Vlasov's Equation on a Great Circle and the Landau Damping Phenomenon

by

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ABSTRACT

Vlasov's equation describes the time evolution of the distribution function for a collisionless physical system of identical particles, such as plasma or galaxies. Together with Poisson's equation, which yields the potential, it forms the Vlasov-Poisson system. In Euclidean space this system has been extensively studied in the past century. It has been recently shown that the Valsov-Poisson system exhibits an interesting, counter-intuitive phenomenon called Landau damping. Our universe, however, may not be flat on a large scale, so it is important to introduce and study a natural extension of the Vlasov-Poisson systems to spaces of constant curvature. Our starting point is the unit sphere $\mathbb{S}^2$, but we further restrict our study to one of its great circles. We show that, even for this reduced model, the potential function has more singularities than in the classical case. Our main result is to derive a Penrose stability criterion for the linear Landau damping phenomenon.

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Chapter 1

Introduction

Vlasov's equation describes the time evolution of the distribution function for a collisionless physical system of identical particles, such as plasma or galaxies. This equation was first derived by Anatoly Vlasov in 1938 to model the behaviour of plasma, [27], [28]. Because of the long-range Coulomb interaction, it is difficult to apply the standard kinetic approach based on Boltzmann's equation to plasma, [17]. Vlasov therefore started with the collisonless Boltzmann equation and adapted it to plasma. In physics, Vlasov's equation usually couples with Maxwell's equation, which takes into account the changing magnetic fields and relativity effects, [1]. When the influence of relativity and the changing magnetic field are negligible, Vlasov's equation is coupled with Poisson's equation. The Vlasov-Poisson system in Euclidean space was widely studied in the past century. The existence of a smooth solution in 1D with the potential of electric field was obtained by Iordanskii, [12]. The global existence in both electric and gravitational cases were obtained in 2D by Ukai and Okabe, [24], and in 3D by Pfaffelmoser, [20], a proof that was later simplified by Shaeffer, [22]. The global existence of the solution in 3D in a cosmological setting with initial data deviated from homogeneous states was obtained by Rein and Rendall, [21].

An interesting phenomenon that takes place in systems described by the Vlasov-Poisson equations is Landau damping[1], according to which the density ρ of the modelled physical system converges to its mean value, while the interaction force F converges to 0. This effect is counter intuitive because, on the one hand, the decay of the density and the force show that the evolution of the physical system is time-irreversible, but, on the other hand, the system is energy preserved and the solution

[1]Landau damping also takes place in Vlasov's equation for a given interaction potential, [25], [26].

is time-reversible (see, e.g., [25]). Landau predicted this effect in 1946 in the case of plasma, [13], and, in 1962, Lynden-Bell stated that a similar phenomenon could occur in galactic dynamics, [15]. Many experts, however, doubted their predictions. But in 1964 Malmberg and Wharton observed Landau damping in one of their experiments, [16]. After that, Landau damping was widely accepted, and mathematicians tried to prove its existence.

In the unbounded Euclidean space, the time decay of the solution of the Vlasov-Poisson system has been characterized by Illner and Rein, [11], Perthame, [19], and Glassey and Shaeffer in the linearized case, [9]. In periodic space, Caglioti and Maffei, [3], gave an inverse scattering approach and Hwang and Velázquez, [10], found some damped solutions. The breakthrough, however, was achieved by Mouhot and Villani in 2009 for the proof of nonlinear Landau damping, [26]. In 2013, Bedrossian, Masmoudi, and Mouhot gave a simpler proof for the existence of this phenomenon and extended the class of potentials, [2]. Recently, Faou and Rousset proved Landau damping for the Vlasov-HMF (Hamiltonian Mean-Field) model, [8].

Most of the work mentioned above is done in Euclidean space. The nature of the physical space, however, is unknown. Physicists agree, nevertheless, that the large-scale universe has constant Gaussian curvature, which could be zero (Euclidean space), positive (elliptic space), or negative (hyperbolic space), [5], [7]. We will therefore further consider the problem in curved space[2]. Although on a small scale we could consider the universe as flat, on a large scale we need to take the curved case into consideration. A universe of nonzero constant Gaussian curvature would correspond to spheres, for positive curvature, and to hyperbolic spheres, for negative curvature. We will treat here the former case, aiming to understand the Landau damping phenomenon.

In [14], by using the equations of motion on the sphere or hyperbolic space, Lind derived the Vlasov-Poisson system on $\mathbb{S}^2$ in the form

$$\begin{cases} \dfrac{\partial f}{\partial t} + x \cdot \nabla_v f + \nabla_0 U \cdot \nabla_0 f = 0 \\ \Delta_0 U = -\rho \\ \rho(t,x) = \displaystyle\int_{\mathbb{R}^2} f(t,x,v)dv \\ x \cdot v = 0, \end{cases}$$

[2]The gravitational interaction potential is no longer the classic one, as explained in [14], [5], [7], [6].

where, ∇_0 and Δ_0 are the gradient and the Laplace operators, respectively, and f is the distribution function, all of them on $\mathbb{S}^2$. Our purpose is to further analyze this system. To avoid the technical difficulties that arise in the general case, we will focus on the 1D problem, i.e. the Vlasov-Poisson system on a great circle of the unit sphere. Without loss of generality, we consider this great circle to be the equator of $\mathbb{S}^2$. So let us assume that the particles lie initially on the equator. By using spherical coordinates,

$$x = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),$$

the reduced system takes the form (see details in Chapter 4 and [14]):

$$\begin{cases} \dfrac{\partial f}{\partial t} + \omega\dfrac{\partial f}{\partial \theta} + \dfrac{\partial U}{\partial \theta}\dfrac{\partial f}{\partial \omega} = 0, \\ U(t,\theta) = \rho * \mathcal{W}, \\ \rho(t,\theta) = \displaystyle\int_{-\infty}^{+\infty} f(t,\theta,\omega)d\omega, \end{cases} \tag{1.1}$$

where $\mathcal{W}(\theta) = \frac{1}{2\pi}\log|\cot\theta|$ is the interaction potential, $\theta \in [0, 2\pi)$, ω is the angular velocity, f is the distribution on the equator, and $*$ denotes the convolution. The classical Vlasov-Poisson system on circle is, cf. [3],

$$\begin{cases} \dfrac{\partial f}{\partial t} + v\dfrac{\partial f}{\partial x} + \dfrac{\partial U}{\partial x}\dfrac{\partial f}{\partial v} = 0, \\ \dfrac{\partial U}{\partial x} = \rho * B, \\ \rho(t,x) = \displaystyle\int_{-\infty}^{+\infty} f(t,x,v)dv, \end{cases}$$

where B is a 2π-periodic function given by $B(x) = \frac{1}{2} - \frac{x}{2\pi}$ for $x \in [0, 2\pi)$, called the derivative of the interaction potential. Since $\mathcal{W}'(\theta) = -\frac{1}{2\pi}\frac{1}{\sin\theta}$ which is not even integrable, our potential is worse than that in the classical case.

Lind showed that the distribution function $f = f(\omega)$ is a homogeneous solution to the above system, [14]. But what happens if we slightly perturb the function f, does Landau damping still appear in this model? Our goal in this thesis is to study stability near a homogeneous solution and prove the result stated below.

Theorem 1.1 (Linear Landau damping on the equator)**.** *Assume that for the above system the stationary solution $f^0(\omega)$ and initial perturbation $h_0(\theta,\omega)$ are both analytic, $(f^0)'(\omega) = O(\frac{1}{|\omega|})$ for large $|\omega|$, and the following Penrose stability condition is* *satisfied:*

$$\forall \omega \in \mathbb{R}, \quad (f^0)'(\omega) = 0 \Longrightarrow |p.v. \int_{-\infty}^{\infty} \frac{(f^0)'(v)}{v-\omega} dv| < 1.$$

Then the solution is linearly stable. And there exists positive constants δ and C which are only dependent on the initial datum such that for large t,

$$\|\rho(t,\theta) - \int\int h_0(\theta,\omega) d\theta d\omega\|_{C^r(\mathbb{T})} \leq Ce^{-\delta t}$$

and

$$\|F(t,\theta)\|_{C^r(\mathbb{T})} \leq Ce^{-\delta t},$$

where $\|u\|_{C^r(\mathbb{T})} = \max_{0 \leq n \leq r, \theta \in \mathbb{T}} |\partial_\theta^n u(\theta)|$ and $r \in \mathbb{N}^+$.

Remark 1.2. *It is easy to exhibit examples of f^0 satisfying the hypothesis of this theorem. See for example in Section 5.5.*

The rest of this thesis is organized as follows. In Chapter 2, we review Vlasov's equation in Euclidean space and state a criterion for Landau damping. In Chapter 3, we introduce Vlasov-Poisson's system on $\mathbb{S}^2$. In Chapter 4, we derive the restriction of these equations to the equator. To do so, we begin with a distribution of matter on the equator and see what the system looks like in a distribution sense. In Chapter 5, we prove our main theorem using a classical method (cf. [25]) and perform some numerical tests to examine the derived criterion. We end the thesis with a summary and with some comments regarding the future study of the Vlasov's equation in curved space.

Chapter 2

The Vlasov Equation in Euclidean Space

2.1 System setting

The classical Vlasov equation describes the motion of a collisionless system of particles. A system of equations that describes the motion of each particle would be too large for a proper analysis. Since there are some 10^{13} stars in a galaxy, and even more particles in small volumes of plasma, it would be impossible to simulate the motion of each particle. A better choice is to describe the state of these particles through the phase distribution function $f(t,x,v)$ for a number of particles $N_{t,x,v} = f(t,x,v)dxdv$, located in $(x, x+dx)$ with velocity in $(v, v+dv)$ at time t, [23]. As long as the phase distribution function is determined, the state of the system is known.

One can derive (at least formally) the classic Vlasov equation by mean field approximation (cf. [25]):

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f + F(t,x) \cdot \nabla_v f = 0 \\ F = -W *_x \rho \\ \rho(t,x) = \displaystyle\int f(t,x,v)dv, \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^d$ or x is in a periodic space $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. The function $\rho = \rho(t,x)$ represents the density of the wave at time t and W is the given potential. For instance, when $d = 3$ the electrostatic interaction potential is $W(r) = const./r$ and the gravitational interaction potential is $W(r) = -const./r$.

2.2 Landau Damping

Landau Damping is an interesting phenomenon exhibited in the Vlasov-Poisson system. A popular explanation is that the damping occurs due to the energy exchange between particles and the wave with phase velocity. A particle whose velocity is close to but less than the wave velocity, gains energy from the wave; otherwise, it looses energy. So if the number of particles with lower speed is relatively large, damping may appear.

Mouhot and Villani gave the criterion for nonlinear damping under some assumption on the interaction potential, [26]. For simplicity, we consider the 1D case.

Let $f^0(v)$ be the homogeneous solution of (2.1) and $f(t,x,v) = f^0(v) + h(t,x,v)$ the solution after perturbation. Then the full nonlinear system is given by

$$\begin{cases} \dfrac{\partial h}{\partial t} + v\dfrac{\partial h}{\partial x} + F(t,x)(\dfrac{\partial f^0}{\partial v} + \dfrac{\partial h}{\partial v}) = 0 \\ F = -W *_x \rho \\ \rho(t,x) = \displaystyle\int h(t,x,v)dv. \end{cases} \tag{2.2}$$

Since h is small, we expect $F(t,x)\frac{\partial h}{\partial v}$ to be negligible. So we remove this term and get the linearized system:

$$\begin{cases} \dfrac{\partial h}{\partial t} + v\dfrac{\partial h}{\partial x} + F(t,x)\dfrac{\partial f^0}{\partial v} = 0 \\ F = -W *_x \rho \\ \rho(t,x) = \displaystyle\int h(t,x,v)dv. \end{cases} \tag{2.3}$$

The following theorems are the criteria for linear and nonlinear Landau damping in periodic flat space (cf. [25], [26]).

Proposition 2.3 (Linear damping). *If W is an interaction potential, such that $\nabla W \in L^1(\mathbb{T})$, $f^0 = f^0(v)$ is a distribution on $\mathbb{R}$ that is an analytic function satisfying $(f^0)'(v) = O(1/|v|)$, and the Penrose stability condition*

$$(f^0)'(\omega) = 0 \Rightarrow \widehat{W}(k) \int \frac{(f^0)'(v)}{v-\omega} dv < 1$$

for all $k \neq 0$. Then the linear Vlasov equation (2.3) is stable near f^0. And there exists positive constants δ and C which are only dependent on the initial datum such that for large t,

$$\|\rho(t,\theta) - \int\int h_0(\theta,\omega)d\theta d\omega\|_{C^r(\mathbb{T})} \leq Ce^{-\delta t}$$

and

$$\|F(t,\theta)\|_{C^r(\mathbb{T})} \leq Ce^{-\delta t},$$

where $h_0(x,v)$ is the initial perturbation and $\widehat{W}$ is the Fourier transform of W in x.

Before stating the result of nonlinear damping, we define the following analytic norm

$$|f|_{\lambda,\mu} = \sup_{k,\eta}\left(e^{2\pi\lambda|\eta|}e^{2\pi\mu|k|/L}|\tilde{f}(k,\eta)|\right),$$

where L is the period of x, $\tilde{f}(k,\eta)$ is the Fourier transform in both x and v.

Theorem 2.4 (Nonlinear damping, [26]). *Let us assume the condition for linear damping is satisfied. And furthermore W satisfies the condition*

$$\forall k \in \mathbb{Z}, \quad |\widehat{W}(k)| \leq \frac{C_W}{|k|^{1+\gamma}}$$

for some constants $C_W > 0$, $\gamma \geq 1$. Suppose also that for some λ and C_0, constant, we have

$$\sup(|\hat{f}^0(\eta)|e^{2\pi\lambda|\eta|}) \leq C_0, \quad \sum_{n\in\mathbb{Z}}\frac{\lambda^n}{n!}|\nabla_v^n f^0|_{L^1} \leq C_0 \leq +\infty,$$

where $\hat{f}^0$ is the Fourier transform of f^0 with respect to v. Then, if for $f_i(x,v) = f(0,x,v)$ and for any $0 < \lambda' < \lambda$, $\beta > 0$, $\mu > 0$, there is ε such that

$$|f_i - f^0|_{\lambda,\mu} + \int\int_{\mathbb{T}\times\mathbb{R}}|f_i - f^0|e^{\beta|v|}dvdx \leq \varepsilon,$$

the nonlinear Vlasov equation (2.2) is stable near f^0. And there exists a constant C such that for large t

$$\|\rho(t,x) - \int\int_{\mathbb{T}\times\mathbb{R}} f_i(x,v) - h_0(x,v)dxdv\|_{C^r(\mathbb{T})} \leq Ce^{-\lambda' t}$$

and

$$\|F(t,x)\|_{C^r(\mathbb{T})} \leq Ce^{-\lambda' t}.$$

Chapter 3

The Vlasov-Possion System on $\mathbb{S}^2$

Thanks to Lind's work [14], we can write the Vlasov-Possion system on $\mathbb{S}^2$ as

$$\begin{cases} \dfrac{\partial f}{\partial t} + x \cdot \nabla_v f + \nabla_0 U \cdot \nabla_0 f = 0 \\ \Delta_0 U = -\rho \\ \rho(t,x) = \displaystyle\int_{\mathbb{R}^2} f(t,x,v)dv \\ x \cdot v = 0. \end{cases} \tag{3.1}$$

However, it is difficult to analyze this system in Cartesian coordinates. It is easier to solve Poisson's equation on $\mathbb{S}^2$ and change the whole system to spherical coordinates. Besides, we will get the reduced model in spherical coordinates (see Chapter 4) which looks similar to the flat case.

Let's first solve Poisson's equation on $\mathbb{S}^2$ (cf. [14]). According to [4], the fundamental solution of the Laplace-Beltrami equation on $\mathbb{S}^2$ is

$$G(x,y) = \frac{1}{2\pi} \log \cot \frac{d(x,y)}{2},$$

where $d(x,y) = \arccos(x \cdot y)$ is the geodesic distance between two points, x, y, on $\mathbb{S}^2$. Thus the solution of Poisson's equation, $\Delta_0 U = -\rho$, is

$$U(x) = \int_{\mathbb{S}^2} G(x,y)\rho(y)dy = \frac{1}{2\pi}\int_{\mathbb{S}^2} \rho(y) \log \cot \frac{d(x,y)}{2} dy. \tag{3.2}$$

Substituting (3.2) back to (3.1) and employing spherical coordinates,

$$x(\theta) = (\sin\theta_1 \cos\theta_1, \sin\theta_1 \sin\theta_2, \cos\theta_1),$$

we can write the Vlasov-Poisson system on $\mathbb{S}^2$ (cf. [14]) as

$$\begin{cases} \dfrac{\partial f}{\partial t} + \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \cdot \nabla_\theta f + \begin{bmatrix} \frac{\partial U}{\partial \theta_1} + \omega_2^2 \sin\theta_1 \cos\theta_1 \\ \frac{1}{\sin\theta_1}\frac{\partial U}{\partial \theta_2} - 2\omega_1\omega_2 \cot\theta_1 \end{bmatrix} \cdot \nabla_\omega f = 0, \\ U(x) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{S}^2} \rho(y) \log\cot\frac{d(x,y)}{2} dy, \\ \rho = \displaystyle\int_{\mathbb{R}^2} f \sin\theta_1 d\omega_1\omega_2, \end{cases} \tag{3.3}$$

with a compatibility condition

$$\int_{\mathbb{S}^2} \rho(x) dx = 0,$$

where (ω_1, ω_2) is the angular velocity. The compatibility condition occurs from the well-posedness of Poisson's equation, [14].

Remark 3.1. *The compatibility condition is meaningless in physics since the density ρ should be positive. We therefore consider the mean of ρ and define*

$$\tilde{\rho} = \rho - \frac{1}{4\pi^2}\int_{\mathbb{S}^2} \rho(x) dx.$$

Then $\tilde{\rho}$ satisfies both Poisson's equation and the compatibility condition (cf. [25], [26]). Without loss of generality, the negative density and the distribution are also acceptable in the rest of this thesis.

Chapter 4

The Vlasov Equation on a Great Circle

Because great circles are invariant of the equations of motion (c.f. [14]), we further consider the restriction of system (3.3) to the equator. However, after the restriction we can see that singularities appear in the potential function. We will find a way to deal with this difficulty and show that properties we are interested still hold in the general case.

4.1 Restriction to $C_{1,2}$

With spherical coordinates:

$$x(\theta) = (\sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2, \cos\theta_1),$$

where $0 \leq \theta_1 \leq \pi$ and $0 \leq \theta_2 < 2\pi$. We define the equator as

$$C_{1,2} := \left\{ x(\theta) = (\theta_1, \theta_2) | \theta_1 = \frac{\pi}{2}, \theta_2 \in (0, 2\pi] \right\}.$$

The distribution function along $C_{1,2}$ can be written as

$$f(t, \theta_1, \theta_2, \omega_1, \omega_2) = \frac{\delta_{\theta_1 - \frac{\pi}{2}}}{\sin^2\theta_1} \otimes \delta_{\omega_1} \otimes \tilde{f}(t, \theta_2, \omega_2), \tag{4.1}$$

where $\tilde{f}$ is a smooth function. Then, the density becomes

$$\rho(t,\theta_1,\theta_2) = \frac{\delta_{\theta_1-\frac{\pi}{2}}}{\sin\theta_1} \otimes \tilde{\rho}(t,\theta_2), \tag{4.2}$$

where we set $\tilde{\rho}(t,\theta_2) = \int_{\mathbb{R}} \tilde{f}(t,\theta_2,\omega_2)d\omega_2$. Using trigonometric identities, the potential (3.2) can be written at any $x \in \mathbb{S}^2$ as:

$$\begin{aligned} U(x) &= \frac{1}{2\pi}\int_{\mathbb{S}^2} \rho(y)\log\cot\frac{d(x,y)}{2}dy \\ &= \frac{1}{2\pi}\int_{\mathbb{S}^2} \rho(y)\log\frac{1+x\cdot y}{\sqrt{1-(x\cdot y)^2}}dy \\ &= \frac{1}{4\pi}\int_{\mathbb{S}^2} \rho(y)\log\frac{1+x\cdot y}{1-x\cdot y}dy. \end{aligned} \tag{4.3}$$

Substituting (4.1), (4.2) and $x = x(\theta) = (\sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2, \cos\theta_1)$, $y = y(\theta') = (\sin\theta_1'\cos\theta_2', \sin\theta_1'\sin\theta_2', \cos\theta_1')$ into (4.3), we get

$$\begin{aligned} U(t,\theta_1,\theta_2) &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{\delta(\theta_1'-\frac{\pi}{2})}{\sin\theta_1'}\tilde{\rho}(t,\theta_2')\log\frac{1+x(\theta)y(\theta')}{1-x(\theta)y(\theta')}\sin\theta_1' d\theta_1' d\theta_2' \\ &= \frac{1}{4\pi}\int_0^{2\pi} \tilde{\rho}(t,\theta_2')\log\frac{1+\sin\theta_1\cos(\theta_2-\theta_2')}{1-\sin\theta_1\cos(\theta_2-\theta_2')}d\theta_2'. \end{aligned} \tag{4.4}$$

This is the potential energy located at $(\theta_1,\theta_2) \in \mathbb{S}^2$ at time t when the particles are distributed on the unit sphere. But we are interested here only in the restriction to $\theta_1 = \frac{\pi}{2}$, i.e. when the particles lie along $C_{1,2}$. This restriction on U gives the potential energy of the reduced model (see details in Proposition 4.2). If we denote $\tilde{U}(t,\theta_2) = U(t,\theta_1,\theta_2)|_{\theta_1=\frac{\pi}{2}}$, then we have

$$\begin{aligned} \tilde{U}(t,\theta_2) &= \frac{1}{4\pi}\int_0^{2\pi} \tilde{\rho}(t,\theta_2')\log\frac{1+\cos(\theta_2-\theta_2')}{1-\cos(\theta_2-\theta_2')}d\theta_2' \\ &= \frac{1}{2\pi}\int_0^{2\pi} \tilde{\rho}(t,\theta_2')\log\cot\frac{d(\theta_2,\theta_2')}{2}d\theta_2', \end{aligned}$$

where

$$d(\theta_2,\theta_2') = d(x(\theta),y(\theta'))|_{\theta_1=\theta_2=\frac{\pi}{2}} = \begin{cases} |\theta_2-\theta_2'|, & \text{if } |\theta_2-\theta_2'| \le \pi \\ 2\pi - |\theta_2-\theta_2'|, & \text{if } |\theta_2-\theta_2'| \ge \pi. \end{cases}$$

Observing that

$$\frac{1}{2\pi}\log\cot\frac{d(\theta-\theta')}{2}=\frac{1}{2\pi}\log|\cot\frac{\theta-\theta'}{2}|,$$

it is natural to define an integral kernel

$$\mathcal{W}(\theta)=\frac{1}{2\pi}\log|\cot\frac{\theta}{2}|. \tag{4.5}$$

Then $\widetilde{U}$ can be written as

$$\widetilde{U}(t,\theta_2)=\tilde{\rho}*\mathcal{W}. \tag{4.6}$$

In the distribution sense (since delta function appears), we put (4.1)∼(4.4) back to the first equation of (3.3) and get the following proposition.

Proposition 4.2. *In the distribution sense, the Vlasov equation on $C_{1,2}$ can be written as*

$$\begin{cases}\dfrac{\partial\tilde{f}}{\partial t}+\omega_2\dfrac{\partial\tilde{f}}{\partial\theta_2}+\dfrac{\partial\tilde{U}}{\partial\theta_2}\dfrac{\partial\tilde{f}}{\partial\omega_2}=0,\\ \tilde{U}(t,\theta_2)=\tilde{\rho}*\mathcal{W},\\ \tilde{\rho}(t,\theta_2)=\displaystyle\int_{-\infty}^{+\infty}\tilde{f}(t,\theta_2,\omega_2)d\omega_2,\end{cases} \tag{4.7}$$

with the compatibility condition

$$\int_0^{2\pi}\tilde{\rho}(t,\theta_2)d\theta_2=0,$$

where

$$\mathcal{W}(\theta)=\frac{1}{2\pi}\log|\cot\frac{\theta}{2}|,$$

$\theta\in\mathbb{T}:=\mathbb{R}/(0,2\pi]$, *and* $\omega\in\mathbb{R}$.

Proof. Let $\phi(\theta_1,\theta_2,\omega_1,\omega_2)$ be a test function in $C_0^\infty([0,\pi]\times(0,2\pi]\times\mathbb{R}\times\mathbb{R})$, and f the distribution on $C_{1,2}$ given as (4.1). Substituting f into the Vlasov equation of (3.3), testing by ϕ, the first term is

$$\begin{aligned}<\frac{\partial f}{\partial t},\phi> &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi}\int_0^{\pi}\delta(\theta_1-\frac{\pi}{2})\delta(\omega_1)\partial_t\tilde{f}(t,\theta_2,\omega_2)\phi d\theta_1 d\theta_2 d\omega_1 d\omega_2\\ &= \int_{\mathbb{R}}\int_0^{2\pi}\frac{\partial\tilde{f}}{\partial t}\phi|_{\theta_1=\frac{\pi}{2},\omega_1=0}d\theta_2 d\omega_2\\ &= <\frac{\partial\tilde{f}}{\partial t},\psi>_{\mathbb{T}\times\mathbb{R}},\end{aligned} \tag{4.8}$$

where $\psi = \phi|_{\theta_1=\frac{\pi}{2},\omega_1=0}$.
Second term: $<\omega_1\frac{\partial f}{\partial\theta_1}+\omega_2\frac{\partial f}{\partial\theta_2},\phi>=<\omega_1\frac{\partial f}{\partial\theta_1},\phi>+<\omega_2\frac{\partial f}{\partial\theta_2},\phi>$ and

$$\begin{aligned}
<\omega_1\frac{\partial f}{\partial\theta_1},\phi> &= <\omega_1 f, -\frac{\partial\phi}{\partial\theta_1}> \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi}\int_0^{\pi} -\frac{\partial\phi}{\partial\theta_1}\delta(\theta_1-\frac{\pi}{2})\delta(\omega_1)\omega_1\tilde{f}d\theta_1 d\theta_2 d\omega_1 d\omega_2 \\
&= 0, \qquad (4.9)
\end{aligned}$$

$$\begin{aligned}
<\omega_2\frac{\partial f}{\partial\theta_2},\phi> &= <\omega_2 f, -\frac{\partial\phi}{\partial\theta_2}> \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi}\int_0^{\pi} -\frac{\partial\phi}{\partial\theta_2}\delta(\theta_1-\frac{\pi}{2})\delta(\omega_1)\omega_2\tilde{f}d\theta_1 d\theta_2 d\omega_1 d\omega_2 \\
&= \int_{\mathbb{R}}\int_0^{2\pi} -\frac{\partial\phi}{\partial\theta_2}|_{\theta_1=\frac{\pi}{2},\omega_1=0}\omega_2\tilde{f}d\theta_2 d\omega_2 \\
&= \int_{\mathbb{R}}\int_0^{2\pi} \omega_2\frac{\partial\tilde{f}}{\partial\theta_2}\phi|_{\theta_1=\frac{\pi}{2},\omega_1=0}d\theta_2 d\omega_2 \\
&= <\omega_2\frac{\partial\tilde{f}}{\partial\theta_2},\psi>_{\mathbb{T}\times\mathbb{R}}\ . \qquad (4.10)
\end{aligned}$$

Together with (4.9) and (4.10), we can write

$$<\omega_1\frac{\partial f}{\partial\theta_1}+\omega_2\frac{\partial f}{\partial\theta_2},\phi>=<\omega_2\frac{\partial\tilde{f}}{\partial\theta_2},\psi>_{\mathbb{T}\times\mathbb{R}}\ . \qquad (4.11)$$

The third term can be divided into four parts:

$$<\frac{\partial U}{\partial\theta_1}\frac{\partial f}{\partial\omega_1},\phi> \qquad (4.12)$$

$$<\omega_2^2\sin\theta_1\cos\theta_1\frac{\partial f}{\partial\omega_1},\phi> \qquad (4.13)$$

$$<\frac{1}{\sin\theta_1}\frac{\partial U}{\partial\theta_2}\frac{\partial f}{\partial\omega_2},\phi> \qquad (4.14)$$

$$<-2\omega_1\omega_2\cot\theta_1\frac{\partial f}{\partial\omega_2},\phi>\ . \qquad (4.15)$$

It is easy to see that (4.13) and (4.15) are both 0. So we focus on (4.12) and (4.14).

$$\begin{aligned}
(4.12) &= <\frac{\partial U}{\partial \theta_1} f, -\frac{\partial \phi}{\partial \omega_1}> \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi}\int_0^{\pi} -\frac{\partial \phi}{\partial \omega_1}\frac{\partial U}{\partial \theta_1}\delta(\theta_1 - \frac{\pi}{2})\delta(\omega_1)\tilde{f} d\theta_1 d\theta_2 d\omega_1 d\omega_2 \\
&= \int_{\mathbb{R}}\int_0^{2\pi} \left(-\frac{\partial \phi}{\partial \omega_1}|_{\theta_1=\frac{\pi}{2},\omega_1=0}\right)\left(\frac{\partial U}{\partial \theta_1}|_{\theta_1=\frac{\pi}{2}}\right)\tilde{f} d\theta_2 d\omega_2 \\
&= 0.
\end{aligned} \tag{4.16}$$

For the last step we use the fact that[1] $\frac{\partial U}{\partial \theta_1}|_{\theta_1=\frac{\pi}{2}} = 0$.

$$\begin{aligned}
(4.14) &= <\frac{1}{\sin\theta_1}\frac{\partial U}{\partial \theta_2} f, -\frac{\partial \phi}{\partial \omega_2}> \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi}\int_0^{\pi} -\frac{\partial \phi}{\partial \omega_2}\frac{1}{\sin\theta_1}\frac{\partial U}{\partial \theta_2}\delta(\theta_1 - \frac{\pi}{2})\delta(\omega_1)\tilde{f} d\theta_1 d\theta_2 d\omega_1 d\omega_2 \\
&= \int_{\mathbb{R}}\int_0^{2\pi} \left(-\frac{\partial \phi}{\partial \omega_2}|_{\theta_1=\frac{\pi}{2},\omega_1=0}\right)\left(\frac{\partial U}{\partial \theta_2}|_{\theta_1=\frac{\pi}{2}}\right)\tilde{f} d\theta_2 d\omega_2 \\
&= \int_{\mathbb{R}}\int_0^{2\pi} \psi\frac{\partial \tilde{U}}{\partial \theta_2}\tilde{f} d\theta_2 d\omega_2 \\
&= <\frac{\partial \tilde{U}}{\partial \theta_2}\tilde{f}, \psi>_{\mathbb{T}\times\mathbb{R}} .
\end{aligned} \tag{4.17}$$

(4.8), (4.11), (4.16) and (4.17) together, we have (4.7). From the compatibility condition, we have that

$$\begin{aligned}
0 &= \int_0^{\pi}\int_0^{2\pi} \frac{\delta(\theta_1 - \frac{\pi}{2})}{\sin\theta_1}\tilde{\rho}(t,\theta_2)\sin\theta_1 d\theta_2 \theta_1 \\
&= \int_0^{2\pi} \tilde{\rho}(t,\theta_2) d\theta_2.
\end{aligned}$$

This remark completes the proof. □

For simplicity, we redenote $\tilde{f}, \tilde{U}, \tilde{\rho}, \theta_2, \omega_2$ by $f, U, \rho, \theta, \omega$, and the the system in the

[1] We refer to [14] for a detailed proof, based on the equations of motion. From the physical point of view, the force acting on the particles is always tangent to the great circle.

distribution sense becomes

$$\begin{cases} \dfrac{\partial f}{\partial t} + \omega \dfrac{\partial f}{\partial \theta} + \dfrac{\partial U}{\partial \theta}\dfrac{\partial f}{\partial \omega} = 0, \\ U(t,\theta) = \rho * \mathcal{W}, \\ \rho(t,\theta) = \displaystyle\int_{-\infty}^{+\infty} f(t,\theta,\omega) d\omega, \end{cases} \tag{4.18}$$

with the compatibility condition

$$\int_0^{2\pi} \rho(t,\theta) d\theta = 0. \tag{4.19}$$

4.3 Singularity in the Force Term

Knowing only U is not enough for our analysis. We still need the expression of F since we need to prove that F tends to 0. Compared with the Vlasov equation in flat space (2.1), the force term in (4.18) is

$$F(t,\theta) = \frac{\partial U}{\partial \theta}. \tag{4.20}$$

Formally, we have

$$F = -\frac{1}{2\pi} \int_0^{2\pi} \rho(\theta') \frac{1}{\sin(\theta - \theta')} d\theta'. \tag{4.21}$$

However, we can easily find that there are two kinds of singular points for F, namely when

$$\theta' = \theta \text{ and } \theta' = \begin{cases} \pi + \theta, & \text{if } \theta \leq \pi, \\ \theta - \pi, & \text{if } \theta > \pi. \end{cases}$$

To overcome this difficulty, we consider the principle value. In this way we can "cancel" the infinity part if there are singular points under the integral as if there is no singularity, and then the principle value becomes the classic integral.

Definition 4.4. *Assuming $K(\theta)$ is a singular integral kernel with singular points $\theta_1, \theta_2 \ldots \theta_n$, then the principle value distribution is defined as follows. For any smooth function f on $\mathbb{T}$,*

$$p.v. \int_0^{2\pi} K(\theta) f(\theta) d\theta := \lim_{\varepsilon \to 0} \int_{(0,2\pi]/\cup_{i=1}^n [\theta_i - \varepsilon, \theta_i + \varepsilon]} f(\theta) K(\theta) d\theta.$$

Let's start to rewrite U in principle value form and F will come after U. The following proposition describes the relation between U in (3.3) and its principle value form.

Proposition 4.5. *Let U be the integral in* (3.3)*, and $\mathcal{W}$ is defined as* (4.5)*. Then*

$$U(\theta) = \rho * \mathcal{W} = p.v.(\rho * \mathcal{W}).$$

Proof. Since now all the calculation is only in the space variable, we can neglect the time t. We fix $\theta \leq \pi$ (the case $\theta > \pi$ is the same) and let r be a constant such that $\varepsilon \leq r \ll \min\{2, \theta\}$. Then

$$\begin{aligned}
& 2\pi \left(p.v. \int_0^{2\pi} \rho(\theta')\mathcal{W}(\theta - \theta') d\theta \right) \\
= \; & \lim_{\varepsilon \to 0} \int_0^{\theta-\varepsilon} + \int_{\theta+\varepsilon}^{\pi+\theta-\varepsilon} + \int_{\pi+\theta+\varepsilon}^{2\pi} \rho(\theta') \log\cot \frac{d(\theta,\theta')}{2} d\theta' \\
= \; & \lim_{\varepsilon \to 0} \int_0^{\theta-\varepsilon} \rho(\theta') \log\cot \frac{\theta - \theta'}{2} d\theta' \qquad (4.22) \\
& + \lim_{\varepsilon \to 0} \int_{\theta+\varepsilon}^{\pi+\theta-\varepsilon} \rho(\theta') \log\cot \frac{\theta' - \theta}{2} d\theta' \qquad (4.23) \\
& + \lim_{\varepsilon \to 0} \int_{\pi+\theta+\varepsilon}^{2\pi} \rho(\theta') \log\cot \frac{2\pi - \theta' + \theta}{2} d\theta'. \qquad (4.24)
\end{aligned}$$

For (4.22),

$$\begin{aligned}
& \lim_{\varepsilon \to 0} \int_0^{\theta-\varepsilon} \rho(\theta') \log\cot \frac{\theta - \theta'}{2} d\theta' \\
= \; & \int_0^{\theta-r} \rho(\theta') \log\cot \frac{\theta - \theta'}{2} d\theta' + \lim_{\varepsilon \to 0} \int_{\theta-r}^{\theta-\varepsilon} \rho(\theta') \log\cot \frac{\theta - \theta'}{2} d\theta'.
\end{aligned}$$

In the first term of the above equation there are no singular points. We thus only need to worry about the second term. If we put $\alpha = \theta - \theta'$ into, we have

$$\begin{aligned}
& \lim_{\varepsilon \to 0} \int_{\theta-r}^{\theta-\varepsilon} \rho(\theta') \log\cot \frac{\theta - \theta'}{2} d\theta' \\
= \; & \lim_{\varepsilon \to 0} \int_\varepsilon^r \rho(\theta - \alpha) \log\cot \frac{\alpha}{2} d\alpha \\
= \; & \int_0^r \rho(\theta - \alpha) \log\cot \frac{\alpha}{2} d\alpha. \qquad (4.25)
\end{aligned}$$

Note that ρ is bounded if ρ is smooth on $\mathbb{T}$, and that $\log\cot\frac{\alpha}{2} \sim -\log\frac{\alpha}{2}$ when $\alpha \to 0$. Furthermore, we have

$$\int_0^r -\log\frac{\alpha}{2}d\alpha \leq \int_0^2 -\log\frac{\alpha}{2}d\alpha = 2.$$

Thus (4.25) is finite and

$$\lim_{\varepsilon\to 0}\int_{\theta-r}^{\theta-\varepsilon}\rho(\theta')\log\cot\frac{\theta-\theta'}{2}d\theta' = \int_{\theta-r}^{\theta}\rho(\theta')\log\cot\frac{\theta-\theta'}{2}d\theta',$$

which means from (4.22) that

$$\lim_{\varepsilon\to 0}\int_{0}^{\theta-\varepsilon}\rho(\theta')\log\cot\frac{\theta-\theta'}{2}d\theta' = \int_{0}^{\theta}\rho(\theta')\log\cot\frac{\theta-\theta'}{2}d\theta'.$$

Similarly, because W is integrable near its singular points and ρ is bounded on $\mathbb{T}$, we have for (4.23) and (4.24) that

$$\lim_{\varepsilon\to 0}\int_{\theta+\varepsilon}^{\pi+\theta-\varepsilon}\rho(\theta')\log\cot\frac{\theta'-\theta}{2}d\theta' = \int_{\theta}^{\pi+\theta}\rho(\theta')\log\cot\frac{\theta'-\theta}{2}d\theta'$$

$$\lim_{\varepsilon\to 0}\int_{\pi+\theta+\varepsilon}^{2\pi}\rho(\theta')\log\cot\frac{2\pi-\theta'+\theta}{2}d\theta' = \int_{\pi+\theta}^{2\pi}\rho(\theta')\log\cot\frac{2\pi-\theta'+\theta}{2}d\theta'.$$

Substituting them back to (4.22)∼(4.24), the proposition is proved. □

We also expect that F can be written in principle value form like U. The next proposition shows that this can be done, indeed.

Proposition 4.6. *Consider $\mathcal{W}(\theta) = \frac{1}{2\pi}\log|\cot\frac{\theta}{2}|$ and for ρ' bounded, define the force*

$$F(\theta) = \rho' * \mathcal{W}.$$

Then

$$F = p.v.(\rho * \mathcal{W}').$$

Proof. Assuming $\theta \le \pi$, for $\theta > \pi$, the proof is the same.

$$\begin{aligned} U(\theta) &= \frac{1}{4\pi}\int_0^{2\pi} \rho(\theta') \log \frac{1+\cos(\theta-\theta')}{1-\cos(\theta-\theta')} d\theta' \\ &= \frac{1}{4\pi}\int_0^{2\pi} \rho(\theta-\theta') \log \frac{1+\cos\theta'}{1-\cos\theta'} d\theta' \\ &= \frac{1}{2\pi}\int_0^{2\pi} \rho(\theta-\theta') \log\cot \frac{d(0,\theta')}{2} d\theta'. \end{aligned}$$

So,

$$\begin{aligned} 2\pi\frac{\partial}{\partial\theta}U &= \int_0^{2\pi} \frac{\partial}{\partial\theta}\rho(\theta-\theta') \log\cot\frac{d(0,\theta')}{2} d\theta' \\ &= \lim_{\varepsilon\to 0}\int_0^{\theta-\varepsilon} + \int_{\theta-\varepsilon}^{\pi+\theta-\varepsilon} + \int_{\pi+\theta+\varepsilon}^{2\pi} \frac{\partial}{\partial\theta}\rho(\theta-\theta')\log\cot\frac{d(0,\theta')}{2}d\theta' \quad (4.26) \\ &\quad + \lim_{\varepsilon\to 0}\int_{\theta-\varepsilon}^{\theta+\varepsilon} + \int_{\pi+\theta-\varepsilon}^{\pi+\theta+\varepsilon} \frac{\partial}{\partial\theta}\rho(\theta-\theta')\log\cot\frac{d(0,\theta')}{2}d\theta'. \quad (4.27) \end{aligned}$$

As long as ρ' is bounded, the integral in (4.27) is well defined according to the discussion in the proof of Proposition 4.5. So (4.27) $= 0$ as $\varepsilon \to 0$. For (4.26),

$$\begin{aligned} &\int_0^{\theta-\varepsilon} + \int_{\theta-\varepsilon}^{\pi+\theta-\varepsilon} + \int_{\pi+\theta+\varepsilon}^{2\pi} \frac{\partial}{\partial\theta}\rho(\theta-\theta')\log\cot\frac{d(0,\theta')}{2}d\theta' \\ =& \int_0^{\theta-\varepsilon} + \int_{\theta-\varepsilon}^{\pi+\theta-\varepsilon} + \int_{\pi+\theta+\varepsilon}^{2\pi} -\rho'(\theta')\log\cot\frac{d(\theta,\theta')}{2}d\theta'. \quad (4.28) \end{aligned}$$

The first term in (4.28) is

$$\begin{aligned} &\int_0^{\theta-\varepsilon} -\rho'(\theta')\log\cot\frac{d(\theta,\theta')}{2}d\theta' \\ =& -\rho(\theta')\log\cot\frac{d(\theta,\theta')}{2}\Big|_{\theta'=0}^{\theta'=\theta-\varepsilon} + \int_0^{\theta-\varepsilon}\rho(\theta')\frac{\partial}{\partial\theta'}\log\cot\frac{d(\theta,\theta')}{2}d\theta' \\ =& \int_0^{\theta-\varepsilon} -\rho(\theta')\frac{1}{\sin(\theta-\theta')}d\theta' \\ & -\rho(\theta-\varepsilon)\log\cot\frac{\varepsilon}{2} + \rho(0)\log\cot\frac{\theta}{2}. \end{aligned}$$

Similarly, the second and third terms in (4.28) are

$$\begin{aligned}
&\int_{\theta+\varepsilon}^{\pi+\theta-\varepsilon} -\rho'(\theta') \log\cot\frac{d(\theta,\theta')}{2} d\theta' \\
= &\int_{\theta+\varepsilon}^{\pi+\theta-\varepsilon} -\rho(\theta') \frac{1}{\sin(\theta-\theta')} d\theta' \\
&-\rho(\pi+\theta-\varepsilon)\log\cot\frac{\pi-\varepsilon}{2} + \rho(\theta+\varepsilon)\log\cot\frac{\varepsilon}{2},
\end{aligned}$$

$$\begin{aligned}
&\int_{\pi+\theta+\varepsilon}^{2\pi} -\rho'(\theta') \log\cot\frac{d(\theta,\theta')}{2} d\theta' \\
= &\int_{\pi+\theta+\varepsilon}^{2\pi} -\rho(\theta') \frac{1}{\sin(\theta-\theta')} d\theta' \\
&-\rho(2\pi)\log\cot\frac{\theta}{2} + \rho(\pi+\theta+\varepsilon)\log\cot\frac{\pi-\varepsilon}{2}.
\end{aligned}$$

Adding these terms together, we get

$$2\pi F(\theta) = \lim_{\varepsilon\to 0}\left(\int_0^{\theta-\varepsilon} + \int_{\theta-\varepsilon}^{\pi+\theta-\varepsilon} + \int_{\pi+\theta+\varepsilon}^{2\pi} -\rho(\theta')\frac{1}{\sin(\theta-\theta')}d\theta' + \mathcal{E}(\varepsilon)\right), \qquad (4.29)$$

where

$$\mathcal{E}(\varepsilon) = (\rho(\theta+\varepsilon)-\rho(\theta-\varepsilon))\log\cot\frac{\varepsilon}{2} + (\rho(\pi+\theta+\varepsilon)-\rho(\pi+\theta-\varepsilon))\log\cot\frac{\pi-\varepsilon}{2}.$$

Then

$$\begin{aligned}
\lim_{\varepsilon\to 0}\mathcal{E}(\varepsilon) &= \lim_{\varepsilon\to 0}\frac{\rho(\theta+\varepsilon)-\rho(\theta-\varepsilon)}{2\varepsilon}2\varepsilon\log\cot\frac{\varepsilon}{2} \\
&\quad + \lim_{\varepsilon\to 0}\frac{\rho(\pi+\theta+\varepsilon)-\rho(\pi+\theta-\varepsilon)}{2\varepsilon}2\varepsilon\log\cot\frac{\pi-\varepsilon}{2} \\
&= \rho'(\theta)\lim_{\varepsilon\to 0}2\varepsilon\log\cot\frac{\varepsilon}{2} + \rho'(\pi+\theta)\lim_{\varepsilon\to 0}2\varepsilon\log\cot\frac{\pi-\varepsilon}{2} \\
&= 0,
\end{aligned}$$

as long as ρ' is bounded. Substituting it back to (4.29), we obtain

$$\begin{aligned}
F &= p.v.\int_0^{2\pi} -\frac{1}{2\pi}\rho(\theta')\frac{1}{\sin(\theta-\theta')}d\theta' \\
&= p.v.(\rho * \mathcal{W}').
\end{aligned}$$

The last step follows from the fact that $\mathcal{W}'(\theta) = -\frac{1}{2\pi}\frac{1}{\sin\theta}$.

This remark completes the proof. $\square$

4.7 The Fourier Transform of F

The Fourier Transform is very important in analysis. In the proof of our main theorem, we will use the Fourier Transform of F. For any $f \in L^1(\mathbb{T})$, the Fourier Transform of f is

$$\hat{f}(k) = \int_{\mathbb{T}} f(x)e^{-ikx}dx,$$

where $k \in \mathbb{N}$. For a two-variable function $f(x,v) \in L^1(\mathbb{T}\times\mathbb{R})$, the Fourier Transform in two variables is

$$\tilde{f}(k,\eta) = \int_{\mathbb{R}}\int_{\mathbb{T}} f(x,v)e^{-ikx}e^{-i\eta v}dxdv,$$

where $k \in \mathbb{N}, \eta \in \mathbb{R}$.

Noting that F can be formally written as a convolution on $\mathbb{T}$, we can write

$$F = \mathcal{W}' * \rho.$$

Then by the convolution theorem, the Fourier transform of F satisfies

$$\widehat{F} = \widehat{\mathcal{W}'}\widehat{\rho}.$$

We expect that this result is still true under Proposition 4.6. Before showing that, let us compute the Fourier transform of $\mathcal{W}'$. However, $\mathcal{W}' = -\frac{1}{2\pi}\frac{1}{\sin\theta} \notin L^1(\mathbb{T})$, we need to redefine its Fourier transform in principle value form:

$$\widehat{\mathcal{W}'} := p.v. \int_0^{2\pi} \mathcal{W}'(\theta)e^{-ik\theta}d\theta.$$

Proposition 4.8. *Let $\mathcal{W} = \frac{1}{2\pi}\log|\cot\frac{\theta}{2}|$, then*

$$\widehat{\mathcal{W}'}(k) = \begin{cases} 0 & k \text{ even} \\ i & k \text{ odd.} \end{cases}$$

Proof.

$$
\begin{aligned}
\widehat{\mathcal{W}'}(k) &= \lim_{\varepsilon\to 0} -\frac{1}{2\pi}\left(\int_{\varepsilon}^{\pi-\varepsilon} + \int_{\pi+\varepsilon}^{2\pi-\varepsilon} \frac{1}{\sin\theta} e^{-i\theta k} d\theta\right) \\
&= \lim_{\varepsilon\to 0} \left[-\frac{1}{2\pi}\int_{\varepsilon}^{\pi-\varepsilon} \frac{1}{\sin\theta} e^{-i\theta k} d\theta \right. \\
&\quad \left. + \frac{1}{2\pi}\int_{\varepsilon}^{\pi-\varepsilon} \frac{1}{\sin\theta}\left(\cos k\theta + i\sin k\theta\right) d\theta\right] \quad \text{(change of variable } \theta' = 2\pi - \theta) \\
&= -\frac{1}{2\pi}\int_{0}^{\pi} -2i\frac{\sin k\theta}{\sin\theta} d\theta \\
&= \frac{i}{\pi}\left(\int_0^{\pi/2} + \int_{\pi/2}^{\pi}\right)\frac{\sin k\theta}{\sin\theta} d\theta \\
&= \frac{i}{\pi}\int_0^{\pi/2}\frac{\sin k\theta}{\sin\theta} d\theta \\
&\quad \frac{i}{\pi}\int_0^{\pi/2}\left(\frac{\sin k\theta + sin(k\pi - k\theta)}{\sin\theta}\right) d\theta \quad \text{(change of variable } \theta' = \pi - \theta) \\
&= \begin{cases} 0 & \text{k even} \\ \frac{2i}{\pi}\int_0^{\pi/2}\frac{\sin k\theta}{\sin\theta} d\theta & \text{k odd.} \end{cases}
\end{aligned}
$$

If k is odd and let $I(k) = \int_0^{\pi/2}\frac{\sin k\theta}{\sin\theta} d\theta$, then

$$
\begin{aligned}
I(k+2) &= \int_0^{\pi/2}\frac{\sin(k\theta + 2\theta)}{\sin\theta} d\theta \\
&= \int_0^{\pi/2}\frac{\sin k\theta\cos 2\theta + \cos k\theta \sin 2\theta}{\sin\theta} d\theta \\
&= \int_0^{\pi/2}\frac{\sin k\theta}{\sin\theta} - 2\sin k\theta\sin\theta + 2\cos k\theta\cos\theta d\theta \\
&= I(k) + 2\int_0^{\pi/2}\cos(k\theta + \theta) d\theta \\
&= I(k)
\end{aligned}
$$

So, $I(k) = I(1) = \frac{\pi}{2}$ if k is odd, which implies

$$
\widehat{\mathcal{W}'}(k) = \begin{cases} 0 & \text{k even} \\ i & \text{k odd} \end{cases}
$$

This remark completes the proof. □

Our next proposition shows that F still satisfies the convolution theorem.

Proposition 4.9. *Let*

$$F = p.v.(\mathcal{W}' * \rho).$$

Then

$$\widehat{F} = \widehat{\mathcal{W}'}\hat{\rho} = \begin{cases} 0 & \text{if } k \text{ even} \\ i\hat{\rho} & \text{if } k \text{ odd.} \end{cases}$$

Proof. According to the definition of F and the Fourier transform,

$$\begin{aligned} \hat{F} &= \int_0^{\pi} \left[\lim_{\varepsilon \to 0} \int_0^{\theta-\varepsilon} + \int_{\theta+\varepsilon}^{\pi+\theta-\varepsilon} + \int_{\pi+\theta+\varepsilon}^{2\pi} -\frac{1}{2\pi} \frac{1}{\sin(\theta-\theta')} \rho(\theta') d\theta' \right] e^{-ik\theta} d\theta \\ &+ \int_{\pi}^{2\pi} \left[\lim_{\varepsilon \to 0} \int_0^{\theta-\pi-\varepsilon} + \int_{\theta-\pi+\varepsilon}^{\theta-\varepsilon} + \int_{\theta+\varepsilon}^{2\pi} -\frac{1}{2\pi} \frac{1}{\sin(\theta-\theta')} \rho(\theta') d\theta' \right] e^{-ik\theta} d\theta. \end{aligned} \tag{4.30}$$

All the areas marked with letters except $B'1, C'1$ in Figure 4.1 are the integral areas of (4.30). Furthermore, all the functions are 2π periodic so that the integrals on $B1, C1$ are equal to the integrals on $B'1, C'1$ respectively. Thus, we have (at the second step we change the variable $\alpha = \theta - \theta'$),

$$\begin{aligned} \hat{F}(k) &= \lim_{\varepsilon \to 0} \int_0^{2\pi} \left[\int_{\theta'-\pi+\varepsilon}^{\theta'-\varepsilon} + \int_{\theta'+\varepsilon}^{\theta'+\pi-\varepsilon} -\frac{1}{2\pi} \frac{1}{\sin(\theta-\theta')} \rho(\theta') e^{-ik\theta} d\theta \right] d\theta' \\ &= \lim_{\varepsilon \to 0} \int_0^{2\pi} \left[\int_{-\pi+\varepsilon}^{-\varepsilon} + \int_{+\varepsilon}^{\pi-\varepsilon} -\frac{1}{2\pi} \frac{1}{\sin(\alpha)} \rho(\theta') e^{-ik(\theta'+\alpha)} d\alpha \right] d\theta' \\ &= \hat{\rho}(k) \lim_{\varepsilon \to 0} \left[\int_{-\pi+\varepsilon}^{-\varepsilon} + \int_{+\varepsilon}^{\pi-\varepsilon} -\frac{1}{2\pi} \frac{1}{\sin(\alpha)} e^{-ik\alpha} d\alpha \right] \\ &= \hat{\rho}(k) \widehat{\mathcal{W}'}(k) \\ &= \begin{cases} 0 & \text{if } k \text{ even} \\ i\hat{\rho} & \text{if } k \text{ odd.} \end{cases} \end{aligned}$$

This remark completes the proof. ∎

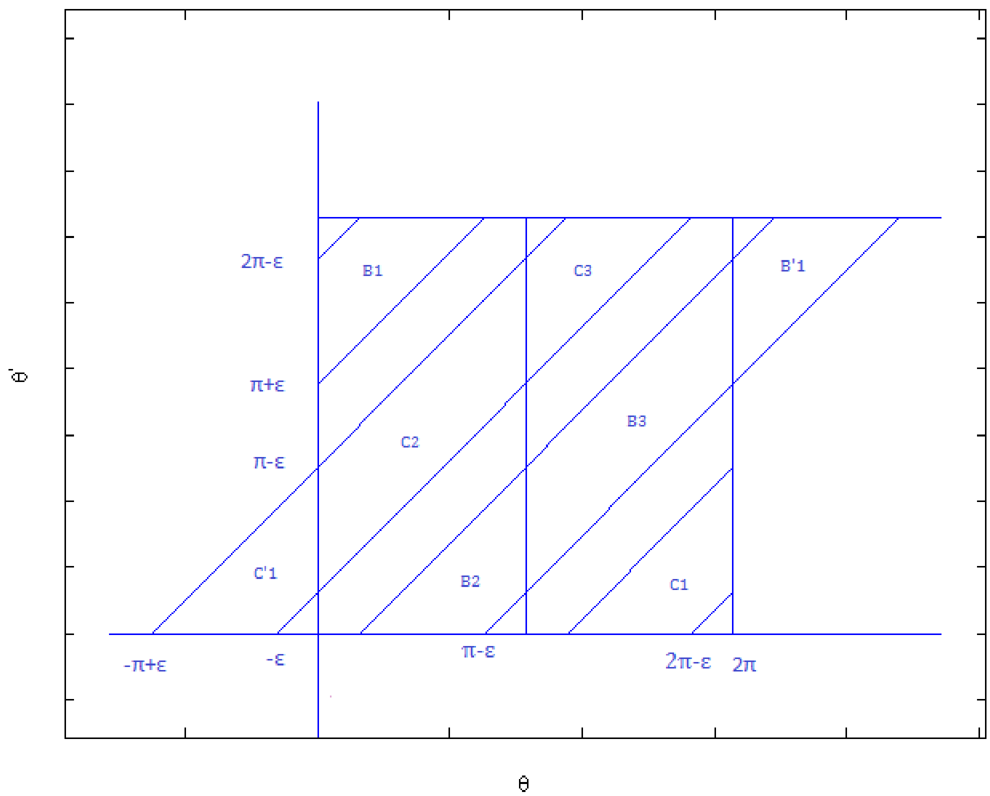

Figure 4.1: Integral area for (4.30)

Chapter 5

Linear Damping on a Great Circle

In this chapter, we prove Theorem 1.1 using a classical method (cf. [25]).

5.1 Linearization around the steady solution

Recall that system (4.18) on $C_{1,2}$ is

$$\begin{cases} \dfrac{\partial f}{\partial t} + \omega \dfrac{\partial f}{\partial \theta} + \dfrac{\partial U}{\partial \theta} \dfrac{\partial f}{\partial \omega} = 0, \\ U(t,\theta) = \rho^f * \mathcal{W}, \\ \rho^f(t,\theta) = \displaystyle\int_{-\infty}^{+\infty} f(t,\theta,\omega) d\omega. \end{cases}$$

Replacing $\frac{\partial U}{\partial \theta}$ by F, we can write this system as

$$\begin{cases} \dfrac{\partial f}{\partial t} + \omega \dfrac{\partial f}{\partial \theta} + F \dfrac{\partial f}{\partial \omega} = 0, \\ F(t,\theta) = p.v.(\mathcal{W}' * \rho^f), \\ \rho^f(t,\theta) = \displaystyle\int_{-\infty}^{+\infty} f(t,\theta,\omega) d\omega. \end{cases} \tag{5.1}$$

Write $f = f^0(\omega) + h(t,\theta,\omega)$, where $f^0 = f^0(\omega)$ is the stationary solution of (4.18) and $h = h(t,\theta,\omega)$ is the small perturbation with the initial value $h_0 = h_0(\theta,\omega)$. Then the corresponding density is $\rho^f = \rho^0 + \rho$, where

$$\rho^0 = \int_{\mathbb{R}} f^0(\omega) d\omega = \text{ constant;}$$

$$\rho(t,\theta,\omega) = \int_{\mathbb{R}} h d\omega.$$

According to proposition 4.6,

$$\begin{aligned} F(t,\theta) &= \mathcal{W} * ((\rho^0)' + \rho') \\ &= \mathcal{W} * \rho' \\ &= p.v.(\mathcal{W}' * \rho). \end{aligned}$$

In the linearization process, the term $F\frac{\partial h}{\partial \omega}$ is assumed to be small and thus neglected to obtain the linearized system:

$$\begin{cases} \dfrac{\partial h}{\partial t} + \omega \dfrac{\partial h}{\partial \theta} + F\dfrac{\partial f^0}{\partial \omega} = 0 \\ F(t,\theta) = p.v.(\mathcal{W}' * \rho) \\ \rho = \displaystyle\int_{-\infty}^{+\infty} h(t,\theta,\omega) d\omega, \end{cases} \tag{5.2}$$

where

$$\mathcal{W}(\theta) = \log|\cot\frac{\theta}{2}|, \theta \in [0, 2\pi).$$

5.2 Proof of Theorem 1.1

Recall that the linear stability around the homogeneous solution $f^0(\omega)$ means for the linear system (5.2) that when $t \to \infty$, ρ converges to its mean value and F decays to 0 exponentially fast. We restate our main result as follows.

Theorem 1.1 *For the linear system* (5.2)*, if the stationary solution* $f^0(\omega)$ *and initial perturbation* $h_0(\theta,\omega)$ *are both analytic functions,* $(f^0)'(\omega) = O(\frac{1}{|\omega|})$ *for large* $|\omega|$*, and if the Penrose stability condition,*

$$\forall \omega \in \mathbb{R}, \quad (f^0)'(\omega) = 0 \Longrightarrow |p.v. \int_{-\infty}^{\infty} \frac{(f^0)'(v)}{v-\omega} dv| < 1, \tag{5.3}$$

is satisfied, then (5.2) *is linearly stable. And there exists positive constants* δ *and* C *which are only dependent on the initial datum such that for large* t*,*

$$\|\rho(t,\theta) - \int\int h_0(\theta,\omega) d\theta d\omega\|_{C^r(\mathbb{T})} \le Ce^{-\delta t}$$

and

$$\|F(t,\theta)\|_{C^r(\mathbb{T})} \leq Ce^{-\delta t},$$

where $\|u\|_{C^r(\mathbb{T})} = \max_{0\leq n\leq r,\theta\in\mathbb{T}} |\partial_\theta^n u(\theta)|$ *and* $r \in \mathbb{N}^+$.

Proof: As in the flat case (cf. [25]), our goal here is to find the condition such that the Fourier transform of $\rho(t,\cdot)$ converges to 0 exponentially fast and uniformly for $k \neq 0$, so that ρ converges to its mean and F converges to 0 exponentially fast. For $k = 0$, we have the following result.

Proposition 5.3. *If* h *is the solution to the linear system* (5.2) *and* $h_0(\theta,\omega)$ *is the initial perturbation, then*

$$\hat{\rho}(t,0) = \tilde{h_0}(0,0),$$

which is preserved in time.

Proof. For simplicity, we use the notation $< g >$ as the spatial average of g:

$$< g >= \int_{\mathbb{T}} g d\theta.$$

Applying $< \cdot >$ to (5.2) yields

$$\begin{aligned} \frac{\partial}{\partial t} < h > &= - < \omega \frac{\partial h}{\partial \theta} > - < F \frac{\partial f^0}{\partial \omega} > \\ &= -\omega < \frac{\partial h}{\partial \theta} > - < F > \frac{\partial f^0}{\partial \omega}. \end{aligned}$$

Since $\frac{\partial h}{\partial \theta}$ and F are derivatives of functions on $\mathbb{T}$,

$$< \frac{\partial h}{\partial \theta} >=< F >= 0.$$

So, $< h >$ is preserved in time. Furthermore,

$$\begin{aligned} \hat{\rho}(t,0) &= \int_{\mathbb{T}} \int_{\mathbb{R}} h(t,\theta,\omega) d\theta d\omega \\ &= \int_{\mathbb{R}} < h > dv, \end{aligned}$$

is also independent on time. Therefore,

$$\hat{\rho}(t,0) = \int_{\mathbb{T}}\int_{\mathbb{R}} h(0,\theta,\omega)d\theta d\omega = \hat{h}_0(0,0).$$

This remark completes the proof. □

For $k \neq 0$, we need to find the equation satisfied for $\hat{\rho}(t,k)$.

Step 1: Solving (5.2).

Let $S = F\frac{\partial f^0}{\partial \omega}$ and apply the method of characteristics and the Duhamel principle to (5.2). We could easily get the solution

$$h(t,\theta,\omega) = h_0(\theta - \omega t, \omega) - \int_0^t S(\tau, \theta - \omega(t-\tau), \omega)d\tau. \tag{5.4}$$

Step 2: Applying the Fourier Transform.

Taking Fourier transform both in θ and ω, we have that

$$\begin{aligned}
\tilde{h}(t,k,\eta) &= \int\int h_0(\theta-\omega t,\omega)e^{-i\theta k}e^{-i\omega\eta}d\theta d\omega \\
&\quad - \int\int\int_0^t S(\tau,\theta-\omega(t-\tau),\omega)d\tau e^{-i\theta k}e^{-i\omega\eta}d\theta d\omega \\
&= \int\int h_0(\theta,\omega)e^{-ik(\theta+\omega t)}e^{-i\omega\eta}d\theta d\omega \\
&\quad - \int_0^t\int\int S(\tau,\theta,\omega)e^{-ik(\theta+\omega(t-\tau))}e^{-i\omega\eta}d\theta d\omega d\tau \\
&= \tilde{h}_0(k,\eta+kt) - \int_0^t \widetilde{S}(\tau,k,\eta+k(t-\tau))d\tau.
\end{aligned} \tag{5.5}$$

On the other hand, we should note that S has the structure of separated variables. Together with Proposition 4.9, we have

$$\begin{aligned}
\widetilde{S}(\tau,k,\eta) &= \widehat{F}(\tau,k)\widehat{\frac{\partial f^0}{\partial\omega}}(\eta) \\
&= i\eta\widehat{F}(\tau,k)\hat{f}^0(\eta) \\
&= \begin{cases} 0 & \text{k even} \\ -\eta\hat{\rho}(\tau,k)\hat{f}^0(\eta) & \text{k odd.} \end{cases}
\end{aligned} \tag{5.6}$$

Substituting (5.6) into (5.5), we get

$$\tilde{h}(t,k,\eta) = \begin{cases} \tilde{h}_0(k,\eta+kt) & \text{k even} \\ \tilde{h}_0(k,\eta+kt) - \displaystyle\int_0^t -(\eta+k(t-\tau))\hat{\rho}(\tau,k)\hat{f}^0(\eta+k(t-\tau))d\tau & \text{k odd.} \end{cases} \tag{5.7}$$

Step 3: $\eta = 0$

In (5.7), $\eta = 0$ implies:

$$\begin{aligned} \hat{\rho}(t,k) &= \tilde{h}(t,k,0) \\ &= \begin{cases} \tilde{h}_0(k,kt) & \text{k even} \\ \tilde{h}_0(k,kt) - \displaystyle\int_0^t -k(t-\tau)\hat{\rho}(\tau,k)\hat{f}^0(k(t-\tau))d\tau & \text{k odd} \end{cases} \\ &= \tilde{h}_0(k,kt) + \int_0^t K(t-\tau)\hat{\rho}(\tau,k)dt, \end{aligned} \tag{5.8}$$

where

$$K(t,k) = \begin{cases} 0 & \text{k even} \\ -kt\hat{f}^0(kt) & \text{k odd.} \end{cases}$$

If k is given, then (5.8) is a Volterra equation. The result below estimates the solution of this equation (see Appendix A.1 for the proof).

Lemma 5.4. *Assume that ϕ is a solution of the Volterra equation*

$$\phi(t) = a(t) + \int_0^t K(t-\tau)\phi(\tau)d\tau,$$

where a and the kernel K satisfy the conditions

(i) $|K(t)| \leq C_0 e^{-\lambda_0 t}$;

(ii) $|K^L(\xi) - 1| \geq \kappa > 0$ *for* $0 \leq Re\xi \leq \Lambda$;

(iii) $|a(t)| \leq \alpha e^{-\lambda t}$.

Then for any $\lambda' \leq \min(\lambda, \lambda_0, \Lambda)$,

$$|\phi(t)| \leq Ce^{-\lambda' t},$$

where

$$C = \alpha + \frac{C_0\alpha}{2\sqrt{(\lambda_0 - \lambda')(\lambda - \lambda')}},$$

and $K^L(\xi)$ *stands for the complex Laplace transform defined as*

$$K^L(\xi) = \int_0^\infty e^{\bar{\xi}t} K(t)dt \quad \textit{for } \xi \in \mathbb{C}.$$

Apply this lemma to (5.8), if at first we choose the stationary solution f^0 and initial perturbation h_0 analytic, then for large t, $|K(t,k)| = O(e^{-\lambda_0|k|t})$ and $|\tilde{h}_0(k,kt)| = O(e^{-\lambda|k|t})$, condition (i) and (iii) are satisfied. Furthermore, if (ii) is satisfied, we have the decay of $\hat{\rho}$

$$|\hat{\rho}(t,k)| = O(e^{-\lambda'|k|t}),$$

where $\lambda < \min\{\lambda, \lambda_0\}$ is a constant. Therefore, for any $r \in \mathbb{N}^+$ and large t, there exists a positive constant C which is only depended on initial datum such that

$$\begin{aligned} |\partial_\theta^r \left(\rho(t,k) - \int\int h_0(\theta,\omega)d\theta d\omega\right)| &\leq \sum_{k\neq 0} |k|^r |\hat{\rho}(t,k)| \\ &\leq C\sum_{k\neq 0} |k|^r e^{-\lambda'|k|t}, \end{aligned}$$

When t is large, there also exists a positive constant $\delta < \lambda'$ such that for all $k \neq 0$ and $r \in \mathbb{N}^+$

$$e^{-(\lambda'-\delta)|k|t} \leq |k|^{-r-2}.$$

Then

$$\begin{aligned} |\partial_\theta^r \left(\rho(t,k) - \int\int h_0(\theta,\omega)d\theta d\omega\right)| &\leq \sum_{k\neq 0} |k|^r e^{-\lambda'|k|t} \\ &\leq C\sum_{k\neq 0} |k|^{-2} e^{-\delta|k|t} \\ &\leq \frac{C}{2}\sum_{k\geq 1} k^{-2} e^{-\delta t} \\ &\leq \frac{C\bar{C}}{2} e^{-\delta t}, \end{aligned}$$

where $\bar{C} = \sum_{k=1}^\infty k^{-2}$.

For the force, according to Proposition 4.9,

$$\partial_\theta^r F(t,\theta) = \sum_{k \text{ odd}} (ik)^r i\hat{\rho} e^{ik\theta}.$$

So

$$|\partial_\theta^r F(t,\theta)| \leq \sum_{k \text{ odd}} |k|^r |\hat{\rho}(t,k)|.$$

Arguing as for ρ, we obtain for large t,

$$\|F(t,\theta)\|_{C^r(\mathbb{T})} \leq \frac{C\bar{C}}{2} e^{-\delta t}.$$

Now, the only thing we need to check is condition (ii) if we want the exponential decay of ρ. Condition (ii) basically means when $Re\xi$ is located at the positive neighborhood of 0, the Laplace transform of the kernel $K^L(\xi)$ should be away from 1.
When k is even, $K^L(\xi) \equiv 0$, so it's always away from 1. When k is odd, we evaluated the Laplace transform of K in time at $\xi = (\lambda - i\omega)k$:

$$\begin{aligned}
K^L(\xi) &= -\int_0^\infty kt\hat{f}^0(kt)e^{(\lambda+i\omega)tk}dt \\
&= -\frac{1}{k}\int_0^\infty t\hat{f}^0(t)e^{(\lambda+i\omega)t}dt \\
&= -\frac{1}{k}\int_0^\infty \int_{\mathbb{R}} tf^0(v)e^{-ivt}e^{(\lambda+i\omega)t}dvdt \\
&= \frac{1}{ik}\int_0^\infty \int_{\mathbb{R}} f^0(v)\frac{d}{dv}(e^{-ivt})dve^{(\lambda+i\omega)t}dt \\
&= \frac{i}{k}\int_0^\infty \int_{\mathbb{R}} (f^0)'(v)e^{-ivt}e^{(\lambda+i\omega)t}dvdt \\
&= \frac{1}{k}\int_{\mathbb{R}} \frac{(f^0)'(v)}{i\lambda + (v-\omega)}dv.
\end{aligned} \tag{5.9}$$

If furthermore, the stationary solution f^0 satisfies that $(f^0)'(v)$ decays at least like $O(\frac{1}{|v|})$, then with (5.9) we obtain

$$|K^L(\xi)| \leq C\left|\frac{1}{k}\int_{\mathbb{R}} \frac{1}{v}\frac{1}{i\lambda - \omega + v}dv\right|. \tag{5.10}$$

Let $L(\varepsilon)$ be upper half circle centered at 0 with radius ε. Some simple computations show that

$$\begin{aligned}\frac{1}{k}\int_{\mathbb{R}}\frac{1}{v}\frac{1}{i\lambda-\omega+v}dv &= \frac{1}{k}\lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}+\int_{\varepsilon}^{+\infty}+\int_{L(\varepsilon)}\frac{1}{v}\frac{1}{i\lambda-\omega+v}dv \\ &= \frac{\pi i}{k(i\lambda-\omega)}+\frac{1}{k}\lim_{\varepsilon\to 0}\int_{\pi}^{0}\frac{1}{\varepsilon e^{i\phi}}\frac{1}{i\lambda-\omega+\varepsilon e^{i\phi}}d(\varepsilon e^{i\phi}) \quad (\text{let } v=\varepsilon e^{i\phi}) \\ &= \frac{2\pi i}{k(i\lambda-\omega)},\end{aligned}$$

which means that (5.9) decays at least like $O(\frac{1}{|\omega|})$ as $\omega\to\infty$, uniformly for $\lambda\in[0,\lambda_0]$; so we can only consider the case when $|\omega|$ is bounded. Hence, assume that $|\omega|\le\Omega$. If (5.9) does not go to 1 in the limit $\lambda\to 0^+$, by the continuity there exists $\Lambda>0$ such that (5.9) is away from 1 in the domain $\{|\omega|\le\Omega, 0\le\lambda\le\Lambda\}$. Thus, we could only focus on the limit $\lambda\to 0^+$. In order to compute this limit, we would like to introduce the Plemelj formula

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0+iy}dx = p.v\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}dx - i\pi f(x_0). \tag{5.11}$$

Applying (5.11) to (5.9),we have:

$$\lim_{\lambda\to 0^+}K^L((\lambda-i\omega)k,k)=\frac{1}{k}\left(p.v\int_{-\infty}^{\infty}\frac{(f^0)'(v)}{v-\omega}dv - i\pi(f^0)'(\omega)\right). \tag{5.12}$$

Now, we need to find conditions such that (5.12) does not approach 1. First, if the imaginary part of (5.12) stays away from 0, then it is all fine. However, the imaginary part goes to 0 only if k goes to ∞, in which case the real part also goes to 0 or if $(f^0)'(\omega)\to 0$. So we only need to consider the case when $(f^0)'(\omega)$ approaches 0. Hence, we get a condition:

$$\forall\omega\in\mathbb{R},\quad (f^0)'(\omega)=0\Longrightarrow \frac{1}{k}p.v.\int_{-\infty}^{\infty}\frac{(f^0)'(v)}{v-\omega}dv<1. \tag{5.13}$$

If $1/k$ and $p.v.\int_{-\infty}^{\infty}\frac{(f^0)'(v)}{v-\omega}dv$ have different sign, (5.13) is always true. While if they have the same sign, it is sufficient to consider the case $|k|=1$. Eventually, the stability criterion becomes:

$$\forall\omega\in\mathbb{R},\quad (f^0)'(\omega)=0\Longrightarrow |p.v.\int_{-\infty}^{\infty}\frac{(f^0)'(v)}{v-\omega}dv|<1.$$

5.5 Numerical Simulation

5.5.1 Numerical Scheme

In this section, we run a numerical test to examine the linear stability criterion (5.3). Here is our setting:

- The stationary solution: $f^0(\omega) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\omega^2}{2\sigma^2}}$.
- Initial perturbation: $h_1 = 0.001\cos(\theta) f^0(\omega)$, and $h_2 = 0.001\cos(\theta) e^{\sin(\theta)} f^0(\omega)$.
- Grid size: $N \times N$, $\theta \in [0, 2\pi]$, $\omega \in [-10, 10]$, $N = 512$ or 1024.
- Time step: $dt = 0.0005$; Space step: $d\theta = 2\pi/N$; Velocity step: $d\omega = 20/N$.

Denoting g_{ij}^n as the numerical solution of g at $(ndt, id\theta, jd\omega)$. Recall our system is

$$\begin{cases} \dfrac{\partial h}{\partial t} + \omega \dfrac{\partial h}{\partial \theta} + F \dfrac{\partial f^0}{\partial \omega} = 0 \\ F(t,\theta) = -\dfrac{1}{2\pi} \left[p.v. \left(\dfrac{1}{\sin(\theta - \cdot)} * \rho(\cdot) \right) \right] \\ \rho = \displaystyle\int_{-\infty}^{+\infty} h(t,\theta,\omega) d\omega \end{cases} \tag{5.14}$$

We could assume h has spatial mean value 0, otherwise we take $\tilde{h} = h - \frac{1}{2\pi}\int_0^{2\pi} h d\theta$ so that $\tilde{h}$ still satisfies (5.14) and spatial mean value 0. Furthermore, the compatibility condition is automatically satisfied under this assumption.

The numerical method we use is inspired by [29]. The idea to solve (5.14) numerically is to obtain a linear ODE in time and then use the classic Runge-Kutta method (see [18], pp, 26-27) to get the solution. The details are the following.

Step1, Discretize θ-derivative

Noting that h is periodic in θ and quite smooth, the Fast Fourier transform (FFT) can be used to discretize the θ-derivative. Given h_{ij}^n, fixing n and j, after applying FFT to h_{ij}^n for $i = 1, 2 \ldots N$, we get $\hat{h}_{ij}^n$, the discrete Fourier transform (DFT) of h_{ij}^n. Since h_{ij}^n can be considered as a real signal, then

- For $i \le N/2 + 1$, $\hat{h}_{ij}^n$ represent the positive frequency part,
- For $i > N/2 + 1$, $\hat{h}_{ij}^n$ represent the negative frequency part.

In order to get the DFT of $(\frac{\partial h}{\partial \theta})^n_{ij}$, we rescale $\hat{h}^n_{ij}$ by multiplying $2\pi\sqrt{-1}(i-1)/N$ for $i \leq N/2+1$ and $-2\pi\sqrt{-1}(i-1)/N$ for $i > N/2+1$. At last apply inverse Fast Fourier transform (IFFT) to get $(\frac{\partial h}{\partial \theta})^n_{ij}$.

Step2, Compute ρ^n_i

Rectangular rule is used to compute ρ^n_i

$$\rho^n_i = \sum_{j=1}^{N} h^n_{ij} d\omega.$$

Here the domain of velocity is $|v| \leq v_{max} = 10$. The cut-off velocity v_{max} is carefully chosen such that h is well below round off zero for all t.

Step3, Compute F^n_i

Again, using rectangular rule. Since the definition of F is in principle value sense

$$F(t,\theta) = -\frac{1}{2\pi}\left[p.v.\left(\frac{1}{\sin(\theta-\cdot)} * \rho(\cdot)\right)\right]$$

We could take off the singular point in our scheme and make the numerical result finite. The scheme is

$$F^n_i = \sum_{\substack{j=1 \\ j\neq k_1, j\neq k_2}}^{n} \rho^n_j \frac{1}{\sin((i-j)d\theta)}$$

where

$$k_1 = i, \quad k_2 = \begin{cases} i - N/2 & \text{if } i > N/2 \\ i + N/2 & \text{if } i \leq N/2 \end{cases}$$

are two singular points.

Step4, Solve for h^{n+1}_{ij}

Now we have the line ODE

$$\frac{\partial h}{\partial t} = Q(h)$$

where

$$Q(h) = -\omega\frac{\partial h}{\partial \theta} - F\frac{\partial f^0}{\partial \omega}$$

By Runge-Kutta 4th method

$$
\begin{aligned}
K_1 &= Q(h_{ij}^n) \\
K_2 &= Q(h_{ij}^n + K_1 dt/2) \\
K_3 &= Q(h_{ij}^n + K_2 dt/2) \\
K_4 &= Q(h_{ij}^n + K_3 dt) \\
h_{ij}^{n+1} &= h_{ij}^n + (K_1 + 2K_2 + 2K_3 + K_4)dt/6.
\end{aligned}
$$

We obtain h_{ij}^{n+1}
The collection of codes is in Appendix A.2. We should mention that all the computation is done through GPU acceleration. Our codes only run well on a computer with Nvidia video card[1].

5.5.2 Result

According to the stability criterion (5.3), we deduce that for $f^0(\omega) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\omega^2}{2\sigma^2}}$ the linear system is stable when $\sigma > 1$. The following figures are numerical results. x-axis stands for time and y-axis stands for $\log \max F$.

From Figure 5.1, we can see the force decays exponentially fast. For Figure

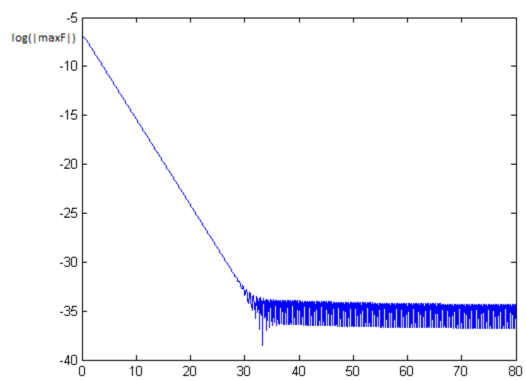

Figure 5.1: $\sigma = 1.5$, $N = 512$, $h_1 = 0.001\cos(\theta)f^0(\omega)$

5.2, we use h_2 as the initial perturbation and the force still decay fast but we also observe the numerical noise which will reduce and finally go away with better grid refinement(Figure 5.3). For $\sigma < 1$, system is unstable, see Figure 5.4 and 5.5.

[1]We use matlab code and GPU computing in matlab only supports Nvidia video card now.

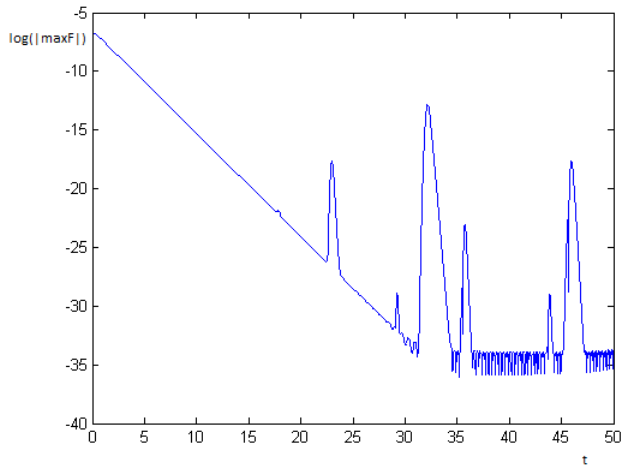

Figure 5.2: $\sigma = 1.5$, $N = 512$, $h_2 = 0.001 \cos(\theta) e^{\sin(\theta)} f^0(\omega)$

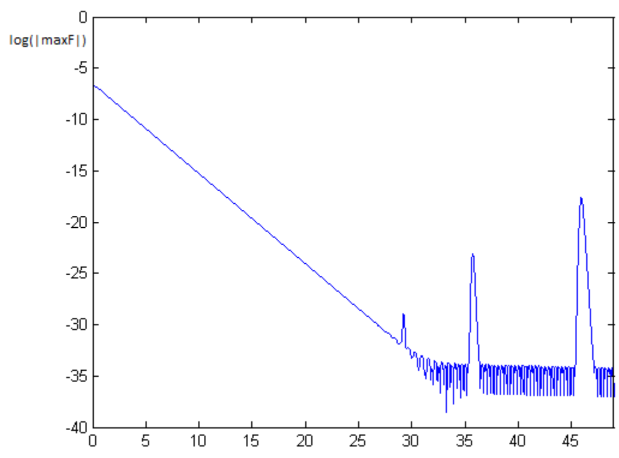

Figure 5.3: $\sigma = 1.5$, $N = 1024$, $h_2 = 0.001 \cos(\theta) e^{\sin(\theta)} f^0(\omega)$

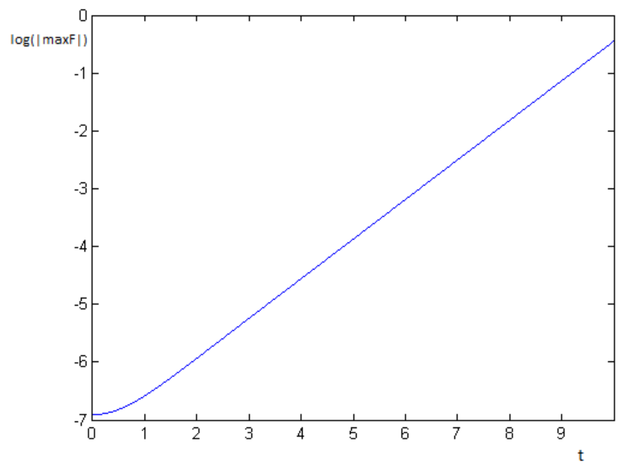

Figure 5.4: $\sigma = 0.5,\ N = 512,\ h_1 = 0.001\cos(\theta)f^0(\omega)$

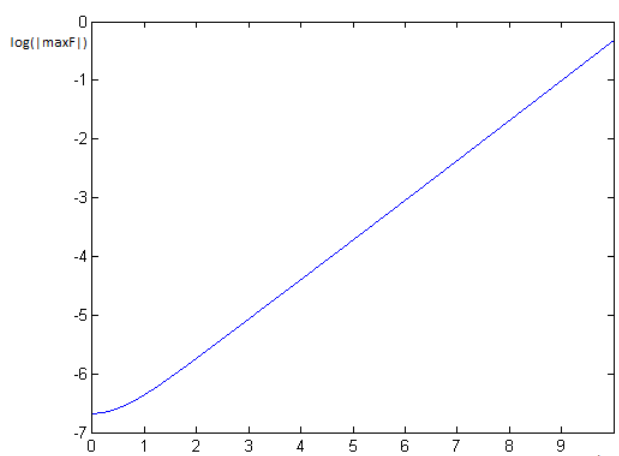

Figure 5.5: $\sigma = 0.5,\ N = 512,\ h_2 = 0.001\cos(\theta)e^{\sin(\theta)}f^0(\omega)$

Chapter 6

Conclusions and Extensions

In this thesis, we explore the Vlasov equation on curved space obtained in [14]. We analyze the singularity that appears after the restriction of the equation to a great circle and rewrite the equation properly by using the principle value. Thanks to [25] and [29], we derive the criterion for linear damping of our reduced model and perform a numerical test.

We only consider here the linear damping, but hope to prove the existence of nonlinear damping of the reduced model or the linear stability in the whole $\mathbb{S}^2$. More generally, we can consider the equation in spaces of negative curvature, namely on the hyperbolic sphere $\mathbb{H}^2$. We expect that our result can be extended to higher dimensional curved spaces, especially $\mathbb{S}^3$ and $\mathbb{H}^3$, cases which would be relevant to our universe.

Appendix A

A.1 Proof of Lemma 5.4

One can also see the proof in [25]. We prove it here for readers' convenience.
Lemma 5.4: If we have an equation like this

$$\phi(t) = a(t) + \int_0^t K(t-\tau)\phi(\tau)d\tau \tag{A.1}$$

and the kernel $K(t)$ satisfies:

(i) $|K(t)| \leq C_0 e^{-\lambda_0 t}$;

(ii) $|K^L(\xi) - 1| \geq \kappa > 0$ for $0 \leq Re\xi \leq \Lambda$;

(iii) $|a(t)| \leq \alpha e^{-\lambda t}$.

Then for any $\lambda' \leq \min(\lambda, \lambda_0, \Lambda)$,

$$|\phi(t)| \leq C\alpha e^{-\lambda' t},$$

where

$$C = \alpha + \frac{C_0 \alpha}{2\sqrt{(\lambda_0 - \lambda')(\lambda - \lambda')}}.$$

$K^L(\xi)$ stands for the complex Laplace transform defined as

$$\text{for } \xi \in \mathbb{C},\ K^L(\xi) = \int_0^\infty e^{\bar{\xi} t} K(t)dt.$$

Proof. Define $\Phi = e^{\lambda' t}\phi(t)$, $A(t) = e^{\lambda' t}a(t)$, then extend Φ, A, K by 0 for $t < 0$.

Multiplying (A.1) by $e^{\lambda' t}$ and we obtain

$$\Phi(t) = A(t) + \Big(K(\cdot) e^{\lambda' \cdot} * \Phi(\cdot) \Big) \tag{A.2}$$

Taking Fourier transform in time,

$$\begin{aligned} \widehat{\Phi}(\xi) &= \widehat{A}(\xi) + \widehat{\Phi}(\xi) \int_{\mathbb{R}} K(t) e^{\lambda' t} e^{-it\xi} dt \\ &= \widehat{A}(\xi) + \widehat{\Phi}(\xi) K^L(\lambda' + i\xi) \end{aligned}$$

Hence

$$\widehat{\Phi} = \frac{\widehat{A}}{1 - K^L(\lambda' + i\xi)}.$$

Taking L^2 norm and by Plancherel's identity and assumption (ii), one get an estimate of Φ,

$$\|\Phi\|_{L^2} \le \frac{\|A\|_{L^2}}{\kappa} \tag{A.3}$$

With assumption (iii),

$$\begin{aligned} \|A\|_{L^2} &= \left(\int_0^\infty e^{2\lambda' t} a^2(t) \right)^{1/2} \\ &\le \alpha \left(\int_0^\infty e^{-2(\lambda - \lambda')t} \right)^{1/2} \\ &\le \frac{\alpha}{\sqrt{2(\lambda - \lambda')}} \end{aligned} \tag{A.4}$$

Similarly, with assumption (i)

$$\|K e^{\lambda' t}\|_{L^2} \le \frac{C_0}{\sqrt{2(\lambda_0 - \lambda')}}$$

(A.3) and (A.4) together yield

$$\|\Phi\|_{L^2} \le \frac{\alpha}{\sqrt{2(\lambda - \lambda')}}$$

At last, taking L^∞ norm on (A.2) and applying Young's inequality

$$\begin{aligned}\|\Phi\|_{L^\infty} &\leq \|A\|_{L^\infty} + \|K(\cdot)e^{\lambda'\cdot} * \Phi(\cdot)\|_{L^\infty} \\ &\leq \|A\|_{L^\infty} + \|K(\cdot)e^{\lambda'\cdot}\|_{L^2}\|\Phi(\cdot)\|_{L^2} \\ &\leq \alpha + \frac{C_0}{\sqrt{2(\lambda_0-\lambda')}}\frac{\alpha}{\sqrt{2(\lambda-\lambda')}}\end{aligned}$$

Let $C = \alpha + \frac{C_0\alpha}{2\sqrt{(\lambda_0-\lambda')(\lambda-\lambda')}}$., we finish the proof. $\square$

A.2 Collection of codes

This code is for $\sigma = 1.5$, $N = 512$, and initial perturbation h_1.

```
% % % % vlasov.m % % %
% This is the main code file

% % % initial setting for all parameter
clear all;
sigma=1.5;
alpha=0.001;
% period in theta
T=2*pi;
% % % lengh the velocity space, max velocity is 10
L=20;
N=2^9;
th=[0:T/N:T-T/N]';
w=[-L/2:L/N:L/2-L/N]';
dth=T/N;
dw=L/N;
[W,TH]=meshgrid(w,th);
% % % turn into gpu array
W=gpuArray(W);
TH=gpuArray(TH);
% % % initial perturabtion h1
f0=h1(alpha, sigma, TH,W);
% % % g is the stationary solution, dgdw is its derivative
```

```
dgdw=dg(sigma, W);
% % % This matrix is for computing of Force
A=zeros(N);
for k = 1:N
for j=1:N
A(k,j)=1/sin(th(k)-th(j));
end
A(k,k)=0;
if k > N/2
A(k,k-N/2)=0;
else
A(k,k+N/2)=0;
end
end

Start=0; %start time
End=50; % end time
fi=f0;
dt=0.0005;
[t, maxF, f0, initial]=sol(Start, End, fi, W, dgdw, dth, dw, dt, A);
plot(t, log(maxF))
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %

% % % % sol.m % % % %
% % main solving function to step n+1
function [t, maxF, f0, initial]=sol(Start, End, f, W, dgdw, dth, dw, dt, A)
initial = f;
t=[0: dt: End]';
maxF=t;
maxF=gpuArray(maxF);

[n,m]=size(f);
dM=[(2*pi*1i/n*[0:n/2]'*ones(1,m));
(2*pi*1i/n*([n/2+1:n-1]'-n)*ones(1,m))];
intM=[ones(1,m); (2*pi*1i/n*[1:n/2]'*ones(1,m));
```

```
(2*pi*1i/n*([n/2+1:n-1]'-n)*ones(1,m))];
I=ones(1, m);
dM=gpuArray(dM);
intM=gpuArray(intM);
I=gpuArray(I);

f0=f;
time=Start;
n=1;
% % RK4 method
while time <= End
[k1, Fh] = Q(f0, W, dth, dw, dM, intM, I, dgdw, A);
[k2, temp] = Q(f0+dt/2*k1, W, dth, dw, dM, intM, I, dgdw, A);
[k3, temp] = Q(f0+dt/2*k2, W, dth, dw, dM, intM, I, dgdw, A);
[k4, temp] = Q(f0+dt*k3, W, dth, dw, dM, intM, I, dgdw, A);
f1=f0+dt/6*(k1+2*k2+2*k3+k4);
f0=f1;
time=time+dt;
maxF(n)=max(abs(Fh));
n=n+1;
end
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %

% % % % Q.m % % % %
% % function of Q, output is Q(f) and the corresponding force
function [y, force]=Q(f, W, dth, dw, dM, intM, I, dgdw, A)
% % % derivative of f w.r to theta.
dfdth=d(f, dth, dM);
% % % density
rho=sum(f,2)*dw;
% % % Force
F=-A*rho*dth*I/2/pi;
force=F(:,1);

y=-W.*dfdth-F.*dgdw;
```

```
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %

% % % %d.m % % % %
% % derivative of X, using FFT and IFFT
function y=d(X,d,dM)
FX=fft(X);
%rescale the fourier coefficient
FdX=FX.*dM;
dX=ifft(FdX);
dX=dX/d;
dX=real(dX);
y=dX;
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %

% % % %h1.m % % % %
function y=h1(alpha, sigma, x, w)
y=alpha*cos(x).*g(sigma, w);
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %

% % % %g.m % % % %
function y=g(sigma, x)
y=1/sqrt(2*pi)/sigma*exp(-x.^2/2/sigma^2)
% % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % % %

% % % %dg.m % % % %
% %derivative of g % %
function y=dg(sigma,x)
y=1/sqrt(2*pi)/sigma*exp(-x.^2/2/sigma^2).*(-x)/sigma^2;
```

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