Carleman-Sobolev classes and Green's potentials for weighted Laplacians

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Abstract

This thesis is based on two papers: the first one concerns Carleman-Sobolev classes for small exponents and the other solves Poisson's equation for the standard weighted Laplacian in the unit disc.

In the first paper we start by noting that for small L^p-exponents, i.e. $0 < p < 1$, the way we usually define Sobolev spaces is very unsatisfactory, which was illustrated by Peetre in 1975. In an attempt to remedy this we introduce completions of a class of smooth functions, which we call Carleman-Sobolev classes since they generalize Sobolev spaces and uses a norm inspired by Carleman classes. If the class is restricted with a growth condition on the supremum norms of the derivatives, we prove that there exists a condition on the weight sequence in the norm which guarantees that the resulting completion can be embedded into $C^\infty(\mathbb{R})$. This condition is even sharp up to some regularity on the weight sequence, in the sense that the norm inequality required for continuity no longer holds. We also show that the growth condition is necessary, in the sense that if we drop it entirely we can naturally embed L^p into this class's completion. Hence in this case we cannot consider the completion as a proper generalization of a Sobolev space.

In the second paper we find Green's function for the standard weighted Laplacian and give conditions on the Riesz-mass such that we can use Green's potential to solve Poisson's equation with zero boundary values in the sense of radial L^1-means. The weight here comes from the theory of weighted Bergman spaces and from this context it gets the label as the standard weight.

Sammanfattning

Den här avhandlingen är baserad på två artiklar: den första handlar om Carleman-Sobolev-klasser för små exponenter och den andra löser Poissons ekvation för den standardviktade Laplacianen i enhetsskivan.

I den första artikeln börjar vi med att notera att för små L^p-exponenter, dvs $0 < p < 1$, så är metoden man vanligen använder för att definiera Sobolevrum väldigt otillfredsställande, detta illustrerades av Peetre i en artikel från 1975. I ett försök att förbättra situationen introducerar vi tillslutningar av klasser av släta funktioner, som vi kallar Carleman-Sobolev-klasser eftersom de generaliserar Sobolevrum och använder en norm inspirerad av Carlemanklasser. Om klassen inskränks med ett växtkrav på derivatornas supremumnormer, så visar vi att det finns ett krav på viktsekvensen i normen som garanterar att den resulterande tillslutningen går att inbädda i $C^\infty(\mathbb{R})$. Detta krav är dessutom skarpt upp till viss regularitet hos viktsekvensen, i meningen att normolikheten som krävs för kontinuitet inte längre håller. Vi visar också att växtkravet är nödvändigt, i meningen att om vi utelämnar detta krav så kan vi inbädda L^p naturligt i tillslutningen av denna klass. Alltså, utan kravet kan vi inte betrakta tillslutningen som en äkta generalisering av ett Sobolevrum.

I den andra artikeln hittar vi Greens funktion för den standardviktade Laplacianen och ger krav på Rieszmassan som tillåter oss att använda Greens potential för att lösa Poissons ekvation med noll på randen i betydelsen av radiella L^1-medelvärden. Vikten kommer från teorin om viktade Bergmanrum och det är där den kallas för standardvikten.

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Part II: Scientific papers

Paper A
Carleman-Sobolev classes for small exponents
(joint with Aron Wennman)
Preprint. arXiv:1404.3127 (2014).

Paper B
Solving Poisson's equation for the standard weighted Laplacian in the unit disc
Preprint. arXiv:1306.2199 (2013).

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Chapter 1

Introduction

This thesis is based on two separate papers in two different areas, therefore we supply two separate introductions.

1.1 Introduction to Carleman-Sobolev classes for small exponents

In the usual case, when $p \geq 1$, there are several equivalent ways of defining Sobolev spaces. The two most commonly considered are the spaces H, the completion of smooth functions with respect to the Sobolev norm, and W, which is the subset of distributions having finite norm. As mentioned, these are equivalent when $p \geq 1$. Indeed, this is even the name of the famous theorem "H=W" (see [2] or the original paper [11]).

For small L^p-exponents, i.e $0 < p < 1$, the situation is far worse. To begin with, the spaces L^p are now only quasi-Banach spaces and have a trivial dual, see [5]. The failure of the triangle inequality will not hinder us significantly, but because of the latter fact we are deterred from trying to define Sobolev spaces as spaces of distributions.

However, the viewpoint of completions is also riddled with pitfalls. In a paper by Peetre [14] we can read a counterexample by Douady showing that if we choose to view the Sobolev space $W^{1,p}$ as a completion of smooth functions with respect to the ordinary Sobolev norm, then we can find objects which cannot even be regarded as functions: he constructs a "function", for a lack of a better word, which is identically zero yet has derivative equal to one almost everywhere!

One approach to remedy this is found in Peetre's book [15, Chapter 11] where he uses ideas from Besov spaces as a starting point.

We will instead try to look at another norm and examine what we get when we take the completion of a class of smooth functions. There is then a couple of possible outcomes, depending on the strength of our norm and on other restrictions on the class:

- nothing new happens and we end up where we started, i.e with smooth functions,
- we get too much, i.e we get all of L^p,
- or something completely different.

As an example, consider what happens when $p \geq 1$ where we end up in the last scenario with the different spaces $W^{k,p}$, which neither are smooth functions nor all of L^p.

The norm we will use comes from the so-called Carleman classes. These classes are defined by a weight sequence $(M_k)_{k\geq 0}$ so that $f \in C^\infty$ belong to the Carleman class $C\{M_k\}$ if there exists some $b > 0$ for which:

$$\frac{\left\|f^{(k)}\right\|_\infty}{M_k} \leq b^k, \quad k \geq 0.$$

Due to the classical theorem of Denjoy-Carleman (see [4]) these classes are quasi-analytic when

$$\sum_{k=0}^{\infty} \frac{M_k}{M_{k+1}} = \infty.$$

A quasi-analytic function is a function which cannot vanish together with all its derivatives at a common point unless it is the zero function. (Where we can read this as: if its Taylor expansion is nothing but zeros at some point, then it must be zero everywhere.) So we ask ourselves if we can find some similar condition, or at least in the same spirit, which in some sense classifies our completions.

Towards that end, we note that in many cases the constant b can be removed by considering $g(x) = f(b^{-1}x)$, so that $\left\|g^{(k)}\right\|_\infty = b^{-k}\left\|f^{(k)}\right\|_\infty$. Then we consider f to belong to the Carleman class if

$$\sup_{k\geq 0} \frac{\left\|f^{(k)}\right\|_\infty}{M_k} \leq 1.$$

To make this expression more Sobolev-flavored, we change the supremum norms to L^p-norms. This might seem like a radical thing to do. In some sense it is, but when $p \geq 1$ there is a connection between the supremum of a function f and the L^p-norms of f and f', at least it is clear for compactly supported f. For some discussion in this direction and for the classical scenario of Carleman classes see [10, Chapter V].

Here the reader might interject: but what about the "infinite" Sobolev norm:

$$\|f\| = \left(\frac{\|f\|_p}{M_0} + \frac{\|f'\|_p}{M_1} + \frac{\|f''\|_p}{M_2} + \dots \right)^{1/p} ?$$

Perhaps we can then choose the weight sequence to ensure that a nice subset have finite norm. So why do we not choose this norm? In some sense the norm we will choose is a drastic simplification of this, chosen in the hope of simplifying the ensuing calculations.

We are now finally ready to reveal the norm for our Carleman-Sobolev classes. For a fixed sequence $\mathcal{M} = (M_k)_{k\geq 0}$ of numbers $M_k \geq 1$ we define the quasi-norm

$$\|f\|_{\mathcal{M}} = \sup_{k\geq 0} \frac{\left\|f^{(k)}\right\|_p}{M_k}, \quad f \in C^\infty(\mathbb{R}).$$

We will frequently drop the quasi-prefix since in our results the failure of the triangle inequality can quite easily be overcome.

The results we present are in the direction of classifying the completions of these classes. Furthermore, we will find that a growth restriction on the derivatives is crucial if we want to consider the completion as a generalized Sobolev space.

1.2 Introduction to Green's potential for weighted Laplacians

Let Ω be a connected domain in $\mathbb{C}$ and let ρ be a positive weight function. Then a weighted Laplacian is a complex second order partial differential operator defined as

$$L_\rho = -\overline{\partial}_z \rho^{-1} \partial_z$$

where ∂_z and $\overline{\partial}_z$ are the Wirtinger-derivatives defined as

$$\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \overline{\partial}_z = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad z = x + iy \in \mathbb{C}.$$

When $\rho = 1$ we get an unweighted or classical case, named so because L_ρ then is the normalized Laplacian $-\frac{1}{4}\Delta$.

The main object of study will be Green's function $G_\rho(z, w)$ corresponding to these operators. These are the fundamental solutions to L_ρ with zero boundary values, that is, $G_\rho(z, w)$ is the function satisfying:

1. $L_\rho G_\rho(z, w) = \delta_0(z - w)$ in sense of distributions on Ω,
2. $G_\rho(z, w) \to 0$ when $z \to \zeta \in \partial\Omega$,
3. $G_\rho(z, w) = \overline{G_\rho(w, z)}$.

One of the reasons for studying these Green's functions is their connection to the Bergman space weighted by ρ, that is, the space of analytic functions with norm:

$$\left(\int_\Omega |f(z)|^2 \rho(z) dA(z)\right)^{1/2}, \quad dA = \frac{1}{\pi} dx dy.$$

This connection was shown in 1951 in Paul Garabedian's paper [6] where he studied the case when ρ is continuously differentiable in $\overline{\Omega}$. He shows in particular that the Bergman kernel $K_\rho(z, w)$ of these spaces are given by Green's function $G_\rho(z, w)$ of the weighted Laplacian by the formula:

$$K_\rho(z, w) = -\frac{1}{\rho(z)\rho(w)} \overline{\partial}_w \partial_z G_\rho(z, w), \quad z \neq w.$$

This means of course that given G_ρ we can find K_ρ. Then since the Poisson kernel for this operator can also be calculated by knowing G_ρ, we see that G_ρ sits in a nice place between these two domain functions.

Another natural question to consider when faced with an differential operator is the corresponding boundary value problem. That is, the question of existence and uniqueness of both the Dirichlet problem:

$$L_\rho u = 0 \quad \text{and} \quad u = f \text{ on } \partial\Omega$$

and Poisson's equation:

$$L_\rho u = g \quad \text{and} \quad u = 0 \text{ on } \partial\Omega.$$

Here we need to be careful and make sure that these equations make sense.

For a physical interpretation one can choose to see these operators as a complex equivalent of the conductivity equation for the electric potential u inside a material with conductivity $\sigma = \frac{1}{\rho}$:

$$\nabla \cdot \sigma \nabla u = 0.$$

They might be considered equivalent since this is exactly the real part of the equation $L_\rho u = 0$, where our weight function now serves the purpose of describing the resistance.

In this setting these operators appear in Calderón's inverse problem, which studies the problem of identifying the conductivity if we know the current through the boundary for arbitrary boundary data. More precisely, if we know the Dirichlet to Neumann map:

$$\Lambda_\sigma : u|_{\partial\Omega} \mapsto \sigma \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}$$

then can we find the conductivity σ? This was answered affirmatively for the plane in 2006 by Astala and Päivärinta, see [3].

These operators have another cousin, namely the weighted bi-Laplacian which is much more well-studied, see e.g. [9] and [8]. That operator is connected much more directly to physics, it is real in contrast to L_ρ and can be used to describe the elasticity of a thin plate.

In Paper B we consider the weight $(1-|z|^2)^\alpha$ in the unit disc $\mathbb{D}$, where α is a real parameter. This weight is commonly referred to as the standard weight in the context of weighted Bergman spaces. Here we refer the reader to [7] and in particular the results concerning the boundedness of the Bergman projection, which describe how nicely these weights fit into the Bergman space theory.

Therefore we denote this standard weighted Laplacian as:

$$L_\alpha = -\overline{\partial}_z (1-|z|^2)^{-\alpha} \partial_z, \quad \alpha > -1,$$

and its corresponding Green function by $G_\alpha(z, w)$. Note that this weight is not so well-behaved towards the boundary when $\alpha < 0$ so the considerations of Garabedian do not directly apply without some extra worries.

Prior to the work of Paper B the Dirichlet problem was solved by Olofsson and Wittsten in [13] where they find the Poisson kernel:

$$P_\alpha(z) = \frac{(1-|z|^2)^{\alpha+1}}{(1-z)(1-\overline{z})^{\alpha+1}}.$$

Using this they show both uniqueness and existence for distributional boundary values. Therefore, since the Bergman kernel is well-known (see [7]), the missing piece was Poisson's equation.

In Paper B we find $G_\alpha(z,w)$ and use its Green's potential

$$G_\alpha^\mu(z) = \int_{\mathbb{D}} G_\alpha(z,w)d\mu(w)$$

to solve

$$L_\alpha G_\alpha^\mu = \mu, \quad \mathbb{D}$$

with zero boundary values, in the sense of radial L^1-means:

$$\int_0^{2\pi} |G_\alpha^\mu(re^{i\theta})|\frac{d\theta}{2\pi} \to 0, \quad r \nearrow 1,$$

where the Riesz-mass μ is any complex Borel measure satisfying

$$\int_{\mathbb{D}} (1-|w|^2)^{\alpha+1} d|\mu|(w) < \infty.$$

Chapter 2

Summary of results

2.1 Paper A

This paper is the result of a collaboration with Aron Wennman where we study two classes of functions defined in relation with the following quasi-norm which we discussed above.

Definition. For a fixed sequence $\mathcal{M} = (M_k)_{k\geq 0}$ of numbers $M_k \geq 1$ we define the quasi-norm

$$\|f\|_{\mathcal{M}} = \sup_{k\geq 0} \frac{\left\|f^{(k)}\right\|_p}{M_k}, \quad f \in C^\infty(\mathbb{R}).$$

With this norm we now define our two Carleman-Sobolev classes as:

Definition. Let $\mathcal{C}$ and $\mathcal{S}$ be the classes of functions defined by

$$\mathcal{C} = \{f \in C^\infty(\mathbb{R}) : \|f\|_{\mathcal{M}} < \infty\}$$

and

$$\mathcal{S} = \left\{f \in C^\infty(\mathbb{R}) : \|f\|_{\mathcal{M}} < \infty \quad \text{and} \quad \overline{\lim_{k\to\infty}} \left\|f^{(k)}\right\|_\infty^{(1-p)^k} \leq 1\right\},$$

respectively. Note that both classes depend on the number $0 < p < 1$ and the choice of the sequence $\mathcal{M}$.

The main objects of study will be the abstract completions of $\mathcal{C}$ and $\mathcal{S}$, which we denote by $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{S}}$ respectively. The starting point of our examination of these is the following result which was communicated to the authors by Håkan Hedenmalm.

Proposition (Hedenmalm). *If $f \in \mathcal{S}$ then*

$$\|f\|_\infty \leq \left(\prod_{k=0}^{\infty} M_k^{p(1-p)^{k-1}} \right) \|f\|_{\mathcal{M}} .$$

It is mainly this result and the methods used in the proof that led to the particular conditions for the class $\mathcal{S}$. Building on this we went on to show the following result regarding $\widehat{\mathcal{S}}$.

Theorem 1. *Suppose the sequence $\mathcal{M}$ satisfies*

$$\prod_{k=0}^{\infty} M_k^{p(1-p)^{k-1}} < \infty.$$

Then $\widehat{\mathcal{S}}$ can be continuously embedded in $C^\infty(\mathbb{R})$.

Hence we now know that in some sense the two conditions

$$\prod_{k=0}^{\infty} M_k^{p(1-p)^{k-1}} < \infty \quad \text{and} \quad \overline{\lim_{k\to\infty}} \left\| f^{(k)} \right\|_\infty^{(1-p)^k} \leq 1$$

are too strong to give something new in the completion. So the natural question is what happens if we drop one or both of these conditions.

The first result in this direction is one where we replace the first condition with the converse and keep the last. To make the following theorem's conditions more readable note that the logaritmized invertion of the first condition above is:

$$p \sum_{k=0}^{\infty} (1-p)^{k-1} \log M_k = \infty.$$

The following theorem's statement will involve the asymptotic behavior of the terms, that is, different ways for the sum to diverge. All of the other conditions are the promised regularity conditions.

Theorem 2. *Suppose*

$$\prod_{k=0}^{\infty} M_k^{p(1-p)^{k-1}} = \infty.$$

Assume either that

$$\varliminf_{k\to\infty} (1-p)^k \log M_k > 0$$

or that all of the following hold:

$$\begin{aligned}
&\log M_k \text{ is an increasing and convex sequence,}\\
&\lim_{k\to\infty} (1-p)^k \log M_k = 0,\\
&\varliminf_{k\to\infty} \frac{\log M_k}{P(k)} = \infty, \quad \text{for any polynomial } P.
\end{aligned}$$

Then there can be no constant C such that

$$\|f\|_\infty \le C \|f\|_{\mathcal{M}}, \quad f \in \mathcal{S}.$$

This we choose to interpret as the impossibility of considering the completion $\widehat{\mathcal{S}}$ as smooth functions in this case, or more specifically the impossibility to continuously embed $\widehat{\mathcal{S}}$ in any space with the supremum norm.

The proof uses ideas from spline approximation and this connects our results to Peetre's original considerations in [14] since it was these approximations that sparked his original interest in the Sobolev spaces $W^{k,p}$ for small exponents. Furthermore, the constructions used in the proof can also be seen hinted at in Cohen's proof of the Denjoy-Carleman theorem (see [4]), which ties these classes even closer to the Carleman classes.

Lastly we show that the second condition mentioned above:

$$\varlimsup_{k\to\infty} \left\| f^{(k)} \right\|_\infty^{(1-p)^k} \le 1,$$

is even more crucial to ensure that we get something interesting in the completion. More specifically we show that:

Theorem 3. *There exists a canonical, continuous embedding*

$$L^p(\mathbb{R}) \hookrightarrow \widehat{\mathcal{C}}.$$

Explicitly one can map $f \in L^p$ to a Cauchy sequence (f_i) in $\mathcal{C}$ such that $f_i \to f$ in L^p and for each derivative we have $f_i^{(n)} \to 0$ in L^p.

Hence we cannot do without this growth restriction on the derivatives if we want to consider the completion as a proper Sobolev space, meaning that we want our space to be a proper subset of L^p in analogy with Sobolev spaces for $p \geq 1$. This motivates that the conditions we have put on $\mathcal{S}$ are the correct ones in this situation.

2.2 Paper B

The following is the first result of Paper B and the starting point for considering the corresponding Green's potential.

Theorem 4. *For the principal branch of the complex exponential, Green's function $G_\alpha(z,w)$ for the operator $L_\alpha = -\overline{\partial}_z(1-|z|^2)^{-\alpha}\partial_z$ for $\alpha > -1$ in $\mathbb{D}$ is given by*

$$G_\alpha(z,w) = (1-\overline{z}w)^\alpha h \circ g(z,w), \quad z \neq w,$$

where

$$h(s) = \int_0^s \frac{t^\alpha}{1-t}dt = \sum_{n=0}^{\infty} \frac{s^{\alpha+1+n}}{\alpha+1+n}, \quad 0 \leq s < 1,$$

and

$$g(z,w) = 1 - \left|\frac{z-w}{1-\overline{z}w}\right|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}.$$

Here we strongly encourage the reader to view the function $h(s)$ as a generalization of the logarithmic expression $-\log(1-s)$. Note also that $G_\alpha(z,w)$ is in general a complex-valued function.

In an effort to make this expression more familiar (at least to the reader who is well-versed in special functions) we mention that it is possible to express Green's function above using the incomplete Beta function as

$$G_\alpha(z,w) = (1-\overline{z}w)^\alpha B\left(\frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}; \alpha+1, 0\right).$$

Therefore by using the zero-balanced Gauss' hypergeometric function ${}_2F_1$ can we get:

$$\begin{aligned} G_\alpha(z,w) =& \frac{1}{\alpha+1} \frac{(1-|z|^2)^{\alpha+1}(1-|w|^2)^{\alpha+1}}{(1-\overline{z}w)(1-z\overline{w})^{\alpha+1}} \\ &\times {}_2F_1\left(1, \alpha+1; \alpha+2; \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}\right). \end{aligned}$$

For definitions and formulas for these special functions see [1].

Now using this function and estimates for it we can show the main result. In particular we mention that a nice estimate for the absolute values of the α-harmonic Poisson kernel by Olofsson was used (see [12]) to obtain the boundary values.

Theorem 5. *Given a complex Borel measure μ on $\mathbb{D}$ which satisfies*

$$\int_{\mathbb{D}} (1 - |w|^2)^{\alpha+1} d|\mu|(w) < \infty$$

then Green's potential, defined as:

$$G_\alpha^\mu(z) = \int_{\mathbb{D}} G_\alpha(z, w) d\mu(w),$$

where $G_\alpha(z, w)$ is Green's function for L_α, is the unique solution to Poisson's equation for L_α ($\alpha > -1$):

1. $L_\alpha G_\alpha^\mu = \mu$ *in sense of distributions on* $\mathbb{D}$ *and*
2. $\displaystyle\int_0^{2\pi} |G_\alpha^\mu(re^{i\theta})| \frac{d\theta}{2\pi} \to 0$ *as* $r \nearrow 1$.

Chapter 3

Discussion and remaining questions

3.1 Paper A

In [14] Peetre also showed that

$$W^{k,p} \cong L^p \oplus L^p \oplus \ldots \oplus L^p \cong L^p,$$

in the sense that the successive copies of L^p represents the derivatives of a function in $W^{k,p}$. (Of course the objects in $W^{k,p}$ are not really functions, but we ask the reader to accept this abuse of terminology.) Then the isomorphism implies that one can choose the derivatives entirely uncoupled from each other.

Since this paper served as a main source of inspiration for the work behind Paper A one naturally asks if we can get similar results in our infinite setting. That is, we ask ourselves if one can show the following two conjectures.

Conjecture 1. For the space $\widehat{\mathcal{C}}$ we have

$$\widehat{\mathcal{C}} \cong L^p \times L^p \times L^p \times \cdots$$

This does not seem too far-fetched if we look back to the embedding of L^p in $\widehat{\mathcal{C}}$ in Theorem 3. There we embedded any f in a way which suggests that this object would be $(f, 0, 0, \ldots)$ in $L^p \times L^p \times L^p \times \cdots$. If this holds true then the conclusion would be that the growth restriction in $\mathcal{S}$, i.e.

$$\overline{\lim_{k\to\infty}} \left\| f^{(k)} \right\|_{\infty}^{(1-p)^k} \leq 1,$$

cannot be dropped if we want something other than L^p in the completion.

If one manages to do all this for the bigger class $\mathcal{C}$ then one wonders if it is possible to achieve the same isomorphism with $\mathcal{S}$.

Conjecture 2. Assume that

$$\prod_{k=0}^{\infty} M_k^{p(1-p)^{k-1}} = \infty$$

and that the regularity assumptions of Theorem 2 are satisfied. Then

$$\widehat{\mathcal{S}} \cong L^p \times L^p \times L^p \times \cdots$$

If both these conjectures would hold true then the further conclusion would be that, with this specific norm, the finiteness of the product above classifies the completions fully up to regularity. Perhaps we can even do without the regularity, but on this both authors feel uncertain.

3.2 Paper B — Extension to singular weights

Here the natural question, I feel, is to try to find similar results with differently weighted Laplacians. In [9] Hedenmalm studied the weighted bi-Laplacian with the singular weight $|z|^{2\alpha}$. Therefore it was a natural weight to try and so I would like to take this moment to state some partial results without proof, or conjectures if you will, for the interested reader.

Because of the singularity we encounter difficulties and ambiguities already in the definition of this weighted Laplacian. We cannot directly adopt the same idea used for the bi-Laplacian in [9], since the ∂_z-derivative of Green's function will not be locally integrable when both arguments are close to the origin. For $\alpha > -\frac{1}{2}$, at least, these difficulties can be overcome by extending the definition of the weighted Laplacian to allow the first derivative to be considered as a principal value.

Using the method of reflection I believe that Green's function $G_\alpha(z, w)$ for the operator $L_\alpha = -\overline{\partial}_z |z|^{-2\alpha} \partial_z$ for $\alpha > -\frac{1}{2}$ in $\mathbb{D}$ is given by

$$G_\alpha(z, w) = \int_{|z|^2|w|^2}^{\min(|z|^2,|w|^2)} \frac{t^\alpha}{\overline{z}w - t} dt - \int_{\max(|z|^2,|w|^2)}^{1} \frac{t^\alpha}{\overline{z}w - t} dt, \quad z \neq w.$$

The min and max here is to avoid the slit-singularity which otherwise would appear in the ∂_z-derivative.

By expanding the integrands we also get the following series expansion of Green's function. Set

$$e_n(z,w) = \frac{1}{\alpha+1+n}\left(1-|w|^{2(\alpha+1+n)}\right)|z|^{2\alpha}\frac{z^{n+1}}{w^{n+1}}$$

and

$$f_n(z,w) = \begin{cases} \frac{1}{\alpha-n}\left(1-|w|^{2(\alpha-n)}\right)\overline{z}^n w^n, & n \neq \alpha, \\ -\overline{z}^\alpha w^\alpha \log|w|^2, & n = \alpha, \end{cases}$$

then we can expand $G_\alpha(z,w)$ for $|z|<|w|$ as

$$G_\alpha(z,w) = \sum_{n=0}^{\infty} e_n(z,w) + \sum_{n=0}^{\infty} f_n(z,w).$$

Basically, this expression is the primitive function of

$$\partial_z G_\alpha(z,w) = \frac{|z|^{2\alpha}}{w-z} - \frac{|z|^{2\alpha}|w|^{2\alpha}\overline{w}}{1-z\overline{w}}$$

which can be used to calculate the corresponding weighted Bergman kernel:

$$K_\alpha(z,w) = \frac{1}{(1-z\overline{w})^2} + \frac{\alpha}{1-z\overline{w}}.$$

Furthermore, we can follow [9] and consider the so-called Almansi-expansion of a function u satisfying $\overline{\partial}|z|^{-2\alpha}\partial u = 0$ and use it to ensure uniqueness in the Dirichlet problem. This expansion can also be used to deduce that for this weight the Poisson kernel is given by

$$P_\alpha(z) = \frac{z|z|^{2\alpha}}{1-z} + \frac{1}{1-\overline{z}}.$$

Therefore, we can solve the Dirichlet problem with the usual Poisson integral.

But what about Poisson's equation? Here one needs to find estimates of the L^1-norm of Green's function but because of the singular weight this becomes much trickier. I have managed to find estimates that ensure that Green's potential G_α^μ exists and allows us to compute $-\overline{\partial}|z|^{-2\alpha}\partial G_\alpha^\mu = \mu$ if

$$\int_{\mathbb{D}} (1-|w|^{2(\alpha+1)})d|\mu| < \infty, \quad \alpha > -\frac{1}{2}.$$

However, the boundary condition, i.e. the radial L^1-means, have proven too elusive for me.

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