Traces, One-Parameter Flows and K-Theory

by

Michael Francis
B.A., University of Victoria, 2011

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ABSTRACT

Given a C*-algebra A endowed with an action α of $\mathbb{R}$ and an α-invariant trace τ, there is a canonical dual trace $\widehat{\tau}$ on the crossed product $A \rtimes_\alpha \mathbb{R}$. This dual trace induces (as would any suitable trace) a real-valued homomorphism $\widehat{\tau}_* : K_0(A \rtimes_\alpha \mathbb{R}) \to \mathbb{R}$ on the even K-theory group. Recall there is a natural isomorphism $\phi^i_\alpha : K_i(A) \to K_{i+1}(A \rtimes_\alpha \mathbb{R})$, the Connes-Thom isomorphism. The attraction of describing $\widehat{\tau}_* \circ \phi^1_\alpha$ directly in terms of the generators of $K_1(A)$ is clear. Indeed, the paper where the isomorphisms $\{\phi^0_\alpha, \phi^1_\alpha\}$ first appear sees Connes show that $\widehat{\tau}_* \phi^1_\alpha[u] = \frac{1}{2\pi i}\tau(\delta(u)u^*)$, where $\delta = \frac{d}{dt}\big|_{t=0}\alpha_t(\cdot)$ and u is any appropriate unitary. A careful proof of the aforementioned result occupies a central place in this thesis. To place the result in its proper context, the right-hand side is first considered in its own right, i.e., in isolation from mention of the crossed-product. A study of 1-parameter dynamical systems and exterior equivalence is undertaken, with several useful technical results being proven. A connection is drawn between a lemma of Connes on exterior equivalence and projections, and a quantum-mechanical theorem of Bargmann-Wigner. An introduction to the Connes-Thom isomorphism is supplied and, in the course of this introduction, a refined version of suspension isomorphism $K_1(A) \to K_0(\mathrm{S}A)$ is formulated and proven. Finally, we embark on a survey of unbounded traces on C*-algebras; when traces are allowed to be unbounded, there is inevitably a certain amount of hard, technical work needed to resolve various domain issues and justify various manipulations.

Contents

List of Tables

List of Figures

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Chapter 1

Introduction

Consider a unital C*-algebra A together with a continuous action α of $\mathbb{R}$ on A by $*$-automorphisms. Let τ be a trace on A. It is standard that τ induces a homomorphism $\tau_* : K_0(A) \to \mathbb{R}$ such that

$$\tau_*([e]) = \tau(e)$$

for every projection $e \in A$. It is appropriate to think of τ_* as an analytical index, the point being that τ acts as a surrogate for the rank function.

If τ is also α-invariant, so that $\tau \circ \delta = 0$ where δ is the derivation associated to the flow α, then there furthermore arises a homomorphism $\mathrm{ind}_\alpha^\tau : K_1(A) \to \mathbb{R}$ such that

$$\mathrm{ind}_\alpha^\tau([u]) = \frac{1}{2\pi i}\tau(\delta(u)u^{-1})$$

for every invertible element u in the domain of δ. One can think of the homotopy invariant quantity $\frac{1}{2\pi i}\tau(\delta(u)u^{-1})$ as a sort of C*-dynamical winding number because of its formal similarity to the winding number formula $\frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t)}\,dt$. Indeed, this analogy is not idle; the latter can be realized as a particular case of the former. Considering this connection, it is appropriate to think of ind_α^τ as a topological index.

The conventional wisdom being that traces pair with K_0, it would seem to be something of a curiosity that, when τ is α-invariant, a sort of "secondary pairing" with K_1 appears. Since one has the suspension isomorphism $s_A^1 : K_1(A) \to K_0(\mathrm{S}A)$ lying around, a natural question might be:

Question. *Is the topological index* ind_α^τ *on* $K_1(A)$ *really a disguised form of the analytical index on* $K_0(\mathrm{S}A)$ *associated to some auxiliary trace on* $\mathrm{S}A$*?*

The answer to this question is a resounding "no". Although it is easy to see that the trace τ on A induces a trace $\mathrm{s}\,\tau$ on the suspension $\mathrm{S}A$, this construction does not involve the flow, and there is generally no relationship between ind_α^τ and $(\mathrm{s}\,\tau)_* \circ s_A^1$. In fact, if A, and hence $\mathrm{S}A$, is commutative, then *every* trace on $\mathrm{S}A$ pairs trivially with K_0 (see Proposition B.11), even though ind_α^τ can be nonzero. Already, this occurs when α is the translation flow on $A = C(\mathbb{T})$ and τ is the Riemann integral.

In spite of the harsh rebuke dealt above, theorems such as the Gohberg-Krein Index Theorem [13], its brethren [22], and its generalizations [26] give credible evidence that it should be possible to identify ind_α^τ with an analytical index of some sort. To be specific, the references above recover ind_α^τ as the index of an associated "Toeplitz operator". These approaches use Breuer's extension of Fredholm theory to the von Neumann algebra setting [4], [5]. The pure C*-algebra K-theory resolution to this problem comes when one involves the dynamics by replacing the suspension $\mathrm{S}A \cong A \otimes C_0(\mathbb{R})$ by the crossed-product $A \rtimes_\alpha \mathbb{R}$, roughly, a twisted version of $A \otimes C_0(\mathbb{R})$ that is generally noncommutative even when A is commutative. An α-invariant trace τ on A still induces a *dual trace* $\widehat{\tau}$ on $A \rtimes_\alpha \mathbb{R}$ by which one stands a reasonable chance to recover ind_α^τ. All that is missing is a device for relating the K-theory of A with that of its crossed-product $A \rtimes_\alpha \mathbb{R}$. Famously, such a device exists. In [6], Connes constructed natural isomorphisms $\phi_\alpha^i : K_i(A) \to K_{i+1}(A \rtimes_\alpha \mathbb{R})$, $i = 0, 1$. Moreover, he proved that

$$\widehat{\tau}_* \circ \phi_\alpha^1 = \mathrm{ind}_\alpha^\tau,$$

thereby showing that the topological index $\mathrm{ind}_{\alpha}^{\tau}$ and the analytical index $\widehat{\tau}_*$ are, modulo an application of his analogue for the Thom map, one and the same.

It is largely accurate to classify this thesis as an exposition of the aforementioned result of Connes, and surrounding theory. Such an undertaking is desirable since, in spite of the disarming simplicity the above formula, the technical work needed to achieve a rigorous formulation and proof is quite substantial. It is to be hoped that our efforts to gather the details in one place may be of use to persons needing access to some aspect or other of the theoretical underpinning. We now summarize the organization of topics.

Chapter 2 contains pertinent material on (1-parameter) automorphism groups. To make a proper study of automorphism groups, we must also study families of unitaries in the multiplier algebra $M(A)$. We consider both unitary groups, which are the implementors of inner automorphism groups, and unitary 1-cocycles, which are the mediators of exterior equivalences between automorphism groups. To a limited extent, we also consider the encompassing notion of groups of Banach space isometries, mainly to the end of giving meaning to the phrase *infinitesimal generator* in a reasonably general context. A major feature of our approach is the emphasis we have placed on casting the families encountered as solutions of differential equations.

In Chapter 3, we make a study of the topological index $\mathrm{ind}_{\alpha}^{\tau}$ in its own right. A few novelties are to be mentioned. In Proposition 3.6 it is shown that a closed, densely defined derivation δ of a unital Banach algebra must have $1 \in \mathrm{dom}(\delta)$. This proposition, although comforting to know, is of rather limited use since as it is difficult to imagine any particular example of a derivation for which this conclusion is not obvious. We work out the index for translation flow on $\mathbb{R}$, which yields the classical winding number, for linear flows on tori, and in particular the Kronecker flow on the 2-torus.

In Chapter 4, we prove a lemma (Theorem 4.22, in our numbering) of Connes to the effect that, for any C*-dynamical system $(A, \mathbb{R}, \alpha)$, for every projection $e \in A$, the projection e

is fixed by a flow in the same exterior equivalence class as α. This result is crucial to the construction of Connes-Thom isomorphism. On the way, we precisely characterize, in Theorem 4.4, the C^1-smooth unitary 1-coycles of an arbitrary C*-dynamical system. We close the section by drawing attention to a particularly nice application of Connes' lemma: to showing that every continuous 1-parameter group of $*$-automorphisms of the algebra of compact operators on separable Hilbert space is unitarily implemented. This is an old result in the mathematical formalism of quantum mechanics, related to, but distinct from, Stone's theorem on 1-parameter unitary groups. As the document [33] attributes a nearby result to Valentine Bargmann and Eugene Wigner, it seems appropriate to use the designation "Bargmann-Wigner theorem" for this statement.

In Chapter 5, we discuss the suspension isomorphisms of C*-algebra K-theory, beginning with mention of their historical antecedent in the commutative setting. Inspired by our hijinks in the commutative case, we show, in Theorem 5.15, that the suspension isomorphism $K_1(A) \to K_0(\mathrm{S}A)$ admits a more refined statement. Roughly, we show there already exists a homotopy bijection on generators, prior to making the passage to K-groups. The result obtained is precisely the C*-analogue of Lemma 1.4.9 in [1]. It is interesting to note that there is no corresponding result for the (more significant) isomorphism $K_0(A) \to K_1(\mathrm{S}A)$, the Bott map. The latter isomorphism depends vitally on the relations imposed during the passage to K-groups.

In Chapter 6, we give a rapid introduction to the Connes-Thom isomorphism, first dealing with axiomatics, and then deriving the explicit formulae for ϕ^0_α and $s^0 \circ \phi^1_\alpha$ appearing in [6] from the axioms.

In Chapter 7, we finally return our attention to the formula $\widehat{\tau}_* \circ \phi^1_\alpha = \mathrm{ind}^\tau_\alpha$ which was discussed at the outset. The proof of Theorem 7.1 is the culmination of our efforts and is the focal point of this document. We then apply the theorem in the case of Kronecker flow on the 2-torus along lines of irrational slope. The chapter closes with brief mention of a "severed thread". The original goal of this project was to give an analogue of the above formula for KMS$_\beta$-states. Ultimately, the results obtained in this direction were unsatisfactory, for reasons we explain here.

The bulk of the appendices is occupied by an introductory account of the theory of unbounded traces on C*-algebra, including the dual trace on the crossed-product. We follow the example of [26] by stubbornly refusing to resort to von Neumann algebra methods whenever practical. Most of this material is probably "standard" for those in the know, or for those already well-versed in the corresponding theory for von Neumann algebra, but it is still rather difficult to track down references for the C*-algebra case. As mentioned above, most of the material in this chapter is probably known, but let us draw attention to a few results which we have not encountered elsewhere. Corollary A.16 shows that, for every C*-algebra A, $x \sim y \Leftrightarrow \exists a \in A : x = a^*a, y = aa^*$ defines an equivalence relation on the positive cone of A. Theorem A.27 generalizes the following statement: "If S, T are bounded operators on a separable Hilbert space such that ST and TS are both trace-class, then $\mathrm{tr}(ST) = \mathrm{tr}(TS)$" a discussion of which can be found at [16]. Proposition A.25 is a completeness result for the "Hilbert-Schmidt elements" associated to an unbounded trace. We prescribe Propositions A.34 and A.37 as remedies for those afflicted with the impression that the dual trace resists being used in concrete computations. Last of all, a (shorter) appendix contains some inevitable lemmas with regards to doing K-theory over suitable dense subalgebras.

Notations and conventions

If $\mathcal{A}$ is an (associative) algebra over $\mathbb{C}$, then $\widetilde{\mathcal{A}}$ denotes the algebra obtained by adjoining a unit to $\mathcal{A}$, even if one already exists. The scalar map $\widetilde{\mathcal{A}} \to \mathbb{C}$ (of which $\mathcal{A}$ is the kernel) is denoted $\varepsilon_{\mathcal{A}}$, or simply ε when confusion seems unlikely.

If A is a C*-algebra with 1, and $e \in A$ is projection, we write $e^\perp$ for the complementary projection $1 - e$.

Given $f \in C_c(\mathbb{R})$, its *Fourier transform* $\widehat{f} \in C_0(\mathbb{R})$ shall be given as $\widehat{f}(s) = \int_{-\infty}^{\infty} f(t)e^{its}\, dt$. Completing the convolution algebra $C_c(\mathbb{R})$ in the largest C*-norm dominated by $\|\cdot\|_1$, one obtains the group C*-algebra $C^*(\mathbb{R})$. The Fourier transform extends uniquely to a C*-algebra isomorphism $C^*(\mathbb{R}) \to C_0(\mathbb{R})$ which we denote simply by $f \mapsto \widehat{f}$ and refer to as the *Fourier isomorphism*. For reasons explained later in Theorem 2.15, we sometimes denote the inverse isomorphism $C_0(\mathbb{R}) \to C^*(\mathbb{R})$ by $g \mapsto g(i\frac{d}{dt})$.

Remark 1.1. If A and B are unital C*-algebras, then a unital $*$-homomorphism $\varphi : A \to B$ obviously restricts to a homomorphism of their unitary groups. If A and B are unital but φ is *nonunital*[1], φ still induces a homomorphism on the unitary groups sending a unitary $u \in A$ to the unitary $\varphi(u) + \varphi(1_A)^{\perp} = \varphi(u) + 1_B - \varphi(1_A) \in B$. It is rather awkward that this natural mapping from unitaries in A to unitaries in B is not always the restriction of φ to the unitaries.

To mitigate the above awkwardness, when we write $\mathrm{U}(A)$ for any C*-algebra A, unital or nonunital, we shall always mean (often implicitly) the group of unitaries in $\widetilde{A}$ whose scalar part equals 1. Similarly, when we write $\mathrm{U}_n(A)$, we mean $\mathrm{U}(M_n(A))$. In the case where A is already unital, this makes no difference, since the group of unitaries in A is canonically isomorphic to $\mathrm{U}(A)$. With this convention, if A and B are any C*-algebras (unital or not) and $\varphi : A \to B$ is any $*$-homomorphism (unital or not) then φ induces a mapping $\mathrm{U}(A) \to \mathrm{U}(B)$ given by restricting the unitized homomorphism $\widetilde{\varphi} : \widetilde{A} \to \widetilde{B}$ to $\mathrm{U}(A)$. This recovers the homomorphism discussed in the above remark, and has the advantage of treating all cases in a homogeneous manner. Analogous comments apply for groups of invertibles i.e. $\mathrm{GL}(A)$ implicitly denotes the group of invertible elements in $\widetilde{A}$ which have scalar part 1.

[1]Such homomorphisms are unavoidable in the context of K-theory, for instance, consider the corner inclusion of $M_n(A)$ into $M_k(A)$ for $n < k$.

Chapter 2

One-parameter dynamics

2.1 Isometric flows in Banach spaces

Definition 2.1. A **flow** Φ on a Banach space X is a strongly-continuous[1] action of $\mathbb{R}$ on X by linear isometries.

In general, of course, there is not a compelling reason to restrict to attention to isometric flows, but such flows will suffice for our purposes. This restriction affords certain conveniences. For instance, the following shows the "strong continuity" coincides with the usual continuity property imposed on a topological group action.

Proposition 2.2. *If Φ is a flow on a Banach space X, then the map $(t, x) \mapsto \Phi_t(x) : \mathbb{R} \times X \to X$ is jointly continuous (as opposed to merely when the second component is held fixed, as strong continuity would seem to dictate).*

Proof. Let $t, s \in \mathbb{R}$ and $x, y \in X$. Note

$$\|\Phi_t(x) - \Phi_s(y)\| \leq \|\Phi_t(x) - \Phi_s(x)\| + \|\Phi_s(x) - \Phi_s(y)\| = \|\Phi_t(x) - \Phi_s(x)\| + \|x - y\|$$

[1] That is $t \mapsto \Phi_t(x)$ is a continuous curve for every $x \in X$. By exploitation of the group law, it suffices to require that $\lim_{t \to 0} \Phi_t(x) = x$ for all $x \in X$.

and the latter clearly vanishes as $t \to s$ and $\|x - y\| \to 0$. □

The presence of a flow immediately gives rise to a stratification of elements by the smoothness of their orbits.

Definition 2.3. Let Φ be a flow on a Banach space X. A $\mathbf{C^k}$ **element** for Φ is an element $x \in X$ such that the curve $t \mapsto \Phi_t(x)$ is C^k smooth. The **infinitesimal generator** of Φ is the partially-defined linear transformation D of X with $\mathrm{dom}(D)$ the subspace of C^1 elements and given by $D(x) = \frac{d}{dt}\Phi_t(x)\big|_{t=0} = \lim_{t\to 0} \frac{\Phi_t(x)-x}{t}$, for all $x \in \mathrm{dom}(D)$.

Proposition 2.4. *Let Φ be a flow on a Banach space X and fix $x \in X$. If $t \mapsto \Phi_t(x)$ is differentiable at $t = 0$, then $\Phi_t(x)$ is a C^1 smooth element for every $t \in \mathbb{R}$ and*

$$\tfrac{d}{dt}\Phi_t(x) = \Phi_t(D(x)) = D(\Phi_t(x)).$$

Proof. Briefly, apply $\frac{d}{ds}\big|_{s=t}$ to $\Phi_s(x) = \Phi_t(\Phi_{s-t}(x)) = \Phi_{s-t}(\Phi_t(x))$. □

Corollary 2.5. *The closed subspace of elements fixed by a Banach space flow equals the kernel of the infinitesimal generator of the flow.*

A routine smoothing argument, given below, shows that C^∞ elements always exist in abundance. Note that, canonically associated to a flow $t \mapsto \Phi_t : \mathbb{R} \to \mathrm{Isom}(X)$, there is a contractive homomorphism $f \mapsto \Phi_f : C_c(\mathbb{R}) \to \mathbb{B}(X)$ given by

$$\Phi_f(x) = \int_{-\infty}^{\infty} f(t)\Phi_t(x)\; dt \qquad\qquad \forall\; f \in C_c(\mathbb{R}), x \in X$$

where $C_c(\mathbb{R})$ is considered as a normed algebra with under convolution and the 1-norm. Note as well the equivariance condition

$$\Phi_{\lambda_t f} = \Phi_t \Phi_f$$

where $(\lambda_t f)(s) = f(s - t)$.

Proposition 2.6. *For any flow Φ on a Banach space X, the C^∞ elements are dense.*

Proof. Let $f \in C_c(\mathbb{R})$ be continuously differentiable. We claim that, for any $x \in X$, the element $\Phi_f(x)$ is smooth and $D(\Phi_f x) = -\Phi_{f'}(x)$. Indeed, since

$$\left\| \frac{\Phi_t \Phi_f(x) - \Phi_f(x)}{t} + \Phi_{f'}(x) \right\| \leq \left\| \frac{\lambda_t f - f}{t} + f' \right\|_1 \|x\|,$$

one just needs to know that $\frac{\lambda_t f - f}{t} \to -f'$ in the 1-norm. Obviously $\frac{\lambda_t f - f}{t} \to -f'$ pointwise. Assuming $|t| \leq 1$ with no harm done, the supports of the $\frac{\lambda_t f - f}{t}$ all lie in a single bounded interval. Furthermore, noting $\frac{f(s-t)-f(s)}{t}$ equals the average value of f' between s and $s-t$, it follows that each function $\frac{\lambda_t f - f}{t}$ is dominated by $\|f'\|_\infty$ times the characteristic function of a fixed interval so that $\left\|\frac{\lambda_t f - f}{t} + f'\right\|_1 \to 0$ by the dominated convergence theorem, proving the claim.

Inductively, it follows that, if $f \in C_c(\mathbb{R})$ has derivatives of every order, then $\Phi_f(x)$ is a C^∞ element for each $x \in X$ and $D^n(\Phi_f(x)) = (-1)^n \Phi_{f^{(n)}}(x)$.

Now, letting $f_n \geq 0$ be a C^∞ bump function with support contained in $[0, 1/n]$ and $\int f_n(t)\, dt = 1$, we note that

$$\|x - \Phi_{f_n})\| = \left\| \int_0^{1/n} f_n(s)(x - \Phi_s(x))\, ds \right\| \leq \max_{0 \leq s \leq 1/n} \|x - \Phi_s(x)\| \to 0$$

as $n \to \infty$ by strong continuity, so the C^∞ elements are dense as desired. □

In particular, the infinitesimal generator of a Banach space flow is densely-defined. With straightforward adjustments to the proof of the standard, single-variable calculus result on interchange of limit and derivative, we get as well that the infinitesimal generator is a closed operator.

Proposition 2.7. *If Φ is a flow on a Banach space X, then its infinitesimal generator D is a closed operator.*

Proof. Suppose $x_n \to x$, $D(x_n) \to y$ where $x_n \in \mathrm{dom}(D)$ and $x, y \in X$. Consider

$$\Phi_t(x_n) = x_n + \int_0^t \tfrac{d}{ds}\Phi_s(x_n)\ ds = x_n + \int_0^t \Phi_s(D(x_n))\ ds$$

for fixed t, and let n tend to ∞. On the LHS we get $\Phi_t(x)$. On the RHS, noting $\Phi_s(z_n)$ goes uniformly in s to $\Phi_s(z)$ when $z_n \to z$, we get $x + \int_0^t \Phi_s(y)\ ds$. Now, let t vary. Applying $\frac{d}{dt}\big|_{t=0}$ to both sides of $\Phi_t(x) = x + \int_0^t \Phi_s(y)\ ds$ gives $D(x) = y$, as desired. □

Lastly, we show that a Banach space flow can be recovered from its infinitesimal generator. Note that, as the generator is generally unbounded, the uniqueness result Theorem 4.2 does not apply.

Theorem 2.8. *Let Φ be a flow on a Banach space X with infinitesimal generator D. Fix $x_0 \in \mathrm{dom}(D)$. Then, the only C^1 curve $\mathbb{R} \to \mathrm{dom}(D)$ that solves the initial value problem*

$$\dot{x}(t) = D(x(t)) \qquad\qquad x(0) = x_0$$

is the orbit map $t \mapsto \Phi_t(x_0)$.

Proof. Let x be any solution to the above initial value problem. Observe that

$$\tfrac{d}{dt}\left(\Phi_{-t}(x(t))\right) = -\Phi_{-t}(D(x(t)) + \Phi_{-t}\dot{x}(t) = 0$$

which means $t \mapsto \Phi_{-t}(x(t))$ is constant. Thus, for all $t \in \mathbb{R}$, $\Phi_{-t}x(t) = \Phi_0 x(0) = x_0$ which, after applying Φ_t to both sides, gives the conclusion. □

Remark 2.9. Note the above proof is a direct generalization of the proof from elementary single-variable calculus that $x(t) = x_0 e^{at}$ is the unique solution to $\dot{x}(t) = ax(t)$, $x(0) = x_0$.

The preceding theorem shows that the generator D of a Banach space flow Φ determines

Φ uniquely on dom(D). Since D is densely-defined, and since our flows are always assumed isometric, we get the desired corollary by continuous extension.

Corollary 2.10. *A Banach space flow is uniquely determined by its infinitesimal generator.*

2.2 Strictly-continuous 1-parameter unitary groups

In this section, we consider strictly-continuous 1-parameter unitary groups in the multiplier algebra $M(A)$ of a C*-algebra A. Our favoured construction[2] of $M(A)$ is as the C*-algebra of "adjointable operators" on A, as explained in [2], II.7.3. From this point of view, the *strict topology* on $M(A)$ equals the $*$-*strong topology* i.e. a net (T_i) in $M(A)$ converges strictly to $T \in M(A)$ if and only if $T_i a \to Ta$ and $T_i^* a \to T^* a$ in norm, for every $a \in A$. In particular, taking adjoints is a strictly continuous operation.

If $U \in M(A)$ is unitary, then $\|Ua - a\| = \|a - U^* a\|$ for all $a \in A$. Thus, the *strict topology* and *strong topology* coincide on $\mathrm{U}(M(A))$. If $\mathcal{H}$ is a Hilbert space and $A = \mathbb{K}(\mathcal{H})$, so that $M(A) = \mathbb{B}(\mathcal{H})$ and $\mathrm{U}(M(A)) = \mathrm{U}(\mathcal{H})$, then these topologies also agree with the strong operator topology on $\mathrm{U}(\mathcal{H})$. Thus, a *strictly continuous unitary group* is, in every sense, the same as a *strongly continuous unitary group.* Nonetheless, we shall only speak of strictly continuous groups as we feel the latter terminology suggests some Hilbert space is at hand, which may not be the case. We remark that the multiplication maps

$$\mathrm{U}(M(A)) \times \mathrm{U}(M(A)) \to \mathrm{U}(M(A)) \qquad\qquad \mathrm{U}(M(A)) \times A \times \mathrm{U}(M(A)) \to A$$

are jointly continuous when $\mathrm{U}(M(A))$ has the strict topology and A has the norm topology.

If A is a unital C*-algebra, then $M(A) = A$ and the strict topology equals the norm topology. In this case, all the strictly continuous 1-parameter unitary groups which can arise

[2] Briefly, an element of $M(A)$ is a function $T : A \to A$ possessing an *adjoint* $T^* : A \to A$ that satisfies $(Ta)^* b = a^*(T^* b)$, for all $a, b \in A$. Automatically, $T \in \mathbb{B}(A)$, the bounded linear operators on A.

are rather trivial by the following proposition, which we adapted from material in [11].

Proposition 2.11. *If (U_t) is a norm-continuous 1-parameter unitary group in $M(A)$, then there is a (unique) self-adjoint element $H \in M(A)$ such that $U_t = e^{itH}$ for all $t \in \mathbb{R}$.*

The following simple-minded example is included in hopes of rendering the proof of the above proposition more transparent.

Example 2.12. It is an elementary fact that every continuous group homomorphism $u : \mathbb{R} \to \mathbb{T}$ has the form $u(t) = e^{ith}$ for some $h \in \mathbb{R}$. Consider the problem of determining h from u, without a priori knowledge of this fact. The obvious approach is differentiation: $ih = \frac{d}{dt}u(t)\big|_{t=0}$. However, as we began only by assuming u is continuous, this tactic requires further justification. On the other hand, we could integrate: $ih \cdot \int_0^{t_0} u(t)\ dt = (u(t_0) - 1)$, for any $t_0 > 0$. Assuming the integral on the left is nonzero (which, by continuity, holds for sufficiently small t_0), we have the formula $ih = \frac{u(t_0)-1}{\int_0^{t_0} u(t)\ dt}$ which makes sense immediately, with no recourse to additional regularity properties u.

Proof of Proposition 2.11. Since (U_t) is norm-continuous and $U_0 = 1$, we note that $\frac{1}{t_0}\int_0^{t_0} U_s\, ds \to 1$ as $t_0 \to 0$. Thus, for $t_0 > 0$ sufficiently small, $\int_0^{t_0} U_s\ ds$ is invertible. Fix some such $t_0 > 0$. Given $t \neq 0$, write

$$\begin{aligned} \frac{U_t - 1}{t}\int_0^{t_0} U_s\ ds &= \frac{1}{t}\int_0^{t_0} U_{s+t}\ ds - \frac{1}{t}\int_0^{t_0} U_s\ ds \\ &= \frac{1}{t}\int_t^{t_0+t} U_s\ ds - \frac{1}{t}\int_0^{t_0} U_s\ ds \\ &= \frac{1}{t}\int_{t_0}^{t_0+t} U_s\ ds - \frac{1}{t}\int_0^{t} U_s\ ds. \end{aligned}$$

Letting $t \to 0$, note the above converges in norm to $U_{t_0} - 1$. Therefore,

$$\lim_{t \to 0} \frac{U_t - 1}{t} = (U_{t_0} - 1)\left(\int_0^{t_0} U_s \, ds\right)^{-1}$$

and we have proved $iH := \frac{d}{dt}U_t\big|_{t=0} \in M(A)$ exists in norm. Exploiting the group law, we get that (U_t) is differentiable and $\frac{d}{dt}U_t = iH \cdot U_t$ for all $t \in \mathbb{R}$. □

Having characterized the norm-continuous unitary groups in $M(A)$, we now consider their strictly continuous counterparts. In our view, $M(A) \subset \mathbb{B}(A)$. It also holds that $\mathrm{U}(M(A)) \subset \mathrm{Isom}(A)$. In particular, a strictly continuous unitary group (U_t) in $M(A)$ is a special kind of flow on the Banach space A, and so has an infinitesimal generator D by preceding section's results. It turns out that our primary interest is in $H = \frac{1}{i}D$.

Definition 2.13. The **Hamiltonian** H of a strictly continuous unitary group (U_t) in $M(A)$ is such that iH is the infinitesimal generator of (U_t). That is, $\mathrm{dom}(H)$ consists of all $x \in A$ such that $t \mapsto U_t x$ is C^1, and $H(x) = \frac{1}{i}\lim_{t \to 0} \frac{U_t x - x}{t}$, for all $x \in \mathrm{dom}(H)$.

By the previous section's work, H is a closed, densely-defined operator uniquely associated to (U_t). Because we normalized by i in the above definition, it is easy to check that $(Hx)^* y = x^*(Hy)$ for all $x, y \in \mathrm{dom}(H)$. Thus, it is correct to think of H as some sort of self-adjoint (or, at least, symmetric) unbounded multiplier of A.

Remark 2.14. As it happens, we shall never have any direct need for H itself, only its "functional calculus" $f \mapsto f(H)$ which we construct directly below. In other words, should the reader desire, the role of H in this thesis can even be relegated to that of a formal symbol, a notational crutch for a map $C_b(\mathbb{R}) \to M(A)$ associated to the group (U_t).

We state this section's main theorem. The result is surely standard, but, since a reference could not be located, we include a proof.

Theorem 2.15. *Let (U_t) be a strictly continuous unitary group in $M(A)$ for some C^*-algebra A, and let H be the Hamiltonian of the group. Then, there is a unique strictly continuous $*$-homomorphism $f \mapsto f(H) : C_b(\mathbb{R}) \to M(A)$ such that $e^{itH} = U_t$ for all $t \in \mathbb{R}$.*

Remark 2.16. To speak of the strict topology on $C_b(\mathbb{R})$, we implicitly identify the latter algebra with $M(C_0(\mathbb{R}))$. The strict topology on $C_b(\mathbb{R})$ is somewhat finer than the topology of uniform convergence on bounded intervals. Indeed, a net (f_i) in $C_b(\mathbb{R})$ converges to $f \in C_b(\mathbb{R})$ uniformly on every bounded interval if and only if $f_i g \to fg$ uniformly for every $g \in C_c(\mathbb{R}) \subset C_0(\mathbb{R})$. The two topologies are equal on any norm-bounded subset of $C_b(\mathbb{R})$.

For the proof of Theorem 2.15, we use the following lemma.

Lemma 2.17. *Let A and B be C^*-algebras and let $\pi : A \to M(B)$ be a $*$-homomorphism. If there exists a bounded approximate unit[3] $(e_\lambda)_{\lambda\in\Lambda}$ in A such that $\pi(e_\lambda)$ converges to 1 strictly in $M(B)$, then π extends uniquely to a unital $*$-hmorphism $\overline{\pi} : M(A) \to M(B)$. Moreover, $\overline{\pi}$ is continuous with respect to the strict topologies on $M(A)$ and $M(B)$.rphism $\overline{\pi} : M(A) \to M(B)$. Moreover, $\overline{\pi}$ is continuous with respect to the strict topologies on $M(A)$ and $M(B)$.*

Remark 2.18. Lemma 2.17 is standard and we omit its proof. See, for instance, Lemma 1.1 in [20]. We comment that the proof in [20] uses the Cohen factorization theorem. As the authors point out, this technology can be avoided, to some extent, by using approximate factorizations instead. For example, the action of $\overline{\pi}(x)$, where $x \in M(A)$, on the right ideal $\pi(A)B = \operatorname{span}\{\pi(a) \cdot b : a \in A, b \in B\} \subset B$ is obviously determined by the equality $\overline{\pi}(x)\cdot\pi(a)b = \pi(xa)b$. The hypotheses in Lemma 2.17 imply $\pi(A)B$ is dense in B, and one can define $\overline{\pi}(x) : B \to B$ by continuously extending. The drawback to this elementary approach, however, is that is not clear how to prove that $\overline{\pi} : M(A) \to M(B)$ is strictly continuous,

[3]For the present application, this can mean that (e_λ) is a net in A such that $\|e_\lambda\| \leq 1$ and e_λ converges strictly to 1 in $M(A)$.

the problem being that strictly convergent nets need not be bounded, obstructing attempts make a simple "$\epsilon/3$" estimate. All that is clear, using this simpler approach, is that $\overline{\pi}$ is strictly continuous on norm-bounded subsets of $M(A)$. On the other hand, using Cohen factorization, one sees that in fact $B = \{\pi(a_1) \cdot b \cdot \pi(a_2) : a_1, a_2 \in A; b \in B\}$ making the strict continuity obvious. As case in point, compare our Lemma 2.17 with Proposition 2.5 in [19].

Proof of Theorem 2.15. Fix some $f \in C_c(\mathbb{R})$. Recycling the notation of the previous section, we let $U_f : A \to A$ be defined by $U_f a = \int_{-\infty}^{\infty} f(t) U_t a \; dt$, for each $a \in A$. To see that $U_f \in \mathrm{M}(A)$, we show its adjoint is U_{f^*}, where f^* is determined by $f^*(t) = \overline{f}(-t)$. Indeed, for any $a, b \in A$, we have

$$(U_f a)^* b = \int_{-\infty}^{\infty} \overline{f}(t)(U_t a)^* b \; dt = \int_{-\infty}^{\infty} \overline{f}(t) a^*(U_{-t} b) \; dt = \int_{-\infty}^{\infty} \overline{f}(-t) a^*(U_t b) \; dt = a^*(U_{f^*} b).$$

It is straightforward to show that $f \mapsto U_f$ is a $\|\cdot\|_1$-contractive $*$-homomorphism of the convolution algebra $C_c(\mathbb{R})$ into $M(A)$. By definition of $C^*(\mathbb{R})$ as the completion of $C_c(\mathbb{R})$ with respect to the largest C^*-norm dominated by $\|\cdot\|_1$, this homomorphism extends uniquely to a $*$-homomorphism $\pi : C^*(\mathbb{R}) \to M(A)$. Let $f_n = f_n^* \in C_c(\mathbb{R})$ be a nonnegative function supported in $[-1/n, 1/n]$ with $\|f_n\|_1 = \int_{-\infty}^{\infty} f_n(t) \; dt = 1$. Given any $x \in A$, we have

$$\|x - \pi(f_n)x\| = \|\int f_n(t)(x - U_t x) \; dt\| \le \int f_n(t) \|x - U_t x\| \; dt.$$

Since $\|x - U_t x\| \to 0$ as $t \to 0$ and $f_n(t)$ is only supported near to $t = 0$ as $n \to \infty$, we see that $\pi(f_n)x = \pi(f_n)^* x \to x$. Thus, $\pi(f_n) \to 1$ strictly and, by Lemma 2.17, π extends (uniquely) to a strictly continuous, unital $*$-homomorphism $\overline{\pi} : M(C^*(\mathbb{R})) \to M(A)$.

Now, let (λ_t) be the canonical strictly continuous 1-parameter group of unitary multipliers of $C^*(\mathbb{R})$ whose action on $C_c(\mathbb{R})$ is determined by $(\lambda_t f)(s) = f(s - t)$. It's straightforward to check the equivariance condition $\pi(\lambda_t f) = U_t \pi(f)$ for all $f \in C_c(\mathbb{R})$, $t \in \mathbb{R}$. Since $C_c(\mathbb{R})$ contains an approximate unit and $\overline{\pi}$ is strictly continuous, it follows that $\overline{\pi}(\lambda_t) = U_t$ for all $t \in \mathbb{R}$. The final step of the construction (purely cosmetics, in effect) is to extend the Fourier transform $C_c(\mathbb{R}) \to C_0(\mathbb{R}) : f \mapsto \widehat{f}$ with $\widehat{f}$ determined by $\widehat{f}(s) = \int_{-\infty}^{\infty} e^{its} f(t)\, dt$ to a C*-algebra isomorphism $C^*(\mathbb{R}) = C_0(\mathbb{R})$. Under this identification, the group $(\lambda_t)_{t\in\mathbb{R}}$ becomes the exponential group $(s \mapsto e^{its})_{t\in\mathbb{R}}$, completing the proof of existence.

It remains to see why there is only one strictly continuous $*$-homomorphism $f \mapsto f(H) : C_b(\mathbb{R}) \to M(A)$ satisfying $e^{itH} = U_t$ for all $t \in \mathbb{R}$. The point here is that the span of all the trigonometric polynomials $s \mapsto e^{its}$, $t \in \mathbb{R}$ is a $*$-algebra[4] in $C_b(\mathbb{R}) = M(C_0(\mathbb{R}))$ which is dense in the strict topology. Indeed, suppose $f \in C_b(\mathbb{R})$ and take a bounded interval $[-\frac{M}{2}, \frac{M}{2}]$. The trigonometric monomial $z(s) = e^{\frac{2\pi is}{M}}$ separates the points of $[-\frac{M}{2}, \frac{M}{2}]$ so, by Weierstrass approximation, there is a polynomial $p(z, \overline{z})$ which closely approximates f on $[-\frac{M}{2}, \frac{M}{2}]$. Moreover, p has M as a period, so the norm of p does not much exceed that of f. Since the strict topology and the topology of uniform convergence on compact sets agree on any norm bounded subset of $C_b(\mathbb{R})$, the trigonometric polynomials are strictly dense in $C_b(\mathbb{R})$ as claimed. □

Remark 2.19. We shall often find it convenient, when declaring a strictly-continuous unitary group (U_t), to write it as $(e^{itH})_{t\in\mathbb{R}}$ immediately, and leave it implicit that H is the Hamiltonian of the group. In light of Corollary 2.10, this practice can never cause ambiguity.

2.3 C*-dynamical systems

When a Banach space possesses some extra structure, one naturally has a heightened interest in the flows which respect that structure. For instance, in the preceding, section we could be said to have been studying the flows on a C*-algebra A which preserved its right Hilbert

[4]The norm-closure of this $*$-algebra is the C*-algebra of so-called *almost-periodic functions*

A-module structure. So, when we speak of a "Banach algebra flow", it shall be assumed to act by (isometric) Banach algebra automorphisms. For a "Banach $*$-algebra flow" the automorphisms will be taken to be $*$-preserving. One easily gets that, in such situations, the infinitesimal generator of the flow reflects the extra structure.

Proposition 2.20. *If δ is the infinitesimal generator of a Banach algebra flow, then* $\mathrm{dom}(\delta)$ *is subalgebra and δ is a derivation in the sense that $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in$* $\mathrm{dom}(\delta)$*. For a Banach $*$-algebra flow,* $\mathrm{dom}(\delta)$ *is a $*$-subalgebra and δ is also $*$-preserving.*

Our chief concern is the C*-algebra case.

Definition 2.21. A **C*-dynamical system** is a triple $(A, \mathbb{R}, \alpha)$ where A is a C*-algebra, and α is strongly continuous action of $\mathbb{R}$ on A by $*$-automorphisms.

The reason for the (apparently redundant) inclusion of $\mathbb{R}$ in this triple is that the term "C*-dynamical system" typically refers to the more general situation wherein any, e.g., locally compact group is allowed to act.

Definition 2.22. If (U_t) is a strictly continuous unitary group in the multiplier algebra $M(A)$ of a C*-algebra A, as considered in the preceding section, then $\alpha_t = \mathrm{Ad}(U_t)$ defines a C*-algebra flow α on A.

Obviously not all flows are inner (implemented by a unitary group); only the trivial flow is inner when A is commutative.

Example 2.23. Suppose X is a locally compact Hausdorff space and α is C*-algebra flow on $C_0(X)$ with infinitesimal generator δ. Then, there is a continuous action ϕ of $\mathbb{R}$ on X by homeomorphisms such that $\alpha_t(f) = f \circ \phi_t$ for all $f \in C_0(\mathbb{R})$, $t \in \mathbb{R}$. If $f \in C_0(X)$ is a C^1 element for α, then $t \mapsto f(\phi_t(x))$ is continuously differentiable for each $x \in X$ and $(\delta(f))(x) = \frac{d}{dt} f(\phi_t(x))\big|_{t=0}$. If, furthermore, $X = M$ is a smooth, compact manifold, V is a smooth vector field on M, and ϕ is the flow on M associated to V, then $\mathrm{dom}(\delta)$ contains $C^\infty(M)$ and $\delta(f) = Vf$ for all $f \in C^\infty(M)$.

The preceding example gives some indication of why strong continuity is the correct continuity condition for C*-algebra flows. Indeed, suppose we asked for continuity with respect to the relative norm-topology on automorphisms. Then, we would get no nonontrivial flows on commutative C*-algebras whatsoever by the following proposition.

Proposition 2.24. *If* $A = C_0(X)$ *where* X *is a locally compact Hausdorff space and* $\mathrm{Aut}_*(A)$ *is its group of* $*$*-automorphisms, then the relative norm topology on* $\mathrm{Aut}_*(A)$ *inherited from* $\mathbb{B}(A)$ *is discrete.*

Proof. Indeed, let $\alpha, \beta \in \mathrm{Aut}_*(A)$ be the $*$-automorphisms defined by precompostion with distinct $g, h \in \mathrm{Homeo}(X)$. Fix $x \in X$ with $g(x) \neq h(x)$. From Urysohn's lemma follows the existence of an $f \in A$ with $f(x) = \|f\| = 1$ such that $\alpha(f)$ and $\beta(f)$ have disjoint supports. Then $\|\alpha(f) - \beta(f)\| = 1$, witnessing $\|\alpha - \beta\| \geq 1$, whence $\mathrm{Aut}_*(A)$ is norm-discrete. □

We now list some important commutative flows. We shall return to these examples intermittently.

Example 2.25 (The translation flow on $\mathbb{R}$). Defining $(\alpha_t f)(s) = f(s+t)$ for all $f \in C_0(\mathbb{R})$; $s, t \in \mathbb{R}$, one obtains a C*-algebra flow α on $C_0(\mathbb{R})$. The infinitesimal generator δ of α is given by $\delta(f) = f'$ for all $f \in C_0^1(\mathbb{R})$ such that $f' \in C_0(\mathbb{R})$. The Riemann integral is a densely-defined, lower semicontinuous, α-invariant trace on $C_0(\mathbb{R})$, in the sense of Section A.3.

Example 2.26 (Linear flows on Tori). Fix a vector $\vec{\theta} = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$ and view it as a vector field on the d-dimensional torus, $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Identify $C(\mathbb{T}^d)$ with the C*-algebra of $\mathbb{Z}^d$-periodic functions on $\mathbb{R}^d$. The linear flow α on $C(\mathbb{T}^d)$ associated to $\vec{\theta}$ is given by $(\alpha_t f)(x_1, \ldots, x_d) = (x_1 + t\theta_1, \ldots, x_d + t\theta_d)$. The infinitesimal generator δ of α has dom$(\delta \subset C^1(\mathbb{T}^d)$ and is given by $\delta(f) = \nabla f \cdot \vec{\theta}$. In particular, if $f_j \in C(\mathbb{T}^d)$ is the jth coordinate projection $f_j(x_1, \ldots, x_n) = e^{2\pi i x_j}$, then $\delta(f_j) = 2\pi i \theta_j \cdot f_j$.

Example 2.27 (The Kronecker flow on $\mathbb{T}^2$). Specializing to the case $d = 2$ above and taking

$\vec{\theta} = (1, \theta)$ for some irrational number θ, we arrive at the Kronecker flow α on $\mathbb{T}^2$ given by $\alpha_t f)(x, y) = f(x + t, y + \theta t)$.

Finally, we discuss C*-algebra flows and multiplier algebras. Since each $*$-automorphism θ of a C*-algebra A extends uniquely[5] to a $*$-automorphism of $M(A)$, denoted simply by θ, one might be tempted to guess that applying this extension principle along a C*-algebra flow on A ought to yield a C*-algebra flow on $M(A)$. Unfortunately, things aren't so simple. The action so obtained is generally discontinuous.

Example 2.28. Let $A = C_0(\mathbb{R})$ and let α be translation flow. Then, $M(A) = C_b(\mathbb{R})$ and the extension of α to $C_b(\mathbb{R})$ is still given by translation in the variable. However, $t \mapsto \alpha_t(f)$ is not norm-continuous for all $f \in C_b(\mathbb{R})$. To see the continuity fail, consider the translates of any function with sufficiently bad oscillatory behavior at ∞.

It is sometimes interesting to ask which multipliers *do* evolve continuously under the extended flow. For instance, in the above example, one sees the translates of any almost-periodic function vary continuously.

Definition 2.29. Let $(A, \mathbb{R}, \alpha)$ be a C*-dynamical system, let $x \in M(A)$, and let k be a nonnegative integer. We say that x is a $\mathbf{C^k}$ **multiplier** for α if $t \mapsto \alpha_t(x)$ is a C^k curve with respect to the C*-algebra norm of $M(A)$.

If $(A, \mathbb{R}, \alpha)$ is a C*-dynamical system, then the C^0 multipliers for α constitute an α-invariant C*-subalgebra of $M(A)$ the restriction of α to which is a C*-algebra flow. The basic reason for having bothered with the above definition is the following example.

Example 2.30. Every C*-dynamical system $(A, \mathbb{R}, \alpha)$ has a crossed-product $A \rtimes_\alpha \mathbb{R}$. There is a canonical strictly continuous unitary group (e^{itH}) in $M(A \rtimes_\alpha \mathbb{R})$ and a canonical embedding $a \mapsto a : A \to M(A \rtimes_\alpha \mathbb{R})$ such that $e^{itH} a e^{-itH} = \alpha_t(a)$. If β denotes the flow on

[5]Indeed, A being an essential ideal in $M(A)$, any $*$-homomorphism out of $M(A)$ is determined by its restriction to A.

$A \rtimes_\alpha \mathbb{R}$ unitarily implemented by (e^{itH}), then the extension of β to a 1-parameter group of automorphisms of $M(A \rtimes_\alpha \mathbb{R})$ is still conjugation by the group (e^{itH}). Thus, A sits in $M(A \rtimes_\alpha \mathbb{R})$ as C^0 multipliers for β.

It's easy to see that no such difficulties arise if one is only passing to the unitization. Every C*-dynamical system can be "unitized" in a unique way.

2.4 Unitary 1-cocycles

Definition 2.31. A **unitary 1-cocycle** u of a C*-dynamical system $(A, \mathbb{R}, \alpha)$ is a strictly continuous family $(u_t)_{t\in\mathbb{R}}$ of unitaries in $M(A)$ obeying the *cocycle law*:

$$u_{s+t} = u_s \alpha_s(u_t) \qquad \forall s, t \in \mathbb{R}.$$

It is easy to check that, if α is a C*-algebra flow, and u is a unitary 1-cocycle of α, then $\mathrm{Ad}(u)\alpha$ given by $t \mapsto u_t \alpha_t(\cdot) u_t^{-1}$ is another C*-algebra flow.

Definition 2.32. If α is a C*-algebra flow, and u is a unitary 1-cocycle of α, then we refer to $\mathrm{Ad}(u)\alpha$ as the **perturbation of α by u**. We call two C*-algebra flows **exterior equivalent** whenever one is a perturbation of the other.

One can check that:

- If u is a unitary 1-cocycle of α, then u^{-1} is a unitary 1-cocycle of $\mathrm{Ad}(u)\alpha$.
- If u is a unitary 1-cocycle of α, and v is a unitary 1-cocycle of $\mathrm{Ad}(u)\alpha$, then vu is a unitary 1-cocycle of α.

and it follows that exterior equivalence is an equivalence relation on C*-algebra flows.

Example 2.33. Let (U_t) and (V_t) be two strictly continuous unitary groups in $M(A)$ and let α and β be the corresponding unitarily implemented flows. Then $u_t = V_t U_t^{-1}$ defines a unitary cocycle of α, and the perturbation of α by u is β. Conversely, if u is any unitary cocycle of α, then $(u_t U_t)$ is a strictly continuous unitary group in $M(A)$. Thus, the set of unitarily implemented flows on A precisely equals the exterior equivalence class of the trivial flow. In particular the property of being unitarily implemented is preserved by exterior equivalence.

Definition 2.34. If, above, H is the Hamiltonian of (U_t), then we denote the Hamiltonian of $(u_t U_t)$ by H_u and call H_u the **perturbation of H by u**. Thus, by definition, $u_t e^{itH} = e^{itH_u}$ for all $t \in \mathbb{R}$.

It is no great shock that cocycles can be pushed forward through equivariant homomorphisms. The following simple result in this direction shall suffice for our purposes.

Proposition 2.35. *Let $(A, \mathbb{R}, \alpha)$ and $(B, \mathbb{R}, \beta)$ be unital*[6] *C*-dynamical systems and let $\varphi : A \to B$ be a unital equivariant homomorphism. If u is a unitary 1-cocycle of α, then $\varphi(u)$ is a unitary 1-cocycle for β.*

We end this section with a sufficient condition for two flows to be exterior equivalence

Definition 2.36. Two C*-algebra flows α and β on A are **conjugate** if there is a $*$-automorphism θ of A such that $\beta_t = \theta\alpha_t\theta^{-1}$ for all $t \in \mathbb{R}$. If the latter can be accomplished with $\theta = \mathrm{Ad}(U)$ for some unitary $U \in M(A)$, then we say α and β are **unitarily conjugate**.

Proposition 2.37. *If α is a C*-algebra flow on A and U is a unitary in $M(A)$, then $u_t = U\alpha_t(U^*)$ defines a unitary 1-cocycle u of α and $\mathrm{Ad}(u)\alpha = \mathrm{Ad}(U)\alpha\,\mathrm{Ad}(U)^{-1}$. Thus, for C*-algebra flows, "unitarily conjugate" implies "exterior equivalent".*

Proof. Writing $\alpha_t(U^*)a = \alpha_t(U^*\alpha_{-t}(x))$, where $a \in A$ is arbitrary, shows that $t \mapsto \alpha_t(U^*)$ is strictly continuous. Thus, $t \mapsto u_t$ is strictly continuous. The rest is algebra. □

[6] Thus, $M(A) = A$, and the strict topology is the norm topology

Remark 2.38. Note that, if α in the above proposition is unitarily implemented by a strictly continuous group (e^{itH}), then $u_t e^{itH} = Ue^{itH}U^* e^{-itH} e^{itH} = Ue^{itH}U^*$ for all $t \in \mathbb{R}$. Thus, in this case, the perturbation of the Hamiltonian H by u is $\mathrm{Ad}(U) \circ H$.

2.5 Crossed Products by $\mathbb{R}$

Associated to each C*-dynamical system $(A, \mathbb{R}, \alpha)$ is a C*-algebra $A \rtimes_\alpha \mathbb{R}$, the **crossed-product** of the system. For our purposes, the following picture of $A \rtimes_\alpha \mathbb{R}$ is convenient:

(C1) There is a canonical strictly continuous unitary group (e^{itH}) in $M(A \rtimes_\alpha \mathbb{R})$.

(C2) There is a canonical embedding $a \mapsto a$ of A in $M(A \rtimes_\alpha \mathbb{R})$.

(C3) (e^{itH}) implements α in the sense that $e^{itH} a e^{-itH} = \alpha_t(a)$ for all $a \in A$, $r \in \mathbb{R}$.

(C4) $A \rtimes_\alpha \mathbb{R}$ is generated by "elementary products" $a \cdot f(H)$ where $a \in A$, $f \in C_0(\mathbb{R})$.

We shall refer to the Hamiltonian H above as the **Hamiltonian of the crossed-product**. One may think of $A \rtimes_\alpha \mathbb{R}$ as a twisted analogue of $A \otimes C_0(\mathbb{R})$, the latter being generated by commuting products $a \cdot f$ where $a \in A$, $f \in C_0(\mathbb{R})$. Indeed, $A \rtimes_\alpha \mathbb{R} \cong A \otimes C_0(\mathbb{R})$ when α is trivial. A typical construction of $A \rtimes_\alpha \mathbb{R}$ begins with the normed $*$-algebra $C_c(\mathbb{R}, A)$ with product, involution, and norm given by

$$(xy)(s) = \int_{-\infty}^{\infty} x(t)\alpha_t(y(-t+s))\ dt$$

$$x^*(t) = \alpha_t(x(-t)^*)$$

$$\|x\|_1 = \int_{-\infty}^{\infty} \|x(t)\|\ dt$$

and then completes it with respect to the largest C*-norm that is dominated by $\|\cdot\|_1$. The group (e^{itH}) in $M(A \rtimes_\alpha \mathbb{R})$ of (C1) is determined by

$$(e^{itH} \cdot x)(s) = \alpha_t(x(-t+s)) \qquad (x \cdot e^{itH})(s) = x(s-t) \qquad \forall x \in C_c(\mathbb{R}, A).$$

The embedding of A in $M(A \rtimes_\alpha \mathbb{R})$ of (C2) is determined by

$$(a \cdot x)(t) = ax(t) \qquad (x \cdot a)(t) = x(t)\alpha_t(a) \qquad \forall a \in A, x \in C_c(\mathbb{R}, A)$$

One can check that (e^{itH}) implements α in the sense of (C3). Indeed, one way to think of the crossed-product construction is as an enlarging procedure by which an arbitrary flow can be implemented by a group of unitaries in a weak sense[7] – perhaps not by a group in $M(A)$, but certainly by a group in $M(A \rtimes_\alpha \mathbb{R})$. The criterion (C4) follows from

Proposition 2.39. *Let $(A, \mathbb{R}, \alpha)$ be a C*-dynamical system, and H the Hamiltonian of the crossed-product. If $a \in A$ and $f = \widehat{g}$ where $g \in C_c(\mathbb{R})$, then $a \cdot f(H)$ belongs to $C_c(\mathbb{R}, A) \subset A \rtimes_\alpha \mathbb{R}$ and is given by $t \mapsto g(t)a$.*

Proof. Let $f = \widehat{g}$ where $g \in C_c(\mathbb{R})$. First we note that, if $x \in C_c(\mathbb{R}, A)$, then $f(H) \cdot x$ belongs to $C_c(\mathbb{R}, A)$ and is given by $s \mapsto \int_{-\infty}^{\infty} g(t)\alpha_t(x(s-t))\, dt$. Indeed, recalling the definitions $f(H) \cdot x = \int_{-\infty}^{\infty} g(t)e^{itH}x\, dt$, and $(e^{itH}x)(s) = \alpha_t(x(s-t))$, it is straightforward to verify that the approximants to the integral $\int_{-\infty}^{\infty} g(t)e^{itH}x\, dt$ converge in L^1, and hence in the C^*-norm, to $s \mapsto \int_{-\infty}^{\infty} g(t)\alpha_t(x(s-t))\, dt$. If $a \in A$, we have also $a \cdot f(H) \cdot x$ belonging to $C_c(\mathbb{R}, A)$ and given by $s \mapsto a \int_{-\infty}^{\infty} g(t)\alpha_t(x(s-t))\, dt = \int_{-\infty}^{\infty} g(t)a\alpha_t(x(s-t))$. Since, $a \cdot f(H)$ and $s \mapsto g(t)a$ give the same multiplier of $C_c(\mathbb{R}, A)$, they are equal as claimed. □

[7]This is directly analogous to a corresponding point of view on the construction of the group theoretic semidirect product. Given an action θ of a group G on a second group H by automorphisms, one constructs the semidirect product $H \rtimes_\theta G$ as an ambient group containing both H and G, H normally so, in such a way that the action θ is realized as the conjugation action of G on H inside $H \rtimes_\theta G$.

Example 2.40. The crossed-product of $\mathbb{C}$ (by the trivial flow) is the same thing as the group C*-algebra $C^*(\mathbb{R})$ of $\mathbb{R}$. The canonical strictly continuous group in $M(C^*(\mathbb{R}))$ is the group (λ_t) given by $(\lambda_t f)(s) = f(s-t)$ when $f \in C_c(\mathbb{R})$. The Hamiltonian of (λ_t) is just $i\frac{d}{dt}$. The map $f \mapsto f(i\frac{d}{dt})$ is the inverse of the Fourier isomorphism $f \mapsto \widehat{f} : C^*(\mathbb{R}) \to C_0(\mathbb{R})$ determined on $C_c(\mathbb{R}) \subset C^*(\mathbb{R})$ by $\widehat{f}(s) = \int_{-\infty}^{\infty} f(t)e^{its}\, dt$.

We revisit two of the systems from Section 2.3 and identify the crossed-products.

Example 2.41 (The translation flow on $\mathbb{R}$). If α is the translation flow on $C_0(\mathbb{R})$ determined by $(\alpha_t f)(x) = f(x+t)$, as considered in Example 2.25, then

$$C_0(\mathbb{R}) \rtimes_\alpha \mathbb{R} \cong \mathbb{K}(\mathcal{H})$$

where $\mathcal{H} = L^2(\mathbb{R})$. To see why, identify each integral kernel $k \in C_c(\mathbb{R}^2)$ with its corresponding Hilbert-Schmidt operator, given on $\xi \in \mathcal{H}$ by $(k\xi)(x) = \int_{-\infty}^{\infty} k(x,y)\xi(y)\, dy$, $\xi \in L^2(\mathbb{R})$, so that $C_c(\mathbb{R}^2) \subset \mathbb{K}(L^2(\mathbb{R}))$. A reasonably canonical choice of isomorphism $\mathbb{K}(\mathcal{H}) \to C_0(\mathbb{R}) \rtimes_\alpha \mathbb{R}$ sends $k \in C_c(\mathbb{R}^2)$ to $f \in C_c(\mathbb{R}, \mathbb{C}_0(\mathbb{R}))$ given by $f(x,y) = k(y, x+y)$. Making the identifications $M(C_0(\mathbb{R}) \rtimes_\alpha \mathbb{R})) = M(\mathbb{K}(\mathcal{H})) = \mathbb{B}(\mathcal{H})$, the canonical strictly continuous group (e^{itH}) of the crossed-product is given on $\xi \in \mathcal{H}$ by $(e^{itH}\xi)(x) = \xi(x+t)$. Thus, one may think of the Hamiltonian H of the crossed-product as the momentum operator $\frac{1}{i}\frac{d}{dx} : \mathcal{H} \to \mathcal{H}$, in this case. The embedding of $C_0(\mathbb{R})$ in $M(C_0(\mathbb{R}) \rtimes_\alpha \mathbb{R}))$ is just the usual embedding of $C_0(\mathbb{R})$ into $\mathbb{B}(\mathcal{H})$ as multiplication operators.

Example 2.42 (The Kronecker flow on $\mathbb{T}^2$). Identify $C(\mathbb{T}^2)$ with the C*-algebra of $\mathbb{Z}^2$-periodic functions on $\mathbb{R}^2$. Fix an irrational number θ and let α be the Kronecker flow on $C(\mathbb{T}^2)$ determined by $(\alpha_t f)(x,y) = f(x+t, y+\theta t)$. In this case,

$$C(\mathbb{T}^2) \rtimes_\alpha \mathbb{R} \cong \mathbb{K}(L^2(\mathbb{T})) \otimes A_\theta$$

where A_θ is the irrational rotation algebra. The existence of an isomorphism follows from the much more general Corollary 2.8 in [14], but let us sketch an elementary approach. Our strategy is to embed both algebras as operators on $\mathcal{H} = L^2(\mathbb{T}^2) = L^2(\mathbb{T}) \otimes L^2(\mathbb{T})$, and then show the embedded algebras are conjugate by a unitary transformation W of $\mathcal{H}$.

Recall that A_θ can be defined as the sub-C*-algebra of $\mathbb{B}(L^2(\mathbb{T}))$ generated by the copy of $C(\mathbb{T})$ (embedded as multiplication operators) and the unitary transformation R_θ given by $(R_\theta \xi)(x) = \xi(x+\theta)$. Since, obviously, $\mathbb{K}(L^2(\mathbb{T})) \subset \mathbb{B}(L^2(\mathbb{T}))$ as well, it is manifestly the case that $\mathbb{K}(L^2(\mathbb{T})) \otimes A_\theta$ is faithfully represented on $\mathcal{H} = L^2(\mathbb{T}) \otimes L^2(\mathbb{T})$.

Define a covariant representation, (μ, U) of $(C(\mathbb{T}^2), \mathbb{R}, \alpha)$ on $\mathcal{H} = L^2(\mathbb{T}^2)$ by

$$(\mu(f)\xi)(x,y) = f(x,y)\xi(x,y) \qquad\qquad (U_t\xi)(x,y) = \xi(x+t, y+\theta t).$$

One can check that the *integrated form* (see [34] for more information) of this covariant representation

$$(\pi(F)\xi)(x,y) = \int_{-\infty}^{\infty} F(t,x,y)\xi(x+t, y+\theta t)\ dt \qquad \forall F \in C_c(\mathbb{R}, C(\mathbb{T}^2)), \xi \in \mathcal{H}$$

is a faithful representation of $C(\mathbb{T}^2) \rtimes_\alpha \mathbb{R}$ on $\mathcal{H}$. Now, define $W : \mathcal{H} \to \mathcal{H}$ by

$$(W\xi)(x,y) = \xi(x, y + \{x\}\theta)$$

where $\{r\} \in [0,1)$ denotes the fractional part of a real number r. To be sure, $(x,y) \mapsto (x, y + \{x\}\theta)$ is a discontinuous map of $\mathbb{T}^2$. Nonetheless, as the mapping is measure-preserving, W is a unitary transformation, and so $\widetilde{\pi} = W\pi(\cdot)W^{-1}$ is a another faithful representation of $C(\mathbb{T}^2) \rtimes_\alpha \mathbb{R}$ on $\mathcal{H}$. With some care, one checks that

$$(\widetilde{\pi}(F)\xi)(x,y) = \int_{-\infty}^{\infty} F(t-x, x, y+x\theta)\xi(t, y+\lfloor t \rfloor \theta)\ dt \quad \forall F \in C_c(\mathbb{R}, C(\mathbb{T}^2)), \xi \in \mathcal{H} \quad (2.1)$$

where $\lfloor r \rfloor$ denotes the greatest integer not exceeding a real number r, and it is assumed that $0 \leq x < 1$ for simplicity's sake. The constraint on x is not essential, since x really represents a point on $\mathbb{T}$, but the formula (2.1) is cleaner this way.

Fixing an element $a = f \cdot R_\theta^n \in A_\theta$, where $f \in C(\mathbb{T})$, $n \in \mathbb{Z}$, and an integral kernel $k \in C(\mathbb{T}^2) \subset \mathbb{K}(L^2(\mathbb{T}))$, we define $F_{k,a} : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{C}$ by

$$F_{k,a}(t, x, y) = \begin{cases} k(x, x+t) \cdot f(y - \{x\}\theta) & \text{if } n \leq x + t < n+1 \\ 0 & \text{otherwise} \end{cases} .$$

Although $F_{k,a}$ does not belong to $C_c(\mathbb{R}, C(\mathbb{T}^2))$, if one uses it in equation (2.1), one gets

$$(\widetilde{\pi}(F_{a,k})\xi)(x, y) = f(y) \int_0^1 k(x, t)\xi(t, y + n\theta)\, dt \qquad\qquad \forall \xi \in \mathcal{H}$$

which says exactly that

$$\widetilde{\pi}(F_{k,a}) = k \otimes a \in \mathbb{K}(L^2(\mathbb{T})) \otimes A_\theta.$$

One can show that, when $F \in C_c(\mathbb{R}, C(\mathbb{T}^2))$ is a good approximation of $F_{a,k}$, then $\widetilde{\pi}(F)$ approximates $\widetilde{\pi}(F_{k,a}) = k \otimes a$ in operator norm and it follows that $\mathbb{K}(L^2(\mathbb{T})) \otimes A_\theta \subset \widetilde{\pi}(C(\mathbb{T}^2) \rtimes_\alpha \mathbb{R})$. The reverse inclusion follows from similarly unattractive arguments, so $\widetilde{\pi}$ is an isomorphism of $C(\mathbb{T}^2) \rtimes_\alpha \mathbb{R}$ onto $\mathbb{K}(L^2(\mathbb{T})) \otimes A_\theta \subset \mathbb{B}(\mathcal{H})$.

The most important maps between crossed-products are induced from the underlying dynamical systems. We consider two such classes of: those arising from equivariant homomorphisms, and those arising from exterior equivalences.

Definition 2.43. If $(A, \mathbb{R}, \alpha)$ and $(B, \mathbb{R}, \beta)$ are two $\mathbb{R}$-dynamical systems and $\varphi : A \to B$ is an equivariant $*$-homomorphism, then the **dual homomorphism** is the unique $*$-homomorphism $\widehat{\varphi} : A \rtimes_\alpha \mathbb{R} \to B \rtimes_\beta \mathbb{R}$ satisfying $\widehat{\varphi}(x) = \varphi \circ x$ for all $x \in C_c(\mathbb{R}, A)$.

In fact, as one can easily check, the crossed-product construction is a functor from the category of C*-dynamical systems and equivariant $*$-homomorphisms to the category of C*-algebras and $*$-homomorphisms. Given our preferred picture of the crossed-product, it is worthwhile to make note of the dual map's behavior on elementary elements.

Proposition 2.44. *Let $(A, \mathbb{R}, \alpha)$ and $(B, \mathbb{R}, \beta)$ and let $\varphi : A \to B$ be an equivariant $*$-homomorphism. Then, the dual homomorphism $\widehat{\varphi}$ is determined by*

$$\widehat{\varphi}(a \cdot f(H)) = \varphi(a) \cdot f(K) \qquad\qquad \forall a \in A, f \in C_0(\mathbb{R}).$$

Here, H is the Hamiltonian of $A \rtimes_\alpha \mathbb{R}$ and K is the Hamiltonian of $B \rtimes_\beta \mathbb{R}$.

Proof. When $f = \widehat{g}$ for $g \in C_c(\mathbb{R})$, this follows from Proposition 2.39 and the definition of $\widehat{\varphi}$. The general statement follows by continuity. □

Example 2.45. If $(A, \mathbb{R}, \alpha)$ is a C*-dynamical system, and $e \in A$ is an α-invariant projection, then $\varphi_e : \mathbb{C} \to A$ determined by $\varphi_e(1) = e$ is an equivariant[8] $*$-homomorphism (nonunital if $e \neq 1$). The dual homomorphism $\widehat{\varphi_e} : C^*(\mathbb{R}) \to A \rtimes_\alpha \mathbb{R}$ is such that

$$\widehat{\varphi}_e \left(f\left(i\tfrac{d}{dt}\right)\right) = \widehat{\varphi}_e\left(1 \cdot f\left(i\tfrac{d}{dt}\right)\right) = \varphi_e(1) \cdot f(H) = e \cdot f(H) \qquad\qquad \forall f \in C_0(\mathbb{R})$$

where $i\frac{d}{dt}$ is Hamiltonian of $C^*(\mathbb{R})$ and H is the Hamiltonian of $A \rtimes_\alpha \mathbb{R}$.

Exterior equivalent flows, on the other hand, give canonically isomorphic crossed-products.

Proposition 2.46. *Let $(A, \mathbb{R}, \alpha)$ be a C*-dynamical system, u a unitary 1-cocycle for α, and $\alpha' = \mathrm{Ad}(u)\alpha$ the adjusted flow. Then, there is unique $*$-isomorphism $\iota_u : A \rtimes_{\alpha'} \mathbb{R} \to A \rtimes_\alpha \mathbb{R}$ given by $(\iota_u(x))(t) = x(t)u_t$ for all $x \in C_c(\mathbb{R}, A)$, $t \in \mathbb{R}$.*

[8]With respect to the trivial dynamics on $\mathbb{C}$, which are the only dynamics that $\mathbb{C}$ supports.

Proof. The point is that ι_u is already a $\|\cdot\|_1$-isometric $*$-isomorphism from $C_c(\mathbb{R}, A)$ with $*$-algebra structure coming from α' to $C_c(\mathbb{R}, A)$ with $*$-algebra structure coming from α. Indeed, given $x, y \in C_c(\mathbb{R}, A)$, $s \in \mathbb{R}$, we have the following.

$$\|\iota_u x\|_1 = \int \|x(t)u_t\| \; dt = \int \|x(t)\| \; dt = \|x\|_1$$

$$(\iota_u x)^*(s) = \alpha_s(x(-s)u_{-s})^* = (u_s^* \alpha'_s(x(-s))u_s\alpha_s(u_{-s}))^* = (u_s^*\alpha'(x(-s)))^* = \iota_u(x^*)(s)$$

$$\begin{aligned}(\iota_u x)(\iota_u y)(s) &= \int x(t)u_t\alpha_t(y(s-t)u_{s-t}) \; dt \\ &= \int x(t)\alpha'_t(y(s-t))u_t\alpha_t(u_{s-t}) \; dt \\ &= \int x(t)\alpha'_t(y(s-t))u_s \; dt \\ &= \iota_u(xy)(s)\end{aligned}$$

It is clear how to construct the inverse map. □

Just as for $\widehat{\varphi}$, we would like to know the behavior of ι_u on our elementary elements. That is, we want the analogue of Proposition 2.44. First, we check

Lemma 2.47. *For any C^*-dynamical system $(A, \mathbb{R}, \alpha)$, the embedding of A into $M(A \rtimes_\alpha \mathbb{R})$ extends uniquely to embedding of $M(A)$ into $M(A \rtimes_\alpha \mathbb{R})$. Moreover, the extension is unital, faithful, and continuous with respect to the strict topologies.*

Proof. To show the extension exists, is unique and is continuous for the strict topologies, we apply Lemma 2.17. Let (e_λ) be a net in A such that $\|e_\lambda\| \leq 1$ and $e_\lambda \to 1$ strictly in $M(A)$. Suppose that $x \in C_c(\mathbb{R}, A)$ and fix $\epsilon > 0$. Find a finite set $\{t_1, \ldots, t_n\} \subset \mathbb{R}$ such that, for every $t \in \mathbb{R}$, there is an i (depending on t) such that $\|x(t) - x(t_i)\| < \epsilon/3$. Now, find λ_0 so that $\lambda \geq \lambda_0$ implies $\|e_\lambda x(t_i) - x(t_i)\| \leq \epsilon/3$ for $i = 1, \ldots, n$. Then, for every $t \in \mathbb{R}$,

$$\|e_\lambda x(t) - x(t)\| \leq \|e_\lambda x(t) - e_\lambda x(t_i)\| + \|e_\lambda x(t_i) - x(t_i)\| + \|x(t_i) - x(t)\| < \epsilon$$

Thus, $e_\lambda x \to x$ uniformly over $\mathbb{R}$ and so, noting x and $e_\lambda x$ have the same compact support, $\|e_\lambda x - x\|_1 \to 0$ as well. From a trivial $\epsilon/3$ argument, we get $e_\lambda x \to x$ for all $x \in A \rtimes_\alpha \mathbb{R}$. Similarly, $x e_\lambda \to x$ for all $x \in A \rtimes_\alpha \mathbb{R}$ so that $e_i \to 1$ strictly in $M(A \rtimes_\alpha \mathbb{R})$ and Lemma 2.17 applies as desired. Since A is an essential ideal in A, the faithfulness of the extension $M(A) \to M(A \rtimes_\alpha \mathbb{R})$ follows from its faithfulness on A. □

The strict continuity of the containment $M(A) \to M(A \rtimes_\alpha \mathbb{R})$ permits the following observation.

Corollary 2.48. *Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system, and H the Hamiltonian of $A \rtimes_\alpha \mathbb{R}$. If u is a unitary 1-cocycle of α, then u is also a unitary 1-cocycle for the flow on $A \rtimes_\alpha \mathbb{R}$ implemented by (e^{itH}).*

We now describe the isomorphism ι_u of Proposition 2.46 in terms of elementary elements.

Proposition 2.49. *Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system, u a unitary 1-cocycle of α, and $\alpha' = \mathrm{Ad}(u)\alpha$ the perturbation of α by u. Then, the isomorphism $\iota_u : A \rtimes_{\alpha'} \mathbb{R} \to A \rtimes_\alpha \mathbb{R}$ is determined by*

$$\iota_u(a \cdot f(H')) = a \cdot f(H_u) \qquad \forall a \in A,\ f \in C_0(\mathbb{R})$$

where H is the Hamiltonian of $A \rtimes_\alpha \mathbb{R}$, H' is the Hamiltonian of $A \rtimes_{\alpha'} \mathbb{R}$, and H_u is the perturbation of H by u.

Proof. We shall in fact show that $\iota_u(a) = a$ and $\iota_u(f(H')) = f(H_u)$ for all $a \in A$, $f \in C_0(\mathbb{R})$ where we have implicitly extended ι_u to an isomorphism between the multiplier algebras.

It's easy to see $a \cdot \iota_u(x) = \iota_u(a \cdot x)$ for all $x \in C_c(\mathbb{R}, A)$, and it follows that $\iota_u(a) = a$. Similarly, the calculation $(\iota_u(x) \cdot u_t \cdot e^{itH})(s) = (\iota_u(x) \cdot u_t)(s-t) = x(s-t)u_{s-t}\alpha_{s-t}(u_t) = x(s-t)u_s = \iota_u(xe^{itH'})(s)$ shows $\iota_u(x) \cdot e^{itH_u} = \iota_u(xe^{itH'})$ for all $x \in C_c(\mathbb{R}, A)$, and it follows that $\iota_u(e^{itH'}) = e^{itH_u}$. From the uniqueness in Theorem 2.15, we get $\iota_u(f(H')) = f(H_u)$ for all $f \in C_0(\mathbb{R})$, and we are finished. $\square$

The above proposition shows one has an perturbed picture of $A \rtimes_\alpha \mathbb{R}$ as the algebra generated by elementary products $a \cdot f(H_u)$, $a \in A$, $f \in C_0(\mathbb{R})$, whenever u is a fixed unitary cocycle of α. Referring to Proposition 2.39, and using the definition of ι_u on $C_c(\mathbb{R}, A)$, we also have

Corollary 2.50. *Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system, u a unitary 1-cocycle of α, H the Hamiltonian $A \rtimes_\alpha \mathbb{R}$, and H_u the perturbation of H by u. If $a \in A$ and $f = \widehat{g}$ where $g \in C_c(\mathbb{R})$, then $a \cdot f(H_u)$ belongs to $C_c(\mathbb{R}, A) \subset A \rtimes_\alpha \mathbb{R}$ and is given by $t \mapsto g(t)au_t$.*

Using the above Corollary, the definition of $\widehat{\varphi}$ on $C_c(\mathbb{R}, A)$ and referring to Proposition 2.35, we get the following version of Proposition 2.44 to account for the perturbed picture of the crossed-product.

Corollary 2.51. *Let $(A, \mathbb{R}, \alpha)$ and $(B, \mathbb{R}, \beta)$ be unital*[9] *C^*-dynamical systems, let u be a unitary 1-cocycle of α, and let $\varphi : A \to B$ be a unital equivariant homomorphism. Then, the dual homomorphism $\widehat{\varphi}$ is determined by*

$$\widehat{\varphi}(a \cdot f(H_u)) = \varphi(a) \cdot f(K_{\varphi(u)}) \qquad \forall a \in A, f \in C_0(\mathbb{R}).$$

Here, H is the Hamiltonian of $A \rtimes_\alpha \mathbb{R}$ and H_u is the perturbation of H by u. Likewise, K is the Hamiltonian of $B \rtimes_\beta \mathbb{R}$ and $K_{\varphi(u)}$ is its perturbation by $\varphi(u)$.

[9]Thus, $M(A) = A$, and the strict topology is the norm topology

Chapter 3

Winding number-type invariants in operator algebras

3.1 Paradigm

Let Γ be the group of all smooth loops γ in the punctured plane $\mathbb{C} \setminus \{0\}$, based so that $\gamma(0) = \gamma(1) = 1$, and with group law given by pointwise multiplication. Recall, from elementary complex analysis that

$$\mathrm{ind}(\gamma) := \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(t)}{\gamma(t)}\, dt$$

is the winding number of a loop $\gamma \in \Gamma$. In particular, ind is a homotopy-invariant group homomorphism $\Gamma \to \mathbb{Z}$. Interestingly, many properties of ind can be deduced using nothing beyond this integral formula, elementary calculus, and a bit of algebraic trickery. For example, the Leibniz rule for products shows

$$\frac{(\gamma_1\gamma_2)'}{\gamma_1\gamma_2} = \frac{\gamma_1'\gamma_2 + \gamma_1\gamma_2'}{\gamma_1\gamma_2} = \frac{\gamma_1'}{\gamma_1} + \frac{\gamma_2'}{\gamma_2}$$

which implies ind is a homomorphism. Now, suppose $(s,t) \mapsto \gamma(s,t)$ is a smooth homotopy of loops in Γ, that is, a smooth map such that $\gamma(s,0) = \gamma(s,1) = 1$ for all $s \in [0,1]$. For any holomorphic function f on $\mathbb{C} \setminus \{0\}$, we have

$$\frac{\partial}{\partial s}\left(\frac{\partial \gamma}{\partial t} \cdot f(\gamma(s,t))\right) = \frac{\partial^2 \gamma}{\partial s \partial t} \cdot f(\gamma(s,t)) + \frac{\partial \gamma}{\partial s} \cdot \frac{\partial \gamma}{\partial t} \cdot f'(\gamma(s,t)) = \frac{\partial}{\partial t}\left(\frac{\partial \gamma}{\partial s} \cdot f(\gamma(s,t))\right)$$

and differentiating under the integral sign then shows

$$\frac{d}{ds}\int_0^1 \frac{\partial \gamma}{\partial t} \cdot f(\gamma(s,t))\ dt = \int_0^1 \frac{\partial}{\partial t}\left(\frac{\partial \gamma}{\partial s} \cdot f(\gamma(s,t))\right)\ dt = \left[\frac{\partial \gamma}{\partial s} \cdot f(\gamma(s,t))\right]_{t=0}^{t=1} = 0$$

where the right-hand side vanishes because $\gamma(s,t)$ is constant in s when $t = 0, 1$. In other particular, putting $\gamma_s = \gamma(s,\cdot)$ and using $f(z) = \frac{1}{z}$, we get that $\mathrm{ind}(\gamma_s)$ is independent of the homotopy parameter $s \in [0,1]$.

The decidedly formal character of the computations above suggests it may be possible to abstract them in order to obtain homotopy invariant homomorphisms in different settings. In this thesis, we are specifically interested in generalizations for operator algebras, and C*-algebras in particular. In order to illustrate the paradigm, we prove here a simple theorem of this type assuming unrealistically strong hypotheses so as to keep technical clutter to a minimum.

Definition 3.1. Let B be a Banach algebra. A **bounded derivation** of B is bounded linear map $\delta : B \to B$ satisfying $\delta(xy) = \delta(x)y + y\delta(x)$ for all $x, y \in B$. A **bounded trace** on B is a bounded linear functional $\tau : B \to \mathbb{C}$ such that $\tau(xy) = \tau(yx)$ for all $x, y \in B$.

If δ is a bounded derivation of a unital Banach algebra B, then some simple algebraic manipulations give the expected identities

$$\delta(1) = 0 \qquad\qquad \delta(x^{-1}) = -x^{-1}\delta(x)x^{-1} \qquad\qquad \forall x \in \mathrm{GL}(B).$$

Theorem 3.2. *Suppose B is a unital Banach algebra with bounded derivation δ and bounded trace τ. Then, $x \mapsto \tau(\delta(x)x^{-1})$ is a group homomorphism $\mathrm{GL}(B) \to \mathbb{C}$. Furthermore, if $\tau \circ \delta = 0$, then the homomorphism is constant on connected-components.*

Proof. The group law comes from an easy manipulation using the defining properties of δ and τ:

$$\tau(\delta(xy)y^{-1}x^{-1}) = \tau(\delta(x)yy^{-1}x^{-1}) + \tau(x\delta(y)y^{-1}x^{-1}).$$

For the second part, it is useful to first record the expected identities

$$\delta(1) = 0 \qquad\qquad \delta(x^{-1}) = -x^{-1}\delta(x)x^{-1} \quad \forall x \in \mathrm{GL}(B).$$

whose verifications are simple algebraic manipulations. Adding the assumption $\tau \circ \delta = 0$ we get, from $\delta(xy) = \delta(x)y + x\delta(y)$, the additional identity

$$\tau(\delta(x)y) + \tau(\delta(y)x) = 0 \quad \forall x \in B.$$

Now let $t \mapsto x_t : [0,1] \to \mathrm{GL}(B)$ be a smooth path. We have

$$\begin{aligned}
\tfrac{d}{dt}\tau\left((\delta(x_t)x_t^{-1})\right) &= \tau(\delta(\dot{x}_t)x_t^{-1}) - \tau(\delta(x_t)x_t^{-1}\dot{x}_t x_t^{-1}) \\
&= \tau(\delta(\dot{x}_t)x_t^{-1}) - \tau(x_t^{-1}\delta(x_t)x_t^{-1}\dot{x}_t) \\
&= \tau(\delta(\dot{x}_t)x_t^{-1}) + \tau(\delta(x_t^{-1})\dot{x}_t) \\
&= 0
\end{aligned}$$

where, for convenience, Newton's dot notation has been employed to indicate differentiation with respect to the parameter. We conclude that $\tau(\delta(x_t)x_t^{-1})$ does not depend on t.

Thus far, we have only shown the values of the homomorphism coincide at the endpoints of any smooth path. However, as $\mathrm{GL}(B)$ is open in B, two points in the same connected-component of B are joined by a piecewise linear path. □

Given a unital Banach algebra B, we always assume $\|1\| = 1$ which implies that the left regular representation of B on itself is isometric. The matrix algebra $\mathrm{M}_n(B)$ acts on B^n, and can be given the operator norm corresponding to this action. Thus, the $M_n(B)$ are Banach algebras, and the corner embeddings $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right) : M_n(B) \to M_{n+1}(B)$ are isometric. A bounded trace τ on B extends to bounded trace τ_n on each matrix algebra by application down the diagonal followed by summation. A bounded derivation δ of B extends to a bounded derivation δ_n of each $\mathrm{M}_n(B)$ by entry-wise application. It's easy to see that, if $n < k$ and $x, y \in \mathrm{M}_n(B) \subset \mathrm{M}_k(B)$, then $\tau_n(\delta_n(x)y) = \tau_k(\delta_k(x)y)$. If B is *nonunital*, then we can first make the minimal unitization $\widetilde{B}$ a Banach algebra using the norm $\|(x, \lambda)\|_1 = \|x\| + |\lambda|$, and extend τ and δ by $\delta(x, \lambda) := \delta(x)$, $\tau(x, \lambda) := \tau(x)$ to obtain equivalent data on $\widetilde{B}$. By this discussion, we get a mild improvement of Theorem 3.2 by allowing the algebra to be nonunital, and prolonging the homomorphism to K-theory.

Corollary 3.3. *Suppose B is a Banach algebra with a bounded derivation δ and a bounded trace τ satisfying $\tau \circ \delta = 0$. Then, there is a homomorphism $K_1(B) \to \mathbb{C}$ sending the class of $x \in \mathrm{GL}_n(B)$ to the number $\tau_n(\delta_n(x)x^{-1})$.*

Expressions of the form $\tau(\delta(x)x^{-1})$ and their properties have a long history. The earliest instance which I was able to locate occurs in the proof Theorem 2.1 from [27]. The authors attribute the relevant portion of the argument to Huzihiro Araki.

The boundedness assumptions in this section are rather unreasonable since the operations being modeled, differentiation and integration, are, themselves, not bounded in many contexts. We expend some effort in weakening these assumptions to obtain more substantial versions of Theorem 3.2. See, in this vein, Theorem 3.8, Theorem 3.11 and Theorem B.9.

3.2 Unbounded derivations

In this section, we prove a simple extension of Theorem 3.2. Precisely, we relax the boundedness assumption on the derivation δ.

Definition 3.4. A closed, densely-defined derivation of a Banach algebra B is a closed, densely-defined linear transformation $\delta : B \to B$ such that $\mathrm{dom}(\delta)$ is an algebra and $\delta(xy) = \delta(x)y + x\delta(y)$ is satisfied for all $x, y \in \mathrm{dom}(\delta)$.

Example 3.5. If α is a continuous action of $\mathbb{R}$ on a Banach algebra B by isometric automorphisms, then the infinitesimal generator $\delta = \frac{d}{dt}(\cdot)\big|_{t=0}$ of α is a closed, densely-defined derivation of B. In this case, a bounded trace τ on B satisfies $\tau \circ \delta = 0$ if and only if τ is α-invariant. Many examples of closed densely-defined derivations are of this form, but not all. Consider the commutative C*-algebra $C([0,1])$ and let δ be differentiation with $\mathrm{dom}(\delta)$ consisting of all continuously differentiable $f \in C([0,1])$. Then, δ is a (self-adjoint) closed, densely-defined derivation of $C([0,1])$, but does not generate an automorphism group.

Note that, when the B in Definition 3.4 is unital, we did not assume that $1 \in \mathrm{dom}(\delta)$. However, this turns out to follow automatically, as we now show. Moreover, in this case, $\mathrm{dom}(\delta)$ is also closed under taking inverses.

Proposition 3.6. *Let δ be a closed, densely-defined derivation of a unital Banach algebra B. Then the following hold:*

1. *The unit belongs to* $\mathrm{dom}(\delta)$. *Moreover,* $\delta(1) = 0$.

2. *If $x \in \mathrm{dom}(\delta) \in \mathrm{GL}(B)$, then $x^{-1} \in \mathrm{dom}(\delta)$. Moreover, $\delta(x^{-1}) = -x^{-1}\delta(x)x^{-1}$.*

Proof. Obviously, if $1 \in \mathrm{dom}(\delta)$, then $\delta(1) = \delta(1 \cdot 1) = \delta(1) + \delta(1)$ so that $\delta(1) = 0$. Thus, we just need to show that 1, or any nonzero scalar for that matter, belongs to $\mathrm{dom}(\delta)$. Use density of $\operatorname{dom}(\delta)$ to produce an $x \in \operatorname{dom}(\delta)$ such that $\|x+1\| < 1$. Note that

$$(x+1)^n - 1 = \sum_{k=1}^{n} \binom{n}{k} x^k$$

belongs to $\operatorname{dom}(\delta)$ and $(x+1)^n - 1 \to -1$ as $n \to \infty$. We claim that $\delta((x+1)^n - 1) \to 0$ as $n \to \infty$ so that, by closedness of δ, $-1 \in \operatorname{dom}(\delta)$ and (1) will be proved. We prove $\delta((x+1)^n - 1) \to 0$, by establishing the identity

$$\delta((x+1)^{n+1} - 1) = \delta(x)(x+1)^n + (x+1)\delta(x)(x+1)^{n-1} + \ldots + (x+1)^n\delta(x). \qquad (3.1)$$

Note (3.1) is trivially obtained if $1 \in \operatorname{dom}(\delta)$, but the latter is what we are trying to prove. Once (3.1) is established, we shall have the estimate

$$\|\delta((x+1)^{n+1} - 1)\| \leq (n+1)\|\delta(x)\|\|x+1\|^n$$

which implies $\delta((x+1)^n - 1) \to 0$, as desired. To establish (3.1), we first expand out the left-hand side

$$\delta((x+1)^{n+1}-1) = \sum_{k=1}^{n} \binom{n+1}{k} \delta(x^k) = \sum_{k=1}^{n} \binom{n+1}{k} \left(\delta(x)x^{k-1} + x\delta(x)x^{k-2} + \ldots + x^{k-1}\delta(x)\right)$$

The coefficient of $x^p\delta(x)x^q$ above, where $0 \leq p+q \leq n$, is $\binom{n+1}{p+q+1}$. Meanwhile the coefficient of $x^p\delta(x)x^q$ on the right-hand side of (3.1) is $\sum_{i=p}^{n-q} \binom{i}{p}\binom{n-i}{q}$. So, we are reduced to proving the binomial identity

$$\sum_{i+j=n} \binom{i}{p}\binom{j}{q} = \binom{n+1}{p+q+1} \qquad 0 \leq p+q \leq n \qquad (3.2)$$

which can be done by recognizing the right-hand side as counting the number of $(p+q+1)$-

element subsets S of an $(n+1)$-element set according to the position of the $(p+1)$st element of S. The proof of (1) is now complete. As an aside, we remark that (3.2) is formally similar to, but distinct from, the Vandermonde convolution formula

$$\sum_{i+j=n} \binom{p}{i}\binom{q}{j} = \binom{p+q}{n} \qquad 0 \leq n \leq p+q.$$

Towards (2), suppose that $x \in \mathrm{dom}(\delta) \cap GL(B)$. If we already have $x^{-1} \in \mathrm{dom}(\delta)$, then it's easy to see $0 = \delta(xx^{-1}) = \delta(x)x^{-1} + x\delta(x^{-1})$ and therefore that $\delta(x^{-1}) = -x^{-1}\delta(x)x^{-1}$. Thus, we just need to show that $x^{-1} \in \mathrm{dom}(\delta)$ is a necessity.

Case 1: First, assume in addition that $\|x-1\| < 1$ so that x^{-1} is given by the norm-convergent series $\sum_{n=0}^{\infty}(1-x)^n$ where $(1-x)^n \in \mathrm{dom}(\delta)$ for all $n \geq 0$. Moreover, for each n, we have the easily obtained estimate

$$\|\delta((1-x)^n)\| \leq \|\delta(x)\|\|1-x\|^{n-1}$$

which shows $\delta\left(\sum_{n=0}^{N}(1-x)^n\right)$ converges in norm as $N \to \infty$. Since, δ is closed, it follows that $x^{-1} \in \mathrm{dom}(\delta)$ and, indeed, that $\delta(x^{-1}) = \sum_{n=1}^{\infty} \delta((1-x)^n)$.

Case 2: Now let $x \in \mathrm{dom}(\delta) \cap \mathrm{GL}(B)$ be arbitrary. Since δ is densely-defined, we can find $y \in \mathrm{dom}(\delta)$ very close to x^{-1}. Since $\mathrm{GL}(B)$ is open and $x^{-1} \in \mathrm{GL}(B)$, we may also suppose that $y \in \mathrm{GL}(B) \cap \mathrm{dom}(\delta)$. But now, $xy \in \mathrm{GL}(B) \cap \mathrm{dom}(\delta)$ and we may suppose $\|xy - 1\| < 1$ (since y is close to x^{-1}) so, by Case 1 above, $(xy)^{-1} = y^{-1}x^{-1} \in \mathrm{dom}(\delta)$. Thus, $x^{-1} = yy^{-1}x^{-1} \in \mathrm{dom}(\delta)$ as desired. □

By the above proposition and Lemma B.10, we have the corollary

Corollary 3.7. *If δ is a closed, densely-defined derivation of a unital Banach algebra B, then the inclusion* $\mathrm{GL}(\mathrm{dom}(\delta)) \hookrightarrow \mathrm{GL}(B)$ *is a π_0-equivalence*[1].

[1] That is, induces a bijection on path components

It is easy to check that, if δ is a closed, densely-defined derivation, then the graph norm $\|\cdot\| + \|\delta(\cdot)\|$ is submultiplicative. Thus, $\mathrm{dom}(\delta)$ is a Banach algebra in its own right, to which Theorem 3.2 can be applied, and we get the following incremental improvement.

Theorem 3.8. *Let δ be a closed, densely-defined derivation of a unital Banach algebra B, let τ be a bounded trace on B, and suppose that $\tau \circ \delta = 0$. Then there is a unique group homomorphism $\mathrm{GL}(B) \to \mathbb{C}$ sending x to $\tau(\delta(x)x^{-1})$ whenever $x \in \mathrm{GL}(\mathrm{dom}(\delta))$. This homomorphism is constant on connected components of $\mathrm{GL}(B)$.*

Example 3.9. Consider the C*-algebra $C(\mathbb{T})$ of $\mathbb{Z}$-periodic functions on $\mathbb{R}$. Let δ be given by ordinary differentiation on the continuously differentiable functions in $C(\mathbb{T})$. Let $\tau = \int_0^1 (\cdot)\, dt : C(\mathbb{T}) \to \mathbb{C}$. Then, δ is a closed, densely-defined derivation, τ is a bounded trace, and $\tau \circ \delta = 0$. In this case, a loop $\gamma \in \mathrm{GL}(C(\mathbb{T}))$ is a loop in the punctured plane $\mathbb{C} \setminus \{0\}$ and, if $\gamma \in \mathrm{dom}(\delta)$ as well, one has $\tau(\delta(\gamma)\gamma^{-1}) = \int_0^1 \frac{\gamma'(t)}{\gamma(t)}\, dt$ i.e. the homomorphism of Theorem 3.8 is just $2\pi i$ times the winding number in this example.

We can extend a closed, densely-defined derivation δ of a Banach algebra B to a closed densely-defined derivation δ_n of each matrix algebra $\mathrm{M}_n(B)$ by putting $\mathrm{dom}(\delta_n) = \mathrm{M}_n(\mathrm{dom}(\delta))$ and applying δ entrywise. One also has an extension to the unitzation which vanishes on the scalars. By a similar discussion to that which followed Theorem 3.2, we have the following upgrade of Corollary 3.3.

Corollary 3.10. *Suppose B is a Banach algebra with a closed, densely-defined derivation δ and a bounded trace τ satisfying $\tau \circ \delta = 0$. Then, there is a homomorphism $K_1(B) \to \mathbb{C}$ sending the class of $x \in \mathrm{GL}_n(\mathrm{dom}(\delta))$ to the number $\tau_n(\delta_n(x)x^{-1})$.*

We have improved on the results of previous section by allowing unbounded derivations, provided they are closed and densely-defined. We desire a further improvement which will allow unbounded traces as well. Conspicuously, however, we must decide what continuity property of τ will replace the boundedness[2]. One route to resolving this difficulty is to involve positivity. Thus, one is naturally led to the C*-algebraic setting.

3.3 The C*-dynamical winding number

In this section, we construct the homomorphism $\operatorname{ind}_\alpha^\tau : K_1(A) \to \mathbb{R}$ that was discussed in the introductory chapter, and discuss examples. As the unital case is effectively subsumed by the last section's results, we only consider a nonunital C*-algebra A and a densely-defined, lower semicontinuous trace $\tau : A_+ \to [0, \infty]$, as discussed in Appendix A. For convenience, let us gather a few key facts:

1. $A_1^\tau = \{x \in A : \tau(|x|) < \infty\} = \operatorname{span}\{x \in A_+ : \tau(x) < \infty\}$ is a dense, $*$-invariant ideal in A, meanwhile a Banach $*$-algebra for the norm $\|x\|_\tau = \|x\| + \tau(|x|)$.

2. τ extends uniquely to a self-adjoint functional $\tau : A_1^\tau \to \mathbb{C}$. Moreover, $|\tau(x)| \leq \tau(|x|)$ whenever $x \in A_1^\tau$, so τ belongs to the continuous dual of A_1^τ.

3. $\tau(xy) = \tau(yx)$ holds when $x, y \in A$ are such that both xy and yx belong to A_1^τ, so τ is a bounded trace on the Banach algebra A_1^τ.

If, furthermore, $(A, \mathbb{R}, \alpha)$ is a C*-dynamical system, with δ the infinitesimal generator of α, and τ is α-invariant:

4. α restricts to a continuous action α^τ on the Banach $*$-algebra A_1^τ by isometric $*$-automorphisms.

5. $\operatorname{dom}(\delta^\tau) = \{x \in A_1^\tau \cap \operatorname{dom}(\delta) : \delta(x) \in A_1^\tau\}$, where δ^τ is the generator of α^τ.

See Proposition A.18, Theorem A.27, Proposition A.31 and Corollary A.32 and for details. Our goal is the following result.

[2]Note that a densely-defined, closed linear functional on a Banach space is automatically bounded.

Theorem 3.11. *If $(A, \mathbb{R}, \alpha)$ is a C^*-dynamical system, and τ is a densely-defined, lower semicontinuous, α-invariant trace on A, then there is a unique group homomorphism $\mathrm{ind}_\alpha^\tau : K_1(A) \to \mathbb{R}$ given by*

$$\mathrm{ind}_\alpha^\tau([x]) = \frac{1}{2\pi i}\tau_n(\delta_n(x)x^{-1})$$

for all $x \in \mathrm{GL}_n(A_1^\tau)$ with $\delta_n(x) \in \mathrm{M}_n(A_1^\tau)$.

By Example 3.5 and Corollary 3.10, we have already a homomorphism $K_1(A_1^\tau) \to \mathbb{C}$ sending $[x] \mapsto \tau_n(\delta_n(x)x^{-1})$ for every $x \in \mathrm{GL}_n(\mathrm{dom}(\delta^\tau))$. Thus, to prove, Theorem 3.11 above, we just need to see why A_1^τ carries all the K-theory of A, and why $\tau_n(\delta_n(x)x^{-1})$ is pure imaginary. To these ends, we supply the following lemmas.

Lemma 3.12. *If $\mathcal{I}$ is an ideal in an associative algebra $\mathcal{A}$, and n is a positive integer, then $\mathrm{M}_n(\widetilde{\mathcal{I}})$ is inverse-closed in $\mathrm{M}_n(\widetilde{\mathcal{A}})$.*

Proof. Suppose $x \in \mathcal{I}$ and $x+1$ is invertible in $\widetilde{\mathcal{A}}$. Write $(x+1)^{-1} = y+1$ where $y \in A$. Since $xy + x + y = 0$ and $\mathcal{I}$ is an ideal in A, we see $y \in \mathcal{I}$.

Now suppose that $x+\lambda$ is invertible in $\mathrm{M}_n(\widetilde{\mathcal{A}})$ where $x \in \mathrm{M}_n(\mathcal{I})$ and $\lambda \in \mathrm{GL}(n)$. Observe $(x+\lambda)^{-1} - \lambda^{-1} = (\lambda^{-1}x + 1_n)^{-1}\lambda^{-1} - \lambda^{-1} = ((\lambda^{-1}x+1_n)^{-1} - 1_n)\lambda^{-1}$. Since $\lambda^{-1}x \in \mathrm{M}_n(\mathcal{I})$ and $\mathrm{M}_n(\mathcal{I})$ is an ideal in $\mathrm{M}_n(\mathcal{A})$, the preceding paragraph implies $(\lambda^{-1}x + 1_n)^{-1} - 1_n \in \mathrm{M}_n(\mathcal{I})$, whence $(x+\lambda)^{-1} \in \mathrm{M}_n(\widetilde{\mathcal{I}})$. □

Lemma 3.13. *If A is a unital C^*-algebra, and $x \in \mathrm{GL}(A)$, then x^* is path-equivalent to x^{-1} in $\mathrm{GL}(A)$.*

Proof. Since x^*x is positive and invertible, it is path-equivalent to 1 by spectral theory. Thus $x^* = x^*x \cdot x^{-1}$ is path-equivalent to $x^{-1} = 1 \cdot x^{-1}$. □

Proof of Theorem 3.11. Since A_1^τ is a dense ideal in A, Lemmas 3.12 and B.10 show the inclusion $A_1^\tau \hookrightarrow A$ induces an isomorphism on K_1.

Now suppose $x \in \mathrm{GL}_n(A_1^\tau)$ and $\delta_n(x) \in \mathrm{M}_n(A_1^\tau)$. By Lemma 3.12, $x^{-1} \in \mathrm{GL}_n(A_1^\tau)$. Since A_1^τ is a $*$-algebra, $x^* \in \mathrm{GL}_n(A_1^\tau)$. By the calculation

$$\begin{aligned}
\overline{\tau_n(\delta_n(x)x^{-1})} &= \tau_n((x^*)^{-1}\delta_n(x^*)) && \text{since } \tau \text{ and } \delta \text{ are } *\text{-preserving} \\
&= \tau_n(\delta_n(x^*)(x^*)^{-1}) && \text{tracial property} \\
&= \tau_n(\delta_n(x^{-1})x) && [x^*] = [x^{-1}] \text{ in } K\text{-theory} \\
&= -\tau(\delta_n(x)x^{-1}), && \text{group law}
\end{aligned}$$

$\tau_n(\delta_n(x)x^{-1})$ is pure-imaginary, so the index ind_α^τ is real. □

Remark 3.14. An alternative approach to the index ind_α^τ is to define $K_1(A)$ using the stabilization $\mathrm{K}A$. See Theorem 1.8 and Corollary 3.10 in [26] for more information.

3.4 Applications

In this section, we assess the performance of the homomorphism $\mathrm{ind}_\alpha^\tau : K_1(A) \to \mathbb{R}$ when the base algebra is commutative, i.e. $A = C(X)$, and the trace is bounded, i.e. τ is a probability measure. We shall conclude that ind_α^τ can, at best, detect the cohomology group $H^1(X) \subset K^1(X)$. We shall also see that this end is achieved for well-chosen flows on the d-dimensional torus $\mathbb{T}^d$. Let $X = (X, p)$ be a compact Hausdorff space with basepoint. In the ensuing discussion, we only work with basepoint-preserving maps. Thus, for example,

$C(X)$ denotes the C*-algebra of continuous maps on X which vanish at p,

$C(X, \mathrm{U}(n))$ denotes the group of continuous based maps $X \to \mathrm{U}(n)$,

$[X, \mathrm{U}(n)]$ denotes group of homotopy classes of continuous based maps $X \to \mathrm{U}(n)$, and so on. Identify $\mathrm{U}_n(C(X)) = C(X, \mathrm{U}(n))$ so that

$$K^1(X) = K_1(C(X)) = \varinjlim\, [X, \mathrm{U}(n)].$$

Recall that the 1st Čech cohomology group $H^1(X) = H^1(X, \mathbb{Z})$ is isomorphic[3] to $[X, \mathbb{T}]$. In fact, since we do not need the higher-order cohomology groups, let us view this as the definition of $H^1(X)$. As $\mathrm{U}(1) = \mathbb{T}$, we certainly have a map

$$H^1(X) \longrightarrow K^1(X).$$

Moreover, it is not difficult to see that the determinant mappings $\det : \mathrm{U}(n) \to \mathbb{T}$ induce a homomorphism

$$\det_* : K^1(X) \to H^1(X) \qquad\qquad [u] \mapsto [\det \circ u]$$

which is a right inverse for the above map. In particular, we can consider $H^1(X)$ as a subgroup of $K^1(X)$.

Now, a C*-algebra flow α on $C(X)$ is necessarily of the form $\alpha_t(f)(x) = f(\sigma_t(x))$ where σ is a continuous (basepoint-preserving) action of $\mathbb{R}$ on X. A state ω on X is necessarily given by $\omega(f) = \int_X f\, d\mu$ for some (regular, Borel) probability measure μ on X. The state ω is α-invariant if and only if μ is an invariant measure for σ in the usual sense. If $f \in C(X)$ is a C^1 element for α, then $f \circ \sigma : \mathbb{R} \to \mathbb{C}$ is a C^1 curve and the derivation δ associated to α is given by $\delta(f)(x) = \frac{d}{dt} f(\sigma_t(x))\big|_{t=0}$. In this context, we write ind_σ^μ instead of ind_α^τ for the homomorphism $K^1(X) \to \mathbb{R}$ constructed in Theorem 3.11 by ind_σ^μ to make the dependence on the commutative data more explicit.

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[3]More generally, for a group G and a positive integer n, [17] shows the Čech cohomology group $H^n(X, G)$ is in natural bijection with $[X, K(G, n)]$, where $K(G, n)$ denotes an Eilenberg-Maclane space.

Example 3.15. If, above, X is a smooth manifold, V is a smooth vector field on X, and σ is the flow on X associated to V, then the derivation δ of $C(X)$ associated to the flow is given by differentiation along V, that is to say $\delta(f) = Vf$, on smooth functions f.

Proposition 3.16. *Let $X = (X, p)$ be a pointed compact Hausdorff space, σ a continuous, basepoint-preserving action of $\mathbb{R}$ on X, and μ a σ-invariant (regular, Borel) probability measure on X. Then, the index $\operatorname{ind}_\sigma^\mu : K^1(X) \to \mathbb{R}$ assumes all its values on $H^1(X) \subset K^1(X)$. Indeed,*

$$\operatorname{ind}_\sigma^\mu = \operatorname{ind}_\sigma^\mu \circ \det_*$$

where $\det_*$ is the retraction $K^1(X) \to H^1(X)$ induced by the determinant, so $\operatorname{ind}_\sigma^\mu$ is completely determined by its restriction to $H^1(X)$.

Proof. Let τ be the tracial state on $C(X)$ associated to μ. The canonical extension of τ to trace on $\mathrm{M}_n(C(X)) = C(X, \mathrm{M}_n(\mathbb{C}))$, which shall be denoted simply by μ, is given by

$$\tau(f) = \int_X \operatorname{tr}(f(x))\, d\mu \qquad\qquad \forall f \in C(X, \mathrm{M}_n(\mathbb{C}))$$

where tr is the canonical trace on $\mathrm{M}_n(\mathbb{C})$. Now, if a_t is a C^1 path in $\mathrm{GL}(n)$, it is elementary to see that

$$\tfrac{d}{dt} \det(a_t) = \det(a_t) \operatorname{tr}(a_t^{-1} \tfrac{d}{dt} a_t).$$

Thus, if $u \in C(X, \mathrm{U}(n))$ is a C^1 element for α, we see

$$\begin{aligned}
\tau(\delta(u)u^{-1}) &= \int_X \operatorname{tr}(\tfrac{d}{dt} u(\sigma_t(x))\big|_{t=0} u(x)^{-1})\, d\mu \\
&= \int_X \frac{\tfrac{d}{dt} \det(u(\sigma_t(x)))\big|_{t=0}}{\det(u(x))}\, d\mu \\
&= \tau(\delta(\det \circ u)(\det \circ u)^{-1})
\end{aligned}$$

in other words, $\operatorname{ind}_\sigma^\mu([u]) = \operatorname{ind}_\sigma^\mu(\det_*([u]))$. □

By the above, to determine $\mathrm{ind}^{\mu}_{\sigma}$, one just needs to calculate it on $H^1(X) \subset K^1(X)$. Shortly, we shall do just that for linear flows on $X = \mathbb{T}^d$, but let us first show that $H^1(\mathbb{T}^d) \cong \mathbb{Z}^d$ with d commuting generators given by the d coordinate projections $\mathbb{T}^d \to \mathbb{T}$.

Lemma 3.17. *Let $X_1, \dots, X_n$ be a compact Hausdorff spaces* with basepoint. *Then, we have a split exact sequence of abelian groups*

$$0 \longrightarrow H^1(\bigwedge_i X_i) \xrightarrow{q^*} H^1(\prod_i X_i) \xrightarrow{\iota^*} \bigoplus_i H^1(X_i) \longrightarrow 0$$

where q^* is induced by the map $q : \prod X_i \to \bigwedge_i X_i$ from the cartesian product to the smash product which collapses the wedge sum $\bigvee_i X_i$, and ι^* is induced, factor-wise, by the inclusion of X_i into $\prod_i X_i$.

Proof. As $q \circ \iota_i$ is constant, we have $\iota_i^* \circ q^* = (q \circ \iota_i)^* = 0$ showing $\mathrm{ran}(q^*) \subset \ker(\iota^*)$. Define a homomorphism $d : \bigoplus_i H^1(X_i) \to H^1(\prod_i X_i)$ by sending the element represented by an n-tuple of circle-valued maps $(f_1, \dots, f_n)$ to the map $(x_1, \dots, x_n) \mapsto f_1(x_1) \cdots f_n(x_n)$. It's easy to check $\iota^* \circ d = \mathrm{id}$. Thus, ι^* is surjective, and we have a right split. It's also easy to check $\gamma \cdot d(\iota^*(\gamma))^{-1} \in \mathrm{ran}(q^*)$ for all $\gamma \in H^1(\prod_i X_i)$ and, from this observation, the exactness in the middle follows. It remains to see why q^* is injective. To see this, suppose that $f : \prod X_i \to \mathbb{T}$ is identically 1 on $\bigvee_i X_i$, and that f_t is a null-homotopy of f through based maps $\prod_i X_i \to \mathbb{T}$. Let $g_t : \prod_i X_i$ be given by $g_t(x_1, \dots, x_n) = f(\iota_1(x_1)) \cdots f(\iota_n(x_n))$. Then, $f_t g_t^{-1}$ is a null-homotopy of f through maps which are identically 1 on $\bigvee_i X_i$. □

Corollary 3.18. *If, in the above set-up, $\bigwedge_i X_i$ is locally-path connected, and simply connected, then $H^1(\prod_i X_i) \cong \bigoplus_i H^1(X_i)$.*

Proof. We just need to show that $H^1(\prod_i X_i) = 0$. The point here is that, if $f : \bigwedge_i X_i \to \mathbb{T}$, then the hypotheses above imply f lifts through the cover of $\mathbb{T}$ by $\mathbb{R}$. Thus, f is null-homotopic because $\mathbb{R}$ is contractible. □

Corollary 3.19. *For each d, $H^1(\mathbb{T}^d) \cong \mathbb{Z}^d$ with generators the d different projections $f_j : \mathbb{T}^d \to \mathbb{T}$.*

Fix a vector of real parameters $\theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$ and consider the constant vector field V_θ on $\mathbb{R}^d$. The linear flow σ on $\mathbb{R}^d$ associated to V_θ is such that

$$\sigma_t(s_1, \ldots, s_d) = (s_1 + t\theta_1, \ldots, s_d + t\theta_d)$$

Since this just a translation flow, it descends to the quotient $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Given a smooth function $f \in C(\mathbb{T}^d)$, viewed as a $\mathbb{Z}^d$-periodic function on $\mathbb{R}^d$, the derivation δ associated to σ is given simply by

$$\delta(f) = \nabla f \cdot \theta.$$

In particular, if f is the jth coordinate projection $f_j(s_1, \ldots, s_n) = e^{2\pi i s_j}$, we get simply $\delta(f_j) = 2\pi i \theta_j \cdot f_j$. Thus, if μ is the d-dimensional Lebesgue measure on $\mathbb{T}^d$, then

$$\mathrm{ind}_\sigma^\mu([f_j]) = \frac{1}{2\pi i} \int_{\mathbb{T}^d} 2\pi i f_j f_j^{-1} \, d\mu = \theta_j.$$

Thus, identifying $H^1(\mathbb{T}^d) = \mathbb{Z}^d$, we see that ind_θ^μ is just the homomorphism

$$(n_1, \ldots, n_d) \mapsto \sum_{i=1}^{d} n_i \theta_i.$$

If the $\theta_1, \ldots, \theta_d \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$, then this map has trivial kernel. So, it is possible for $\mathrm{ind}_\theta^\mu : H^1(X) \to \mathbb{R}$ to be an isomorphism onto its range.

One can specialize to $d = 2$, and consider the Kronecker flow on $\mathbb{T}^2$ along lines of irrational slope. That is, for some fixed irrational number θ, consider the flow σ on $\mathbb{T}^2$ given by

$$\sigma_t((x, y)) = (x + t, y + \theta t) \qquad \forall t \in \mathbb{R}, (x, y) \in \mathbb{T}^2.$$

By the above discussion, $\mathrm{ind}_\sigma^\mu : K^1(\mathbb{T}^2) \to \mathbb{R}$ has range $\mathbb{Z} + \theta\mathbb{Z}$ in this case.

Remark 3.20. In this section, we have confined ourselves to the stance that $\mathrm{ind}_\sigma^\mu : H^1(X, \mathbb{Z}) \to \mathbb{R}$ is a map out of the first Čech cohomology group. If X is a compact manifold, then such a functional is just an element of $H_1(X, \mathbb{R})$. A concise geometric description of a representing cycle exists: the *asymptotic cycle* C_σ^μ introduced in [32].

Remark 3.21. In spite of our efforts to maintain a high level of generality, allowing for both noncommutative A and unbounded τ, we have not included any calculations of the index ind_α^τ when the algebra A is noncommutative. In the interest of space, let us simply assert that the index ind_α^τ is nontrivial, for example, for some invariant flows on the irrational rotation algebra A_θ (with its unique tracial state).

Chapter 4

Smooth perturbations

4.1 An ODE uniqueness theorem

By a (time-dependent) **uniformly Lipschitz vector field** on a Banach space X, we mean a continuous mapping $V : \mathbb{R} \times X \to X$ such that, for some constant $K > 0$, the bound $\|V(t,x) - V(t,y)\| \leq K\|x - y\|$ holds for all $t \in \mathbb{R}$ and $x, y \in X$. In this section, we establish that the integral curves of such a vector field are unique when they exist. This shall follow as a corollary of the following elementary lemma. ent) **uniformly Lipschitz vector field** on a Banach space X, we mean a continuous mapping $V : \mathbb{R} \times X \to X$ such that, for some constant $K > 0$, the bound $\|V(t,x) - V(t,y)\| \leq K\|x - y\|$ holds for all $t \in \mathbb{R}$ and $x, y \in X$. In this section, we establish that the integral curves of such a vector field are unique when they exist. This shall follow as a corollary of the following elementary lemma.

Lemma 4.1. *Let $f : [0, \infty) \to [0, \infty)$ be a continuously differentiable function. If*

$$f' \leq Kf \qquad\qquad f(0) = 0$$

is satisfied for some $K \geq 0$, then f is identically zero.

Proof. Differentiating $e^{-Kt} f(t)$ yields $e^{-Kt}(f'(t) - Kf(t))$ which, by assumption, is nonpositive. Thus, $t \mapsto e^{-Kt}f(t)$ is nonincreasing. The conclusion follows. □

Roughly speaking, the above lemma asserts that, in the instant where a function first becomes positive, the growth is faster than exponential. The same conclusion is a simple consequence of Gronwall's inequality.

Theorem 4.2. *Let $V : \mathbb{R} \times X \to X$ be a uniformly Lipschitz vector field on a Banach space X, and let $x_0 \in X$. Then, the initial value problem*

$$\dot{x}(t) = V(t, x(t)) \qquad\qquad x(0) = x_0 \tag{4.1}$$

has at most one solution $x : \mathbb{R} \to X$.

Proof. Suppose that x and y both solve the IVP. We shall prove uniqueness for forward time only. On the face of it, our solutions need only be differentiable, but from (4.1), we can see they are actually C^1. In particular, the fundamental theorem of calculus applies and so

$$x(t) - y(t) = \left(x_0 + \int_0^t \dot{x}(s)\ ds\right) - \left(x_0 + \int_0^t \dot{y}(s)\ ds\right) = \int_0^t \left(V(s, x(s)) - V(s, y(s))\right)\ ds.$$

From here follows (for some $K > 0$) the bound

$$\|x(t) - y(t)\| \leq \int_0^t \|V(s, x(s)) - V(s, y(s))\|\ dt \leq K \int_0^t \|x(s) - y(s)\|\ ds.$$

The rest follows from Lemma 4.1 applied to the function $f(t) = \int_0^t \|x(s) - y(s)\|\ ds$. □

4.2 C^1 unitary 1-cocycles

We saw in Theorem 2.11 that there is a bijective correspondence between the norm-continuous unitary groups (U_t) in $M(A)$ and the self-adjoint elements $H \in M(A)$. Each norm-continuous group (U_t) is already C^∞ and has the form $U_t = e^{itH}$ for a unique self-adjoint element $H \in M(A)$. In this section, we derive an analogous result for the unitary 1-cocycles of a C*-dynamical system. Unfortunately, it is not the case that a norm-continuous unitary 1-cocycle is automatically smooth, as the following example shows.

Example 4.3. Let $(A, \mathbb{R}, \alpha)$ be a C*-dynamical system with A unital. Recall from Example 2.37 that, for any unitary $U \in A$, $u_t = U\alpha_t(U^*)$ defines a unitary 1-cocycle u of α. This cocycle is always norm-continuous, but only C^1 when U is C^1 for α.

Given a C*-dynamical system $(A, \mathbb{R}, \alpha)$, recall (Definition 2.29) that a C^0*-multiplier* is an element $x \in M(A)$ such that $t \mapsto \alpha_t(x)$ is norm-continuous for the unique extension of α to an action of $\mathbb{R}$ on $M(A)$ by $*$-automorphisms.

Theorem 4.4. *For every C*-dynamical system* $(A, \mathbb{R}, \alpha)$*, there is a bijective correspondence between the self-adjoint* C^0 *multipliers of* α*, and the the* C^1 *unitary* 1*-cocycles of* α*. The bijection pairs each* C^0 *multiplier* $P \in M(A)$ *with the unique cocycle* u^P *satisfying*

$$\tfrac{d}{dt} u_t^P \big|_{t=0} = iP.$$

Remark 4.5. Note that, already if u is a norm-continuous unitary 1-cocycle for $(A, \mathbb{R}, \alpha)$ then, from $u_{s+t} = u_s \alpha_s(u_t)$, one sees that, for all $t \in \mathbb{R}$, u_t belongs to the unital C*-algebra $B \subset M(A)$ of C^0 multipliers. Thus, u is also a cocycle of the unital C*-dynamical system $(B, \mathbb{R}, \alpha)$. The effect is that, in order to prove Theorem 4.4 above, one need only consider the case where A is unital so that $M(A) = A$ and the strict-topology coincides with the norm topology.

Remark 4.6. Since the unitary 1-cocycle u^P above depends, not just on the self-adjoint $P \in A$, but also on the flow α, it would be more correct to denote it by $u^{P,\alpha}$. This more extravagant notation is usually unnecessary since there tends to be a distinguished flow α.

On the rare occasions where we use a unitary 1-cocycle of some auxiliary flow, we do use the extra adornment. This is done in Proposition 4.16, to follow.

The idea behind Theorem 4.4 is to uncover u^P as the unique solution to the initial value problem

$$\dot{x}(t) = ix(t)\alpha_t(P) \qquad\qquad x(0) = 1. \tag{4.2}$$

Since, $V(t,x) = ix\alpha_t(P)$ is a uniformly Lipschitz vector field in the sense of Theorem 4.2, we know (4.2) has at most one solution. Existence is got explicitly from a "Duhamel series". See [12] for more information on this terminology.

Proposition 4.7. *If u is a C^1 unitary 1-cocycle for a unital C^*-dynamical system $(A, \mathbb{R}, \alpha)$, then $\frac{d}{dt}u_t\big|_{t=0}$ is anti-self-adjoint and u_t solves (4.2) for the self-adjoint $P = \frac{1}{i}\frac{d}{dt}u_t\big|_{t=0}$.*

Proof. In fact, if w_t is any C^1 path of unitaries, with $w_0 = 1$, then $\dot{w}_0$ is anti-self-adjoint[1], as follows from differentiating $1 = w_t^* w_t$ and evaluating at $t = 0$.

For any $s, t \in \mathbb{R}$, we have $u_{s+t} = u_s\alpha_s(u_t)$. Differentiating this expression in t gives $\dot{u}_{s+t} = u_s\alpha_s(\dot{u}_t)$. Taking $t = 0$ gives $\dot{u}_s = u_s\alpha_s(\dot{u}_0) = iu_s\alpha_s(P)$, as desired. □

Proposition 4.8. *Let $(A, \mathbb{R}, \alpha)$ be a unital C^*-dynamical system. Suppose that (4.2) has a (necessarily unique) solution u for some self-adjoint $P \in A$. Then, u is a C^1 cocycle for α.*

Proof. First we verify the cocycle law: $u_{s+t} = u_s\alpha_s(u_t)$. Notice that

$$\tfrac{d}{dt}u_{s+t} = iu_{s+t}\alpha_{s+t}(P)$$

and also that

$$\tfrac{d}{dt}u_s\alpha_s(u_t) = iu_s\alpha_s(u_t\alpha_t(P) = i(u_s\alpha_s(u_t))\alpha_{s+t}(P)$$

[1]Compare this observation with the one following Corollary 4.20.

so that both $x(t) = u_{s+t}$ and $x(t) = u_s\alpha_s(u_t)$ satisfy the initial value problem

$$\dot{x}(t) = ix(t)\alpha_{s+t}(P) \qquad\qquad x(0) = u_s.$$

The desired cocycle law follows from Theorem 4.2. It remains to show each u_t is unitary. First, note each u_t is at least invertible with inverse $\alpha_t(u_{-t})$.

$$u_t\alpha_t(u_{-t}) = u_0 = 1 \qquad\qquad \alpha_t(u_{-t})u_t = \alpha_t(u_{-t}\alpha_{-t}(u_t)) = u_0 = 1$$

Now we just need to show, e.g., that $u_tu_t^* = 1$. We have

$$\tfrac{d}{dt}(u_tu_t^*) = iu_t\alpha_t(P)u_t^* + u_t(iu_t\alpha_t(P))^* = iu_t\alpha_t(P)u_t^* - iu_t\alpha_t(P)u_t^* = 0$$

and so, since $u_0u_0^* = 1$, we get $u_tu_t^* = 1$ for all t. □

Proposition 4.9. *For every unital C^*-dynamical system $(A, \mathbb{R}, \alpha)$, for every self-adjoint $P \in A$, the initial value problem* (4.2) *has a (unique) solution.*

Proof. We recursively define a sequence of C^1 paths $I_n^P : \mathbb{R} \to A$, $n \geq 0$:

$$I_0^P \equiv 1 \qquad\qquad I_{n+1}^P(t) = \int_0^t I_n^P(s)\alpha_s(P)\ ds.$$

By design, $\frac{d}{dt}I_{n+1}^P(t) = I_n^P(t)\alpha_t(P)$. We have also the direct formula

$$I_n^P(t) = \int\cdots\int_{0\leq s_1\leq\ldots\leq s_n\leq t} \alpha_{s_1}(P)\cdots\alpha_{s_n}(P)\ ds_1\cdots ds_n \tag{4.3}$$

for $n > 0$. Strictly speaking, the above formula only makes good sense when $t \geq 0$. It would be more correct to change variables and use

$$I_n^P(t) = t^n \int \cdots \int_{0 \le s_1 \le \ldots \le s_n \le 1} \alpha_{ts_1}(P) \cdots \alpha_{ts_1}(P) \; ds_1 \cdots ds_n$$

which makes sense for all $t \in \mathbb{R}$, but we ignore this detail. Since the volume of the simplex $0 \le s_1 \ldots \le s_n \le t$ is $t^n/n!$ (a good way to see this is to note this simplex is, up to measure zero, a fundamental domain for the symmetric group action on the cube $[0,t]^n$ which permutes coordinates), and since $\|\alpha_{s_1}(P) \cdots \alpha_{s_n}(P)\| \le \|P\|^n$, we have the estimate

$$\|I_n^P(t)\| \le \frac{(|t|\|P\|)^n}{n!}. \tag{4.4}$$

Thus, we can, and do, define u^P by the absolutely convergent series

$$u_t^P = \sum_{n=0}^{\infty} i^n I_n^P(t). \tag{4.5}$$

Notice $u_0^P = 1$, i.e. the initial condition is correct. Since the convergence is uniform for t in bounded subsets of $\mathbb{R}$, we may differentiate the series (4.5) term-by-term to obtain

$$\dot{u}_t^P = \sum_{n=0}^{\infty} i^n \tfrac{d}{dt} I_n^P(t) = \sum_{n=1}^{\infty} i^n I_{n-1}^P(t)\alpha_t(P) = i \left(\sum_{n=1}^{\infty} i^{n-1} I_{n-1}^P(t) \right) \alpha_t(P) = i u_t^P \alpha_t(P),$$

and we have our solution to (4.2). □

Combining Propositions 4.7, 4.8 and 4.9 and referring to Remark 4.5, we see that Theorem 4.4 is proved.

It is occasionally possible to describe the cocycle of Theorem 4.4 more explicitly. In Example 2.37, we noted that unitarily conjugate flows are exterior equivalent. We show that, if the conjugating unitary is C^1 for the flow, then the mediating cocycle is also C^1.

Proposition 4.10. *If $(A, \mathbb{R}, \alpha)$ is a C^*-dynamical system and $U \in M(A)$ is a C^1 unitary multiplier for α, then $P = i\delta(U)U^*$ is a self-adjoint C^0 multiplier for α. The C^1 cocycle u^P is given by $u_t^P = U\alpha_t(U^*)$ for all $t \in \mathbb{R}$. The adjusted flow $\mathrm{Ad}(u^P)\alpha$ is simply $\mathrm{Ad}(U)\alpha\,\mathrm{Ad}(U)^{-1}$.*

Proof. We already know $u_t = U\alpha_t(U^*)$ defines a unitary 1-cocycle u of α such that $\mathrm{Ad}(u)u = \mathrm{Ad}(U)\alpha\,\mathrm{Ad}(U)^{-1}$. Clearly u is C^1 and

$$\tfrac{d}{dt}\left[U\alpha_t(U^*)\right]_{t=0} = U\delta(U^*) = -UU^*\delta(U)U^* = iP$$

so $u = u^P$ by Theorem 4.4. □

Corollary 4.11. *Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system, $U \in M(A)$ a C^1 unitary multiplier, and put $P = i\delta(U)U^*$. If $x \in A$ is fixed by α, then $\mathrm{Ad}(U)x$ is fixed by $\mathrm{Ad}(u^P)\alpha$.*

It is worthwhile to keep track of how the infinitesimal data changes when one adjusts by a C^1 unitary 1-cocycle. The next two propositions are in this vein.

Proposition 4.12. *If A is a C^*-algebra, (e^{itH}) is a strictly-continuous group in $M(A)$, and u^P is a C^1 cocycle for the flow implemented by (e^{itH}), the the perturbation of H by u^P is $H + P$. That is, $u_t^P e^{itH} = e^{it(H+P)}$ for all $t \in \mathbb{R}$.*

Proof. Apply $\frac{d}{dt}\big|_{t=0}$ to $u_t^P e^{itH} a$ when $a \in \mathrm{dom}(H)$. □

Proposition 4.13. *Let $(A, \mathbb{R}, \alpha)$ be C^*-dynamical system and let u^P be a C^1 cocycle for α. Then, the infinitesimal generator of $\alpha^P = \mathrm{Ad}(u^P)\alpha$ is $\delta + i[P, \cdot]$, where δ is the infinitesimal generator of α. In particular, α and α^P have the same smooth elements.*

Proof. Apply $\frac{d}{dt}\big|_{t=0}$ to $u_t^P \alpha_t(a)(u_t^P)^*$ when $a \in \mathrm{dom}(\delta)$. □

A simple, but importance, consequence of Proposition 4.13 is the following.

Corollary 4.14. *Let $(A, \mathbb{R}, \alpha)$ be C*-dynamical system, let δ be the infinitesimal generator of α, and let $x \in \mathrm{dom}(\delta)$. Then, $\mathrm{Ad}(u^P)\alpha$ fixes x if and only if the self-adjoint C^0 multiplier P satisfies $i[P, x] = -\delta(x)$. Meanwhile, as soon as one such P_0 is found, the rest are given by $P = P_0 + Q$ as Q ranges over self-adjoint C^0 multipliers commuting with x.*

Proof. Consider the linear mapping $P \mapsto i[P, x]$, defined on the real vector space of self-adjoint C^0 multipliers. Combining the preceding proposition with Proposition 2.5, we see that P does the job if and only if $i[P, x] = -\delta(x)$. Meanwhile, the kernel of $P \mapsto i[P, x]$ is the commutant of x, so the second claim follows. □

Our C^1 cocycles $t \mapsto u_t^P$ are so-called because they are C^1 in the variable t. The next proposition tells us a sense in which they also vary smoothly in P.

Proposition 4.15. *Fix a self-adjoint C^0 multiplier $P \in M(A)$. For fixed $t \in \mathbb{R}$, the map $\lambda \mapsto u_t^{\lambda P} : \mathbb{R} \to M(A)$ is C^1. In fact, for t confined to a bounded interval,*

$$\left\| \frac{u_t^{\mu P} - u_t^{\lambda P}}{\mu - \lambda} - \frac{d}{d\lambda} u_t^{\lambda P} \right\| \to 0 \tag{4.6}$$

uniformly in t as $\mu \to \lambda$. At $\lambda = 0$, we have the simple formula

$$\left. \frac{d}{d\lambda} u_t^{\lambda P} \right|_{\lambda=0} = i \int_0^t \alpha_s(P)\, ds. \tag{4.7}$$

Proof. Without loss of generality, $\|P\| = 1$, since otherwise we may absorb $\|P\|$ into the parameter λ. Using the notation from the proof of Proposition 4.9, notice

$$I_n^{\lambda P}(t) = \int \cdots \int_{0 \le s_1 \le \ldots \le s_n \le t} \alpha_{s_1}(\lambda P) \cdots \alpha_{s_n}(\lambda P)\, ds_1 \cdots ds_n = \lambda^n I_n^P(\lambda).$$

Thus,

$$u_t^{\lambda P} = \sum_{n=0}^{\infty} i^n \lambda^n I_n^P(t). \tag{4.8}$$

If it is further assumed that $|t| \leq M$ for some fixed $M > 0$, then the estimate (4.4) gives the bound

$$\|I_n^P(t)\| \leq \frac{(|t|\|P\|)^n}{n!} \leq \frac{M^n}{n!}.$$

So, differentiability and the uniform convergence in (4.6) follows from standard "power series"-type analysis. The formula (4.7) is clear from (4.8). □

Given two perturbations P, Q, one can either adjust a flow by $P + Q$ all in one go, or one can first adjust by P and then adjust the resulting flow by Q; the end result is the same.

Proposition 4.16. *Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system and let $P, Q \in M(A)$ be self-adjoint C^0 multipliers. Then*

$$u^{Q,\alpha^P} u^{P,\alpha} = u^{P+Q,\alpha},$$

where $\alpha^P = \mathrm{Ad}(u^{P,\alpha})\alpha$, and the cocycle notation is explained in Remark 4.6.

Proof. One knows already knows that the C^1 family of unitaries $u^{Q,\alpha^P} u^{P,\alpha}$ is a unitary 1-cocycle for α from the general principle for strictly continuous cocycles. Since

$$\tfrac{d}{dt}\left[u_t^{Q,\alpha^P} u_t^{P,\alpha}\right]_{t=0} = \left[\left(\tfrac{d}{dt} u_t^{Q,\alpha^P}\right) u_t^{P,\alpha} + u_t^{Q,\alpha^P}\left(\tfrac{d}{dt} u_t^{P,\alpha}\right)\right]_{t=0} = i(P+Q),$$

we have $u^{Q,\alpha^P} u^{P,\alpha} = u^{P+Q,\alpha}$ by Theorem 4.4. □

We end this section with the analogues of Proposition 2.35 and Corollary 2.51 for C^1 cocycles.

Proposition 4.17. *Let $(A, \mathbb{R}, \alpha)$ and $(B, \mathbb{R}, \beta)$ be C^*-dynamical systems with A and B*

unital[2], $\varphi : A \to B$ *be a unital, equivariant homomorphism, and let* $P \in A$ *be self-adjoint. Then,* $\varphi(u^{P,\alpha}) = u^{\varphi(P),\beta}$, *with notation as in Remark 4.6.*

Proof. One already knows the C^1 family $\varphi(u^{P,\alpha})$ is a β cocycle from Proposition 2.35. Since $\frac{d}{dt}\varphi(u^{P,\alpha})\big|_{t=0} = \varphi\left(\frac{d}{dt}u^{P,\alpha}\big|_{t=0}\right) = i\varphi(P)$, Theorem 4.4 implies $\varphi(u^{P,\alpha}) = u^{\varphi(P),\beta}$. □

Proposition 4.18. *Let* $(A, \mathbb{R}, \alpha)$ *and* $(B, \mathbb{R}, \beta)$ *be unital*[3] *C*-dynamical systems, let* $P \in A$ *be self-adjoint, and let* $\varphi : A \to B$ *be a unital equivariant homomorphism. Then, the dual homomorphism* $\widehat{\varphi} : A \rtimes_\alpha \mathbb{R} \to B \rtimes_\alpha \mathbb{R}$ *is determined by*

$$\widehat{\varphi}(a \cdot f(H + P)) = \varphi(a) \cdot f(K + \varphi(P)) \qquad \forall a \in A, f \in C_0(\mathbb{R}).$$

Here, H *is the Hamiltonian of* $A \rtimes_\alpha \mathbb{R}$ *and* K *is the Hamiltonian of* $B \rtimes_\beta \mathbb{R}$.

Proof. Combine the above proposition with Corollary 2.51 and Proposition 4.12. □

4.3 Connes' projection lemma

In this section, $(A, \mathbb{R}, \alpha)$ denotes a C*-dynamical $\mathbb{R}$-dynamical system with A *unital*. We derive a lemma of Connes to the effect that every projection in A is fixed by a flow in the same exterior equivalence class as α. This lemma is crucially important for the construction of Connes' Thom isomorphism, see Chapter 6. We give another interesting application of this lemma in Chapter 4.5.

Given some projection $e \in A$, recall that A is (topologically, but not as a C*-algebra) the direct sum of the closed subspaces $eAe, eAe^\perp, e^\perp Aa, e^\perp Ae^\perp$ where $e^\perp := 1 - e$. The unique representation of any $x \in A$ is $x = x_{11} + x_{12} + x_{21} + x_{22}$ where $x_{11} = exe, x_{12} = exe^\perp, x_{21} = e^\perp xe, x_{22} = e^\perp xe^\perp$. This is the so-called *Pierce decomposition* of x with respect

[2]Thus, $M(A) = A$, and the strict topology is the norm topology
[3]Thus, $M(A) = A$, and the strict topology is the norm topology

to e. One tends to write $x = \left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right)$ in this situation because, through this representation, the expected operations on these matrices agree with the operations on the C*-algebra A. If $a, b \in A$, we use the notation $\{a, b\} = ab + ba$ for the *anticommutator* of a and b. We record three simple identities regarding the formation of commutators and anticommutators with projections.

Fact 4.19. *Let $e, x \in A$ with e a projection. Then,*

$$[x, e] = e^{\perp}xe - exe^{\perp} \qquad [[x, e], e] = e^{\perp}xe + exe^{\perp} \qquad \{e, x\} = 2exe + exe^{\perp} + e^{\perp}xe.$$

Each identity above is trivially obtained from the Pierce decomposition. For example, the first identity is essentially the matrix formula $\left(\begin{smallmatrix} x_{11} & 0 \\ x_{21} & 0 \end{smallmatrix}\right) - \left(\begin{smallmatrix} x_{11} & x_{12} \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -x_{12} \\ x_{21} & 0 \end{smallmatrix}\right)$ and the second is really just the observation that applying the matrix operation $\left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 0 & -x_{12} \\ x_{21} & 0 \end{smallmatrix}\right)$ twice yields $\left(\begin{smallmatrix} 0 & x_{12} \\ x_{21} & 0 \end{smallmatrix}\right)$.

From the first identity, one reads off the well-known fact that an element x is "diagonal" with respect to a projection e if and only if $[x, e] = 0$. Similarly, each of the latter two identities leads to its own characterization of "off-diagonality", which we gather below.

Corollary 4.20. *Let $e, x \in A$ with e a projection. Then, the following are equivalent:*

1. $exe = e^{\perp}xe^{\perp} = 0$.[4]

2. $x = [[x, e], e]$.

3. $x = \{x, e\}$.

These basic observations about matrix arithmetic have an interesting geometric consequence. Suppose that $t \mapsto e_t$ is a C^1 path of projections in A. From

$$0 = \tfrac{d}{dt}\left(e_t - e_t^2\right) = \dot{e}_t - \dot{e}_t e_t - e_t \dot{e}_t = \dot{e}_t - \{\dot{e}_t, e_t\}$$

[4] That is, the diagonal of the Pierce decomposition of x with respect to e is zero.

we see that the equivalent hypotheses above are satisfied with $x = \dot{e}_t, e = e_t$. That is, for all t, $\dot{e}_t = \{\dot{e}_t, e_t\} = [[\dot{e}_t, e_t], e_t]$ and $\dot{e}_t$ has zeros on the diagonal of its Pierce decomposition with respect to e_t. In other words, to remain inside the set of projections, one is forced to move "off diagonal" with respect to the present projection.[5] Since the set of projections in A is α-invariant, so that $t \mapsto \alpha_t(e)$ is a path of projections, we get

Corollary 4.21. *If $e \in A$ is a C^1 projection for α, then $\delta(e) = [[\delta(e), e], e]$.*

We now state and prove Connes' lemma.

Theorem 4.22. *Let $(A, \mathbb{R}, \alpha)$ be an $\mathbb{R}$-dynamical system with A unital. Let $e \in A$ be a C^1 projection for α. Then, $P = i[\delta(e), e]$ is self-adjoint[6] and the adjusted flow $\mathrm{Ad}(u^P)\alpha$ fixes e.*

Though the proof of this theorem is a simple verification, we first pause to motivate the choice of P. If α is implemented by a group (e^{itH}), then, at least formally, $\delta = i[H, \cdot]$ and $x \in A$ will be α-invariant if it commutes with H. In this case, $\alpha' = \mathrm{Ad}(u^P)\alpha$ is unitarily implemented by $(e^{itH'})$ where $H' = H + P$ (Proposition 4.12). We want P such that

1. $H' = H + P$ commutes with e.

2. The definition of P makes no direct reference to H.

The first criterion ensures $\mathrm{Ad}(u^P)\alpha$ will fix e, and the second is there to facilitate a straightforward generalization to the case where α is not unitarily implemented. We take

$$P = -eHe^{\perp} - e^{\perp}He$$

[5] A nice application of this idea is to showing that central projections are always fixed or, in other words, that, if $A = B \oplus C$, then α is the direct sum of a flow on B and a flow on C.

[6] One can check the Pierce decomposition of $i[\delta(e), e]$ with respect to e is $P = \begin{pmatrix} 0 & -ie\delta(e) \\ i\delta(e)e & 0 \end{pmatrix}$. Since the generic self-adjoint in the commutant of e looks like $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, we see from Proposition 4.14 that all perturbations P such that $\mathrm{Ad}(u^P)\alpha$ fixes e look like $\begin{pmatrix} a & -ie\delta(e) \\ i\delta(e)e & b \end{pmatrix}$. Thus, Connes' choice of P is reasonably canonical.

so that $H' = H + P = eHe + e^{\perp}He^{\perp}$. Obviously the first criterion is satisfied. To see that the second criterion is also satisfied, use the second identity in Fact 4.20 to write

$$-eHe^{\perp} - e^{\perp}He = -[[H, e], e] = i[i[H, e], e] = i[\delta(e), e]$$

Armed with a candidate P whose definition only depends on the flow, we proceed to the

Proof of Theorem 4.22. Since e and $\delta(e)$ are self-adjoint, and since a commutator of two self-adjoints is anti-self-adjoint, $P = i[\delta(e), e]$ is self-adjoint. To see e is fixed by $\mathrm{Ad}(u^P)\alpha$, observe $i[P, e] = -[[\delta(e), e], e] = -\delta(e)$ by Corollary 4.21, and then apply by Corollary 4.14. □

Remark 4.23. The standing assumption that A was unital throughout this section was not critical. Indeed, suppose $(B, \mathbb{R}, \beta)$ is an $\mathbb{R}$-dynamical system and B is nonunital. Let $e \in B$ be a C^1 projection. Then, e is still C^1 for the unitized system $(\widetilde{B}, \mathbb{R}, \widetilde{\beta})$ which has a C^1 unitary 1-cocycle $t \mapsto u_t^P : \mathbb{R} \to \widetilde{B}$ for which the adjusted dynamics fix P. Noting $\widetilde{B} \subset M(B)$, it is easy to check u_t^P is also a cocycle for the original, nonunital system.

Remark 4.24. Theorem 4.22 says that every smooth projection is fixed by a flow in the same smooth exterior equivalence class. As observed in [29], it is actually true that every projection is fixed by a flow in the same norm-continuous exterior equivalence class. The point is that arbitrary projections are approximated by smooth ones, and that nearby projections are unitarily conjugate. Thus, the cocycle in Example 2.37 can be combined with the one in Theorem 4.22 to deal with the general case.

4.4 The function $(s+i)^{-1}$

Recall our convention that, for $f \in L^1(\mathbb{R})$, the *Fourier transform* $\widehat{f} \in C_0(\mathbb{R})$ is given by $\widehat{f}(s) = \int_{-\infty}^{\infty} f(t)e^{its}\, dt$. With this choice, it holds that

$$\widehat{f}(s) = \frac{i}{s+i} \qquad \text{when} \qquad f(t) = \begin{cases} e^{-t} & t > 0 \\ 0 & t < 0 \end{cases}. \tag{4.9}$$

The functional calculus of $(s+i)^{-1}$ has some convenient properties which we shall rely at two points in this thesis.

Theorem 4.25. *Let A be a C^*-algebra, and let (e^{itH}) be a strictly continuous 1-parameter group in* $\mathrm{M}(A)$*. Then, there holds*

$$i[(H+i)^{-1}, x] = -(H+i)^{-1}\delta(x)(H+i)^{-1}$$

whenever $x \in M(A)$ is a C^1 multiplier for the associated flow $\alpha_t = \mathrm{Ad}(e^{itH})$.

Formally, speaking the explanation for this identity is that

$$\begin{aligned} i[(H+i)^{-1}, x] &= i(H+i)^{-1}(x(H+i) - (H+i)x)(H+i)^{-1} \\ &= -(H+i)^{-1} i[H, x](H+i)^{-1} \\ &= -(H+i)^{-1}\delta(x)(H+i)^{-1}. \end{aligned}$$

However, as have been working with the functional calculus of H, rather than directly with H, writing, $1 = (H+i)(H+i)^{-1}$, for example, is not wholly justified. Fortunately, we can obtain the result by direct integration.

Proof of Theorem 4.25. Let f be as in Equation (4.9). The desired identity follows from the following computation.

$$
\begin{aligned}
-(H+i)^{-1}\delta(x)(H+i)^{-1} &= \widehat{f}(H)\delta(x)\widehat{f}(H) \\
&= \int_0^\infty e^{-s}e^{isH}\ ds\ \delta(x)\int_0^\infty e^{-t}e^{itH}\ dt \\
&= \int_0^\infty e^{isH}\int_0^\infty e^{-(s+t)}\delta(x)e^{itH}\ dtds \\
&= \int_0^\infty e^{isH}\int_s^\infty e^{-t}\delta(x)e^{-isH}e^{itH}\ dt\ ds \quad \text{(replacing } t \text{ by } -s+t) \\
&= \int_0^\infty e^{-t}\int_0^t \alpha_s(\delta(x))\ ds\ e^{itH}\ dt \\
&= \int_0^\infty e^{-t}\int_0^t \tfrac{d}{ds}\alpha_s(x)\ ds\ e^{itH}\ dt \\
&= \int_0^\infty e^{-t}(\alpha_t(x)-x)e^{itH}\ dt \\
&= \int_0^\infty e^{-t}(e^{itH}x - xe^{itH})\ dt \\
&= \int_0^\infty e^{-t}e^{itH}\ dtx - x\int_0^\infty e^{-t}e^{itH}\ dt \\
&= [\widehat{f}(H), x] \\
&= i[(H+i)^{-1}, x].
\end{aligned}
$$

Here, if $g \in L^1(\mathbb{R})$ and h is a strictly continuous, norm-bounded function $\mathbb{R} \to M(A)$, then the integral $\int_{-\infty}^{\infty} g(t)h(t)\ dt \in M(A)$ is defined so that $\int_{-\infty}^{\infty} g(t)h(t)\ dt \cdot a = \int_{-\infty}^{\infty} g(t)h(t)a\ dt$ for all $a \in A$. □

The importance of the above result rests with the following corollary, which we shall need later when dealing with *dual traces* on $A \rtimes_\alpha \mathbb{R}$.

Corollary 4.26. *Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system, let $a \in A$ be C^1 for α. Then, we have the following identity in which all three terms belong to $A \rtimes_\alpha \mathbb{R}$*

$$ia(H+i)^{-1} = i(H+i)^{-1}a + (H+i)^{-1}\delta(a)(H+i)^{-1}.$$

Here δ is the infinitesimal generator of α, and H is the Hamiltonian of $A \rtimes_\alpha \mathbb{R}$.

Recall we have been picturing $A \rtimes_\alpha \mathbb{R}$ as generated by "elementary products" $a \cdot f(H)$ where $a \in A$, $f \in C_0(\mathbb{R})$. This description of $A \rtimes_\alpha \mathbb{R}$ in terms of generators is not particularly compelling unless we also have relations on those generators. To be sure, we have already discussed the relations

$$e^{itH} a e^{-itH} = \alpha_t(a) \qquad\qquad i[H, a] \text{"}=\text{"} \delta(a),$$

(the second of these, we have only been using formally), but these are somewhat unsatisfying since $s \mapsto e^{its}$ and $s \mapsto s$ do not belong to $C_0(\mathbb{R})$. The relation in Corollary 4.26 is more compatible with our picture. Moreover, since $s \mapsto (s+i)^{-1}$ is injective, it generates $C_0(\mathbb{R})$ by the Weierstrass theorem, so one is led to suspect the relation in Corollary 4.26 actually determines the crossed-product.

Question. *Can one construct the crossed product $A \rtimes_\alpha \mathbb{R}$ as the universal C^*-algebra generated by A and $C_0(\mathbb{R})$ subject to the relation $[g, a] = g\delta(a)g$ where $g(s) = \frac{i}{s+i}$ and $a \in \operatorname{dom}(\delta)$, the domain of the infinitesimal generator of α? Indeed, given any closed, self-adjoint derivation δ of A, can one construct a "crossed-product" $A \rtimes_\delta \mathbb{R}$ in this way?*

As it happens, the integral trickery in the proof of Theorem 4.25 above shall serve us rather well; we use the same maneuvers to prove the following.

Theorem 4.27. *Let A be a C^*-algebra, and let (e^{itH}) be a strictly continuous 1-parameter group in $\mathrm{M}(A)$. Then, there holds*

$$\frac{d}{d\lambda}(H + \lambda P + i)^{-1} = -(H + \lambda P + i)^{-1} P (H + \lambda P + i)^{-1} \qquad \text{(}\lambda \text{ a real parameter)}$$

whenever $P \in M(A)$ is a C^0 multiplier for the associated action $\alpha_t = \mathrm{Ad}(e^{itH})$. The differentiation is intended in the sense of norm-convergent difference quotients.

Again, formally this obvious when $(H+\lambda P+i)^{-1}$ is thought of as the inverse of $H+\lambda P+i$, rather than as the image of $s \mapsto \frac{1}{s+i}$ under the $*$-homomorphism $C_0(\mathbb{R}) \to M(A)$ associated to the group $(e^{it(H+\lambda P)})$. The rigorous justification is as follows.

Proof. It suffices to prove the identity when $\lambda = 0$, since one may replace H with $H + \lambda_0 P$. As before, we use a special function and its Fourier transform.

$$f(t) = \begin{cases} e^{-t} & t > 0 \\ 0 & t < 0 \end{cases} \qquad\qquad \widehat{f}(s) = \frac{i}{s+i}. \tag{4.10}$$

The desired identity follows from the calculation

$$\begin{aligned}
-(H+i)^{-1}P(H+i)^{-1} &= \widehat{f}(H)P\widehat{f}(H) \\
&= \int_0^\infty e^{-s}e^{isH}\ ds\ P \int_0^\infty e^{-t}e^{itH}\ dt \\
&= \int_0^\infty e^{isH} \int_0^\infty e^{-(s+t)}Pe^{itH}\ dtds \\
&= \int_0^\infty e^{isH} \int_s^\infty e^{-t}Pe^{-isH}e^{itH}\ dt\ ds \qquad \text{(replacing } t \text{ by } -s+t) \\
&= -i \int_0^\infty e^{-t} \left(i \int_0^t \alpha_s(P)\ ds \right)\ e^{itH}\ dt \\
&= -i \int_0^\infty e^{-t} \left(\frac{d}{d\lambda} u_t^{\lambda P} \Big|_{\lambda=0} \right)\ e^{itH}\ dt \qquad \text{(Proposition 4.15)} \\
&= -i \frac{d}{d\lambda} \int_0^\infty e^{-t}u_t^{\lambda P}e^{itH}\ dt\ \Big|_{\lambda=0} \\
&= -i \frac{d}{d\lambda} \int_0^\infty e^{-t}e^{it(H+\lambda P)}\ dt\ \Big|_{\lambda=0} \\
&= -i \frac{d}{d\lambda} \widehat{f}(H+\lambda P)\ \Big|_{\lambda=0} \\
&= \frac{d}{d\lambda}(H+\lambda P+i)^{-1}\ \Big|_{\lambda=0}.
\end{aligned}$$

□

4.5 The Bargmann-Wigner theorem

A famous theorem of Marshall Stone assigns an infinitesimal generator to every strongly-continuous 1-parameter group of unitary operators on a separable Hilbert space.

Theorem 4.28 (Stone, 1930). *To each strongly-continuous 1-parameter group $(U_t)_{t\in\mathbb{R}}$ of unitary operators on a Hilbert space $\mathcal{H}$, there corresponds exactly one self-adjoint operator*[7] *$H : \mathcal{H} \to \mathcal{H}$ such that $U_t = e^{itH}$ for all $t \in \mathbb{R}$. Conversely, each self-adjoint operator $H : \mathcal{H} \to \mathcal{H}$ determines a strongly continuous 1-parameter group of unitaries by exponentiation.*

In addition to being a useful theoretical tool, Stone's result is historically important because of its role in the mathematical formulation of quantum mechanics. Specifically, it links the classical formulation of the canonical commutation relations[8] to the Weyl formulation[9]. With this accomplished, the way is paved for the Stone-von Neumann Theorem which shows the canonical commutation relations arise in only one way, up to a suitable notion of unitary equivalence. This has the highly desirable consequence that analysis of the corresponding physical system does not depend on a particular choice of representation (e.g. position or momentum representation). A somewhat less famous, but closely related, theorem of Bargmann and Wigner is as follows.

Theorem 4.29 (Bargmann-Wigner, 1960s). *For any strongly continuous action α of $\mathbb{R}$ on $\mathbb{K}(\mathcal{H})$ by $*$-automorphism, there is a strongly continuous 1-parameter group of unitaries*[10] *(U_t) such that $\alpha_t = \mathrm{Ad}(U_t)$ for all $t \in \mathbb{R}$.*

Thus, an application of Stone's theorem immediately shows all C*-algebra flows on $\mathbb{K}(H)$ have the form $t \mapsto \mathrm{Ad}(e^{itH})$ for some self-adjoint operator $H : \mathcal{H} \to \mathcal{H}$ (with H unique up

[7] H can be unbounded. In fact, boundedness of H is equivalent to norm-continuity of (U_t) is equivalent to smoothness of (U_t).

[8] Self-adjoint operators x and P on a separable Hilbert space satisfy the commutation law $[x, P] = i$.

[9] Strongly continuous 1-parameter unitary groups U and V acting jointly irreducibly on a separable Hilbert space satisfy the braiding relation $V_sU_t = e^{ist}U_sV_t$.

[10] Unique up to multiplication by a character $\mathbb{R} \to \mathbb{T}$.

to an additive real constant). We give a conceptually simple proof of the Bargmann-Wigner Theorem by applying Connes' projection lemma (Theorem 4.22, in our numbering system). Since the Bargmann-Wigner theorem predates Connes' lemma by twenty some odd years, this approach may well be new. The first thing to realize is that, with the introduction of a single extra hypothesis:

"There exists an α-invariant rank-1 projection $e \in \mathbb{K}(\mathcal{H})$".

the proof of Theorem 4.29 reduces to a computation. This is an old trick due to Kaplansky (cf. [18]).

Proof of Theorem 4.29: special case. Choose a unit vector ξ in the range of e so that $e = \xi \otimes \overline{\xi}$. Define $U_t : \mathcal{H} \to \mathcal{H}$ by

$$U_t(\eta) = \alpha_t(\eta \otimes \overline{\xi})\xi$$

Obviously U_t is a linear map $\mathcal{H} \to \mathcal{H}$. An easy computation shows each U_t is an isometry

$$\begin{aligned}
\langle U_t(\varphi), U_t(\psi)\rangle &= \langle \alpha_t(\varphi \otimes \overline{\xi})\xi, \alpha_t(\psi \otimes \overline{\xi})\xi\rangle \\
&= \langle \xi, \alpha_t(\xi \otimes \overline{\varphi})\alpha_t(\psi \otimes \overline{\xi})\xi\rangle \\
&= \langle \xi, \alpha_t(\xi \otimes \overline{\varphi} \cdot \psi \otimes \overline{\xi})\xi\rangle \\
&= \langle \xi, \alpha_t(\langle \varphi, \psi\rangle e)\xi\rangle \\
&= \langle \varphi, \psi\rangle \langle \xi, \alpha_t(e)\xi\rangle \\
&= \langle \varphi, \psi\rangle \langle \xi, e\xi\rangle \\
&= \langle \varphi, \psi\rangle \langle \xi, \xi\rangle \\
&= \langle \varphi, \psi\rangle
\end{aligned}$$

Another easy computation gives the group law for the U_t:

$$\begin{aligned} U_{t+s}(\eta) &= \alpha_{t+s}(\eta \otimes \overline{\xi})\xi \\ &= \alpha_{t+s}(\eta \otimes \overline{\xi})\alpha_t(e)\xi \\ &= \alpha_t(\alpha_s(\eta \otimes \overline{\xi}) \cdot \xi \overline{\otimes} \xi)\xi \\ &= \alpha_t((\alpha_s(\eta \otimes \overline{\xi})\xi) \otimes \overline{\xi})\xi \\ &= U_t(U_s(\eta)) \end{aligned}$$

Since $U_0 = \mathrm{id}_{\mathcal{H}}$ clearly holds, it follows from the group law that the isometries U_t are invertible, so (U_t) is a 1-parameter group of unitaries. From the definition

$$U_t(\eta) = \alpha_t(\eta \otimes \overline{\xi})\xi$$

and from strong continuity of α_t, it is clear that $t \mapsto U_t$ is strongly continuous. The final thing is to check that the U_t implement the dynamics. Indeed, if $a \in A$ and if $\eta \in \mathcal{H}$ we have

$$(U_t a)\eta = \alpha_t((a\eta) \otimes \overline{\xi})\xi = \alpha_t(a \cdot \eta \otimes \overline{\xi})\xi = \alpha_t(a)\alpha_t(\eta \otimes \overline{\xi})\xi = \alpha_t(a)U_t\eta$$

so $U_t a U_t^* = \alpha_t(a)$ and we are finished. □

The above does argument is insufficient to prove Theorem 4.29, as there does not always exist an α-invariant rank-1 projection.

Example 4.30. Let $H \in B(\mathcal{H})$ be a self-adjoint operator, and let α be the flow on $\mathbb{K}(H)$ unitarily implemented by the 1-parameter unitary group (e^{itH}). A rank-1 projection $\xi \otimes \overline{\xi}$ is fixed by α if and only if ξ is an eigenvalue of H. So, if H has empty point spectrum, then α doesn't fix any rank-1 projection.

We now ask ourselves what can be done to recover the general case. The basic observation, which should immediately make us think to use Connes' projection lemma, is that the relation of exterior equivalence preserves the flows which are unitarily implemented. See Example 2.33.

Proof of Theorem 4.29: general case. First of all, there exists a rank-1 projection e which is α-C^1. Indeed, the domain $\mathrm{dom}(\delta)$ of the infinitesimal generator of α is closed under holomorphic functional calculus so, by Lemma B.1, any rank-1 projection e_0 is Murray-von Neumman equivalent to a (necessarily rank-1) projection $e \in \mathrm{dom}(\delta)$. Next, Connes' projection lemma (Theorem 4.22 in our numbering) implies (see Remark 4.23) that α is exterior equivalent to a flow α' on $\mathbb{K}(\mathcal{H})$ which fixes e. By the preceding proof, α' is unitarily implemented. Thus, the original flow α is unitarily implemented as well. □

We conclude the section by reproving Theorem 4.29 in finite dimension, where easier methods suffice. Let (α_t) be a 1-parameter group of automorphisms of $\mathrm{M}_n(\mathbb{C})$. By the Bargmann-Wigner theorem, we know there is a self-adjoint $H \in \mathrm{M}_n(\mathbb{C})$ such that $\alpha_t = \mathrm{Ad}(e^{itH})$. By the finite-dimensional spectral theorem, H diagonalizes over an orthonormal basis of eigenvectors $\xi_1, \ldots, \xi_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Then e^{itH} diagonalizes over the same basis, but with corresponding eigenvalues $e^{it\lambda_1}, \ldots, e^{it\lambda_n}$. In particular, e^{itH} commutes with each of the rank-1 projections $\xi_j \otimes \overline{\xi}_j$, whence (α_t) fixes a whole orthonormal system of rank-1 projections. We are encouraged, then, to prove the existence of a fixed rank-1 projection directly, which suffices to prove the Bargmann-Wigner theorem in finite dimension.

Proposition 4.31. *Every C^*-algebra flow on $\mathrm{M}_n(\mathbb{C})$ fixes a rank-1 projection.*

Proof. Let (α_t) be a flow on $M_n(\mathbb{C})$. Considering that, for each $t \in \mathbb{R}$, the particular automorphism α_t is unitarily implemented and that unitaries are orthogonally diagonalizable, it is at least clear that each particular automorphism fixes a rank-1 projection. For $n = 1, 2, \ldots$, let e_n be a rank-1 projection fixed by $\alpha_{1/2^n}$. From the group law for the flow, we see that $\alpha_{1/2^n}$ actually fixes all of $e_n, e_{n+1}, e_{n+2}, \ldots$. Since the space of rank-1 projections in $\mathrm{M}_n(\mathbb{C})$ is (norm) compact, the sequence $e_1, e_2, \ldots$ accumulates at some rank-1 projection e. By continuity, e is fixed by $\alpha_{1/2^n}$ for $n = 1, 2, \ldots$. It follows that e is fixed by α_t whenever t is a dyadic rational. By continuity, the whole flow fixes e and we are finished. □

Chapter 5

The suspension isomorphisms

The **cone** and **suspension** of a C*-algebra A are the (necessarily nonunital) C*-algebras:

$$\mathrm{C}A := \{x \in C([0,1], A) : x(0) = 0\} \qquad \mathrm{S}A := \{x \in C([0,1], A) : x(0) = x(1) = 0\}.$$

These constructions are functorial, with a $*$-homomorphism $\varphi : A \to B$ inducing $*$-homomorphisms $\mathrm{C}\varphi : \mathrm{C}A \to \mathrm{C}B$ and $\mathrm{S}\varphi : \mathrm{S}A \to \mathrm{S}B$ by composition. Meanwhile, for fixed A, we have an exact sequence

$$\mathrm{S}A \hookrightarrow \mathrm{C}A \twoheadrightarrow A$$

where the first map is inclusion and the second is evaluation at 1. It is helpful to have $\mathrm{C}A$ as a sort of mediator between A and $\mathrm{S}A$ because $\mathrm{C}A$ is *contractible* in the sense that there is a strongly continuous 1-parameter family of $*$-endomorphism (φ_t) of $\mathrm{C}A$ such that $\varphi_0 \equiv 0$ and $\varphi_1 = \mathrm{id}$ corresponding to the existence of a homotopy from the collapsing map $[0,1] \to \{0\}$ to the identity map $[0,1] \to [0,1]$ through basepoint-preserving self-maps of $[0,1]$.

It is a deep topological fact that the K-theory of a suspension agrees with the K-theory of the original algebra – up to dimension shift. More precisely, for each C*-algebra A, there are a pair isomorphisms $s^i_A : K_i(A) \to K_{i+1}(\mathrm{S}A)$, natural in A. The isomorphism $K_1(A) \to K_0(\mathrm{S}A)$ stems from an operator-algebraic generalization of a clutching construction for vector bundles. The isomorphism $K_0(A) \to K_1(\mathrm{S}A)$ is one manifestation of Bott's periodicity theorem. Since the suspension isomorphisms are intimately linked with Connes' Thom isomorphism and the latter plays an important role in this thesis, we feel it is worthwhile to take a moment to describe the maps and meditate briefly on their geometric significance.

5.1 Clutching

The notions of suspension and cone for C*-algebras come from the corresponding notions for spaces. Let $X = (X, x_0)$ be a pointed compact Hausdorff space. The **suspension** $\mathrm{S}X$ and **cone** $\mathrm{C}X$ of X are the (necessarily connected) pointed compact Hausdorff spaces

$$\mathrm{C}X = \frac{X \times [0,1]}{(X \times \{0\}) \cup (\{x_0\} \times [0,1])} \qquad \mathrm{S}X = \frac{X \times [0,1]}{(X \times \{0,1\}) \cup (\{x_0\} \times [0,1])}$$

with basepoints equal to the collapsed subspaces. There is a canonical pointed embedding of X into $\mathrm{C}X$, roughly as $X \times \{1\}$, and the quotient $(\mathrm{C}X)/X$ by this subspace identifies with $\mathrm{S}X$. In other words, there is an "exact sequence" of spaces

$$X \hookrightarrow \mathrm{C}X \twoheadrightarrow \mathrm{S}X.$$

These are the sensible definitions to make if one desires compatibility with the duality between compact pointed spaces and commutative C*-algebras:

$$X = (X, x_0) \longrightarrow C_0(X) := \{f \in C(X) : f(x_0) = 0\}.$$

That is to say, these definitions lead to natural isomorphisms $\mathrm{S}C_0(X) \cong C_0(\mathrm{S}X)$, $\mathrm{C}C_0(X) \cong C_0(\mathrm{C}X)$, and so forth.

Example 5.1. If $X = S^1$, then $\mathrm{C}X \cong \mathbb{D}^2$ with basepoint on the boundary, and $\mathrm{S}X \cong S^2$. If $X = S^1$, plus an isolated basepoint, then $\mathrm{C}X \cong \mathbb{D}^2$ with basepoint in the interior, and $\mathrm{S}X$ is S^2 with two identified points.

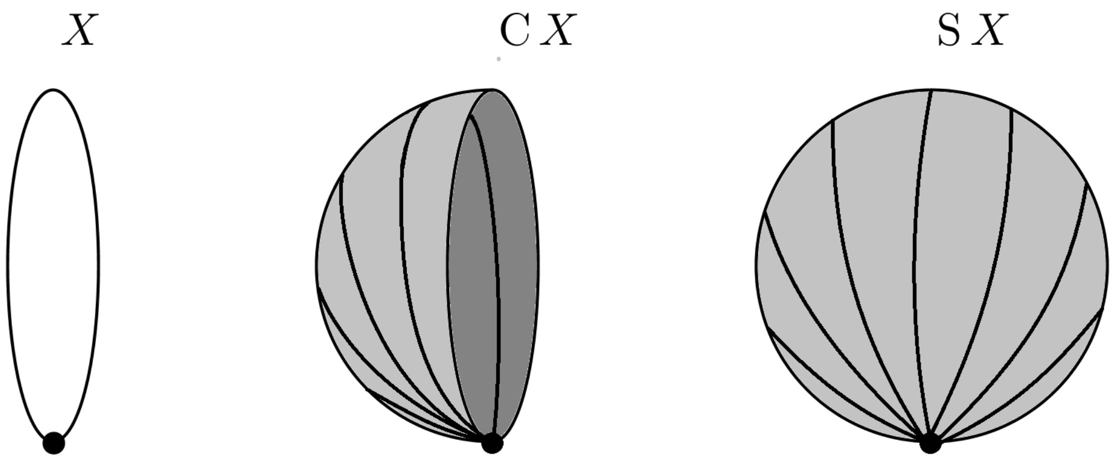

Figure 5.1: The case $X = S^1$, with basepoint on the circle.

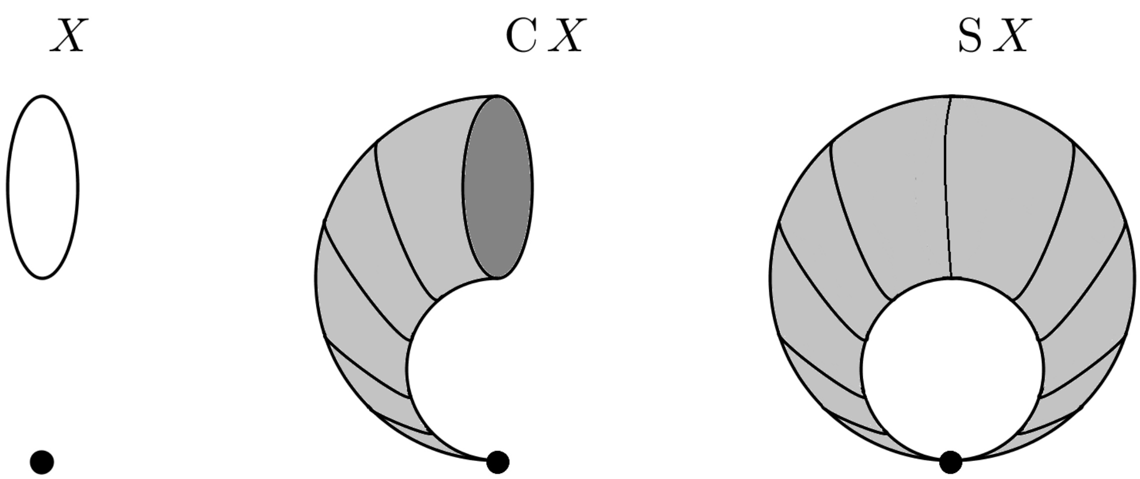

Figure 5.2: The case $X = S^1$, with a disjoint basepoint.

As already mentioned, the K-theory isomorphism $K_1(A) \to K_0(\mathrm{S}A)$ is the operator theoretic manifestation of a highly geometric construction from homotopy theory which we outline here for general culture.

Problem 5.2. *Given a pointed compact Hausdorff space X, describe the isomorphism classes of complex vector bundles over the suspension* $\mathrm{S}X$*.*

When X is a pointed sphere, $\mathrm{S}X$ is the pointed sphere of one higher dimension, so this problem is of obvious interest in connection with the study of complex vector bundles on spheres.

The idea is that, since $\mathrm{S}X$ is obtained from $\mathrm{C}X$ by collapsing X into the basepoint, the bundles over $\mathrm{S}X$ should similarly arise as quotients of bundles on $\mathrm{C}X$. Since $\mathrm{C}X$ is contractible, every bundle over $\mathrm{C}X$ is trivial. Thus, it would seem we just need to decide which are the ways to collapse the fibres over X of the trivial bundle $\mathbb{C}^n \times \mathrm{C}X$ into a single fibre, and thus obtain a bundle over $\mathrm{S}X$. The precise gluing data needed is a family of coordinate-change functions dictating how each fibre over X is identifies with the fibre over the basepoint; in other words, a map $X \to \mathrm{U}_n(\mathbb{C})$. This train of thought leads to a solution, as it were, to Problem 5.2.

Fact 5.3. *For every pointed compact Hausdorff X, for every positive integer n, there is a 1-1 correspondence between:*

1. *Homotopy classes of maps $X \to \mathrm{U}_n(\mathbb{C})$.*

2. *Isomorphism classes of n-dimensional (complex) vector bundles over* $\mathrm{S}X$*.*

There is little sense improving the rather scanty formulation of the above fact since we shall obtain a more general result in the following section. See §1.4 of [1] for a treatment of the classical case.

5.2 C*-algebraic clutching

In this section we show that, for every C*-algebra A, for every positive integer n, there is a 1-1 correspondence between the path components in $\mathrm{U}_n(A)$ and the path components in

$\mathrm{Vec}_n(\mathrm{S}A)$. This homotopy bijection is the instigator of a K-theoretic isomorphism $K_1(A) \to K_0(\mathrm{S}A)$.

Let A be a C*-algebra. When $k \leq \ell$, regard $\mathrm{M}_k(A)$ as a subalgebra of $\mathrm{M}_\ell(A)$ via the (necessarily nonunital) corner inclusion $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$. One can regard the union $\mathrm{M}_\infty(A) := \bigcup_{k=1}^\infty \mathrm{M}_k(A)$ as the $*$-algebra of infinite matrices with finitely many nonzero entries in A. We do not bother completing $\mathrm{M}_\infty(A)$ (which would give the stabilization of A), or even topologizing it. Instead, by a path in $\mathrm{M}_\infty(A)$, we shall always understand a path in one of the C*-algebras $\mathrm{M}_k(A)$ where k is arbitrarily large, but finite. A similar discussion applies to the union $\mathrm{U}_\infty(A) = \bigcup_{k=1}^\infty \mathrm{U}_k(A)$ i.e. the group of unitaries of the form $1 + a$ where $a \in \mathrm{M}_\infty(A)$.

Definition 5.4. For a C*-algebra A and a positive integer n, we write $\mathrm{Vec}_n(A)$ for the set of all projections $e \in \mathrm{M}_\infty(\widetilde{A})$ with scalar part $s(e) = 1_n$. That is, all projections of the form $1_n + a$ where $a \in \mathrm{M}_\infty(A)$.

Example 5.5. . Let $X = (X, x_0)$ be a pointed compact Hausdorff space, and let $A = \{f \in C(X) : f(x_0) = 0\}$. Then, an element of $\mathrm{U}_k(A)$ is a map $u : X \to \mathrm{U}(k)$ such that $u(x_0) = 1$. In other words, u continuously selects a coordinate system in each fibre of the trivial bundle $X \times \mathbb{C}^k$, choosing the standard basis in the fibre over x_0. An element $e \in \mathrm{Vec}_n(A)$ is a projection-valued map $e : X \to \mathrm{M}_k(\mathbb{C})$, for some k, satisfying $e(x_0) = 1_n$. In other words, e is a subbundle of the trivial bundle $X \times \mathbb{C}^k$ such that the fibre over x_0 is $\mathbb{C}^n \times \{0\}$.

The following technical lemma is a mild generalization of Proposition 5.2.6 in [23].

Lemma 5.6. *Fix a projection p in a unital C*-algebra B. Let X be the set of projections $q \in B$ with $\|p - q\| < 1$. Then, there is a continuous, unitary-valued map $q \mapsto u_q : X \to \mathrm{U}(B)$ such that $u_1 = 1$ and $u_q p u_q^* = q$ for all $q \in X$. Additionally, if $\varphi : B \to C$ is a unital C*-algebra homomorphism, then $\varphi(u_q) = 1$ whenever $q \in X$ has $\varphi(q) = \varphi(p)$.*

Proof. For each projection $q \in B$, $v_q := q - q^\perp = 2q - 1$ is a self-adjoint involution (roughly, reflection though q) from which q is recovered by $q = \frac{v_q+1}{2}$. For each $q \in X$, put $z_q = \frac{v_q v_p + 1}{2}$. Clearly $z_p = 1$. Additionally, if $\varphi(q) = \varphi(p)$, then $\varphi(z_q) = \varphi\left(\frac{z_p^2+1}{2}\right) = \varphi(1) = 1$. Since

$$\|z_q - 1\| = \left\|\frac{v_p v_q - 1}{2}\right\| \underset{v_p^2=1}{=} \left\|\frac{v_q - v_p}{2}\right\| = \|p - q\| < 1,$$

each z_q is invertible. Moreover,

$$z_q p = \frac{v_q v_p p + p}{2} \underset{v_p p = p}{=} \frac{v_q + 1}{2} \cdot p = qp \qquad q z_p = \frac{q v_q v_p + q}{2} \underset{q v_q + q}{=} q \cdot \frac{v_p + 1}{2} = qp$$

so $z_q p z_q^{-1} = q$. Noting that p commutes with $z_q^* z_q$, it follows that p commutes with $|z_q|^{-1}$ and therefore that the unitary $u_q := z_q |z_q|^{-1}$ satisfies $u_q p u_q^* = q$. Additionally, if $q \in X$ has $\varphi(q) = \varphi(p)$, then $\varphi(u_q) = \varphi(z_q)|\varphi(z_q)|^{-1} = 1$ □

By a straightforward subdivision argument, one gets the following path-lifting result.

Corollary 5.7. *If $\varphi : B \to C$ is a unital C*-algebra homomorphism and e_t is a path of projections in B such that $\varphi(e_t)$ is constant, then there exists a path of unitaries u_t in B with $u_0 = 1$ and $e_t = u_t e_0 u_t^*$ such that, in addition, $\varphi(u_t) \equiv 1$.*

We apply this corollary to show that homotopy equivalence in $\mathrm{Vec}_n(A)$ is exactly orbit equivalence under the action of the identity component in $\mathrm{U}_\infty(A)$.

Proposition 5.8. *Suppose A is a C*-algebra and n is a positive integer. If e_t is a path of projections in $\mathrm{Vec}_n(A)$, then there exists a path of unitaries $u_t \in \mathrm{U}_\infty(A)$ such that $u_0 = 1$ and $u_t e_0 u_t^* = e_t$ for all t.*

Proof. Let k be large enough that the path e_t stays inside $\mathrm{M}_k(A) + 1_n$. In the preceding corollary, take $B = \mathrm{M}_k(\widetilde{A})$, $C = \mathrm{M}_k(\mathbb{C})$ and φ to be the homomorphism $\mathrm{M}_k(\widetilde{A}) \to \mathrm{M}_k(\mathbb{C})$ which reads off the scalar part of each entry. □

Example 5.9. If $X = (X, x_0)$ is a pointed compact Hausdorff space, and $A = \{f \in C(X) : f(x_0) = 0\}$, then the preceding theorem says that any homotopy of subbundles of $X \times \mathbb{C}^k$ such that the fibre over x_0 is always $\mathbb{C}^n$ can be realized by continuously changing the coordinate system in each fibre, all the while leaving coordinates of $\mathbb{C}^k$ over x_0 unchanged.

Definition 5.10. We call a continuous map $f : X \to Y$ a **strong π_0-equivalence** in the event the following conditions are met.

1. *Path-lifting:* For every path y_t in Y, there is a path x_t in X such that $f(x_t) = y_t$
2. *Connected fibres:* For every $y \in Y$, the "fibre" $f^{-1}(y)$ is path connected.

Obviously a strong π_0-equivalence is a π_0-equivalence, that is, induces a bijection on path-components.

Example 5.11. Suppose a group G acts continuously on a space X and $x \in X$. If

1. For every path x_t in X with $x_0 = x$, there is a path $g_t \in G$ with $g_0 = 1$, $g_t x = x_t$ (in particular, the path-component of x is contained in its orbit under G).
2. The stabilizer subgroup $\mathrm{Stab}(x)$ is connected.

then it is easy to see that $g \mapsto gx : G \to \mathrm{Orb}(x)$ is a strong π_0-equivalence.

Strong π_0-equivalences are more robust than ordinary π_0-equivalences in that they can be "restricted" in the following sense.

Proposition 5.12. *Let $f : X \to Y$ be a strong π_0-equivalence. If $Y_0 \subset Y$ is some subspace, then the restricted mapping $f : f^{-1}(Y_0) \to Y_0$ is also a strong π_0 equivalence.*

The above need not hold for ordinary π_0-equivalence, as the following example shows.

Example 5.13. Projection onto the real axis is a π_0-equivalence $S^1 \to [-1, 1]$ (both spaces are path-connected). However, the restricted mapping $S^1 \setminus \{\pm 1\} \to (-1, 1)$ is not. The problem here is that the fibres over the points in the interior of the interval are not connected.

Theorem 5.14. *Let A be a C^*-algebra, n a positive integer. Put*

$$G = \{U \in \mathrm{U}_\infty(\mathrm{C}A) : U(1) \in \mathrm{U}_n(A) \times \mathrm{U}_\infty(A)\}$$

That is, G consists of all $U \in \mathrm{U}_\infty(\mathrm{C}A)$ for which $U(1)$ has the form $\left(\begin{smallmatrix} u & 0 \\ 0 & * \end{smallmatrix}\right)$ where $u \in \mathrm{U}_n(A)$. Then, the maps

$$G \to \mathrm{Vec}_n(\mathrm{S}A) : U \mapsto U1_nU^* \qquad\qquad G \to \mathrm{U}_n(A) : U \to 1_nU(1)1_n$$

are both strong π_0-equivalences.

Proof. First we attend to the mapping $G \to \mathrm{Vec}_n(\mathrm{S}A)$. Consider the action of $\mathrm{U}_\infty(\mathrm{C}A)$ on $\mathrm{Vec}_n(\mathrm{C}A)$ by conjugation. Given any path $e_t \in \mathrm{Vec}_n(\mathrm{C}A)$ with $e_0 = 1_n$, Theorem 5.8 guarantees a path $U_t \in \mathrm{U}_\infty(\mathrm{C}A)$ such that $U_0 = 1$, $U_t1_nU_t^* = e_t$. Meanwhile, the stabilizer of 1_n is exactly $\mathrm{U}_n(\mathrm{C}A) \times \mathrm{U}_\infty(\mathrm{C}A)$ which is, not only path-path connected, but deformation retracts (by a strongly-continuous 1-parameter family of $*$-endomorphisms) onto $\mathrm{U}(n) \times \mathrm{U}(\infty)$, which is path-connected. Thus, we are in the situation of Example 5.11 and $U \mapsto U1_nU^*$ is a strong π_0 equivalence from $\mathrm{U}_\infty(\mathrm{C}A)$ to the orbit of $1_n \in \mathrm{Vec}_n(\mathrm{C}A)$ under this conjugation action which contains the path component of 1_n. However, $\mathrm{Vec}_n(\mathrm{C}A)$ deformation retracts onto $\{1_n\}$, so is path-connected whence $U \mapsto U1_nU^* : \mathrm{U}_\infty(\mathrm{C}A) \to \mathrm{Vec}_n(A)$ is a strong π_0-equivalence (of two path-connected spaces). Meanwhile, we recognize G as exactly the stabilizer of $\mathrm{Vec}_n(\mathrm{S}A) \subset \mathrm{Vec}_n(\mathrm{C}A)$ of this action. So, the mapping $G \to \mathrm{Vec}_n(\mathrm{S}A) : U \mapsto U1_nU^*$ is also a strong π_0-equivalence by Proposition 5.12.

Next, we examine the map $G \to \mathrm{U}_n(A)$. Since this is actually a group homrphism, we can consider it as a group action where $\mathrm{U}_n(A)$ acts on itself by left multiplication. Now, fixing the point $1 \in \mathrm{U}_n(A)$, suppose there is some path u_t in $\mathrm{U}_n(A)$ with $u_0 = 1$. Then, define $U_t \in \mathrm{U}_n(\mathrm{C}A) \subset G$ by $U_t(s) = u_{ts}$ so that $U_0 \equiv 1$ and $U_t(1) = 1_nU_t(1)1_n = u_t$. So, this action has path-lifting over $1 \in \mathrm{U}_n(A)$. morphism, we can consider it as a group action where $\mathrm{U}_n(A)$ acts on itself by left multiplication. Now, fixing the point $1 \in \mathrm{U}_n(A)$, suppose there is some path u_t in $\mathrm{U}_n(A)$ with $u_0 = 1$. Then, define $U_t \in \mathrm{U}_n(\mathrm{C}A) \subset G$ by $U_t(s) = u_{ts}$ so that $U_0 \equiv 1$ and $U_t(1) = 1_n U_t(1) 1_n = u_t$. So, this action has path-lifting over $1 \in \mathrm{U}_n(A)$.

The crux, then, is to show that the stabilizer of $1 \in \mathrm{U}_n(A)$ is connected. That is, we are looking at the subgroup $H \subset G$ consisting of all $U \in G$ such that $U(1)$ has the form $\left(\begin{smallmatrix} 1_n & 0 \\ 0 & v \end{smallmatrix}\right)$. Take some $U \in H$. Now, it is at least clear that U is homotopic to 1 in $\mathrm{U}_\infty(\mathrm{C}A)$ (which deformation retracts onto the path-connected $\mathrm{U}(\infty)$). It follows that U is homotopic to $\left(\begin{smallmatrix} U & 0 \\ 0 & U^* \end{smallmatrix}\right)$ in H. Now, there is a $V \in \mathrm{U}_\infty(\mathrm{C}A)$ such that $V(1) = \left(\begin{smallmatrix} v^* & 0 & 0 \\ 0 & 1_n & 0 \\ 0 & 0 & v \end{smallmatrix}\right)$ and this V is homotopic to 1 in $\mathrm{U}_\infty(\mathrm{C}A)$. Thus, $\left(\begin{smallmatrix} 1_n & 0 \\ 0 & V \end{smallmatrix}\right)$ is homotopic to 1 in G. Thus, U is homotopic in H to $W = \left(\begin{smallmatrix} U & 0 \\ 0 & U^* \end{smallmatrix}\right)\left(\begin{smallmatrix} 1_n & 0 \\ 0 & V \end{smallmatrix}\right)$ which satisfies $W(1) = 1$. In other words, $W \in \mathrm{U}_\infty(\mathrm{S}A) \subset G$. Meanwhile, W is homotopic to 1 in $\mathrm{U}_\infty(\mathrm{C}A)$, so W is homotopic to $\left(\begin{smallmatrix} W & 0 \\ 0 & W^* \end{smallmatrix}\right)$ in H. Finally, the latter is homotopic to 1 in $\mathrm{U}_\infty(\mathrm{S}A) \subset H$. □

As a quick corollary to the above, we have the desired homotopy bijection.

Theorem 5.15. *For every C*-algebra A, for every positive integer n, there is bijection between the path components of $\mathrm{U}_n(A)$ and the path components of $\mathrm{Vec}_n(\mathrm{S}A)$ sending the class of $u \in \mathrm{U}_n(A)$ to the class of $U 1_n U^* \in \mathrm{Vec}_n(A)$ where $U \in \mathrm{U}_\infty(\mathrm{C}A)$ is chosen arbitrarily subject to $U(1) = \left(\begin{smallmatrix} u & 0 \\ 0 & * \end{smallmatrix}\right)$.*

5.3 Bott Periodicity

In C*-algebra K-theory, Bott's periodicity theorem manifests as a natural isomorphism $s_A^0 : K_0(A) \to K_1(\mathrm{S}A)$ for every C*-algebra A. When A is unital, s_A^0 is such that the class of a projection e in $\mathrm{M}_n(A)$ is sent to the class of the unitary loop[1] $z \mapsto ze + e^\perp$ in $\mathrm{U}_n(\mathrm{S}A)$. If A is nonunital, s_A^0 is got from the naturality. A proof in the case where A is commutative

[1]In this section, we switch conventions and use $\mathrm{S}A = \{x \in C(S^1, A) : x(1) = 0\}$.

appears in [1] and adapts readily to the C*-algebraic setting (see, for instance, [31]). The crux is to establish surjectivity of s_A^0; in essence, the following claim.

Theorem 5.16 (Bott periodicity)**.** *For every unital C*-algebra A, the K-group $K_1(\mathrm{S}A)$ is generated by classes of elementary loops $z \mapsto ze + e^\perp$ for e a projection in $M_n(A)$.*

For a C*-algebra A, let us agree, at least in this section, that

$$K_1(A) = \varinjlim \pi_0(\mathrm{GL}_n(A)).$$

For suspensions, we have $\mathrm{GL}_n(\mathrm{S}A) = \Omega\,\mathrm{GL}_n(A)$, the space of based loops[2] $u : S^1 \to \mathrm{GL}_n(A)$, $u(1) = 1_n$. Thus, $\pi_0(\mathrm{GL}_n(\mathrm{S}A)) = \pi_1(\mathrm{GL}_n(A))$ so that $K_1(\mathrm{S}A)$ can also be viewed as follows:

$$K_1(\mathrm{S}A) = \varinjlim \pi_1(\mathrm{GL}_n(A)).$$

We shall prove Theorem 5.16 in the special case $A = \mathbb{C}$. Admittedly, the exercise is somewhat idle since, as shown in the following section, a stronger claim can be proved by easier methods. However, even in this very simple case, the main ideas of the general case are brought to the forefront, so the expenditure of effort seems worthwhile. The claim to be proven is:

Theorem 5.17. *The direct limit of homotopy groups $\varinjlim \pi_1(\mathrm{GL}(n))$ is generated by classes of elementary loops $z \mapsto ze + e^\perp$ where e is a projection in $\mathrm{M}_n(\mathbb{C})$.*

We divide the proof into three lemmas, beginning with the following approximation lemma.

Lemma 5.18. *For each n, the component group of $\mathrm{GL}_n(C(S^1)) = C(S^1, \mathrm{GL}(n))$ is generated by homotopy classes of polynomial loops of the form $z \mapsto a_0 + a_1 z + \dots a_k z^k$ where $a_i \in \mathrm{M}_n(\mathbb{C})$.*

[2] We needn't be particularly mindful of basepoints. Recall $K_1(A) \cong K_1(\widetilde{A})$ for every C*-algebra A.

Proof. Fix a loop $u : S^1 \to \mathrm{GL}(n)$. We want to show u is homotopic to a product of polynomial loops and their inverses. Each entry of u belongs to $C(S^1)$ so, by Weierstrass approximation, u can be uniformly approximated by a loop u' whose entries are polynomials in z and $\overline{z} = z^{-1}$. Concisely, u' is a "Laurent loop" of the form $z \mapsto \sum_{i=-\ell}^{k} z^i a_i$, $a_i \in \mathrm{M}_n(\mathbb{C})$. Moreover, once the approximation is good enough, u and u' are homotopic elements of $\mathrm{GL}_n(C(S^1))$. Since u' factors as the product of the loops

$$z \mapsto (z^\ell 1_n)^{-1} \qquad\qquad z \mapsto a_{-\ell} + z a_{-\ell+1} + \ldots + z^{k+\ell} a_k,$$

the claim is proved. □

The inspired step in the proof is to combine the polynomial approximation above with the following linearization lemma. It is worthy of note that the size of the linearized matrix L below depends on the degree of the polynomial p. Since there is no telling how high a degree polynomial may be needed at the approximation stage of the argument, there is therefore no telling how large the matrices involved may become. Note as well that the construction of the matrix L below is modeled on the procedure by which the order of an ODE is reduced at the cost of increasing the dimension of the system.

Lemma 5.19 (Higmann linearization)**.** *Let A be a unital C^*-algebra. Let z and $a_0, a_1, \ldots, a_k$ be elements of A. Suppose that $p := a_0 + a_1 z + \ldots a_k z^k \in \mathrm{GL}(A)$. Then,*

$$L := \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_k \\ -z & 1 & 0 & \cdots & 0 \\ 0 & -z & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

is in $\mathrm{GL}_{k+1}(A)$ and belongs to the same path component as $p \oplus 1_k$.

Proof. We have the factorization

$$\underbrace{\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_k \\ -z & 1 & 0 & \cdots & 0 \\ 0 & -z & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}}_{L} = \begin{pmatrix} 1 & p_1 & p_2 & \cdots & p_k \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \underbrace{\begin{pmatrix} p & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}}_{p \oplus 1_k} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -z & 1 & 0 & \cdots & 0 \\ 0 & -z & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

where

$$\begin{aligned} p_1 &= a_1 + a_2 z + \ldots a_k z^{k-1} \\ p_2 &= a_2 + a_3 z + \ldots a_k z^{k-2} \\ &\vdots \\ p_{k-1} &= a_{k-1} + a_k z \\ p_k &= a_k. \end{aligned}$$

Since the three matrices on the right are obviously invertible, L is too. In fact, the first and third factors on the right belong to the identity component of $\mathrm{GL}_{k+1}(A)$, just continuously decay the off-diagonal entries to zero. Thus, L is homotopic to $p \oplus 1_k$ as desired. $\square$

Our third lemma is concerned with non-normal perturbations of projections. Prior to this point, our proofs have possessed more or less straightforward generalizations for C*-algebras different from $\mathbb{C}$. In the next proof, we rely on special facts about $A = \mathbb{C}$. To contend with more general cases, the holomorphic functional calculus should be used.

Lemma 5.20. *For $a \in \mathrm{M}_n(\mathbb{C})$, the following are equivalent:*

1. *No eigenvalue of a has real part $1/2$.*

2. For every $z \in S^1$*,* $za + (1-a)$ *is invertible.*

Moreover, letting X *denote the set of all such matrices* $a \in \mathrm{M}_n(A)$*, every path-component of* X *contains precisely one of the projections* 1_k*,* $k = 0, 1, \ldots, n$*.*

Proof. Note statement (2) holds vacuously for $z = 1$. If $z \in S^1$ is different from 1, then

$$\begin{aligned} za + (1-a) \text{ is noninvertible } &\iff 0 \text{ is an eigenvalue of } za + (1-a) = 1 - (1-z)a \\ &\iff \frac{1}{1-z} \text{ is an eigenvalue of } a. \end{aligned}$$

The equivalence of (1) and (2) now follows because $z \mapsto \frac{1}{1-z}$ sends S^1 to the line with real part 1/2. To see this last fact quickly, note $z \mapsto \frac{1}{z-1}$ is a fractional linear transformation, and it sends $1 \mapsto \infty$, so the image of the circle S^1 is a line. Since $-1 \mapsto \frac{1}{2}$, the image is a line through $\frac{1}{2}$. Finally, $\overline{\frac{1}{1-z}} = \frac{1}{1-\bar{z}}$, so the image is symmetric across the real axis.

Since similar matrices have identical spectra, and since similarity is precisely orbit equivalence under the conjugation action of the path-connected group $\mathrm{GL}(n)$, we see that every matrix is connected to, say, its Jordan normal form by a path along which the spectrum remains constant. Since the spectrum of an upper-triangular matrix is its set of diagonal entries, we see that each upper-triangular matrix is connected to its diagonal matrix by a path along which the spectrum remains constant (just decay the off-diagonal entries to zero). Putting these two observations together, we conclude that each matrix in X is connected, by a path in X, to a diagonal matrix. Obviously any diagonal matrix in X is connected to one of the matrices 1_k by moving each eigenvalue λ along a straight line path to 0 or 1 according to the side of the line $\mathrm{Re}(z) = 1/2$ that λ falls on. We do not actually need the converse statement that 1_k is not connected to 1_ℓ in X when $k \neq \ell$, though this is a rather simple consequence of continuity of the spectrum (with multiplicity). □

Having gathered our forces we are now able to give

Proof of Theorem 5.17. Fix a loop $u : S^1 \to \mathrm{GL}(n)$. Combining Lemmas 5.18 and 5.19, we get that, for some k, $u \oplus 1_k : S^1 \to \mathrm{GL}(n+k)$ is homotopic in $C(S^1, \mathrm{GL}(n+k))$ to a loop L of the form $z \mapsto b + zc$ where $b, c \in \mathrm{M}_{n+k}(\mathbb{C})$. Since $b + c$ is invertible (take $z = 1$) and $\mathrm{GL}(n+k)$ is path connected, we have that L is homotopic to $L' = (b+c)^{-1}L$. Putting $a := (b+c)^{-1}b$, we get $L'(z) = za + (1-a)$. Applying Lemma 5.20 above, L' is homotopic to $z \mapsto z1_r + 1_r^{\perp}$ for some r. This proves the theorem, modulo some easy basepoint issues (technically, the loops in $\pi_1(\mathrm{GL}(n))$ are based). □

5.4 Computation of $\pi_1(U(n))$

In the previous section, we admitted that our restriction to the the case $A = \mathbb{C}$ was a bit frivolous. Indeed, Theorem 5.17 can be proven by easier methods, as we now show.

Proposition 5.21. *For every integer $n \geq 2$, $\mathrm{U}(n)$ is a $\mathrm{U}(n-1)$-bundle over S^{2n-1}.*

Example 5.22. Note $\mathrm{U}(n) \cong \mathrm{SU}(n) \times S^1$ for all n. Since $S^3 \cong \mathrm{SU}(2)$ by $(z, w) \mapsto \left(\begin{smallmatrix} z & -\overline{w} \\ w & \overline{z} \end{smallmatrix}\right)$, we have in fact $\mathrm{U}(2) \cong S^3 \times S^1 \cong S^3 \times \mathrm{U}(1)$, so $\mathrm{U}(2)$ is a trivial $\mathrm{U}(1)$-bundle over S^3.

Proof. View S^{2n-1} as the unit sphere in $\mathbb{C}^n$. Our bundle map $p : \mathrm{U}(n) \to S^{2n-1}$ is induced by the action of $\mathrm{U}(n)$ on S^{2n-1}. Take $p(u) = ue_n$ where $e_n = (0, \ldots, 0, 1)$. Notice $p^{-1}(e_n) = \mathrm{U}(n-1)$ where we identify $\mathrm{U}(n-1)$ with the upper left corner of $\mathrm{U}(n)$. It suffices to produce a local trivialization at $e_n \in S^{2n-1}$ since then a local trivialization exists at ve_n for each $v \in \mathrm{U}(n)$ by conjugation. We trivialize over the neighbourhood $W = \{(z_1, \ldots, z_n) \in S^{2n-1} : z_n \neq 0\}$ of e_n. Note that it suffices to construct a section $s : W \to p^{-1}(W)$ of $p : p^{-1}(W) \to W$ since then $\Phi : W \times \mathrm{U}(n-1) \to p^{-1}(W)$ given by $\Phi(x, v) = s(x)v$ will give the desired trivialization. For these purposes, let $\Gamma : \mathrm{GL}(n) \to \mathrm{U}(n)$ be the map given by applying the Gramm-Schmidt algorithm to the columns of $a \in \mathrm{GL}(n)$ in *reverse order* (right to left). Let $\iota : p^{-1}(W) \to \mathrm{GL}(n)$ be the map which sends x to the matrix obtained by replacing the rightmost column of the $n \times n$ identity matrix with with x (this is invertible because it is upper-triangular with nonvanishing diagonal). Then, $\Gamma \circ \iota : W \to p^{-1}(W)$ is our section. □

Since fibre bundles have homotopy lifting (see [15], Proposition 4.48), we have the following corollary.

Corollary 5.23. *Suppose f_t is a homotopy of pointed maps $X \to S^{2n-1}$ and that $g : X \to \mathrm{U}(n)$ is a lift of f_0 through the bundle projection $p : \mathrm{U}(n) \to S^{2n-1}$ (consider $1 \in \mathrm{U}(n)$ to be the basepoint). Then, the whole homotopy lifts to a homotopy $\widetilde{f_t}$ of maps $X \to U(n)$ with $\widetilde{f_0} = g$.*

Proposition 5.24. *For every integer $n \geq 2$, the inclusion $u \mapsto \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) : \mathrm{U}(n-1) \to \mathrm{U}(n)$ induces an isomorphism on fundamental groups[3].*

Proof. We check separately injectivity and surjectivity.

Surjectivity: We should take a based map $\varphi : S^1 \to \mathrm{U}(n)$ and show it can be homotoped to one with range contained in $\mathrm{U}(n-1)$. The composite $\psi := p \circ \varphi : S^1 \to S^{2n-1}$ is null-homotopic since $2n - 1 \geq 3$ so there is a homotopy ψ_t of maps $S^1 \to S^{2n-1}$ such that $\psi_0 = \psi$ and $\psi_1 \equiv e_n$. Lifting, we obtain a homotopy φ_t of maps $S^1 \to \mathrm{U}(n)$ such that $\varphi_0 = \varphi$ and φ_1 has range contained in $\mathrm{U}(n-1)$, as desired.

Injectivity: Suppose some based map $\varphi : S^1 \to \mathrm{U}(n-1) \subset \mathrm{U}(n)$ is null-homotopic as a map $S^1 \to \mathrm{U}(n)$. We need to show the null-homotopy can occur inside $\mathrm{U}(n-1)$. Since φ is null-homotopic in $\mathrm{U}(n)$, there is a map $\Phi : \mathbb{D}^2 \to \mathrm{U}(n)$ whose restriction to the boundary $\partial\mathbb{D}^2 = S^1$ is φ. The composition $\Psi := p \circ \Phi : \mathbb{D}^2 \to S^{2n-1}$ collapses S^1 to the basepoint e_n, so Ψ is, in essence, a map $S^2 \to S^{2n-1}$. As $2n - 1 \geq 3$, any map $S^2 \to S2n - 1$ is null-homotopic. In other words, there is a null-homotopy

[3] Assume all loops are based at the identity matrix

$\Psi_t : \mathbb{D}^2 \to S^{2n-1}$ with $\Psi_0 = \Psi$, $\Psi_1 \equiv e_n$ such that Ψ_t maps S^1 to e_n for every $t \in [0,1]$. Lifting, we get a homotopy of maps $\Phi_t : \mathbb{D}^2 \to S^{2n-1}$ with $\Phi_0 = \Phi$, $(\Phi_1) \subset \mathrm{U}(n-1)$ such that Φ_t maps S^1 into $\mathrm{U}(n-1)$ for all $t \in [0,1]$. Then, the restrictions $\varphi_t := \Phi_t\big|_{S^1}$ are a homotopy between φ and φ_1 through maps $S^1 \to \mathrm{U}(n-1)$. Moreover, φ_1 is null-homotopic as a map $S^1 \to \mathrm{U}(n-1)$, because it has the extension $\Phi_1 : \mathbb{D}^2 \to \mathrm{U}(n-1)$.

□

As corollary, we get $\pi_1(\mathrm{U}(n)) \cong \mathbb{Z}$ for all n.

Corollary 5.25. *For every positive integer n, the determinant* det : $\mathrm{U}(n) \to S^1$ *and the inclusion* $z \mapsto \left(\begin{smallmatrix} z & 0 \\ 0 & 1_{n-1} \end{smallmatrix}\right) : S^1 \to \mathrm{U}(n)$ *induce inverse isomorphisms between* $\pi_1(\mathrm{U}(n))$ *and* $\pi_1(S^1)$.

Proof. We have the sequence of pointed maps

$$S^1 \hookrightarrow \mathrm{U}(2) \hookrightarrow \mathrm{U}(3) \hookrightarrow \cdots \hookrightarrow \mathrm{U}(n) \stackrel{\det}{\to} S^1$$

whose full composite is the identity. Applying π_1 and using functoriality, the result follows from Proposition 5.24 above. □

Theorem 5.17 follows trivially from the above corollary, so we have an the alternative, more elementary, proof as alluded to before.

Chapter 6

The Connes-Thom Isomorphism

The Connes-Thom isomorphism is really a pair of isomorphisms which together show the K-theory of a 1-parameter crossed-product is naturally isomorphic to the K-theory of the original algebra, with a dimension shift.

$$K_i(A \rtimes \mathbb{R}) \cong_\alpha K_{i+1}(A) \qquad\qquad i = 0, 1$$

In particular, the K-theory of the crossed-product doesn't depend on the action α. In the case where the action is trivial, so that $A \rtimes \mathbb{R} \cong A \otimes C^*(\mathbb{R}) \cong A \otimes C_0(\mathbb{R})$, one recovers the suspension isomorphism on K-theory.

By now, there are several proofs of Connes' theorem (see, for instance, [10], [29]). For the purposes of this thesis, it is important to cast the isomorphisms in a relatively explicit form. With this goal in mind, the best resource is still probably the original work of Connes [6]. The present chapter is adapted from Connes' paper.

6.1 Overview and axiomatics

Definition 6.1. A **Thom map** is a selection, for each C*-dynamical system $(A, \mathbb{R}, \alpha)$, of a pair of group homomorphisms

$$\phi^0_\alpha : K_0(A) \to K_1(A \rtimes_\alpha \mathbb{R}) \qquad \phi^1_\alpha : K_1(A) \to K_0(A \rtimes_\alpha \mathbb{R})$$

such that the following axioms are satisfied:

1. *Orientation:* If $A = \mathbb{C}$, then $\phi^0_{\mathrm{id}} : K_0(\mathbb{C}) \to K_1(C^*(\mathbb{R}))$ is such that the image of $\phi^0_{\mathrm{id}}([1]) \in K_1(C^*(\mathbb{R}))$ by the Fourier isomorphism[1] $C^*(\mathbb{R}) \to C_0(\mathbb{R})$ is the class in $K_1(C_0(\mathbb{R})) = K^1(\mathbb{R})$ represented by a loop of winding number 1.

2. *Naturality*: If $(A, \mathbb{R}, \alpha)$ and $(B, \mathbb{R}, \beta)$ are C*-dynamical systems, and $\varphi : A \to B$ is an equivariant $*$-homomorphism, then the following square commutes.

$$\begin{array}{ccc} K_i(A) & \xrightarrow{\phi^i_\alpha} & K_{i+1}(A \rtimes_\alpha \mathbb{R}) \\ {\scriptstyle \varphi_*}\downarrow & & \downarrow{\scriptstyle (\widehat{\varphi})_*} \\ K_i(B) & \xrightarrow[\phi^i_\beta]{} & K_{i+1}(B \rtimes_\beta \mathbb{R}) \end{array} \qquad i \in 0, 1$$

3. *Suspension*: Let $A = (A, \mathbb{R}, \alpha)$ be a C*-dynamical system. Then, diagram

$$\begin{array}{ccc} K_i(A) & \xrightarrow{\phi^i_\alpha} & K_{i+1}(A \rtimes_\alpha \mathbb{R}) \\ {\scriptstyle s^i_A}\downarrow & & \downarrow{\scriptstyle s^{i+1}_{A \rtimes_\alpha \mathbb{R}}} \\ K_{i+1}(\mathrm{S}A) & \xrightarrow[\phi^{i+1}_{\mathrm{S}\alpha}]{} & K_{i+1}(\mathrm{S}(A \rtimes_\alpha \mathbb{R})) \end{array} \qquad i \in 0, 1$$

   is commutative. Here, $\mathrm{S}A = (\mathrm{S}A, \mathbb{R}, \mathrm{S}\alpha)$ is the suspended system and we are conflating $\mathrm{S}(A \rtimes_\alpha \mathbb{R})$ with $(\mathrm{S}A) \rtimes_{\mathrm{S}\alpha} \mathbb{R}$ by way of the obvious C*-algebra isomorphism.

[1] We take $f \mapsto \widehat{f} : C^*(\mathbb{R}) \to C_0(\mathbb{R})$ to be determined on $C_c(\mathbb{R})$ by $\widehat{f}(s) = \int_{-\infty}^{\infty} f(t) e^{its}\, dt$.

These three axioms also imply that a Thom map must respect, in an appropriate sense, the operations of unitization, and tensor product with $M_n(\mathbb{C})$. Significantly, exterior equivalence must be respected as well in the following sense (Proposition II.3 in [6]).

Lemma 6.2. *If* $(A, \mathbb{R}, \alpha)$ *is a C*-dynamical system,* u *a unitary 1-cocycle of* α*, and* $\alpha' = \mathrm{Ad}(u)\alpha$ *the adjusted flow, then the isomorphism* $\iota_u : A \rtimes_{\alpha'} \mathbb{R} \to A \rtimes_\alpha \mathbb{R}$ *of Proposition 2.49 is such that*

$$(\iota_u)_* \circ \phi^i_{\alpha'} = \phi^i_\alpha \qquad\qquad i = 0, 1$$

for any Thom map, $\{\phi^0_\cdot, \phi^1_\cdot\}$*.*

Using the above axioms and their consequences, it is shown in [6] that

Theorem 6.3 (Connes)**.** *A Thom map exists, is unique, and each map* ϕ^i_α *is an isomorphism.*

In this thesis, we assume the isomorphisms ϕ^i_α exist and content ourselves with deducing their form from the axioms.

6.2 The isomorphism $K_0(A) \to K_1(A \rtimes \mathbb{R})$

Let A be a unital C*-algebra and let the suspended C*-algebra be $\mathrm{S}A = A \otimes C_0(\mathbb{R})$. Thus, $\mathrm{S}A$ is generated by commuting products $a \cdot f$ where $a \in A$, $f \in C_0(\mathbb{R})$. The suspension isomorphism $\mathrm{susp}^0_A : K_0(A) \to K_1(\mathrm{S}A)$ is such that, for any projection $e \in A$,

$$\mathrm{s}^0_A([e]) = [e \cdot b + e^\perp],$$

where $b \in \mathrm{GL}(C_0(\mathbb{R}))$ is a loop[2] of winding number 1. Since the crossed-product $A \rtimes_\alpha \mathbb{R}$ is generated by (generally noncommuting) products $a \cdot f(H)$ where $a \in A$, $f \in C_0(\mathbb{R})\}$ and

[2]The Cayley transform $b(t) = \frac{t-i}{t+i}$ is one such loop.

H is the Hamiltonian of $A \rtimes_\alpha \mathbb{R}$, one expects ϕ^0_α will be defined in similar manner to s^0_A. One knows, however, that some extra ingredient is needed since, if e does not commute with $b(H)$, then there is no reason why $e \cdot b(H) + e^\perp$ should be invertible.

Theorem 6.4. *Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system with A unital and let e be a projection in $\mathrm{dom}(\delta)$, where δ is the infinitesimal generator of α. If P is a self-adjoint element of A such that $i[P, e] = -\delta(e)$, then $e \cdot b(H + P) + e^\perp \in \mathrm{GL}(A \rtimes_\alpha \mathbb{R})$ and*

$$\phi^0_\alpha([e]) = [e \cdot b(H + P) + e^\perp].$$

Here, H is the Hamiltonian of the crossed-product and $b = 1 + \ell \in \mathrm{GL}(C_0(\mathbb{R}))$ is a loop of winding number 1, $\ell \in C_0(\mathbb{R})$.

Proof. We can drop the smoothness hypothesis on e and prove something stronger: $\phi^0_\alpha([e]) = [e \cdot b(H_u) + e^\perp]$ when u is a unitary cocycle of α such that $\mathrm{Ad}(u)\alpha$ fixes e, and H_u is the perturbation of H by u. By Proposition 4.12 and Corollary 4.14, this is more general than the desired theorem. Put $\alpha' = \mathrm{Ad}(u)\alpha$. Since e is fixed by α', the $*$-homomorphism $\varphi_e : \mathbb{C} \to A$ sending $1 \mapsto e$ is equivariant with respect to α' (and the trivial dynamics on $\mathbb{C}$). Thus, by the naturality axiom, the square

$$\begin{array}{ccc} K_0(\mathbb{C}) & \xrightarrow{\varphi_e} & K_0(A) \\ \phi^0_{\mathrm{id}} \downarrow & & \phi^0_{\alpha'} \downarrow \\ K_1(C^*(\mathbb{R})) & \xrightarrow{\widehat{\varphi_e}} & K_1(A \rtimes_{\alpha'} \mathbb{R}) \end{array}$$

is commutative. We chase $[1] \in K_0(\mathbb{C})$ both ways around the diagram. Going clockwise, one gets to $\phi^0_{\alpha'}([e]) \in K_1(A \rtimes_{\alpha'} \mathbb{R})$. Recall (Example 2.40) that $i\frac{d}{dt}$ is the Hamiltonian of $C^*(\mathbb{R})$ and its functional calculus $f \mapsto f(i\frac{d}{dt})$ inverts the Fourier isomorphism $C^*(\mathbb{R}) \to C_0(\mathbb{R})$. Thus, by the orientation axiom, $\phi^0_{\mathrm{id}}([1])$ is represented by $b(i\frac{d}{dt}) = \ell(i\frac{d}{dt}) + 1$, where we write $b = \ell + 1$ so that $\ell \in C_0(\mathbb{R})$. As in Example 2.45, one has $\widehat{\varphi}_e(\ell(i\frac{d}{dt})) = \widehat{\varphi}_e(1 \cdot \ell(i\frac{d}{dt})) = \varphi_e(1) \cdot$

$\ell(H') = e\cdot\ell(H')$ where H' denotes the Hamiltonian of $A \rtimes_{\alpha'} \mathbb{R}$. Thus, going counterclockwise, we get to the class of $e \cdot \ell(H') + 1 = e \cdot b(H') + e^{\perp} \in \mathrm{GL}(A \rtimes_{\alpha'} \mathbb{R})$ proving

$$\phi^0_{\alpha'}([e]) = [e \cdot b(H') + e^{\perp}].$$

By Lemma 6.2, one has $\phi^0_\alpha = (\iota_u)_* \circ \phi^0_{\alpha'}$. Applying Proposition 2.49, we get

$$\iota_u([e \cdot b(H') + e^{\perp}) = e \cdot b(H_u) + e^{\perp},$$

which completes the proof. □

By exploiting the compatibility of the Thom map with unitization and tensor product by $\mathrm{M}_n(\mathbb{C})$, one sees how to calculate ϕ^0_α on general K-theory classes.

Theorem 6.5. *Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system (possibly nonunital), let e be a projection in $\mathrm{M}_n(\widetilde{A})$, and let $e_0 = \varepsilon(e)$ be the scalar part of e so that $[e]-[e_0] \in K_0(A) \subset K_0(\widetilde{A})$. If $e \in \mathrm{dom}(\delta)$, where δ is the infinitesimal generator of the flow $\beta = \widetilde{\alpha} \otimes \mathrm{id}$ on $\mathrm{M}_n(\widetilde{A}) = \widetilde{A} \otimes \mathrm{M}_n(\mathbb{C})$, and $P \in M_n(A)$ is a self-adjoint element[3] such that $i[P, e] = -\delta(e)$, then*

$$\left(e \cdot b(H + P) + e^{\perp}\right) \left(e_0 \cdot b(H) + e_0^{\perp}\right)^{-1} \in \mathrm{GL}_n(A \rtimes_\alpha \mathbb{R}) \subset \mathrm{GL}_n(\widetilde{A} \rtimes_{\widetilde{\alpha}} \mathbb{R})$$

and this element represents $\phi^0_\alpha([e]) \in K_1(A \rtimes_\alpha \mathbb{R})$. Here, H denotes the Hamiltonian of $\mathrm{M}_n(\widetilde{A}) \rtimes_\beta \mathbb{R} = \mathrm{M}_n(A \rtimes_{\widetilde{\alpha}} \mathbb{R})$ and $b = \ell + 1 \in \mathrm{GL}(C_0(\mathbb{R}))$ is a loop of winding number 1.

Sketch of proof. We know from the preceding theorem that

$$\varphi^0_\beta([e]) = [e \cdot b(H + P) + e^{\perp}] \qquad \varphi^0_\beta([e_0]) = [e_0 \cdot b(H) + e_0^{\perp}]$$

[3] By Connes' projection lemma, the particular choice $P = i[\delta(e), e]$ always does the job.

in $K_1(M_n(\widetilde{A})) \rtimes_\beta \mathbb{R})$. The compatibility of the Thom map with inflation to matrices (Proposition II.2 in [6]) implies that, indeed,

$$\varphi^0_{\tilde{\alpha}}([e]) = [e \cdot b(H+P) + e^\perp] \qquad\qquad \varphi^0_{\tilde{\alpha}}([e_0]) = [e_0 \cdot b(H) + e_0^\perp]$$

in $K_1(\widetilde{A} \rtimes_\alpha \mathbb{R})$. The compatibility of the Thom map with unitization (Proposition II.1 in [6]) then implies that $\left(e \cdot b(H+P) + e^\perp\right)\left(e_0 \cdot b(H) + e_0^\perp\right)^{-1}$ represents the image of $\phi^1_\alpha([e])$ by $\widehat{\iota}_* : K_1(A \rtimes_\alpha \mathbb{R}) \to K_1(A \rtimes_{\tilde{\alpha}} \mathbb{R})$. Thus, it only remains to see this representative already belongs to the image of $\widehat{\iota} : GL_n(A \rtimes_\alpha \mathbb{R}) \to \mathrm{GL}_n(\widetilde{A} \rtimes_{\tilde{\alpha}} \mathbb{R})$. Using the Takesaki-Takai duality theorem, it can be shown (Lemma I.1 in [6]) that the exactness of the equivariant sequence

$$0 \longrightarrow M_n(A) \xrightarrow{\iota} M_n(\widetilde{A}) \xrightarrow{\varepsilon} \mathrm{M}_n(\mathbb{C}) \longrightarrow 0$$

implies the exactness of the dual sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_n(A) \rtimes_{\alpha\otimes\mathrm{id}} \mathbb{R} & \xrightarrow{\widehat{\iota}} & M_n(\widetilde{A}) \rtimes_\beta \mathbb{R} & \xrightarrow{\widehat{\varepsilon}} & \mathrm{M}_n(\mathbb{C}) \rtimes_{\mathrm{id}} \mathbb{R} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & M_n(A \rtimes_\alpha \mathbb{R}) & \xrightarrow{\widehat{\iota}} & M_n(\widetilde{A} \rtimes_{\tilde{\alpha}} \mathbb{R}) & \xrightarrow{\widehat{\varepsilon}} & M_n(C^*(\mathbb{R})) & \longrightarrow & 0. \end{array}$$

Thus, we just need to see $e \cdot b(H+P) + e^\perp = e \cdot \ell(H+P) + 1$ and $e_0 \cdot b(H) + e_0^\perp = e_0 \cdot \ell(H) + 1$ have the same image under $\widehat{\varepsilon} : \mathrm{GL}_n(A \rtimes_{\tilde{\alpha}} \mathbb{R}) \to \mathrm{GL}_n(C^*(\mathbb{R}))$. Since $\varepsilon(P) = 0$ and $\varepsilon(e) = e_0$, this follows from Proposition 2.51. $\square$

6.3 The isomorphism $K_1(A) \to K_0(A \rtimes_\alpha \mathbb{R})$

It turns out that, in general, the odd Thom map $\phi^1_\alpha : K_1(A) \to K_0(A \rtimes_\alpha \mathbb{R})$ is difficult to describe. The composition

$$K_1(A) \xrightarrow{\phi^1_\alpha} K_0(A \rtimes_\alpha \mathbb{R}) \xrightarrow{s^0_{A \rtimes_\alpha \mathbb{R}}} K_1(S(A \rtimes_\alpha \mathbb{R})),$$

however, does admit a satisfactory description in terms of the generators of the K-groups. The following is Proposition III.1 in [6] together with a mild variation of its proof.

Theorem 6.6. *Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system, and let $u \in \mathrm{U}(A)$ be a C^1 unitary. Then, the image of $\phi^1_\alpha([u]) \in K_0(A \rtimes_\alpha \mathbb{R})$ by the suspension isomorphism $K_0(A \rtimes_\alpha \mathbb{R}) \to K_1(\mathrm{S}(A \rtimes_\alpha \mathbb{R}))$ is represented by the loop $\mathscr{C} \in \mathrm{GL}_2(\mathrm{S}(A \rtimes_\alpha \mathbb{R}))$ given as the concatenate of the two paths:*

$$\mathscr{C}_1(\lambda) = \begin{pmatrix} b(H + \lambda P) b(H)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \qquad \lambda \in [0, 1]$$

$$\mathscr{C}_2(t) = W(t) \begin{pmatrix} b(H) & 0 \\ 0 & 1 \end{pmatrix} W(t)^{-1} \begin{pmatrix} b(H)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \qquad t \in [0, 1]$$

where:

H is the Hamiltonian of $A \rtimes_\alpha \mathbb{R}$,

$b \in \mathrm{GL}(C_0(\mathbb{R}))$ is a loop with winding number 1,

$P = i\delta(u^*)u$, δ the infinitesimal generator of A,

$W = R \left(\begin{smallmatrix} u^* & 0 \\ 0 & 1 \end{smallmatrix}\right) R^*$ for $R : [0, 1] \to \mathrm{U}(2)$ a smooth path from $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Note that $P = i\delta(u^*)u$ in the statement is a perturbation implementing an exterior equivalence from α to $\mathrm{Ad}(u^*)\alpha\,\mathrm{Ad}(u)$.

Proof. By the suspension axiom for the Thom map, the square

$$\begin{array}{ccc} K_1(A) & \xrightarrow{\phi^1_\alpha} & K_0(A \rtimes_\alpha \mathbb{R}) \\ {\scriptstyle s^1_A}\downarrow & & \downarrow{\scriptstyle s^0_{A\rtimes_\alpha\mathbb{R}}} \\ K_0(\mathrm{S}A) & \xrightarrow[\phi^0_{\mathrm{S}\alpha}]{} & K_1(\mathrm{S}(A \rtimes_\alpha \mathbb{R})) \end{array}$$

is commutative. Thus, the problem is reduced to calculating the composition of the two maps s^1_A and $\phi^0_{\mathrm{S}\alpha}$. By definition, $\mathrm{s}^1_A([u]) = [e] - [e_0] \in K_0(\mathrm{S}A) \subset K_0(\widetilde{\mathrm{S}A})$ where $e_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $e = Ue_0U^* \in \mathrm{M}_2(\widetilde{\mathrm{S}A})$ and $U \in \mathrm{U}_2(\widetilde{\mathrm{C}A})$ has $U(0) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $U(1) = \left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right)$. Since $\left(\begin{smallmatrix} u^* & 0 \\ 0 & 1 \end{smallmatrix}\right)$ commutes with e_0, one also has $e = We_0W^*$ where $W = U\left(\begin{smallmatrix} u^* & 0 \\ 0 & 1 \end{smallmatrix}\right)$ has $W(0) = \left(\begin{smallmatrix} u^* & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $W(1) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & u^* \end{smallmatrix}\right)$. We assume the projection loop e is "lazy" at the start and end of its journey. To be more clear about this, we henceforth subdivide all loops in this proof into three segments so that, for instance, an element of $x \in \mathrm{S}A$ is the concatenate of three paths

$$\lambda \mapsto x_1(\lambda) \qquad t \mapsto x_2(t) \qquad \mu \mapsto x_3(\mu) \qquad [0,1] \to A$$

satisfying $x_1(0) = 0$, $x_1(1) = x_2(0)$, $x_2(1) = x_3(0)$, $x_3(1) = 0$. We choose e so that

$$e_1(\lambda) = e_0,\ \forall\lambda \in [0,1] \qquad e_2(t) = W(t)e_0W(t),\ \forall t \in [0,1] \qquad e_3(\mu) = e_0,\ \forall\mu \in [0,1].$$

We now calculate $\phi^0_{\mathrm{S}\alpha}([e] - [e_0])$ using Theorem 6.5. Let β be the natural dynamics on $\mathrm{M}_2(\widetilde{\mathrm{S}A})$. We need to produce a self-adjoint $Q = Q_1 \cup Q_2 \cup Q_3 \in \mathrm{M}_2(\mathrm{S}B)$ such that $i[Q, e] =$

$-\Delta(e)$, where Δ is the infinitesimal generator of β. In other words, we need to choose Q so

$$i[Q_1(\lambda), e_0] = 0 \qquad \forall \lambda \in [0,1]$$

$$i[Q_2(t), e_2(t)] = -\delta(e_2(t)) \qquad \forall t \in [0,1]$$

$$i[Q_3(\mu), e_0)] = 0 \qquad \forall \mu \in [0,1]$$

Since $e_2(t) = W(t)e_0W(t)^*$ where e_0 is α-invariant, we follow Corollary 4.11 and take

$$Q_2(t) = i\delta(W(t))W(t)^* \qquad \forall t \in [0,1].$$

In particular, we have $Q_2(0) = \left(\begin{smallmatrix} P & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $Q_2(1) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & P \end{smallmatrix}\right)$ where $P = i\delta(u^*)u$. The simplest choices for Q_1 and Q_3 are then

$$Q_1(\lambda) = \begin{pmatrix} \lambda P & 0 \\ 0 & 0 \end{pmatrix} \qquad Q_3(\mu) = \begin{pmatrix} 0 & 0 \\ 0 & (1-\mu)P \end{pmatrix} \qquad \forall \lambda, \mu \in [0,1]$$

Now, we can find the image of $[e] - [e_0]$ in $K_1(\mathrm{S}A \rtimes_{\mathrm{S}\alpha} \mathbb{R})$ using Theorem 6.5. We need to compute $b(H'+Q)$ where H' is the Hamiltonian of $\mathrm{M}_2(\widetilde{\mathrm{S}}A \rtimes_{\mathrm{S}\alpha} \mathbb{R})$. We get

$$\begin{pmatrix} b(H+\lambda P) & 0 \\ 0 & b(H) \end{pmatrix} \qquad \forall \lambda \in [0,1]$$

$$W(t)\begin{pmatrix} b(H) & 0 \\ 0 & b(H) \end{pmatrix} W(t)^* \qquad \forall t \in [0,1]$$

$$\begin{pmatrix} b(H) & 0 \\ 0 & b\left(H+(1-\mu)P\right) \end{pmatrix} \qquad \forall \mu \in [0,1]$$

so that $eb(H'+Q)+e^{\perp}$ is equal to

$$\begin{pmatrix} b(H+\lambda P) & 0 \\ 0 & 1 \end{pmatrix} \qquad \forall \lambda \in [0,1]$$

$$W(t)\begin{pmatrix} b(H) & 0 \\ 0 & 1 \end{pmatrix} W(t)^* \qquad \forall t \in [0,1]$$

$$\begin{pmatrix} b(H) & 0 \\ 0 & 1 \end{pmatrix} \qquad \forall \mu \in [0,1]$$

and, finally, $s^0_{A\rtimes_\alpha\mathbb{R}}(\phi^1_\alpha([u]))$ is represented by $\mathscr{C} = (e\cdot b(H'+Q)+e^{\perp})(e_0\cdot b(H')+e_0^{\perp})^{-1}$

where $(e_0 \cdot b(H') + e_0^{\perp}) \equiv \left(\begin{smallmatrix} b(H) & 0 \\ 0 & 1 \end{smallmatrix}\right)$ so that $\mathscr{C}$ is the path

$$\mathscr{C}_1(\lambda) = \begin{pmatrix} b(H+\lambda P)b(H))^{-1} & 0 \\ 0 & 1 \end{pmatrix} \qquad \forall \lambda \in [0,1]$$

$$\mathscr{C}_2(t) = W(t) \begin{pmatrix} b(H) & 0 \\ 0 & 1 \end{pmatrix} W(t)^* \begin{pmatrix} b(H)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \qquad \forall t \in [0,1]$$

$$\mathscr{C}_3(\mu) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \forall \mu \in [0,1].$$

Obviously, we can homotope the constant third portion of the path away without changing the K-theory class of $[\mathscr{C}]$, so we get the stated result. □

Since the suspension isomorphism s^0 is generally difficult to invert, the above the Theorem gives little indication how one might go about computing ϕ^1 itself. In Section 7.2, we consider a few simple cases where the calculation can be made.

Chapter 7

Conne's trace formula

7.1 The formula $\widehat{\tau}_* \circ \phi^1_\alpha = \mathrm{ind}^\tau_\alpha$

We now prove the formula of Connes which was discussed in the introductory chapter and which has motivated much of the work done in this document.

Theorem 7.1 ([6], Theorem 3)**.** *Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system, and let τ be an α-invariant, densely-defined, lower semicontinuous trace on A. If $u = z + 1$ is a unitary in $\widetilde{A}$, where $z \in A$ is a C^1 element for α such that z and $\delta(z)$ are in A^τ_1, then*

$$\widehat{\tau}_*(\phi^1_\alpha([u])) = \frac{1}{2\pi i}\tau(\delta(u)u^*)$$

where $\widehat{\tau}$ denotes the dual trace on $A \rtimes_\alpha \mathbb{R}$.

There are not substantial differences between the proof appearing below and the original proof in [6], although we do work harder to deal with the domain issues which crop up when unbounded traces are allowed onto the field. Potentially, this is because we use "lower semi-continuous, densely-defined traces", whereas "semi-continuous, semi-finite traces" are used in [6]. It is presumed that, to speak of a semi-finite trace on a C*-algebra, one implicitly assumes the trace extends to the enveloping von Neumann algebra. However, the author of this thesis is mostly ignorant of von Neumann algebra theory, and feels unqualified to comment as to whether the distinction has mathematical significance, or is purely terminological.

Proof. We organize the proof into three parts.

1. Preliminaries.

For brevity, write $B = A \rtimes_\alpha \mathbb{R}$. We aim to calculate $\widehat{\tau}_*(\phi^1_\alpha([u]))$ by an application of Theorem B.9. From Theorem 6.6, we have

$$s^0_B(\phi^1_\alpha([u])) = [\mathscr{C}] \in K_1(\mathrm{S}B)$$

where $\mathscr{C} \in \mathrm{GL}_2(\mathrm{S}B)$ equals the concatenate of the two paths:

$$\mathscr{C}_1(\lambda) = \begin{pmatrix} b(H+\lambda P)b(H)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \qquad \lambda \in [0,1]$$

$$\mathscr{C}_2(t) = W(t)\begin{pmatrix} b(H) & 0 \\ 0 & 1 \end{pmatrix} W(t)^{-1}\begin{pmatrix} b(H)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \qquad t \in [0,1],$$

$b = 1 + \ell \in \mathrm{GL}(C_0(\mathbb{R}))$ is a loop of winding number 1,

$P = i\delta(u^*)u$ (note $P \in A^\tau_1$, since $\delta(u^*) = \delta(z)^* \in A^\tau_1$),

$W = R\left(\begin{smallmatrix} u^* & 0 \\ 0 & 1 \end{smallmatrix}\right)R^*$,

$R : [0,1] \to \mathrm{U}(2)$ is smooth path from $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

A careful choice of ℓ is needed to ensure $\mathscr{C}_1$ and $\mathscr{C}_2$ are C^1 with respect to the trace-class norm, and to make the integrals tractable. Of course, $b = 1 + \ell$ must be a loop of winding number 1, but we also impose the following constraints.

1. ℓ has the form $s \mapsto \sum_{i=1}^{n} \frac{\lambda_i}{s-p_i}$ where $\lambda_i, p_i \in \mathbb{C}$, and the p_i are non-real.

2. $s \mapsto s \cdot \ell(s)$ is in $C_0(\mathbb{R}) \cap L^2(\mathbb{R})$. In particular, $\ell \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$.

Fortunately, these two constraints are compatible. To satisfy (1), the natural choice would seem to be $\ell(s) = \frac{-2i}{s+i}$ so that $b(s) = 1 + \ell(s) = \frac{s-i}{s+i}$ is the Cayley transform which winds $\mathbb{R}$ around the circle, but this ℓ is not in $L^1(\mathbb{R})$, so (2) fails. On the other hand, we could use $\ell(s) = \frac{2}{(s+i)^2}$, which parametrizes the cardioid with polar equation $r = 1 - \cos(\theta)$. This

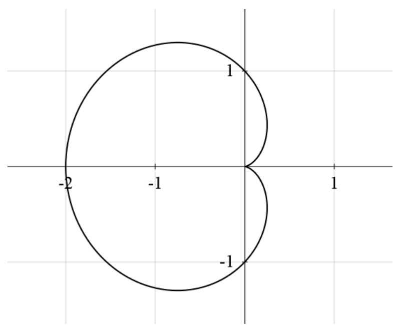

Figure 7.1: The cardioid $\ell(s) = \frac{2}{(s+i)^2}$.

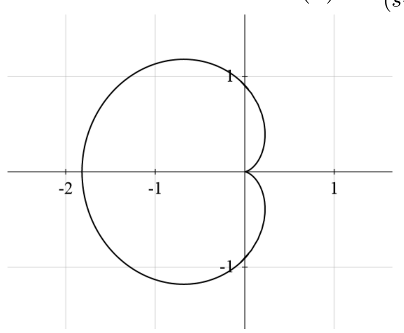

Figure 7.2: The "cardioidoid" $\ell(t) = \frac{2}{(t+i)(t+(1.1)i)}$.

time, (2) holds, but, because of the double pole at $-i$, the partial fraction decomposition of $\ell(t)$ is not into linear factors, so (1) fails. However, one can simply perturb one pole a small amount and use

$$\ell(s) = \frac{2}{(s+i)(s-p)}$$

for some $p \in \mathbb{C}$ sufficiently close to $-i$ to get a loop satisfying both (1) and (2).

For the rest of the proof, we occupy ourselves by showing, for $i = 1, 2$, that $\mathscr{C}_i$ values in $\mathrm{GL}_2(B_1^{\widehat{\tau}})$ and is C^1 with respect to the trace class norm, and calculating that

$$\int_0^1 \widehat{\tau}\left(\frac{d\mathscr{C}_1}{d\lambda}\mathscr{C}_1(\lambda)^{-1}\right) d\lambda = \tau(\delta(u)u^*) \qquad\qquad \int_0^1 \widehat{\tau}\left(\frac{d\mathscr{C}_2}{dt}\mathscr{C}_2(t)^{-1}\right) dt = 0.$$

Once this is achieved, Theorem B.9 implies $\frac{1}{2\pi i}\tau(\delta(u)u^*) = \widehat{\tau}_*(\phi^1_\alpha([u]))$, as desired.

2. Calculation of $\int_0^1 \widehat{\tau}\left(\frac{d\mathscr{C}_1}{d\lambda}\mathscr{C}_1(\lambda)^{-1}\right) d\lambda$.

First, we must check that $\lambda \mapsto b(H+\lambda P)b(H)^{-1}$ values in $\mathrm{GL}(B_1^{\widehat{\tau}})$ and is C^1 with respect to the trace-class norm. As $\lambda = 0$ maps to $1 \in \mathrm{GL}(B_1^{\widehat{\tau}})$, it suffices to check:

1. $\lambda \mapsto b(H + \lambda P)$ is C^1 with respect to the C*-algebra norm.

2. $\frac{d}{d\lambda}b(H + \lambda P)$ values in $B_1^{\widehat{\tau}}$ and is continuous with respect to the trace class norm.

By the choice of ℓ, we have $b(H + \lambda P) = 1 + \sum_{i=1}^n \lambda_i(H + \lambda P - p_i)^{-1}$. By Theorem 4.27, $\lambda \mapsto b(H + \lambda P)$ is C^1 for the norm of $\widetilde{A} \rtimes_{\widetilde{\alpha}} \mathbb{R}$, proving (1), and, moreover,

$$\frac{d}{d\lambda}b(H + \lambda P) = -\sum_{i=1}^n \lambda_i(H + \lambda P - p_i)^{-1}P(H + \lambda P - p_i)^{-1}.$$

Since $s \mapsto \frac{1}{s-p_i}$ is the Fourier transform of a function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and $P \in A_1^\tau$, Proposition A.39 implies that $\lambda \mapsto (H+\lambda P-p_i)^{-1}P(H+\lambda P-p_i)^{-1} : \mathbb{R} \to B_1^{\widehat{\tau}}$ is continuous with respect to the trace class norm for each i, proving (2).

Next, we show $\widehat{\tau}\left(\frac{d}{d\lambda}b(H + \lambda P)\big|_\lambda b(H + P)^{-1}\right) = \tau(\delta(u)u^*)$, irrespective of λ. Applying Corollary A.35 and Remark A.38, we get for every i that

$$\widehat{\tau}\left((H - p_i)^{-1}P(H - p_i)^{-1}b(H)^{-1}\right) = \tau(P) \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(s - p_i)^2}\frac{1}{b(s)}\, ds.$$

As $b'(s) = \sum_{i=1}^{n} \frac{-\lambda_i}{(s-p_i)^2}$ and $P = i\tau(\delta(u^*)u)$, summing over i gives

$$\begin{aligned}\widehat{\tau}\left(\frac{d}{d\lambda} b(H+\lambda P)\Big|_{\lambda=0} b(H)^{-1}\right) &= \tau(P)\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b'(s)}{b(s)}\,ds \\ &= -\tau(\delta(u^*)u)\cdot(\text{winding number of } b) \\ &= \tau(\delta(u)u^*).\end{aligned}$$

3. Calculation of $\int_0^1 \widehat{\tau}\left(\frac{d\mathscr{C}_2}{dt}\mathscr{C}_2(t)^{-1}\right) dt$.

First, let us see why $\mathscr{C}_2 = W\left(\begin{smallmatrix} b(H) & 0 \\ 0 & 1\end{smallmatrix}\right)W^*\left(\begin{smallmatrix} b(H)^{-1} & 0 \\ 0 & 1\end{smallmatrix}\right)$ has values in $\mathrm{GL}(B_1^{\widehat{\tau}})$ and is C^1 with respect to the trace-class norm. Put $Z = R\left(\begin{smallmatrix} z & 0 \\ 0 & 0\end{smallmatrix}\right)R^*$ so that $W = 1 + Z^*$. Observe,

$$\begin{aligned} W\left(\begin{smallmatrix} b(H) & 0 \\ 0 & 1\end{smallmatrix}\right)W^* &= 1 + W\left(\begin{smallmatrix} \ell(H) & 0 \\ 0 & 0\end{smallmatrix}\right)W^* \\ &= \left(\begin{smallmatrix} b(H) & 0 \\ 0 & 1\end{smallmatrix}\right) + W\left(\begin{smallmatrix} \ell(H) & 0 \\ 0 & 0\end{smallmatrix}\right)W^* - \left(\begin{smallmatrix} \ell(H) & 0 \\ 0 & 0\end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} b(H) & 0 \\ 0 & 1\end{smallmatrix}\right) + (1+Z^*)\left(\begin{smallmatrix} \ell(H) & 0 \\ 0 & 0\end{smallmatrix}\right)(1+Z) - \left(\begin{smallmatrix} \ell(H) & 0 \\ 0 & 0\end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} b(H) & 0 \\ 0 & 1\end{smallmatrix}\right) + \left[Z^*\left(\begin{smallmatrix} \ell(H) & 0 \\ 0 & 0\end{smallmatrix}\right) + \left(\begin{smallmatrix} \ell(H) & 0 \\ 0 & 0\end{smallmatrix}\right)Z + Z^*\left(\begin{smallmatrix} \ell(H) & 0 \\ 0 & 0\end{smallmatrix}\right)Z\right].\end{aligned}$$

Since $z, \delta(z) \in A_1^{\tau}$ and ℓ is such that $s \mapsto s\cdot\ell(s)$ is in $C_0(\mathbb{R})\cap L^2(\mathbb{R})$, Proposition A.37 shows the bracketed term above is a curve in $\mathrm{M}_2(B_1^{\widehat{\tau}})$. Thus,

$$\mathscr{C}_2 = 1 + \left[Z^*\left(\begin{smallmatrix} \ell(H) & 0 \\ 0 & 0\end{smallmatrix}\right) + \left(\begin{smallmatrix} \ell(H) & 0 \\ 0 & 0\end{smallmatrix}\right)Z + Z^*\left(\begin{smallmatrix} \ell(H) & 0 \\ 0 & 0\end{smallmatrix}\right)Z\right]\left(\begin{smallmatrix} b(H)^{-1} & 0 \\ 0 & 1\end{smallmatrix}\right)$$

has values in $\mathrm{GL}_2(B_1^{\widehat{\tau}})$. Moreover, as the only dependence on the parameter t is in the smoothly varying scalar matrix R, $\mathscr{C}_2$ is clearly C^1 with respect to the trace class norm.

Another simple calculation shows that

$$\begin{aligned}\frac{d\mathscr{C}_2}{dt}\mathscr{C}_2^{-1} &= \left[\frac{dW}{dt}\left(\begin{smallmatrix} b(H) & 0 \\ 0 & 1\end{smallmatrix}\right)W^* - W\left(\begin{smallmatrix} b(H) & 0 \\ 0 & 1\end{smallmatrix}\right)W^*\frac{dW}{dt}W^*\right]W\left(\begin{smallmatrix} b(H)^{-1} & 0 \\ 0 & 1\end{smallmatrix}\right)W^* \\ &= W\left[W^*\frac{dW}{dt} - \left(\begin{smallmatrix} b(H) & 0 \\ 0 & 1\end{smallmatrix}\right)W^*\frac{dW}{dt}\left(\begin{smallmatrix} b(H)^{-1} & 0 \\ 0 & 1\end{smallmatrix}\right)\right]W^* \\ &= -W\left[\left(\begin{smallmatrix} b(H) & 0 \\ 0 & 1\end{smallmatrix}\right)X\left(\begin{smallmatrix} b(H)^{-1} & 0 \\ 0 & 1\end{smallmatrix}\right) - X\right]W^*\end{aligned}$$

where $X := W^*\frac{dW}{dt}$ has entries which are scalar multiplies of products of z and z^*. Put $k = -\frac{\ell}{1+\ell}$ so that $b^{-1} = 1 + k$. and observe that $s \mapsto s \cdot k(s)$ is in $C_0(\mathbb{R}) \cap L^2(\mathbb{R})$ as well. Then, the bracketed term above becomes

$$\left(\begin{smallmatrix} \ell(H) & 0 \\ 0 & 0\end{smallmatrix}\right)X + X\left(\begin{smallmatrix} k(H) & 0 \\ 0 & 0\end{smallmatrix}\right) + \left(\begin{smallmatrix} \ell(H) & 0 \\ 0 & 0\end{smallmatrix}\right)X\left(\begin{smallmatrix} k(H) & 0 \\ 0 & 0\end{smallmatrix}\right)$$

which, again by Proposition A.37, has values in $\mathrm{M}_2(B_1^{\widehat{\tau}})$. Thus, we get

$$\begin{aligned}\widehat{\tau}\left(\frac{d\mathscr{C}_2}{dt}\mathscr{C}_2^{-1}\right) &= -\widehat{\tau}\left(\left(\begin{smallmatrix} \ell(H) & 0 \\ 0 & 0\end{smallmatrix}\right)X + X\left(\begin{smallmatrix} k(H) & 0 \\ 0 & 0\end{smallmatrix}\right) + \left(\begin{smallmatrix} \ell(H) & 0 \\ 0 & 0\end{smallmatrix}\right)X\left(\begin{smallmatrix} k(H) & 0 \\ 0 & 0\end{smallmatrix}\right)\right) \\ &= -\widehat{\tau}\left(\left(\begin{smallmatrix} \ell(H) & 0 \\ 0 & 0\end{smallmatrix}\right)X + \left(\begin{smallmatrix} k(H) & 0 \\ 0 & 0\end{smallmatrix}\right)X + \left(\begin{smallmatrix} k(H) & 0 \\ 0 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} \ell(H) & 0 \\ 0 & 0\end{smallmatrix}\right)X\right) \\ &= 0\end{aligned}$$

where we used $1 = bb^{-1} = (1+\ell)(1+k)$.

□

By exploiting the compatibility of the Connes-Thom isomorphism with tensor product by $\mathrm{M}_n(\mathbb{C})$, one obtains from Theorem 7.1 the formula

$$\widehat{\tau}_* \circ \phi_\alpha^1 = \mathrm{ind}_\alpha^\tau$$

which was discussed in the introduction, where ind_α^τ is as in Theorem 3.11.

7.2 Applications

Recall that, when $(A, \mathbb{R}, \alpha)$ is a C*-dynamical system, we generally do not have an explicit description of the Connes-Thom isomorphism $\phi^1_\alpha : K_1(A) \to K_0(A \rtimes_\alpha \mathbb{R})$; Theorem 6.6 only describes the composite $\mathrm{s}^0_{A\rtimes_\alpha\mathbb{R}} \circ \phi^1_\alpha : K_1(A) \to K_1(\mathrm{S}A \rtimes_\alpha \mathbb{R})$. However, if τ is a densely-defined, lower semicontinuous α-invariant trace on A, and the the homomorphism $\mathrm{ind}^\tau_\alpha : K_1(A) \to \mathbb{R}$ happens to be injective, then $\widehat{\tau}_* : K_0(A \rtimes_\alpha \mathbb{R}) \to \mathbb{R}$ is injective too and the equality $\widehat{\tau}_* \circ \phi^1_\alpha = \mathrm{ind}^\tau_\alpha$ uniquely determines ϕ^1_α. Thus, in certain cases, the results of the preceding section enable us to calculate ϕ^1_α. We carry out this paradigm for two examples.

Example 7.2. Let α be the translation flow on $C_0(\mathbb{R})$ determined by $(\alpha_t f)(s) = f(s + t)$. In this case, $C_0(\mathbb{R}) \rtimes_\alpha \mathbb{R} \cong \mathbb{K}(L^2(\mathbb{R}))$, the C*-algebra of compact operators on $L^2(\mathbb{R})$. The Riemann integral is a densely-defined, lower semicontinous, α-invariant trace τ on $C_0(\mathbb{R})$. The associated homomorphism ind^τ_α is just the classical winding number, and carries $K^1(\mathbb{R})$ isomorphically onto $\mathbb{Z} \subset \mathbb{R}$. It is not difficult to see that, up to nonnegative scalar multiple, the only densely-defined, lower semicontinuous trace on $\mathbb{K}(L^2(\mathbb{R}))$ is the standard one. By the equality $\mathrm{ind}^\tau_\alpha = \widehat{\tau}_* \circ \phi^1_\alpha$, the range of $\widehat{\tau}_*$ is $\mathbb{Z}$, so the dual trace $\widehat{\tau}$ must be the standard trace with the standard normalization. It follows that $\phi^1_\alpha([b]) = [e]$ where $b \in \mathrm{GL}(C_0(\mathbb{R}))$ is a loop of winding number 1, and $e \in \mathbb{K}(L^2(\mathbb{R}))$ is a rank-1 projection, and this determines the Connes-Thom isomorphism $\phi^1_\alpha : K^1(\mathbb{R}) \to K_0(\mathbb{K}(L^2(\mathbb{R}))$ completely.

Example 7.3. Fix an irrational number θ and consider the Kronecker flow α on the C*-algebra $C(\mathbb{T}^2)$ of $\mathbb{Z}^2$-periodic functions on $\mathbb{R}^2$ determined by $(\alpha_t f)(x, y) = f(x + t, y + \theta t)$. Observe that the 2-dimensional Riemann integral is an α-invariant tracial state τ on $C(\mathbb{T}^2)$. It was shown in Section 3.4 that the associated homomorphism ind^τ_α carries $K^1(\mathbb{T}^2) = H^1(\mathbb{T}^2)$ isomorphically onto $\mathbb{Z} + \theta\mathbb{Z} \subset \mathbb{R}$, sending $[f_1] \mapsto 1$ and $[f_2] \mapsto \theta$ where $f_1, f_2 : \mathbb{T}^2 \to \mathbb{T}$ are the two coordinate projections. In Example 2.27, we sketched the construction of an explicit isomorphism between $A_\theta \otimes \mathbb{K}(L^2(\mathbb{T}))$ and $C(\mathbb{T}^2) \rtimes_\alpha \mathbb{R}$, where A_θ is the irrational rotation algebra. It is not difficult to show that, up to nonnegative scalar multiple, the only densely-defined, lower semicontinuous trace on $A_\theta \otimes K(L^2(\mathbb{T}))$ is the stabilization of the unique (see [28]) tracial state τ_θ on A_θ. In particular, the dual trace $\widehat{\tau}$ is one of these multiples. By the equality $\mathrm{ind}_\alpha^\tau = \widehat{\tau}_* \circ \phi_\alpha^1$, the range of $\widehat{\tau}_*$ is $\mathbb{Z} + \theta\mathbb{Z}$, but this does not suffice to determine the normalization of $\widehat{\tau}$. For instance, notice that $\frac{1}{\varphi} \cdot (\mathbb{Z} + \varphi\mathbb{Z}) = \mathbb{Z} + \varphi\mathbb{Z}$ when φ is the Golden ratio. Recall however that, a least formally, the isomorphism $A_\theta \otimes K(L^2(\mathbb{T})) \to C(\mathbb{T}^2) \rtimes_\alpha \mathbb{R}$ sketched in Example 2.27 sends $1 \otimes e_0 \mapsto F$, where e_0 is the rank-1 projection onto the constant function $1 \in L^2(\mathbb{T})$ and $F \in C(\mathbb{T}^2) \rtimes_\alpha \mathbb{R}$ "is" the function

$$F(t, x, y) = \begin{cases} 1 & \text{if } 0 \leq \{x\} - t < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Morally, $\widehat{\tau}(F) = \tau(F(0)) = \int_{\mathbb{T}^2} 1 \; dxdy = 1$. By writing $F = F^*F$ and using some approximation arguments, this can be made rigorous. Having determined the normalization, we now know $\widehat{\tau}$ is just the stabilization of τ_θ, as expected. If we assume for convenience that $0 < \theta < 1$, then we can completely determine the Connes-Thom isomorphism ϕ_α^1 from $K_1(C(\mathbb{T}^2)) = K^1(\mathbb{T}^2)$ to $K_0(C(\mathbb{T}^2) \rtimes_\alpha \mathbb{R}) = K_0(A_\theta \otimes \mathbb{K}) = K_0(A_\theta)$ as the map sending $[f_1] \mapsto [1]$ and $[f_2] \mapsto [p]$ where $f_i : \mathbb{T}^2 \to \mathbb{T}$ is the ith coordinate projection and $p \in A_\theta$ is a Powers-Rieffel projection [30] such that $\tau_\theta(p) = \theta$.

Chapter 8

Conclusion

Conclusion

This thesis investigated the relationship of 1-parameter flows and traces to the K-theory of a C*-algebra A. The overarching goal was to give a detailed and accessible account of the formula:

$$\tfrac{1}{2\pi i}\tau(\delta(u)u^{-1}) = \widehat{\tau}_*(\phi_\alpha^1([u])) \tag{8.1}$$

appearing in Theorem 3 of Connes' paper [6], Theorem 7.1 in our numbering. Above, δ is the derivation of A associated to a 1-parameter flow α, u is an appropriately chosen unitary, ϕ_α^1 is the Connes-Thom isomorphism $K_1(A) \to K_0(A \rtimes_\alpha \mathbb{R})$, and $\widehat{\tau}$ is the dual trace on $A \rtimes_\alpha \mathbb{R}$ induced from an α-invariant trace τ on A. The left hand side of (8.1), as explained in Chapter 3, is topological in nature, whereas the right hand side is analytic, resembling a Fredholm index. Connes' result was subsequently generalized in [10] to the pairing between K-theory and cyclic cohomology. In a sense, (8.1) is the the starting point of cyclic cohomology.

Realizing the above goal presented an interesting challenge. The arguments and constructions (indeed, even the definitions) underpinning (8.1) occasionally proved difficult to track down, leaving little recourse but to prove things from scratch. For example, a subtle domain issue in the proof of Theorem 7.1 stymied the author for months until the relation in Theorem 4.25 was found and used as a means to the technical result Proposition A.37. It is hoped that the considerable effort expended gathering the relevant details and definitions may be useful to other researchers. In addition, our investigations of (8.1) led to a number of interesting diversions, for example:

- In the course of setting up the machinery needed for Connes' Thom isomorphism, we classified all smooth unitary 1-cocycles of a given flow α using only differential equations methods (Theorem 4.4).

- We showed that Proposition 4 of [6] leads to a "modern" proof of a quantum mechanical theorem of Bargmann-Wigner (Theorem 4.29).

- Our discussion of the K-theoretic suspension isomorphisms led us to Theorem 5.15, a more refined form of the isomorphism $K_1(A) \to K_0(\mathrm{S}A)$, roughly speaking, an isomorphism at the semigroup level.

Questions

Finally, we gather a few questions whose answers the author would like to know.

- Let δ be a closed, densely-defined derivation of a C*-algebra A, *not*[1] of the form $\frac{d}{dt}\alpha_t(\cdot)\big|_{t=0}$ for any C*-algebra flow α. Can one construct a sensible "crossed product" $A \rtimes_\delta \mathbb{R}$ as the universal C*-algebra generated by A and $C_0(\mathbb{R})$ subject to the single relation $[f, a] = f\delta(a)f$ where $f(s) = \frac{i}{s+i}$ and $a \in \mathrm{dom}(\delta)$? See Section 4.4.

[1]Consider, for example, differentiation of C^1 functions in $C([0,1])$. For a set of necessary and sufficient conditions under which a derivation is a generator, see Theorem 3.2.50 in [3].

- Let τ be a densely-defined, lower semicontinuous trace on a nonunital C*-algebra A. If $x, y \in M(A)$ are such that both products xy and yx belong to A_1^τ, does then $\tau(xy) = \tau(yx)$? See Theorem A.27.

- For the construction of the dual trace $\widehat{\tau}$ associated to an invariant trace τ, we deferred to [9] and [26]. As the appeal to [9] represents the sole point where von Neumann algebra methods are relied upon in this thesis, it is natural to ask exactly how much von Neumann algebra theory, if any, is needed to construct $\widehat{\tau}$? See Section A.3.

- Suppose α is a C*-algebra flow on A and τ is an α-invariant densely-defined, lower semicontinuous trace on A. Let H be the Hamiltonian of the crossed-product $A \rtimes_\alpha \mathbb{R}$. If $f \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$ and $a \in A_1^\tau$, does it follow that $a \cdot f(H) \in (A \rtimes_\alpha \mathbb{R})_1^{\widehat{\tau}}$, where $\widehat{\tau}$ is the dual trace? See Proposition A.37.

Chapter A

Unbounded Traces on C*-algebras

In this appendix, we develop the theory of unbounded traces on C*-algebras. Much of what appears is cobbled together from the three sources [25], [24] and [26], though there is some independent work as well. An effort has been made to expunge usage of von Neumann algebra techniques, wherever possible.

A.1 Hereditary cones

A **hereditary cone** in a C*-algebra A is a nonempty subset P of the positive cone A_+ that is closed under the algebraic operations of addition and multiplication by nonnegative scalars, as well as closed downward in the order-theoretic sense. That is, whenever $x \in P$ and $0 \leq y \leq x$, then $y \in P$ as well. We associate to P the two collections

$A_1^P = \mathrm{span}(P)$, the (not necessarily closed) $\mathbb{C}$-linear span of P,

$A_2^P = \{x \in A : x^*x \in P\}$.

It is useful to keep some simple commutative example in mind, such as:

$$A = C_0(\mathbb{R}) \qquad P = C_0(\mathbb{R})_+ \cap L^1(\mathbb{R}) \qquad A_1^P = C_0(\mathbb{R}) \cap L^1(\mathbb{R}) \qquad A_2^P = C_0(\mathbb{R}) \cap L^2(\mathbb{R}).$$

Proposition A.1. *For any hereditary cone $P \subset A_+$, the following hold.*

1. A_2^P *is a left-ideal in* A.

2. $P \subset A_1^P \subset A_2^P$.

3. $\operatorname{span}\{x^*y : x, y \in A_2^P\} = A_1^P$

4. A_1^P *is a* $*$*-subalgebra of* A.

5. $(A_1^P)_+ = P$.

In particular, notice from (3) and (5) that P can be recovered from either A_1^P or A_2^P.

Proof.

1. For any $x, y \in A_2^P$ and $\lambda, \mu \in \mathbb{C}$, we have

$$|\lambda x + \mu y|^2 \leq |\lambda x + \mu y|^2 + |\lambda x - \mu y|^2 = 2|\lambda|^2|x|^2 + 2|\mu|^2|y|^2 \in P$$

   whence $|\lambda x + \mu y|^2 \in P$ by heredity proving that A_2^P is a linear subspace of A. Furthermore, if $x \in A, y \in A_2^P$, we have

$$(xy)^*(xy) = y^*(x^*x)y \leq \|x\|^2 y^*y \in P$$

   which shows A_2^P is a left-ideal.

2. If $x \in P$, then $x^2 \in P$ (consider, with no harm done, the case where $x \leq 1$ so that $x^2 \leq x$ by spectral theory). So, $P \subset A_2^P$. Then $A_1^P \subset A_2^P$ too since A_1^P is the smallest linear space containing P.

3. Since for any $x \in P$ we have $x^{\frac{1}{2}} \in A_2^P$, we have $P \subset \{x^*y : x, y \in A_2^P\}$ whence $A_1^P \subset \operatorname{span}\{x^*y : x, y \in A_2^P\}$. In the other direction, suppose that $x, y \in A_2^P$. Then

$x^*y = \frac{1}{4}\sum_{k=0}^{3} i^k|i^kx+y|^2$ where $|i^kx+y|^2 \le |i^kx+y|^2 + |i^kx-y|^2 = 2|x|^2+2|y|^2 \in P$ so that $|i^kx+y|^2 \in P$ by heredity. So, $x^*y \in A_1^P = \mathrm{span}(P)$ and we are finished.

4. Since the elements of P are positive, it is evident that $A_1^P = \mathrm{span}(P)$ is a $*$-invariant subspace of A. Closure under multiplication follows from (2) and (3): $A_1^P \cdot A_1^P = (A_1^P)^* \cdot A_1^P \subset (A_2^P)^* \cdot A_2^P \subset A_1^P$.

5. Obviously $P \subset (A_1^P)_+$. In the other direction, if $x = \sum \lambda_i x_i \in (A_1^P)_+$ with the $x_i \in P$, then $x = \frac{x+x^*}{2} = \sum \mathrm{Re}(\lambda_i)x_i \le \sum |\lambda_i| x_i \in P$ so that $x \in P$ by heredity.

□

We shall say that a hereditary cone $P \subset A$ is **unitarily invariant** if $uPu^* \subset P$ for every unitary $u \in \widetilde{A}$. Some equivalent characterizations are given below.

Proposition A.2. *For a hereditary cone $P \subset A_+$, the following are equivalent.*

1. *$a^*Pa \subset P$ for every $a \in \widetilde{A}$.*
2. *$u^*Pu \subset P$ for every unitary $u \in \widetilde{A}$ (P is unitarily invariant)*
3. *A_1^P is a 2-sided ideal in A.*

Moreover, when these conditions hold, A_2^P is a 2-sided ideal in A as well[1].

Proof. Obviously (1) ⇒ (2).

(2) ⇒ (3): Since A_1^P is $*$-invariant, it is enough to check that A_1^P is a, say, left ideal. In fact, it is enough to show that $u^*a \in A_1^P$ whenever $u \in \widetilde{A}$ is unitary and $a \in A_1^P$ (because every element in $\widetilde{A}$ is a sum of unitaries). By polarizing the sesquilinear map $(x, y) \mapsto x^*ay$, we establish

$$u^*a = u^*a1 = \frac{1}{4}\sum_{k=0}^{4}(i^ku+1)^*a(i^ku+1).$$

[1] Possibly, A_2^P being a 2-sided ideal also belongs on this list of equivalent conditions as well, but I did not pursue this question.

But now,

$$\begin{aligned}(i^k u+1)^* a(i^k u+1) &\leq (i^k u+1)^* a(i^k u+1)+(i^k u-1)^* a(i^k u-1)\\ &= 2(i^k u)^* a(i^k u)+2\cdot 1^* a 1\\ &= 2u^* a u+2a\in P\end{aligned}$$

and, applying heredity, it follows that $u^*a \in A_1^P$ as desired.

(3) ⇒ (1): Suppose $a \in \widetilde{A}$, $x \in P$. Since $P \subset A_1^P$, we get $a^*xa \in A_1^P$ by assumption that A_1^P is an ideal. But, a^*xa is also positive and $P = (A_1^P)_+$, so $a^*xa \in P$.

For the "moreover" statement, we just need to check A_2^P is a right ideal (since it is always a left ideal). Suppose that $x \in A_2^P$ and $y \in A$. Since $x \in A_2^P$, by definition $x^*x \in P$. Then, by (3), $(xy)^*(xy) = y^*(x^*x)y$ is in P as well i.e. $xy \in A_2^P$. □

The next proposition can be viewed as a supplement to Proposition 5.2.2 in [24] (misnumbered in my copy as Proposition 5.5.2) in that it justifies the existence of the approximate unit appearing in the author's proof.

Proposition A.3. *If hereditary cone P is unitarily invariant, and dense in A_+, then:*

1. *For all $a \in A_+$ and all nonnegative functions f with compact support contained in $(0, \infty)$, one has $f(a) \in P$.*

2. *P contains an approximate unit. That is, there is an increasing net $(u_\lambda)_{\lambda\in\Lambda}$ in P such that $0 \leq u_\lambda \leq 1$ and such that $u_\lambda x \to x$ for all $x \in A$.*

Proof. Let a, f be as in (1). Find $g \in C_c(0, \infty)_+$ such that $g = 1$ on the support of f. Choose $\epsilon > 0$ small (any $\epsilon < 1$ will do) and find $y \in P$ such that $\|y - g(a)\| \leq \epsilon$. Then,

$$f(a) = f(a)^{1/2}(g(a) - y)f(a)^{1/2} + f(a)^{1/2}yf(a)^{1/2} \leq \epsilon \cdot f(a) + f(a)^{1/2}yf(a)^{1/2}$$

so that $f(a) \leq \frac{1}{1-\epsilon} \cdot f(a)^{1/2} y f(a)^{1/2} \in P$, whence $f(a) \in P$ by heredity.

Before proceeding to the proof of (2), recall the operation of *quasi-inversion*:

$$x \mapsto x^{\mathrm{qi}} := 1 - (1-x)^{-1} = x(x-1)^{-1}.$$

which is an involution of the $x \in A$ with $1 \notin \operatorname{spec}(x)$. By basic spectral considerations, quasi-inversion furthermore exchanges $\{x \in A : 0 \leq x < 1\}$ with $\{x \in A : -\infty < x \leq 0\}$ in an order-reversing manner.[2]

We take $\Lambda = \{u \in P : \|u\| < 1\}$ so the net is its own index set. In particular, the net is vacuously increasing. To see Λ is directed we use the fact that A_1^P is closed under quasi-inversion which follows from the formula for x^{qi} and the fact that A_1^P is an ideal in $\widetilde{A}$. Suppose $u, v \in \Lambda$. Since $0 \leq u, v < 1$, we have $-\infty < u^{\mathrm{qi}} + v^{\mathrm{qi}} \leq u^{\mathrm{qi}}, v^{\mathrm{qi}} \leq 0$ whence $0 \leq u, v \leq (u^{\mathrm{qi}} + v^{\mathrm{qi}})^{\mathrm{qi}} < 1$.

Now, in checking that $\lim_{u \in \Lambda} ux = x$, it suffices to consider $x \in A_+$, since such x span A. One has

$$0 \leq (x - ux)^*(x - ux) = x(1-u)^2 x \leq x(1-u)x$$

so it suffices to show $x(1-u)x$ approaches 0 as u increases. The latter expression has the virtue of being decreasing in u (recall that squaring does not generally preserve the order on A_+). So it suffices to show there exist choices of $u \in \Lambda \subset P$ which make $x(1-u)x$ small. One just finds $f \in C_c(0,\infty)_+$ so that $|f| < 1$ and $t^2(1 - f(t))$ is uniformly small over the spectrum of x. Then $u = f(x) \in \Lambda$ does the job. □

[2]Some trivia: the map $x \mapsto x(x-1)^{-1}$ is also the one which produces the "dual exponent" in the context of L^p spaces. It interchanges the conditions $1 < x \leq 2$ and $2 \leq x < +\infty$.

A.2 Unbounded traces

Definition A.4. A **weight** on a C*-algebra A is a map $\phi : A_+ \to [0, \infty]$ such that $\phi(x+y) = \phi(x) + \phi(y)$ and $\phi(\lambda x) = \lambda\phi(x)$ for all $x, y \in A_+$ and $\lambda \in [0, \infty)$.

Observe that a weight is necessarily increasing: if $0 \leq x \leq y$ then $\phi(x) \leq \phi(x)+\phi(y-x) = \phi(y)$. Notice that $\{x \in A_+ : \phi(x) < \infty\}$ is a hereditary cone[3] in A naturally associated to ϕ. Mimicking the notation of the previous section, we associate the following "domains" to ϕ.

$$(A_1^\phi)_+ = \{x \in A_+ : \phi(x) < \infty\} \qquad A_1^\phi = \operatorname{span}((A_1^\phi)_+) \qquad A_2^\phi = \{x \in A : \phi(x^*x) < \infty\}$$

The introduction of $(A_1^\phi)_+$ before A_1^ϕ may seem rather "cart before the horse" notationally speaking but, considering Proposition A.1 part (5), the designation makes good sense. It is not difficult to see that ϕ extends[4] uniquely to a linear functional on A_1^ϕ. We frequently conflate ϕ with this extension so that A_1^ϕ is also a natural domain of ϕ. On the other hand, ϕ need not be defined on A_2^ϕ. Rather, one should view A_2^ϕ as a semi-inner product space with positive sesquilinear form $(x, y) \mapsto \phi(x^*y)$.

Definition A.5. A weight ϕ is **lower semi-continuous** if $\{a \in A_+ : \phi(a) \leq t\}$ is closed in A_+ for every $t \in [0, \infty)$. A weight ϕ is called **densely-defined** if A_1^ϕ is dense in A (or, equivalently, if $(A_1^\phi)_+$ is dense in A_+).

Lower semicontinuous, densely-defined weights are a, rather coarse, noncommutative analogue for the Radon measures of locally compact measure theory, as the following examples make clear.

[3]Trivially, every hereditary cone $P \subset A_+$ arises this way; the function which is 0 on P and ∞ on the rest of A_+ is a weight.

[4]It is perhaps cleanest to first extend ϕ to a real linear functional on $(A_1^\phi)_{\mathrm{sa}}$

Example A.6. If X is a locally compact Hausdorff space and μ is a regular, Borel measure[5] on X, then $\phi_\mu(f) = \int_X f \, d\mu$, $\forall f \in C_0(X)_+$, defines a lower-semicontinuous, densely-defined weight on $C_0(X)$. Every lower semicontinuous, densely-defined weight on $C_0(X)$ is uniquely of this form.

Example A.7. Define $\phi : C_0(\mathbb{R})_+ \to [0, \infty]$ by

$$\phi(f) = \begin{cases} \int_{-\infty}^{\infty} f(s) \, ds & \text{if f has compact support} \\ \infty & \text{otherwise} \end{cases}$$

This is a densely-defined weight on $C_0(\mathbb{R})$, but not a lower semicontinuous one. Given $f \in C_0(\mathbb{R})_+ \cap L^1(\mathbb{R})$ not compactly supported, it is easy to give a sequence $f_n \in C_c(\mathbb{R})_+$ converging uniformly to f from below. And yet, one has $\phi(f) = \infty > \int_{\infty}^{\infty} f(s) \, ds = \lim_{n\to\infty} \tau(f_n)$.

Lower semicontinuous weights can be thought of as the weights such that Fatou's lemma holds.

Proposition A.8 (Fatou's lemma). *For a weight ϕ on a C^*-algebra A, the following conditions are equivalent:*

1. *ϕ is lower semi-continuous.*

2. *For every norm convergent net $x_\lambda \to x$ in A_+, one has $\phi(x) \leq \liminf_\lambda \phi(x_\lambda)$ where, by definition, $\liminf_\lambda \phi(x_\lambda) = \sup_{\lambda\in\Lambda} \inf_{\alpha\geq\lambda} \phi(x_\alpha)$.*

3. *For every norm convergent sequence $x_n \to x$ in A_+, one has $\phi(x) \leq \liminf_n \phi(x_n)$.*

Proof. Suppose (1) holds and $x_\lambda \to x$, with an eye towards proving (2). With no harm done, $\phi(x) < \infty$. By (1), $U_\epsilon = \{y \in A_+ : \phi(y) > \phi(x) - \epsilon\}$ is an open neighbourhood of x for every

[5] That is, a positive measure μ defined on the Borel σ-algebra of X such that (1) $\mu(K) < \infty$ when K is compact, (2) $\mu(E) = \sup_{K\subset E} \mu(K)$, K compact, when E is Borel and $\mu(E) < \infty$ (3) $\mu(E) = \inf_{U\supset E} \mu(U)$, U open, when E is Borel.

$\epsilon > 0$. Since $x_\lambda \to x$, U_ϵ captures every tail of the net and it follows that $\liminf x_\lambda \geq \phi(x) - \epsilon$. Taking $\epsilon \to 0$ establishes (2). Clearly (2) implies (3). Finally, suppose (3) holds and fix $t \in [0, \infty)$. Let x be a point in the norm closure of $\{a \in A_+ : \phi(a) \leq t\}$. Then, there is a sequence x_n converging to x with $\phi(x_n) \leq t$ for all n. By (3), $\phi(x) \leq \liminf \phi(x_n) \leq t$ and (1) is established. □

As a corollary, lower semi-continuous weights satisfy a Beppo Levi-type result.

Corollary A.9 (Beppo Levi's theorem)**.** *If $\phi : A_+ \to [0, \infty]$ is a lower semi-continuous weight, and $x_\lambda \to x$ is a norm-convergent net in A_+ satisfying $x_\lambda \leq x$ for all λ (in particular, there is the case of an increasing net), then $\phi(x_\lambda) \to \phi(x)$.*

Proof. For all λ, $\phi(x_\lambda) \leq \phi(x)$, so $\limsup \phi(x_\lambda) \leq \phi(x) \leq \liminf \phi(x_\lambda)$. □

Our interest in weights is only incidental. We now specialize to traces[6].

Definition A.10. A **trace** τ on a C*-algebra A is a weight such that $\tau(u^*xu) = \tau(x)$ for every $x \in A_+$ and every unitary $u \in \widetilde{A}$.

This is the definition used in [26]. The next proposition shows this definition is at least as permissive as another conventional one, see 6.1.1 in [8].

Proposition A.11. *If τ is a weight such that $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$, then $\tau(w^*xw) = \tau(x)$ for all unitary multipliers w and all $x \in A_+$. In particular, τ is a trace.*

Proof. Let τ be as above. Take x a positive element in A and w a unitary element in $M(A)$. Then, writing $x = a^*a$, for some $a \in A$, we get

$$\tau(w^*xw) = \tau((aw)^*(aw)) = \tau(aw(aw)^*) = \tau(aa^*) = \tau(a^*a) = \tau(x)$$

as desired. □

[6]For commutative C*-algebras, there is no difference i.e. one still has "densely-defined, lower semicontinuous trace" ↔ "Radon measure".

For densely-defined, lower semicontinuous traces, which are the only pedigree of any interest in this thesis, the two competing definitions agree.

Lemma A.12. *Let τ be a densely-defined, lower semicontinuous trace, viewed as linear functional on the dense, self-adjoint ideal A_1^τ (see Propositions A.1, A.2). Then, for any $x \in A_1^\tau, y \in A$, one has $\tau(xy) = \tau(yx)$.*

Proof. Since unitaries span $\widetilde{A}$, it suffices to check the claim when $y = u$ is a unitary in $\widetilde{A}$. Then, $\tau(ux) = \tau(u^*(ux)u) = \tau(xu)$. □

In fact, the above Lemma can generalized slightly so as to allow $y \in M(A)$. The point is that x can be factored as $x = x_1 x_2$ where $x_1 \in A_1^\tau$ and $x_2 \in A$. To see this, apply the Cohen factorization theorem to the action of A on A_1^τ. This argument uses Proposition A.19.

Proposition A.13. *Let $\tau : A_+ \to [0, \infty]$ be a densely-defined, lower semicontinuous weight on A. Then, the following are equivalent:*

1. *$\tau(a^*a) = \tau(aa^*)$ for all $a \in A$.*

2. *$\tau(wxw^*) = \tau(a)$ for all $a \in A_+$, and all unitaries $w \in M(A)$.*

3. *τ is a trace.*

Proof. We have seen that (1) implies (2) implies (3), even for general weights. Hence, we assume (3) holds and show $\tau(aa^*) \leq \tau(a^*a)$, which suffices to prove (1). By Proposition A.3, there is an approximate unit u_λ for A whose terms are in $(A_1^\tau)_+$. We have

$$\tau(aa^*) \leq \liminf \tau(u_\lambda a(u_\lambda a)^*) = \liminf \tau(a^* u_\lambda^2 a)$$

where the inequality follows from lower semicontinuity and the equality follows from Lemma A.12. As $a^* u_\lambda^2 a \leq a$ for all λ, Corollary A.9 says $\tau(a^* u_\lambda^2 a) \to \tau(a^*a)$, so we are finished. □

If Lemma A.12 is an assertion about "trace-class" elements, then the following is the corresponding assertion about "Hilbert-Schmidt" elements.

Corollary A.14. *Let τ be a densely-defined, lower semicontinuous trace. Then, for all x, y in the dense, self-adjoint ideal A_2^τ (see Propositions A.1, A.2), one has $\tau(xy) = \tau(yx)$.*

Proof. Essentially, we want to show that two positive sesquilinear forms

$$(x, y) \mapsto \tau(x^* y) \qquad (x, y) \mapsto \tau(yx^*) \qquad A_2^\tau \times A_2^\tau \to \mathbb{C}$$

are equal. By polarization, one need only show the corresponding quadratic forms

$$x \mapsto \tau(x^* x) \qquad x \mapsto \tau(xx^*) \qquad A_2^\tau \to [0, \infty)$$

agree, but this follows from Proposition A.13 above. □

We use the following observation from [26], the proof of which is a simple polynomial approximation argument. If $a \in A$ where A is a C*-algebra embedded in $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and $a = w|a|$ is the polar decomposition of a in $B(\mathcal{H})$, then $wf(|a|) \in A$ for any $f \in C_0(0, \infty)$. The reason for the fuss, of course, is that w may well not belong to A. This observation leads us to the following factoring lemma.

Lemma A.15. *For every C*-algebra A, for every $a \in A$, there is a factorization $a = a_1|a|^{1/2}$ where $a_1 \in A$ is such that $a_1^* a_1 = |a|$ and $a_1 a_1^* = |a^*|$.*

Proof. By the Gelfand-Naimark theorem, we may assume A sits in $B(\mathcal{H})$ for some Hilbert space. Polar decompose a as $a = w|a|$ where $w \in B(\mathcal{H})$ a uniquely determined partial isometry mapping the closed range of $|a|$ isometrically onto the closed range of a. Note $a_1 = w|a|^{1/2} \in A$. By design, $a = a_1|a|^{1/2}$. Moreover, $a_1^* a_1 = |a|^{1/2}(w^* w)|a|^{1/2} = |a|$ and $(a_1 a_1^*)^2 = a_1|a|a_1^* = w|a||a|w^* = aa^*$ so that $a_1 a_1^* = |a^*|$. □

Let us digress to point out a cute corollary. Note the obviousness of the claim for C*-algebras that are closed under polar decomposition.

Corollary A.16. *For every C*-algebra A, the relation $\sim$ on A_+ defined by $x \sim y$ if and only if there exists $a \in A$ such that $x = a^*a$ and $y = aa^*$ is an equivalence relation.*

Proof. Symmetry and reflexivity follow trivially replacing a by a^* and using the existence of positive square roots. The subtle point is transitivity. Suppose that $x \sim y$ and $y \sim z$. That is, there are $a, b \in A$ with

$$x = a^*a \qquad aa^* = y = b^*b \qquad z = bb^*.$$

Using the preceding proposition, factor a, b as $a = a_1|a|^{1/2}$, $b = b_1|b|^{1/2}$ where

$$x^{1/2} = a_1^*a_1 \qquad a_1a_1^* = y^{1/2} = b_1^*b_1 \qquad z^{1/2} = b_1b_1^*.$$

Put $c = b_1a_1$ and observe

$$c^*c = (b_1a_1)^*(b_1a_1) = a_1^*(b_1^*b_1)a_1 = a_1^*(a_1a_1^*)a_1 = x^{1/2}x^{1/2} = x$$

$$cc^* = (b_1a_1)(b_1a_1)^* = b_1(a_1a_1^*)b_1^* = b_1(b_1^*b_1)b_1^* = z^{1/2}z^{1/2} = z$$

so that $x \sim z$. □

The equivalence relation just introduced is an extension of Murray-von Neumann equivalence from the projections, to the whole of the positive cone. Note the resemblance to Cuntz-Pedersen equivalence, set down in [7]. A nice application is to the following corollary.

Corollary A.17. *If τ is a densely-defined, lower semicontinuous trace on A, and $a \in A$, then $\tau(|a|) = \tau(|a^*|)$.*

Proof. By Proposition A.13, the densely-defined, lower semicontinuous weights which are traces are exactly the ones which factor through the equivalence $\sim$. By Lemma A.15, one has $|a| \sim |a^*|$, so we are done. □

The below proposition is, by and large, a reproduction of Proposition A1 in [26] and its sneaky proof.

Proposition A.18. *Let τ be a densely-defined, lower semicontinuous trace.*

1. *If $a \in A_1^\tau$, then $|\tau(x)| \leq \tau(|x|)$.*

2. *$\|a\|_1 := \tau(|a|)$ defines a seminorm on A_1^τ.*

3. *$\{a \in A : \tau(|a|) < \infty\} = \{xy : x, y \in A_2^\tau\} = A_1^\tau$ (Recall that, by definition, A_1^τ equals the linear span of all $a \in A_+$ with $\tau(a) < \infty$).*

Proof. We first prove the bound in (1) holds for $a \in A$ with $\tau(|a|) < \infty$. Thus, (1) as it is stated will follow when (3) is established. Observe that, if $a \in A$ has $\tau(|a|) < \infty$, then the factorization $a = a_1|a|^{1/2}$ of Lemma A.15 is into elements of A_2^τ (since $a_1^* a_1 = |a|$). Thus

$$|\tau(a)| = |\tau(a_1|a|^{1/2})| \leq \tau(a_1 a_1^*)^{1/2} \tau(|a|)^{1/2} = \tau(a_1^* a_1)^{1/2} \tau(|a|) = \tau(|a|)$$

where the inequality is an application of the Cauchy-Schwartz inequality for the positive, sesquilinear form $(x, y) \mapsto \tau(xy^*) : A_2^\tau \times A_2^\tau \to \mathbb{C}$.

The positive homogeneity needed for (2) is clear. That this seminorm is finite will follow from (3). So, to prove (2) we just need to take $x, y \in A_1^\tau$ and establish the triangle inequality. With no harm done, suppose $\tau(|x|), \tau(|y|) < \infty$ and as well that $\|x + y\| \leq 1$. Polar decompose $x + y$ in some ambient operator algebra as $x + y = w|x + y|$ (and so also

$|x+y| = w^*(x+y)$). Now, for any $\epsilon > 0$, write

$$|x+y|^{1+\epsilon} = |x+y|^{\epsilon} w^*(x+y) = \underbrace{(w|x+y|^{\epsilon})^*}_{z_\epsilon}(x+y) = z_\epsilon x + z_\epsilon y$$

where $z_\epsilon \in A$ (so also $z_\epsilon x, z_\epsilon y \in A_1^\tau$) and $\|z_\epsilon\| \leq 1$. Note that $|z_\epsilon x|^2 \leq \|z_\epsilon\|^2 |x|^2 \leq |x|^2$ and so, by operator monotonicity of square root, $|z_\epsilon x| \leq |x|$. Similarly, $|z_\epsilon y| \leq |y|$. So, one gets the estimate

$$\tau(|x+y|^{1+\epsilon}) = |\tau(z_\epsilon x) + \tau(z_\epsilon y)| \leq |\tau(z_\epsilon x)| + |\tau(z_\epsilon y)| \leq \tau(|z_\epsilon x|) + \tau(|z_\epsilon y|) \leq \tau(|x|) + \tau(|y|).$$

Taking $\epsilon \to 0$ and applying upper semicontinuity of τ establishes the desired bound.

Finally, we prove (3). The containment $\{x \in A : \tau(|x|) < \infty\} \subset \{xy : x, y \in A_2^\tau\}$ is immediate from Lemma A.15. The containment $\{xy : x, y \in A_2^\tau\} \subset A_1^\tau$ holds because, for x, y in the $*$-algebra A_2^τ, one may apply polarization to x^*y. The bulk of our effort, then, is concentrated in showing that $A_1^\tau \subset \{x \in A : \tau(|x|) < \infty\}$. Luckily, since $A_1^\tau = \text{span}\{x \in A_+ : \tau(x) < \infty\}$ by definition, every $a \in A_1^\tau$ is trivially a sum of $x \in A_1^\tau$ with $\tau(|x|) < \infty$ so triangle inequality in the preceding paragraph gives the result that $\tau(|a|) < \infty$. □

It is easy to see that, if τ is a densely-defined, lower semicontinuous trace, then A_1^τ is a normed $*$-algebra in the norm $\|a\|_\tau := \|a\| + \tau(|a|)$, with Corollary A.17 showing the involution is isometric. In fact, completeness also holds. This is exactly Proposition A4 in [26]. As we have nothing to add to the authors' proof, we simply quote this result.

Proposition A.19. *Let τ be a densely-defined, lower semicontinuous trace on a C^*-algebra A. Then A_1^τ is a Banach $*$-algebra in the norm $\|x\|_\tau := \|x\| + \tau(|x|)$.*

Remark A.20. Lest the trivial go overlooked, we hasten to point out that, by Proposition A.18 (1), if τ is a densely-defined, lower semicontinuous trace, then τ belongs to the continuous dual of the Banach $*$-algebra A_1^τ.

We now record a useful analogue of Lebesgue's dominated convergence theorem. To be sure, a version of this could have been stated and proved for lower semicontinuous weights and put with its brethren Proposition A.8 (Fatou's lemma) and Corollary A.9 (Beppo Levi's theorem). We chose, however, not to do this.

Proposition A.21 (Lebesgue dominated convergence)**.** *Let τ be a densely-defined, lower semicontinuous trace on a C^*-algebra A. Let $x_\lambda \to x$ be a norm-convergent net in A. If there exists $b \in A_1^\tau$ such that $x_\lambda^* x_\lambda \leq b^* b$ for all λ, then x and all the x_λ belong to A_1^τ, and $x_\lambda \to x$ in the norm of A_1^τ. In particular, $\tau(x_\lambda) \to \tau(x)$.*

Proof. Of course, $x^* x \leq b^* b$ too by continuity. By operator monotonicity of the square root $|x|, |x_\lambda| \leq |b|$, proving $x, x_\lambda \in A_1^\tau$. Observe

$$(x - x_\lambda)^*(x - x_\lambda) \leq (x - x_\lambda)^*(x - x_\lambda) + (x + x_\lambda)^*(x + x_\lambda) = 2x_\lambda^* x_\lambda + 2x^* x \leq 4b^* b$$

so that $|x - x_\lambda| \leq 2|b|$. Thus, $0 \leq 2|b| - |x - x_\lambda| \leq 2|b|$. Applying Corollary A.9, we get that $2\tau(|b|) - \tau(|x - x_\lambda|) = \tau(2|b| - |x - x_\lambda|) \to 2\tau(|b|)$ so that $\tau(|x - x_\lambda|) \to 0$. ☐

An application is part (1) of the following proposition.

Proposition A.22. *Let τ be a densely-defined, lower semicontinuous trace on a C^*-algebra A, and let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate unit*[7] *in A.*

1. *If $x \in A_1^\tau$, then $\|x - e_\lambda x\|_1 \to 0$ where $\|y\|_1 := \tau(|y|)$.*

2. *If $x \in A_2^\tau$, then $\|x - e_\lambda x\|_2 \to 0$ where $\|y\|_2 := \sqrt{\tau(y^* y)}$.*

Proof.

1. Since $(e_\lambda x)^*(e_\lambda x) = x^* e_\lambda^2 x \leq x^* x$ and $x \in A_1^\tau$, we are done by Proposition A.21.

2. One has $|x - e_\lambda x|^2 = x^*(1 - e_\lambda)^2 x \leq x^*(1 - e_\lambda)x = x^* x - x^* e_\lambda x$ so that $\tau(|x - e_\lambda x|^2) \leq \tau(x^* x) - \tau(x^* e_\lambda x)$. By Corollary A.9, $\tau(x^* e_\lambda x) \to \tau(x^* x)$.

[7] That is, an increasing net in A_+ with $0 \leq e_\lambda \leq 1$ such that $e_\lambda x \to x$ in norm for every $x \in A$.

□

Since the ideal A_1^τ contains an approximate unit by Proposition A.1, we get

Corollary A.23. *If τ is a densely-defined, lower semicontinuous trace on a C^*-algebra A, then $A_1^\tau \subset A_2^\tau$ densely with respect to the norm $\|\cdot\| + \|\cdot\|_2$ of A_2^τ.*

If τ is a densely-defined, lower semicontinuous trace, we write $\mathcal{H}^\tau$ for the Hilbert space completion of the semi-inner product space A_2^τ. We denote by $x \mapsto \xi_x$ the dense mapping (injective, if τ is faithful) $A_2^\tau \to \mathcal{H}^\tau$. It is easy to see that A has a GNS representation π^τ on $\mathcal{H}^\tau$ determined by $\pi^\tau(a)\xi_x = \xi_{ax}$ for all $a \in A$, $x \in A_2^\tau$. As a quick corollary to Proposition A.22 above, we have

Lemma A.24. *The GNS representation corresponding to a densely-defined, lower semicontinuous trace is nondegenerate.*

We use the above corollary to prove the analogue for A_2^τ of Proposition A.19. Almost surely, Proposition A.25 is known, but, since a reference could not be located, a proof is given just the same.

Proposition A.25. *Let τ be a densely-defined, lower semicontinuous trace on A. Then, A_2^τ is complete in the norm $x \mapsto \|x\| + (\tau(x^*x))^{1/2}$. In different words, $x \mapsto \xi_x : A_2^\tau \to \mathcal{H}^\tau$ is a closed transformation.*

Proof. We first prove $x \mapsto \xi_x$ is closable, and then prove it equals its closure. To see it is closable, assume x_n is a sequence in A_2^τ such that $\|x_n\| \to 0$ and such that ξ_{x_n} converges to some vector $\xi \in \mathcal{H}^\tau$. We should show that, necessarily, $\xi = 0$. The GNS representation being nondegenerate, we just need to check $\pi^\tau(A) \cdot \xi = 0$. Indeed, as A_2^τ is dense in A (it contains A_1^τ), we just need to check $\pi^\tau(y)\xi = 0$ for arbitrary $y \in A_2^\tau$. We have $\|\pi^\tau(y)\xi\|_2 = \lim_{n\to\infty} \|\pi^\tau(y)\xi_{x_n}\|_2$ where

$$\|\pi^\tau(y)\xi_{x_n}\|_2^2 = \tau((yx_n)^*yx_n) = \tau(yx_n(yx_n)^*) \leq \|x_n\|^2\tau(yy^*) \to 0$$

and so $\xi = 0$ and $x \mapsto \xi_x$ is indeed closable.

Next, we show the domain of the closure is no larger. Indeed, suppose that x is in the domain of the closure. Thus, there are $x_n \in A_2^\tau$ such that $x_n \to x$ in A while the ξ_{x_n} converge to $\xi \in \mathcal{H}^\tau$. But then, $x_n^* x_n \to x^* x$ so that, by the lower semicontinuity,

$$\tau(x^*x) \leq \lim \tau(x_n^* x_n) = \lim \|\xi_{x_n}\|_2^2 = \|\xi\|_2^2 < \infty$$

whence $x \in A_2^\tau$. □

The above proposition gives a means of proving particular elements of A belong to A_2^τ. Typically, one is only certain of some elementary elements in A_2^τ, but, using Proposition A.25, one can recover more via limits. In combination with the factorization $A_1^\tau = A_2^\tau \cdot A_2^\tau$, one can also produce elements of A_1^τ.

At this point, we are essentially finished with the technical development of the theory of densely-defined, lower semicontinuous traces. As a sort of "survey result" we prove such traces extend over matrices in precisely the manner one would expect.

Proposition A.26. *Let τ be a densely-defined, lower semicontinuous trace on A. Then, there is a densely-defined, lower semicontinuous trace τ_n on $\mathrm{M}_n(A)$ satisfying*

$$\tau_n(x) = \sum_{i=1}^{n} \tau(x_{ii}) \qquad \forall x \in \mathrm{M}_n(A)_+$$

$$M_n(A)_1^{\tau_n} = M_n(A_1^\tau) \qquad M_n(A)_2^{\tau_n} = M_n(A_2^\tau)$$

Proof. If $x \in M_n(A)_+$, then the diagonal entries of x are positive (use, for instance, that $x = y^*y$ for some $y \in M_n(A)$), so its easy to see that τ_n is a weight. Lower semicontinuity of τ_n follows from lower semicontinuity of τ and fact that norm convergence in $M_n(A)$ coincides with entrywise norm convergence in A. Given $x \in M_n(A)$, we have

$$\tau_n(x^*x) = \sum_{ij} \tau(x_{ij}^* x_{ij}) \tag{A.1}$$

and so $x^*x \in (M_n(A)_1^{\tau_n})_+ \Leftrightarrow \sum_{ij} x_{ij}^* x_{ij} \in (A_1^\tau)_+ \Leftrightarrow x_{ij} \in (A_1^\tau)_+ \forall i,j$. Thus, $M_n(A)_2^{\tau_n} = M_n(A_2^\tau)$. Now, since A_2^τ is dense in A we have $M_n(A)_2^{\tau_n}$ dense in $M_2(A)$ so that $M_n(A)_1^\tau = \text{span}\{x^*y : x, y \in M_n(A)_2^{\tau_n}\}$ (see Proposition A.1 (3)) is dense, which proves τ_n is densely-defined. Now, using Equation A.1 above, we see that $\tau_n(x^*x) = \tau_n(xx^*)$ for all $x \in M_n(A)$ so that, by Proposition A.13, τ_n is a densely-defined, lower semicontinuous trace.

It remains only to check $M_n(A)_1^{\tau_n} = M_n(A_1^\tau)$. One the one hand, we have

$$\mathrm{M}_n(A)_1^{\tau_n} = \mathrm{M}_n(A)_2^{\tau_n} \cdot \mathrm{M}_n(A)_2^{\tau_n} = \mathrm{M}_n(A_2^\tau) \cdot \mathrm{M}_n(A_2^\tau) \subset \mathrm{M}_n(A_1^\tau),$$

using Proposition A.18 (3) and the fact that $M_n(A)_2^{\tau_n} = M_n(A_2^\tau)$. In the other direction, fix some $a \in A_1^\tau$ and some index ij. We show that the matrix $x \in M_n(A)$ with an a in the ij and zeros elsewhere is in $M_n(A)_1^{\tau_n}$, which completes the proof. Indeed, $|x|$ has $|a|$ in the jj spot and zeros elsewhere, so we are finished by A.18 part (3). □

We have shown, thus far, that the following identities are satisfied by a densely-defined, lower semicontinuous trace τ on a C*-algebra A.

Property:	Domain assumptions:
$\tau(xx^*) = \tau(x^*x)$	$x \in A$
$\tau(\lvert x\rvert) = \tau(\lvert x^*\rvert)$	$x \in A$
$\tau(xy) = \tau(yx)$	$x \in A_1^\tau$, $y \in M(A)$
$\tau(xy) = \tau(yx)$	$x, y \in A_2^\tau$

Table A.1: Catalogue of tracial identities.

We conclude this section by proving one further result in this direction.

Theorem A.27. *Let τ be a densely-defined, lower semicontinuous trace on a C^*-algebra A. If $a, b \in A$ are such that both products ab and $ba \in A$ are in A_1^τ, then $\tau(ab) = \tau(ba)$ holds.*

The above result settles many questions about unbounded traces and their values on cyclic shifts, but we do leave the following question.

Question. *Let τ be a densely-defined, lower semicontinuous trace on a C^*-algebra A. If multipliers $x, y \in \mathrm{M}(A)$ are such that xy and yx belong to A_1^τ, is it true that $\tau(xy) = \tau(yx)$?*

The case $A = K(\mathcal{H})$ is known. See [21]. We divide the proof of Theorem A.27 into a series of lemmas.

Lemma A.28. *Let τ be a densely-defined, lower semicontinuous trace on A. Let $a, b \in A$ with $a \geq 0$. If $ab \in A_1^\tau$ and $ba \in A_1^\tau$, then $\tau(ab) = \tau(ba)$.*

Proof. Let $f_n \geq 0$ be the "cut-off function" with $f_n = 0$ on $[0, \frac{1}{n}]$, $f_n = 1$ on $[\frac{2}{n}, \infty)$, and linear interpolation on the interface. Noting that

$$|f_n(a)ab|^2 \leq (ab)^* f_n(a)^2 (ab) \leq (ab)^*(ab) = |ab|^2$$

and that $f_n(a)ab \to ab$ in norm, we can apply Theorem A.21 and conclude that $f_n(a)ab \to ab$ in the Banach $*$-algebra norm of A_1^τ. Since $ab^* = (ba)^* \in A_1^\tau$, we get that $f_n(a)ab^* \to ab^*$ in A_1^τ by the same argument. Taking the adjoint of the preceding limit, we get that $baf_n(a) \to ba$ in A_1^τ as well. On the other hand, $f_n(a) \in A_1^\tau$ by Proposition A.3, so that

$$\tau(\underbrace{f_n(a)a}_{A_1^\tau} b) = \tau(bf_n(a)a) = \tau(baf_n(a))$$

using tracial property (2) from our catalogue, and the fact that a and $f(a)$ commute. Since τ is a continuous functional on A_1^τ, we get $\tau(ab) = \tau(ba)$ by taking $n \to \infty$. □

An easy polynomial approximation argument gives the following.

Lemma A.29. *If a is an element of a C^*-algebra A, and f is a continuous function on $\mathbb{R}$, then $f(aa^*)a = af(a^*a)$. In particular, $|a^*|^p a = a|a|^p$ for every exponent $p > 0$.*

Lemma A.30. *Let τ be a densely-defined, lower semicontinuous trace on a C^*-algebra A. If $a, b \in A$ are such that both products ab and ba belong to A_1^τ then, for every $\epsilon > 0$,*

$$\tau(|a^*|^\epsilon ab) = \tau(ba|a|^\epsilon).$$

Proof. Embed A into $B(H)$ for some Hilbert space H. Let $a = w|a|$ be the polar decomposition of a there. Recall that $w|a|^\epsilon$ belongs to A. Thus, $|a|^{1+\epsilon}b = (w|a|^\epsilon)^*ab \in A_1^\tau$ and $c := bw|a|^\epsilon \in A$. By the preceding lemma, $|a^*|^{2\epsilon}ab = a|a|^{2\epsilon} = w|a|^\epsilon|a|^{1+\epsilon}b$ so, using tracial property (2) from our catalogue, we get

$$\tau(|a^*|^{2\epsilon}ab) = \tau(w|a|^\epsilon|a|^{1+\epsilon}b) = \tau(|a|^{1+\epsilon}bw|a|^\epsilon) = \tau(|a|^{1+\epsilon}c).$$

Meanwhile, $ba|a|^{2\epsilon} = bw|a|^\epsilon|a|^{1+\epsilon} = c|a|^{1+\epsilon}$. The equality $\tau(c|a|^{\epsilon+1}) = \tau(|a|^{\epsilon+1}c)$ follows from Lemma A.28 above so, replacing 2ϵ by ϵ, we are done. □

Finally, we are able to give

Proof of Theorem A.27. Suppose $a, b \in A$ and $ab, ba \in A_1^\tau$. Since $|a^*|^\epsilon ab \to ab$ in norm as $\epsilon \to 0$ and since

$$(|a^*|^\epsilon ab)^*(|a^*|^\epsilon ab) = (ab)^*|a^*|^{2\epsilon}(ab) \leq \text{constant} \cdot |ab|^2,$$

Theorem A.21 applies and shows that $\tau(|a^*|^\epsilon ab) \to \tau(ab)$ as $\epsilon \to 0$. Similarly, one concludes that $\tau(ba|a|^\epsilon) \to \tau(ba)$ as $\epsilon \to 0$, so we are done by the preceding lemma. □

A.3 The dual trace

If $(A, \mathbb{R}, \alpha)$ is a C*-dynamical system, and τ is an α-invariant, densely-defined, lower semicontinuous trace on A, then the crossed product $A \rtimes_\alpha \mathbb{R}$ supports a trace $\widehat{\tau}$, the *dual trace* of τ. In this section, we outline the construction of $\widehat{\tau}$ and prove various technical results. First, note the flow α restricts sensibly to the domain of definition of τ.

Proposition A.31. *If $(A, \mathbb{R}, \alpha)$ is a C*-dynamical system, and τ is an α-invariant, densely-defined, lower semicontinuous trace on a A, then α restricts to a continuous action α^τ of $\mathbb{R}$ on A_1^τ by isometric $*$-automorphisms.*

The above is Lemma 1.1 in [26]. We have nothing to add to the proof, but we do pause to derive a corollary.

Corollary A.32. *Let $x \in A^\tau$ be C^1 for the flow on A and $\delta(x) \in A^\tau$, then x is also C^1 for the induced flow on A^τ. In particular, $\tau(\delta(x)) = 0$.*

Proof. Suppose that x is as above. Write

$$\frac{\alpha_t(x) - x}{t} = \frac{1}{t}\int_0^t \frac{d}{ds}\alpha_s(x)\ ds = \frac{1}{t}\int_0^t \alpha_s(\delta(x))\ ds$$

where, by the above proposition, the integrand is continuous for the norm of A_1^τ. We have then

$$\begin{aligned}\|\frac{\alpha_t(x) - x}{t} - \delta(x)\|_\tau &= \frac{1}{t}\|\int_0^t (\alpha_s(\delta(x)) - \delta(x))\ ds\|_\tau \\ &\leq \frac{1}{t}\int_0^t \|\alpha_s(\delta(x)) - \delta(x)\|_\tau\ ds.\end{aligned}$$

The RHS goes to 0 as $t \to 0$ by the preceding proposition. For the second statement, since $(1/t)(\alpha_t(x) - x) \to \delta(x)$ in the norm of A_1^τ, and τ belongs to the continuous dual of A_1^τ, we get $\tau(\delta(x)) = \lim_{t\to 0} \tau((1/t)(\alpha_t(x) - x)) = 0$. □

We do not construct the dual trace in this thesis. Instead, we take as our starting point the construction in [26] using the theory of Hilbert algebras, as laid out in [9].

Let $(A, \mathbb{R}, \alpha)$ be a C*-dynamical system, and let τ be an α-invariant, densely-defined, lower semicontinuous trace on A. Then, the **dual trace**, denoted $\widehat{\tau}$, on the crossed product $A \rtimes_\alpha \mathbb{R}$ exists and is such that $C_c(\mathbb{R}, A_1^\tau)$ is dense in $(A \rtimes_\alpha \mathbb{R})_2^{\widehat{\tau}}$ and

$$\widehat{\tau}(x^* y) = \int_{-\infty}^{\infty} \tau(x(t)^* y(t))\ dt. \qquad \forall x, y \in C_c(\mathbb{R}, A_1^\tau) \tag{A.2}$$

From a theoretical point of view (i.e. one would like to show the dual trace exists, is densely-defined, lower semicontinuous, and so on) the above approach is quite convenient. The major drawback is that it is not clear how to go about proving some specific element x belongs to $(A \rtimes_\alpha \mathbb{R})_1^{\widehat{\tau}}$ or calculate $\widehat{\tau}(x)$, unless a specific factorization of x can be found. Nonetheless, some things are still easy to see. For instance, the following proposition follows directly from the definitions.

Proposition A.33. *Let $(A, \mathbb{R}, \alpha)$ be a C*-dynamical system, and let τ be an α-invariant, densely-defined, lower semicontinuous trace on A. Let u be a unitary cocycle of the system, and let $\beta = \mathrm{Ad}(u)\alpha$ be the exterior equivalent flow. Then, τ is also β-invariant and*

$$\widehat{\tau}_\alpha(\iota_u(x)) = \widehat{\tau}_\beta(x) \qquad \forall x \in (A \rtimes_\beta \mathbb{R})_+$$

where τ_α and τ_β are the dual traces on $A \rtimes_\alpha \mathbb{R}$ and $A \rtimes_\beta \mathbb{R}$, and ι_u is the isomorphism $A \rtimes_\beta \mathbb{R} \to A \rtimes_\alpha \mathbb{R}$ of Proposition 2.49. In particular, ι_u induces an isomorphism on the Banach algebras $(A \rtimes_\alpha \mathbb{R})_1^{\widehat{\tau}_\alpha}$ and $(A \rtimes_\alpha \mathbb{R})_1^{\widehat{\tau}_\beta}$.

Our preferred point of view is that $A \rtimes_\alpha \mathbb{R}$ is a twisted version of $A \otimes C_0(\mathbb{R})$, generated by "elementary products" $a \cdot f(H)$ where $a \in A$, $f \in C_0(\mathbb{R})$. Naturally, we want to know how the dual trace interacts with these products. We give two results in this vein: Propositions A.34 and A.37.

Proposition A.34. *Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system, and let (e^{itH}) be the canonical unitary group in $M(A \rtimes_\alpha \mathbb{R})$ implementing α. Let τ be an α-invariant densely-defined, lower semicontinuous trace on A. If $f_i \in C_0(\mathbb{R}) \cap L^2(\mathbb{R})$ and $a_i \in A_2^\tau$, $i = 1, 2$, then $x_i = a_i \cdot f_i(H) \in (A \rtimes_\alpha \mathbb{R})_2^{\widehat{\tau}}$ for $i = 1, 2$, and*

$$\widehat{\tau}(x_1^* x_2) = \tau(a_1^* a_2) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{f_1(s)} f_2(s)\ ds.$$

Proof. First, suppose that, in fact, $f_i = \widehat{g_i}$, where $g_i \in C_c(\mathbb{R})$, and $a_i \in A_1^\tau \subset A_2^\tau$. Then, x_i belongs to $C_c(\mathbb{R}, A_1^\tau)$ and is given by $x_i(t) = g_i(t) a_i$. In this case, it is easy to see, using the defining equation (A.2), that

$$\widehat{\tau}(x_1^* x_2) = \tau(a_1^* a_2) \cdot \int_{-\infty}^{\infty} \overline{g_1(t)} g_2(t)\ dt = \tau(a_1^* a_2) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{f_1(s)} f_2(s)\ ds,$$

as desired. So, we are just looking to extend the range of applicability of this formula a small amount. This, we shall achieve using the completeness of A_2^τ in the norm $\|\cdot\| + \|\cdot\|_2$ (Proposition A.25).

Choose now $a \in A_2^\tau$, $f \in C_0(\mathbb{R}) \cap L^2(\mathbb{R})$ and put $x = a \cdot f(H)$. By Corollary A.23, there is a sequence $a_n \in A_1^\tau$ such that $\|a_n - a\| + \|a_n - a\|_1 \to 0$. By Lemma A.36 below, there is a sequence $g_n \in C_c(\mathbb{R})$ such that $\|\widehat{g_n} - f\| + \|\widehat{g_n} - f\|_2 \to 0$. Putting $x_n = a_n \cdot \widehat{g_n}(H)$, we obviously have $\|x_n - x\| \to 0$. By the above paragraph, we get

$$\|x_m - x_m\|_2^2 = \tau(a_m^* a_m) \cdot \frac{1}{2\pi} \int |f_m|^2 - 2\,\mathrm{Re}\left(\tau(a_m^* a_n) \cdot \frac{1}{2\pi} \int \overline{f_m} f_n\right) + \tau(a_n^* a_n) \cdot \frac{1}{2\pi} \int |f_n|^2$$

which vanishes as $m, n \to \infty$. It follows from the completeness of Proposition A.25 that $x \in A_2^\tau$. Using similar argumentation, one deduces the desired trace formula on the general elements by continuity. □

Corollary A.35. *Let A, α, H and τ be as above.*

1. *If $a, b \in A_2^\tau$ and $f \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$, then $a \cdot f(H) \cdot b \in (A \rtimes_\alpha \mathbb{R})_1^{\widehat{\tau}}$ and*

$$\widehat{\tau}\,(a \cdot f(H) \cdot b) = \tau(ab) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s)\ ds.$$

2. *If $a \in A_1^\tau$ and $f, g \in C_0(\mathbb{R}) \cap L^2(\mathbb{R})$, then $f(H) \cdot a \cdot g(H) \in (A \rtimes_\alpha \mathbb{R})_1^{\widehat{\tau}}$ and*

$$\widehat{\tau}\,(f(H) \cdot a \cdot g(H)) = \tau(a) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) g(s)\ ds.$$

In the course of proving Proposition A.34, we appealed to a simple fact from Fourier analysis, which we now prove.

Lemma A.36. *The image of $C_c(\mathbb{R})$ under the Fourier transform is dense in $C_0(\mathbb{R}) \cap L^2(\mathbb{R})$ for the norm $\|\cdot\| + \|\cdot\|_2$.*

Proof. Any $f \in C_0(\mathbb{R}) \cap L^2(\mathbb{R})$ can be approximated by a C^∞ bump function g in the norm $\|\cdot\| + \|\cdot\|_2$. The bump function can be written as $g = \widehat{h}$ where $h \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$. Now, by truncation, h can be approximated by $k \in C_c(\mathbb{R})$ in the norm $\|\cdot\|_1 + \|\cdot\|_2$. As the Fourier transform is $\|\cdot\|_1 \to \|\cdot\|$ contractive, and $\|\cdot\|_2 \to \|\cdot\|_2$ isometric, $\widehat{k}$ closely approximates the original f in $\|\cdot\| + \|\cdot\|_2$. □

We now consider the analogue of Proposition A.34 for $(A \rtimes_\alpha \mathbb{R})_1^{\widehat{\tau}}$. Unfortunately, this involves some side hypotheses which we have, so far, been unable to remove.

Proposition A.37. *Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system, and let (e^{itH}) be the canonical unitary group in $M(A \rtimes_\alpha \mathbb{R})$. Let τ be an α-invariant, densely-defined, lower semicontinuous* *trace on A. If $a \in A_1^\tau$ and $f \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$, then $a \cdot f(H)$ and $f(H) \cdot a$ are in $(A \rtimes_\alpha \mathbb{R})_1^{\widehat{\tau}}$ and*

$$\widehat{\tau}\,(a \cdot f(H)) = \widehat{\tau}\,(f(H) \cdot a) = \tau(a) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s)\; ds$$

provided we also assume that:

1. *The element a is C^1 for α and $\delta(a) \in A_1^\tau$.*

2. *The function f is such that $s \mapsto sf(s)$ is in $C_0(\mathbb{R}) \cap L^2(\mathbb{R})$.*

Proof. The hypothesis (2) says exactly that f factors like $f(s) = (s+i)^{-1}g(s)$ where $g \in C_0(\mathbb{R}) \cap L^2(\mathbb{R})$. We want to show $a \cdot f(H) = a \cdot (H+i)^{-1} g(H)$ is in $(A \rtimes_\alpha \mathbb{R})_1^{\widehat{\tau}}$. Using Corollary 4.26 we have

$$a \cdot f(H) = (H+i)^{-1} \cdot a \cdot g(H) - i(H+i)^{-1} \cdot \delta(a) \cdot f(H),$$

and the right hand side is in $(A \rtimes_\alpha \mathbb{R})_1^{\widehat{\tau}}$ by Corollary A.35 (2). With domain issue resolved, it is straightforward to apply the formula of Corollary A.35 (2) and show $\widehat{\tau}(a \cdot f(H)) = \tau(a) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s)\; ds$, recalling that $\tau(\delta(a)) = 0$ by CorollaryA.32. Since $\widehat{\tau}$, τ and $\int$ are $*$-preserving, one gets the same value for $\widehat{\tau}\,(f(H) \cdot a)$. □

Recall that, in the proof of Proposition A.34, we began with extra regularity hypotheses on the generators, but were able to remove them by a continuity argument. Thus, it is natural to wonder the following.

Question. *Can the technical hypotheses at the end of Proposition A.37 above be removed? Or else, to what extent can they be relaxed?*

Remark A.38. Applying Proposition A.33, one sees Propositions A.34 and A.37 still hold true if H is replaced by H_u where u is a unitary 1-cocycle in $M(A)$, and H_u generates the unitary group $(u_t e^{itH})$ in $M(A \rtimes_\alpha \mathbb{R})$.

In light of the above remark, one can then ask in what sense does an elementary element $a \cdot f(H_u)$ vary continuously with respect to u? The final result of this section answers two such questions.

Proposition A.39. *Let $(A, \mathbb{R}, \alpha)$ be a C^*-dynamical system, (e^{itH}) the canonical unitary group in $M(A \rtimes_\alpha \mathbb{R})$, τ an α-invariant, densely-defined, lower semicontinuous trace on A, and P a self-adjoint element of A.*

1. *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,[8] and $a \in A_2^\tau$, then*

$$\lambda \mapsto a \cdot \widehat{f}(H + \lambda P) : \mathbb{R} \to (A \rtimes_\alpha \mathbb{R})_2^{\widehat{\tau}}$$

   is continuous with respect to the norm of $(A \rtimes_\alpha \mathbb{R})_2^{\widehat{\tau}}$.

2. *If $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and $a \in A_1^\tau$, then*

$$\lambda \mapsto \widehat{f}(H + \lambda P) \cdot a \cdot \widehat{g}(H + \lambda P) : \mathbb{R} \to (A \rtimes_\alpha \mathbb{R})_1^{\widehat{\tau}}$$

   is continuous with respect to the norm of $(A \rtimes_\alpha \mathbb{R})_1^{\widehat{\tau}}$.

Proof. By Proposition A.33, we only need to check the continuity at $\lambda = 0$. Observe that $x_\lambda = a \cdot \widehat{f}(H + \lambda P) - a \cdot \widehat{f}(H) \in L^1(\mathbb{R}, A)$ is given by $t \mapsto f(t)(u_t^{\lambda P} - 1)a$ so

$$\widehat{\tau}(x_\lambda^* x_\lambda) = \int_{-\infty}^{\infty} \tau(\|f(t)\|^2 a^*(u_t^{\lambda P} - 1)^*(u_t^{\lambda P} - 1)a)\ dt \leq \tau(a^*a) \cdot \int_{-\infty}^{\infty} |f(t)|^2 \|u_t^{\lambda P} - 1\|^2\ dt$$

The right hand side vanishes as $\lambda \to 0$ since $\lim_{\lambda \to 0} \|u_t^{\lambda P} - 1\| = 0$ for all t and, obviously, $\|u_t^{\lambda P} - 1\| \leq 2$, so the first assertion is proved. The second assertion follows from the first by factoring a as $a = bc$ where $b, c \in A_2^\tau$ and using the continuity of the product $(A \rtimes_\alpha \mathbb{R})_2^{\widehat{\tau}} \times (A \rtimes_\alpha \mathbb{R})_2^{\widehat{\tau}} \to (A \rtimes_\alpha \mathbb{R})_1^{\widehat{\tau}}$. □

[8] Which, in particular, implies that $\widehat{f} \in C_0(\mathbb{R}) \cap L^2(\mathbb{R})$.

Chapter B

K-theory and dense subalgebras

B.1 Spectral embeddings

It is well-known that closedness under holomorphic functional calculus is a convenient property in the context of K-theory. The following elaboration of Lemma 4 in [6] is case in point.

Lemma B.1. *Let $\mathscr{A}$ be a norm-dense $*$-subalgebra of a unital C^*-algebra A such that $\mathscr{A}$ is closed under the holomorphic functional calculus*[1] *of A. Then,*

1. *Any projection $e \in A$ is Murray-von Neumann equivalent to a projection $e' \in \mathscr{A}$.*
2. *If projections $e, e' \in \mathscr{A}$ are Murray-von Neumann equivalent in A, then the equivalence can be mediated by a partial isometry in $\mathscr{A}$.*
3. *Any invertible $x \in A$ is path-equivalent to a unitary $u' \in \mathscr{A}$.*
4. *If invertibles $x, x' \in \mathscr{A}$ are path-equivalent in $\mathrm{GL}(A)$, then they are path-equivalent in $\mathrm{GL}(\mathscr{A})$.*

[1] In particular $\mathscr{A}$ is a unital subalgebra of A that is closed under inversion

Proof. 1. Find $x \in \mathscr{A}$ as close as desired to e. Then, $x^* \in \mathscr{A}$ is also close to $e = e^*$. So, replacing x by $(1/2)(x + x^*)$, we may suppose that x is self-adjoint. Provided x is sufficiently near to e, its spectrum (in A) will be concentrated near $\{0, 1\}$. Then, with $f(z) = 1$ for $\mathrm{Re}(z) > 1/2$, $f(z) = 0$ for $\mathrm{Re}(z) < 1/2$, we have $e' = f(x)$ a projection as close as we want to e. By the assumption of closedness by holomorphic functional calculus, we have $e' \in \mathscr{A}$. In a C*-algebra, sufficiently close projection are even unitarily equivalent, so (1) follows.

2. We know there is a $v \in A$ such that $v^*v = e$ and $vv^* = e'$. Find $x \in \mathscr{A}$ as close as desired to v. Then $e'xe \in \mathscr{A}$ is also close to $v = e've$. So, we may suppose that $x \in e'Ae$. Since x^*x is in eAe and close to e, it will be invertible in eAe. Similarly, xx^* will be invertible in $e'Ae'$. Let $y = (x^*x)^{-1/2} \in eAe$, using the functional calculus of the corner eAe. We can also write $y = (x^*x + e^\perp)^{-1/2} - e^\perp$, using the functional calculus of A. Since $x^*x + e^\perp$ is close to $1 \in A$, its spectrum is concentrated near $1 \in \mathbb{C}$ where $z \mapsto z^{-1/2}$ is holomorphic, so the latter formula for y shows that $y \in \mathscr{A}$. We claim the element $w = xy$ of $\mathscr{A}$ mediates a Murray-von Neumann equivalence from e to e', which will prove (2). Indeed, we have $w^*w = (x^*x)^{-1/2}x^*x(x^*x)^{-1/2} = (x^*x)^0 = e$, using the functional calculus of the corner eAe. Meanwhile, we have $ww^*xx^* = x(x^*x)^{-1}x^*xx^* = xex^* = xx^*$, using the functional calculus of the corner eAe. Multiplying the latter equality on the right by the inverse of xx^* in $e'Ae'$ shows that $ww^* = e'$.

3. Take a ball centred on x and contained in $\mathrm{GL}(A)$. Select an element y of $\mathscr{A}$ in that ball. Obviously x is connected to y, the straight line path does just fine. Meanwhile, y is connected to the unitary $u' = y|y|^{-1}$. But, $y^*y \in \mathscr{A}$, since y is a $*$-algebra. Without loss of generality, $\mathrm{spec}(y^*y) \subset (0, 1)$, or else continuously scale y down a bit. Thus, $|y|^{-1} = (y^*y)^{-1/2} \in \mathscr{A}$ by closedness under holomorphic functional calculus, since $\mathrm{spec}(y^*y)$ is contained in the disk of convergence of the relevant binomial series.

4. Immediate from Lemma B.10.

□

Corollary B.2. *If $\mathscr{A}$ is a norm-dense $*$-subalgebra of a unital C*-algebra A such that, for every n, $\mathrm{M}_n(\mathscr{A})$ is closed under the holomorphic functional calculus of $\mathrm{M}_n(A)$, then the inclusion $\mathscr{A} \hookrightarrow A$ induces isomorphisms on both K-groups.*

In view of the above results, it shall obviously be useful to know of practical situations where closedness under holomophic functional calculus holds.

Definition B.3. Let A be unital C*-algebra, and $\mathscr{A}$ a unital Banach $*$-algebra. By a **spectral embedding** of $\mathscr{A}$ into A, we mean a unital embedding $\mathscr{A} \subset A$ such that $\mathrm{GL}_n(\mathscr{A}) = \mathrm{GL}_n(A) \cap \mathrm{M}_n(\mathscr{A})$ for every n.

It is well known that, if one only assumes $\mathrm{GL}(\mathscr{A}) = \mathrm{GL}(A) \cap \mathscr{A}$ above, then the same conclusion for all $n > 1$ follows automatically. We don't bother optimizing the definition in this way since, for all examples of interest, it shall be clear that $\mathrm{GL}_n(\mathscr{A}) = \mathrm{GL}(A) \cap \mathrm{M}_n(\mathscr{A})$ for all n. In case that nonunital algebras are afoot, note the following.

Lemma B.4. *Let $\mathscr{B}$ be a subalgebra of a nonunital algebra B. If $\widetilde{\mathrm{M}_n(\mathscr{B})}$ is inverse-closed in $\widetilde{\mathrm{M}_n(B)}$, then $\mathrm{M}_n(\widetilde{\mathscr{B}})$ is inverse closed in $\mathrm{M}_n(\widetilde{B})$.*

Proof. Take $x = b + \lambda \in \mathrm{M}_n(\widetilde{B})$ where $b \in \mathrm{M}_n(B)$, $\lambda \in \mathrm{M}_n(\mathbb{C})$. If x is invertible, then

$$x^{-1} = \lambda^{-1}(b\lambda^{-1} + 1_n)^{-1} = \lambda^{-1}(y + 1_n) = \lambda^{-1}y + \lambda^{-1}$$

for $y \in \mathrm{M}_n(\mathscr{B})$. Thus, $x^{-1} \in \mathrm{M}_n(\widetilde{\mathscr{B}})$, as desired. □

Most of our interest in spectral embeddings derives from the following lemma.

Lemma B.5. *If $\mathscr{A} \subset A$ is a spectral embedding of a Banach $*$-algebra $\mathscr{A}$ into a C*-algebra A, then $\mathrm{M}_n(\mathscr{A})$ is closed under the holomorphic functional calculus of $\mathrm{M}_n(A)$ for every n.*

Proof. Endow $\mathrm{M}_n(\mathscr{A})$ with a Banach $*$-algebra norm, for instance by making it act on $\mathscr{A} \oplus \ldots \oplus \mathscr{A}$. By basic spectral considerations, $*$-homomorphisms from Banach $*$-algebras to C*-algebras are automatically norm-decreasing, so the embedding $\mathrm{M}_n(\mathscr{A}) \subset \mathrm{M}_n(A)$ is norm-decreasing when $\mathrm{M}_n(A)$ is made a C*-algebra in the unique way. It follows that $f(x)$ is unambiguously defined when $x \in \mathrm{M}_n(\mathscr{A})$ and f is holomorphic on a neighbourhood of $\mathrm{spec}_{\mathscr{A}}(x) \supset \mathrm{spec}_A(x)$. But, since $\mathrm{spec}_A(x) = \mathrm{spec}_{\mathscr{A}}(x)$ by assumption, the result follows. $\square$

Combining Lemma B.1 with Corollary B.2, one sees that a dense spectral embedding carries all the K-theory of the ambient C*-algebra. Two especially pertinent examples of dense, spectral embeddings are given below.

Example B.6. Let $\delta : A \to A$ be a closed, densely-defined, self-adjoint derivation of a unital C*-algebra A. Then $\mathrm{dom}(\delta) \subset A$ is a dense, spectral embedding. Indeed, noting

$$\|xy\| + \|\delta(xy)\| \leq \|x\|\|y\| + \|\delta(x)\|\|y\| + \|x\|\|\delta(y)\| \leq (\|x\| + \|\delta(x)\|)(\|y\| + \|\delta(y)\|)$$

one sees that $\mathrm{dom}(\delta)$ is Banach $*$-algebra for the graph norm $\|x\|_\delta = \|x\| + \|\delta(x)\|$. To see that $\mathrm{GL}_n(\mathrm{dom}(\delta)) = \mathrm{GL}_n(A) \cap \mathrm{M}_n(\mathrm{dom}(\delta))$ for each n, note that δ extends to closed, densely-defined, self-adjoint derivation of each matrix algebra $\mathrm{M}_n(A)$ with domain $\mathrm{M}_n(\mathrm{dom}(\delta))$ by entry-wise application. Then, apply Proposition 3.6.

Example B.7. Let τ be a (lower semicontinuous, densely-defined) tracial weight of a nonunital C*-algebra B. Let $\mathscr{B} = B_1^\tau \subset B$ be the ideal of definition of τ. Then, $\widetilde{\mathscr{B}} \subset \widetilde{B}$ is a dense, spectral embedding. Indeed, by Proposition A4 in [26], $\mathscr{B}$ is a Banach $*$-algebra for the norm $\|x\|_\tau = \|x\| + \tau(|x|)$. Thus the unitization $\widetilde{\mathscr{B}}$ is also a Banach $*$-algebra. Certainly, $\widetilde{\mathscr{B}}$ is densely embedded in $\widetilde{B}$, so it just remains to check, for every $x \in M_n(\widetilde{\mathscr{B}})$ invertible in $\mathrm{M}_n(\widetilde{B})$, that $x^{-1} \in \mathrm{M}_n(\widetilde{\mathscr{B}})$. Indeed, write $x = y + \lambda$ where $y \in \mathrm{M}_n(\mathscr{B})$ and $\lambda \in \mathrm{M}_n(\mathbb{C})$. Observe $(y+\lambda)^{-1} - \lambda^{-1} = (y+\lambda^{-1})(1_n - (y+\lambda)\lambda^{-1}) = -(y+\lambda^{-1})y\lambda^{-1}$ belongs to $\mathrm{M}_n(\mathscr{B})$, the latter being a 2-sided ideal in $\mathrm{M}_n(\widetilde{B})$. So, $x^{-1} = y' + \lambda^{-1}$ where $y' \in \mathrm{M}_n(\mathscr{B})$ and

$\lambda^{-1} \in \mathrm{M}_n(\mathbb{C})$ i.e. $x^{-1} \in \mathrm{M}_n(\widetilde{\mathscr{B}})$.

B.2 Relevance to traces

By a **bounded trace** on a Banach algebra $\mathscr{A}$ we mean a bounded linear functional $\tau : \mathscr{A} \to \mathbb{C}$ such that $\tau(xy) = \tau(yx)$ for all $x, y \in \mathscr{A}$. If $\mathscr{A}$ is a Banach $*$-algebra and $\tau(x^*) = \overline{\tau(x)}$ for all $x \in \mathscr{A}$, then τ is **self-adjoint**. Trivially, τ extends to every matrix algebra $\mathrm{M}_n(\mathscr{A})$ by applying τ down the diagonal and summing. Moreover, the extension is self-adjoint if τ is. As a corollary to Lemmas B.5 and B.1 above, we have

Corollary B.8. *Let $\mathscr{A} \subset A$ be a dense, spectral embedding of a unital Banach $*$-algebra $\mathscr{A}$ into a unital C*-algebra A. Let τ be a bounded, self-adjoint trace on $\mathscr{A}$. Then, there is a homomorphism $\tau_* : K_0(A) \to \mathbb{R}$ such that $\tau_*([e]) = \tau(e)$ for every projection $e \in \mathrm{M}_n(\mathscr{A})$.*

Bott periodicity amounts to an isomorphism $K_0(A) \cong K_1(\mathrm{S}A)$ for every C*-algebra A. Thus, if $\mathscr{A} \subset A$ is a dense, spectral embedding of a unital Banach $*$-algebra $\mathscr{A}$ into a unital C*-algebra A, then a bounded, self-adjoint trace τ on $\mathscr{A}$ induces homomorphism on $K_1(\mathrm{S}A)$ as well. It stands to reason that τ_* should admit a direct description in terms of the generators of $K_1(\mathrm{S}A)$, essentially, invertible loops in $\mathrm{GL}_n(A)$. Indeed, we have the following, based on Lemma 5 in [6].

Theorem B.9. *Let $\mathscr{A} \subset A$ be a dense, spectral embedding of a unital Banach $*$-algebra $\mathscr{A}$ into a unital C*-algebra A. Let τ be a bounded, self-adjoint trace on $\mathscr{A}$. Suppose $x \in K_0(A)$ has $s_A^0(x) = [\mathscr{C}]$, where s_A^0 is the suspension isomorphism $K_0(A) \to K_1(\mathrm{S}A)$ and $\mathscr{C}$ is a piecewise-C^1 (for the norm of $\mathscr{A}$) loop $[0,1] \to \mathrm{GL}_n(\mathscr{A})$ with $\mathscr{C}(0) = \mathscr{C}(1)$. Then,*

$$\tau_*(x) = \frac{1}{2\pi i}\int_0^1 \tau\left(\frac{d\mathscr{C}}{dt}\mathscr{C}^{-1}(t)\right) dt.$$

Before proceeding to the proof of the above proposition, we set down a simple technical lemma, a particular case of Lemma A9 on pp. 167 of [1].

Lemma B.10. *Let X be a Banach space, D a dense subspace of X, and U a nonempty open subset of X. Then, the inclusion $D \cap U \hookrightarrow U$ is a π_0-equivalence.*

Proof. Since U is locally path-connected, its path components are open, hence each contains an element of D. If $x, y \in D \cap U$ are connected by a path in U, then, by a straightforward compactness argument, the path can be chosen to be polygonal and to have vertices in D. Since D is linearly closed, this curve actually stays in $D \cap U$. □

Proof of Theorem B.9. For the purposes of this proof, $K_1(\mathrm{S}A)$ is taken to be a quotient of $\mathrm{GL}_\infty(\mathrm{S}A) = \bigcup_{n=1}^{\infty} \mathrm{GL}_n(\mathrm{S}A)$. Let $\Gamma_n \subset \mathrm{GL}_n(\mathrm{S}A)$ be the subgroup of piecewise-C^1 loops $\mathscr{C} : [0,1] \to \mathrm{GL}_n(\mathscr{A})$ with $\mathscr{C}(0) = \mathscr{C}(1) = 1$ and put $\Gamma_\infty = \bigcup_{n=1}^{\infty} \Gamma_n \subset \mathrm{GL}_\infty(\mathrm{S}A)$. As a consequence[2] of Lemma B.10 above, we get that every path component in $\mathrm{GL}_n(\mathrm{S}A)$ intersects Γ_n. Therefore, $K_1(\mathrm{S}A)$ can also be viewed as a quotient of Γ_∞ where $\mathscr{C}_0, \mathscr{C}_1 \in \Gamma_\infty$ are put equivalent when they are homotopic in $\mathrm{GL}_n(\mathrm{S}A)$ for some n.

Using the tracial property of τ, it's easy to see that $I(\mathscr{C}) = \frac{1}{2\pi i} \int_0^1 \tau\left(\frac{d\mathscr{C}}{dt} \mathscr{C}^{-1}\right) dt$ defines a group homomorphism $I : \Gamma_\infty \to \mathbb{C}$. We claim that I descends to a map $\overline{I} : K_1(\mathrm{S}A) \to \mathbb{C}$. Indeed, suppose that $\mathscr{C}_0, \mathscr{C}_1 \in \Gamma_\infty$ are homotopic in $\mathrm{GL}_n(\mathrm{S}A)$ for some n, with an eye to showing $I(\mathscr{C}_1) = I(\mathscr{C}_2)$. Again, as a consequence of Lemma B.10, we may assume they are joined by a homotopy which remains in Γ_n. With the latter observation, we are reduced to the case where $\mathscr{C}_0$ and $\mathscr{C}_1$ are very close as elements of $\mathrm{GL}_n(\mathrm{S}A)$. In this case, we may assume the straight-line homotopy $\mathscr{C}_s(t) = (1-s)\mathscr{C}_0(t) + s\mathscr{C}_1(t)$ stays inside $\mathrm{GL}_n(\mathrm{S}A)$, and

[2] Since, technically $\mathrm{GL}_n(\mathrm{S}A) \subset 1 + \mathrm{M}_n(\mathrm{S}A) \subset \mathrm{M}_n(\widetilde{\mathrm{S}A})$, this needs a small fiddle. One simply translates the problem and uses $X = \mathrm{M}_n(\mathrm{S}A)$, the Banach space of loops $[0,1] \to \mathrm{M}_n(A)$ based at 0, $D \subset X$ the dense subspace of piecewise-C^1 loops $[0,1] \to \mathrm{M}_n(\mathscr{A})$, and $U \subset X$ the open subset of loops which value in $\mathrm{QI}_n(A) = 1_n - \mathrm{GL}_n(A)$.

therefore in Γ_n as well. But, then

$$\begin{aligned}\frac{d}{ds}\int_0^1 \tau\left(\mathscr{C}_s^{-1}\frac{\partial\mathscr{C}_s}{\partial t}\right)\,dt &= \int_0^1 \tau\left(\mathscr{C}_s^{-1}\frac{\partial^2\mathscr{C}_s}{\partial s\partial t} - \mathscr{C}_s^{-1}\frac{\partial\mathscr{C}_s}{\partial s}\mathscr{C}_s^{-1}\frac{\partial\mathscr{C}_s}{\partial t}\right)\,dt\\ &= \int_0^1 \tau\left(\mathscr{C}_s^{-1}\frac{\partial^2\mathscr{C}_s}{\partial t\partial s} - \mathscr{C}_s^{-1}\frac{\partial\mathscr{C}_s}{\partial t}\mathscr{C}_s^{-1}\frac{\partial\mathscr{C}_s}{\partial s}\right)\,dt\\ &= \int_0^1 \frac{\partial}{\partial t}\tau\left(\mathscr{C}_s^{-1}\frac{\partial\mathscr{C}_s}{\partial s}\right)\,dt\\ &= \tau\left(\mathscr{C}_s^{-1}\frac{\partial\mathscr{C}_s}{\partial s}\right)\Bigg|_{t=0}^{t=1}\end{aligned}$$

which vanishes since $\frac{\partial\mathscr{C}_s}{\partial s} \equiv 0$ at $t = 0, 1$. Thus, the quantity $I(\mathscr{C}_s)$ does not depend on the homotopy parameter s and $I(\mathscr{C}_0) = I(\mathscr{C}_1)$, as desired.

At this point, we have two homomorphisms $K_0(A) \to \mathbb{C}$ at our disposal. Namely, τ_* and the composite of $s_A^0 : K_0(A) \to K_1(\mathrm{S}A)$ with $\overline{I} : K_1(\mathrm{S}A) \to \mathbb{C}$. It remains to show these two homomorphisms are the same. In view of Lemma B.1, we just need to check they agree on $[e] \in K_0(A)$ where e is a projection in $\mathrm{M}_n(\mathscr{A})$. In this case, $\tau_*([e]) = \tau_n(e)$. On the other hand, the suspension isomorphism $K_0(A) \to K_1(\mathrm{S}A)$ sends $[e] \mapsto [\mathscr{C}]$ where $\mathscr{C} \in \mathrm{GL}_n(\mathrm{S}A)$ is the unitary loop $t \mapsto \exp(2\pi i t)e + e^\perp$. As it happens, this $\mathscr{C}$ belongs to Γ_n and so

$$\overline{I}([\mathscr{C}]) = I(\mathscr{C}) = \frac{1}{2\pi i}\int_0^1 \tau\left(2\pi i\cdot\exp(2\pi i t)e\cdot(\exp(-2\pi i t)e + e^\perp)\right)\,dt = \tau(e)$$

completing the proof. □

For a densely-defined lower semicontinuous trace $\tau = \tau_\mu$ (see Proposition A.6) on a nonunital, commutative C*-algebra, the map τ_* is always trivial.

Proposition B.11. *Let X be a locally compact Hausdorff space, and let $\tau = \tau_\mu$ be a lower semicontinuous, densely-defined trace on $A = C_0(\mathbb{R})$. If X is noncompact and connected, then the map $\tau_* : K_0(A) \to \mathbb{R}$ is trivial. In particular, τ_* is trivial if X is the suspension of another locally compact Hausdorff space.*

Proof. Consider a generator $[e] \in K_0(\widetilde{A})$ where $e = e_0 + a \in \mathrm{M}_n(\widetilde{A})$ is a projection, $e_0 \in \mathrm{M}_n(\mathbb{C})$, $a \in \mathrm{M}_n(A) = C_0(X, \mathrm{M}_n(\mathbb{C}))$. We claim that $\mathrm{tr}(a(x)) = 0$ for all $x \in X$. Indeed, since X is connected, $\{e(x) : x \in X\} \subset \mathrm{M}_n(\mathbb{C})$ is connected. Thus, all the $e(x)$ are equivalent projections. In fact they are all equivalent to e_0 since $e(x) \to e_0$ as x escapes to infinity. Thus $\mathrm{tr}(e(x)) = \mathrm{tr}(a(x)) + \mathrm{tr}(e_0) = \mathrm{tr}(e_0)$ for all $x \in X$ so that $\mathrm{tr}(a(x)) = 0$. Now, by definition of τ_*, $\tau_*([e]) = \widetilde{\tau}(e) = \tau(a) = \int_X \mathrm{tr} \circ a \; d\mu = 0$ as desired. $\square$

Bibliography

[1] M.F. Atiyah. *K-Theory.* Benjamin, 1967.

[2] B. Blackadar. *Operator Algebras: Theory of C*-Algebras and von Neumann Algebras.* Encyclopaedia of Mathematical Sciences. Springer, 2006.

[3] O. Bratteli and D.W. Robinson. *Operator algebras and quantum statistical mechanics.* Springer, 1981.

[4] M. Breuer. Fredholm theories in von Neumann algebras I. *Math. Ann.*, 178:243–254, 1968.

[5] M. Breuer. Fredholm theories in von Neumann algebras II. *Math. Ann.*, 180:313–325, 1969.

[6] A. Connes. An analogue of the Thom isomorphism for crossed products of a C*-algebra by an action of R. *Adv. in Math*, 39(1):31–55, 1981.

[7] J. Cuntz and G. K. Pedersen. Equivalence and traces on C*-algebras. *J. Func. Anal.*, 33(2):135–164, 1979.

[8] J. Diximier. *C*-algebras.* North-Holland Mathematical Library. North-Holland, 1977.

[9] J. Dixmier. *von Neumann algebras.* North-Holland mathematical library. North-Holland Pub. Co., 1981.

[10] G.A. Elliott, T. Natsume, and R. Nest. The Heisenberg group and K-theory. *K-Theory*, 7(5):409–428, 1993.

[11] K.J. Engel and R. Nagel. *A Short Course on Operator Semigroups*. Springer, 2006.

[12] D.S. Freed. Geometry of Dirac operators. `http://www.ma.utexas.edu/users/dafr/`. Accessed: 2013-11-02.

[13] I.C. Gohberg and M.G. Krein. Fundamental aspects of defect numbers, root numbers and indices of linear operators. *Usp. Mat. Nauk.*, 12(2):43–118, 1957.

[14] P. Green. The structure of imprimitivity algebras. *J. Func. Anal.*, 36(1):88–104, 1980.

[15] A. Hatcher. *Algebraic Topology*. Electronic, 2002.

[16] A. Henriques. tr(ab) = tr(ba). `http://mathoverflow.net/questions/76386/trab-trba`. Accessed: 2014-05-30.

[17] P.J. Huber. Homotopical cohomology and Čech cohomology. *Math. Annalen.*, 144:73–76, 1961.

[18] I. Kaplansky. Algebras of type I. *Ann. of Math.*, 56(3):460–472, 1952.

[19] E.C. Lance. *Hilbert C*-Modules: A Toolkit for Operator Algebraists*. Cambridge University Press, 1995.

[20] M.B. Landstad, J. Phillips, I. Raeburn, and C.E. Sutherland. Representations of crossed products by coactions and principal bundles. *Trans. Amer. Math. Soc.*, 299(2), 1987.

[21] C. Laurie, E. Nordgren, H. Radjavi, and P. Rosenthal. On triangularization of algebras of operators. *Usp. Mat. Nauk.*, 12(2):43–118, 1957.

[22] M. Lesch. On the index of the infinitesimal generator of a flow. *J. Operator Theory*, 25:73–92, 1991.

[23] N.E.W. Olsen. *K-Theory and C*-Algebras: A Friendly Approach.* Oxford University Press, Incorporated, 1993.

[24] G. Pedersen. *C*-algebras and their automorphism groups.* Academic Press, 1979.

[25] G.K. Pedersen. Measure theory for C*-algebras I. *Math. Scand.*, 1966.

[26] J. Phillips and I. Raeburn. An index theorem for Toeplitz operators with noncommutative symbol space. *J. Funct. Anal.*, 120(2):239–263, 1994.

[27] W. Pusz and S. L. Woronowicz. Passive states and KMS states for general quantum systems. *Comm. Math. Phys.*, 58:273–290, 1978.

[28] M.A. Rieffel. C*-algebras associated with irrational rotations. *Pacific J. Math.*, 93(2):415–429, 1981.

[29] M.A. Rieffel. Connes' analogue for crossed products of the Thom isomorphism. *Contemp. Math.*, 10:143–154, 1981.

[30] M.A. Rieffel. Projective modules over higher-dimensional noncommutative tori. *Canad. J. Math.*, 40:257–338, 1988.

[31] M. Rørdam, F. Larsen, and N. Laustsen. *An introduction to K-theory for C*-algebras.* Cambridge University Press, 2000.

[32] S. Schwartzman. Asymptotic cycles. *Ann. Math.*, 66(2):270–284, 1957.

[33] B. Simon. Quantum dynamics: From automorphism to Hamiltonian. In *Studies in Mathematical Physics: Essays in Honor of Valentine Bargmann.* Princeton University Press, 1976.

[34] D.P. Williams. *Crossed products of C*-algebras.* American Mathematical Society, 2007.