover. Additionally, it is possible to include several different excess of loss contracts by introducing costs and binary variables for each one.

The optimization problem resulting from combining the models (4.7) and (4.4) is a discrete programming problem, which is much harder to solve than the linear program [8]. The optimal solution can be obtained by exhaustive search over the set of possible solutions (or by a more sophisticated algorithm).

Remark 3 (On Complexity). Combining proportional with non-proportional retrocession by using (4.7) and (4.4) yields a model where the optimal solution can be found, at least theoretically. It should be noted, however, that in case a number of simulations, J, is used to estimate the distribution of D_γ, then there will be J binary variables that each depends on the simulated value, $D_{\gamma j}$, and the decision variable z_γ. For the exhaustive search approach this means that the complexity increases exponentially in the number of simulations, which is an extremely unattractive feature of the model.

To conclude, the most feasible model for non-proportional retrocession appears to be (4.5), assuming that there is no overlap with proportional retrocession contracts. In that case, exhaustive search can be performed by simply calculating the optimal value of the proportional retrocession linear program for every possible combination of excess of loss contracts.

4.3 Optimization With Risk Constraints

This section will give details on how some of the risk measures presented in Section 2.2 can be incorporated in optimization problems when an approximate distribution of the portfolio has been obtained. The choice to focus on constraining risk is not necessary as a constraint can be converted to an objective function for minimization of risk. The implementation of quantile risk measures will be presented first, then proceeding to loss size risk measures and the generalized case.

4.3.1 Implementation of Quantile Risk Measures

Rockafellar and Uryasev [16] present an implementation of TVaR within the framework of linear programming. The linear programming formulation of TVaR builds on definition (2.16). Krokhmal et al. [13] extend the implementation to optimization where risk is used either as a constraint or as an objective.

To formulate TVaR for use in a linear programming constraint, let X denote the loss function of the portfolio and suppose that X is linear in the decision variables, $x = \{x_1, x_2, \dots, x_{n_\Gamma}\}$. Assume that an approximate distribution of the portfolio constituents has been obtained and let J denote the number of observations making up the approximate distribution. As definition (2.16) is an equivalent definition of TVaR it holds that

$$\begin{aligned} \mathrm{TVaR}_\alpha(X) &= \inf_{\phi \in \mathbb{R}} \left\{ \phi + \frac{1}{1-\alpha} \mathbb{E}[X - \phi]^+ \right\} \\ &\leq \phi + \frac{1}{1-\alpha} \mathbb{E}[X - \phi]^+ \approx \phi + \frac{1}{J(1-\alpha)} \sum_{j=1}^{J} [X_j - \phi]^+, \end{aligned} \tag{4.8}$$

where the approximate equality refers to the use of the approximate distribution to estimate the expected value. X_j denotes the value of the loss function in the j:th sample observation in the approximate distribution.

If risk is constrained by $\mathrm{TVaR}_\alpha \leq f$, the approximate constraint can be obtained by using (4.8) and letting ϕ be an unconstrained decision variable. The resulting constraint is

$$\phi + \frac{1}{J(1-\alpha)} \sum_{j=1}^{J} [X_j - \phi]^+ \leq f. \tag{4.9}$$

(4.9) contains the non-linear function $[\cdot]^+$ but can be converted to a linear constraint. To do so, introduce the auxiliary variable z_j and impose the following constraints

$$\begin{aligned} &\phi + \frac{1}{J(1-\alpha)} \sum_{j=1}^{J} z_j \leq f \\ &z_j \geq X_j - \phi \text{ for } j = 1, 2, \dots, J \\ &z_j \geq 0 \text{ for } j = 1, 2, \dots, J. \end{aligned} \tag{4.10}$$

The auxiliary variables z_j always fulfill $z_j \geq [X_j - \phi]^+$ due to the last two groups of constraints. Thus, when the left-hand-side of the risk constraint takes its maximum value it holds that $z_j = [X_j - \phi]^+$. Finally, (4.10) can be incorporated in a linear program if X is linear in the decision variables x.

It has already been noted that the spectral risk measures presented in the section on quantile risk measures can be approximated as a convex combination of simulated outcomes,

$$M_{\phi}^{(N)}(X) = \sum_{i=1}^{N} X_{i:N} \phi_i \tag{4.11}$$

where $X_{i:N}$ are ordered realizations of the stochastic variable X and

$$\phi_i = \frac{\phi(\frac{i}{N})}{\sum_{k=1}^{N} \phi(\frac{k}{N})}, \qquad i = 1, \ldots, N. \tag{4.12}$$

4.3.2 Implementation of Loss Size Risk Measures

The loss size risk measure (3.1) has a particularly straightforward implementation when an approximate distribution for the portfolio constituents has been obtained. The implementation results in what is typically a non-linear optimization problem.

According to (3.1) the weighted outcome risk measure is

$$\rho(X) = \int_{-\infty}^{\infty} L(x) x f(x) dx = \mathbb{E}[L(X)X]. \tag{4.13}$$

The expected value $\mathbb{E}[L(X)X]$ can be approximated by the mean of the observed values of $L(X)X$ in the approximate distribution. If $\rho(X)$ in (4.13) is constrained by $\rho(X) \leq f$ then the constraint can be expressed as

$$\frac{1}{J} \sum_{j=1}^{J} L(X_j) X_j \leq f \tag{4.14}$$

by using the approximate distribution.

From (4.14) it can be seen that for non-constant forms of $L(\cdot)$, the optimization problem will be non-linear.

4.3.3 Implementation of General Risk Measures

In this section the implementations of the previous two sections are extended to general risk measures introduced in section 3.2. Implementation of the generalized risk measures builds on their expression as the TVaR of a modified stochastic variable, $XL(X)$.

First consider the fundamental case of the weighted tail risk, as defined by (3.1). The weighted tail risk can be expressed as

$$\rho(X) = \text{TVaR}_{\alpha}(XL(X)), \tag{4.15}$$

so combining the approaches of the two previous sections yields

$$\begin{aligned} &\phi + \frac{1}{J(1-\alpha)} \sum_{j=1}^{J} z_j \leq R \\ &z_j \geq X_j L(X_j) - \phi \text{ for } j = 1, 2, \ldots, J \\ &z_j \geq 0 \text{ for } j = 1, 2, \ldots, J. \end{aligned} \tag{4.16}$$

As in the case of the loss size risk measure, (4.16) is non-linear for non-constant $L(\cdot)$.

The weighted quantile risk measure defined by (3.21) is a sum of risk measures of the type (4.15). Thus it can be implemented by introducing several variables $\phi^{(\alpha)}$ as well as several classes of auxiliary variables, $\left\{z_j^{(\alpha)}\right\}_{j=1}^{J}$. If each of the auxiliary variables fulfill that $z_j^{(\alpha)} \geq [X_j - \phi^{(\alpha)}]^+$ then the constraint on risk is satisfied with respect to the simulated distribution.

4.4 Assessing the Optimal Decisions

When applying optimization models to practical problems there might be a need to provide rich information regarding the optimal decisions encountered. This section will focus on two sets of information that are important in this study. First, the uniqueness of the optimal solutions will be considered. Second, techniques for assessing the impact of stochastic variables on the optimal decisions will be studied. The two areas of focus are determined to be the most relevant to this study but numerous other sensitivity analyses are available in literature [19].

4.4.1 Uniqueness of the Optimal Decisions

There can be many optimal solutions of a linear program [8]. The simplex algorithm can find an optimal solution but it does not generally answer the question of uniqueness [14, 19]. Mangasarian [14] introduces necessary and sufficient conditions for uniqueness of solutions in linear programming. The paper defines the linear optimization problem

$$\begin{aligned} \min \quad & p^T x \\ \text{subject to} \quad & Ax = b \\ & Cx \geq d \end{aligned} \tag{4.17}$$

and the corresponding dual problem

$$\begin{aligned} \max \quad & b^T u + d^T v \\ \text{subject to} \quad & A^T u + C^T v = p \\ & v \geq 0. \end{aligned} \tag{4.18}$$

Using over-bar to identify optimal solutions the paper then proceeds to define

$$\begin{aligned} J &= \{i : C_i \bar{x} = d_i\}, \\ K &= \{i : \bar{v}_i > 0\} \\ L &= \{i : C_i \bar{x} = d_i, \bar{v}_i = 0\}. \end{aligned} \tag{4.19}$$

In order to express the conditions for uniqueness, let C_J, C_L and C_K be the matrix whose rows are i:th row of C,C_i, where i is in J, L or K respectively. Several necessary and sufficient conditions for uniqueness are introduced. Condition (v) in the paper states that the optimization problem (4.17) has a unique solution if and only if the rows of $[A^T \; C_K^T \; C_L^T]$ are linearly independent and there is no x such that

$$Ax = 0, \qquad C_K x = 0, \qquad C_L x \geq 0. \tag{4.20}$$

As a suggestion for testing (4.20) the author suggests solving the optimization problem

$$\max \quad \{(1, \ldots, 1) \cdot C_L x : Ax = 0, C_K x = 0, C_L x \geq 0\}, \tag{4.21}$$

which has a zero maximum if the solution is unique.

The test of uniqueness above requires solving two additional linear programs (if the optimization algorithm does not immediately provide the dual solution). In addition to testing for uniqueness a more detailed analysis involving the identification of the non-unique decision variables and the extent to which they can be varied while maintaining optimality may be of interest. To investigate the details it is possible to use an approach that is more computationally expensive than the uniqueness test above but which has approximately the same computer memory usage as does solution of the original optimization problem.

Suppose that an optimal solution, denoted x^*, to the problem (2.2) has been obtained. The question of interest is whether any of the components, say x_i^* for some i, of the decision vector x^* is non-unique and if so, to what extent the decision x_i^* can be varied while maintaining optimality. In other words, what is the maximum variation of x_i given that the optimal value $z^T x$ is maintained. Thus formulated, the question can be answered by solving the optimization problem

$$\begin{aligned} \max \quad & x_i \\ \text{subject to} \quad & \\ & Ax \leq b, \\ & z^T x \leq (1 + \text{sign}(z^T x^*)\epsilon) z^T x^*. \end{aligned} \tag{4.22}$$

The maximum is taken in (4.22) but a full analysis of the possible variation should also solve the corresponding minimization problem with respect to x_i. The constant $\epsilon \in [0, 1)$ is introduced to allow for analysis of the possible variation in x_i when allowing for a small degree of sub-optimality. That is, instead of requiring strict optimality one can take $\epsilon = 0.01$ to analyze the possible variation in x_i when allowing an objective function value that is 1% below the optimum.

4.4.2 Reliability of the Optimal Decisions

Optimization models may include stochastic variables. In that case, implementation of the models may require obtaining an approximate probability distribution by some means, such as Monte Carlo simulation. As the approximation is finite the question arises of whether any inference can be made based on the optimal solution obtained. One way to approach the question of inference is to consider whether the optimal decisions for one finite sample are optimal or almost optimal for another finite sample. More generally, the *bootstrap* technique can be used to estimate the sampling distribution of estimated parameters, such as the optimal decisions.

Let $\theta(X)$ be a parameter that depends on the stochastic variable X. Suppose that a sample of n realizations of X have been obtained, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. The sampling distribution of $\theta(X)$ can then be estimated by taking random samples with replacement of size n from $\mathbf{x}$. Properties of the sampling distribution of $\theta(X)$ can then be estimated by calculating $\theta(\mathbf{x}_s)$ for $s = 1, 2, \ldots, S$ [5]. S here denotes the number of resamplings performed and $\mathbf{x}_s$ denotes the random sample.

Ferrier and Hirschberg [6] apply the bootstrap technique to estimating confidence intervals for linear programming efficiency scores. The topic of the paper is not directly relevant to this thesis but it provides an example of how bootstrapping can be applied to linear programs. The efficiency scores are a measure of resource utilization obtained as the optimal value of the objective function in a linear program. In the setting of a general linear program, it can also be of interest to consider how constraint satisfaction and the optimal decisions vary between samples.

The technique of bootstrapping is applied to the problem of reinsurance portfolio optimization in two main ways in this thesis. First, the optimal decisions are bootstrapped by performing the optimization many times on re-samples of the portfolio distribution. Second, the optimal portfolio is evaluated by bootstrapping the expected result and the risk measure used in the optimization.

5 Results from Optimization of a Reinsurance Portfolio

This section presents the results of applying the linear model developed above to the problem of retrocession. The section begins with a description of the portfolio being optimized and proceeds to describe the details and assumptions that constitute the business situation being evaluated. Finally, the results of the optimization are presented in the form of optimal decisions as well as several sensitivity analyses.

5.1 Details on the Portfolio

The portfolio being optimized is that of Sirius International. Sirius International is a reinsurance firm headquartered in Stockholm [2]. Sirius International Insurance Group has business in the property, casualty, accident & health and marine reinsurance.

Public figures show that the group holding company had gross written premiums of 1,120 million USD in 2013 and pre-tax income of 189 million USD.

5.2 The Retrocession Situation

The optimization is carried out to determine optimal retrocession decisions for the current gross portfolio, that is, the portfolio before the impact of current retrocession. The result and properties of the optimal portfolio can then be compared with the current portfolio net of retrocession.

Optimizing the portfolio requires knowledge of the objective of the retrocession as well as firm preferences regarding risk. Out of the measures presented it is assumed that $\text{TVaR}_{0.995}$ reflects the risk preferences. This choice is not meant to reflect actual firm risk preferences but rather to serve as an example of the implementations presented above.

5.2.1 Objective of Retrocession

A hypothetical scenario is used as the starting point for optimization of the portfolio. First, suppose that it would be possible to start over with protection of the gross reinsurance portfolio. Second, suppose that it is required that risk as measured by $\text{TVaR}_{0.995}(X)$, where X is the portfolio loss function, be the same as for the current portfolio net of retrocession. Third, suppose that the objective of retrocession is to fulfill the risk constraint and maximize the expected result. The optimal retrocession in that situation can then be compared to the current portfolio net of retrocession.

5.2.2 Supply of Protection

In order to optimize the portfolio it is also necessary to assume that there is supply of retrocession protection. The specific characteristics of the contracts available dictate which retrocession decisions are feasible.

It is assumed that it is possible to buy retrocession cover for each business unit individually. Furthermore, it is assumed that the cover available is of the proportional type and that it is allowable to retrocede up to 70% of the exposure to the gross reinsurance portfolio.

Two different forms of the reinsurance premium principle are tested. The first premium principle is to only have an overriding commission as defined in (1.1). It is assumed that

[2] Sirius International is an operating unit of Sirius International Insurance Group Ltd, which is a wholly owned subsidiary of White Mountains Insurance Group (WTM-NYSE).

the overriding commission amounts to 5% of ceded premiums. The second premium principle includes both an overriding commission and a profit commission as defined in (1.2). The profit commission is evaluated because the stochastic nature of the profit commission might have an impact on the optimization. The parameters of the profit commission are assumed to be $c_1 = 0.05$, $c_2 = 0.15$ and $c_3 = 0.1$, as suggested by expertise at Sirius International.

5.2.3 Statement of the Optimization Problem

The above assumptions can be implemented using the linear optimization model defined by (4.1) and the linear programming formulation of TVaR. The 20 business units each make up one class of contracts and thus $\Gamma = \{1, 2, \ldots, 20\}$. The final model with the first premium principle is

$$\begin{aligned}
&\min_{(x_1,x_2,\ldots,x_{20})} \quad \mathbb{E}[X] \\
&\text{subject to} \\
&\qquad 0.3 \leq x_\gamma \leq 1 \text{ for } \gamma = 1, 2, \ldots, 20 \\
&\qquad \mathrm{TVaR}_{0.995}(X) \leq \mathrm{TVaR}_{0.995}(-P_{\text{Current Portfolio}} + D_{\text{Current Portfolio}}) \\
&\qquad X = -\sum_{\gamma=1}^{20} P_\gamma(1 - (1-c)x_\gamma) + \sum_{\gamma=1}^{20} D_\gamma(1 - x_\gamma).
\end{aligned} \tag{5.1}$$

The final model with the second premium principle is

$$\begin{aligned}
&\min_{(x_1,x_2,\ldots,x_{20})} \quad \mathbb{E}[X] \\
&\text{subject to} \\
&\qquad 0.3 \leq x_\gamma \leq 1 \text{ for } \gamma = 1, 2, \ldots, 20 \\
&\qquad \mathrm{TVaR}_{0.995}(X) \leq \mathrm{TVaR}_{0.995}(-P_{\text{Current Portfolio}} + D_{\text{Current Portfolio}}) \\
&\qquad X = -\sum_{\gamma=1}^{20} P_\gamma(1 - x_\gamma) - D_\gamma(1 - x_\gamma) + x_\gamma c_1 P_\gamma \\
&\qquad\qquad + c_2 x_\gamma \max(P_\gamma - D_\gamma - c_1 P_\gamma - c_3 P_\gamma, 0)
\end{aligned} \tag{5.2}$$

5.3 Details on the Analysis

To perform the optimization, sample sets with 10000 simulated years are drawn, with replacement, from a base sample set that approximates the modelled one-year portfolio distribution. It is assumed that the modelled one-year portfolio distribution is a correct representation of the actual one-year portfolio distribution.

Optimal retrocession is calculated once for each sample set drawn. 1000 sample sets are drawn for the first premium principle and 100 sample sets are drawn for the second premium principle. Optimization based on the sample sets gives a bootstrapped distribution of the optimal decisions and the optimal value of the objective function, as described in section 4.4.2. The optimal portfolio for a premium principle is taken to be the portfolio obtained by averaging the optimal retained share for all sample sets. The performance of that portfolio is evaluated by bootstrapping $\mathbb{E}[X]$ and $\mathrm{TVaR}_{0.995}(X)$ using 10000 sample sets for each statistic.

The optimization is performed by implementing the optimization models in MATLAB and using the interior-point algorithm of `linprog` to find the optimal solution.

5.4 Results

Results for the optimal portfolio when there is supply of cover with an overriding commission are presented first. Then the results on the optimal portfolio when there is supply of cover with both overriding commission and profit commission are presented and finally, the results of sensitivity tests are presented.

5.4.1 Premium Principle: Overriding Commission

This section presents the optimal portfolio resulting from applying the model (5.1). First, tables and figures showing the distribution of the bootstrapped optimal retained share are presented. Second, the performance of the portfolio obtained by applying the average optimal retained share is studied by showing the distribution of the bootstrapped mean result and the bootstrapped $\text{TVaR}_{0.995}$.

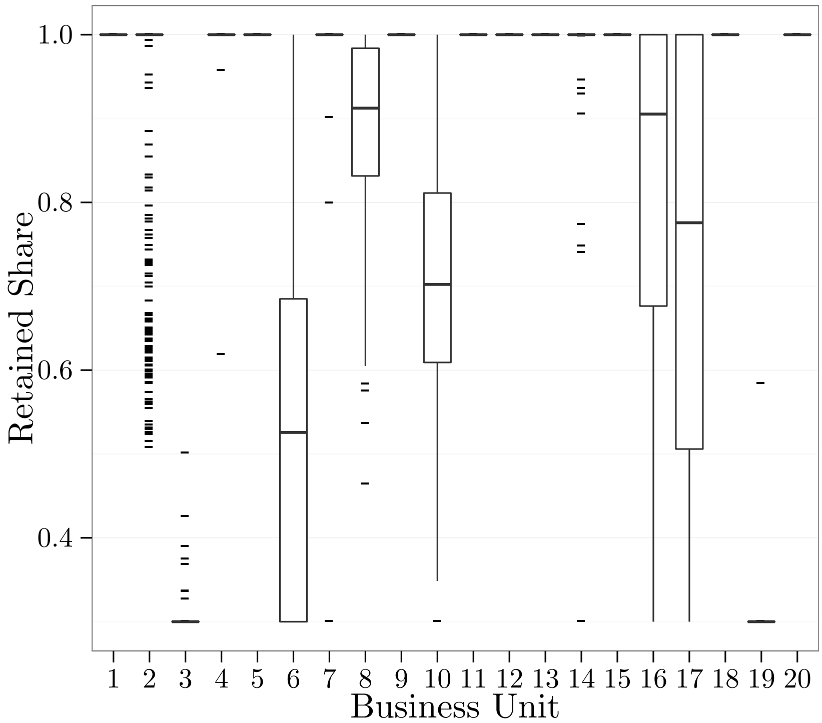

Figure 5.1: Boxplot of the optimal retained share obtained for each business unit in 1000 optimizations. Each optimization is based on resampling with replacement from the base sample set obtained from the portfolio distribution. The whiskers extend 1.5 times the interquartile range. Outliers are represented by short lines.

Figure 5.1 shows that a majority of the business units have no retrocession at all in almost all of the 1000 optimizations performed. In contrast, maximum possible retrocession cover (70%) is bought for business units 3 and 19 in almost all of the optimizations. This indicates that the business units 3 and 19 are driving most of the Tail-Value-at-Risk of the portfolio. Furthermore, there appears to be considerable uncertainty regarding what the optimal retrocession is for business units 6,16 and 17. Uncertainty regarding the optimal decisions can be an issue as it may indicate that the optimization has adapted to the sample rather than the underlying distribution.

Table 5.1 allows further investigation of the distribution of the bootstrapped optimal retained

	Minimum	1st Quartile	Median	Mean	3rd Quartile	Maximum
BU 2	0.5076	1.0000	1.0000	0.9536	1.0000	1.0000
BU 3	0.3000	0.3000	0.3000	0.3009	0.3000	0.5014
BU 4	0.6188	1.0000	1.0000	0.9992	1.0000	1.0000
BU 6	0.3000	0.3000	0.5257	0.5550	0.6850	1.0000
BU 7	0.3000	1.0000	1.0000	0.9874	1.0000	1.0000
BU 8	0.4641	0.8316	0.9124	0.8946	0.9838	1.0000
BU 10	0.3000	0.6091	0.7022	0.7038	0.8112	1.0000
BU 14	0.3000	1.0000	1.0000	0.9960	1.0000	1.0000
BU 16	0.3000	0.6764	0.9053	0.8059	1.0000	1.0000
BU 17	0.3000	0.5059	0.7757	0.7300	1.0000	1.0000
BU 19	0.3000	0.3000	0.3000	0.3006	0.3000	0.5841

Table 5.1: Descriptive statistics for the optimal retained share for each business unit that has an optimal retained share below 1 in at least one of 1000 optimizations.

share. Left out of the table are the 9 business units for which no retrocession occurs in any of the 1000 optimizations. The 9 units left out probably contribute very little to the Tail-Value-at-Risk.

The means presented in table 5.1, together with a retained share $x_\gamma = 1$ for the business units left out of the table, represent the average optimal retained share of each business unit.

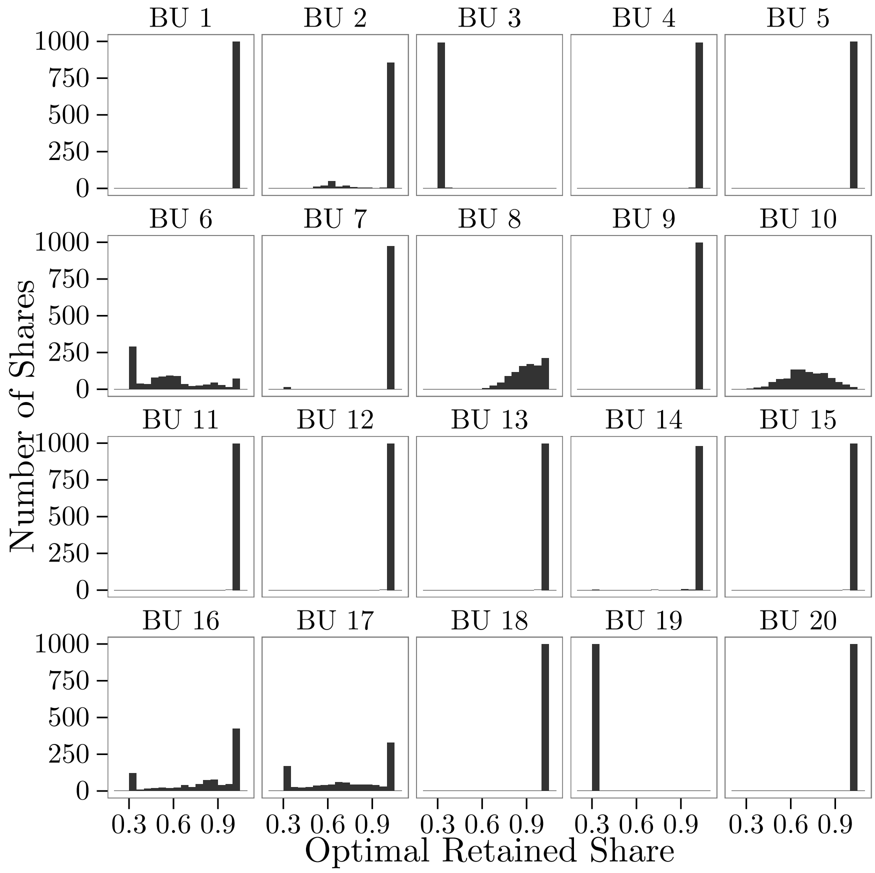

Figure 5.2: Histograms of the optimal retained share obtained for each Business Unit (BU) in 1000 optimizations.

As with figure 5.1 and table 5.1, figure 5.2 expounds the sampling distribution of the optimal retained shares. The histograms highlight the uncertainty associated with business units 6, 16 and 17 especially well. Business units 13 and 15 are also somewhat uncertain but for business units 6, 16 and 17, most of the probability is concentrated to the diametrically opposite retained shares, 1 and 0.3. It appears there is indeed an issue of adaptation to the portfolio distribution sample for the business units 6, 16 and 17.

The optimal retained share calculated for 1000 sample sets is presented in figure 5.1, figure 5.2 and table 5.1. The average optimal retained share for each business unit across the 1000 optimizations is taken to construct the optimal portfolio. The performance of this optimal portfolio is shown in figure 5.4 and figure 5.5.

Figure 5.3 presents the mean result (or indeed the value of the objective function) obtained in each of 1000 optimizations. It is closely related to figure 5.4.

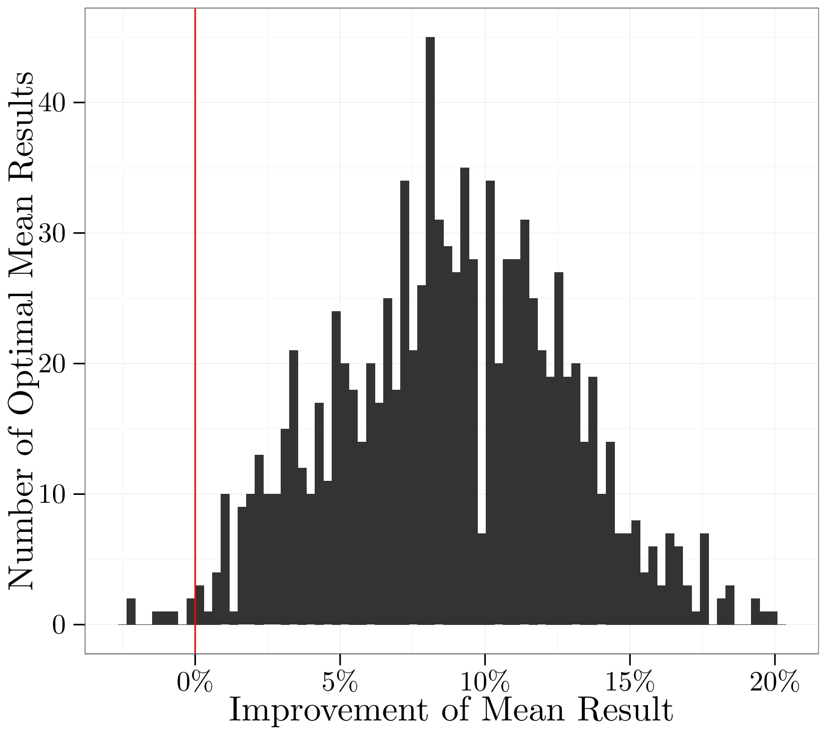

Figure 5.3: Histogram of the optimal mean result obtained in 1000 optimizations, expressed as a percentage of the current mean result after retrocession. The red line shows the current mean result after retrocession.

Both figure 5.4 and figure 5.3 suggest that a considerable benefit might be obtained by applying optimal retrocession to a reinsurance portfolio. An 8% improvement of annual expected returns, as figure 5.4 suggests, would probably seem quite attractive. It is paramount to observe that the results are dependent on several assumptions. The assumptions will be further discussed below.

The difference between figure 5.4 and figure 5.3 is attributable to the fact that the latter is the direct result of the optimization and thus satisfies the risk constraint *for every sample* used in the optimization. Figure 5.4 presents the *bootstrapped* mean result of the portfolio obtained by applying the average optimal retained share, which does not guarantee that the portfolio satisfies the risk constraint for a sample drawn from the portfolio distribution. This point is apparent when considering figure 5.5.

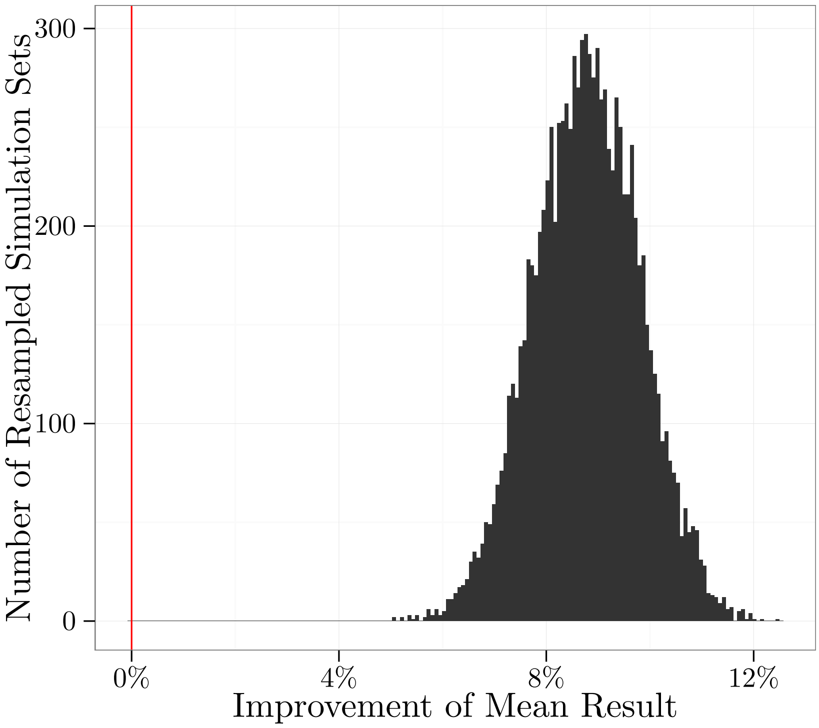

Figure 5.4: The optimal portfolio is obtained using the average optimal retained share. The histogram shows the mean result of that portfolio, calculated for each of 10000 sample sets. The red line shows the current mean result after retrocession.

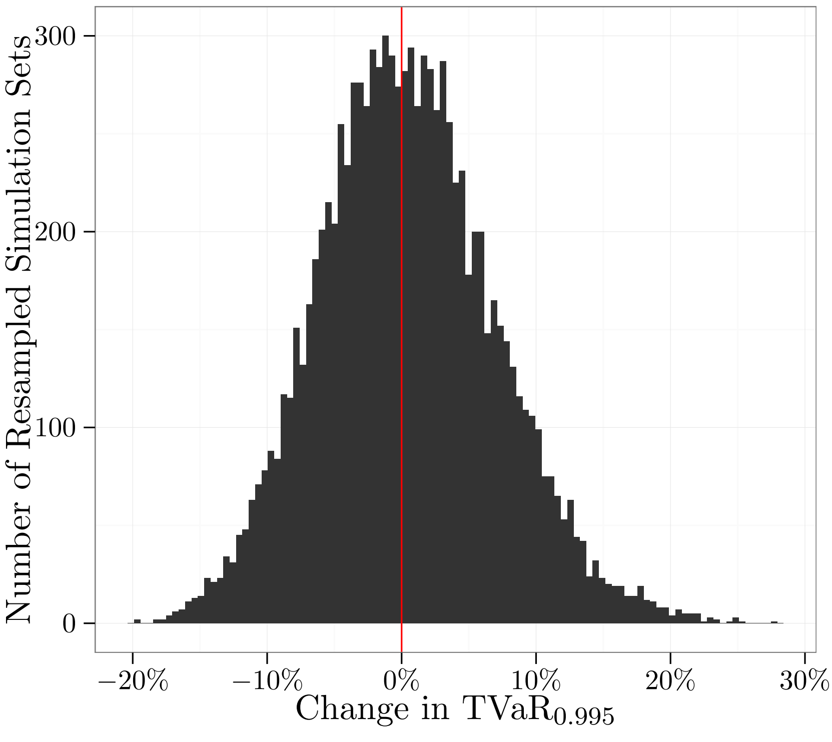

Figure 5.5: The optimal portfolio is obtained using the average optimal retained share. The histogram shows the $\text{TVaR}_{0.995}$ of that portfolio, calculated for each of 10000 sample sets. The red line shows the current $\text{TVaR}_{0.995}$ after retrocession.

Figure 5.5 shows that there is some uncertainty associated with the actual risk of the optimal portfolio. A large portion of the sampled $\text{TVaR}_{0.995}$ lie within ±10% of the target $\text{TVaR}_{0.995}$, which probably can be considered acceptable. The histogram is centered around the red line because the red line is $\text{TVaR}_{0.995}(-P_{\text{Current Portfolio}} + D_{\text{Current Portfolio}})$, which is the constraint in the optimization problem.

5.4.2 Premium Principle: Overriding Commission and Profit Commission

The results from optimizing the portfolio when retrocession contracts are assumed to include a profit commission are largely the same as when no profit commission is included. The profit commission makes retrocession more profitable and thus the mean result of the optimal portfolio is slightly better.

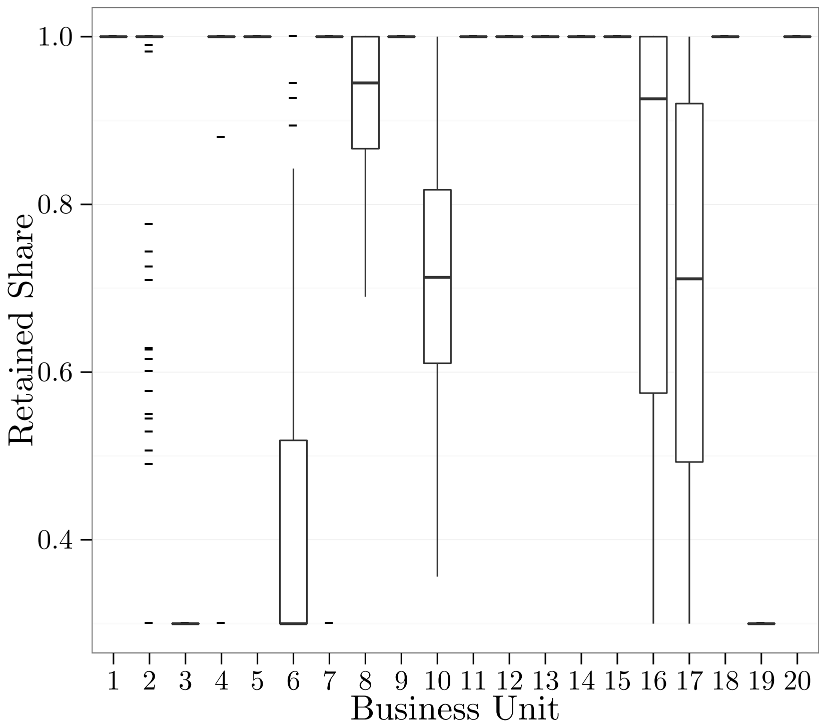

Figure 5.6: Boxplot of the optimal retained share obtained for each business unit in 100 optimizations. The premium principle is assumed to include a profit commission. The whiskers extend 1.5 times the interquartile range. Outliers are represented by short lines.

The optimal retained shares for the two different premium principles are very similar as becomes apparent when comparing figure 5.6 with figure 5.1. There is a tendency to retain less of the exposure to business unit 6 when including a profit commission but that hardly signifies much because of the uncertainty related to business units 6, 16 and 17.

The optimal portfolio is constructed in the same way as in the previous section. The performance of the optimal portfolio is also evaluated in the same way and is presented in figure 5.7 and figure 5.8.

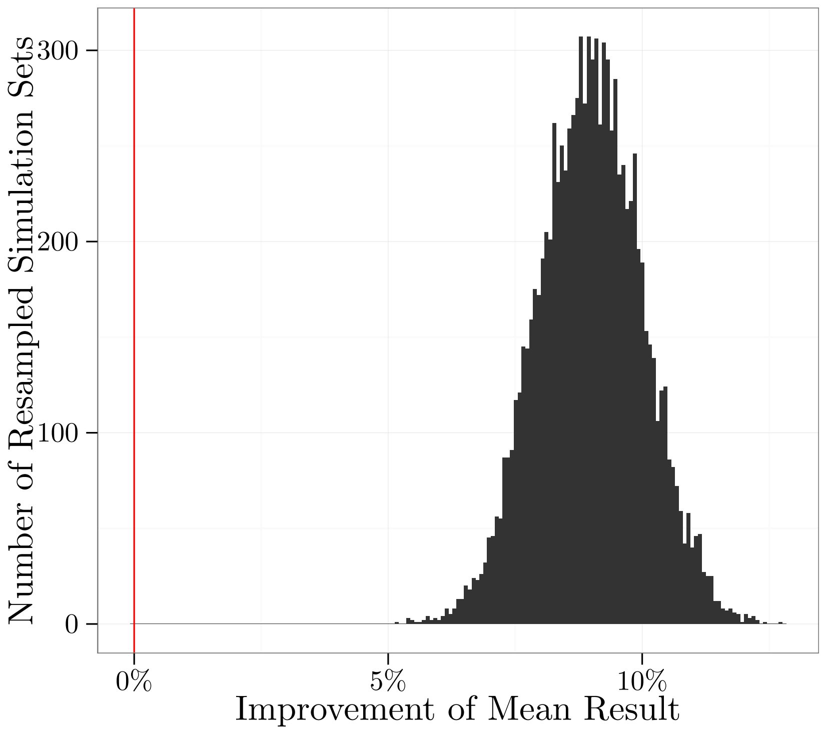

Figure 5.7: The optimal portfolio is obtained using the average optimal retained share. The histogram shows the mean result of that portfolio, calculated for each of 10000 sample sets. The red line shows the current mean result after retrocession.

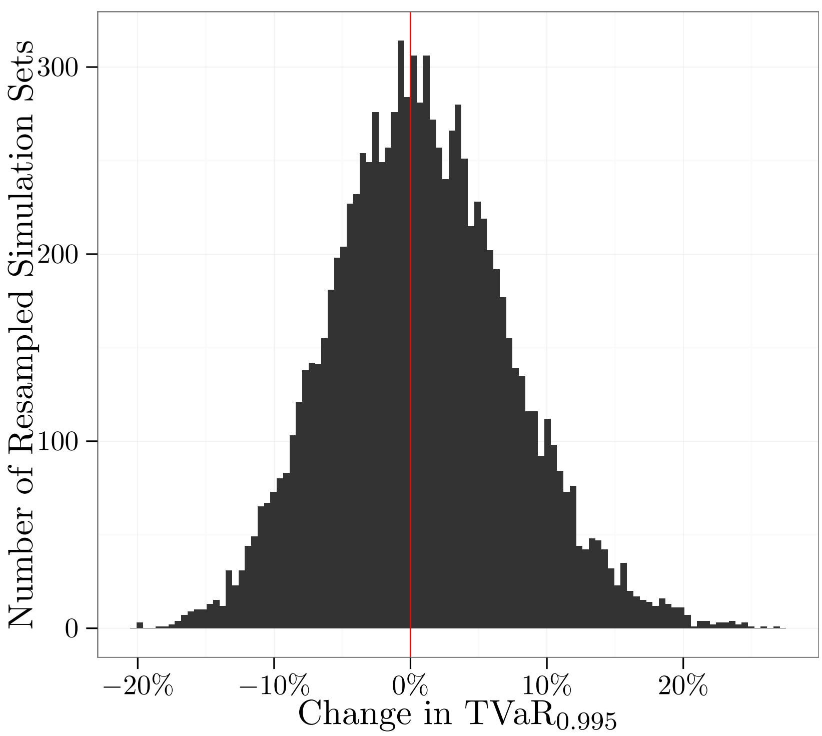

Figure 5.8: The optimal portfolio is obtained using the average optimal retained share. The histogram shows the $\text{TVaR}_{0.995}$ of that portfolio, calculated for each of 10000 sample sets. The red line shows the current $\text{TVaR}_{0.995}$ after retrocession.

5.4.3 Uniqueness of the Optimal Retrocession Decisions

Two approaches to testing uniqueness of the optimal retrocession are suggested in section 4.4.1. Both approaches have been applied to the optimization problem defined by (5.1) and are consistent with a unique optimal solution. The analysis is not conclusive, however, as the analysis necessitated a scaling down of the number of samples drawn from the portfolio distribution due to memory constraints on the computational server.

5.4.4 Discussion of the Assumptions

The results obtained may partly be explained by the assumptions made when formulating the retrocession situation. The assumptions regarding supply of protection are likely to impact the results and are also uncertain as they pertain to the market conditions for retrocession. The objective of retrocession could also deviate from the assumptions in both objective function and the constraints. This section will focus on the supply of protection to analyze the impact of external market conditions on the optimization results.

Several assumptions are made regarding the supply of protection. First, non-proportional retrocession might be available. This would mean more flexibility in retrocession, which, if not harnessed, could lead to sub-optimal results. Second, it is assumed that proportional retrocession is available for each business unit. If that is not the case, the above retrocession might not be feasible, implying that the results above are overstated. Third, proportional retrocession up to 70% is assumed feasible. More or less flexibility might be available but, as the optimization problem is linear, a small change in this constraint is unlikely to result in a large change in the objective function. This has been tested for a few bootstrapped samples and seems to hold. Fourth, it is assumed that there is an overriding commission of 5%.

The overriding commission involves a lot of uncertainty as it may vary from contract to contract. Furthermore, it is likely to have a large impact on the portfolio result. For this reason, an additional bootstrapped optimization has been performed with a reverse commission, that is, a commission going from the cedent to the retrocessionaire.

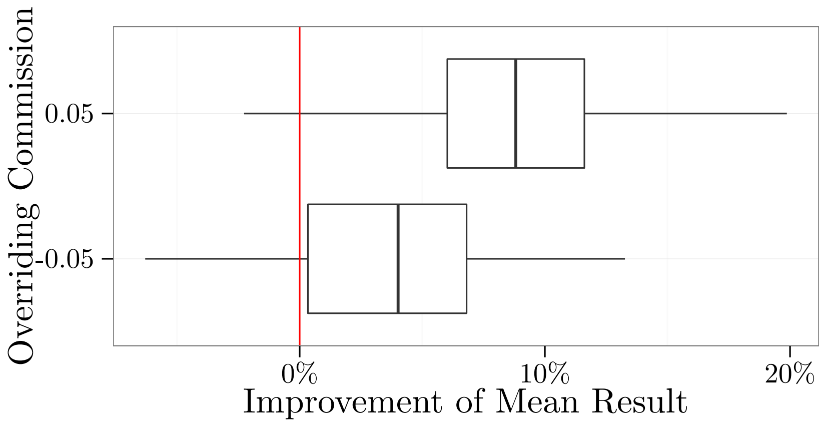

Figure 5.9: Boxplot of the mean result in bootstrapped optimizations when the premium principle includes an overriding commission of 5% and −5% respectively. The red line shows the current mean result after retrocession. The whiskers extend 1.5 times the interquartile range.

Figure 5.9 shows that there is indeed a considerable difference in result when applying a commission of −5% instead of 5%. The result still appears to be considerably better than that of the current mean result, however. This indicates that the result improvement cannot be explained by the commission assumption, as the -5% commission arguably is quite extreme. Gustafsson [9] states that the overriding commission usually is in the range 2.5% - 5%.

5.4.5 Limitations of the Analysis

There are some limitations of the analysis that cannot be resolved by obtaining more information on company risk preferences or the market supply of protection.

For the optimization above, it is assumed that the model of the one-year portfolio distribution is a correct representation of the actual losses that the portfolio could suffer. This makes the analysis very dependent on having a correct model of the portfolio. For example, if some inputs to the model originate in the business units, then it is paramount that the methodology used to obtain the inputs is uniform across all business units. Otherwise, the optimal decisions might reflect differences in the methodology for obtaining model inputs rather than an optimal trade-off between mean result and risk.

Another limitation of the analysis is that it assumes that risk and expected result are the only two factors that are affected by retrocession. In reality, the impact on the reinsurance business might be more complex. In that case, some aspects of the impact are lost by applying the optimization model.

5.4.6 Beyond Retrocession

The models introduced can be used to experiment with a shift in focus from optimal retrocession to optimal underwriting. The conditions in the optimization problem defined by (5.1) can be altered to allow “retention” of more than 100% of the current portfolio. A “retained” share of more than 100% amounts to expanding the portfolio by exactly replicating the contracts held in the business unit.

Some tests have been performed, allowing up to 200% of the current portfolio for each business unit. The tests hint (perhaps unsurprisingly) at optimal underwriting being a more powerful instrument of result improvement than optimal retrocession. However, the assumption of portfolio expansion through replication is a big leap of faith.

6 Conclusions

The purpose of this thesis has been to investigate the possibility of modeling retrocession as an optimization problem. The conclusion is that proportional retrocession is very apt to be modeled as an optimization problem. Non-proportional retrocession presents some difficulties but a practically oriented approach, that may be satisfactory if the number of available non-proportional contracts is relatively small, has been introduced.

The model for proportional retrocession has been applied to optimization of a large portfolio of reinsurance contracts. The conclusion from that optimization is that the optimal portfolio appears reasonably well-determined and reliable. The results further indicate that there may be a substantial benefit to optimizing a reinsurance portfolio using retrocession.

Finally, the quantile-based approach and the outcome-based approach to measuring risk have been further developed to allow comparison. The theoretical results highlight that there are fundamental differences between the two approaches and that the characteristics that differ may be desirable or undesirable depending on the situation.

6.1 Suggestions for Further Research

It would be of interest to further investigate when underwriting, as opposed to retrocession, can be used to form an optimal portfolio. Such an investigation might include modeling market conditions in addition to the portfolio.

The model of non-proportional retrocession presented in this thesis is not nearly as apt for optimization as the model for proportional retrocession. A more thorough study of discrete optimization modeling techniques and algorithms could yield a more tractable model of non-proportional retrocession.

A theoretical extension is also possible. Sufficient conditions for convexity are given for the risk measures developed but the question of necessary conditions for convexity could also be investigated.

7 Bibliography

[1] Acerbi, C. (2002). Spectral measures of risk: A coherent representation of subjective risk aversion. *Journal of Banking and Finance*, 26.

[2] Acerbi, C. and Tasche, D. (2002). On the coherence of expected shortfall. *Journal of Banking and Finance*, 26.

[3] Arrow, K. J. (1963). Uncertainty and the welfare economics of medical care. *The American Economics Review*, 103.

[4] Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9:203–228.

[5] Efron, B. (1979). Bootstrap methods: Another look at the jackknife. *The Annals of Statistics*, 7.

[6] Ferrier, G. D. and Hirschberg, J. G. (1997). Bootstrapping confidence intervals for linear programming efficiency scores: With an illustration using Italian banking data. *Journal of Productivity Analysis*, 8.

[7] Föllmer, H. and Scheid, A. (2008). Convex and coherent risk measures.

[8] Griva, I., Nash, S. G., and Sofer, A. (2009). *Linear and Nonlinear Optimization.* SIAM.

[9] Gustafsson, B. (2000). *Återförsäkring.* Ifu.

[10] Kałuszka, M. (2003). Mean-variance optimal local reinsurance contracts. *Control and Cybernetics*, 32.

[11] Kałuszka, M. and Okolewski, A. (2008). An extension of Arrow's result on optimal reinsurance contract. *The Journal of Risk and Insurance*, 75.

[12] Kreps, R. (2005). Riskiness leverage models. *Guy Carpenter Instrat.*

[13] Krokhmal, P., Palmquist, J., and Uryasev, S. (2002). Portfolio optimization with conditional value-at-risk objective and constraints. *Journal of Risk*, 4.

[14] Mangasarian, O. L. (1979). Uniqueness of solution in linear programming. *Linear Algebra and its Applications*, 25.

[15] Swiss Re (2002). An introduction to reinsurance.

[16] Rockafellar, R. T. and Uryasev, S. (2000). Optimization of conditional value-at-risk. *Innovations in risk management*, 2.

[17] Rockafellar, R. T., Uryasev, S., and Zabarankin, M. (2006). Generalized deviation in risk analysis. *Finance and Stochastics*, 10.

[18] Uryasev, S. (2000). Conditional value-at-risk: Optimization algorithms and applications. *Financial Engineering News*, 14.

[19] Winston, W. L. (2003). *Operations Research: Algorithms and Applications.* Cengage Learning.

Appendix A Comments on the Riskiness Leverage

Kreps [12] presents the riskiness leverage risk measure that depends on the loss size.

Definition 24 (Riskiness Leverage)**.** Consider a stochastic variable X that takes positive values for negative portfolio results and that has mean μ. The riskiness leverage risk measure is then defined by

$$\rho(X) = \int_{-\infty}^{\infty} L(x)(x - \mu)f(x)dx, \tag{A.1}$$

where $L(x)$ is the riskiness leverage, which is a function

$$L : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto L(x)$$

that weights the deviation, $X - \mu$, depending on the outcome for the entire portfolio.

The riskiness leverage measure defined by (A.1) can be convex or non-convex depending on the choice of riskiness leverage function $L(\cdot)$. This thesis considers an alteration of (A.1) that measures the absolute outcome X, or the deviation $X - \mu_X$, but not both in the same measure.

Rockafellar et al. [17] introduce the concept of a deviation measure. Such a measure is defined as the risk measure of the deviation variable, $X - \mu_X$. Defining the loss-size risk measure without mixing X and $X - \mu_X$ enables the measurement of deviation as suggested in [17]. No clear reason to mix the two has been encountered.

Appendix B Further Statistics on Optimal Retained Shares

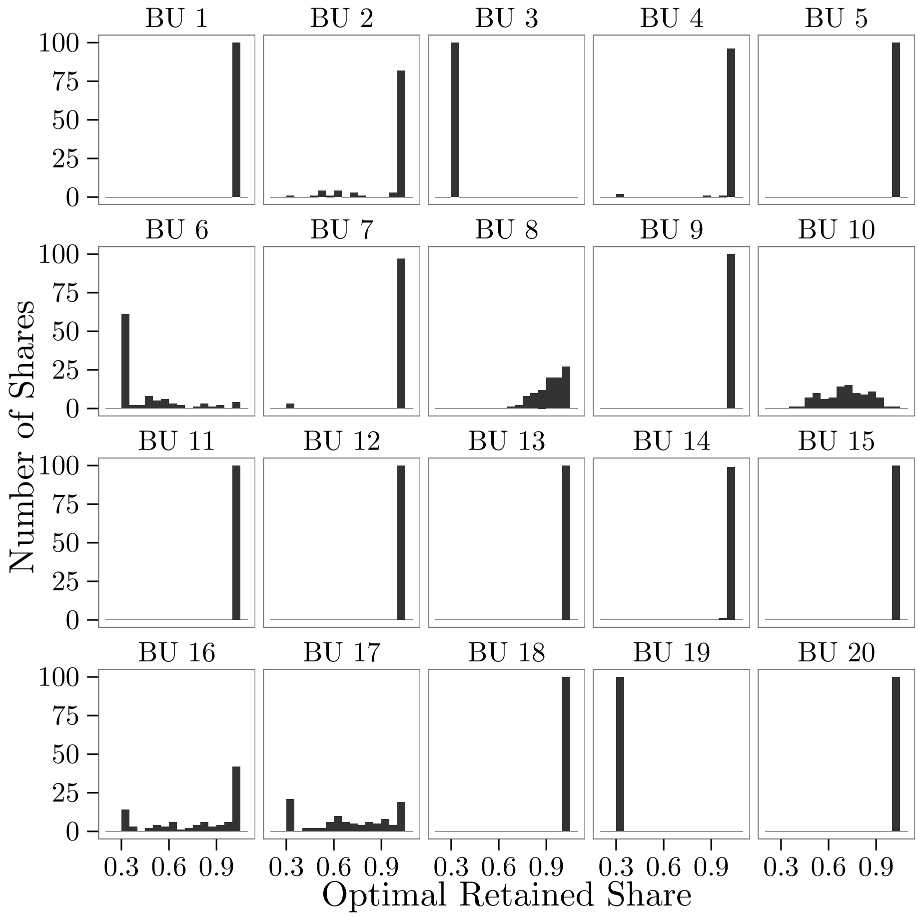

Figure B.1: Histograms of the optimal retained share obtained for each Business Unit (BU) in 100 optimizations, when a profit commission is included in the premium principle.

	Minimum	1st Quartile	Median	Mean	3rd Quartile	Maximum
BU 2	0.3000	1.0000	1.0000	0.9389	1.0000	1.0
BU 3	0.3000	0.3000	0.3000	0.3000	0.3000	0.3
BU 4	0.3000	1.0000	1.0000	0.9848	1.0000	1.0
BU 6	0.3000	0.3000	0.3000	0.4305	0.5186	1.0
BU 7	0.3000	1.0000	1.0000	0.9790	1.0000	1.0
BU 8	0.6898	0.8664	0.9448	0.9206	1.0000	1.0
BU 10	0.3561	0.6105	0.7130	0.7076	0.8174	1.0
BU 16	0.3000	0.5750	0.9260	0.7839	1.0000	1.0
BU 17	0.3000	0.4928	0.7113	0.6925	0.9202	1.0
BU 19	0.3000	0.3000	0.3000	0.3000	0.3000	0.3

Table B.1: Descriptive statistics for the optimal retained share for each business unit that has an optimal retained share below 1 in at least one of 100 optimizations, when a profit commission is included in the premium principle.

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