Random Walks on Graphs

by

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Declaration

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Abstract

We study random walks on finite graphs. The reader is introduced to general Markov chains before we move on more specifically to random walks on graphs. A random walk on a graph is just a Markov chain that is time-reversible. The main parameters we study are the hitting time, commute time and cover time. We find novel formulas for the cover time of the subdivided star graph and broom graph before looking at the trees with extremal cover times.

Lastly we look at a connection between random walks on graphs and electrical networks, where the hitting time between two vertices of a graph is expressed in terms of a weighted sum of effective resistances. This expression in turn proves useful when we study the cover cost, a parameter related to the cover time.

Opsomming

Ons bestudeer toevallige wandelings op eindige grafieke in hierdie tesis. Eers word algemene Markov kettings beskou voordat ons meer spesifiek aanbeweeg na toevallige wandelings op grafieke. 'n Toevallige wandeling is net 'n Markov ketting wat tyd herleibaar is. Die hoof paramaters wat ons bestudeer is die treftyd, pendeltyd en dektyd. Ons vind oorspronklike formules vir die dektyd van die verdeelde stergrafiek sowel as die besemgrafiek en kyk daarna na die twee bome met uiterste dektye.

Laastens kyk ons na 'n verband tussen toevallige wandelings op grafieke en elektriese netwerke, waar die treftyd tussen twee punte op 'n grafiek uitgedruk word in terme van 'n geweegde som van effektiewe weerstande. Hierdie uitdrukking is op sy beurt weer nuttig wanneer ons die dekkoste bestudeer, waar die dekkoste 'n paramater is wat verwant is aan die dektyd.

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Contents

Introduction

Random walks on graphs have diverse applications in fields such as computer science, physics and economics. We study them from a mathematical point of view and look at some of the following questions: starting from a vertex how long does it take on average to return to that vertex? how long does it take to visit a given vertex? how long does it take to visit all the vertices?

We start by taking a slightly more general view in the first chapter by considering time-homogeneous Markov chains on a countable state space. The most important part of this chapter is that an irreducible, positive recurrent Markov chain, has a unique stationary distribution (see Section 1.3). Moreover if we assume that the Markov chain is aperiodic, then the stationary distribution plays a role in limiting results as well.

In the short second chapter we show that we can consider a random walk on a graph as a Markov chain that satisfies the additional property of reversibility. Moreover we look at some examples of the main parameters we study during this thesis.

The third chapter deals with random walks on trees, which are in general easier to work with than random walks on graphs because of the special structure of a tree. In Section 3.1 formulas for the hitting time and commute time between any two vertices of a tree are given. We then find novel formulas for the cover time of two classes of trees. Section 3.2 looks at the two trees with extremal cover times. Not surprisingly the cover time is minimised from the center of the star, a result proved by Brightwell and Winkler in [6]. On the other hand the cover time is maximised from the central vertex or one of the two adjacent central vertices of a path. This result is due to Feige [12].

In Chapter 4 we look at a connection between electrical networks and random walks on graphs. A harmonic parameter related to the hitting time is defined. Since the voltages in an electrical network are harmonic this helps us to write the hitting time between two vertices in a graph as a weighted sum of effective resistances, a result due to Tetali [19]. This result generalises the hitting time formula for a tree in Section 3.1.

In the fifth and final chapter we consider partial sums of hitting times. One of these partial sums, the cover cost, can be used to provide upper and lower bounds for the cover time of a graph. In Section 5.2 these partial sums are expressed in terms of well known graph theoretical indices. Finally we consider some of their symmetry properties.

Chapter 1

General Markov Chains

1.1 Introductory facts and notation

This section introduces a few introductory, yet important, results that will be frequently required in calculations during the remainder of the text.
A Markov chain is characterised by the Markov property: given the present state of the chain the next state we visit is conditionally independent of the past history of the chain. Moreover we require a Markov chain to possess the property of time homogeneity: the probability of visiting a particular future state from the same present state remains the same irrespective of the number of states we visited in the past.

Definition 1.1. A *Markov chain* is a sequence of random variables $Z_0, Z_1, Z_2, \dots$, abbreviated as (Z_n), with the following properties:

1. *Markov Property.* $\Pr(Z_{n+1} = x_{n+1} \mid Z_n = x_n, Z_{n-1} = x_{n-1}, \dots, Z_0 = x_0)$ $= \Pr(Z_{n+1} = x_{n+1} \mid Z_n = x_n)$ for $x_0, x_1, \dots x_{n+1} \in X$.
2. *Time homogeneity.* For all $x, y \in X$ and $m, n \in \mathbb{N}_0$ which satisfy $\Pr(Z_m = x) > 0$ and $\Pr(Z_n = x) > 0$, one has $\Pr(Z_{m+1} = y \mid Z_m = x) =$ $\Pr(Z_{n+1} = y \mid Z_n = x)$.

The set of possible values that the $Z_{i,i\geq 0}$ can take forms a countable set X called the *state space* of the Markov chain. Let $P = (p(x,y))_{x,y\in X}$ denote the matrix of *one-step transition probabilities* of a Markov chain where $p(x,y) = \Pr(Z_{n+1} = y \mid Z_n = x)$. Let $p^{(n)}(x,y)$ be the probability to be at y at the n-th step starting from x. That is $p^{(n)}(x,y) = \Pr(Z_n = y \mid Z_0 = x)$.

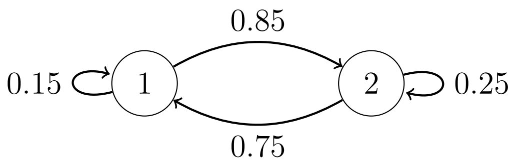
We obtain the *graph of a Markov chain* by considering the state space X as the vertices of a graph and letting xy be an edge between x and y if and only if $p(x,y) > 0$. The edges are oriented and we allow loops. Every edge xy is weighted by $p(x, y)$.

We illustrate the concept of a Markov chain with the following simple example:

Example 1.2. Jonathan is a lazy student, he dislikes studying for two days in a row. Indeed $\Pr(\text{Jonathan studies tomorrow} \mid \text{he studies today}) = 0.15$ while $\Pr(\text{Jonathan studies tomorrow} \mid \text{he does not study today}) = 0.75$. If Z_n denotes whether Jonathan studies or not on day n, where $n \geq 0$, then (Z_n) is a 2-state Markov chain with transition matrix

$$P = \begin{pmatrix} 0.15 & 0.85 \\ 0.75 & 0.25 \end{pmatrix}$$

where the first state corresponds to the event of Jonathan studying and the second state corresponds to the event of Jonathan not studying. The graph of this Markov chain looks as follows:



We will return to this example at a later stage.

Lemma 1.3 (cf. [21, Lemma 1.21])**.**

1. *The number* $p^{(n)}(x, y)$ *is the element at position* (x, y) *in the* n*-th power* P^n *of the transition matrix,*
2. $p^{(m+n)}(x, y) = \sum_{w \in X} p^{(m)}(x, w) p^{(n)}(w, y)$.
3. P^n *is a stochastic matrix, that is* $\sum_{y \in X} p^{(n)}(x, y) = 1$.

Note that to determine the probability of visiting a state after n steps we only need to know the initial distribution as well as the transition matrix of a Markov chain since $\Pr(Z_n = x) = \sum_{y \in X} \Pr(Z_0 = y) \Pr(Z_n = x \mid Z_0 = y)$.

We write $\Pr_x(\cdot)$ and $E_x(\cdot)$ for probabilities and expectations of the chain started at state x. More generally given an intial distribution θ, we write $\Pr_\theta(\cdot)$ and $E_\theta(\cdot)$. For a set of states $A \subset X$ and any distribution let $\theta(A) = \sum_{x \in A} \theta(x)$.

Definition 1.4. A *stopping time* with respect to a sequence of random variables $Z_0, Z_1, Z_2, \ldots$ is a random time τ such that for each $n \geq 0$, the event $\{\tau = n\}$ is completely determined by $\{Z_0, Z_1, \ldots, Z_n\}$.

So a stopping time is a random time that only depends on the past and present. If a random time depends on a future event then it is not a stopping time.

Let $T_x = \min\{n \geq 0 : Z_n = x\}$ and $T_x^+ = \min\{n \geq 1 : Z_n = x\}$ be the *first hitting time* and the *first return time* to state x respectively. More generally for a set of states $A \subset X$ set $T_A = \min\{n \geq 0 : Z_n \in A\}$ and $T_A^+ = \min\{n \geq 1 : Z_n \in A\}$. It is easy to see that the first hitting time is a stopping time: $\{T_x = 0\} = \{Z_0 = x\}$ depends only on $\{Z_0\}$. For $n \geq 1$ $\{T_x = n\} = \{Z_n = x, Z_n \neq x \text{ for } i = 0, 1, \ldots, n-1\}$ only depends on $\{Z_0, Z_1, \ldots, Z_n\}$. Similarly the first return time $T_x^+ = \min\{n \geq 1 : Z_n = x\}$ is a stopping time with respect to $\{Z_1, \ldots, Z_n\}$.

Theorem 1.5 (cf. [5, Theorem 4])**.** *Let (Z_n) be a Markov chain and τ a stopping time for (Z_n) with* $\Pr(\tau < \infty) = 1$*. Then the relation*

$$\Pr\left(Z_{\tau+m} = y \mid Z_0 = x_0, Z_1 = x_1, \ldots, Z_\tau = x_\tau\right) = \Pr\left(Z_m = y \mid Z_0 = x_\tau\right)$$

holds for all $m \in \mathbb{N}$ and for all $x_0, \ldots, x_\tau, y \in X$.

Proof. We have

$$\begin{aligned}
&\Pr\left(Z_{\tau+m} = y \mid Z_0 = x_0, Z_1 = x_1, \ldots, Z_\tau = x_\tau\right) \\
&= \sum_{n=0}^{\infty} \Pr\left(\tau = n, Z_{\tau+m} = y \mid Z_0 = x_0, Z_1 = x_1, \ldots, Z_\tau = x_\tau\right) \\
&= \sum_{n=0}^{\infty} \Pr\left(Z_{\tau+m} = y \mid \tau = n, Z_0 = x_0, Z_1 = x_1, \ldots, Z_\tau = x_\tau\right) \\
&\times \Pr\left(\tau = n \mid Z_0 = x_0, Z_1 = x_1, \ldots, Z_\tau = x_\tau\right) \\
&= \sum_{n=0}^{\infty} \Pr\left(Z_{n+m} = y \mid Z_n = x_\tau\right) \Pr\left(\tau = n \mid Z_0 = x_0, \ldots, Z_\tau = x_\tau\right) \\
&= \Pr\left(Z_m = y \mid Z_0 = x_\tau\right) \sum_{n=0}^{\infty} \Pr\left(\tau = n \mid Z_0 = x_0, \ldots, Z_\tau = x_\tau\right)
\end{aligned}$$

which yields the statement, since τ is finite with probability one. $\square$

Lemma 1.6. *For a set of states $A \subset X$*

$$E_x T_A = \begin{cases} 0 & \textit{if } x \in A \\ 1 + \sum_{y \notin A} p(x,y) E_y T_A & \textit{if } x \notin A. \end{cases}$$

Proof. Let $Z_0, Z_1, Z_2, \ldots$ be a Markov chain with $Z_0 = x$. If $x \in A$ then $T_A = 0$ so that $E_x T_A = 0$. If $x \notin A$ then $T_A = 1 + T_A^{'}$ where $T_A^{'}$ is the remaining time after the first step until A is hit. Now $E_xT_A = 1 + E_xT_A'$ where

$$\begin{aligned}
E_xT_A' &= \sum_{n=1}^{\infty} n \Pr\left(T_A' = n \mid Z_0 = x\right) \\
&= \sum_{n=1}^{\infty} n \sum_{y\in X} \Pr\left(T_A' = n \mid Z_1 = y, Z_0 = x\right) \Pr\left(Z_1 = y \mid Z_0 = x\right) \\
&= \sum_{n=1}^{\infty} n \sum_{y\in X} \Pr\left(T_A' = n \mid Z_1 = y\right) p(x,y) \\
&= \sum_{n=1}^{\infty} n \sum_{y\in X} \Pr\left(T_A = n \mid Z_0 = y\right) p(x,y) \\
&= \sum_{y\notin A} p(x,y) E_yT_A.
\end{aligned}$$

□

Set $f^{(n)}(x,y) = \Pr_x(T_y = n) = \Pr_x(Z_n = y, Z_i \neq y, \text{ for } i = 0,\ldots,n-1)$ and $u^{(n)}(x,y) = \Pr_x\left(T_y^+ = n\right) = \Pr_x(Z_n = y, Z_i \neq y, \text{ for } i = 1,\ldots,n-1)$. Let

$$G(x,y|z) = \sum_{n=0}^{\infty} p^{(n)}(x,y)z^n,$$

$$F(x,y|z) = \sum_{n=0}^{\infty} f^{(n)}(x,y)z^n,$$

and

$$U(x,y|z) = \sum_{n=0}^{\infty} u^{(n)}(x,y)z^n$$

be the *complex generating functions* of $p^{(n)}(x,y)$, $f^{(n)}(x,y)$ and $u^{(n)}(x,y)$ respectively. $G(x,y|z)$ is called the *Green Kernel* of the Markov chain. Let V^y denote the number of times a Markov chain visits $y \in X$. Setting

$$V_n^y = \begin{cases} 1 & \text{if } Z_n = y \\ 0 & \text{if } Z_n \neq y \end{cases}$$

we have $V^y = \sum_{n=0}^{\infty} V_n^y$. Note that

$$G(x,y) := G(x,y|1) = \sum_{n=0}^{\infty} p^{(n)}(x,y) = E_x(V^y)$$

whereas

$$F(x,y) := F(x,y|1) = \sum_{n=0}^{\infty} f^{(n)}(x,y) = \Pr_x(T_y < \infty)$$

and

$$U(x,y) := U(x,y|1) = \sum_{n=0}^{\infty} u^{(n)}(x,y) = \Pr_x\left(T_y^+ < \infty\right).$$

Also, $F(x,y) = U(x,y)$ when $x \neq y$, whereas $U(x,x)$ is the probability to ever return to x, while $F(x,x) = 1$.

The respective *radii of convergence* of $U(x,y|z)$ and $G(x,y|z)$ are given by

$$r(x,y) = \frac{1}{\limsup_{n\to\infty}\left(p^{(n)}(x,y)\right)^{\frac{1}{n}}} \text{ and } s(x,y) = \frac{1}{\limsup_{n\to\infty}\left(u^{(n)}(x,y)\right)^{\frac{1}{n}}}$$

(see [18] Section V.12.). Note that $s(x,y) \geq r(x,y) \geq 1$ since $u^{(n)}(x,y) \leq p^{(n)}(x,y) \leq 1$.

Theorem 1.7 (cf. [21, Theorem 1.38])**.** *1.* $G(x,x|z) = \frac{1}{1-U(x,x|z)}, |z| < r(x,x)$.

2. $G(x,y|z) = F(x,y|z)G(y,y|z), |z| < r(x,y)$.

3. $U(x,x|z) = \sum_y p(x,y)zF(y,x|z), |z| < s(x,x)$.

4. *If* $y \neq x$ *then* $F(x,y|z) = \sum_w p(x,w)zF(w,y|z), |z| < s(x,y)$.

Proof. 1. Let $n \geq 1$. If $Z_0 = x$ and $Z_n = x$ then there must be an instant $k \in \{1,\ldots,n\}$ such that $T_x^+ = k$. The events $\{T_x^+ = k\}$, $k = 1,\ldots,n$ are pairwise disjoint. Hence

$$\begin{aligned} p^{(n)}(x,x) &= \sum_{k=1}^{n} \Pr_x\left(Z_n = x, T_x^+ = k\right) \\ &= \sum_{k=1}^{n} \Pr_x\left(Z_n = x \mid T_x^+ = k\right)\Pr_x\left(T_x^+ = k\right) \\ &= \sum_{k=1}^{n} p^{(n-k)}(x,x)u^{(k)}(x,x). \end{aligned}$$

Since $u^{(0)}(x,x) = 0$, we have

$$p^{(n)}(x,x) = \sum_{k=0}^{n} p^{(n-k)}(x,x)u^{(k)}(x,x) \text{ for } n \geq 1.$$

If $n = 0$ then $p^{(0)}(x,x) = 1$, while $\sum_{k=0}^{n} p^{(n-k)}(x,x)u^{(k)}(x,x) = 0$. It follows that

$$\begin{aligned}
G(x,x|z) &= \sum_{n=0}^{\infty} p^{(n)}(x,x)z^n \\
&= 1 + \sum_{n=1}^{\infty}\sum_{k=0}^{n} p^{(n-k)}(x,x)u^{(k)}(x,x)z^n \\
&= 1 + \sum_{n=0}^{\infty}\sum_{k=0}^{n} p^{(n-k)}(x,x)u^{(k)}(x,x)z^n \\
&= 1 + U(x,x|z)G(x,x|z)
\end{aligned}$$

as long as $|z| < r(x,x)$ in which case both power series converge absolutely.

2. Let $n \geq 0$. If $Z_0 = x$ and $Z_n = y$ then there must be an instant $k \in \{0,1,\ldots,n\}$ such that $T_y = k$. The events $\{T_y = k\}, \quad k = 0,1,\ldots,n$ are pairwise disjoint. Hence

$$\begin{aligned}
p^{(n)}(x,y) &= \sum_{k=0}^{n} \mathrm{Pr}_x\left(Z_n = y, T_y = k\right) \\
&= \sum_{k=0}^{n} \mathrm{Pr}_x\left(Z_n = y \mid T_y = k\right)\mathrm{Pr}_x\left(T_y = k\right) \\
&= \sum_{k=0}^{n} p^{(n-k)}(y,y)f^{(k)}(x,y).
\end{aligned}$$

Note that

$$\frac{p^{(n)}(x,y)}{f^{(k)}(x,y)} \geq p^{(n-k)}(y,y)$$

for $k \leq n$ so that $r(x,y) \leq r(y,y)$. It follows that

$$\begin{aligned}
G(x,y|z) &= \sum_{n=0}^{\infty} p^{(n)}(x,y)z^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} p^{(n-k)}(y,y)f^{(k)}(x,y)z^n \\
&= \sum_{k=0}^{\infty} f^{(k)}(x,y)z^k \sum_{n-k=0}^{\infty} p^{(n-k)}(y,y)z^{n-k} \\
&= F(x,y|z)G(y,y|z)
\end{aligned}$$

if $|z| < r(x,y)$.

3. If $n \geq 1$ then the events $\{T_x^+ = n, Z_1 = y\}, \quad y \in X$, are pairwise disjoint with union $\{T_x^+ = n\}$, whence

$$
\begin{aligned}
u^{(n)}(x,x) &= \sum_{y \in X} \Pr_x\left(T_x^+ = n, Z_1 = y\right) \\
&= \sum_{y \in X} p(x,y) \Pr_x\left(T_x^+ = n \mid Z_1 = y\right) \\
&= \sum_{y \in X} p(x,y) \Pr_y\left(T_x = n-1\right) \\
&= \sum_{y \in X} p(x,y) f^{(n-1)}(y,x).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
U(x,x|z) &= \sum_{n=1}^{\infty} u^{(n)}(x,x) z^n \\
&= \sum_{y \in X} p(x,y) z \sum_{n=1}^{\infty} f^{(n-1)}(y,x) z^{n-1} \\
&= \sum_{y \in X} p(x,y) z F(y,x|z).
\end{aligned}
$$

Note that

$$\frac{u^{(n)}(x,x)}{p(x,y)} \geq f^{(n-1)}(y,x)$$

when $p(x,y) > 0$ so that $s(y,x) \geq s(x,x)$ and thus the formula holds for $|z| < s(x,x)$.

4. $f^{(0)}(x,y) = 0$ since $x \neq y$. If $n \geq 1$ then the events $\{T_y = n, Z_1 = w\}, \quad w \in X$, are pairwise disjoint with union $\{T_y = n\}$, whence

$$
\begin{aligned}
f^{(n)}(x,y) &= \sum_{w \in X} \Pr_x\left(T_y = n, Z_1 = w\right) \\
&= \sum_{w \in X} \Pr_x\left(Z_1 = w\right) \Pr_x\left(T_y = n \mid Z_1 = w\right) \\
&= \sum_{w \in X} \Pr_x\left(Z_1 = w\right) \Pr_w\left(T_y = n-1\right) \\
&= \sum_{w \in X} p(x,w) f^{(n-1)}(w,y).
\end{aligned}
$$

Therefore

$$\begin{aligned}
F(x, y|z) &= \sum_{n=1}^{\infty} f^{(n)}(x, y) z^n \\
&= \sum_{w \in X} p(x, w) z \sum_{n=1}^{\infty} f^{(n-1)}(w, y) z^{n-1} \\
&= \sum_{w \in X} p(x, w) z F(w, y|z).
\end{aligned}$$

Note that

$$\frac{f^{(n)}(x, y)}{p(x, w)} \geq f^{(n-1)}(w, y)$$

when $p(x, w) > 0$ so that $s(w, y) \geq s(x, y)$ and thus the formula holds for $|z| < s(x, y)$.

□

Definition 1.8. Let $G = (V, E)$ be an oriented graph. For $x, y \in V$, a *cut point* between x and y is a vertex $w \in X$ such that every path from x to y must pass through w.

Proposition 1.9 (cf. [21, Proposition 1.43])**.** *Suppose the state w is a cut point between $x, y \in X$ in the graph of a Markov chain. Then $F(x, y|z) = F(x, w|z)F(w, y|z)$ for all $z \in \mathbb{C}$ with $|z| < s(x, y)$.*

Proof. If w is a cut point between x and y, then the Markov chain must visit w before it can reach y. Therefore

$$\begin{aligned}
f^{(n)}(x, y) &= \Pr_x (T_y = n) \\
&= \Pr_x (T_y = n, T_w \leq n) \\
&= \sum_{k=0}^{n} \Pr_x (T_y = n, T_w = k) \\
&= \sum_{k=0}^{n} \Pr_x (T_w = k) \Pr_x (T_y = n \mid T_w = k) \\
&= \sum_{k=0}^{n} f^{(k)}(x, w) f^{(n-k)}(w, y).
\end{aligned}$$

The equality $F(x, y|z) = F(x, w|z)F(w, y|z)$ holds when $F(x, w|z) = 0$ or $F(w, y|z) = 0$. So suppose $f^{(k)}(x, w) > 0$ for some k. Then

$$f^{(n-k)}(w, y) \leq \frac{f^{(n)}(x, y)}{f^{(k)}(x, w)}$$

for all $n \geq k$, whence $s(w, y) \geq s(x, y)$. Similarly we may suppose that $f^{(l)}(w, y) > 0$ for some l, which implies $s(x, w) \geq s(x, y)$. Then

$$f^{(n)}(x, y)z^n = \sum_{k=0}^{n} f^{(k)}(x, w)z^k f^{(n-k)}(w, y)z^{n-k}$$

for $|z| < s(x, y)$, and the product formula for power series implies the result. □

Corollary 1.10 (cf. [21, Exercise 1.45])**.** *If w is a cut point between x and y then* $E_x\left(T_y \mid T_y < \infty\right) = E_x\left(T_w \mid T_w < \infty\right) + E_w\left(T_y \mid T_y < \infty\right).$

Proof. If w is a cut point between x and y, then we have

$$\frac{F'(x, y|z)}{F(x, y|z)} = \frac{F'(x, w|z)}{F(x, w|z)} + \frac{F'(w, y|z)}{F(w, y|z)}$$

after differentiating $F(x, y|z) = F(x, w|z)F(w, y|z)$. Furthermore

$$\begin{aligned} E_x\left(T_y \mid T_y < \infty\right) &= \sum_{n=1}^{\infty} n \Pr_x\left(T_y = n \mid T_y < \infty\right) \\ &= \sum_{n=1}^{\infty} n \frac{\Pr_x\left(T_y = n\right)}{\Pr_x\left(T_y < \infty\right)} \\ &= \sum_{n=1}^{\infty} \frac{n f^{(n)}(x, y)}{F(x, y|1)} \\ &= \frac{F'(x, y|1)}{F(x, y|1)}. \end{aligned}$$

More precisely, in the case when $z = 1$ is on the boundary of the disk of convergence of $F(x, y|z)$, we can apply the Theorem of Abel: $nf^{(n)}(x, y)$ is non-negative for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} nf^{(n)}(x, y)z^n$ is finite for all $|z| < 1$ so that $F'(x, y|1) = F'(x, y|1^-)$ completing the proof. □

Note that if the state space is the set of vertices of a finite and connected graph then $F(x, y|1) = 1$ and $E_xT_y = E_xT_w + E_wT_y$.

1.2 Classes of a Markov chain

For $x, y \in X$ and $n = 0, 1, 2, \ldots$ we write

1. $x \xrightarrow{n} y$, if $p^{(n)}(x, y) > 0$,

2. $x \to y$, if there is $n \geq 0$ such that $x \xrightarrow{n} y$,
3. $x \not\to y$, if there is no $n \geq 0$ such that $x \xrightarrow{n} y$,
4. $x \leftrightarrow y$, if $x \to y$ and $y \to x$.

Lemma 1.11. *$\leftrightarrow$ is an equivalence relation on X.*

Proof. This relation is clearly symmetric. The relation $\to$ is reflexive since $p^{(0)}(x,x) = 1$ so that $\leftrightarrow$ is reflexive. Suppose $x \xrightarrow{m} w$ and $w \xrightarrow{n} y$ then $p^{(m+n)}(x,y) = \sum_{w \in X} p^{(m)}(x,w)p^{(n)}(w,y) \geq p^{(m)}(x,w)p^{(n)}(w,y) > 0$. Hence $x \xrightarrow{m+n} y$ so that $\to$ and in particular $\leftrightarrow$ is transitive. □

The relation $\leftrightarrow$ divides X into equivalence classes. If $x \leftrightarrow y$, we say that the states x and y *communicate.*

Definition 1.12. An *irreducible class* is an equivalence class with respect to $\leftrightarrow$.

The Markov chain itself is called *irreducible* if there is a unique irreducible class. In this case all elements communicate.

Definition 1.13. The *period* of an irreducible class C is the number $P = P(C) = \gcd(\{n > 0 : p^{(n)}(x,x) > 0\})$, where $x \in C$.

Lemma 1.14 (cf. [21, Lemma 2.20])**.** *The number $P(C)$ does not depend on the specific choice of $x \in C$.*

Proof. Let $x, y \in C$ with $x \neq y$. Write $P(x) = \gcd(\mathbb{N}_x)$, where $\mathbb{N}_x = \{n > 0 : p^{(n)}(x,x) > 0\}$ and we write $\mathbb{N}_y$ and $P(y)$ analogously. Since C is irreducible there are $k, l > 0$ such that $p^{(k)}(x,y) > 0$ and $p^{(l)}(y,x) > 0$. Now we have $p^{(k+l)}(x,x) \geq p^{(k)}(x,y)p^{(l)}(y,x) > 0$, and hence $P(x)$ divides $k + l$.

Let $n \in \mathbb{N}_y$. Note that $p^{(k+n+l)}(x,x) \geq p^{(k)}(x,y)p^{(n)}(y,y)p^{(l)}(y,x) > 0$, whence $P(x)$ divides $k + n + l$. It follows that $P(x)$ divides each $n \in \mathbb{N}_y$. Hence $P(x)$ divides $P(y)$. By symmetry we can interchange x and y so that $P(y)$ also divides $P(x)$. Hence $P(x)$=$P(y)$ completing the proof. □

If $P(C) = 1$ then C is called an *aperiodic class.* Note that C is aperiodic when $p(x,x) > 0$ for some $x \in C$. An irreducible Markov chain is called *aperiodic* if its irreducible class has period 1. The Markov chain in Example 1.2 is an aperiodic Markov chain.

Lemma 1.15 (cf. [21, Lemma 2.22])**.** *Let C be an irreducible class and $d = P(C)$. For each $x \in C$ there is $m_x \in \mathbb{N}$ such that $p^{(md)}(x,x) > 0$ for all $m \geq m_x$.*

Proof. Set $\mathbb{N}_x = \{n > 0 : p^{(n)}(x,x) > 0\}$. Now note that

$$n_1, n_2 \in \mathbb{N}_x \Rightarrow n_1 + n_2 \in \mathbb{N}_x. \tag{1.1}$$

We know from elementary number theory that the greatest common divisor of a set of positive integers can always be written as a finite linear combination of elements of the set with integer coefficients. Thus there are $n_1, \ldots, n_l \in \mathbb{N}_x$ and $a_1, \ldots, a_l \in \mathbb{Z}$ such that

$$d = \sum_{i=1}^{l} a_i n_i.$$

Let

$$n^+ = \sum_{i:a_i>0} a_i n_i \quad \text{and} \quad n^- = \sum_{i:a_i<0} (-a_i) n_i.$$

Now $n^+, n^- \in \mathbb{N}_x$ by (1.1), and $d = n^+ - n^-$. Set $k^+ = \frac{n^+}{d}$ and $k^- = \frac{n^-}{d}$. Then $k^+ - k^- = 1$. We define

$$m_x = k^-(k^- - 1).$$

Let $m \geq m_x$. We can write $m = qk^- + r$ where $q \geq k^- - 1$ and $0 \leq r \leq k^- - 1$. Hence, $m = qk^- + r(k^+ - k^-) = (q-r)k^- + rk^+$ with $(q-r) \geq 0$ and $r \geq 0$. Now by (1.1)

$$md = (q-r)n^- + rn^+ \in \mathbb{N}_x.$$

□

Definition 1.16. A state $x \in X$ is called *recurrent* if $U(x,x) = \Pr_x(T_x^+ < \infty) = 1$ and *transient* otherwise.

Theorem 1.17 (cf. [21, Theorem 3.4(a)]). *The state x is recurrent if and only if $G(x,x) = \infty$ and transient if and only if $G(x,x) < \infty$.*

Proof. Note that $U(x,x) = \lim_{z\to 1^-} U(x,x|z)$ and $G(x,x) = \lim_{z\to 1^-} G(x,x|z)$. Therefore by Theorem 1.7

$$G(x,x) = \lim_{z\to 1^-} \frac{1}{1 - U(x,x|z)} = \begin{cases} \infty & \text{if } U(x,x) = 1 \\ \frac{1}{1-U(x,x)} & \text{if } U(x,x) < 1. \end{cases}$$

so that the result follows. □

Theorem 1.18. *Recurrence and transience of states are class properties with respect to the relation $x \leftrightarrow y$. In other words communicating states are all either recurrent or transient.*

Proof. Suppose x is recurrent and $x \leftrightarrow y$. There are $k, l \geq 0$ such that $p^{(k)}(x, y) > 0$ and $p^{(l)}(y, x) > 0$. Now

$$G(y, y) \geq \sum_{n=k+l}^{\infty} p^{(n)}(y, y) \geq p^{(l)}(y, x) \sum_{m=0}^{\infty} p^{(m)}(x, x) p^{(k)}(x, y) = \infty$$

so thay y is recurrent by Theorem 1.17.

Similarly suppose x is transient and $x \leftrightarrow y$. There are $k, l \geq 0$ such that $p^{(k)}(x, y) > 0$ and $p^{(l)}(y, x) > 0$. Now

$$p^{(k)}(x, y) p^{(l)}(y, x) \sum_{m=0}^{\infty} p^{(m)}(y, y) \leq \sum_{n=0}^{\infty} p^{(k+n+l)}(x, x) \leq G(x, x) < \infty$$

so that y is recurrent by Theorem 1.17. ☐

Theorem 1.19 (cf. [21, Exercise 3.7])**.** *If a state $y \in X$ is transient, then $\lim_{n\to\infty} p^{(n)}(x, y) = 0$, regardless of the initial state $x \in X$.*

Proof. If y is transient, then we have by Theorem 1.7

$$\sum_{n=0}^{\infty} p^{(n)}(x, y) = G(x, y) = F(x, y) G(y, y) \leq G(y, y) < \infty$$

so that $\lim_{n\to\infty} p^{(n)}(x, y) = 0$. ☐

Lemma 1.20 (cf. [21, Theorem 3.4(b)])**.** *If x is recurrent and $x \to y$ then $U(y, x) = \Pr_y(\exists n > 0 : Z_n = x) = 1$.*

Proof. We prove this lemma by induction: if $x \xrightarrow{n} y$ then $U(y, x) = 1$. Since state x is recurrent we have $U(x, x) = 1$ so that the induction statement is true for $n = 0$. Suppose that it holds for n. If $x \xrightarrow{n+1} y$ then there is a $w \in X$ such that $x \xrightarrow{n} w \xrightarrow{1} y$. By the induction hypothesis $U(w, x) = 1$. Now by Theorem 1.7

$$1 = U(w, x) = p(w, x) + \sum_{v \neq x} p(w, v) U(v, x).$$

Since $\sum_{v \in X} p(w, v) = 1$ we have

$$0 = \sum_{v \neq x} p(w, v)(1 - U(v, x)) \geq p(w, y)(1 - U(y, x)) \geq 0.$$

Since $p(w, y) > 0$, we must have $U(y, x) = 1$, completing the proof by induction. ☐

Definition 1.21. A recurrent state x is called *positive recurrent*, if $E_x(T_x^+) < \infty$.

Theorem 1.22 (cf. [21, Theorem 3.9]). *If x is positive recurrent and $x \leftrightarrow y$ then y is positive recurrent.*

Proof. We know from Theorem 1.7 that

$$\frac{1-U(x,x|z)}{1-U(y,y|z)} = \frac{G(y,y|z)}{G(x,x|z)} \text{ for } 0<z<1.$$

Now x is recurrent and y is recurrent by Theorem 1.18. Hence as $z \to 1^-$, the right hand side of the equality above becomes an expression of type $\frac{\infty}{\infty}$. Since $0 < U'(y,y|1^-) \leq \infty$ we can apply de l'Hospital's rule to this equality. This yields

$$\lim_{z\to 1^-} \frac{G(y,y|z)}{G(x,x|z)} = \frac{U'(x,x|z)}{U'(y,y|z)} = \frac{E_x\left(T_x^+\right)}{E_y\left(T_y^+\right)}.$$

Since $x \leftrightarrow y$ there are $k,l>0$ such that $p^{(k)}(x,y)>0$ and $p^{(l)}(y,x)>0$. Therefore, if $0<z<1$,

$$\begin{aligned} G(y,y|z) &= \sum_{n=0}^{k+l-1} p^{(n)}(y,y)z^n + \sum_{n=k+l}^{\infty} p^{(n)}(y,y)z^n \\ &\geq \sum_{n=0}^{k+l-1} p^{(n)}(y,y)z^n + p^{(l)}(y,x)G(x,x|z)p^{(k)}(x,y)z^{k+l}. \end{aligned}$$

Hence

$$\lim_{z\to 1^-} \frac{G(y,y|z)}{G(x,x|z)} = \frac{E_x\left(T_x^+\right)}{E_y\left(T_y^+\right)} \geq p^{(l)}(y,x)p^{(k)}(x,y) > 0.$$

In particular $E_y\left(T_y^+\right) < \infty$ so that y is positive recurrent. □

Theorem 1.23 (cf. [21, Theorem 3.10]). *An irreducible Markov chain, with finite state space X, is positive recurrent.*

Proof. Since $\sum_{y\in X} p^{(n)}(x,y) = 1$, we have for each $x \in X$ and for $0 \leq z < 1$

$$\sum_{y\in X} G(x,y|z) = \sum_{n=0}^{\infty}\sum_{y\in X} p^{(n)}(x,y) = \frac{1}{1-z}.$$

By Theorem 1.7 we can write $G(x,y|z) = \frac{F(x,y|z)}{1-U(y,y|z)}$. Thus we obtain the following identity for each $x \in X$ and $0 \leq z < 1$:

$$\sum_{y\in X} F(x,y|z)\frac{1-z}{1-U(y,y|z)} = 1.$$

Note that since we only have a finite number of states while we have an infinite time frame, it is certain that at least one of the states will be visited infinitely many times. This state is recurrent and since all of the states communicate it follows from Theorem 1.18 that all states are recurrent so that $U(y, y|1^-) = 1$. We also have by Lemma 1.20 that $F(x, y|1^-) = 1$. Now since X is finite we can exchange the sum and limit and apply de l'Hospital's rule:

$$1 = \lim_{z \to 1^-} \sum_{y \in X} F(x, y|z) \frac{1 - z}{1 - U(y, y|z)} = \sum_{y \in X} \frac{1}{U'(y, y|1^-)}.$$

Therefore there must be $y \in X$ such that $U'(y, y|1^-) < \infty$. Hence y, and by the preceding theorem, the whole of X, is positive recurrent. □

1.3 The stationary distribution

In general for, $x \in X$, the probability that the Markov chain visits x at time n changes as n changes, where $n \geq 0$. If there is a probability distribution θ such that $\Pr_\theta (Z_n = x)$ remains constant for all n, irrespective of x, then this distribution is called the stationary distribution (see Proposition 1.25). The main result in this section is that if a Markov chain is irreducible and positive recurrent the stationary distribution at a state x is given by the inverse of the mean return time to x. Furthermore if the Markov chain is aperiodic as well, then the distribution of Z_n tends to the stationary distribution as $n \to \infty$.

Definition 1.24. Let (Z_n) be a Markov chain with state space X and transition matrix P. The *stationary distribution* of (Z_n) is a row vector $\pi = (\pi(x), x \in X)$ where $\sum_{x \in X} \pi(x) = 1$ and $\pi(x) \geq 0$ for all $x \in X$ such that $\pi P = \pi$. In other words $\pi(y) = \sum_{x \in X} \pi(x)p(x, y)$ for all $y \in X$.

We quickly find the stationary distribution for the Markov chain in Example 1.2. The two equations $\pi(1) = 0.15\pi(1) + 0.75\pi(2)$ and $\pi(1) + \pi(2) = 1$ yield the stationary distribution $\pi(1) = 0.46875, \pi(2) = 0.53125$.

Proposition 1.25. *If the probability distribution of Z_0 is π, the stationary distribution, then the probability distribution of Z_n is π for all $n \geq 1$.*

Proof. Set $\pi(x) = \pi_0(x) = \Pr (Z_0 = x)$ and $\pi_n(x) = \Pr (Z_n = x)$ for $n \geq 1$. Then

$$\begin{aligned}
\pi_n(y) &= \Pr (Z_n = y) \\
&= \sum_{y \in X} \Pr (Z_n = y \mid Z_0 = x) \Pr (Z_0 = x) \\
&= \sum_{y \in X} p^{(n)}(x, y)\pi(x).
\end{aligned}$$

By Lemma 1.2. $\pi_n = \pi P^n$ in matrix notation. Hence

$$\begin{aligned}\pi_n &= (\pi P)P^{n-1}\\ &= \pi P^{n-1}\\ &\vdots\\ &= \pi P\\ &= \pi.\end{aligned}$$

□

Proposition 1.26. *Suppose a Markov chain has a stationary distribution π. If the state y is transient then $\pi(y) = 0$.*

Proof. Since π is stationary, we have $\pi P^n = \pi$ for all n, so that

$$\pi(y) = \sum_{x \in X} \pi(x) p^{(n)}(x, y) \text{ for all n.}$$

Since y is transient, Theorem 1.19 says that $\lim_{n\to\infty} p^{(n)}(x, y) = 0$ for any $x \in X$. Thus

$$\sum_{x \in X} \pi(x) p^{(n)}(x, y) \to 0 \text{ as } n \to \infty$$

which implies $\pi(y) = 0$. □

Corollary 1.27. *If an irreducible Markov chain has a stationary distribution, then the chain is recurrent.*

Proof. Being irreducible, the chain must be either recurrent or transient, by Theorem 1.18. However if the chain were transient then the previous proposition would imply that $\pi(x) = 0$ for all $x \in X$ which would contradict the fact that π is a probability distribution that must sum to one. □

Theorem 1.28 (cf. [5, Theorem 13])**.** *Let (Z_n) be an irreducible Markov chain with state space X and transition matrix P. Let $x \in X$ be positive recurrent and let $m_x = E_x T_x^+$ be the mean return time to x. Then a stationary distribution π is given by*

$$\pi(y) = m_x^{-1} \sum_{n=0}^{\infty} \Pr_x \left(Z_n = y, T_x^+ > n \right)$$

for all $y \in X$. In particular $\pi(x) = m_x^{-1}$.

Proof. First of all note that $\sum_{y \in X} \pi(y) = 1$, since

$$\begin{aligned}
\sum_{y \in X} \sum_{n=0}^{\infty} \mathrm{Pr}_x\left(Z_n = y, T_x^+ > n\right) &= \sum_{n=0}^{\infty} \sum_{y \in X} \mathrm{Pr}_x\left(Z_n = y, T_x^+ > n\right) \\
&= \sum_{n=0}^{\infty} \mathrm{Pr}_x\left(T_x^+ > n\right) \\
&= \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \mathrm{Pr}_x\left(T_x^+ = k\right) \\
&= \sum_{k=0}^{\infty} k \, \mathrm{Pr}_x\left(T_x^+ = k\right) \\
&= m_x.
\end{aligned}$$

Furthermore $\pi(x) = m_x^{-1}$ since

$$\mathrm{Pr}_x\left(Z_n = x, T_x^+ > n\right) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$$

We proceed to show that π is stationary. Note that

$$\begin{aligned}
\pi(y) &= m_x^{-1} \sum_{n=0}^{\infty} \mathrm{Pr}_x\left(Z_n = y, T_x^+ > n\right) \\
&= m_x^{-1} \sum_{n=1}^{\infty} \mathrm{Pr}_x\left(Z_n = y, T_x^+ \geq n\right) \\
&= m_x^{-1} \sum_{n=1}^{\infty} \mathrm{Pr}_x\left(Z_n = y, T_x^+ > n-1\right)
\end{aligned}$$

since

$$\sum_{n=0}^{\infty} \mathrm{Pr}_x\left(Z_n = x, T_x^+ > n\right) = 1 = \sum_{n=1}^{\infty} \mathrm{Pr}_x\left(T_x^+ = n\right) = \sum_{n=1}^{\infty} \mathrm{Pr}_x\left(Z_n = x, T_x^+ \geq n\right)$$

by positive recurrence of x if $x = y$ and

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathrm{Pr}_x\left(Z_n = y, T_x^+ > n\right) &= \sum_{n=1}^{\infty} \mathrm{Pr}_x\left(Z_n = y, T_x^+ > n\right) \\
&= \sum_{n=1}^{\infty} \mathrm{Pr}_x\left(Z_n = y, T_x^+ \geq n\right)
\end{aligned}$$

if $x \neq y$. We now have

$$\begin{aligned}
\Pr_x\left(Z_n = y, T_x^+ > n-1\right) &= \frac{\Pr\left(Z_n = y, T_x^+ > n-1, Z_0 = x\right)}{\Pr\left(Z_0 = x\right)} \\
&= \sum_{z\in X} \frac{\Pr\left(Z_n = y, Z_{n-1} = z, T_x^+ > n-1, Z_0 = x\right)}{\Pr\left(Z_0 = x\right)} \\
&= \sum_{z\in X} \frac{\Pr\left(Z_n = y, Z_{n-1} = z, T_x^+ > n-1, Z_0 = x\right)}{\Pr\left(Z_{n-1} = z, T_x^+ > n-1, Z_0 = x\right)} \\
&\times \frac{\Pr\left(Z_{n-1} = z, T_x^+ > n-1, Z_0 = x\right)}{\Pr\left(Z_0 = x\right)} \\
&= \sum_{z\in X} p(z,y) \Pr_x\left(Z_{n-1} = z, T_x^+ > n-1\right)
\end{aligned}$$

so that

$$\begin{aligned}
\pi(y) &= m_x^{-1} \sum_{n=1}^{\infty} \sum_{z\in X} p(z,y) \Pr_x\left(Z_{n-1} = z, T_x^+ > n-1\right) \\
&= \sum_{z\in X} p(z,y) m_x^{-1} \sum_{n=0}^{\infty} \Pr_x\left(Z_n = z, T_x^+ > n\right) \\
&= \sum_{z\in X} \pi(z) p(z,y)
\end{aligned}$$

which completes the proof. □

Theorem 1.29 (cf. [5, Theorem 14]). *Let (Z_n) be an irreducible, positive recurrent Markov chain. Then (Z_n) has a unique stationary distribution.*

Proof. Existence has been shown in Theorem 1.28. We are left to show that this stationary distribution is unique. Let π denote the stationary distribution constructed in Theorem 1.28 and x the positive recurrent state that served as recurrence point for π. Let α denote any stationary distribution of (Z_n). Then there is a state $y \in X$ with $\alpha(y) > 0$ and some $m \in \mathbb{N}$ with $p^{(m)}(y,x) > 0$, since (Z_n) is irreducible. It follows from Proposition 1.25 that

$$\alpha(x) = \sum_{z\in X} \alpha(z) p^{(m)}(z,x) \geq \alpha(y) p^{(m)}(y,x) > 0.$$

Hence we can multiply α by a scalar c such that $c \cdot \alpha(x) = \pi(x) = \frac{1}{m_x}$. Let $\tilde{\alpha} = c \cdot \alpha$ and let $\tilde{P}$ be the transition matrix P without the x^{th} column, i.e. we define the (y,z)th entry of $\tilde{P}$ by $\tilde{p}(y,z) = p(y,z)$ if $z \neq x$ and 0 otherwise. Let δ^x be the $1 \times |X|$ matrix with $\delta^x_y = 1$ if $x = y$ and 0 otherwise. Note that the xy^{th} entry of $\tilde{P}^n$ is equal to $\Pr_x\left(Z_n = y, T_x^+ > n\right)$. Now the stationary distribution π can be represented by $\pi = m_x^{-1} \cdot \delta^x \sum_{n=0}^{\infty} \tilde{P}^n$ where we multiply by δ^x since we are only interested in the x^{th} row of $\tilde{P}$.

We claim that $m_x\tilde{\alpha} = \delta^x + m_x\tilde{\alpha}\tilde{P}$. For the entry $\tilde{\alpha}(x)$ we have $m_x\tilde{\alpha}(x) = 1 = \delta^x$ since $(\tilde{\alpha}\tilde{P})_x = 0$. For $\tilde{\alpha}(y)$ with $x \neq y$ we have $(\tilde{\alpha}\tilde{P})_y = c \cdot (\alpha P)_y = \tilde{\alpha}(y)$ so that the claim holds for this case as well. We can proceed with the same argument to see that

$$\begin{aligned} m_x\tilde{\alpha} &= \delta^x + (m_x\tilde{\alpha}\tilde{P})\tilde{P} = \delta^x + \delta^x\tilde{P} + m_x\tilde{\alpha}\tilde{P}^2 = \dots \\ &= \delta^x \sum_{n=0}^{\infty} \tilde{P}^n \\ &= m_x\pi. \end{aligned}$$

Hence $\tilde{\alpha} = \pi$ so that $c = 1$, proving that the stationary distribution is unique. □

Theorem 1.30 (cf. [5, Theorem 15])**.** *Let (Z_n) be an irreducible, positive recurrent Markov chain. Then the stationary distribution π of (Z_n) is given by*

$$\pi(x) = m_x^{-1} = \frac{1}{E_x T_x^+}$$

for all $x \in X$.

Proof. Since all states in X are positive recurrent, the stationary distribution constructed in Theorem 1.28 can be pursued for any initial state x. This yields $\pi(x) = \frac{1}{E_x T_x^+}$ for all $x \in X$. Since the stationary distribution is unique the statement follows. □

To prove the final result in this chapter, we first need to introduce the notion of coupling. We are given the law of a stochastic process $(X_n, n \geq 0)$ and the law of a stochastic process $(Y_n, n \geq 0)$, i.e. we know what is $\Pr(X_n = x)$ and $\Pr(Y_n = x)$ for all n, but we are not told what the joint probabilities are. A coupling of them refers to a joint construction on the same probability space.

Suppose now we have two stochastic processes as above that have been coupled. A random time T is called a *meeting time* between the two processes if

$$X_n = Y_n \text{ for all } n \geq T.$$

If the processes never meet then T takes on the value ∞.

Proposition 1.31 (cf. [16, Proposition 4])**.** *Let T be a meeting time of two coupled stochastic processes (X_n) and (Y_n). Then, for all $n \geq 0$, and all $x \in X$, where X is the set of possible values that (X_n) and (Y_n) can assume,*

$$|\Pr(X_n = x) - \Pr(Y_n = x)| \leq \Pr(T > n).$$

If in particular $\Pr(T < \infty) = 1$, *then*

$$\sup_{x \in X} |\Pr(X_n = x) - \Pr(Y_n = x)| \to 0 \text{ as } n \to \infty.$$

Proof. We have

$$\begin{aligned}
\Pr(X_n = x) &= \Pr(X_n = x, n < T) + \Pr(X_n = x, n \geq T) \\
&= \Pr(X_n = x, n < T) + \Pr(Y_n = x, n \geq T) \\
&\leq \Pr(n < T) + \Pr(Y_n = x),
\end{aligned}$$

and hence $\Pr(X_n = x) - \Pr(Y_n = x) \leq \Pr(T > n)$. Similarly by interchanging X_n and Y_n we obtain $\Pr(Y_n = x) - \Pr(X_n = x) \leq \Pr(T > n)$. Combining these two inequalities we obtain the first assertion:

$$|\Pr(X_n = x) - \Pr(Y_n = x)| \leq \Pr(T > n).$$

Since this holds for all $x \in X$ and since the right hand side does not depend on x, we can write this as

$$\sup_{x \in X} |\Pr(X_n = x) - \Pr(Y_n = x)| \leq \Pr(T > n), \quad n \geq 0.$$

Now if $\Pr(T < \infty) = 1$, then

$$\lim_{n \to \infty} \Pr(T > n) = 0.$$

Therefore

$$\sup_{x \in X} |\Pr(X_n = x) - \Pr(Y_n = x)| \to 0 \text{ as } n \to \infty.$$

□

Theorem 1.32 (cf. [16, Theorem 17])**.** *If a Markov chain (X_n) is irreducible, positive recurrent and aperiodic with (unique) stationary distribution π, then*

$$\lim_{n \to \infty} \Pr(X_n = x) = \pi(x),$$

for any initial distribution.

Proof. We start by considering the laws of two processes. The first process, (X_n), has the law of a Markov chain with transition probabilities $p(x, y)$, where X_0 is distributed according to some arbitrary distribution μ. The second process, (Y_n), has the law of a Markov chain with transition probabilities $p(x, y)$, where Y_0 is distributed according to the stationary distribution π. We assume both processes are positive recurrent, aperiodic Markov chains, on the same state space X. Note that they have the same transition probabilities and differ only in how the initial states are chosen. We couple them by assuming that (X_n) is independent of (Y_n). Having coupled them we can now define joint probabilities. Let

$$T = \inf\{n \geq 0 : X_n = Y_n\}.$$

Consider the process

$$W_n = (X_n, Y_n).$$

Then $(W_n, n \geq 0)$ is a Markov chain with state space $X \times X$. Its initial state $W_0 = (X_0, Y_0)$ has distribution $\Pr(W_0 = (x, y)) = \mu(x)\pi(y)$. Its 1-step transition probabilities are

$$\begin{aligned} q((x,y),(x',y')) &= \Pr(W_{n+1} = (x',y') \mid W_n = (x,y)) \\ &= \Pr(X_{n+1} = x' \mid X_n = x) \Pr(Y_{n+1} = y' \mid Y_n = y) \\ &= p(x,x')p(y,y'). \end{aligned}$$

Its n-step transition probabilities are

$$q^{(n)}((x,y),(x',y')) = p^{(n)}(x,x')p^{(n)}(y,y').$$

From Lemma 1.15 with $d = 1$ and the aperiodicity assumption, we know that $p^{(n)}(x, x') > 0$ and $p^{(n)}(y, y') > 0$ for all large enough n, implying that $q^{(n)}((x, y), (x', y')) > 0$ for all large enough n. Hence (W_n) is an irreducible Markov chain. Note that $\sigma(x, y) = \pi(x)\pi(y), (x, y) \in X \times X$ is a stationary distribution for (W_n). (Had we started with both X_0 and Y_0 independent and distributed according to π, then for all $n \geq 1$, X_n and Y_n would be independent and distributed according to π.) Hence by Corollary 1.27 (W_n) is recurrent. Since (W_n) is irreducible and recurrent, it follows from Lemma 1.20 that (W_n) hits the diagonal $\{(x, x) : x \in X\}$ in $X \times X$ in finite time with probability one. Hence $\Pr(T < \infty) = 1$.

We now define a third process by $Z_n = X_n I(n < T) + Y_n I(n \geq T)$, where I denotes the indicator function. Thus Z_n equals X_n before T and $Z_n{=}Y_n$ on or after time T. We have $Z_0{=}X_0$, where X_0 has arbitrary distribution μ by assumption. Furthermore (Z_n) is a Markov chain with transition probabilities $p(x, y)$. Hence for all n and x, $\Pr(Z_n = x) = \Pr(X_n = x)$.

Note that T is a meeting time between (Z_n) and (Y_n). It follows from Proposition 1.31 that

$$|\Pr(Z_n = x) - \Pr(Y_n = x)| \leq \Pr(T > n).$$

Since $\Pr(Z_n = x) = \Pr(X_n = x)$, and since $\Pr(Y_n = x) = \pi(x)$, we have

$$|\Pr(X_n = x) - \pi(x)| \leq \Pr(T > n).$$

Since $\Pr(T < \infty) = 1$, $\Pr(T > n)$ converges to 0 as $n \to \infty$, so that the statement follows. □

1.4 Hitting time identities

In this section we look at a few important identities involving the hitting time. Throughout we work with an irreducible, positive recurrent and aperiodic Markov chain (Z_n) with state space X, transition matrix P and unique stationary distribution π. The results in this section are all taken from the second chapter of [2].

Proposition 1.33 (cf. [2, Proposition 4])**.** *Let θ be a probability distribution on the state space X. Let $0 < S < \infty$ be a stopping time such that $\theta(y) = \Pr_\theta(Z_S = y)$ for all $y \in X$ and $E_\theta S < \infty$. Let x be an arbitrary state, then*

$$E_\theta\,(\textit{number of visits to } x \textit{ before time } S) = \pi(x) E_\theta S.$$

Proof. First note that $\Pr_\theta(Z_S = y) = \Pr_\theta(Z_0 = y)$ for any $y \in X$, by definition. Write $\rho(x) = E_\theta\,(\text{number of visits to } x \text{ before time S})$. Then

$$\begin{aligned}
\rho(y) &= \sum_{t=0}^{\infty} \Pr_\theta\left(Z_t = y, S > t\right) \\
&= \Pr_\theta(Z_S = y) + \sum_{t=1}^{\infty} \Pr_\theta\left(Z_t = y, S > t\right) \\
&= \sum_{t=1}^{\infty} \Pr_\theta(Z_S = y, S = t) + \sum_{t=1}^{\infty} \Pr_\theta\left(Z_t = y, S > t\right) \\
&= \sum_{t=1}^{\infty} \Pr_\theta\left(Z_t = y, S \geq t\right) \\
&= \sum_{t=0}^{\infty} \Pr_\theta\left(Z_{t+1} = y, S > t\right) \\
&= \sum_{t=0}^{\infty} \sum_{x\in X} \Pr_\theta\left(Z_t = x, S > t, Z_{t+1} = y\right) \\
&= \sum_{t=0}^{\infty} \sum_{x\in X} \Pr_\theta\left(Z_t = x, S > t\right) \Pr_\theta\left(Z_{t+1} = y \mid Z_t = x\right) \\
&= \sum_{x\in X} \rho(x) p(x, y).
\end{aligned}$$

Let $\tau(y) = \frac{\rho(y)}{\sum_{y\in X}\rho(y)}$. Then $\sum_{y\in X} \tau(y) = 1$ and $\tau(y) = \sum_{x\in X} \tau(x)p(x, y)$. It follows that $\tau = \pi$ since the stationary distribution is unique. Since $\sum_{y\in X} \rho(y) = E_\theta S$ the result follows. □

In the previous proposition instead of starting from an arbitrary distribution we can start from w and choose S to be T_w^+ so that

$$E_w\left(\text{number of visits to } x \text{ before time } T_w^+\right) = \pi(x) E_w T_w^+.$$

Setting $w = x$ gives $1 = \pi(w) E_w T_w^+$ so that we obtain the familiar result $E_w T_w^+ = \frac{1}{\pi(w)}$. For general x we have

$$E_w\left(\text{number of visits to } x \text{ before time } T_w^+\right) = \frac{\pi(x)}{\pi(w)}.$$

If we start from w and choose S to be the first return time to w, after the first visit to x, then $E_w S = E_w T_x + E_x T_w$ and by Proposition 1.33 $\pi(x) E_w S = E_w\,(\text{number of visits to } x \text{ before time S})$. Since there are no visits to x before time T_x we have

$$E_x\,(\text{number of visits to } x \text{ before time } T_w) = \pi(x)\left(E_w T_x + E_x T_w\right). \tag{1.2}$$

We set

$$Z(x, y) = \sum_{n=0}^{\infty} \left(p^{(n)}(x, y) - \pi(y)\right).$$

Note that $\sum_{y \in X} Z(x, y) = 0$ for all i since $\sum_{y \in X} p^{(n)}(x, y) = \sum_{y \in X} \pi(y) = 1$.

Lemma 1.34 (cf. [2, Lemma 11]). *$\pi(x) E_\pi T_x = Z(x, x)$ for any $x \in X$.*

Proof. Suppose we start from state x. Fix a time $t_0 \geq 1$ and define a stopping time S as follows:

1. wait time t_0,
2. then wait (if necessary) until the chain next hits x.

Then by Proposition 1.33 we have

$$\sum_{t=0}^{t_0 - 1} p^{(t)}(x, x) = \pi(x) \left(t_0 + E_\rho T_x\right)$$

where $\rho(\cdot) = \Pr_x\left(Z_{t_0} = \cdot\right)$. Hence

$$\sum_{t=0}^{t_0 - 1} \left(p^{(t)}(x, x) - \pi(x)\right) = \pi(x) E_\rho T_x.$$

If $t_0 \to \infty$ then $\rho \to \pi$ by Theorem 1.32 so that the result follows. □

Lemma 1.35 (cf. [2, Lemma 12]). *$\pi(x) E_y T_x = Z(x, x) - Z(y, x)$ for any $x, y \in X$.*

Proof. If $x = y$ then $E_x T_x = 0$ so that the result holds. For $x \neq y$ we prove this lemma in a similar fashion as the previous lemma. Suppose we start from state x. Fix a time $t_0 \geq 1$ and define a stopping time S as follows:

1. wait until the chain hits y,
2. then wait a further time t_0,
3. then wait (if necessary) until the chain next hits x.

Then by Proposition 1.33 we have

$$E_x\left(\text{number of visits to } x \text{ before time } T_y\right) + \sum_{t=0}^{t_0 - 1} p^{(t)}(y, x) = \pi(x)\left(E_x T_y + t_0 + E_\rho T_x\right),$$

where $\rho(\cdot) = \Pr_y\left(Z_{t_0} = \cdot\right)$. By equation (1.2)

$$E_x\left(\text{number of visits to } x \text{ before time } T_y\right) = \pi(x)\left(E_y T_x + E_x T_y\right)$$

so that

$$\sum_{t=0}^{t_0-1}\left(p^{(t)}(y,x)-\pi(x)\right)=\pi(x)\left(E_\rho T_x-E_y T_x\right).$$

If $t_0\to\infty$ then $\rho\to\pi$ by Theorem 1.32 so that $Z(y,x)=\pi(x)\left(E_\pi T_x-E_y T_x\right)$. Since $\pi(x)E_\pi T_x=Z(x,x)$ by the previous lemma the result follows. $\square$

Corollary 1.36 (cf. [2, Corollary 13])**.** *The sum* $\sum_{x\in X}\pi(x)E_y T_x$*, known as Kemeny's constant, is independent of* y*. In particular*

$$\sum_{x\in X}\pi(x)E_y T_x=\sum_{x\in X}Z(x,x).$$

Proof. Since $\sum_{x\in X}Z(y,x)=0$ the result follows from Lemma 1.35. $\square$

Chapter 2

Random walks on graphs

2.1 Reversible Markov chains

Definition 2.1. Let π be the stationary distribution of an irreducible and positive recurrent Markov chain. Call the Markov chain *reversible* if $\pi(x)p(x,y) = \pi(y)p(y,x)$ for all states x and y.

The name reversible comes from the following: Suppose a Markov chain has a unique stationary distribution and starts at this distribution. This means $Z_0, Z_1, Z_2, \ldots$ all have distribution π. Consider the time reversed conditional probability

$$\begin{aligned}
&\Pr\left(Z_k = x \mid Z_{k+1} = y, Z_{k+2} = z_{k+2}, \ldots, Z_n = z_n\right)\\
&= \frac{\Pr\left(Z_k = x, Z_{k+1} = y, \ldots, Z_n = z_n\right)}{\Pr\left(Z_{k+1} = y, \ldots, Z_n = z_n\right)}\\
&= \frac{\pi(x)p(x,y)p\left(y, z_{k+2}\right)\ldots p\left(z_{n-1}, z_n\right)}{\pi(y)p\left(y, z_{k+2}\right)\ldots p\left(z_{n-1}, z_n\right)}\\
&= \frac{\pi(x)p(x,y)}{\pi(y)}.
\end{aligned}$$

Note that this probability only depends on x and y. So if a Markov chain starts at its stationary distribution then the time reversed chain satisfies the Markov property. If a Markov chain and its time reversed chain have the same transition probability then

$$\Pr\left(Z_{k+1} = x \mid Z_k = y\right) = p(y,x) = \frac{\pi(x)p(x,y)}{\pi(y)}.$$

Now $Z_0, Z_1, \ldots, Z_n$ has the same joint probability distribution as $Z_n, Z_{n-1}, \ldots, Z_0$ for every n since both start at the same stationary distribution.

Proposition 2.2. *Any probability distribution α that satisfies $\alpha(x)p(x,y) = \alpha(y)p(y,x)$ is the stationary distribution.*

Proof. $\sum_x \alpha(x)p(x,y) = \alpha(y)\sum_x p(y,x) = \alpha(y)$. The result follows from Theorem 1.29, which tells us the stationary distribution is unique. □

Let $G = (V(G), E(G))$ be an undirected, weighted graph. If it is clear with which underlying graph we are working we simply write E for the set of edges. Write $w(x,y) = w(y,x)$ for the weight of an edge between vertices x and y. Set $w(x,y) = 0$ if there is no edge between x and y. Let $p(x,y) = \frac{w(x,y)}{\sum_z w(x,z)}$ be the probability of moving from vertex x to vertex y. Let $w = \sum_{x,y\in V} w(x,y)$ be the sum of all the weights. Note that each edge is weighted twice, once in each direction. Let $\pi(x) = \frac{\sum_z w(x,z)}{w}$. Then $\pi(x)p(x,y) = \frac{w(x,y)}{w} = \frac{w(y,x)}{w} = \pi(y)p(y,x)$.

So we can consider a random walk on an undirected graph to be a reversible Markov chain with transition matrix $P = (p(x,y))_{x,y\in V}$ where $p(x,y) = \frac{w(x,y)}{\sum_z w(x,z)}$. Note that if a graph is connected, finite and non-bipartite then the corresponding Markov chain is respectively irreducible, positive recurrent and aperiodic.

If we assign a weight of one to every edge and disallow loops, then we have a random walk on an unweighted graph. In this case

$$p(x,y) = \begin{cases} \frac{1}{d(x)} & \text{if } xy \text{ is an edge} \\ 0 & \text{if not} \end{cases}$$

where $d(x)$ is the degree of vertex x. The stationary distribution becomes $\pi(x) = \frac{d(x)}{2|E|}$. A random walk on an unweighted graph is called a *simple random walk*. In the remainder of this thesis we assume a random walk on a graph is simple unless stated otherwise.

Example 2.3. Start a king at a corner of a standard chessboard. The king randomly makes a legal king-move. What is the expected number of moves until the king returns to the starting square if it moves to any legal square with equal probability?

We can consider the king's walk on the chessboard as a random walk on a graph with 64 vertices (the squares of the chessboard) and the possible king-moves are the edges. The 4 corner squares have degree 3. The 24 other squares on the side have degree 5 and the remaining 36 central squares have degree 8 so that

$$2|E| = \sum_{v\in G} d(v) = (3)(4) + (6)(4)(5) + (8)(36) = 420.$$

So the expected number of steps to return to the same corner c is $E\left(T_c^+\right) = \frac{1}{\pi(c)} = \frac{420}{3} = 140$.

2.2 Main Parameters

1. The expected number of steps on a graph G to visit vertex y for the first time after we started at vertex x is called the *hitting time* and is denoted by $H_{x,y}(G)$. If it is clear on which graph we are working we simply write $H_{x,y}$. Note that $H_{x,y} = E_x T_y$ for $x \neq y$.

2. The sum $C_{x,y} = H_{x,y} + H_{y,x}$ is called the *commute time* whereas $D_{x,y} = H_{x,y} - H_{y,x}$ is called the *difference time*.

3. The *cover time* C_x is the expected number of steps to visit every vertex starting from x.

4. The *cover and return time* CR_x is the expected number of steps to visit every vertex and return to the starting vertex x. $CR_x = C_x + R_x$ where the *return time* R_x is an average of hitting times $H_{z,x}$, weighted for each z by the probability that z is the last vertex visited when covering the graph starting at x.

Example 2.4. Consider the complete graph K_n, on vertices $v_1, \ldots, v_n$. By symmetry H_{v_i,v_j} is the same for any $i \neq j$. We determine H_{v_1,v_2}. The probability of moving to v_2 for the first time after k steps is $\left(\frac{n-2}{n-1}\right)^{k-1}\left(\frac{1}{n-1}\right)$. Setting $x = \frac{n-2}{n-1}$ yields:

$$\begin{aligned} H_{v_1,v_2} &= \sum_{k=1}^{\infty} k x^{k-1} \left(\frac{1}{n-1}\right) \\ &= \left(\frac{1}{n-1}\right) \frac{d}{dx} \left(\sum_{k=1}^{\infty} x^k\right) \\ &= n-1. \end{aligned}$$

Then by symmetry the commute time is $H_{v_1,v_2} + H_{v_2,v_1} = 2(n-1)$.

Let S_i be the first time i vertices are visited. $S_1 = 0$ since we start at some vertex. $S_{i+1} - S_i$ is the number of steps to visit $i+1$ vertices given that we have visited i vertices. The cover time remains the same irrespective of the starting vertex.

$$\begin{aligned} C_{v_1} &= E\left(S_n\right) \\ &= \sum_{i=1}^{n-1} E\left(S_{i+1} - S_i\right) \\ &= \sum_{i=1}^{n-1} \sum_{t=1}^{\infty} t \left(\frac{i-1}{n-1}\right)^{t-1} \left(\frac{n-i}{n-1}\right) \\ &= \sum_{i=1}^{n-1} \frac{n-1}{n-i} \end{aligned}$$

The return time to v_1 is just $n-1$ since the last vertex to be visited is, with equal probability, any of the $n-1$ non-starting vertices.

Although it is not true in general that $H_{x,y} = H_{y,x}$, the following symmetry property holds for random walks on graphs:

Theorem 2.5 (cf. [8, Lemma 2])**.** *For any three vertices x, y and z, $H_{x,y} + H_{y,z} + H_{z,x} = H_{x,z} + H_{z,y} + H_{y,x}$.*

We defer the proof of this theorem to chapter 4.

Corollary 2.6 (cf. [17, Corollary 2.5])**.** *We can define a preorder on the vertices by $x \leq y$ if and only if $H_{x,y} \leq H_{y,x}$. Such an ordering can be obtained by fixing any vertex t and ordering the vertices according to the value of $H_{x,t} - H_{t,x}$.*

Proof. First note that if $H_{x,t} - H_{t,x} \leq H_{y,t} - H_{t,y}$ then $H_{x,t} + H_{t,y} \leq H_{y,t} + H_{t,x}$ so that $H_{x,y} \leq H_{y,x}$ by Theorem 2.5. This order is clearly reflexive since $H_{x,x} \leq H_{x,x}$. Furthermore if $H_{x,y} \leq H_{y,x}$ and $H_{y,z} \leq H_{z,y}$ then $H_{x,y} + H_{y,z} \leq H_{y,x} + H_{z,y} \iff H_{x,z} \leq H_{z,x}$ by Theorem 2.5, so that this order is transitive as well. □

This ordering is not unique, because of the ties. However by the preceding theorem we can define an equivalence relation on the vertices by $x \sim y$ if and only if $H_{x,y} = H_{y,x}$. Now there is a unique ordering of the equivalence classes. Vertices in the highest class are easy to reach but difficult to get out of, while those in the lowest class are difficult to reach but easy to get out of.

Chapter 3

Random walks on trees

3.1 Hitting and cover times

In this section we find exact formulas for the hitting time and commute time between any two vertices of a tree. However, there is no known formula for the cover time of an arbitrary tree. For most trees one should be satisfied with obtaining good upper and lower bounds on the cover time. Initial work done in this regard include papers by Aldous [1] and Devroye and Sbihi [10]. Exact formulas for certain trees can be found. We find novel formulas for the star graph of arbitrary radius (see Figure 3.1) and the broom graph (Figure 3.2).

Let T be a tree with n vertices. For adjacent vertices x and y and edge $e = xy$ let $T_{x:y}$ be the component of $T - \{e\}$ that contains x. Let T' be the induced subtree on the vertices $T_{x:y} \cup \{y\}$. Let $d'(k)$ be the T'-degree and $d(k)$ the T-degree of k. Then by Theorem 1.30 we get the following hitting time formula appearing in [3]:

$$\begin{aligned}
H_{x,y}(T) = H_{x,y}(T') &= \frac{1}{\pi(y)} - 1 \\
&= \sum_{k \in V(T')} d'(k) - 1 \\
&= \sum_{k \in V(T_{x:y})} d(k) \\
&= 2\,|E| \sum_{k \in V(T_{x:y})} \pi(k).
\end{aligned}$$

For any two adjacent vertices x and y we have $H_{x,y} + H_{y,x} = 2\,|E| = 2(n-1)$. More generally if $d(x, y) = r$ and $(x = x_0, x_1, \ldots, x_r = y)$ is the (x, y)-path then

$$\begin{aligned} H_{x,y} + H_{y,x} &= \sum_{i=0}^{r-1} \left(H_{x_i,x_{i+1}} + H_{x_{i+1},x_i}\right) \\ &= 2\,|E|\,d(x,y) \\ &= 2r(n-1). \end{aligned} \tag{3.1}$$

So there is a very simple formula for the commute time between any two vertices of a tree. To determine an explicit formula for the hitting time from x to y on a path $(x = x_0, x_1, \ldots, x_r = y)$ we first require the following lemma:
Lemma 3.1. *For any vertex x of a tree T with n edges, we have*

$$\sum_{y \in V(T)} d(y)d(x,y) = 2 \sum_{y \in V(T)} d(x,y) - n.$$

Proof. We prove this lemma by induction on the number of edges n. If $n = 1$ our tree is a path of length 1. Then

$$\sum_{y \in V(T)} d(y)d(x,y) = 1 = 2 \sum_{y \in V(T)} d(x,y) - 1$$

where x is any one of the two leaves of this path. Suppose for $n = k - 1$ that

$$\sum_{y \in V(T)} d(y)d(x,y) = 2 \sum_{y \in V(T)} d(x,y) - (k-1).$$

A tree T with k edges consists of a tree T' with $k-1$ edges with a leaf, say l, adjoined to any of the k possible vertices of T'. Let m denote the single vertex adjacent to l. Let $d'(k)$ be the T'-degree and $d(k)$ the $T-$degree of a vertex k. Note that $d'(y)d(x,y) = d(y)d(x,y)$ for $y \in T' \setminus \{m\}$ and $d(m)d(x,m) = (d'(m)+1)d(x,m)$. Hence

$$\begin{aligned} \sum_{y \in V(T)} d(y)d(x,y) &= d(l)d(x,l) + d(x,m) + \sum_{y \in V(T')} d'(y)d(x,y) \\ &= d(x,l) + d(x,m) + 2 \sum_{y \in V(T')} d(x,y) - (k-1) \\ &= d(x,l) - 1 + d(x,l) + 2 \sum_{y \in V(T')} d(x,y) - (k-1) \\ &= 2 \sum_{y \in V(T)} d(x,y) - k \end{aligned}$$

completing the induction. $\square$

Set $l(x,k,y) = \frac{1}{2}(d(x,y) + d(k,y) - d(x,k))$, the length of the intersection of the (x,y)-path and the (k,y)-path. Then by the previous lemma

$$\begin{aligned}
H_{x,y} &= \sum_{i=0}^{r-1} H_{x_i,x_{i+1}} \\
&= \sum_{i=0}^{r-1} \sum_{k\in V\left(T_{x_i:x_{i+1}}\right)} d(k) \\
&= \sum_{k\in V(T)} l(x,k,y)d(k) \\
&= d(x,y)\,|E| + \frac{1}{2}\left(2\sum_{k\in V(T)} d(k,y) - |E| - 2\sum_{k\in V(T)} d(k,x) + |E|\right) \\
&= r(n-1) + \sum_{k\in V(T)} d(k,y) - \sum_{k\in V(T)} d(k,x), \qquad (3.2)
\end{aligned}$$

simplifying the formula given in Beveridge [3].

An equivalent hitting time formula, given in [6], can be found on a path $(x = x_0, x_1, \ldots, x_r = y)$ by noting that $\sum_{k\in V(T_{x:y})} d(k) = 2m + 1$ where m is the number of edges in $T_{x:y}$. For $i = 1, \ldots, r-1$ let T_i be the component of $T - \{x_{i-1}x_i\} - \{x_i x_{i+1}\}$ that contains vertex x_i and T_0 the component of $T - \{x_0 x_1\}$ containing x_0. Let m_i be the number of edges in T_i. Then $\sum_{k\in V(T_0)} d(k) = 2m_0 + 1$ and $\sum_{k\in V(T_i)} d(k) = 2m_i + 2$ for $i = 1, \ldots, r-1$. Now

$$\begin{aligned}
H_{x,y} &= \sum_{i=0}^{r-1} H_{x_i,x_{i+1}} \\
&= \sum_{i=0}^{r-1} \sum_{k\in V\left(T_{x_i:x_{i+1}}\right)} d(k) \\
&= \sum_{k\in V(T_0)} d(k)r + \sum_{i=1}^{r-1} \sum_{k\in V(T_i)} d(k)(r-i) \\
&= r(2m_0+1) + \sum_{i=1}^{r-1}(2m_i+2)(r-i) \\
&= 2\sum_{i=0}^{r-1} m_i(r-i) + r + 2\sum_{i=1}^{r-1} i \\
&= r^2 + 2\sum_{i=0}^{r-1} m_i(r-i). \qquad (3.3)
\end{aligned}$$

This yields the following corollary mentioned by Brightwell and Winkler in [6].

Corollary 3.2. *The hitting time between two leaves of an n-vertex tree, where $n \geq 3$, is minimised by pairs of leaves at distance two.*

Proof. Suppose x and y are leaves of a tree where $d(x, y) = r$. By equation (3.3)

$$H_{x,y} = r^2 + 2\sum_{i=0}^{r-1} m_i(r-i).$$

Now r of the $n-1$ edges in the tree lie on the path connecting x and y. The remaining $n-1-r$ edges lie in $T_1, \ldots, T_{r-1}$ (note that $m_0 = 0$ since x is a leaf). The sum

$$2\sum_{i=0}^{r-1} m_i(r-i)$$

is minimised if all of the remaining $n-1-r$ edges lie in T_{r-1}. In this case $m_1 = \ldots = m_{r-2} = 0$ and $m_{r-1} = n-1-r$ so that

$$H_{x,y} = r^2 + 2(n-1-r) = 2(n-1) + r^2 - 2r.$$

For $r \geq 2$, $r^2 - 2r$ is minimised when $r = 2$.(If $r = 1$ then x and y can only both be leaves if $n = 2$.)It follows that $H_{x,y} \geq 2(n-1)$ where the minimum is obtained for pairs of leaves at distance 2. □

Corollary 3.3. *The hitting time between two vertices of an n-vertex tree is maximised by the endpoints of the path of length $n-1$.*

Proof. An n-vertex tree has $n-1$ edges, with r edges lying on the path connecting $T_0, T_1, \ldots, T_r$ so that $\sum_{i=0}^{r-1} m_i \leq n-1-r$. It follows that $0 \leq \sum_{i=0}^{r-1} m_i(r-i) \leq (n-1-r)r$. Then by equation (3.3) we have $r^2 \leq H_{x,y} \leq r^2 + 2(n-1-r)r$ where $d(x, y) = r$. The upper bound $r^2 + 2(n-1-r)r = 2r(n-1) - r^2$ is maximised when $r = n-1$ with $H_{x,y} = (n-1)^2$ giving us the desired result.

□

Example 3.4. Consider the path of length n on vertices $x_0, x_1, \ldots, x_n$. For

$i \leq j$:

$$\begin{aligned} H_{x_i,x_j} &= \sum_{k\in V(T)} d(k,x_j) - \sum_{k\in V(T)} d(k,x_i) + |E|\, d(x_i,x_j) \\ &= \sum_{k=1}^{j} k + \sum_{k=1}^{n-j} k - \sum_{k=1}^{i} k - \sum_{k=1}^{n-i} k + n(j-i) \\ &= \frac{j(j+1) + (n-j)(n-j+1) - i(i+1) - (n-i)(n-i+1) + 2n(j-i)}{2} \\ &= j^2 - i^2. \end{aligned}$$

For most n-vertex trees it is difficult to find an exact formula for the cover time. We will look at a few examples where it is possible to find an exact formula.

For a tree with k leaves let L_i be the expected number of steps before i distinct leaves are visited. Hence L_k is the cover time of the tree. Note that $L_k = L_1 + \sum_{i=1}^{k-1}(L_{i+1} - L_i)$ where $L_{i+1} - L_i$ is the expected number of steps to visit $i+1$ leaves after i leaves have been visited. It follows that the cover time of a tree is minimised from one of its leaves, since if we do not start from a leaf it takes at least one step to visit the first leaf. The remaining time to cover the tree is a weighted average (weighed by the probability of visiting a given leaf first) of the cover times from the leaves, which takes at least as many steps as the cover time from the leaf with least cover time.

Example 3.5. Let us first find the cover time for the path of length n on vertices $0, 1, 2, \ldots, n$. $C_0 = C_n = n^2$ is just the hitting time of one of the leaves when starting from the other.

To find the cover time from an internal vertex $i \in \{1, 2, \ldots, n-1\}$ we first need to find E_iT_A where $A = \{0, n\}$ is the set of leaves. Note that $E_0T_A = E_nT_A = 0$ and $E_iT_A = 1 + \frac{1}{2}E_{i-1}T_A + \frac{1}{2}E_{i+1}T_A$ for $i = 1, 2, \ldots, n-1$. By computing the first few values of E_iT_A it looks like $E_iT_A = i + \frac{i}{i+1}E_{i+1}T_A$ for $i = 1, 2, \ldots, n-1$. We verify this by induction:

$E_1T_A = 1 + \frac{1}{2}E_2T_A$. Suppose $E_{i-1}T_A = i - 1 + \frac{i-1}{i}E_iT_A$. Then

$$\begin{aligned} E_iT_A &= 1 + \frac{1}{2}E_{i-1}T_A + \frac{1}{2}E_{i+1}T_A \\ &= 1 + \frac{1}{2}\left(i - 1 + \frac{i-1}{i}E_iT_A\right) + \frac{1}{2}E_{i+1}T_A. \end{aligned}$$

Hence $E_iT_A\frac{i+1}{2i} = \frac{i+1}{2} + \frac{1}{2}E_{i+1}T_A$ so that $E_iT_A = i + \frac{i}{i+1}E_{i+1}T_A$. In particular $E_{n-1}T_A = n-1$. We prove by induction that $E_iT_A = i(n-i)$ for $i = 1, \ldots, n-1$:

$E_1T_A = E_{n-1}T_A = 1(n-1)$. Suppose $E_{i-1}T_A = (i-1)(n-(i-1))$. Then $E_iT_A\frac{i-1}{i} + (i-1) = (i-1)(n-(i-1))$. Hence $E_iT_A = i(n-i)$. The expected number of steps to visit the second leaf after the first is n^2 in both cases so that the cover time from an internal vertex i is $C_i = i(n-i) + n^2$.

C_i is a function of i that achieves its maximum at $i = \frac{n}{2}$. So for a path of even length the cover time is maximised from the central vertex while it is maximised from the two adjacent central vertices if the path length is odd.

Example 3.6. Let $S = S_{n,r}$ be the subdivided n-star of radius r. This tree consists of n paths of length r that are connected at the central vertex 0.

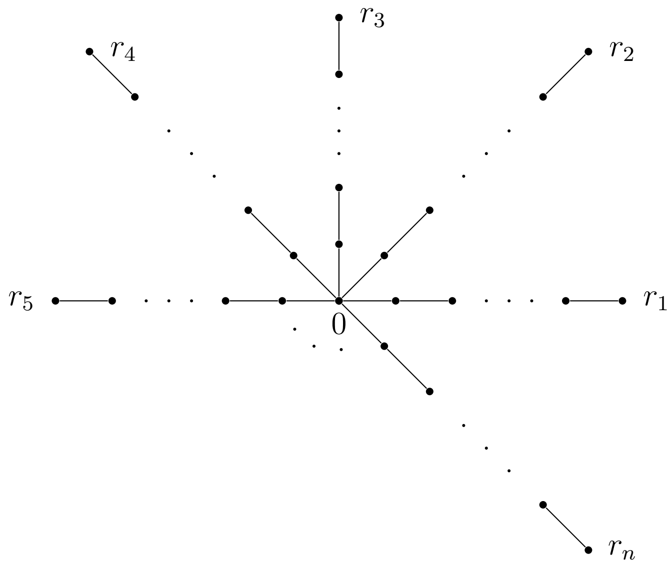


Figure 3.1: $S_{n,r}$

Let P_{2r} be the path of length $2r$ on vertices $0, 1, \ldots, 2r$. We start by finding $H_{0,2r}(P_{2r})$ where we assume the random walk moves with equal probability to any neighbour except at the central vertex r where $p(r, r+1) = p$ and $p(r, r-1) = q = 1-p$. We already know that $H_{0,r}(P_{2r}) = r^2$ so we only need to find $H_{r,2r}(P_{2r})$. We proceed by finding $F(r, 2r|z)$.
$F(i, r|z) = \frac{1}{2}zF(i-1, r|z) + \frac{1}{2}zF(i+1, r|z)$ for $i = 1, \ldots, r-1$. This is a linear recursion with associated characteristic polynomial $\frac{1}{2}z\lambda^2 - \lambda + \frac{1}{2}z$ with roots $\lambda_1(z) = \frac{1}{z}(1-\sqrt{1-z^2})$ and $\lambda_2(z) = \frac{1}{z}(1+\sqrt{1-z^2})$. We study the case $|z| < 1$ where the roots are distinct and $F(i, r|z) = a\lambda_1(z)^i + b\lambda_2(z)^i$.
We can find a and b by noting that $F(0, r|z) = zF(1, r|z)$ and $F(r, r|z) = 1$. Hence $a + b = az\lambda_1(z) + bz\lambda_2(z)$ and $a\lambda_1(z)^r + b\lambda_2(z)^r = 1$. Solving for a and b yields

$$a = \frac{z\lambda_2(z) - 1}{\lambda_1(z)^r (z\lambda_2(z) - 1) + \lambda_2(z)^r (1 - z\lambda_1(z))}$$

and

$$b = \frac{1 - z\lambda_1(z)}{\lambda_1(z)^r\left(z\lambda_2(z)-1\right) + \lambda_2(z)^r\left(1 - z\lambda_1(z)\right)}.$$

Substituting in a and b gives for $i = 0, 1, \ldots, r-1$

$$\begin{aligned} F(i, r|z) &= \frac{\left(z\lambda_2(z)-1\right)\lambda_1(z)^i + \left(1 - z\lambda_1(z)\right)\lambda_2(z)^i}{\lambda_1(z)^r\left(z\lambda_2(z)-1\right) + \lambda_2(z)^r\left(1 - z\lambda_1(z)\right)} \\ &= \frac{z^{r-i}\left((1-\sqrt{1-z^2})^i + (1+\sqrt{1-z^2})^i\right)}{(1-\sqrt{1-z^2})^r + (1+\sqrt{1-z^2})^r}. \end{aligned}$$

Furthermore $F(r, r+1|z) = pz + qzF(r-1, r|z)F(r, r+1|z)$. Hence

$$F(r, r+1|z) = \frac{pz}{1 - qz^2\frac{(1-\sqrt{1-z^2})^{r-1} + (1+\sqrt{1-z^2})^{r-1}}{(1-\sqrt{1-z^2})^r + (1+\sqrt{1-z^2})^r}}.$$

To simplify calculations we consider the second half of the path on vertices $r, r+1, \ldots, 2r$ separately as a path of length r on vertices $0, 1, \ldots, r$ with $F^*(i, j|z)$ the probability generating function of the hitting time for any two vertices i, j on this path. So $F(r, r+1|z) = F^*(0, 1|z)$ and $F(r, 2r|z) = F^*(0, r|z)$.

Since $F^*(i, r|z) = \frac{1}{2}zF^*(i-1, r|z) + \frac{1}{2}zF^*(i+1, r|z)$ for $i = 1, \ldots, r-1$ we again have the same linear recurrence relation with characteristic polynomial $\frac{1}{2}z\lambda^2 - \lambda + \frac{1}{2}z$ with roots $\lambda_1(z)$ and $\lambda_2(z)$. So $F^*(i, r|z) = a\lambda_1(z)^i + b\lambda_2(z)^i$ with $F^*(r, r|z) = 1$ and $F^*(0, r|z) = F^*(0, 1|z)F^*(1, r|z)$. Hence $a\lambda_1(z)^r + b\lambda_2(z)^r = 1$ and $a + b = F^*(0, 1|z)(a\lambda_1(z) + b\lambda_2(z))$. Set $A = F^*(0, 1|z)$. Solving for a and b yields

$$a = \frac{1 - A\lambda_2(z)}{\lambda_1(z)^r - \lambda_2(z)^r + A\lambda_1(z)\lambda_2(z)\left(\lambda_2(z)^{r-1} - \lambda_1(z)^{r-1}\right)}$$

and

$$b = \frac{A\lambda_1(z) - 1}{\lambda_1(z)^r - \lambda_2(z)^r + A\lambda_1(z)\lambda_2(z)\left(\lambda_2(z)^{r-1} - \lambda_1(z)^{r-1}\right)}.$$

Since $\lambda_1(z)\lambda_2(z) = 1$ we have for $i = 0, 1, \ldots, r-1$

$$F^*(i, r|z) = \frac{\lambda_1(z)^i - \lambda_2(z)^i - A\left(\lambda_1(z)^{i-1} - \lambda_2(z)^{i-1}\right)}{\lambda_1(z)^r - \lambda_2(z)^r - A\left(\lambda_1(z)^{r-1} - \lambda_2(z)^{r-1}\right)}.$$

In particular

$$F^*(0, r|z) = \frac{-\frac{2}{z}\sqrt{1-z^2}A}{\lambda_1(z)^r - \lambda_2(z)^r - A\left(\lambda_1(z)^{r-1} - \lambda_2(z)^{r-1}\right)}.$$

By the binomial theorem

$$\lambda_1(z)^r - \lambda_2(z)^r = \frac{1}{z^r}(-2\sqrt{1-z^2}) \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor - 1} \binom{r}{2k+1} \left(1-z^2\right)^k$$

and

$$\lambda_1(z)^r + \lambda_2(z)^r = \frac{2}{z^r} \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r}{2k} \left(1-z^2\right)^k .$$

This gives

$$F^*(0,r|z) = \frac{A}{\frac{1}{z^{r-1}} \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor - 1} \binom{r}{2k+1} \left(1-z^2\right)^k - \frac{A}{z^{r-2}} \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor - 1} \binom{r-1}{2k+1} \left(1-z^2\right)^k}$$

where

$$A = \frac{pz}{1 - qz^2 \dfrac{\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} \binom{r-1}{2k} \left(1-z^2\right)^k}{\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r}{2k} \left(1-z^2\right)^k}} .$$

Set

$$C = \frac{1}{z^{r-1}} \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor - 1} \binom{r}{2k+1} \left(1-z^2\right)^k - \frac{A}{z^{r-2}} \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor - 1} \binom{r-1}{2k+1} \left(1-z^2\right)^k .$$

Then $\frac{d}{dz} F^*(0,r|z) = \frac{A'C - AC'}{C^2}$ where

$$A' = \frac{p}{1 - qz^2 \dfrac{\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} \binom{r-1}{2k} \left(1-z^2\right)^k}{\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r}{2k} \left(1-z^2\right)^k}} + 2pqz^2 \frac{\dfrac{\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} \binom{r-1}{2k} \left(1-z^2\right)^k}{\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r}{2k} \left(1-z^2\right)^k}}{\left(1 - qz^2 \dfrac{\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} \binom{r-1}{2k} \left(1-z^2\right)^k}{\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r}{2k} \left(1-z^2\right)^k}\right)^2}$$

$$+\frac{pqz^3B}{\left(1-qz^2\frac{\sum_{k=0}^{\left\lfloor\frac{r-1}{2}\right\rfloor}\binom{r-1}{2k}\left(1-z^2\right)^k}{\sum_{k=0}^{\left\lfloor\frac{r}{2}\right\rfloor}\binom{r}{2k}\left(1-z^2\right)^k}\right)^2}$$

with

$$B=\frac{\sum_{k=0}^{\left\lfloor\frac{r}{2}\right\rfloor}\binom{r}{2k}\left(1-z^2\right)^k\sum_{k=1}^{\left\lfloor\frac{r-1}{2}\right\rfloor}\binom{r-1}{2k}k\left(1-z^2\right)^{k-1}(-2z)}{\left(\sum_{k=0}^{\left\lfloor\frac{r}{2}\right\rfloor}\binom{r}{2k}\left(1-z^2\right)^k\right)^2}$$

$$-\frac{\sum_{k=0}^{\left\lfloor\frac{r-1}{2}\right\rfloor}\binom{r-1}{2k}\left(1-z^2\right)^k\sum_{k=1}^{\left\lfloor\frac{r}{2}\right\rfloor}\binom{r}{2k}k\left(1-z^2\right)^{k-1}(-2z)}{\left(\sum_{k=0}^{\left\lfloor\frac{r}{2}\right\rfloor}\binom{r}{2k}\left(1-z^2\right)^k\right)^2}$$

and

$$C'=(1-r)z^{-r}\sum_{k=0}^{\left\lfloor\frac{r}{2}\right\rfloor-1}\binom{r}{2k+1}\left(1-z^2\right)^k+\frac{1}{z^{r-1}}\sum_{k=1}^{\left\lfloor\frac{r}{2}\right\rfloor-1}\binom{r}{2k+1}k\left(1-z^2\right)^{k-1}(-2z)$$

$$-\frac{A'}{z^{r-2}}\sum_{k=0}^{\left\lfloor\frac{r-1}{2}\right\rfloor-1}\binom{r-1}{2k+1}\left(1-z^2\right)^k-A(2-r)z^{1-r}\sum_{k=0}^{\left\lfloor\frac{r-1}{2}\right\rfloor-1}\binom{r-1}{2k+1}\left(1-z^2\right)^k$$

$$-\frac{A}{z^{r-2}}\sum_{k=1}^{\left\lfloor\frac{r-1}{2}\right\rfloor-1}\binom{r-1}{2k+1}k\left(1-z^2\right)^{k-1}(-2z).$$

Now

$$\begin{aligned}
&\lim_{z\to 1^-}\frac{d}{dz}F^*(0,r|z)\\
&=\frac{\frac{p(1-q)-p(-2q-q(-2\binom{r-1}{2}+2\binom{r}{2}))}{(1-q)^2}\left(r-\frac{p}{1-q}(r-1)\right)}{\left(r-\frac{p}{1-q}(r-1)\right)^2}\\
&-\frac{\frac{p}{1-q}\left((1-r)r-2\binom{r}{3}-(r-1)\frac{p(1-q)-p(-2q-q(-2\binom{r-1}{2}+2\binom{r}{2}))}{(1-q)^2}\right)}{(r-\frac{p}{1-q}(r-1))^2}\\
&-\frac{\frac{p}{1-q}\left(-\frac{p}{1-q}(2-r)(r-1)+2\frac{p}{1-q}\binom{r-1}{3}\right)}{(r-\frac{p}{1-q}(r-1))^2}\\
&=\frac{p(1-q)+2pq-pq(r-1)(r-2)+pqr(r-1)}{(1-q)^2}+r(r-1)+\frac{r(r-1)(r-2)}{3}\\
&+(r-1)\frac{p(1-q)+2pq-pq(r-1)(r-2)+pqr(r-1)}{(1-q)^2}-(r-2)(r-1)\\
&-\frac{(r-1)(r-2)(r-3)}{3}\\
&=r(r-1)+r\frac{p(1-q)+2pq-pq(r-1)(r-2)+pqr(r-1)}{(1-q)^2}\\
&=-r^2+\frac{2r^2}{p}
\end{aligned}$$

We can now compute the cover time of S. Let A be the set of leaves of this tree. Let vertex i be the vertex whose distance to the closest leaf is $i = 0, 1, \ldots, r$. So $i = 0$ if we start from a leaf and $i = r$ if we start from the central vertex. Note that i can be any one of n different vertices on one of the n branches but by symmetry of S (we can just rotate S without changing its structure) the cover time is the same for vertices of equal distance to their closest leaf.
Recall that $L_{j+1} - L_j$ is the expected number of steps to visit $j+1$ leaves after visiting j leaves. We have $C_i = E_iT_A + \sum_{j=1}^{n-1}(L_{j+1} - L_j)$. $E_iT_A = H_{i,0}(P_r) = r^2 - (r-i)^2$ since moving from the central vertex of S to an adjacent vertex corresponds by symmetry to moving from vertex r to $r-1$ on a path of length r on vertices $0, 1, \ldots, r$ whilst hitting A on S corresponds to hitting 0 on P_r.

To find $L_{j+1} - L_j$ we consider the path P_{2r} where we move from a vertex to an adjacent vertex with equal probability except at the midpoint of the path where we have $p(r, r+1) = \frac{n-j}{n}$ and $p(r, r-1) = \frac{j}{n}$. Then the expected number of steps from 0 to $2r$ in P_{2r} is the expected number of steps to visit $j+1$ leaves after having visited j leaves in S. Since we have visited j leaves, in other words covered j of the n paths, $H_{r,2r}(P_{2r}) = \frac{2r^2}{\frac{n-j}{n}} - r^2$. Furthermore

$H_{0,r}(P_{2r}) = r^2$. Now

$$L_{j+1} - L_j = r^2 + \frac{2r^2}{\frac{n-j}{n}} - r^2 = \frac{2nr^2}{n-j}$$

for $j = 1, \ldots, n-1$. So

$$\begin{aligned} C_i &= r^2 - (r-i)^2 + 2nr^2 \sum_{j=1}^{n-1} \frac{1}{n-j} \\ &= i(2r-i) + 2nr^2 \sum_{j=1}^{n-1} \frac{1}{j}. \end{aligned}$$

The cover time is maximised from the central vertex of S as expected.

Example 3.7. Let $B = B_{n,r}$ be the tree that consists of a star with n leaves $l_1, l_2, \ldots, l_n$ centered at vertex r, together with a path of length r on vertices $0, 1, \ldots, r$. Similarly let W be the tree that consists of a path of length r on vertices $0, 1, \ldots, r$ with two leaves k_1 and k_2 connected to r.

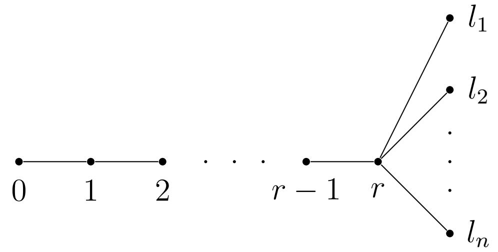


Figure 3.2: B

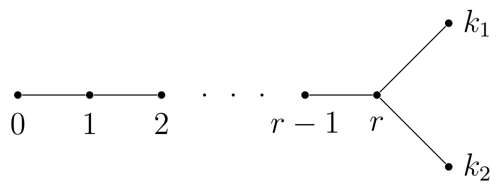


Figure 3.3: W

To find the cover time from an arbitrary vertex of B we do some auxiliary calculations on W. We start by finding E_iT_A for $A = \{0, k_2\}$.

- Suppose we move from any vertex with equal probability to any of its neighbours except at vertex r where $p(r, r-1) = \frac{1}{n+1}$, $p(r, k_1) = \frac{p}{n+1}$

and $p(r, k_2) = \frac{n-p}{n+1}$. We already know from Example 3.5 that $E_{r-1}T_A = r - 1 + \frac{r-1}{r}E_rT_A$. Furthermore

$$\begin{aligned} E_rT_A &= 1 + \frac{1}{n+1}E_{r-1}T_A + \frac{p}{n+1}E_{k_1}T_A + \frac{n-p}{n+1}E_{k_2}T_A \\ &= 1 + \frac{1}{n+1}\left(r - 1 + \frac{r-1}{r}E_rT_A\right) + \frac{p}{n+1}\left(1 + E_rT_A\right) \end{aligned}$$

so that $E_rT_A = \frac{r(n+r+p)}{r(n-p)+1}$. More generally for a vertex $i \in \{1, \ldots, r-1, r\}$ of distance $r - i$ from r we check by induction on the distance d from r that

$$E_iT_A = i(r-i) + i\frac{n+r+p}{r(n-p)+1}.$$

If $d = 0$ then $E_rT_A = \frac{r(n+r+p)}{r(n-p)+1}$. Suppose for $d = r-(i+1)$ that $E_{i+1}T_A = (i+1)(r-(i+1)) + (i+1)\frac{n+r+p}{r(n-p)+1}$ where $i + 1 = 2, 3, \ldots, r$. Then for $d = r - i$:

$$\begin{aligned} E_iT_A &= i + \frac{i}{i+1}E_{i+1}T_A \\ &= i + i(r-(i+1)) + i\frac{n+r+p}{r(n-p)+1} \\ &= i(r-i) + i\frac{n+r+p}{r(n-p)+1} \end{aligned}$$

completing the induction.

- Suppose $p(k_2, k_2) = 1$ and $p(k_2, k) = 0$ for $k \neq k_2$. For the remaining vertices we still suppose we move with equal probability to any adjacent vertex except at r where $p(r, r-1) = \frac{1}{n+1}$, $p(r, k_1) = \frac{p}{n+1}$ and $p(r, k_2) = \frac{n-p}{n+1}$. We proceed to find $F(i, 0) = F(i, 0|1)$ for $i = 1, \ldots, r$. Note that $F(i, 0) = \Pr_i(T_0 < \infty)$ is the probability of visiting leaf 0 before leaf k_2.

We prove by induction that $F(i, 0) = \frac{1}{i+1} + \frac{i}{i+1}F(i+1, 0)$ for $i = 1, \ldots, r-1$: $F(1, 0) = \frac{1}{2}F(0, 0) + \frac{1}{2}F(2, 0) = \frac{1}{2} + \frac{1}{2}F(2, 0)$. Suppose $F(i-1, 0) = \frac{1}{i} + \frac{i-1}{i}F(i, 0)$. Then

$$\begin{aligned} F(i, 0) &= \frac{1}{2}F(i-1, 0) + \frac{1}{2}F(i+1, 0) \\ &= \frac{1}{2}\left(\frac{1}{i} + \frac{i-1}{i}F(i, 0)\right) + \frac{1}{2}F(i+1, 0) \end{aligned}$$

so that $F(i, 0) = \frac{1}{i+1} + \frac{i}{i+1}F(i+1, 0)$ completing the induction.

In particular $F(r-1, 0) = \frac{1}{r} + \frac{r-1}{r}F(r, 0)$. Since $F(r, 0) = \frac{1}{n+1}F(r-1, 0) + \frac{p}{n+1}F(r, 0)$ we have $F(r, 0) = \frac{1}{n+1-p}F(r-1, 0)$ so that $F(r, 0) = \frac{1}{r(n-p)+1}$.

More generally for a vertex $i = 1, \dots, r$ of distance $r - i$ from r we check by induction on the distance d from r that $F(i, 0) = \frac{r-i}{r} + \frac{i}{r}\frac{1}{r(n-p)+1}$:

If $d = 0$ then $F(r, 0) = \frac{1}{r(n-p)+1}$. Suppose for $d = r - (i + 1)$ that $F(i+1, 0) = \frac{r-(i+1)}{r} + \frac{i+1}{r}\frac{1}{r(n-p)+1}$ where $i + 1 = 2, 3, \dots, r$. Then

$$\begin{aligned} F(i, 0) &= \frac{1}{i+1} + \frac{i}{i+1}\left(\frac{r-(i+1)}{r} + \frac{i+1}{r}\frac{1}{r(n-p)+1}\right) \\ &= \frac{r-i}{r} + \frac{i}{r}\frac{1}{r(n-p)+1}. \end{aligned}$$

- Associate the following weights to the edges of W: $w(r, k_1) = w(k_1, r) = p$, $w(r, k_2) = w(k_2, r) = n - p$ and $w(i, i+1) = w(i+1, i) = 1$ for $i = 0, 1, \dots, r-1$. Let $w = 2(n + r)$ be the sum of the weights. Then

$$\begin{aligned} E_0 T_{k_2} &= E_0 T_r + E_r T_{k_2} \\ &= r^2 + \left(E_{k_2} T_{k_2}^+ - 1\right) \\ &= r^2 + \frac{2(n+r)}{n-p} - 1. \end{aligned}$$

  We will use this weighted version of W to calculate expectations after we have already visited 0. If we still need to visit 0 we use the unweighted version of W to calculate expectations.

The leaves adjacent to r in B that have already been visited and those that still need to be visited are represented by k_1 and k_2 in W respectively. For $j = 0, 1, \dots, n$ suppose we have hit j leaves adjacent to r in B. Assume without loss of generality we visit the leaves in order $l_1, l_2, \dots, l_j$. Set $p(r, k_1) = \frac{j}{n+1}$. Now $E_i T_{\{0, l_{j+1}, \dots, l_n\}} = E_i T_{\{0, k_2\}}$ for $i = 1, \dots, r$ where $E_i T_{\{0, l_{j+1}, \dots, l_n\}}$ is a hitting time in B and $E_i T_{\{0, k_2\}}$ is a hitting time in W.

Similarly for given j from a starting vertex i the probability of visiting leaf 0 before leaf k_2 in W is the same as the probability of visiting 0 before $\{l_{j+1}, \dots, l_n\}$ in B. We can now compute the cover time from any vertex in B. If we start from 0 then $L_{j+1} - L_j = 1 + E_r T_{k_2}$ with $p = j$ for $j = 1, \dots, n-1$. Now

$$\begin{aligned} C_0(B) &= E_0 T_{\{l_1, l_2, \dots, l_n\}} + \sum_{j=1}^{n-1} (L_{j+1} - L_j) \\ &= r^2 + \frac{2(n+r)}{n} - 1 + \sum_{j=1}^{n-1} \frac{2(n+r)}{n-j} \\ &= r^2 - 1 + 2(n+r) \sum_{i=1}^{n} \frac{1}{i}. \end{aligned}$$

$C_i(B) = E_iT_A + L_2 - L_1 + \sum_{j=2}^{n} (L_{j+1} - L_j)$ for $i = 1, 2, \ldots, r$ and $A = \{0, l_1, l_2, \ldots, l_n\}$ where $E_iT_A = i(r-i) + i\frac{n+r}{rn+1}$. Let L be the event of visiting leaf 0 before any of the leaves adjacent to r then

$$\begin{aligned} L_2 - L_1 &= \Pr_i(L)\, E_0T_{A\setminus\{0\}} + (1 - \Pr_i(L))\, E_{l_1}T_{A\setminus\{l_1\}} \\ &= \left(\frac{r-i}{r} + \frac{i}{r}\frac{1}{rn+1}\right)\left(r^2 + \frac{2(n+r)}{n} - 1\right) \\ &+ \left(1 - \left(\frac{r-i}{r} + \frac{i}{r}\frac{1}{rn+1}\right)\right)\left(1 + \frac{r(n+r+1)}{r(n-1)+1}\right). \end{aligned}$$

To calculate $L_{j+1} - L_j$ for $j = 2, \ldots, n$ we have three possibilities with different hitting times:

1. We never visit 0. This happens with probability $\left(1 - \frac{r-i}{i} - \frac{i}{r}\frac{1}{rn+1}\right)\prod_{p=1}^{j-1}\left(1 - \frac{1}{r(n-p)+1}\right)$ and gives $E_{l_j}T_{A\setminus\{l_1,\ldots,l_j\}} = 1 + \frac{r(n+r+j)}{r(n-j)+1}$.

2. The j^{th} leaf we visit is 0. This happens with probability $\left(1 - \frac{r-i}{i} - \frac{i}{r}\frac{1}{rn+1}\right)\prod_{p=1}^{j-2}\left(1 - \frac{1}{r(n-p)+1}\right)\left(\frac{1}{r(n-(j-1))+1}\right)$ and gives $E_0T_{A\setminus\{0,l_1,\ldots,l_{j-1}\}} = r^2 + \frac{2(n+r)}{n-j+1} - 1$.

3. 0 is the i^{th} leaf we visit where $i = 1, \ldots, j-1$. This happens with probability $1 - \left(1 - \frac{r-i}{i} - \frac{i}{r}\frac{1}{rn+1}\right)\prod_{p=1}^{j-2}\left(1 - \frac{1}{r(n-p)+1}\right)$ and gives $E_{l_{j-1}}T_{A\setminus\{0,l_1,\ldots,l_{j-1}\}} = \frac{2(n+r)}{n-j+1}$.

Putting this together gives the cover time from $i = 1, \ldots, r$:

$$
\begin{aligned}
C_i &= i(r-i) + i\frac{n+r}{rn+1} + \left(\frac{r-i}{r} + \frac{i}{r}\frac{1}{rn+1}\right)\left(r^2 + \frac{2(n+r)}{n} - 1\right) \\
&+ \left(1 - \left(\frac{r-i}{r} + \frac{i}{r}\frac{1}{rn+1}\right)\right)\left(1 + \frac{r(n+r+1)}{r(n-1)+1}\right) \\
&+ \sum_{j=2}^{n}\left(1 + \frac{r(n+r+j)}{r(n-j)+1}\right)\left(1 - \frac{r-i}{i} - \frac{i}{r}\frac{1}{rn+1}\right)\prod_{p=1}^{j-1}\left(1 - \frac{1}{r(n-p)+1}\right) \\
&+ \sum_{j=2}^{n}\left(r^2 + \frac{2(n+r)}{n-j+1} - 1\right)\left(1 - \frac{r-i}{i} - \frac{i}{r}\frac{1}{rn+1}\right)\prod_{p=1}^{j-2}\left(1 - \frac{1}{r(n-p)+1}\right) \\
&\left(\frac{1}{r(n-(j-1))+1}\right) \\
&+ \sum_{j=2}^{n}\left(\frac{2(n+r)}{n-j+1}\right)\left(1 - \left(1 - \frac{r-i}{i} - \frac{i}{r}\frac{1}{rn+1}\right)\prod_{p=1}^{j-2}\left(1 - \frac{1}{r(n-p)+1}\right)\right) \\
&= -i^2 + i\left(r + \frac{n - r - r^2 n + r^2 - n^2 r^3 + n r^3 + 2rn^2}{(rn+1)(r(n-1)+1)}\right) \\
&+ i\frac{n(r-1)(r(2n+r)+1)}{rn+1}\sum_{j=2}^{n}\frac{1}{(r(n-j+1)+1)(r(n-j)+1)}\prod_{p=1}^{j-2}\frac{r(n-p)}{r(n-p)+1} \\
&+ 2(n+r)\sum_{j=1}^{n}\frac{1}{j} + r^2 - 1 \qquad (3.4)
\end{aligned}
$$

where (3.4) follows after some simplification.

Similarly for $n \geq 2$ the cover time from any vertex adjacent to r, say l_1 is

$$\begin{aligned} C_{l_1} &= 1 + \frac{r(n+r+1)}{r(n-1)+1} + \frac{1}{r(n-1)+1}\left(r^2 + \frac{2(n+r)}{n-1} - 1\right) \\ &+ \left(1 - \frac{1}{r(n-1)+1}\right)\left(1 + \frac{r(n+r+2)}{r(n-2)+1}\right) \\ &+ \sum_{j=2}^{n-1}\left(1 + \frac{r(n+r+j+1)}{r(n-(j+1)+1}\right)\prod_{p=1}^{j}\left(1 - \frac{1}{r(n-p)+1}\right) \\ &+ \sum_{j=2}^{n-1}\frac{r^2 + \frac{2(n+r)}{n-j} - 1}{r(n-j)+1}\prod_{p=1}^{j-1}\left(1 - \frac{1}{r(n-p)+1}\right) \\ &+ \sum_{j=2}^{n-1}\frac{2(n+r)}{n-j}\left(1 - \prod_{p=1}^{j-1}\left(1 - \frac{1}{r(n-p)+1}\right)\right) \\ &= \frac{(r(2n+r)+1)(r(2n-3)+1)}{(r(n-1)+1)(r(n-2)+1)} + \frac{(r^2-1)(n-1)+2(n+r)}{(n-1)(r(n-1)+1)} \\ &+ \sum_{j=2}^{n-1}\frac{2(n+r)}{n-j} + \sum_{j=2}^{n-1}\frac{(r-1)(r^2+2rn+1)}{(r(n-j)+1)(r(n-j-1)+1)}\prod_{p=1}^{j-1}\frac{r(n-p)}{r(n-p)+1}. \end{aligned} \tag{3.5}$$

3.2 Extremal cover times

In this section we find the n-vertex trees with minimum and maximum cover and cover and return time respectively. Brightwell and Winkler proved that the star is the n-vertex tree with minimum cover time in [6], while Feige [12] proved that the path is the n-vertex tree with maximum cover time. At this point one might want to recall the definitions of the main parameters mentioned in Section 2.2. We start by finding the tree with minimum cover time.

Lemma 3.8 (cf. [6, Lemma 3])**.** *Let G be a graph with $m-1$ edges, and fix a vertex x of G. Let a new graph G' be defined by adding a vertex y adjacent to x in G. Then*

$$C_y(G') = 1 + \frac{m}{m-1}C_x(G) + \frac{1}{m-1}R_x(G).$$

Proof. If A is the expected number of steps spent on the edge xy during a covering tour of G' from y then $C_y(G') = C_x(G) + A$. Let B be the expected number of times x is hit (including the initial visit) during a covering tour from x in G. After the i^{th} such hit the random walk in G' may waste some time U_i on the edge xy before reentering $G - \{x\}$. If we set $U = E(U_i)$ then

$A = 1 + B \cdot U$ since we spend an initial step on xy when moving from y to x whilst the remaining visits to xy all correspond to hitting x "from the G side" and then possibly spending some time on xy.

B is also equal to the number of times x is hit during a cover and return tour of G from x, provided the initial visit to x is not counted as a hit. An infinite random walk on G can be partitioned into cover and return tours. The probability of being at x at a random point in the i^{th} tour is the same irrespective of i where $i \in \mathbb{N}$. We can choose i to be arbitrarily large so that this probability is $\frac{d(x)}{2(m-1)}$ as in the stationary distribution by Theorem 1.32. Hence

$$B = \frac{d(x)\,(C_x(G) + R_x(G))}{2(m-1)}.$$

Once at x the edge xy will be taken with probability $\frac{1}{1+d(x)}$. In this case we spend two steps on xy and return to x. Otherwise we move with probability $\frac{d(x)}{1+d(x)}$ to one of the neighbours of x in G. In this case if we hit x again it will be for the $(i+1)^{th}$ time from the G side and contribute no further to visiting xy during the i^{th} visit. Hence

$$U = \frac{d(x)}{d(x)+1}(0) + \frac{1}{d(x)+1}(2+U)$$

from which it follows that $U = \frac{2}{d(x)}$. Now,

$$A = 1 + \frac{2}{d(x)}.\frac{d(x)\,(C_x(G) + R_x(G))}{2(m-1)} = 1 + \frac{C_x(G) + R_x(G)}{m-1}$$

so that

$$\begin{aligned} C_y(G') &= C_x(G) + 1 + \frac{C_x(G) + R_x(G)}{m-1} \\ &= 1 + \frac{m}{m-1}C_x(G) + \frac{1}{m-1}R_x(G) \end{aligned}$$

as required. □

Theorem 3.9 (cf. [6, Theorem 1]). *For any fixed $n-1 \geq 0$ the star S_{n-1} (consisting of a central vertex and $n-1$ adjacent leaves), with a leaf v as the starting vertex, has the least cover time among all trees on n vertices. (For $n = 4$ the star S_3 and P_3, both starting at leaves are equally fast. For all other n the star is strictly fastest.)*

Proof. Let $C_{\min}(n) = \min\{C_v(T) : T \text{ a tree on } n \text{ vertices}, v \text{ a vertex of } T\}$. We show by induction on n that $C_{\min}(n) = C_{\text{leaf}}(S_{n-1})$, for all positive n. For $n \leq 3$ all trees are stars, so there is nothing left to prove so we can assume henceforth that $n \geq 4$. Suppose that the n-vertex tree T and vertex v satisfy $C_{\min}(n) = C_v(T)$. We may, as noted just before Example 3.5, assume that v is a leaf of T, attached to some vertex u. Let $U = T - \{v\}$, we consider two cases, (1) where u is a leaf of U, and (2) otherwise. We know from the previous lemma that $C_v(T)$ depends positively on both $C_u(U)$ and $R_u(U)$, in particular

$$C_v(T) = 1 + \frac{n-1}{n-2} C_u(U) + \frac{1}{n-2} R_u(U).$$

We first consider case (1). By induction, a leaf of the star S_{n-2} has the least cover time of any vertex in a tree on $n-1$ vertices. The return time from a leaf u of U is an average of hitting times from the other leaves to u, weighted by the probability of visiting a given leaf last. We know from Corollary 3.2 that the hitting time between leaves of a tree are minimised between pairs of leaves of distance 2. Since the only tree where every other leaf is at distance only 2 is a star, it follows that a leaf of the star S_{n-2} has strictly least return time in a tree with $n-2$ vertices. So this return time is just equal to the hitting time between any two leaves of S_{n-2}, which is $2(n-2)$ by Corollary 3.2.

Thus in case (1), T must be a star with additional pendant edge, that is a star with $n-2$ leaves where $n-3$ leaves are adjacent to the central vertex and one leaf is at distance 2 from the central vertex. For this case we have

$$C_v(T) = 1 + \frac{n-1}{n-2} C_{\text{leaf}}(S_{n-2}) + \frac{1}{n-2} 2(n-2).$$

For case (2) we can also conclude from the induction hypothesis that the unique least cover time from a nonleaf in an $(n-1)$-vertex tree is from the centre of the star, since only in this case is the expected time to reach a leaf as small as 1 and after having reached a leaf the cover time from any of the leaves are minimal by the induction hypothesis. The return time from the centre of a star is just 1 so that the star-centre also has strictly least return time among all vertices of any $(n-1)$-vertex tree. For case (2) we conclude that v is attached to the centre of the star S_{n-2} so that T is just S_{n-1}. Now we have

$$\begin{aligned} C_v(T) &= 1 + \frac{n-1}{n-2} C_{\text{centre}}(S_{n-2}) + \frac{1}{n-2}.1 \\ &= 1 + \frac{n-1}{n-2} \left(C_{\text{leaf}}(S_{n-2}) + 1 \right) + \frac{1}{n-2} \\ &= 1 + \frac{n-1}{n-2} C_{\text{leaf}}(S_{n-2}) + \frac{n}{n-2}. \end{aligned}$$

For $n = 4$ cases (1) and (2) produce the same cover time, so that $C_{\text{leaf}}(P_3) = C_{\text{leaf}}(S_3)$. For $n > 4$ case (2) gives the smaller cover time so that only the stars remain, completing the proof. □

Corollary 3.10 (cf. [6, Theorem 2]). *For trees on n vertices, the cover and return time $CR_v(T)$ is minimised when T is a star and v is its centre.*

Proof. Starting from the centre of a star it takes one step to visit the first leaf and one step after covering the star to return to the centre of the star, that is $CR_{\text{centre}}(S_{n-1}) = C_{\text{leaf}}(S_{n-1}) + 2$. We know from the preceding theorem that the cover time of an n-vertex tree is minimised from any one of the leaves of S_{n-1}. Hence $CR_{\text{centre}}(S_{n-1})$ has the minimum cover and return time of all n-vertex trees starting from a nonleaf.

If we start from a leaf we end at any of the other leaves after covering the tree. We know from Corollary 3.2 that the hitting time between different leaves is at least $2(n-1)$. Hence the return time to a leaf is at least $2(n-1)$ so that the cover and return time from a leaf is at least $C_{\text{leaf}}(S_{n-1})+2(n-1)$. Since $n \geq 3$ for a nonleaf to exist this number exceeds $C_{\text{leaf}}(S_{n-1}) + 2 = CR_{\text{centre}}(S_{n-1})$, completing the proof. $\square$

We proceed to find the n-vertex tree with maximum cover time.

Lemma 3.11 (cf. [12, Lemma 2.1]). *Let T_1 and T_2 be two random times. If $T_1 \vee T_2 = \min(T_1, T_2)$ denotes the occurrence of one of the times (whichever happens first) and $T_1 \wedge T_2 = \max(T_1, T_2)$ the occurrence of both of the times then $E(T_1) + E(T_2) = E(T_1 \vee T_2) + E(T_1 \wedge T_2)$.*

Proof. We have

$$E(T_1) + E(T_2) = \sum_{i \in \mathbb{N}} i\{\Pr(T_1 = i) + \Pr(T_2 = i)\} \quad \text{and}$$

$$E(T_1 \vee T_2) + E(T_1 \wedge T_2) = \sum_{i \in \mathbb{N}} i\{\Pr(T_1 \vee T_2 = i) + \Pr(T_1 \wedge T_2 = i)\}.$$

Now we have three possibilities:

1. $T_1 = T_2$. In this case $T_1 = T_2 = T_1 \vee T_2 = T_1 \wedge T_2$.
2. $T_1 < T_2$. In this case $T_1 = T_1 \vee T_2$ and $T_2 = T_1 \wedge T_2$.
3. $T_2 < T_1$. In this case $T_2 = T_1 \vee T_2$ and $T_1 = T_1 \wedge T_2$.

We see for each case that $T_1 + T_2 = T_1 \vee T_2 + T_1 \wedge T_2$, so that $E(T_1) + E(T_2) = E(T_1 \vee T_2) + E(T_1 \wedge T_2)$. $\square$

Before we can find the n-vertex tree with maximum cover time, we need to define some additional terminology. Let T be an n-vertex tree defined on a subset of the vertices of a graph G (recall we assume graphs to be connected, finite and non-bipartite). Define the *weight* of the tree as the sum of the

commute times in G along the edges of T. That is $W(T) = \sum_{xy \in E(T)} C_{x,y}(G)$, where summation is taken over the endpoints of the $n-1$ edges of T. Note that if we take $G = T$ then $W(T) = 2(n-1)^2$ since the commute time between adjacent vertices x and y in a tree is equal to $2(n-1)$.
For vertices $u, v \in T$, let $P_{u,v}$ denote the set of directed edges on the unique path leading from u to v along the edges of T. Set $H^T_{u,v}(G) = \sum_{xy \in P_{u,v}} H_{x,y}(G)$, that is the sum of hitting times in G along the unique path from u to v in T. Note that $H^T_{u,v}(G) \geq H_{u,v}(G)$ since there are possibly other paths in G but not in T that lead from u to v. Let $C_u(T : G)$ be the expected number of steps to cover the vertices in T, starting from u, where we are allowed to visit any vertex in G before covering T.

Lemma 3.12 (cf. [12, Lemma 2.2])**.** *For any tree T defined on the vertices of a graph G, and for any starting vertex $u \in T$,*

$$C_u(T : G) \leq W(T) - \max_{v \in V(T)} H_{v,u}(G).$$

Proof. Suppose the vertex with maximum hitting time to u is x, that is $\max_{v \in V(T)} H_{v,u}(G) = H_{x,u}(G)$. Instead of using $H_{x,u}(G)$ we use the upper bound $H^T_{x,u}(G) \geq H_{x,u}(G)$. We may assume that x is a leaf of T, since $H^T_{x,u}(G)$ is maximised when x is a leaf of T (if x is not a leaf of T then there is a vertex adjacent to x whose distance is farther away from u with greater hitting time to u). There is a path $(u = x_0, x_1, \ldots, x_r = x)$ of length r between u and x. As before let T_0 be the component of $T - \{x_0 x_1\}$ containing x_0 and for $i = 1, \ldots, r-1$ let T_i be the component of $T - \{x_{i-1} x_i\} - \{x_i x_{i+1}\}$ that contains vertex x_i. Let m_i be the number of edges in T_i.

For $i = 0, 1, \ldots, r-1$ there is a path $P = P_{2m_i}$ of length $2m_i$ beginning and ending at $u = x_i$ which traverses each edge of T_i once in each direction and in particular visits each vertex in T_i. We can now visit all the vertices in T as follows: Starting from x_0 we cover T_0 along the path P_{2m_0}. Upon returning to x_0 having covered T_0 we move to x_1. We repeat this process of covering T_i starting and ending from x_i along P_{2m_i} for $i = 1, \ldots, r-1$ and moving to x_{i+1} after covering T_i. In this way each edge of T is traversed in both directions, except for the path leading from u to x which is only traversed once when moving from u to x. Hence

$$C_u(T : G) \leq W(T) - H^T_{x,u}(G) \leq W(T) - \max_{v \in V(T)} H_{v,u}(G).$$

□

Recalling the definition of the difference time from Section 2.2 we now prove the following theorem:

Theorem 3.13 (cf. [12, Theorem 1.2]). *Let G be a connected graph on n vertices. Let T be a tree defined on a subset of the vertices of G, where the edges of T need not be edges of G. Then*

$$C_u(T:G) \leq \frac{1}{2}\left(W(T) + \max_{v \in V(T \setminus \{u\})} D_{u,v}(G)\right).$$

Proof. We will prove this theorem for any graph G and any tree T by induction on the number of vertices in T. For the base case, T only contains two vertices, u and v. Then $C_u(T:G) = H_{u,v}(G)$ whilst $\frac{1}{2}(W(T) + D_{u,v}(G)) = \frac{1}{2}(C_{u,v}(G) + D_{u,v}(G)) = H_{u,v}(G)$ so that we have equality in the statement of the theorem.

For the inductive step, we make the inductive hypothesis that the theorem holds for any u', and any T' with at most k vertices, and prove the theorem for any u and any T with $k+1$ vertices. We distinguish between two cases, according to the degree of u in T. If the degree of u in T is 1, then by removing u from T we are left with a tree T' with k vertices. If $y \in T'$ is the neighbour of $u \in T$ then $C_u(T:G) \leq H_{u,y}(G) + C_y(T':G)$. Now T' only has k vertices so by the induction hypothesis,

$$C_y(T':G) \leq \frac{1}{2}\left(W(T') + \max_{v \in V(T' \setminus \{y\})} D_{y,v}(G)\right).$$

Note that $H_{u,y}(G) = \frac{1}{2}(C_{u,y}(G) + D_{u,y}(G))$ and $W(T) = C_{u,y}(G) + W(T')$. Furthermore it follows from Theorem 2.5 that $D_{u,v}(G) = D_{u,y}(G) + D_{y,v}(G)$. Hence

$$\begin{aligned} C_u(T:G) &\leq \frac{1}{2}\left(W(T') + C_{u,y}(G) + D_{u,y}(G) + \max_{v \in V(T' \setminus \{y\})} D_{y,v}(G)\right) \\ &= \frac{1}{2}\left(W(T) + \max_{v \in V(T' \setminus \{y\})} D_{u,v}(G)\right) \\ &\leq \frac{1}{2}\left(W(T) + \max_{v \in V(T \setminus \{u\})} D_{u,v}(G)\right) \end{aligned}$$

completing the induction proof for the case when the degree of u in T is 1.

In the other case the degree of u in T is greater than 1. We can partition T into two subtrees, T_1 and T_2, that intersect only at u. Let $H_{T_1,u} = \max_{x \in V(T_1)} H_{x,u}$ and $H_{T_2,u} = \max_{x \in V(T_2)} H_{x,u}$. Let $C_u(T_1 \wedge T_2 : G)$ denote the time to cover T_1 and T_2 in G and $C_u(T_1 \vee T_2 : G)$ denote the time to cover T_1 or T_2 in G (whichever tree is covered first).

The time to cover T_1 and T_2 is equal to the time to cover the first tree, then return from the last vertex in the covering tour of the first tree to u and finally to cover the second tree from u. If we cover T_2 first then $C_u(T_1 \wedge T_2 : G) \leq$

$C_u(T_1 \vee T_2 : G) + H_{T_2,u}(G) + C_u(T_1 : G)$. Similarly if we cover T_1 first then $C_u(T_1 \wedge T_2 : G) \leq C_u(T_1 \vee T_2 : G) + H_{T_1,u}(G) + C_u(T_2 : G)$. Hence $C_u(T_1 \wedge T_2 : G) \leq C_u(T_1 \vee T_2 : G) + \max\left(H_{T_2,u}(G) + C_u(T_1 : G), H_{T_1,u}(G) + C_u(T_2 : G)\right)$.

We assume w.l.o.g. that $H_{T_2,u}(G) + C_u(T_1 : G) \geq H_{T_1,u}(G) + C_u(T_2 : G)$ so that $C_u(T_1 \wedge T_2 : G) \leq C_u(T_1 \vee T_2 : G) + H_{T_2,u}(G) + C_u(T_1 : G)$. We have by Lemma 3.11 that $C_u(T_1 \vee T_2 : G) = C_u(T_1 : G) + C_u(T_2 : G) - C_u(T_1 \wedge T_2 : G)$ so that $C_u(T_1 \vee T_2 : G) \leq C_u(T_1 : G) + \frac{1}{2}\left(C_u(T_2 : G) + H_{T_2,u}(G)\right)$. Now $C_u(T_2 : G) + H_{T_2,u}(G) \leq W(T_2 : G)$ by Lemma 3.12 so that

$$\begin{aligned}
C_u(T : G) &= C_u(T_1 \wedge T_2 : G) \\
&\leq C_u(T_1 : G) + \frac{1}{2}W(T_2 : G) \\
&\leq \frac{1}{2}\left(W(T_1 : G) + W(T_2 : G) + \max_{x \in V(T_1)} D_{u,x}(G)\right) \\
&= \frac{1}{2}\left(W(T : G) + \max_{x \in V(T_1)} D_{u,x}(G)\right) \\
&\leq \frac{1}{2}\left(W(T : G) + \max_{x \in V(T)} D_{u,x}(G)\right).
\end{aligned}$$

This completes the proof by induction. □

Theorem 3.14 (cf. [12, Lemma 3.4])**.** *For any fixed $n \geq 1$ the path of length $n-1$, with the midpoint of the path as starting vertex for odd n or any of the two midpoints of the path as starting vertex for even n, has the maximum cover time among all trees on n vertices.*

Proof. Let T be an n-vertex tree with vertices u and $v \in V(T)$ where $d(u,v) = r$. We have $C_{u,v} = 2r(n-1)$ and $H_{v,u} \geq r^2$. We know from (3.3) that $H_{v,u} = r^2$ if and only if the only edges in the component of $T - \{v\}$ containing u are all edges along the path between u and v. Now $D_{u,v} = C_{u,v} - 2H_{v,u} \leq 2r(n-1) - r^2$. This is maximised when $r = \frac{n-1}{2}$ for odd n, and $r = \frac{n}{2}$ or $\frac{n-2}{2}$ for even n. It follows that

$$\max_{u,v \in V(T)} D_{u,v} \leq \left\lfloor \frac{(n-1)^2}{2} \right\rfloor.$$

Since $W(T) = 2(n-1)^2$ for any tree, it follows by the previous theorem, with $G = T$, that

$$\max_{u \in V(T)} C_u(T) \leq \left\lfloor \frac{5(n-1)^2}{4} \right\rfloor.$$

We know from Example 3.5 that

$$C_{\frac{n-1}{2}}(P_{n-1}) = \frac{5(n-1)^2}{4}$$

for odd n and

$$C_{\frac{n}{2}}(P_{n-1}) = C_{\frac{n-2}{2}}(P_{n-1}) = \left\lfloor \frac{5(n-1)^2}{4} \right\rfloor$$

for even n. We still need to check that P_{n-1} is the unique tree with this cover time. We can partition T into two subtrees, T_1 and T_2, that intersect only at u, with $v \in V(T_2)$. To attain the maximum difference time T_1 can be an arbitrary tree, while T_2 must be a path of length $\frac{n-1}{2}$ if n is odd or a path of length $\frac{n}{2}$ or $\frac{n-2}{2}$ if n is even.

However to attain the maximum cover time we require additionally as in the proof of Theorem 3.13, this time with strict equality, that $C_u(T_1 \wedge T_2 : T) = C_u(T_1 \vee T_2 : T) + \max\left(H_{T_2,u}(T) + C_u(T_1 : T), H_{T_1,u}(T) + C_u(T_2 : T)\right)$. Now $C_u(T_1 : T) \leq \frac{1}{2}\left(W(T) + \max_{x \in V(T_1)} D_{u,x}(T)\right)$ where we can only have strict equality if T_1 is a path of length $\frac{n-1}{2}$ for odd n and a path of length $\frac{n}{2}$ or $\frac{n-2}{2}$ for even n. Hence $C_u(T_1 \vee T_2 : T)$ can only be maximised if both T_1 and T_2 are paths so that P_{n-1} is the unique tree with maximum cover time among all trees on n vertices. □

We conclude this section by finding the tree with maximum cover and return time.

Lemma 3.15 (cf. [6, Lemma 4]). *Let G be a graph with $m-1$ edges, and fix a vertex x of G. Let a new graph G' be defined by adding a vertex y adjacent to x in G. Then*

$$CR_y(G') = \frac{m}{m-1} CR_x(G) + 2m.$$

Proof. The probability of visiting a given vertex last is the same from x (in G) as from y (in G') so that $R_y(G') = R_x(G) + H_{x,y}(G') = R_x(G) + 2m - 1$. It follows from Lemma 3.8 that

$$\begin{aligned} CR_y(G') &= C_y(G') + R_y(G') \\ &= 1 + \frac{m}{m-1} C_x(G) + \frac{1}{m-1} R_x(G) + R_x(G) + 2m - 1 \\ &= \frac{m}{m-1} CR_x(G) + 2m. \end{aligned}$$

□

Lemma 3.16 (cf. [6, Lemma 5]). *On any n-vertex tree T, the maximum value of the cover and return time $CR_v(T)$ can be reached only when v is a leaf.*

Proof. Suppose that u is not a leaf with adjacent vertices $v_1, \ldots, v_k$ where $k \geq 2$. Let T_i be the component of $T - \{u\}$ containing v_i, for $i = 1, 2, \ldots, k$. Let m_i be the number of vertices in T_i, then $m = \sum_{i=1}^{k} m_i$ is the total number of edges in T. For each i, let p_i be the probability that the last leaf hit in a covering tour beginning at u lies in T_i. Let p'_i be the same probability, but for a covering tour starting from v_i.

We claim that $p'_i < p_i$. This is equivalent to claiming that the probability of covering T_i before $T - T_i$ is greater starting from v_i than from u. This is true since

$$\begin{aligned}&\Pr_u(\text{covering } T_i \text{ before } T \setminus T_i)\\&= \Pr_u(\text{visiting } v_i \text{ before covering } T \setminus T_i) \Pr_{v_i}(\text{covering } T_i \text{ before } T \setminus T_i).\end{aligned}$$

For $i = 1, \ldots, k$ suppose a random walk on T begins by stepping from u to v_i. Let s_i be the number of steps to cover T and return to u, and let t_i be the number of steps to cover and return to v_i. Note that $E(t_i) = CR_{v_i}(T)$. If B_i is the event that the last leaf we visit during a covering tour is in T_i, then $E(s_i - t_i|B_i) = H_{v_i,u}(T)$ and $E(s_i - t_i|\text{not } B_i) = -H_{u,v_i}(T)$ so that

$$\begin{aligned}E(s_i - t_i) &= E(s_i - t_i|B_i)\Pr(B_i) + E(s_i - t_i|\text{not } B_i)\Pr(\text{not } B_i)\\&= H_{v_i,u}(T)p'_i - H_{u,v_i}(T)(1 - p'_i)\\&= (2m_i - 1)p'_i - (2m - 2m_i + 1)(1 - p'_i).\end{aligned}$$

Now

$$\begin{aligned}CR_u(T) &= 1 + \frac{1}{k}\sum_{i=1}^{k} E(s_i)\\&= 1 + \frac{1}{k}\sum_{i=1}^{k}\left(CR_{v_i}(T) + (2m_i - 1)p'_i - (2m - 2m_i + 1)(1 - p'_i)\right)\end{aligned}$$

so that

$$\begin{aligned}kCR_u(T) - \sum_{i=1}^{k} CR_{v_i}(T) &= k + \sum_{i=1}^{k}(2m_i - 1) - 2m\sum_{i=1}^{k}(1 - p'_i)\\&= 2m - 2mk + 2m\sum_{i=1}^{k} p'_i\\&< 2m(2 - k)\\&\leq 0.\end{aligned}$$

Hence $CR_u(T) < \frac{1}{k}\sum_{i=1}^{k} CR_{v_i}(T)$ so that $CR_{v_i}(T) > CR_u(T)$ for at least one v_i, proving the lemma. □

Corollary 3.17 (cf. [6, Theorem 2]). *For trees on n vertices, the cover and return time $CR_v(T)$ is maximised when T is a path and v is one of its endpoints.*

Proof. We prove this corollary by induction on n. If $n = 1$ our tree is a path (of empty length). Since this is the only tree with one vertex the result holds trivially. Suppose that the cover and return time for trees with $k-1$ vertices is maximised from one of the endpoints of a path of length $k-2$. If T is a k-vertex tree, then by the previous lemma $CR_v(T)$ can only be maximal when v is a leaf. In that case let u be the only neighbour of v. Then by Lemma 3.15 $CR_u(T-\{v\})$ is also maximal. Now by our induction hypothesis $T-\{v\}$ must be a path of length $k-2$ where u is one of its endpoints. Hence T is a path of length $n-1$ with v as one of its endpoints, completing the proof by induction. $\square$

Chapter 4

The electrical connection

4.1 Introductory facts and notation

Any graph can be associated to an electrical network by replacing vertices by nodes and edges by electrical resistances. A good first reference with respect to this connection is the book Random Walks and Electrical Networks by Doyle and Snell [11]. The *effective resistance* between any two nodes x and y, denoted by R_{xy}, is defined as the voltage between x and y, V_{xy}, when a unit current enters x and leaves y. The effective resistance between two vertices of a graph is defined as the effective resistance between those two nodes in the associated electrical network. It is sufficient for our purposes to assume throughout that we are dealing with unit resistances. The main result proved later in this chapter, due to Tetali [19], relates the hitting time between two vertices to a weighted sum of effective resistances.

In this chapter we assume that graphs are finite and connected. Furthermore graphs may have multiple edges between two vertices but no loops. As mentioned, we can associate a graph G with n vertices and m edges to an electrical network with n nodes and m branches. When performing a random walk on G we assume as usual that we move to any neighbour of a vertex with equal probability.

Before continuing we mention some facts from electric network theory that we will refer to later in this chapter. Kirchhoff's current law states that the sum of all currents flowing into a node is zero whereas Ohm's law deals with the relationship between voltage and current in an ideal conductor. It states that the potential difference, or voltage, across an ideal conductor is proportional to the current flowing through it. So for nodes a and b in an electrical network Ohm's law states that $i_{ab} = \frac{V_{ab}}{r_{ab}} = \frac{v(a)-v(b)}{r_{ab}}$, where i_{ab} is the current flowing from a to b, V_{ab} is the voltage between a and b, r_{ab} is the resistance (in our case we always have unit resistance) between a and b and $v(a)$ and $v(b)$ are the voltages at a and b.

It follows from Ohm's law that the current flowing between two nodes is a linear function of the voltage between those two nodes (or the voltage is a linear function of the current). The *superposition principle* states that the voltage across (or current through) an element in a linear circuit is the sum of the voltages across (or currents through) that element due to each independent source acting alone.

If we fix a graph, with current $i_{xy} \in \mathbb{R}$ flowing from x to y, where $i = i_{xy}$ units of current enter x and leave out of y, then i_{xy} is unique (we give the same argument as in the second chapter of [4]): if there is another current j entering at x and leaving from y we can take the superposition of these two circuits, one with i units of current flowing from x to y and the other with j units of current flowing from y to x, to obtain a network in which no current enters or leaves the network at any point. If in this network there is a positive current in some edge from x_1 to x_2, i.e. $i \neq j$, then by Kirchhoff's current law a positive current flows from x_2 to x_3, then from x_3 to x_4 and so on. Since the network is finite we will return to some point previously visited, say w.l.o.g. the last new vertex we visit before returning to x_1 is x_n, obtaining a circuit in whose edges positive currents flow in one direction. But then by Ohm's law $v(x_1) > v(x_2) > v(x_3) > \ldots > v(x_n) > v(x_1)$, which is impossible, so that $i = j$.

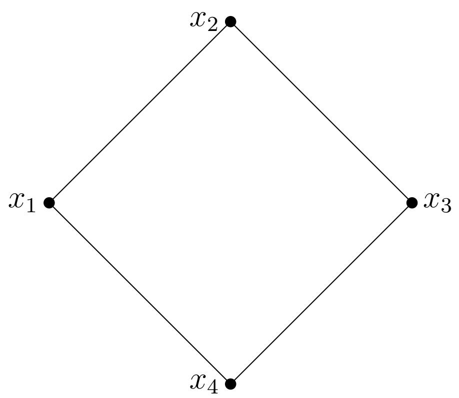


Figure 4.1: C_4

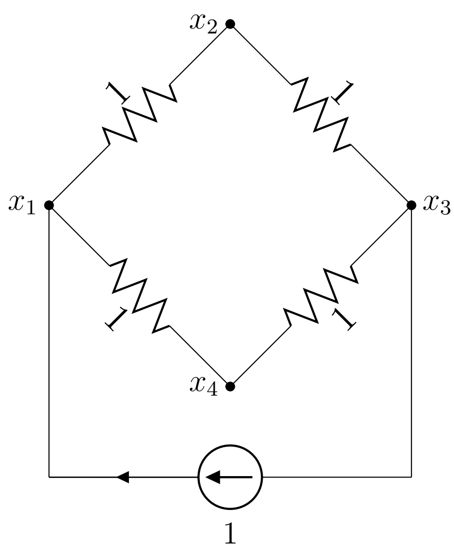


Figure 4.2: Electrical network corresponding to C_4

Example 4.1. Consider the cycle of length 4, C_4, on vertices $x_1, \ldots, x_4$ (see figure 4.1). Suppose a unit current flows into x_1 and out of x_3. Since $r_{x_1x_2} = r_{x_1x_4}$ it follows from Kirchhoff's current law that half a unit of current flows from x_1 to x_2 as well as from x_1 to x_4. Hence $V_{x_1x_2} = V_{x_1x_4} = \frac{1}{2}$ by Ohm's Law.

We proceed to calculate the effective resistance between two non-adjacent nodes, say x_1 and x_3. Before doing so recall from electric circuit theory that two resistors with resistances R_1 and R_2, connected in series, may be replaced by a single resistor with resistance R_1+R_2. If two resistors with resistances R_1 and R_2 are connected in parallel, we may replace them with a single resistor of resistance R if $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.

The resistances between x_1 and x_2 and x_2 and x_3 in figure 4.2 are connected in series and can be replaced by a single resistor with resistance 2. Similarly the resistances between x_1 and x_4 and x_4 and x_3 can be replaced by a single resistor with resistance 2. Hence we have two resistors with resistance 2 connected in parallel between x_1 and x_4. The resistors in parallel may in turn be replaced by a single resistor of resistance $\left(\frac{1}{2} + \frac{1}{2}\right)^{-1} = 1$ so that $R_{x_1x_3} = 1$.

If $x, y \in V(G)$ are the vertices from which a current enters and exits in the corresponding electrical network, we call the set $I = V(G) \setminus \{x, y\}$ the interior of $V(G)$ and $B = \{x, y\}$ the boundary of $V(G)$. A function $f(x)$ defined on $V(G)$ is *harmonic* if, at points of I, it satisfies the averaging property $f(x) = \sum_{y \in N(x)} \frac{f(y)}{d(x)}$ where $N(x)$ is the set of vertices adjacent to x and $d(x)$ is the degree of x.

Proposition 4.2. *A harmonic function defined on $V(G)$ takes its maximum and minimum value on the boundary.*

Proof. Let M be the maximum of f. If $f(a) = M$ for $a \in I$, then $f(b) = M$ for any neighbour b of a, since $f(a)$ is the average of its neighbours and can not assume a value strictly greater than M at any of the neighbouring vertices. Similarly $f(c) = M$ for any neighbour c of b. Continuing in this way, we conclude that $f(x) = M$ for any $x \in V(G)$, including the boundary. The same argument holds for the minimum value as well. □

Corollary 4.3. *If $f(x)$ and $g(x)$ are harmonic functions on $V(G)$ such that $f(x) = g(x)$ for all $x \in B$, then $f(x) = g(x)$ for all $x \in V(G)$.*

Proof. Let $h(x) = f(x) - g(x)$. For any $x \in I$,

$$\begin{aligned}\sum_{y \in N(x)} \frac{h(y)}{d(x)} &= \sum_{y \in N(x)} \frac{f(y) - g(y)}{d(x)} \\ &= f(x) - g(x) \\ &= h(x),\end{aligned}$$

so that $h(x)$ is harmonic. Since $h(x) = 0$ on the boundary, it follows from Proposition 4.2 that the maximum and minimum values of h is 0. Thus $h(x) = 0$ for all x so that $f(x) = g(x)$ for all x. □

4.2 Hitting time correspondence

Theorem 4.4 (cf. [19, Theorem 1]). *The effective resistance between nodes x and y is equal to the expected number of traversals out of x along any specific edge xz in a random walk starting at x and ending at y.*

Proof. Let U_z^{xy} be the expected number of visits to z before reaching y in a random walk from x to y. Hence $U_y^{xy} = 0$ and for $x \neq y \neq z$ we have

$$\begin{aligned}U_z^{xy} &= \sum_{t=0}^{T_y - 1} p^{(t)}(x, z) \\ &= \sum_{t=1}^{T_y} p^{(t)}(x, z) \\ &= \sum_{t=1}^{T_y} \sum_{w \in V(G)} p^{(t-1)}(x, w) p(w, z) \\ &= \sum_{t=0}^{T_y - 1} \sum_{w \in V(G)} p^{(t)}(x, w) p(w, z) \\ &= \sum_{w \in V(G)} U_w^{xy} p(w, z) \\ &= \sum_{w \in N(z)} \frac{U_w^{xy}}{d(w)}.\end{aligned}$$

Note that the function

$$\frac{U_z^{xy}}{d(z)} = \sum_{w \in N(z)} \frac{1}{d(z)} \frac{U_w^{xy}}{d(w)} \qquad (z \neq y \neq x)$$

is harmonic. By Kirchhoff's current law $\sum_{w\in N(z)} i_{wz} = 0$ for $z \in V(G)\setminus\{x,y\}$ so that $v(z)d(z) = \sum_{w\in N(z)} v(w)$ by Ohm's law for $z \neq x, y$. Hence

$$\begin{aligned}
V_{zy} &= \frac{1}{d(z)}\left(\sum_{w\in N(z)} v(w) - d(z)v(y)\right) \\
&= \frac{1}{d(z)}\sum_{w\in N(z)} (v(w) - v(y)) \\
&= \sum_{w\in N(z)} \frac{1}{d(z)} V_{wy}
\end{aligned}$$

for $z \neq x, y$. Note that V_{zy} is harmonic. By defining $V_{yy} = 0$ and $V_{xy} = U_x^{xy}/d(x)$ it follows from Corollary 4.3 that

$$V_{zy} = \frac{U_z^{xy}}{d(z)} \qquad \text{for all } z. \tag{4.1}$$

Since $U_x^{xy}/d(x)$ is the expected number of traversals out of x along a specific edge and since the effective resistance R_{xy} is by definition equal to V_{xy}, when a unit current enters x and leaves y, we will be done by proving that a unit current indeed enters x and leaves y. By Ohm's law

$$\begin{aligned}
i_{wz} &= V_{wy} - V_{zy} \\
&= \frac{U_w^{xy}}{d(w)} - \frac{U_z^{xy}}{d(z)},
\end{aligned}$$

in other words the current in any branch wz is the expected number of net traversals along wz. In a random walk from x to y the sum of the net number of traversals along the possible edges out of x will be one, that is

$$\sum_{w\in N(x)} \left(\frac{U_x^{xy}}{d(x)} - \frac{U_w^{xy}}{d(w)}\right) = \sum_{w\in N(x)} i_{xw} = 1,$$

since this net number of traversals will be 0 every time we return to x before reaching y, while it is 1 if we visit y after returning to x for the last time. Hence a unit current leaves from x and by Kirchhoff's current law a unit current enters x. Similarly $\sum_{w\in N(y)} i_{wy} = 1$, since we traverse any edge leading to y only once before the random walk ends upon reaching y. Hence a unit current leaves y by Kirchhoff's current law. □

Theorem 4.5 (cf. [19, Theorem 2])**.** *The effective resistance between nodes x and y is also the expected number of traversals out of $z \neq x, y$ along any specific edge zw in a random walk from x to y and then back to x.*

Proof. Let $U_z = U_z^{xy} + U_z^{yx}$ denote the expected number of visits to z in a random walk from x to y and back to x. We know from (4.1) that $U_z^{xy} = V_{zy}d(z)$ when a unit current enters x and leaves y. It also follows from (4.1) that $U_z^{yx} = V_{zx}^*d(z)$ where V_{zx}^* is the voltage between z and x when a unit current enters y and leaves x. By taking the superposition of these two circuits, one with unit current flowing from x to y and other with unit current flowing from y to x, the currents cancel out and we have a circuit with no current. Hence $V_{zx} + V_{zx}^* = 0$ for all z so that $U_z^{yx} = -V_{zx}d(z)$. Now

$$\begin{aligned} U_z &= V_{zy}d(z) - V_{zx}d(z) \\ &= V_{xy}d(z) \\ &= R_{xy}d(z). \end{aligned}$$

This completes the proof, since $U_z/d(z)$ is the expected number of traversals out of z in a random walk from x to y and then back to x. □

Corollary 4.6 (cf. [19, Corollary 1])**.** *Let G be a finite, connected graph without loops, with n vertices and m edges. The commute time satisfies $C_{x,y} = 2mR_{xy}$ for any $x, y \in V(G)$.*

Proof. As usual we have one unit of current entering x and leaving y. For U_z, as in the previous theorem, we have $C_{x,y} = \sum_{z\in V(G)} U_z$ where $U_x = U_x^{xy} + U_x^{yx} = U_x^{xy}$ and $U_y = U_y^{xy} + U_y^{yx} = U_y^{yx}$.

It follows from (4.1) that $U_x^{xy} = V_{xy}d(x) = R_{xy}d(x)$ while $U_y^{yx} = -d(y)V_{yx}$, by (4.1) and the superposition principle, so that $U_y^{yx} = d(y)V_{xy} = d(y)R_{xy}$. Furthermore $U_z = R_{xy}d(z)$ for $z \neq x, y$ as we saw in the previous theorem. We now have

$$\begin{aligned} C_{x,y} &= \sum_{z\in V(G)} U_z \\ &= R_{xy} \sum_{z\in V(G)} d(z) \\ &= 2mR_{xy}. \end{aligned}$$

□

A different proof for the previous corollary is given in [7].

Corollary 4.7. *The effective resistance between any two vertices $x, y \in V(T)$, where T is an n-vertex tree, is just equal to the usual distance, that is $R_{xy} = d(x, y)$.*

Proof. This follows directly from the previous corollary and equation (3.1). □

Before we can prove the main result in this chapter we state the following reciprocity theorem, illustrated in Figure 4.3, for electrical networks. We only state this theorem since its proof can be found in many standard circuit theory textbooks (e.g. see [15] or [9]).

Theorem 4.8 (Reciprocity)**.** *The voltage V across any branch of a network, due to a single current source anywhere else in the network, will equal the voltage across the branch at which the source was originally located if the source is placed at the branch across which the voltage V was originally measured.*

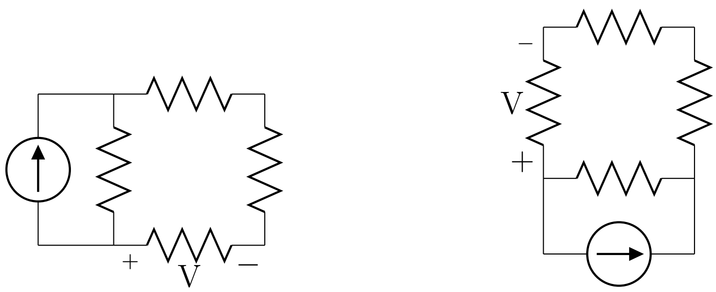


Figure 4.3: Equivalent voltages

So we may interchange the location of the current source and the voltage, without a change in voltage as long as the direction of the current source corresponds to the polarity of the branch voltage in each position. Recall that we interpret a voltage as $V_{wy} = U_w^{xy}/d(w)$, when a unit current flows into x and out of y, for any $w \in V(G)$.

The voltage between z and y, V_{zy}, with unit current flowing into x and out of y is equal to V_{xy}, with unit current flowing into z and out of y by the reciprocity theorem so that

$$\frac{U_z^{xy}}{d(z)} = \frac{U_x^{zy}}{d(x)}. \tag{4.2}$$

Proposition 4.9 (cf. [19, Corollary 3])**.** *Any three vertices x, y, z of a graph satisfy*

$$R_{xz} + R_{zy} - R_{xy} = \frac{U_x^{yz}}{d(x)} + \frac{U_y^{xz}}{d(y)} = 2\frac{U_x^{yz}}{d(x)} = 2\frac{U_y^{xz}}{d(y)}.$$

Proof. By Theorem 4.5 and equation (4.2),

$$\begin{aligned}
R_{xz} + R_{zy} - R_{xy} &= \left(\frac{U_y^{xz}}{d(y)} + \frac{U_y^{zx}}{d(y)}\right) + \left(\frac{U_x^{zy}}{d(x)} + \frac{U_x^{yz}}{d(x)}\right) - \left(\frac{U_z^{xy}}{d(z)} + \frac{U_z^{yx}}{d(z)}\right) \\
&= \left(\frac{U_y^{xz}}{d(y)} + \frac{U_z^{yx}}{d(z)}\right) + \left(\frac{U_z^{xy}}{d(z)} + \frac{U_x^{yz}}{d(x)}\right) - \left(\frac{U_z^{xy}}{d(z)} + \frac{U_z^{yx}}{d(z)}\right) \\
&= \frac{U_y^{xz}}{d(y)} + \frac{U_x^{yz}}{d(x)} \\
&= 2\frac{U_y^{xz}}{d(y)} \\
&= 2\frac{U_x^{yz}}{d(x)}.
\end{aligned}$$

□

Note that the triangle inequality for effective resistances, $R_{xz} + R_{zy} \geq R_{xy}$, follows immediately from the previous proposition. We can now prove the main result in this chapter.

Theorem 4.10 (cf. [19, Theorem 5])**.** *For any $x, y \in V(G)$, the hitting time satisfies*

$$H_{x,y} = \frac{1}{2} \sum_{z \in V(G)} d(z) \left(R_{xy} + R_{yz} - R_{xz}\right).$$

Proof. As in Proposition 4.9 we have

$$R_{xy} + R_{yz} - R_{xz} = 2\frac{U_z^{xy}}{d(z)}.$$

Rearranging and summing over $z \in V(G)$ yields

$$\sum_{z \in V(G)} U_z^{xy} = \frac{1}{2} \sum_{z \in V(G)} d(z) \left(R_{xy} + R_{yz} - R_{xz}\right).$$

However, $H_{x,y} = \sum_{z \in V(G)} U_z^{xy}$, proving the theorem. □

Note. This is the generalisation of (3.2), the hitting time formula for trees.

It follows directly from the previous theorem that

$$D_{x,y} = H_{x,y} - H_{y,x} = \sum_{z \in V(G)} d(z) \left(R_{yz} - R_{xz}\right), \tag{4.3}$$

since $R_{xy} = R_{yx}$. The proof of Theorem 2.5 also follows easily from Theorem 4.10, indeed

$$\begin{aligned} H_{x,y} + H_{y,z} + H_{z,x} &= \frac{1}{2} \sum_{k \in V(G)} d(k) \left(R_{xy} + R_{yz} + R_{zx} \right) \\ &= \frac{1}{2} \sum_{k \in V(G)} d(k) \left(R_{xz} + R_{zy} + R_{yx} \right) \\ &= H_{x,z} + H_{z,y} + H_{y,x}. \end{aligned}$$

Example 4.11. We want to calculate the hitting time between any two vertices of the complete graph K_n, on vertices $v_1, \ldots, v_n$, using the methods developed in this chapter. By symmetry we can calculate the hitting time between any two vertices, say v_1 and v_n. We start by finding $R_{v_1 v_n}$.

First consider a circuit where 1 unit of current enters into v_1 and $\frac{1}{n-1}$ units of current leave at each of the remaining $n-1$ vertices. By Kirchhoff's current law, a unit current flows out of v_1. Hence by symmetry $\frac{1}{n-1}$ units of current flows into each of the remaining $n-1$ vertices from v_1, in particular $i_{v_1 v_n} = \frac{1}{n-1}$. Now $V_{v_1 v_n} = i_{v_1 v_n} r_{v_1 v_n} = \frac{1}{n-1}$.

Next consider a circuit where 1 unit of current leaves from v_n and $\frac{1}{n-1}$ units of current enter at each of the remaining $n-1$ vertices. By Kirchhoff's current law, a unit current flows into v_n. Hence by symmetry $\frac{1}{n-1}$ units of current flows from each edge connected to v_n, in particular $i_{v_1 v_n} = \frac{1}{n-1}$. Now $V_{v_1 v_n} = i_{v_1 v_n} r_{v_1 v_n} = \frac{1}{n-1}$.

Superimposing these two circuits on each other, we have a circuit where $\frac{n}{n-1}$ units of current enter into v_1 and $\frac{n}{n-1}$ units of current leaves from v_2, while no units of current enter or leave from any of the remaining vertices. Furthermore the voltage between v_1 and v_n for this circuit is $V_{v_1 v_n} = \frac{1}{n-1} + \frac{1}{n-1} = \frac{2}{n-1}$. Recall that $R_{v_1 v_n}$ is the voltage between v_1 and v_n when a unit current enters into v_1 and leaves out of v_2. Hence by Ohm's law $R_{v_1 v_n} = \left(\frac{n-1}{n}\right)\left(\frac{2}{n-1}\right) = \frac{2}{n}$.

It follows by symmetry that $R_{v_i v_j} = \frac{2}{n}$ for $i \neq j$, while $R_{v_i v_i} = 0$. Since $d(k) = n-1$ for all $k \in V(K_n)$ it follows from Theorem 4.10 that

$$\begin{aligned} H_{v_1, v_n} &= \frac{1}{2} \sum_{z \in V(K_n)} d(z) R_{v_1 v_n} + \frac{1}{2} \sum_{z \in V(K_n)} d(z) \left(R_{v_n z} - R_{v_1 z} \right) \\ &= \left(\frac{n(n-1)}{2} \right) \left(\frac{2}{n} \right) + 0 \\ &= n - 1. \end{aligned}$$

This confirms the answer we obtained for the hitting time in Example 2.4.

Chapter 5

Cover Cost

5.1 Introductory remarks

Let G be an n-vertex, connected graph with no loops and m edges. For $x \in V(G)$ the sum $CC_x = \sum_{y \in V(G)} H_{x,y}(G)$ is called the *cover cost* from x. The notion of cover cost was introduced by Georgakopoulus in [13]. Let U_k be the first time N such that the set of vertices the walk has visited by time N contains exactly k different vertices. Note that $U_1 = 0$ and $E_i U_n = C_i$, where C_i is the time to cover the graph starting from i. If T_k, as usual, denotes the first time we visit a vertex k, then

$$\sum_{k \in V(G)} T_k = \sum_{k=1}^{n} U_k.$$

Indeed $\sum_{k \in V(G)} T_k$ denotes the sum of the times we first hit each vertex and since we hit every vertex of the graph in some order this equals $\sum_{k=1}^{n} U_k$. Starting from x and taking expectations on both sides we have

$$\sum_{k \in V(G)} H_{x,k} = \sum_{k=1}^{n} E_x U_k.$$

Since

$$U_k = (U_k - U_{k-1}) + (U_{k-1} - U_{k-2}) + \ldots + (U_2 - U_1)$$

for any k, it follows that

$$\sum_{k=1}^{n} U_k = (U_n - U_{n-1}) + 2\,(U_{n-1} - U_{n-2}) + \ldots + (n-1)\,(U_2 - U_1)\,.$$

Note that

$$C_x = E_x U_n = E_x (U_n - U_{n-1}) + E_x (U_{n-1} - U_{n-2}) + \ldots + E_x (U_2 - U_1)$$

by the linearity of expectation, so that $\frac{1}{n-1} CC_x < C_x < CC_x$. It can be useful to bound the cover time with the cover cost since, as we will see later in this chapter, in most cases it is much simpler to calculate the cover cost of a graph.

5.2 Related identities

We look at identities similar to the cover cost in this section. The results in this section are all from a paper by Wagner and Georgakopoulus [14]. In analogy to the cover cost the *reverse cover cost* is defined by $RC_x = \sum_{y \in V(G)} H_{y,x}$. Similarly the *weighted cover cost* and *weighted reverse cover cost* is defined respectively by

$$WCC_x = \sum_{y \in V(G)} d(y) H_{x,y} \quad \text{and} \quad WRC_x = \sum_{y \in V(G)} d(y) H_{y,x}.$$

Define the *centrality* of a vertex by $D(x) = \sum_{y \in V(G)} d(x,y)$ and the *weighted centrality* by $WD(x) = \sum_{y \in V(G)} d(y) d(x,y)$. Similarly we define the *resistance centrality* and *weighted resistance centrality* by

$$R(x) = \sum_{y \in V(G)} R_{xy} \quad \text{and} \quad WR(x) = \sum_{y \in V(G)} d(y) R_{xy}$$

where R_{xy} is the effective resistance between x and y, as defined in the previous chapter. The *Wiener index* $W(G)$ of a graph G is defined as

$$W(G) = \frac{1}{2} \sum_{x \in V(G)} \sum_{y \in V(G)} d(x,y)$$

while the *Kirchhoff index* $K(G)$ is defined as

$$K(G) = \frac{1}{2} \sum_{x \in V(G)} \sum_{y \in V(G)} R_{xy}.$$

Finally we define the weighted variants

$$WK(G) = \frac{1}{2} \sum_{x \in V(G)} \sum_{y \in V(G)} d(y) R_{xy} \quad \text{and} \quad WWK(G) = \frac{1}{2} \sum_{x \in V(G)} \sum_{y \in V(G)} d(x) d(y) R_{xy}.$$

Theorem 5.1 (cf. [14, Theorem 4])**.** *For any vertex $x \in V(G)$, we have*

$$\begin{aligned}
&CC_x = mR(x) - \frac{n}{2}WR(x) + WK(G),\\
&RC_x = mR(x) + \frac{n}{2}WR(x) - WK(G),\\
&WRC_x = 2mWR(x) - WWK(G) \text{ and}\\
&WCC_x = WWK(G).
\end{aligned}$$

Proof. Recall from the previous chapter that we can express the hitting time in terms of effective resistances as

$$H_{x,y} = mR_{xy} + \frac{1}{2}\left(WR(y) - WR(x)\right). \tag{5.1}$$

Hence

$$\begin{aligned}
CC_x &= \sum_{y \in V(G)} H_{x,y}\\
&= mR(x) + \frac{1}{2}\sum_{y \in V(G)} WR(y) - \frac{1}{2}\sum_{y \in V(G)} WR(x)\\
&= mR(x) - \frac{n}{2}WR(x) + WK(G).
\end{aligned}$$

Similarly

$$\begin{aligned}
RC_x &= \sum_{y \in V(G)} H_{y,x}\\
&= mR(x) + \frac{1}{2}\sum_{y \in V(G)} WR(x) - \frac{1}{2}\sum_{y \in V(G)} WR(y)\\
&= mR(x) + \frac{n}{2}WR(x) - WK(G).
\end{aligned}$$

We also have

$$\begin{aligned}
WRC_x &= \sum_{y \in V(G)} d(y)H_{y,x}\\
&= mWR(x) + \frac{1}{2}\sum_{y \in V(G)} d(y)WR(x) - \frac{1}{2}\sum_{y \in V(G)} d(y)WR(y)\\
&= mWR(x) + mWR(x) - WWK(G)\\
&= 2mWR(x) - WWK(G)
\end{aligned}$$

and

$$\begin{aligned}
WCC_x &= \sum_{y\in V(G)} d(y)H_{x,y} \\
&= mWR(x) + \frac{1}{2}\sum_{y\in V(G)} d(y)WR(y) - \frac{1}{2}\sum_{y\in V(G)} d(y)WR(x) \\
&= mWR(x) + WWK(G) - mWR(x) \\
&= WWK(G).
\end{aligned}$$

□

We already saw in Section 1.4 that WCC_x is independent of the starting vertex x. Indeed it follows from the previous theorem and Corollary 1.36 that $WCC_x = WWK(G) = 2m\sum_{x\in V(G)} Z(x,x)$ for any $x \in V(G)$, where $Z(x,x) = \sum_{n=0}^{\infty}\left(p^{(n)}(x,x) - \pi(x)\right)$.

The effective resistance R_{xy} is equal to the ordinary distance $d(x,y)$ on any tree T. This helps us to simplify the quantities in the previous theorem and yields the following corollary.

Corollary 5.2 (cf. [14, Theorems 1 and 2]). *For any vertex $x \in V(T)$, where T is a tree, we have*

$$\begin{aligned}
CC_x &= 2W(T) - D(x), \\
RC_x &= (2n-1)D(x) - 2W(T) \text{ and} \\
WRC_x &= 4(n-1)D(x) - 4W(T) + n - 1.
\end{aligned}$$

Proof. Recall from Lemma 3.1 that $WD(x) = 2D(x) - m$. We also know that the number of edges in a finite tree is just one less than the number of vertices, that is $m = n - 1$. Hence

$$\begin{aligned}
CC_x &= mR(x) - \frac{n}{2}WR(x) + WK(G) \\
&= (n-1)D(x) - \frac{n}{2}\left(2D(x) - (n-1)\right) + \frac{1}{2}\sum_{x\in V(T)}\left(2D(x) - (n-1)\right) \\
&= -D(x) + \sum_{x\in V(T)} D(x) \\
&= 2W(T) - D(x).
\end{aligned}$$

Similarly

$$\begin{aligned}
RC_x &= mR(x) + \frac{n}{2}WR(x) - WK(G) \\
&= (n-1)D(x) + \frac{n}{2}\left(2D(x) - (n-1)\right) - \frac{1}{2}\sum_{x \in V(T)} \left(2D(x) - (n-1)\right) \\
&= (n-1)D(x) + nD(x) - \sum_{x \in V(T)} D(x) \\
&= (2n-1)D(x) - 2W(T).
\end{aligned}$$

Finally

$$\begin{aligned}
WRC_x &= 2mWR(x) - WWK(G) \\
&= 2(n-1)\sum_{y \in V(T)} d(y)d(x,y) - \frac{1}{2}\sum_{x \in V(T)} d(x)WD(x) \\
&= 2(n-1)\left(2D(x) - (n-1)\right) - \frac{1}{2}\sum_{x \in V(T)} d(x)\left(2D(x) - (n-1)\right) \\
&= 4(n-1)D(x) - 2(n-1)^2 - \sum_{y \in V(T)} WD(y) + \frac{1}{2}(n-1)\sum_{x \in V(T)} d(x) \\
&= 4(n-1)D(x) - 2(n-1)^2 - 2\sum_{y \in V(T)} D(y) + n(n-1) + (n-1)^2 \\
&= 4(n-1)D(x) - 4W(T) + n - 1.
\end{aligned}$$

□

The following theorem is a generalisation of Corollary 2.6 for trees.

Theorem 5.3 (cf. [14, Theorem 3])**.** *For any tree T, and any vertices $x, y \in V(T)$, the following are equivalent:*
(i) $D(x) \leq D(y)$; *(ii) $WD(x) \leq WD(y)$;*
(iii) $H_{y,x} \leq H_{x,y}$; *(iv) $WRC_x \leq WRC_y$;*
(v) $RC_x \geq RC_y$; *(vi) $CC_x \geq CC_y$.*

Proof. The equivalence of (i) and (ii) follows immediately from Lemma 3.1, that of (i) and (iii) follows from equation (3.2), while the equivalence of (iii) and (iv) follows from equation (5.1). It follows from Corollary 5.2 that $CC_x + D(x) = 2W(T)$, so that (i) and (vi) are equivalent. Finally it also follows from Corollary 5.2 that $RC_x + (2n-1)CC_x = 4(n-1)W(T)$, so that (v) and (vi) are equivalent. □

5.3 Some symmetry properties

In this final section we look at the restrictions we need to impose on graphs, so that the cover cost, reverse cover cost and weighted reverse cover cost remains constant, irrespective of the starting vertex. We already saw in the previous section that the weighted cover cost, WCC_x, remains constant irrespective of x for any graph.

Theorem 5.4 (cf. [14, Corollary 5]). *The cover cost CC_x is independent of the starting vertex x if and only if G is regular.*

Proof. If G is d-regular then

$$CC_x = \frac{WCC_x}{d} = \frac{WWK(G)}{d}.$$

On the other hand suppose CC_x remains constant irrespective of x. For every $x \in V(G)$ we have

$$\begin{aligned}\sum_{z\in N(x)} (CC_x - CC_z) &= \sum_{z\in N(x)} \sum_{y\in V(G)} (H_{x,y} - H_{z,y}) \\ &= \sum_{z\in N(x)} \sum_{y\neq x} (H_{x,y} - H_{z,y}) + \sum_{z\in N(x)} (0 - H_{z,x}).\end{aligned}$$

It follows from Lemma 1.6 that $H_{x,y} = 1 + \frac{1}{d(x)} \sum_{z\in N(x)} H_{z,y}$ for $y \neq x$ so that $\sum_{z\in N(x)} (H_{x,y} - H_{z,y}) = d(x)$, after rearranging. We know that the expected return time to x, $H^+_{x,x}$, is equal to $\frac{1}{\pi(x)} = \frac{2m}{d(x)}$. It also follows from the same argument as in Lemma 1.6 that $H^+_{x,x} = 1 + \frac{1}{d(x)} \sum_{z\in N(x)} H_{z,x}$. Hence $\sum_{z\in N(x)} -H_{z,x} = d(x) - 2m$ after rearranging. It follows that

$$\begin{aligned}\sum_{z\in N(x)} (CC_x - CC_z) &= \sum_{y\neq x} d(x) + d(x) - 2m \\ &= nd(x) - 2m.\end{aligned}$$

Since CC_x remains constant, irrespective of x by assumption, it follows that $d(x) = \frac{2m}{n}$ for all x, so that G is regular. □

Proposition 5.5. *The weighted reverse cover cost WRC_x is independent of the starting vertex x if and only if $H_{y,z} = H_{z,y}$ for all vertices $y, z \in V(G)$.*

Proof. It follows from Theorem 5.1 that WRC_x is constant for all $x \in V(G)$ if and only if $WR(x)$ is constant for all $x \in V(G)$. Since the effective resistance between x and y, R_{xy}, is symmetric it is clear from equation (5.1) that $WR(x)$ is constant for all $x \in V(G)$ if and only if $H_{y,z} = H_{z,y}$ for all $y, z \in V(G)$. □

It is well known that a regular graph need not have symmetric hitting times, that is $H_{x,y} = H_{y,x}$ for all $x, y \in V(G)$. We illustrate this with the following example.

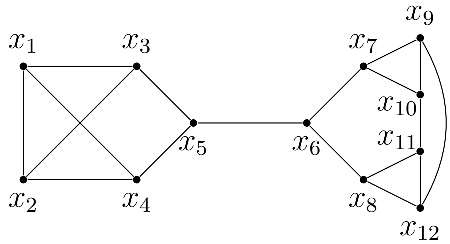


Figure 5.1: Regular graph with $H_{x_1,x_3} \neq H_{x_3,x_1}$

Example 5.6. The expected return time to any vertex x in a regular graph is just equal to the number of vertices in the graph. Indeed if the graph is d-regular, then $H^+_{x,x} = \frac{2m}{d} = \frac{nd}{d} = n$. In particular for the graph in Figure 5.1 we have,

$$12 = H^+_{x_1,x_1} = 1 + \frac{1}{3}H_{x_2,x_1} + \frac{1}{3}H_{x_4,x_1} + \frac{1}{3}H_{x_3,x_1}.$$

We also have $H_{x_2,x_1} = 1 + \frac{1}{3}H_{x_3,x_1} + \frac{1}{3}H_{x_4,x_1}$ so that $12 = \frac{4}{3} + \frac{4}{9}H_{x_4,x_1} + \frac{4}{9}H_{x_3,x_1}$. Looking at the first step a random walk can take from x_3 and x_4 it is clear that $H_{x_3,x_1} = H_{x_4,x_1}$ so that $H_{x_3,x_1} = 12$.

On the other hand $H_{x_2,x_3} = 1 + \frac{1}{3}H_{x_1,x_3} + \frac{1}{3}H_{x_4,x_3}$ and $H_{x_1,x_3} = 1 + \frac{1}{3}H_{x_2,x_3} + \frac{1}{3}H_{x_4,x_3}$ so that $H_{x_1,x_3} = H_{x_2,x_3}$ and hence $H_{x_1,x_3} = \frac{3}{2} + \frac{1}{2}H_{x_4,x_3}$. The two equations $12 = H^+_{x_3,x_3} = 1 + \frac{1}{3}H_{x_1,x_3} + \frac{1}{3}H_{x_2,x_3} + \frac{1}{3}H_{x_5,x_3}$ and $H_{x_4,x_3} = 1 + \frac{1}{3}H_{x_1,x_3} + \frac{1}{3}H_{x_2,x_3} + \frac{1}{3}H_{x_5,x_3}$ imply that $H_{x_4,x_3} = 12$. It follows that $H_{x_1,x_3} = \frac{3}{2} + (\frac{1}{2})(12) = 7\frac{1}{2} \neq H_{x_3,x_1}$.

Note that the previous example also implies that the weighted reverse cover cost need not be constant on a regular graph. It is still an open question whether a graph G with $H_{x,y} = H_{y,x}$ for all $x, y \in V(G)$ is regular. Remember we are assuming that the random walk is simple and that the underlying graph has no loops. An example given in [20] shows that an arbitrary weighted graph with symmetric hitting times need not be regular.

If $H_{x,y} = H_{y,x}$ for all $x, y \in V(G)$, then WRC_x remains constant for all

$x \in V(G)$, as noted in Proposition 5.5. In particular for any $x \in V(G)$

$$\begin{aligned} WRC_x &= WCC_x \\ &= 2m \sum_{y \in V(G)} \pi(y) H_{x,y} \\ &= 2m \sum_{y \in V(G)} Z(y,y) \end{aligned}$$

by Corollary 1.36. Also note that $WRC_x = 2m \sum_{y \in V(G)} \pi(y) H_{y,x} = 2m E_\pi T_x$ so that we have $E_\pi T_x = \sum_{y \in V(G)} Z(y,y)$ for any $x \in V(G)$. Lemma 1.34 states that $\pi(x) E_\pi T_x = Z(x,x)$ so that $\frac{Z(x,x)}{\pi(x)} = \sum_{y \in V(G)} Z(y,y)$ for any $x \in V(G)$ if G has symmetric hitting times.

Lastly note that if a graph is regular and $H_{x,y} = H_{y,x}$ for all $x, y \in V(G)$ then RC_x remains constant, independent of x.

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