The Risk Parity Approach to Asset Allocation

by

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Abstract

We consider the problem of portfolio's asset allocation characterised by risk and return. Prior to the 2007-2008 financial crisis, this important problem was tackled using mainly the Markowitz mean-variance framework. However, throughout the past decade of challenging markets, particularly for equities, this framework has exhibited multiple drawbacks.

Today many investors approach this problem with a 'safety first' rule that puts risk management at the heart of decision-making. Risk-based strategies have gained a lot of popularity since the recent financial crisis. One of the 'trendiest' of the modern risk-based strategies is the Risk Parity model, which puts diversification in terms of risk, but not in terms of dollar values, at the core of portfolio risk management.

Inspired by the works of Maillard *et al.* (2010), Bruder and Roncalli (2012), and Roncalli and Weisang (2012), we examine the reliability and relationship between the traditional mean-variance framework and risk parity. We emphasise, through multiple examples, the non-diversification of the traditional mean-variance framework. The central focus of this thesis is on examining the main Risk-Parity strategies, i.e. the Inverse Volatility, Equal Risk Contribution and the Risk Budgeting strategies.

Lastly, we turn our attention to the problem of maximizing the absolute expected value of the logarithmic portfolio wealth (sometimes called the drift term) introduced by Oderda (2013). The drift term of the portfolio is given by the sum of the expected price logarithmic growth rate, the expected cash flow, and half of its variance. The solution to this problem is a linear combination of three famous risk-based strategies and the high cash flow return portfolio.

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Opsomming

Ons kyk na die probleem van batetoewysing in portefeuljes wat gekenmerk word deur risiko en wins. Voor die 2007-2008 finansiele krisis, was hierdie belangrike probleem deur die Markowitz gemiddelde-variansie raamwerk aangepak. Gedurende die afgelope dekade van uitdagende markte, veral vir aandele, het hierdie raamwerk verskeie nadele getoon.

Vandag, benader baie beleggers hierdie probleem met 'n 'veiligheid eerste' reël wat risikobestuur in die hart van besluitneming plaas. Risiko-gebaseerde strategieë het baie gewild geword sedert die onlangse finansiële krisis. Een van die gewildste van die moderne risiko-gebaseerde strategieë is die Risiko-Gelykheid model wat diversifikasie in die hart van portefeulje risiko bestuur plaas.

Geïnspireer deur die werke van Maillard *et al.* (2010), Bruder and Roncalli (2012), en Roncalli and Weisang (2012), ondersoek ons die betroubaarheid en verhouding tussen die tradisionele gemiddelde-variansie raamwerk en Risiko-Gelykheid. Ons beklemtoon, deur middel van verskeie voorbeelde, die nie-diversifikasie van die tradisionele gemiddelde-variansie raamwerk. Die sentrale fokus van hierdie tesis is op die behandeling van Risiko-Gelykheid strategieë, naamlik, die Omgekeerde Volatiliteit, Gelyke Risiko-Bydrae en Risiko Begroting strategieë.

Ten slotte, fokus ons aandag op die probleem van maksimering van absolute verwagte waarde van die logaritmiese portefeulje welvaart (soms genoem die drif term) bekendgestel deur Oderda (2013). Die drif term van die portefeulje word gegee deur die som van die verwagte prys logaritmiese groeikoers, die verwagte kontantvloei, en die helfte van die variansie. Die oplossing vir hierdie probleem is 'n lineêre kombinasie van drie bekende risiko-gebaseerde strategieë en die hoë kontantvloei wins portefeulje.

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Last but not the least, I would like to thank my family for their unconditional love, support and understanding of what I have been going through.

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Dedication

This thesis is dedicated to my daughter 'Kgadi Happy Moremi'.

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Contents

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Stellenbosch University http://scholar.sun.ac.za

List of Figures

Stellenbosch University http://scholar.sun.ac.za

List of Tables

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Chapter 1

Introduction

In the aftermath of the 2007/2008 financial crisis, which was characterized by low-interest rates and high risks of draw-down on the capital markets, most institutional investment companies experienced a large number of subprime investors defaulting their loans. Academics believed that this event was due to the deterioration of housing prices, which led to home owners owning more than their property's worth. The problem persisted to an extent that central banks of several developed countries resorted to coordinating action to provide liquidity support to financial institutions. Specifically, the US Federal Reserve Bank (FED) slashed both the discount and the fund rates. However, none of these actions turned inflation down and even the volume of investment in the equity market remained red[1].

This detrimental behaviour of the economy led to almost all investment strategies performing very poorly. The performance of the markets left an indelible impression on investors about the strategies they had implemented. The question was, 'What went wrong with the strategies we used to believe in?' This triggered a search by both academic researchers and market practitioners to find an alternative investment strategy that would perform well during all kinds of market scenarios.

Before we commence the search here, one needs to understand the fundamentals of the existing strategies. The origin of market analysis, in particular, the stock price movements, was first introduced by Bachelier (1900) in his PhD thesis entitled 'The Theory of Speculation'. Markowitz (1952)[2] contributed to market analysis by incorporating multiple assets that form a portfolio and developing the mean-variance strategy[3]. He determined the con-

[1] The total number of contracts or shares that have been recorded as an activity in the equity marketplace for a period of time

[2] 1990 Economics Nobel prize winner.

[3] An investment model that combines the expected return and risk of the portfolio and gives decision on allocation of assets through mathematical optimization techniques.

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cept of an efficient portfolio[4] and showed that there exist multiple efficient portfolios that form the efficient frontier[5]. The Markowitz strategy requires three input parameters, namely the expected return, correlation matrix and the covariance matrix of asset returns. The precise estimation of these parameters is often difficult and subjected to significant errors. These problems lead to many investors implementing some 'rule of thumb' as an alternative investment strategy. The most commonly used rule is 60/40 strategy which simply allocates 60% of investment wealth to stocks and 40% to bonds.

Contradicting this idea of anticipating asset price movement in order to beat the market prices, Fama (1995)[6] argued in his work entitled 'Random walks in stock market prices' that asset expected returns follow Martingale expectation. He developed the efficient-market hypothesis (EMH), illustrating the fact that the price of an asset is an accurate reflection of all available information in the market. However, Lo and MacKinlay (2011) disputed the idea of Fama, and argued that the EMH is not completely valid. Incorporating asset cross autocorrelation returns, one could still be able to predict future asset returns.

More recent research focuses on risk-based asset allocations to protect investments against significant losses, with diversification controlling the investment decision. Institutional investment reports show that risk-based portfolio allocations were the only strategies that performed exceptionally during the recent crisis, see Peters (2010), Podkaminer (2013), Rappoport and Nottebohm (2012) and Romahi and Santiago (2012). In particular, the so-called Risk Parity scheme (i.e., an investment strategy that has a constant level of risk that is equally divided amongst the components in a portfolio) gained popularity since the recent financial crisis.

The main objective of this thesis is to study the risk-based strategies with more emphasis on the risk parity strategy. This strategy has been dominating the investment media, particularly in the journal of investing and the journal of portfolio management. In addition, Roncalli (2013) and Lussier (2013) published separately books detailing this concept. The former devoted his book to the concept of risk parity while the latter seeks to identify the structural qualities or characteristics required when building a portfolio to reliably increase the likelihood of excess performance.

We study the risk parity strategy in comparison with the other three risk-based strategies, namely, Minimum Variance, Equal Weighted and Maximum Diversification. The main advantage of these strategies is that they diminish the input parameters of the traditional mean-variance strategy. In particular,

[4] The only portfolio that offers maximum return for a given level of risk.

[5] A curve characterised by all efficient portfolios.

[6] One of the recipient of the 2013 economics Nobel Prize laureate.

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the estimation of expected return is not accountable for portfolio's compositions.

However, in order to determine the efficient risk-based strategy, it is important to understand the traditional mean-variance strategy. We show how risk-based strategies are linked with the mean-variance strategy. From the efficient portfolio[7] we decompose the covariance matrix into the product of diagonal matrix and correlation matrix. Applying the same approach to the risk parity approach, we find that the risk contribution of a component is actually the square of the component Sharpe ratio.

1.1 Origin of Asset Allocation

In this section, we present the genesis of the popular investment strategy that builds on the ideas of optimizing the trade-off between returns and risks. Portfolio management, or asset allocation, is never trivial, particularly when there is risk associated with the choice of assets. An excellent portfolio design is characterized by basic concepts, such as safe investment, high income of return and a potential for capital appreciation in the future. Safe investment refers to a strategy that holds a variety of asset classes. The intuition behind this is that if one class is performing badly, the entire portfolio performance could still be compensated for by the remaining asset classes. Investors in this case compose their portfolios based on their return objectives, liability requirements, risk tolerance and some taxation.

Most portfolio allocations rest on the mean-variance framework which is now described here. Markowitz (1952) noted the return of the mean-variance portfolio, a desirable thing, while risk is considered undesirable. A quadratic optimization technique is typically implemented to determine the optimum portfolio that will serve the interest of investors.

A series of successful research studies have been conducted with attempts to improve the original mean-variance model. Among the researchers are Tobin (1958) who introduced risk-free assets to balance portfolio return and developed the separation theorem; Sharpe (1964), Mossin (1966), Treynor (1962), Lintner (1965) who developed the Capital Asset Pricing model and Black and Litterman (1992) who extended that model to incorporate investors' views. We detail more of the mean-variance strategy in the next subsection.

1.1.1 Mean-Variance Framework

We begin the description of this strategy by first introducing the following notations. We consider a situation where the investor wants to invest a unit

[7] To be defined in the next section.

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Figure 1.1: Adjusted Daily Stock Prices of PAYX

amount, say $x_0 > 0$, in n-risky assets (or components). This implies that x_0 has to be distributed amongst all n-assets according to the investors' preferences and requirements. The main objective of investment, in general, is to obtain profit from the future performance of these n-assets. However, this performance is not known in advance, and all components are subjected to random future prices. For instance, Figure 1.1 shows the degree of randomness a stock ticker PAYX has undergone since 26 March 1990.

Since portfolios are held for a period of time, we define the standard return[8] of the i^{th} security as

$$R_{i,t} = \frac{x_{i,t}}{x_{i,t-1}}, \quad t = 1, \dots, K, \tag{1.1.1}$$

where $x_{i,t-1}$ and $x_{i,t}$ are the unit closing and opening prices of the security in the market at times $t-1$ and t, respectively. However, asset-return measurements are not necessarily determined from daily stock prices. For example, they could be determined using security prices taken hourly, weekly, monthly, yearly, etc. The rate of return (or simply, the arithmetic-price return), $r_{i,t}$ for the i^{th} security is given by

$$r_{i,t} = \frac{x_{i,t} - x_{i,t-1}}{x_{i,t-1}}, \quad t = 1, \dots, K. \tag{1.1.2}$$

Thus, the value of $x_{i,t}$ is expressible as

$$x_{i,t} = (1 + r_{i,t})x_{i,t-1}, \tag{1.1.3}$$

which resembles the geometric return of the i^{th} asset (analysis of assets or portfolios using this notation is covered in chapter 8). Note that from here on,

[8] Another measure of returns is logarithmic, which does not posses linearity property and is precisely used for stock prices analysis.

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throughout the rest of the thesis, we will omit the subscript t for presentation purpose and we shall use it only when necessary.

In order to determine the first two moments of the i^{th} asset, we consider its K-trailing arithmetic returns,

$$\begin{pmatrix} r_{i1} \\ r_{i2} \\ \vdots \\ r_{iK} \end{pmatrix}. \tag{1.1.4}$$

The expected return, $\bar{r}_i$, and variance, σ_i^2, of this component, are,

$$\bar{r}_i = \frac{1}{K}\sum_{k=1}^{K} r_{ik}, \tag{1.1.5}$$

$$\sigma_i^2 = \frac{1}{K-1}\sum_{k=1}^{K}(r_{ik} - \bar{r}_i)^2. \tag{1.1.6}$$

The volatility of the i^{th} component, σ_i, is the square root of its variance. Very often, investors invest a quantity of a security, and thus each component weight can be expressed as

$$z_i = \frac{x_i}{x_0} \quad i = 1, \ldots, n, \tag{1.1.7}$$

where x_i denotes the quantity of security i. Thus, the vectors of weights and returns of n-assets are denoted by $\mathbf{z}$ and $\mathbf{r}$, respectively.

Definition 1.1. *A portfolio is a collective set of n-random pay-off assets that can be expressed as a linear combination of the vector of weights $\mathbf{z} \in \mathbb{R}^n$ fulfilling the budget constraint $\mathbf{z}^T\mathbb{1} = 1$, where $\mathbb{1} \in \mathbb{R}^n$ is a vector of ones and the return of the portfolio is given by $\mathbf{z}^T\mathbf{r}$.*

These assets are believed to hedge the initial invested amount over time. The challenge is how to distribute the investor's wealth among these assets in a portfolio [9]. Note that z_i denotes the proportion of the investor's wealth in asset i. The risk (volatility) of the portfolio is given by,

$$\sigma(\mathbf{z}) = \sqrt{\mathbf{z}^T\Sigma\mathbf{z}}, \tag{1.1.8}$$

where $\Sigma \in \mathbb{R}^{n\times n}$ is a positive-definite covariance matrix of asset returns. We denote σ_{ij} as the covariance constant between asset i and j in the market.

[9] Assets can be bought or sold in shares, e.g. if one rand buys 0.25 shares of security and gives a profit of 3 rands, then for 10 rands, one can purchase 2.5 shares and provide profit of 30 rands.

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More details about portfolio-risk measures are given in section (2.1). It is important to note that the weights of assets in a portfolio are not strictly positive. Some may be negative, indicating the borrowing of a risk-free asset. Since the objective of investment is to hedge funds, the selection of asset-weights is crucial. Assuming that all investors are rational, i.e, all investors will appreciate an optimal portfolio[10], and in addition, their risk tolerance is heterogeneous, we can infer the objective of investment. The investor either wishes to maximize the expected return for a given level of risk (volatility), or to minimize the risk for a given level of expected return. We denote by $\bar{\mathbf{r}} \in \mathbb{R}^n$, a vector of asset expected returns in a portfolio. The portfolio expected return is given by,

$$\mu(\mathbf{z}) = \mathbf{z}^T \bar{\mathbf{r}}. \tag{1.1.9}$$

Consider the former objective. Following Roncalli (2013), the system with only budget being constrained for this problem is expressed mathematically as

$$\mathbf{z}^{MVO} = \arg \max_{\mathbf{z} \in \mathbb{R}^n} \left(\mathbf{z}^T \bar{\mathbf{r}} - \frac{\lambda}{2} \mathbf{z}^T \Sigma \mathbf{z} \right) \tag{1.1.10}$$

such that

$$\mathbf{z}^T \mathbb{1} = 1,$$

where λ is considered the risk-aversion parameter[11]. The Lagrangian function for this system is

$$\mathcal{L}(\mathbf{z}, \lambda_0) = \mathbf{z}^T \bar{\mathbf{r}} - \frac{\lambda}{2} \mathbf{z}^T \Sigma \mathbf{z} + \lambda_0 (\mathbb{1}^T \mathbf{z} - 1). \tag{1.1.11}$$

The first order differential equations are:

$$\frac{\partial \mathcal{L}(\mathbf{z}, \lambda_0)}{\partial \mathbf{z}} = \bar{\mathbf{r}} - \lambda \Sigma \mathbf{z} + \lambda_0 \mathbb{1} = 0, \tag{1.1.12}$$

$$\frac{\partial \mathcal{L}(\mathbf{z}, \lambda_0)}{\partial \lambda_0} = \mathbf{z}^T \mathbb{1} - 1 = 0 \tag{1.1.13}$$

From equation (1.1.12), it follows that

$$\mathbf{z}^{MVO} = \lambda^{-1} \Sigma^{-1} (\bar{\mathbf{r}} + \lambda_0 \mathbb{1}). \tag{1.1.14}$$

Substituting the above equation into equation (1.1.13), we have

$$\lambda_0 = \frac{\lambda - \mathbb{1}^T \Sigma^{-1} \bar{\mathbf{r}}}{\mathbb{1}^T \Sigma^{-1} \mathbb{1}}. \tag{1.1.15}$$

[10] Portfolio z^* is optimal if, for any other attainable portfolio, there does not exist a portfolio z such that $\mu_{z^*} < \mu_z$ and $\sigma_{z^*} \geq \sigma_z$ or $\mu_{x^*} = \mu_x$ and $\sigma_{x^*} > \sigma_x$.

[11] Parameter used to scale the acceptable risk level of an investor.

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Thus, equation (1.1.12) is given by

$$\begin{aligned}
\mathbf{z}^{MVO} &= \lambda^{-1}\Sigma^{-1}\bar{\mathbf{r}} + \lambda^{-1}\Sigma^{-1}\lambda_0\mathbb{1} \\
&= \lambda^{-1}\Sigma^{-1}\mathbf{r} + \lambda^{-1}\Sigma^{-1}\left[\frac{\lambda_0 - \mathbb{1}^T\Sigma^{-1}\bar{\mathbf{r}}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}\right]\mathbb{1} \\
&= \frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}} + \lambda^{-1}\Sigma^{-1}\left[\bar{\mathbf{r}} - \mathbb{1}\frac{\mathbb{1}^T\Sigma^{-1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}\bar{\mathbf{r}}\right]. \qquad (1.1.16)
\end{aligned}$$

The solution in equation (1.1.16) is interpreted as follows: The first term is called the global minimum variance portfolio (discussed in the next chapter). The second term determines the portfolio's expected return relative to individual-asset expected returns, see Lee (2011).

In the absence of budget constraint, the Lagrange function of the above mathematical problem (1.1.10) with target variance, σ_0^2, is given by,

$$\mathcal{L}(\mathbf{z}) = \mathbf{z}^T\bar{\mathbf{r}} - \tfrac{\lambda}{2}(\mathbf{z}^T\Sigma\mathbf{z} - \sigma_0^2). \qquad (1.1.17)$$

Also, the first-order condition of the above function (1.1.17) is:

$$\tfrac{\partial\mathcal{L}(\mathbf{z})}{\partial\mathbf{z}} = \bar{\mathbf{r}} - \lambda\Sigma\mathbf{z} = 0, \qquad (1.1.18)$$

which implies that the solution to the unconstrained mean-variance portfolio is given by:

$$\mathbf{z}^{MVO} = \lambda\Sigma^{-1}\bar{\mathbf{r}}. \qquad (1.1.19)$$

Alternatively, for an investor who wants to minimize the portfolio variance given the level of expected return and adding more constrains, for example, the short-selling constrain, the problem can be specified mathematically as follows,

$$\mathbf{z}^{MVO} = \arg\min_{\mathbf{z}\in\mathbb{R}^n} \tfrac{1}{2}\mathbf{z}^T\Sigma\mathbf{z} \qquad (1.1.20)$$

such that

$$\begin{cases} \mathbf{z}^T\mathbb{1} = 1, \\ \mathbf{z}^T\bar{\mathbf{r}} = a, \\ \mathbf{0} \leq \mathbf{z} \leq \mathbb{1}, \end{cases}$$

where $\mathbf{0} \in \mathbb{R}^n$ is a vector of zeros. The first constraint means that the investor has fully utilized his or her wealth in an investment. The second constraint denotes the target of the expected return, while the last constraint means that there is no short-selling of securities during the period of investment. The standard mean-variance solution to this problem is detailed in Appendix A.1.

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To understand more about the mean-variance strategy, we consider Figure 1.2 as an example. The blue curve is called the efficient frontier[12]. The portfolio marked 'x' is called the global minimum variance portfolio and is obtained by minimizing the variance of the portfolio with no assumption of the expected return considered. An investor targeting 14% risk will prefer portfolio 'A' to 'C' because the former has almost 15% expected return while the latter has 12% expected return. Also, portfolio 'A' is by assumption preferred to 'B' because 'A' has lower risk compared to portfolio 'B'.

By studying liquidity preference, Tobin (1958) showed that the balance between the risk-returns of the portfolio can be obtained by the lending or borrowing of assets at risk-free rates[13]. This technique is often referred to as leverage and such portfolio boundary conditions are defined as

$$\mathbf{z}_{\text{leverage}} = \{\mathbf{z} \in \mathbb{R}^n_+ : \mathbf{z}^T \mathbb{1} = \ell\}, \tag{1.1.21}$$

where $\ell \geq 1$, is the size of leveraged portfolio. We denote by

$$z_0 = 1 - \mathbf{z}^T \mathbb{1}, \tag{1.1.22}$$

the weight of the risk-free asset and by r_0, the associated rate of return. In particular, the case $z_0 < 0$ indicates that investors have borrowed the risk-free asset. Similarly, $z_0 > 0$ indicates that they were over the budget and hence lent the remaining wealth at a risk free rate. Thus the return of such portfolio consisting of risk-free asset is defined as

$$r(\mathbf{z}) = r_0 + \mathbf{z}^T(\mathbf{r} - r_0\mathbb{1}), \tag{1.1.23}$$

and the expected return is

$$\bar{r}(\mathbf{z}) = r_0 + \mathbf{z}^T(\bar{\mathbf{r}} - r_0\mathbb{1}). \tag{1.1.24}$$

Another boundary condition that is under consideration is called threshold and is defined as follows:

$$\mathbf{z}_{\text{threshold}} = \{\mathbf{z} \in [a, b]^n : \mathbf{z}^T \mathbb{1} = 1\}, \tag{1.1.25}$$

such that asset weights are given as

$$0 \leq \text{a} \leq z_i \leq \text{b} \leq 1 \quad \text{for} \quad i = 1, \ldots, n.$$

The above constraint means that some shares can be given boundaries, that are within the common, 'no short-selling' constraint.

Fixed income assets in this strategy alter the objectives of the investors significantly. Investors in this case hold the tangency portfolio (often known

[12] A curve representing all optimal portfolios.
[13] A special type of asset with zero variance (equivalent to fixed deposit).

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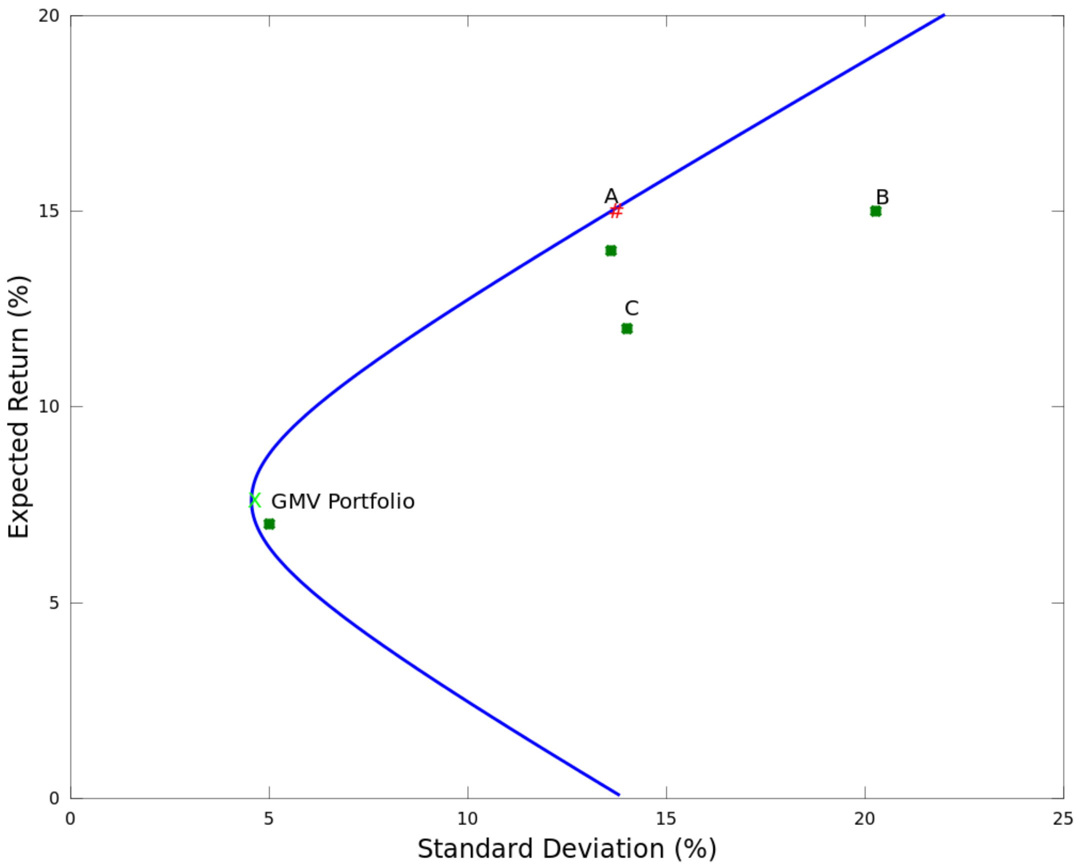
Figure 1.2: Markowitz Efficient Frontier



as market portfolio) which is a blend of risky assets and risk-free assets; see Figure 1.3. This portfolio is obtained by maximizing the Sharpe ratio defined as follows:

$$SR(\mathbf{z}) = \frac{\mathbf{z}^T\bar{\mathbf{r}} - r_0}{\sqrt{\mathbf{z}^T\Sigma\mathbf{z}}}. \tag{1.1.26}$$

Although the mean-variance strategy provides optimum portfolios, it has suffered a lot of criticisms around stability issues. Most of this criticism revolves around the required plug-in parameters. The mean-variance strategy tends to maximize the errors associated with an estimation of these input parameters which lead to portfolio's instability; see Michaud (1989).

Several techniques have been proposed to deal with the problem of estimating parameters that are reliant on statistical measures. The most important input parameters in the mean-variance framework is Σ, which describes the asset movement with respect to each other in terms of returns and a vector of expected returns $\bar{\mathbf{r}}$; see Satchell (2011). For any significant change in the

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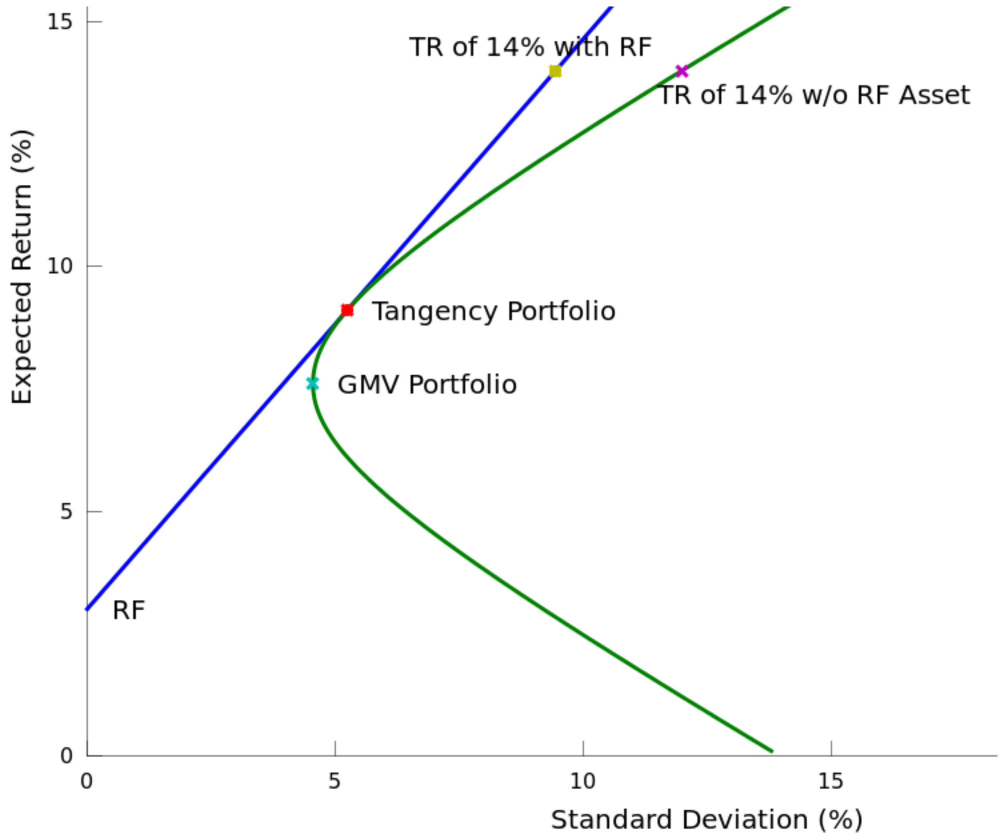
Figure 1.3: Efficient Frontier with Risk Free Asset



input parameters, the entire allocation changes dramatically.

Hanoch and Levy (1969) indicate the error of mean-variance optimization when investors are almost certain about the future performance of assets.

Example 1.2. *Consider two portfolios X and Y as denoted in the following table: The expected return of portfolio X is larger than that of portfolio Y.*

Table 1.1: Statistical Analysis of Strategies

x	$\mathbb{P}(X = x)$	y	$\mathbb{P}(Y = y)$
5	0.8	50	0.99
500	0.2	5000	0.01
$\bar{x}$	104	$\bar{y}$	99.5
$var(x)$	7844	$var(y)$	242574.75

Also, we observe that the variance of portfolio Y is larger than that of portfolio X. Thus, following the mean-variance criterion, one would definitely choose portfolio X. However, with portfolio Y, we are almost certain about the return

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of 50. *In this case one would consider portfolio* Y, *which disputes the decision given by the mean-variance strategy.*

From the Black and Litterman (1992) (BL) model, the estimation of the input parameters of the mean-variance strategy were improved by incorporating the investor views[14] of the economy, in which the expected return and variance of the portfolio were given as

$$\mu^{\mathrm{BL}} = \left[(\tau\Sigma)^{-1} + \mathrm{P}^T\Omega^{-1}\mathrm{P}\right]^{-1}\left[(\tau\Sigma)^{-1}\boldsymbol{\pi} + \mathrm{P}^T\Omega^{-1}\mathbf{q}\right], \tag{1.1.27}$$

$$\sigma^2_{\mathrm{BL}} = \left[(\tau\Sigma)^{-1} + \mathrm{P}^T\Omega^{-1}\mathrm{P}\right]^{-1}, \tag{1.1.28}$$

respectively. The notations are described as follows,

1. Σ denotes the $n \times n$ covariance matrix.
2. P is the $k \times 1$ matrix of views.
3. Ω is the $k \times k$ diagonal matrix of views.
4. $\mathbf{q}$ is a views vector of expected return; see Salomons (2007).

In practise, this model is difficult to implement since it involves unknown parameters such as τ (denoting variance scaling parameter according to Black and Litterman (1992)) which is difficult to predict and also, specifying views about assets requires experience.

1.2 On Risk Parity

In recent years, the risk-parity concept has been introduced in the investment realm and most investors have already shown interest in it, not only because of its flexibility, but also because of its improvement of investment principles. In a typical portfolio that one might deploy, say 60% allocation to equities and 40% to bonds which is a common asset allocation for simple portfolio, how much risk is contributed by equities and bonds, respectively? It turns out that equities dominate in terms of risk contribution, almost 90% of risk of the portfolio is from equities. The idea of risk parity is that, having several categories of risk, say bonds, equities, real estate, hedge funds, etc, one can allocate assets based on their respective risk contributions (preferably equalizing their risk contributions), see Rappoport and Nottebohm (2012).

The question arises, 'What are the consequences of allocating assets with the objective of equalizing their risk contributions?' This suggests one invests

[14] Investors information about the markets, sectors or specific company performance.

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more in bonds (about 90%) as well as 70% in equities which implies that the portfolio is exposed to leverage risk; see Lee (2011). Another observation is that when interest rates goes down, the value of the bond goes up. Thus, bonds with longer expiration dates, carry greater risk.

The origin of this strategy dates back to the question put to Bob Prince[15] by Ray Dalio[16] in 1996 as 'What kind of investment strategy will perform well across all economic scenarios?' His intention was to develop a self-maintained investment strategy that would manage his family wealth under different market regimes in his absence. While trying to figure out which strategy would suit their objective, Prince identified a portfolio that has half inflation and half deflation, and this was improved to incorporate growth markets. Since then, the fund investment firm 'Bridgewater associates' has been using this approach under the name 'All Weather Strategy' for managing funds; see Table 1.2.

Table 1.2: All Weather Assets Portfolio of Ray Dalio

	Growth	Inflation	
Rising	Equities	Inflation Linked Bonds Commodities	Market Expectation
Falling	Nominal Bonds Inflation Linked Bonds	Nominal Bonds Equities	

The term 'risk parity' was first introduced by Edward Qian[17] in his work entitled 'Efficient Portfolios Through True Diversification'. Naively, this phenomenon can be described as a strategy that determines risk for the entire portfolio and divides this risk equally amongst components. This strategy caught the attention of many investors in the recent financial crisis. Analysis during this period shows that the alpha strategy's performance was poor than the risk parity portfolios.

Definition 1.3. *Risk parity is an innovative investment approach that allocates the weight of portfolio components through their risk contributions to the risk of the portfolio.*

The application of the RP concept in theory is somewhat confusing. Others refer to RP as the Equal Risk Contribution (ERC) strategy. However, it should be emphasized that the two strategies resemble each other if they consist of only two assets or all pair-wise correlations are the same. This approach

[15] An employee at Bridgewater Associates.

[16] The founder of Bridgewater Associates.

[17] Chief investment officer and head of research at PanAgora Asset Management Inc. in Boston

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provides investors with a 'spanner'[18] to the detrimental behavioural tendency borne in the traditional asset allocation. The portfolio in this case is diversified by risk, not by capital.

Indeed, the risk-parity approach, especially when compared to traditional investment concepts, has shown good performance. Instead of picking a static stock-bond, the investors decide how much volatility they are willing to take for a specific asset. The following example illustrates the risk contributions of components in a typical portfolio.

Example 1.4. *Consider the 60/40 strategy with annual volatility of both stock and bond being 15% and 5%, respectively, and their correlation determined to be 20%. Generally, stock returns are more volatile than bonds. Thus, the risk contribution*[19] *of the stock is*

$$\begin{aligned}\mathcal{RC}_{stock} &= \frac{(0.6)^2(0.15)^2 + (0.20)(0.15)(0.05)(0.40)(0.60)}{(0.60)^2(0.15)^2 + 2((0.20)(0.15)(0.05)(0.40)(0.60)) + (0.40)^2(0.05)^2}\\ &= 0.918 = 91.8\%.\end{aligned}$$

This implies that the risk contribution of a bond to the portfolio risk is

$$\mathcal{RC}_{bond} = 1 - \theta_{stock} = 0.082 = 8.24\%.$$

Similarly, when we alter the weight allocations, say 40/60 *strategy, stock still dominates bonds in terms of risk contribution; see Figure 1.4. The stock contributes* 75.9% *while bonds only contribute* 24.1% *to the entire portfolio risk. Stocks in reality are more volatile than bonds, hence a* 40/60 *or* 60/40 *allocation strategy is not diversified as intended.*

The risk-parity approach is not restricted to asset allocation. We can still apply risk parity in derivative instruments such as options, futures, swaps and forwards. Features of a risk-parity portfolio are not new, some of its theoretical components come from the Markowitz mean-variance strategy. The main task of investment managers applying risk parity is first to manage risk in which the optimized function includes constraints for

1. short positions, and
2. long constraints, which are sometimes relaxed to accommodate leverage.

This strategy has shown good performance in the rising interest rates environment against the traditional 60/40 strategy which was considered the most balanced strategy by investors prior to the 2007/2008 financial crisis, see Figure 1.5.

[18] Full tool kit that merges investment theory, robust optimization and the risk budgeting contrast to the traditional approach.

[19] Formally defined in section (2.4).

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Figure 1.4: Asset Allocation Based on MVO

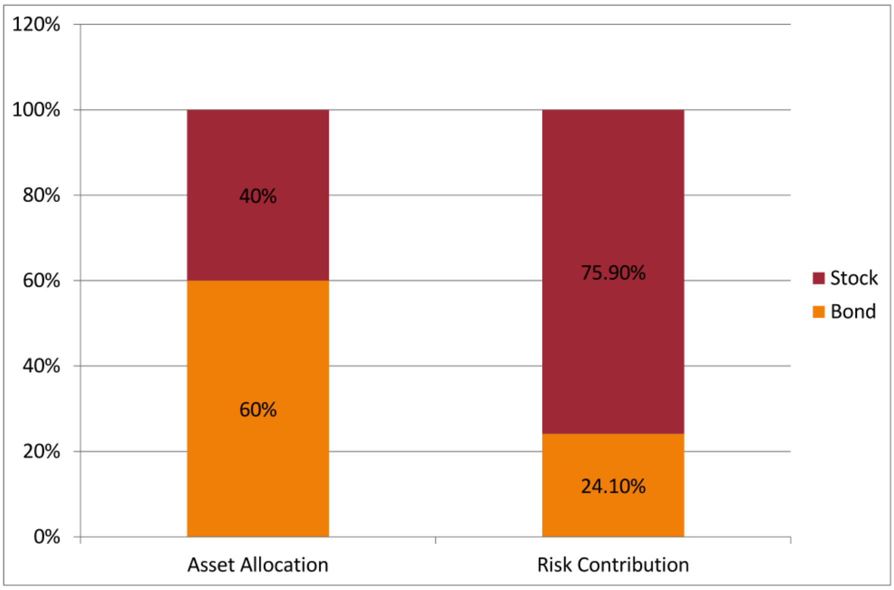
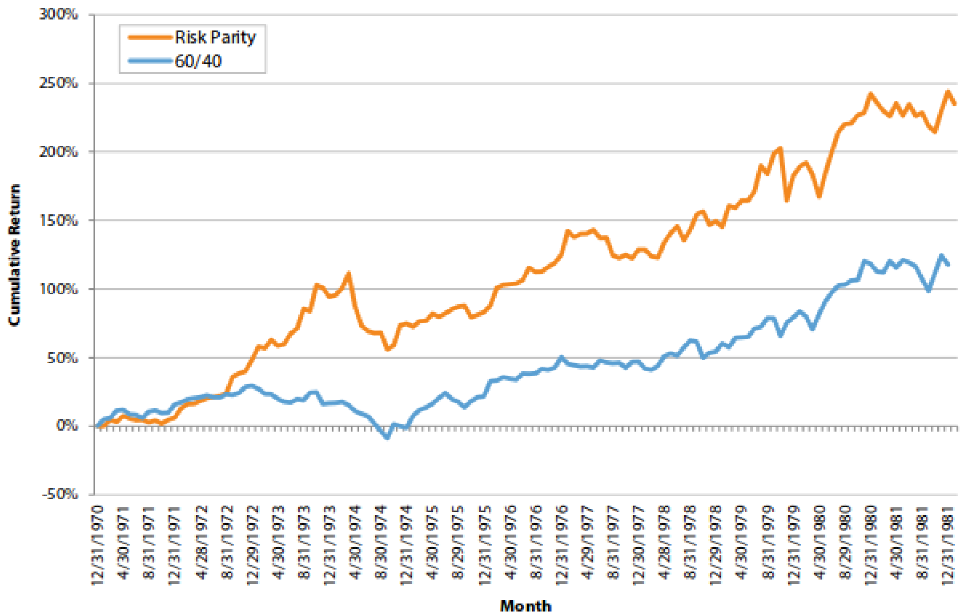


Figure 1.5: 60/40 Strategy vs Risk Parity. Source: Sallient Investment Institution.



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1.2.1 Asset Classes

Asset classes are groups of securities that are bound by the same rules and regulations of investment and have the same behaviour in the market environment. Examples of asset classes are equities (i.e. stocks), bonds (fixed-income assets), real estates, commodities and cash.

The most important feature of risk parity is the way assets are grouped together to form a portfolio. This allows investors to diversify portfolios by asset classes through their risk contribution; Bhansali *et al.* (2012). The large amount of weight allocation is targeted on the low-correlated asset classes, examples are equities versus exchange traded funds, OTC swaps versus listed future; see Kunz (2011).

1.2.2 Diversification

Diversification of a portfolio refers to a blending of a variety of asset classes such that the performance of the portfolio remains balanced under different economic climates. The term 'diversification' in finance is sometimes described as 'Don't put all your eggs in one basket'. This means that one needs to invest money in different asset classes with the idea of not being affected by only one risk factor. [20]

For instance, consider a basket carrying two types of eggs (i.e large eggs and small eggs). One can think of large eggs as stocks and small eggs as bonds. Because bonds are less volatile compared to stocks (except during a period of hyperinflation or when there is a danger of a government default), we assume that each type of egg yields the relative return of 1 and 9, respectively[21]. Implementing a 60/40 strategy, this basket yields an equivalent of 58, i.e $(6 \times 9) + (4 \times 1)$. The large eggs contribute 93.1% to this basket while small eggs contribute 6.9%. Clearly, one can infer that this basket is diversified in terms of reward and not the deviation from this reward.

Fund managers believe that holding different types of assets is more appealing investment than active investments. The main idea here is that if one market drops in performance, it could be that the other market(s) appreciates in performance value. Risk embedded in a risk-parity portfolio can also be demystified and even its drivers could be exposed. If the investor were to diversify such portfolio, s/he should first have to implement the following recipe,

1. Understand how to group components according to their classifications.

[20] The investor believes in holding different types of assets from different markets than active investor who trace markets performance and tries to hedge from mispricing of markets products.

[21] In reality stock are more compensated than bonds

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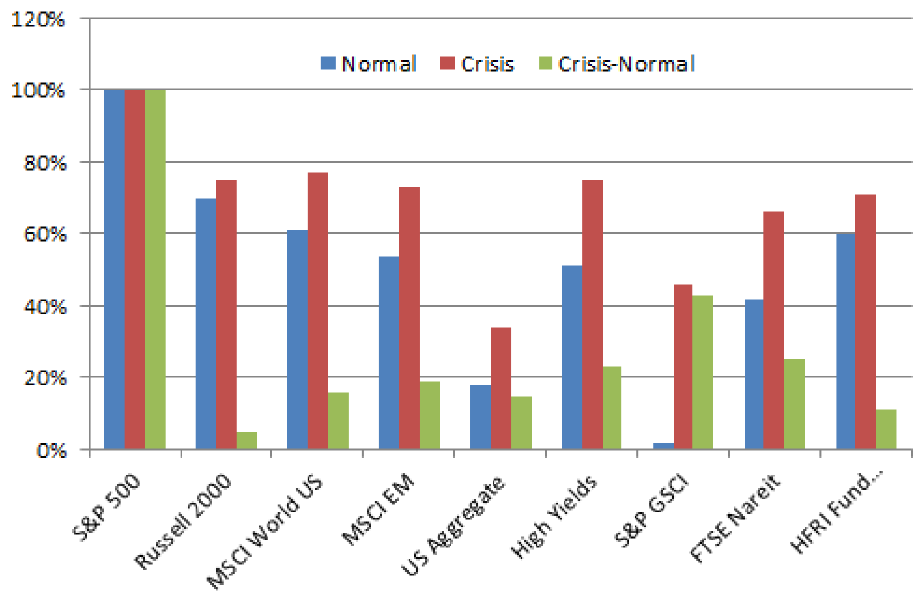
2. Find the relation between the component classifications and their economic sources of risk.

Podkaminer (2013) and Bhansali *et al.* (2012) used the same technique for diversifying their portfolios termed the 'Risk factor approach'. This approach allocates capital across a range of uncorrelated assets, such that if a specific asset class declines, it will be compensated by the other and hence the return of the portfolio is maintained.

1.2.3 Correlation between Asset Classes

Investors first tool to diminish risk is typically through diversification among asset classes with low pair-wise correlation. This is attained during the normal economic scenarios. However, at the times of the financial crisis, correlation amongst asset classes increases, failing diversification. The typical 60/40 strategy exhibit over 90% of risk coming from equity market class; see Qian (2011). Figure 1.6 illustrates the difference of the correlations at the normal and crisis state[22] for the components against the S& P 500. It follows that component correlations during financial crisis increases significantly, resulting in an increase of portfolio's risk.

Figure 1.6: Effect of Correlation during Crisis



[22] A state is considered normal if component returns are above −5%, otherwise is referred to crisis state; Benson *et al.* (2012).

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1.2.4 Leverage

Although risk parity seems to be more diversified strategy, very often practitioners are not satisfied with the return of this strategy. The strategy provides lower returns compare to the traditional mean-variance strategy. In order to enhance the same portfolio return as a traditional mean-variance strategy, leverage is introduced; see Romahi and Santiago (2012) and Bhansali *et al.* (2012). Leverage in a portfolio comes in different formats. The institutional investors might wish to leverage the entire portfolio by borrowing money from a plan-wide borrowing facility, or alternatively by applying the derivative instrument to leverage the portfolio to the optimum Sharpe ratio.

1.3 Implementation of Risk Parity Strategy

Many of the financial institutions have already started offering this product to their clients. Examples are:

1. Global asset allocation (e.g IBRA fund of Invesco or the All Weather Strategy of Bridgewater);
2. Commodity allocation (e.g the Lyxor Commodity Active Fund);
3. Bond Indexation (e.g the RB EGBI index sponsored by Lyxor and calculated by Citigroup);
4. Equity indexation (e.g the SmartIX ERC indexes sponsored by Lyxor and calculated by FTSE).

About this thesis

This work is divided into eight chapters. The first four chapters seek to demystify the analytical frameworks of the investment strategies. We discuss the risk-based strategies and their relationships with the traditional mean-variance efficient portfolio. More precisely, Chapter 1 highlights the background of the financial crisis and the insight of the traditional mean-variance strategy and its flaws. Furthermore, the appealing of the recent investment direction, i.e., risk parity approach, is discussed.

Chapter 2 is dedicated to the introduction of risk measures and the theoretical frameworks of the risk-based strategies with more emphasis on the risk parity strategy. We give a distinction between the naive risk parity and the equal risk contribution strategies under volatility as a risk measure. Chapter 3 illustrates the general proof for risk parity strategy being mean-variance

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efficient. Lastly, Chapter 4 is dedicated to the risk budgeting strategy, an extension of the equal risk contribution strategy which integrate investor's views about the risk budget.

The second part is dedicated to risk parity using downside risk measures. In Chapter 5, we discuss the risk parity using expected shortfall and risk factors. Chapter 6 is devoted to the discussion of portfolio's rebalancing, transaction costs and leverage. In Chapter 7 we do the empirical simulations of the introduced investment strategies.

Lastly, in Chapter 8 we discuss Oderda (2013)'s approach to portfolio's asset allocation. This approach maximizes the expected return of the logarithmic wealth which yields an interesting solution that is actually a linear combination of risk-based strategies and the market portfolio.

In Appendix A, we provide the analytical derivation of the standard mean-variance portfolio solution and the proofs of the properties of the risk-parity portfolio (known as the equal-risk contribution properties of Maillard *et al.* (2010)) .

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Chapter 2

Understanding Risk-Based Strategies

In this chapter we present an overview of the theoretical framework of risk-parity (RP) strategies and some robust investment strategies, namely, equal weighting (EW), global minimum variance (GMV), and maximum diversification (MD). These strategies are often classified as risk-based, or risk-controlled, or μ-free, or Smart-beta, or Non-market cap strategies, and have gained huge attention in the aftermath of the recent 2007-2008 financial crisis. Perhaps this is because the definition of portfolio diversification has been reviewed and put on the heart of these investment strategies.

In contrast, portfolios constructed based on diversification of wealth[1] showed pessimistic results during this period. In particular, equity markets performed poorly, with returns recorded at -50%. The Johannesburg Stock Exchange (JSE)[2] also recorded a 40% drop of its All Share Index. The greatest incentive for using risk-based strategies in the investment realm is the ability to determine asset allocations without the need to estimate portfolio expected return.

Risk-based strategies are often called robust in the literature because of their good performance during the recent crisis. However, the term robust is actually over-used; see Poddig and Unger (2012). Generally, a strategy is called robust if its optimum solution under uncertain input parameters is consistent or stable with the objective value. This is called 'solution robustness', and examples are EW, MV and MD strategies. Other methods for determining the robustness of strategies focus on the sensitivity of input parameters, and this is called 'Structured robustness'. RP is an example of such strategies because it is less sensitive to changes of input parameters than the traditional mean-variance strategy.

[1] Builds based on the Markowitz mean variance strategy.
[2] The leading African trade market.

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Risk-based strategies exclude any information regarding the expected return in the composition of portfolios. Another constraint unusual in the mean-variance strategy is that components have equal amounts of risk contributions to the entire portfolio risk. The portfolio allocates risk (known as the beta chaser) instead of capital allocation (known as the alpha chaser). More intuitively, risk parity can be thought of as a way to construct a portfolio that has a constant level of risk that is equally divided amongst asset.

The strategy accounts more asset classes as in traditional mixed funds, often making a pure weighting of shares and pensions, any of these asset classes share percentage of the total risk (and not in total assets). For instance, Scherer (2012) used risk parity approach to analyse the US Futures. Portfolios constructed in this manner generate a better Sharpe ratio and lower setbacks in falling markets.

We start by discussing some risk measures that are under consideration for portfolio constructions. These are Volatility, Semi-Variance, Value at Risk (VaR) and Conditional Value at Risk (CVaR). The use of these measures depends on the investor's ability to perform the necessary computations.

2.1 Risk Measures

Although the main aim of investment is to achieve positive returns during any kind of the economic cycle, these returns are subjected to risk. Risk plays an important role in portfolio's decision making. In particular, it is the first step to determine in portfolio's risk management. It is a positive and increasing function defined on the domain of $\mathbb{R}$ and is bounded below by zero. For instance, if $\xi(\mathbf{z})$ denotes the risk of the portfolio, then for any $\epsilon > 0$, we have

$$\xi^2(\mathbf{z}) \geq \epsilon||\mathbf{z}||^2, \tag{2.1.1}$$

which corresponds to a non-degenerating function. A practitioner tries by all means to diminish this, but risk is relative. An investment with more risk will be more compensated when the markets are in favourable conditions and will perform severely in unfavourable market conditions. In this section, we highlight a variety of portfolio risk measures.

2.1.1 Variance

Variance measures the deviation of component returns from the mean (or expected return) of the portfolio. The square root of this measure is known as volatility, defined as in (1.1.8). This measure is the most popular risk measure in the investment realm. It dominates other risk measures because of its computational simplicity and ease of interpretation.

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2.1.2 Semi-Variance

Although variance is a popular measure of risk in the investment industries, it has been criticised for incorporating both lower and upper returns of the mean when determining risk. Semi-Variance excludes returns above a given benchmark (often the mean) and concentrates on the lower level of returns (known as portfolio loss). Therefore, for any continuous distribution of returns, portfolio Semi-Variance, SV_p, is defined as

$$SV_p^2 = \int_{-\infty}^{\bar{\mathbf{r}}} (\bar{\mathbf{r}} - r)^2 f(r) dr. \tag{2.1.2}$$

The function f denotes the distribution of the return, r. The integral limits can be interpreted as the range of returns investors dislike that are less than a given benchmark return $\bar{\mathbf{r}}$. In the discrete case, the Semi-Variance portfolio is given by

$$SV_p^2 = \sum_{r<\bar{\mathbf{r}}} (\bar{\mathbf{r}} - r)^2 \mathbb{P}(\mathbf{r} = r). \tag{2.1.3}$$

2.1.3 Value at Risk (VaR)

This measure of risk generalizes the likelihood of a portfolio under-performing through downside statistical measures. For random portfolio returns, VaR can be determined as follows,

$$\mathrm{VaR}_\alpha(r(\mathbf{z})) = \inf\{\ell \in \mathbb{R} : \mathbb{P}(r(\mathbf{z}) > \ell) \leq 1 - \alpha\}, \tag{2.1.4}$$

where α denotes the confidence level. This measure of risk assesses the potential losses of a portfolio over a given future time period with a given degree of confidence. Commonly used confidence levels are 95% to 99%. Since we assess the potential losses, it is important to use these levels of confidence, particularly when marketing for the company. This measure will be detailed in Chapter 3 for further portfolio analysis.

2.1.4 Conditional Value at Risk (CVaR)

This risk measure provides the probability of returns falling below VaR. It is often argued that VaR provides the threshold not to be exceeded by portfolio returns, and thus does not precisely give the amount exceeding this threshold. However, CVaR, which is often called expected shortfall (ES), provides the expected amount exceeding VaR:

$$\mathrm{CVaR}_{\mathbf{z}}(\alpha) = -\mathbb{E}[r(\mathbf{z})|r(\mathbf{z}) \leq -\mathrm{VaR}_\alpha(r(\mathbf{z}))], \tag{2.1.5}$$

where $\mathbb{E}$ denotes the conditional expectation operator, $\mathrm{VaR}_{\mathbf{z}}(\alpha)$ is the threshold not to be exceeded for a given portfolio $\mathbf{z}$ and α denotes the confidence level.

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2.2 Important Properties of Risk-Based Strategies

As with any risk-based strategy, it is important to define the properties of risk and its sources from assets. In this section, we present these important properties and then discuss the theoretical frameworks and the previous studies of the risk-based strategies in the latter sections.

2.2.1 Marginal Contribution

Marginal contribution in a portfolio is a quantitative measure that determines the significant impact of components to the entire portfolio risk. There are two approaches of determining this measure, i.e., the discrete and continuous marginal risk contributions.

2.2.1.1 Discrete Marginal Contribution (DMC)

The DMC of a component is determined by taking the difference between the risk measure of the portfolio and the portfolio risk measured without that component. If $\xi(\tilde{\mathbf{z}})$ denotes the risk measure computed without component i, i.e.,

$$\tilde{\mathbf{z}} = \mathbf{z} \setminus z_i, \tag{2.2.1}$$

then the marginal risk contribution of this component is given by:

$$\mathcal{MC}_i(\mathbf{z}) = \xi(\mathbf{z}) - \xi(\tilde{\mathbf{z}}). \tag{2.2.2}$$

DMC is mainly used in the simulation of VaR models for trading and simulation-based stochastic analysis. The disadvantage of DMC is that it is not additive, and thus can not be applied for risk decomposition.

2.2.1.2 Continuous Marginal Contribution (CMC)

The CMC of the components is obtained by taking the partial derivative to the entire portfolio risk with each and every component in the portfolio. For instance, the marginal risk contribution of the i^{th} component is defined as follows:

Definition 2.1. *Let $\xi(\mathbf{z})$ be the risk measure of the portfolio, then the marginal risk contribution of the i^{th} asset is:*

$$\mathcal{MC}_i(\mathbf{z}) = \frac{\partial \xi(\mathbf{z})}{\partial z_i}. \tag{2.2.3}$$

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In particular, $\dfrac{\partial \xi(\mathbf{z})}{\partial z_i}$ denotes a small change in the entire portfolio risk possessed by the i^{th} component. We denote by $\mathcal{MC}(\mathbf{z})$ a vector of asset marginal contributions in a portfolio, i.e.,

$$\mathcal{MC}(\mathbf{z}) = \frac{\partial \xi(\mathbf{z})}{\partial \mathbf{z}}. \tag{2.2.4}$$

The risk of the portfolio is required to be homogeneous function such that the decomposition is possible. Recall the definition of homogeneous function as follows:

Definition 2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. We say that f is homogeneous of degree $d \in \mathbb{R}$ if $f(az) = a^d f(z)$ for $a \in \mathbb{R}$ and $z \in \mathbb{R}^n$.*

Proposition 2.3. *The volatility $\sigma(\mathbf{z})$ of a portfolio is a homogeneous function of degree one. Moreover, the marginal risk contributions of components are given by,*

$$\frac{\partial \sigma(\mathbf{z})}{\partial \mathbf{z}} = \frac{\Sigma \mathbf{z}}{(\mathbf{z}^T \Sigma \mathbf{z})^{\frac{1}{2}}}, \quad \textit{for} \quad \mathbf{z} \in \mathbb{R}^n. \tag{2.2.5}$$

Proof. First, we show that volatility is a homogeneous function of degree one. By considering portfolio's volatility as defined in equation (1.1.8) and let $c \in \mathbb{R}$, we can write

$$\begin{aligned} 0 \le \sigma(c\mathbf{z}) = \left((c\mathbf{z})^T \Sigma (c\mathbf{z})\right)^{\frac{1}{2}} &= \left(c^2 \mathbf{z}^T \Sigma \mathbf{z}\right)^{\frac{1}{2}} \\ = |c| \left(\mathbf{z}^T \Sigma \mathbf{z}\right)^{\frac{1}{2}} &= |c|\, \sigma(\mathbf{z}) = c\sigma(\mathbf{z}), \end{aligned}$$

which proves the first statement. To show equation (2.2.5), we simply take the partial derivative of the volatility, i.e.,

$$\frac{\partial \sigma(\mathbf{z})}{\partial \mathbf{z}} = \frac{\partial \left(\mathbf{z}^T \Sigma \mathbf{z}\right)^{\frac{1}{2}}}{\partial \mathbf{z}} = \frac{1}{2} \left(\mathbf{z}^T \Sigma \mathbf{z}\right)^{-\frac{1}{2}} 2\Sigma \mathbf{z} = \frac{\Sigma \mathbf{z}}{(\mathbf{z}^T \Sigma \mathbf{z})^{\frac{1}{2}}}. \tag{2.2.6}$$

□

Clearly, the marginal risk contribution of a particular asset is directly proportional to the i^{th} row of the product matrices $\Sigma \mathbf{z}$, i.e.,

$$\frac{\partial \sigma(\mathbf{z})}{\partial z_i} \propto (\Sigma \mathbf{z})_i = z_i \sigma_i^2 + \sigma_i \sum_{i \ne j}^{n} z_j \sigma_j \rho_{ij}, \tag{2.2.7}$$

where ρ_{ij} denotes the correlation between the i^{th} and the j^{th} components. Normalizing this by portfolio risk, we get,

$$\frac{\partial \sigma\left(\mathbf{z}\right)}{\partial z_i} = \frac{(\Sigma \mathbf{z})_i}{\sigma(\mathbf{z})}. \tag{2.2.8}$$

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2.2.1.3 Beta Contributions

An alternative to determine the sensitivity or significant change to the portfolio risk due to the change in component weights is to use beta defined as follows,

$$\beta_i = \frac{\text{cov}(r_i, r(\mathbf{z}))}{\sigma^2(\mathbf{z})} \quad i = 1, \ldots, n, \tag{2.2.9}$$

where $\text{cov}(r_i, r(\mathbf{z}))$ denotes the covariance of the component return and the return of the portfolio, see Salomons (2007). Moreover, equations (2.2.8) and (2.2.9) simplify the marginal risk contributions. By expanding the numerator of equation (2.2.9), we have,

$$\begin{aligned}\text{cov}(r_i, r(\mathbf{z})) &= \text{cov}(r_i, z_1 r_1 + z_2 r_2 + \cdots + z_i r_i + \cdots + z_n r_n) \\ &= \text{cov}(r_i, z_1 r_1) + \cdots + \text{cov}(r_i, z_i r_i) + \cdots + \text{cov}(r_i, z_n r_n) \\ &= z_1 \sigma_{i1} + \cdots + z_i \sigma_i^2 + \cdots + z_n \sigma_{in}. \end{aligned} \tag{2.2.10}$$

Hence,

$$\beta_i = \frac{\text{cov}(r_i, r(\mathbf{z}))}{\sigma^2(\mathbf{z})}, \tag{2.2.11}$$

which implies that

$$\text{cov}(r_i, r(\mathbf{z})) = \beta_i \sigma^2(\mathbf{z}). \tag{2.2.12}$$

We showed in Proposition (2.3) that,

$$\frac{\partial \sigma(\mathbf{z})}{\partial(\mathbf{z})} = \frac{\Sigma \mathbf{z}}{(\mathbf{z}^T \Sigma \mathbf{z})^{\frac{1}{2}}} = \frac{(\Sigma \mathbf{z})}{\sigma(\mathbf{z})}. \tag{2.2.13}$$

Clearly, $\frac{\Sigma \mathbf{z}}{\sigma(\mathbf{z})}$ is a vector of component marginal contributions. Now we consider $\Sigma \mathbf{z}$,

$$\Sigma \mathbf{z} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} z_1\sigma_1^2 + z_2\sigma_{12} + \cdots + z_n\sigma_{1n} \\ z_1\sigma_{21} + z_2\sigma_2^2 + \cdots + z_2\sigma_{2n} \\ \vdots \\ z_1\sigma_{n1} + z_2\sigma_{n2} + \cdots + z_n\sigma_n^2 \end{pmatrix}. \tag{2.2.14}$$

This implies that the i^{th} row corresponds to:

$$(\Sigma \mathbf{z})_i = z_1\sigma_{i1} + z_2\sigma_{i2} + \cdots + z_i\sigma_i^2 + \cdots + z_n\sigma_{in}. \tag{2.2.15}$$

Since, $\rho_{ii}\sigma_i\sigma_i = \sigma_i^2$, from equation (2.2.11) we deduce that

$$\mathcal{MC}_i(\mathbf{z}) = \frac{\partial \sigma(\mathbf{z})}{\partial z_i} = \frac{(\Sigma \mathbf{z})_i}{\sigma(\mathbf{z})} = \frac{\beta_i \sigma^2(\mathbf{z})}{\sigma(\mathbf{z})} = \beta_i \sigma(\mathbf{z}). \tag{2.2.16}$$

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Remark 2.4. *The marginal contribution of the i^{th} asset can be expressed as the product of its volatility and linear correlation between its return and the return of the portfolio, i.e.,*

$$\begin{aligned}\mathcal{MC}_i(\mathbf{z}) = \beta_i\sigma(\mathbf{z}) &= \frac{(\Sigma\mathbf{z})_i}{\sigma^2(\mathbf{z})} \\ &= \frac{cov(r_i, r(\mathbf{z}))}{\sigma(\mathbf{z})} = \frac{\rho_{i,\mathbf{z}}\sigma_i\sigma(\mathbf{z})}{\sigma(\mathbf{z})} \\ &= \rho_{i,\mathbf{z}}\sigma_i, \end{aligned} \tag{2.2.17}$$

where $\rho_{i,\mathbf{z}}$ is a correlation between the return of the i^{th} component and the portfolio.

This leads to the conclusion that sensitivity of the i^{th} asset in a portfolio is:

$$\beta_i = \frac{\text{cov}(r_i, r(\mathbf{z}))}{\sigma^2(\mathbf{z})} = \frac{\sigma_{i\mathbf{z}}}{\sigma(\mathbf{z})} \times \frac{1}{\sigma(\mathbf{z})} = \frac{\mathcal{MC}_i(\mathbf{z})}{\sigma(\mathbf{z})}. \tag{2.2.18}$$

Similarly, correlation of the i^{th} component with respect to the portfolio is,

$$\rho_{i,\mathbf{z}} = \beta_i \times \frac{\sigma(\mathbf{z})}{\sigma_i} = \frac{\mathcal{MC}_i(\mathbf{z})}{\sigma(\mathbf{z})} \times \frac{\sigma(\mathbf{z})}{\sigma_i} = \sigma_i^{-1}\mathcal{MC}_i(\mathbf{z}). \tag{2.2.19}$$

2.2.2 Risk Contribution

In the literature, component risk contribution is defined as the weighted marginal contribution. It is classified into two, namely the absolute and the relative risk contributions.

Definition 2.5. *Let $\sigma(\mathbf{z})$ be the risk measure of the portfolio $\mathbf{z}$. Then the relative risk contribution of the i^{th} component is:*

$$\mathcal{RC}_i(\mathbf{z}) = z_i\mathcal{MC}_i(\mathbf{z}), \tag{2.2.20}$$

where $\mathcal{MC}_i$ is given as in equation (2.2.3).

2.2.2.1 Absolute Risk Contribution

The absolute risk contribution of the i^{th} component is given by,

$$\mathcal{RC}_i^{\text{abs}} = z_i(\Sigma\mathbf{z})_i = z_i\sum_{i=1}^{n} z_j\text{cov}(r_i, r_j) = z_i\text{cov}(r_i, r(\mathbf{z})). \tag{2.2.21}$$

Considering all components in a universe, equation (2.2.21) is expressed as follows:

$$\mathcal{RC}^{\text{abs}} = D_{\mathbf{z}}\Sigma\mathbf{z}, \tag{2.2.22}$$

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where $D_{\mathbf{z}}$ denotes the diagonal matrix with entries in the main diagonal representing a vector of component weights. Moreover, the absolute risk contribution of the i^{th} component can be related to the standard deviation of the portfolio. In this case, we have

$$\mathcal{RC}_i^{\text{abs}} = \frac{z_i \sum_{j=1}^{n} z_j \text{cov}(r_i, r_j)}{\sigma(\mathbf{z})} = \frac{z_i \text{cov}(r_i, r(\mathbf{z}))}{\sigma(\mathbf{z})}. \tag{2.2.23}$$

2.2.2.2 Relative Risk Contribution

The relative risk contributions require that their respective sum equal to the total portfolio volatility. In order to obtain the risk measure of the portfolio as the sum of component risk contributions, we use the following theorem for the additive decomposition of continuous function which state as follows,

Theorem 2.6 (Euler's Theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Then f is homogeneous of degree r if and only if for all $z \in \mathbb{R}^n$ it satisfies Euler's partial differential equation*

$$rf(\mathbf{z}) = \sum_{i=1}^{n} z_i \frac{\partial f(\mathbf{z})}{\partial z_i}. \tag{2.2.24}$$

See Fleming (1977) for the proof of this Theorem. In the case of volatility, the portfolio risk can be expressed as a linear combination of asset relative risk contributions, i.e.,

$$\begin{aligned} \sigma(\mathbf{z}) &= z_1 \cdot \frac{\partial \sigma(\mathbf{z})}{\partial z_1} + z_2 \cdot \frac{\partial \sigma(\mathbf{z})}{\partial z_2} + \cdots + z_n \cdot \frac{\partial \sigma(\mathbf{z})}{\partial z_n} \\ &= \sum_{i=1}^{n} z_i \mathcal{MC}_i(\mathbf{z}) \\ &= \mathbf{z}^T \mathcal{MC}(\mathbf{z}) \\ &= \mathbb{1}^T \mathcal{RC}(\mathbf{z}), \end{aligned} \tag{2.2.25}$$

where $\mathcal{MC}(\mathbf{z})$ and $\mathcal{RC}(\mathbf{z})$ are $n \times 1$ vector of marginal and relative risk contributions, respectively.

The percentage risk contribution is simply expressed as the ratio of component risk contribution to the overall portfolio risk, i.e.,

$$\%\mathcal{RC}_i(\mathbf{z}) = \frac{\mathcal{RC}_i(\mathbf{z})}{\xi(\mathbf{z})}. \tag{2.2.26}$$

Alternatively, the marginal and relative risk contribution of the i^{th} component respectively are given by:

$$\mathcal{MC}_i(\mathbf{z}) = \beta_i \sigma(\mathbf{z}), \tag{2.2.27}$$

$$\mathcal{RC}_i^{\text{rel}}(\mathbf{z}) = z_i \beta_i \sigma(\mathbf{z}) \quad i = 1, \ldots, n. \tag{2.2.28}$$

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Also, the percentage contribution is

$$\%\mathcal{RC}_i(\mathbf{z}) = z_i\beta_i \quad i = 1, \ldots, n. \tag{2.2.29}$$

2.2.3 Diversification Index

Another important concept in the risk-based strategies is called diversification, a concept that has various definition in the literature. In the case of volatility as risk measure, Choueifaty and Coignard (2008) defined diversification index as the ratio between two different risk measures, the component-weighted volatilities and the total portfolio volatility, i.e.

$$DR(\mathbf{z}) = \frac{\mathbf{z}^T\boldsymbol{\sigma}}{\sqrt{\mathbf{z}^T\Sigma\mathbf{z}}}. \tag{2.2.30}$$

A portfolio with the highest ratio in this case is considered better diversified in terms of risk. We detail more about this concept in the next section.

Another way to deem diversification is to consider portfolio's concentration. The commonly used concentration measure is called Herfindahl Hirschman Index (HHI) defined as follows,

$$\text{HHI} = \frac{\left(\sum_i^n z_i^2\right) - \frac{1}{n}}{1 - \frac{1}{n}}, \; n \geq 2. \tag{2.2.31}$$

It is the normalized Herfindahl index, which is given by,

$$\text{HI}(\mathbf{z}) = \sum_{i=1}^{n} z_i^2. \tag{2.2.32}$$

HHI takes values between 0 and 1. If the value determined is zero, the corresponding portfolio is equally-weighted[3] and a portfolio with only one components yields the value one. Other measures of portfolio's risk diversification include the Gini index and the Shannon entropy; see Roncalli (2013).

2.2.4 Stability

Portfolio stability, determined as the sum of the absolute values of the difference between each position at time t_{reb^+} and t_{reb^-}, is a measure of change in portfolio weights during rebalancing. This measure, often termed portfolio turnover, is useful to determine the transaction cost[4]. Mathematically, portfolio turnover is given by,

$$\text{Turnover}(t_{reb}) = \sum_{i}^{n} |z_i(t_{reb^+}) - z_i(t_{reb^-})|. \tag{2.2.33}$$

[3] Investment strategy that will be more detailed in the next section.
[4] More detail of transaction cost is discussed in Chapter 6.

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It is sometimes included as a constrain in the optimization problem where investor assign some constant not to be exceeded, typically between 0 and 1. The higher the value of portfolio turnover, the more expensive is the rebalancing. The average portfolio turnover is given by,

$$\text{Average Turnover} = \frac{1}{H} \sum_{t_{reb}=1}^{H} \text{Turnover}(t_{reb}), \tag{2.2.34}$$

where H is the number of rebalancing terms.

2.3 Risk-Based Strategies

The primary question to address in the investment industry is 'what is the proportion of wealth one has to allocate to a particular asset?' To help resolve this important problem, several strategies have been established with the intention to help investors make the right choice in the financial industry. In particular, the traditional mean-variance strategy has been dominating since the last mid-century. However, due to the flaws associated with this strategy, the recent investment direction is focusing on the risk-based strategies, an investment strategies that put diversification of risk at the heart of components allocation. The most incentives of these strategies is that the estimation of the expected return does not play a role in the portfolio's composition and hence they focus on risk management.

In this section, we present the three famous strategies, namely Equal-Weighted, Global Minimum Variance and Maximum Diversification strategies. The investor using any of these strategies strives to minimize the portfolio's risk. Balancing risk of the components better prepares for unknown future events. Moreover, these strategies share one common characteristic which is the requirement of risk model as the input parameter.

2.3.1 Equal Weighted Strategy

The Equal Weighted (EW) strategy is a type of strategy where investors are pleased to hold equal proportions of asset weights in their portfolio. It is often referred to as a 'rule of thumb' strategy because it does not rely on any available optimization models and it is easy to establish. This strategy excludes the use of parameter estimations in the classical mean-variance optimization strategy for the allocation of assets. Merton (1980) noted that the estimation of such additional parameters is difficult and also subjected to errors. Hence, EW strategy is considered well-diversified in terms of asset weights and often referred to as robust in a sense that no estimation parameters are needed for

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the allocation of weights. Also, it does not take into account the trends of the economy[5] for portfolio's composition.

Weights in this case are determined by the number of assets available in a portfolio. For instance, if we have n-securities in a portfolio, then each component capital weight is:

$$z_i^{EW} = n^{-1}, \quad i = 1, \dots, n. \tag{2.3.1}$$

Thus, it is apparent that the more assets available in a portfolio, the lower is the fractional allocation. The number of components in a portfolio play a major role in the allocation of components. This is the reason Lussier (2013) describe the EW strategy as a simple strategy which benefits from the law of large numbers. The author argues that over the long-run, this strategy performs better for atleast three reasons: First the strategy's benefits is from the small-cap bias. Secondly, it yields efficient diversification of idiosyncratic risk. And thirdly, it is more concerned about the efficient smoothing of component-price fluctuations.

The return and volatility of this portfolio depend on the number of assets included. For instance, if we have n assets in a portfolio, then the return and volatility of the portfolio becomes:

$$r^{EW} = \frac{1}{n}\sum_{i=1}^{n} r_i, \tag{2.3.2}$$

$$\sigma_{EW} = \sqrt{(\tfrac{\mathbb{1}}{n})^T \Sigma \tfrac{\mathbb{1}}{n}} = \frac{1}{n}\sqrt{\mathbb{1}^T \Sigma \mathbb{1}}. \tag{2.3.3}$$

Also, the marginal and risk contribution of components in a portfolio are the same, i.e

$$\mathcal{MC}^{EW} = \frac{\Sigma \mathbb{1}}{\sqrt{\mathbb{1}^T \Sigma \mathbb{1}}} \tag{2.3.4}$$

$$\mathcal{RC}^{EW} = D_{\frac{1}{n}} \mathcal{MC}, \tag{2.3.5}$$

where $D_{\frac{1}{n}}$ is the diagonal matrix of $\mathbf{z} = \frac{1}{n}$. In addition, the percentage contribution of components is:

$$\%\mathcal{RC}^{EW} = D_{\frac{1}{n}} \frac{\mathcal{MC}}{\sigma_{EW}} = D_{\frac{1}{n}} \frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T \Sigma^{-1} \mathbb{1}}. \tag{2.3.6}$$

The percentage contribution of the EW portfolio can be described as the product of EW portfolio solution and the GMV solution. Hence, the percentage contribution of the i^{th} component correspond to

$$\%\mathcal{RC}_i^{EW} = \frac{1}{n} \frac{(\Sigma^{-1}\mathbb{1})_i}{\mathbb{1}^T \Sigma^{-1} \mathbb{1}} \quad i = 1, \dots, n. \tag{2.3.7}$$

[5] The information related to asset returns and volatilities

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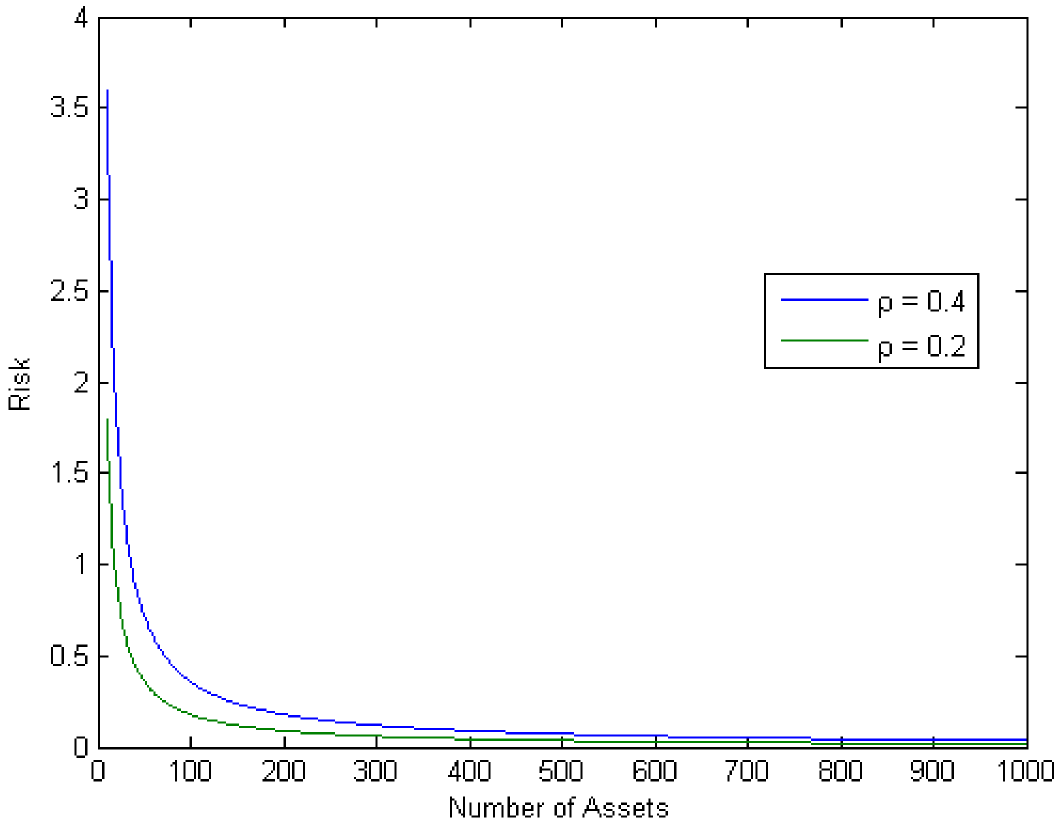
Lee (2011) noted that the EW strategy is mean-variance efficient if correlations of the components is the same and volatilities and returns are identical. Although this strategy seems to be the most simplest in an investment industry, it has been criticized for being illiquid[6] and also lacks economic representation. Investors in this case take some risk that is not compensated at any circumstance. Thus, its performance could sometimes be outperformed by capital-weighted strategies. However, Lussier (2013) concluded that if EW strategy is implemented for diversification and balanced universe, then the approach is better than the capital-weighted strategy.

To illustrate the benefit of diversification of the EW portfolio, we report in Figure 2.1 the risk versus the number of assets. By considering constant correlation matrices, $\rho = 40\%$ and $\rho = 20\%$, and also constant volatility of assets, $\sigma_i = 30\%$, we confirm, as noted by Lussier (2013), that the benefit of diversification is realized as n (the number of assets)[7] start from 100 and more.

Figure 2.1: Risk of EW Portfolio over n-Assets



[6] It does not offer the opportunity to rebalance the portfolio
[7] In this simulation, we consider n between 100 and 1000.

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2.3.2 Global Minimum Variance Strategy

The Global Minimum Variance (GMV) portfolio is a strategy that focusses on obtaining the lowest risk of the portfolio on the efficient frontier, see Best and Grauer (1992). It is found on the left-tip of the efficiency frontier exhibiting that it is the only portfolio with the minimum risk in a given universe, hence the name global minimum variance. Allocations of capital weights in this strategy do not involve the target expected return and it is the only strategy on the efficiency frontier to do so. It uses quadratic optimization technique to obtain asset capital allocations such that the risk of the portfolio is the minimum one. The only input parameters required in the optimized solution is the correlations and volatilities. The unconstrained global minimum variance portfolio problem is expressed as follows:

$$\mathbf{z}^{\mathrm{GMV}_{un}} = \arg \min_{\mathbf{z}\in\mathbb{R}^n} \tfrac{1}{2}\mathbf{z}^T\Sigma\mathbf{z} \qquad (2.3.8)$$
$$\text{such that } \mathbf{z}^T\mathbb{1} = 1.$$

Thus, following the same approach in Appendix A.1, we obtain the solution to the above system as:

$$\mathbf{z}^{\mathrm{GMV}_{un}} = \frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}. \qquad (2.3.9)$$

This solution exhibits that weights in a GMV portfolio are inversely proportional to the covariance matrix, i.e.

$$\mathbf{z}^{\mathrm{GMV}_{un}} \propto \Sigma^{-1}\mathbb{1}. \qquad (2.3.10)$$

Moreover, the volatility of the global minimum variance portfolio is given as follows:

$$\begin{aligned}\sigma_{\mathrm{GMV}_{un}} &= \sqrt{\left[\frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}\right]^T \Sigma \frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}} \\ &= \sqrt{\frac{\mathbb{1}^T\Sigma^{-1}\Sigma\Sigma^{-1}\mathbb{1}}{[\mathbb{1}^T\Sigma^{-1}\mathbb{1}]^2}} = \frac{1}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}. \qquad (2.3.11)\end{aligned}$$

For the constrained global minimum variance portfolio, in particular, long-only, the problem is expressed as follows:

$$\mathbf{z}^{\mathrm{GMV}_{c}} = \arg \min_{\mathbf{z}\in\mathbb{R}^n} \tfrac{1}{2}\mathbf{z}^T\Sigma\mathbf{z} \qquad (2.3.12)$$

such that

$$\begin{cases} \mathbf{z}^T\mathbb{1} = 1 \\ \mathbf{0} \leq \mathbf{z} \leq \mathbb{1}. \end{cases}$$

Below, we illustrate analytical expression for this approach over the two assets universe.

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2.3.2.1 Global Minimum Variance over Two-Assets Universe

The above optimization problem is expressed as follows:

$$\mathbf{z}^{\text{GMV}_c} = \arg\min_{z_1, z_2} \sigma^2_{GMV,c} \tag{2.3.13}$$

such that

$$\begin{cases} \mathbf{z}^T \mathbb{1} = 1 \\ \mathbf{0} \leq \mathbf{z} \leq \mathbb{1}. \end{cases}$$

where the variance is given by,

$$\begin{aligned} \sigma^2_{GMV,c} &= z_1^2\sigma_1^2 + z_2^2\sigma_2^2 + 2z_1z_2\rho_{1,2}\sigma_1\sigma_2 \\ &= z_1^2\sigma_1^2 + (1-z_1)^2\sigma_2^2 + 2z_1(1-z_1)\rho_{1,2}\sigma_1\sigma_2. \end{aligned} \tag{2.3.14}$$

Taking the derivative of equation (2.3.14) with respect to z_1, yield the following,

$$\frac{\partial \sigma^2_{GMV,c}}{\partial z_1} = 2z_1\sigma_1^2 + 2(1-z_1)(-1)\sigma_2^2 + 2(1-z_1)\rho_{1,2}\sigma_1\sigma_2 - 2z_1\rho_{1,2}\sigma_1\sigma_2 = 0.$$

Rearranging, we have

$$z_1\left(2\sigma_1^2 + 2\sigma_2^2 - 4\rho_{1,2}\sigma_1\sigma_2\right) = 2\sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2. \tag{2.3.15}$$

Thus,

$$z_1^{GMV,c} = \frac{2\sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2}{2\sigma_1^2 + 2\sigma_2^2 - 4\rho_{1,2}\sigma_1\sigma_2} = \frac{\sigma_2^2 - \rho_{1,2}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2}. \tag{2.3.16}$$

Also, the weight of the second component is,

$$z_2^{GMV,c} = 1 - z_1^{GMV,c}. \tag{2.3.17}$$

In Chapter 7, we illustrate numerically that a portfolio based on the minimum variance approach minimizes both component volatilities and correlations by equalizing the marginal risk contributions. This result is also verified by Linzmeier (2011). Thus, we can think of GMV strategy as an investment approach that determines weights such that component marginal risk contributions are equal. Although GMV incorporates the estimation of parameters, the allocation of the long-only portfolio seems to be more concentrated in few assets, which does not reflect the idea of risk-diversification through various asset classes.

Lee (2011) proved an interesting property of the GMV strategy which simplify the analysis of the marginal and risk contributions of components in

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this strategy. The most interesting property is that the covariance between any portfolio or asset with GMV portfolio is simply the variance of the GMV portfolio. Considering arbitrary portfolio $\mathbf{z}^*$, this property is expressed mathematical as follows:

$$\begin{aligned}\sigma_{\mathbf{z}^{GMV},\mathbf{z}^*} &= \mathbf{z}_{GMV}^T \Sigma \mathbf{z}^* \\ &= \left(\frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}\right)^T \Sigma \mathbf{z}^* = \frac{\mathbb{1}^T\Sigma^{-1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}\Sigma \mathbf{z}^* \\ &= \frac{1}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}. \end{aligned} \tag{2.3.18}$$

Thus, from equation (2.2.9), we deduce that beta is equal to one. Moreover, the marginal and risk contribution of the i^{th} asset as defined in equations (2.2.27) and (2.2.28), respectively, are,

$$\mathcal{MC}_i = \sigma_{GMV}, \tag{2.3.19}$$

$$\mathcal{RC}_i = z_i \sigma_{GMV}, \quad i = 1, \ldots, n, \tag{2.3.20}$$

since beta of the i^{th} asset is one, see equation (2.3.18), the percentage contribution is as follows

$$\%\mathcal{RC}_i = \mathbf{z}_i^{GMV}. \tag{2.3.21}$$

Recall the solution to the mean-variance optimization as given in equation (1.1.16). That is,

$$\mathbf{z}^{MVO} = \frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}} + \lambda^{-1}\Sigma^{-1}\left[\bar{\mathbf{r}} - \mathbb{1}\frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}\bar{\mathbf{r}}\right]. \tag{2.3.22}$$

As Lee (2011) noted, when components have the same expected returns, then the next term in equation (2.3.22) is zero and the solution to the mean-variance optimization is the same as the global minimum variance portfolio. In other words, the GMV strategy is mean variance efficient if all components in the universe have identical expected returns.

2.3.3 Maximum Diversification Strategy

Another risk-based investment strategy that has recently come under consideration is the maximum diversification (MD). Choueifaty and Coignard (2008) defined a quantitative measure of portfolio diversification as the ratio between the weighted average volatilities to the volatility of the portfolio. Mathematically, this ratio is expressed as follows:

$$DR(\mathbf{z}) = \frac{\mathbf{z}^T\boldsymbol{\sigma}}{\sqrt{\mathbf{z}^T\Sigma\mathbf{z}}}, \tag{2.3.23}$$

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where $\boldsymbol{\sigma}$ is a vector of component volatilities. Equation (2.3.23) can be interpreted as the ratio between non-diversified portfolio's risk (determined in a universe of uncorrelated assets) to the total portfolio risk, see Meucci (2009). The main idea is to obtain portfolio's composite that maximizes this ratio. In other words, this ratio measures the distance between two portfolio volatilities.

Unlike the MVO, the MD approach diminishes the impact of exogenous shocks that might come from concentrated assets. The strategy often incorporate components with high volatility, as long as they have low pair-wise correlations. Since the objective of this strategy is to attain the highest diversified portfolio in a given universe, we have the following relationship,

$$\mathbf{z}^T\boldsymbol{\sigma} \geq \sqrt{\mathbf{z}^T\Sigma\mathbf{z}}. \tag{2.3.24}$$

It is clear that the ratio between the weighted average volatilities and the portfolio's risk will always be greater or equal to one. In addition, for a single component investment, $DR(\mathbf{z})$ equals to one. According to Choueifaty and Coignard (2008), this measure of portfolio diversification means that the higher is this ratio, the more diversified is the portfolio.

Alternatively, Lussier (2013) formulated portfolio diversification benefit defined as follows,

$$DB(\mathbf{z}) = 1 - \frac{\sqrt{\mathbf{z}^T\Sigma\mathbf{z}}}{\mathbf{z}^T\boldsymbol{\sigma}}, \tag{2.3.25}$$

which is zero when invested in only one component or when pair-wise correlations of components is the same. However, in the case where pair-wise correlations is negative one, the diversification benefit is one. According to Lussier (2013), the objective of an investor is to maximize $DR(\mathbf{z})$ which is equivalent to minimizing equation (2.3.25).

Consider the maximization problem as the objective of an investor. Choueifaty *et al.* (2013) described the maximum diversification portfolio as a solution to the following quadratic optimization problem:

$$\mathbf{z}^{MD} = \arg\max_{\mathbf{z}\in\mathbb{R}^n} DR(\mathbf{z}) \tag{2.3.26}$$

such that

$$\begin{cases} \mathbf{z}^T\mathbb{1} = 1 \\ \mathbf{0} \leq \mathbf{z} \leq \mathbb{1}. \end{cases}$$

Note that this strategy looks similar to the Sharpe ratio (well known as tangency portfolio) in the absence of risk-free asset except that instead of a vector expected returns, we use a vector of volatilities. Thus the solution to this

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strategy can be obtained by making direct substitution to the solution of the portfolio maximum Sharpe ratio and is given by the following:

$$\mathbf{z}^{MD} = \left(\frac{\boldsymbol{\sigma}^2}{\boldsymbol{\sigma}_{\mathrm{A}}}\right)\Sigma^{-1}\boldsymbol{\sigma}, \tag{2.3.27}$$

where $\boldsymbol{\sigma}_A$ is the weighted average component volatilities. Roncalli (2013) expressed the optimization problem of system (2.3.26) as follows,

$$\mathbf{z}^{MD} = \arg\max_{\mathbf{z}\in\mathbb{R}^n} \ln DR(\mathbf{z}) \tag{2.3.28}$$

such that

$$\begin{cases} \mathbf{z}^T\mathbb{1} = 1 \\ \mathbf{0} \leq \mathbf{z} \leq \mathbb{1}. \end{cases}$$

Note that the natural logarithm is introduced for simplifying the objective function. This is equivalent to maximizing the diversification ratio, see Choueifaty *et al.* (2013). The Lagrangian function of the above system is

$$\mathcal{L}(\lambda, \lambda_c, \mathbf{z}) = \ln DR(\mathbf{z}) + \lambda_c(\mathbf{z}^T\mathbb{1} - 1) + \lambda^T\mathbf{z}, \tag{2.3.29}$$

where $\lambda \in \mathbb{R}^n$ and $\lambda_c \in \mathbb{R}$. The term $\ln DR(\mathbf{z})$ can be expanded as

$$\ln(\mathbf{z}^T\boldsymbol{\sigma}) - \frac{1}{2}\ln(\mathbf{z}^T\Sigma\mathbf{z}). \tag{2.3.30}$$

The first-order derivative of equation (2.3.29) with respect to portfolio $\mathbf{z}$ yields the following:

$$\frac{\partial\mathcal{L}(\lambda, \lambda_c, \mathbf{z})}{\partial\mathbf{z}} = \frac{\boldsymbol{\sigma}}{\mathbf{z}^T\boldsymbol{\sigma}} - \frac{\Sigma\mathbf{z}}{\mathbf{z}^T\Sigma\mathbf{z}} + \lambda_c\mathbb{1} + \lambda = 0. \tag{2.3.31}$$

The above equation is satisfied for the minimum values of λ's and $\mathbf{z}$. In addition, Roncalli (2013) extended the analysis of this strategy by determining the threshold correlation of components with the market portfolio such that the weights remain positive. The portfolio maximizing the above objective function is given by:

$$z_i^{MD} = DR(\mathbf{z}^{MD})\frac{\sigma_i\sigma_{MD}}{\sigma_i^2}\left(1 - \frac{\rho_{i,m}}{\rho^*}\right) \quad i = 1, \ldots, n, \tag{2.3.32}$$

where σ_i^2 denotes the idiosyncratic variance and $\rho_{i,m}$ is the correlation between the i^{th} asset and the market portfolio and is given as:

$$\rho_{i,m} = \frac{\beta_i\sigma_m}{\sigma_i}. \tag{2.3.33}$$

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Note that β_i denotes the sensitivity of the i^{th} component to the portfolio. It is clear that the allocations will remain positive if the following condition is always true: $\rho_{i,m} < \rho^*$. The volatility of the i^{th} asset in this case is given by:

$$\sigma_i = \sqrt{\beta_i^2 \sigma_m^2 + \sigma_i^2}, \tag{2.3.34}$$

ρ^* denotes the threshold correlation given by:

$$\rho^* = \frac{1 + \sum_{i=1}^{n} \frac{\rho_{i,m}^2}{1-\rho_{i,m}^2}}{\sum_{i=1}^{n} \frac{\rho_{i,m}}{1-\rho_{i,m}^2}}. \tag{2.3.35}$$

Several correlation properties for both the portfolio that satisfies equation (2.3.31) and the under-performing portfolio have been established, see Roncalli (2013). These properties adhere to what Choueifaty *et al.* (2013) described as 'the core properties of Maximum Diversification strategy'. In order to determine these properties, equation (2.3.31) is express as follows,

$$\Sigma \mathbf{z} = \frac{\sigma_{MD}^2}{\mathbf{z}^T \boldsymbol{\sigma}} \boldsymbol{\sigma} + \lambda \sigma_{MD}^2 = \frac{\sigma_{MD}^2}{DR(\mathbf{z})} \boldsymbol{\sigma} + \lambda \sigma_{MD}^2, \tag{2.3.36}$$

since $DR_{MD} = \dfrac{\mathbf{z}^T \boldsymbol{\sigma}}{\sigma_{MD}^2}$. Considering the portfolio that under-performs the MD with the same fixed budgetary constraint, the correlation between these portfolios is given by,

$$\begin{aligned} \rho_{\mathbf{z},\mathbf{z}^{MD}} &= \frac{\mathbf{z}^T \Sigma \mathbf{z}^{MD}}{\sigma(\mathbf{z})\sigma(\mathbf{z}^{MD})} = \frac{\mathbf{z}^T}{\sigma(\mathbf{z})\sigma(\mathbf{z}^{MD})} \left[\frac{\sigma(\mathbf{z}^{MD})}{DR(\mathbf{z}^{MD})} \boldsymbol{\sigma} + \lambda \sigma_p^2(\mathbf{z}^{MD}) \right] \\ &= \frac{DR(\mathbf{z})}{DR(\mathbf{z}^{MD})} + \frac{\sigma_p(\mathbf{z}^{MD})}{\sigma(\mathbf{z})} \mathbf{z}^T \lambda, \end{aligned} \tag{2.3.37}$$

with $\mathbf{z}$ denoting an arbitrary portfolio that is under-performing. This highlights that correlation between these portfolios is dependent on the budgetary constraint for as long as $\lambda > 0$. When $\lambda = 0$, correlation is directly proportional to the ratio of their diversification ratios. Some of the interesting observations on this strategy are,

1. Component correlations to MD portfolio are the same when considering the fixed budget constraint, i.e., if $\mathbf{z}^{MD}$ denotes the solution to the MD portfolio, then for any other asset, z_i, we have

$$\rho_{z_i,\mathbf{z}^{MD}} = \rho_{z_j,\mathbf{z}^{MD}} \quad i, j = 1, \ldots, n. \tag{2.3.38}$$

2. MD strategy is the same as the maximizing Sharpe ratio when all components have same Sharpe ratio.

3. The marginal risk contributions of portfolio constituents are all equal, see Choueifaty and Coignard (2008) and Clarke *et al.* (2013).

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2.4 Risk Parity Strategy

Risk Parity (RP) approach is a general concept for investment strategies that allocate component weights based on their risk features. The idea behind this approach is that by allocating stocks, bonds, commodities and other investment securities by their respective amount of risk, the common concentration of portfolio risk coming from one market regime may be diminished. This approach requires only the covariance matrix as an input parameter, the same as the minimum variance portfolio. However, the difference between the two strategies is that the former does not require optimization technique for the composition of portfolio (often referred to as an ad-hoc rule).

From literature perspective, the attraction of this strategy is the believe that diversification is a form of added value in the investment realm. Indeed when considering diversification as the spread of portfolio risk to the available components in a universe, RP strategy is the right candidate. This is because all components in the universe play an important role in the portfolio's performance. However, there is a slight misunderstanding of this concept in the literature. Practitioners often refer to RP as Equal Risk Contribution (ERC). It worth mentioning that the two are different. Fisher *et al.* (2012*a*) described RP as the inverse volatility strategy and the ERC strategy of Maillard *et al.* (2010) as the beta strategy. In this section we present the formulations and possible solutions to the famous risk parity strategy (i.e Inverse Volatility), and we shall discuss the ERC strategy in the next sections.

2.4.1 Inverse Volatility Strategy

Inverse Volatility (IV) strategy, often called naive Risk Parity, weights component assets inversely to their standalone volatilities (normalized so that weights sum to one). This is a simply investment strategy in which component weights are determined based on their respective volatilities. Investors implementing this strategy make assumption that all pair-wise correlations are the same. In contrast to other risk-based strategies, correlation of the components does not play any role in the allocation of weights for this strategy. In this case, the weight of the i^{th} component is given as follows:

$$z_i^{IV} \propto \sigma_i^{-1}, \quad i = 1, \ldots, n. \tag{2.4.1}$$

More precisely, the weight of components are normalized such that their sum add up to one, i.e.,

$$z_i^{IV} = \frac{\sigma_i^{-1}}{\sum_{j=1}^{n} \sigma_j^{-1}}, \quad \text{for } i = 1, \ldots, n. \tag{2.4.2}$$

This is called unlevered risk parity.

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However, the return of the IV strategy seems to be lower than that of the heuristic mean-variance approach. In addition, the risk parity portfolios do not lie on the efficient frontier of the Markowitz mean-variance strategy. Thus, practitioners implementing these approaches use leverage to obtain the same level of portfolio risk (volatility) as mean-variance approach.

2.5 Equal Risk Contribution Strategy

The equal risk contribution (ERC) is defined as one of the risk-based strategies in which weights are determined by equalizing component risk contributions. Maillard *et al.* (2010) studied the properties of equal risk contribution and find that its volatility lies between the volatility of the equal weighted and the minimum variance portfolios. This strategy mimics the EW strategy, but instead of equalizing component exposures, investors equalize component risk contributions. More precisely, if $\mathcal{RC}_i$ denotes the risk contribution of the i^{th} asset in a portfolio of n-assets, then for any other asset, say j, the allocations of the ERC strategy are such that:

$$\mathcal{RC}_i = \mathcal{RC}_j \quad i, j = 1, \ldots, n. \tag{2.5.1}$$

In this case, components with high risk contribution will be given less preference in terms of weights than low risk contribution.

Thus, the ERC strategy can be seen as the strategy that shares its total risk to the available components in the investment universe. In particular, each component risk contribution is given by:

$$\mathcal{RC}_i = \frac{\sigma_{ERC}}{n} \quad i = 1, \ldots, n, \tag{2.5.2}$$

where σ_{ERC} is the volatility of the ERC portfolio. Thus, the volatility of the ERC portfolio is given by the total sum of the risk contributions, i.e.

$$\sigma_{ERC} = \sum_{i=1}^{n} \mathcal{RC}_i. \tag{2.5.3}$$

The idea of ERC is to find a portfolio that is well diversified in terms of risk, not capital, across all asset classes. This strategy can also be extended such that the allocations is determined through their respective factor contributions, see Chapter 5.

2.5.1 Specification of ERC

We consider a portfolio of n-assets and mimic the notation of Maillard *et al.* (2010) on the specification of equal risk contribution strategy defined as follows:

$$\mathbf{z}^{ERC} = \{\mathbf{z} \in [0,1]^n : \mathbf{z}^T \mathbb{1} = 1, \mathcal{RC}_i = \mathcal{RC}_j \quad i, j = 1, \ldots, n\}. \tag{2.5.4}$$

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The domain of this problem is a set of all positive numbers less than or equal to one. This illustrates that no short-selling of assets is allowed which make it easier to compare with other heuristic risk-based strategies, because they do not possess negative weights. The first constraint indicates full budget utilization and the second one ensures that components have equal risk contributions.

Considering the relative risk contributions in a universe of n-assets, one can think of ERC portfolio as the solution to the following non-linear system

$$\begin{aligned} z_1 \sum_{i=1}^{n} z_i \text{cov}(r_1, r_i) &= z_2 \sum_{i=1}^{n} z_i \text{cov}(r_2, r_i) \\ z_1 \sum_{i=1}^{n} z_i \text{cov}(r_1, r_i) &= z_3 \sum_{i=1}^{n} z_i \text{cov}(r_3, r_i) \\ &\vdots \\ z_1 \sum_{i=1}^{n} z_i \text{cov}(r_1, r_i) &= z_n \sum_{i=1}^{n} z_i \text{cov}(r_n, r_i), \end{aligned} \tag{2.5.5}$$

such that

$$\begin{cases} \mathbf{z}^T \mathbb{1} = 1 \\ \mathbf{0} \leq \mathbf{z} \leq \mathbb{1}. \end{cases}$$

Remark 2.7. *For $n = 2$, the ERC portfolio weights are independent of the correlation parameter ρ, i.e., i.e*

$$\begin{aligned} z_1 \left(\Sigma z\right)_1 &= z_2 \left(\Sigma z\right)_2 \\ z_1 \left(z_1 \sigma_1^2 + \rho_{12} \sigma_1 \sigma_2 z_2\right) &= z_2 \left(z_2 \sigma_2^2 + \rho_{12} \sigma_1 \sigma_2 z_1\right) \\ z_1^2 \sigma_1^2 &= z_2^2 \sigma_2^2, \end{aligned}$$

The weights of the portfolio are given by,

$$z_1 = \frac{\sigma_1^{-1}}{\sigma_1^{-1} + \sigma_2^{-1}} \tag{2.5.6}$$

and

$$z_2 = \frac{\sigma_2^{-1}}{\sigma_1^{-1} + \sigma_2^{-1}}. \tag{2.5.7}$$

This solution is similar to the IV portfolio. However, for multiple asset portfolios (i.e., $n > 2$), it is not easy to find the analytical solution. Below here we highlight alternative direction in obtaining the solution to the ERC.

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2.5.2 Allocation of ERC Strategy

Maillard *et al.* (2010) showed that for ERC portfolio, component weights are directly proportional to their respective inverse beta normalized by the number of portfolio assets, and hence called the beta strategy, see Fisher *et al.* (2012*a*).

Proposition 2.8. *Component weights of the ERC are directly proportional to their inverse beta normalised to the number of constituents in a ERC portfolio, i.e.,*

$$z_i^{ERC} = \frac{\beta_i^{-1}}{n}, \quad i = 1, \ldots, n. \tag{2.5.8}$$

Proof. Consider equation (2.2.15) with the i^{th} component, which gives $(\Sigma \mathbf{z})_i = \sigma_{i\mathbf{z}}$. This implies that beta in equation (2.2.18) is given by,

$$\beta_i = \frac{\sigma_{i\mathbf{z}}}{\sigma^2(\mathbf{z})}, \quad i = 1, \ldots, n, \tag{2.5.9}$$

where $\sigma^2(\mathbf{z})$ denotes portfolio variance. Making the covariance the subject, we have,

$$\sigma_{i\mathbf{z}} = \beta_i \sigma^2(\mathbf{z}), \quad i = 1, \ldots, n. \tag{2.5.10}$$

We recall that ERC allocate assets such that their corresponding risk contributions are equal. This implies that the risk contribution of the i^{th} component is given as follows:

$$\mathcal{RC}_i = \frac{\sigma(\mathbf{z})}{n}, \quad i = 1, \ldots, n. \tag{2.5.11}$$

From equation (2.2.20) we can write the covariance of the i^{th} component and ERC portfolio as:

$$\sigma_{i,\mathbf{z}^{\text{ERC}}} = \frac{\sigma(\mathbf{z})\mathcal{RC}_i}{z_i}, \quad i = 1, \ldots, n. \tag{2.5.12}$$

Replacing equations (2.5.12) and (2.5.11) into equation (2.5.10) gives the following:

$$z_i^{ERC} = \frac{\beta_i^{-1}}{n}, \quad i = 1, \ldots, n. \tag{2.5.13}$$

In particular, if we consider the budget constraint, $\mathbb{1}^T \mathbf{z} = 1$, equation (2.5.13) becomes,

$$z_i^{ERC} = \frac{\beta_i^{-1}}{\sum_{j=1}^{n} \beta_j^{-1}} \quad i = 1, \ldots, n. \tag{2.5.14}$$

□

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Equation (2.5.13) can be interpreted as follows, the more dominant is the component risk (volatility), the lower the allocation of weight or vice versa. This makes sense because according to Qian (2009), the goal of risk-based investors is to protect their investment against severe losses of either the stocks or bonds over a long-term investment. However, equation (2.5.13) is considered endogenous because it turns out that we are using the unknowns to find the unknowns. Beta is a function of z_i which is not known yet. Thus, a solution to problem (2.5.4) is not a closed-form which means that the total sum of portfolio weights do not add up to one.

One of the main scepticism that many investors have about the RP or ERC is the lack of an explicit objective function, see Lee (2011). It is well known that in order to determine the optimum solution of a particular problem, a well defined objective function is required. Fortunately, for the ERC approach, Maillard *et al.* (2010) provided two approaches to determine the optimal solution. The first approach uses sequential quadratic programming (SQP) algorithm where the system to be solved is defined as follows:

$$\mathbf{z}^{ERC} = \arg\min_{\mathbf{z}\in\mathbb{R}^n} f(\mathbf{z}) \tag{2.5.15}$$

such that

$$\begin{cases} \mathbf{z}^T\mathbb{1} = 1 \\ \mathbf{0} \leq \mathbf{z} \leq \mathbb{1}, \end{cases}$$

and

$$f(\mathbf{z}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(z_i(\Sigma\mathbf{z})_i - z_j(\Sigma\mathbf{z})_j\right)^2, \tag{2.5.16}$$

which measures the square of the difference between all pairs of component risk contributions. Alternatively, this function can be expressed as the square or absolute of the average value of the differences between component risk contributions and their respective risk budgets. Thus, equation (2.5.16) can be expressed as follows:

$$f(\mathbf{z}) = \sum_{i=1}^{n}\left(\mathcal{RC}_i - \frac{1}{n}\right)^2, \tag{2.5.17}$$

and the absolute form is given as:

$$f(\mathbf{z}) = \sum_{i=1}^{n}|\mathcal{RC}_i - \frac{1}{n}|. \tag{2.5.18}$$

These are the special cases of the risk budgeting strategy where the budgets are the same for all components. By assumption, investors in this case minimize

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any of these functions as their objective for investment. If $f(\mathbf{z})$ is equal to zero, then the solution to the above system is an ERC, i.e., $\mathcal{RC}_i = \mathcal{RC}_j$ for all i and j. Unfortunately, none of the above defined functions is convex and thus obtaining the minimum solution might be difficult. Also, Griveau-Billion *et al.* (2013) argue that SQP algorithm is time-consuming and does not always converge, especially for $n > 200$.

Maillard *et al.* (2010) suggest an alternative to system (2.5.15), which involves optimization technique defined as follows,

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \sqrt{\mathbf{x}^T\Sigma\mathbf{x}} \tag{2.5.19}$$

such that

$$\begin{cases} \mathbf{x}^T\mathbb{1} \geq c \\ \mathbf{x} \geq \mathbf{0}, \end{cases}$$

where $\mathbf{x}$ is an arbitrary portfolio and c denotes a constant to which the budget constrain has been relaxed to obtain the condition $f(\mathbf{x}^*) = 0$. Hence, the weights of the ERC strategy in this case are given by,

$$z_i^{ERC} = \frac{x_i^*}{\sum_{i=1}^n x_i^*} \quad i = 1, \ldots, n. \tag{2.5.20}$$

The solution in this case exists and is unique.

2.5.3 Alternative Approach for Solving ERC Weights

Apart from Maillard *et al.* (2010) approaches, several algorithms that dispute the use of optimization technique for fast and efficient computation of ERC portfolios have been suggested. Chaves *et al.* (2012) adapted Newton's Algorithm to determine the optimal solution of the ERC strategy.

The Newton's algorithm generates a sequence of approximated solutions around a point, say a, of non-linear system using Taylor's expansion defined as follows,

$$f(\mathbf{z}) \approx f(a) + J(a)(\mathbf{z} - a), \tag{2.5.21}$$

in which higher-order terms are excluded. $J(a)$ denotes the Jacobian matrix. Because we are interested in finding the solution such that $f(\mathbf{z}) = 0$, solving for $\mathbf{z}$ in the approximation (2.5.21) gives,

$$\mathbf{z} = a - \left[J(a)\right]^{-1} f(a). \tag{2.5.22}$$

The idea behind this approach is to repeatedly iterate solution (2.5.22) until a desired or close solution is found. In particular, given the initial guess solution,

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say $\mathbf{z}^k$, one can approximate the optimal solution as,

$$\mathbf{z}^{k+1} = \mathbf{z}^k - \left[J(\mathbf{z}^k)\right]^{-1} f(\mathbf{z}^k) \quad k = 0, 1, \ldots \tag{2.5.23}$$

The solution in this case is the one that is close to $f(\mathbf{z}) = 0$. This implies that at each iterative step, the algorithm computes the inverse Jacobian matrix and the function $f(\mathbf{z})$ to determine the new solution $\mathbf{z}^{k+1}$. This solution is used to test whether $f(\mathbf{z}) = 0$ or not.

To express the problem of risk parity weights in Newton's algorithm, we consider $n > 1$ assets and note that the total risk contributions is given by,

$$D_{\mathbf{z}} \Sigma \mathbf{z} = c\mathbb{1}, \tag{2.5.24}$$

which can simply expressed as

$$\Sigma \mathbf{z} - \frac{c}{\mathbf{z}} = 0. \tag{2.5.25}$$

An alternative to the ERC solution is to use similar approach as in the mean-variance strategy. As noted by Kaya and Lee (2012), the objective of investment is to maximize the generalized non-linear utility function subject to a non-linear constrain. This approach has a special characteristic which allows components to have different risk contributions. On the other hand, Maillard *et al.* (2010) minimize the volatility of the portfolio subject to the non-linear constraint. The optimization problem is then defined as follows:

$$\mathbf{z}^{ERC} = \arg \min_{\mathbf{z} \in \mathbb{R}^n} \mathbf{z}^T \Sigma \mathbf{z} \tag{2.5.26}$$

such that

$$\begin{cases} \mathbb{1}^T \ln \mathbf{z} \geq c \\ \mathbf{z} \geq 0, \end{cases}$$

where $\ln \mathbf{z}$ is a vector denoting natural logarithm of components weights and $c \in \mathbb{R}$. The associated Lagrange function is

$$\mathcal{L}(\mathbf{z}, \lambda) = \mathbf{z}^T \Sigma \mathbf{z} - \lambda(\mathbb{1}^T \ln \mathbf{z} - c). \tag{2.5.27}$$

The solution, $\mathbf{z}^{ERC}$, verifies the following first-order condition,

$$\frac{\partial \mathcal{L}(\mathbf{z}, \lambda)}{\partial \mathbf{z}} = 2\Sigma \mathbf{z} - \mathbf{z}^{-1} = 0, \tag{2.5.28}$$

where $\lambda = 1$ and,

$$\mathbf{z}^{-1} = \begin{pmatrix} z_1^{-1} \\ z_2^{-1} \\ \vdots \\ z_n^{-1} \end{pmatrix}. \tag{2.5.29}$$

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Thus, for the i^{th} component, we have

$$2(\Sigma\mathbf{z})_i - z_i^{-1} = 0, \quad i = 1, \dots, n, \tag{2.5.30}$$

which can be expressed as follows

$$z_i(\Sigma\mathbf{z})_i = \frac{1}{2}, \quad i = 1, \dots, n. \tag{2.5.31}$$

In addition, if equation (2.5.31) holds, then we have the following

$$\mathbf{z}^T\Sigma\mathbf{z} = \frac{n}{2}. \tag{2.5.32}$$

Hence, from the two equations, i.e (2.5.32) and (2.5.31), we deduce that the risk contribution of the i^{th} component is

$$\mathcal{RC}_i = \frac{z_i(\Sigma\mathbf{z})_i}{\mathbf{z}^T\Sigma\mathbf{z}} = \frac{1/2}{n/2} = \frac{1}{n}. \tag{2.5.33}$$

This result indicates that the risk contribution of a component is inversely proportional to the total number of the components in a universe. In particular, the absolute risk contribution of a two assets portfolio is derived as follows. Recall the weights of this portfolio as illustrated in example (2.7). We know that the absolute risk contribution of a component is given as,

$$\mathcal{RC}^{abs} = \mathbf{z}^T\frac{\Sigma\mathbf{z}}{\sigma(\mathbf{z})},$$

and the percentage contribution is,

$$\%\mathcal{RC}^{abs} = \mathbf{z}^T\frac{\Sigma\mathbf{z}}{\sigma^2(\mathbf{z})}. \tag{2.5.34}$$

Expanding the above equation (2.5.34) for the two assets portfolio and consider the absolute risk contribution of the first component, we have

$$\%\mathcal{RC}_1^{abs} = \frac{z_1^2\sigma_1^2 + z_1 z_2\rho_{1,2}\sigma_1\sigma_2}{\sigma^2(\mathbf{z})}, \tag{2.5.35}$$

where the variance in this case is given by,

$$\sigma^2(\mathbf{z}) = z_1^2\sigma_1^2 + z_2^2\sigma_2^2 + 2z_1 z_2\rho_{1,2}\sigma_1\sigma_2. \tag{2.5.36}$$

Substituting the corresponding weights as given in example (2.7) into equation (2.5.35), yield the following absolute risk contribution,

$$\begin{aligned}
\%\mathcal{RC}_1^{asb} &= \frac{\left(\frac{\sigma_1^{-1}}{(\sigma_1^{-1}+\sigma_2^{-1})}\right)^2\sigma_1^2 + \left(\frac{\sigma_1^{-1}}{(\sigma_1^{-1}+\sigma_2^{-1})}\right)\left(\frac{\sigma_2^{-1}}{(\sigma_1^{-1}+\sigma_2^{-1})}\right)\sigma_1\sigma_2\rho_{1,2}}{\left(\frac{\sigma_1^{-1}}{(\sigma_1^{-1}+\sigma_2^{-1})}\right)^2\sigma_1^2 + \left(\frac{\sigma_2^{-1}}{(\sigma_1^{-1}+\sigma_2^{-1})}\right)^2\sigma_2^2 + 2\left(\frac{\sigma_1^{-1}}{(\sigma_1^{-1}+\sigma_2^{-1})}\right)\left(\frac{\sigma_2^{-1}}{(\sigma_1^{-1}+\sigma_2^{-1})}\right)\rho_{1,2}\sigma_1\sigma_2} \\
&= \frac{1+\rho_{1,2}}{(\sigma_1^{-1}+\sigma_2^{-1})^2}\frac{(\sigma_1^{-1}+\sigma_2^{-1})^2}{2(1+\rho_{1,2})} \\
&= \frac{1}{2}.
\end{aligned} \tag{2.5.37}$$

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This result verifies that for the two assets portfolio, the percentage contribution of a component is 50%.

2.6 Dilemma of Risk Parity

Although risk parity seems to be more robust portfolio allocation strategy in terms of diversifying risk, it has its own downfall. The expected return and risk of the portfolio constructed in this manner is usually lower than that of a typical 60/40. To enhance the same risk as the 60/40 portfolio, investors introduce leverage, a technique that raised many sceptics to many investors in the implementation of risk parity strategy; see Bhansali *et al.* (2012).

2.6.1 Leverage Concern

Risk parity is assumed to be suitable strategy for the time of recent financial crisis, particularly when the growth market is declining. During such period, leverage portfolio is highly required. Usually, leverage is implemented through fixed income assets such as government bonds. The problem with risk parity arises as the growth markets start giving potential results and investors become interested in it. For risk parity, this implies that leverage is no longer necessary. Thus, maintaining return of risk parity tantamount mean variance becomes difficult. Also, when borrowing becomes more expensive, the risk parity strategy might produce bad results. The portfolio tends to be more concentrated in the fixed income assets.

2.6.2 Correlation Concern

Another way of constructing a well diversified portfolio is to hold assets that are less correlated to each other. However, selection of asset classes requires care since markets incorporate convertible assets. Careless selection of assets may lead to highly correlated portfolio constituent and thus result with an under-diversified portfolio. Romahi and Santiago (2012) introduced an alternative approach on portfolio construction using risk parity assumptions. Instead of diversifying portfolio through component risk contributions, investors implement a different strategy that focuses on the risk factor. This approach is detailed in Chapter 5.

2.7 Summary

In this chapter, we presented various risk measures that are under consideration for portfolio's construction. We introduced the popular risk-based strategies with more attention on the risk parity strategy. The empirical study which support the theory of these strategies is presented in Chapter 7.

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By considering a simple example, we illustrate the comparison of the risk-based strategies. We observe that if components have the same volatilities and all pair-wise correlations are also equal, then all strategies lead to the same allocation. Lussier (2013), also identified the same results. In addition, he discovered that in the case of identical asset returns and volatilities, all strategies lead to the same allocation only in the two assets case. For $n > 2$, the EW (which benefits from the law of large number) and ERC strategy produce different allocation. Note that for the two assets case, only single correlation is available. Thus, the EW strategy will be more appropriate if practitioners are not certain about their estimation of the required input parameters.

Unlike the GMV portfolio, the ERC strategy includes all the components in a universe. However, its volatility is significantly higher than that of the GMV portfolio. In particular, Maillard *et al.* (2010) showed analytically that the volatility of the ERC lies between the volatility of the GMV and EW portfolios. The ERC strategy produces a portfolio such that component risk contributions is the same, while the EW strategy determined the allocation based on the number of component in the universe. The latter makes sense if we believe that neither stock returns nor risk can be forecast. The GMV strategy allocation is such that components marginal contributions are the same, while the MD strategy maximizes the diversification ratio. Moreover, MD is the same as the maximum Sharpe ratio strategy if all components have the same marginal volatility.

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Chapter 3

Link between Risk Parity and Efficient Mean-Variance Portfolio

In this Chapter, we discuss the relationship between risk parity and the mean variance strategy. In practice, there are many questions evolving around the risk parity strategy, mainly the out-performance of this strategy against other investment strategies, see Romahi and Santiago (2012), Rappoport and Nottebohm (2012). We illustrate the general condition for which risk parity strategy is mean-variance efficient. In particular, for the conditions, say all pair-wise correlations are the same and components have the same Sharpe ratios, risk parity is still mean-variance efficient.

3.1 Decomposition of the MV Input Parameters

The covariance matrix can be decomposed as the matrix multiplication of correlation matrix and the vector matrix of asset standard deviations. i.e.,

$$\Sigma = D_\sigma C D_\sigma, \tag{3.1.1}$$

where D_σ is a diagonal matrix with a vector of volatilities σ in the main diagonal, i.e.,

$$D_\sigma = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_n \end{pmatrix}, \tag{3.1.2}$$

which can be generalised for any vector $\mathbf{x}$, i.e., $D_{\mathbf{x}}$. C is a symmetric correlation matrix, i.e., $\rho_{ij} = \rho_{ji}$, that explains the strength of the relationship between

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relative asset classes. That is,

$$C = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho_{n-1n} \\ \rho_{n1} & \cdots & \rho_{nn-1} & 1 \end{pmatrix}. \tag{3.1.3}$$

Definition 3.1. *If r_0 does not play any role in a portfolio, then the security Sharpe ratio, s_i, is defined as,*

$$\mathbf{s}_i = \frac{\bar{r}_i}{\sigma_i}, \quad i = 1, \ldots, n, \tag{3.1.4}$$

where $\bar{r}_i$ denotes the expected return of the i^{th} security and σ_i is the corresponding volatility.

We denote by $\mathbf{s}$ a vector of security Sharpe ratios, i.e.,

$$\mathbf{s}^T = (\mathbf{s}_1, \ldots, \mathbf{s}_n). \tag{3.1.5}$$

This vector can be decomposed as a diagonal matrix of inverse volatilities and a vector of expected returns, i.e.,

$$\mathbf{s} = D_{\sigma^{-1}}\bar{\mathbf{r}} = \begin{pmatrix} \sigma_1^{-1} & 0 & \cdots & 0 \\ 0 & \sigma_2^{-1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \sigma_n^{-1} \end{pmatrix} \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \vdots \\ \bar{r}_n \end{pmatrix}. \tag{3.1.6}$$

When risk free asset is not included in a portfolio, the Sharpe ratio is then given as a function of weights, expected return and the covariance matrix of the risky assets.

Definition 3.2. *Let $\mathbf{z} \in \mathbb{R}^n$ be an arbitrary portfolio of only risky assets with volatility $0 < \sigma(\mathbf{z}) < \infty$ and a vector of components excess return $\bar{\mathbf{r}}$. Then the portfolio's Sharpe ratio is given by:*

$$S(\mathbf{z}, \bar{\mathbf{r}}, \Sigma) = \frac{\mathbf{z}^T\bar{\mathbf{r}}}{\sqrt{\mathbf{z}^T\Sigma\mathbf{z}}}. \tag{3.1.7}$$

Let us consider the following useful properties,

$$\Sigma = D_\sigma C D_\sigma \tag{3.1.8}$$
$$\Sigma^{-1} = D_{\sigma^{-1}} C^{-1} D_{\sigma^{-1}} \tag{3.1.9}$$
$$(\sigma^{-1})^T \Sigma \sigma^{-1} = \mathbb{1}^T C \mathbb{1}. \tag{3.1.10}$$

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Remark 3.3. *The inverse of a diagonal matrix $D_{\sigma^{-1}}$ is actually the diagonal matrix D_σ, i.e.,*

$$D_\sigma^{-1} = D_{\sigma^{-1}}. \tag{3.1.11}$$

Proposition 3.4. *The decomposition of the portfolio's covariance matrix leads to the Sharpe ratio expressed as the product of the vector of security Sharpe ratios and weights per weighted correlation of securities, i.e.*

$$\frac{\mathbf{z}^T\bar{\mathbf{r}}}{\sqrt{\mathbf{z}^T\Sigma\mathbf{z}}} = \frac{\mathbf{z}^T\mathbf{s}}{\sqrt{\mathbf{z}^T C\mathbf{z}}} \tag{3.1.12}$$

Proof. We consider the properties of the inner product vectors, i.e.,

$$\mathbf{z}^T\bar{\mathbf{r}} = \langle\mathbf{z}, \bar{\mathbf{r}}\rangle. \tag{3.1.13}$$

Using this property we can express the left hand side of equation (3.1.12) as:

$$\frac{\mathbf{z}^T\bar{\mathbf{r}}}{\sqrt{\mathbf{z}^T\Sigma\mathbf{z}}} = \frac{\langle\mathbf{z}, \bar{\mathbf{r}}\rangle}{\sqrt{\langle\mathbf{z}, \Sigma\mathbf{z}\rangle}}. \tag{3.1.14}$$

Note that the following property holds, $\bar{\mathbf{r}} = D_\sigma\mathbf{s}$. Thus, equation (3.1.14) can be expressed as follows:

$$\begin{aligned}\frac{\langle\mathbf{z}, \bar{\mathbf{r}}\rangle}{\sqrt{\langle\mathbf{z}, \Sigma\mathbf{z}\rangle}} &= \frac{\langle\mathbf{z}, D_\sigma\mathbf{s}\rangle}{\sqrt{\langle\mathbf{z}, D_\sigma C D_\sigma\mathbf{z}\rangle}} \\ &= \frac{\langle D_\sigma\mathbf{z}, \mathbf{s}\rangle}{\sqrt{\langle D_\sigma\mathbf{z}, C D_\sigma\mathbf{z}\rangle}}.\end{aligned} \tag{3.1.15}$$

Setting $\tilde{\mathbf{z}} = D_\sigma\mathbf{z}$, we have

$$\frac{\langle\tilde{\mathbf{z}}, \mathbf{s}\rangle}{\sqrt{\langle\tilde{\mathbf{z}}, C\tilde{\mathbf{z}}\rangle}} = \frac{\tilde{\mathbf{z}}^T\mathbf{s}}{\sqrt{\tilde{\mathbf{z}}^T C\tilde{\mathbf{z}}}}. \tag{3.1.16}$$

□

Remark 3.5. *The portfolio that uniquely maximizes the mean-variance is given by:*

$$\mathbf{z}^{MVO} = \lambda\Sigma^{-1}\bar{\mathbf{r}} = \lambda\begin{pmatrix}\Sigma_1^{-1}\\ \Sigma_2^{-1}\\ \vdots\\ \Sigma_n^{-1}\end{pmatrix}\begin{pmatrix}\bar{r}_1\\ \bar{r}_2\\ \vdots\\ \bar{r}_n\end{pmatrix}, \tag{3.1.17}$$

where Σ_i^{-1} correspond to the i^{th} row of the inverse covariance matrix Σ and

$$\lambda = \frac{1}{\mathbb{1}^T\Sigma^{-1}\bar{\mathbf{r}}},$$

see Glombek (2012) for a proof. This means that the i^{th} component is given as follows:

$$\mathbf{z}_i^{MVO} = \lambda\Sigma_i^{-1}\bar{\mathbf{r}} \quad i = 1, \ldots, n.$$

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3.2 Risk Parity and Mean-Variance Efficient

As many questions are still asked based on the performance of risk parity, Kaya and Lee (2012) and Kaya (2012) answer the one relevant to efficiency. The former illustrates efficient risk parity on a special condition, while the latter gives the general case.

3.2.1 Conditional Efficiency of Risk Parity

We consider the situation where components have identical Sharpe ratios and their pair-wise correlations is the same.

Proposition 3.6. *Let* $\mathbf{z} \in \mathbb{R}^n$ *be a risk parity portfolio. Then* $\mathbf{z}$ *is mean variance efficient if components have identical Sharpe ratios and correlations are the same.*

Proof. Consider the efficient mean variance solution given in Remark (3.5), and assume without loss of generality that $\lambda = 1$. This means

$$\mathbf{z}^{MVO} = \Sigma^{-1}\bar{\mathbf{r}}. \tag{3.2.1}$$

If $D_{\mathbf{z}}$ denotes the diagonal matrix of n-component weights, i.e

$$D_{\mathbf{z}} = \begin{pmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n \end{pmatrix}, \tag{3.2.2}$$

then the risk contribution of components is:

$$\mathcal{RC}(\mathbf{z}) = D_{\mathbf{z}}\frac{\Sigma\mathbf{z}}{\sigma(\mathbf{z})}. \tag{3.2.3}$$

Substituting $\mathbf{z}$ by equation (3.2.1), we have

$$\mathcal{RC}(\mathbf{z}) = D_{\mathbf{z}}\frac{\Sigma(\Sigma^{-1}\bar{\mathbf{r}})}{\sigma(\mathbf{z})} = \frac{D_{\mathbf{z}}\bar{\mathbf{r}}}{\sigma(\mathbf{z})}. \tag{3.2.4}$$

Thus, the risk contribution of the i^{th} component is given as follows:

$$\mathcal{RC}_i(\mathbf{z}) = \frac{\bar{r}_i(\Sigma^{-1}\bar{\mathbf{r}})_i}{\sigma(\mathbf{z})} \tag{3.2.5}$$

Using property (3.1.9), and letting b_{ij} denote the entries of the inverse correlation matrix C^{-1}, it follows that the entries of the inverse covariance matrix Σ^{-1} are given by:

$$\sigma_{ij}^{-1} = \frac{b_{ij}}{\sigma_i\sigma_j}. \tag{3.2.6}$$

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Thus, equation (3.2.5) becomes:

$$\begin{aligned}\mathcal{RC}_i(\mathbf{z}) &= \frac{\bar{r}_i\left[\sigma_{i1}^{-1}\bar{r}_1 + \sigma_{i2}^{-1}\bar{r}_2 + \cdots + \sigma_{in}^{-1}\bar{r}_n\right]}{\sigma(\mathbf{z})} \\ &= \frac{\bar{r}_i\left[\frac{\bar{r}_1 b_{i1}}{\sigma_i\sigma_1} + \frac{\bar{r}_2 b_{i2}}{\sigma_i\sigma_2} + \cdots + \frac{\bar{r}_n b_{in}}{\sigma_i\sigma_n}\right]}{\sigma(\mathbf{z})} \\ &= \frac{\left(\frac{\bar{r}_i}{\sigma_i}\right)^2}{\sigma(\mathbf{z})}\left[\frac{\bar{r}_1\sigma_i b_{i1}}{\bar{r}_i\sigma_1} + \frac{\bar{r}_2\sigma_i b_{i2}}{\bar{r}_i\sigma_2} + \cdots + \frac{\bar{r}_n\sigma_i b_{in}}{\bar{r}_i\sigma_n}\right].\end{aligned} \tag{3.2.7}$$

Letting $\mathbf{s}_i = \dfrac{\bar{r}_i}{\sigma_i}$, the above equation becomes:

$$\mathcal{RC}_i(\mathbf{z}) = \frac{\mathbf{s}_i^2}{\sigma(\mathbf{z})}\left[\frac{\mathbf{s}_1}{\mathbf{s}_i}b_{1i} + \frac{\mathbf{s}_2}{\mathbf{s}_i}b_{2i} + \cdots + \frac{\mathbf{s}_n}{\mathbf{s}_i}b_{ni}\right]. \tag{3.2.8}$$

Thus, the ERC portfolio satisfies the following equation:

$$\frac{\mathcal{RC}_i(\mathbf{z})}{\mathcal{RC}_j(\mathbf{z})} = \left(\frac{\mathbf{s}_i}{\mathbf{s}_j}\right)^2\left[\frac{\frac{\mathbf{s}_1}{\mathbf{s}_i}b_{1i} + \frac{\mathbf{s}_2}{\mathbf{s}_i}b_{2i} + \cdots + \frac{\mathbf{s}_n}{\mathbf{s}_i}b_{ni}}{\frac{\mathbf{s}_1}{\mathbf{s}_j}b_{1j} + \frac{\mathbf{s}_2}{\mathbf{s}_j}b_{2j} + \cdots + \frac{\mathbf{s}_n}{\mathbf{s}_j}b_{nj}}\right]. \tag{3.2.9}$$

For the case of equal Sharpe ratios and correlations, i.e., $\mathbf{s}_i = s$ and $b_{ij} = b$, respectively, and s and b are arbitrary constants, then the ERC strategy satisfies the following:

$$\frac{\mathcal{RC}_i(\mathbf{z})}{\mathcal{RC}_j(\mathbf{z})} = 1, \tag{3.2.10}$$

which proves that the ERC portfolio is mean-variance efficient if the component Sharpe ratios and correlations are the same. □

3.2.2 General Efficient Risk Parity

Proposition (3.6) illustrates the efficiency of risk parity to the mean-variance portfolio under some specified conditions. This gives bounds to the understanding of efficient risk parity in a general context. However, Kaya (2012) extends the proof to the general case when correlations and Sharpe ratios are different.

Theorem 3.7. *Let $\mathbf{z} \in \mathbb{R}^n$ be a risk parity portfolio. Then, $\mathbf{z}$ is mean-variance efficient if and only if the following condition holds:*

$$D_{\mathbf{s}}C^{-1}\mathbf{s} = \tilde{c}\mathbb{1}, \tag{3.2.11}$$

where $\mathbf{s}$ is a column vector of component Sharpe ratios, $D_{\mathbf{s}}$ is a diagonal matrix with component Sharpe ratios in the main diagonal, C^{-1} an inverse correlation matrix, $\tilde{c}$ is an arbitrary constant and $\mathbb{1}$ a column vector of ones.

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Proof. ($\Rightarrow$) Suppose $\mathbf{z}^{MVO} \in \mathbb{R}^n$ is an efficient mean-variance portfolio. We need to show that if $\mathbf{z}^{MVO}$ satisfy the risk parity, then condition (3.2.11) holds. Recall that the total absolute risk contributions of the risk parity portfolios satisfy the following:

$$D_{\mathbf{z}}\Sigma^{-1}\mathbf{z} = c\mathbb{1}. \tag{3.2.12}$$

Using Remark (3.5), the left hand side of equation (3.2.12) can be written as follows:

$$D_{\mathbf{z}}\Sigma\mathbf{z} = D_{\mathbf{z}}\Sigma(\lambda\Sigma^{-1}\bar{\mathbf{r}}) = \lambda \begin{pmatrix} z_1\bar{r}_1 \\ z_2\bar{r}_2 \\ \vdots \\ z_n\bar{r}_n \end{pmatrix}. \tag{3.2.13}$$

Thus, for the i^{th} component, we have

$$(D_{\mathbf{z}}\Sigma\mathbf{z})_i = \lambda z_i\bar{r}_i \quad i = 1, \ldots, n. \tag{3.2.14}$$

Substituting again the $\mathbf{z}_i^{MVO}$ in the equation (3.2.14), we have,

$$(D_{\mathbf{z}}\Sigma\mathbf{z})_i = \lambda(\lambda\Sigma^{-1}\bar{\mathbf{r}})_i\bar{r}_i = \lambda^2\bar{r}_i\Sigma_i^{-1}\bar{\mathbf{r}} \quad i = 1, \ldots, n. \tag{3.2.15}$$

Define by $D_{\bar{\mathbf{r}}}$, a diagonal matrix of component expected returns, i.e.,

$$D_{\bar{\mathbf{r}}} = \begin{pmatrix} \bar{r}_1 & 0 & \cdots & 0 \\ 0 & \bar{r}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{r}_n \end{pmatrix}. \tag{3.2.16}$$

Now, equation (3.2.15) can be expressed in a matrix form equation, i.e.,

$$D_{\mathbf{z}}\Sigma\mathbf{z} = \lambda^2 D_{\bar{\mathbf{r}}}\Sigma^{-1}\bar{\mathbf{r}}. \tag{3.2.17}$$

Using property (3.1.9) in equation (3.2.17), we have:

$$\lambda^2 D_{\bar{\mathbf{r}}} D_{\sigma^{-1}} C^{-1} D_{\sigma^{-1}}\bar{\mathbf{r}} = \lambda^2 D_{\mathbf{s}} C^{-1}\mathbf{s}, \tag{3.2.18}$$

where $D_{\mathbf{s}} = D_{\bar{\mathbf{r}}}D_{\sigma^{-1}}$ and $\mathbf{s} = D_{\sigma^{-1}}\bar{\mathbf{r}}$. Thus, equation (3.2.12) is equivalent to:

$$\lambda^2 D_{\mathbf{s}} C^{-1}\mathbf{s} = c\mathbb{1}. \tag{3.2.19}$$

If we let $\dfrac{c}{\lambda^2} = \tilde{c}$, we have

$$D_{\mathbf{s}} C^{-1}\mathbf{s} = \tilde{c}\mathbb{1}. \tag{3.2.20}$$

Hence, if $\mathbf{z}$ is mean-variance efficient and satisfy the risk parity portfolio, then the condition (3.2.11) holds.

($\Leftarrow$) Suppose the Sharpe ratios and correlations satisfy condition (3.2.11). We need to show that:

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1. If $\mathbf{z}$ is mean-variance efficient, then $\mathbf{z}$ is the risk parity portfolio.
2. If $\mathbf{z}$ is a risk parity portfolio, then $\mathbf{z}$ is also efficient.

Consider the first condition, i.e $\mathbf{z} \in \mathbb{R}^n$ is mean-variance efficient. That is,

$$\mathbf{z} = \lambda\Sigma^{-1}\bar{\mathbf{r}}. \tag{3.2.21}$$

From equation (3.2.20), it follows that the risk contribution satisfy:

$$D_{\mathbf{z}}\Sigma\mathbf{z} = D_{\mathbf{s}}C^{-1}\mathbf{s} = \tilde{c}\mathbb{1}, \tag{3.2.22}$$

which implies that risk parity is also mean-variance efficient.

To prove the second condition, we assume that $\mathbf{z}^* \in \mathbb{R}^n$ is a risk parity portfolio with fixed volatility. According to Kaya and Lee (2012), $\mathbf{z}^*$ exists and is unique. We need to show that $\mathbf{z}^* = \lambda\Sigma^{-1}\bar{\mathbf{r}}$ also satisfies the following condition

$$D_{\mathbf{z}^*}\Sigma\mathbf{z}^* = \tilde{c}\mathbb{1}, \tag{3.2.23}$$

for some constant $\tilde{c} \geq 0$. Consider the above equation (3.2.23). This could be written as,

$$\tilde{c}\mathbb{1} = D_{\bar{\mathbf{r}}}D_{\boldsymbol{\sigma}^{-1}}C^{-1}D_{\sigma^{-1}}\bar{\mathbf{r}} = D_{\bar{\mathbf{r}}}\Sigma^{-1}\bar{\mathbf{r}}.$$

Alternatively,

$$\bar{\mathbf{r}} = \tilde{c}\Sigma D_{\bar{\mathbf{r}}}^{-1}\mathbb{1}. \tag{3.2.24}$$

Thus,

$$\mathbf{z}^* = \lambda\Sigma^{-1}\bar{\mathbf{r}} = \lambda\Sigma^{-1}(\tilde{c}\Sigma D_{\bar{\mathbf{r}}}^{-1}\mathbb{1}) = \tilde{c}\lambda D_{\bar{\mathbf{r}}}^{-1}\mathbb{1}. \tag{3.2.25}$$

Substituting the above equation into the total absolute risk contribution, we have

$$D_{\mathbf{z}^*}\Sigma\mathbf{z}^* = D_{\mathbf{z}^*}\Sigma(\tilde{c}\lambda D_{\bar{\mathbf{r}}}^{-1}\mathbb{1}) = \lambda D_{\mathbf{z}^*}(\tilde{c}\Sigma D_{\bar{\mathbf{r}}}^{-1}\mathbb{1}).$$

Replacing equation (3.2.24) into the above equation, yields

$$D_{\mathbf{z}^*}\Sigma\mathbf{z}^* = \lambda D_{\mathbf{z}^*}\bar{\mathbf{r}} = \lambda\begin{pmatrix} z_1^*\bar{r}_1 \\ z_2^*\bar{r}_2 \\ \vdots \\ z_n^*\bar{r}_n \end{pmatrix}. \tag{3.2.26}$$

But, equation (3.2.25) implies that the i^{th} component is given by,

$$z_i^* = \tilde{c}\lambda\frac{1}{\bar{r}_i} \quad i = 1, \ldots, n.$$

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Thus, the total absolute risk contribution in equation (3.2.26) is given by,

$$D_{\mathbf{z}^*}\Sigma\mathbf{z}^* = \tilde{c}\lambda^2 \begin{pmatrix} \frac{1}{\bar{r}_1}\bar{r}_1 \\ \frac{1}{\bar{r}_2}\bar{r}_2 \\ \vdots \\ \frac{1}{\bar{r}_n}\bar{r}_n \end{pmatrix} = \tilde{c}\lambda^2\mathbb{1}.$$

Setting $\tilde{c}\lambda^2 = c$ proves that $\mathbf{z}^* = \lambda\Sigma^{-1}\bar{\mathbf{r}}$ is a risk parity and is also efficient portfolio. □

3.2.3 Risk Parity and Tangency Portfolio

Instead of focussing on the risk and return of the portfolios, investors may consider maximizing the portfolio Sharpe ratio which changes the objective of the standard mean variance strategy. The portfolio that maximizes the Sharpe ratio in the mean-variance framework is known as the tangency portfolio and is given by the following:

$$\mathbf{z}^{TP} = \frac{\Sigma^{-1}\bar{\mathbf{r}}}{\mathbb{1}^T\Sigma^{-1}\bar{\mathbf{r}}}, \quad \mathbb{1}^T\Sigma^{-1}\bar{\mathbf{r}} \neq 0. \tag{3.2.27}$$

The portfolio's maximum Sharpe ratio is given by:

$$\max_{\mathbf{z}\in\mathbb{R}^n} S(\mathbf{z}, \bar{\mathbf{r}}, \Sigma) = \max_{\tilde{\mathbf{z}}\in\mathbb{R}^n} S(\tilde{\mathbf{z}}, \mathbf{s}, C) = \sqrt{\bar{\mathbf{r}}^T\Sigma^{-1}\bar{\mathbf{r}}},$$

with the optimum weights being proportional to the inverse of the covariance matrix and a vector of expected returns, i.e

$$\mathbf{z}^{TP} \propto \Sigma^{-1}\bar{\mathbf{r}}, \tag{3.2.28}$$

under a fixed budgeting constraint:

$$\mathbf{z}^T\mathbb{1} = 1, \tag{3.2.29}$$

see Glombek (2012) and Fisher *et al.* (2012*a*). In particular, we can express the portfolio weights as:

$$\mathbf{z}^{MSR} = D_\sigma^{-1}C^{-1}\mathbf{s}, \tag{3.2.30}$$

when λ is one.

Below we distinguish two cases related to the Sharpe ratio in portfolio selection. First, we notice that when components have identical Sharpe ratio, then the MVO portfolio is the same as the most diversified portfolio, i.e.

$$\mathbf{z}^{MSR} = D_\sigma^{-1}C^{-1}\mathbb{1}, \tag{3.2.31}$$

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and this solution can be expressed as:

$$CD_\sigma \mathbf{z}^{MSR} = \mathbb{1}. \tag{3.2.32}$$

Secondly, if we consider the case of uncorrelated assets, and assume that Sharpe ratios are equal, the results are equivalent to the volatility portfolio and weights are given by:

$$\mathbf{z}^{MSR} = D_\sigma^{-1}\mathbb{I}_n^{-1}\mathbb{1}, \tag{3.2.33}$$

which can be represented as:

$$\mathbb{I}_n D_\sigma \mathbf{z}^{MSR} = \mathbb{1}, \tag{3.2.34}$$

where $\mathbb{I}$ is n-dimensional identity matrix.

Thus the risk parity portfolio satisfy the following equation:

$$(\mathbb{I}D_\sigma \mathbf{z}^*) \bullet (CD_\sigma \mathbf{z}^*) = \mathbb{1} \bullet \mathbb{1}, \tag{3.2.35}$$

where $\bullet$ denotes the *Hadamard* element-by-element product (also called the *Schur* product) of two identically-sized matrices with the following properties:

1. $A \bullet B = B \bullet A$,
2. $\mathbb{1}_n \bullet A = A$,

for n-dimensional matrices A, B and $\mathbb{1}_n$ (i.e an n-dimension matrix of ones).

When Sharpe ratios are different, the weights of the risk parity satisfy the following equation:

$$(\mathbb{I}D_\sigma \mathbf{z}^*) \bullet (CD_\sigma \mathbf{z}^*) = \mathbf{s} \bullet \mathbf{s}, \tag{3.2.36}$$

where the left hand side indicates the total risk contributions. This implies that the risk contribution of asset i is:

$$\mathcal{RC}_i(\mathbf{z}) = \mathbf{s}_i^2. \tag{3.2.37}$$

Alternatively, this result could be seen as a direct consequence of equation (3.2.8), where the correlation between components is zero. Thus, the component risk contributions in a portfolio are directly proportional to their respective Sharpe ratio squared if Sharpe ratios are identical and correlation is zero.

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3.2.4 RP Outperform MVO

According to Mossin (1966), Sharpe (1964), Treynor (1962) and Lintner (1965), the portfolio that has the highest return in the mean variance framework is the one with the highest Sharpe ratio.

In practice, there are still many doubts about risk parity implementation. In particular, practitioners lodge concern about the future performance of RP, Inker (2011). We illustrate the condition under which RP outperforms other investment strategies, particularly, the MVO. We determine this condition based on Fisher *et al.* (2012*a*) approach which requires two portfolios, say $\mathbf{w}$ and $\mathbf{z}$, with the vector of component expected returns denoted by $\mathbf{m}$ and the future covariance matrix Ω. We know that risk parity portfolio weights, in particular, the IV portfolio weights, are inversely proportional to their respective volatilities. If we denote by $\mathbf{w}$ the vector of risk parity weights, then one can just substitute this vector by the corresponding vector of component volatilities and obtain the following Sharpe ratio:

$$S(\mathbf{w},\mathbf{m},\Omega) = \frac{\mathbf{m}^T\mathbf{w}}{\sqrt{\mathbf{w}^T\Omega\mathbf{w}}} = \frac{\mathbf{m}^T\boldsymbol{\sigma}^{-1}}{\sqrt{(\boldsymbol{\sigma}^{-1})^T\Omega\boldsymbol{\sigma}^{-1}}}, \tag{3.2.38}$$

where $\boldsymbol{\sigma}^{-1} \in \mathbb{R}^n$ is a vector of component inverse volatilities. Similarly, for any other mean-variance portfolio, say $\mathbf{z} \in \mathbb{R}^n$, the Sharpe ratio[1] is:

$$S(\mathbf{z},\mathbf{m},\Omega) = \frac{\mathbf{m}^T\mathbf{z}}{\sqrt{\mathbf{z}^T\Omega\mathbf{z}}} = \frac{\mathbf{m}^T\Sigma^{-1}\bar{\mathbf{r}}}{\sqrt{(\Sigma^{-1}\bar{\mathbf{r}})^T\Omega\Sigma^{-1}\bar{\mathbf{r}}}}, \tag{3.2.39}$$

where $\mathbf{z}$ is given as in equation (3.2.28).

The RP strategy outperforms the mean-variance strategy if and only if the following holds:

$$\mathbf{m}^T\left(\frac{\boldsymbol{\sigma}^{-1}}{\sqrt{(\boldsymbol{\sigma}^{-1})^T\Omega\boldsymbol{\sigma}^{-1}}} - \frac{\Sigma^{-1}\bar{\mathbf{r}}}{\sqrt{(\Sigma^{-1}\bar{\mathbf{r}})^T\Omega\Sigma^{-1}\bar{\mathbf{r}}}}\right) > 0. \tag{3.2.40}$$

In other words, risk parity portfolio outperforms the tangency portfolio if and only if its Sharpe ratio is superior. But, from the theory of the mean-variance optimization, the return of the portfolio is not determined through future covariance estimations. Using the solution of the two strategies, RP outperforms tangency portfolio if and only if the following hold:

$$\mathbf{m}^T\left(\frac{\boldsymbol{\sigma}^{-1}}{\sqrt{\mathbb{1}^T\boldsymbol{\sigma}^{-1}}} - \frac{\Sigma^{-1}\bar{\mathbf{r}}}{\sqrt{\mathbb{1}^T\Sigma^{-1}\bar{\mathbf{r}}}}\right) > 0. \tag{3.2.41}$$

[1]Determined in the absent of risk-free asset.

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Interpreting this solution, the RP outperforms the tangency portfolio if the linear combination of RP solution (i.e., a vector of component weights) and the vector of component expected returns is greater than that of the tangency portfolio. If they are equal, then the two portfolios are the same. Otherwise, risk parity has underperformed.

However, when the future covariance estimation is similar to the historical covariance, and components have different Sharpe ratios, then the condition for RP outperforming tangency portfolio changes. Using properties (3.1.8), (3.1.9) and (3.1.10) and setting:

$$\mathbf{s} = \{\mathbf{s}_i = \frac{\bar{r}_i}{\sigma_i}\}, \tilde{\mathbf{s}} = \{\tilde{\mathbf{s}}_i = \frac{m_i}{\sigma_i}\} \quad \text{and} \quad \mathbf{v} = \mathbf{z} \bullet \boldsymbol{\sigma} = \{z_i \sigma_i\} \quad i = 1, \ldots, n,$$

the inequality (3.2.40) can be written as:

$$\tilde{\mathbf{s}}\boldsymbol{\sigma}^T \left(\frac{\boldsymbol{\sigma}^{-1}}{\sqrt{\mathbb{1}^T C \mathbb{1}}} - \frac{\frac{\mathbf{v}}{\boldsymbol{\sigma}}}{\sqrt{\left(\frac{\mathbf{v}}{\boldsymbol{\sigma}}\right)^T D_{\boldsymbol{\sigma}} C D_{\boldsymbol{\sigma}} \left(\frac{\mathbf{v}}{\boldsymbol{\sigma}}\right)}} \right) = \tilde{\mathbf{s}}\boldsymbol{\sigma}^T \left(\frac{\boldsymbol{\sigma}^{-1}}{\sqrt{\mathbb{1}^T C \mathbb{1}}} - \frac{\frac{\mathbf{v}}{\boldsymbol{\sigma}}}{\sqrt{(\mathbf{v}^T C \mathbf{v})}} \right) > 0.$$

Distributing the multiplication of $\boldsymbol{\sigma}^T$, we have:

$$\tilde{\mathbf{s}} \left(\frac{\mathbb{1}}{\sqrt{\mathbb{1}^T C \mathbb{1}}} - \frac{\mathbf{v}}{\sqrt{(\mathbf{v}^T C \mathbf{v})}} \right) > 0. \tag{3.2.42}$$

But, the solution of the tangency portfolio is given by:

$$\frac{\Sigma^{-1}\bar{\mathbf{r}}}{\sqrt{\bar{\mathbf{r}}^T \Sigma^{-1} \bar{\mathbf{r}}}} = \frac{D_{\boldsymbol{\sigma}}^{-1} C^{-1} D_{\boldsymbol{\sigma}}^{-1} (\mathbf{s}\boldsymbol{\sigma})}{\sqrt{(\mathbf{s}\boldsymbol{\sigma})^T D_{\boldsymbol{\sigma}^{-1}} C^{-1} D_{\boldsymbol{\sigma}^{-1}} (\mathbf{s}\boldsymbol{\sigma})}} = \frac{D_{\boldsymbol{\sigma}^{-1}} C^{-1} \mathbf{s}}{\sqrt{\mathbf{s}^T C^{-1} \mathbf{s}}}. \tag{3.2.43}$$

Hence, the risk parity portfolio outperform the tangency portfolio if and only if the following is satisfied:

$$\mathbf{s} \left(\frac{\mathbb{1}}{\sqrt{\mathbb{1}^T C \mathbb{1}}} - \frac{D_{\boldsymbol{\sigma}^{-1}} C^{-1} \mathbf{s}}{\sqrt{\mathbf{s}^T C^{-1} \mathbf{s}}} \right) > 0. \tag{3.2.44}$$

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Chapter 4

Risk Budgeting Approach

In this Chapter we present the risk-budgeting (RB) strategy of Bruder and Roncalli (2012) used to construct a portfolio of multi-asset classes. Unlike other risk parity approaches discussed in Chapter 2, the RB approach integrates investor's views about component risk budgets in a universe. Investors, in this case, use the information of components in the markets and specify the level of risk contribution they are willing to take. More specifically, the risk budgeting portfolio, $\mathbf{z} = (z_1, z_2, \ldots, z_n)^T$, is such that each component risk contribution matches the corresponding risk budget, $\mathbf{b} = (b_1, b_2, \ldots, b_n)^T$. Contrary to the ERC strategy, the risk contributions of a two assets portfolio is not necessarily equal.

Definition 4.1. *Risk budgeting is a process in which the risk of a portfolio is measured, divided into some component risk contributions using the theory of asset allocation, each component is assigned risk budget, and the sum of component risk budgets adds up to one, see Pearson (2011).*

The decomposition of risk helps investors to conceptualize and also to be decisive[1] on risk allocation whenever portfolio is to be established. Thus, the risk budget portfolio consists of quantitative constraint of asset classes, manager's views about the risk budgets, factors influencing the performance of assets, maintenance of risk budget constraint through leverage, continuous monitoring of risk budgeting for sources of risk and allocation of assets based on the defined risk budget.

4.1 Specification of Risk Budgeting Portfolio

In Chapter 2, we defined the risk contribution of assets in a portfolio. Bruder and Roncalli (2012) took the definition of risk contribution further by intro-

[1] Investors are able to filter assets based on their risk contribution. Thus the combination of risk decomposition and set of risk sources generate the subject of risk budgeting.

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ducing the constraint of each asset risk contribution called risk budget. For a portfolio of n-assets, it is very convenient to define the risk budgeting portfolio as a solution to the following system:

$$\begin{aligned} \mathcal{RC}_1(z_1, z_2, \ldots, z_n) &= b_1 \\ \mathcal{RC}_2(z_1, z_2, \ldots, z_n) &= b_2 \\ &\vdots \\ \mathcal{RC}_i(z_1, z_2, \ldots, z_n) &= b_i \\ &\vdots \\ \mathcal{RC}_n(z_1, z_2, \ldots, z_n) &= b_n. \end{aligned}$$

This system is not based on any optimization technique and also does not integrate the expected return estimation in the portfolio's composition.

But the system above has few drawbacks. First, it does not explicitly reflect component exposures and small risk budget. Secondly, specifying that other assets have negative risk budget implies that the risk is concentrated in the other assets which does not reflect diversification objective. Thus, the above system could be expressed precisely as a non-linear system, i.e.,

$$\begin{cases} z_i(\Sigma \mathbf{z})_i = b_i(\mathbf{z}^T \Sigma \mathbf{z}) \\ \sum_{i=1}^{n} z_i = 1 \\ \sum_{i=1}^{n} b_i = 1 \\ z_i \geq 0 \\ b_i \geq 0, \end{cases} \tag{4.1.1}$$

to explicitly reflect the marginal risk contribution for each asset.

The only quandary that arises from system (4.1.1) is when an investor specifies zero risk budget for a specific asset. Bruder and Roncalli (2012) showed that there exist two solutions in this case. We denote by $(\Sigma \mathbf{z})_i$, the i^{th} row of the product of the covariance matrix and vector of exposure. Recall that $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$ denotes the covariance between the return of the i^{th} and j^{th} assets where ρ_{ij} is the corresponding correlation. The marginal risk contribution of the i^{th} asset is then given by:

$$\mathcal{MC}_i(\mathbf{z}) = \frac{\partial \sigma(\mathbf{z})}{\partial z_i} = \frac{z_i\sigma_i^2 + \sigma_i \Sigma_{j \neq i} z_j \sigma_j \rho_{ij}}{\sigma(\mathbf{z})}. \tag{4.1.2}$$

Thus, the absolute risk contribution of the k^{th}-asset assigned zero risk budget in a portfolio of n-assets is given as:

$$\mathcal{RC}_k(\mathbf{z}) = z_k \mathcal{MC}_k(\mathbf{z}) = z_k \frac{z_k\sigma_k^2 + \sigma_k \Sigma_{j \neq k} z_j \sigma_j \rho_{kj}}{\sigma(\mathbf{z})} = 0. \tag{4.1.3}$$

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Solving the above equation, we obtain two solutions, i.e.

$$z_k^* = 0 \quad \text{or} \quad z_k^{**} = -\frac{\sigma_k \Sigma_{j\neq k} z_j \sigma_j \rho_{kj}}{\sigma_k^2} \tag{4.1.4}$$

and z^{**} can only be positive if $\rho_{kj} < 0$, since $\sigma_i > 0$ by definition. However, setting asset risk budget to zero, the investor has an intuition that this asset will not form part of the portfolio. Moreover, this makes the solution to system (4.1.1) to be difficult to obtain. In general, the positive solution of this problem depends on the structure of the covariance (or information) matrix. Bruder and Roncalli (2012) suggest that instead of setting component risk budgets to zero, one removes the corresponding assets and determine the composition with only non-zero risk budget components. In this case, the non-linear system (4.1.1) is written as follows:

$$\mathbf{z}^{RB} = \{\mathbf{z} \in [0,1]^n : \sum_{i=1}^{n} z_i = 1, \mathcal{RC}_i(\mathbf{z}) = b_i \sigma(\mathbf{z})\}, \tag{4.1.5}$$

where $b_i \in (0,1]^n$ for $i = 1, \ldots, n$ and their sum is one.

4.1.1 Some Analytical Solutions of the RB Portfolio

In determining the analytical solution of the RB portfolio in a general case, Bruder and Roncalli (2012) observed that it is impossible and only few special cases can be obtained. In what follows we illustrate these special cases.

4.1.1.1 RB in a Two-Assets Universe

In a two assets case, Roncalli (2013) found that the risk budgeting solution of system (4.1.1) is a complex function characterised by component volatilities, correlation and risk budget. The solution in this case satisfy the following system:

$$\begin{pmatrix} \mathcal{RC}_1 \\ \mathcal{RC}_2 \end{pmatrix} = \begin{pmatrix} z^2\sigma_1^2 + z(1-z)\rho\sigma_1\sigma_2 \\ (1-z)^2\sigma_2^2 + z(1-z)\rho\sigma_1\sigma_2 \end{pmatrix} = \begin{pmatrix} b\sigma(\mathbf{z}) \\ (1-b)\sigma(\mathbf{z}) \end{pmatrix}, \tag{4.1.6}$$

where z and $(1-z)$ are the weights of the respective assets, b and $(1-b)$ their corresponding risk budgets, and σ_1 and σ_2 represent their volatilities while $\sigma(\mathbf{z})$ is the total portfolio volatility.

To solve system (4.1.6), we first express it as a function in which quadratic formula will be efficient, i.e

$$\frac{z^2\sigma_1^2 + z(1-z)\rho\sigma_1\sigma_2}{b} = \frac{(1-z)^2\sigma_2^2 + z(1-z)\rho\sigma_1\sigma_2}{1-b}, \tag{4.1.7}$$

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which can simply be written as,

$$\begin{aligned}&z^2\sigma_1^2 + z\rho\sigma_1\sigma_2 - z^2\rho\sigma_1\sigma_2 - z^2\sigma_1^2 b - z\rho\sigma_1\sigma_2 b\\ &+ z^2\rho\sigma_1\sigma_2 b - \sigma_2^2 b + 2b\sigma_2^2 z - b\sigma_2^2 z^2 - b\rho\sigma_1\sigma_2 z + b\rho\sigma_1\sigma_2 z = 0.\end{aligned}$$

Rearranging terms in order to apply the quadratic formula, we have

$$\begin{aligned}z^2 &: (1-b)\sigma_1^2 + (2b-1)\rho\sigma_1\sigma_2 - b\sigma_2^2\\ z &: (1-2b)\rho\sigma_1\sigma_2 + 2b\sigma_2^2\\ C &: -b\sigma_2^2.\end{aligned}$$

We determine separately the square root part of the formula as follows. First we make substitution for $B^2 - 4AC$, i.e.,

$$B^2 - 4AC = 4(b-\tfrac{1}{2})^2\rho^2\sigma_1^2\sigma_2^2 + 4b(1-b)\sigma_1^2\sigma_2^2.$$

Substituting everything in the quadratic formula yield the following,

$$z = \frac{-\left[(1-2b)\rho\sigma_1\sigma_2 + 2b\sigma_2^2\right] + \sqrt{4(b-\frac{1}{2})^2\rho^2\sigma_1^2\sigma_2^2 + 4b(1-b)\sigma_1^2\sigma_2^2}}{2\left[(1-b)\sigma_1^2 + (2b-1)\rho\sigma_1\sigma_2 - b\sigma_2^2\right]}.$$

Rearranging and taking out the common factor of $4\sigma_1^2\sigma_2^2$ from the square root, yields,

$$z = \frac{2(b-\frac{1}{2})\rho\sigma_1\sigma_2 - 2b\sigma_2^2 + 2\sigma_1\sigma_2\sqrt{(b-\frac{1}{2})^2\rho^2 + b(1-b)}}{2\left[(1-b)\sigma_1^2 + 2(b-\frac{1}{2})\rho\sigma_1\sigma_2 - b\sigma_2^2\right]}, \qquad (4.1.8)$$

and taking out the common factor of two, gives the following,

$$z = \frac{(b-\frac{1}{2})\rho\sigma_1\sigma_2 - b\sigma_2^2 + \sigma_1\sigma_2\sqrt{(b-\frac{1}{2})^2\rho^2 + b(1-b)}}{(1-b)\sigma_1^2 + (b-\frac{1}{2})\rho\sigma_1\sigma_2 - b\sigma_2^2}, \qquad (4.1.9)$$

which confirms the complexity of the risk budgeting solution. Thus, determining the general solution will get even worse than the two-assets case because of the increase in parameters such as n-volatilities and budgets, $\frac{n(n-1)}{2}$ pair-wise correlations. For some constant correlations matrix, say $\rho \in \{-1, 0, 1\}$, the solution, $z_{(\rho)}$, to equation (4.1.9) is

$$z_{(-1)} = \frac{\sigma_2}{\sigma_1 + \sigma_2}, \qquad (4.1.10)$$

$$z_{(0)} = \frac{\sqrt{b}\sigma_2}{\sqrt{1-b}\sigma_1 + \sqrt{b}\sigma_2}, \qquad (4.1.11)$$

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$$z_{(1)} = \frac{b\sigma_2}{\sigma_1(1-b) - b\sigma_2}, \tag{4.1.12}$$

Note that we only considered the positive sign in the preceding square root because in the case of negative, we often find negative solutions which violates the weight constrains. See Appendix (A.2.1) for some of the proofs. More precisely, in the long-only portfolio, the solution to this problem is a set of all positive weights.

4.1.1.2 RB in the General Universe (i.e Case $n > 2$)

The two-assets case has already shown complexity of how the solution appears which suggests that for the general case, the solution might be even more difficult. Again, Roncalli (2013) determine the RB solution for special cases of uniform correlations, $\rho_{i,j} = \rho$, amongst the components in a universe. Below we detail how to obtain such analytical solution.

For the case $\rho = 0$, which simply says that there is no correlation amongst the n-assets of the portfolio, the solution to system (4.1.1) can be obtained as follows. Considering the marginal contribution as defined in equation (4.1.2). The risk contribution of the i^{th} asset is then given by:

$$\mathcal{RC}_i(\mathbf{z}) = z_i \frac{z_i\sigma_i^2 + \sigma_i \Sigma_{j\neq i} z_j \rho_{ij}\sigma_j}{\sigma(\mathbf{z})} = \frac{z_i^2\sigma_i^2}{\sigma(\mathbf{z})}. \tag{4.1.13}$$

But from equation (4.1.1), we have

$$b_i(\mathbf{z}^T\Sigma\mathbf{z}) = z_i(\Sigma\mathbf{z})_i = z_i^2\sigma_i^2 = \mathcal{RC}_i(\mathbf{z})\sigma(\mathbf{z}). \tag{4.1.14}$$

We know that

$$\sigma^2(\mathbf{z}) = \mathbf{z}^T\Sigma\mathbf{z}, \tag{4.1.15}$$

hence we can deduce that:

$$b_i(\mathbf{z}^T\Sigma\mathbf{z}) = \mathcal{RC}_i(\mathbf{z})\sigma(\mathbf{z}), \tag{4.1.16}$$

which implies that,

$$\mathcal{RC}_i(\mathbf{z}) = b_i\sigma(\mathbf{z}). \tag{4.1.17}$$

Hence

$$b_i\sigma_p^2(\mathbf{z}) = z_i^2\sigma_i^2, \tag{4.1.18}$$

which yields the following for both the i^{th} and j^{th} assets

$$\sqrt{b_i}\sigma(\mathbf{z}) = z_i\sigma_i, \qquad \sqrt{b_j}\sigma(\mathbf{z}) = z_j\sigma_i. \tag{4.1.19}$$

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Thus, the solution of the i^{th} component is given by:

$$z_i = \frac{\sigma_i^{-1}\sqrt{b_i}}{\sum_{j=1}^{n}\sigma_j^{-1}\sqrt{b_j}}, \tag{4.1.20}$$

since the weights add up to one. Equation (4.1.20) illustrates that component exposure is directly proportional to the product of the square root of their risk budget and the inverse volatility when pair-wise correlations is zero. This implies that, if a component, say z_i, increases (decreases) in volatility, then there should be a decrease (increase) of weight.

In the same approach, we can show that for the case $\rho = 1$, we have

$$b_j z_i \sigma_i = b_i z_j \sigma_j, \tag{4.1.21}$$

and we can deduce that

$$z_i = \frac{\sigma_i^{-1} b_i}{\sum_{j=1}^{n}\sigma_j^{-1} b_j}. \tag{4.1.22}$$

Also for $\rho = -\dfrac{1}{n-1}$ which indicates perfect negative correlation, we have

$$z_i = \frac{\sigma_i^{-1}}{\sum_{j=1}^{n}\sigma_j^{-1}}, \tag{4.1.23}$$

which simply exhibits that component exposures are inversely proportional to the volatility of the returns, i.e.,

$$z_i \propto \sigma_i^{-1}. \tag{4.1.24}$$

See Appendix (A.2) for the proofs. This result implies that the RB portfolio is IV efficient if the constant correlation reaches its lower bound. Moreover, Bruder and Roncalli (2012) showed that the variance of the portfolio in this case is zero. Recall the definition of variance as in equation (4.1.15). This implies that,

$$\begin{aligned}\sigma^2(\mathbf{z}) &= z_1^2\sigma_1^2 + z_1\sigma_1\sum_{j\neq 1}\rho\sigma_j z_j + z_2^2\sigma_2^2 + z_2\sigma_2\sum_{j\neq 2}\rho\sigma_j z_j + \cdots \\ &\quad + z_i^2\sigma_i^2 + z_i\sigma_i\sum_{j\neq i}\rho\sigma_j z_j + \cdots + z_n^2\sigma_n^2 + z_n\sigma_n\sum_{j\neq n}\rho\sigma_j z_j,\end{aligned}$$

which can be written as

$$\begin{aligned}\sigma^2(\mathbf{z}) &= \sum_{i=1}^{n}\sigma_i^2 z_i^2 + \sum_{i=1}^{n}\left(\sigma_i z_i \sum_{j\neq i}\rho\sigma_j z_j\right) \\ &= \sum_{i=1}^{n}\sigma_i^2 z_i^2 + \sum_{i=1}^{n}\left(\rho\sigma_i z_i\left(\sum_{j=1}^{n}\sigma_j z_j - \sigma_i z_i\right)\right).\end{aligned}$$

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Rearranging, we have

$$\sigma^2(\mathbf{z}) = (1-\rho)\sum_{i=1}^{n}\sigma_i^2 z_i^2 + \rho\sum_{i=1}^{n}\sigma_i z_i\left(\sum_{j=1}^{n}\sigma_j z_j\right). \tag{4.1.25}$$

Setting $\sigma_i z_i = \sigma_j z_j = \varpi$ and substituting for $\rho = -\frac{1}{n-1}$, the above equation (4.1.25) becomes,

$$\begin{aligned}\sigma^2(\mathbf{z}) &= (1-(-\tfrac{1}{n-1}))\sum_{i=1}^{n}\varpi^2 + \rho\sum_{i=1}^{n}\varpi(\sum_{j=1}^{n}\varpi)\\ &= \tfrac{n^2}{n-1}\varpi^2 - \tfrac{n^2}{n-1}\varpi^2 = 0.\end{aligned} \tag{4.1.26}$$

The quandary of finding the general solution arises when ρ is not a constant matrix. However, Bruder and Roncalli (2012) deduced the financial interpretation of the general solution to system (4.1.5) by first considering the fact that beta is given by,

$$\beta_i = \frac{(\Sigma\mathbf{z})_i}{\sigma^2(\mathbf{z})}, \tag{4.1.27}$$

see (2.2.18) and (2.2.9). This implies that the i^{th}-row of the covariance matrix is given as follows,

$$(\Sigma\mathbf{z})_i = \beta_i\sigma^2(\mathbf{z}), \tag{4.1.28}$$

and the risk contribution is

$$\mathcal{RC}_i = b_i\beta_i z_i. \tag{4.1.29}$$

The risk budgeting portfolio is such that,

$$\frac{\beta_i z_i\sigma^2(\mathbf{z})}{b_i} = \frac{\beta_j z_j\sigma^2(\mathbf{z})}{b_j}.$$

We can now infer that

$$z_i = \frac{b_i\beta_i^{-1}}{\sum_{j=1} b_j\beta_j^{-1}} \quad i = 1,\ldots,n. \tag{4.1.30}$$

Equation (4.1.30) indicates that the exposure of component i is directly proportional to the product of its inverse beta and the risk budget. It is similar to the solution given in equation (2.5.14), except that in this case, the risk budgets are incorporated. However, it does not provide a close-form solution because of endogeneity, see Maillard *et al.* (2010), and thus finding the solution to the RB portfolio requires the use of numerical solution.

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4.2 Optimization of Risk Budget Portfolio

In this section, we present the optimization techniques for finding the risk budgeting portfolio. Bruder and Roncalli (2012) suggested two approaches in determining the optimal RB portfolio. These are the optimization program and the search algorithms.

4.2.1 RB using Optimization Program

In order to determine the optimal solution of system (4.1.5), we follow Bruder and Roncalli (2012) approach in which the optimization program for the RB portfolio is defined as follows,

$$\mathbf{z}^* = \arg\min_{\mathbf{z}\in\mathbb{R}^n} \sqrt{\mathbf{z}^T\Sigma\mathbf{z}}$$
$$s.t \quad \begin{cases} \sum_{i=1}^{n} b_i \ln z_i \geq c \\ \mathbf{z} \geq \mathbf{0}, \end{cases} \tag{4.2.1}$$

where the arbitrary constant, c, is scaled to satisfy the budget constrain. In other words, the program minimizes a convex function (i.e portfolio's volatility) subject to convex constrain. The Lagrange function for this optimization problem is,

$$\mathcal{L}(\mathbf{z},\lambda,\lambda_c) = (\mathbf{z}\Sigma\mathbf{z})^{\frac{1}{2}} - \lambda_c(\mathbf{b}^T \ln \mathbf{z} - c) - \lambda^T\mathbf{z} \tag{4.2.2}$$

where $\lambda \in \mathbb{R}^n$ and $\lambda_c \in \mathbb{R}$ are the Lagrange multipliers. The first-order conditions satisfy the following,

$$\frac{\partial\mathcal{L}(\mathbf{z},\lambda,\lambda_c)}{\partial\mathbf{z}} = \frac{\partial\sigma(\mathbf{z})}{\partial\mathbf{z}} - \lambda\mathbb{1} - \lambda_c\left(\frac{\mathbf{b}}{\mathbf{z}}\right) = 0, \tag{4.2.3}$$

and the Karush Kuhn-Tucker conditions are,

$$\min(\lambda,\mathbf{z}) = 0 \tag{4.2.4}$$
$$\min\Big(\lambda_c, \mathbf{b}^T\ln\mathbf{z} - c\Big) = 0. \tag{4.2.5}$$

Note that the first-order derivative is taken with respect to all components in a universe, i.e

$$\frac{\partial\mathcal{L}(\mathbf{z},\lambda,\lambda_c)}{\partial\mathbf{z}} = \frac{\partial\mathcal{L}(\mathbf{z},\lambda,\lambda_c)}{\partial\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}} \quad \text{and} \quad \left(\frac{\mathbf{b}}{\mathbf{z}}\right) = \begin{pmatrix} \frac{b_1}{z_1} \\ \frac{b_2}{z_2} \\ \vdots \\ \frac{b_n}{z_n} \end{pmatrix}. \tag{4.2.6}$$

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Moreover, we notice that $\mathbf{z}$ should be strictly positive as $\ln(z)$ is undefined for $\mathbf{z} = 0$ and therefore condition (4.2.4) will be satisfied for $\lambda = 0$. In particular, the solution to the i^{th} component is,

$$\frac{\partial\sigma(\mathbf{z})}{\partial z_i} - \lambda_c \frac{b_i}{z_i} = 0, \tag{4.2.7}$$

which implies the following,

$$z_i \frac{\partial\sigma(\mathbf{z})}{\partial z_i} = \lambda_c b_i. \tag{4.2.8}$$

This solution indicates that the risk contribution of a component is directly proportional to the respective risk budget. The scaling factor λ_c is interpreted as the adjustment parameter for constant c to obtain the solution $\mathbf{z}^*$.

Thus, the optimal solution to the RB portfolio is obtained by normalizing the solution $\mathbf{z}^*$. In particular, the i^{th}-component weight is given by,

$$z_i^{RB} = \frac{z_i^*}{\sum_{j=1}^n z_j^*} \quad i = 1, \dots, n. \tag{4.2.9}$$

In this case, the optimal RB portfolio exists and is unique. However, this result is only valid in the case of system (4.1.5).

In the case where some risk budgets are set to zero, the investment universe can be divided into two, i.e. the set where component risk budgets are zeros, denoted by $\mathcal{N}$, and the set of non-zero risk budgets. The Lagrange function of the optimization problem (4.2.1) is,

$$\mathcal{L}(\mathbf{z}, \lambda, \lambda_c) = (\mathbf{z}\Sigma\mathbf{z})^{\frac{1}{2}} - \lambda_c(\sum_{i \notin \mathcal{N}} b_i \ln z_i - c) - \lambda^T \mathbf{z} \tag{4.2.10}$$

and the respective first-order condition is,

$$\mathcal{L}(\lambda, \lambda_c, \mathbf{z}) = \begin{cases} \dfrac{\partial\sigma(\mathbf{z})}{\partial \mathbf{z}} - \lambda\mathbb{1} - \lambda_c \dfrac{\mathbf{b}}{\mathbf{z}} = 0 & \text{if } i \notin \mathcal{N} \\ \dfrac{\partial\sigma(\mathbf{z})}{\partial \mathbf{z}} - \lambda\mathbb{1} = 0 & \text{if } i \in \mathcal{N}. \end{cases}$$

Thus, if $i \notin \mathcal{N}$, we still have the solution as described above. However, for the case $i \in \mathcal{N}$, we have two cases to consider. First, if $z_i = 0$, it implies that $\lambda_i > 0$ and $\frac{\partial\sigma(\mathbf{z})}{\partial \mathbf{z}} > 0$. Secondly, if $z_i > 0$, it implies that $\lambda_i = 0$ and $\frac{\partial\sigma(\mathbf{z})}{\partial \mathbf{z}} = 0$. As noted in the specification of the risk budgeting in section (4.1), the positivity of the risk budgeting solution when some risk budgets are set to zero depends on the structure of the covariance (or information) matrix.

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4.2.2 Algorithmic Approach

The first approach minimizes the sum of square of the difference between the risk contributions of components. The system to be solved in this case is,

$$\mathbf{z}^{RB} = \arg\min_{\mathbf{z}\in\mathbb{R}^n} f(\mathbf{z}, b)$$
$$s.t \quad \begin{cases} \mathbb{1}^T\mathbf{z} = 1 \\ \mathbf{0} \leq \mathbf{z} \leq \mathbb{1}, \end{cases} \tag{4.2.11}$$

where the function,

$$f(\mathbf{z}, b) = \sum_{i=1}^{n} \left(\frac{z_i(\Sigma\mathbf{z})_i}{\sum_{j=1}^{n} z_j(\Sigma\mathbf{z})_j} - b_i \right)^2 .$$

Note that this function, $f(\mathbf{z}, b)$, is deduced from system (4.1.1), where the specification for the i^{th} and j^{th} components, respectively, is

$$z_i(\Sigma\mathbf{z})_i = b_i\sigma(\mathbf{z}), \quad \text{and} \quad z_j(\Sigma\mathbf{z})_j = b_j\sigma(\mathbf{z}). \tag{4.2.12}$$

Equating the two component specifications, and taking the j^{th}-sum both sides gives the following,

$$\frac{z_i(\Sigma\mathbf{z})_i}{\sum_{j=1}^{n} z_j(\Sigma\mathbf{z})_j} = b_i.$$

Rearranging, and squaring both side yield:

$$\left(\frac{z_i(\Sigma\mathbf{z})_i}{\sum_{j=1}^{n} z_j(\Sigma\mathbf{z})_j} - b_i \right)^2 = 0.$$

Now taking the sum with respect to the i^{th} elements, gives

$$\sum_{i=1}^{n} \left(\frac{z_i(\Sigma\mathbf{z})_i}{\sum_{j=1}^{n} z_j(\Sigma\mathbf{z})_j} - b_i \right)^2 = 0.$$

Thus, the function to be optimized under the risk budgeting constraints becomes:

$$f(\mathbf{z}, b) = \sum_{i=1}^{n} \left(\frac{z_i(\Sigma\mathbf{z})_i}{\sum_{j=1}^{n} z_j(\Sigma\mathbf{z})_j} - b_i \right)^2 .$$

Alternatively, Roncalli (2013) considered the following function

$$f(\mathbf{z}, b) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{z_i(\Sigma\mathbf{z})_i}{b_i} - \frac{z_j(\Sigma\mathbf{z})_j}{b_j} \right)^2$$

as the appropriate function for defining the optimization problem of the risk budgeting portfolio. The solution of the risk budgeting portfolio in this case may be found by using the sequential quadratic programming (SQP) algorithm, see Roncalli (2013).

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4.2.2.1 Jacobi Power Method

Despite the endogenous of solution (4.1.30) for system (4.1.5), Roncalli (2013) suggests the implementation of Jacobi algorithm for finding the optimum solution of the risk budgeting portfolio. The algorithm iterates the following function,

$$z^{k+1} = \frac{b_i/\beta_i^k}{\sum_{j=1}^{n} b_j/\beta_j^k}, \tag{4.2.13}$$

until a certain error tolerance is met. In the case of Chaves *et al.* (2012), the algorithm terminates if the following condition is satisfied,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{z_i(\Sigma\mathbf{z})_i}{b_i} - \frac{z_j(\Sigma\mathbf{z})_j}{b_j}\right)^2 < \varepsilon, \tag{4.2.14}$$

where ε is a tolerance level. According to Roncalli (2013), this algorithm converges well for a small universe, and also when the starting point is well chosen[2]. However, for large universe, it often fails.

4.2.2.2 Cyclical Coordinate Descent (CCD) Algorithm

Griveau-Billion *et al.* (2013) present a recent algorithm for the computation of the RB portfolio. This algorithm design is based on the capabilities of handling an enormous amount of data and the speed of getting the results. The minimized function is required to be convex and differentiable. Unlike the classical descent algorithm, the CCD algorithm searches the value for each component which minimizes the objective function. The great incentives about this algorithm is its simplicity and easy to implement.

To illustrate the insight of this algorithm, we follow Griveau-Billion *et al.* (2013) approach which resume from the solution given in equation (4.2.7) of the risk budgeting system (4.2.1). If we assume without loss of generality that $\lambda_c = 1$, solution (4.2.7) then is expressed as,

$$\frac{\partial\mathcal{L}(\mathbf{z},\lambda)}{\partial z_i} = \frac{(\Sigma\mathbf{z})_i}{\sigma_(\mathbf{z})} - \frac{b_i}{z_i} = 0. \tag{4.2.15}$$

This implies that,

$$z_i(\Sigma\mathbf{z})_i - b_i\sigma(\mathbf{z}) = 0. \tag{4.2.16}$$

Expanding the left hand side, we have

$$z_i^2\sigma_i^2 + z_i\sigma_i\sum_{j\neq i} z_j\sigma_j\rho_{i,j} - b_j\sigma(\mathbf{z}) = 0. \tag{4.2.17}$$

[2] The required starting point for facilitating the convergence of this algorithm is any of the solutions obtained with the constant correlation matrix.

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Thus, the solution to this problem can be solved using the quadratic formula, i.e

$$z_i^{RB} = \frac{-\sigma_i \sum_{j\neq i} z_j\sigma_j\rho_{i,j} + \sqrt{(\sigma_i \sum_{j\neq i} z_j\sigma_j\rho_{i,j})^2 + 4b_i\sigma_i^2\sigma(\mathbf{z})}}{2\sigma_i^2}. \tag{4.2.18}$$

Since the solution to this problem is restricted to only positive weights, then only positive roots given by the above equation (4.2.18) will be considered. This approach will be iterated amongst all components in a given universe. However, to ease the input parameter at each iteration stage, the above equation is expressed as follows,

$$z_i^{RB} = \frac{-(\Sigma\mathbf{z})_i + z_i\sigma_i^2 + \sqrt{((\Sigma\mathbf{z})_i + z_i\sigma_i^2)^2 + 4\sigma_i^2 b_i\sigma(\mathbf{z})}}{2\sigma_i^2}, \tag{4.2.19}$$

because the covariance matrix and the volatility of the portfolio can be easily obtained. Since the algorithm determines the values of components that minimizes the objective function, to keep track of the updated components, we define two portfolios,

$$\mathbf{z} = (z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_n) \tag{4.2.20}$$

$$\text{and,} \quad \tilde{\mathbf{z}} = (z_1, \ldots, z_{i-1}, z_i^{RB}, z_{i+1}, \ldots, z_n), \tag{4.2.21}$$

to reflect the portfolio before and after the update of the i^{th}-component. The updates of the vector, $\Sigma\mathbf{z}$, and the portfolio's volatility, $\sigma(\mathbf{z})$, respectively are given by,

$$\Sigma(\tilde{\mathbf{z}}) = \Sigma\mathbf{z} - \Sigma_{.,i}z_i + \Sigma_{.,i}\tilde{z}_i, \tag{4.2.22}$$

and

$$\sigma(\tilde{\mathbf{z}}) = \sqrt{\sigma^2(\mathbf{z}) - 2z_i\Sigma_{i,.}z_i + z_i^2\sigma_i^2 + 2\tilde{z}_i\Sigma_{i,.}\tilde{z}_i - \tilde{z}^2\sigma_i^2}, \tag{4.2.23}$$

where $\Sigma_{i,.}$ and $\Sigma_{.,i}$ denote the i^{th} row and column of the covariance matrix.

4.3 Analytical Comparison of the GMV, EW and RB Portfolios

Bruder and Roncalli (2012) adopted the approach of Maillard *et al.* (2010) in the ranking of risk for the above mentioned strategies. However, the latter integrate the risk budget in the analysis. In this case, the EW strategy is such that each component weight is equal to the risk budget, i.e.,

$$z_i^{EW} = b_i \quad i = 1, \ldots, n, \tag{4.3.1}$$

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which implies that the EW is as follows,

$$\frac{z_i^{EW}}{b_i} = \frac{z_j^{EW}}{b_j}. \tag{4.3.2}$$

Also, we recall that the GMV portfolio is such that any pair of the component marginal-risk contributions in a universe is identical, i.e.,

$$\frac{\partial\sigma(\mathbf{z})}{\partial z_i} = \frac{\partial\sigma(\mathbf{z})}{\partial z_j}. \tag{4.3.3}$$

According to Roncalli (2013), the volatility of the risk budgeting portfolio lies between the volatility of the GMV and the EW portfolio and this can be shown analytically by considering the optimization problem (4.2.1) with an additional budget constrain. In this case, the Lagrange function becomes,

$$\mathcal{L}(\lambda, \lambda_0, \lambda_c, \mathbf{z}) = \sigma(\mathbf{z}) - \lambda_0(\mathbb{1}^T\mathbf{z} - 1) - \lambda^T\mathbf{z} - \lambda_c\left(\sum_{i=1}^{n} b_i \ln z_i - c\right), \tag{4.3.4}$$

where the corresponding first-order condition satisfies,

$$\mathbf{z}\frac{\partial\sigma(\mathbf{z})}{\partial\mathbf{z}} = \lambda_0^T\mathbf{z} + \lambda_c^T\mathbf{b}, \tag{4.3.5}$$

because $\lambda_0 \in \mathbb{R}$ is set to zero in order to satisfy the RB portfolio. In particular, for the i^{th}-component, we have

$$z_i\frac{\partial\sigma(\mathbf{z})}{\partial z_i} = \lambda_0 z_i + \lambda_c b_i. \tag{4.3.6}$$

Thus, the solution to this problem is a function of c, and if we consider $c \in \{c_1, c_2\}$, with $c_1 \le c_2$, then the constrain $\sum_{i=1}^{n} b_i \ln z_i - c \ge 0$ implies that $\sum_{i=1}^{n} b_i \ln z_i - c_2 \le \sum_{i=1}^{n} b_i \ln z_i - c_1$. In other words, the constrain with $c = c_1$ is less expensive than the one with $c = c_2$. Thus, the volatilities are ranked as follows,

$$\sigma(\mathbf{z}^*(c_1)) \le \sigma(\mathbf{z}^*(c_2)). \tag{4.3.7}$$

In particular, if $c = -\infty$, then the result of the optimization problem (4.2.1) correspond to minimum variance portfolio. Also, if $c = \sum_{i=1}^{n} b_i \ln b_i$, the solution is the equal weighted portfolio.

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However, for $c \in (-\infty, \sum_{i}^{n} b_i \ln b_i)$, the solution to the optimization problem (4.2.1) may be called the risk budgeting with c scaled to obtain the desired constrains. Thus, the ranking of portfolio's volatility is as follows,

$$\sigma(\mathbf{z}^*(c_1)) \leq \sigma(\mathbf{z}^*(c)) \leq \sigma(\mathbf{z}^*(c_2)), \tag{4.3.8}$$

which shows that the volatility of the risk budgeting portfolio lies between the volatility of the equal budgeting portfolio and the global minimum variance portfolio.

4.4 Generalized Risk-Based Strategy

In this section, we present a generic risk-based investment strategy proposed by Jurczenko *et al.* (2013) which encapsulate the characteristics of the other risk-based strategies discussed in Chapter 2. This strategy consists of two calibration parameters, γ and δ, and are adjusted to obtain the characteristic of a specific risk-based strategy. The former controls sensitivity to the covariance estimate and the later takes care of the risk tolerance of an investors. Thus, the generalised risk-based strategy is the solution to the following system,

$$\frac{z_i^{\gamma}}{\sigma_i^{\delta}}\mathcal{MC}_i = \frac{z_j^{\gamma}}{\sigma_j^{\delta}}\mathcal{MC}_j = \tau \quad i,j = 1,\ldots,n \tag{4.4.1}$$
$$\text{such that,} \quad \mathbf{z}^T\mathbb{1} = 1.$$

The marginal risk contributions $\mathcal{MC}$ are defined as in equation (2.2.3) and τ is just targeted constant. Table (4.1) details the characteristics of the risk-based strategies for specific values of γ and δ.

Table 4.1: Calibration of (γ, δ) and Characteristics of Risk-Based Strategies

Portfolio	MV	MD	ERC	EW
(γ, δ)	(0.0)	$(0, 1)$	(1.0)	$(\infty, 0)$
Characteristics	$\mathcal{MC}_i(\mathbf{z})=\mathcal{MC}_j(\mathbf{z})$	$\frac{\mathcal{MC}_i(\mathbf{z})}{\sigma_i} = \frac{\mathcal{MC}_j(\mathbf{z})}{\sigma_j}$ $\rho_{i,p} = \rho_{j,p}$	$\mathcal{RC}_i=\mathcal{RC}_j$	$z_i = z_j = \frac{1}{n}$

The term $\frac{z_i^{\gamma}}{\sigma_i^{\delta}}\mathcal{MC}_i$ looks similar to the risk contribution of the i^{th} component except that the weight are now modified and hence called modified risk contribution. The crucial task in this general approach is to determine the existence and uniqueness of solution and interpret it. Since risk-based investors

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are opt to diminish risk of the portfolio, the above system can be re-written to reflect explicitly their objective, i.e

$$\mathbf{z}^* = \arg\min_{\mathbf{z}\in\mathbb{R}^n} \frac{1}{2}\mathbf{z}^T\Sigma\mathbf{z} \tag{4.4.2}$$

such that,

$$\sum_{i=1}^{n} \frac{\sigma_i^\delta (z_i^{1-\gamma} - 1)}{1-\gamma} \geq c\ ,$$

where c is an arbitrary constant. The associate Lagrange function is given as follows:

$$\mathcal{L}(\mathbf{z}, \lambda_c) = \frac{1}{2}\mathbf{z}^T\Sigma\mathbf{z} - \lambda_c \left[\sum_{i=1}^{n}\left(\frac{\sigma_i^\delta (z_i^{1-\gamma} - 1)}{1-\gamma}\right) - c\right], \tag{4.4.3}$$

where $\lambda_c \geq 0$. Now, the first order derivative gives:

$$\frac{\partial \mathcal{L}(\mathbf{z}, \lambda_c)}{\partial \mathbf{z}} = \Sigma\mathbf{z} - \lambda_c \begin{pmatrix} \sigma_1^\delta z_1^{-\gamma} \\ \sigma_2^\delta z_2^{-\gamma} \\ \vdots \\ \sigma_n^\delta z_n^{-\gamma} \end{pmatrix} = 0. \tag{4.4.4}$$

Multiply equation (4.4.4) by inverse portfolio volatility, $\sigma^{-1}(\mathbf{z})$, gives the following:

$$\frac{z_i^\gamma}{\sigma_i^\delta}\mathcal{MC}_i = \frac{\lambda_c}{\sigma(\mathbf{z})}. \tag{4.4.5}$$

This provides the characteristics of the solution to system (4.4.1). The optimal solution, $\mathbf{z}^*$, may not necessarily satisfy the budgetary constrain. Thus the optimal solution of the generalised risk-based (GRB) strategy is normalised as follows,

$$z_i^{GRB} = \frac{z_i^*}{\mathbb{1}^T\mathbf{z}^*}, \quad i = 1\ldots, n. \tag{4.4.6}$$

In this case the solution exists and is unique.

Alternatively, Jurczenko *et al.* (2013) show the existence and uniqueness of the solution by considering the second-order condition of system (4.4.4), i.e.

$$\frac{\partial^2 \mathcal{L}(\mathbf{z}, \lambda_c)}{\partial \mathbf{z}^2} = \Sigma + \lambda_c\gamma \begin{pmatrix} \frac{\sigma_1^\delta}{z_1^{1+\gamma}} \\ \frac{\sigma_2^\delta}{z_2^{1+\gamma}} \\ \vdots \\ \frac{\sigma_n^\delta}{z_n^{1+\gamma}} \end{pmatrix}. \tag{4.4.7}$$

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We notice that for any value of δ and $\gamma \in \{0, 2k+1, +\infty\}$, the solution will always be attained when the covariance is positively definite and $k \in \mathbb{Z}^+$.

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Chapter 5

Alternative Risk Measures and Risk Parity

In this chapter, we present other risk parity approaches called tail risk parity and factor risk parity. The former allocates components based on the downside risk measure while the latter considers the primitive sources of risk underlying asset returns.

5.1 Tail Risk Parity (TRP)

This is an asset allocation strategy that reduces drawdowns and allows to retain a good portion of the upside by giving better diversification of risk and return sources and protects investment against systemic crisis at a price significantly lower than the options market. It is a very transparent strategy, in other words we can implement the strategy with vanilla cash instrument, regular stocks and bonds. Also, it is robust strategy in a sense that it uses non-parametric portfolio construction. Similar to risk parity strategies, TRP is also flexible in a sense that it accommodates a variety of asset classes and risk appetite.

Alankar *et al.* (2012) argue that volatility measures normal dispersion of portfolio returns. It is regarded as a simple measure of risk in terms of computation. However, it does not consider extreme losses of the portfolio. Further more, it does not differentiate between profits and losses. However, when volatility of the portfolio is the same as the risk of the TRP, often called expected tail loss (ETL), then risk parity strategy is equivalent to tail risk parity.

The idea of TRP is to diversify the portfolio through the use of extreme losses. The red shaded area in Figure 5.1 illustrates the expected tail loss of the portfolio given a specific confidence level. The return of JCP, DF, NEM, and CLF are below a specific level of investors preference. For example, one

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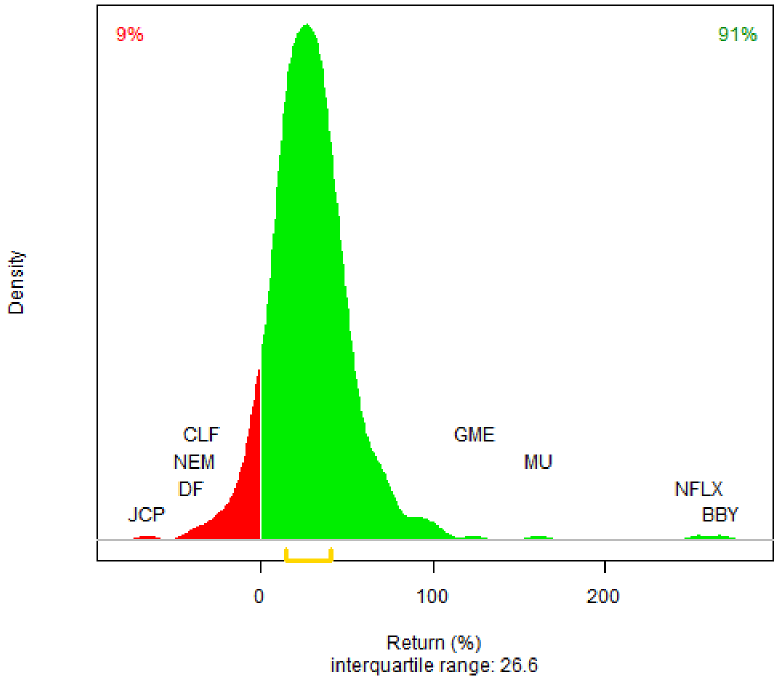
Figure 5.1: Expected Tail Loss



may prefer to measure returns below 1% level of confidence. This distinguishes between profit and losses as losses are defined as the negative returns of the portfolios. Academics believe that this risk is the one investors are not aware of and cause severe impact on portfolio's performance in the event of financial crisis.

The advantage of the TRP is that it is a forward looking strategy. The expected return of the portfolio is anticipated prior the worst market scenarios using the confidence level as probability. Investors in this case optimize the portfolio based on level of preference on the expected return and the expected tail loss. We notice that the expected tail loss (or conditional value at risk) is a success measure of the value at risk. Thus, we begin by discussing the value at risk and later continue with expected tail loss.

5.1.1 Value-at-Risk

In this subsection, we highlight briefly an alternative risk measure called Value at Risk $\mathrm{VaR}_\alpha(r(\mathbf{z}))$ for portfolio construction. This measure is the threshold that determines the possible portfolio losses under confidence level (known as probability) for a specific time horizon. It answers questions, such as 'what is the most one can lose in an investment over the next trading period with certain confident level α'? In particular, this confident level is the probability

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of getting the return, $r(\mathbf{z})$, of the portfolio, below a specific benchmark, i.e.,

$$\mathbb{P}(r(\mathbf{z}) \leq -\mathrm{VaR}_\alpha(r(\mathbf{z})) = \alpha. \tag{5.1.1}$$

The α value normally ranges between 0.01 and 0.05. These are convenient measures of level of confidence used to communicate across board members and clients. The time horizon is considered one in order to simplify the calculation. The α-quantile of portfolio's return is given by,

$$\mathrm{VaR}_\alpha(r(\mathbf{z})) = \inf\{\ell \in \mathbb{R} | \mathbb{P}(r(\mathbf{z}) > \ell) \leq 1 - \alpha\}. \tag{5.1.2}$$

The most appealing with this approach is that it is forward looking (i.e., investors are able to anticipate the possible loss of the portfolio for the next coming day, week, month, or any specific trading period). It illustrates the concern that investors might be facing, such as the loss of portfolio value. This loss is classified as depreciation of portfolio return.

In general, VaR is a measure of the potential loss of risky portfolio value over a defined period for a given probability known as confidence level. For any random portfolio return that is normally distributed with mean $\bar{r}(\mathbf{z})$ and variance $\sigma^2(\mathbf{z})$, the portfolio value at risk is given by,

$$\mathrm{VaR}_\alpha = -(\bar{r}(\mathbf{z}) + z_\alpha \times \sigma(\mathbf{z})) \times x, \tag{5.1.3}$$

where z_α is the left-tail of the α percentile of the standard normal distribution[1], σ the volatility (or standard deviation) and x is the value of the portfolio.

Example 5.1. *Suppose the investor computed the mean and the standard deviation as* $(\bar{r}(\mathbf{z}) = 0.5\%, \sigma(\mathbf{z}) = 6\%)$ *of the portfolio worth one million rands. Then the* 99% *analytical VaR for the next trading day is:*

$$\begin{aligned} VaR_\alpha(r(\mathbf{z})) &= -(0.005 + (-2.3263 \times 0.06)) \times R1,000000.00 \\ &= R134,578.00 \end{aligned}$$

This tells us that there is 1% chance of losing R134, 578.00 in the next trading day. There are three designated approaches to determine VaR namely analytical (also known as Parametric), Non-parametric (historical simulations) and lastly the Monte Carlo simulations. The distinctions of these methodologies are illustrated in the lecture notes of the Federal Reserve Bank of Boston lead by Embrechts *et al.* (2005).

Like any other risk measure, VaR has pros and cons that arises in practice. The good side of VaR is that it predicts the future loss of portfolio value under certain probability by assuming that asset returns are normally distributed.

[1] The right tail of the distribution is considered the profit of the investment

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The main idea of determining the efficient risk measure is to construct portfolio allocations strategy that diversify risk through component risk contributions, such that component with highest risk contributions are given less preference in weighting than those with less risk contribution.

Artzner *et al.* (1999) noted that VaR is not a coherent risk measure since it fails to satisfy some of the properties of coherent measure. In particular, Sub-additivity does not hold for this measure. Constructing a portfolio based on this risk measure may lead to a concentration of portfolio risk and when one is considering maximizing return as the objective of investment, the resulting portfolio may be riskier. Thus, an alternative downside risk measure which conform with the objective of risk decomposition is called conditional value-at-risk (CVaR).

5.2 Conditional Value-at-Risk

This measure diminish the lack of adequacy of VaR and adheres to the properties of coherent risk measures as defined by Artzner *et al.* (1999). The term CVaR and Expected Shortfall were named separately in the works of Rockafellar *et al.* (2002) and Clark and Siems (2002), respectively. CVaR can be decomposed into component risk contributions since it adheres to the properties of coherent risk measures, see Boudt *et al.* (2013*a*). We assume that component returns of the portfolio are normally distributed. Below we present in details this risk measure and its application in risk budgeting portfolio.

Definition 5.2. *CVaR is a probabilistic risk measure that provide the expected risk (often regarded as loss) of the portfolio under confidence level exceeding the threshold VaR, defined as:*

$$CVaR_{\alpha}(r(\mathbf{z})) = -\mathbb{E}[r(\mathbf{z})|r(\mathbf{z}) \leq -VaR_{\alpha}(r(\mathbf{z}))], \tag{5.2.1}$$

where $\mathbb{E}$ *denote the expectation operator and* $VaR_{\alpha}(r(\mathbf{z}))$ *is the threshold not to be exceeded.*

Recall that the expected return of the portfolio is given as the sum of the product of components expected returns, $\bar{r}(\mathbf{z})$, and quantity held on each component, i.e.,

$$\bar{r}(\mathbf{z}) = \sum_{i=1}^{n} z_i \bar{r}_i. \tag{5.2.2}$$

Boudt *et al.* (2013*a*) showed that portfolios constructed using CVaR as risk measure provide less interest in buying assets that have risk above the VaR. Optimization of such portfolios without constraints yield similar results of risk parity portfolio of Bruder and Roncalli (2012).

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The marginal contribution of the i^{th} component is:

$$\mathcal{MC}_i^{CVaR} = \frac{\partial \text{CVaR}_\alpha(r(\mathbf{z}))}{\partial z_i}. \tag{5.2.3}$$

This denotes the significant change in CVaR possessed by component i. It can also be denoted by:

$$\mathcal{MC}_i^{CVaR} = \frac{\partial \left(-\mathbb{E}[r(\mathbf{z})|r(\mathbf{z}) \leq -\text{VaR}_\alpha(r(\mathbf{z}))]\right)}{\partial z_i}. \tag{5.2.4}$$

The component risk contribution to the CVaR as a risk measure is the weighted marginal contribution given by:

$$\mathcal{RC}_i^{CVaR} = z_i \frac{\partial \left(-\mathbb{E}[r(\mathbf{z})|r(\mathbf{z}) \leq -\text{VaR}_\alpha(r(\mathbf{z}))]\right)}{\partial z_i}. \tag{5.2.5}$$

Proposition 5.3. *The risk contribution of components in a portfolio can be expressed as the negative conditional expectation of component returns given that the returns are lower than a given threshold, i.e*

$$\mathcal{RC}_i^{CVaR} = -\mathbb{E}[z_i r_i | \bar{r}(\mathbf{z}) \leq -VaR_\alpha(r(\mathbf{z}))], \tag{5.2.6}$$

where $VaR_{\mathbf{z}}(\alpha)$ represents the threshold not to be exceeded given by:

$$\mathbb{P}[\mathbf{z}^T\mathbf{r} \leq -VaR_\alpha(r(\mathbf{z}))] = \alpha. \tag{5.2.7}$$

Proof. The proof follows from Scaillet (2004). Suppose $X = -\sum_{j=1}^{n} z_j r_j$ and $Y_i = -r_i$, $i = 1, \ldots, n$, so that equation (5.2.1) can be written as:

$$\text{CVaR} = \mathbb{E}[X + z_i Y_i | X + z_i Y_i \leq -\text{VaR}_\alpha(r(\mathbf{z}))]. \tag{5.2.8}$$

□

The results in equation (5.2.6) is a direct consequence of the following Lemma.

Lemma 5.4. *If (X, Y) represents a pair of vectors, and*

$$CVaR_\alpha(\epsilon) = \mathbb{E}[X + \epsilon Y | X + \epsilon Y \leq -VaR_\alpha(\epsilon)], \tag{5.2.9}$$

is the conditional expectation of the portfolio where α-quantile is given by:

$$\mathbb{P}[X + \epsilon Y \leq -VaR_\alpha(\epsilon)] = \alpha, \tag{5.2.10}$$

then the marginal contribution of component i to the portfolio risk is:

$$\frac{\partial CVaR_\alpha(\epsilon)}{\partial \epsilon} = \mathbb{E}[Y | X + \epsilon Y \leq -VaR_\alpha(\epsilon)]. \tag{5.2.11}$$

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Proof. We consider a pair of vectors (X, Y) and set the function $f(x, y)$ to be the probability density function. Now, the conditional expectation of portfolio return given that a particular threshold has been exceeded (i.e. $\text{VaR}_\alpha(\epsilon)$) is:

$$\begin{aligned}\mathbb{E}[X + \epsilon Y | X + \epsilon Y \leq -\text{VaR}_\alpha(\epsilon)] &= \frac{\mathbb{E}[X + \epsilon Y]\Delta_{X+\epsilon Y \leq \mathbb{Q}(\epsilon,\alpha)}}{\alpha} \\ &= \frac{1}{\alpha}\int\left[\int_{-\infty}^{\mathbb{Q}(\epsilon,\alpha)-\epsilon Y}(x+\epsilon y)f(x,y)dx\right]dy,\end{aligned}$$

where $\Delta_{X+\epsilon Y \leq \mathbb{Q}(\epsilon,\alpha)}$ is an indicator. It provides the value of asset negative returns when a given threshold is exceeded, otherwise it gives zero. Recall that the objective of CVaR is to concentrate on the negative return because they are the ones considered undesirable thing. Taking the derivative of the above equation with respect to ϵ yields:

$$\frac{1}{\alpha}\int\left[\int_{-\infty}^{\mathbb{Q}(\epsilon,\alpha)-\epsilon Y} yf(x,y)dx + \mathbb{Q}(\epsilon,\alpha)f(\mathbb{Q}(\epsilon,\alpha)-\epsilon y, y)\left(\frac{\partial\mathbb{Q}(\epsilon,\alpha)}{\partial\epsilon} - y\right)\right]dy.$$

□

Thus, the percentage contribution of component i is:

$$\%\mathcal{RC}_i^{CVaR} = \frac{\mathcal{RC}_i^{CVaR}}{\text{CVaR}_{\alpha(\mathbf{z})}} = \frac{\mathbb{E}[z_i r_i | \bar{r}(\mathbf{z}) \leq -\text{VaR}_\alpha(\mathbf{z})]}{\mathbb{E}[\bar{r}(\mathbf{z}) | \bar{r}(\mathbf{z}) \leq -\text{VaR}_\alpha(\mathbf{z})]}, \quad (5.2.12)$$

where $[z_i r_i | \bar{r}(\mathbf{z}) \leq -\text{VaR}_\alpha(\mathbf{z})]$ denotes an absolute CVaR contribution of a component i, see Boudt *et al.* (2013*a*). Market practitioners and academic researchers have already noted that the risk of the portfolio is dominated by one market regime, usually stock/equity market, see Levell *et al.* (2010) and Bhansali *et al.* (2012). We define below the largest component condition value at risk concentration in a portfolio as:

$$C_z(\alpha) = \max_{i=1,\dots,n} \mathcal{RC}_i^{CVaR}. \quad (5.2.13)$$

One can think of this component as the one with the highest negative return in a universe. For the ERC strategy, such component is restricted to identical amount of risk contribution of the remaining assets.

5.2.1 Optimization with CVaR

Resuming from the assumption of investors rationality, the optimum portfolio using CVaR incorporate expected return. The investor's objective in this case is a function of three components, namely, the maximum expected return, minimum CVaR and maximum diversification of the downside risk concentration.

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Boudt *et al.* (2013*a*) introduced two strategies of optimization such function, namely, the minimum conditional value at risk concentration (MCC) and the percentage constraint contribution (PCC) based portfolios.

The objective function of the MCC strategy is to maximize diversification of the downside risk concentration. Thus, the problem can be defined as:

$$\mathbf{z}^{\text{MCC}} = \arg\min_{\mathbf{z}\in\mathbb{R}^n} C_z(\alpha) \tag{5.2.14}$$

such that

$$\mathbf{z}^T\mathbb{1} = 1.$$

Portfolios optimized using this strategy yield a balance between the downside risk diversification and the minimum CVaR. Embedding portfolio targeted return into this strategy, the optimization problem becomes:

$$\mathbf{z} = \arg\min_{\mathbf{z}\in\mathbb{R}^n} C_z(\alpha) \tag{5.2.15}$$

such that,

$$\begin{cases} \mathbf{z}^T\mathbb{1} = 1 \\ \mu^T\mathbf{z} \geq a. \end{cases}$$

This strategy is often referred to as the mean-CVaR because it yields an efficient frontier for various target of returns and risks.

The ERC based on the CVaR constrain the percentage contributions of components often known as risk budgets in the work of Bruder and Roncalli (2012). It has a special property which requires all the component risk contributions to be equal, i.e.,

$$\%\mathcal{RC}_1^{CVaR} = \%\mathcal{RC}_2^{CVaR} = \cdots = \%\mathcal{RC}_n^{CVaR}. \tag{5.2.16}$$

Thus, the relative weights of such portfolio components are inversely proportional, i.e.,

$$\frac{z_i}{z_j} = \frac{\partial\text{CVaR}/\partial z_j}{\partial\text{CVaR}/\partial z_i}. \tag{5.2.17}$$

However, the fact that ERC does not incorporate expected return in the portfolio composition, leads to the MCC strategy given more consideration in practise, Boudt *et al.* (2013*b*).

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5.2.2 Findings of CVaR Portfolios

Based on the US bond, S&P 500, NAREIT and GSCI asset class index, Boudt *et al.* (2013*b*) analysed the minimum CVaR, MCC, ERC and EW portfolios over the period January 1984 to June 2010. The minimum CVaR portfolio allocates asset risk contributions efficiently to the total risk of the portfolio. However, in the absence of budgetary constrain, risk of components still seem to be concentrated on other asset classes which yield high portfolio turnover. Compared to ERC, minimum CVaR yield exceptional results. ERC portfolio provides a portfolio with lowest risk concentration of assets and lowest turnover, but it suffers from the highest total risk.

Unlike ERC portfolio of Qian (2013*b*), the MCC portfolio allows investors to include more investment objectives and constraints. In terms of turnover, MCC is lower than minimum CVaR. It provides a balance between the overall risk of the portfolio, expected return and high diversification with low portfolio turnover. The incorporation of expected return in portfolio optimization as objective or constraint serve a good advantage to this approach.

In their empirical results, Boudt *et al.* (2013*a*) showed that EW and 60/40 portfolios fail to produce diversified portfolio using ex-ante. The new strategy MCC that incorporates both the downside risk diversification and low CVaR portfolio, provides the largest CVaR diversification, with the total portfolio CVaR significantly higher than that of minimum CVaR portfolio, but the returns are superior than other strategies.

Based on different market regime and sampling data, minimum CVaR portfolio performed well during the bear market, but suffers good side of return during normal or bull market. Its performance is less affected when market switch to more negative observation (i.e., when markets are declining). Thus, the minimum CVaR is recommended in the bear market and MCC in a normal or bull market.

5.2.3 Minimum CVaR versus other $\mu-$Free Strategies

In this subsection, we compare the minimum CVaR strategy with other risk-based strategies using data collected from finance.yahoo website. This comprises MSCI index from eight countries denoted by the following tickers over the period April 2003 to August 2014: EWA, EWJ, EWY, EWG, EWW, EFA, EEM and EWZ. Table (5.1) depicts the summary statistics of these strategies performance. We find that the volatility of the minimum CVaR (MCVaR) portfolio is significantly higher than the global minimum variance portfolio (GMV). Also, the cumulative performance of a portfolio has been consistently outperforming the GMV portfolio until the beginning of 2011 and since then, the result has been favourable for GMV, see Figure (5.2).

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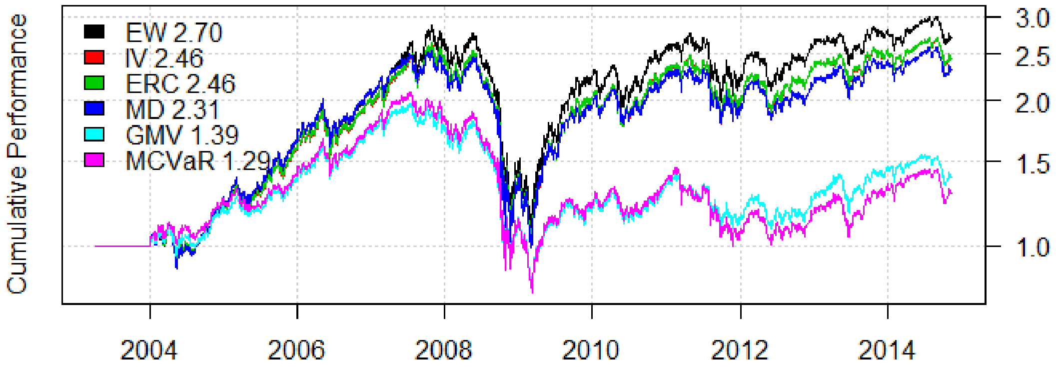
Table 5.1: Performance Statistics of the μ-free Strategies vs MCVaR

	Strategies					
	EW	IV	ERC	MD	GMV	MCVaR
$\bar{\mathbf{r}}(\mathbf{z})$	8.96	8.06	8.06	7.51	2.91	20.21
$\sigma(\mathbf{z})$	26.31	25.28	25.22	24.8	20.96	21.57
CVaR	-3.98	-3.84	-3.82	-3.72	-3.13	-3.21
$DR(\mathbf{z})$	0.32	0.28	0.28	0.23	0	0
Sharpe	0.46	0.43	0.43	0.42	0.24	0.21

Figure 5.2: Comparison of Minimum CVaR and other $\mu-$Free Portfolios



5.3 Factor Risk Parity

Contrary to the previous analysis of portfolio's strategies based on asset classes, we present in this section, risk parity approach which seeks diversification based on the primitive sources of risk underlying the asset returns. Bhansali *et al.* (2012) show through example that risk parity portfolio designed using asset classes is not truly diversified.

In practice, there are three types of factor models. The first one is the macroeconomic factor model in which factors are observed from the macro-financial variables. The second one is the fundamental factor model where factors are created from observed asset characteristics. Lastly, we have statistical factor model which assumes that factors are unobservable and can be determined from the asset returns. These approaches provide better explanation of asset returns that is distinguishable from other asset return models. In addition, they provide intense analysis of risks of the returns.

We study the improved risk parity strategy that seeks to equalize factors instead of asset class risk contributions and hence the name factor risk parity. Many articles have been published and most intend to address the shortcomings of original risk parity strategy; see Roncalli and Weisang (2012) and

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Bhansali *et al.* (2012). The believe is that there is an overlap of correlations amongst asset classes which exhibit poor diversification of strategies. Benson *et al.* (2012) show that during the time of financial crisis, correlations increases significantly. Thus, risk parity portfolio may not sound diversified if assets in a portfolio are dominated by equity-like assets class which leads the portfolio risk still dominated by the growth market.

The promising diversification of strategies is through factor models. Factors are the fundamental building blocks that make up asset classes, for instance, equity asset class may constitute US equity, Non-US equity, and bond class may constitute US government bond, SA government bond and so on. These constituents are called factors. To understand more of factors, we recommend an example given by Bhansali *et al.* (2012) where asset classes were considered foods and factors were considered nutrients. It states that 'Although the body needs food, it actually needs nutrients to build strong bones and muscles'. Thus, it is more important to consume nutritious food than just a basket full of food. The believe is that portfolios constructed based on factors yield a more diversified and efficient portfolio than asset class portfolios.

Factor risk parity strategy mimics the ordinary risk parity portfolio construction[2]. Instead of diversifying portfolio risk based on asset class risk contributions, we use factor risk contributions. It has been shown that careless selection of asset classes may still lead to the entire portfolio risk dominated by unique risk of market regime; see Bhansali *et al.* (2012) and Podkaminer (2013). In addition, the existence of convertible assets fuel this problem. There are a variety of risk factors in the investment environment and example are growth, inflation, liquidity, volatility and momentum.

Example 5.5. *Consider the dominant risk factors, namely growth and inflation as the drivers of asset returns, then the portfolio return is given by:*

$$r_p(\mathbf{Z}) = Z_e r_e + Z_b r_b + E, \tag{5.3.1}$$

where Z_e *and* Z_b *are the weights associated with growth and inflation risk factor, respectively. The term* E *is the residual and* r_e *and* r_b *are the returns of the respective risk factors. The risk of the portfolio is then given by:*

$$\sigma_p^f(\mathbf{Z}) = \sqrt{\sigma_e^2 Z_e^2 + \sigma_b^2 Z_b^2 + 2\rho_{eb}\sigma_e\sigma_b Z_e Z_b + \sigma_E^2}. \tag{5.3.2}$$

5.3.1 Framework of Factor Based Risk Parity

In constructing the framework of factor risk parity, we consider approaches of Zivot (2011), Roncalli and Weisang (2012) and Peeters (2013). We consider a portfolio of n-assets, i.e., $\{a_1, \dots, a_n\}$ and assume that their variations are explained by K-observable macroeconomic factors denoted by the

[2] Asset class based risk parity portfolio.

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set $\{f_1, \ldots, f_K\}$ which are independent and identically distributed (*i.i.d.*) with mean $\bar{r}_f$ and variance σ_f^2. The general factor model for asset returns is given as follows:

$$\mathbf{r} = \bar{\mathbf{a}} + \mathbf{B}\mathbf{f}_t + D_\sigma \varepsilon_t, \tag{5.3.3}$$

which shows that asset returns are dependent on factors. In a matrix form, we have:

$$\begin{pmatrix} r_{1t} \\ r_{2t} \\ \vdots \\ r_{nt} \end{pmatrix} = \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_n \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1K} \\ b_{21} & b_{22} & \cdots & b_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nK} \end{pmatrix} \begin{pmatrix} f_{1t} \\ f_{2t} \\ \vdots \\ f_{Kt} \end{pmatrix} + \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{pmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \vdots \\ \varepsilon_{nt} \end{pmatrix}.$$

This implies that the return of the i^{th} asset is given by the following linearly equation:

$$r_{it} = \bar{a}_i + \sum_{k=1}^{K} b_{ki} f_{kt} + \sigma_i \varepsilon_{it} \quad i = 1, \ldots, n, \tag{5.3.4}$$

where $\bar{a}_i$ is a constant of the i^{th} asset (often set to zero), b_{ki} is the factor loading to the i^{th} and ε_i denotes the error term at time t, see Roncalli and Weisang (2012) and Bhansali *et al.* (2012). Because the error terms are uncorrelated, their covariance is given as follows:

$$\sigma_{\varepsilon_{it}\varepsilon_{js}} = \begin{cases} \sigma_i & \text{if } i = j \text{ and } t = s \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.5}$$

Also, the covariance of a specific error term with risk factors is zero. That is,

$$\sigma_{f_k, \varepsilon_{it}} = 0, \tag{5.3.6}$$

and this is because the asset returns in a linear factor model are explained by the risk of factors and risk not coming from factors (often called idiosyncratic risk of assets). This means that factors do not span the entire risk of the portfolio. The covariance matrix of factor-based portfolio return is:

$$\Sigma_t = \mathbf{B}^T S \mathbf{B} + D_\sigma, \tag{5.3.7}$$

where S denotes the $K \times K$ covariance matrix of factor returns. We define the variance of the i^{th} asset as follows:

$$\sigma_{it}^2 = \mathbf{B}_i^T S \mathbf{B}_i + \sigma_i^2, \quad i = 1, \ldots, n. \tag{5.3.8}$$

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Also, the covariance and correlation between assets in the factor-based strategies are:

$$\sigma_{ij,t} = \mathbf{B}_i^T S \mathbf{B}_j, \tag{5.3.9}$$

$$\rho_{ij,t} = \frac{\sigma_{ij,t}}{(\sigma_{it}^2 \sigma_{jt}^2)^{\frac{1}{2}}}, \quad i, j = 1, \ldots, n. \tag{5.3.10}$$

respectively. Multiplying equation (5.3.3) from the left by a transpose vector of component weights $\mathbf{z}$, we have:

$$\mathbf{z}^T \mathbf{r} = \mathbf{z}^T \bar{\mathbf{a}}_t + \mathbf{z}^T \mathbf{B} \mathbf{f}_t + \mathbf{z}^T D_\sigma \varepsilon. \tag{5.3.11}$$

Setting $\alpha_t = \mathbf{z}^T \bar{\mathbf{a}}_t$, $\beta_t = \mathbf{z}^T \mathbf{B}$ and $\varsigma_t = \mathbf{z}^T D_\sigma$, we can write, respectively, the return and variance of the linear factor model as:

$$r(\mathbf{z}) = \alpha_t + \beta_t \mathbf{f}_t + \varsigma_t \varepsilon_t, \tag{5.3.12}$$

$$\sigma_t^2 = \beta^T S \beta + \sigma_{\varepsilon_t}^2. \tag{5.3.13}$$

Note that $\sigma_{\varepsilon_t}^2 = \mathbf{z}^T D_\sigma \mathbf{z}$. Peeters (2013) and Zivot (2011) combined the two covariance matrices by first assuming that $\varepsilon_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_{\varepsilon t}^2)$. Thus, the factor model return is:

$$r(\mathbf{z}) = \alpha_t + \begin{pmatrix} \beta_t^T \\ \sigma_\varepsilon \end{pmatrix}^T \begin{pmatrix} \mathbf{f}_t \\ e_t \end{pmatrix} = \alpha_t + \gamma^T \begin{pmatrix} \mathbf{f}_t \\ e_t \end{pmatrix}, \tag{5.3.14}$$

where $e_t = \dfrac{\varepsilon_t}{\sigma_\varepsilon} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$ and the expression of the covariance matrix is:

$$\Pi = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1k} & 0 \\ s_{21} & s_{22} & \cdots & s_{2k} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{k1} & s_{k2} & \cdots & s_{kk} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{5.3.15}$$

Therefore the volatility of the portfolio under linear factor model is given by:

$$\sigma(\gamma) = \sqrt{\gamma^T \Pi \gamma}. \tag{5.3.16}$$

5.3.2 Risk Decomposition of Factor-Based Risk Parity

From relation (5.3.16), it is clear that the homogeneity property of coherent risk measures is achieved. Thus, we mimic the ERC of asset classes in defining

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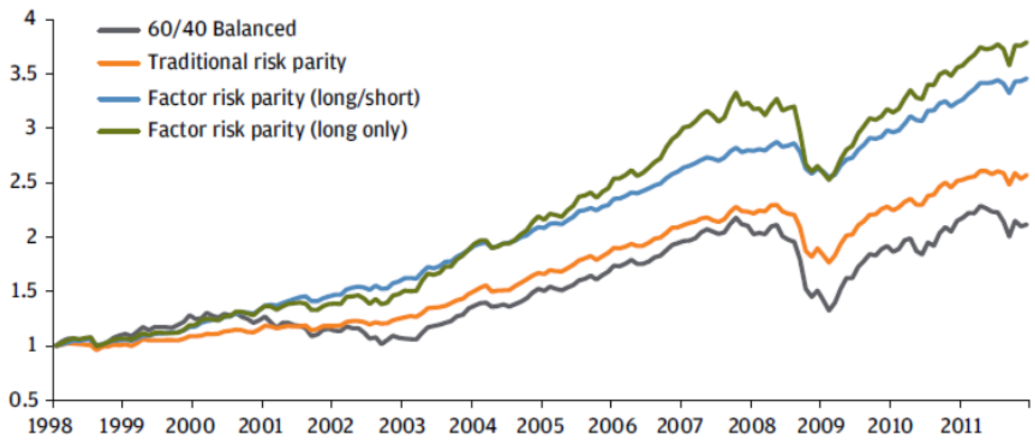
Figure 5.3: Backtesting of Factor-Based, Traditional Risk Parity and 60/40 Strategy. Source: JPMorgan Asset



the marginal and risk contribution of factors. Using Theorem (2.6), we denote factor-based portfolio volatility as follows:

$$\begin{aligned}\sigma(\gamma) &= \gamma_1 \frac{\partial\sigma(\gamma)}{\partial\gamma_1} + \gamma_2 \frac{\partial\sigma(\gamma)}{\partial\gamma_2} + \cdots + \gamma_{k+1} \frac{\partial\sigma(\gamma)}{\partial\gamma_{k+1}} \\ &= \beta_1 \frac{\partial\sigma(\gamma)}{\partial\beta_1} + \beta_2 \frac{\partial\sigma(\gamma)}{\partial\beta_2} + \cdots + \beta_k \frac{\partial\sigma(\gamma)}{\partial\beta_k} + \sigma_\varepsilon \frac{\partial\sigma(\gamma)}{\partial\sigma_\varepsilon}. \end{aligned} \tag{5.3.17}$$

The marginal contribution, $\mathcal{MC}^f_{i,t}$, of the i^{th} asset is denoted as:

$$\mathcal{MC}^f_{i,t} = \frac{\partial\sigma(\gamma)}{\partial\gamma_{i,t}}, \tag{5.3.18}$$

and the risk contribution, $\mathcal{RC}^f_i(\gamma)$, as:

$$\mathcal{RC}^f_i(\gamma) = \gamma_i \frac{\partial\sigma(\gamma)}{\partial\gamma_i}. \tag{5.3.19}$$

Figure (5.3) illustrates the performance of factor based risk parity (long only and long-short) against the traditional risk parity and the 60/40 asset allocation strategies. It is clear that the factor-based risk parity strategies performed well over the period 2001 until 2012 with the long-only factor risk parity dominating the performance. Moreover, factor-based risk parity, whether long or short, performs better than the traditional risk parity as well as 60/40 strategy.

5.3.3 Draw-Backs of Factor Risk Parity

Although factor-based risk parity seems well constructed in terms of managing diversification of portfolios, there are still drawbacks related to this approach.

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Podkaminer (2013) highlighted some of these drawbacks which include implementation of factors for portfolio construction. The main challenge is that there is no natural way to invest in many factors directly. Furthermore, the weight allocation of factors and forward looking assumptions are difficult to determine.

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Chapter 6

Rebalancing, Transaction Cost and Leverage

As noted by Lussier (2013), asset allocation is a process in which participants in the investment industry keep track of the evolution of markets, and bet according to their liability requirements, return objectives, risk tolerance and some taxation. A well performing investment strategy consists of a periodical rebalancing which is subjected to the transaction cost. Investors implementing rebalancing methodology in their portfolio compositions believe that rebalancing is a source of some added value. In this chapter, we discuss the implementation of rebalancing portfolios, their corresponding transaction cost and one of the 'trendiest' problem in the risk-based asset allocation called leverage.

6.1 Portfolio Rebalancing

Portfolio rebalancing in an investment world is the action of redefining the weights of the portfolio such that its performance is equivalent to the corresponding benchmark portfolio. Although this technique diminish significantly the total risk of the portfolio, it is often exposed to transaction costs.

6.1.1 Portfolio's Trading Cost

Although risk parity strategy mitigates detrimental behavioural tendency of the traditional asset allocation, it can be improved by incorporating the rebalancing penalties. That is, the cost of buying or selling of assets in order to rebalance the portfolio and avoid turnover, see Darolles *et al.* (2012). The impact of transaction cost leads to a substantial drag on the performance of strategies.

In order to trace the cost of rebalancing portfolios, we consider the notations of Anderson *et al.* (2012) in which components of the portfolio are defined

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as functions of time. Investors in this case specify the waiting period for the next rebalancing day, say h. We denote by $\mathbf{z}_{t-1} \in \mathbb{R}^n$, the portfolio before rebalancing, and by $\mathbf{z}_t \in \mathbb{R}^n$, the portfolio at the rebalancing date. Volatility of a component in this case is a function of the trailing returns over the waiting period and is defined as in equation (1.1.6). The covariance matrix of components is given as:

$$\Sigma_t = \begin{pmatrix} \sigma^2_{(1),t} & \sigma_{(1,2),t} & \cdots & \sigma_{(1,n),t} \\ \sigma_{(2,1),t} & \sigma^2_{(2),t} & \cdots & \sigma_{(2,n),t} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{(n,1),t} & \sigma_{(n,2),t} & \cdots & \sigma^2_{(n),t} \end{pmatrix}, \tag{6.1.1}$$

where $\sigma_{(i,j),t}$ is the covariance between asset i and j in a portfolio at time t.

The cost of rebalancing the portfolio is defined as:

$$k_t \sum_{i=1}^{n} z_{t-1,i} \ln\left(\frac{z_{t-1,i}}{z_{t,i}}\right), \tag{6.1.2}$$

where $k_t \in \mathbb{R}^+$ is the dynamic trading cost of reallocating component $z_{t-1,i}$ to $z_{t,i}$ in a portfolio. In practice, the cost of rebalancing a portfolio changes over time due to the dynamics of security prices. In the event where borrowing is prohibited, expression (6.1.2) exhibits that portfolio or asset rebalancing is directly proportional to its ratio of security weights. In finance, we have a variety of trading penalties (or cost). We have commission which is the amount charged for making trading, the bid or ask spread which is the different in prices for buying an asset and immediately sell it, and the last one is market impact which is the cost of trading multiple stocks.

For the case where the investor is willing to minimize the risk of the portfolio, the objective function of the portfolio embroiling trading cost becomes:

$$\mathbf{z}^* = \arg\min_{\mathbf{z}\in\mathbb{R}} \left(\sigma(\mathbf{z}_t) + \gamma k_t \sum_{i=1}^{n} z_{t-1,i} \ln\left(\frac{z_{t-1,i}}{z_{t,i}}\right)\right). \tag{6.1.3}$$

Taking the first order derivative of the above equation with respect to $z_{t,i}$, we have:

$$\frac{\partial\sigma(\mathbf{z}_t)}{\partial z_{t,i}} + \lambda k_t z_{t-1,i}\left(-\frac{z_{t-1,i}}{z^2_{t,i}} \times \frac{1}{\frac{z_{t-1,i}}{z_{t,i}}}\right) = 0,$$

which implies that:

$$\frac{\partial\sigma(\mathbf{z}_t)}{\partial z_{t,i}} = \frac{\lambda k_t z_{t-1,i}}{z_{t,i}}.$$

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Thus,

$$z_{t,i}\frac{\partial\sigma(\mathbf{z}_t)}{\partial z_{t,i}} = \gamma k_t z_{t-1,i},$$

indicating that the risk contribution of the i^{th} component at time t is proportional to the preceding allocation. Darolles *et al.* (2012) showed that for small adjustment of portfolio reallocation, this trading cost is inversely proportional to the preceding allocation.

Expanding equation (6.1.2), we have:

$$k_t\sum_{i=1}^{n} z_{t-1,i}\ln\left(\frac{z_{t-1,i}}{z_{t,i}}\right) = -k_t\sum_{i=1}^{n} z_{t-1,i}\ln\left(1+\frac{z_{t,i}-z_{t-1,i}}{z_{t-1,i}}\right). \tag{6.1.4}$$

Setting $x = \dfrac{z_{t,i}-z_{t-1,i}}{z_{t-1,i}}$, we have:

$$\ln\left(1+\frac{z_{t,i}-z_{t-1,i}}{z_{t-1,i}}\right) = \ln(1+x).$$

We know from Taylor's series that the expansion of the above function around point zero gives the following:

$$\ln(1+x) = \sum_{i=1}^{\infty}(-1)^{1+i}\frac{x^i}{i} \quad \forall\,|x| < 1.$$

Thus, omitting the higher order terms and substituting the above expansion into equation (6.1.4), yields the following:

$$\begin{aligned} k_t\sum_{i=1}^{n} z_{t-1,i}\ln\left(\frac{z_{t-1,i}}{z_{t,i}}\right) &\approx -k_t\sum_{i=1}^{n} z_{t-1,i}\left(x-\frac{x^2}{2}\right)\\ &= -k_t\sum_{i=1}^{n} z_{t-1,i}\left(\frac{z_{t,i}-z_{t-1,i}}{z_{t-1,i}} - \frac{\left(z_{t,i}-z_{t-1,i}\right)^2}{2z_{t-1,i}^2}\right). \end{aligned}$$

Applying full budget constraint to the above equation, yields the following:

$$-k_t\sum_{i=1}^{n} z_{t-1,i}\left(\frac{z_{t,i}-z_{t-1,i}}{z_{t-1,i}} - \frac{\left(z_{t,i}-z_{t-1,i}\right)^2}{2z_{t-1,i}^2}\right) = \frac{k_t}{2}\sum_{i=1}^{n}\frac{(z_{t,i}-z_{t-1,i})^2}{z_{t-1,i}}. \tag{6.1.5}$$

Usually, equities are the ones dominating in terms of risk and thus are given low weight as to diversify the portfolio risk. From equation (6.1.5), we notice

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that assets with small (large) amount of weights will be more (less) expensive in terms of trading costs.

Goldberg and Mahmoud (2013) compared several strategies including minimum variance, risk parity, beta, traditional 60/40, and equal weighted based on portfolio turnover. The authors find that on average, minimum variance strategy dominated all other strategies in terms of portfolio turnover, followed by low beta (ERC), and then risk parity strategy. However, the rise of turnover seems to be more reliant on market regimes. This is because the demands of rebalancing is highly experienced during the bear markets than bull markets.

6.2 Leverage and Inverse-Volatility Portfolio

In this subsection, we discuss the use of leverage in the risk parity strategies, particularly, for the Inverse-Volatility (IV) strategy. Practitioners using risk parity strategies often use leverage to enhance desired risk of their respective benchmark portfolio[1]. Sebastian (2012) argued that risk parity provides lower return and risk than any of these benchmark portfolios, and thus could be levered to match the respective benchmark portfolio risk.

Since component weights of the IV portfolio are inversely proportional to their respective volatilities, for the time dependent variables, these weights are given as follows:

$$z_{i,t} = \frac{\sigma_{i,t}^{-1}}{\sum_{j=1}^{n} \sigma_{j,t}^{-1}}, \quad i = 1, 2, \ldots, n. \tag{6.2.1}$$

The subscript t represents the time a particular measure is taken. The question is 'how to leverage risk parity given its benchmark portfolio?' Levered portfolio refers to a combination of risky assets and money in the bank (i.e., cash, either borrowed or lent). The weights of unlevered risk parity portfolio are multiplied by the leverage ratio, a constant term that controls leverage as defined by Asness *et al.* (2012) from equation (6.2.1) as follows:

$$L_t = \frac{1}{\sum_{j=1}^{n} \sigma_{j,t}^{-1}}. \tag{6.2.2}$$

When leverage is not applied, this term is identical for all components. For the levered RP portfolio, this leverage term is set to a constant which Anderson *et al.* (2012) define as the ratio between the volatility of the benchmark portfolio to the volatility of unlevered risk parity. That is,

$$L_t = \frac{\sigma_{B,t}}{\sigma_{IV,t}}, \tag{6.2.3}$$

[1] This could be value-weighted market such as 60/40 or the mean variance portfolio with target risk/return.

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where $\sigma_{B,t}$ and $\sigma_{IV,t}$, respectively, are the volatilities of the benchmark and the Inverse Volatility portfolio (or unlevered risk parity) at time t. Levering a portfolio simply means that one borrow or lends money at risk free rate and invest in a specific asset class in order to enhance desired expected return. The common class where leverage is applied is fixed-income asset, such as government bonds, particularly treasury bond. Applying leverage on risk parity portfolio, we multiply the weights of the assets in a unlevered portfolio by the leverage ratio. Thus, the levered portfolio is then given by

$$z_{i,t}^{L} = L_t z_{i,t} \quad i = 1, \ldots, n, \tag{6.2.4}$$

where $z_{i,t}$ is the weight of the i^{th} component in a unlevered portfolio. This means that if the volatility of the IV portfolio is lower than a specific benchmark portfolio, then we determine the leverage ratio and multiply it to each and every component.

To illustrate the levered IV strategy, we consider a numerical example for a universe of two asset classes (i.e., CRSP stock and bond) depicted in Table 6.1 with monthly returns over the period 1926-2010. Stock has a volatility of 5.42% and is allocated 14.91% while the bond with volatility 0.95% is allocated 85.09%. It is clear that the higher (or lower) the volatility of the component, the lower (or higher) is the allocation. The most interesting aspect about the IV strategy is that all components in the investment universe contribute to the performance of the portfolio. This reflects better diversification, unlike the long-only GMV and MD strategies which often have little or no exposure to some components in their optimization solutions.

Levering IV, we get a portfolio with expected return significantly higher than both IV and its respective benchmark portfolio (i.e. 60/40), exhibit in Table 6.1.

Table 6.1: Leverage Inverse Volatility vs. 60/40 Portfolio

	Summary Statistics			Strategies		
Assets	$\bar{r}_i$	σ_i	ρ	IV	60/40	Levered IV
CRSP Stock	0.91%	5.42%	10.80%	14.91%	60%	41.07%
CRSP Bond	0.42%	0.95%		85.09%	40%	234.59%
Volatility				1.20%	3.31%	3.31%
Expected return				0.50%	0.72%	1.37%
Leverage ratio						2.76

The return of risk parity in this case is given by:

$$r_t^{RP} = \sum_{i=1}^{n} z_{i,t} r_{i,t}. \tag{6.2.5}$$

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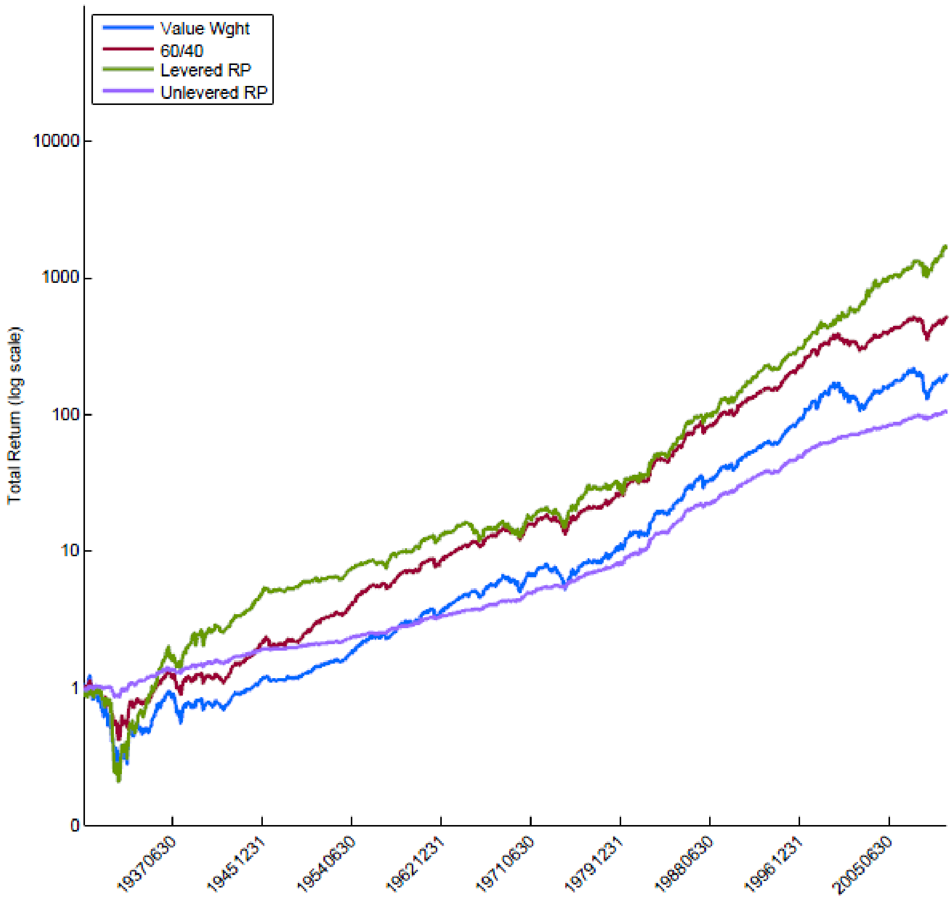
Figure 6.1: Levered vs. Unlevered Risk Parity Portfolio over the Period (1926-2010): Source: Anderson *et al.* (2012).



However, for the levered risk parity portfolio, the return is given by,

$$r_{L,t} = \mathbf{z}_t^T \mathbf{r}_t + \mathbb{1}^T(\mathbf{z}_t^L - \mathbf{z}_t)(\mathbf{r} - r_{b,t}), \tag{6.2.6}$$

where $\mathbf{z}_t^* \in \mathbb{R}^n$ is vector of risk parity weights with leverage ratio applied on each and every component at time t. The rate of borrowed assets at time t is denoted by $r_{b,t}$. The second term in the above equation is the one that provide additional return on risk parity approach and can be considered leverage term. Levered risk parity will resemble unlevered risk parity if $\mathbf{z}_t^L = \mathbf{z}_t$ or $\mathbf{r} = r_{b,t}$. Thus, the excess return of the levered risk parity is given by:

$$r_{L,t}^e = r_{L,t} - r_0. \tag{6.2.7}$$

Figure 6.1 illustrates the performance of levered risk parity against value weighted, unlevered risk parity and 60/40 strategies. Levered risk parity in this case outperformed other three strategies on several occasions. However, Anderson *et al.* (2012) showed that period of backtesting has a significant impact on the results exhibited in this figure. They divided the period of

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backtesting into four and analyse each subdivision. The authors conclude that for the periods (1926-1945, 2001-2010), levered risk parity performed exceptionally well. 60/40 and the value weighted strategies only outperformed during the (1946-1982), and all strategies showed consistence growth during the period (1983-2000) except for unlevered risk parity.

Naranjo (2009) discussed the importance of leverage in a portfolio and indicated that during the demand of frequent rebalancing, fixed-income asset is no longer the best asset to consider as it requires long-term investment. It follows that investment in bond is risky because the longer it takes to the expiration, the higher the chances of government to fail to repay the bond. An alternatives to the fixed income asset in this case are derivative instruments such as option futures, and any other short term traded contracts.

6.2.1 Levered Portfolio and Lower Risk

To determine the condition under which levered portfolio obtains significant risk, we consider two portfolios, levered and unlevered. These portfolios consist of two asset classes, say bond and equity as in Ruban and Melas (2010). Since leverage is applied at a specific time period, we denote elements that build these portfolios as function of time. We denote by $\sigma^2_{L,t}(\mathbf{z})$ and $\sigma^2_{u,t}(\mathbf{z})$, respectively, the variance of the levered and unlevered portfolios. Asset class weights at time t are $z_{b,t}$ for the bond and $z_{e,t}$ for equity. Now the variance of each of the two portfolios is:

$$\sigma^2_{L,t}(\mathbf{z}_t) = L_t^2 z^2_{b,t}\sigma^2_{b,t} + z^2_{e,t}\sigma^2_{e,t} + 2L_t\rho_t z_{e,t} z_{b,t}\sigma^2_{e,t}\sigma^2_{b,t} \quad (6.2.8)$$

$$\sigma^2_{u,t}(\mathbf{z}_t) = z^2_{b,t}\sigma^2_{b,t} + z^2_{e,t}\sigma^2_{e,t} + 2\rho_t z_{e,t} z_{b,t}\sigma^2_{e,t}\sigma^2_{b,t}, \quad (6.2.9)$$

where $\mathbf{z}_t$ is the vector of weights at time t. Since we expect the levered portfolio to exhibit lower risk than unlevered portfolio, we have the following condition:

$$\sigma^2_{L,t}(\mathbf{z}_t) < \sigma^2_{u,t}(\mathbf{z}_t). \quad (6.2.10)$$

Substituting equations (6.2.8) and (6.2.9) in the above equation, yields:

$$L_t^2 z^2_{b,t}\sigma^2_{b,t} + z^2_{e,t}\sigma^2_{e,t} + 2L_t\rho_t z_{e,t} z_{b,t}\sigma^2_{e,t}\sigma^2_{b,t} - (z^2_{b,t}\sigma^2_{b,t} + z^2_{e,t}\sigma^2_{e,t} + 2\rho_t z_{e,t} z_{b,t}\sigma^2_{e,t}\sigma^2_{b,t}) < 0$$

which implies that

$$(L_t^2 - 1)z^2_{b,t}\sigma^2_{b,t} + 2(L_t - 1)\rho_t z_{b,t} z_{e,t}\sigma_{b,t}\sigma_{e,t} < 0. \quad (6.2.11)$$

After performing some algebra, we have:

$$(L_t + 1)z_{b,t}\sigma_{b,t} + 2\rho_t z_{e,t}\sigma_{e,t} < 0. \quad (6.2.12)$$

Rearranging, we find that:

$$\rho_t < -\frac{1}{2}(L_t + 1)\frac{z_{b,t}}{z_{e,t}}\frac{\sigma_{b,t}}{\sigma_{e,t}}. \quad (6.2.13)$$

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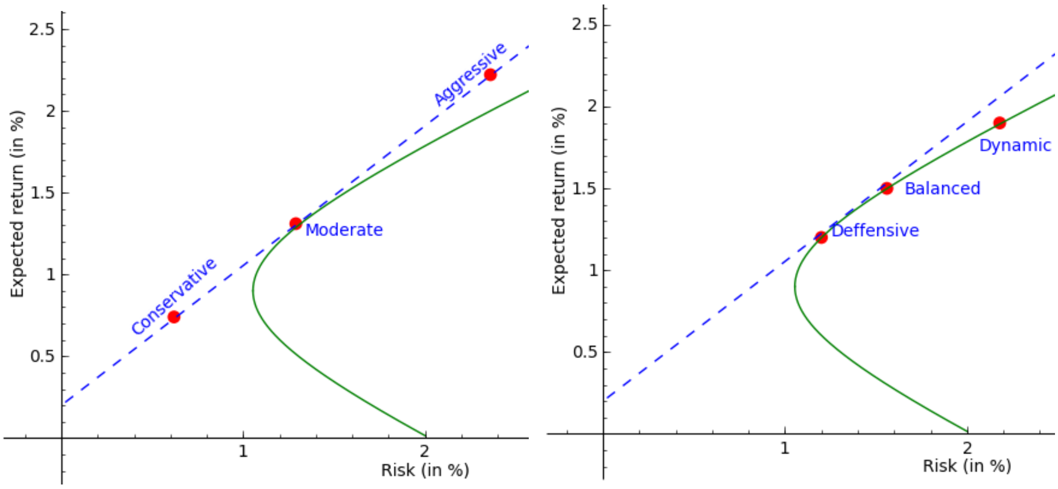
Thus, levered portfolio provides lower volatility whenever the above condition is met. Although the introduction of leverage in a portfolio is to reduce risk while achieving desired portfolio return, this technique has limitation. Sebastian (2012) studied the benefits of leverage in a portfolio and argued that until a certain point of risk-return tradeoff of the traditional efficient frontier is exceeded, the benefits of levered portfolio will always be zero. Hence it is so apparent that the benefits of leverage will only be accumulated when the investors operate in high risk levels.

6.3 Diversified Fund Strategies

Diversification of fund strategies originates from the Markowitz (1952) mean-variance strategy. The author suggests that investors should hold a variety of asset classes in their portfolios; see Chapter 1. Tobin (1958) deployed this theory and incorporates risk-free asset which lead to separation theorem. The optimal portfolio is then a combination of the risk-free asset and the efficient portfolio and is called the tangent portfolio. Since investors have different appetite of risk, their allocation will also be different. Bruder and Roncalli (2012) classified risk tolerance of fund investors into three categories. These are conservative (low risk tolerance), moderate (medium risk tolerance) and aggressive (risk lover); see Figure 6.2a.

Figure 6.2: Diversification of Modern Portfolio Constructions



(a) Illustration of Investors Risk Profile (b) Diversified Fund Strategies

The conservative investor may allocates 20% of his wealth in the risky assets and 80% in the risk-free asset. On the other hand, the moderate and aggressive investors allocate the same amount of wealth to the risky portfolio and risk-free asset. However, the latter deploys leverage technique in his portfolio to have

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high risk profile. Bruder and Roncalli (2012) named the portfolios allocated in these manners the diversified fund strategies, well know as 'lifestyle portfolios'. The conservative, moderate and aggressive investor's portfolio, respectively, corresponds to the defensive, balanced and dynamic fund strategies; see Figure 6.2b.

While these funds allocation strategies are well established in terms of investors risk tolerance, they are not well diversified in practice. Roncalli (2013) simulated these strategies based on their risk contributions and observed that defensive and balanced strategies are not stable. The risk of the dynamic strategy is almost explained by the equity market. It follows that risk management of fund investment is difficult due to the fact that there is no simple relationship between the investor's risk tolerance and the risk of the diversified fund.

Alternatively, the risk parity portfolios provide a balance allocation of assets through the risk contributions. However, this strategy also faces some drawbacks. By definition, it overweights low-volatile components, particularly, the fixed-income asset. This allocation provides low return and risk of the portfolio which push investors to deploy leverage technique. Also, Roncalli (2013) argue that this approach involves practitioners that are in the asset management industry.

As Qian (2013*a*) noted, risk parity portfolio is well diversified in terms of risk. Practitioners implementing this strategy incorporate large number of components[2]. They use strategic asset allocation (SAA), i.e., a technique that involves the selection of asset classes for the long-term investment.

[2] Example is the Korea Investment Corporation (KIC)

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Chapter 7

An Empirical Study of Risk-Based Strategies

In this Chapter, we conduct a statistical analysis of the risk-based strategies discussed in Chapter 2. In order to do a comparison, we consider the long-only constrain case.

7.1 Toy Example

We consider a simple universe of five components, $\mathbf{z} = \{z_1, \ldots, z_5\}$, where the input parameters of these strategies are volatilities and correlations of components, given in each part of the tables, 7.1 and 7.2, below. We provide in every strategy of each part the individual weights, the volatility, the risk contributions, and the diversification ratio; see Subsection 2.3.3.

Part A of Table 7.1 considers identical volatilities and equal pair-wise correlations of components. The results of these strategies are identical. However, part B exhibits different results after considering different volatilities of the components. ERC, IV and MD portfolios exhibit the same diversification ratios. Also, the composition of the portfolios are almost the same exhibiting identical volatilities as well as the Sharpe ratio. The MD and GMV strategies have little or no exposure to the highly volatile components. Furthermore, the GMV portfolio exhibits lower volatility and diversification ratio compared to other risk-based strategies.

We notice that for the cases of equal volatilities and different pair-wise correlations, the MD and GMV strategies exhibit the same results. Also the IV and EW strategies have the same results; see part C of Table 7.2. However, for the cases of different pair-wise correlations and volatilities, the GMV and MD strategies still short some components and they have large exposure in low-volatile components. Obviously, the EW strategy is exhibiting high risk in all cases where inputs are not all the same though the allocation is consistent.

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Table 7.1: Performance Analysis of the Risk-Based Strategies with Simple Input Parameters (part A and B)

	Input Parameters								Strategies														
	Assets	Volatilities (%)	Correlations (%)					GMV			MD			ERC			IV			EW		
			z_1	z_2	z_3	z_4	z_5	z_i	$\mathcal{MC}_i$	$\mathcal{RC}_i$	z_i	$\mathcal{MC}_i$	$\mathcal{RC}_i$	z_i	$\mathcal{MC}_i$	$\mathcal{RC}_i$	z_i	$\mathcal{MC}_i$	$\mathcal{RC}_i$	z_i	$\mathcal{MC}_i$	$\mathcal{RC}_i$
Part A	z_1	20	100	20	20	20	20	20	12	2.40	20	12	2.40	20	12	2.40	20	12	2.40	20	12	2.40
	z_2	20	20	100	20	20	20	20	12	2.40	20	12	2.40	20	12	2.40	20	12	2.40	20	12	2.40
	z_3	20	20	20	100	20	20	20	12	2.40	20	12	2.40	20	12	2.40	20	12	2.40	20	12	2.40
	z_4	20	20	20	20	100	20	20	12	2.40	20	12	2.40	20	12	2.40	20	12	2.40	20	12	2.40
	z_5	20	20	20	20	20	100	20	12	2.40	20	12	2.40	20	12	2.40	20	12	2.40	20	12	2.40
	$\mathbf{z}^T\mathbb{1}$							100			100			100			100			100		
	$\sigma(\mathbf{z})$							12			12			12			12			12		
	DR							1.67			1.67			1.67			1.67			1.67		
Part B	z_1	30	100	20	20	20	20	1.12	8.96	0.10	12.58	18	2.26	12.5	17.96	2.25	12.6	18	2.26	20	19.17	3.83
	z_2	40	20	100	20	20	20		10.76		9.43	24	2.26	9.43	24.01	2.26	9.43	24	2.26	20	29.83	5.97
	z_3	10	20	20	100	20	20	70.31	8.96	6.30	37.74	6	2.26	37.74	6	2.27	37.74	6	2.26	20	4.26	0.85
	z_4	15	20	20	20	100	20	24.55	8.96	2.20	25.16	9	2.26	25.16	9.01	2.27	25.16	9	2.26	20	7.19	1.44
	z_5	25	20	20	20	20	100	4.02	8.96	0.36	15.09	15	2.26	15.09	15.01	2.26	15.09	15	2.26	20	14.65	2.93
	$\mathbf{z}^T\mathbb{1}$							100			100			100			100			100		
	$\sigma(\mathbf{z})$							8.96			11.32			11.31			11.32			15.02		
	DR							1.34			1.67			1.67			1.67			1.60		

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Table 7.2: Performance Analysis of the Risk-Based Strategies with Simple Input Parameters (part C and D)

	Input Parameters								Strategies														
	Assets	Volatilities (%)	Correlations (%)					GMV			MD			ERC			IV			EW		
			z_1	z_2	z_3	z_4	z_5	z_i	$\mathcal{MC}_i$	$\mathcal{RC}_i$	z_i	$\mathcal{MC}_i$	$\mathcal{RC}_i$	z_i	$\mathcal{MC}_i$	$\mathcal{RC}_i$	z_i	$\mathcal{MC}_i$	$\mathcal{RC}_i$	z_i	$\mathcal{MC}_i$	$\mathcal{RC}_i$
Part C	z_1	20	100	80	20	30	50	33.23	14.28	4.75	33.23	14.28	4.75	20.25	14.67	2.97	20	14.91	2.98	20	14.91	2.98
	z_2	20	80	100	25	45	65	2.42	14.28	0.35	2.42	14.28	0.35	18.05	16.46	2.97	20	16.78	3.36	20	16.78	3.36
	z_3	20	20	25	100	30	75	34.89	14.28	4.98	34.89	14.28	4.98	21.95	13.53	2.97	20	13.32	2.66	20	13.32	2.66
	z_4	20	30	45	30	100	35	29.46	14.28	4.21	29.46	14.28	4.21	22.39	13.26	2.97	20	12.78	2.56	20	12.78	2.56
	z_5	20	50	65	75	35	100		15.31			15.31		17.36	17.11	2.97	20	17.31	3.46	20	17.31	3.46
	$\mathbf{z}^T\mathbb{1}$							100			100			100			100			100		
	$\sigma(\mathbf{z})$							14.28			14.28			14.85			15.02			15.02		
	DR							1.4			1.4			1.35			1.33			1.33		
Part D	z_1	30	100	80	20	30	50	1.18	9.32	0.11	16.73	21.42	3.58	12.25	22	2.7	12.58	22.37	2.81	20	25.23	5.05
	z_2	40	80	100	25	45	65		15.96		0.91	28.56	0.26	8.19	32.91	2.7	9.43	33.56	3.17	20	37.17	7.43
	z_3	10	20	25	100	30	75	76.07	9.33	7.10	52.76	7.14	3.76	39.84	6.77	2.7	37.74	6.66	2.51	20	5.11	1.02
	z_4	15	30	45	30	100	35	22.75	9.33	2.12	29.66	10.71	3.18	27.10	9.95	2.7	25.16	9.59	2.41	20	8.37	1.67
	z_5	25	50	65	75	35	100		18.97			19.14		12.61	21.38	2.7	15.09	21.64	3.27	20	20.44	4.09
	$\mathbf{z}^T\mathbb{1}$							100			100			100			100			100		
	$\sigma(\mathbf{z})$							9.33			10.78			13.48			14.17			19.26		
	DR							1.22			1.4			1.35			1.33			1.25		

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7.2 Analysis of Risk-Based Strategies with Real Data

The second analysis uses two datasets taken from the yahoo.finance website. The first dataset, Dataset1, is a universe of the following tickers: SPY, MDY, IWM, EFA, VEIEX, VWEHX, VFSUX and TLT over the period July 2002 to August 2014. In Table 7.3, we provide a descriptive statistics of these tickers. Note that VFSUX is the only component exhibiting low-volatility during this period.

Table 7.3: Descriptive Statistics for Dataset1

	Assets							
	SPY	MDY	IWM	EFA	VEIEX	VWEHX	VFSUX	TLT
Mean (%)	0.77	1.00	1.00	0.78	1.29	0.65	0.29	0.64
Std. Error	0.0035	0.0041	0.0046	0.0044	0.0056	0.0020	0.0006	0.0032
Median	0.0127	0.0144	0.0169	0.012	0.0109	0.0088	0.0037	0.0075
Std. Dev (%)	4.21	4.93	5.56	5.29	6.68	2.45	0.71	3.88
Sample Variance (%)	0.18	0.24	0.31	0.28	0.45	0.06	0.00	0.15
Kurtosis	1.93	2.66	1.20	1.73	2.02	13.62	9.64	2.60
Skewness	-0.7890	-0.7375	-0.5066	-0.7185	-0.6400	-1.8296	-1.3569	0.3891
Range	27.45	36.32	36.34	34.04	45.83	23.98	6.27	27.41
Minimum (%)	-16,52	-21.55	-20.96	-20.83	-27.67	-15.50	-3.46	-13.07
Maximum (%)	10.92	14.78	15.39	13.21	18.16	8.49	2.82	14.34
Sum	1.1230	1.4529	1.4437	1.1348	1.8759	0.9464	0.4204	0.9316
Count	145	145	145	145	145	145	145	145

Table 7.4 depicts the correlation matrix of these component arithmetic returns, with TLT exhibiting negative correlation with almost all the components in a universe except VFSUX. The pair-wise correlations of components range from -0.3566 to 0.9660. Their corresponding covariance matrix is given in Table 7.5 and we notice that this matrix is not positively definite.

Table 7.4: Correlation Matrix of Monthly Asset Returns

Assets	SPY	MDY	IWM	EFA	VEIEX	VWEHX	VFSUX	TLT
SPY	1							
MDY	0.9403	1						
IWM	0.9125	0.9660	1					
EFA	0.8978	0.8560	0.8193	1				
VEIEX	0.8120	0.8136	0.7673	0.8854	1			
VWEHX	0.6594	0.6903	0.6285	0.6736	0.6643	1		
VFSUX	0.3192	0.3349	0.2420	0.4130	0.4326	0.6869	1	
TLT	-0.3214	-0.3362	-0.3566	-0.2212	-0.2506	-0.1288	0.1155	1

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Table 7.5: Covariance Matrix of Monthly Asset Returns

Assets	SPY	MDY	IWM	EFA	VEIEX	VWEHX	VFSUX	TLT
SPY	0.00176	0.00194	0.00212	0.00199	0.00277	0.00068	0.00009	-0.00052
MDY	0.00194	0.00241	0.00263	0.00222	0.00266	0.00083	0.00012	-0.00064
IWM	0.00212	0.00263	0.00307	0.00239	0.00283	0.00085	0.00009	-0.00076
EFA	0.00199	0.00222	0.00239	0.00278	0.00311	0.00087	0.00015	-0.00045
VEIEX	0.00227	0.00266	0.00283	0.00311	0.00444	0.00108	0.00020	-0.00065
VWEHX	0.00068	0.00083	0.00085	0.00087	0.00108	0.00060	0.00012	-0.00012
VFSUX	0.00009	0.00012	0.00009	0.00015	0.00020	0.00012	0.00005	0.00003
TLT	-0.00052	-0.00064	-0.00076	-0.00045	-0.00065	-0.00012	0.00003	0.00149

Table 7.6: Statistical Analysis of Risk-Based Strategies for Dataset1

	Strategies				
Assets	GMV	ERC	MD	IV	EW
SPY	0%	6.26%	1.93%	7.80%	12.5%
MDY	0%	5.32%	0%	6.67%	12.5%
IWM	0%	5.12%	16.80%	5.91%	12.5%
EFA	0%	4.67%	0%	6.21%	12.5%
VEIEX	0%	3.90%	2.36%	4.92%	12.5%
VWEHX	0%	10.55%	0%	13.40%	12.5%
VFSUX	98.79%	43.68%	49.04%	46.62%	12.5%
TLT	1.21%	20.50%	29.87%	8.47%	12.5%
$\mathbf{z}^T\boldsymbol{\sigma}$	0.74	2.68	2.68	2.63	4.21
$DR(\mathbf{z})$	1.06	1.68	1.84	1.38	1.28
$\sigma(\mathbf{z})$	0.70%	1.60%	1.46%	1.91%	3.28%

Table 7.7: Component Marginal and Risk Contributions

	Strategies									
	GMV		IV		ERC		MD		EW	
Assets	$\mathcal{MC}_i(\mathbf{z})$	$\%\mathcal{RC}_i$	$\mathcal{MC}_i(\mathbf{z})$	$\%\mathcal{RC}_i$	$\mathcal{MC}_i(\mathbf{z})$	$\%\mathcal{RC}_i$	$\mathcal{MC}_i(\mathbf{z})$	$\%\mathcal{RC}_i$	$\mathcal{MC}_i(\mathbf{z})$	$\%\mathcal{RC}_i$
SPY	0.0124	0	0.0377	15.4	0.0319	12.5	0.0229	3	0.0394	15
MDY	0.0152	0	0.0444	15.6	0.0375	12.5	0.0280	0	0.0463	17.7
IWM	0.0120	0	0.0474	14.7	0.0392	12.6	0.0303	34.9	0.0504	19.2
EFA	0.0208	0	0.0483	15.7	0.0427	12.5	0.0312	0	0.0498	19.0
VEIEX	0.0274	0	0.0587	15.1	0.0512	12.5	0.0364	5.9	0.0608	23.2
VWEHX	0.0164	0	0.0205	14.4	0.0189	12.5	0.0139	0	0.0187	7.1
VFSUX	0.0070	98.8	0.0043	10.5	0.0045	12.4	0.0038	12.9	0.0033	1.2
TLT	0.0070	1.2	−0.0033	−1.5	0.0097	12.5	0.0211	43.3	−0.0062	−2.3

As shown analytically by Scherer (2011), the minimum variance portfolio tends to overweight low-volatile components; see Table 7.6. It dominates in terms of low volatility compared to other risk-based strategies. Also, we report in Figure 7.4 and 7.5, respectively, the time series of portfolio allocations and observe a strong variation of the GMV strategy in Dataset2, exhibiting the highest portfolio turnover; see Figure 7.7.

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We confirm Choueifaty *et al.* (2013)'s result that the MD strategy allocates to low-volatile and often to low pair-wise correlation components. In terms of stability, the MD strategy shows strong variations of the allocation of portfolios in both the datasets. It shares the similarity with the GMV strategy in that they both exhibit high portfolio turnover in both the datasets.

In contrast, the ERC, IV and EW strategies consider all components in a universe for the portfolio's compositions. In particular, the ERC and IV strategies allocate more wealth to low-volatile components; see Table 7.6. The EW portfolio dominates in terms of risk and perhaps this is because practitioners using this approach often take risk that is not compensated.

Figures 7.1, 7.2 and 7.3 we present backtesting performance of these strategies using the same period of dataset except on the last figure which is determined through the MSCI Indices of 22-countries. Since 2002, all these strategies show gradual growth until the impact of the 2007-2008 financial crisis.

The second dataset, Dataset2, is composed of MSCI Index of the following ticker: EWA, EWJ, EWY, EWG, EWW, EFA, EEM, and EWZ. Table 7.9 depicts the analysis of both the marginal and risk contributions of components for different strategies mentioned above. Note that, in this case, C denotes the normalised risk contributions of components. We observe, as noted in theory that the GMV strategy turns to equalize marginal risk contributions of components. Also the ERC strategy preserves its constraint of equalizing components risk contributions. Lastly, EW strategy has different structure of risk contributions and this is because it does not take into account the parameter estimate for the allocation of asset weights.

Table 7.8: Covariance Matrix of Monthly Asset Returns for Dataset 2

Assets	EWA	EWJ	EWY	EWG	EWW	EFA	EEM	EWZ
EWA	0.0050	0.0024	0.0048	0.0041	0.0040	0.0034	0.0045	0.0009
EWJ	0.0024	0.0026	0.0027	0.0024	0.0024	0.0022	0.0024	0.0006
EWY	0.0048	0.0027	0.0073	0.0047	0.0045	0.0036	0.0053	0.0010
EWG	0.0041	0.0024	0.0047	0.0050	0.0039	0.0036	0.0042	0.0010
EWW	0.0040	0.0024	0.0045	0.0039	0.0050	0.0032	0.0043	0.0006
EFA	0.0034	0.0022	0.0036	0.0036	0.0032	0.0029	0.0034	0.0008
EEM	0.0045	0.0024	0.0053	0.0042	0.0043	0.0034	0.0050	0.0009
EWZ	0.0009	0.0006	0.0010	0.0010	0.0006	0.0008	0.0009	0.0087

In Figure 7.3, we back-tested these strategies using monthly returns of the MSCI index from 15-countries. The period of this simulation is from April 1996 to February 2014. We observe that prior 2005, MD strategy has not shown good performance in terms of cumulative return against other risk-based strategies. However, starting from 2005 to February 2014 of the simulation,

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Table 7.9: Marginal and Risk Contribution of Assets

	Strategies								
	GMV			ERC			EW		
Assets	$(\Sigma\mathbf{z})_i$	$\mathcal{RC}_i(\mathbf{z})$	C_i	$(\Sigma\mathbf{z})_i$	$\mathcal{RC}_i(\mathbf{z})$	C_i	$(\Sigma\mathbf{z})_i$	$\mathcal{RC}_i(\mathbf{z})$	C_i
EWA	0.0521	0.0000	0.00%	0.0627	0.0067	12.5%	0.0644	0.0080	14.2%
EWJ	0.0459	0.0258	56.2%	0.0392	0.0067	12.5%	0.0387	0.0048	8.5%
EWY	0.0570	0.0000	0.00%	0.0722	0.0067	12.5%	0.0750	0.0094	16.6%
EWG	0.0529	0.0000	0.00%	0.0622	0.0067	12.5%	0.0637	0.0080	14.1%
EWW	0.0499	0.0000	0.00%	0.0601	0.0067	12.5%	0.0619	0.0077	13.7%
EFA	0.0459	0.0120	26.0%	0.0502	0.0067	12.5%	0.0509	0.0064	11.2%
EEM	0.0517	0.0000	0.00%	0.0642	0.0067	12.5%	0.0662	0.0083	14.6%
EWZ	0.0459	0.0082	17.8%	0.0399	0.0067	12.5%	0.0322	0.0040	7.1%

Table 7.10: Statistical Analysis of Strategies

	Strategies						
Assets	GMV	ERC	EW	MD	Expected Return	Volatility	Sharpe Ratio
EWA	0.00%	10.73%	12.5%	3%	1.0135	7.09%	14.2912
EWJ	56.17%	17.16%	12.5%	29%	1.0071	5.08%	19.8275
EWY	0.00%	9.32%	12.5%	14%	1.0147	8.59%	11.8098
EWG	0.00%	10.83%	12.5%	4%	1.0119	7.07%	14.3109
EWW	0.00%	11.19%	12.5%	16%	1.0167	7.11%	14.3039
EFA	26.03%	13.41%	12.5%	0%	1.0089	5.40%	18.6866
EEM	0.00%	10.49%	12.5%	0%	1.0139	7.08%	14.3213
EWZ	17.81%	16.88%	12.5%	34%	1.0180	9.35%	10.8856
$\mathbf{z}^T\bar{\mathbf{r}}$	1.0095	1.0129	1.0131	1.0138			
$\sigma(\mathbf{z})$	4.59%	5.38%	5.66%	5.25%			
$\mathbf{s}$	21.97707	18.81496	17.89291	19.2951			
$DR(\mathbf{z})$	1.2895	1.3075	1.2534	1.4236			

MD performed better than the remaining strategies. Around 2005, ERC and Risk Parity (IV) have shown good performance against the EW strategy. We should note that the analysis was based on the indices that are affected by the same market risk.

We notice that prior to the year 2007, the RP strategy has been outperformed by the MD strategy. But, from 2009 until 2014, RP shows good performance, outperforming the MD strategy. However, Anderson *et al.* (2012) noted that period of analysis has a significant impact on the results exhibited by back-testing. This is due to the fact that other assets are more recent and as such longer periods might lead to uneven results.

In contrast, RP strategies, i.e., the ERC and IV, seem to be more diversified as all assets play a role in its performance. Looking at the volatilities of components in Table 7.10, EWJ seems to be dominating and hence allocated significant weight. For the EW strategy, the risk contribution of asset EWZ

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Figure 7.1: Back-Testing of Risk-Based Strategies for Dataset1

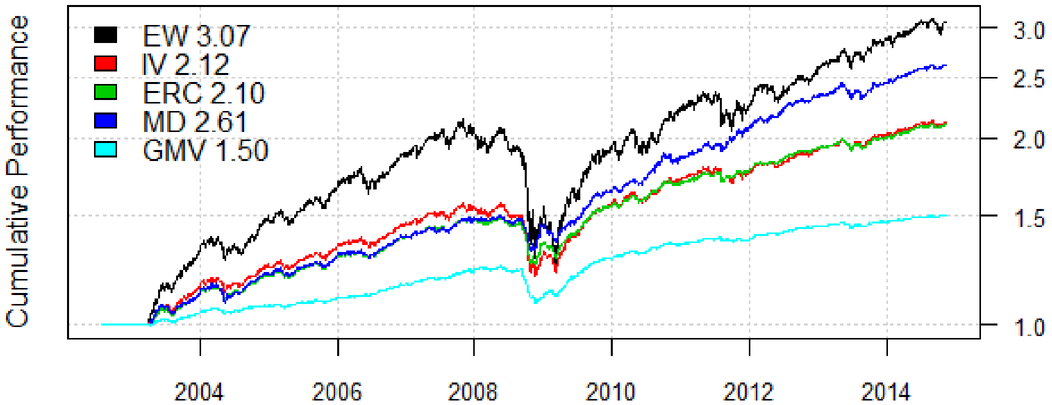


Figure 7.2: Back-Testing of Risk-Based Strategies for Dataset2

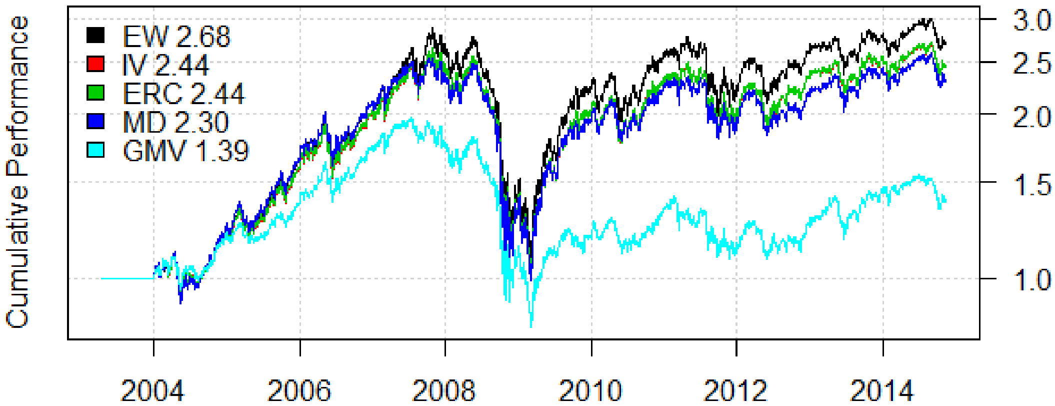
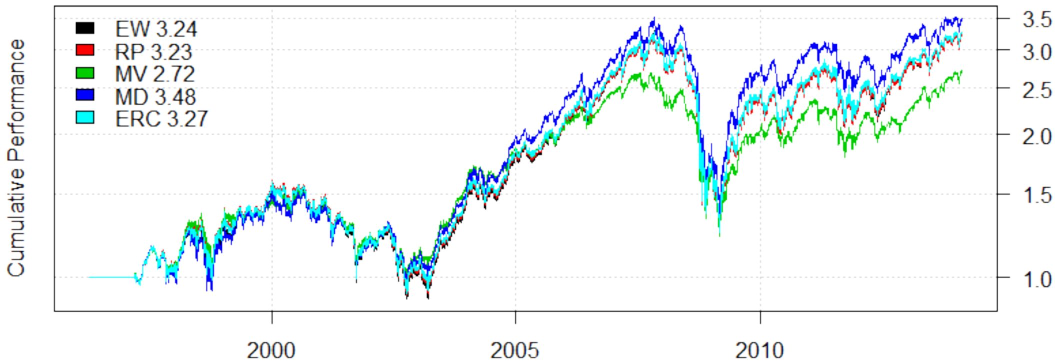


Figure 7.3: Back-Testing of Risk-Based Strategies for MSCI Index of 15-Countries



is significant even though its volatility is high. We remark that indeed[1] the volatility of the RP strategy lies between the volatility of the other strategies, in particular, the EW and the GMV portfolios as noted by Maillard *et al.* (2010).

[1] Based on the choosen assets.

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Figure 7.4: Time Series Portfolio Weights of Risk-Based Strategies for Dataset1

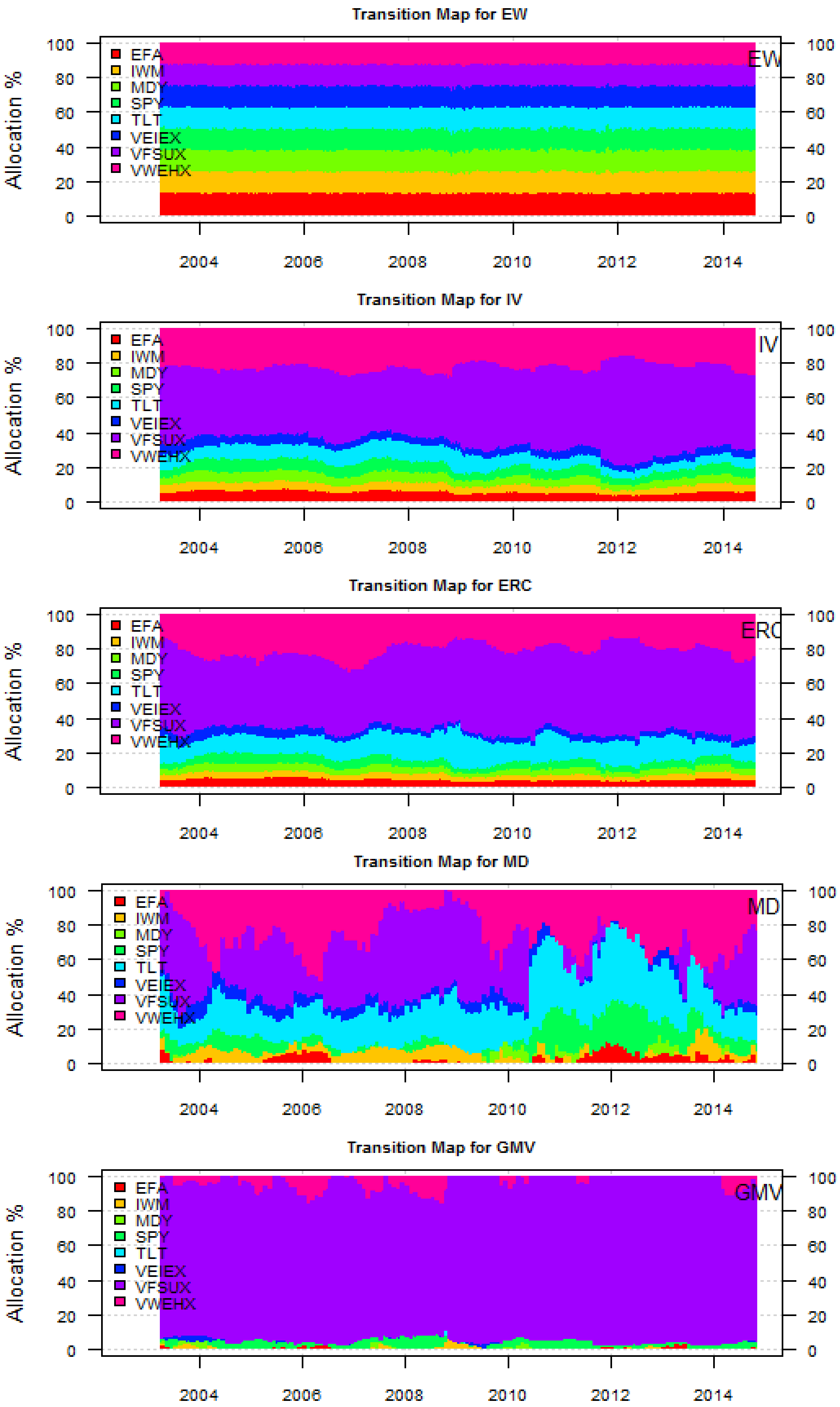

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Figure 7.5: Time Series Portfolio Weights of Risk-Based Strategies for Dataset2

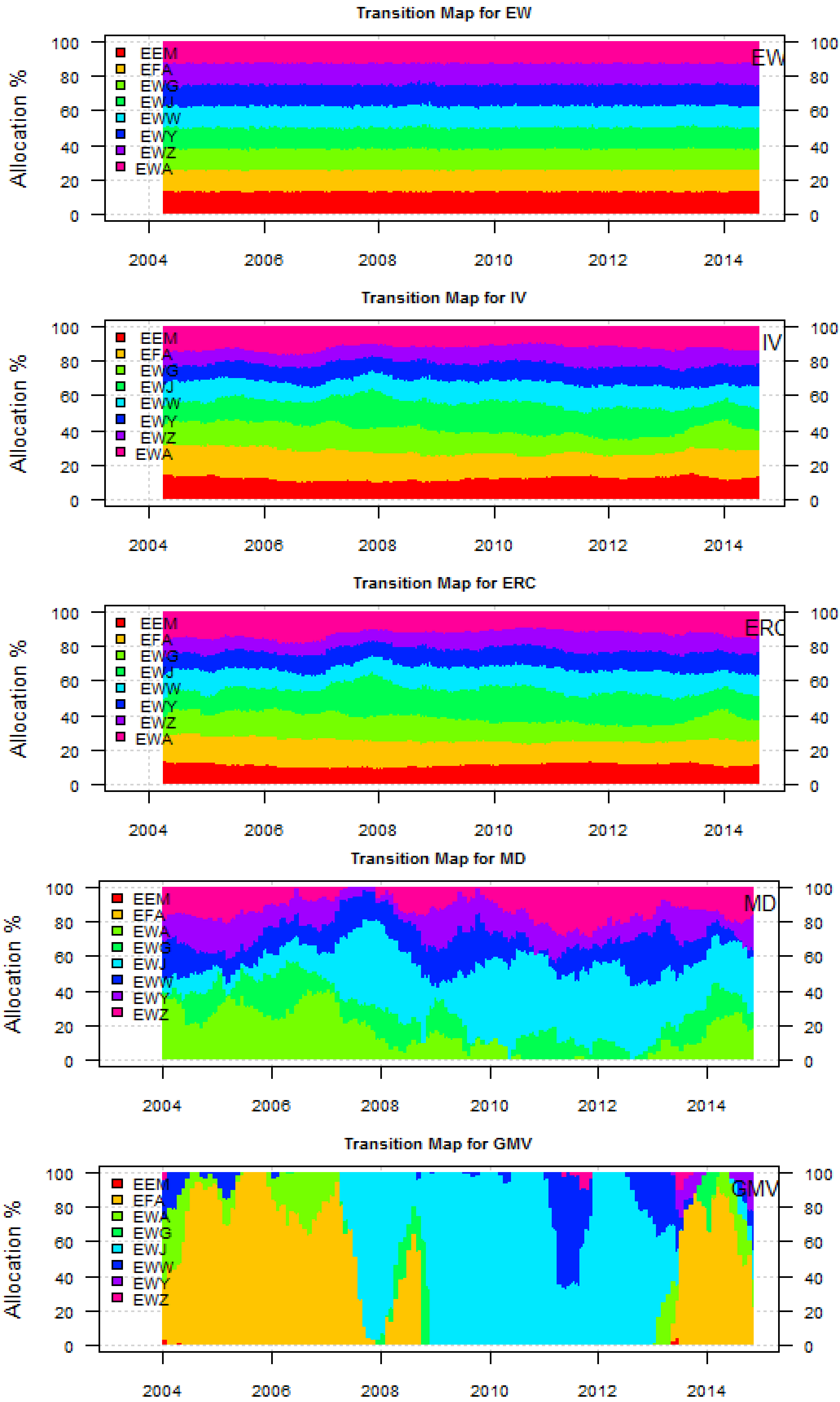

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Figure 7.6: Annual Average Turnover of Risk-Based Strategies for Dataset1

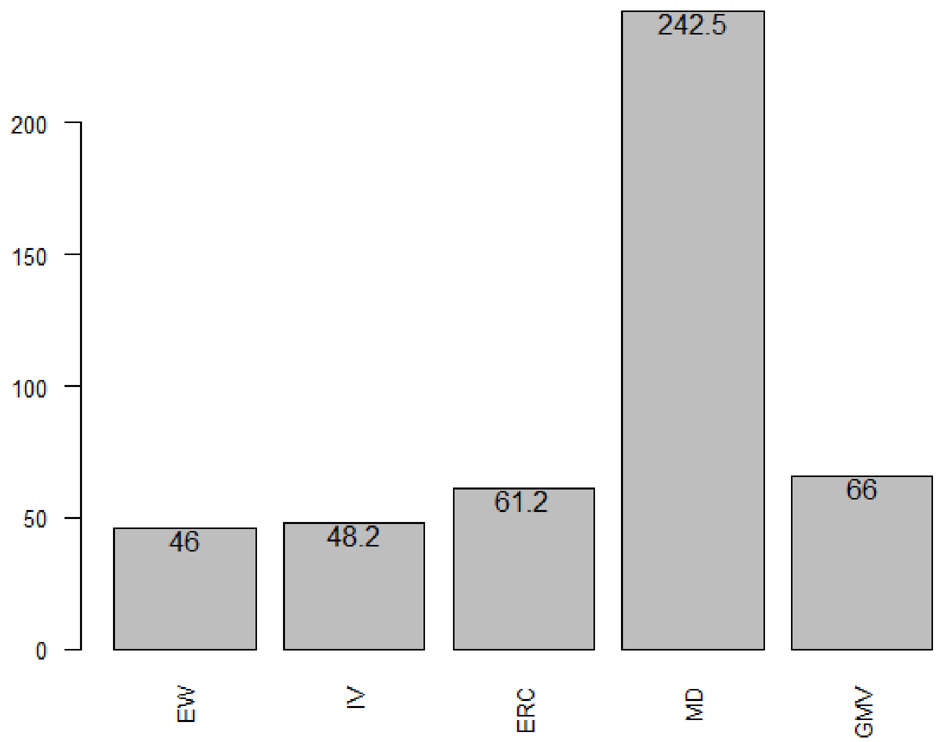
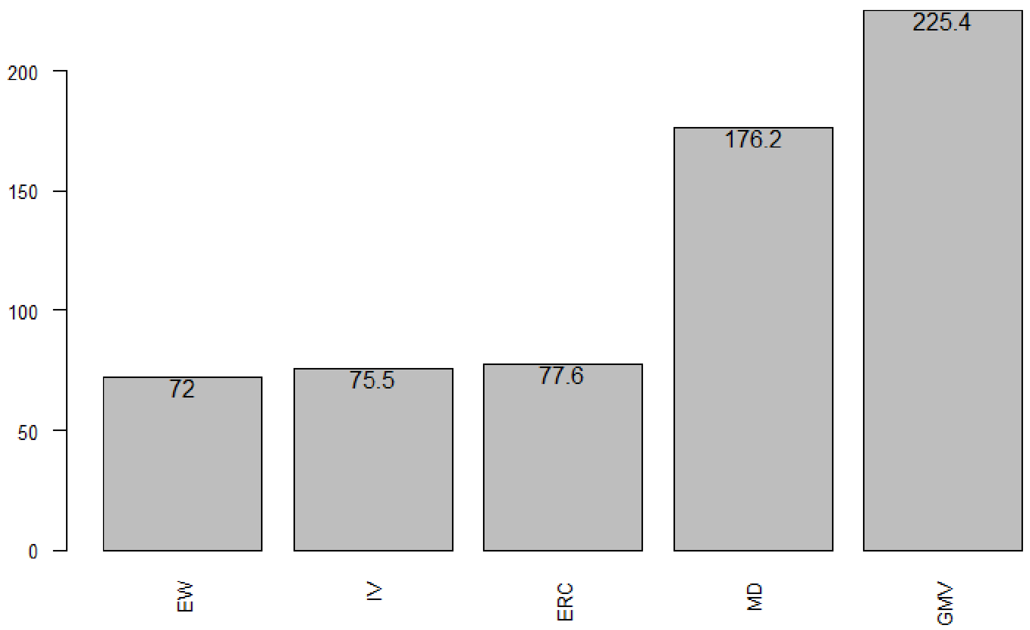


Figure 7.7: Annual Average Turnover of Risk-Based Strategies for Dataset2



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Chapter 8

Risk Parity and Stochastic Portfolio Theory

In this chapter, we turn our attention to the problem of maximizing portfolio's absolute log wealth, at a fixed tracking error risk level. We present Oderda (2013)'s approach to determine the optimum solution to this problem. The author deploys stochastic portfolio theory (SPT) as another version of investment approach in which the focus is on the markets behaviour and arbitrage opportunities. This theory is based on the work of Fernholz (2002) for the optimization of stochastic portfolios. It originates from the idea of Fernholz and Shay (1982) for the analysis of long term portfolio performance in continuous time. Oderda (2013) showed that the solution to the portfolio maximizing relative log wealth is a linear combination of the market portfolio, and of four alternative allocation strategies:

1. The maximum expected cash flow rate of return portfolio.
2. The equally weighted portfolio.
3. The risk parity portfolio.
4. The global minimum variance portfolio.

This provides an immediate link between the risk-based strategies and the utility theory. The latter could be strategies depending on the focus of expected returns of dividend or coupons paying assets.

8.1 Stochastic Portfolio Theory

In this section we present the basic structures of stochastic portfolio model from Fernholz (2002).

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8.1.1 The Stochastic Model

The standard model $\mathcal{M}$ of financial markets is usually described under filtered probability space $(\Omega, \mathbb{F}, \mathbb{P}, \mathcal{F}_t)$ with the notations described as follows,

1. Ω denotes a set containing all different states[1] of the economy.
2. $\mathcal{F}_t$ is the information exhibited by the Brownian motion $W(t) \in \mathbb{R}^d$ at time $t \in [0, \infty)$.
3. $\mathbb{F}$ denotes a set of all information available in the market.

Considering a set of n-risky assets and the risk-free asset with the price process given as $x_i(t)$ for $i = 1, \ldots, n$ and $x_f(t)$, respectively. The dynamics of assets prices are given by:

$$dx_i(t) = x_i(t)\left(\bar{r}_i(t)dt + \boldsymbol{\sigma}_i^T(t)dW(t)\right) \tag{8.1.1}$$

and

$$dx_f(t) = x_f(t)r_f(t)dt \quad t \in [0, \infty)\,. \tag{8.1.2}$$

The notations are described as follows:

1. $\boldsymbol{\sigma}_i^T(t) \in \mathbb{R}^{1\times d}$ is a vector of stock price volatilities on d-dimensional Brownian motion W.
2. $\boldsymbol{\sigma}(t) \in \mathbb{R}^{n\times d}$ is the diffusion matrix of the Brownian motion.
3. $r_f(t)$ and $\bar{r}_i(t)$ denotes the risk-free rate and the mean rate of returns respectively.

More intuitively, the positive definite covariance matrix is defined as:

$$\Sigma(t) = \boldsymbol{\sigma}(t)\boldsymbol{\sigma}^T(t). \tag{8.1.3}$$

The growth rate process of the stocks is defined as:

$$\gamma_i(t) = \bar{r}_i(t) - \frac{\sigma_i^2(t)}{2}. \tag{8.1.4}$$

The logarithmic return of the i^{th} asset is defined as:

$$d\ln x_i(t) = \gamma_i(t)dt + \boldsymbol{\sigma}^T(t)dW(t), \tag{8.1.5}$$

[1] Historical evolution of security prices.

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8.1.2 Settings of Stochastic Portfolio Processes

We define in this subsection a portfolio as stochastic process with weights given as function of time. We restrict ourself to n-risky assets and fixed budgeting constraint and denote the portfolio as:

$$\mathbf{z}(t) = (z_1(t), \ldots, z_n(t))^T. \tag{8.1.6}$$

The logarithmic return of the portfolio is given as:

$$d \ln V_{\mathbf{z}}(t) = \gamma_{\mathbf{z}}(t)dt + \boldsymbol{\sigma}_{\mathbf{z}}^T(t)dW(t), \tag{8.1.7}$$

where $V_{\mathbf{z}}(t)$ is the value of the portfolio at time t, $\gamma_{\mathbf{z}}(t) = \mathbf{z}^T(t)\gamma(t) + \gamma_{\mathbf{z}}^*(t)$ and $\boldsymbol{\sigma}_{\mathbf{z}}^T(t) = \mathbf{z}^T(t)\boldsymbol{\sigma}(t)$ denote the growth rate and volatility of the value process, respectively.

The value process of the portfolio is given as the sum of the martingale component and the drift term given by both the rate of growth of the assets and the portfolio, including portfolio cash flow. It is given by:

$$\begin{aligned} d \ln V_{\mathbf{z}}(t) = \mathbf{z}^T(t)\gamma(t)dt + \frac{1}{2}\left[\mathbf{z}^T D_{\boldsymbol{\sigma}(t)} - \mathbf{z}^T(t)\Sigma(t)\mathbf{z}(t)\right] dt \\ + \mathbf{z}^T(t)\delta(t)dt + \boldsymbol{\sigma}_{\mathbf{z}}(t)dW(t), \end{aligned}$$

where $D_{\boldsymbol{\sigma}(t)}$ is a diagonal matrix with entries in the main diagonal given by as component volatilities.

8.1.2.1 Optimization of log-Wealth Portfolio

We assume that the investor objective is to maximize the portfolio wealth (or value) for a fixed volatility. The problem can be specified mathematically as:

$$\begin{aligned} \mathbf{z}^*(t) &= \arg\max_{\mathbf{z}\in\mathbb{R}^n} U(\mathbf{z}(t), \boldsymbol{\sigma}(t), \gamma(t), \delta(t)) \\ \text{Subject to} &\quad \begin{cases} \mathbf{z}^T(t)\mathbb{1} = 1 \\ \mathbf{z}^T(t)\Sigma(t)\mathbf{z}(t) = \sigma_0(t), \end{cases} \end{aligned} \tag{8.1.8}$$

where $U(\cdot)$ denotes the function to be maximized. Since investors will be interested in maximizing the log-wealth of the portfolio, we define this function as:

$$U(\cdot) = \mathbf{z}^T(t)\gamma(t) + \frac{1}{2}\left[\mathbf{z}^T D_{\boldsymbol{\sigma}(t)} - \mathbf{z}^T(t)\Sigma(t)\mathbf{z}(t)\right] + \mathbf{z}^T(t)\delta(t). \tag{8.1.9}$$

We notice that the diffusion term in our utility function is not present and this is because we are maximizing the drift term, not the noise. Thus, the Lagrange function of the above problem is:

$$\begin{aligned} \mathcal{L} = \mathbf{z}^T(t)\gamma(t) + \frac{1}{2}\left[\mathbf{z}^T D_{\boldsymbol{\sigma}(t)} - \mathbf{z}^T(t)\Sigma(t)\mathbf{z}(t)\right] + \mathbf{z}^T(t)\delta(t) \\ - \lambda_1(\mathbf{z}^T(t)\mathbb{1} - 1) - \lambda_2(\mathbf{z}^T(t)\Sigma(t)\mathbf{z}(t) - \sigma_0(t)), \end{aligned}$$

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where $\mathbf{z}^T(t)\mathbb{1} - 1$ and $\mathbf{z}^T(t)\Sigma(t)\mathbf{z}(t) - \sigma_0(t)$ are budget and risk constraints respectively. λ_1 and λ_2 are the Lagrange multipliers. The first derivative of the above equation with respect to $\mathbf{z}(t)$ yields:

$$\gamma(t) + \frac{1}{2}\left[D_{\boldsymbol{\sigma}(t)} - 2\Sigma(t)\mathbf{z}(t)\right] + \delta(t) - \lambda_1\mathbb{1} - 2\lambda_2\Sigma(t)\mathbf{z}(t) = 0. \tag{8.1.10}$$

We have,

$$\mathbf{z}^*(t) = \frac{\Sigma^{-1}(t)\left[\gamma(t) + \frac{1}{2}D_{\boldsymbol{\sigma}(t)} + \delta(t)\right]}{1 + 2\lambda_2} - \frac{\lambda_1\Sigma^{-1}(t)\mathbb{1}}{1 + 2\lambda_2} \tag{8.1.11}$$

This solution could be written as:

$$\mathbf{z}^*(t) = A\frac{\Sigma^{-1}(t)\left[\gamma(t) + \frac{1}{2}D_{\boldsymbol{\sigma}(t)} + \delta(t)\right]}{\mathbb{1}^T\Sigma^{-1}(t)\left[\gamma(t) + \frac{1}{2}D_{\boldsymbol{\sigma}(t)} + \delta(t)\right]} + B\frac{\Sigma^{-1}(t)\mathbb{1}}{\mathbb{1}^T\Sigma^{-1}(t)\mathbb{1}}, \tag{8.1.12}$$

where the coefficients A and B are as follows:

$$A = \frac{\mathbb{1}^T\Sigma^{-1}(t)\left[\gamma(t) + \frac{1}{2}D_{\boldsymbol{\sigma}(t)} + \delta(t)\right]}{1 + 2\lambda_2} \quad \text{and } B = \frac{\lambda_1\mathbb{1}^T\Sigma^{-1}(t)\mathbb{1}}{1 + 2\lambda_2}.$$

8.2 Link between the Risk-Based Strategies and the Portfolio Maximizing Log-Wealth

In order to determine the link between the risk-based strategies and the portfolio maximizing the log-wealth, we follow Oderda (2013)'s approach in which the covariance matrix is the key parameter that enable us to deduce the relationship. We have seen that correlations and components volatilities play a major role in asset allocation. In particular, several characteristics of strategies are observed using correlations and variances of assets.

Our interest lies on the product of the covariance matrix and vector of components volatilities denoted by $\Sigma^{-1}(t)D_{\sigma(t)}$ in equation (8.1.12). We begin by decomposing our covariance as a product of two matrices using Hadamard criteria for matrix multiplication. This is an entry-wise multiplication of matrices. If we denote by $C(t) \in \mathbb{R}^{n\times n}$ the original correlation matrix of asset returns at time t and $\Gamma(t)$ a matrix of product of component volatilities, then the covariance matrix can be written as:

$$\Sigma(t) = \Gamma(t)\bullet C(t). \tag{8.2.1}$$

Since $\Gamma(t)$ is given by the product of volatilities, its main diagonal entries are variances of component returns.

Correlation also plays an important role in the characterization of a portfolio. For instance, the MD strategy allocates components such that their

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return correlations with the portfolio is uniform. Thus, it becomes relevant to decompose also $C(t)$ into uniform correlation matrix and the matrix formed by taking the difference (component-wise) of the original correlation matrix and the uniform correlation matrix. The decomposition is given as:

$$C(t) = C_0(t) + \Delta C(t), \tag{8.2.2}$$

where

$$C_0(t) = \begin{pmatrix} 1 & \bar{\rho}(t) & \cdots & \bar{\rho}(t) \\ \bar{\rho}(t) & 1 & \cdots & \bar{\rho}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\rho}(t) & \bar{\rho}(t) & \cdots & 1 \end{pmatrix}, \tag{8.2.3}$$

and $\bar{\rho}(t)$ is determine as follows:

$$\bar{\rho}(t) = \frac{1}{n(n-1)} \sum_{i \neq j}^{n} \rho_{i,j}. \tag{8.2.4}$$

$$\Delta C(t) = \begin{pmatrix} 0 & \rho_{1,2}(t) - \bar{\rho}(t) & \cdots & \rho_{1,n}(t) - \bar{\rho}(t) \\ \rho_{2,1}(t) - \bar{\rho}(t) & 0 & \cdots & \rho_{2,n}(t) - \bar{\rho}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n,1}(t) - \bar{\rho}(t) & \rho_{n,2}(t) - \bar{\rho}(t) & \cdots & 0 \end{pmatrix} \tag{8.2.5}$$

where $\rho_{i,j}$ denote the correlation between the i^{th} and j^{th} components in the $C(t)$. The inverse covariance matrix can now be expressed as:

$$\Sigma^{-1}(t) = \Gamma^{-1}(t) \bullet [C_0(t) + \Delta C(t)]^{-1}, \tag{8.2.6}$$

where entries of the matrix $\Gamma^{-1}(t)$ are given by:

$$\Gamma^{-1}_{i,j}(t) = [\sigma_i(t)\sigma_j(t)]^{-1} \quad i, j = 1, \dots, n. \tag{8.2.7}$$

Proposition 8.1. *The inverse of the decomposed correlation matrix satisfy the following expression:*

$$\left[C_0(t) + \Delta C(t)\right]^{-1} = C_0^{-1}(t) - \left[\mathbb{I} + C_0^{-1}(t)\Delta C(t)\right]^{-1} C_0^{-1}(t)\Delta C(t) C_0^{-1}(t), \tag{8.2.8}$$

where $C_0(t)$ is a uniform correlation matrix and $\Delta C(t)$ is as defined in equation (8.2.5).

Proof. Consider equation (8.2.2), and suppose that $C_0(t)$ and $\Delta C(t)$ are square invertible matrices. Then the inverse $C(t)$ follows the intuition below:

$$\left[C_0(t) + \Delta C(t)\right]^{-1} = C_0^{-1}(t) + X, \tag{8.2.9}$$

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where X is an unknown matrix. To find X, we multiply both side from the right by $\left[C_0(t) + \Delta C(t)\right]$, which yield the following:

$$\left[C_0^{-1}(t) + X\right]\left[C_0(t) + \Delta C(t)\right] = \left[C_0(t) + \Delta C(t)\right]^{-1}\left[C_0(t) + \Delta C(t)\right] = \mathbb{I}, \quad (8.2.10)$$

where $\mathbb{I}$ is the identity matrix. Expanding the left hand side, we have:

$$C_0^{-1}(t)C_0(t) + XC_0(t) + C_0^{-1}(t)\Delta C(t) + X\Delta C(t) = \mathbb{I}. \quad (8.2.11)$$

We know that any square invertible matrix multiplied by its inverse yield a diagonal matrix. Thus $\mathbb{I}$ can be written as follows:

$$\mathbb{I} = C_0^{-1}(t)C_0(t). \quad (8.2.12)$$

Thus,

$$X\left[C_0(t) + \Delta C(t)\right] = -C_0^{-1}(t)\Delta C(t). \quad (8.2.13)$$

Now, multiplying both side by $\left[C_0(t) + \Delta C(t)\right]^{-1}$, we have

$$\begin{aligned} X &= -C_0^{-1}(t)\Delta C(t)\left[C_0(t) + \Delta C(t)\right]^{-1} \\ &= -C_0^{-1}(t)\Delta C(t)\left[C_0^{-1}(t) + X\right] \\ &= -C_0^{-1}(t)\Delta C(t)C_0^{-1}(t) - C_0^{-1}(t)\Delta C(t)X \end{aligned} \quad (8.2.14)$$

Now, rearranging with subject to be X, we have

$$\left[\mathbb{I} + C_0^{-1}(t)\Delta C(t)\right]X = -C_0^{-1}(t)\Delta C(t)C_0^{-1}(t). \quad (8.2.15)$$

Multiplying both side from the left by $\left[\mathbb{I} + C_0^{-1}(t)\Delta C(t)\right]^{-1}$ gives:

$$X = -\left[\mathbb{I} + C_0^{-1}(t)\Delta C(t)\right]^{-1}C_0^{-1}(t)\Delta C(t)C_0^{-1}(t). \quad (8.2.16)$$

Thus, substituting X in equation (8.2.9), yields the following results:

$$\left[C_0(t) + \Delta C(t)\right]^{-1} = C_0^{-1}(t) - \left[\mathbb{I} + C_0^{-1}(t)\Delta C(t)\right]^{-1}C_0^{-1}(t)\Delta C(t)C_0^{-1}(t). \quad (8.2.17)$$

□

Our main interest is to deduce the characteristics of the risk-based strategies from this term $\Sigma^{-1}D_{\boldsymbol{\sigma}(t)}$. This can be written as follows:

$$\begin{aligned} \Sigma^{-1}D_{\boldsymbol{\sigma}(t)} &= \left[\Gamma^{-1}(t)\bullet C^{-1}(t)\right]D_{\boldsymbol{\sigma}(t)} \\ &= \left[\Gamma^{-1}(t)\bullet\left[C_0(t) + \Delta C^{-1}(t)\right]^{-1}\right]D_{\boldsymbol{\sigma}(t)} \end{aligned} \quad (8.2.18)$$

But, $C_0(t)$ satisfies the following expression:

$$C_0(t) = \tilde{\rho}(t)\Lambda_1 + (1 - \tilde{\rho}(t))\mathbb{I}, \quad (8.2.19)$$

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where Λ_1 denotes n-dimensional matrix of ones and $\mathbb{I}$ is the identity matrix. In matrix form, we have:

$$C_0(t) = \begin{pmatrix} \tilde{\rho}(t) & \tilde{\rho}(t) & \cdots & \tilde{\rho}(t) \\ \tilde{\rho}(t) & \tilde{\rho}(t) & \cdots & \tilde{\rho}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\rho}(t) & \tilde{\rho}(t) & \cdots & \tilde{\rho}(t) \end{pmatrix} + \begin{pmatrix} 1-\tilde{\rho}(t) & 0 & \cdots & 0 \\ 0 & 1-\tilde{\rho}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-\tilde{\rho}(t) \end{pmatrix}. \tag{8.2.20}$$

Thus, $C_0(t)$ is commutative, therefore, there exist constants such that the inverse of $C_0(t)$ is given by,

$$C_0^{-1}(t) = \phi\Lambda_1 + \psi\mathbb{I}. \tag{8.2.21}$$

Using the following properties,

$$\begin{aligned} \Lambda_1\Lambda_1 &= n\Lambda_1, \\ \Lambda_1\mathbb{I} &= \Lambda_1, \end{aligned}$$

we can determine equation (8.2.12) by multiplying equations (8.2.21) and (8.2.19), i.e.,

$$\begin{aligned} C_0^{-1}(t)C_0(t) &= [\phi\Lambda_1 + \psi\mathbb{I}]\left[\tilde{\rho}(t)\Lambda_1 + (1-\tilde{\rho}(t))\mathbb{I}\right] \\ &= \phi\tilde{\rho}(t)\Lambda_1\Lambda_1 + \phi(1-\tilde{\rho}(t))\Lambda_1\mathbb{I} + \psi\tilde{\rho}(t)\mathbb{I}\Lambda_1 + \psi(1-\tilde{\rho}(t))\mathbb{I}\mathbb{I} \\ &= n\tilde{\rho}(t)\phi\Lambda_1 + \phi(1-\tilde{\rho}(t))\Lambda_1 + \psi\tilde{\rho}(t)\Lambda_1 + \psi(1-\tilde{\rho}(t))\mathbb{I} \end{aligned} \tag{8.2.22}$$

This implies that $\mathbb{I}$ is given by:

$$\mathbb{I} = \left[n\phi\tilde{\rho}(t) + \psi\tilde{\rho}(t) + \phi(1-\tilde{\rho}(t))\right]\Lambda_1 + \psi(1-\tilde{\rho}(t))\mathbb{I}, \tag{8.2.23}$$

and this equation is satisfied for:

$$\psi = \frac{1}{1-\tilde{\rho}(t)}, \quad \text{and} \tag{8.2.24}$$

$$\phi = -\frac{\tilde{\rho}(t)}{(1-\tilde{\rho}(t))\left[n\tilde{\rho}(t) + (1-\tilde{\rho}(t))\right]}. \tag{8.2.25}$$

Substituting these equations in equation (8.2.21), yield the following:

$$C_0^{-1}(t) = -\frac{\tilde{\rho}(t)}{(1-\tilde{\rho}(t))\left[n\tilde{\rho}(t) + (1-\tilde{\rho}(t))\right]}\Lambda_1 + \frac{1}{1-\tilde{\rho}(t)}\mathbb{I}. \tag{8.2.26}$$

Now, looking at equations (8.2.6) and (8.1), we can express equation (8.2.18) as follows:

$$\left[\Gamma^{-1}(t)\bullet\left[C_0^{-1}(t) - \left[1 + C_0^{-1}(t)\Delta C(t)\right]^{-1} C_0^{-1}(t)\Delta C(t)C_0^{-1}(t)\right]\right] D_{\boldsymbol{\sigma(t)}} \tag{8.2.27}$$

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We extract the properties of the risk-based strategies from the above equation by first considering the following term:

$$\left[\Gamma^{-1}(t)\bullet C_0^{-1}(t)\right] D_{\boldsymbol{\sigma}(t)} = \left[\Gamma^{-1}(t)\bullet\left(-\frac{\tilde{\rho}(t)}{(1-\tilde{\rho}(t))\left[n\tilde{\rho}(t)+(1-\tilde{\rho}(t))\right]}\Lambda_1 + \frac{1}{1-\tilde{\rho}(t)}\mathbb{I}\right)\right] D_{\boldsymbol{\sigma}(t)}.$$

But,

$$\begin{aligned}\left[\Gamma^{-1}(t)\bullet\Lambda_1\right] D_{\boldsymbol{\sigma}(t)} &= \left[\begin{pmatrix} \frac{1}{\sigma_1^2} & \frac{1}{\sigma_1\sigma_2} & \cdots & \frac{1}{\sigma_1\sigma_n} \\ \frac{1}{\sigma_2\sigma_1} & \frac{1}{\sigma_2^2} & \cdots & \frac{1}{\sigma_2\sigma_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sigma_n\sigma_1} & \frac{1}{\sigma_n\sigma_2} & \cdots & \frac{1}{\sigma_n^2} \end{pmatrix} \bullet \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}\right] \begin{pmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \vdots \\ \sigma_n^2 \end{pmatrix} \\ &= \sum_{i=1}^{n} \sigma_i \begin{pmatrix} \frac{1}{\sigma_1} \\ \frac{1}{\sigma_2} \\ \vdots \\ \frac{1}{\sigma_n} \end{pmatrix}.\end{aligned} \tag{8.2.28}$$

Also,

$$\begin{aligned}\left[\Gamma^{-1}(t)\bullet\mathbb{I}\right] \Lambda_{\boldsymbol{\sigma}(t)} &= \left[\begin{pmatrix} \frac{1}{\sigma_1^2} & \frac{1}{\sigma_1\sigma_2} & \cdots & \frac{1}{\sigma_1\sigma_n} \\ \frac{1}{\sigma_2\sigma_1} & \frac{1}{\sigma_2^2} & \cdots & \frac{1}{\sigma_2\sigma_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sigma_n\sigma_1} & \frac{1}{\sigma_n\sigma_2} & \cdots & \frac{1}{\sigma_n^2} \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}\right] \begin{pmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \vdots \\ \sigma_n^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.\end{aligned} \tag{8.2.29}$$

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Thus,

$$\left[\Gamma^{-1}(t)\bullet C_0^{-1}(t)\right] D_{\boldsymbol{\sigma}(t)} = \frac{1}{(1-\tilde{\rho}(t))}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} - \frac{n\tilde{\rho}(t)\sum_{i=1}^{n}\sigma_i}{(1-\tilde{\rho}(t))\left[n\tilde{\rho}(t)+(1-\tilde{\rho}(t))\right]}\begin{pmatrix}\frac{1}{\sigma_1}\\\frac{1}{\sigma_2}\\\vdots\\\frac{1}{\sigma_n}\end{pmatrix}, \quad (8.2.30)$$

which reflects the basic property of the equal-weighted and risk parity strategies. We have seen that component weights in a risk parity portfolio are directly proportional to their inverse volatilities. Thus, the solution (8.1.12) can be interpreted as a linear combination of the three risk-control strategies, namely the minimum variance, equal weighted and risk parity strategies, with additional high cash flow rate of portfolio return.

8.3 Conclusion

The risk parity (RP) strategies, as an alternative to a traditional portfolio of 60 percent equities and 40 percent bond, have been widely adopted by investors. RP is a set of asset allocation techniques for genuinely diversified risk portfolio. It is intuitively appealing and empirically attractive.

These strategies are a good starting point for investors willing to manage risk of their portfolios. Risk of the portfolio is diversified across all components in a universe. The incentive of these strategies in the investment realm is the exclusion of the expected return estimate. Investor who deploys any of the risk parity strategies assumes that estimation of expected return is difficult to obtain. In addition, they can be extended with the factor risk parity of Roncalli and Weisang (2012).

We studied the risk parity strategies in line with the other risk-based strategies, i.e., the equal weighted, minimum variance and maximum diversification. Using an empirical study in Chapter 7, we observe that the risk parity strategies exhibit low portfolio turnover compared to the MD, MV. The EW portfolio dominates in terms of low portfolio turnover.

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Appendices

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Appendix A

Proofs of Risk Budget Properties

A.1 Standard Mean Variance Portfolio Solution

Consider the problem of minimizing portfolio's risk for a given level of expected return formulated as follows,

$$\mathbf{z}^{MVO} = \arg\min_{\mathbf{z}\in\mathbb{R}^n} \ \frac{1}{2}\mathbf{z}^T\Sigma\mathbf{z}, \tag{A.1.1}$$

subject to,

$$g(\mathbf{z}) = \mathbf{z}^T\mathbf{1} - 1 = 0, \tag{A.1.2}$$

$$h(\mathbf{z}) = \mathbf{z}^T\bar{\mathbf{r}} - a = 0. \tag{A.1.3}$$

The first constrain is called the budget constraint and the second constraint represents the level of expected return of portfolio. This problem is solved by applying the Lagrangian multipliers method in order to find the critical points that will give the solution to the given constraint. Thus, the Lagrangian function is given as,

$$L(\mathbf{z},\lambda,\gamma) = \frac{1}{2}\mathbf{z}^T\Sigma\mathbf{z} - \lambda(\mathbf{z}^T\bar{\mathbf{r}} - a) - \gamma(\mathbf{z}^T\mathbf{1} - 1). \tag{A.1.4}$$

The critical or optimum points to this problem are found by first order differential equation. Now we solve the following equations,

$$\frac{\partial L(\mathbf{z},\lambda,\gamma)}{\partial \mathbf{z}} = \Sigma\mathbf{z} - \lambda\bar{\mathbf{r}} - \gamma\mathbf{1} = 0, \tag{A.1.5}$$

and

$$\frac{\partial L(\mathbf{z},\lambda,\gamma)}{\partial \lambda} = a - \mathbf{z}^T\bar{\mathbf{r}} = 0, \tag{A.1.6}$$

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$$\frac{\partial L(\mathbf{z}, \lambda, \gamma)}{\partial \gamma} = 1 - \mathbf{z}^T \mathbf{1} = 0. \tag{A.1.7}$$

From equation (A.1.5), we have

$$\mathbf{z}^{MVO} = \Sigma^{-1} (\gamma \mathbf{1} + \lambda \bar{\mathbf{r}}) = \lambda \Sigma^{-1} \bar{\mathbf{r}} + \gamma \Sigma^{-1} \mathbf{1} \tag{A.1.8}$$

as a solution to (A.1.1) with the parameter λ and γ. By substituting (A.1.8) into (A.1.6) and (A.1.7), we have a system of linear equations,

$$\left(\bar{\mathbf{r}}^T \Sigma^{-1} \bar{\mathbf{r}}\right) \lambda + \left(\bar{\mathbf{r}}^T \Sigma^{-1} \mathbf{1}\right) \gamma = a \tag{A.1.9}$$

$$\left(\mathbf{1}^T \Sigma^{-1} \bar{\mathbf{r}}\right) \lambda + \left(\mathbf{1}^T \Sigma^{-1} \mathbf{1}\right) \gamma = 1, \tag{A.1.10}$$

which can be written as,

$$\begin{bmatrix} B & A \\ A & C \end{bmatrix} \begin{bmatrix} \lambda \\ \gamma \end{bmatrix} = \begin{bmatrix} a \\ 1 \end{bmatrix}. \tag{A.1.11}$$

Where

$$\mathfrak{M} = \begin{bmatrix} B & A \\ A & C \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{r}}^T \Sigma^{-1} \bar{\mathbf{r}} & \bar{\mathbf{r}}^T \Sigma^{-1} \mathbf{1} \\ \mathbf{1}^T \Sigma^{-1} \bar{\mathbf{r}} & \mathbf{1}^T \Sigma^{-1} \mathbf{1} \end{bmatrix}. \tag{A.1.12}$$

The solution to equation (A.1.11) is unique if and only if $D = det(\mathfrak{M}) \neq 0$. We first show that $\mathfrak{M}$ is positive. Since Σ is a symmetric and positively definite matrix (i.e. for each $\mathbf{z} \neq 0, \mathbf{z}^T \Sigma \mathbf{z} = \sigma_p^2 > 0$), so is Σ^{-1}. This implies $A, B, C > 0$. To show that $D = BC - A^2 \neq 0$, we consider the vector $A\bar{\mathbf{r}} - B\mathbf{1}$. By assumption, $A\bar{\mathbf{r}} - B\mathbf{1} \neq 0$ (since $A\bar{\mathbf{r}} - B\mathbf{1} = 0$ if and only if $\bar{\mathbf{r}} = \mathbf{1}$, which is forbidden by our assumption that $\bar{\mathbf{r}}$ is a vector of random variables that can not all be equal). So positive definiteness of Σ^{-1}, we have

$$\begin{aligned} 0 &< (A\bar{\mathbf{r}} - B\mathbf{1})^T \Sigma^{-1} (A\bar{\mathbf{r}} - B\mathbf{1}) \\ &= A^2 \bar{\mathbf{r}}^T \Sigma^{-1} \bar{\mathbf{r}} - AB\bar{\mathbf{r}}^T \Sigma \mathbf{1} - BA\mathbf{1}^T \Sigma \bar{\mathbf{r}} + B^2 \mathbf{1}^T \Sigma^{-1} \mathbf{1} \\ &= -BA^2 + B^2 C = B(BC - A^2). \end{aligned} \tag{A.1.13}$$

Since $B > 0$, we have $D > 0$ and in particular $D \neq 0$.

$$\begin{bmatrix} \lambda \\ \gamma \end{bmatrix} = \frac{1}{D} \left(\begin{bmatrix} C & -A \\ -A & B \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix} \right) \tag{A.1.14}$$

Thus the solution to λ and γ is,

$$\lambda = \frac{Ca - A}{D} \quad \text{and} \quad \gamma = \frac{-Aa + B}{D}. \tag{A.1.15}$$

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The value of λ and γ can now be substituted to find the weights of the portfolio. i.e.

$$\begin{aligned}\mathbf{z}^{MVO} &= \left(\frac{Ca - A}{D}\right)\Sigma^{-1}\overline{\mathbf{r}} + \left(\frac{-Aa + B}{D}\right)\Sigma^{-1}\mathbf{1} \\ &= \frac{1}{D}\left[B\Sigma^{-1}\mathbf{1} - A\Sigma^{-1}\overline{\mathbf{r}}\right] + \frac{1}{D}\left[C\Sigma^{-1}\overline{\mathbf{r}} - A\Sigma^{-1}\mathbf{1}\right]a \qquad \text{(A.1.16)}\end{aligned}$$

Let $g = \frac{1}{D}\left[B\Sigma^{-1}\mathbf{1} - A\Sigma^{-1}\overline{\mathbf{r}}\right]$ and $h = \frac{1}{D}\left[C\Sigma^{-1}\overline{\mathbf{r}} - A\Sigma^{-1}\mathbf{1}\right]$. Then we can write equation (A.1.8) as,

$$\mathbf{z}^{MVO} = g + ha \qquad \text{(A.1.17)}$$

A.2 Analysis of Risk Budgeting Solutions for Special Cases of ρ.

A.2.1 Two-Assets Based RB-Solutions with Constant Correlations

Consider the solution as given in equation (4.1.9). For $\rho = -1$, we have

$$z = \frac{-(b-\frac{1}{2})\sigma_1\sigma_2 - b\sigma_2^2 + \sigma_1\sigma_2\sqrt{(b-\frac{1}{2})^2 + b(1-b)}}{(1-b)\sigma_1^2 - 2(b-1)\sigma_1\sigma_2 - b\sigma_2^2}.$$

Rearranging, we find the following,

$$\begin{aligned}z &= \frac{\sigma_2[\sigma_1 - b\sigma_1 - b\sigma_2]}{(\sigma_1+\sigma_2)[\sigma_1 - b\sigma_1 - b\sigma_2]} \\ &= \frac{\sigma_2}{\sigma_1+\sigma_2}. \qquad \text{(A.2.1)}\end{aligned}$$

For the case $\rho = 0$, we have

$$\begin{aligned}z &= \frac{-b\sigma_2^2 + \sigma_1\sigma_2\sqrt{b(1-b)}}{(1-b)\sigma_1^2 - b\sigma_2^2} \\ &= \frac{\sqrt{b}\sqrt{1-b}\sigma_1\sigma_2 - b\sigma_2^2}{(1-b)\sigma_1^2 - b\sigma_2^2} \\ &= \frac{\sqrt{b}\sigma_2(\sqrt{1-b}\sigma_1 - \sqrt{b}\sigma_2)}{(\sqrt{1-b}\sigma_1 + \sqrt{b}\sigma_2)(\sqrt{1-b}\sigma_1 - \sqrt{b}\sigma_2)} \\ &= \frac{\sqrt{b}\sigma_2}{(\sqrt{1-b}\sigma_1 + \sqrt{b}\sigma_2)}, \qquad \text{(A.2.2)}\end{aligned}$$

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and when $\rho = 1$, we have

$$\begin{aligned} z &= \frac{(b-\frac{1}{2})\sigma_1\sigma_2 - b\sigma_2^2 + \frac{1}{2}\sigma_1\sigma_2}{(1-b)\sigma_1^2 - b\sigma_2^2 + 2(b-\frac{1}{2})\sigma_1\sigma_2} \\ &= \frac{b\sigma_1\sigma_2 - b\sigma_2}{\sigma_1^2 - b\sigma_1^2 - b\sigma_2^2 + 2b\sigma_1\sigma_2 - \sigma_1\sigma_2} \\ &= \frac{b\sigma_2(\sigma_1-\sigma_2)}{\sigma_1(\sigma_1-\sigma_2) - b(\sigma_1-\sigma_2)^2} \\ &= \frac{b\sigma_2}{(1-b)\sigma_1 + b\sigma_2}. \end{aligned} \tag{A.2.3}$$

A.2.2 When $\rho = 1$.

$$(\Sigma\mathbf{z})_i = z_i\sigma_i^2 + \sigma_i\sum_{j\neq i}\sigma_j z_j. \tag{A.2.4}$$

Thus,

$$b_i\sigma^2(\mathbf{z}) = z_i(\Sigma\mathbf{z})_i = z_i\left(z_i\sigma_i^2 + \sigma_i\sum_{j\neq i} z_j\sigma_j\right), \tag{A.2.5}$$

alternatively,

$$b_j\sigma^2(\mathbf{z}) = z_j\left(z_j\sigma_j^2 + \sigma_j\sum_{i\neq j} z_i\sigma_i\right). \tag{A.2.6}$$

Dividing (A.2.5) with (A.2.6) yields,

$$\frac{b_i}{b_j} = \frac{z_i\left(z_i\sigma_i^2 + \sigma_i\sum_{j\neq i} z_j\sigma_j\right)}{z_j\left(z_j\sigma_j^2 + \sigma_j\sum_{j\neq i} z_i\sigma_i\right)}. \tag{A.2.7}$$

Hence,

$$b_i z_j^2\sigma_j^2 + b_i\sigma_j z_j\sum_{j\neq i} z_i\sigma_i = b_j z_i^2\sigma_i^2 + b_j\sigma_i z_i\sum_{j\neq i} z_j\sigma_j, \tag{A.2.8}$$

which could be written as,

$$b_i z_j\sigma_j(z_j\sigma_j + \sum_{j\neq i} z_i\sigma_i) = b_j z_i\sigma_i(z_i\sigma_i + \sum_{j\neq i} z_j\sigma_j) \tag{A.2.9}$$

$$b_i z_j\sigma_j(\sum_{j=i} z_j\sigma_j) = b_j z_i\sigma_i(\sum_{j=i} z_j\sigma_j). \tag{A.2.10}$$

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Thus,

$$b_i z_j \sigma_j = b_j z_i \sigma_i. \tag{A.2.11}$$

Rearranging and using the budget constraint for the j^{th} components, i.e,

$$\sum_{j=1}^{n} z_j = \sum_{j=1}^{n} \frac{b_j z_i \sigma_i}{b_i \sigma_j}, \quad i = 1, \dots, n \tag{A.2.12}$$

gives the following,

$$\frac{z_i \sigma_i}{b_i} \sum_{j=1}^{n} \frac{b_j}{\sigma_j} = 1, \quad i = 1, \dots, n. \tag{A.2.13}$$

Thus,

$$z_i = \frac{b_i \sigma_i^{-1}}{\sum_{j=1}^{n} b_j \sigma_j^{-1}}, \quad i = 1, \dots, n. \tag{A.2.14}$$

A.2.3 The Perfect Negative Correlation

Note that the perfect opposite correlation general term is given as $\rho = -\dfrac{1}{n-1}$. This leads the variance of the portfolio to be zero (i.e., $\sigma_p^2(\mathbf{z}) = 0$). Thus,

$$\begin{aligned} z_i(\Sigma \mathbf{z})_i &= z_i \left(z_i \sigma_i^2 - \frac{\sigma_i}{n-1} \sum_{j \neq i} \sigma_j z_j \right) \\ &= z_i \sigma_i \left(z_i \sigma_i - \frac{1}{n-1} \sum_{j \neq i} \sigma_j z_j \right) = 0. \end{aligned} \tag{A.2.15}$$

Dividing both side by $z_i \sigma_i$ and including the i^{th} term in the summation yield,

$$\begin{aligned} & z_i \sigma_i - \left(\frac{1}{n-1} \sum_{j=1}^{n} \sigma_j z_j - \frac{z_i \sigma_i}{n-1} \right) \\ &= z_i \sigma_i \left(\frac{n}{n-1} \right) - \frac{1}{n-1} \sum_{j=1}^{n} \sigma_j z_j = 0. \end{aligned} \tag{A.2.16}$$

Thus,

$$z_i \sigma_i \left(\frac{n}{n-1} \right) = \frac{1}{n-1} \sum_{j=1}^{n} \sigma_j z_j. \tag{A.2.17}$$

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Similarly, for the j^{th} asset, we have,

$$z_j\sigma_j\left(\frac{n}{n-1}\right) = \frac{1}{n-1}\sum_{j=1}^{n}\sigma_j z_j. \tag{A.2.18}$$

Dividing equation (A.2.17) by equation (A.2.18), gives,

$$\frac{z_i\sigma_i}{z_j\sigma_j} = 1. \tag{A.2.19}$$

Setting z_j to be the subject, and taking the sum both side yield,

$$\sum_{j=1}^{n} z_j = \sum_{j=1}^{n}\frac{z_i\sigma_i}{\sigma_j}. \tag{A.2.20}$$

This implies that,

$$z_i\sigma_i\sum_{j=1}^{n}\frac{1}{\sigma_j} = 1. \tag{A.2.21}$$

Hence,

$$z_i = \frac{\sigma_i^{-1}}{\sum_{j=1}\sigma_j^{-1}}. \tag{A.2.22}$$

A.2.4 Risk Budgeting Portfolio Formula

Consider the risk budgeting portfolio in equation (4.1.1)

$$z_i(\Sigma\mathbf{z})_i = b_i\sigma^2(\mathbf{z}) \tag{A.2.23}$$

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Now for ρ not a constant we have,

$$L.H.S = z_i(\Sigma\mathbf{z})_i \tag{A.2.24}$$

$$= z_i \begin{pmatrix} \rho_1\sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{21}\sigma_2\sigma_1 & \rho_2\sigma_2^2 & \cdots & \rho_{2n}\sigma_2\sigma_n \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1}\sigma_n\sigma_1 & \rho_{n2}\sigma_n\sigma_2 & \cdots & \rho_n\sigma_n^2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

$$= z_i \begin{pmatrix} \rho_{1,1}\sigma_1^2 z_1 + \sigma_1 \sum_{j\neq 1} \rho_{1,j}\sigma_j z_j \\ \rho_{2,2}\sigma_2^2 z_2 + \sigma_2 \sum_{j\neq 2} \rho_{2,j}\sigma_j z_j \\ \vdots \\ \rho_{i,i}\sigma_n^2 z_n + \sigma_i \sum_{j\neq i} \rho_{i,j}\sigma_j z_j \\ \vdots \\ \rho_{n,n}\sigma_n^2 z_n + \sigma_n \sum_{j\neq n} \rho_{n,j}\sigma_j z_j \end{pmatrix}$$

$$= \rho_{i,i}\sigma_i^2 z_i^2 + z_i\sigma_i \sum_{j\neq i}^{n} \rho_{i,j}\sigma_j z_j \quad i = 1\ldots,n. \tag{A.2.25}$$

Thus, for constant correlation matrix, $\rho_{i,j} = \rho$, we have

$$L.H.S = \sigma_i^2 z_i^2 + \rho\sigma_i z_i \sum_{j=1}^{n} \sigma_j z_j - \rho\sigma_j^2 z_j^2$$

$$= \sigma_i z_i \left((1-\rho)\sigma_i z_i + \rho \sum_{j=1}^{n} \sigma_j z_j \right) \tag{A.2.26}$$

Hence, the risk contribution of the i^{th} component is given by,

$$\sigma_i z_i \left((1-\rho)\sigma_i z_i + \rho \sum_{j=1}^{n} \sigma_j z_j \right) = b_i\sigma^2(\mathbf{z}). \tag{A.2.27}$$

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