The Gravitational Vlasov-Poisson System on the Unit 2-Sphere with Initial Data along a Great Circle

by

Crystal Lind
B.Sc., University of Northern British Columbia, 2007

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

Master of Science

in the Department of Mathematics and Statistics

The Gravitational Vlasov-Poisson System on the Unit 2-Sphere with Initial Data along a Great Circle

by

Crystal Lind
B.Sc., University of Northern British Columbia, 2007

Supervisory Committee

Dr. F. Diacu, Co-Supervisor
(Department of Mathematics and Statistics)

Dr. S. Ibrahim, Co-Supervisor
(Department of Mathematics and Statistics)

Dr. R. Illner, Departmental Member
(Department of Mathematics and Statistics)

Supervisory Committee

Dr. F. Diacu, Co-Supervisor
(Department of Mathematics and Statistics)

Dr. S. Ibrahim, Co-Supervisor
(Department of Mathematics and Statistics)

Dr. R. Illner, Departmental Member
(Department of Mathematics and Statistics)

ABSTRACT

The Vlasov-Poisson system is most commonly used to model the movement of charged particles in a plasma or of stars in a galaxy. It consists of a kinetic equation known as the Vlasov equation coupled with a force determined by the Poisson equation. The system in Euclidean space is well-known and has been extensively studied under various assumptions. In this paper, we derive the Vlasov-Poisson equations assuming the particles exist only on the 2-sphere, then take an in-depth look at particles which initially lie along a great circle of the sphere. We show that any great circle is an invariant set of the equations of motion and prove that the total energy, number of particles, and entropy of the system are conserved for circular initial distributions.

Contents

List of Figures

ACKNOWLEDGEMENTS

I would like to thank:

my husband, Bob Lomas, and the rest of our wonderful extended family for all their love and encouragement.

Dr. Florin Diacu and Dr. Slim Ibrahim, for mentoring, wisdom, support, and patience.

Shengyi Shen, for helpful discussions.

The University of Victoria, for both academic and financial support.

The Natural Sciences and Engineering Research Council of Canada, for funding in the form of a CGS M award.

Chapter 1

Introduction

Since there are many charged particles in a plasma and many stars in a galaxy, it is impractical to model their movement individually. Instead, we often assume the moving parts to be point particles and describe them all at once using a distribution function. We then take what we know about the forces involved to derive a set of equations which can be used to accurately predict how the distribution will evolve in time. In Euclidean space, the equations have been established for many years so we simply choose the equations that are appropriate according to the assumptions required, and proceed to study the equations in whatever way we wish. For example, in the study of plasmas, both relativity and changing magnetic fields are taken into account by the Vlasov-Maxwell system [1]. If the magnetic fields and effects of relativity are negligible, then the Vlasov-Poisson system proves to be a good model. In self-gravitating systems (those considered in stellar dynamics or cosmology), a fully relativistic treatment requires the use of the Einstein-Vlasov system [20]. Again, if relativity is ignored, the Vlasov-Poisson equations are suitable. In between the fully relativistic Einstein-Vlasov and the fully Newtonian Vlasov-Poisson systems lies the Vlasov-Manev system, in which the Newtonian gravitational potential is replaced by a slightly perturbed potential in the Vlasov-Poisson equation [6]. This treatment is particularly interesting because it shows the advance of the perihelion of Mercury without using relativity. In all of these models, the physical system is assumed to be collisionless; however it should be mentioned that collisions can be taken into account via the Boltzmann equation and its variations. These are just a few of the many systems used to describe the movement of ions in plasma or stars in space. In this thesis, we shall reduce our scope to consider only the stellar dynamical case (i.e. a group of self-gravitating stars in space) and make the following physical assumptions:

1. Our stars have unit mass and are identical to each other.
2. There are no collisions[1].
3. Relativistic effects are negligible (i.e. $v \ll c$, where c is the speed of light).

In response to these assumptions, we identify the Vlasov-Poisson system as the logical set of equations to study. However, our next requirement will prevent us from using the equations already established–we assume the stars are within a curved space. Our goal for this thesis is to extend some of the results from flat space to non-Euclidean spaces, and find out what happens under the assumption that physical space is not flat. If space has positive curvature like a sphere, how does this change the Vlasov-Poisson equations? Do the properties of the system in Euclidean space also exist in the curved problem?

On small scales, the curvature of the universe is negligible and therefore until recently there has been no reason to suspect that our ambient space is anything but flat (Euclidean). However, with recent studies of the cosmic background radiation some researchers have concluded that a curved universe may better fit the acquired data. Therefore any work done on curved spaces is very relevant and could even hold the key to testing whether the universe is curved at all. The study of any equations that might be useful for predicting the motions of stars and galaxies is particularly interesting since on such large scales the differences between flat and non-Euclidean space become relevant.

Although the Vlasov-Poisson system has been studied extensively in the Euclidean setting, as far as we can tell it has never been considered on curved spaces without the use of general relativity. Therefore, a crucial first step in our work is to derive the systems of equations in such a way that they agree with everything we know physically about the space. Consequently, we begin by making some preliminary assumptions that must be satisfied by our equations so that they can be applied to real-life situations. The most fundamental assumption is on the gravitational force between two masses: it is attractive, proportional to the masses, directed along the geodesic connecting the masses, and it depends on the geodesic distance between the masses. As we shall see, any differences between the resulting equations in curved space and the corresponding equations in Euclidean space are due to this difference in the gravitational force.

[1] we will also need to exclude antipodal positions – configurations on the sphere in which particles are separated by a geodesic distance of π – for the same reason.

This thesis is organized as follows. In Chapter 2, we begin by giving an introduction to the mathematics of the Vlasov-Poisson system in Euclidean space, including a brief discussion of the literature surrounding the subject. We supplement this discussion with a slightly more detailed description of the interesting phenomenon known as Landau damping to show the usefulness of studying these equations. After this, we compile some relevant technical parts of geometry, such as the coordinates and metrics we will use throughout the thesis. In Chapter 3, we derive the Vlasov-Poisson system on the 2-sphere. To motivate the section, we begin with the simple example of a gravitational field due to a point mass on the sphere. We then extend this case and derive the Poisson equation for an arbitrary mass distribution using Gauss's Law. After obtaining the Poisson equation, we solve it using the known expression for the fundamental solution of the Laplacian to get the gravitational potential. The next step is finding the equations of motion for an individual particle using Lagrangian mechanics. The form of the equations is previously known but the calculation is reproduced here for completion. We finally put together all of this information in the form of the Vlasov-Poisson system on the 2-sphere. Now having all the information we need, we proceed to Chapter 3, in which we consider another special case: we assume our initial distribution is such that all particles lie along a great circle of the sphere. We prove that any great circle is an invariant set for the equations of motion on the sphere and then re-derive the Vlasov-Poisson system for this distribution. From it, we determine that several quantities are conserved along solutions: total number of particles, total mechanical energy, Casimirs, and entropy. We believe these conservation laws will be helpful in determining the existence of global solutions in the future. After this we linearize the system in preparation for future work. We finish the thesis with a summary and some comments about future explorations into the world of the Vlasov-Poisson system on curved spaces.

Chapter 2

Background

2.1 The Euclidean Vlasov-Poisson system

Within the framework of kinetic theory, the Vlasov-Poisson system for stellar dynamics describes the time-evolution of a group of collisionless particles, whose motion is determined by the gravitational field the particles collectively create, and models the behaviour of large groups of stars or galaxies[1]. We are interested in the initial value problem of the system; namely, given an initial distribution $f_0 = f(0, x, v)$, can we find the phase space distribution f at any time $t \in \mathbb{R}$? The simplest place to start seems to be with what is called the Liouville equation

$$\frac{df}{dt} = 0. \tag{2.1}$$

This equation means physically that if one follows a single particle through phase space, the phase-space density surrounding the particle will not change. This is somewhat difficult to imagine since we are accustomed to seeing only physical space; however, in [5], Binney and Tremaine give an illuminating analogy to this situation. Imagine a large footrace in which all the runners move at constant speeds. Initially, the runners will be clumped together at the start line but their speeds will be widely distributed. At the end of the race, the number of runners crossing the finish line within a short time of each other will be much smaller, but the difference in their speeds will be much smaller as well, thereby conserving the phase-space density.

We can use this conservation equation as a starting place for our derivation. If we

[1]a typical galaxy may contain hundreds of billions of stars and planets, so it is reasonable to use these equations to model them.

substitute $f = f(t, x, v)$ so that f depends on time, position, and velocity, and then use the chain rule we get

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial x}{\partial t} \cdot \nabla_x f + \frac{\partial v}{\partial t} \cdot \nabla_v f = 0. \tag{2.2}$$

Now we have a partial differential equation. But what determines $\frac{\partial x}{\partial t}$ and $\frac{\partial v}{\partial t}$? These functions describe the motion of a single particle in the system, so they are found by solving the equations of motion

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= F(x), \end{aligned} \tag{2.3}$$

where a dot indicates a derivative with respect to time, and F is the sum of the forces acting on a particle of position x. In our system, the only force acting on a particle is the force generated by the whole group of particles. Therefore, the force will be the gradient of the gravitational force function generated by the particle distribution, according to the Poisson equation

$$\Delta U = -4\pi\rho(x) \tag{2.4}$$

coupled with the condition $\rho(x) = \int f dv$, where $F = \nabla U$. In 3-dimensional Euclidean space, for example, the closed set of equations is

$$\begin{gathered} \tfrac{\partial}{\partial t} f(t, x, v) + v \cdot \nabla_{\mathbb{R}^3} f(t, x, v) + \nabla_{\mathbb{R}^3} U(t, x) \cdot \nabla_v f(t, x, v) = 0, \\ -\Delta_{\mathbb{R}^3} U(x) = 4\pi\rho(x), \\ \rho(x) := \textstyle\int_{\mathbb{R}^3} f(x, v) dv, \end{gathered} \tag{2.5}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is particle position, $v = (v_1, v_2, v_3) = (\dot{x}_1, \dot{x}_2, \dot{x}_3) \in \mathbb{R}^3$ is particle velocity, $t \in \mathbb{R}$ is time, $f = f(t, x, v)$ is the distribution function (or phase-space density), $U(t, x)$ is gravitational force function, and $\rho(t, x)$ is spatial density. The symbol $\nabla_{\mathbb{R}^3}$ indicates the gradient operator and $\Delta_{\mathbb{R}^3}$ represents the Laplacian operator so that

$$\nabla_{\mathbb{R}^3} f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right) \tag{2.6}$$

and

$$\Delta_{\mathbb{R}^3} f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}. \tag{2.7}$$

The operator ∇_v is

$$\nabla_v = \left(\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}, \frac{\partial}{\partial v_3} \right), \tag{2.8}$$

and the symbol $\cdot$ stands for the dot product in $\mathbb{R}^3$ so that $x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3$. A classical solution to the system is described in the following definition, taken from [19].

Definition 1. *A function $f : I \times \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ is a classical solution of the Vlasov Poisson system on the open interval $I \subset \mathbb{R}$ if the following holds:*

(i) The function f is continuously differentiable with respect to all its variables.

(ii) The induced spatial density $\rho = \rho_f$ and force function $U = U_f$ exist on $I \times \mathbb{R}^3$. They are continuously differentiable, and U is twice continuously differentiable with respect to x.

(iii) For every compact subinterval $J \subset I$ the field $\nabla_x U$ is bounded on $J \times \mathbb{R}^3$.

(iv) The functions f, ρ, U satisfy the Vlasov-Poisson system on $I \times \mathbb{R}^3 \times \mathbb{R}^3$.

Here, $\nabla_x U$ is the gradient of U with respect to the position variable. Since the force function U is dependent on the spatial density ρ, which in turn is dependent on the phase-space density f, the system reduces to a non-linear partial differential equation on f. As such, it took many years and many researchers to prove the existence of global solutions to the Vlasov-Poisson system in Euclidean space. Before the global existence problem could be settled, several less general problems were considered. For instance, in 1952 Kurth gave the first proof of local existence of solutions, [3]. In 1977 Batt proved global existence for spherically symmetric solutions [2], and in 1985 Bardos and Degond proved global existence of solutions with the assumption of small initial data, [12]. The existence of global solutions with general initial data was proved by Pfaffelmoser in 1990 and independently by Lions and Perthame in 1991, [18] [15].

2.2 Linear Landau Damping

One of the most interesting qualities of the Vlasov-Poisson system is its ability to predict a phenomenon known as Landau damping which occurs in isolated, collisionless particle systems. Physically speaking, the system of particles can be thought of as having two parts: the first part is the background field generated by the moving particles, and the second is the particles themselves. If a particle has a speed which is close to but greater than the wave speed of the field, then it will lose energy to the wave and the field will be anti-damped. Conversely, if the particle has a speed which is close to but lesser than the wave speed of the field, then it will gain energy from the wave and the field will be damped. This phenomenon is highly counter-intuitive on a macroscopic scale since there are no external forces (or collisions) present to damp the field; nonetheless, in 1946, Landau predicted this behaviour through his purely mathematical study of the Vlasov-Poisson system and his results were confirmed experimentally for the plasma case in 1964 by Malmberg and Wharton, [16]. In the gravitational case, Landau damping and other so-called "violent relaxation" processes[2] lend an explanation for the short relaxation times of galaxies. We now briefly summarize[3] the derivation of Landau damping as presented in [22] in the plasma physics setting, starting with the usual Vlasov-Poisson system

$$\begin{gathered} \partial_t f + v \cdot \partial_x \cdot f + F(t,x) \cdot \partial_v f = 0, \\ F = -\partial_x W *_x \rho, \\ \rho(t,x) := \textstyle\int_{\mathbb{R}} f(t,x,v) dv, \end{gathered} \tag{2.9}$$

where the subscript x, for instance, denotes that the derivative or convolution is taken with respect to the spatial variable. If one compares this set of equations to the set presented in the last section, there is a slight difference– the equations use F, the force field due to the particles, rather than U, the force function due to the particles. Thus, in this discussion, we assume the solution to the Poisson equation has the form $U = W * \rho$ and the force is then given by $F = -\partial_x U$. Spatially homogeneous stationary solutions to (2.9) are found to be of the form $f^0(v)$ and the equations are linearized about these solutions by setting $f = f^0(v) + h(t,x,v)$, where $\|h\| << 1$ in some sense. Since f^0 does not contribute to the force field[4], substituting this perturbed f into (2.9) yields

$$\partial_t h + v \cdot \partial_x h + F[h] \cdot \partial_v (f^0 + h) = 0, \tag{2.10}$$

[2] see [17].

[3] we take here the one-dimensional case but note that in [22], the calculations are based in d dimensional space.

[4] indeed, $F = -\partial_x W * \int f^0(v)dv = -W * \int \partial_x f^0(v)dv = 0$.

where $F[h] = -\partial_x W *_{x,v} h$. Since h is small, the quadratic term $F[h] \cdot \partial_v h$ will be small compared to the other terms, so our linearized equation is

$$\partial_t h + v \cdot \partial_x h + F[h] \cdot \partial_v f^0 = 0. \tag{2.11}$$

Solving this linear equation by the method of characteristics, taking the Fourier transform in x and v, then integrating in v results in an equation on the Fourier transform of the spatial density associated with h in which the solution modes evolve in time independently of each other. As such, we can fix the mode and study the equation[5] using the Fourier-Laplace transform. The final outcome is eventually a stability condition on f^0, taken from Proposition 3.7 in [22]:

Proposition 1 (Sufficient condition for stability in dimension 1)**.** *If W is an even potential with $\nabla W \in L^1(\mathbb{T})$, and $f^0 = f^0(v)$ is an analytic[6] profile on $\mathbb{R}$ such that $(f^0)'(v) = O(1/|v|))$, then the Vlasov equation with interaction W, linearized near f^0, is linearly stable under analytic perturbations as soon as the condition*

$$\forall \omega \in \mathbb{R}, \qquad (f^0)'(\omega) = 0 \implies \widehat{W}(k) \int \frac{(f^0)'(v)}{v - \omega} dv < 1 \tag{2.12}$$

is satisfied for all $k \neq 0$.

Here $\widehat{W}$ is the Fourier transform of W in the position variable and $L^1(\mathbb{T})$ stands for the space of Lebesgue integrable functions on the one-dimensional torus. The stability mentioned in the proposition indicates that the force due to the perturbed distribution, $F[h]$, decays exponentially with time.

As an example, take a Newtonian gravitational interaction, $\widehat{W}(k) = -\dfrac{1}{|k|^2}$, and a Gaussian stationary solution $f^0(v) = \rho^0 \sqrt{\dfrac{\beta}{2\pi}} e^{-\beta v^2/2}$. As long as $\rho^0 \beta < |k|^2$, the conditions in the proposition are satisfied and we get stability. The result of numeric simulations of W, ρ combinations similar to those completed in [23] produce the images in Figure 2.1, which clearly show the exponential decay of $F[h]$ in time. The damping makes sense according to our physical understanding since in the Gaussian distribution the particles having speeds slightly smaller than the speed of the wave are more numerous than the particles having speeds slightly larger than the speed of

[5] it turns out to be a Volterra equation.

[6] locally given by a convergent power series

the wave, therefore resulting in a net energy loss of the wave, and a damping of the force.

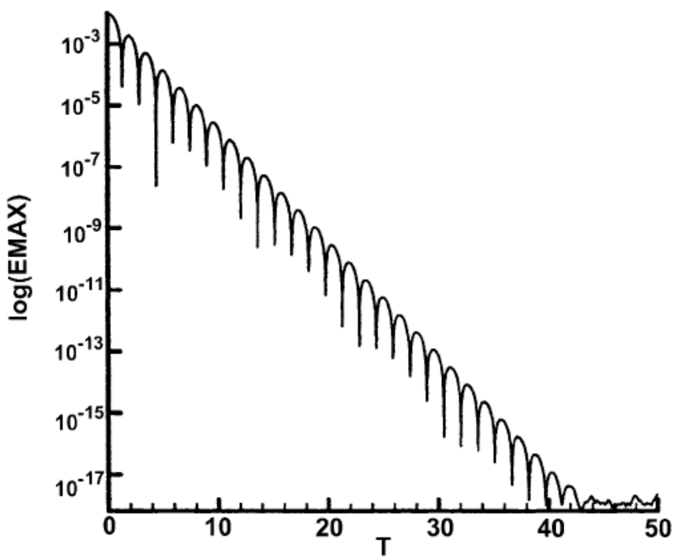

Figure 2.1: Data from numerical solutions to the linear Vlasov-Poisson system for Coulomb interactions and Gaussian equilibrium solutions, taken from [23]. Here, the vertical axis measures the logarithm of the maximum electric force field and the horizontal axis is time. From the figure it is clear that the maximum electric field decays to zero exponentially with time.

2.3 Local Coordinates on $\mathbb{S}^2$

In our problem, we are interested in the movement of a group of particles on the 2-sphere, so that each particle's position vector $x = (x_1, x_2, x_3)$ is contained in $\mathbb{S}^2$ where

$$\mathbb{S}^2 := \left\{ x \,\middle|\, x_1^2 + x_2^2 + x_3^2 = 1 \right\}.$$

We embed the 2-sphere in 3-dimensional Euclidean space, $\mathbb{R}^3$, but note that the particles cannot move into this external space and no forces they generate can cross into $\mathbb{R}^3 \setminus \mathbb{S}^2$. Instead, all movement is within $\mathbb{S}^2$ and all force field lines are along the geodesics of the 2-sphere as in Figure 2.2. This means a particle at position x has a velocity which exists in the tangent space of the sphere at x, denoted by $T_x\mathbb{S}^2$. Another way to describe this property is that for each particle on the sphere with phase space coordinates (x, v), we have $x \cdot v = 0$. Since we are working on a two

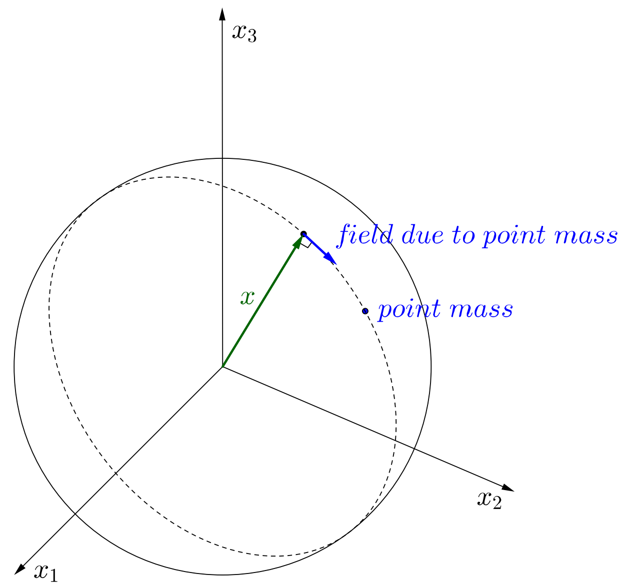

Figure 2.2: Gravitational field at $x \in \mathbb{S}^2$ due to a point mass.

dimensional spherical space, calculations will often be simplified if we use spherical coordinates and parametrize the surface using two angular variables. In the following, we denote the position vector by

$$x = x(\theta) = (x_1, x_2, x_3) = (\sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2, \cos\theta_1) \tag{2.13}$$

where θ_1 is the zenith angle and θ_2 is the azimuthal angle as in Figure 2.3. Using these definitions, we can write the angles θ_1, θ_2 in terms of the rectangular coordinates x_1, x_2, x_3 according to[7]

$$\begin{aligned} \theta_1 &= \arccos\left(\frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\right) \\ \theta_2 &= \arctan\left(\frac{x_2}{x_1}\right) \end{aligned} \tag{2.14}$$

[7]Here we note that we technically must extend the arctan function periodically so that its range is $(-\pi, \pi]$.

for $\theta_1 \in [0, \pi]$ and $\theta_2 \in (-\pi, \pi]$. We define $\hat{\mathrm{e}}_r$, $\hat{\mathrm{e}}_1$, and $\hat{\mathrm{e}}_2$ to be orthogonal unit vectors in the direction of increasing r, θ_1, and θ_2, respectively according to

$$\begin{bmatrix} \hat{\mathrm{e}}_r \\ \hat{\mathrm{e}}_1 \\ \hat{\mathrm{e}}_2 \end{bmatrix} = \begin{bmatrix} \sin\theta_1\cos\theta_2 & \sin\theta_1\sin\theta_2 & \cos\theta_1 \\ \cos\theta_1\cos\theta_2 & \cos\theta_1\sin\theta_2 & -\sin\theta_1 \\ -\sin\theta_2 & \cos\theta_2 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} \tag{2.15}$$

where $\hat{x}_1, \hat{x}_2$, and $\hat{x}_3$ are the usual rectangular Cartesian unit vectors, and define

$$v = \omega_1\hat{\mathrm{e}}_1 + \omega_2\sin\theta_1\hat{\mathrm{e}}_2,$$

where $\omega_1 \in \mathbb{R}$ and $\omega_2 \in \mathbb{R}$. Acceleration is then

$$a = -(\omega_2^2\sin^2\theta_1 + \omega_1^2)\hat{\mathrm{e}}_r + (\dot{\omega}_1 - \omega_2^2\sin\theta_1\cos\theta_1)\hat{\mathrm{e}}_1 + (\dot{\omega}_2\sin\theta_1 + 2\omega_1\omega_2\cos\theta_1)\hat{\mathrm{e}}_2. \tag{2.16}$$

The ambient Euclidean space $\mathbb{R}^3$ in which the sphere is embedded induces a natural

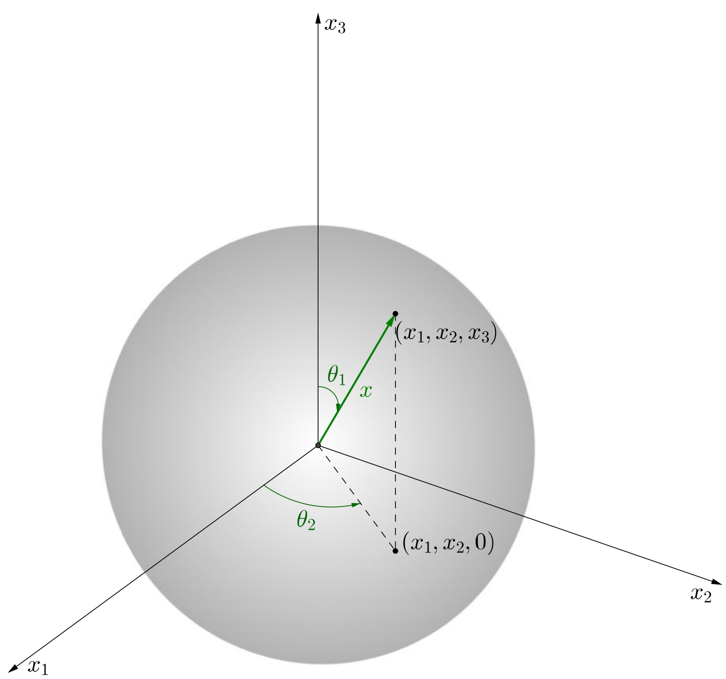

Figure 2.3: Spherical coordinates for the unit 2-sphere.

Riemannian metric on the sphere. We know the Euclidean metric is given by the quadratic form

$$ds^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 \tag{2.17}$$

and since our Riemannian metric is induced by the ambient space $\mathbb{R}^3$, we can obtain it by restricting (2.17) to $\mathbb{S}^2$ through calculating the differential of (2.13), i.e.

$$\begin{aligned} ds^2 \mid_{\mathbb{S}^2} &= (\cos\theta_1 \cos\theta_2 d\theta_1 - \sin\theta_1 \sin\theta_2 d\theta_2)^2 \\ &\quad + (\cos\theta_1 \sin\theta_2 d\theta_1 + \sin\theta_1 \cos\theta_2 d\theta_2)^2 + (-\sin\theta_1 d\theta_1)^2 \\ &= (d\theta_1)^2 + \sin^2\theta_1 (d\theta_2)^2. \end{aligned} \tag{2.18}$$

Based on this, we can define the matrix formed by the components of the standard metric tensor on the 2-sphere as

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta_1 \end{bmatrix} \tag{2.19}$$

with inverse

$$g^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{\sin^2\theta_1} \end{bmatrix}. \tag{2.20}$$

Tangent vectors are given by

$$\begin{aligned} \hat{\theta}_1 &= \hat{\mathrm{e}}_1 \\ \hat{\theta}_2 &= \sin\theta_1 \ \hat{\mathrm{e}}_2. \end{aligned} \tag{2.21}$$

The gradient on $\mathbb{S}^2$ is then

$$\nabla_{\mathbb{S}^2} f = g^{11} \frac{\partial f}{\partial \theta_1} \hat{\theta}_1 + g^{22} \frac{\partial f}{\partial \theta_2} \hat{\theta}_2 = \begin{bmatrix} \dfrac{\partial f}{\partial \theta_1} \\ \left(\dfrac{1}{\sin\theta_1}\right) \dfrac{\partial f}{\partial \theta_2} \end{bmatrix}, \tag{2.22}$$

the divergence on $\mathbb{S}^2$ is

$$\begin{aligned} \operatorname{div}_{\mathbb{S}^2} F &= \sum_{j=1}^{2} \left(\frac{1}{\sqrt{\det g}}\right) \frac{\partial}{\partial \theta_j} \left(\sqrt{\det g}\ F_j\right) \\ &= \frac{1}{\sin\theta_1} \frac{\partial}{\partial \theta_1} (\sin\theta_1 F_1) + \frac{1}{\sin\theta_1} \frac{\partial}{\partial \theta_2} (\sin\theta_1 F_2) \\ &= \frac{1}{\sin\theta_1} \frac{\partial}{\partial \theta_1} (\sin\theta_1 F_1) + \frac{\partial}{\partial \theta_2} F_2, \end{aligned} \tag{2.23}$$

and the Laplacian (Laplace-Beltrami operator) on $\mathbb{S}^2$ is

$$\begin{aligned}
\Delta_{\mathbb{S}^2} f = \mathrm{div}_{\mathbb{S}^2}\left(\nabla_{\mathbb{S}^2} f\right) &= \frac{1}{\sqrt{\det g}} \sum_{i,j} \frac{\partial}{\partial \theta_i}\left(\sqrt{\det g}\; g^{ij}\, \frac{\partial f}{\partial \theta_j}\right) \\
&= \frac{1}{\sin\theta_1}\left[\frac{\partial}{\partial\theta_1} \sin\theta_1 \frac{\partial f}{\partial \theta_1} + \frac{\partial}{\partial \theta_2} \sin\theta_1 \left(\frac{1}{\sin^2\theta_1}\right) \frac{\partial f}{\partial\theta_2}\right] \\
&= \frac{1}{\sin\theta_1} \frac{\partial}{\partial\theta_1}\left(\sin\theta_1 \frac{\partial f}{\partial\theta_1}\right) + \frac{1}{\sin^2\theta_1} \frac{\partial^2 f}{\partial \theta_2^2} \\
&= \cot\theta_1 \frac{\partial f}{\partial\theta_1} + \frac{\partial^2 f}{\partial\theta_1^2} + \frac{1}{\sin^2\theta_1}\frac{\partial^2 f}{\partial\theta_2^2}
\end{aligned} \tag{2.24}$$

in local coordinates, where $\det g$ is the determinant of (2.19). The volume form on $\mathbb{S}^2$ is calculated via

$$\Omega = \sqrt{|\det g|}\; d\theta_1 \wedge d\theta_2 = \sin\theta_1 d\theta_1 d\theta_2, \tag{2.25}$$

see [21], for instance. A note about calculations on $\mathbb{S}^2$: there are in general two ways to complete calculations on the sphere. The first is to extend the functions by homogeneity to $\mathbb{R}^3$ (i.e. replace x with $x/|x|$ where the form of the function allows), do computations using the standard $\mathbb{R}^3$ operators, then restrict back to the space $\mathbb{S}^2$ using $|x|^2 = 1,\ x \cdot v = 0$. The second method is to use the local operators (as defined above) directly.

Chapter 3

Gravitation and the Vlasov-Poisson system on the unit 2-sphere

In studying the Vlasov-Poisson system on any space, it is crucial to understand the physical laws that govern movement on the space. Therefore, we dedicate a substantial amount of time to understanding how gravity works on the 2-sphere. We begin with a very simple example involving a point mass on the sphere in the hope that examining this problem will help us when we attempt to derive the expressions for gravitational fields due to arbitrary distributions. After our exploration of gravity on the sphere, we derive the equations of motion of a particle due to a gravitational field and conclude the chapter with the new form of the Vlasov-Poisson system, which can be applied to a wide range of mass distributions on the sphere.

3.1 Gravity on the unit 2-sphere

In the last chapter, we explained that gravitational field lines on the sphere are bent so that the gravitational force between any two masses lies along the geodesic connecting them. In this section, we use this property to develop the tools we need to derive the Poisson equation and subsequently find the form of the gravitational potential on the sphere that is required to close the Vlasov-Poisson system.

3.1.1 Gravitational field due to a point mass

As a motivational example, consider a particle of mass m placed at the north pole of the unit sphere (i.e. at $\theta_1 = 0$). Gauss's law says if we choose any Gaussian

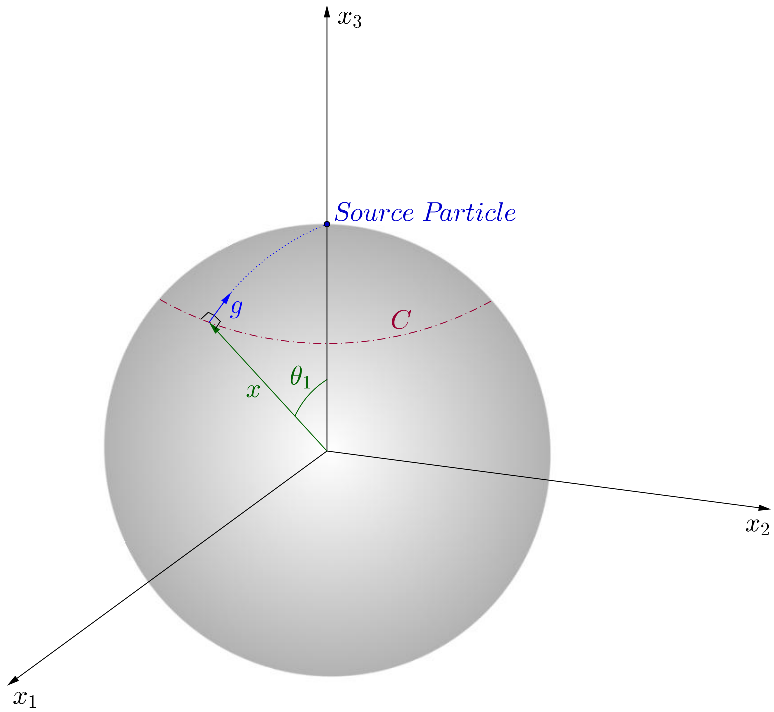

Figure 3.1: Gaussian curve C for a point mass located at the north pole of $\mathbb{S}^2$.

surface surrounding the particle, then the negative of the gravitational flux through the Gaussian surface will be proportional to the enclosed mass[1]. So we have

$$-\Phi = -\oint g \cdot dl = m, \tag{3.1}$$

where $\Phi = \oint g \cdot dl$ is the flux through the Gaussian surface, g is the gravitational field strength due to the mass, and m is the mass of the particle. Since everything is constrained to the sphere, including the gravitational field, our Gaussian surface will be a closed curve in $\mathbb{S}^2$. Let us choose our curve, C, to be a circle as pictured in Figure 3.1 so that the circle's equation in $\mathbb{S}^2$ is $\theta_1 = \textit{constant} \leq \pi/2$. Then the gravitational field at every point on the circle is perpendicular to C and constant due to symmetry, so we can simplify our expression in (3.1) to

$$|g||C| = m. \tag{3.2}$$

[1]Note here that we have chosen units such that the proportionality constant is equal to one.

Noticing that the circumference of our circle is $2\pi \sin\theta_1$ allows us to write

$$2\pi \sin\theta_1 |g| = m, \tag{3.3}$$

which implies

$$|g| = \frac{m}{2\pi \sin\theta_1}.$$

As already stated, since the point mass has circular symmetry and is positioned at the north pole, the gravitational field will be directed along longitudinal lines, so our gravitational field as a vector is

$$g = -\frac{m}{2\pi \sin\theta_1}\hat{\mathrm{e}}_1 \tag{3.4}$$

where the negative sign indicates that gravitation is attractive. This gives some motivation for the following derivations of Gauss's law and the Poisson equation on $\mathbb{S}^2$.

3.1.2 Gauss's Law

Gauss's law is a physical law that relates the amount of mass in a region to the flux of the gravitational field out of the region. In the calculation above, we used the integral form of the law, but there is also a differential form:

$$\left(-\mathrm{div}_{\mathbb{S}^2}\, F\right)(x) = \rho(x) \tag{3.5}$$

for all $x \in \mathbb{S}^2$. The equation means that the number of gravitational field lines leading to any position x is equal to the mass density at that point.

3.1.3 The Poisson equation

In this section, we derive the Poisson equation on $\mathbb{S}^2$ so that we have access to the most general way to find the gravitational field on the sphere.

Lemma 1. *The Poisson equation on* $\mathbb{S}^2$ *is*

$$-\Delta_{\mathbb{S}^2}\, U = \rho, \tag{3.6}$$

where $U : \mathbb{S}^2 \to \mathbb{R}$, $\Delta_{\mathbb{S}^2}$ *is the Laplace-Beltrami operator on* $\mathbb{S}^2$ *given by* (2.24), *and*

$\rho : \mathbb{S}^2 \to \mathbb{R}$ is a C^1 function of x.

Proof. Any conservative force, such as gravitation, on a Riemannian manifold can be written as the gradient of some force function[2]. So we have

$$F = \nabla_{\mathbb{S}^2} U. \tag{3.7}$$

Substituting this into (3.5) yields

$$-\mathrm{div}_{\mathbb{S}^2} (\nabla_{\mathbb{S}^2} U) = \rho. \tag{3.8}$$

For $U \in C^\infty$, we have $\mathrm{div}_{\mathbb{S}^2} (\nabla_{\mathbb{S}^2} U) = \Delta_{\mathbb{S}^2} U$, where Δ is the Laplace-Beltrami operator on the sphere given by (2.24). Therefore we can rewrite the above as

$$-\Delta_{\mathbb{S}^2} U = \rho, \tag{3.9}$$

which we refer to as the Poisson equation on $\mathbb{S}^2$. □

Let's consider the well-posedness of this equation. If we integrate our Poisson equation (3.6) over the sphere, we get

$$-\int_{\mathbb{S}^2} \Delta_{\mathbb{S}^2} U = \int_{\mathbb{S}^2} \rho \tag{3.10}$$

and rewriting $\Delta_{\mathbb{S}^2} = \mathrm{div}_{\mathbb{S}^2} (\nabla_{\mathbb{S}^2} U)$ gives us

$$-\int_{\mathbb{S}^2} \mathrm{div}_{\mathbb{S}^2} (\nabla_{\mathbb{S}^2} U) = \int_{\mathbb{S}^2} \rho. \tag{3.11}$$

Now the divergence theorem[3] on $\mathbb{S}^2$ says that the left hand side must be equal to

$$-\int_{\partial\mathbb{S}^2} \nabla_{\mathbb{S}^2} U \cdot \hat{n} \, dL \tag{3.12}$$

but $\partial\mathbb{S}^2$ is empty, so the left hand side of (3.10) is zero. Substituting this yields

$$0 = \int_{\mathbb{S}^2} \rho. \tag{3.13}$$

[2]see for instance Proposition 5.60 of [10]
[3]see Appendix A.1.

Thus, in order for our Poisson equation to be well defined, we require $\int_{\mathbb{S}^2} \rho = 0$, a property which is physically impossible since we identify ρ with a mass density. However, if we take out the mean of ρ, then (3.6) makes sense. This trick is known as *Jean's swindle* in galactic dynamics, see [17] or [5].

3.1.4 Solution to the Poisson equation

Lemma 2. *A solution to* (3.6)*, the Poisson equation on* $\mathbb{S}^2$*, is given by*

$$U(x) = \frac{1}{2\pi} \int_{\mathbb{S}^2} \log\cot\left[\frac{d(x,y)}{2}\right] \rho(y)dy, \tag{3.14}$$

where $d(x,y)$ *is the geodesic distance between* x *and* y *on* $\mathbb{S}^2$*.*

Proof. According to [7], a spherically symmetric fundamental solution to the Laplacian on the unit 2-sphere is given by

$$\mathcal{G}(x,y) = \frac{1}{2\pi} \log\cot\left[\frac{d(x,y)}{2}\right], \tag{3.15}$$

where $d(x,y) = \cos^{-1}(x \cdot y)$ is the geodesic distance between x and y on the unit sphere. In other words, $\mathcal{G}$ solves

$$-\Delta_{\mathbb{S}^2}\mathcal{G}(x,y) = \frac{\delta(\theta_1 - \theta_1') \otimes \delta(\theta_2 - \theta_2')}{\sin\theta_1}, \tag{3.16}$$

where $x = (\sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2, \cos\theta_1)$, $y = (\sin\theta_1'\cos\theta_2', \sin\theta_1'\sin\theta_2', \cos\theta_1')$. Since a fundamental solution of the Laplacian must satisfy

$$\int_{\mathbb{S}^2} (-\Delta_{\mathbb{S}^2}\varphi)(y)\mathcal{G}(x,y)dy = \varphi,$$

for any test function[4] φ, a solution to the Poisson equation is

$$U(x) = \int_{\mathbb{S}^2} \mathcal{G}(x,y)\rho(y)dy. \tag{3.17}$$

Substituting (3.15) into this expression yields the desired result. $\square$

[4] a C^∞ function with compact support.

If we wish, we can use $d(x,y) = \cos^{-1}(x \cdot y)$ to rewrite U as

$$U(x) = \frac{1}{2\pi} \int_{\mathbb{S}^2} \rho(y) \log \cot \left[\frac{1}{2} \cos^{-1}(x \cdot y)\right] dy \tag{3.18}$$

and use trigonometric identities to write

$$\cot \left[\frac{1}{2} \cos^{-1}(x \cdot y)\right] = \frac{1 + \cos\cos^{-1}(x \cdot y)}{\sin\cos^{-1}(x \cdot y)} = \frac{1 + (x \cdot y)}{\sqrt{1 - (x \cdot y)^2}}, \tag{3.19}$$

so we get

$$U(x) = \frac{1}{2\pi} \int_{\mathbb{S}^2} \rho(y) \log \left(\frac{1 + (x \cdot y)}{\sqrt{1 - (x \cdot y)^2}}\right) dy, \tag{3.20}$$

for $x, y \in \mathbb{S}^2$.

Corollary. *The solution to the Poisson equation on $\mathbb{S}^2$ in local spherical coordinates is given by*

$$U\left(x(\theta)\right) = \frac{1}{2\pi} \iint \rho(y(\theta')) \log \left(\frac{1 + (x(\theta) \cdot y(\theta'))}{\sqrt{1 - (x(\theta) \cdot y(\theta'))^2}}\right) \sin\theta_1' d\theta_1' d\theta_2'. \tag{3.21}$$

Proof. Parametrizing x and y in (3.20) using $x = (\sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2, \cos\theta_1)$ and $y = (\sin\theta_1' \cos\theta_2', \sin\theta_1' \sin\theta_2', \cos\theta_1')$ gives the desired expression, where the area unit dy has been transformed according to

$$\begin{aligned} dy &= \left|\left(\frac{\partial y}{\partial \theta_1}\right) \times \left(\frac{\partial y}{\partial \theta_2}\right)\right| d\theta_1' d\theta_2' \\ &= \left|(\cos\theta_1' \cos\theta_2', \cos\theta_1' \sin\theta_2', -\sin\theta_1') \times (-\sin\theta_1' \sin\theta_2', \sin\theta_1' \cos\theta_2', 0)\right| d\theta_1' d\theta_2' \\ &= \sin\theta_1' d\theta_1' d\theta_2'. \end{aligned}$$

$\square$

3.1.5 Gravitational potential: Homogeneity and Euler's Formula

The force function U given above is defined only for $x \in \mathbb{S}^2$; however, we can easily extend it to values of x that are in $\mathbb{R}^3 \setminus \mathbb{S}^2$. From the above expressions for U, we see that $U(kx) = U(x)$ for all $k \neq 0$, which by definition means that U is a homogeneous function of degree 0 for all $x \in \mathbb{S}^2$. We extend the force function as

$$\bar{U}(x) = U\left(\frac{x}{|x|}\right) = \frac{1}{2\pi}\int_{\mathbb{S}^2} \rho\left(\frac{y}{|y|}\right) \log\left(\frac{|x||y| + (x\cdot y)}{\sqrt{|x|^2|y|^2 - (x\cdot y)^2}}\right) dy \tag{3.22}$$

so that $\bar{U}$ is defined on $x \in \mathbb{R}^3$, coincides with U for $x \in \mathbb{S}^2$, and is also homogeneous of degree 0. Due to this homogeneity, we can apply Euler's formula for homogeneous functions[5] to get

$$x \cdot \nabla_{\mathbb{R}^3}\bar{U}(x) = 0, \tag{3.23}$$

a result we will use later. If we restrict x back to $\mathbb{S}^2$, the formula reads

$$x \cdot \nabla_{\mathbb{S}^2} U(x) = 0 \tag{3.24}$$

and means physically that the gravitational force is perpendicular to x for $x \in \mathbb{S}^2$.

3.1.6 Gravitational field due to a point mass, revisited

As a check, let us calculate the gravitational field due to a point mass at the north pole via our formula for the gravitational force function, (3.21) – it should match (3.4). Let the point mass be located at $\theta = (0, 0)$. The distribution describing a particle of unit mass at this point is

$$\rho(y) = \frac{\delta(\theta_1')}{2\pi\sin\theta_1'} \tag{3.25}$$

so that

$$\int_{\mathbb{S}^2} \rho(y)dy = \int_{-\pi}^{\pi}\int_0^{\pi} \rho(y(\theta'))\sin\theta_1' d\theta_1' d\theta_2' = 1.$$

Substituting this expression for ρ into (3.21) gives us

$$\begin{aligned} U(x(\theta)) &= \frac{1}{2\pi}\iint \frac{\delta(\theta_1')}{2\pi\sin\theta_1'} \log\left[\frac{1 + x(\theta)\cdot y(\theta')}{\sqrt{1 - (x(\theta)\cdot y(\theta'))^2}}\right] \sin\theta_1' d\theta_1' d\theta_2' \\ &= \frac{1}{2\pi}\iint \frac{\delta(\theta_1')}{2\pi} \log\left[\frac{1 + x(\theta)\cdot y(\theta')}{\sqrt{1 - (x(\theta)\cdot y(\theta'))^2}}\right] d\theta_1' d\theta_2'. \end{aligned} \tag{3.26}$$

[5] see for example [8].

When $\theta_1' = 0$, we have

$$x \cdot y = \sin\theta_1 \cos\theta_2 \sin(0)\cos(\theta_2') + \sin\theta_1 \sin\theta_2 \sin(0)\sin(\theta_2') + \cos\theta_1 \cos(0) = \cos\theta_1$$

so that (3.26) becomes

$$\begin{aligned} U(x(\theta)) &= \frac{1}{2\pi}\log\left[\frac{1+\cos\theta_1}{\sqrt{1-\cos^2\theta_1}}\right] \\ &= \frac{1}{2\pi}\log\left[\frac{1+\cos\theta_1}{\sin\theta_1}\right] \\ &= \frac{1}{2\pi}\log\left[\cot\left(\frac{\theta_1}{2}\right)\right]. \end{aligned} \tag{3.27}$$

Using (2.22) we then calculate the field generated by U to be

$$\begin{aligned} \nabla_{\mathbb{S}^2} U(x(\theta)) &= \frac{\partial U}{\partial \theta_1}\hat{\mathrm{e}}_1 \\ &= \frac{1}{2\pi}\left[\left(\tan\frac{\theta_1}{2}\right)\left(-\csc^2\frac{\theta_1}{2}\right)\left(\frac{1}{2}\right)\right]\hat{\mathrm{e}}_1 \\ &= -\frac{1}{2\pi}\left(\frac{1}{\sin\theta_1}\right)\hat{\mathrm{e}}_1 \\ &= -\frac{m}{2\pi\sin\theta_1}\hat{\mathrm{e}}_1, \end{aligned} \tag{3.28}$$

which agrees with (3.4).

3.1.7 Gravitational force due to an arbitrary distribution

Proposition 2. *The gravitational force on a unit mass particle located at $x \in \mathbb{S}^2$ due to a spatial distribution $\rho : \mathbb{S}^2 \to \mathbb{R}$ is given by*

$$\nabla_{\mathbb{S}^2} U = \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{y - (x\cdot y)x}{[1-(x\cdot y)^2]}\rho(y)dy. \tag{3.29}$$

Proof 1. We can extend U by homogeneity as in (3.22) by replacing x with $\dfrac{x}{|x|}$ so that (3.20) becomes

$$\bar{U}(x) = \frac{1}{2\pi}\int_{\mathbb{S}^2} \rho(y)\log\left(\frac{|x| + (x\cdot y)}{\sqrt{|x|^2 - (x\cdot y)^2}}\right) dy. \tag{3.30}$$

From this we can calculate using the usual gradient in $\mathbb{R}^3$ that

$$\nabla_{\mathbb{R}^3}\bar{U} = \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{|x|^2 y - (x\cdot y)x}{[|x|^2 - (x\cdot y)^2]}\rho\,(y)\,dy, \tag{3.31}$$

so that after restricting to the sphere (i.e. invoking $|x|^2 = 1$), we get

$$\nabla_{\mathbb{S}^2} U = \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{y - (x\cdot y)x}{[1 - (x\cdot y)^2]}\rho(y)dy, \tag{3.32}$$

$\square$

as required.

Proof 2. This force can also be calculated using $\nabla_{\mathbb{S}^2}$ directly. Applying the gradient on $\mathbb{S}^2$ from (2.22) to (3.21) yields

$$\begin{aligned} \nabla_{\mathbb{S}^2} U(x(\theta)) &= \nabla_{\mathbb{S}^2}\left[\frac{1}{2\pi}\iint \rho(y)\log\left(\frac{1+(x\cdot y)}{\sqrt{1-(x\cdot y)^2}}\right)\sin\theta_1' d\theta_1' d\theta_2'\right] \\ &= \frac{1}{2\pi}\iint \rho(y)\nabla_{\mathbb{S}^2}\left[\log\left(\frac{1+(x\cdot y)}{\sqrt{1-(x\cdot y)^2}}\right)\right]\sin\theta_1' d\theta_1' d\theta_2', \end{aligned} \tag{3.33}$$

where x and y are functions of θ_1, θ_2 and θ_1', θ_2', respectively, and the gradient is taken with respect to the unprimed coordinate system. Using (2.22) we get that

$$\begin{aligned}
&\nabla_{\mathbb{S}^2}\left[\log\left(\frac{1+(x\cdot y)}{\sqrt{1-(x\cdot y)^2}}\right)\right] \\
=&\ \left[\frac{\sqrt{1-(x\cdot y)^2}}{1+x\cdot y}\right]\left[\frac{\sqrt{1-(x\cdot y)^2}+(1+x\cdot y)\left[1-(x\cdot y)^2\right]^{-1/2}(x\cdot y)}{1-(x\cdot y)^2}\right]\frac{\partial(x\cdot y)}{\partial\theta_1}\hat{\mathrm{e}}_1 \\
&+\frac{1}{\sin\theta_1}\left[\frac{\sqrt{1-(x\cdot y)^2}}{1+x\cdot y}\right]\left[\frac{\sqrt{1-(x\cdot y)^2}+(1+x\cdot y)\left[1-(x\cdot y)^2\right]^{-1/2}(x\cdot y)}{1-(x\cdot y)^2}\right]\frac{\partial(x\cdot y)}{\partial\theta_2}\hat{\mathrm{e}}_2 \\
=&\ \left[\frac{1-(x\cdot y)^2+x\cdot y+(x\cdot y)^2}{\left[1-(x\cdot y)^2\right]\left[1+(x\cdot y)\right]}\right]\left[\frac{\partial(x\cdot y)}{\partial\theta_1}\hat{\mathrm{e}}_1+\frac{1}{\sin\theta_1}\frac{\partial(x\cdot y)}{\partial\theta_2}\hat{\mathrm{e}}_2\right] \\
=&\ \left[\frac{1}{1-(x\cdot y)^2}\right]\nabla_{\mathbb{S}^2}(x\cdot y).
\end{aligned} \tag{3.34}$$

A short calculation[6] shows $\nabla_{\mathbb{S}^2}(x\cdot y) = y-(x\cdot y)x$ for $x, y \in \mathbb{S}^2$, so substituting into (3.33) we get (3.29) and our proposition is proved. □

3.2 Equations of Motion on the unit 2-sphere

The equations of motion are expressions of Newton's second law: the acceleration of a particle is proportional to the net force exerted upon it. In this section, we use Lagrangian mechanics to derive the equations of motion for a single particle in a gravitational field $\nabla_{\mathbb{S}^2}U$ on the unit 2-sphere.

Proposition 3. *The equations of motion for a particle with position $x = x(\theta)$ on the sphere $\mathbb{S}^2$ under the effect of a force function $U : \mathbb{S}^2 \to \mathbb{R}$ are*

$$\begin{aligned}
\dot{\theta}_1 &= \omega_1 \\
\dot{\theta}_2 &= \omega_2 \\
\dot{\omega}_1 &= \frac{\partial U}{\partial\theta_1}+\omega_2^2\sin\theta_1\cos\theta_1 \\
\dot{\omega}_2 &= \left(\frac{1}{\sin^2\theta_1}\right)\frac{\partial U}{\partial\theta_2}-2\omega_1\omega_2\cot\theta_1
\end{aligned} \tag{3.35}$$

in local coordinates.

[6]see Appendix A.4 for proof.

Proof 1. Let $x = x(\theta)$ be the coordinate of a single particle on our manifold $\mathbb{S}^2$ and assume the only force acting on the particle is the gradient of our gravitational force function, U. Assume further that the particle has unit mass. Then the kinetic energy, T, of the particle is

$$T = \frac{1}{2}|v|^2,$$

where v is the velocity of the particle. We can parametrize the particle's spatial coordinates using

$$x = \hat{\mathrm{e}}_r = (\sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2, \cos\theta_1).$$

Differentiating with respect to time[7] yields $v = \dot{\theta}_1\hat{\mathrm{e}}_1 + \dot{\theta}_2\sin\theta_1\hat{\mathrm{e}}_2$ and so we can write the kinetic energy as

$$T = \frac{1}{2}(\dot{\theta}_2^2\sin^2\theta_1 + \dot{\theta}_1^2). \tag{3.36}$$

We denote the gravitational force function at x by $U(x(\theta))$ and define potential energy, V, to be the negative of this force function, so we have

$$V = -U(x(\theta)). \tag{3.37}$$

Now we use Lagrangian dynamics[8] to derive the equations of motion of the particle at x. The Lagrangian, L, is defined as

$$L = T - V$$

and the equations of motion are given by the Euler-Lagrange equations

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} &= 0 \qquad \text{and} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) - \frac{\partial L}{\partial\theta_2} &= 0. \end{aligned} \tag{3.38}$$

Substituting our expressions for the kinetic and potential energies, (3.36) and (3.37),

[7]see Appendix A.5 for calculation.
[8]see for instance [11].

respectively, gives us $L = \frac{1}{2}(\dot{\theta}_2^2 \sin^2\theta_1 + \dot{\theta}_1^2) + U(x(\theta))$ so that

$$\begin{aligned}
\frac{\partial L}{\partial \theta_1} &= \dot{\theta}_2^2 \sin\theta_1 \cos\theta_1 + \frac{\partial U}{\partial \theta_1}, & \frac{\partial L}{\partial \theta_2} &= \frac{\partial U}{\partial \theta_2}, \\
\frac{\partial L}{\partial \dot{\theta}_1} &= \dot{\theta}_1, & \frac{\partial L}{\partial \dot{\theta}_2} &= \dot{\theta}_2 \sin^2\theta_1, \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) &= \ddot{\theta}_1, & \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) &= \ddot{\theta}_2 \sin^2\theta_1 + 2\dot{\theta}_1\dot{\theta}_2 \sin\theta_1 \cos\theta_1,
\end{aligned} \tag{3.39}$$

and finally the equations of motion are

$$\begin{aligned}
\ddot{\theta}_1 &= \frac{\partial U}{\partial \theta_1} + \dot{\theta}_2^2 \sin\theta_1 \cos\theta_1 \\
\ddot{\theta}_2 &= \left(\frac{1}{\sin^2\theta_1}\right)\frac{\partial U}{\partial \theta_2} - 2\dot{\theta}_1\dot{\theta}_2 \cot\theta_1,
\end{aligned} \tag{3.40}$$

or as a first-order system,

$$\begin{aligned}
\dot{\theta}_1 &= \omega_1 \\
\dot{\theta}_2 &= \omega_2 \\
\dot{\omega}_1 &= \frac{\partial U}{\partial \theta_1} + \omega_2^2 \sin\theta_1 \cos\theta_1 \\
\dot{\omega}_2 &= \left(\frac{1}{\sin^2\theta_1}\right)\frac{\partial U}{\partial \theta_2} - 2\omega_1\omega_2 \cot\theta_1.
\end{aligned} \tag{3.41}$$

□

Proof 2. Alternatively, we could derive the equations of motion in extrinsic coordinates as is done in [8]. For this method, we will need to make use of constrained Lagrangian mechanics[9] since the particles are restricted to positions on $\mathbb{S}^2$. For such a situation the equations of motion will be given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) - \frac{\partial L}{\partial x_i} - \lambda\frac{\partial f}{\partial x_i} = 0 \tag{3.42}$$

for $i = 1, 2, 3$, where L is the Lagrangian, $f = 0$ is the constraint equation, and

[9]again [11] is a good reference.

λ is the Lagrange multiplier[10]. For us, the kinetic energy[11] is $T = \frac{1}{2}m|v|^2|x|^2$, the potential energy is $V = -\bar{U}$ with $\bar{U}$ defined in (3.22), and we have the constraint equation $f = x_1^2 + x_2^2 + x_3^2 - 1 = 0$. Our Lagrangian is then

$$L = T - V = \frac{1}{2}|v|^2|x|^2 + \bar{U}$$

so that we calculate

$$\begin{aligned} \frac{\partial L}{\partial x_i} &= |v|^2 x_i + \partial_{x_i}\bar{U}, & \frac{\partial L}{\partial v_i} &= v_i|x|^2, \\ \frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) &= \dot{v}_i|x|^2 + 2v_i^2 x_i & \frac{\partial f}{\partial x_i} &= 2x_i, \end{aligned} \tag{3.43}$$

for $i = 1, 2, 3$. Substituting these into (3.42) yields

$$\dot{v}_i|x|^2 + 2v_i^2 x_i - |v|^2 x_i - \partial_{x_i}\bar{U} - \lambda(2x_i) = 0, \tag{3.44}$$

for $i = 1, 2, 3$, or in vector form

$$\dot{v}|x|^2 + 2v(x \cdot v) - |v|^2 x - \nabla_{\mathbb{R}^3}\bar{U} - 2\lambda x = 0, \tag{3.45}$$

where λ is the Lagrange multiplier and $\nabla_{\mathbb{R}^3}$ is the usual gradient in $\mathbb{R}^3$. Now we need to find λ. First take the scalar product of (3.45) with x to get

$$(x \cdot \dot{v})|x|^2 + 2(x \cdot v)(x \cdot v) - |v|^2|x|^2 - x \cdot \nabla_{\mathbb{R}^3}\bar{U} - 2\lambda|x|^2 = 0. \tag{3.46}$$

Since our particle is constrained to the sphere, we can differentiate the equation $f = x_1^2 + x_2^2 + x_3^2 - 1 = 0$ with respect to time twice to get $2|v|^2 + 2x \cdot \dot{v} = 0$. In addition, we have $x \cdot \nabla_{\mathbb{R}^3}\bar{U} = 0$ from Euler's formula, (3.23). Using these properties, we can simplify (3.46) to

$$-|v|^2|x|^2 - |v|^2|x|^2 - 2\lambda|x|^2 = 0 \tag{3.47}$$

from which we get $\lambda = -|v|^2$. Substituting $\lambda = -|v|^2$ and $|x|^2 = 1$ into (3.45) gives

[10] we need a single multiplier because we have one constraint equation taking us from three degrees of freedom to two.

[11] the unexpected factor of $|x|^2$ is a requirement evident from Hamiltonian mechanics, see Section 3.6 of [8].

us

$$\ddot{x} = \nabla_{\mathbb{R}^3}\bar{U}(x) - |v|^2 x \tag{3.48}$$

which is equivalent to

$$\ddot{x} = \nabla_{\mathbb{S}^2}U(x) - |v|^2 x. \tag{3.49}$$

We can recover (3.35) from (3.48) by changing to spherical coordinates in $\mathbb{R}^3$ and restricting back to $\mathbb{S}^2$ using $|x|^2 = 1$. After some calculations, this gives us

$$\ddot{r} = \frac{d}{dt}|x|^2 = 0$$

in the $\hat{\mathrm{e}}_r$-direction,

$$\ddot{\theta}_1 - \dot{\theta}_2^2 \sin\theta_1 \cos\theta_1 = \frac{\partial U}{\partial \theta_1} \implies \ddot{\theta}_1 = \frac{\partial U}{\partial \theta_1} + \dot{\theta}_2^2 \sin\theta_1 \cos\theta_1$$

in the $\hat{\mathrm{e}}_1$-direction, and

$$\ddot{\theta}_2 \sin\theta_1 + 2\dot{\theta}_1\dot{\theta}_2 \cos\theta_1 = \frac{1}{\sin\theta_1}\frac{\partial U}{\partial \theta_2} \implies \ddot{\theta}_2 = \left(\frac{1}{\sin^2\theta_1}\right)\frac{\partial U}{\partial \theta_2} - 2\dot{\theta}_1\dot{\theta}_2 \cot\theta_1$$

in the $\hat{\mathrm{e}}_2$-direction, equations which are equivalent to (3.35). □

3.2.1 The 2-sphere as an invariant set

In this section, we prove invariance of the 2-sphere; that is, we show that if a particle starts on the sphere with velocity tangent to the sphere, then it will remain on the sphere for all later time[12]. This is an important result since if our equations of motion allowed particles off of the sphere, they would not be consistent with our main assumption.

Proposition 4. *If* $(x, v) \in \mathbb{R}^4$ *is a solution to* (3.48) *with initial conditions such that* $|x(t_0)|^2 = 1$ *and* $(x \cdot v)(t_0) = 0$*, then* $|x|^2 = 1$ *and* $x \cdot v = 0$ *for all* $t > t_0$.

Proof. Write

$$\frac{d}{dt}(|x|^2) = 2x \cdot v,$$

so that

$$\frac{d^2}{dt^2}(|x|^2) = 2|v|^2 + 2x \cdot \dot{v}.$$

[12] Actually, the particle moves along geodesics of the sphere, but here we are only concerned with the particle staying on the sphere.

Since we assume (x, v) satisfies (3.49), this is the same as

$$\frac{d^2}{dt^2}(|x|^2) = 2|v|^2 + 2x \cdot (\nabla_{\mathbb{R}^3}\bar{U}\,(x) - |v|^2 x),$$

which is equivalent to

$$\frac{d^2}{dt^2}(|x|^2) = 2x \cdot \nabla_{\mathbb{R}^3}\bar{U} + 2|v|^2(1 - |x|^2).$$

Appealing to Euler's Formula, (3.23), we have $x \cdot \nabla_{\mathbb{R}^3}\bar{U} = 0$ and so

$$\frac{d^2}{dt^2}(|x|^2) = 2|v|^2(1 - |x|^2)$$

for $x \in \mathbb{R}^3$.

Let $y = |x|^2$. Then we have

$$\ddot{y} = 2|v|^2(1 - y),$$

which can be written as the following first-order system:

$$\begin{aligned} \dot{y} &= z \\ \dot{z} &= 2|v|^2(1 - y) \\ y(0) &=,\ z(0) = 0 \end{aligned} \tag{3.50}$$

for $y, z, |v|^2 \in \mathbb{R}$. At the point $(y, z) = (1, 0)$, we have $\dot{y} = 0, \dot{z} = 0$ so this point is by definition an equilibrium solution. Since $y := |x|^2$ and $z := \dot{y} = 2x \cdot v$, we therefore have $|x|^2 = 1$ and $x \cdot v = 0$ for all $t \geq t_0$. $\square$

3.3 The Vlasov-Poisson system on the unit 2-sphere

3.3.1 The Vlasov Equation

According to kinetic theory (see for instance, [14]), the governing equation for the motion of a continuous particle distribution with no collisions is

$$\frac{d}{dt}f = 0, \tag{3.51}$$

where f is the phase-space distribution function. Since our f depends on the independent coordinates $t, \theta_1, \theta_2, \omega_1, \omega_2$, we can use the chain rule to rewrite this equation as

$$\partial_t f + \dot{\theta}_1\ \partial_{\theta_1} f + \dot{\theta}_2\ \partial_{\theta_2} f + \dot{\omega}_1\ \partial_{\omega_1} f + \dot{\omega}_2\ \partial_{\omega_2} f = 0, \tag{3.52}$$

where a dot indicates a derivative with respect to time. Using the equations of motion,

$$\begin{aligned}
\dot{\theta}_1 &= \omega_1 \\
\dot{\theta}_2 &= \omega_2 \\
\dot{\omega}_1 &= \frac{\partial U}{\partial \theta_1} + \omega_2^2 \sin\theta_1 \cos\theta_1 \\
\dot{\omega}_2 &= \left(\frac{1}{\sin^2\theta_1}\right)\frac{\partial U}{\partial \theta_2} - 2\omega_1\omega_2 \cot\theta_1,
\end{aligned} \tag{3.53}$$

we write (3.51) as

$$\partial_t f + \omega_1\ \partial_{\theta_1} f + \omega_2\ \partial_{\theta_2} f + \frac{\partial U}{\partial \theta_1} + (\omega_2^2 \sin\theta_1 \cos\theta_1)\ \partial_{\omega_1} f + (\sin^{-2}\theta_1 \frac{\partial U}{\partial \theta_2} - 2\omega_1\omega_2\cot\theta_1)\ \partial_{\omega_2} f = 0, \tag{3.54}$$

or equivalently as

$$\frac{\partial f}{\partial t} + \begin{bmatrix}\omega_1 \\ \omega_2\end{bmatrix} \cdot \nabla_\theta f + \begin{bmatrix} \dfrac{\partial U}{\partial \theta_1} + \omega_2^2 \sin\theta_1\cos\theta_1 \\ \left(\dfrac{1}{\sin^2\theta_1}\right)\dfrac{\partial U}{\partial \theta_2} - 2\omega_1\omega_2\cot\theta_1 \end{bmatrix} \cdot \nabla_\omega f = 0, \tag{3.55}$$

where $\nabla_\theta f = \left(\dfrac{\partial f}{\partial \theta_1}, \dfrac{\partial f}{\partial \theta_2}\right)$ and $\nabla_\omega f = \left(\dfrac{\partial f}{\partial \omega_1}, \dfrac{\partial f}{\partial \omega_2}\right)$. We call this equation the Vlasov equation for $\mathbb{S}^2$.

3.3.2 Density condition

In our solution of the Poisson equation, we use the spatial density ρ whereas in the Vlasov equation we have the phase space density f. These two densities describe the same particles and so they must agree. Therefore, we require that

$$\rho = \int_{\mathbb{R}^3} f\ dv. \tag{3.56}$$

Since our motion is restricted to the sphere, we can parametrize velocity space by $v = \varphi(\omega)$, where

$$\varphi(\omega_1, \omega_2) = \begin{bmatrix} \cos\theta_1 \cos\theta_2 \omega_1 - \sin\theta_1 \sin\theta_2 \omega_2 \\ \cos\theta_1 \sin\theta_2 \omega_1 + \sin\theta_1 \cos\theta_2 \omega_2 \\ \sin\theta_1 \omega_1 \end{bmatrix}, \tag{3.57}$$

and change (3.56) to spherical coordinates via[13]

$$\rho = \iint f \left| \frac{\partial \varphi}{\partial \omega_1} \times \frac{\partial \varphi}{\partial \omega_2} \right| d\omega_1 d\omega_2. \tag{3.58}$$

A short calculation yields

$$\left| \frac{\partial \varphi}{\partial \omega_1} \times \frac{\partial \varphi}{\partial \omega_2} \right| = \sin\theta_1,$$

so our density condition for $\mathbb{S}^2$ becomes

$$\rho = \iint f \sin\theta_1 d\omega_1 d\omega_2, \tag{3.59}$$

where the integrals are taken over all possible values of ω_1, ω_2 such that extrinsic velocity variable v belongs to the tangent space of $\mathbb{S}^2$.

3.3.3 The Vlasov-Poisson system

Now that we have derived the new Poisson and Vlasov equations, we can put them together to form the closed Vlasov-Poisson system on $\mathbb{S}^2$

$$\begin{cases} \dfrac{\partial f}{\partial t} + \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \cdot \nabla_\theta f + \begin{bmatrix} \dfrac{\partial U}{\partial \theta_1} + \omega_2^2 \sin\theta_1 \cos\theta_1 \\ \left(\dfrac{1}{\sin^2\theta_1} \right) \dfrac{\partial U}{\partial \theta_2} - 2\omega_1\omega_2 \cot\theta_1 \end{bmatrix} \cdot \nabla_\omega f = 0, \\ -\Delta_{\mathbb{S}^2} U = \rho, \\ \rho = \iint f \sin\theta_1 \, d\omega_1 d\omega_2, \\ f(0,\theta,\omega) = f_0(\theta,\omega), \end{cases} \tag{3.60}$$

[13] see for instance [10].

where $\nabla_\theta f = \left(\frac{\partial f}{\partial \theta_1}, \frac{\partial f}{\partial \theta_2}\right)$, $\nabla_\omega f = \left(\frac{\partial f}{\partial \omega_1}, \frac{\partial f}{\partial \omega_2}\right)$, $(\theta_1, \theta_2) \in [0, \pi] \times (-\pi, \pi]$. Alternatively, we could write this using our known form of U from (3.21) as

$$\begin{cases} \dfrac{\partial f}{\partial t} + \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \cdot \nabla_\theta f + \begin{bmatrix} \dfrac{\partial U}{\partial \theta_1} + \omega_2^2 \sin\theta_1 \cos\theta_1 \\ \left(\dfrac{1}{\sin^2\theta_1}\right) \dfrac{\partial U}{\partial \theta_2} - 2\omega_1\omega_2 \cot\theta_1 \end{bmatrix} \cdot \nabla_\omega f = 0, \\ U(x) = \dfrac{1}{2\pi} \iint \rho(y(\theta')) \log\left(\dfrac{1 + (x(\theta)\cdot y(\theta')}{\sqrt{1 - (x(\theta)\cdot y(\theta'))^2}}\right) \sin\theta_1' d\theta_1' d\theta_2', \\ \rho = \iint f \sin\theta_1 \; d\omega_1 d\omega_2, \\ f(0, \theta, \omega) = f_0(\theta, \omega). \end{cases} \tag{3.61}$$

Chapter 4

The Vlasov-Poisson system for circular initial data

A great circle of a unit 2-sphere is any circle contained in the sphere that has a unit radius. If we require that the particles initially lie along the same great circle within $\mathbb{S}^2$ with initial velocities directed tangent to the great circle, then we expect they will remain on the great circle. In this chapter, we use the equations of motion to prove this property and explore the Vlasov-Poisson system on one particular[1] great circle in $\mathbb{S}^2$.

Definition 2. *We define the great circle $C_{1,2}$ to be*

$$C_{1,2} := \left\{x \mid x_1^2 + x_2^2 = 1, x_3 = 0,\ x \in \mathbb{R}^3\right\}, \tag{4.1}$$

or equivalently in local coordinates as

$$C_{1,2} := \left\{(\theta_1, \theta_2) \mid \theta_1 = \frac{\pi}{2}, \theta_2 \in (-\pi, \pi]\right\}, \tag{4.2}$$

which is also known as the equator of $\mathbb{S}^2$.

[1] we restrict our discussion to a specific great circle, but this does not result in a loss of generality since the same arguments can be applied to any great circle. In fact, any other great circle can be obtained from $C_{1,2}$ through a simple rotation of coordinates, due to the symmetry of 2-spheres.

4.1 Great circles as invariant sets

Consider a phase space distribution of particles f in $\mathbb{S}^2$ given by

$$f(t,\theta_1,\theta_2,\omega_1,\omega_2) = \frac{1}{\sin^2\theta_1}\delta(\theta_1 - \frac{\pi}{2}) \otimes \delta(\omega_1) g(t,\theta_2,\omega_2), \tag{4.3}$$

so that the particles are distributed along the great circle $C_{1,2}$ with velocities only in the $\hat{e}_2$ direction. The spatial density ρ is then given by

$$\rho = \iint f \sin\theta_1 d\omega_1 d\omega_2 = \frac{1}{\sin\theta_1}\delta\left(\theta_1 - \frac{\pi}{2}\right)\rho_g(t,\theta_2), \tag{4.4}$$

where $\theta_2' \in (-\pi,\pi]$, and $\rho_g = \iint g(t,\theta_2,\omega_2)d\omega_2$. From (3.32), we can write the force on a particle at any position $x \in \mathbb{S}^2$ due to the distribution ρ as

$$\nabla_{\mathbb{S}^2}U(x) = ((\nabla_{\mathbb{S}^2}U)_1, (\nabla_{\mathbb{S}^2}U)_2, (\nabla_{\mathbb{S}^2}U)_3)(x)$$

where

$$\begin{aligned}
(\nabla_{\mathbb{S}^2}U)_1(x) &= \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{y_1 - (x\cdot y)x_1}{[1-(x\cdot y)^2]}\rho(y)\,dy,\\
(\nabla_{\mathbb{S}^2}U)_2(x) &= \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{y_2 - (x\cdot y)x_2}{[1-(x\cdot y)^2]}\rho(y)\,dy,\\
(\nabla_{\mathbb{S}^2}U)_3(x) &= \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{y_3 - (x\cdot y)x_3}{[1-(x\cdot y)^2]}\rho(y)\,dy.
\end{aligned} \tag{4.5}$$

Substituting (4.4) we get

$$\begin{aligned}
(\nabla_{\mathbb{S}^2}U)_1(x(\theta)) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos\theta_2' - (x\cdot y)\sin\theta_1\cos\theta_2}{[1-(x\cdot y)^2]}\rho_g(y(\theta_2'))\,d\theta_2',\\
(\nabla_{\mathbb{S}^2}U)_2(x(\theta)) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\theta_2' - (x\cdot y)\sin\theta_1\sin\theta_2}{[1-(x\cdot y)^2]}\rho_g(y(\theta_2'))\,d\theta_2',\\
(\nabla_{\mathbb{S}^2}U)_3(x(\theta)) &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(x\cdot y)\cos\theta_1}{[1-(x\cdot y)^2]}\rho_g(y(\theta_2'))\,d\theta_2',
\end{aligned} \tag{4.6}$$

where $x\cdot y = \sin\theta_1\cos\theta_2\cos\theta_2' + \sin\theta_1\sin\theta_2\sin\theta_2'$ since $\theta_1' = \pi/2$.

Proposition 5. *Each great circle on $\mathbb{S}^2$ is an invariant set for the equations of motion,* (3.35).

We shall give two proofs of this proposition.

Proof 1. Consider the equations of motion (3.49) for a single particle located at $x \in \mathbb{S}^2$. In the x_3-direction we have

$$\begin{aligned} \dot{x}_3 &= v_3 \\ \dot{v}_3 &= (\nabla_{\mathbb{S}^2} U)_3 - |v|^2 x_3 \end{aligned} \tag{4.7}$$

where $(\nabla_{\mathbb{S}^2} U)_3$ is the force acting on the particle in the x_3-direction. Changing to spherical coordinates and using (4.6), we can rewrite (4.7) as

$$\begin{aligned} \dot{x}_3 &= -\omega_1 \sin\theta_1 \\ \dot{v}_3 &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(x \cdot y)\cos\theta_1}{[1-(x\cdot y)^2]} \rho_g(y(\theta_2'))\, d\theta_2' - \left(\omega_2^2 \sin^2\theta_1 + \omega_1^2\right)\cos\theta_1. \end{aligned} \tag{4.8}$$

We wish to find an equilibrium solution for this system, i.e. a point for which $(\dot{x}_3, \dot{v}_3) = (0,0)$. When $(\theta_1, \omega_1) = \left(\frac{\pi}{2}, 0\right)$, we calculate

$$\dot{x}_3 = (0)\sin\frac{\pi}{2} = 0$$

and

$$\dot{v}_3 = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(x \cdot y)\cos\frac{\pi}{2}}{[1-(x\cdot y)^2]} \rho_g(y(\theta_2'))\, d\theta_2' - \left(\omega_2^2 \sin^2\frac{\pi}{2} + (0)^2\right)\cos\frac{\pi}{2} = 0,$$

with $x \cdot y \neq \pm 1$. Therefore, we conclude that $(\theta_1, \omega_1) = \left(\frac{\pi}{2}, 0\right)$ is an equilibrium solution to the equations of motion with initial conditions $(x_3, v_3)(0) = (0,0)$ and so the great circle $C_{1,2}$ is an invariant set for the equations of motion. Since the choice of great circle $C_{1,2}$ was arbitrary, the above argument holds for any great circle on $\mathbb{S}^2$ with a simple rotation of coordinates. $\square$

Proof 2. Consider the quantity

$$u = x_1^2 + x_2^2$$

with $x \in \mathbb{S}^2$. Taking the time derivative of u and using our equations of motion, (3.49), for $\dot{x}_1$ and $\dot{x}_2$ yields

$$\dot{u} = \frac{d}{dt}(x_1^2 + x_2^2) = 2x_1 v_1 + 2x_2 v_2 \tag{4.9}$$

for the first derivative. For the second derivative, we get

$$\ddot{u} = \frac{d}{dt}(2x_1 v_1 + 2x_2 v_2) = 2(v_1^2 + v_2^2) + 2x_1 \dot{v}_1 + 2x_2 \dot{v}_2 \tag{4.10}$$

and using our equations of motion,(3.49), again to replace $\dot{v}_1, \dot{v}_2$ we have

$$\ddot{u} = 2(v_1^2 + v_2^2) + 2x_1 \left[(\nabla_{\mathbb{S}^2} U)_1 - |v|^2 x_1 \right] + 2x_2 \left[(\nabla_{\mathbb{S}^2} U)_2 - |v|^2 x_2 \right].$$

We can write this second order system as a first order system by introducing the new variable $w := \dot{u}$. After doing this, the system becomes

$$\begin{aligned} &\dot{u} = w \\ &\dot{w} = 2(v_1^2 + v_2^2) + 2x_1 \left[(\nabla_{\mathbb{S}^2} U)_1 - |v|^2 x_1 \right] + 2x_2 \left[(\nabla_{\mathbb{S}^2} U)_2 - |v|^2 x_2 \right], \end{aligned} \tag{4.11}$$

where

$$u = u(x_1(\theta), x_2(\theta))$$

and

$$w = w(x_1(\theta), x_2(\theta), v_1(\theta, \omega), v_2(\theta, \omega)).$$

We are looking for equilibrium solutions to (4.11), so we need to know under which conditions $\dot{u} = \dot{w} = 0$. This happens when $(\theta_1, \omega_1) = (\frac{\pi}{2}, 0)$ since at this point we have

$$x_1 = \sin\frac{\pi}{2}\cos\theta_2 = \cos\theta_2$$

$$x_2 = \sin\frac{\pi}{2}\sin\theta_2 = \sin\theta_2$$

$$v_1 = (0)\cos\frac{\pi}{2}\cos\theta_2 - \omega_2 \sin\frac{\pi}{2}\sin\theta_2 = -\omega_2 \sin\theta_2$$

$$v_2 = (0)\cos\frac{\pi}{2}\sin\theta_2 + \omega_2 \sin\frac{\pi}{2}\cos\theta_2 = \omega_2 \cos\theta_2$$

$$v_1^2 + v_2^2 = (-\omega_2 \sin\theta_2)^2 + (\omega_2 \cos\theta_2)^2 = \omega_2^2$$

$$|v|^2 = (0)^2 + \omega_2^2 \sin^2\frac{\pi}{2} = \omega_2^2$$

$$x\cdot y = \sin\frac{\pi}{2}\cos\theta_2\sin\frac{\pi}{2}\cos\theta_2' + \sin\frac{\pi}{2}\sin\theta_2\sin\frac{\pi}{2}\sin\theta_2' + \cos\frac{\pi}{2}\cos\frac{\pi}{2} = \cos\theta_2\cos\theta_2' + \sin\theta_2\sin\theta_2'$$

$$\begin{aligned}
(\nabla_{\mathbb{S}^2}U)_1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\theta_2' - (x\cdot y)\sin\frac{\pi}{2}\cos\theta_2}{[1-(x\cdot y)^2]}\rho_g(y(\theta_2'))\,d\theta_2' \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\theta_2' - (x\cdot y)\cos\theta_2}{[1-(x\cdot y)^2]}\rho_g(y(\theta_2'))\,d\theta_2' \\
(\nabla_{\mathbb{S}^2}U)_2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin\theta_2' - (x\cdot y)\sin\frac{\pi}{2}\sin\theta_2}{[1-(x\cdot y)^2]}\rho_g(y(\theta_2'))\,d\theta_2' \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin\theta_2' - (x\cdot y)\sin\theta_2}{[1-(x\cdot y)^2]}\rho_g(y(\theta_2'))\,d\theta_2',
\end{aligned}$$

so that

$$\dot{u} = 2x_1v_1 + 2x_2v_2 = -2\cos\theta_2\omega_2\sin\theta_2 + 2\sin\theta_2\omega_2\cos\theta_2 = 0$$

and

$$\dot{w} = 2(v_1^2+v_2^2) + 2x_1\left[(\nabla_{\mathbb{S}^2}U)_1 - |v|^2x_1\right] + 2x_2\left[(\nabla_{\mathbb{S}^2}U)_2 - |v|^2x_2\right] = 0.$$

This means the set of points for which $(\theta_1,\omega_1) = \left(\frac{\pi}{2},0\right)$ is an equilibrium solution to (4.11). Therefore, we can conclude that if all particles start on $C_{1,2}$ with initial velocities directed along $C_{1,2}$, then the equations of motion (3.35) keep them there for all time. □

4.2 The Vlasov-Poisson system

Armed with the knowledge that particles starting on $C_{1,2}$ remain on $C_{1,2}$, we can confidently restrict our problem to this class of distributions and discover the form that the Vlasov-Poisson system has for them. Before deriving these equations, we extend all functions of θ_2 periodically so they are defined on $\theta_2 \in \mathbb{R}$ with period 2π.

Lemma 3. *The Vlasov-Poisson system on the 2-sphere with initial distribution along the great circle $C_{1,2}$ is*

$$\frac{\partial f}{\partial t} + \omega_2\frac{\partial f}{\partial\theta_2} + \left(-\frac{1}{2\pi}\int_{-\pi}^{\pi}\rho(\theta_2')\left[\frac{1}{\sin(\theta_2-\theta_2')}\right]d\theta_2'\right)\frac{\partial f}{\partial\omega_2} = 0. \tag{4.12}$$

Proof. The spatial density for distributions on the great circle is from (4.3) and (4.4) given by

$$\rho(t,\theta_1,\theta_2) = \frac{1}{\sin\theta_1}\delta\left(\theta_1 - \frac{\pi}{2}\right)\int g(t,\theta_2,\omega_2)d\omega_2 = \frac{1}{\sin\theta_1}\delta\left(\theta_1 - \frac{\pi}{2}\right)\rho_g(t,\theta_2). \quad (4.13)$$

Next, we must calculate the gravitational force function at x due to the distribution ρ. Starting with the solution to Poisson's equation on the sphere from (3.14),

$$U(x) = \frac{1}{2\pi}\int_{\mathbb{S}^2} \log\cot\left[\frac{d(x,y)}{2}\right]\rho(y)dy$$

and substituting our ρ from (4.13) yields

$$\begin{aligned} U(\theta_1,\theta_2) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^{\pi}\left[\frac{1}{\sin\theta_1'}\delta(\theta_1' - \frac{\pi}{2})\int g(t,\theta_2',\omega_2)d\omega_2\right]\log\cot\left[\frac{d(x,y)}{2}\right]\sin\theta_1' d\theta_1' d\theta_2' \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\rho_g(\theta_2')\log\cot\left[\frac{d(x,y)}{2}\right]d\theta_2', \end{aligned} \quad (4.14)$$

where now the θ_1 coordinate of y is $\pi/2$. We have on the sphere that

$$d(x,y) = \cos^{-1}(x\cdot y), \quad (4.15)$$

so substituting this into (4.14) gives us

$$U(\theta_1,\theta_2) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\rho_g(\theta_2')\log\cot\left[\frac{\cos^{-1}(x\cdot y)}{2}\right]d\theta_2'. \quad (4.16)$$

We can rewrite this as

$$\begin{aligned} U(\theta_1,\theta_2) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\rho_g(\theta_2')\log\left[\sqrt{\frac{1+x\cdot y}{1-x\cdot y}}\right]d\theta_2' \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\rho_g(\theta_2')\log\left[\frac{1+\sin\theta_1\cos(\theta_2-\theta_2')}{1-\sin\theta_1\cos(\theta_2-\theta_2')}\right]d\theta_2', \end{aligned} \quad (4.17)$$

since $x\cdot y = \sin\theta_1\cos\theta_2\cos\theta_2' + \sin\theta_1\sin\theta_2\sin\theta_2'$. Now we must calculate the force at x due to the distribution ρ for use in the Vlasov equation. Applying the gradient on $\mathbb{S}^2$ to (4.17) yields

$$\begin{aligned}
\nabla_{\mathbb{S}^2} U(\theta_1, \theta_2) &= \frac{\partial U}{\partial \theta_1} \hat{\mathrm{e}}_1 + \left(\frac{1}{\sin\theta_1} \right) \frac{\partial U}{\partial \theta_2} \hat{\mathrm{e}}_2 \\
&= \frac{1}{4\pi} \int_{-\pi}^{\pi} \rho_g(\theta_2') \frac{\partial}{\partial \theta_1} \log \left[\frac{1 + \sin\theta_1 \cos(\theta_2 - \theta_2')}{1 - \sin\theta_1 \cos(\theta_2 - \theta_2')} \right] d\theta_2' \, \hat{\mathrm{e}}_1 \\
&+ \frac{1}{4\pi} \int_{-\pi}^{\pi} \rho_g(\theta_2') \frac{1}{\sin\theta_1} \frac{\partial}{\partial \theta_2} \log \left[\frac{1 + \sin\theta_1 \cos(\theta_2 - \theta_2')}{1 - \sin\theta_1 \cos(\theta_2 - \theta_2')} \right] d\theta_2' \, \hat{\mathrm{e}}_2 \\
&= \frac{1}{4\pi} \int_{-\pi}^{\pi} \rho_g(\theta_2') \frac{2\cos\theta_1 \cos(\theta_2 - \theta_2')}{1 - \sin^2\theta_1 \cos^2(\theta_2 - \theta_2')} d\theta_2' \, \hat{\mathrm{e}}_1 \\
&+ \frac{1}{4\pi} \int_{-\pi}^{\pi} \rho_g(\theta_2') \frac{-2\sin(\theta_2 - \theta_2')}{1 - \sin^2\theta_1 \cos^2(\theta_2 - \theta_2')} d\theta_2' \, \hat{\mathrm{e}}_2 \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\rho_g(\theta_2')}{1 - \sin^2\theta_1 \cos^2(\theta_2 - \theta_2')} \left[\cos\theta_1 \cos(\theta_2 - \theta_2') \, \hat{\mathrm{e}}_1 - \sin(\theta_2 - \theta_2') \, \hat{\mathrm{e}}_2 \right] d\theta_2',
\end{aligned} \tag{4.18}$$

so that the force on a particle on the circle $C_{1,2}$ is

$$\begin{aligned}
\nabla_{\mathbb{S}^2} U\left(\frac{\pi}{2}, \theta_2\right) &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \rho_g(\theta_2') \left[\frac{\sin(\theta_2 - \theta_2')}{1 - \cos^2(\theta_2 - \theta_2')} \right] d\theta_2' \, \hat{\mathrm{e}}_2 \\
&= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \rho_g(\theta_2') \left[\frac{1}{\sin(\theta_2 - \theta_2')} \right] d\theta_2' \, \hat{\mathrm{e}}_2
\end{aligned} \tag{4.19}$$

and we see that there is no force in the $\hat{\mathrm{e}}_1$ direction, as expected. To get our Vlasov equation, we substitute (4.3) and (4.19) into (3.60) with $\omega_1 = 0$ so that for particles on the circle $C_{1,2}$, the Vlasov-Poisson system reduces to

$$\frac{\partial g}{\partial t} + \omega_2 \frac{\partial g}{\partial \theta_2} + \left(-\frac{1}{2\pi} \int_{-\pi}^{\pi} \rho_g(\theta_2') \left[\frac{1}{\sin(\theta_2 - \theta_2')} \right] d\theta_2' \right) \frac{\partial g}{\partial \omega_2} = 0 \tag{4.20}$$

with $\rho_g(t, \theta_2') = \displaystyle\int g(t, \theta_2, \omega_2) d\omega_2$. Re-labelling g as f and ρ_g as ρ yields the desired equation. □

In what follows, we will often use an alternate form of the system, given by the next corollary.

Corollary. *The Vlasov-Poisson system on $C_{1,2}$ can be written as*

$$\begin{cases} \dfrac{\partial f}{\partial t} + \omega_2 \dfrac{\partial f}{\partial \theta_2} + F[\rho] \dfrac{\partial f}{\partial \omega_2} = 0, \\ F[\rho] = \dfrac{\partial U}{\partial \theta_2} = -\dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} \rho(\theta_2') \left[\dfrac{1}{\sin(\theta_2 - \theta_2')} \right] d\theta_2' \\ \rho = \displaystyle\int f \, d\omega_2. \end{cases} \tag{4.21}$$

Following Definition 1 for solutions in $\mathbb{R}^3$, we impose the following definition for classical solutions on $C_{1,2}$.

Definition 3. *A function $f : I \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is a classical solution of the Vlasov-Poisson system for circular initial data on the open interval $I \subset \mathbb{R}$ if the following hold:*

(i) The function f is continuously differentiable with respect to all its variables.

(ii) The induced spatial density ρ and force function U exist on $I \times \mathbb{R}$. They are continuously differentiable, and U is twice continuously differentiable with respect to θ_2.

(iii) For every compact subinterval $J \subset I$ the field $\nabla_{\mathbb{S}^2} U$ is bounded on $J \times \mathbb{R}^3$.

(iv) The functions f, ρ, U satisfy the Vlasov-Poisson system (4.12) on $I \times \mathbb{R} \times \mathbb{R}$.

In addition to these conditions, we require that f is compactly supported in ω_2. This is physically justified since we are assuming relativistic effects are negligible, and therefore the speeds of the particles must be small compared to the speed of light.

4.2.1 Conserved quantities

There are several quantities that are conserved along solutions of the Vlasov-Poisson system in $\mathbb{R}^3$ including total number of particles, total mechanical energy, Casimirs, and entropy, [17], [6]. In this section, we explore which, if any, of these quantities are also constants of the Vlasov-Poisson system with mass initially distributed on $C_{1,2}$. We assume f is a classical solution to the Vlasov-Poisson system as defined in Definition 3 and also impose the condition that f is compactly supported in ω_2.

Lemma 4. *The total number of particles of the system,*

$$\int \int_{-\pi}^{\pi} f(t, \theta_2, \omega_2) d\theta_2 d\omega_2, \tag{4.22}$$

is conserved.

Proof. Integrate the Vlasov-Poisson system (4.21)

$$\begin{aligned}
0 &= \iint \left[\frac{\partial f}{\partial t} + \omega_2 \frac{\partial f}{\partial \theta_2} + F[\rho] \frac{\partial f}{\partial \omega_2} \right] \, d\theta_2 d\omega_2 \\
&= \iint \frac{\partial f}{\partial t} \, d\theta_2 d\omega_2 + \iint \omega_2 \frac{\partial f}{\partial \theta_2} \, d\theta_2 d\omega_2 + \iint F[\rho] \frac{\partial f}{\partial \omega_2} \, d\theta_2 d\omega_2 \\
&= \frac{d}{dt} \int \int_{-\pi}^{\pi} f \, d\theta_2 d\omega_2 + \int \int_{-\pi}^{\pi} \omega_2 \frac{\partial f}{\partial \theta_2} \, d\theta_2 d\omega_2 + \int_{-\pi}^{\pi} \int F[\rho] \frac{\partial f}{\partial \omega_2} \, d\omega_2 d\theta_2 \\
&= \frac{d}{dt} \int \int_{-\pi}^{\pi} f \, d\theta_2 d\omega_2 - \int \int_{-\pi}^{\pi} f \frac{\partial \omega_2}{\partial \theta_2} \, d\theta_2 d\omega_2 - \int_{-\pi}^{\pi} \int f \frac{\partial F[\rho]}{\partial \omega_2} \, d\omega_2 d\theta_2 \\
&= \frac{d}{dt} \int \int_{-\pi}^{\pi} f \, d\theta_2 d\omega_2,
\end{aligned}$$

where we have used the fact that f has compact support in ω_2 and is 2π-periodic in θ_2. Therefore the total number of particles in the system is conserved as required. □

Lemma 5. *The total mechanical energy of the system of particles,*

$$E := T + V = \frac{1}{2} \int \int_{-\pi}^{\pi} f \omega_2^2 \, d\theta_2 d\omega_2 - \int_{-\pi}^{\pi} U \rho \, d\theta_2, \tag{4.23}$$

is conserved.

Proof. The total mechanical energy of the system is by definition the sum of the total kinetic energy and the total potential energy. In order to get the total kinetic energy, we integrate the kinetic energy of each particle over phase space

$$T = \int \int_{-\pi}^{\pi} f \omega_2^2 \, d\theta_2 d\omega_2. \tag{4.24}$$

The total potential energy is similarly obtained by integrating the potential energy of each particle over position space

$$V = - \int_{-\pi}^{\pi} U(\theta_1, \theta_2) \rho(\theta_2) \, d\theta_2, \tag{4.25}$$

so that the total mechanical energy is

$$E := T + V = \frac{1}{2}\int\int_{-\pi}^{\pi} f\omega_2^2 \; d\theta_2 d\omega_2 - \int_{-\pi}^{\pi} U\rho \; d\theta_2. \tag{4.26}$$

To get our conservation law, we multiply the Vlasov equation,

$$\frac{\partial f}{\partial t} + \omega_2 \frac{\partial f}{\partial \theta_2} + \frac{\partial U}{\partial \theta_2}\frac{\partial f}{\partial \omega_2} = 0, \tag{4.27}$$

by $\frac{1}{2}\omega_2^2$ and integrate over phase space. The first term becomes

$$\frac{1}{2}\int\int_{-\pi}^{\pi} \omega_2^2 \frac{\partial f}{\partial t} \; d\theta_2 d\omega_2 = \frac{d}{dt}\left(\frac{1}{2}\int\int_{-\pi}^{\pi} \omega_2^2 f \; d\theta_2 d\omega_2\right) = \frac{d}{dt}T, \tag{4.28}$$

and the second term is

$$\frac{1}{2}\int\int_{-\pi}^{\pi} \omega_2^3 \frac{\partial f}{\partial \theta_2} d\theta_2 d\omega_2 = \frac{1}{2}\int \omega_2^3 \left(\int_{-\pi}^{\pi} \frac{\partial f}{\partial \theta_2} d\theta_2\right) d\omega_2 = 0 \tag{4.29}$$

since $f(\pi) = f(-\pi)$. The last term is

$$\begin{aligned} & \frac{1}{2}\int\int_{-\pi}^{\pi} \omega_2^2 \left(\frac{\partial U}{\partial \theta_2}\right) \frac{\partial f}{\partial \omega_2} \; d\theta_2 d\omega_2 \\ = \;& \frac{1}{2}\int_{-\pi}^{\pi} \left(\frac{\partial U}{\partial \theta_2}\right) \int \omega_2^2 \frac{\partial f}{\partial \omega_2} \; d\omega_2 d\theta_2 \\ = \;& -\frac{1}{2}\int_{-\pi}^{\pi} \left(\frac{\partial U}{\partial \theta_2}\right) \int f \frac{\partial \omega_2^2}{\partial \omega_2} \; d\omega_2 d\theta_2 \\ = \;& -\int_{-\pi}^{\pi} \left(\frac{\partial U}{\partial \theta_2}\right) \int \omega_2 f \; d\omega_2 d\theta_2 \end{aligned} \tag{4.30}$$

since f has compact support in ω_2. We can write this as

$$-\int \omega_2 \int_{-\pi}^{\pi} \left(\frac{\partial U}{\partial \theta_2}\right) f \; d\omega_2 d\theta_2 = \int \omega_2 \int_{-\pi}^{\pi} \left(\frac{\partial f}{\partial \theta_2}\right) U \; d\omega_2 d\theta_2 \tag{4.31}$$

by again using the fact that $f(-\pi) = f(\pi)$. If we substitute $\omega_2 \frac{\partial f}{\partial \theta_2} = -\frac{\partial f}{\partial t} - \frac{\partial U}{\partial \theta_2}\frac{\partial f}{\partial \omega_2}$

from the Vlasov equation we get

$$
\begin{aligned}
& -\int\int_{-\pi}^{\pi}\left[\frac{\partial f}{\partial t}+\frac{\partial U}{\partial \theta_2}\frac{\partial f}{\partial \omega_2}\right]U\ d\theta_2 d\omega_2 \\
= & -\int\int_{-\pi}^{\pi}\frac{\partial f}{\partial t}U d\theta_2 d\omega_2-\int\int_{-\pi}^{\pi}\frac{\partial U}{\partial \theta_2}\frac{\partial f}{\partial \omega_2}U\ d\theta_2 d\omega_2 \\
= & -\frac{d}{dt}\int_{-\pi}^{\pi}U\int f\ d\omega_2 d\theta_2+\int_{-\pi}^{\pi}\int f\frac{\partial}{\partial \omega_2}\left(U\frac{\partial U}{\partial \theta_2}\right)\ d\omega_2 d\theta_2 \qquad (4.32)\\
= & -\frac{d}{dt}\int_{-\pi}^{\pi}U\rho\ d\theta_2 \\
= & \ \frac{d}{dt}V
\end{aligned}
$$

So that putting all three terms together yields

$$\frac{d}{dt}E=\frac{d}{dt}(T+V)=0, \tag{4.33}$$

and our conservation is proved. □

There are many other quantities which are conserved along solutions of the Vlasov-Poisson system. In fact, the integral of any function of a stationary solution will be conserved. These integrals are commonly called *Casimirs* or *Casimir functionals* and are given by the following definition.

Definition 4. *The Casimirs of the Vlasov-Poisson system are defined in* $\mathbb{R}^3$ *to be*

$$\iint A(f(x,v))\ dxdv, \tag{4.34}$$

where A is any arbitrary smooth function.

Mathematicians will often study the stability of solutions of the Vlasov-Poisson system compared to that of special stationary solutions given by the minimizers of the total energy under Casimir constraints[2].

Lemma 6. *The Casimirs of the Vlasov-Poisson system on the sphere with initial spatial density on a great circle,*

$$\iint A(f(\theta_2,\omega_2))\ d\theta_1 d\omega_2, \tag{4.35}$$

[2]see for instance [9].

are conserved along solutions.

Proof. Substituting $f = f(\theta_2, \omega_2), dx = d\theta_2, dv = d\omega_2$ in Definition 4 yields the Casimirs of the system with initial spatial density on a great circle,

$$\iint A(f(\theta_2, \omega_2)) d\theta_2 d\omega_2. \tag{4.36}$$

The function $A(f)$ satisfies the Vlasov equation (4.20) since by the chain rule we have

$$\frac{\partial A(f)}{\partial t} + \omega_2 \frac{\partial A(f)}{\partial \theta_2} + \frac{\partial U}{\partial \theta_2}\frac{\partial A(f)}{\partial \omega_2} = A'(f)\left[\frac{\partial f}{\partial t} + \omega_2 \frac{\partial f}{\partial \theta_2} + \frac{\partial U}{\partial \theta_2}\frac{\partial f}{\partial \omega_2}\right] = 0. \tag{4.37}$$

So we can integrate over position and velocity space and write

$$\begin{aligned}
0 &= \int\int_{-\pi}^{\pi}\left[\frac{\partial A(f)}{\partial t} + \omega_2 \frac{\partial A(f)}{\partial \theta_2} + \frac{\partial U}{\partial \theta_2}\frac{\partial A(f)}{\partial \omega_2}\right] d\theta_2 d\omega_2 \\
&= \int\int_{-\pi}^{\pi}\frac{\partial A(f)}{\partial t}\, d\theta_2 d\omega_2 + \int\int_{-\pi}^{\pi}\omega_2 \frac{\partial A(f)}{\partial \theta_2}\, d\theta_2 d\omega_2 + \int\int_{-\pi}^{\pi}\frac{\partial U}{\partial \theta_2}\frac{\partial A(f)}{\partial \omega_2}\, d\theta_2 d\omega_2 \\
&= \frac{d}{dt}\int\int_{-\pi}^{\pi} A(f)\, d\theta_2 d\omega_2 - \int\int_{-\pi}^{\pi} A(f)\frac{\partial \omega_2}{\partial \theta_2}\, d\theta_2 d\omega_2 - \int_{-\pi}^{\pi}\int \frac{\partial}{\partial \omega_2}\frac{\partial U}{\partial \theta_2} A(f)\, d\theta_2 d\omega_2 \\
&= \frac{d}{dt}\int\int_{-\pi}^{\pi} A(f)\, d\theta_2 d\omega_2,
\end{aligned}$$

where we have used the fact that A is a function of f and therefore 2π-periodic in θ_2. We conclude that the Casimirs are conserved along solutions. □

A special type of Casimir functional is the entropy of the system, which is a measure of the randomness or disorder of a system. Conservation of entropy reflects the preservation of information of the system in that whatever information is given about the system initially remains for all time.

Definition 5. *The entropy of the system in* $\mathbb{R}^3$ *is defined to be*

$$S = -\iint f \log f \; dx dv. \tag{4.38}$$

Lemma 7. *The entropy of the system on the sphere with initial spatial density on a great circle*

$$S = -\iint f(\theta_2, \omega_2) \log(f(\theta_2, \omega_2)) \; d\theta_2 d\omega_2 \tag{4.39}$$

is constant along solutions.

Proof. Substituting $f = f(\theta_2, \omega_2), dx = d\theta_2, dv = d\omega_2$ for the circle yields the entropy for initial distribution on a great circle,

$$S = -\iint f(\theta_2, \omega_2) \log(f(\theta_2, \omega_2)) \, d\theta_2 d\omega_2. \tag{4.40}$$

Setting $A(f) = -f \log f$ in Lemma 6 yields the desired result. □

It is interesting to note that although conservation of entropy occurs in the Vlasov-Poisson system, entropy is not generally conserved in the closely-related Boltzmann equation. Instead, entropy exclusively increases with time due to the inclusion of collisions in the model[3].

4.2.2 Equilibria

Proposition 6. *Any distribution $f(t, \theta_2, \omega_2) = f^0(\omega_2)$ is a spatially homogeneous equilibrium solution of* (4.20)*, the Vlasov-Poisson system on $C_{1,2}$.*

Proof. Any stationary solution by definition must satisfy (4.20) with $\dfrac{\partial f}{\partial t} = 0$, i.e.

$$\omega_2 \frac{\partial f}{\partial \theta_2} + \left(\frac{\partial U}{\partial \theta_2}\right) \frac{\partial f}{\partial \omega_2} = 0. \tag{4.41}$$

Consider a spatially homogeneous distribution function $f = f^0(\omega_2)$. For this form of f, we get

$$\frac{\partial f^0}{\partial \theta_2} = 0, \tag{4.42}$$

so the first term in (4.41) is 0. We can use (4.13) to calculate

$$\rho = \int f^0 \, d\omega_2 = \rho^0, \tag{4.43}$$

where ρ^0 is a constant. From (4.21) we get that the force due to the homogeneous distribution is

$$F[\rho^0] \;=\; -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\sin(\theta_2 - \theta_2')} \rho^0 = 0, \tag{4.44}$$

since ρ^0 is a constant. Therefore the second term in (4.41) vanishes and we conclude that $f(\theta_2, \omega_2) = f^0(\omega_2)$ is a spatially homogeneous equilibrium (stationary) solution to (4.20). □

[3] see [22].

For the next equilibrium solution, we will require a definition.

Definition 6. *The microscopic energy of the system of particles is defined to be*

$$E(x,v)=\frac{|v|^2}{2}-U(x), \tag{4.45}$$

or in spherical coordinates

$$E(x,v)=\frac{1}{2}\omega_2^2-U(x(\theta_2)), \tag{4.46}$$

where $U=\int_{-\pi}^{\pi}\mathcal{G}(\theta_2-\theta_2')\int f(\theta_2',\omega_2)d\omega_2 d\theta_2'$.

Proposition 7. *Any function of the microscopic energy is a stationary solution of* (4.20)*, the Vlasov-Poisson system on* $C_{1,2}$*.*

Proof. Any stationary solution by definition must satisfy (4.20) with $\dfrac{\partial f}{\partial t}=0$, i.e.

$$\omega_2\frac{\partial f}{\partial\theta_2}+\left(\frac{\partial U}{\partial\theta_2}\right)\frac{\partial f}{\partial\omega_2}=0. \tag{4.47}$$

Consider a function of the microscopic energy defined in Definition 6 so that $f=\bar{f}(E)$, where $\bar{f}$ is an arbitrary function $\bar{f}:\mathbb{R}\to\mathbb{R}$. For this form of f, we get

$$\begin{aligned} & \omega_2\frac{\partial\bar{f}(E)}{\partial\theta_2}+\left(\frac{\partial U}{\partial\theta_2}\right)\frac{\partial\bar{f}(E)}{\partial\omega_2} \\ = \;& \bar{f}'(E)\left[\omega_2\frac{\partial E}{\partial\theta_2}+\left(\frac{\partial U}{\partial\theta_2}\right)\frac{\partial E}{\partial\omega_2}\right] \\ = \;& \bar{f}'(E)\left[-\omega_2\frac{\partial U}{\partial\theta_2}+\left(\frac{\partial U}{\partial\theta_2}\right)\omega_2\right] \\ = \;& 0 \end{aligned} \tag{4.48}$$

as long as the condition $U=\int_{-\pi}^{\pi}\mathcal{G}(\theta_2-\theta_2')\int f(\theta_2',\omega_2)d\omega_2 d\theta_2'$ is satisfied. Therefore we conclude that $f=\bar{f}(E)$ is a spatially homogeneous equilibrium (stationary) solution to (4.20). □

4.3 The Linear Vlasov-Poisson system

In the study of the Vlasov-Poisson system it is common to first linearize the equation about an equilibrium solution before studying the non-linear version. For the Euclidean case at least, important behaviour such as Landau damping becomes evident even from the simpler linearized system. Therefore, in this section we linearize the system derived for $C_{1,2}$ in order to help with further studies.

4.3.1 Linearization of the Vlasov equation on $C_{1,2}$

Let $f^0 = f^0(\omega_2)$ be a homogeneous equilibrium solution to (4.21) as in the previous section and suppose $f(t, \theta_2, \omega_2) = f^0(\omega_2) + h(t, \theta_2, \omega_2)$ where h is 2π-periodic and $\|h\| << 1$. We have

$$\frac{\partial U^0}{\partial \theta_2} = 0, \qquad \frac{\partial f^0}{\partial t} = 0, \qquad \text{and} \quad \frac{\partial f^0}{\partial \theta_2} = 0, \tag{4.49}$$

and from (4.13)

$$\begin{aligned} \rho &= \int \left[f^0 + h(t, \theta_2, \omega_2) \right] d\omega_2 \\ &= \rho^0 + \int h \, d\omega_2 \\ &= \rho^0 + \rho^h, \end{aligned} \tag{4.50}$$

where we have used ρ^h to denote $\displaystyle\int h \, d\omega_2$. From this and (4.21) we calculate the force on a particle at $\left(\frac{\pi}{2}, \theta_2\right)$ to be

$$\begin{aligned} F[\rho] &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\sin(\theta_2 - \theta_2')} \left(\rho^0 + \rho^h\right) d\theta_2' \\ &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\sin(\theta_2 - \theta_2')} \left(\rho^h\right) d\theta_2' \\ &=: F[\rho^h]. \end{aligned} \tag{4.51}$$

Therefore the Vlasov equation, $\dfrac{\partial f}{\partial t} + \omega_2 \dfrac{\partial f}{\partial \theta_2} + F[\rho] \dfrac{\partial f}{\partial \omega_2} = 0$, becomes

$$\begin{aligned} \frac{\partial (f^0 + h)}{\partial t} + \omega_2 \frac{\partial (f^0 + h)}{\partial \theta_2} + F[\rho] \frac{\partial (f^0 + h)}{\partial \omega_2} &= 0 \\ \frac{\partial h}{\partial t} + \omega_2 \frac{\partial h}{\partial \theta_2} + F[\rho^h] \frac{\partial (f^0 + h)}{\partial \omega_2} &= 0. \end{aligned} \tag{4.52}$$

For distributions in which the quadratic term $F[\rho^h] \dfrac{\partial h}{\partial \omega_2}$ is negligible compared to the linear terms for small h, we can further reduce to

$$\frac{\partial h}{\partial t} + \omega_2 \frac{\partial h}{\partial \theta_2} + F[\rho^h] \frac{\partial f^0}{\partial \omega_2} = 0, \tag{4.53}$$

where

$$F[\rho^h] = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\sin(\theta_2 - \theta_2')} \left(\rho^h\right) d\theta_2'.$$

We refer to (4.53) as the linearized Vlasov equation on $C_{1,2}$.

Chapter 5

Conclusions and Extensions

In this thesis, we took the first steps in exploring the Vlasov-Poisson system in curved spaces. We completed numerous preliminary calculations to accurately determine an acceptable form of the Vlasov-Poisson system on the unit 2-sphere. These included deriving the Poisson equation (with help from [7]), writing an expression for the form of the solution of it, and finding the gravitational potential due to an arbitrary distribution. We then took a special distribution in which the particles were arranged on a great circle of the sphere and moving with a velocity directed along the same great circle. This led us to a nice one-dimensional problem that we could study in a variety of ways. We determined the new form of the gravitational potential and force, and proved a number of conservation laws.

In Diacu's book [8], it is mentioned that the qualitative aspects of the stellar dynamics of constant curvature spaces can be studied by ignoring the magnitude of the curvature and including only the sign[1]. Therefore, we expect that our results here can be applied easily to spaces of any positive constant curvature. In addition to this extension, there are countless avenues left to explore within the umbrella of the Vlasov-Poisson system on spaces of constant curvature. Some of these are:

- Existence and uniqueness of global solutions for circular initial data.
- Stability criteria (including criteria for Landau damping) for the linearized equation with circular initial data.
- Initial distributions other than those on great circles.
- Extension to the 3-sphere; this would be the most relevant to our physical space.

[1] this is justified through a coordinate change and rescaling of the time variable.

- Comparison of results using Vlasov-Poisson to results from Einstein-Vlasov (or other methods involving relativity).

We expect the topics in this thesis can be explored equally well under the assumption of a negatively curved space, in which the particles move on the hyperbolic unit sphere. Given the time it took to solve the analogous problem in Euclidean space, we expect proving the existence (or non-existence) and uniqueness of global solutions for general initial data will likely be a difficult but rewarding problem. A proof of the existence or non-existence of non-linear Landau damping, a challenging problem in Euclidean space, would be another great result.

Appendix A

Additional Information

A.1 Spherical Coordinates for $\mathbb{R}^3$

In the following, we denote the position vector in spherical coordinates by $x(r, \theta_1, \theta_2)$ where r is Euclidean distance measured from the origin, θ_1 is the zenith angle (measured from the positive z-axis), and θ_2 is the azimuthal angle (measured from the positive x-axis in the xy-plane) as in Figure 2.3. We define the coordinate change $\varphi : (x_1, x_2, x_3) \to (r, \theta_1, \theta_2)$ by:

$$\begin{aligned} x_1 &= r \sin\theta_1 \cos\theta_2 \\ x_2 &= r \sin\theta_1 \sin\theta_2 \\ x_3 &= r \cos\theta_1 \end{aligned} \tag{A.1}$$

so that the inverse relationship is

$$\begin{aligned} r &= \sqrt{x_1^2 + x_2^2 + x_3^2} \\ \theta_1 &= \arccos\left(\frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\right) \\ \theta_2 &= \arctan\left(\frac{x_2}{x_1}\right) \end{aligned} \tag{A.2}$$

with $r \in [0,\infty)$, $\theta_1 \in [0,\pi]$, $\theta_2 \in [0,2\pi)$. We define $\hat{\mathrm{e}}_r$, $\hat{\mathrm{e}}_1$, and $\hat{\mathrm{e}}_2$ to be orthogonal unit vectors in the direction of increasing r, θ_1, and θ_2, respectively so that

$$\begin{aligned}
\hat{\mathrm{e}}_r &:= \frac{\partial x}{\partial r} \Big/ \left|\frac{\partial x}{\partial r}\right| && = \sin\theta_1 \cos\theta_2 \; \hat{x}_1 + \sin\theta_1 \sin\theta_2 \; \hat{x}_2 + \cos\theta_1 \; \hat{x}_3 \\
\hat{\mathrm{e}}_1 &:= \frac{\partial x}{\partial \theta_1} \Big/ \left|\frac{\partial x}{\partial \theta_1}\right| && = \cos\theta_1 \cos\theta_2 \; \hat{x}_1 + \cos\theta_1 \sin\theta_2 \; \hat{x}_2 - \sin\theta_1 \; \hat{x}_3 \\
\hat{\mathrm{e}}_2 &:= \frac{\partial x}{\partial \theta_2} \Big/ \left|\frac{\partial x}{\partial \theta_2}\right| && = -\sin\theta_2 \; \hat{x}_1 + \cos\theta_2 \; \hat{x}_2.
\end{aligned} \tag{A.3}$$

Taking the time derivative of (A.1) and simplifying yields

$$\begin{aligned}
\dot{x}_1 &= \dot{r}\sin\theta_1\cos\theta_2 + r\dot{\theta}_1\cos\theta_1\cos\theta_2 - r\dot{\theta}_2\sin\theta_1\sin\theta_2 \\
\dot{x}_2 &= \dot{r}\sin\theta_1\sin\theta_2 + r\dot{\theta}_1\cos\theta_1\sin\theta_2 + r\dot{\theta}_2\sin\theta_1\cos\theta_2 \\
\dot{x}_3 &= \dot{r}\cos\theta_1 - r\dot{\theta}_1\sin\theta_1
\end{aligned} \tag{A.4}$$

so we define

$$\begin{aligned}
v_1 &= \omega_r\sin\theta_1\cos\theta_2 + r\omega_1\cos\theta_1\cos\theta_2 - r\omega_2\sin\theta_1\sin\theta_2 \\
v_2 &= \omega_r\sin\theta_1\sin\theta_2 + r\omega_1\cos\theta_1\sin\theta_2 + r\omega_2\sin\theta_1\cos\theta_2 \\
v_3 &= \omega_r\cos\theta_1 - r\omega_1\sin\theta_1
\end{aligned} \tag{A.5}$$

where $\omega_r = \dot{r}, \omega_1 = \dot{\theta}_1$, and $\omega_2 = \dot{\theta}_2$, so that $v = v_1 \; \hat{x}_1 + v_2 \; \hat{x}_2 + v_3 \; \hat{x}_3$. The angular velocities are given in terms of v by:

$$\begin{aligned}
\omega_r &= \sin\theta_1\cos\theta_2 v_1 + \sin\theta_1\sin\theta_2 v_2 + \cos\theta_1 v_3 \\
\omega_1 &= \frac{1}{r}\cos\theta_1\cos\theta_2 v_1 + \frac{1}{r}\cos\theta_1\sin\theta_2 v_2 - \frac{1}{r}\sin\theta_1 v_3 \\
\omega_2 &= -\frac{\sin\theta_2}{r\sin\theta_1}v_1 + \frac{\cos\theta_2}{r\sin\theta_1}v_2.
\end{aligned} \tag{A.6}$$

Taking derivatives and simplifying again yields

$$\begin{aligned}
\dot{v}_1 &= \dot{\omega}_r \sin\theta_1 \cos\theta_2 + 2\omega_r\omega_1 \cos\theta_1 \cos\theta_2 - 2\omega_r\omega_2 \sin\theta_1 \sin\theta_2 \\
&\quad + r\dot{\omega}_1 \cos\theta_1 \cos\theta_2 - r\omega_1^2 \sin\theta_1 \cos\theta_2 - 2r\omega_1\omega_2 \cos\theta_1 \sin\theta_2 \\
&\quad - r\dot{\omega}_2 \sin\theta_1 \sin\theta_2 - r\omega_2^2 \sin\theta_1 \cos\theta_2 \\
\dot{v}_2 &= \dot{\omega}_r \sin\theta_1 \sin\theta_2 + 2\omega_r\omega_1 \cos\theta_1 \sin\theta_2 + 2\omega_r\omega_2 \sin\theta_1 \cos\theta_2 \\
&\quad + r\dot{\omega}_1 \cos\theta_1 \sin\theta_2 - r\omega_1^2 \sin\theta_1 \sin\theta_2 + 2r\omega_1\omega_2 \cos\theta_1 \cos\theta_2 \\
&\quad + r\dot{\omega}_2 \sin\theta_1 \cos\theta_2 - r\omega_2^2 \sin\theta_1 \sin\theta_2 \\
\dot{v}_3 &= \dot{\omega}_r \cos\theta_1 - 2\omega_r\omega_1 \sin\theta_1 - r\dot{\omega}_1 \sin\theta_1 - r\omega_1^2 \cos\theta_1 .
\end{aligned} \tag{A.7}$$

The gradient, divergence, and Laplacian in spherical coordinates are, respectively:

$$\begin{aligned}
&\nabla_{\mathbb{R}^3} f = \frac{\partial f}{\partial r}\hat{\mathrm{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta_1}\hat{\mathrm{e}}_1 + \frac{1}{r\sin\theta_1}\frac{\partial f}{\partial \theta_2}\hat{\mathrm{e}}_2 \\
&\mathrm{div}_{\mathbb{R}^3} F = \hat{\mathrm{e}}_r \cdot \frac{\partial F}{\partial r} + \frac{1}{r}\hat{\mathrm{e}}_1 \cdot \frac{\partial F}{\partial \theta_1} + \frac{1}{r\sin\theta_1}\hat{\mathrm{e}}_2 \cdot \frac{\partial F}{\partial \theta_2} \\
&\Delta_{\mathbb{R}^3} f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta_1}\frac{\partial}{\partial \theta_1}\left(\sin\theta_1 \frac{\partial f}{\partial \theta_1}\right) + \frac{1}{r^2 \sin^2\theta_1}\frac{\partial^2 f}{\partial \theta_2^2}
\end{aligned} \tag{A.8}$$

where f is any scalar function of θ and F is any vector-valued function of θ. If we wish to hold r constant, then all the above simplify to

A.2 Non-unit spheres

In the following, we assume $x \in \mathbb{S}^2_R$ where $\mathbb{S}^2_R$ is the R-radius 2-sphere. We denote the position vector in spherical coordinates by $x(R, \theta_1, \theta_2)$ where R is the (constant) Euclidean distance measured from the origin, θ_1 is the zenith angle (measured from the positive x_3-axis), and θ_2 is the azimuthal angle (measured from the positive x_1-axis in the x_1x_2-plane) as in Figure 2.3. We define the coordinate change $\varphi : (x_1, x_2, x_3) \to (R, \theta_1, \theta_2)$ by:

$$\begin{aligned}
x_1 &= R\sin\theta_1 \cos\theta_2 \\
x_2 &= R\sin\theta_1 \sin\theta_2 \\
x_3 &= R\cos\theta_1
\end{aligned} \tag{A.9}$$

so that the inverse relationship is

$$\begin{aligned} R &= \sqrt{x_1^2 + x_2^2 + x_3^2} \\ \theta_1 &= \arccos\left(\frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\right) \\ \theta_2 &= \arctan\left(\frac{x_2}{x_1}\right) \end{aligned} \tag{A.10}$$

with $R \in [0, \infty)$, $\theta_1 \in [0, \pi]$, $\theta_2 \in [0, 2\pi)$. We define $\hat{\mathrm{e}}_r$, $\hat{\mathrm{e}}_1$, and $\hat{\mathrm{e}}_2$ to be orthogonal unit vectors in the direction of increasing r, θ_1, and θ_2, respectively so that

$$\begin{aligned} \hat{\mathrm{e}}_r &:= \frac{\partial x}{\partial r} / \left|\frac{\partial x}{\partial r}\right| &&= \sin\theta_1 \cos\theta_2 \; \hat{x}_1 + \sin\theta_1 \sin\theta_2 \; \hat{x}_2 + \cos\theta_1 \; \hat{x}_3 \\ \hat{\mathrm{e}}_1 &:= \frac{\partial x}{\partial \theta_1} / \left|\frac{\partial x}{\partial \theta_1}\right| &&= \cos\theta_1 \cos\theta_2 \; \hat{x}_1 + \cos\theta_1 \sin\theta_2 \; \hat{x}_2 - \sin\theta_1 \; \hat{x}_3 \\ \hat{\mathrm{e}}_2 &:= \frac{\partial x}{\partial \theta_2} / \left|\frac{\partial x}{\partial \theta_2}\right| &&= -\sin\theta_2 \; \hat{x}_1 + \cos\theta_2 \; \hat{x}_2. \end{aligned} \tag{A.11}$$

Taking the time derivative of (A.9) and simplifying yields

$$\begin{aligned} \dot{x}_1 &= R\dot{\theta}_1 \cos\theta_1 \cos\theta_2 - R\dot{\theta}_2 \sin\theta_1 \sin\theta_2 \\ \dot{x}_2 &= R\dot{\theta}_1 \cos\theta_1 \sin\theta_2 + R\dot{\theta}_2 \sin\theta_1 \cos\theta_2 \\ \dot{x}_3 &= \dot{R} \cos\theta_1 - R\dot{\theta}_1 \sin\theta_1 \end{aligned} \tag{A.12}$$

so we define

$$\begin{aligned} v_1 &= R\omega_1 \cos\theta_1 \cos\theta_2 - R\omega_2 \sin\theta_1 \sin\theta_2 \\ v_2 &= R\omega_1 \cos\theta_1 \sin\theta_2 + R\omega_2 \sin\theta_1 \cos\theta_2 \\ v_3 &= R\omega_1 \sin\theta_1 \end{aligned} \tag{A.13}$$

where $\omega_1 = \dot{\theta}_1$, and $\omega_2 = \dot{\theta}_2$, so that $v = v_1 \; \hat{x}_1 + v_2 \; \hat{x}_2 + v_3 \; \hat{x}_3$. The angular velocities are given in terms of v by:

$$\begin{aligned} \omega_1 &= \frac{1}{R} \cos\theta_1 \cos\theta_2 v_1 + \frac{1}{R} \cos\theta_1 \sin\theta_2 v_2 - \frac{1}{R} \sin\theta_1 v_3 \\ \omega_2 &= -\frac{\sin\theta_2}{R \sin\theta_1} v_1 + \frac{\cos\theta_2}{R \sin\theta_1} v_2. \end{aligned} \tag{A.14}$$

Taking derivatives and simplifying again yields

$$\begin{aligned}
\dot{v}_1 &= +R\dot{\omega}_1 \cos\theta_1 \cos\theta_2 - R\omega_1^2 \sin\theta_1 \cos\theta_2 - 2R\omega_1\omega_2 \cos\theta_1 \sin\theta_2 \\
&\quad -R\dot{\omega}_2 \sin\theta_1 \sin\theta_2 - R\omega_2^2 \sin\theta_1 \cos\theta_2 \\
\dot{v}_2 &= +R\dot{\omega}_1 \cos\theta_1 \sin\theta_2 - R\omega_1^2 \sin\theta_1 \sin\theta_2 + 2R\omega_1\omega_2 \cos\theta_1 \cos\theta_2 \\
&\quad +R\dot{\omega}_2 \sin\theta_1 \cos\theta_2 - R\omega_2^2 \sin\theta_1 \sin\theta_2 \\
\dot{v}_3 &= -R\dot{\omega}_1 \sin\theta_1 - R\omega_1^2 \cos\theta_1.
\end{aligned} \tag{A.15}$$

The gradient, divergence, and Laplacian in spherical coordinates are, respectively:

$$\begin{aligned}
\nabla_R f &= \frac{1}{R}\frac{\partial f}{\partial \theta_1}\hat{\mathrm{e}}_1 + \frac{1}{R\sin\theta_1}\frac{\partial f}{\partial \theta_2}\hat{\mathrm{e}}_2 \\
\mathrm{div}_R F &= \frac{1}{R}\hat{\mathrm{e}}_1 \cdot \frac{\partial F}{\partial \theta_1} + \frac{1}{R\sin\theta_1}\hat{\mathrm{e}}_2 \cdot \frac{\partial F}{\partial \theta_2} \\
\Delta_R f &= \frac{1}{R^2 \sin\theta_1}\frac{\partial}{\partial \theta_1}\left(\sin\theta_1 \frac{\partial f}{\partial \theta_1}\right) + \frac{1}{R^2 \sin^2\theta_1}\frac{\partial^2 f}{\partial \theta_2^2}
\end{aligned} \tag{A.16}$$

The Poisson equation then is

$$-\Delta_R U = \rho. \tag{A.17}$$

The solution from [7] is given by

$$U_R = \int_{\mathbb{S}_R^2} \frac{1}{2\pi} \log\cot\left(\frac{d(x/R, y/R)}{2}\right) \rho(y)\, dy \tag{A.18}$$

where $x, y \in \mathbb{S}_R^2$. The equations of motion for a particle on $\mathbb{S}_R^2$ are

$$\begin{aligned}
\ddot{\theta}_1 &= \dot{\theta}_2^2 \sin\theta_1 \cos\theta_1 + \frac{1}{R^2}\frac{\partial U_R}{\partial \theta_1} \\
\ddot{\theta}_2 &= \frac{1}{R^2 \sin^2\theta_1}\frac{\partial U_R}{\partial \theta_2} - 2\dot{\theta}_1\dot{\theta}_2 \cot\theta_1.
\end{aligned} \tag{A.19}$$

The Vlasov equation in local coordinates turns into

$$\frac{\partial f}{\partial t} + \omega \cdot \nabla_\theta f + \begin{bmatrix} \frac{1}{R^2}\frac{\partial U_R}{\partial \theta_1} + \dot{\theta}_2^2 \sin\theta_1 \cos\theta_1 \\ \frac{1}{R^2 \sin^2\theta_1}\frac{\partial U_R}{\partial \theta_2} - 2\dot{\theta}_1\dot{\theta}_2 \cot\theta_1 \end{bmatrix} \cdot \nabla_\omega f = 0, \tag{A.20}$$

and the density condition is

$$\int_{T_x\mathbb{S}^2} f\ dv = \int_{\mathbb{R}^2} f \left| \frac{\partial v}{\partial \omega_1} \times \frac{\partial v}{\partial \omega_2} \right| d\omega_1 d\omega_2 = \int_{\mathbb{R}^2} R \sin\theta_1 f\ d\omega_1 d\omega_2. \tag{A.21}$$

Therefore, our new system for non-unit spheres is

$$\begin{cases} \dfrac{\partial f}{\partial t} + \omega \cdot \nabla_\theta f + \begin{bmatrix} \dfrac{1}{R^2}\dfrac{\partial U_R}{\partial \theta_1} + \dot{\theta}_2^2 \sin\theta_1 \cos\theta_1 \\ \dfrac{1}{R^2 \sin^2\theta_1}\dfrac{\partial U_R}{\partial \theta_2} - 2\dot{\theta}_1\dot{\theta}_2 \cot\theta_1 \end{bmatrix} \cdot \nabla_\omega f = 0 \\ U_R = \displaystyle\int_{\mathbb{S}^2_R} \frac{1}{2\pi} \log\cot\left(\frac{d(x/R, y/R)}{2}\right) \rho(y)\ dy \\ \rho = \displaystyle\int_{\mathbb{R}^2} R\sin\theta_1 f\ d\omega_1 d\omega_2. \end{cases} \tag{A.22}$$

A.3 Theorems on Manifolds

Theorem 1 (Divergence Theorem)**.** *Let X be a Riemannian manifold with volume form Ω_X, ∂X be the boundary of X with volume form $\Omega_{\partial X}$ and let ξ be a compactly supported vector field on X. Then*

$$\int_{\partial X} (\xi \cdot \hat{n})\, \Omega_{\partial X} = \int_X (\mathrm{div}_{\Omega_X} \xi)\ \Omega_X \tag{A.23}$$

Proof. Classical result, see for instance [13]. □

Theorem 2. *Let X be a Riemannian manifold with volume form Ω_X, ∂X be the boundary of X with volume form $\Omega_{\partial X}$, ψ be a scalar function on X and F be a compactly supported vector field on X. Then*

$$\int_{\partial X} \psi(F \cdot \hat{n})\Omega_{\partial X} = \int_X \left[(\nabla_{\Omega_X}\psi) \cdot F + (\mathrm{div}_{\Omega_X} F)\psi\right]\ \Omega_X \tag{A.24}$$

Proof. Use $\xi = \psi F$ in Theorem 1 where ψ is a scalar function on X and F is a vector field on X. □

A.4 Proof of $\nabla_{\mathbb{S}^2}(x \cdot y) = y - (x \cdot y)x$ for $x, y \in \mathbb{S}^2$

Proposition 8. *For $x, y \in \mathbb{S}^2$, we have*

$$\nabla_{\mathbb{S}^2}(x \cdot y) = y - (x \cdot y)x. \tag{A.25}$$

Proof. The definiton of $\nabla_{\mathbb{S}^2}$ from (2.22) is

$$\nabla_{\mathbb{S}^2}\, f = \left(\frac{\partial f}{\partial \theta_1}\right)\hat{\mathrm{e}}_1 + \left(\frac{1}{\sin\theta_1}\right)\frac{\partial f}{\partial \theta_2}\hat{\mathrm{e}}_2 \tag{A.26}$$

so we can write

$$\begin{aligned} \nabla_{\mathbb{S}^2}(x \cdot y) &= \frac{\partial (x \cdot y)}{\partial \theta_1}\hat{\mathrm{e}}_1 + \left(\frac{1}{\sin\theta_1}\right)\frac{\partial (x \cdot y)}{\partial \theta_2}\hat{\mathrm{e}}_2 \\ &= \frac{\partial x}{\partial \theta_1} \cdot y\ \hat{\mathrm{e}}_1 + \left(\frac{1}{\sin\theta_1}\right)\frac{\partial x}{\partial \theta_2} \cdot y\ \hat{\mathrm{e}}_2 \end{aligned} \tag{A.27}$$

We have from (2.15)

$$\begin{aligned} \hat{\mathrm{e}}_1 &= \cos\theta_1\cos\theta_2\hat{x}_1 + \cos\theta_1\sin\theta_2\hat{x}_2 - \sin\theta_1\hat{x}_3 \\ \hat{\mathrm{e}}_2 &= -\sin\theta_2\hat{x}_1 + \cos\theta_2\hat{x}_2 \end{aligned} \tag{A.28}$$

and from (2.13)

$$x = (\sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2, \cos\theta_1)\,. \tag{A.29}$$

Using these, (A.27) becomes

$$\begin{aligned} \nabla_{\mathbb{S}^2}(x \cdot y) &= \frac{\partial x}{\partial \theta_1} \cdot y\ \left(\cos\theta_1\cos\theta_2\hat{x}_1 + \cos\theta_1\sin\theta_2\hat{x}_2 - \sin\theta_1\hat{x}_3\right) \\ &\quad + \left(\frac{1}{\sin\theta_1}\right)\frac{\partial x}{\partial \theta_2} \cdot y\ \left(-\sin\theta_2\hat{x}_1 + \cos\theta_2\hat{x}_2\right) \\ &= \left(\cos\theta_1\cos\theta_2 y_1 + \cos\theta_1\sin\theta_2 y_2 - \sin\theta_1 y_3\right) \\ &\quad \times \left(\cos\theta_1\cos\theta_2\hat{x}_1 + \cos\theta_1\sin\theta_2\hat{x}_2 - \sin\theta_1\hat{x}_3\right) \\ &\quad + \left(\frac{1}{\sin\theta_1}\right)\left(-\sin\theta_1\sin\theta_2 y_1 + \sin\theta_1\cos\theta_2 y_2\right) \\ &\quad \times \left(-\sin\theta_2\hat{x}_1 + \cos\theta_2\hat{x}_2\right) \end{aligned} \tag{A.30}$$

so that in the x_1-direction we have

$$\begin{aligned}
& \cos^2\theta_1\cos^2\theta_2 y_1 + \cos^2\theta_1\sin\theta_2\cos\theta_2 y_2 - \sin\theta_1\cos\theta_1\cos\theta_2 y_3 + \sin^2\theta_2 y_1 - \sin\theta_2\cos\theta_2 y_2 \\
= \; & y_1\left(\cos^2\theta_1\cos^2\theta_2 + \sin^2\theta_2\right) + y_2\left(\cos^2\theta_1 - 1\right)\left(\sin\theta_2\cos\theta_2\right) - y_3\left(\sin\theta_1\cos\theta_1\cos\theta_2\right) \\
= \; & y_1\left(\left(1-\sin^2\theta_1\right)\cos^2\theta_2 + \sin^2\theta_2\right) + y_2\left(-\sin^2\theta_1\right)\left(\sin\theta_2\cos\theta_2\right) - y_3\left(\sin\theta_1\cos\theta_1\cos\theta_2\right) \\
= \; & y_1\left(1-\sin^2\theta_1\cos^2\theta_2\right) + y_2\left(-\sin^2\theta_1\right)\left(\sin\theta_2\cos\theta_2\right) - y_3\left(\sin\theta_1\cos\theta_1\cos\theta_2\right) \\
= \; & y_1 - \left(\sin\theta_1\cos\theta_2\right)\left(y_1\sin\theta_1\cos\theta_2 + y_2\sin\theta_1\sin\theta_2 + y_3\cos\theta_1\right) \\
= \; & y_1 - x_1\left(y_1x_1 + y_2x_2 + y_3x_3\right) \\
= \; & y_1 - x_1\left(x\cdot y\right),
\end{aligned}$$

in the x_2-direction we have

$$\begin{aligned}
& \cos^2\theta_1\sin\theta_2\cos\theta_2 y_1 + \cos^2\theta_1\sin^2\theta_2 y_2 - \sin\theta_1\cos\theta_1\sin\theta_2 y_3 - \sin\theta_2\cos\theta_2 y_1 + \cos^2\theta_2 y_2 \\
= \; & y_1\left(\cos^2\theta_1\sin\theta_2\cos\theta_2 - \sin\theta_2\cos\theta_2\right) + y_2\left(\cos^2\theta_1\sin^2\theta_2 + \cos^2\theta_2\right) - y_3\sin\theta_1\cos\theta_1\sin\theta_2 \\
= \; & y_1\left(-\sin^2\theta_1\sin\theta_2\cos\theta_2\right) + y_2\left(\left(1-\sin^2\theta_1\right)\sin^2\theta_2 + \cos^2\theta_2\right) - y_3\sin\theta_1\cos\theta_1\sin\theta_2 \\
= \; & y_1\left(-\sin^2\theta_1\sin\theta_2\cos\theta_2\right) + y_2\left(1-\sin^2\theta_1\sin^2\theta_2\right) - y_3\sin\theta_1\cos\theta_1\sin\theta_2 \\
= \; & y_2 - \sin\theta_1\sin\theta_2\left(y_1\sin\theta_1\cos\theta_2 + y_2\sin\theta_1\sin\theta_2 + y_3\cos\theta_1\right) \\
= \; & y_2 - x_2\left(y_1x_1 + y_2x_2 + y_3x_3\right) \\
= \; & y_2 - x_2\left(x\cdot y\right),
\end{aligned}$$

and in the x_3-direction we have

$$\begin{aligned}
& -\sin\theta_1\cos\theta_1\cos\theta_2 y_1 - \sin\theta_1\cos\theta_1\sin\theta_2 y_2 + \sin^2\theta_1 y_3 \\
= \; & -\sin\theta_1\cos\theta_1\cos\theta_2 y_1 - \sin\theta_1\cos\theta_1\sin\theta_2 y_2 + (1-\cos^2\theta_1)y_3 \\
= \; & y_3 - \cos\theta_1\left(\sin\theta_1\cos\theta_2 y_1 + \sin\theta_1\sin\theta_2 y_2 + \cos\theta_1 y_3\right) \\
= \; & y_3 - x_3\left(y_1x_1 + y_2x_2 + y_3x_3\right) \\
= \; & y_3 - x_3\left(x\cdot y\right).
\end{aligned}$$

Putting these components together yields our desired expression. $\square$

A.5 Calculation of $v = \dot{\theta}_1 \hat{\mathrm{e}}_1 + \dot{\theta}_2 \sin\theta_1 \hat{\mathrm{e}}_2$

We have $x = (\sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2, \cos\theta_1)$. Differentiating x with respect to time yields

$$\begin{aligned} \frac{dx}{dt} &= \dot{\theta}_1 \cos\theta_1 \cos\theta_2 \hat{x}_1 - \dot{\theta}_2 \sin\theta_1 \sin\theta_2 \hat{x}_1 + \dot{\theta}_1 \cos\theta_1 \cos\theta_2 \hat{x}_2 + \dot{\theta}_2 \sin\theta_1 \cos\theta_2 \hat{x}_2 - \dot{\theta}_1 \sin\theta_1 \hat{x}_3 \\ &= \dot{\theta}_1 \hat{e}_1 + \dot{\theta}_2 \sin\theta_1 \hat{e}_2 \end{aligned} \tag{A.31}$$

where we have used the definition of unit vectors given in (2.15).

Bibliography

[1] Håkan Andréasson. The Einstein-Vlasov system/Kinetic Theory. *Living Reviews in Relativity*, 14(4):5 – 55, 2011.

[2] Jürgen Batt. Global symmetric solutions of the initial value problem in stellar dynamics. *Journal of Differential Equations*, 25:342–364, 1977.

[3] Jürgen Batt. Recent developments in the mathematical investigation of the initial value problem of stellar dynamics and plasma physics. *Annals of Nuclear Energy*, 7(45):213 – 217, 1980.

[4] Jacob Bedrossian, Nader Masmoudi, and Clement Mouhot. Landau damping: paraproducts and Gevrey regularity. arXiv:1311.2870v1 [math.AP], 2013.

[5] James Binney and Scott Tremaine. *Galactic Dynamics*. Princeton University Press, Princeton [NJ], 2008.

[6] A.V. Bobylev, P. Dukes, R. Illner, and H.D. Victory Jr. On Vlasov Manev Equations I: Foundations, Properties, and Nonglobal Existence. *Journal of Statistical Physics*, 88(3-4):885–911, 1997.

[7] Howard S. Cohl. Fundamental Solution of Laplace's Equation in Hyperspherical Geometry. *Symmetry, Integrability and Geometry: Methods and Applications*, 108, 2011.

[8] Florin Diacu. *Relative Equilibria of the Curved N-Body Problem*, volume 1. Atlantis Press (Zeger Karssen), Amsterdam, 2012.

[9] J. Dolbeault, J. Fernández, and O. Sánchez. Stability for the Gravitational Vlasov-Poisson System in Dimension Two. *Communications in Partial Differential Equations*, 31(10):1425–1449, 2006.

[10] Gerald B. Folland. *Advanced Calculus*. Prentice-Hall, Inc., New Jersey, 2002.

[11] Grant R. Fowles and George L. Cassiday. *Analytical Mechanics*. Thomson Brooks/Cole, Belmont, CA, USA, Seventh edition, 2005.

[12] R.T. Glassey. *The Cauchy Problem in Kinetic Theory*. Miscellaneous Titles Series. Society for Industrial and Applied Mathematics, 1987.

[13] Serge Lang. *Introduction to Differentiable Manifolds*. Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY, Second edition, 2002.

[14] Richard L. Liboff. *Kinetic Theory: Classical, Quantum, and Relativistic Descriptions*. Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY, Third edition, 2003.

[15] P.L. Lions and B. Perthame. Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. *Inventiones Mathematicae*, 105(1):415–430, 1991.

[16] J. H. Malmberg and C. B. Wharton. Collisionless Damping of Electrostatic Plasma Waves. *Phys. Rev. Lett.*, 13:184–186, Aug 1964.

[17] Villani C. Mouhot, C. On Landau Damping. *Acta Mathematica*, 207(1):29–201, 2011.

[18] K. Pfaffelmoser. Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data. *Journal of Differential Equations*, 95(2):281–303, 1992.

[19] Gerhard Rein. *Collisionless Kinetic Equations from Astrophysics - The Vlasov-Poisson System*. Handbook of Differential Equations: Evolutionary Equations. North-Holland, 2007.

[20] Alan D. Rendall. The Einstein-Vlasov system. arXiv:gr-qc/0208082v1, 2002.

[21] Gerardo F. Torres del Castillo. *Differentiable Manifolds: A Theoretical Physics Approach*. Springer Science+Business Media, LLC, 223 Spring Street, New York, NY, 2012.

[22] C. Villani. Landau Damping. *Lecture Notes, CEMRACS 2010*.

[23] Tie Zhou, Yan Guo, and Chi-Wang Shu. Numerical study on Landau damping. *Physica D: Nonlinear Phenomena*, 157(4):322 – 333, 2001.