Graph Convexity and Vertex Orderings

by

Rachel Jean Selma Anderson
B.Sc., McGill University, 2002
B.Ed., University of Victoria, 2007

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ABSTRACT

In discrete mathematics, a *convex space* is an ordered pair $(V, \mathcal{M})$ where $\mathcal{M}$ is a family of subsets of a finite set V, such that: $\emptyset \in \mathcal{M}$, $V \in \mathcal{M}$, and $\mathcal{M}$ is closed under intersection. The elements of $\mathcal{M}$ are called *convex sets*. For a set $S \subseteq V$, the *convex hull* of S is the smallest convex set that contains S. A point x of a convex set X is an *extreme point* of X if $X \setminus \{x\}$ is also convex. A convex space $(V, \mathcal{M})$ with the property that every convex set is the convex hull of its extreme points is called a *convex geometry*. A graph G has a P-elimination ordering if an ordering $v_1, v_2, ..., v_n$ of the vertices exists such that v_i has property P in the graph induced by vertices $v_i, v_{i+1}, ..., v_n$ for all $i = 1, 2, ..., n$. Farber and Jamison [18] showed that for a convex geometry $(V, \mathcal{M})$, $X \in \mathcal{M}$ if and only if there is an ordering $v_1, v_2, ..., v_k$ of the points of $V - X$ such that v_i is an extreme point of $\{v_i, v_{i+1}, ..., v_k\} \cup X$ for each $i = 1, 2, ..., k$. With these concepts in mind, this thesis surveys the literature and summarizes results regarding graph convexities and elimination orderings. These results include classifying graphs for which different types of convexities give convex geometries, and classifying graphs for which different vertex ordering algorithms result in a P-elimination ordering, for P the characteristic property of the extreme points of the convexity. We consider the geodesic, monophonic, m^3, 3-Steiner and 3-monophonic convexities, and the vertex ordering algorithms LexBFS, MCS, MEC and MCC. By considering LexDFS, a recently introduced vertex ordering algorithm of Corneil and Krueger [11], we obtain new results: these are characterizations of graphs for which all LexDFS orderings of all induced subgraphs are P-elimination orderings, for every characteristic property P of the extreme vertices for the convexities studied in this thesis.

Contents

List of Figures

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My parents (Bill and Dianne), for your faith in me, and mom for your unused math gene.

My friends and family for your love, support, and laughter. And thanks for, every once in a while, asking me about the “dots and lines”.

DEDICATION

This thesis is dedicated to my Dad, Bill Anderson, who passed away on July 11, 2013.

Dad, you instilled curiosity and love of learning into me at an early age. You always believed in me and my various, sometimes random, new pursuits. Who knew that waitressing and motorcycling would lead to a masters in math? You did! You always saw that anything is possible, and life is what you make of it.

Dad, you showed me that learning is a means and not an ends. You were a patient teacher. I wish you could be here to see me finish this, but I know that you trusted that I would and could. I will always carry your faith in me like the precious treasure that it is. I love you and miss you. And I will click my heels on graduation day!

Chapter 1

Introduction

Ordering the vertices of a graph is a useful and powerful tool in executing algorithms. Algorithms can often be performed more efficiently if the vertices are first ordered in a certain way. While such an ordering could be random, often the algorithm can run more efficiently when the vertex ordering satisfies some criteria.

For a fixed ordering $\alpha : v_1, v_2, ..., v_n$ of the vertices of a graph G, let G_i be the graph induced on the vertices $v_i, v_{i+1}, ..., v_n$. The position of a vertex v in the ordering is denoted by $\alpha(v)$. Moreover, if $\alpha(u) < \alpha(v)$ for some $u, v \in V(G)$, then we write $u < v$. Let P be a property that a vertex may have within a graph. We say that G has a *P-elimination ordering* if an ordering $v_1, v_2, ..., v_n$ of the vertices exists such that v_i has property P in the graph G_i for all $i = 1, 2, ..., n$. Elimination orderings are also referred to as *elimination schemes* or *dismantling schemes* in the literature.

There are graph classes that can be completely characterized by the presence or absence of a specific elimination ordering. Suppose, for example, that we would like to determine if a given graph G is a *forest* (acyclic). Instead of searching for cycles in the graph, the following elimination scheme may be employed. Search for a vertex of degree 0 or 1 in G and let it be v_1. Next, search for a vertex of degree 0 or 1 in $G - v_1$ and let it be v_2. Continue this process of listing a found vertex of degree 0 or 1, followed by deleting the vertex from the remaining graph. If an elimination ordering $v_1, v_2, ..., v_n$ can be found in this way, then G is a forest. If, at any point, a vertex of degree 0 or 1 cannot be found, then G is not a forest. That is to say, G has a {degree at most 1}-elimination ordering if and only if G is a forest.

If every induced subgraph of a graph G contains a vertex of degree at most k, then G is a *k-degenerate* graph. This graph class was introduced by Lick and White [25] in 1970. Forests are *1*-degenerate, as described above. It is well known that the planar graphs (those that can be drawn in the plane with no edges crossing) always have a vertex of degree at most 5, and are therefore 5-degenerate. In general, a graph is k-degenerate if and only if it has a {degree at most k}-elimination ordering. To colour the vertices of a {degree at most k}-elimination ordering greedily, at most $k+1$ colours are needed. Therefore, a k-degenerate graph will have chromatic number at most $k+1$. This result is best possible, as seen by taking the complete graph on $k+1$ vertices.

Elimination orderings require each vertex v_i to possess specific local properties within the induced subgraph G_i. Algorithms can move along a vertex ordering using these local properties in a greedy way. Consider the complete bipartite graph $K_{n,n}$ with the edges of a perfect matching removed, known as a *crown* graph (see Fig. 1.1). Let α be an ordering of the vertices of the crown graph such that the end vertices of the removed edges of the matching are consecutive. If the vertices are coloured greedily in the order of α, then n colours will be used. For example, in the crown graph of Fig. 1.1, a greedy colouring of the vertices in the order $\alpha : s, w, t, x, u, y, v, z$ uses four colours. On the other hand, there exists a vertex ordering of any bipartite graph (such as the crown graph) which can be coloured greedily by two colours; for example s, t, u, v, w, x, y, z. Vertex orderings can be used to optimize algorithm efficiency and

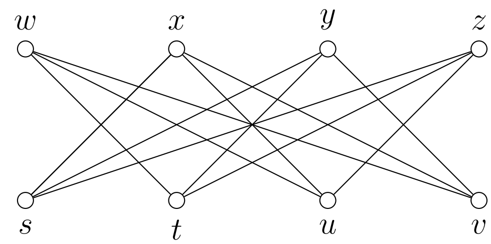


Figure 1.1: A crown graph on 8 vertices.

in proving results.

1.1 An Overview

Abstract convexity arises in several areas of mathematics and has its origins in Euclidean convexity. In Euclidean space, a set of points is considered convex if, for every pair of points within the set, the line segment joining the pair of points lies entirely within the set. This thesis will look at formal definitions of convexity in the discrete context of graphs. Elimination orderings on the vertices arise naturally from notions of convexity in graphs and will be our primary focus.

In Chapter 2 we define terms, describe several vertex ordering algorithms, and formalize the notions of convexity and convex geometries. In Chapter 3 we consider the relationships between perfect elimination orderings and the vertex ordering algorithms, and explore the geodesic and monophonic convexities. Next we examine semiperfect elimination orderings and their relationship to the various algorithms, which naturally leads us to explore the m^3-convexity in Chapter 4. Chapter 5 deviates slightly from our topic of convexity to consider elimination orderings of distance hereditary graphs. In Chapter 6 we examine generalizations of the geodesic and monophonic convexities, namely the 3-Steiner and 3-monophonic convexities, where the $3SS$ vertices are the extreme vertices of these convexities.

A vertex is *simplicial* if every two of its neighbours induce a connected graph. We see different relaxations of this property throughout this thesis. In Chapter 7 we define *nearly simplicial* vertices to be those for which every three of its neighbours induce a connected graph. We pose some open problems with respect to the property of being nearly simplicial and point out connections between nearly simplicial elimination orderings and k-independence orderings introduced by Akcoglu et al in [1] and studied further by Ye and Borodin [36].

Throughout the thesis we present several new results, as well as original proofs of known results. We offer new and simple proofs for two well known theorems on convex geometries, Theorems 2.2 and 2.3. Theorem 2.1 and Corollary 6.8 are new results using the MEC and MCC algorithms. We also obtain new results using the LexDFS algorithm, specifically Theorems 4.12 and 6.9, as well as Theorem 3.3 which offers a new proof for a known result.

Chapter 2

Preliminaries

2.1 Definitions

This thesis will consider simple, undirected, connected, finite graphs. Graph theory concepts and definitions that are not stated may be found in Bondy and Murty [4]. Various graph classes will be defined throughout this thesis in appropriate chapters. The book *Graph Classes: A Survey* by Brandstädt, Le and Spinrad [6] is an invaluable resource in obtaining a better understanding of the definitions and properties of these graph classes.

For a graph G and a subset of its vertices $X \subseteq V(G)$, the notation $\langle X \rangle$ denotes the subgraph *induced* by the vertices of X. That is to say, $\langle X \rangle$ has vertex set X and uv is an edge in $\langle X \rangle$ precisely when uv is an edge in G, for all $u, v \in X$.

A *complete graph* on n vertices, K_n, is a graph for which every pair of vertices is adjacent. A *clique* is a subset C of the vertices such that $\langle C \rangle$ is a complete subgraph. A *maximal clique* is a clique that is not included in any larger clique. A *vertex cut* or *separator* is a subset of the vertices of a connected graph whose removal results in a disconnected graph. A *minimal clique separator* is a subset of the vertices that is both a clique and a separator, and does not properly contain a vertex cut.

Graph G is *isomorphic* to H, denoted by $G \cong H$, if there is a bijection f from $V(G)$ to $V(H)$ such that, for all vertices $u, v \in V(G)$, u and v are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. The *subdivision* of an edge of a graph consists of replacing an edge uv by the edges uw and wv, where w is a new vertex. A subdivision of a graph G is a graph which results from a sequence of subdivisions of edges in G. Graph G is *homeomorphic* with H if there exist subdivisions of G and H that are isomorphic.

For any two vertices $u, v \in V(G)$ of some graph G, the *distance* between u and v, denoted by $d_G(u, v)$, or simply $d(u, v)$ if the context is clear, is the number of edges in a shortest path connecting u and v in G. A subgraph H of a graph G is *isometric* if it preserves distances; that is to say if $d_H(u, v) = d_G(u, v)$ for all vertices $u, v \in V(H)$. A graph is *distance hereditary* if it is connected and if every connected induced subgraph is isometric.

The *open neighbourhood* of a vertex v in a graph G, denoted by $N_G(v)$, is the set of all vertices adjacent to v in G. We write $N(v)$ if the context is clear. The *closed neighbourhood* of v, denoted by $N[v]$, is the open neighbourhood of v together with v itself, i.e., $N[v] = N(v) \cup \{v\}$. The k^{th} *neighbourhood* of v, $N^k(v)$, is the set of all vertices u such that $d(u, v) = k$. The *disk of radius* k centred at v, $D(v, k)$, is the set of all vertices u such that $d(u, v) \leq k$. Clearly, $D(v, k) = N^0(v) \cup N^1(v) \cup N^2(v) \cup ... \cup N^k(v)$. A *universal vertex* v is a vertex that is adjacent to all other vertices in the graph, i.e., $N[v] = V(G)$. The k^{th} *power* of a graph G, denoted by G^k, is a graph for which $V(G^k) = V(G)$ and $uv \in E(G^k)$ if and only if $d_G(u, v) \leq k$. A set of vertices is *homogeneous* if every pair of vertices in the set has an identical neighbourhood outside of the set, i.e., S is homogeneous if $N(x) \setminus S = N(y) \setminus S$ for all $x, y \in S$. A single vertex and $V(G)$ are both *trivial homogeneous sets*. A *proper homogeneous set*

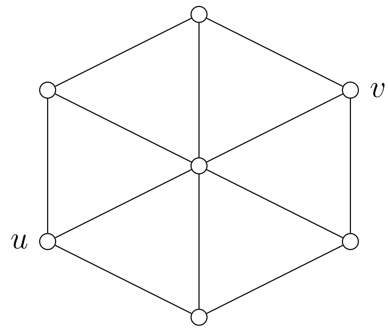


Figure 2.1: The wheel on 7 vertices is bridged but not chordal.

is one that is not trivial.

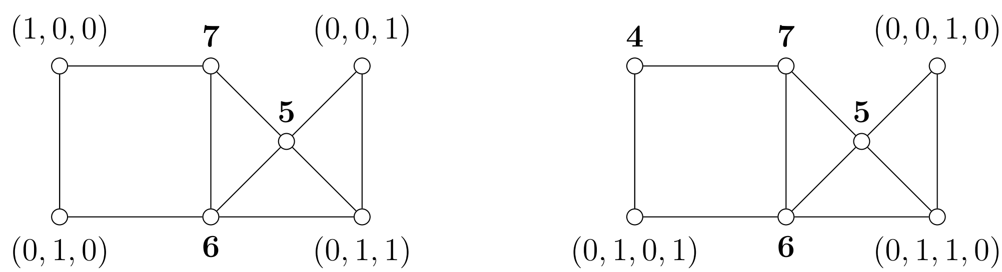
The notation P_n or C_n denotes a path or cycle, respectively, on n vertices. A *chord* of a path or cycle is an edge uv such that $d(u, v) > 1$ within the path or cycle. A graph is *chordal* if every cycle of length at least four contains a chord. For C, a cycle of length at least four, C has a *bridge* if there exists a pair of vertices $u, v \in V(C)$ such that $d_C(u, v) > d_G(u, v)$. A graph is *bridged* if every cycle of length at least four has a bridge. Every chordal graph is bridged; however, the converse is not true. For example, the wheel in Fig. 2.1 is a bridged graph since $d_C(u, v) = 3 > d_G(u, v) = 2$, where C is the outer cycle. The wheel is not chordal as the outer cycle of the wheel is chordless.

A vertex v is *simplicial* if its neighbourhood $N(v)$ induces a complete graph or, equivalently, if it is not the centre vertex of an induced P_3. A *simplicial elimination ordering*, more commonly referred to as *perfect elimination ordering*, is an ordering $v_1, v_2, ..., v_n$ for which v_i is simplicial in G_i for $i = 1, 2, ..., n$.

2.2 Vertex Ordering Algorithms

This section discusses several vertex ordering algorithms and their properties. Lexicographic Breadth First Search (LexBFS), was developed in 1976 by Rose, Tarjan and

Leuker [32] in order to recognize chordal graphs and find perfect elimination orderings of their vertices in linear time. In 1984, Tarjan and Yannakakis [34] developed Maximum Cardinality Search (MCS) as another simple algorithm capable of recognizing chordal graphs in linear time. Maximum Cardinality Neighbourhood in Component (MCC) and Maximal Element in Component (MEC), algorithms capable of finding every perfect elimination ordering of a chordal graph, were developed by Shier [33] in 1984 for this purpose. The final algorithm described, Lexicographic Depth First Search (LexDFS), was developed in 2005 by Krueger and Corneil [11] as a depth first search analog to LexBFS.



(a) Partially labelled by LexBFS. (b) The LexBFS labelling one step further.

Figure 2.2: An example of one step in a LexBFS labelling.

Lexicographic Breadth First Search, abbreviated *LexBFS*, is a vertex ordering algorithm which gives integer labels to the vertices of an n-vertex graph G in the order $n, n-1, ..., 2, 1$. All vertices have an associated binary vector, which is initially empty, that changes as vertices receive their labels. The algorithm starts by selecting any vertex as the initial vertex and assigns to it the label n. Suppose that labels $n, n-1, ..., i+1$ have been assigned. For each unlabelled vertex v, the associated binary vector $(j_n, j_{n-1}, ..., j_{i+1})$ is constructed by letting $j_k = 1$ if the vertex labelled k is adjacent to v and $j_k = 0$ otherwise, for $i+1 \leq k \leq n$. The next available label, namely i, is assigned to an, as yet, unlabelled vertex with lexicographically largest associated binary vector. Ties are broken arbitrarily. Refer to Fig. 2.2 for an example of one step in a LexBFS labelling. Fig. 2.2(a) shows the resulting binary vectors of the vertices after three vertices have been labelled. Fig. 2.2(b) reflects the change in the binary vectors of the unlabelled vertices after the fourth vertex has been labelled.

A LexBFS ordering of the vertices always has the following property:

P1: If $a < b < c$, and ac is an edge and bc is not an edge, then there exists some vertex $d > c$ such that bd is an edge and ad is not an edge.

Maximum Cardinality Search, abbreviated *MCS*, is another well known vertex ordering algorithm that gives integer labels to the vertices of an n-vertex graph G in the order $n, n-1, ..., 2, 1$. The algorithm selects any vertex to be the initial vertex and assigns to it the label n. Each unlabelled vertex u has a weight equal to the number of labelled vertices in its neighbourhood $N(u)$. Suppose that labels $n, n-1, ..., i+1$ have been assigned. Then the next label, namely i, is assigned to an unlabelled vertex of largest weight. Ties are broken arbitrarily.

An MCS ordering of the vertices always has the following property:

P2: If $a < b < c$, and ac is an edge and bc is not an edge, then there exists some vertex $d > b$ such that bd is an edge and ad is not an edge.

It is readily observed that that P2 is a weaker property than P1.

Maximum Cardinality Neighbourhood in Component, abbreviated *MCC*, is a variation of the MCS algorithm, and labels the vertices of an n-vertex graph G in the order $n, n-1, ..., 2, 1$. At each step in both MCS and MCC, the algorithm produces a new set of candidate vertices that are eligible to be labeled next. For MCS this set consists of all vertices adjacent to a maximum number of labeled vertices. For MCC the connected components of the graph induced by the unlabelled vertices are considered. For each component, those vertices adjacent to a maximum number of labelled vertices (as compared to other vertices in the same component) are included in the set of vertices eligible to be labelled next. Thus, at each step, the set of eligible vertices produced by MCS is a subset of those produced by MCC.

Maximal Element in Component, abbreviated *MEC*, labels the vertices of an n-vertex graph G in the order $n, n-1, ..., 2, 1$. At each step, the connected components of the graph induced by the unlabelled vertices are considered. To be labelled next, a vertex must be adjacent to a maximal set of labelled vertices relative to those in its component of unlabelled vertices. That is to say, an unlabelled vertex may be labelled next if its neighbourhood in the set of labelled vertices is not properly contained in the neighbourhood in the set of labelled vertices of any other vertex belonging to its component of unlabelled vertices. At each step, the set of vertices eligible to be labelled next produced by MCC is a subset of those produced by MEC.

An MEC or MCC ordering of the vertices always has the following property:

P3: If (i) $a < b < c$, (ii) ac is an edge and bc is not an edge, and (iii) a and b are in the same component of $G - S$ where S is the set of vertices with labels greater than b, then there exists some vertex $d > b$ such that bd is an edge and ad is not an edge.

An example that illustrates the difference between the algorithms MCS, MCC and MEC is shown in Fig. 2.3. When labelling with the MCS algorithm, vertex w would receive the next label, as it is adjacent to the greatest number of labelled vertices. When labelling with the MCC algorithm, both v and w are candidates to be labelled next. When labelling with the MEC algorithm, any one of u, v or w could be labeled next as within the unlabelled vertices w induces its own component, and neither neighbourhood of u or v (within the labelled vertices) is properly contained in the neighbourhood of the other.

Breadth First Search prioritizes visiting neighbours of the least recently visited vertex. Depth First Search prioritizes visiting neighbours of the most recently visited

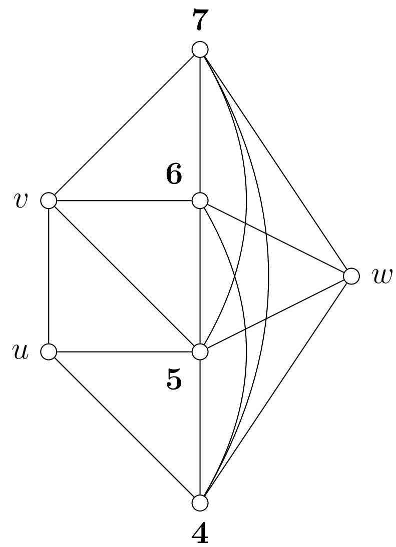
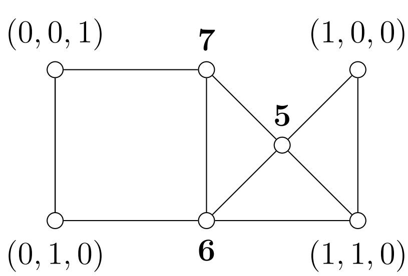


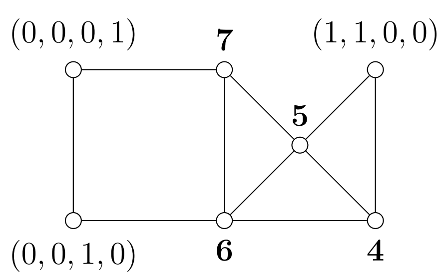
Figure 2.3: A graph partially labelled by MCS, MCC or MEC.

vertex.

LexDFS is comparable to the LexBFS algorithm except that the associated binary vectors are built in the reverse order. For each unlabelled vertex v, the associated binary vector $(j_{i+1}, ..., j_{n-1}, j_n)$ is constructed by letting $j_k = 1$ if the vertex labelled k is adjacent to v and $j_k = 0$ otherwise, for $i + 1 \leq k \leq n$. The next available label, namely i, is assigned to an, as yet, unlabelled vertex with lexicographically largest associated binary vector.



(a) Partially labelled by LexDFS.



(b) The LexDFS labelling one step further.

Figure 2.4: An example of a step in a LexDFS labelling.

Fig. 2.4 shows an example of one step in a LexDFS ordering and may be compared to one step in a LexBFS ordering, as shown in Fig. 2.2.

A LexDFS ordering of the vertices always has the following property:

P4: If $a < b < c$, and ac is an edge and bc is not an edge, then there exists some vertex d for which $b < d < c$ such that bd is an edge and ad is not an edge.

As a relatively new algorithm, LexDFS has the potential to give new insights into old problems. One such example is the problem of finding the minimum number of vertex disjoint paths that will cover all the vertices of a given graph, known as the minimum path cover problem. We now introduce several graph classes that are relevant to the minimum path cover problem.

A graph is an *interval graph* if its vertices correspond to intervals of a line, and edges are present precisely when the two intervals intersect. The interval graphs may be characterized by an elimination ordering: G is an interval graph if and only if an ordering α of the vertices of G exists such that for every triple $x, y, z \in V(G)$ such that $x < y < z$ and xz is an edge, then xy is also an edge [12]. A *comparability graph* is an undirected graph which admits a transitive orientation; that is, if there are directed edges from x to y and from y to z then there is a directed edge from x to z. A graph is a *cocomparability graph* if it is the complement of a comparability graph. Equivalently, a graph G is a cocomparability graph if and only if there exists a poset on $V(G)$ such that two vertices are adjacent in G if and only if they are not comparable in the poset. The cocomparability graphs may also be characterized by an elimination ordering: G is a cocomparability graph if and only if a *cocomparability ordering* of the vertices of G exists such that for every triple $x, y, z \in V(G)$ such that $x < y < z$ and xz is an edge, y is adjacent to at least one of x or z [12]. These elimination ordering characterizations allow us to see that the interval graphs are a subclass of the cocomparability graphs.

Until recently, the minimum path cover problem could be solved on the cocomparable graphs, but only by the indirect method of first finding the corresponding poset structure. A solution was sought that would use only the structure of the graph. In 2013, Corneil, Dalton and Habib [10] were able to find such a solution to the minimum path cover problem on cocomparability graphs, and thus also interval graphs, which uses the graph structure only. The solution consists of three steps: (1) Running a known algorithm to obtain a cocomparability ordering of the vertices; (2) Running the LexDFS algorithm, using results from the cocomparability ordering to choose a starting vertex and to break any ties; and (3) Interpreting the LexDFS ordering to obtain a list of paths which are a certifiable minimum path cover.

By comparing the algorithms, one finds that every MCS ordering is an MCC ordering, and every MCC ordering is an MEC ordering. Also, both LexBFS and LexDFS orderings are specific types of MEC orderings. As such, we see that MEC is the most general search algorithm that we consider, and MCS, LexBFS and LexDFS the most specific.

Brandstädt, Dragan and Nicolai [5] (Krueger and Corneil [11]) show that an ordering α of the vertices of a graph G is a LexBFS (LexDFS) ordering if and only if α has property P1 (P4). That is to say, LexBFS and LexDFS are completely characterized by their respective properties. Shier [33] developed the MEC and MCC algorithms but it was Olariu [30] who first concisely stated their property P3. Since every MCC search is an MEC search, and there exist graphs with MEC searches not obtainable by MCC, P3 does not completely characterize the MCC algorithm. Likewise, P2 does not completely characterize the MCS algorithm. The MEC algorithm is completely characterized by P3, and we prove this shortly.

The MCS algorithm is a specific type of MCC algorithm, where the former compares every two unlabelled vertices in the graph, and the latter compares an unlabelled vertex only to those in its unlabelled component. Analogous to this, the *Maximal Neighbourhood Search* (MNS) algorithm is a specific type of MEC algorithm, where the former compares every two unlabelled vertices in the graph, and the latter compares an unlabelled vertex only to those in its unlabelled component. That is to say, a vertex is eligible to be labelled next by the MNS algorithm when its neighbourhood in the labelled vertices is not properly contained by the neighbourhood in the labelled vertices of any other unlabelled vertex. At each step, the set of vertices eligible to be labelled next produced by MNS is a subset of those produced by MEC. When labelling Fig. 2.3 with the MNS algorithm, vertex w would receive the next label, as the neighbourhoods of u and v (within the labelled vertices) are properly contained by the neighbourhood of w. While MNS is an interesting algorithm, we will not further investigate it in this thesis. The following characterization of MEC has not previously been shown; however, our proof is based directly on the proof of the analogous result for the MNS algorithm by Corneil and Krueger [11].

Theorem 2.1. *An ordering α of the vertices of a graph G is an MEC ordering if and only if α has property P3.*

Proof. Suppose $\alpha : v_1, v_2, ..., v_n$ is an MEC ordering of a graph G for which property P3 does not hold. Let $a, b, c \in V(G)$ be three vertices that satisfy (i), (ii), and (iii) of the hypothesis of P3 but for which the conclusion of P3 does not hold. Suppose $b = v_i$. Since $c \in V(G_{i+1})$ and c is adjacent to a but not to b, $N(b) \cap V(G_{i+1}) \subsetneq N(a) \cap V(G_{i+1})$. But then b can not be labelled next (as v_i). This contradiction shows that P3 holds for α, establishing the sufficiency.

Suppose that $\phi : v_1, v_2, ..., v_n$ is an ordering of a graph G for which property P3 holds, but suppose that ϕ is not an MEC ordering. Let v_j be the greatest vertex in the ordering that could not have been chosen next by the MEC algorithm. Then there is some vertex $u < v_j$ in the same component of $G - V(G_{j+1})$ as v_j, such that

$N_G(v_j) \cap V(G_{j+1}) \subsetneq N_G(u) \cap V(G_{j+1})$. Let $w \in V(G_{j+1})$ be a neighbour of u that is not adjacent to v_j. Since $u < v_j < w$, uw is an edge and v_jw is not an edge, and u and v_j are in the same unlabelled component, by P3 there exists a vertex $x > v_j$ such that v_jx is an edge and ux is not an edge. However, since $x \in V(G_{j+1})$, this contradicts the fact that $N(v_j) \cap V(G_{j+1}) \subsetneq N(u) \cap V(G_{j+1})$. Thus, ϕ is an MEC ordering, establishing the necessity. □

In this thesis we will use the vertex ordering algorithms described in this section in theorems of the following type: Every *given algorithm* vertex ordering of G is a *given vertex property* ordering if and only if G is *given induced subgraph*-free. The given vertex property will be related to specific convexites, as described in further chapters.

2.3 Convexity

Convexity is a broadly used mathematical term which extends into geometry, topology, analysis and graph theory. The extensive study of convexity in geometry gives us intuitive notions of the subject. The study of convexity in graph theory, the area of this thesis, allows for the abstraction of the ideas of convexity into a discrete setting.

Let V be a finite set, and suppose that $\mathcal{M}$ is a family of subsets of V with the following three properties:

1. $\emptyset \in \mathcal{M}$.

2. $V \in \mathcal{M}$.

3. $\mathcal{M}$ is closed under intersection.

Then $\mathcal{M}$ is referred to as a *convexity* or, equivalently, an *alignment* and V is the *ground set* for the convexity. The subsets of V contained in the family $\mathcal{M}$ are called *convex sets*. Such a pair $(V, \mathcal{M})$ is referred to as a *convex space* or, equivalently, an *aligned space*.

Example 1. *If* $V = \{v_1, v_2, v_3, v_4\}$ *and*
$\mathcal{M} = \big\{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3\},$
$\{v_1, v_3, v_4\}, \{v_1, v_2, v_3, v_4\}\big\}$,
then $(V, \mathcal{M})$ *is an aligned space.*

Suppose that X is a convex set of a convex space $(V, \mathcal{M})$ and let $x \in X$. If $X - \{x\} \in \mathcal{M}$, then x is an *extreme point* of X. We denote the set of all extreme points of a convex set X by $ex(X)$. In Example 1, $ex(\{v_1, v_2, v_3, v_4\}) = \{v_2, v_4\}$.

Let $(V, \mathcal{M})$ be an alignment and $Y \subseteq V$. The *convex hull* of Y, denoted by $CH(Y)$, is the smallest convex set of which Y is a subset. When a specific type of convexity is being referred to, a subscript may be used in the convex hull notation; for example, $CH_\tau(X)$ denotes the smallest τ-convex set of which X is a subset. The subscript is omitted when the context is clear. Since $\mathcal{M}$ is closed under intersections, the convex hull of any set Y will be unique. In Example 1, $CH(\{v_2, v_4\}) = \{v_1, v_2, v_3, v_4\}$.

If $CH(ex(X)) = X$ for every convex set X of a convex space $(V, \mathcal{M})$, then $(V, \mathcal{M})$ is a *convex geometry*. In other words, a convex geometry consists of a finite set V and a family $\mathcal{M}$ of subsets of V such that:

1. $(V, \mathcal{M})$ is a convex space, and

2. Every convex set is the convex hull of its extreme points.

The latter property is referred to as the *Minkowski-Krein-Milman property*. We are familiar with the property as it holds for all closed and bounded convex sets in Euclidean space. We say that the *anti-exchange property* holds for an aligned space $(V, \mathcal{M})$ if for any convex set X and any two distinct points $y, z \notin X$, if y is in the convex hull of $X \cup \{z\}$ then z is not in the convex hull of $X \cup \{y\}$. As we now see, an alternative definition of a convex geometry is a convex space $(V, \mathcal{M})$ for which the anti-exchange property holds. While the equivalence of the Minkowski-Krein-Milman and anti-exchange properties is well known, the proof given is the author's own.

Theorem 2.2. *For a convex space $(V, \mathcal{M})$, the Minkowski-Krein-Milman property holds if and only if the anti-exchange property holds.*

Proof. Let $(V, \mathcal{M})$ be a convex space for which the anti-exchange property does not hold. Then there exists a convex set X and points $y, z \notin X$ such that $y \in CH(X \cup \{z\})$ and $z \in CH(X \cup \{y\})$. By closure under intersections, $CH(X \cup \{z\}) = CH(X \cup \{y\}) = U$, and the extreme points of U must be contained in X. Therefore $CH(ex(U)) \subseteq X \subsetneq U$ and the Minkowski-Krein-Milman property does not hold. This establishes the sufficiency.

Let $(V, \mathcal{M})$ be a convex space for which the Minkowski-Krein-Milman property does not hold. Consequently, there exists a convex set that is not the convex hull of its extreme points. Let Y be such a convex set, such that $ex(Y) = \{e_1, e_2, ..., e_r\}$. By closure under intersection, $CH(ex(Y)) = CH(\{e_1, e_2, ..., e_r\}) = Y^* \subsetneq Y$.

Let X be a largest convex set such that $Y^* \subseteq X \subsetneq Y$. Note that $|X| + 2 \leq |Y|$, as otherwise there are extreme points of Y which are not contained in $\{e_1, e_2, ..., e_r\}$. Let w and z be distinct points of Y not in X. If $CH(X \cup \{w\}) \neq Y$, then $CH(X \cup \{w\}) \cap Y$ would violate our choice of X. For this reason, $CH(X \cup \{w\}) = Y$ and, likewise, $CH(X \cup \{z\}) = Y$. Thus the anti-exchange property does not hold, establishing the necessity. □

Convexities and convex geometries arise in graph theory by choosing the vertex set $V(G)$ of a graph G to be the ground set. While it is possible to find convexities and convex geometries on $V(G)$ without using the structure of G, it is more interesting to define convexities based on the structure of the graph. For example, we say that a set of vertices is *geodesically convex*, or *g-convex*, if it contains all shortest paths between any two vertices in the set. Similarly, a set of vertices is *monophonically convex*, or *m-convex*, if it contains all chordless paths between any two vertices in the set. These specific types of convexities, as well as others, will be further explored in later chapters.

Farber and Jamison [18] were the first to relate convex geometries and elimination orderings in the following classic result. They state that the theorem follows directly from Jamison's work on antimatroids [22] and Edelman's work on meet-distributive lattices [17]. We give here our own proof.

Theorem 2.3. *Let $(V, \mathcal{M})$ be a convex geometry for a finite set V. Then $X \in \mathcal{M}$ if and only if there is an ordering $v_1, v_2, ..., v_k$ of the points of $V - X$ such that v_i is an extreme point of $\{v_i, v_{i+1}, ..., v_k\} \cup X$ for each $i = 1, 2, ..., k$.*

Proof. For $(V, \mathcal{M})$ a convex geometry, suppose there exists a set X of points and an ordering $v_1, v_2, ..., v_k$ of the points of $V - X$ such that v_i is an extreme point of $X_i = \{v_i, v_{i+1}, ..., v_k\} \cup X$ for each $i = 1, 2, ..., k$. Since v_k is an extreme point of $X_k = \{v_k\} \cup X$, the set X is convex. This establishes the necessity.

We prove sufficiency by induction on $k = |V - X|$ for X a convex set. Note that there is a convex set of each cardinality $t = 0, 1, ..., |V|$. This follows from the fact that $(V, \mathcal{M})$ is a convex geometry, therefore each non-empty convex set has at least one extreme point whose deletion leaves a convex set of order one less. Suppose X is a convex set of order $|V| - 1$, i.e., $k = 1$. Let v_1 be the single point not in X. Then v_1 is an extreme point of $X \cup \{v_1\} = V$. For our inductive hypothesis, suppose that for every convex set X such that $|V - X| = k > 1$, the points of $V - X$ can be ordered $v_1, v_2, ..., v_k$ such that v_i is an extreme point of $\{v_i, v_{i+1}, ..., v_k\} \cup X$ for $i = 1, 2, ..., k$.

Let Y be a convex set such that $|V - Y| = k \geq 1$. Since Y is non-empty and $(V, \mathcal{M})$ is a convex geometry, Y contains at least one extreme point y. Let $Y' = Y - \{y\}$

and note that Y' is convex and $|V - Y'| = k + 1$. By the inductive hypothesis, there exists an ordering $v_1, v_2, ..., v_k$ of $V - Y$ such that v_i is an extreme point of $\{v_i, v_{i+1}, ..., v_k\} \cup Y$ for every $i = 1, 2, ..., k$. Let $y = v_{k+1}$. Then v_i is an extreme point of $\{v_i, v_{i+1}, ..., v_k, v_{k+1}\} \cup Y'$ for $i = 1, 2, ..., k+1$. The result follows by induction. $\square$

Suppose $(V(G), \mathcal{M})$ is a convex geometry for a graph G . Since the empty set is convex, the vertices of G may always be ordered as $v_1, v_2, ..., v_n$ such that v_i is an extreme vertex of G_i for each $i = 1, 2, ..., n$. If P is a property that characterizes the extreme vertices of G, with respect to the convex space $(V(G), \mathcal{M})$, then G has a P-elimination ordering.

Chapter 3

Perfect Elimination Orderings

As previously mentioned, a vertex is simplicial when it is not the centre of an induced P_3. A simplicial elimination ordering is more commonly referred to as a perfect elimination ordering. In this chapter we see that graphs for which the g-convexity and the m-convexity form convex geometries have interesting connections with classes of graphs having perfect elimination orderings. We pointed out in Chapter 2 that the MCC and MEC algorithms were introduced in an attempt to capture all perfect elimination orderings of chordal graphs, and in this chapter we prove this to be the case. Furthermore, we offer a new proof for a characterization of chordal graphs based on LexDFS orderings.

3.1 Chordal Graphs

It is well known that the chordal graphs can be completely classified by perfect elimination orderings.

Theorem 3.1. [14], [19], [31] *A graph is chordal if and only if it has a perfect elimination ordering.*

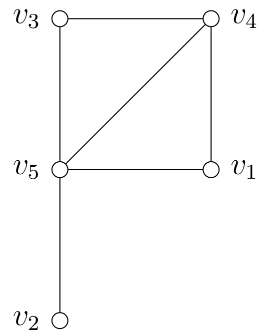


Figure 3.1: A labelling of the dart graph.

Rose, Tarjan and Leuker [31, 32] showed that every LexBFS or MCS ordering of a chordal graph is a perfect elimination ordering. LexBFS and MCS can be used to create candidate elimination orderings which can then be checked to see if they are perfect elimination orderings. This allows for recognition of the chordal graphs in linear-time. In fact, LexBFS and MCS were specifically developed for this purpose.

Not all perfect elimination orderings of a chordal graph, however, can be generated by LexBFS or MCS. For example, the perfect elimination ordering v_1, v_2, v_3, v_4, v_5 of the dart graph shown in Fig. 3.1 cannot be generated by LexBFS or MCS. For many chordal graphs, the two algorithms LexBFS and MCS cannot even generate every perfect elimination ordering generated by the other. Shier [33] developed two algorithms, MEC and MCC (described in Section 2.2), each capable of generating all perfect elimination orderings of any chordal graph.

Recall that a set of vertices is m-convex if it contains all chordless paths between any two vertices in the set. An ordering of the vertices $v_1, v_2, ..., v_n$ is a perfect elimination ordering if and only if $\{v_i, v_{i+1}, ..., v_n\}$ is m-convex for all i, $1 \leq i \leq n$.

Theorem 3.2. [33]

For a chordal graph the following are equivalent:

(1) α is a perfect elimination ordering.

(2) α is an MCC ordering.

(3) α is an MEC ordering.

Proof. Let α be an ordering of the vertices of a chordal graph G.

(2) $\implies$ (3): All MCC orderings are MEC orderings.

(3) $\implies$ (1): Suppose α is an MEC ordering that is not simplicial. Then there exists a chordless path $u_0, u_1, ..., u_m$ such that $u_i < min\{u_0, u_m\}$ for some $0 < i < m$. Choose P to be such a path so that $min\{u_0, u_m\}$ is maximized. Without loss of generality, suppose $u_m < u_0$ and that u_i receives the minimum label of all vertices of P. Let u_m receive label k. We want to show that the MEC algorithm could not label u_m before u_i.

Suppose $u_m = k$ has a neighbour z in $V(G_{k+1})$. Let u_j be the smallest neighbour of z in $V(P)$ for $i \leq j \leq m$. Then zu_{i-1} is an edge, as otherwise $\{u_{i-1}, u_i, ..., u_j, z\}$ induces a path which contradicts our choice of P. Vertex u_i is adjacent to both z and u_m ($i = j = m-1$), as otherwise $\{u_{i-1}, u_i, ..., u_j, z\}$ induces a chordless cycle on 4 or more vertices. Thus, every neighbour of u_m in $V(G_{k+1})$ is also a neighbour of u_i. Vertices u_i and $u_m = k$ are in the same connected component of $G - V(G_{k+1})$. Vertex $u_m < u_{i-1}$, as otherwise the path $u_0, u_1, ..., u_{i-1}, z$ would contradict our choice of P. That is to say, the neighbourhood of u_m in $V(G_{k+1})$ is properly contained in the neighbourhood of u_i in $V(G_{k+1})$, and the MEC algorithm can not label u_m before u_i. As a result of this contradiction, α must be a perfect elimination ordering.

(1) $\implies$ (2): Let $\alpha : v_1, v_2, ..., v_n$ be a perfect elimination ordering. Let C be the component containing v_{k-1} in the subgraph $G - V(G_k)$. Suppose there is a vertex $y \in V(C)$ that has a greater number of neighbours in G_k than v_{k-1}. Let z be a vertex of G_k that is adjacent to y but not to v_{k-1}. Let P^* be a chordless v_{k-1}, y-path in C. Let $y' \in V(P^*)$ be the vertex of minimum distance to v_{k-1} that is adjacent to z. The chordless path composed of the v_{k-1}, y'-subpath of P^* together with the

edge $y'z$ contains $y' < v_{k-1}$ as an internal vertex. Consequently, $\{v_{k-1}, v_k, ..., v_n\}$ is not m-convex. As observed prior to the theorem, this is a contradiction to α being a perfect elimination ordering. Thus, there is no vertex in $V(C)$ that has a greater number of neighbours in G_k than v_{k-1}. By induction, α is an MCC ordering. □

The proof for the following result on the LexDFS algorithm and perfect elimination orderings is new, but the result is not. Corneil and Krueger [11] were the first to define the MNS algorithm; however, Tarjan and Yannakakis [34] described a characteristic property for MNS in 1984, and showed that any ordering of the vertices of G with this property is a perfect elimination ordering if and only if G is chordal. Since every LexDFS ordering is an MNS ordering, the result follows. As is the case for LexBFS and MCS, this result shows LexDFS to be an algorithm that may be used to certify that a graph is chordal, or to find a perfect elimination ordering of a graph known to be chordal.

Theorem 3.3. *Every LexDFS ordering of G is a perfect elimination ordering if and only if G is chordal.*

Proof. Suppose G is not chordal. Let C_k, $k \geq 4$, be an induced cycle in G. For a LexDFS ordering $v_1, v_2, ..., v_n$ of the vertices of G, let v_i be the vertex of C_k to receive the least label. Then v_i is the centre vertex of an induced $P_3 : u, v_i, w$ in G_i where u and w are the neighbours of v_i in C_k. This establishes the sufficiency.

Suppose G is a chordal graph and that there exists a LexDFS ordering α : $v_1, v_2, ..., v_n$ of the vertices that is not a perfect elimination ordering. Let v_i be the vertex of largest label that is not simplicial in G_i. Let $\{v_i, u, w\}$ induce a P_3 in G_i and, without loss of generality, suppose $v_i < u < w$. Since $v_i w$ is an edge and uw is not an edge, by P4 there exists a vertex x_1 such that $u < x_1 < w$, and ux_1 is an edge and $v_i x_1$ is not an edge. Since G is chordal, $x_1 w$ is not an edge. Since $v_i < x_1 < w$, and $v_i w$ is

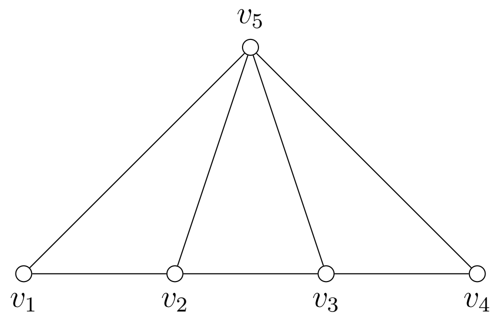


Figure 3.2: A 3-fan graph.

an edge and x_1w is not an edge, by P4 there exists a vertex x_2 such that $x_1 < x_2 < w$, and x_1x_2 is an edge and v_ix_2 is not an edge. If x_2w is an edge, then ux_2 is also an edge, as otherwise $\{w, v_i, u, x_1, x_2\}$ induces a C_5; however, now $\{w, v_i, u, x_2\}$ induces a C_4, a contradiction to G being chordal. Thus x_2w is not an edge. Repeatedly applying the LexDFS property P4 to vertices v_i, x_j, w for $j = 2, 3, ...$ ensures the existence of a vertex x_{j+1} such that $x_j < x_{j+1} < w$, and x_jx_{j+1} is an edge and v_ix_{j+1} is not an edge. After each such application of property $P4$, it can be determined that $x_{j+1}w$ is not an edge since G is chordal. This leads to an infinite path $w, v_i, u, x_1, x_2, ...$; a contradiction since G is a finite graph. This establishes the necessity. □

3.2 Geodesic Convexity

Geodesic convexity is the type of graph convexity most similar to Euclidean convexity, and is therefore the most intuitive. Recall that a subset of vertices X is considered to be geodesically convex, or g-convex, if for any two vertices $u, v \in X$ all vertices on any shortest u, v-path are also contained in X. We refer to any shortest u, v-path in G as a *u, v-geodesic*. More formally, let the *geodesic u, v-interval*, $I_g[u, v]$, be the set of all vertices that lie on a u, v-geodesic. A set X of vertices is geodesically convex precisely when $I_g[u, v] \subseteq X$ for all $u, v \in X$.

For a graph G with vertex set $V(G)$, let $\mathcal{M}_g$ be the family of all g-convex subsets of $V(G)$. The empty set and $V(G)$ are clearly both g-convex sets and the properties of g-convexity imply closure under intersection. Thus, $(V(G), \mathcal{M}_g)$ is necessarily a convex space. However, there are graphs G for which such a convex space $(V(G), \mathcal{M}_g)$ is not a convex geometry. Take, for example, the 3-fan of Fig. 3.2.

Example 2. *The family of g-convex sets for the 3-fan shown in Fig. 3.2 is:*
$\mathcal{M}_g = \{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_5\}, \{v_2, v_5\}, \{v_3, v_5\}, \{v_4, v_5\}, \{v_1, v_2, v_5\}, \{v_2, v_3, v_5\}, \{v_3, v_4, v_5\}, \{v_1, v_4, v_5\}, \{v_1, v_2, v_3, v_5\}, \{v_2, v_3, v_4, v_5\}, \{v_1, v_2, v_3, v_4, v_5\}\}$.

The Minkowski-Krein-Milman property does not hold for all sets in $\mathcal{M}_g$. For example, $CH(ex(\{v_1, v_2, v_3, v_4, v_5\})) = CH(\{v_1, v_4\}) = \{v_1, v_4, v_5\} \neq \{v_1, v_2, v_3, v_4, v_5\}$. For this reason, $(V(G), \mathcal{M}_g)$ is not a convex geometry.

The graphs for which the g-convex sets do form a convex geometry, as we will see, are precisely the *Ptolemaic graphs* [18]. The definition of a Ptolemaic graph given by Kay and Chartrand [24] is a connected graph for which any four vertices u, v, w, y satisfy the *Ptolemaic inequality*: $d(u, v)d(w, y) \leq d(u, w)d(v, y) + d(v, w)d(u, y)$. Howorka [21] gave another characterization of the Ptolemaic graphs: The Ptolemaic graphs are precisely those graphs that are both chordal and distance hereditary. They are also known to be the chordal graphs that do not contain an induced 3-fan [6].

Theorem 3.4. [18] *A graph G is Ptolemaic if and only if the geodesic convexity of G is a convex geometry.*

We will use Theorem 3.7 in the proof of this result. As such we delay the proof until Theorem 3.7 has been established.

There are problems that can be solved in polynomial time for the Ptolemaic graphs but remain NP-hard for the chordal graphs. To this end, we define an *odd chord* of a cycle to be an edge connecting two vertices of odd distance on the cycle. A *strongly chordal graph* is a chordal graph for which every even cycle of length at least 6 contains an odd chord [6]. The Ptolemaic graphs are a subclass of the strongly chordal graphs.

One example of a problem that is NP-hard for the chordal graphs but can be solved in polynomial time for the strongly chordal graphs, and thus the Ptolemaic graphs, is the *Steiner tree problem* [35] which can be stated as follows: Given a subset of the vertices of a graph, find a tree of minimum size containing the subset.

3.3 Monophonic Convexity

Recall that a subset X of vertices is considered to be monophonically convex, or m-convex, if for any two vertices $u, v \in X$ all vertices on any chordless u, v-path are also contained in X. More formally, let the *monophonic* u, v*-interval*, $I_m[u, v]$, be the set of all vertices that lie on a chordless u, v-path. A set X of vertices is monophonically convex precisely when $I_m[u, v] \subseteq X$ for all $u, v \in X$.

The extreme vertices of both g-convex and m-convex sets are precisely the simplicial vertices. To illustrate the difference between the monophonic and the geodesic convexity consider the following example.

Example 3. *For the cycle C_5, labelled clockwise v_1, v_2, v_3, v_4, v_5, the vertex set $\{v_1, v_2, v_3\}$ is g-convex but not m-convex since v_4 and v_5 lie on a chordless (but not shortest) v_1, v_3-path.*

It is well known that the following graph properties are equivalent [6]:

1. G is chordal.

2. Every minimal vertex cut of every induced subgraph of G induces a complete graph.

3. Every induced subgraph of G has a simplicial vertex.

We now state two known lemmas about chordal graphs and simplicial vertices, and give a proof of the second lemma only, as it is less well known than the first of these two.

Lemma 3.5. [14, 31] *Every chordal graph is complete or contains at least two non-adjacent simplicial vertices.*

Lemma 3.6. [18] *Every non-simplicial vertex of a chordal graph lies on a chordless path between two simplicial vertices.*

Proof. Let G be a chordal graph on n vertices. If $n = 1$ or $n = 2$, no non-simplicial vertex exists. If $n = 3$, then G is either K_3 (all vertices are simplicial) or P_3, for which the lemma clearly holds.

For our inductive hypothesis, suppose the lemma holds for chordal graphs on fewer than n vertices. Suppose v is a non-simplicial vertex in G, i.e., v is the centre vertex of an induced $P_3 : u, v, w$. Let Y be the set of internal vertices of all u, w-paths. The set Y is a vertex cut of G, and $v \in Y$. Let U and W be the sets of vertices of the two components of $G - Y$ that contain u and w, respectively. The set Y is a minimal vertex cut of $\langle U \cup Y \cup W \rangle$. Therefore, $\langle Y \rangle$ is a complete graph. By the inductive hypothesis, u is either simplicial or lies on a chordless path between two simplicial vertices found in $\langle U \cup Y \rangle$. In the latter case, since $\langle Y \rangle$ is a complete graph, at least one of the simplicial vertices is in U. Consequently, there is a simplicial vertex $u' \in U$. Likewise, there is a simplicial vertex $w' \in W$. By taking the chordless u', v-path followed by the chordless v, w'-path we obtain a chordless u', w'-path with internal vertex v. The result follows by induction. □

As with the g-convexity, the m-convexity defines an alignment on the vertices of a graph, but only produces a convex geometry for certain graphs. Specifically, the graphs for which the m-convex sets produce a convex geometry are precisely the chordal graphs.

Theorem 3.7. [18] *A graph G is chordal if and only if the monophonic convexity of G is a convex geometry.*

Proof. Let G be a chordal graph and X an m-convex subset of the vertices. The extreme vertices of X are precisely the simplicial vertices of $\langle X \rangle$. Thus, $x \in CH(ex(X))$ for all simplicial vertices $x \in X$. The non-extreme vertices of X are precisely the non-simplicial vertices of $\langle X \rangle$. By Lemma 3.6, every non-simplicial vertex $v \in X$ lies on a chordless path between two simplicial vertices $u, w \in X$. Since $u, w \in CH(ex(X))$ and $v \in I_m[u, w]$, $X \subseteq CH(ex(X))$. By closure under intersection, $X = CH(ex(X))$. Thus the Minkowski-Krein-Milman property holds. This establishes the sufficiency.

Suppose the monophonic convexity of a graph G is a convex geometry. Since the extreme vertices of an m-convex set are precisely the simplicial vertices, by Theorem 2.3, G has a perfect elimination ordering and, by Theorem 3.1, G is chordal. □

We now have the tools needed to prove Theorem 3.4; namely, that *a graph G is Ptolemaic if and only if the geodesic convexity of G is a convex geometry.*

Proof (of Theorem 3.4). Let G be a Ptolemaic graph, i.e., a chordal graph with no induced 3-fan. Suppose that G contains some chordless path which is not a shortest path. Let $P : v_0, v_1, v_2, ..., v_k$ be such a chordless path, chosen to have minimum length. Then $d_P(v_0, v_k) = k$ and $d_G(v_0, v_k) = j < k$. All proper subpaths of P must be shortest paths. Therefore, $d_G(v_0, v_{k-1}) = d_G(v_1, v_k) = k - 1$ and $d_G(v_1, v_{k-1}) = k - 2$.

For the distance $j \leq k - 1$, the Ptolemaic inequality

$$d(v_0, v_{k-1})d(v_k, v_1) \leq d(v_0, v_k)d(v_{k-1}, v_1) + d(v_{k-1}, v_k)d(v_0, v_1)$$

is true only if $k = 1$ or 2. However, since k is the path length of P, a chordless but not shortest path, both of these values of k lead to contradiction. Thus, every chordless path of G is a shortest path, and the monophonic convexity is equivalent to the geodesic convexity. By Theorem 3.7, the monophonic convexity (and thus the geodesic convexity) of G is a convex geometry. This establishes the sufficiency.

Suppose the geodesic convexity of a graph G is a convex geometry. Since the extreme vertices of a g-convex set are precisely the simplicial vertices, by Theorem 2.3, G has a perfect elimination ordering and, by Theorem 3.1, G is chordal.

Suppose G contains an induced 3-fan, as labelled in Fig. 3.2.
Let $X = CH(\{v_1, v_2, v_3, v_4, v_5\})$. Then $\{v_1, v_2, v_3, v_4, v_5\} \subseteq X$ and $ex(X) \subseteq \{v_1, v_4\}$. If $ex(X) \subsetneq \{v_1, v_4\}$, then $CH(ex(X)) = ex(X) \neq X$. Thus $ex(X) = \{v_1, v_4\}$ and v_1 and v_4 are simplicial in $\langle X \rangle$. Since $d(v_1, v_4) = 2$, all common neighbours of v_1 and v_4 must induce a complete graph. However, then $v_2, v_3 \notin CH(\{v_1, v_4\})$ since the only vertices in $CH(\{v_1, v_4\})$ are v_1, v_4 and their common neighbours. This contradicts the Minkowski-Krein-Milman property. Consequently, G is Ptolemaic. This establishes the necessity. □

3.4 Concluding Remarks

Apart from the convex geometries for the m-convexity and the g-convexity considered in this chapter, there are many other important subclasses of chordal graphs that have been widely studied in the literature. One such class is the class of k-trees.

A *k-tree* is a chordal graph for which every maximal clique has $k + 1$ vertices, and every minimal clique separator has k vertices. A k-tree may be constructed by starting with a complete graph on k vertices and, at each step, adding a new vertex which is adjacent to a k-clique. By this construction definition, we see that a k-tree clearly has a perfect elimination ordering. Indeed, we could go further and define a *k-clique simplicial* vertex to be a vertex for which every set of k neighbours is a clique. By this definition a k-tree has a k-clique simplicial elimination ordering.

A *partial k-tree* is a subgraph of a k-tree. There are problems known to be NP-hard for general graphs that are solvable in linear time for partial k-trees for bounded values of k [2]. Some examples include:

(1) The 3-colouring problem. Given a graph, decide if its vertices may be coloured by three colours.

(2) Hamiltonicity. Given a graph, decide if there exists a cycle which passes through every vertex exactly once.

(3) The dominating set problem. Given a graph, find a smallest subset of vertices D such that every vertex not in D is adjacent to a vertex in D.

Chapter 4

Semiperfect Elimination Orderings

A vertex is simplicial if it is not the center of an induced P_3. Jamison and Olariu [23] relaxed the simplicial condition to define a *semisimplicial vertex* as one that is not the center of an induced P_4. Their focus was on graphs for which all LexBFS orderings and all MCS orderings (of all induced subgraphs) are semisimplicial elimination orderings. Dragan, Nicolai and Brandstädt [15] posed the question: For what type of convexity might the semisimplicial vertices characterize the extreme points? Semisimplicial elimination orderings are often referred to as semiperfect elimination orderings in the literature. We use both terms interchangeably.

In this chapter we take the approach of Dragan, Nicolai and Brandstädt [15] of first introducing the convexity for which the semisimplicial vertices are the extreme vertices, and then showing how these ideas can be used to obtain the results of Jamison and Olariu [23]. We follow this with characterizations of graphs for which every MEC or MCC ordering is semisimplicial. We close by presenting a new result, namely a characterization of those graphs for which all LexDFS orderings are semisimplicial.

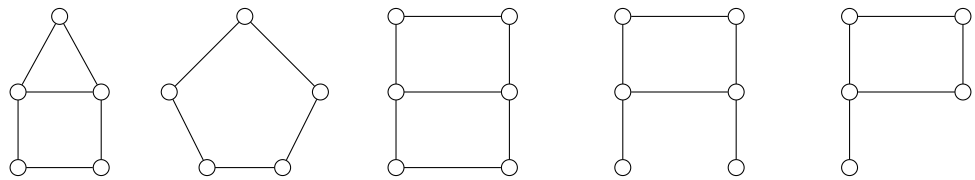


Figure 4.1: From left to right: A house, hole (of size 5), domino, A, and P graph.

4.1 m^3-Convexity

Monophonic convexity can be modified in the following way: A subset X of vertices is defined to be m^3-*convex* if for any two vertices $u, v \in X$ all vertices on a chordless u, v-path of length at least 3 are also contained in X. The idea of m^3-convexity was introduced by Dragan, Nicolai and Brandstädt [15]. For any graph G, the set of all m^3-convex sets is a convexity. Unlike m-convex sets, an m^3-convex set need not induce a connected subgraph. For example, a non-adjacent pair of vertices of a C_4 form an m^3-convex set. Since every pair of non-adjacent vertices u, v for which there is no induced u, v-path of length at least 3 is m^3-convex, the semisimplicial vertices are precisely the extreme vertices of the m^3-convex sets.

Lemma 4.1. [15] *A vertex ordering* $v_1, v_2, ..., v_n$ *is semisimplicial if and only if* $\{v_i, v_{i+1}, ..., v_n\}$ *is* m^3*-convex for* $i = 1, 2, ..., n$.

Recall that a set of vertices is homogeneous if every pair of vertices in the set has an identical neighbourhood outside of the set, i.e., H is homogeneous if $N(x) \setminus H = N(y) \setminus H$ for all $x, y \in H$. A homogeneous set is proper if it is neither a single vertex nor the set of all vertices.

We next state a useful result by Olariu [29] without proof. See Fig. 4.1 for diagrams of the subgraphs: house, hole, domino, A and P.

Lemma 4.2. [29] *A graph G is HHDA-free ({house, hole, domino, A}- free) if and* *only if each induced subgraph of G is chordal or contains a proper homogeneous set.*

The following lemmas are useful in dealing with homogeneous sets.

Lemma 4.3. [15] *A vertex v of a graph G is semisimplicial in G if and only if each connected component of the complement of $\langle N(v)\rangle$ is homogeneous in G.*

Proof. Suppose v is a centre vertex of an induced $P_4 : v_1, v, v_2, v_3$. Vertices v_1 and v_2 are adjacent (and thus in the same component C) in the complement of $\langle N(v)\rangle$. Since v_3 is adjacent to v_2 but not v_1, and $v_3 \notin N(v)$, C is not homogeneous in G. This establishes the necessity.

Let B be a connected component of the complement of $\langle N(w)\rangle$ for some vertex w, and suppose that $V(B)$ is not a homogeneous set in G. Then there exist vertices $x, y \in V(B)$ and $z \notin V(B)$ such that z is adjacent to x but not to y. Choose such vertices x, y, z such that $d(x, y)$ in B is minimized. Every vertex in $N(w) \setminus V(B)$ is adjacent in G to every vertex in B. Thus $z \notin N(w)$. If x is not adjacent to y in G, then $\{z, x, w, y\}$ induces a P_4 for which w is a centre vertex, and the sufficiency holds. Assume then that x is adjacent to y in G. Let $x, u_1, u_2, ..., u_k, y$ be a shortest x, y-path in B, for $k \geq 1$. Since u_1 is not adjacent to x in G, u_1 is not adjacent to z, as otherwise this would contradict our choice of x, y, and z since $d_B(u_1, y) < d_B(x, y)$. Vertex w is a centre vertex of the $P_4 : z, x, w, u_1$, which establishes the sufficiency. □

Lemma 4.4. [15] *Let H be a proper homogeneous set of vertices of a graph G. Let $T = V(G) \setminus (H \setminus \{v\})$ for some $v \in H$. If $x \in T$ is semisimplicial in $\langle T\rangle$, but not in G, then $x = v$.*

Proof. Let H be a proper homogeneous set of vertices of a graph G. Let $T = V(G) \setminus (H \setminus \{v\})$ for some $v \in H$. Suppose x is a vertex which is semisimplicial in $\langle T\rangle$, but not in G. Let P^* be an induced P_4 in G (but not in T) for which x is a centre vertex.

An induced P_4 may intersect a homogeneous set at 0, 1, or all 4 vertices. Suppose $x \notin H$. Since $x \in T$ is semisimplicial in $\langle T \rangle$ but not G, exactly one vertex of P^* is in H. However, if this vertex is replaced with v then x is a midpoint of this new P_4 in $\langle T \rangle$, giving a contradiction. Therefore, $x \in H$, i.e., $x = v$. □

Lemma 4.5. [15] *Let H be a proper homogeneous set of vertices of a graph G. Let $T = V(G) \setminus (H \setminus \{v\})$ for some $v \in H$. If $x \in H$ is semisimplicial in $\langle H \rangle$, but not in G, then no vertex of H is semisimplicial in G, and v is not semisimplicial in $\langle T \rangle$.*

Proof. Let H be a proper homogeneous set of vertices of a graph G. Let $T = V(G) \setminus (H \setminus \{v\})$ for some $v \in H$. Suppose $x \in H$ is a vertex which is semisimplicial in $\langle H \rangle$, but not in G. Let $P^* : w, x, y, z$ be an induced P_4 in G, but not $\langle H \rangle$.

Since $x \in H$ is semisimplicial in $\langle H \rangle$ and since an induced P_4 may intersect a homogeneous set at 0, 1, or all 4 vertices, $P^* \cap H = \{x\}$. Since H is homogeneous, $\{w, u, y, z\}$ induces a P_4 in G for all $u \in H$. Thus, no $u \in H$ is semisimplicial in G. The path w, v, y, z is induced in $\langle T \rangle$, therefore v is not semisimplicial in $\langle T \rangle$. □

The graphs for which the m^3-convexity produces a convex geometry can be characterized by forbidden subgraphs.

Theorem 4.6. [15] *For a graph G, the following are equivalent:*

1. *G is HHDA-free ({house, hole, domino, A}-free).*
2. *For every induced subgraph F of G, every non-semisimplicial vertex of F lies on an induced path of length at least 3 between two semisimplicial vertices of F.*
3. *The m^3-convex sets of vertices of G form a convex geometry.*

Proof. (1) $\implies$ (2):

Suppose G is a HHDA-free graph on n vertices. If $n \leq 4$, then either G is the P_4 graph, in which case the implication holds, or G is not the P_4 graph and does not contain a non-semisimplicial vertex. For our inductive hypothesis, suppose $n > 4$ and that (1) implies (2) for all graphs on fewer than n vertices. If G contains only semisimplicial vertices, then the result holds vacuously. Suppose x is a non-semisimplicial vertex in G, and thus a centre vertex of an induced P_4: u, x, v, w.

First, suppose that G is chordal.

Claim I: There exist vertices u_i, $i = 1, 2, ..., s$, and w_j, $j = 1, 2, ..., t$, for $s, t \geq 1$, such that $u_1, u_2, ..., u_s, x, y, w_t, ..., w_2, w_1$ is an induced path in G and both u_1 and w_1 are simplicial (and thus semisimplicial).

If both u and w are simplicial, then we are done. Suppose then that u is the centre vertex of an induced P_3. Let M be the convex hull of $\{u, x, v, w\}$ with respect to the m-convexity (note that this is not with respect to the m^3-convexity). Let S be the set of neighbours of u in M, i.e., $S = N_M(u)$. Every u, v-path in $\langle M \rangle$ contains a vertex of S. This is also true in G since all vertices of each chordless u, v-path of G are contained in M.

Since $\langle M \rangle$ is chordal but not complete, it follows from Lemma 3.5 that $\langle M \rangle$ contains two non-adjacent simplicial vertices. The vertices x and v are clearly not simplicial in $\langle M \rangle$. Moreover, no vertices of $M \setminus \{u, x, v, w\}$ are simplicial in $\langle M \rangle$, as otherwise, if $z \in M \setminus \{u, x, v, w\}$ is simplicial in $\langle M \rangle$ then $M \setminus \{z\}$ is an m-convex set of smaller cardinality that contains $\{u, x, v, w\}$. By elimination, u and w must be the only non-adjacent simplicial vertices of $\langle M \rangle$. Thus $\langle S \rangle$ is complete. Recall, however, that u is not simplicial in G. Let K be the component of $\langle V(G) \setminus S \rangle$ which contains vertex u, and let R be the induced subgraph of G on the vertices of $K \cup S$. The subgraph R is chordal but not complete and therefore, by Lemma 3.5, it contains at least two non-adjacent simplicial vertices, at most one of which is in S. Let u_1 be a simplicial vertex in K. Note that u_1 is also simplicial in G.

Let $P : u_1, u_2, ..., u_s, x$ be a chordless path connecting u_1 and x in R. Then

$P^* : u_1, u_2, ..., u_s, x, v$ is an induced path. A symmetrical argument holds if w is not simplicial. Thus, there exist vertices u_i, $i = 1, 2, ..., s$, and w_j, $j = 1, 2, ..., t$, such that $s, t \geq 1$, and $u_1, u_2, ..., u_s, x, v, w_t, ..., w_2, w_1$ is an induced path in G, and both of the vertices u_1 and w_1 are simplicial, and thus semisimplicial. So (2) follows if G is chordal.

Now suppose that G is not chordal. By Lemma 4.2, G contains a proper homogeneous set H.

Case 1: $x \in H$.

Let $T = V(G) \setminus (H \setminus \{x\})$. If x is semisimplicial in $\langle T \rangle$ then, by Lemma 4.5, x is not semisimplicial in $\langle H \rangle$. By the inductive hypothesis, x lies on an induced path of length at least 3 between two vertices $y, z \in H$ that are semisimplicial in $\langle H \rangle$. By Lemma 4.5, y and z are semisimplicial in G. So (2) follows in this case.

If x is not semisimplicial in $\langle T \rangle$ then, by the inductive hypothesis, x lies on an induced path of length at least 3 between two vertices $y, z \in T$ that are semisimplicial in $\langle T \rangle$. By Lemma 4.4, y and z are also semisimplicial in G. So (2) follows in this case.

Case 2: $x \notin H$.

Then, by Lemma 4.4, x is not semisimplicial in $\langle S \rangle$ for $S = V(G) \setminus (H \setminus \{v'\})$, for v' some vertex in H. By the inductive hypothesis, x lies on an induced path of length at least 3 between two vertices $y, z \in S$ that are semisimplicial in $\langle S \rangle$. If y is not semisimplicial in G then, by Lemma 4.4, $y = v'$. However, if $y = v'$ then, by Lemma 4.5, y is not semisimplicial in $\langle S \rangle$, a contradiction. The identical argument applies to vertex z. Thus both y and z are semisimplicial in G. So (2) follows in this case as well.

(2) $\implies$ (3): Suppose statement (2) holds. Let X be an m^3-convex subset of the vertices of G. The extreme vertices of X are precisely the semisimplicial vertices of

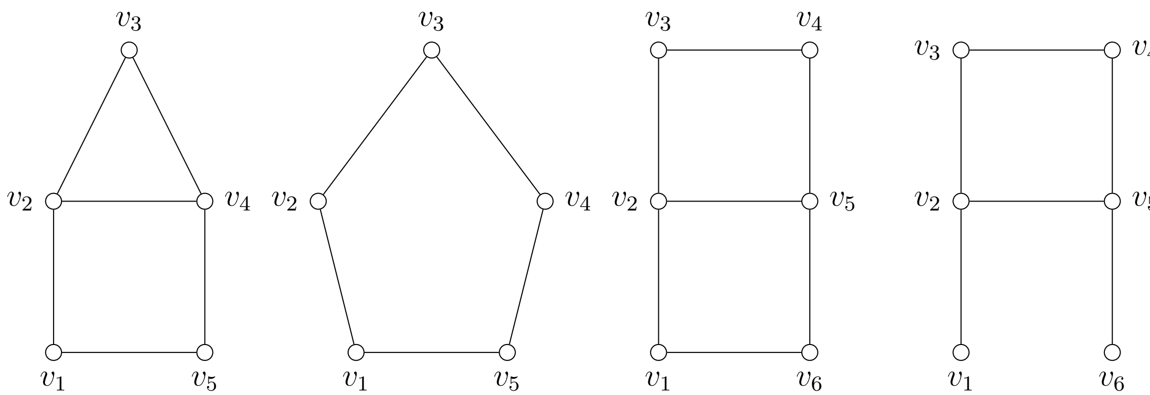


Figure 4.2: From left to right: A labelled house, hole (of size 5), domino and A graph.

$\langle X \rangle$. Thus, $x \in CH(ex(X))$ for all vertices $x \in X$ that are semisimplicial in $\langle X \rangle$. The non-extreme vertices of X are precisely the non-semisimplicial vertices of $\langle X \rangle$. Every non-semisimplicial vertex $v \in X$ lies on an induced path of length at least 3 between two vertices $u, w \in X$ that are semisimplicial in $\langle X \rangle$. Since $u, w \in CH(ex(X))$ and $v \in I_{m^3}[u, w]$, $X \subseteq CH(ex(X))$. By closure under intersection, $X = CH(ex(X))$; i.e., the Minkowski-Krein-Milman property holds, and the m^3-convex sets of G form a convex geometry.

(3) $\implies$ (1): Suppose the m^3-convexity of a graph G is a convex geometry. Since the extreme vertices of an m^3-convex set are precisely the semisimplicial vertices, by Theorem 2.3, G has a semiperfect elimination ordering. Since all vertices of the induced hole and domino subgraphs (see Fig. 4.2) are non-semisimplicial, G does not contain either as an induced subgraph.

Consider the house as labelled in Fig. 4.2. Let $X = CH(\{v_1, v_2, v_3, v_4, v_5\})$. Vertex v_3 is the only vertex that is semisimplicial within the house subgraph, therefore $ex(X) \subseteq \{v_3\}$. Thus, with respect to the m^3-convexity, $CH(ex(X)) = ex(X) \neq X$, i.e. the Minkowski-Krein-Milman property does not hold. From this contradiction we conclude that G is house-free, and thus HHD-free.

We have established that (3) implies that G is HHD-free. To make the case that G

is A-free requires considerably more detail. We proceed by induction on $n = |V(G)|$. For a graph G on 6 or fewer vertices, either G is A-free and the implication holds vacuously, or G is the A graph. In the latter case, for the A labelled as in Fig. 4.2, $CH(ex(\{v_1, v_2, v_3, v_4, v_5, v_6\})) = CH(\{v_1, v_6\}) = \{v_1, v_2, v_5, v_6\}$ and the Minkowski-Krein-Milman property does not hold; a contradiction. Thus (3) implies (1) for graphs on 6 or fewer vertices. For our inductive hypothesis, suppose (3) implies (1) for all graphs on fewer than n vertices.

Case 1: G contains a proper homogeneous set H.

Let $v \in H$, let $T = V(G) \setminus (H \setminus \{v\})$, and let $S \subseteq T$ be an m^3-convex set in G. If $v \in S$ then let $S' = S \cup H$. If $v \notin S$ then let $S' = S$.

Claim I: S' is m^3-convex in G.

Suppose S' is not m^3-convex in G. Then there exist vertices $x, y \in S'$ and an induced x, y-path P of length at least 3 such that some vertex of P is not in S'. Since S is m^3-convex in G, $S' = S \cup H$ and at most one of the vertices x, y is in S. Without loss of generality, suppose $x \in H$. If $P \cap H = \{x\}$, then the path induced by $(V(P) \setminus \{x\}) \cup \{v\})$ is a chordless of path of length at least 3 with both endpoints in S; a contradiction. Thus $|V(P) \cap H| \geq 2$. Let x_1 be the vertex of P closest to x (in P) that is not in H. Since H is a homogeneous set and x_1 is adjacent to some vertex of H, xx_1 is an edge. Since P is a chordless path, x and x_1 are adjacent in P. Let $x_2 \neq x$ be the vertex of P closest to x_1 in P that is in H. Since P is a chordless path, x_1 and x_2 are adjacent in P. Let $P : x, x_1, x_2, x_3, ..., x_{t-1}, x_t = y$ for some $t \geq 3$. If $x_3 \in H$ then x_1x_3 is an edge, and if $x_3 \notin H$ then xx_3 is an edge. Since P is chordless, both cases give a contradiction. Thus Claim I follows.

Since, by Claim I, S' is m^3-convex in G, by Theorem 2.3 there is an ordering $\alpha : v_1, v_2, ..., v_k$ of the vertices of $V(G) \setminus S'$ such that v_i is semisimplicial in $\langle \{v_i, v_{i+1}, ..., v_k\} \cup S' \rangle$ for $i = 1, 2, ..., k$. Let $\beta : u_1, u_2, ..., u_j$ be the vertex ordering of the vertices of $T \setminus S$ such that β is a subordering of α. Then u_i is semisimplicial in $\langle \{u_i, u_{i+1}, ..., u_j\} \cup S \rangle$ for $i = 1, 2, ..., j$. Thus, by Theorem 2.3, S is m^3-convex in $\langle T \rangle$. By the inductive hypothesis, $\langle T \rangle$ is HHDA-free.

If we let $S \subseteq H$ be an m^3-convex set in G, we may apply a similar argument to show that $\langle H \rangle$ is HHDA-free. Since H is a proper homogeneous set, we find by inspection that at most one vertex of the A subgraph may be contained in H. However, if $a \in H$ is a vertex of an A subgraph then a may be replaced with v to find an A subgraph in $\langle T \rangle$; a contradiction. Thus, in this case, G is HHDA-free.

Case 2: G does not contain a proper homogeneous set.

Suppose G contains an A as labelled in Fig. 4.2. Let v be a semisimplicial vertex of G.

Claim II: Vertex v is simplicial in G.

Suppose that v is the centre vertex of an induced $P_3 : u_1, v, u_2$. Vertices u_1 and u_2 are in the same component C of the complement of $\langle N(v) \rangle$. From Lemma 4.3 it follows that C is homogeneous in G; a contradiction since G does not contain a proper homogeneous set. Thus Claim II follows.

Claim III: If t is a simplicial vertex in G, then any m^3-convex set of $G - t$ is m^3-convex in G.

Let t be a simplicial vertex. The neighbourhood of t induces a clique, thus t is not an internal vertex of any induced path of length at least 3 and Claim III follows.

From Claims II and III it follows that the m^3-convexity of $G - v$ is a convex geometry. The graph $G - v$ is, by the inductive hypothesis, HHDA-free. Since G is HHD-free but contains an A subgraph, and since v is simplicial in G, either $v = v_1$ or $v = v_6$ (for the A subgraph as labelled in Fig. 4.2). Moreoever, v_1 and v_6 are the only semisimplicial (and thus simplicial) vertices of G.

Claim IV: Any common neighbour of v_1 and v_6 is also a neighbour of all other vertices of the A subgraph.

If v_1 and v_6 have a common neighbour z, then z is adjacent to both v_2 and v_5, as otherwise $\{z, v_1, v_2, v_5, v_6\}$ induces a house or hole. Vertex z is adjacent to at least one of the vertices v_3 or v_4, as otherwise $\{z, v_2, v_3, v_4, v_5\}$ induces a house. If z is adjacent to v_4 but not v_3, then $\{z, v_1, v_2, v_3, v_4\}$ induces a house; and if z is adjacent to v_3 but not v_4 then $\{z, v_3, v_4, v_5, v_6\}$ induces a house. Thus z is adjacent to both v_3 and v_4. Thus Claim IV follows.

Claim V: $N(v_1) \subseteq N(v_3)$ and $N(v_6) \subseteq N(v_4)$.

Suppose some vertex w is adjacent to v_1 but not v_3. By Claim IV, w is not adjacent to v_6. Since v_1 is simplicial, w is adjacent to v_2. Vertex w is adjacent to at least one of the vertices v_4 or v_5, as otherwise $\{w, v_2, v_3, v_4, v_5, v_6\}$ induces an A in $G - v_1$. If w is adjacent to v_5, then w is also adjacent to v_4, as otherwise $\{w, v_2, v_3, v_4, v_5\}$ induces a house. Thus wv_4 is an edge. However, now $\{w, v_1, v_2, v_3, v_4\}$ induces a house, a contradiction. Thus $N(v_1) \subseteq N(v_3)$ and, by symmetry, $N(v_6) \subseteq N(v_4)$.

Claim VI: Every vertex in $N(v_1)$ is adjacent to every vertex in $N(v_6)$.

If $w \in N(v_1) \cap N(v_6)$ then, since v_1 and v_6 are both simplicial, w is adjacent to all vertices in $N(v_1) \cup N(v_6)$. Suppose then that $z \in N(v_1) \setminus N(v_6)$, $w \in N(v_6) \setminus N(v_1)$, and suppose z and w are not adjacent. Suppose $z = v_2$. Then $w \neq v_5$ and, since v_6 is simplicial, wv_5 is an edge. By Claim V, wv_4 is an edge. Vertex w is adjacent to v_3, as otherwise $\{w, v_2, v_3, v_4, v_5\}$ induces a house. However, now $\{w, z, v_3, v_5, v_6\}$ induces a house, a contradiction, thus $z \neq v_2$ and, by symmetry, $w \neq v_5$.

Since v_1 and v_6 are simplicial, zv_2 and wv_5 are edges. It follows from Claim V that zv_3 and wv_4 are edges. Vertex z is adjacent to at least one of the vertices v_4 or v_5, as otherwise $\{z, v_2, v_3, v_4, v_5\}$ induces a house. If z is adjacent to v_4, then z is also adjacent to v_5, as otherwise $\{z, v_1, v_2, v_4, v_5\}$ induces a house. Thus, zv_5 is an edge. By symmetry, wv_2 is an edge. Vertices w and v_3 are not adjacent, as otherwise

$\{w, z, v_3, v_5, v_6\}$ induces a house. By symmetry, vertices z and v_4 are not adjacent. However, now the vertex set $\{w, z, v_3, v_4, v_5\}$ induces a house, a contradiction. Thus Claim VI follows.

Claim VII: $M = N[v_1] \cup N[v_6]$ is m^3-convex in G.

Let $M = N[v_1] \cup N[v_6]$ and suppose M is not m^3-convex in G. Let $w, z \in M$ be vertices such that there is a chordless w, z-path P of length at least 3 with an internal vertex of P not in M. Since v_1 and v_6 are simplicial, and since every vertex in $N(v_1)$ is adjacent to every vertex in $N(v_6)$, $\{w, z\} \cap \{v_1, v_6\} \neq \emptyset$. Without loss of generality, suppose that $z = v_1$. Let z' be the neighbour of z in P. If $w \in N(v_1)$ or if $w \in N(v_6)$, then w is adjacent to z'; a contradiction. Thus $w = v_6$. Let w' be the neighbour of w in P. Then $z'w'$ is an edge and every vertex of P is in M; a contradiction. Thus Claim VII follows.

Since $M = N[v_1] \cup N[v_6]$ is m^3-convex in G, by Theorem 2.3 there exists an ordering $u_1, u_2, ..., u_k$ of the vertices of $V(G) \setminus M$ such that u_i is semisimplicial in $\langle \{u_i, u_{i+1}, ..., u_k\} \cup M \rangle$ for $i = 1, 2, ..., k$. However, since $v_3, v_4 \in V(G) \setminus M$ and both are non-semisimplical vertices in $\langle \{v_1, v_2, v_3, v_4, v_5, v_6\} \rangle$, no such ordering exists. Thus G is A-free.

We finally conclude that G is HHDA-free and (3) implies (1). □

4.2 Semisimplicial Elimination and Ordering Algorithms

Jamison and Olariu [23] were the first to show that every LexBFS ordering of a graph G is semisimplicial if and only if G is HHD-free, and that any MCS ordering of any induced subgraph of G is semisimplicial if and only if G is HHP-free. These results were later again obtained by Dragan, Nicolai and Brandstädt [15] using their results on convexity and it is their approach that we use in the proof of the following theorem.

Theorem 4.7. [15] *If* $v_1, v_2, ..., v_n$ *is a LexBFS ordering of the vertices of an HHD-free graph* G*, then* $\{v_i, v_{i+1}, ..., v_n\}$ *is an* m^3*-convex set for each* $i = 1, 2, ..., n$*.*

Proof. Let G be an HHD-free graph and $v_1, v_2, ..., v_i, v_{i+1}, ..., v_n$ a LexBFS ordering of the vertices. We proceed by induction on i. When $i = n$ the statement is true as $\{v_n\}$ is an m^3-convex set. Suppose that $\{v_j, v_{j+1}, ..., v_n\}$ is m^3-convex for all $j > i$, but that $\{v_i, v_{i+1}, ..., v_n\}$ is not m^3-convex. Then there exists a vertex $y \in V(G_{i+1})$, such that there is an induced v_i, y-path of length at least 3 with at least one internal vertex not in $V(G_i)$. From the shortest of these paths choose P to be the one for which y has the largest label. Let x be the neighbour of y in P.

Case 1: $x < v_i$.

Suppose $P = v_i, u_1, u_2, ..., u_t, x, y$. Since $x < v_i < y$, and xy is an edge and $v_i y$ is not an edge, by P1 there exists a vertex $z > y$ such that zv_i is an edge and xz is not an edge. Suppose that $z = u_1$. Since by the inductive hypothesis $\{v_{i+1}, ..., v_n\}$ is an m^3-convex set, and $z, u_2, u_3, ..., u_t, x, y$ is a chordless path such that $z, y \in V(G_{i+1})$ and $x \notin V(G_{i+1})$; such a z, y-path must have length less than 3, i.e., $P : v_i, z, x, y$. However, x and z are not adjacent, therefore $z \neq u_1$. By the inductive hypothesis, $Q = z, v_i, u_1, u_2, ..., u_t, x, y$ cannot be an induced path, as $z, y \in V(G_{i+1})$ and $x \notin V(G_{i+1})$. However, since P is an induced path, any chord of Q must be incident to z. Let u_r, $1 \leq r \leq t$, be the vertex closest to y on P that is adjacent to z. Then zy is an edge since $z, u_r, u_{r+1}, ..., u_t, x, y$ cannot be an induced path. Since G is hole-free, $u_t z$ must be a chord (i.e. $r = t$). If $t = 1$, $\{x, y, z, v_i, u_1\}$ induces a house. If $t = 2$, $\{x, y, z, v_i, u_1, u_2\}$ induces a house or domino. If $t \geq 3$, $\{x, y, z, v_i, u_1, u_2, ..., u_t\}$ induces a house, domino or hole. All values of t lead to a contradiction.

Case 2: $x > v_i$.

By our choice of P to have minimum length, $P = v_i, w, x, y$ and w is the only vertex of P not in $V(G_i)$. Since $w < v_i < x$, and wx is an edge and $v_i x$ is not an edge, by P1 there exists a vertex $v > x$ such that vv_i is an edge and vw is not an edge. Let v be the largest such vertex. Since, by the inductive hypothesis, $\{v_{i+1}, v_{i+2}, ..., v_n\}$ is m^3-convex, vx is an edge. Since G does not contain a house, vy is not an edge.

Suppose that $y < v$. Since $v_i < y < v$, and $v_i v$ is an edge and yv is not an edge, by P1 there exists a vertex $u > v$ such that yu is an edge and $v_i u$ is not an edge. If uw is an edge, then P1 would imply the existence of some vertex $t > u > v$ adjacent to v_i but not w, contradicting our maximum choice of v. Therefore, uw is not an edge. Since $y < v < u$, so not to contradict the inductive hypothesis, the path v, x, y, u is not induced. Either or both of the chords uv and ux are present. If both chords are present, $\{v, v_i, w, x, u\}$ induces a house. If only uv is present, $\{v, v_i, w, x, y, u\}$ induces a domino. Therefore ux is an edge and uv is not. However, since $u > y$, this is a contradiction to our original choice of a maximum y.

Suppose then that $v < y$. Since $x < v < y$, and xy is an edge and vy is not an edge, by P1 there exists a vertex $s > y$ such that sv is an edge and sx is not an edge. If sv_i were an edge then, so not to contradict the inductive hypothesis, the paths s, v_i, w, x, y and s, w, x, y are not induced. As a result, sw and sy are edges; however, then $\{v_i, w, x, y, s\}$ would induce a house. Hence, sv_i is not an edge. If sw is an edge then, as in the previous case ($y < v$), P1 would imply a vertex which contradicts our choice of v as the largest vertex adjacent to v_i but not w. Thus, sw is not an edge. By the inductive hypothesis, sy is an edge. However, now $\{v_i, w, x, y, s, v\}$ induces a domino, leading again to a contradiction. □

Corollary 4.8. [23][15] *Every LexBFS ordering of a graph G is semisimplicial if and only if G is HHD-free.*

Proof. Suppose G contains an induced house. Referring to Fig. 4.3, choose vertex z

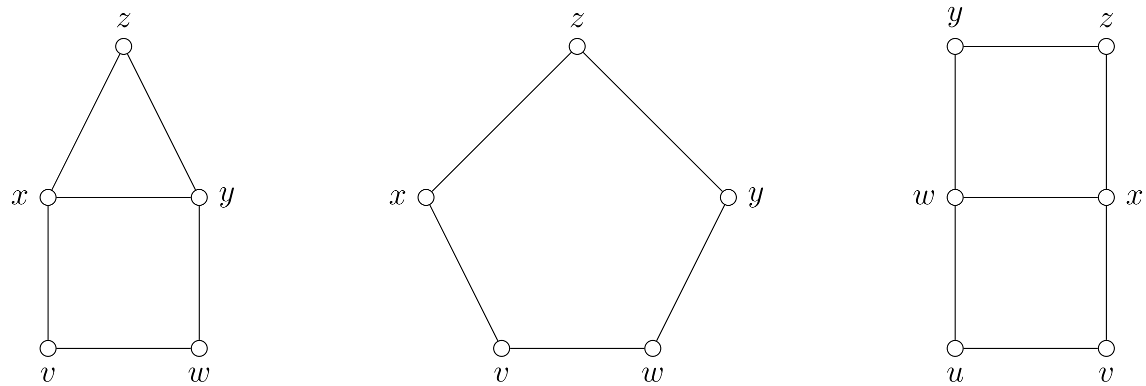


Figure 4.3: From left to right: A house, hole (of size 5) and domino graph labelled: $(u), v, w, x, y, z$ by LexBFS such that the vertex labelled v (or u) is the centre vertex of an induced P_4.

to receive the first label n of a LexBFS ordering. Any neighbour of n may receive the next label $n-1$, so choose vertex y. Once these two vertices have been labelled, the order in which the remaining vetices of the house are labeled is forced. Thus, $i = v < w < x < y < z$ for some integer i such that $1 \leq i \leq n-4$. Then v is the centre of the induced $P_4 : w, v, x, z$ in G_i, a contradiction.

Suppose G contains a hole or domino subgraph. Since every vertex of the hole or domino is the centre of an induced P_4, the vertex of the hole or domino that receives the smallest label i of a LexBFS ordering is the centre of an induced P_4 in G_i, a contradiction. This establishes the sufficiency.

Let $v_1, v_2, ..., v_n$ be a LexBFS ordering of an HHD-free graph G. By Theorem 4.7, $\{v_i, v_{i+1}, ..., v_n\}$ is m^3-convex for $i = 1, 2, ..., n$. Thus, v_i is an extreme vertex of $\{v_i, v_{i+1}, ..., v_n\}$ for $i = 1, 2, ..., n$. Since the extreme vertices of the m^3-convex sets are precisely the semisimplicial vertices, $v_1, v_2, ..., v_n$ is a semisimplicial elimination ordering. This establishes the necessity. □

Theorem 4.9. [15] *If $v_1, v_2, ..., v_n$ is an MCS ordering of the vertices of an HHP-free graph G, then $\{v_i, v_{i+1}, ..., v_n\}$ is an m^3-convex set for each $i = 1, 2, ..., n$.*

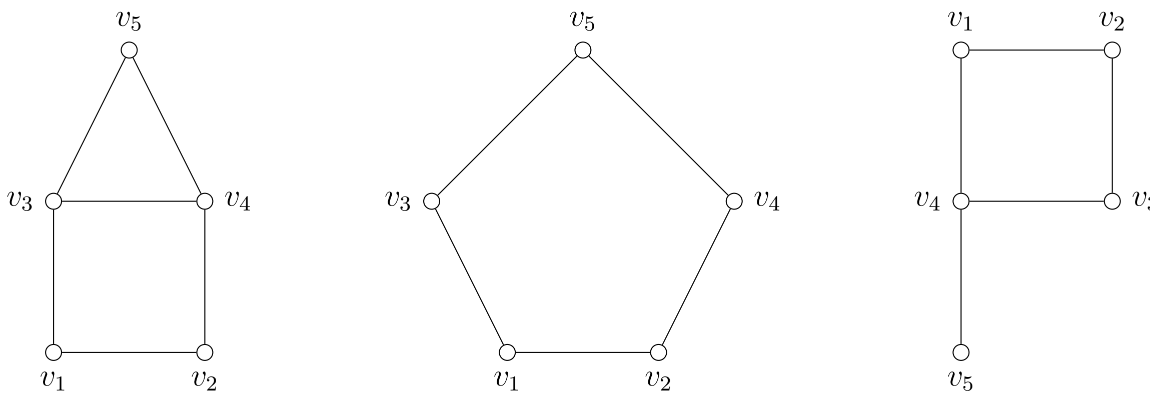


Figure 4.4: From left to right: House, hole (of size 5) and P graphs labelled by MCS such that the vertex labelled v_1 is the centre vertex of an induced P_4.

The proof of this theorem is very similar to that of Theorem 4.7, with P2 replacing P1, and the subgraph P replacing the domino. As such we will omit it.

Corollary 4.10. [23][15] *Any MCS ordering of any induced subgraph F of G is a semisimplicial elimination ordering of F if and only if G is HHP-free.*

Proof. The house, hole and P subgraphs shown in Fig. 4.4 may be ordered as v_1, v_2, v_3, v_4, v_5 by the MCS algorithm. The vertex labelled v_1 is the centre of an induced P_4 in all cases. This establishes the sufficiency.

Let $v_1, v_2, ..., v_n$ be an MCS ordering of an HHP-free graph G. By Theorem 4.9, $\{v_i, v_{i+1}, ..., v_n\}$ is m^3-convex for $i = 1, 2, ..., n$. Thus, v_i is an extreme vertex of $\{v_i, v_{i+1}, ..., v_n\}$ for $i = 1, 2, ..., n$. Since the extreme vertices of the m^3-convex sets are precisely the semisimplicial vertices, $v_1, v_2, ..., v_n$ is a semisimplicial elimination ordering. This establishes the necessity. □

It follows from Corollary 4.10 that if G contains a house, hole or P, then there is an MCS ordering of some induced subgraph of G (namely the house, hole or P) that is not a semisimplicial ordering. In fact, if G contains a house or hole, then there exists an MCS ordering of the vertices of G that is not semisimplicial. However, there

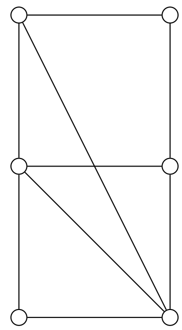


Figure 4.5: A graph containing a P as a subgraph and for which every MCS ordering of its vertices is semisimplical.

exist graphs G which contain P as a subgraph for which every MCS ordering of the vertices of G is semisimplicial. Fig. 4.5 is an example of such a graph.

The algorithms LexBFS and MCS generate perfect elimination orderings of chordal graphs; however, they do not generate all of them, as noted in Chapter 3. For a chordal graph, α is a perfect elimination ordering if and only if α is an MCC or MEC ordering (Theorem 3.2). We now ask: For what class of graphs do MCC and MEC generate semisimplicial elimination orderings?

We define the term *increasing path* for use in the proof that follows. A path $p_1, p_2, ..., p_k$ is an increasing path if $p_1 < p_2 < ... < p_k$ for a fixed ordering of the vertices.

Lemma 4.11. [30] *Let G be an HHP-free graph and let α be a vertex ordering generated by MEC or MCC . Suppose that a, b, c and d are vertices such that $a < b < c$; $a < d$; ab, ac, and bd are edges; and bc and ad are not edges. Then cd is an edge.*

Proof. Let G be HHP-free and let α be a vertex ordering generated by MEC or MCC. Let a, b, c and d be vertices such that $a < b < c$; $a < d$; ab, ac, and bd are edges; and bc and ad are not edges. If α is a semisimplicial elimination ordering, then vertex a is not the centre of an induced P_4, therefore cd is an edge and we are done. Assume then that α is not a semisimplicial elimination ordering and there exist vertices a,

b, c, and d which satisfy the given properties but cd is not an edge. Choose such a set of vertices $\{a, b, c, d\}$ such that the vertex labels are as large as possible, first considering a, then c, then b, and finally d.

Claim I: Vertices b and c have no common neighbour e such that $a < e$ and ae is not an edge.

Suppose, to the contrary, that b and c have a common neighbour e such that $a < e$ and ae is not an edge. If ed is an edge, then $\{a, b, c, d, e\}$ induces a house. If ed is not an edge, then $\{a, b, c, d, e\}$ induces a P. Either case gives a contradiction.

Claim II: $b < d$.

Since $a < b < c$, ac is an edge and bc is not an edge, and a and b are in the same unlabelled component, by P3 there exists a vertex $d' > b$ such that bd' is an edge and ad' is not an edge. It follows from Claim I that cd' is not an edge. Vertex d' must be vertex d, so not to contradict our choice of d. Thus, $b < d$.

We define B to be the set containing precisely the vertices x for which there is an increasing b, x-path such that vertex a is not adjacent to any vertices of the path (with the exception of vertex b), i.e., $x \in B$ if and only if there is an increasing path $b = u_0, u_1, ..., u_s = x$ such that $au_i \notin E(G)$ for $i = 1, 2, ..., s$. We define the set of vertices C in an identical way, i.e., $y \in C$ if and only if there is an increasing path $c = w_0, w_1, ..., w_t = y$ such that $aw_i \notin E(G)$ for $i = 1, 2, ..., t$. Note that $b, d \in B$ and $c \in C$.

Let b' and c' be the vertices of largest label in B and C, respectively. Let $b = b_0, b_1, ..., b_p = b'$ and $c = c_0, c_1, ..., c_q = c'$ be increasing chordless paths that are not adjacent to vertex a (with the exception of vertices b and c). If cb_1 was an edge, then $\{a, b, b_1, c, d\}$ would either induce a house (if db_1 was an edge) or P subgraph (if db_1 was not an edge). Suppose $i \geq 2$ is the smallest value for which cb_i was an edge. Then $\{a, b, b_1, b_2, ..., b_i, c, \}$ would induce a hole. Thus, vertex c is non-adjacent to b_i

for $i = 0, 1, ..., p$. If $b' < c$, then property P3 applied to vertices a, b' and c would imply the existence of a vertex b^*, adjacent to b' but not to a such that $b' < b^*$. This would contradict that b' is the largest labelled vertex in B; therefore, $c < b'$.

Claim III: $C \neq \{c\}$.

Let k be the smallest value such that $c < b_k$. Since $b_{k-1} < c < b_k$, $b_{k-1}b_k$ is an edge and cb_k is not an edge, and b_{k-1} and c are in the same unlabelled component, by P3 there exists a vertex $x > c$ such that cx is an edge and $b_{k-1}x$ is not an edge. If x is not adjacent to a, then $x \in C$ and we are done. Assume then that xa is an edge. If xd is an edge, then xb is also an edge, as otherwise $\{a, b, c, d, x\}$ induces a house. If xd is not an edge, then xb is still an edge so not to contradict our choice of c. Let j, $0 \leq j \leq k-1$, be the largest value such that xb_j is an edge. However, b_j now contradicts our choice of a (for $b_j = a$, $x = b$, $b_{j+1} = c$ and $c = d$). Thus, x is not adjacent to a and $x \in C$.

Claim IV: $B \cap C \neq \emptyset$.

If v is adjacent to y for some $v \in B$ and $y \in C$, then either $v \in C$ or $y \in B$ and we are done. Assume then that none of the vertices of B are adjacent to those of C.

Recall that $b < c < b'$. Without loss of generality, suppose that $b' < c'$. Let i be the smallest value such that $b' < c_i$. Since $c_{i-1} < b' < c_i$, $c_{i-1}c_i$ is an edge and $b'c_i$ is not an edge, and c_{i-1} and b' are in the same unlabelled component, by P3 there exists a vertex $b'' > b'$ such that $b'b''$ is an edge and $c_{i-1}b''$ is not an edge. This vertex b'' is adjacent to a so not to contradict the maximality of b'.

If c_1b'' is an edge, then cb'' is an edge as otherwise $\{a, c, c_1, b'', b'\}$ induces a P. If c_1b'' is not an edge, then cb'' is still an edge as to not contradict our original choice of vertices b, c and d (by choosing c, b'', and c_1, respectively). Let j, $0 \leq j < i-1$ be the largest value for which $b''c_j$ is an edge. However, c_j now contradicts our choice of a (for $c_j = a$, $c_{j+1} = b$, $b'' = c$, $c_{j+2} = d$). Therefore, there is an edge vy for some

$v \in B$ and $y \in C$ and, thus, $B \cap C \neq \emptyset$.

Let w be the smallest vertex in $B \cap C$. Let P_B be an increasing chordless b, w-path and let P_C be an increasing chordless c, w-path. Since a is not adjacent to w, it follows from Claim I that w is adjacent to at most one of the vertices b or c. Thus, vertices a, b and c together with P_B and P_C induce a hole. From this final contradiction we conclude that cd is an edge. □

Theorem 4.12. [30] *Any MEC or MCC ordering of any induced subgraph F of G is a semisimplicial elimination ordering of F if and only if G is HHP-free.*

Proof. The vertices of the house, hole and P subgraphs shown in Fig. 4.4 may be ordered as v_1, v_2, v_3, v_4, v_5 by the MEC and MCC algorithms. The vertex labelled v_1 is the centre of an induced P_4 in all cases. This establishes the sufficiency.

Let $\alpha : v_1, v_2, ..., v_n$ be an MEC or MCC ordering of an HHP-free graph G, and suppose α is not a semisimplicial elimination ordering. Let v_i be a vertex in the ordering that is not semisimplicial in G_i.

Let P^* be an induced P_4 in G_i that contains v_i as a centre vertex. Without loss of generality, let the neighbours of v_i in P^* be r and s such that $r < s$. Let $t \in V(G_i)$ be the vertex such that $\langle \{v_i, r, s, t\} \rangle = P^*$. If t was the neighbour of r in P^* then, by Lemma 4.11, t and s would also be adjacent; a contradiction since P^* is an induced P_4. Thus, $P^* : r, v_i, s, t$. However, since $v_i < r < s$, $v_i s$ is an edge and rs is not an edge, and v_i and r are in the same unlabelled component, by P3 there exists a vertex $u > r$ such that ur is an edge and uv_i is not an edge. When applied to vertices v_i, r, s, u (respectively as a, b, c, d in the lemma), it follows from Lemma 4.11 that u is adjacent to s. However, $\{v_i, r, s, t, u\}$ now induces a house (if ut is an edge) or a P (if ut is not an edge). This contradiction establishes the necessity. □

We now consider graph characterizations based on the LexDFS algorithm and

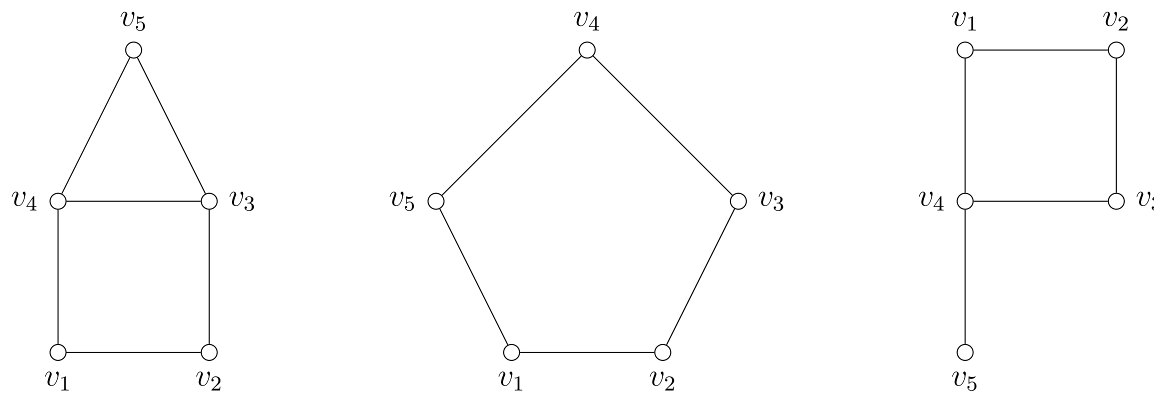


Figure 4.6: From left to right: A house, hole (of size 5) and P labelled by LexDFS such that the vertex labelled v_1 is the centre vertex of an induced P_4.

semisimplicial elimination orderings. This is a new result. Note that the characterization is identical to those for MEC, MCC and MCS.

Theorem 4.13. *Any LexDFS ordering of any induced subgraph F of G is a semisimplicial elimination ordering of F if and only if G is HHP-free.*

Proof. The vertices of the house, hole and P subgraphs shown in Fig. 4.6 may be ordered as v_1, v_2, v_3, v_4, v_5 by the LexDFS algorithm. The vertex labelled v_1 is the centre of an induced P_4 in all cases. This establishes the sufficiency.

Let $\alpha : v_1, v_2, ..., v_n$ be a LexDFS ordering of an HHP-free graph G, and suppose α is not a semisimplicial elimination ordering. Let P^* be an induced P_4 in G_i that contains v_i as a centre vertex. Without loss of generality, let the neighbours of v_i in P^* be x and w such that $x < w$. Since $v_i < x < w$, and v_iw is an edge and xw is not an edge, by P4 there exists a vertex y_1 such that $x < y_1 < w$, and xy_1 is an edge and v_iy_1 is not an edge. Choose y_1 to be such a vertex of maximum label.

Suppose wy_1 is an edge. Since v_i is a centre vertex of an induced P_4 (containing vertices x and w) in G_i, either x or w has a neighbour $z \in V(G_i)$ such that $\{z, x, w, v_i\}$ induces a P_4. Suppose z is adjacent to w. If zy_1 is not an edge, then

$\{z, w, v_i, x, y_1\}$ induces a P. If zy_1 is an edge, then $\{z, w, v_i, x, y_1\}$ induces a house. From this contradiction it follows that wy_1 is not an edge.

Since $v_i < y_1 < w$, and $v_i w$ is an edge and $y_1 w$ is not an edge, by P4 there exists a vertex y_2 such that $y_1 < y_2 < w$, and $y_1 y_2$ is an edge and $v_i y_2$ is not an edge. Choose y_2 to be such a vertex of maximum label. Vertex y_2 is not adjacent to x, as this would contradict our choice of y_1. Vertex y_2 is also not adjacent to w, as otherwise $\{w, v_i, x, y_1, y_2\}$ would induce a hole.

This pattern may be followed for $j = 2, 3, 4, \ldots$. Since $v_i < y_j < w$, and $v_i w$ is an edge and $y_j w$ is not an edge, by P4 there exists a vertex y_{j+1} such that $y_j < y_{j+1} < w$, and $y_j y_{j+1}$ is an edge and $v_i y_{j+1}$ is not an edge. Choose y_{j+1} to be such a vertex of maximum label. Vertex y_{j+1} is not adjacent to x, as this would contradict our choice of y_1. For $k = 1, 2, \ldots, j-1$, vertex y_{j+1} is not adjacent to y_k, as this would contradict our choice of y_{k+1}. Finally, vertex y_{j+1} is not adjacent to w, as otherwise $\{w, v_i, w, y_1, y_2, \ldots, y_{j+1}\}$ would induce a hole. This contradiction establishes the necessity. □

4.3 Concluding Remarks

For G an HHP-free graph, every MCS, MEC, MCC and LexDFS ordering of $V(G)$ is semisimplicial, as shown in this chapter. However, there exist HHP-free graphs with semisimplicial elimination orderings that may be obtained by LexDFS but not by MCS or MCC. The true-twin C_4 graph of Fig. 4.7 provides an example of such a LexDFS ordering $(v_1, v_2, v_3, v_4, v_5)$.

As noted in Chapter 3, a graph is chordal if and only if it has a perfect elimination ordering. The graphs which have a semisimplicial elimination ordering have not yet been characterized. We have seen that the HHD-free and HHP-free graphs have

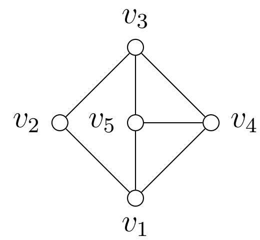
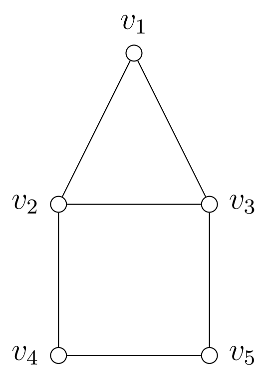
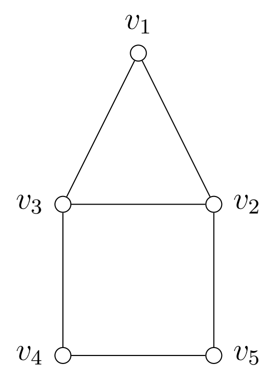


Figure 4.7: A true-twin C_4 labelled by LexDFS.

semiperfect elimination orderings. However, the house is an example of a graph that is not HHD or HHP-free, but does have semiperfect elimination orderings. Fig. 4.8 gives examples of semisimplicial elimination orderings of the house that may be generated by LexBFS or LexDFS (and both orderings may also be generated by any of MCS, MCC or MEC).



(a) A LexBFS (and MCS, MCC, MEC) ordering of the house.



(b) A LexDFS (and MCS, MCC, MEC) ordering of the house.

Figure 4.8: The ordering v_1, v_2, v_3, v_4, v_5 of the house is a semiperfect elimination ordering in both (a) and (b).

The m^3-convexity has a geodesic analog, the g^3-convexity. The g^3-convexity was defined by Nielsen and Oellermann [27] in 2009, but has not been studied subsequently. A subset X of vertices is defined to be *g^3-convex* if for any two vertices $u, v \in X$ such that $d(u, v) \geq 3$, all vertices on a u, v-geodesic are also contained in X. A vertex v is *weakly semisimplicial* if it is either semisimplicial or, if v is the centre of an induced P_4, then the end vertices of the P_4 are distance 2 apart. The weakly semisimplicial vertices are precisely the extreme vertices of the g^3-convex sets [27].

Chapter 5

Elimination Orderings of Distance Hereditary Graphs

Recall that distance hereditary graphs are connected graphs for which every connected induced subgraph is isometric, i.e., distances are preserved. We saw in Chapter 3 that the class of graphs for which the family of g-convex sets form a convex geometry are precisely the Ptolemaic graphs, i.e., the chordal distance hereditary graphs. The class of distance hereditary graphs was first characterized by Howorka [20], who showed that a graph is distance hereditary if and only if every cycle of length at least 5 has at least two crossing chords. Distance hereditary graphs can also be characterized by forbidden subgraphs. A graph is distance hereditary if and only if it is {house, hole, domino, 3-fan}-free [3]. Because the distance hereditary graphs are a subclass of the HHD-free graphs, it follows from Corollary 4.8 that every LexBFS ordering of a distance hereditary graph is semisimplicial.

In this chapter we will see that distance herediatry graphs can be characterized in terms of elimination orderings. The first section focuses on the characterization of Bandelt and Mulder [3], namely that a graph is distance herediatry if and only if it has a P-elimination ordering, where P is the property of being a leaf or twin. In the second section we explore interesting connections between LexBFS orderings and distance herediatry graphs, due to Dragan and Nicolai [15].

5.1 A Characterization of Distance Hereditary Graphs by Elimination Orderings

A vertex v is a *true twin* of a vertex u if $N[v] = N[u]$, and a *false twin* if $N(v) = N(u)$. When this difference is not important, u and v are referred to as *twins*. Notice that if u and v are twins in G then $G - u \cong G - v$. We say that a graph has been obtained by *splitting* a vertex u of a graph G when it is obtained by adding a new vertex v to $V(G)$, as well as the edges vw for all $w \in N(u)$, so that u and v are twins. The edge uv may or may not be added, i.e., we allow v to be either a true twin or a false twin of u.

A *leaf* or *pendant vertex* is a vertex of degree 1. We say that a graph has been obtained by *attaching a pendant vertex* at vertex u of a graph G when it is obtained by adding a new vertex v to $V(G)$, and the edge uv. Thus, v is a leaf in the resulting graph.

Distance hereditary graphs can be classified by the following elimination ordering. A graph is distance hereditary if and only if its vertices can be ordered $v_1, v_2, ..., v_n$ such that v_i is a leaf or a twin in G_i for all $i = 1, 2, ..., n$. This follows from the next theorem.

Theorem 5.1. [3] *A non-trivial graph G is distance hereditary if and only if G is obtained from K_2 by a sequence of operations of either attaching a pendant vertex or splitting a vertex of the previous graph in the sequence.*

Proof. Let G be a graph obtained from K_2 by a sequence of operations of either attaching a pendant vertex or splitting a vertex of the previous graph in the sequence. The graph K_2 is distance hereditary. For our inductive hypothesis, assume G', the graph on $n-1 \geq 2$ vertices which directly precedes G in the sequence, is distance hereditary .

Suppose G is obtained by attaching a pendant vertex to some vertex of G'. Clearly G will also be distance hereditary.

Suppose G is obtained by splitting a vertex $u \in V(G')$. Let u^* be the vertex of G that is not in G'. We want to show that distance is preserved for every connected induced subgraph of G.

Let H be a connected induced subgraph of G and $x, y \in V(H)$. If $u^* \notin V(H)$, then H is a subgraph of G', and thus, by the inductive hypothesis, H is distance hereditary. Since $G - u^* \cong G' \cong G - u$, it follows that if H does not contain both u and u^*, then $d_H(x, y) = d_G(x, y)$. Suppose $u, u^* \in V(H)$. Since $N(u) \setminus \{u^*\} = N(u^*) \setminus \{u\}$, the only shortest x, y-path in H containing both u and u^* is the u, u^*-path. If u and u^* are true twins, then $d_H(u, u^*) = d_G(u, u^*) = 1$. If u and u^* are false twins, then $d_H(u, u^*) = d_G(u, u^*) = 2$. Any other shortest x, y-path in H either contains neither of the vertices u, u^*, or exactly one of them. Thus G is distance hereditary. This establishes the necessity.

We now prove a result stronger than the sufficiency: *Every non-trivial distance hereditary graph contains either two leaves or a pair of twin vertices.* Clearly the statement is true for K_2. Let G be a distance hereditary graph on $n \geq 3$ vertices. For the inductive hypothesis, assume that every distance hereditary graph on fewer than n vertices contains either two leaves or a pair of twin vertices.

Case 1: G contains at least 2 leaves.

The statement is true.

Case 2: G contains exactly one leaf.

Let vertex z' be the leaf attached to vertex z. The graph $G - z'$ contains at most one leaf. By the inductive hypothesis, $G - z'$ contains a pair of twin vertices x and x'. Suppose that $G - z'$ contains exactly one leaf, z. If z had a twin vertex in $G - z'$, then that vertex would be a leaf in G. Therefore z is neither x nor x' and these two vertices are twins in G.

Suppose then that $G - z'$ has no leaves. If z is neither x nor x', then these vertices are twins in G and we are done. So assume that $z = x$. Consider now the graph $G - x'$, in which z' is a leaf. If there exist two neighbours of x' in G that are leaves in $G - x'$, then two such vertices are twins in G and we are done.

Suppose that no neighbours of x' in G are leaves in $G - x'$. Then z' is the only leaf of $G - x'$ and, by the inductive hypothesis, $G - x'$ contains a pair of twin vertices y and y'. The vertex z is the only vertex adjacent to z' so $z \neq y, y'$. The vertex x' is either adjacent to both y and y' or neither vertex (since z and x' are twins in $G - z'$). Therefore y and y' are twins in G.

Now suppose that exactly one neighbour x^* of x' in G is a leaf in $G - x'$. Then $G - x' - x^*$ has exactly one leaf, z', and therefore, by the inductive hypothesis, contains twin vertices w and w'. In G, neither w nor w' is adjacent to x^*. By the properties of twins, if w is adjacent to x', then it is adjacent to z, and thereby w' is adjacent to z, and thus also to x'. Thus w and w' are twins in G.

Case 3: G contains no leaves and there exists a vertex $z \in V(G)$ such that $G - z$ has at least 2 leaves.

Let u' and v' be leaves in $G - z$, adjacent to vertices u and v respectively. Suppose u' and v' belong to the same component H of $G - z$. Since $d_G(u', v') = 2$ and G is distance hereditary, $d_H(u', v') = 2$, implying that $N(u') = N(v') = \{z, u\}$, i.e., $u = v$ and thus u' and v' are twins in G and we are done. Assume then that all leaves of $G - z$ belong to different components. Let H be the component containing the leaf

u', adjacent to $u \in V(H)$. The subgraph H has at least two vertices, exactly one leaf, and is distance hereditary; therefore, by the inductive hypothesis, H contains a pair of twin vertices x and x'. Note that vertices x, x' are distinct from u, u'. If z is adjacent to both x and x', or neither vertex, then x and x' are twins in G and we are done. Assume then that z is adjacent to x but not x'. Let H' be the subgraph induced by the vertices $V(H) \cup \{z\}$, and note that H' has no leaves. By the inductive hypothesis, H' contains a pair of twin vertices y and y'. If $z \neq y$ or y', then y and y' are twins in G and we are done. Assume then that $z = y$. Since z and u are the only vertices in H' adjacent to u', we conclude that $y' = u$. Since z and u are twins in H', then u and x are adjacent. Now x' is adjacent to u since x and x' are twins in H. However, z and x' must be adjacent, since z and u are twins in H', a contradiction to our assumption that z is adjacent to x but not x'.

Case 4: G contains no leaves and for every vertex $z \in V(G)$ the subgraph $G - z$ has at most 1 leaf.

For every vertex $z \in V(G)$, each component of $G - z$ contains at least three vertices, as otherwise $G - z$ has at least two leaves. By the inductive hypothesis, each component of $G - z$ contains a pair of twin vertices. Note that these twin pairs do not include a leaf (if present), nor the vertex to which any leaf is adjacent. If any of these twin pairs are twins in G, then we are done. Assume then that for every pair of twin vertices in a component of $G - z$, z is adjacent to exactly one vertex of the twin pair. Choose z to be a vertex of maximum degree in G, and let u and u' be twins in $G - z$ such that z is adjacent to u' but not u.

Subcase 4.1: Vertices u and u' are adjacent.

Then $d(u, z) = 2$ and, since G contains no leaves, u, and thus also u', is adjacent to some vertex distinct from z and u'. By the inductive hypothesis, let v and v' be twins in $G - u$ such that u is adjacent to v' but not v.

Suppose that $v' = u'$. If $v = z$, then $v' = u'$ would have degree one greater than the degree of z, a contradiction to our choice of z; therefore, $v \neq z$. Since v and v' are twins in $G - u$, v must be adjacent to z. Since v and u are not adjacent in G, and u and u' are twins in $G - z$; v and $v' = u'$ are also not adjacent. Since G has no leaves, there is a vertex y, distinct from $v' = u'$, that is adjacent to u. By properties of twins, y is adjacent to both $v' = u'$ and v. However, now the vertices $\{u, v' = u', z, v, y\}$ induce either a house (if zy is not an edge) or a 3-fan (if zy is an edge), both forbidden subgraphs of G.

Suppose then that $v' \neq u'$. Then, by properties of twins, u' is adjacent to v', and thus also to v. Since v is adjacent to u' but not u, $v = z$. However, now the degree of v' is one greater than the degree of $z = v$, a contradiction to our choice of z.

Subcase 4.2: Vertices u and u' are not adjacent.

By the inductive hypothesis, let v and v' be twins in $G-u$ such that u is adjacent to v' but not v. By properties of twins, u' is also adjacent to v'. Since v and v' are twins in $G - u$, either $v = u'$ or $v = z$.

Suppose that $d(u, z) = 3$. This implies that v' is not adjacent to z. In this case $v = z$ so not to contradict the fact that v and v' are twins in $G - u$. However, now the degree of v' is one greater than the degree of $z = v$, a contradiction to our choice of z.

Suppose that $d(u, z) = 2$. Then z and u (and thus also u') have a common neighbour. In fact, z must be adjacent to all neighbours of u since G is {HHD, 3-fan}-free; thus, z is adjacent to v'. Suppose that v and u' are distinct, i.e., $z = v$. However, since v and v' are twins in $G - u$, the degree of v' is one greater than the degree of $z = v$, a contradiction to our choice of z. Thus, $v = u'$. By the inductive hypothesis there exist twins w and w' in $G - u'$ such that u' is adjacent to w' but not w. Either $w' \neq v'$ is adjacent to u, or $w' = z$. Suppose the former. Since z is adjacent to all neighbours of u, z is adjacent to w', implying that w and u are distinct. Then, by properties of twins, w is adjacent to u, and thus also to u', a contradiction. Assume then that $w' = z$. If $u = w$ then, by properties of twins, the degree of v' is one greater than the degree of z, a contradiction to our choice of z. So $u \neq w$. Twin properties imply that w is adjacent to v', and thereby also adjacent to $v = u'$, a contradiction.

Thus, every non-trivial distance hereditary graph contains either two leaves or a pair of twin vertices. This establishes the sufficiency. □

Since a distance hereditary graph may be obtained from K_2 by successively either attaching a pendant vertex or splitting a vertex of the previous graph in the sequence, one may dismantle a distance hereditary graph by successively removing leaves and twins.

Corollary 5.2. [3] *A graph G is distance hereditary if and only if G is connected and has a P-elimination ordering, where P is the vertex property of being a leaf or a twin.*

Proof. Let G be a non-trivial distance hereditary graph. It follows from Theorem 5.1 that G can be obtained from K_2 by a sequence of operations of either attaching a pendant vertex or splitting a vertex at each step. For a fixed such sequence, let v_n and v_{n-1} be the vertices of the K_2, and let v_{n-2} be the first new vertex added in the sequence, v_{n-3} the next vertex added after v_{n-2}, and so on, until v_1 is the last vertex added. Then $v_1, v_2, ..., v_n$ is a P-elimination ordering, where P is the vertex property of being a leaf or a twin. This establishes the sufficiency.

Suppose now that G is a connected graph with a P-elimination ordering α : $v_1, v_2, ..., v_n$, where P is the vertex property of being a leaf or a twin. The vertex v_{n-1} is a leaf attached to v_n in G_{n-1}, i.e., $G_{n-1} \cong K_2$. For $i = n-2, n-3, ..., 2, 1$, if v_i is a leaf in G_i we may obtain G_i from G_{i+1} by attaching the pendant vertex v_i to a vertex of G_{i+1} and if v_i is a twin in G_i we may obtain G_i by splitting a vertex of G_{i+1}. Thus, by Theorem 5.1, G is distance hereditary. This establishes the necessity. □

5.2 Characterizations of Distance Hereditary Graphs by LexBFS Orderings

A *cograph* is a graph that does not contain an induced P_4. A vertex v is 2-*simplicial* if $D(v, 2)$, the collection of all vertices u such that $d(u, v) \leq 2$, induces a cograph. If a vertex is 2-simplicial then it is also semisimplicial, however the converse is not true. The apex of the 3-fan (the vertex labelled v_5 in Fig. 3.2) provides an example of a vertex that is semisimplicial but not 2-simplicial.

Theorem 5.3. [26] *A graph is distance hereditary if and only if it has a* 2*-simplicial elimination ordering.*

Proof. Any graph that is not distance hereditary contains, as a subgraph, a house, hole, domino or 3-fan (see Fig. 4.1 and 3.2) [3]. These forbidden subgraphs can be inspected to see that none contain a 2-simplicial vertex. This establishes the necessity.

Let G be a distance hereditary graph on $n \geq 3$ vertices. For our inductive hypothesis, suppose every distance hereditary graph on fewer than n vertices has a 2-simplicial elimination ordering, and notice that this is true for K_2. By Theorem 5.1, G contains a leaf u attached to a vertex u', or a pair of twins u, u'. By the inductive hypothesis, $G - u$ has a 2-simplicial elimination ordering. Let $v_1, v_2, ..., v_{n-1}$ be a 2-simplicial elimination ordering of the vertices of $G - u$, and suppose $u' = v_i$.

Case 1: Vertex u is a leaf attached to vertex v_i.

Suppose $D(u, 2)$ does not induce a cograph, i.e., the graph induced by $D(u, 2)$ contains an induced P_4. Since u is a leaf, the induced P_4 is contained in $N(v_i) - u$.

However, this induced P_4 together with vertex v_i induces a 3-fan, a contradiction. Thus, $D(u,2)$ induces a cograph and the ordering $u, v_1, v_2, ..., v_{n-1}$ is 2-simplicial.

Case 2: Vertices u and v_i are twins.

Then $D(u,2) = D(v_i,2)$, so the ordering $v_1, v_2, ..., v_{i-1}, u, v_i, ..., v_{n-1}$ is 2-simplicial.

□

Recall that G^2 is the graph with vertex set $V(G)$ such that two vertices u, v are adjacent in G^2 precisely when $d_G(u,v) \leq 2$. Dragan and Nicolai [16] show that for a distance hereditary graph G, every LexBFS ordering of the vertices of G is a perfect elimination ordering of G^2.

Lemma 5.4. [16] *Every LexBFS ordering of a distance hereditary graph G is a perfect elimination ordering of G^2.*

Proof. Suppose $\alpha : v_1, v_2, ..., v_n$ is a LexBFS ordering of a distance hereditary graph G. Since G is HHD-free, by Corollary 4.8, α is a semisimplicial elimination ordering of G. Suppose that α is not a perfect elimination ordering of G^2. Let v_i be a vertex in the ordering that is not simplicial in G_i^2, and let x and y be vertices such that x, v_i, y is an induced P_3 in G_i^2. Then $x, y \in D_{G_i}(v_i, 2)$ and $d_{G_i}(x,y) \geq 3$. At most one of the vertices x, y is adjacent to v_i in G_i.

Suppose x, v_i, u, y is a path in G_i. Since α is a semisimplicial elimination ordering the path is not induced; however, the presence of any additional edge violates the property $d_{G_i}(x,y) \geq 3$. Therefore, $x, y \in {N_{G_i}}^2(v_i)$. Let a and b be vertices in $N(v_i)$ adjacent, respectively, to x and y in G_i. Since $d_{G_i}(x,y) \geq 3$, none of ay, bx, nor xy are edges in G. However, ab is an edge of G, as otherwise v_i is a center vertex of the induced $P_4 : x, a, v_i, b$ in G_i. Without loss of generality assume that $a < b$. By Lemma 4.1, $V(G_j)$ is m^3-convex in G for $j = 1, 2, ..., n$. For this reason, and since x, a, b, y is an induced path in G, $min\{x, y\} < a$.

Case 1: $x < a < y$.

Since $v_i < x < b$, and v_ib is an edge and xb is not an edge, by P1 there exists a vertex $z > b$ such that zx is an edge and zv_i is not an edge. Since $a < b < z$, by m^3-convexity the path b, a, x, z cannot be induced, i.e., z is adjacent to a or b. If zb is an edge and za is not, then $\{v_i, a, b, x, z\}$ induces a house. If both za and zb are edges, then $\{v_i, a, b, x, z\}$ induces a 3-fan. If za is an edge and zb is not, then m^3-convexity implies that z, a, b, y is not an induced path, i.e., zy is an edge. However, then $\{v_i, a, b, z, y\}$ induces a house in G. All possibilities lead to a contradiction.

Case 2: $y < a < x$.

Since $v_i < y < a$, and v_ia is an edge and ya is not an edge, by P1 there exists a vertex $w > a$ such that wy is an edge and wv_i is not an edge. By m^3-convexity, the path a, b, y, w cannot be induced, i.e., w is adjacent to either a or b. If wa and wb are both edges, then $\{v_i, a, b, w, y\}$ induces a 3-fan. If wa is an edge and wb is not, then $\{v_i, a, b, w, y\}$ induces a house. If wb is an edge and wa is not, then m^3-convexity implies that x, a, b, w is not an induced path, i.e., xw is an edge. However, then $\{v_i, a, b, x, w\}$ induces a house in G. All possibilities lead to a contradiction.

Case 3: $x, y < a$.

Since $v_i < x, y < a < b$, and v_ia and v_ib are edges and neither xa nor yb is an edge, by P1 there exist vertices $t > a$ and $u > b$ such that tx and uy are edges and neither tv_i nor uv_i is an edge. By m^3-convexity, the path t, x, a, b cannot be induced, i.e., t is adjacent to a or b. If t is adjacent to both a and b, then $\{a, b, x, t, v_i\}$ induces a 3-fan. If t is adjacent to only b, then $\{a, b, x, t, v_i\}$ induces a house. Therefore t is adjacent to a and not b. Likewise, u is adjacent to b and not a. By m^3-convexity, t, a, b, u cannot be an induced path, i.e., tu is an edge. However, then $\{a, b, t, u, v_i\}$ induces a house. All possibilities lead to a contradiction.

It follows that α is a perfect elimination ordering of G^2. $\square$

Lemma 5.5. [16] *For a distance hereditary graph G, x is 2-simplicial in G if and only if x is simplicial in G^2.*

Proof. Let G be a distance hereditary graph.

Suppose that x is not simplicial in G^2. Let y and z be vertices such that y, x, z is an induced P_3 in G^2. Since $d_G(y, z) \geq 3$, at most one of y, z is adjacent to x in G.

Case 1: $y \in N_G(x)$ and $z \in N^2{}_G(x)$.

Let v be a vertex in $N_G(x)$ such that y, x, v, z is a path in G. Since $d_G(x, y) \geq 3$, y is not adjacent to v or z. Now y, x, v, z is an induced P_4 in G, and x is not 2-simplicial in G.

Case 2: $y, z \in N^2{}_G(x)$.

Let y' and z' denote the neighbours of y and z, respectively, in $N_G(x)$. Since $d_G(y, z) \geq 3$, y is not adjacent to z' and z is not adjacent to y'. If $y'z'$ is not an edge in G, then x is not 2-simplicial in G and we are done. Assume then that $y'z'$ is an edge in G. However, now y, y', z', z is an induced P_4 in $D_G(x, 2)$, i.e., x is not 2-simplicial in G. This establishes the sufficiency.

Now suppose that x is not 2-simplicial in G. Let q, r, s, t be an induced P_4 in $D_G(x, 2)$. Since G is distance hereditary, $d_G(q, t) = 3$. This implies that vertex x is distinct from both q and t, and that q and t are not adjacent in G^2. Since $q, t \in D_G(x, 2)$, vertex x is adjacent to both q and t in G^2. Thus, q, x, t is an induced P_3 in G^2 and x is not simplicial in G^2. □

Theorem 5.6. [16] *For a distance hereditary graph G and an ordering α of the vertices, α is a 2-simplicial elimination ordering of G if and only if α is a perfect elimination ordering of G^2.*

Proof. For any ordering $\alpha : v_1, v_2, ..., v_n$ of a distance hereditary graph G, the subgraphs G_i are distance hereditary for $i = 1, 2, ..., n$. The result now follows directly from Lemma 5.5. □

Corollary 5.7. [16] *A graph G is distance hereditary if and only if every LexBFS ordering of G is 2-simplicial.*

Proof. Let G be a distance hereditary graph and α a LexBFS ordering of G. By Lemma 5.4 and Theorem 5.6, α is a 2-simplicial ordering of G. This establishes the sufficiency.

Let G be a graph for which every LexBFS ordering of G is 2-simplicial. By Theorem 5.3, G is distance hereditary. □

5.3 Concluding Remarks

As established in this chapter, a distance hereditary graph has a P-elimination ordering, where P is the vertex property of being a leaf or a twin. Such a {leaf or twin}-elimination ordering may be computed for a distance hereditary graph in linear time [13]. For various optimization problems, a greedy algorithm may proceed along the {leaf or twin}-elimination ordering of a distance hereditary graph to more efficiently solve the problem for the graph. Some examples of such optimization problems include:

(1) The maximum weighted independent set problem. Given a weighted graph, find an independent set (no two vertices adjacent) of maximum weight.

(2) The minimum weighted ab-separator problem. Given a weighted graph with two fixed vertices a and b, find a subset of the vertices of minimum weight such that every a, b-path contains a vertex of the subset.

(3) Computing the diameter of a graph.

(4) Computing the average distance of a graph. Given a graph, find the average distance over all pairs of vertices.

For a distance hereditary graph G and a corresponding {leaf or twin}-elimination ordering, algorithms for problems (1), (2), and (3) can run in $O(n)$ time for $n = |V(G)|$ [9], and an algorithm for problem (4) can run in $O(m)$ time for $m = |E(G)|$ [28].

Chapter 6

3-Steiner Simplicial ($3SS$) Elimination Orderings

In this chapter we consider graph convexities defined in terms of intervals between three or more vertices, which generalize the geodesic and monophonic intervals between pairs of vertices. In Sections 6.1 and 6.2 we characterize convex geometries for these graph convexities and in Section 6.3 we consider graphs with P-elimination orderings where P is the property that characterizes the extreme vertices with respect to these convexities.

6.1 k-Steiner Convexity

Recall that for a connected graph G and a subset U of its vertices, a *Steiner tree* for U is a connected subgraph of G with a smallest number of edges that contains U. The *Steiner distance* of U, denoted by $d(U)$, is the number of edges in a Steiner tree for U. If $U = V(G)$, finding a Steiner tree is equivalent to finding a minimum spanning tree. If $|U| = 2$, the Steiner tree problem is the shortest path problem.

A set X of vertices is *k-Steiner convex*, or *kS-convex*, if the vertices of all Steiner

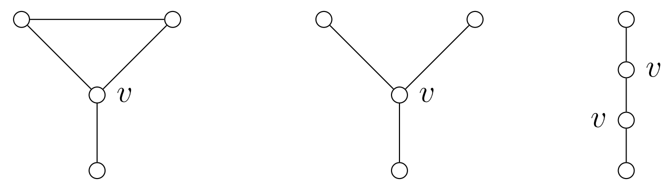


Figure 6.1: From left to right: A paw, claw, and P_4. Vertices labelled v are not 3-Steiner simplicial.

trees for every k-subset of X are also in X. More formally, let $U \subseteq V(G)$. Then the *Steiner interval of* U, denoted by $I_S[U]$, is the set of all vertices on a Steiner tree for U. A set X of vertices is *k-Steiner convex* precisely when $I_S[U] \subseteq X$ for all k-subsets U of X. Geodesic convexity is equivalent to 2-Steiner convexity.

For a graph G, the empty set and $V(G)$ are clearly both kS-convex sets and the properties of kS-convexity imply closure under intersection. Thus, when $\mathcal{M}$ is the family of all kS-convex subsets of the vertices, $(V(G), \mathcal{M})$ is necessarily a convex space.

For the remainder of this section we focus on the $3S$-convexity of graphs. A set X of vertices is $3S$-convex when, for each triple of vertices in X, the vertices of each smallest tree containing the triple is also in X. A vertex which is not a center vertex of an induced claw, paw, or P_4 (see Fig. 6.1) is called *3-Steiner simplicial*, or $3SS$. The extreme vertices of the $3S$-convex sets are precisely the $3SS$ vertices.

Theorem 6.1. [7] *For a $3S$-convex set X of a connected graph G, a vertex $t \in X$ is an extreme point of X if and only if t is $3SS$ in $\langle X \rangle$.*

Proof. Let X be a $3S$-convex set of a connected graph G, and let t be a vertex in X.

Suppose t is not $3SS$ in $\langle X \rangle$, i.e., t is the center of an induced claw, paw or P_4 on vertices s, t, u, v in X. The Steiner distance of $\{s, u, v\}$ in the claw, paw, or P_4 induced by $\{s, t, u, v\}$ is 3. Since these subgraphs are induced, there is at most one edge joining vertices s, u, and v in X. Accordingly, the Steiner distance of $\{s, u, v\}$

in X is 3. If t were an extreme point of X, then the Steiner distance of $\{s, u, v\}$ in $X - t$ would be strictly less than 3. Inspection of the claw, paw, and P_4 subgraphs show this is a contradiction, establishing the sufficiency.

Let w, x, y and z be distinct vertices of a $3S$-convex set X. Suppose that w is both $3SS$ in $\langle X \rangle$ and a vertex of a Steiner tree T of $\{x, y, z\}$, i.e., w is not an extreme point of X. Since w is not a leaf of T, w must be a cut vertex of $\langle V(T) \rangle$; therefore $\langle V(T) \setminus \{w\} \rangle$ has at least two components. If $\langle V(T) \setminus \{w\} \rangle$ has at least three components, then w is the center of an induced claw or paw in $\langle X \rangle$, a contradiction. Therefore $\langle V(T) \setminus \{w\} \rangle$ has exactly two components, T_1 and T_2. Notice that $V(T_1) \cup V(T_2) \subseteq V(T) \subseteq X$.

Without loss of generality, assume $x \in T_1$ and $y, z \in T_2$ and that w_1 and w_2 are neighbours of w in T_1 and T_2 respectively. Since T_2 contains at least two vertices let w_2 be adjacent to a vertex y_2 in T_2. Suppose wy_2 is an edge in G. Since w is not the center of an induced paw, w_1 must be adjacent to w_2 or y_2. However, removing w from T and adding the edge w_1w_2 or w_1y_2 results in a tree of smaller size than T that contains $\{x, y, z\}$, a contradiction to the fact that T is a Steiner tree for $\{x, y, z\}$. Thus, neither w_1w_2 nor w_1y_2 are edges and w is a center vertex of a P_4 or paw induced by $\{w_1, w, w_2, y_2\}$. This contradiction establishes the necessity. □

As we have seen with other types of convexity, the graphs for which the $3S$-convexity produces a convex geometry may be characterized by forbidden subgraphs. The *replicated twin* C_4 is such a subgraph, and is shown in Fig. 6.2. Note that there are two optional edges in the subgraph, and thus four subgraphs which we refer to as replicated twin C_4's.

Theorem 6.2. [27] *A graph G is $\{P_4$, replicated twin $C_4\}$-free if and only if the $3S$-convexity of G is a convex geometry.*

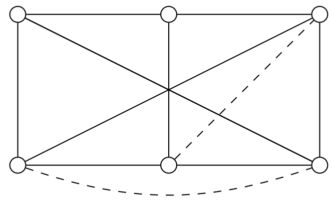


Figure 6.2: A replicated twin C_4 graph. Any subset of the dashed edges may be present.

Proof. For a graph G, let $(V(G), \mathcal{M})$ be the convex space, where $\mathcal{M}$ is the set of all $3S$-convex subsets of $V(G)$.

Suppose G contains an induced $P_4 : u, v, w, x$. Let R be the $3S$-convex hull of $\{u, v, w, x\}$. Since v and w are centre vertices of an induced P_4 in R, it follows from Theorem 6.1 that the set T of $3SS$ vertices of R is some subset of $\{u, x\}$. However, since T does not contain at least three vertices, $CH(ex(R)) = CH(T) = T \neq R$. The Minkowski-Krein-Milman property does not hold, therefore $(V(G), \mathcal{M})$ is not a convex geometry.

Suppose now that G contains H, an induced replicated twin C_4. Note, by inspection, that every vertex of a replicated twin C_4 is the centre of an induced claw or paw. Thus H contains no $3SS$ vertices. Let S be the $3S$-convex hull of the vertices of H. By Theorem 6.1, S contains no $3SS$ vertices. Thus, $CH(ex(S)) = CH(\emptyset) = \emptyset \neq S$. The Minkowski-Krein-Milman property does not hold, therefore $(V(G), \mathcal{M})$ is not a convex geometry. These two cases establish the necessity.

Suppose G is $\{P_4$, replicated twin $C_4\}$-free, but that $(V(G), \mathcal{M})$ is not a convex geometry. Fix G to be such a graph on a minimum number of vertices. Note that since G is P_4-free, G must also be {house, hole, domino, 3-fan}-free, and thus distance hereditary. Also, since G is P_4-free, $diam(G) \leq 2$. If $diam(G) = 1$, then G is a complete graph, $(V(G), \mathcal{M})$ is a convex geometry and we are done. Assume then that $diam(G) = 2$.

For H a connected induced subgraph of G, we will temporarily use the notation $CH_H(X)$ to denote the $3S$-convex hull of a set of vertices $X \subseteq V(H)$ within H. Since we chose G to have a minimum number of vertices, for any proper connected induced subgraph H of G, $CH_H(ex(V(H))) = V(H)$, as otherwise H meets the criteria set for G and has fewer vertices. It can be shown, by inspection, that $(V(G), \mathcal{M})$ is a convex geometry for any P_4-free graph on four or fewer vertices. Thus we assume G has at least five vertices.

Case 1: G has a universal vertex v.

Subcase 1.1: $G - v$ is connected.

Let H be the subgraph $G - v$. Let w be an extreme point of $V(H)$. As such, w is not the centre of a claw or paw in H. Vertex v is universal so it is not a peripheral vertex of a claw or paw in G, thus w is $3SS$ in G as well as H. For this reason, every extreme point of $V(H)$ is also an extreme point of $V(G)$ and $CH(ex(V(H))) \subseteq CH(ex(V(G)))$. If v is an extreme point of $V(G)$, then $CH(ex(V(G))) = V(G)$ and $(V(G), \mathcal{M})$ is a convex geometry. If v is not an extreme point of $V(G)$, then v is the centre vertex of a claw or paw whose three peripheral vertices x, y and z are in H. In this case $v \in CH(ex(V(G))) = V(G)$ since v is in the Steiner interval of x, y and z. Thus, $(V(G), \mathcal{M})$ is a convex geometry.

Subcase 1.2: $G - v$ has at least two components.

Let $H_1, H_2, ..., H_k$ for $k \geq 2$ be the components of $G - v$. By our choice of G, $CH_{H_i}(ex(V(H_i))) = V(H_i)$ for $i = 1, 2, ..., k$. As in Subcase 1.1, since v is a universal vertex, every extreme point of H_i is also an extreme point of G. Thus $CH(ex(V(H_i)))$ is either $V(H_i)$ or $V(H_i) \cup \{v\}$ for $i = 1, 2, ..., k$. If, for some i, $|V(H_i)| \geq 2$, then H_i contains at least two extreme vertices; as otherwise the Minkowski-Krein-Milman property would not hold for $V(H_i)$. Suppose some H_i contains at least two vertices and has extreme points x and y. Let z be an extreme point of H_j for some $j \neq i$. Then

$v \in CH(\{x, y, z\})$ and $CH(ex((V(G))) = V(G)$, i.e., $(V(G), \mathcal{M})$ is a convex geometry and we are done. Assume then that H_i contains a single vertex for $i = 1, 2, ..., k$. Each of these single vertices $V(H_i)$ is an extreme point of G. Since G has at least five vertices, $k \geq 4$, and v is in the Steiner interval of any three of these extreme points. Thus, $v \in CH(ex(V(G))) = V(G)$ and $(V(G), \mathcal{M})$ is a convex geometry.

Case 2: G does not have a universal vertex.

By our choice of G, there exists a vertex v which is the centre of a claw or paw (not $3SS$) in G. Choose v to be a non-$3SS$ vertex of maximum degree in G. Let x, y, z be three peripheral vertices of a claw or paw centred at v such that z is not adjacent to x or y. Since $diam(G) = 2$ and G does not have a universal vertex, there exists a vertex v' distance 2 from v. Let S be the set of common neighbours of v and v'. If $w \notin S$ is adjacent to v, then w is adjacent to every vertex in S, as otherwise, for any $r \in S$ that is nonadjacent to w, $\{w, v, r, v'\}$ induces a P_4. By symmetry, if $w' \notin S$ is adjacent to v', then w' is adjacent to every vertex in S. Since z is not adjacent to x or y, either x, y, z are all contained in S or none of them are in S.

Subcase 2.1: $x, y, z \in S$.

Suppose v has a neighbour $w \notin S$. Vertex w must be adjacent to x, y and z, as stated above; however, $\{v, v', w, x, y, z\}$ now induces a replicated twin C_4, a contradiction. By symmetry $N(v) = N(v')$.

Claim I: $S \cup \{v, v'\} = V(G)$.

Suppose Claim I is false and that there exists some vertex $r \notin S \cup \{v, v'\}$ such that ru is an edge for some $u \in S$. If $u = x$, then rz is an edge; as otherwise $\{r, u, v, z\}$ induces a P_4. However, if rz is an edge then ry is also an edge; as otherwise $\{r, z, v, y\}$ induces a P_4. If $u = y$ or $u = z$ then, by symmetry, r is adjacent to x, y, and z. However, now $\{v, r, v', x, y, z\}$ induces a replicated twin C_4. Consequently, r is not adjacent to x, y, or z. Since r is adjacent to $u \in S \setminus \{x, y, z\}$, and not adjacent to x, y

or z, the edges ux, uy, and uz must be present; as otherwise $\{r, u, v, (xyz)\}$ induces a P_4, where the notation (xyz) denotes any one of these vertices. Now $\{u, x, y, z\}$ induces a claw or a paw where u is the central vertex. The degree of u in G is at least 6. Thus, by our choice of v, there are at least two vertices $w, w' \in S$ that are not adjacent to u. Vertex r is adjacent to both w and w'; as otherwise $\{r, u, v, (ww')\}$ induces a P_4. However, now $\{r, v, v', u, w, w'\}$ induces a replicated twin C_4. Thus Claim I follows.

Let $H = G - v$. By Claim I, the subgraph H is connected. Suppose all vertices that are extreme points of the subgraph H are also extreme points of G. By our choice of G, $CH_H(ex(V(H))) = V(H)$. Since $x, y, z \in CH_H(ex(V(H)))$ and G is distance hereditary, $x, y, z \in CH(ex(V(G)))$. However, since v is in the Steiner interval of x, y and z, this implies that $v \in CH(ex(V(G)))$, and $(V(G), \mathcal{M})$ is a convex geometry; a contradiction. We therefore assume not all vertices that are extreme points of H are extreme points of G. Let u be $3SS$ in H but not in G. Thus u is the central vertex, and v a peripheral vertex, of a claw or paw in G. Let u, v, s, t be the vertices of this claw or paw. If $s, t \in S$, then v is adjacent to both vertices and $\{u, v, s, t\}$ does not induce a claw or paw. If $s = v'$, then $t \in S$, and thus t is adjacent to both v and s, and $\{u, v, s, t\}$ does not induce a claw or paw. By symmetry, $t \neq v'$. This violates Claim I, thus contradicting our assumption that not all vertices that are extreme points of H are extreme points of G.

Subcase 2.2: $x, y, z \notin S$.

For every vertex $u \in S$, ux is an edge; as otherwise $\{v', u, v, x\}$ induces a P_4. Likewise uy and uz are edges for every $u \in S$, and u is adjacent to every vertex in $N(v) \setminus S$.

Claim II: Every vertex in S must be adjacent to at least $|S| - 2$ vertices in S.

Suppose u, u', u'' are three distinct vertices in S such that u is not adjacent to u'

or u''. As stated previously, every vertex in S is adjacent to x, y and z. However, now $\{x, y, z, u, u', u''\}$ induces a replicated twin C_4. Thus Claim II follows.

Since u is the centre vertex of the claw or paw induced by vertices $\{u, x, y, z\}$ and has degree at least 5, by our choice of v there exists a unique vertex $w \in S$ such that u is not adjacent to w. Furthermore, $N[u] \subseteq N[v] \cup \{v'\}$. Hence $N(v') \subseteq N(v)$; as otherwise $\{t, v', u, v\}$ induces a P_4 for a vertex t adjacent to v' but not v.

Claim III: $N[v] \cup \{v'\} = V(G)$.

Suppose there exists a vertex w' such that $w' \notin N[v] \cup \{v'\}$. Since G is connected, w' is adjacent to some vertex $w'' \in N(v) \setminus \{u\}$. If $w'' \in S$, then w'' is the centre of the claw or paw induced by $\{w'', x, y, z\}$ and w'' would have degree 1 greater than v. Thus, $w'' \in N(v) \setminus S$. However, now $\{w', w'', u, v\}$ induces a P_4. Thus Claim III follows.

The subgraph H is connected. As in Subcase 2.1, if every extreme point of H is an extreme point G, then $(V(G), \mathcal{M})$ is a convex geometry, a contradiction to our assumption. Therefore, there exists a vertex u' that is an extreme point of H but not of G. Thus u' is the central vertex, and v a peripheral vertex, of a claw or paw in G. Let u', v, a, b be the vertices of this claw or paw. Since $N[v] \cup \{v'\} = V(G)$ and the subgraph induced by $\{v, a, b\}$ contains at most one edge, without loss of generality $a = v'$ and $b \in N(v) \setminus S$. Thus $u' \in S$. However, then u' is the centre vertex of a claw or paw induced by $\{u', x, y, z\}$ in H, and thus not an extreme point of $V(H)$. This contradicts the criteria for u'. As a result, every vertex that is an extreme point of H is also an extreme point of G and, as in Subcase 2.1, $(V(G), \mathcal{M})$ is a convex geometry. □

We observe that those graphs for which the $3S$-convex sets form a convex geometry are a subclass of the cographs since they are P_4-free.

6.2 k-Monophonic Convexity

We have seen that k-Steiner convexity is a generalization of g-convexity. We now describe a generalization of m-convexity.

For a subset U of vertices of a connected graph G, a *minimal* U*-tree* is a subtree T of G which contains all vertices of U and has the property that every vertex of $V(T) \setminus U$ is a cut vertex of $\langle V(T) \rangle$. The collection of all vertices of G that lie on some minimal U-tree for a given U is the *monophonic interval of* U. A subset X of vertices is considered to be *k-monophonically convex*, or *km-convex*, when it includes the monophonic interval of every k-subset of X.

We see that $2M$-convexity is equivalent to m-convexity. Also, for a subset U of vertices, every Steiner interval of U is a minimal U-tree. We saw in Theorem 6.1 that the extreme points of $3S$-convex sets are precisely the $3SS$ vertices, i.e., those that are not the centre vertex of an induced claw, paw, or P_4. We now show that the $3SS$ vertices are also the extreme vertices of $3M$-convex sets.

Theorem 6.3. [8] *For a* $3M$*-convex set* X *of a connected graph* G*, a vertex* $v \in X$ *is an extreme point of* X *if and only if* v *is* $3SS$ *in* $\langle X \rangle$*.*

Proof. Let X be a $3M$-convex set of a connected graph G, and let v be a vertex in X.

Suppose v is not $3SS$ in $\langle X \rangle$, i.e., v is the center of an induced claw, paw or P_4 on vertices v, u, w, y in X. Then v lies on a minimal tree for $\{u, w, y\}$ and is therefore not an extreme point of X, implying sufficiency.

Suppose that v is not an extreme point of X. Then v is an internal vertex of a minimal U-tree T, for some set of three vertices $U \subseteq X$. Let H be the disconnected subgraph $T - v$. Suppose v has at least three neighbours in H. Let x, y and z be any such three neighbours. If, for every such triple $\langle \{x, y, z\} \rangle$ is connected, then v is not a cut vertex for $\langle V(T)\rangle$; a contradiction. Thus, there is at most one edge in $\langle\{x,y,z\}\rangle$ and $\langle\{v,x,y,z\}\rangle$ induces a paw or claw for which v is the centre vertex, i.e., v is $3SS$.

Assume then that v has exactly two neighbours x,y in H. As before, $\langle\{x,y\}\rangle$ induces a disconnected graph, i.e., x is not adjacent to y. Since T contains at least four vertices, H contains at least three vertices. Thus, one of the components of H containing x or y has at least two vertices. Without loss of generality, suppose it is the component containing y, and let z be a neighbour of y in H. Since v is a cut vertex for $\langle V(T)\rangle$, $\langle\{x,y,z\}\rangle$ is not connected. Thus, x is not adjacent to y or z, and $\langle\{v,x,y,z\}\rangle$ induces P_4 for which v is a centre vertex, i.e., v is $3SS$. This completes the proof of the necessity. □

We now show that the graphs for which the $3M$-convexity produces a convex geometry are characterized by the same forbidden subgraphs as for the $3S$-convexity.

Theorem 6.4. [8] *Let G be a connected graph, and let $\mathcal{M}_{3M}$ and $\mathcal{M}_{3S}$ be the families of all $3M$-convex and $3S$-convex subsets of $V(G)$, respectively. Then $(V(G),\mathcal{M}_{3M})$ is a convex geometry if and only if $(V(G),\mathcal{M}_{3S})$ is a convex geometry.*

Proof. Suppose $(V(G),\mathcal{M}_{3S})$ is a convex geometry. Let X be a $3M$-convex set of vertices. Then X is $3S$-convex since every Steiner interval of U is also a minimal U-tree, for $U\subseteq X$. Since $CH_{3S}(ex(X))=X$, and the $3SS$ vertices of $\langle X\rangle$ are the extreme points for both the $3S$-convexity and the $3M$-convexity, $CH_{3M}(ex(X))=X$ and $(V(G),\mathcal{M}_{3M})$ is a convex geometry. This establishes the necessity.

Suppose $(V(G),\mathcal{M}_{3M})$ is a convex geometry.

Claim I: G is P_4 free.

Suppose G contains the induced $P_4 : q,r,s,t$. Let M be the $3M$-convex hull of $V(P)$. The extreme points of M are some subset of $\{q,t\}$. However, the $3M$-convex hull of any subset of $\{q,t\}$ is itself (as it does not contain at least 3 points). Thus, if G contains a P_4 then $(V(G),\mathcal{M}_{3M})$ is not a convex geometry; a contradiction.

Let X be a $3S$-convex set of vertices and let $\{u, v, w\} \subseteq X$. Let H be a minimal $\{u, v, w\}$-tree. We want to show that H is also a Steiner tree for $\{u, v, w\}$.

The set of leaves of H is a subset of $\{u, v, w\}$ and, as such, H is either a path or homeomorphic to a claw.

Case 1: H is a path.

Without loss of generality, suppose that u and w are the end vertices and v is an internal vertex of the path. Then the u, v-subpath and v, w-subpath are both induced. Since G is P_4-free, each subpath contains at most 3 vertices. If $V(H) = \{u, v, w\}$, then H is the Steiner tree for $\{u, v, w\}$. If $V(H) = \{u, v, w', w\}$, for w' an internal vertex of the v, w-subpath, then H is a Steiner tree for $\{u, v, w\}$, since the Steiner distance of $\{u, v, w\}$ is 3 regardless of whether or not u and w' are adjacent. If $V(H) = \{u, u', v, w', w\}$, for u' an internal vertex of the u, v-subpath, then the only possible additional edges of $\langle V(H) \rangle$ are uw', uw, $u'w'$, and $u'w$. The presence of any of the edges uw', uw, and $u'w$, would violate that u' and w' are both cut vertices of $\langle V(H) \rangle$. Only the edge $u'w'$ does not violate that H is a minimal $\{u, v, w\}$-tree. If the edge $u'w'$ is present, then $\{u, u', w', w\}$ induces a P_4; a contradiction. However, without the edge $u'w'$, the vertex set $\{u, u', v, w'\}$ induces a P_4. Thus H may not take on this configuration.

Case 2: H is homeomorphic to a claw.

Let x be the vertex of degree 3 in H. Each subpath (u, x), (v, x), (w, x) is induced and has at most 1 internal vertex, since G is P_4-free. By the same argument used in Case 1, it is not possible for 2 or more of the subpaths to have an internal vertex. Suppose then that exactly 1 of the subpaths has an internal vertex. Without loss of generality, let u' be an internal vertex of the u, x-subpath. Vertex u' is a cut vertex of $\langle V(H) \rangle$, therefore u is not adjacent to v or w in $\langle V(H) \rangle$. Since G is P_4-free, u' is adjacent to both v and w. However, now x is not a cut vertex of $\langle V(H) \rangle$; a contradiction. Therefore, no subpath of H has an internal vertex. Since $\langle\{u, v, w\}\rangle$ is disconnected, the Steiner distance of $\{u, v, w\}$ is 3, and H is a Steiner tree for $\{u, v, w\}$.

Since X is $3S$-convex and every minimal $\{u, v, w\}$-tree is a Steiner tree for $\{u, v, w\}$ then X is also $3M$-convex. Since $CH_{3M}(ex(X)) = X$, and the $3SS$ vertices of $\langle X\rangle$ are the extreme points for both the $3S$-convexity and the $3M$-convexity, $CH_{3S}(ex(X)) = X$ and $(V(G), \mathcal{M}_{3S})$ is a convex geometry. This establishes the sufficiency. □

Corollary 6.5. *A graph G is $\{P_4$, replicated twin $C_4\}$-free if and only if the $3M$-convexity of G is a convex geometry.*

Proof. The result follows directly from Theorems 6.2 and 6.4. □

6.3 $3SS$-Elimination Orderings

The graphs for which every LexBFS ordering is a $3SS$ elimination ordering have been characterized by Cáceres and Oellermann [7] as those free of the following induced subgraphs: house, hole, domino, true-twin C_4, and false-twin C_4 (see Fig. 6.3). The graphs for which every MCS ordering of every induced sugbraph is a $3SS$ elimination ordering have also been characterized by Cáceres and Oellermann [7] as those free of the following induced subgraphs: house, hole, P, true-twin C_4, and $K_{3,3}$.

When the difference between the true-twin C_4, and false-twin C_4 is not important we will use the general term *twin* C_4 to encompass both subgraphs.

Theorem 6.6. [7] *Every LexBFS ordering of a graph G is a $3SS$ ordering if and only if G is $\{$HHD, twin $C_4\}$-free.*

Proof. Suppose G contains a house, hole or domino subgraph. By Corollary 4.8, G will have a LexBFS ordering that is not semisimplicial, and therefore not $3SS$.

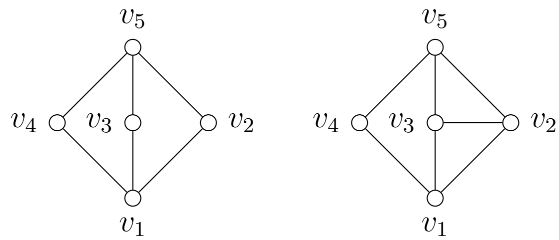


Figure 6.3: From left to right: A false-twin C_4, and a true-twin C_4 with LexBFS labelings.

Suppose G contains a twin C_4. Consider a LexBFS ordering which labels vertex v_5 of the twin C_4 of Fig. 6.3 as n. Since $d(v_5, v_j) = 1$ for $j = 4, 3, 2$, and $d(v_5, v_1) = 2$; the vertices v_4, v_3 and v_2 of both twin C_4's in Fig. 6.3 are labelled before v_1. If v_1 is labelled i, then the set of vertices $\{v_1, v_2, v_3, v_4\}$ induces a claw or paw in G_i, for which v_1 is the center vertex. Such a LexBFS ordering is not $3SS$. This establishes the sufficiency.

Suppose G is $\{$HHD, twin $C_4\}$-free, but that there exists a LexBFS ordering α: $v_1, v_2, ..., v_n$ which is not a $3SS$ ordering. Let v_i be a vertex that is not $3SS$ in G_i. Since G is HHD-free, it follows from Corollary 4.8 that α is a semisimplicial elimination ordering. Thus, v_i is not a centre vertex of a P_4 and v_i is the centre of an induced claw or paw in G_i. Let the neighbours of v_i in the claw or paw be x_1, x_2 and x_3 such that $v_i < x_1 < x_2 < x_3$. If $x_3 = n$, then both x_1 and x_2 are adjacent to x_3 since they are labelled before v_i in the LexBFS ordering. However, then $\{v_i, x_1, x_2, x_3\}$ would not not induce a claw or paw in G_i; therefore, $x_3 \neq n$. Thus, x_3 has a neighbour y such that $x_3 < y$.

Case 1: Suppose x_3 is not adjacent to x_1 or x_2.

If v_iy is not an edge, then both x_2y and x_1y are edges since v_i is not the centre of an induced P_4. However, $\{x_1, x_2, x_3, v_i, y\}$ now induces a twin C_4, a contradiction. Therefore v_iy is an edge.

Since $v_i < x_2 < x_3$, and v_ix_3 is an edge and x_2x_3 is not an edge, by P1 there exists a vertex $z > x_3$ such that zx_2 is an edge and zv_i is not an edge. Since yv_i is an edge, $z \neq y$. Vertices z and x_3 are adjacent, as otherwise $\{z, x_2, v_i, x_3\}$ would induce a P_4. Vertices z and x_1 are also adjacent, as otherwise $\{z, x_3, v_i, x_1\}$ would induce a P_4. We now find, however, that $\{x_1, x_2, x_3, v_i, z\}$ induces a twin C_4; a contradiction.

Case 2: Suppose x_3 is adjacent to x_2.

Since $\{v_i, x_1, x_2, x_3\}$ induces a claw or paw, x_1 is not adjacent to x_2 or x_3. Vertex y is adjacent to v_i or x_1 as otherwise $\{x_1, v_i, x_3, y\}$ induces a P_4. Suppose y is not adjacent to v_i. Then yx_1 is an edge, and yx_2 is also an edge, as otherwise the vertex set $\{x_1, x_2, x_3, y, v_i\}$ induces a house. However, $\{x_1, x_2, x_3, y, v_i\}$ now induces a true-twin C_4. Therefore y is adjacent to v_i.

Since $v_i < x_1 < x_3$, and v_ix_3 is an edge and x_1x_3 is not an edge, by P1 there exists a vertex $u > x_3$ such that ux_1 is an edge and uv_i is not an edge. Since yv_i is an edge, $u \neq y$. The edge ux_3 is present, as otherwise $\{x_3, v_i, x_1, u\}$ induces a P_4. The edge ux_2 is present, as otherwise $\{x_2, v_i, x_1, u\}$ induces a P_4. We now find, however, that $\{v_i, x_1, x_2, x_3, u\}$ induces a true-twin C_4; a contradiction.

Case 3: Suppose x_3 is adjacent to x_1.

The proof of this case is identical to Case 2, with x_1's and x_2's exchanged.

This establishes the necessity. □

Theorem 6.7. [7] *Any MCS ordering of any induced subgraph F of G is a $3SS$ ordering of F if and only if G is $\{$HHP, true-twin C_4, $K_{3,3}\}$-free.*

Proof. Suppose G contains a house, hole or P subgraph. By Corollary 4.10, some induced subgraph of G will have an MCS ordering that not semisimplicial, and thereby not $3SS$.

Suppose G contains a true-twin C_4. The vertex ordering v_1, v_2, v_3, v_4, v_5 of the true-twin C_4 of Fig. 6.3 is an MCS ordering for which v_1 is the centre of an induced

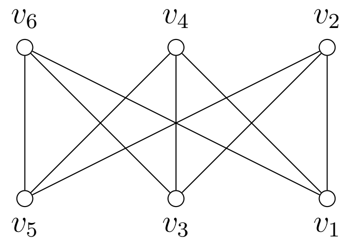


Figure 6.4: $K_{3,3}$ labelled by MCS.

paw. Thus the ordering is not $3SS$.

Suppose G contains a $K_{3,3}$. The vertex ordering $v_1, v_2, v_3, v_4, v_5, v_6$ of the $K_{3,3}$ of Fig. 6.4 is an MCS ordering for which v_1 is the centre of an induced claw. Thus the ordering is not $3SS$. This completes the proof of sufficiency.

Suppose that G is {HHP, true-twin C_4, $K_{3,3}$}-free, but there exists an MCS ordering α: $v_1, v_2, ..., v_n$ which is not a $3SS$ ordering. Let vertex v_i be the center of an induced claw or paw in G_i. Note that α is a semisimplicial elimination ordering by Corollary 4.10, therefore v_i is not a center vertex of a P_4. Let the neighbours of v_i in the claw or paw be x_1, x_2 and x_3 such that $v_i < x_1 < x_2 < x_3$.

Case 1: Suppose x_2 is not adjacent to x_1 or x_3.

Since $v_i < x_1 < x_2$, and $v_i x_2$ is an edge and $x_1 x_2$ is not an edge, by P2 there exists a vertex $y_1 > x_1$ such that $y_1 x_1$ is an edge and $y_1 v_i$ is not an edge. Likewise, x_2 has a neighbour $y_2 > x_2$, non-adjacent to v_i.

Vertices y_1 and x_2 are adjacent, as otherwise $\{y_1, x_1, v_i, x_2\}$ would induce a P_4 in G_i with v_i as a centre vertex. Vertices x_1 and y_2 are adjacent, as otherwise $\{x_1, v_i, x_2, y_2\}$ would induce a P_4 in G_i with v_i as a centre vertex. Vertices x_3 and y_2 are adjacent, as otherwise $\{y_2, x_2, v_i, x_3\}$ would induce a P_4 in G_i with v_i as a centre vertex. Vertices y_1 and x_3 are adjacent, as otherwise $\{y_1, x_1, v_i, x_3\}$ would induce a P_4 in G_i with v_i as a centre vertex.

If neither of the edges $x_1 x_3$ nor $y_1 y_2$ is present, then $\{v_i, x_1, x_2, x_3, y_1, y_2\}$ induces a $K_{3,3}$. If the edge x_1x_3 is present, then $\{v_i, x_1, x_2, x_3, y_2\}$ induces a true-twin C_4. If the edge y_1y_2 is present, then $\{v_i, x_1, x_2, y_1, y_2\}$ induces a true-twin C_4. All options result in a contradiction.

Case 2: Suppose x_2 is adjacent to x_1.

Since v_i is the center of a claw or paw in G_i, x_3 is not adjacent to x_1 or x_2. Since $v_i < x_1 < x_3$, and v_ix_3 is an edge and x_1x_3 is not an edge, by P2 there exists a vertex $y > x_1$ such that yx_1 is an edge and yv_i is not an edge.

Vertices y and x_3 are adjacent, as otherwise $\{y, x_1, v_i, x_3\}$ would induce a P_4 in G_i with v_i as a centre vertex. Vertices y and x_2 are adjacent, as otherwise $\{v_i, x_1, x_2, x_3, y\}$ would induce a house. However, the vertex set $\{v_i, x_1, x_2, x_3, y\}$ now induces a true-twin C_4, a contradiction.

Case 3: Suppose x_2 is adjacent to x_3.

Since v_i is the center of a claw or paw in G_i, x_1 is not adjacent to x_2 or x_3. Since $v_i < x_1 < x_3$, and v_ix_3 is an edge and x_1x_3 is not an edge, by P2 there exists a vertex $z > x_1$ such that zx_1 is an edge and zv_i is not an edge. Vertices z and x_2 are adjacent, as otherwise $\{z, x_1, v_i, x_2\}$ would induce a P_4 in G_i with v_i as a centre vertex. Vertices z and x_3 are adjacent, as otherwise $\{v_i, x_1, x_2, x_3, z\}$ would induce a house. However, the vertex set $\{v_i, x_1, x_2, x_3, z\}$ now induces a true-twin C_4, a contradiction. This completes the proof of necessity. □

Corollary 6.8. *Any MEC or MCC ordering of any induced subgraph F of G is a $3SS$ ordering of F if and only if G is $\{$HHP, true-twin C_4, $K_{3,3}\}$-free.*

Proof. The proof of the sufficiency is identical to that of Theorem 6.7 where MCS is replaced by MEC/MCC, and Corollary 4.10 is replaced with Theorem 4.12.

The proof of the necessity is also identical to that of Theorem 6.7 where property P2 is replaced with P3. Note that for every application of MCS property P2 for $a < b < c$ in Theorem 6.7, vertices a and b are in the same unlabelled component;

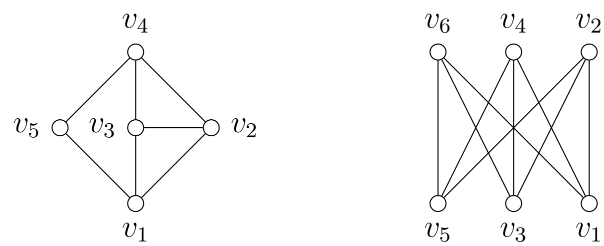


Figure 6.5: From left to right: A true-twin C_4 and a $K_{3,3}$ graph with LexDFS labelings.

therefore MEC/MCC property P3 may be substituted. □

As was the case for semisimplicial elimination orderings, the characterization of graphs for which every LexDFS ordering of every subgraph is a $3SS$ ordering is the same as that of MCS, MEC and MCC. The following result is new.

Theorem 6.9. *Any LexDFS ordering of any induced subgraph F of G is a $3SS$ ordering of F if and only if G is {HHP, true-twin C_4, $K_{3,3}$}-free.*

Proof. Suppose G contains a house, hole, or P subgraph. By Theorem 4.13, some induced subgraph of G will have a LexDFS ordering that is not semisimplicial, and thereby not $3SS$.

Suppose G contains a true-twin C_4 or a $K_{3,3}$. Fig. 6.5 shows LexDFS orderings for a true-twin C_4 and a $K_{3,3}$ for which the vertex labelled v_1 is the centre of an induced claw or paw. This establishes the sufficiency.

Suppose that G is {HHP, true-twin C_4, $K_{3,3}$}-free, but there exists a LexDFS ordering α: $v_1, v_2, ..., v_n$ which is not a $3SS$ ordering. Let v_i be a vertex in the ordering that is not $3SS$ in G_i. By Theorem 4.13, α is a semisimplicial elimination ordering, therefore v_i is not a center vertex of a P_4 in G_i. Thus v_i is the center of an induced claw or paw in G_i. Let the neighbours of v_i in the claw or paw be x_1, x_2 and x_3 such that $x_1 < x_2 < x_3$.

Case 1: Suppose x_1 is not adjacent to x_2 or x_3.

Since $v_i < x_1 < x_2$, and $v_i x_2$ is an edge and $x_1 x_2$ is not an edge, by P4 there exists a vertex y_1 such that $x_1 < y_1 < x_2$, and $y_1 x_1$ is an edge and $y_1 v_i$ is not an edge. Vertices y_1 and x_2 are adjacent, as otherwise $\{y_1, x_1, v_i, x_2\}$ induces a P_4 in G_i with v_i as a centre vertex. Vertices y_1 and x_3 are adjacent, as otherwise $\{y_1, x_1, v_i, x_3\}$ induces a P_4 in G_i with v_i as a centre vertex. If $x_2 x_3$ is an edge, then $\{v_i, x_1, x_2, x_3, y_1\}$ induces a true-twin C_4, a contradiction. Thus, $x_2 x_3$ is not an edge. Since $v_i < x_2 < x_3$, and $v_i x_3$ is an edge and $x_2 x_3$ is not an edge, by P4 there exists a vertex y_2 such that $x_2 < y_2 < x_3$, and $y_2 x_2$ is an edge and $y_2 v_i$ is not an edge. Since $y_1 < x_2 < y_2$, $y_1 \neq y_2$. Vertices y_2 and x_1 are adjacent, as otherwise $\{x_1, v_i, x_2, y_2\}$ induces a P_4 in G_i with v_i as a centre vertex. Vertices y_2 and x_3 are adjacent, as otherwise $\{y_2, x_2, v_i, x_3\}$ induces a P_4 in G_i with v_i as a centre vertex. Vertices y_1 and y_2 are adjacent, as otherwise $\{v_i, y_1, y_2, x_1, x_2, x_3\}$ induces a $K_{3,3}$. However, now $\{v_i, y_1, y_2, x_1, x_2\}$ induces a true-twin C_4, a contradiction.

Case 2: Suppose x_1 is adjacent to x_2.

In this case v_i is the center of a paw in G_i, and x_3 is not adjacent to x_1 or x_2. Since $v_i < x_1 < x_3$, and $v_i x_3$ is an edge and $x_1 x_3$ is not an edge, by P4 there exists a vertex y such that $x_1 < y < x_3$, and $y x_1$ is an edge and $y v_i$ is not an edge.

Vertices y and x_3 are adjacent, as otherwise $\{y, x_1, v_i, x_3\}$ induces a P_4 in G_i with v_i as a centre vertex. Vertices y and x_2 are adjacent, as otherwise the vertex set $\{v_i, x_1, x_2, x_3, y\}$ induces a house. However, $\{v_i, x_1, x_2, x_3, y\}$ now induces a true-twin C_4, a contradiction.

Case 3: Suppose x_1 is adjacent to x_3.

The proof of this case is identical to Case 2, with x_2 and x_3 exchanged. This completes the proof of necessity. □

6.4 Concluding Remarks

In Chapter 4 we studied the m^3-convexity and mentioned its geodesic analog the g^3-convexity. Convexities have been studied that are defined in terms of more than one interval structure. For example, a set X is said to be m^3_3-convex if it is both $3M$- and m^3-convex. Similarly, a set X is said to be g^3_3-convex if it is both $3S$- and g^3-convex.

A tailed-twin C_4 is obtained by attaching a pendant vertex to a twin C_4 at v_1 as labelled in Fig. 6.3. Nielsen and Oellermann [27] proved that the g^3_3-convexity of G is a convex geometry if and only if G is {house, hole, domino, A, 3-fan, replicated twin C_4, tailed-twin C_4}-free. Cáceres, Oellermann, and Puertas [8] proved that the m^3_3-convexity of G is a convex geometry if and only if G is {house, hole, domino, A, replicated twin C_4, tailed-twin C_4}-free.

Chapter 7

Concluding Remarks

In this thesis, we considered P-elimination orderings of the vertices of a graph, where P is a vertex property. We saw that convex geometries naturally give rise to P-elimination orderings when P is a vertex property that characterizes the extreme points relative to a given graph convexity. In addition to the graph classes for which a given convexity is a convex geometry, we saw that there are larger graph classes that may have P-elimination orderings. In particular, we considered graphs for which every LexBFS (or MCS, MCC, MEC, or LexDFS) ordering produced a P-elimination ordering, where P characterizes the extreme vertices for the g-convexity, m-convexity, m^3-convexity, or the $3S$-convexity. In Chapter 5, we noticed interesting connections between elimination orderings and distance hereditary graphs. In particular we saw that for a distance hereditary graph G, a simplicial elimination ordering of G^2 corresponds precisely to a 2-simplicial elimination orderings of G. Moreover, if every LexBFS ordering of a graph G is 2-simplicial then G is distance hereditary.

The vertex property of being semisimplicial is a relaxation of the property of being simplicial. More specifically, a vertex is simplicial if it is not the centre of an induced P_3 and it is semisimplicial if it is not the centre of an induced P_4. Alternatively, a vertex is simplicial if and only if every two of its neighbours induce a connected graph. This latter definition gives rise to a different relaxation of the simplicial property. A vertex is *nearly simplicial* if every three of its neighbours induce a connected graph. That is to say, a vertex is nearly simplicial if it is not the centre of an induced claw or paw. Thus the $3SS$ vertices are precisely those that are both nearly simplicial and semisimplicial. It appears to be a difficult problem to characterize those graphs for which (i) every LexBFS ordering is nearly simplicial, or (ii) any MCS or MEC or MCC or LexDFS ordering of any induced subgraph is nearly simplicial.

Nearly simplicial elimination orderings are, in fact, a special case of a different generalization of simplicial elimination orderings. A set of vertices is an *independent set* if no pair of vertices in the set are adjacent. The *independence number* of a graph is the size of a largest independent set. A *k-independence ordering* is a vertex ordering $v_1, v_2, ..., v_n$ such that $\langle N(v_i) \cap V(G_i) \rangle$ has independence number at most k for $i = 1, 2, ..., n$.

The k-independence orderings were introduced by Akcoglu et al. [1] in 2002 in order to study auctions in which agents place a monetary bid on a subset of the available items, with the goal of maximizing revenue by accepting the greatest valued collection of non-overlapping bids. This is an NP-hard problem, but Akcoglu et al. [1] found that, by using properties of k-independence orderings, good approximation algorithms exist for small values of k. In 2012, Ye and Borodin [36] studied k-independence orderings, developing polynomial time approximation algorithms for graphs with previously found k-independence orderings (for fixed values of k) for several NP-hard problems including:

(1) The minimum vertex cover problem. Given a graph, find a smallest set of vertices incident to all edges of the graph.

(2) The minimum vertex colouring problem. Given a graph, colour its vertex set

with a smallest number of colours such that no two adjacent vertices receive the same colour.

(3) The weighted maximum independent set problem. Given a weighted graph, find an independent set of maximum weight.

For a perfect elimination ordering $v_1, v_2, ..., v_n$, the neighbourhood of v_i in G_i induces a complete graph (with independence number 1). Thus, for a connected graph, perfect elimination orderings are precisely the 1-independence orderings and, as such, the 1-independence orderings characterize the chordal graphs. With the exception of $k = 1$, graphs with k-independence orderings remain largely uncharacterized. If the independence number of a graph G is k, then every ordering of the vertices of G is k-independent. If a vertex ordering $v_1, v_2, ..., v_n$ is not 2-independent, then some v_i has at least three neighbours in G_i with no edges between them, i.e., v_i is the centre of an induced claw and is thereby not nearly simplicial in G_i. Accordingly, all nearly simplicial elimination orderings are 2-independent. We note, however, that a vertex ordering $v_1, v_2, ..., v_n$ for which v_i is not the centre of an induced claw in G_i for $i = 1, 2, ..., n$ but v_i is the centre of an induced paw in G_i for some $i = 1, 2, ..., n$ is a 2-independent ordering that is not a nearly simplicial ordering. Because k-independence is a property of the neighbourhood $N(v_i)$ of a vertex v_i, and the semisimplicial and $3SS$ properties extend to $N^2(v_i)$, semisimplicial and $3SS$ orderings are not readily comparable to k-independent orderings.

7.1 Open Problems

One may ask, for a vertex property P (including the above mentioned k-independent and nearly simplicial relaxations of simplicial): For what type of convexity does the property P characterize the extreme points of the convex sets? For every newly defined type of graph convexity, one may ask: For what class of graphs is this type of convexity a convex geometry? For every newly defined vertex ordering algorithm, one may ask: For what class of graphs does this algorithm always give a P-elimination ordering, for P the property of being an extreme point relative to the convexity? As new and useful types of convexity and algorithms are defined, as recently exemplified by LexDFS, these become tractable questions. For example, Corneil and Krueger [11] also define the vertex ordering algorithm Maximal Neighbourhood Search (MNS), which allows the next vertex to be labelled to be chosen from those with a maximal neighbourhood among the previously labelled vertices.

Bibliography

[1] K. Akcoglu, J. Aspnes, B. DasGupata, M.-Y. Kao. Opportunity cost algorithm for combinatorial auctions. *Applied Optimization* **74** (2002) 455-479.

[2] S. Arnborg, J. Lagergren, D. Seese. Easy problems for tree-decomposable graphs. *J. Algorithms* **12: 2** (1991) 308-340.

[3] H.-J. Bandelt, H.M. Mulder. Distance-hereditary graphs. *J. Combin. Theory Ser. B* **41** (1986) 182-208.

[4] J.A. Bondy, U.S.R. Murty. (2008). *Graduate Texts in Mathematics: Graph Theory.* Springer.

[5] A. Brandstädt, F.F. Dragan, F. Nicolai. LexBFS orderings and powers of chordal graphs. *Discrete Math.* **171** (1997) 27-42.

[6] A. Brandstädt, V.B. Le, J. Spinrad. *Graph Classes: A Survey.* SIAM Monographs on Discrete Mathematics and Applications. (1999).

[7] J. Cáceres, O.R. Oellermann. On 3-Steiner simplicial orderings. *Discrete Math.* **309** (2009) 5828-5833.

[8] J. Cáceres, O.R. Oellermann, M.L. Puertas. Minimal trees and monophonic convexity. *Discussiones Mathematicae. Graph Theory* **32** (2012) 685-704.

[9] O. Cogis, E. Thierry. Greedy Algorithms for Optimization on Distance-Hereditary Graphs. RR. LIRMM no.01049 (2001).

[10] D.G. Corneil, B. Dalton, M. Habib. LDFS-based certifying algorithm for the minimum path cover problem on cocomparability graphs. *SIAM J. Comput.* **42(3)** (2013) 792-807.

[11] D.G. Corneil, R.M. Krueger. A unified view of graph searching. *SIAM J. Discrete Math.* **22(4)** (2008) 1259-1276.

[12] P. Damaschke. (1990) Forbidden ordered subgraphs. In R. Bodendick & R. Henn (Eds.), *Topics in Combinatorics and Graph Theory (219-229)*. Heidelberg: Physica-Verlag.

[13] G. Damiand, M. Habib, C. Paul. A simple paradigm for graph recognition: application to cographs and distance hereditary graphs. *Theor. Comput. Sci.* **263** (2001) 99-111.

[14] G.A. Dirac. On rigid circuit graphs. *Abh. Math. Sem. Univ. Hamburg* **38** (1961) 71-76.

[15] F.F. Dragan, F. Nicolai, A. Brandstädt. Convexity and HHD-Free Graphs. *Discrete Math.* **12:1** (1999) 119-135.

[16] F.F. Dragan, F. Nicolai. LexBFS-orderings of distance hereditary graphs with application to the diametral pair problem. *Discrete Appl. Math..* **98** (2000) 191-207.

[17] P.H. Edelman. Meet-distibutive lattices and the anti-exchange closure. *Algebr. Univ.* **10** (1980) 290-299.

[18] M. Farber, R.E. Jamison. Convexity in graphs and hypergraphs. *SIAM J. Alg. Discrete Methods* **7** (1986) 433-444.

[19] D.R. Fulkerson, O.A. Gross. Incidence matrices and interval graphs. *Pacific J. Math.* **15** (1965) 835-855.

[20] E. Howorka. A characterization of distance hereditary graphs. *Quart. J. Math. Oxford* Ser.2 **28** (1977) 417-420.

[21] E. Howorka. A characterization of ptolemaic graphs. *J. Graph Thoery* **5** (1981) 323-331.

[22] B. Jamison. Copoints in antimatroids. *Congressus Numerantium* **29** (1980) 535-544.

[23] B. Jamison, S. Olariu. On the semi-perfect elimination. *Adv. Appl. Math* **9** (1988) 364-376.

[24] D.C. Kay, G. Chartrand. A characterization of certain ptolemaic graphs. *Canad. J. Math.* **17** (1965) 342-346.

[25] D.R. Lick, A.T. White. k-degenerate graphs. *Canad. J. Math.* **22** (1970) 1082-1096.

[26] F. Nicolai. *A Hypertree Characterization of Distance-herediatry Graphs.*. Tech. Report Gerhard-Mercator-Universitt Gesamthochschule. Duisburg SM-DU-255. (1994).

[27] M.H. Nielsen, O.R. Oellermann. Steiner trees and convex geometries. *Discrete Math.* **23** (2009) 680-693.

[28] O. Oellermann. Computing the average distance of a distance hereditary graph in linear time. *Congr. Numer.* **103**. (1994) 219-223.

[29] S. Olariu. Weak bipolarizable graphs. *Discrete Math.* **74** (1989) 159-171.

[30] S. Olariu. Some aspects of the semi-perfect elimination. *Discrete Appl. Math.* **31** (1991) 291-298.

[31] D. Rose. Triangulated graphs and the elimination process. *J. Math. Anal. Appl.* **32** (1970) 597-609.

[32] D. Rose, R. Tarjan, G. Leuker. Algorithmic aspects of vertex elimination on graphs. *SIAM J. Comput.* **5** (1976) 266-283.

[33] D.R. Shier. Some aspects of perfect elimination orderings in chordal graphs. *Discrete Appl. Math.* **7** (1984) 325-331.

[34] R. E. Tarjan, M. Yannakakis. Simple linear-time algorithms to test chordality of graphs, test acyclicity of hypergraphs, and selectively reduce acyclic hypergraphs. *SIAM J. Comput.* **13** (1984) 566579.

[35] K. White, M. Farber, W.R. Pulleyblank. Steiner trees, connected domination and strongly chordal graphs. *Networks* **15** (1985) 109-124.

[36] Y. Ye, A. Borodin. Elimination graphs. *ACM Trans. on Algorithms* **8(2): 14** (2012).