Gray code numbers of complete multipartite graphs

by

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ABSTRACT

Let G be a graph and $k \geq \chi(G)$ be an integer. The k-colouring graph of G is the graph whose vertices are k-colourings of G, with two colourings adjacent if they colour exactly one vertex differently. We explore the Hamiltonicity and connectivity of such graphs, with particular focus on the k-colouring graphs of complete multipartite graphs. We determine the connectivity of the k-colouring graph of the complete graph K_n for all n, and show that the k-colouring graph of a complete multipartite graph K is 2-connected whenever $k \geq \chi(K) + 1$. Additionally, we examine a conjecture that every connected k-colouring graph is 2-connected, and give counterexamples for $k \geq 4$. As our main result, we show that for all $k \geq 2t$, the k-colouring graph of a complete t-partite graph is Hamiltonian. Finally, we characterize the complete multipartite graphs K whose $(\chi(K) + 1)$-colouring graphs are Hamiltonian.

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Chapter 1

Introduction

Let G be a graph and let k be a positive integer. The focus of our work is on *the k-colouring graph* of G, denoted $\mathcal{C}_k(G)$, which is the graph whose vertices are *proper k-colourings* of G, with two colourings adjacent if and only if they differ in the colour of exactly one vertex. For a graph G, we consider the Hamiltonicity and connectivity of $\mathcal{C}_k(G)$, for various values of k. Primarily, we will give results on the Hamiltonicity and connectivity of k-colouring graphs of complete multipartite graphs.

The problem of determining the Hamiltonicity of $\mathcal{C}_k(G)$ was first considered by Choo [8] in 2003 (also see [9]). Choo has shown that, given a graph G, there is a number $k_0(G)$ such that for all $k \geq k_0(G)$, $\mathcal{C}_k(G)$ is Hamiltonian. The number $k_0(G)$ is referred to as the *Gray code number* of G, as a Hamilton cycle in $\mathcal{C}_k(G)$ is a combinatorial Gray code.

The existence of $k_0(G)$ for any graph G suggests the obvious question: Given G, what is $k_0(G)$? Choo [8] answers this question for complete graphs, trees and cycles. Further work on this problem has been done by Celaya et al. [3], who determine Gray code numbers of complete bipartite graphs. The results of Celaya et al. [3] are a basis for the results of this thesis.

Connectivity of the k-colouring graph has been explored more thoroughly than Hamiltonicity of the k-colouring graph. This is in no small part due to its relevance to the *Glauber dynamics Markov chain* of k-colourings. This is the Markov chain whose states are k-colourings, and a transition between states occurs by selecting a colour c uniformly at random, and a vertex uniformly at random to be coloured with c. Algorithms for random sampling of k-colourings and approximating the number of k-colourings arise from these Markov chains, and connectivity of the k-colouring graph plays a pivotal role. Jerrum [18] gives a fully polynomial randomized approximation scheme for estimating the number of k-colourings of a graph when $k \geq 2\Delta(G) + 1$. Dyer et al. [11] give an algorithm for almost uniformly randomly generating a k-colouring of a random graph G with constant average degree, when k is sufficiently small compared to $\Delta(G)$. Lucier and Molloy [19] give results on Glauber dynamics Markov chains of bounded degree trees.

The problem of determining connectivity of the k-colouring graph of a graph G in general is considered by Cereceda et al. [5] in 2008. This work includes a proof that $\mathcal{C}_k(G)$ is connected whenever $k \geq 1 + Col(G)$, which follows from a result of Dyer et al. [11]. In addition, it is shown that in general there is no function $\phi(\chi(G))$ such that the $\phi(\chi(G))$-colouring graph of G is connected. If $\chi(G) = 2$ or 3, then the $\chi(G)$-colouring graph of G is *not* connected, and when $\chi(G) \geq 4$ there exist graphs for which the $\chi(G)$-colouring graph of G is connected. Connectivity of the 3-colouring graph of a bipartite graph is examined by Cereceda et al. [6]. Given a bipartite graph G, it is shown that $\mathcal{C}_3(G)$ is connected if and only if G is *pinchable* to C_6, where *pinching* refers to indentifying two vertices at distance two, and a graph G is pinchable to H when there is a series of pinches that transforms G into H. Some complexity results are also given. The problem of deciding whether or not the 3-colour graph of a bipartite graph is connected is shown to be coNP-Complete. In contrast, the problem of deciding whether or not two k-colourings are in the same component of $\mathcal{C}_k(G)$ is PSPACE-Complete when $k \geq 4$ [2], and in P when $k = 3$ [4].

Some alternate colour graphs have also been considered. Finbow and MacGillivray [12] consider variations of the k-colouring graph, the *k-Bell colour graph* and the *k-Stirling colour graph*. The k-Bell colour graph of G is the graph whose vertices are the partitions of the vertices of G into at most k independent sets. The k-Stirling colour graph of G is the graph whose vertices are the partitions of the vertices of G into exactly k independent sets. Various results on the Hamiltonicity and connectivity of such graphs are given.

Two colorings are referred to as *non-isomorphic* if they admit different partitions of $V(G)$. In 2012, Haas [16] examined the *canonical k-colouring graph* of G, whose vertices are non-isomorphic k-colourings which are lexographically least under some enumeration π of the vertices of G. Two vertices are adjacent if and only if they differ in the colour of exactly one vertex. It is shown that every graph has a canonical k-colouring graph which is not connected for some π and k. Additionally, it is shown that every tree T has an ordering π of its vertices such that the canonical k-colouring graph of T under π is Hamiltonian for every $k \geq 3$. Finally, it is shown that the canonical k-colouring graph of a cycle C, with $k \geq 4$, will always be connected under some π.

This thesis continues the work of finding Gray code numbers for classes of graphs. In particular, the class of complete multipartite graphs is examined. We also give results on the connectivity of colour graphs of complete multipartite graphs. Chapter 2 gives formal definitions and notation which will be used throughout this thesis, as well as an overview of some theorems that we will commonly reference. In Chapter 3, we discuss the connectivity of the colour graph of complete multipartite graphs. We find the connectivity of the k-colouring graph of a complete graph K_t, for all $k \geq t+1$. We show that the k-colouring graph of a complete multipartite graph has connectivity at least 2 whenever it is connected. We address whether or not a connected k-colouring graph is in general necessarily 2-connected, and show that this is false for $k \geq 4$. In Chapter 4, we examine a class of graphs, a subclass of what we call SDR graphs, which appear as subgraphs of k-colouring graphs of complete multipartite graphs. We show that these graphs will have always have Hamilton paths, and give results on the structure of such paths. In Chapter 5, for complete multipartite graphs K, we give our results regarding the Gray code number $k_0(K)$ of K. We establish an upper-bound on $k_0(K)$, and characterize the graphs K whose $(\chi(K)+1)$-colouring graphs are Hamiltonian. In Chapter 6, we close with a brief discussion of open problems.

Chapter 2

Background

In this chapter, we introduce the definitions and notation which will be used throughout the rest of this thesis. In addition, we present a selection of useful theorems on Hamiltonicity and connectivity of colour graphs.

2.1 Definitions and Notation

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. For u, $v \in V(G)$, we use the notation $u \sim v$ to denote $uv \in E(G)$. A *proper k-colouring* of G is a function $f : V(G) \to \{1, 2, \ldots, k\}$ such that if $v_i \sim v_j$, then $f(v_i) \neq f(v_j)$. We say a proper k-colouring *uses* the colour $c \in \{1, 2, \ldots, k\}$ if for some $v \in V(G)$, $f(v) = c$. The *proper k-colouring graph* of G, $\mathcal{C}_k(G)$ is the graph whose vertex set is the set of *proper k-colourings* of G, with two colourings being adjacent if and only if they differ in the colour of exactly one vertex of G. As we restrict our attention to only proper k-colourings, we will refer to the proper k-colouring graph and proper k-colourings as simply the k-colour graph and k-colourings respectively.

A complete t-partite graph $K_{a_1,a_2,\ldots,a_t}$ is the graph whose vertex set is partitoned by sets $V_1, V_2, \ldots, V_t$, with $|V_i| = a_i$, and for $v \in V_i$ and $u \in V_j$, $u \sim v$ if and only if $i \neq j$. Notice that the complete t-partite graph $K_{1,1,\ldots,1}$ is isomorphic to K_t. Unless otherwise stated, we assume without loss of generality that $a_1 \geq a_2 \geq \cdots \geq a_t$.

The *Gray code number* $k_0(G)$ of G is the smallest number such that $\mathcal{C}_k(G)$ is Hamiltonian for all $k \geq k_0(G)$. The existence of $k_0(G)$ for any graph G was shown by Choo [8], and a proof will be given at the end of this chapter. A Hamilton cycle in $\mathcal{C}_k(G)$ corresponds to a cyclic list of the k-colourings of G such that consecutive colourings in the list differ in the colour of exactly one vertex.

Let G be a graph where $\mathcal{C}_k(G)$ is Hamiltonian, and let $C = f_0, f_1, \ldots, f_{N-1}, f_0$ be a Hamilton cycle in $\mathcal{C}_k(G)$. We say that C has *Property* $\mathcal{A}$ if for all $c \in \{1, 2, \ldots, k\}$, there is an integer s such that, interpreting indices modulo N, neither f_s nor f_{s+1} color any vertex with c. If the integer s is assigned to colours c_1 and c_2, one of c_1 or c_2 can be reassigned the integer $s+1$, as adjacent colourings differ in the colour of exactly one vertex. Therefore, if a Hamilton cycle has Property $\mathcal{A}$, then each colour c can be assigned a *unique* integer s such that neither f_s nor f_{s+1} color any vertex with c. If $\mathcal{C}_k(G)$ has a Hamilton cycle with Property $\mathcal{A}$, we say $\mathcal{C}_k(G)$ is $\mathcal{A}$-*Hamiltonian*. This property is introduced in this thesis, and is used extensively as a construction tool throughout.

The notation $v_1, v_2, \ldots, v_i$ will be used to denote a path from v_1 to v_i, and the notation $v_1, v_2, \ldots, v_i, v_1$ will be used to denote a cycle. If $P_1 = v_1, v_2, \ldots, v_i$, $P_2 = u_1, u_2, \ldots, u_j$, and $v_i \sim u_1$, then P_1P_2 is used to denote the path $v_1, v_2, \ldots, v_i, u_1, u_2, \ldots, u_j$. Similar notation is used to concatenate the path P_1 with the single vertex u_1. That is, P_1u_1 denotes the path $v_1, v_2, \ldots, v_i, u_1$.

Let G be a graph, and let $\pi = v_1, v_2, \ldots, v_n$ be an enumeration of the vertices of G. Let G_i denote the subgraph of G induced by the vertices $\{v_1, v_2, \ldots, v_i\}$, and let $d_{G_i}(v)$ denote the degree of v in G_i. Let $D_\pi = \max_{1 \leq i \leq n} d_{G_i}(v_i)$. The *colouring number of* G, denoted $Col(G)$, is the value $\min_\pi D_\pi + 1$.

Let $\mathcal{G}$ be a group and $X \subset \mathcal{G}$. The *Cayley graph* $Cay(X : \mathcal{G})$ is defined as the graph with vertex set $V(Cay(X : \mathcal{G})) = \mathcal{G}$ and with vertices g and g' adjacent if and only if $g' = gx$ for some $x \in X$. Results on Hamiltonicity of Cayley graphs can be found in [20] and [10].

Any further terminology and notation will be consistent with Bondy and Murty [1].

2.2 Useful Theorems

Among the results of Cereceda et al. [5] regarding connectivity of k-colouring graphs is the following theorem, a slight modification of a theorem by Dyer et al. [11], which shows $\mathcal{C}_k(G)$ is connected for a sufficiently large k.

Theorem 2.2.1 (Cereceda et al. [5])**.** *Let G be a graph. If $k \geq 1 + Col(G)$, then $\mathcal{C}_k(G)$ is connected.*

An analagous result for Hamiltonicity was given by Choo [8].

Theorem 2.2.2 (Choo [8]). *Let G be a graph. If $k \geq 2 + Col(G)$, then $\mathcal{C}_k(G)$ is Hamiltonian.*

This theorem proves the existence of $k_0(G)$ for any graph G. In the next section of this chapter, we will give the proof of this theorem, modified such that the construction produces a Hamilton cycle with Property $\mathcal{A}$. Along with this existence result, Choo [8] establishes Gray code numbers for complete graphs, trees and cycles.

Theorem 2.2.3 (Choo [8]). *$k_0(K_1) = 3$, and $k_0(K_n) = n + 1$ for $n \geq 2$.*

The proof of this theorem shows that $\mathcal{C}_{t+1}(K_t) \cong Cay(X : S_{t+1})$, where X is the generating set of transpositions $X = \{(1, t+1), (2, t+1), \ldots, (t, t+1)\}$. We will see in Chapter 5 that the structure of $\mathcal{C}_{t+1}(K_{a_1,a_2,\ldots,a_t})$ closely depends on the structure of $\mathcal{C}_{t+1}(K_t)$.

Theorem 2.2.4 (Choo [8]). *Let T be a star with $n + 1 \geq 2$ vertices. Then $\mathcal{C}_3(T)$ is Hamiltonian if and only if n is odd.*

Given that a star T with $n+1$ vertices is isomorphic to $K_{n,1}$, this result also has particular relevance to our problem.

Theorem 2.2.5 (Choo [8]). *Let T be a tree. If T is a star with $2k + 1 \geq 3$ vertices, then $k_0(T) = 4$. Otherwise, $k_0(T) = 3$.*

Theorem 2.2.6 (Choo [8]). *For all $n \geq 3$, we have $k_0(C_n) = 4$.*

Further work has been done by Celaya et al. [3], who gave Gray code numbers for complete bipartite graphs. The ideas presented in [3] are a basis for the work done in this thesis. We attempt to generalize these results on complete 2-partite graphs to results on complete t-partite graphs.

Theorem 2.2.7 (Celaya et al. [3]). *For positive integers l and r, $\mathcal{C}_2(K_{l,r})$ is not Hamiltonian, and $\mathcal{C}_3(K_{l,r})$ is Hamiltonian if and only if l, r are both odd.*

In Chapter 5, we generalize this theorem to characterize the complete t-partite graphs $K = K_{a_1,a_2,\ldots,a_t}$ for which $\mathcal{C}_{t+1}(K)$ is Hamiltonian.

Theorem 2.2.8 (Celaya et al. [3]). *Let $1 \leq l \leq r$ and let $k \geq 4$. Then $\mathcal{C}_k(K_{l,r})$ is Hamiltonian.*

The main result of this thesis is the following generalization of this theorem, which we will prove in Chapter 5.

Theorem 2.2.9. *Let $a_1, a_2, \ldots, a_t$ be positive integers such that $a_1 \geq a_2 \geq \cdots \geq a_t$. Then, $\mathcal{C}_k(K_{a_1,a_2,\ldots,a_t})$ is Hamiltonian for all $k \geq 2t$.*

2.3 The Modified Existence Theorem

As a final preliminary, we give a proof of Theorem 2.2.2, modified such that it constructs Hamilton cycles with Property $\mathcal{A}$. To begin, we introduce a useful class of graphs known as C-Graphs. In this section, we consider subscripts to be modulo N.

A *C-Graph* is a graph G whose vertices may be partitioned into sets $F_0, F_1, \ldots, F_{N-1}$ such that for $i \in \{0, 1, \ldots, N-1\}$, $|F_i| \geq 3$ and F_i induces a Hamilton connected subgraph of G. We will now give some conditions under which a C-Graph is Hamiltonian, proofs of which can be found in [9] (Choo and MacGillivray). Let $[F_j, F_{j+1}]$ denote the set of edges with one vertex in F_j, and one vertex in F_{j+1}.

Lemma 2.3.1 (Choo and MacGillivray [9]). *Let G be a C-Graph with vertex partition $F_0, F_1, \ldots, F_{N-1}$. If, for each $i \in \{0, 1, \ldots, N-1\}$, there exist vertex disjoint edges $x_i y_{i+1}$ where $x_i \in F_i$ and $y_{i+1} \in F_{i+1}$, then G is Hamiltonian.*

[Choo and MacGillivray [9]]

Corollary 2.3.2 (Choo and MacGillivray [9]). *Let G be a C-Graph with vertex partition $F_0, F_1, \ldots, F_{N-1}$. Suppose for each $j \in \{0, 1, \ldots, N-1\}$ that $[F_j, F_{j+1}]$ contains at least 2 vertex disjoint edges. If there exists $i \in \{0, 1, \ldots, N-1\}$ such that some vertex $x \in F_i$ has a neighbour in F_{i+1}, and $[F_{i-1}, F_i - \{x\}]$ contains at least two vertex disjoint edges, then G is Hamiltonian.*

Corollary 2.3.3 (Choo and MacGillivray [9]). *Let G be a C-Graph with vertex partition $F_0, F_1, \ldots, F_{N-1}$. Suppose for each $j \in \{0, 1, \ldots, N-1\}$ that $[F_j, F_{j+1}]$ contains at least 2 vertex disjoint edges. If there exists $i \in \{0, 1, \ldots, N-1\}$ such that $[F_i, F_{i+1}]$ contains at least three vertex disjoint edges, then G is Hamiltonian.*

In light of these results, we have all the tools we need to prove the modified existence theorem. The construction used in this proof is identical to the construction used by Choo [8]. This proof merely notes that the Hamilton cycle constructed does in fact have Property $\mathcal{A}$.

Theorem 2.3.4. *Let G be a graph. If $k \geq Col(G) + 2$, then $\mathcal{C}_k(G)$ has a Hamilton cycle with Property $\mathcal{A}$.*

Proof. Let $\sigma = v_1 v_2 \ldots v_n$ be an ordering of $V(G)$ such that $D_\sigma = min_\pi D_\pi$. Let $k \geq 3 + D_\sigma = 2 + Col(G)$. Let G_i denote the subgraph of G induced by $v_1, v_2, \ldots, v_i$. We will show that $\mathcal{C}_k(G_i)$ has a Hamilton cycle with Property $\mathcal{A}$ by induction on i.

Let $\{1, 2, \ldots, k\}$ be our set of colours. Then, $\mathcal{C}_k(G_1) = \mathcal{C}_k(K_1) \cong K_k$. This graph clearly has a Hamilton cycle, and since $k \geq 3$, for each $j \in \{1, 2, \ldots, k\}$ any such Hamilton cycle must have consecutive colourings which do not use j. Thus, Property $\mathcal{A}$ is present.

For some $i \in \{2, 3, \ldots, n-1\}$, let $f_0, f_1, \ldots, f_{N-1}, f_0$ be a Hamilton cycle in $\mathcal{C}_k(G_{i-1})$ which has Property $\mathcal{A}$. Let F_j be the set of colourings in $\mathcal{C}_k(G_i)$ which agree with f_j on $V(G_{i-1})$, for $0 \leq j \leq N-1$. Now, since $k \geq 3 + D_\sigma \geq 3 + d_{G_i}(v_i)$, we have $|F_j| \geq 3$. We also have that F_j induces a complete subgraph of $\mathcal{C}_k(G_i)$. Therefore, since complete graphs are Hamilton connected, $\mathcal{C}_k(G_i)$ is a C-Graph with vertex partition $F_0, F_1, \ldots, F_{N-1}$.

Now, consider some $j \in \{0, 1, \ldots, N-1\}$. For colourings $c_j \in F_j$ and $c_{j+1} \in F_{j+1}$, $c_j \sim c_{j+1}$ if and only if c_j and c_{j+1} colour v_i the same colour. As a result, edges in $[F_j, F_{j+1}]$ are vertex disjoint. Let w_j denote the unique vertex such that $f_j(w_j) \neq f_{j+1}(w_j)$. If $v_i \nsim w_j$, then each vertex in F_j has a neighbour in F_{j+1}. In this case, $[F_j, F_{j+1}]$ contains at least three vertex disjoint edges. If $v_i \sim w_j$, a vertex in F_j which colours v_i with the colour $f_{j+1}(w_j)$ will not have a neighbour in F_{j+1}. Therefore, in this case we may only guarantee that $[F_j, F_{j+1}]$ has at least two vertex disjoint edges. Therefore, if for some j, $w_j \nsim v_i$, then $\mathcal{C}_k(G_i)$ is Hamiltonian by Corollary 2.3.3.

Suppose that $w_j \sim v_i$ for each $j \in \{0, 1, \ldots, N-1\}$. We have already shown that $[F_j, F_{j+1}]$ contains at least two vertex disjoint edges for each j. Let c_{N-1} be a colouring in F_{N-1} which has a neighbour in F_0. Let r be the largest integer such that f_{r-1} uses the colour $c_{N-1}(v_i)$, but f_r does not. Let c_r be the colouring in F_r which assigns v_i the colour $c_{N-1}(v_i)$. By definition of r, f_{r+1} does not use $c_{N-1}(v_i)$. Then, c_r has a neighbour in F_{r+1}, and does not have a neighbour in F_{r-1}. Therefore, $[F_{r-1}, F_r - \{c_r\}]$ has at least two vertex disjoint edges. Then, $\mathcal{C}_k(G_i)$ is Hamiltonian by Corollary 2.3.2.

All that is left is to verify the Property $\mathcal{A}$ holds for these Hamilton cycles. First, it is important to notice that the Hamilton cycle constructed by Corollary 2.3.2, and similarly by Corollary 2.3.3, is the concatenation of Hamilton paths of the F_is. Therefore, our constructed Hamilton cycle visits the vertices of F_j and F_{j+1} consecutively, for each $j \in \{0, 1, \ldots, N-1\}$. By induction, for $l \in \{1, 2, \ldots, k\}$ there exists j_l such that neither f_{j_l} nor f_{j_l+1} use the colour l. Then, F_{j_l} and F_{j_l+1} each contain a single vertex which uses colour l. Since $|F_{j_l}| + |F_{j_l+1}| \geq 6$, and exactly two of these vertices use l, there must be consecutive vertices which *do not* use l. Thus, our Hamilton cycles maintains Property $\mathcal{A}$. $\square$

Chapter 3

Connectivity of Colour Graphs

In an effort to improve our overall understanding of colour graphs, in particular colour graphs of complete multipartite graphs, we considered the connectivity of these graphs. Chartrand and Kapoor [7] show that the cube of a connected graph is Hamiltonian. In the context of colour graphs, for a graph G, a Hamilton cycle in the cube of $\mathcal{C}_k(G)$ corresponds to a cyclic list of the k-colourings of G such that consecutive colourings in the list differ in the colour of at most three vertices. In this chapter, we will give a few basic results on the connectivity of colour graphs. To begin, we examine the connectivity of the colour graph of K_t, the simplest complete t-partite graph. We show that the connectivity of $\mathcal{C}_k(K_t)$ is equal to its minimum degree. We then turn our attention toward $\mathcal{C}_k(K_{a_1,a_2,\ldots,a_t})$, proving that we have connectivity at least 2 whenever $k \geq t+1$, an obvious necessary condition for Hamiltonicity. To finish the chapter, we take a brief look at connectivity of colour graphs in general.

3.1 Connectivity of $\mathcal{C}_k(K)$

The first section of this chapter considers the connectivity of colour graphs of complete multipartite graphs, starting with K_t. In the case of K_t, we are able to establish the connectivity of $\mathcal{C}_k(K_t)$ by proving the following theorem.

Theorem 3.1.1. *$\mathcal{C}_k(K_t)$ has connectivity $\delta(\mathcal{C}_k(K_t)) = t(k-t)$, whenever $k \geq t+1$.*

Proof. We will prove the result by induction on the number of colours, k. As we have previously noted, $\mathcal{C}_{t+1}(K_t) \cong Cay(X : S_{t+1})$, where X is the minimal generating set of transpositions $X = \{(1, t+1), (2, t+1), \ldots, (t, t+1)\}$. It follows from a theorem of

Godsil [14] that $Cay(X : S_{t+1})$ has connectivity t, and thus $\mathcal{C}_{t+1}(K_t)$ has connectivity $t = t((t+1) - t)$ as well.

Suppose for some $k - 1 \geq t + 1$ the result holds, and consider $\mathcal{C}_k(K_t)$. Let x and y be any two non-adjacent vertices in $\mathcal{C}_k(K_t)$. We will prove our result in two cases, based on the number of colours used by x and y.

Case 1: There is a colour c not used by x or y.

In this case, we will show that any set which disconnects x from y must have size at least $t(k-t)$ by describing $t(k-t)$ internally vertex disjoint paths between x and y. Let c be any colour not used by x or y. Let G be the subgraph of $\mathcal{C}_k(K_t)$ induced by the vertices which do not use c. Note that $x, y \in V(G)$. Then, $G \cong \mathcal{C}_{k-1}(K_t)$, and therefore by induction G contains $t((k-1)-t)$ internally vertex disjoint xy-paths. We will utilize the fact that none of these paths contain a vertex which uses the colour c to construct t additional internally vertex disjoint xy-paths. Let $P_0 = x, f_1, f_2, \ldots, f_\alpha, y$ denote any one of our xy-paths contained in G. Let x^i denote the vertex obtained by recolouring v_i in x with colour c, for $1 \leq i \leq t$. Define y^i similarly. Let f_j^i denote the vertex obtained by recolouring v_i in f_j to c, for $1 \leq i \leq t$ and $1 \leq j \leq \alpha$. Then, $x, x^i, f_1^i, f_2^i, \ldots, f_\alpha^i, y^i, y$ is a walk from x to y. Though it may not itself be a path due to the possibility of repeated vertices, it contains a path P_i from x to y. Then $P_1, P_2, \ldots, P_t$ are our t additional internally vertex disjoint xy-paths, and we have a total of $t(k-t)$ paths, as desired.

Case 2: All k colours are used by x or y.

The proof is by contradiction. Let S be a minimal set that separates x from y in $\mathcal{C}_k(K_t)$, and suppose $|S| < t(k-t)$. Let G_x and G_y denote the components of $\mathcal{C}_k(K_t) - S$ which contain x and y respectively. Let S_x and S_y denote the sets of colours *not* used by x and y respectively. Note that we have $S_x \cap S_y = \emptyset$. By Case 1, any vertex which does not use a colour $c_x \in S_x$ is either in G_x or in our cut-set S. Similarly, any vertex which does not use a colour $c_y \in S_y$ is either in G_y or in S. Therefore, any colouring which uses neither c_x nor c_y must be in S.

The number of such colourings is $N = (k-2)(k-3)\cdots(k-(t+1))$. Since $(k-2) \geq t$, we have $N \geq t(k-t)$ when $t \geq 3$, contradicting $|S| < t(k-t)$. When $t = 2$, we must have $k = 4$, as at most four colours may be used by x and y. Without loss of generality, we may assume $S_x = \{1, 2\}$ and $S_y = \{3, 4\}$, and there are 8 vertices, $(2,4), (4,2), (2,3), (3,2), (1,4), (4,1), (1,3)$ and $(3,1)$ which must be in S. Again, a contradiction is reached as $|S| < 4$, and the result is proven.

$\square$

In light of the previous theorem, one might wonder if in general the connectivity of $\mathcal{C}_k(K_{a_1,a_2,\ldots,a_t})$ is equal to its minimum degree. This is, however, not the case. Consider the graph $\mathcal{C}_{t+1}(K_{a_1,1,1,\ldots,a_t=1})$, with $a_1 \geq 3$. This graph has minimum degree a_1, but connectivity at most 2, as any two t-colourings which differ only in the colour of the vertices in V_1 form a cut set. Therefore, there are colour graphs of complete multipartite graphs with arbitrarily large minimum degree, but with connectivity at most 2. What then, *can* we say about the connectivity of these colour graphs in general? The following theorem gives a simple lower bound on the connectivity of such graphs.

Theorem 3.1.2. *For $K = K_{a_1,a_2,\ldots,a_t}$, the graph $\mathcal{C}_k(K)$ has connectivity at least 2 whenever $k \geq t+1$.*

Proof. By Menger's theorem, it is sufficient to show that between any two vertices in $\mathcal{C}_k(K)$ there are two vertex-disjoint paths. To do this we will show that any two vertices lie on a common cycle.

Let $V_1, V_2, \ldots, V_t$ be the t-partition of K. Consider K_t, the complete graph on t vertices, with vertex set $V(K_t) = \{v_1, v_2, \ldots, v_t\}$. By Theorem 2.2.3, $\mathcal{C}_k(K_t)$ is Hamiltonian whenever $k \geq t+1$. Let N denote the number of vertices in $\mathcal{C}_k(K_t)$. For the remainder of this proof, we interpret subscripts modulo N. Let $C = f_0, f_1, \ldots, f_{N-1}, f_0$ be a Hamilton cycle in $\mathcal{C}_k(K_t)$. Let F_i denote the vertex in $\mathcal{C}_k(K)$ where for each $j \in \{1, 2, \ldots, t\}$ and each $u \in V_j$, $F_i(u) = f(v_j)$. That is, F_i is the colouring of K which colours the vertices of V_j with the colour used by f_i to colour v_j. Since f_i and f_{i+1} are adjacent colourings, F_i and F_{i+1} differ only in the colour of vertices in V_j, for some j. Let P_i denote some path in $\mathcal{C}_k(K)$ from F_i to a neighbour of F_{i+1} obtained by successively changing the colour of vertices in V_j from $f_i(v_j)$ to $f_{i+1}(v_j)$. Then, $C' = P_0 P_1 \cdots P_{N-1} F_0$ is a cycle in $\mathcal{C}_k(K)$ which contains every t colouring of K.

We claim that for every vertex x not contained in $V(C')$, there are at least two internally vertex-disjoint paths from x to distinct vertices of C'. Since x uses at least $t+1$ colours, for some j, V_j uses at least two distinct colours, $c_{x,1}$ and $c_{x,2}$, to colour its vertices. Let $P_{x,i}$, for $i = 1$ or 2, be the path obtained by recolouring the vertices of V_j which are not already coloured with $c_{x,i}$ to $c_{x,i}$ one by one, and then recolouring the vertices of each V_h which uses more than one colour until V_h is monocoloured. The paths $P_{x,1}$ and $P_{x,2}$ are internally vertex disjoint, as, aside from x, no vertex of $P_{x,1}$ colours V_j the same as a vertex of $P_{x,2}$. Each path ends in a t-colouring, and the claim is proven.

It is now straightforward to see that any two distinct vertices $x, y \in V(\mathcal{C}_k(K))$ lie on a common cycle. If both x and y lie on C', this is trivial. If exactly one of x or y lies on C', using our two internally vertex disjoint paths, a common cycle is again found immediately. If neither x nor y lie on C', there are three possible cases.

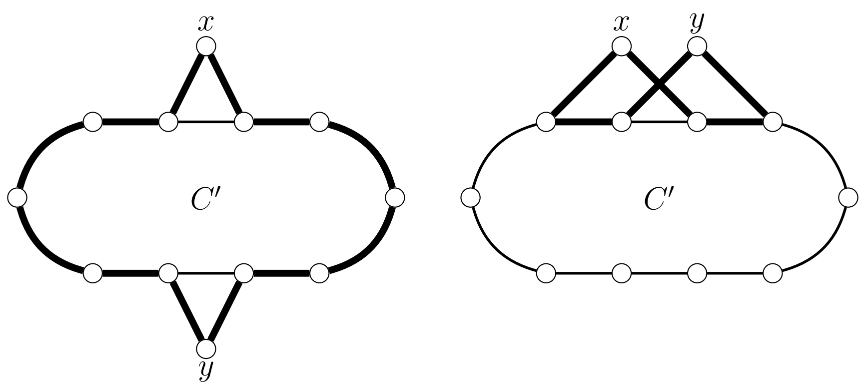

Figure 3.1: $P_{x,1}$ and $P_{x,2}$ do not intersect $P_{y,1}$ or $P_{y,2}$.

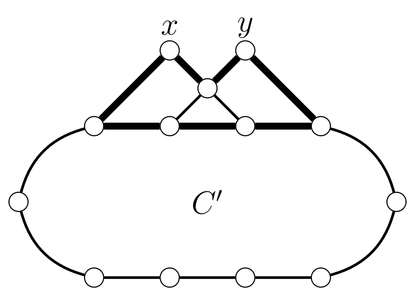

Figure 3.2: One of $P_{x,1}$ or $P_{x,2}$ intersects $P_{y,1}$ or $P_{y,2}$.

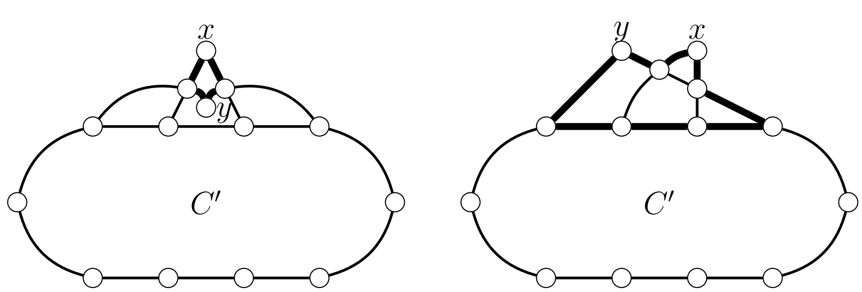

Figure 3.3: Both $P_{x,1}$ and $P_{x,2}$ intersect one of $P_{y,1}$ or $P_{y,2}$.

Figures 3.1-3.3 show examples of these three cases, and how to find our cycle in each. Cycles can be found in each case using methods similar to those shown by our figures.

□

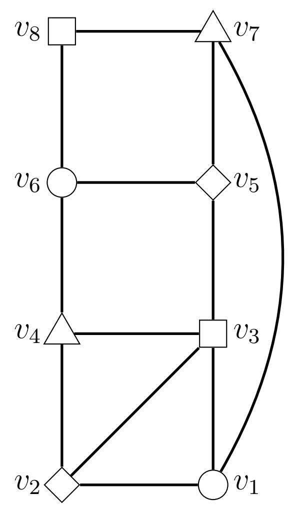

Figure 3.4: The graph H_4 with colouring f, a leaf of the connected colour graph $\mathcal{C}_4(H_4)$.

3.2 2-Connectedness of colour graphs

In this section, we turn our attention to a problem which is only tangentially related to our main focus, but is still worth consideration. Although never published, Horak [17] conjectured that every colour graph which is connected must also be 2-connected. A theorem of Fleischner [13] states that the square of every 2-connected graph is Hamiltonian. For a graph G, a Hamilton cycle in the square of $C_k(G)$ corresponds to a cyclic list of the k-colourings of G, such that consecutive colourings in the list differ in the colour of at most two vertices. If Horak's conjecture is true, the square of every connected colour graph is Hamiltonian. Indeed, this is the case for colour graphs of complete multipartite graphs. However, we will show that for each $k \geq 4$, there is at least one graph G such that $\mathcal{C}_k(G)$ is connected, but *not* 2-connected.

Let H_4 be the graph displayed in Figure 3.4. The colour of a vertex in Figure 3.4 is represented by its shape. For $i \geq 5$, let $H_i = H_{i-1} + \{u_i\}$, where $H_{i-1} + \{u_i\}$ is the graph obtained by adding a dominating vertex u_i to H_{i-1}. Given an ordering $\pi =$

$x_1, x_2, \ldots, x_{i+4}$ of the vertices of H_i, we have $Col(H_i) \leq D_\pi + 1 = \max_{1 \leq j \leq i+4} d_{H_i^j}(x_i) + 1$. Let σ be the ordering $x_1 = u_1$, $x_2 = u_2$, $\ldots$, $x_{i-4} = u_{i-4}$, $x_{i-3} = v_1$, $x_{i-2} = v_2$, $\ldots$, $x_{i+4} = v_8$. Then, $D_\sigma + 1 = i - 1 \geq Col(H_i)$. By Theorem 2.2.1, we know that $\mathcal{C}_k(H_i)$ is connected whenever $k \geq Col(H_i) + 1$. Therefore, $\mathcal{C}_i(H_i)$ is connected. Furthermore, the colouring f in $\mathcal{C}_4(H_4)$ shown in Figure 3.4 has only a single vertex which can change colour: the vertex v_8 may change from square to diamond. Let f' denote the colouring obtained by recolouring v_8 to diamond. Then, f' is a cut vertex in $\mathcal{C}_4(H_4)$, and $\mathcal{C}_4(H_4)$ is therefore not 2-connected. By extending f to an i-colouring of H_i by using the additional $i - 4$ colours to colour u_1 through u_{i-4}, we may use a similar argument to show that H_i is connected, but not 2-connected. Therefore, we arrive at the following conclusion:

Theorem 3.2.1. *For $k \geq 4$, there is a graph G such that $\mathcal{C}_k(G)$ is connected, but not 2-connected.*

The question still remains whether or not Horak's conjecture is true when $k = 3$. When $k = 1$, the conjecture holds vacuously. Let Q_n denotes the $n - cube$, the graph whose vertex set is the set of binary strings of length n, where two strings are adjacent if they differ in exactly one position. When $k = 2$, if $\mathcal{C}_k(G)$ is connected, then G can contain no edges, and $\mathcal{C}_k(G) \cong Q_n$, where $n = |V(G)|$. It is well known that Q_n is Hamiltonian whenever $n \geq 2$, and is therefore 2-connected.

Chapter 4

Hamilton Paths and Cycles in SDR Graphs

In this chapter, we will examine some properties of the following class of graphs. For a collection of sets $S = A_1, A_2, \ldots, A_t$, where $A_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,a_i}\}$, we define a graph G_S corresponding to reconfigurations of SDRs of S. Let $V(G_S) = \{(v_1, v_2, \ldots, v_t) | v_i \in A_i \text{ and } v_i \neq v_j \text{ if } i \neq j\}$, and $E(G_S) = \{((v_1, v_2, \ldots, v_t), (u_1, u_2, \ldots, u_t)) | \exists\, i \text{ such that } v_j = u_j \iff j \neq i\}$. In other words, if u and v are vertices of G_S, then u and v are SDRs of S, where the ith coordinate corresponds to the representative of A_i. We have $u \sim v \iff u$ and v differ in exactly one coordinate. Our study of this class of graphs is motivated by their relation to colour graphs. For example, consider the complete graph K_t with vertex set $\{v_1, v_2, \ldots, v_t\}$. Then, G_S is isomorphic to the graph of vertex colourings of K_t where the colour of v_i is restricted to elements of A_i.

4.1 Preliminaries

Not all collections S produce graphs G_S which are relevant to our study of colour graphs. We define the set $\mathcal{S}_t$, which contains the t-collections of sets we will examine in this chapter.

For $t \geq 2$, let $\mathcal{S}_t$ denote the set of collections $S = A_1, A_2, \ldots, A_t$ for which the following properties hold:

- $A_i = \{x_1, x_2, \ldots, x_l, y_1^i, y_2^i, \ldots, y_{a_i}^i\}$,
- $a_1 \geq a_2 \geq \cdots \geq a_t \geq 1$,

- $A_i \cap A_j = \{x_1, x_2, \ldots, x_l\}$ $\forall i, j$ such that $i \neq j$,
- $|\bigcup_{i=1}^{t} A_i| \geq 2t$.

Let $S \in \mathcal{S}_t$. Then, S is a collection of sets where an element z of $\bigcup_{i=1}^{t} A_i$ is either in every set, or exactly one set. Specifically, each y_i^j is distinct. In the Chapter 5, we will see that the colour graph $C_k(K_{b_1,b_2,\ldots,b_t})$, where $b_i \geq 2$ for every i, can be partitioned into some number of subgraphs, each of which is isomorphic to G_S, for some $S \in \mathcal{S}_t$. The lemmas in this chapter show in a variety of ways that we may always find a Hamilton path in G_S which suits our needs. This is a very difficult task. In order to prove the results of the section, we must consider a property of G_S analagous to Property $\mathcal{A}$ in colour graphs. In the context of an SDR graph G_S, we will say a Hamilton cycle C in G_S has Property $\mathcal{A}$ if for each $i \in \{1, 2, \ldots, l\}$, there exist consecutive vertices in C which do not use x_i. We say G_S is $\mathcal{A}$-Hamiltonian if it contains a Hamilton cycle with Property $\mathcal{A}$. For the remainder of this chapter, when discussing a collection of sets S, it is assumed $S \in \mathcal{S}_t$ unless otherwise stated.

In our examination of S, it is extremely useful to utilize the automorphisms of G_S.

Let $X = \bigcup_{i=1}^{t} A_i$. Let $\pi^x : X \to X$ be any bijection where for every i and j, $\pi^x(y_i^j) = y_i^j$. In other words, π^x is some function which permutes the x_is. For $v = (v_1, v_2, \ldots, v_t) \in V(G_S)$, let $\pi^x(v) = (\pi^x(v_1), \pi^x(v_2), \ldots, \pi^x(v_t))$.

Automorphism Property I: π^x is an automorphism of G_S.

Let $i \in \{1, 2, \ldots, t\}$, and let $\pi^{y^i} : X \to X$ be any bijection where for every k, $\pi^{y^i}(y_k^j) = y_k^j$ when $j \neq i$, and $\pi^{y^i}(x_j) = x_j$. Then, π^{y^i} is a function which permutes the y_j^is for some fixed i. As before, for $v = (v_1, v_2, \ldots, v_t) \in V(G_S)$, let $\pi^{y^i}(v) = (\pi^{y^i}(v_1), \pi^{y^i}(v_2), \ldots, \pi^{y^i}(v_t))$.

Automorphism Property II: π^{y^i} is an automorphism of G_S.

Automorphism Properties I and II utilize the fact that if a and b are elements of the same sets in some collection of sets S, swapping the labels of a and b in any SDR of S will give you an SDR of S. The automorphism of Automorphism Property I permutes the labels of elements which are in every set of a collection of sets S, while the automorphism of Automorphism Property II permutes the labels of elements which are in exactly one set of such a collection. We now give a third automorphism of G_S.

Suppose $|A_i| = |A_j|$ for some $i < j$. We define the function $\phi_{(i\,j)} : V(G_S) \to V(G_S)$ by the following rule:

$$\begin{aligned}\phi_{(i\,j)}&((v_1, v_2, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots, v_t)) \\ &= (v_1, v_2, \ldots, v_{i-1}, v'_j, v_{i+1}, \ldots, v_{j-1}, v'_i, v_{j+1}, \ldots, v_t)\end{aligned}$$

$$v'_i = \begin{cases} v_i & \text{if } v_i \in \{x_1, x_2, \ldots, x_l\} \\ y_k^j & \text{if } v_i = y_k^i \end{cases}$$

$$v'_j = \begin{cases} v_j & \text{if } v_j \in \{x_1, x_2, \ldots, x_l\} \\ y_k^i & \text{if } v_j = y_k^j \end{cases}$$

The function $\phi_{(i\,j)}$ captures the notion of swapping the ith and jth coordinates of every vertex in G_S.

Automorphism Property III: $\phi_{(i\,j)}$ is an automorphism of G_S.

Let X^t be the unique collection in $\mathcal{S}_t$ which satisfies the additional properties $l = 1$, $a_1 = t$, and $a_i = 1$ for $i \in \{2, 3, \ldots, t\}$. Then, X^t is the collection where $A_1 = \{x_1, y_1^1, y_2^1, \ldots, y_t^1\}$, and $A_i = \{x_1, y_1^i\}$. The collection X^t is a special case, and must be separately addressed.

4.2 Hamilton path and cycle constructions

Our first result restricts attention to the case $t = 2$. This result will be used as a base case for induction to prove results for larger values of t. Consider the following example to demonstrate the use of Automorphism Property I and Automorphism Property II.

Suppose $S = A_1, A_2$, with $A_1 = \{x_1, x_2, x_3, y_1^1, y_2^1, y_3^1, y_4^1\}$ and $A_2 = \{x_1, x_2, x_3, y_1^2\}$. Say a Hamilton cycle in G_S which contains the edge $e = (u, v)$, where $u = (y_1^1, x_1)$ and $v = (y_1^1, x_2)$ is required. Consider any edge of the form $e' = (u', v')$ with $u' = (y_i^1, x_j)$ and $v' = (y_i^1, x_k)$ for some $i \in \{1, 2, 3, 4\}$ and $j, k \in \{1, 2, 3\}$, $j \neq k$. By Automorphism Properties I and II, some automorphism of G_S maps e to e'. Then, a Hamilton cycle in G_S which contains the edge e' can be mapped to a Hamilton cycle which contains the edge e by some automorphism.

We refer to the orbit of an edge e under the automorphisms of Automorphism Properties I and II as its *edge type*. Suppose we want to show that for every edge e of G_S, there is a Hamilton cycle in G_S which contains e. In light of the above, we only need to show that an edge from every edge type appears in a Hamilton cycle in G_S.

With this fact in mind, we are now ready to prove our result.

Lemma 4.2.1. *Let $S \in \mathcal{S}_2 - \{X^2\}$. For any edge e in G_S, there is a Hamilton cycle with Property $\mathcal{A}$ which contains e.*

Proof. To prove this result in a reasonably efficient manner, we appeal to Automorphism Properties I and II. Consider an edge e of G_S. Instead of finding a Hamilton cycle which contains e, one may find a Hamilton cycle C' which contains any edge e' which shares an edge type with e. Using an automorphism which maps e' to e, we may transform C' into a Hamilton cycle which contains e.

We use $(x, y) \to (x', y)$ to denote the set of edges $((v_1, v_2), (v_1', v_2))$, where v_1, $v_2 \in \{x_1, x_2, \ldots, x_l\}$, $v_1 \neq v_1'$ and $v_2 \in \{y_1^2, y_2^2, \ldots, y_{a_2}^2\}$. The other edge types listed define sets of edges in a similar manner. The following are the 12 possible edge types, which we label as 1 through 12.

1. $(x, x') \leftrightarrow (x'', x')$
2. $(x, x') \leftrightarrow (x, x'')$
3. $(x, x') \leftrightarrow (y, x')$
4. $(x, x') \leftrightarrow (x, y)$
5. $(y, y') \leftrightarrow (x, y')$
6. $(y, y') \leftrightarrow (y, x)$
7. $(y, y') \leftrightarrow (y'', y')$
8. $(y, y') \leftrightarrow (y, y'')$
9. $(x, y) \leftrightarrow (x', y)$
10. $(x, y) \leftrightarrow (x, y')$

11. $(y, x) \leftrightarrow (y', x)$

12. $(y, x) \leftrightarrow (y, x')$

The following is a list of possible values for l, a_1 and a_2, together with when each edge type will occur.
$l = 0$:

- $a_1 \leq 2,\ a_2 = 1$: No graphs.
- $a_1 \geq 3,\ a_2 = 1 : 7$.
- $a_1 \geq 2,\ a_2 \geq 2 : 7, 8$.

$l = 1$:

- $a_1 = 1,\ a_2 = 1$: No graphs.
- $a_1 = 2,\ a_2 = 1 : G_{X^2}$.
- $a_1 \geq 3,\ a_2 = 1 : 5, 6, 7, 11$.
- $a_1 \geq 2,\ a_2 \geq 2 : 5, 6, 7, 8, 10, 11$.

$l = 2$:

- $a_1 = 1,\ a_2 = 1 : 3, 4, 5, 6, 9, 12$.
- $a_1 \geq 2,\ a_2 = 1 : 3, 4, 5, 6, 7, 9, 11, 12$.
- $a_1 \geq 2,\ a_2 \geq 2 : 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$.

$l \geq 3$:

- $a_1 = 1,\ a_2 = 1 : 1, 2, 3, 4, 5, 6, 9, 12$.
- $a_1 \geq 2,\ a_2 = 1 : 1, 2, 3, 4, 5, 6, 7, 9, 11, 12$.
- $a_1 \geq 2,\ a_2 \geq 2 : 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$.

Figures 4.1-4.10 show the smallest graphs of the possiblities listed above, and Hamilton cycles in those graphs which have the edge types we desire, as well as Property $\mathcal{A}$. The method of generalizing these cycles to larger graphs should be apparent, as additional rows and columns of vertices are easily included. Figures which contain two cycles in two copies of the graph are those whose edge types are not all covered by a single cycle. In each figure, the dashed edges represent the set of edges used to verify Property $\mathcal{A}$ on the cycle. Note that some edges are not shown in these figures. Vertices which share either a row or a column are adjacent. The SDR which a vertex represents is indicated by row and column. For example, if a vertex is in row x_2 and column y_1^2, it represents the SDR (x_2, y_1^2). The rows are labelled with elements of A_1, and the columns are labelled with elements of A_2.

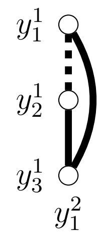

Figure 4.1: $l = 0, a_1 = 3, a_2 = 1$.

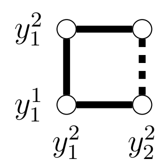

Figure 4.2: $l = 0, a_1 = 2, a_2 = 2$.

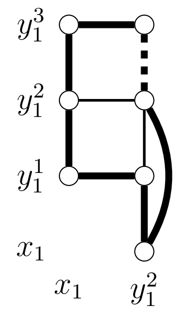

Figure 4.3: $l = 1, a_1 = 3, a_2 = 1$.

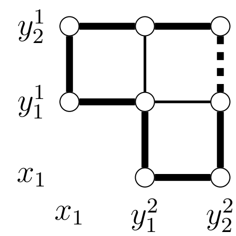

Figure 4.4: $l = 1, a_1 = 2, a_2 = 2$.

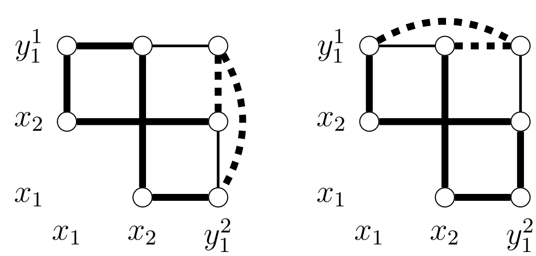

Figure 4.5: $l = 2, a_1 = 1, a_2 = 1$.

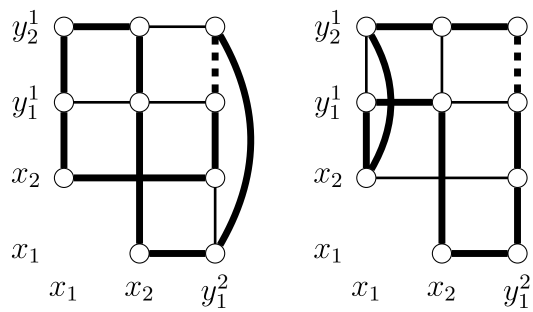

Figure 4.6: $l = 2, a_1 = 2, a_2 = 1$.

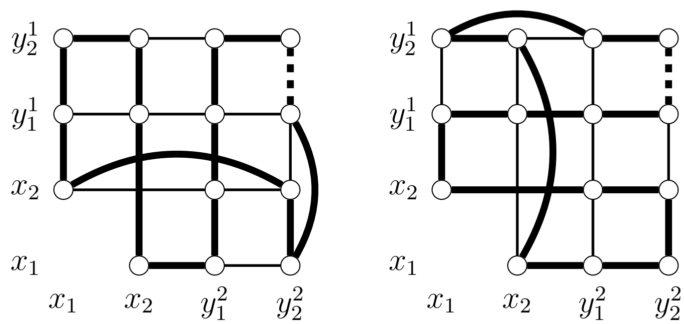

Figure 4.7: $l = 2, a_1 = 2, a_2 = 2$.

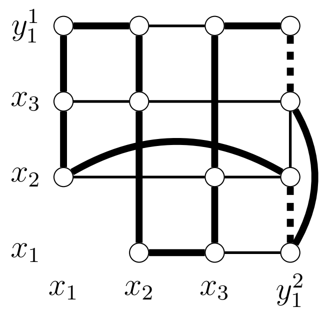

Figure 4.8: $l = 3, a_1 = 1, a_2 = 1$.

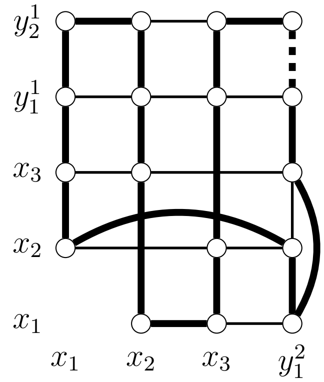

Figure 4.9: $l = 3, a_1 = 2, a_2 = 1$.

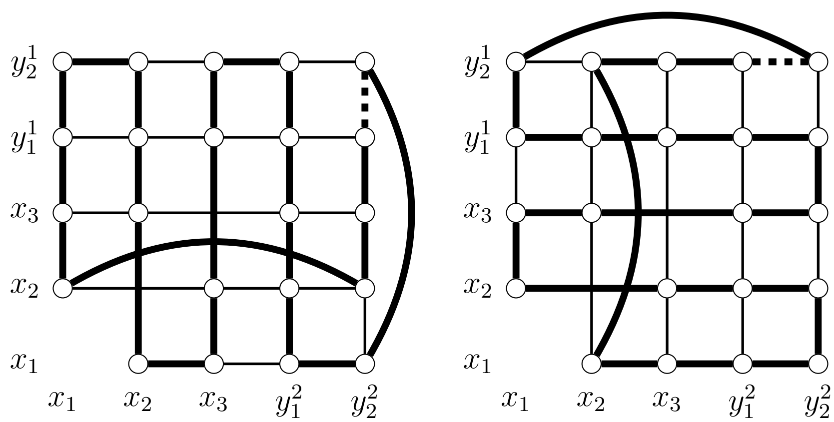

Figure 4.10: $l = 3, a_1 = 2, a_2 = 2$.

□

Our work in the next chapter has more specific requirements for a few particular collections in $\mathcal{S}_t$. We will now resolve one such case. Let J_n denote the graph $K_n \square K_n - \{(1,1),(2,2),\ldots,(n-1,n-1)\}$. Celaya et al. [3] prove the following result regarding J_n.

Lemma 4.2.2 (Celaya et al. [3])**.** *For $n \geq 3$, J_n has a Hamilton path from (n,n) to every other vertex of $J_n - \{(n,n)\}$.*

We use this result as the basis for induction to prove the following lemma.

Lemma 4.2.3. *Given a collection of sets $S = A_1, A_2, \ldots, A_n$, where $n \geq 2$, and $A_i = \{x_1, x_2, \ldots, x_l, y_i\}$, with $l \geq n$, the graph G_S has a Hamilton path from $y = (y_1, y_2, \ldots, y_n)$ to any other vertex in $V(G_S)$.*

Proof. First, note that $S \in \mathcal{S}_t$, as $|\bigcup_{i=1}^{n} A_i| = l + n \geq 2n$. Hence, we turn to the automorphisms of G_S to simplify the problem. By Automorphism Properties I and III, for any vertex $x \in V(G_S)$, there exists an automorphism ϕ such that $\phi(y) = y$ and $\phi(x) = v$, where $v \in X_n = \{(x_1, y_2, y_3, \ldots, y_n),$
$(x_1, x_2, y_3, \ldots, y_n), \ldots, (x_1, x_2, x_3, \ldots, x_n)\}$. Therefore, to prove this lemma it is sufficient to find a Hamilton path beginning at v and ending at y for each $v \in X_n$.

We prove this result by induction on n. Notice that when $n = 2$, $G_S \cong J_{l+1}$. Therefore, by Lemma 4.2.2, the result holds when $n = 2$. Now, suppose for some i, $i \geq 2$, the result holds. Let $n = i + 1$. For $j \in \{1, 2, \ldots, l\}$, let H_j denote the subgraph of G_S induced by vertices in which the nth coordinate is x_j. Let H_0 denote the subgraph of G_S induced by vertices in which the nth coordinate is y_n. For $j \in \{1, 2, \ldots, l\}$, $H_j \cong G_{S_j}$, where $S_j = A_1 - \{x_j\}, A_2 - \{x_j\}, \ldots, A_{n-1} - \{x_j\}$, and by induction G_{S_j} has a Hamilton path from $(y_1, y_2, \ldots, y_{n-1})$ to any other vertex in $V(G_{S_j})$. Similarly, $H_0 \cong G_{S_0}$, where $S_0 = A_1, A_2, \ldots, A_{n-1}$, and G_{S_0} has a Hamilton path from $(y_1, y_2, \ldots, y_{n-1})$ to any other vertex in $V(G_{S_0})$.

For $0 \leq j \leq l$, let u_j denote the vertex in H_j which the kth coordinate is y_k, for each $1 \leq k \leq i$. Notice that $u_0 = y$, and that $u_j \sim u_k$ for each $j, k \in \{0, 1, 2, \ldots, l\}$, with $j \neq k$. For some $Y \subset \{1, 2, \ldots, l\}$, let H_Y denote the subgraph of G_S induced by vertices in which the nth coordinate is an element of $\{x_i | i \in Y\}$. We will now show that for any $j, k \in \{1, 2, \ldots, l\}$, where $j \neq k$, there exists a Hamilton path in $H_{\{j,k\}}$ beginning at u_j and ending at u_k. First, let P_j be a Hamilton path in H_j from u_j to v_j, where v_j is any vertex for which no coordinate is x_k. Let P_k be a Hamilton path in H_k from v_k to u_k, where v_k is the vertex obtained by switching the $(i+1)$st coordinate of v_j from x_j to x_k. Then, $P_j P_k$ is the desired path. By combining several such paths, it follows that for any even subset $I \subset \{1, 2, \ldots, l\}$, the graph H_I has a Hamilton path from u_j to u_k for any $j, k \in I$, with $j \neq k$.

Let $v \in X_n$. We are now ready to construct a Hamilton path from v to $y = u_0$. We consider cases, based on the parity of l, and the nth coordinate of v.

Case 1.1: l odd, nth coordinate x_n.

Let P_0 be a Hamilton path in H_0 from u_0 to v_0, where v_0 is the vertex $(x_n, y_2, y_3, \ldots, y_n)$. Let w_0 be the vertex which follows u_0 in P_0. Let P_0' denote the path from w_0 to v_0 obtained by removing u_0 from P_0. Since w_0 is adjacent to u_0, there is exactly one

$s \in \{1, 2, \ldots, l\}$ such that some coordinate of w_0 is x_s. Let $t \in \{1, 2, \ldots, l\} - \{s, n\}$. Let P_t be a Hamilton path in H_t from u_t to v_t, where v_t is the vertex obtained by switching the nth coordinate of w_0 from y_n to x_t. Let $r \in \{1, 2, \ldots, l\} - \{t, n\}$. Let P_r denote a Hamilton path in H_r from v_r to u_r, where v_r is the vertex obtained by switching the nth coordinate of v_0 from y_n to x_r. Let P_n be a Hamilton path in H_n from u_n to $v_n = v$. Now, since l is odd, $I = \{1, 2, \ldots, l\} - \{n, r, t\}$ is even. Let P_I be a Hamilton path in H_I from u_j to u_k for some $j, k \in I$. Now, a Hamilton path in G_S from $y = u_0$ to $v = v_n$ is $u_0 P_t P'_0 P_r P_I P_n$. (See Figure 4.11.)

Case 1.2: l odd, nth coordinate y_n.

Let P_0 be a Hamilton path in H_0 from u_0 to $v_0 = v$. Define w_0, P'_0 and P_t analogously to the previous case. Again, $I = \{1, 2, \ldots, l\} - \{t\}$ is even, so we may define P_I in a similar fashion to the previous case as well. Then, $u_0 P_I P_t P'_0$ is the desired Hamilton path. (See Figure 4.12.)

Case 2.1: l even, nth coordinate x_n.

Define P_0 and P_n as in Case 1.1. Let $t \in \{1, 2, \ldots, l\} - \{n\}$. Let P_t be a Hamilton path in H_t from v_t to u_t, where v_t is the vertex obtained by switching the nth coordinate of v_0 from y_n to x_t. Then, $I = \{1, 2, \ldots, l\} - \{n, t\}$ is even, and we may define P_I similar to the previous cases. Then, $P_0 P_t P_I P_n$ is the desired Hamilton path. (See Figure 4.13.)

Case 2.2: l even, nth coordinate y_n.

Define P'_0 and P_t as in Case 1.2. Let w_t be the vertex which follows u_t in P_t, and let P'_t be the path from w_t to v_t obtained by removing u_t from P_t. Again, since w_t is adjacent to u_t, there is exactly one $s \in \{1, 2, \ldots, l\}$ such that some coordinate of w_t is x_s. Let $r \in \{1, 2, \ldots, l\} - \{s, t\}$. Let P_r be a Hamilton path in H_r from u_r to v_r, where v_r is the vertex obtained by switching the nth coordinate of w_t from x_t to x_r. Once again, $I = \{1, 2, \ldots, l\} - \{r, t\}$ is even, and we may define P_I analogously to the previous cases. Then, $u_0 u_t P_I P_r P'_t P'_0$ is the desired Hamilton path. (See Figure 4.14.)

The result follows by induction. $\square$

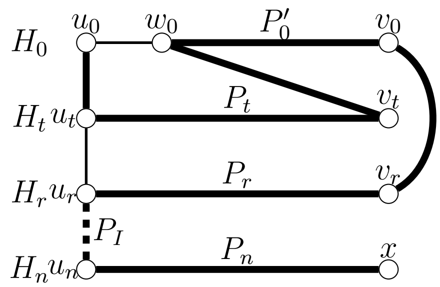

Figure 4.11: l odd, nth coordinate x_i.

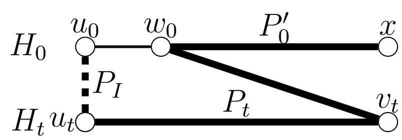

Figure 4.12: l odd, nth coordinate y_i.

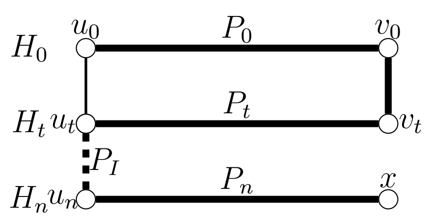

Figure 4.13: l even, nth coordinate x_i.

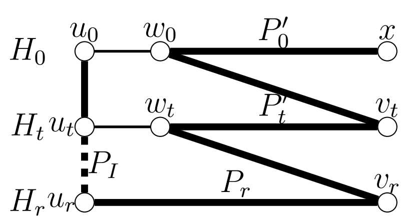

Figure 4.14: l even, nth coordinate y_i.

We now address another special case, the collection of sets X^t. Recall that this is the collection where $A_1 = \{x_1, y_1^1, y_2^1, \ldots, y_t^1\}$ and $A_i = \{x_1, y_1^i\}$ for $i \in \{2, 3, \ldots, t\}$.

Lemma 4.2.4. *For each $t \geq 3$, the graph G_{X^t} contains a Hamilton path. Furthermore, any pair of vertices $u = (u_1, u_2, \ldots, u_t)$ and $v = (v_1, v_2, \ldots, v_t)$ of $V(G_{X^t})$ may be chosen as the endpoints of such a path as long as $u_i = x_1$ or $v_i = x_1$ for some i. If, in addition, $u \nsim v$, then the path has an edge neither of whose end points use x_1.*

Proof. Let H_i be the subgraph of G_{X^t} induced by vertices in which the ith coordinate is x_1. For $i \in \{2, 3, \ldots, t\}$, H_i is isomorphic to K_t, with the t vertices corresponding to the t possible choices for the first coordinate. The choices for the first coordinate are $\{y_1^1, y_2^1, \ldots, y_t^1\}$, with each other coordinate being fixed. Since H_i is isomorphic to K_t, it must contain a Hamilton path from any vertex to any other vertex. The graph H_1 is simply the single vertex $(x_1, y_1^2, y_1^3, \ldots, y_1^t)$.

Let H_0 be the subgraph of G_{X^t} induced by vertices in which no coordinate is x_1. Notice that $\bigcup_{i=0}^{t} V(H_i) = V(G_{X^t})$, and $V(H_i) \cap V(H_j) = \emptyset$ whenever $i \neq j$. H_0 is also isomorphic to K_t. Each vertex in H_0 is adjacent to a vertex in H_i, $i \in \{1, 2, \ldots, t\}$, by switching the ith coordinate to x_1. As such, for every edge (v_1, v_2) in H_0, there exists a path beginning at v_1 and ending at v_2 whose internal vertices are exactly the vertices of H_i.

Let u and v denote the vertices we wish to be the endpoints of our Hamilton path. We now consider two cases, based on whether or not u and v are adjacent.

Case 1: $u \sim v$.

Consider a Hamilton cycle in H_0 with edges $e_1, e_2, \ldots, e_t$, and replace each of e_i with a path to H_i as described above. The result is a Hamilton cycle C of G_{X^t}.

We must now confirm that without loss of generality this Hamilton cycle contains the edge $e = (u, v)$. We know at least one of u or v uses x_1 on some coordinate; therefore, e is not an edge of H_0. For each $i \in \{2, 3, \ldots, t\}$, the cycle we described contains some edge which switches the ith coordinate from x_1 to y_1^i, as well as an edge which fixes the ith coordinate at x_1, changing the first coordinate from y_i^1 to y_j^1 for some $i \neq j$. For H_1, our cycle contains an edge which switches the first coordinate from x_1 to y_j^1 for some j. These edges cover all possible edge types outside of edges within H_0. By Automorphism Property II, we can take our cycle C and permute the y_j^1s to get the edge we want.

Case 2: $u \nsim v$

Suppose $u \in V(H_1)$. Then u is adjacent to every vertex of H_0. Therefore, we must have $v \in V(H_i)$ for some $i \in \{2, 3, \ldots, t\}$. Starting from v, it is simple to construct a path which first visits every vertex of H_i, then visits every vertex in H_0, and finally moves to u. The path we have constructed contains $t - 1$ edges of H_0. Then, since

$|\{2, 3, \ldots, i-1, i+1, \ldots, t\}| = t-2$, for each $j \in \{2, 3, \ldots, i-1, i+1, \ldots, t\}$ we may assign an edge (v_{j-1}, v_j) of our path to be replaced with a path from v_{j-1} to v_j whose internal vertices are the vertices of H_j, completing our Hamilton path from u to v.

Suppose $u \in V(H_0)$. We again must have $v \in V(H_i)$ for some $i \in \{2, 3, \ldots, t\}$, as u is adjacent to the lone vertex of H_1, and also to every other vertex in H_0. Since $t \geq 3$, there exists a Hamilton path in H_i starting at v and ending at a vertex w not adjacent to u. From w, we can take a Hamilton path in H_0 ending at u. Again, this path uses $t-1$ edges from H_0, so we may connect our remaining $t-2$ subgraphs to form a Hamilton path as we have done previously.

Suppose $u \in V(H_i)$, for $i \in \{2, 3, \ldots, t\}$. The last remaining case to check is $v \in H_j$, for $j \in \{2, 3, \ldots, t\}$ and $j \neq i$. Start by taking any Hamilton path in H_i starting at u. Move into H_0, and take a Hamilton path ending at any vertex not adjacent to v, which is always possible as $t \geq 3$. Now, move into H_j, and take a Hamilton path ending at v. The result is a path starting at u and ending at v, which visits every vertex in each of H_i, H_j and H_0. Additionally, this path uses $t-1$ edges within H_0. As such, we can connect the remaining $t-2$ subgraphs in the manner described above.

In each of these three cases, the Hamilton path described contains an edge in H_0, which is an edge that does not use x_1 on either of its end points, and we are done. $\square$

Our final goal for this chapter is to generalize Lemma 4.2.1 for larger values of t. In other words, we want to prove the following theorem.

Theorem 4.2.5. *Let $S \in \mathcal{S}_t - \{X^t\}$. For any edge e in G_S, there is a Hamilton cycle with Property $\mathcal{A}$ in G_S which contains e.*

The proof for this theroem is quite long and involved. We will present the proof as a series of lemmas. The general tactic for the proof is by induction on t, using Lemma 4.2.1 to verify the base case $t = 2$. For induction we assume that the result is true for any $S \in \mathcal{S}_k$, when $2 \leq k \leq t-1$.

Recall that for $S = A_1, A_2, \ldots, A_t \in \mathcal{S}_t$, we define $A_i = \{x_1, x_2, \ldots, x_l, y_1^i, y_2^i, \ldots, y_{a_i}^i\}$, with $a_1 \geq a_2 \geq \cdots \geq a_t$. For $z \in A_t$, let H_z denote the subgraph of G_S induced by vertices in which the t-th coordinate is z. If z is used on the t-th coordinate it cannot be used on any other coordinate, and so we have $H_z \cong G_{S'}$, where $S' = A_1 - \{z\}, A_2 - \{z\}, \ldots, A_{t-1} - \{z\} = A_1', A_2', \ldots, A_{t-1}'$. In order to use induction, we must verify that $S' \in \mathcal{S}_{t-1}$, and we must address that possiblity that $S' = X^{t-1}$.

It is not difficult to see that the only property of sets in $\mathcal{S}_{t-1}$ which S' might not satisfy is the requirement $\alpha = |\bigcup_{i=1}^{t-1} A'_i| \geq 2(t-1)$. Since $S \in \mathcal{S}_t$, we know $|\bigcup_{i=1}^{t} A_i| \geq 2t$. Therefore, if $z \in \{y_1^t, y_2^t, \ldots, y_{a_t}^t\}$ then $A_i - \{z\} = A_i$, and $\alpha \geq 2t - a_t$. If $a_t \leq 2$, we are done. Suppose $a_t \geq 3$. Since $a_1 \geq a_2 \geq \cdots \geq a_t$, in this case $|\bigcup_{i=1}^{t} A_i| \geq a_t t$, $\alpha \geq a_t t - a_t = a_t(t-1) \geq 2(t-1)$, and we are done. If, on the other hand, $z \in \{x_1, x_2, \ldots, x_l\}$, then $\alpha \geq 2t - a_t - 1$. If $a_t \leq 1$, we are done. Suppose $a_t \geq 2$. Here we have $|\bigcup_{i=1}^{t} A_i| \geq a_t t + 1$, as the element z must also be accounted for. Therefore, we have $\alpha \geq a_t t + 1 - a_t - 1 = a_t(t-1) \geq 2(t-1)$, and we are done. Therefore, $S' \in \mathcal{S}_{t-1}$.

Since $S' \in \mathcal{S}_{t-1}$, we may assume by the induction hypothesis that for any edge e in $G_{S'}$, there is a Hamilton cycle in $G_{S'}$ with Property $\mathcal{A}$ which contains e, provided S' is *not* X^{t-1}. We resolve the case $S' = X^{t-1}$ separately.

Suppose $S' = X^{t-1}$. In this case, we have $a_t = 1$. If $z = y_1^t$, then we must have $A_1 = \{x_1, y_1^1, y_2^1, \ldots, y_{t-1}^1\}$, and $A_i = \{x_1, y_1^i\}$ for $i \in \{2, 3, \ldots, t\}$. However, this implies $|\bigcup_{i=1}^{t} A_i| = 2t - 1$, a contradiction. Therefore, we must have $l = 2$, and $z = x_1$ or $z = x_2$. In this case, $A_1 = \{x_1, x_2, y_1^1, y_2^1, \ldots, y_t^1\}$ and $A_i = \{x_1, x_2, y_1^i\}$. Let $Y^t \in \mathcal{S}_t$ denote this collection of sets. We resolve this special case with the following lemma.

Lemma 4.2.6. *Let $S \in \mathcal{S}_k$, $S \neq X^k$, for some k, $2 \leq k \leq t-1$. If, for any edge e of G_S, there is a Hamilton cycle with Property $\mathcal{A}$ in G_S which contains e then, for any edge e in Y^t, there is a Hamilton cycle with Property $\mathcal{A}$ in G_{Y^t} which contains e.*

Proof. The case $t = 3$ will be handled via diagrams.

When $t = 3$, consider the following ten edge types, with notation similar to the notation used in the proof of Lemma 4.2.1. Similarly to the proof of Lemma 4.2.1, we use Automorphism Properties I, II and III in order to restrict our search to Hamilton cycles that contain each of these edge types.

1. $(x, x', y) \to (y, x', y)$

2. $(x, x', y) \to (x, y, y')$

3. $(y, x, x') \to (y', x, x')$

4. $(y, x, x') \to (y, y', x')$

5. $(x, y, y') \to (y'', y, y')$

6. $(x, y, y') \to (x', y, y')$

7. $(y, x, y') \to (y, y'', y')$

8. $(y, x, y') \to (y'', x, y')$

9. $(y, x, y') \to (y, x', y')$

10. $(y, y', y'') \to (y''', y', y'')$

For $\alpha \in \{x_1, x_2, y_1^t\}$, let H_α denote the subgraph of G_{Y^t} induced by the vertices in which the third coordinate is α. See Figure 4.15 and Figure 4.16 for Hamilton cycles C_1 and C_2 with Property $\mathcal{A}$ in G_{Y^3}, which cover all ten possible edge types.

Now, suppose $t \geq 4$. The graph G_{Y^t} may be partitioned into the three disjoint subgraphs H_{x_1}, H_{x_2}, and $H_{y_1^t}$. Since H_{x_1} and H_{x_2} are both isomorphic to G_{X^t}, we may apply Lemma 4.2.4. The graph $H_{y_1^t}$ is isomorphic to $G_{S_{y_1^t}}$, where $S_{y_1^t} = A_1, A_2, \ldots, A_{t-1}$. It can be easily checked that $S_{y_1^t} \in \mathcal{S}_{t-1}$. By assumption, we may find a Hamilton cycle in $H_{y_1^t}$ which contains a prescribed edge, and has Property $\mathcal{A}$. Let $e = (u, v) = ((u_1, u_2, \ldots, u_{k+1}), (v_1, v_2, \ldots, v_{k+1}))$ be any edge in G_{Y^t}. We will construct a Hamilton cycle C in G_{Y^t} which contains the edge e, and then we will verify that C has consecutive vertices not containing x_i, for $i \in \{1, 2\}$. To do this, we will utilize the symmetries of G_{Y^t}.

By Automorphism Properties I, II and III, as before, it suffices to find enough Hamilton cycles with Property $\mathcal{A}$ so that an edge of each edge type (listed below) appears in one of them. We can then apply the appropriate automorphism to find the Hamilton cycle we desire. Note that by Automorphism Property III, since $a_2 = a_3 = \cdots = a_t = 1$, we only need to consider the edges in which the first or the second coordinate changes. The following are the edge categories of G_{Y^t}, grouped by which coordinate changes.

Case 1: First coordinate changes:

a. from y_i^1 to y_j^1, with x_1 and x_2 not used.

b. from y_i^1 to y_j^1, with x_1 used, x_2 not used.

c. from y_i^1 to y_j^1, with x_1 and x_2 used.

d. from y_i^1 to x_1, with x_2 not used.

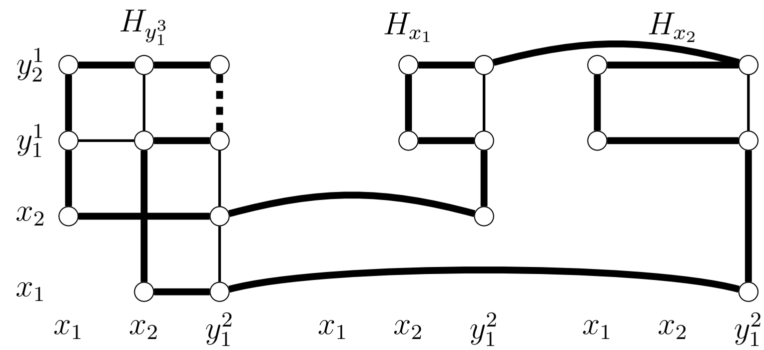

Figure 4.15: Hamilton cycle C_1 in G_S.

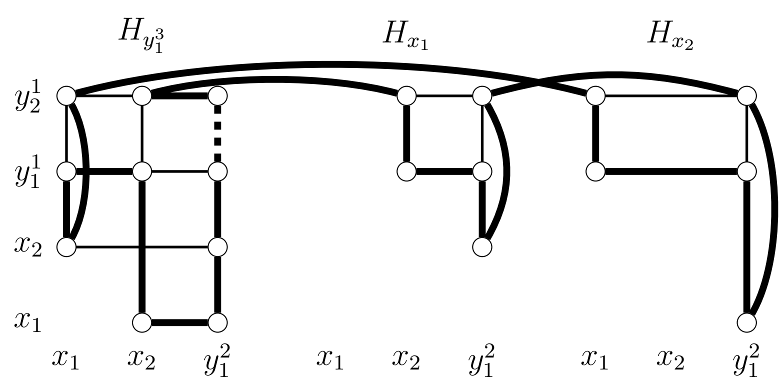

Figure 4.16: Hamilton cycle C_2 in G_S.

e. from y_i^1 to x_1, with x_2 used.

f. from x_1 to x_2.

Case 2: Second coordinate changes:

a. from y_1^2 to x_1, with x_2 not used.

b. from y_1^2 to x_1, with x_2 used on the first coordinate.

c. from y_1^2 to x_1, with x_2 used on the ith coordinate, $i \in \{3, 4, \ldots, t\}$.

d. from x_1 to x_2.

We will now describe a set of Hamilton cycles with Property $\mathcal{A}$ in G_{Y^t} which contain an edge of each of these ten edge categories. First, we will partition $H_{y_1^t}$ into three subgraphs. Let H_{x_1,y_1^t}, H_{x_2,y_1^t}, and $H_{y_1^{t-1},y_1^t}$ be the subgraphs of $H_{y_1^t}$ induced by vertices with $(t-1)$st coordinate x_1, x_2, and y_1^{t-1}, respectively. We will denote these by $H_{x_1,y}$, $H_{x_2,y}$ and $H_{y,y}$. We have $t \geq 4$, and each of these subgraphs is isomorphic to the graph $G_{S'}$ for a collection of $t-2$ sets $S' \in \mathcal{S}_{t-2}$. Therefore, by assumption, $H_{x_1,y}$, $H_{x_2,y}$ and $H_{y,y}$ all have Hamilton cycles with Property $\mathcal{A}$.

Consider a Hamilton cycle $C_{y,y}$ in $H_{y,y}$. Clearly, $C_{y,y}$ must contain a pair of consecutive vertices u_1 and u_2, where u_1 uses neither x_1 nor x_2 on any coordinate, and u_2 uses exactly one of x_1 or x_2 on its coordinates. Without loss of generality, assume some coordinate of u_2 is x_1. Let $P_{y,y}$ be the Hamilton path in $H_{y,y}$ from u_2 to u_1. Let u_1' be the vertex obtained by changing the last coordinate of u_1 to x_1. Let u_2' be the vertex obtained by changing the last coordinate of u_2 to x_2. Notice, $u_1' \in H_{x_1}$ and $u_2' \in H_{x_2}$. Now, by Lemma 4.2.4, let P_{x_1} be a Hamilton path in H_{x_1} starting at u_1' and ending at some vertex v_1', which uses x_2 in the $(t-1)$st coordinate and is *not* adjacent to u_1'. Let P_{x_2} be a Hamilton path in H_{x_2} starting at the vertex w_1', which uses x_1 in the $(t-1)$st coordinate, and is equal to v_1' on coordinates one through $t-2$, and ending at u_2'. Since u_2' uses x_1 on some coordinate other than the $(t-1)$st, we have that w_1' and u_2' are *not* adjacent. In addition, by Lemma 4.2.4, P_{x_1} and P_{x_2} can be constructed to contain a single edge which does not use x_2 or x_1, respectively.

Let v_1 be the vertex obtained by changing the last coordinate of v_1' to y_1^t, and let w_1 be the vertex obtained by changing the last coordinate of w_1' to y_1^t. Notice $v_1 \in H_{x_2,y}$ and $w_1 \in H_{x_1,y}$, and that $v_1 \sim w_1$. Let v_2 be the vertex obtained by switching the first coordinate of v_1 to y_i^1 for some $i \in \{1, 2, \ldots, t-1\}$, and let w_2

be the vertex obtained by switching the first coordinate of w_1 to y_i^1 as well. Then, $v_2 \sim w_2$. Now, by induction, there is a Hamilton path $P_{x_1,y}$ in $H_{x_1,y}$ from w_2 to w_1, and there is a Hamilton path $P_{x_2,y}$ in $H_{x_2,y}$ from v_1 to v_2. Now, $P_{y,y}P_{x_1}P_{x_2,y}P_{x_1,y}P_{x_2}u_2$ is a Hamilton cycle C_{Y^t} in G_{Y^t}. (See Figure 4.17.)

This construction does not depend on which Hamilton cycle we choose for $H_{y,y}$. By induction, there exists a Hamilton cycle $C_{y,y}$ in $H_{y,y}$ which uses any edge of $H_{y,y}$, and therefore the cycle we constructed may contain any such edge. Looking at the list of the ten edge types, this covers almost all of them. The only possible edges not covered are $1c$ and $2c$, and only when t is exactly four (as the $(t-1)$st and t-th coordinates are fixed at y_1^{t-1} and y_1^t respectively within $H_{y,y}$). However, both edge types $1c$ and $2c$ are contained in our Hamilton paths P_{x_1} in H_{x_1}, and P_{x_2} in H_{x_2}. As such, C_{Y^t} may contain any edge in G_{Y^t}. We must now verify that C_{Y^t} has Property $\mathcal{A}$. By Lemma 4.2.4, and since $u_1' \nsim v_1'$ and $u_2' \nsim w_1'$, P_{x_1} and P_{x_2} will contain an edge that fits our needs

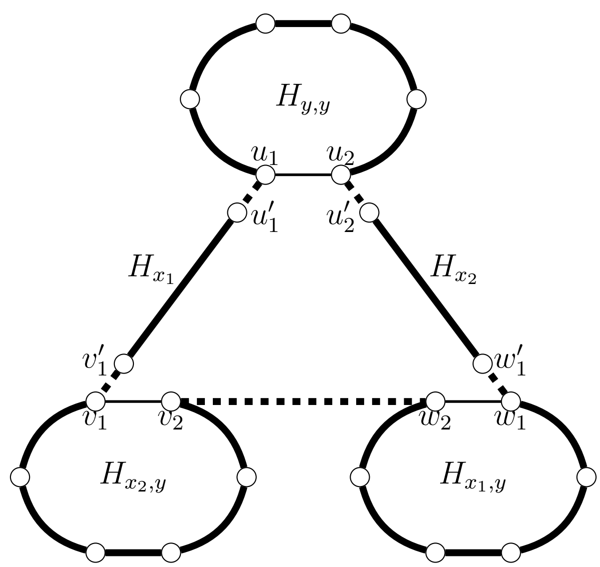

Figure 4.17: A Hamilton cycle in G_{Y^t}.

□

In light of this result, we now consider $S \in \mathcal{S}_t$, $S \neq X^t$ and $S \neq Y^t$. As previously discussed, for $z \in A_t$, we may now assume that for any edge e of H_z, there is a Hamilton cycle with Property $\mathcal{A}$ in H_z which contains e. The problem of proving G_S has the Hamilton cycle we want is still far from easy. We will split the remainder of our proof into three parts, based on the value of l. The first case considers only the collections S for which $l = 0$.

Lemma 4.2.7. *Let $S \in \mathcal{S}_t - \{X^t, Y^t\}$, with $l = 0$ and $t \geq 3$. For any edge e in G_S, there is a Hamilton cycle with Property $\mathcal{A}$ in G_S which contains e.*

Proof. In this case, $G_S \cong K_{a_1} \square K_{a_2} \square \cdots \square K_{a_t}$. This graph has a Hamilton cycle whenever $a_1 + a_2 + \cdots + a_t > t + 1$. Since we have $a_1 + a_2 + \cdots + a_t \geq 2t$, we have a Hamilton cycle. By Automorphism Property II, we may assume that this cycle contains the edge e we desire. When $l = 0$, any Hamilton cycle vacuously satisfies the requirements of Property $\mathcal{A}$. $\square$

Lemma 4.2.8. *Let $S \in \mathcal{S}_t - \{X^t, Y^t\}$, with $l = 1$ and $t \geq 3$. For any edge e in G_S, there is a Hamilton cycle with Property $\mathcal{A}$ in G_S which contains e.*

Proof. Let H_0 denote the subgraph of G_S induced by the vertices which do not use x_1 on any coordinate, and let H_i, $i \in \{1, 2, \ldots, t\}$, denote the subgraph of G_S induced by the vertices which use x_1 on coordinate i. Notice that $H_0 \cong K_{a_1} \square K_{a_2} \square \cdots \square K_{a_t}$, and $H_i \cong K_{a_1} \square K_{a_2} \square \cdots \square K_{a_{i-1}} \square K_{a_{i+1}} \square \cdots \square K_{a_t}$. Recalling that $a_1 \geq a_2 \geq \cdots \geq a_t$, we consider two cases based on the value a_2.

Case 1: $a_2 = 1$.

Here, we have $A_1 = \{x_1, y_1^1, y_2^1, \ldots, y_{a_1}^1\}$, and $A_i = \{x_1, y_1^i\}$ when $i \in \{2, 3, \ldots, t\}$. We may assume $a_1 \geq t + 1$, as $|\bigcup_{i=1}^t A_i| < 2t$ if $a_1 < t$, and $S = X^t$ if $a_1 = t$. We can use the same technique as in Case 1 of the proof of Lemma 4.2.4 to construct a Hamilton cycle C_S in G_S. However, in this case $H_0 \cong K_{a_1}$, which contains $a_1 \geq t+1$ vertices. As such, $E(C_S)$ will contain some edge of H_0. This edge does not use x_1 on any coordinate, and by the construction our cycle may contain any edge not contained within H_0. By Automorphism Property II, we may assume C_S uses a particular edge contained within H_0. Therefore, there is a cycle that contains any edge we want, and will always have some edge contained in H_0. This gives consecutive vertices in C_S which do not use x_1 on any coordinate. Hence, Property $\mathcal{A}$ holds.

Case 2: $a_2 \geq 2$.

First, we show that H_i, $i \in \{1, 2, \ldots, t\}$, contains a Hamilton path from u to v whenever $u \sim v$. Recall that $H_i \cong K_{a_1} \square K_{a_2} \square \cdots \square K_{a_{i-1}} \square K_{a_{i+1}} \square \cdots \square K_{a_t}$. In this case, we have $a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_t \geq t$. If $a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_t > t$, for any edge e in H_i, H_i contains a Hamilton cycle which contains e, for the reasons described in the proof of Lemma 4.2.7. If $a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_t = t$, then $H_i \cong K_2$. In either case, it is clear that H_i contains a Hamilton path from u to v whenever $u \sim v$.

Now, we also have $a_1 + a_2 + \cdots + a_t \geq 2t - 1$, and therefore for any edge e with endpoints in H_0, there is a Hamilton cycle C_0 in H_0 which contains e, for reasons stated in the proof of Lemma 4.2.7. For some edge $e = (u, v)$ in C_0, there are distinct vertices u_i and v_i in H_i such that $u \sim u_i$, $v \sim v_i$, and $u_i \sim v_i$ if and only if u and v do not differ on the ith coordinate. If that is the case for the edge e, we may replace e with a Hamilton path in H_i from u_i to v_i. Therefore, if for each $i \in \{1, 2, \ldots, t\}$, there is an edge e_i with endpoints that do not differ in the ith coordinate, and $e_i \neq e_j$ whenever $i \neq j$, we may construct a Hamilton cycle in G_S by replacing e_i with a Hamilton path in H_i for each i. We now show that such a set of edges exists.

Let $s = \max\{i | a_i \geq 2\}$. In this case, we have $s \geq 2$. Notice that if $i \in \{s+1, s+2, \ldots, t\}$, then there are no edges in H_0 which change coordinate i, so any edge in C_0 may be chosen as e_i. For each $i \in \{1, 2, \ldots, s\}$, there are at least $a_i \geq 2$ edges in C_0 which change the ith coordinate. For $i \in \{1, 2, \ldots, s-1\}$, let e_i be any edge in C_0 which changes the $(i+1)$st coordinate. Let e_s be any edge in C_0 which changes the first coordinate. We will now use a counting argument to show that there are enough "unclaimed" edges of C_0 to assign to the remaining $t - s$ subgraphs H_i. Let m denote the number of edges in C_0. There are then $m - s$ unclaimed edges of C_0. Noting that $m \geq a_1 + a_2 + \cdots + a_s$, consider the following:

$$\begin{aligned} 2t - 1 &\leq a_1 + a_2 + \cdots + a_t \\ 2t - 1 &\leq a_1 + a_2 + \cdots + a_s + (t - s) \\ (t - 1) + s &\leq a_1 + a_2 + \cdots + a_s \end{aligned}$$

Thus, we have $m - s \geq a_1 + a_2 + \cdots + a_s \geq (t-1) + s - s = (t-1)$. We have only $t - s \leq t - 2 = t - 2$ edges left to choose, and at least $(t-1)$ edges from which to choose. Choose any set of $t - s$ of these edges to be e_i for $i \in \{s+1, s+2, \ldots, t\}$. Notice that we have at least one edge e_0 in C_0 where $e_0 \neq e_i$ for each $i \in \{1, 2, \ldots, t\}$. Furthermore, we can choose e_i for $i \in \{s+1, s+2, \ldots, t\}$ so that we can assume e_0 changes the jth coordinate, for some $j \in \{1, 2, \ldots, s\}$. Our Hamilton cycle C_S of G_S is constructed by replacing each e_i in C_0 with a Hamilton path in H_i. The resulting cycle C_S must contain at least one edge of H_0, which gives us consecutive vertices which do not use x_1 on any coordinate. We must now show that for any edge

$e = (u, v)$ in G_S, our construction can produce a Hamilton cycle which contains e. We will do so in three cases, based on the three possible edge types in G_S.

Suppose $u \in V(H_0)$, $v \in V(H_i), i \in \{1, 2, \ldots, t\}$. In this case, the edge e switches the ith coordinate from y^i_j to x_1, for some $j \in \{1, 2, \ldots, a_i\}$. The Hamilton cycle in G_S will always contain *some* edge e' with the same edge type as e, and by Automorphism Property II, we can permute the y^i_ks so that e' maps to e, and we are done.

Suppose u, $v \in V(H_0)$. Let w denote the coordinate in which u and v differ. Note that we must have $w \in \{1, 2, \ldots, s\}$. As described above, we may construct our Hamilton cycle such that the edge e_0 changes coordinate w. Again, by Automorphism Property II, we can permute the y^j_ks so that the edge e_0 maps to our desired edge e, and we are done.

Suppose u,$v \in V(H_i)$ for some $i \in \{1, 2, \ldots, t\}$. Let w denote the coordinate in which u and v differ. Again, we have $w \in \{1, 2, \ldots, s\}$. If H_i has only two vertices, then we are done. So, assume H_i has at least three vertices. If the Hamilton path in H_i we used to construct our Hamilton cycle contains some edge e' which changes coordinate w, we may permute the y^j_ks such that e' maps to e, and we are done. The only way such an e' will not exist is if the endpoints of the Hamilton path in H_i differ in the wth coordinate. This occurs precisely when e_i changes the wth coordinate. If $i \in \{s+1, s+2, \ldots, t\}$, we can alter the choice of e_i to an edge which does not change the wth coordinate. If $i \in \{1, 2, \ldots, s\}$, we alter the choice of e_i easily unless $s = 2$. However, if $s = 2$, each edge in H_i will be an edge which changes the wth coordinate. Therefore we may construct a Hamilton cycle such that it contains an e', which we can map to e under some permutation of the y^j_ks by Automorphism Property II.

These three cases cover all the possible edge types in G_S, and therefore, for any $e \in E(G_S)$, we may construct a Hamilton cycle in G_S which contains e, and has an edge e_0 which does not use x_1 on any coordinate, so we are done. □

Lemma 4.2.9. *Let $S \in \mathcal{S}_t - \{X^t, Y^t\}$, with $l \geq 2$ and $t \geq 3$. For any edge e in G_S, there is a Hamilton cycle with Property $\mathcal{A}$ in G_S which contains e.*

Proof. Let $e = (v, v')$ be any edge in G_S. We will construct a Hamilton cycle in G_S which contains the edge e in two cases, based on whether or not v and v' differ on the t-th coordinate, or some other coordinate.

Case 1: $v = (v_1, v_2, \ldots, v_{t-1}, v_t)$, $v' = (v_1, v_2, \ldots, v_{t-1}, v'_t)$, $v_t \neq v'_t$.

Our first step is to construct a cycle which contains exactly the vertices of H_{v_t} and $H_{v'_t}$, and which contains the edge e. By induction, H_{v_t} and $H_{v'_t}$ will each contain

Hamilton cycles which use prescribed edges. Let C_{v_t} be a Hamilton cycle in H_{v_t} using the edge $(v, u) = ((v_1, v_2, \ldots, v_{t-1}, v_t), (v_1, v_2, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_{t-1}, v_t))$ and let $C_{v_t'}$ be a Hamilton cycle in $H_{v_t'}$ using the edge $(v', u') = ((v_1, v_2, \ldots, v_{t-1}, v_t'), (v_1, v_2, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_{t-1}, v_t'))$.

We must justify that such a v_i' exists. If $a_i > 1$ for some i, one of y_1^i or y_2^i is a valid candidate. Otherwise, if $v_j \neq y_1^j$ for some j, y_1^j is a valid candidate. Now, if $a_i = 1$ and $v_i = y_1^i$ for all i, we must have $l \geq 3$, as $|\bigcup_{i=1}^t A_i| \geq 2t$ and $t - 1 \geq 2$. Then, one of x_1, x_2 or x_3 is a valid candidate. Therefore, we may always find the desired v_i', for some $2 \leq i \leq t-1$.

Now, as $v \sim v'$ and $u \sim u'$, we may simply delete the edges (v, u) and (v', u') and add the edges (v, v') and (u, u') to create the desired cycle. Figure 4.18 displays this process. We will informally refer to this process as *stitching* the cycles C_{v_t} and $C_{v_t'}$ together.

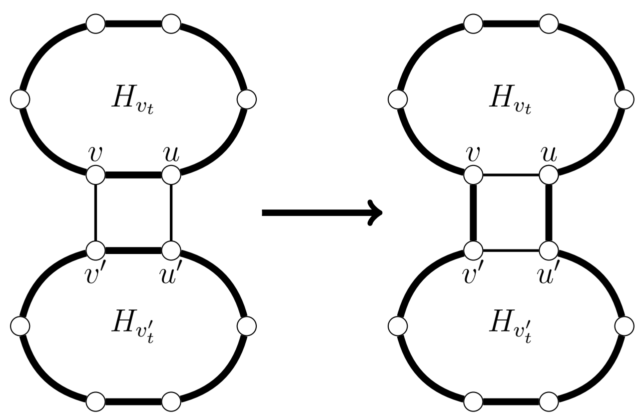

Figure 4.18: Stitching cycles together.

In several steps we will extend our cycle to include the vertices of H_a, for each $a \in A_t - \{v_t, v_t'\}$.

First, we will extend our cycle to include the vertices of H_{x_i}, for $x_i \in \{x_1, x_2, \ldots, x_l\} - \{v_t, v_t'\}$. By induction, the cycle C_{v_t} we have chosen for H_{v_t} will have consecutive vertices which do not use x_i on any coordinate (it is possible that these vertices are the vertices u and v, which is a problem that is addressed later). Let $(v_i, u_i) = ((w_1, w_2, \ldots, w_{t-1}, v_t), (w_1, w_2, \ldots, w_{i-1}, w_i', w_{i+1}, \ldots, w_{t-1}, v_t))$ be an edge of H_{v_t} which does not use x_i. By induction, we may choose a Hamilton cycle C_{x_i} in H_{x_i} which contains the edge $(v_i', u_i') = ((w_1, w_2, \ldots, w_{t-1}, x_i), (w_1, w_2, \ldots, w_{i-1}, w_i', w_{i+1}, \ldots, w_{t-1}, x_i))$. We may now stitch C_{x_i} onto our existing cycle by deleting the edges (v_i, u_i) and

(v_i', u_i'), and adding the edges (v_i, v_i') and (u_i, u_i').

We must now extend our cycle to include the vertices of $H_{y_i^t}$, for $y_i^t \in \{y_1^t, y_2^t, \ldots, y_{a_t}^t\} - \{v_t, v_t'\}$. In this case, any choice of an edge in H_{v_t} which has not been stitched onto can be chosen to attach $H_{y_i^t}$, in a similar manner. Note that the number of edges used by a Hamilton cycle in H_{v_t} is large enough that running out of edges to attach cycles onto is not a concern.

After this process is complete, we are left with a Hamilton cycle C_S in G_S, which contains the edge e. We must verify that C_S has Property $\mathcal{A}$. By induction, $C_{v_t'}$ must contain consecutive vertices which do not use x_i on any coordinate, for each $x_i \in \{x_1, x_2, \ldots, x_l\} - \{v_t, v_t'\}$. If u' and v' are the only such vertices for some x_i, then note that the edge e must not use x_i.

Suppose $v_t \in \{x_1, x_2, \ldots, x_l\}$. If u' and v' are not the only consecutive vertices in $C_{v_t'}$ which do not use v_t on any coordinate, then we have consecutive vertices in C_S which do not use v_t on any coordinate. Suppose they are. Then, for each $x_i \in \{x_1, x_2, \ldots, x_l\} - \{v_t\}$, there must be consecutive vertices in $C_{v_t'}$ which do not use x_i on any coordinate, and which are not both u' and v'. Then, instead of stitching a Hamilton cycle C_a in H_a, for some $a \in A_t - \{v_t, v_t'\}$, to an edge of C_{v_t}, we may stitch it to an edge $e' \neq (u', v')$ of $C_{v_t'}$. By assumption, e' *will* use v_t, and by induction C_a must contain an edge which does not use v_t, our final cycle C_S will contain an edge which does not use v_t, and is contained within H_a. The edge in C_{v_t} we used previously to stitch C_a onto will be in our final cycle, and hence we will still have an edge which does not use a.

Suppose $v_t' \in \{x_1, x_2, \ldots, x_l\}$. We may assume that $v_t \in \{x_1, x_2, \ldots, x_l\}$, otherwise we may switch the roles of v_t and v_t' in the construction and use the previous argument. Since $a_t \geq 1$, we must have $y_1^t \in A_t - \{v_t, v_t'\}$. We may stitch a Hamilton cycle $C_{y_1^t}$ in $H_{y_1^t}$ onto some edge of C_{v_t} which uses v_t' on one of its end vertices, and our cycle C_S is then guaranteed to have consecutive vertices contained within $H_{y_1^t}$ which do not use v_t' for similar reasons as those discussed in the previous case.

In the case that $A_t = \{x_1, x_2, y_{t,1}\}$, with $v_t = x_1$ and $v_t' = x_2$, there is only one subgraph, $H_{y_1^t}$, to attach. Therefore, we cannot use both of the previous arguments, as each requires the attachment of such a subgraph. An alternate construction is given for this case. Here, we have $v = (y_{v_1}^1, y_{v_2}^2, \ldots, y_{v_{t-2}}^{t-2}, y_{v_{t-1}}^{t-1}, x_1)$ and $v' = (y_{v_1}^1, y_{v_2}^2, \ldots, y_{v_{t-2}}^{t-2}, y_{v_{t-1}}^{t-1}, x_2)$. Let:

- $u = (y_{v_1}^1, y_{v_2}^2, \ldots, y_{v_{t-2}}^{t-2}, x_2, x_1)$,

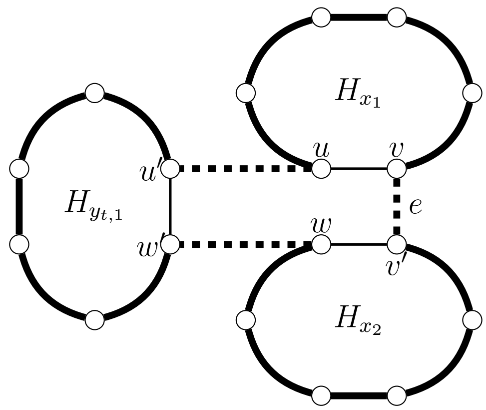

Figure 4.19: A Hamilton Cycle in G_S, when $A_t = \{x_1, x_2, y_1^t\}$.

- $u' = (y^1_{v_1}, y^2_{v_2}, \ldots, y^{t-2}_{v_{t-2}}, x_2, y^t_1)$,
- $w = (y^1_{v_1}, y^2_{v_2}, \ldots, y^{t-2}_{v_{t-2}}, x_1, x_2)$, and
- $w' = (y^1_{v_1}, y^2_{v_2}, \ldots, y^{t-2}_{v_{t-2}}, x_1, y^t_1)$.

Then, we have $u \sim v$, $u \sim u'$, $w \sim v'$, $w \sim w'$, and $u' \sim w'$. Figure 4.19 illustrates how to connect Hamilton cycles in H_{x_1}, H_{x_2}, and $H_{y_1^t}$ into a Hamilton cycle C_S in G_S. Since some coordinate of u' is x_2, and some coordinate of w' is x_1, by induction, there must be consecutive vertices in C_S which do not use x_1 on any coordinate, and consecutive vertices which do not use x_2 on any coordinate, each pair contained within $H_{y_1^t}$. Therefore, we can guarantee that our constructed cycle C_S will have Property $\mathcal{A}$.

We must now address the possibility that, for some x_i, the only consecutive vertices in C_{v_t} which do not use x_i on any coordinate are the vertices v and u. Suppose this is the case. If the vertices v' and u' are not the only consecutive vertices in $C_{v_t'}$ which do not contain x_i, C_{x_i} may simply be stitched onto $C_{v_t'}$ instead of C_{v_t}. Suppose that the vertices v' and u' *are* the only consecutive vertices in $C_{v_t'}$ which do not contain x_i.

We may resolve this case easily when H_{x_i} is not the only subgraph whose vertices we need to include with our cycle in H_{v_t} and $H_{v_t'}$; that is, when $A_t \neq \{v_t, v_t', x_i\}$. In this case, take some $a \in A_t - \{v_t, v_t', x_i\}$, and stitch a cycle C_a in H_a to C_{v_t} in the manner described above. Now, by induction, the cycle C_a will have two consecutive vertices which do not use x_i on any coordinate. Furthermore, those vertices cannot be the vertices used to stitch C_a to C_{v_t}, as then we must have another pair of consecutive vertices in C_{v_t} which do not use x_i on any coordinate, which contradicts the assumption that v and u are the only consecutive vertices in C_{v_t} which do not use x_i on any coordinate. Therefore, by induction we may choose a cycle C_{x_i} in H_{x_i} which will allow us to stitch together C_{x_i} and C_a. Stitching the remaining C_as to our existing cycle can proceed in the manner described above. Note that the edge e must not use x_i because $v_t \neq x_i$ and v does not use x_i. Hence, we may construct our cycle such that Property $\mathcal{A}$ is satisfied.

Now, suppose that $A_t = \{v_t, v_t', x_i\}$. Since we know $a_t \geq 1$, and in this case $l \geq 2$, we must have $A_t = \{x_1, x_2, y_1^t\}$. Without loss of generality, we may assume $v_t = y_1^t$,$v_t' = x_1$, and $x_i = x_2$. Recall that when we stitched C_{v_t} and $C_{v_t'}$ together using the edges (v, u) in C_{v_t} and (v', u') in $C_{v_t'}$, the choice of u was made arbitrarily. We chose any u that was adjacent to both v, and some vertex u' in $H_{v_t'}$. Knowing that v and v' do not use x_i on any coordinate, we may choose u to be a vertex that *does* use x_i on some coordinate. Simply take any coordinate of v other than the last and change it to x_i to get such a u. The choice for u' follows. As before, we can choose $C_{v_t'}$ to contain our new (v', u'), and C_{v_t} to contain our new (v, u), and stitch them together with our prescribed edge $e = (v, v')$ and the edge (u, u'). By induction, both C_{v_t} and $C_{v_t'}$ will have consecutive vertices which do not use x_i on any coordinate, and neither of these vertices can be u or u' respectively, as each use x_i on some coordinate. We can then choose a cycle C_{x_i} in H_{x_i} using the appropriate edge, and stitch it to C_{v_t} as before. We know that $C_{v_t'}$ will have some edge that does not use x_i on any coordinate, and this edge will occur in our final cycle C_S. Now, we must show that our cycle has consecutive vertices which do not use $v_t' = x_1$ on any coordinate. Let (w, w') denote consecutive vertices in C_{v_t} which do not use x_2 on any coordinate. If either w or w' use x_1 on some coordinate, then by stitching C_{x_2} onto (w, w') and appealing to induction, we know that our final cycle C_S will have consecutive vertices which do not use x_1 on any coordinate and which lie in H_{x_i}. Suppose neither w nor w' use x_1 on any coordinate. Consider the vertex w'' following w' on the cycle C_{v_t}. Since w' uses neither x_2 nor x_1 on any coordinate, w'' can use at most one of x_2 and x_1 its coordinates. Thus, a careful selection of one of (w, w') or (w', w'') as the edge to attach C_{x_2} onto will yield a Hamilton cycle C_S in G_S which contains consecutive vertices which do not use x_1 on any coordinate.

We now consider the case where v and v' differ in some coordinate other than the t-th coordinate.

Case 2: $v = (v_1, v_2, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_{t-1}, v_t)$, $v' = (v_1, v_2, \ldots, v_{i-1}, v_i', v_{i+1},$

$\ldots, v_{t-1}, v_t)$, $v_i \neq v_i'$, $i \neq t$.

We will split this case into sub-cases, based on the t-th coordinate of our edge e.

Case 2.1: $v_t = x_i$, for some $i \in \{1, 2, \ldots, l\}$.

By induction, let C_{x_i} be a Hamilton cycle in H_{x_i} which contains the edge e. Let (u, w) be any other edge of C_{x_i}. Let u' and w' be the vertices obtained by switching the t-th coordinate of u and w respectively to y_1^t. By induction, let $C_{y_1^t}$ be a Hamilton cycle with Property $\mathcal{A}$ in $H_{y_1^t}$ which contains the edge (u', w'). As before, we may remove the edges (u, w) from C_{x_i} and (u', w') from $C_{y_1^t}$, and add in the edges (u, u') and (w, w') to create a cycle which uses the vertices of H_{x_i} and $H_{y_1^t}$. By induction, $C_{y_1^t}$ has a distinct edge e_j for every $j \in \{1, 2, \ldots, l\} - \{i\}$, such that neither of the endpoints of e_j use x_j on any coordinate. Let e_j' be the edge of H_{x_j} obtained by switching the t-th coordinate of the vertices of e_j to x_j. By induction, let C_{x_j} be a Hamilton cycle with Property $\mathcal{A}$ in H_{x_j} which contains the edge e_j'. Then, as before, we may stitch each such cycle onto our existing cycle with the appropriate pair of edge deletions and additions.

It is possible that for some j, the only candidate for the edge e_j is the edge (u', w'). If $l = 2$, we may simply choose the edge (u, w) to be an edge which uses x_j, which prevents the possibility that $e_j = (u', w')$. Otherwise, for each $h \in \{1, 2, \ldots, l\} - \{i, j\}$, e_h must use x_j, and therefore the edge e_h' will too. Then, by induction, C_{x_h} will have an edge $e_{j,h}$ which does not use x_j, and that edge cannot be the edge e_h'. We may then switch the t-th coordinates of the vertices of $e_{j,h}$ to obtain the edge $e_{j,h}'$ in H_{x_j}, use induction to create a Hamilton cycle C_{x_j} in H_{x_j} which contains the edge $e_{j,h}'$, and stitch C_{x_j} onto C_{x_h}.

For any $y \in \{y_2^t, y_3^t, \ldots, y_{a_t}^t\}$, we may pick any unused edge e_y of $C_{y_1^t}$. Let e_y' be the edge obtained by switching the t-th coordinates of the vertices of e_y to y. Let C_y be a Hamilton cycle in H_y which contains the edge $e'y$, and stitch C_y to $C_{y_1^t}$.

After this process is complete, we have our Hamilton cycle C_S in G_S which contains the edge e. We must now show that this cycle has Property $\mathcal{A}$. By induction, we know C_{x_i} has an edge which does not use x_j for $j \in \{1, 2, \ldots, l\} - \{i\}$. If (u, w) is such an edge for some x_j, then the edges (u, u') and (w, w') are both edges in C_S which do not use x_j. It remains to show that x_i is not used on an edge of C_S.

Consider any of the edges e_j, for $j \in \{1, 2, \ldots, l\} - \{i\}$. If e_j uses x_i, then e_j' uses x_i as well. Therefore, C_S will have an edge contained within H_{x_j} which does not use x_i, and we are done. Suppose then that e_j does not use x_i. Then, e_j' does not use x_i as well. Therefore, each of the edges used to stitch C_{x_j} to $C_{y_1^t}$ will not use x_i, and we are done.

Case 2.2: $v_t = y_i^t$, for some $i \in \{1, 2, \ldots, a_t\}$.

By induction, let $C_{y_i^t}$ be a Hamilton cycle in $H_{y_i^t}$ which uses the edge e. By induction, $C_{y_i^t}$ has an edge e_j for every $j \in \{1, 2, \ldots, l\}$, such e_j does not use x_j. Let e'_j be the edge of H_{x_j} obtained by switching the t-th coordinate of the vertices of e_j to x_j. By induction, let C_{x_j} be a Hamilton cycle in H_{x_j} which contains the edge e'_j. We may stitch C_{x_j} to $C_{y_i^t}$ on the edge e_j. This possibility that $e_j = e$ for some j is resolved in a similar manner as the possiblity that $e_j = (u', w')$ for some j in the previous case.

For any $y \in \{y_1^t, y_2^t, \ldots, y_{a_t}^t\} - \{y_i^t\}$, we may pick any unused edge e_y of $C_{y_i^t}$. Let e'_y be the edge obtained by switching the t-th coordinates of the vertices of e_y to y. Let C_y be a Hamilton cycle in H_y which contains the edge $e'y$, and stitch C_y to $C_{y_1^t}$ as before.

We are left with a Hamilton cycle C_S in G_S, which contains our edge e. As always, we must confirm C_S has Property $\mathcal{A}$.

First, suppose that $e_j \neq e$, for all $j \in \{1, 2, \ldots, l\}$. If this is the case, then either C_{x_1} or the set of edges used to stitch C_{x_1} and $C_{y_i^t}$ together will contain edges which do not use x_j for $j \in \{2, 3, \ldots, l\}$. Additionally, either C_{x_2} or the set of edges used to stitch C_{x_2} and $C_{y_i^t}$ together will contain an edge which does not use x_1.

Suppose that for some $j \in \{1, 2, \ldots, l\}$, e is the only edge of $C_{y_i^t}$ which does not use x_j. In this case, we will have stitched H_{x_j} onto H_{x_h} for some $h \in \{1, 2, \ldots, l\} - \{j\}$. Then, by induction, C_{x_j} will have consecutive vertices which do not use x_m on any coordinate, for $m \in \{1, 2, \ldots, l\} - \{j, h\}$. Since by assumption e does not use x_j, we just need to find consecutive vertices in C_S which do not use x_h on any coordinate. This can be done in the same manner as finding an edge which does not use x_i from the previous case. However, in this case, there is the possibility that when $l = 2$ this cannot be done. If this is the case, without loss of generality we may say that C_{x_1} is stitched to $C_{y_1^t}$, C_{x_2} is stitched to C_{x_1}, and the edge e is the only edge in $C_{y_1^t}$ which does not use x_2. Then, consider any vertex u of $C_{y_1^t}$ which uses neither x_1 nor x_2 on any coordinate. Since e is the only edge which does not use x_2, the neighbours w and w' of u in $C_{y_1^t}$ will not use x_2. Therefore, neither (u, w) nor (u, w') will use x_1 on any coordinate. Since we only use one edge of $C_{y_1^t}$ to stitch C_{x_1} onto, our Hamilton cycle C_S will use at least one of (u, w) and (u, w'), and we are done. $\square$

This completes the proof of Theorem 4.2.5.

Chapter 5

Hamiltonicity of $\mathcal{C}_k(K)$

Recall that we denote a complete multipartite graph $K_{a_1,a_2,\ldots,a_t}$ by K. In this chapter, we address our main problem: determining whether or not $\mathcal{C}_k(K)$ is Hamiltonian for various values of k. In the first section, we present two "construction" theorems, which given a graph G and subgraph H of G, under particular circumstances allow us to construct a Hamilton cycle in the colour graph of G from a Hamilton cycle in the colour graph of H. We then use these theorems to give our main result, an upper bound on $k_0(K)$, the Gray code number of a complete multipartite graph K. We give a result on the lower bound on $k_0(K)$ with respect to the number of parts of K with size two, and we fully characterize the graphs K for which $\mathcal{C}_{t+1}(K)$ is Hamiltonian.

5.1 Construction Theorems

The first of our construction theorems examines the symmetries between the colour graph $\mathcal{C}_k(G)$, and the colour graph $\mathcal{C}_{k+1}(G+\{v\})$, where $G+\{v\}$ is the graph obtained by adding a dominating vertex to G. This is of particular use for the purpose of constructing Hamilton cycles in complete multipartite graphs, as $K_{a_1,a_2,\ldots,a_t} + \{v\} \cong K_{a_1,a_2,\ldots,a_t,1}$.

Theorem 5.1.1. *For any graph G, if $\mathcal{C}_k(G)$ is $\mathcal{A}$-Hamiltonian and $k \geq \chi(G) + 2$, then $\mathcal{C}_{k+1}(G + \{v\})$ is $\mathcal{A}$-Hamiltonian.*

Proof. Let H_i denote the subgraph of $\mathcal{C}_{k+1}(G + \{v\})$ induced by the vertices which colour v with i. Notice that each $H_i \cong \mathcal{C}_k(G)$, and is therefore $\mathcal{A}$-Hamiltonian by our hypothesis. Let $C_1 = f_0^1, f_1^1, \ldots, f_{N-1}^1, f_0^1$ be a Hamilton cycle in H_1 with Property $\mathcal{A}$. For $i \in \{2, 3, \ldots, k+1\}$, let s_i be an integer such that the colour i is not used in both of $f^1_{s_i}$ and $f^1_{s_{i+1}}$. By Property $\mathcal{A}$, such an s_i must exist, and we may assume $s_i = s_j \iff i = j$. We may also assume $s_i < s_j$ whenever $i < j$, by relabeling our colours if necessary.

For $i \in \{2, 3, \ldots, k+1\}$, let $C_i = \pi_i(C_1) = \pi_i(f^1_0), \pi_i(f^1_1), \ldots, \pi_i(f^1_{N-1}), \pi_i(f^1_0) = f^i_0, f^i_1, \ldots, f^i_{N-1}, f^i_0$, where $\pi_i = (1\ i)$. It is not difficult to see that if $f^1_j \sim f^1_k$, then $f^i_j \sim f^i_k$. Therefore, C_i is a Hamilton cycle in H_i. Furthermore, $f^i_{s_i} \sim f^1_{s_i}$ as $f^1_{s_i}$ colours no vertex with i, and therefore the two colourings differ only in the colour of v. Similarly, $f^i_{s_i+1} \sim f^1_{s_i+1}$. Let $P_i = f^i_{s_i}, f^i_{s_{i-1}}, \ldots, f^i_{s_{i+1}}$. P_i is a Hamilton path in H_i from $f^i_{s_i}$ to $f^i_{s_{i+1}}$. For $i \in \{2, 3, \ldots, k\}$, let $P'_i = f^1_{s_i+1}, f^1_{s_i+2}, \ldots, f^1_{s_{i+1}}$, and let $P'_{k+1} = f^1_{s_{k+1}+1}, f^1_{s_{k+1}+2}, \ldots, f^1_{s_2}$. Then, the following is a Hamilton cycle in $\mathcal{C}_{k+1}(G + \{v\})$: $C = P_2P'_2P_3P'_3 \cdots P_{k+1}P'_{k+1}f^2_{s_2}$.

To complete the proof, we must show that C has Property $\mathcal{A}$. We know that $H_1 \cong \mathcal{C}_k(G)$, and therefore C_1 contains consecutive vertices which do not use the colour i, for $i \in 2, 3 \ldots, k+1$. Therefore, since $C_i = \pi_i(C_1)$, C_i must contain consecutive vertices which do not use the colour j, for $j \in \{1, 2, \ldots, j-1, j, j+1, \ldots, k+1\}$. However, P_i is C_i with the edge $e_i = (f^i_{s_i}, f^i_{s_{i+1}})$ removed. By construction, e_i does not use colour 1, and therefore P_i may not contain consecutive vertices which do not use 1. At this point, we know that $P_i \cup P_j$, with $j \neq i$ and $i, j \in \{2, 3, \ldots, k+1\}$, must contain consecutive vertices which do not use colour h, for $h \in \{2, 3, \ldots, k+1\}$. Therefore, all that remains to verify Property $\mathcal{A}$ is to find consecutive vertices on C which do not use colour 1.

To find such vertices, we appeal to our assumption that $k \geq \chi(G) + 2$. For some $i \in \{2, 3, \ldots, k+1\}$, consider integers p and q such that f^1_p and f^1_q use exactly $\chi(G)$ colours, and do not use the colour i. Let c_p be another colour not used by f^1_p, and let c_q be another colour not used by f^1_q. For some $c \in \{i, c_p, c_q\}$, there exist $s, t \in \mathbb{Z}$, $s \neq t$, such that $f^1_s, f^1_{s+1}, f^1_t, f^1_{t+1}$ all do not use the colour c. Therefore, as $H_c = \pi_c(H_1)$, $f^c_s, f^c_{s+1}, f^c_t, f^c_{t+1}$ all do not use the colour 1, and so C_c has at least two edges which do not use colour 1. Therefore, P_c must contain at least one edge which does not use colour 1. Therefore, C has Property $\mathcal{A}$, and we have shown $\mathcal{C}_{k+1}(G + \{v\})$ is $\mathcal{A}$-Hamiltonian, as desired. □

Corollary 5.1.2. *If $\mathcal{C}_k(K_{a_1,a_2,\ldots,a_t})$ is $\mathcal{A}$-Hamiltonian, and $k \geq t+2$, then $\mathcal{C}_{k+1}(K_{a_1,a_2,\ldots,a_t,1})$ is $\mathcal{A}$-Hamiltonian.*

The second of our two construction theorems is the one which necessitated our work in Chapter 4. Unlike our first construction theorem, this theorem is particular to complete multipartite graphs. We show that, given a sufficiently large number of colours, k, adding a vertex to each part of a complete multipartite graph whose k-colour graph is $\mathcal{A}$-Hamiltonian will result in a graph whose k-colour graph is *also* $\mathcal{A}$-Hamiltonian. The result is a generalization of work done by Celaya et al. [3] on colour graphs of complete bipartite graphs.

Lemma 5.1.3. *If $\mathcal{C}_k(K_{a_1,a_2,\ldots,a_t})$ is $\mathcal{A}$-Hamiltonian and $k \geq 2t$, $\mathcal{C}_k(K_{a_1+1,a_2+1,\ldots,a_t+1})$ is $\mathcal{A}$-Hamiltonian.*

Proof. Let $K' = K_{a_1,a_2,\ldots,a_t}$, and $K = K_{a_1+1,a_2+1,\ldots,a_t+1}$. Let $\{V_1, V_2, \ldots, V_t\}$ be the t-partition of the vertices of K', and let $v_1, v_2, \ldots, v_t$ be vertices such that $\{V_1 \cup \{v_1\}, V_2 \cup \{v_2\}, \ldots, V_t \cup \{v_t\}\}$ is the t-partition of K. Let $f_0, f_1, \ldots, f_{N-1}, f_0$ be a Hamilton cycle with Property $\mathcal{A}$ in $\mathcal{C}_k(K')$. For $j \in \{0, 1, \ldots, N-1\}$, let F_j denote the set of colourings in $\mathcal{C}_k(K)$ which agree with f_j on the colours of $V(K')$. Let $A_{j,i}$ denote the set of colours that could be assigned to v_i to extend f_j to a colouring of K. Since the v_is induce a clique, each must be assigned a different colour. Therefore, colourings in F_j correspond to SDRs of the collection $S_j = A_{j,1}, A_{j,2}, \ldots, A_{j,t}$. Additionally, two colourings in F_j are adjacent if and only if their corresponding SDRs are adjacent in the SDR graph of S_j. Therefore, the subgraph induced by F_j is isomorphic to the SDR graph of S_j. We will now show that S_j must be isomorphic to one of the SDR graphs we examined in Chapter 4.

Consider some colour $c \in \{1, 2, \ldots, k\}$. If c is used in f_j, it must be used on exactly one part, say V_n. Then, $c \in A_{j,i}$ if and only if $i = n$. If c is not used in f_j, then $c \in A_{j,i}$ for $i \in \{1, 2, \ldots, t\}$. In other words c is either available to exactly one of the v_is, or it is available to every v_i. Let $X_j = \{x_1, x_2, \ldots, x_{l_j}\}$ be the set of colours such that $x_n \in A_{j,i}$ for $1 \leq n \leq l_j$ and $1 \leq i \leq t$. Let $Y_{j,i} = \{y_{i,1}, y_{i,2}, \ldots, y_{i,b_{j,i}}\}$ for $i \in \{1, 2, \ldots, t\}$ be the set of colours such that, for $1 \leq n \leq b_{j,i}$, $y_{i,n} \in A_{j,m}$ if and only if $m = i$. Then, $A_{j,i} = X_j \cup Y_{j,i}$.

Clearly, we must have $b_{j,i} \geq 1$, for each i, as at least one colour must be used in f_j to colour the vertices of V_i. Additionally, we have $A_{j,h} \cap A_{j,i} = X_j$ for $h, i \in \{1, 2, \ldots, t\}$. Since $k \geq 2t$, we must have $|\bigcup_{i=1}^{t} A_{j,i}| \geq 2t$. Since $t \geq 2$, we have all the conditions we need to guarantee $S_j \in \mathcal{S}_t$, and we may therefore apply either Lemma 4.2.4 or Theorem 4.2.5.

For any $j \in \{1, 2, \ldots, t\}$, we call a vertex $w \in F_j$ a *sink* if it is adjacent to a vertex in F_{j+1}. If a vertex $w \in F_j$ is *not* a sink, this means that for some i, the colour used by w to colour v_i is not available to v_i to extend f_{j+1}. That is, $w(v_i) \in A_{j,i}$, but

$w(v_i) \notin A_{j+1,i}$. Since f_j and f_{j+1} differ in the colour of only one vertex, if w is not a sink, f_j colours no vertex with $w(v_i)$, and f_{j+1} uses the colour $w(v_i)$ on some vertex in $V_1 \cup V_2 \cup \cdots \cup V_{i-1} \cup V_{i+1} \cup \cdots \cup V_t$.

We will find a Hamilton cycle in $\mathcal{C}_k(K)$ using a similar idea as in the proof of Lemma 2.3.1 (See [9]). For $i \in \{0, 1, \ldots, N-1\}$, we will define vertices t_i and s_i such that there exists a Hamilton path in F_i from s_i to t_i, and $t_i \sim s_{i+1}$. Without loss of generality, we may assume f_0 is the colouring which assigns each vertex in V_j colour j, for $1 \leq j \leq t$. Let $t_0 \in F_0$ be the vertex such that $t_0(v_j) = j$. Since f_0 uses colour j on V_j for each $j \in \{1, 2, \ldots, t\}$, $A_{1,j}$ must contain the colour j. Therefore, t_0 is a sink. We define t_j and s_j for $1 \leq j \leq N-1$ inductively. Suppose $t_0, s_1, t_1, s_2, t_2, \ldots, s_{j-1}, t_{j-1}$ have been defined. Let s_j be the neighbour of t_{j-1} in F_j.

If $S_j \in \mathcal{S} - \{X^t\}$, then by Theorem 4.2.5 we know that F_j contains a Hamilton cycle which contains some prescribed edge in F_j. As a result, F_j contains a Hamilton path from s_j to any vertex adjacent to s_j. We now show that some neighbour of s_j in F_j must be a sink. If a neighbour of s_j is *not* a sink, then there exists a colour c which is not used by f_j but is used by f_{j+1}. Say c is used by f_{j+1} to colour a vertex in V_i. Then, any vertex $w \in F_j$ which either does not use c, or uses c to colour v_i is a sink. Since $|A_{j,h}| \geq 2$ for each $h \in \{1, 2, \ldots, t\}$, clearly some neighbour of s_j satisfies one of those conditions, and is therefore a sink. Let t_j be such a sink. By Theorem 4.2.5, there is a Hamilton path in F_j from s_j to t_j.

If $S_j = X^t$, then $A_{j,i} = \{x_1, y_{i,1}, y_{i,2}, \ldots, y_{i,t}\}$ for some $i \in \{1, 2, \ldots, t\}$, and $A_{j,h} = \{x_1, y_{h,1}\}$ for all $h \in \{1, 2, \ldots, t\} - \{i\}$. In this case, we know by Lemma 4.2.4 that F_j contains a Hamilton path between any two vertices, so long as at least one of them uses the colour x_1. Since x_1 is the only colour not used by f_j, there exists $\alpha \in \{1, 2, \ldots, t\}$ such that every vertex in F_j which uses x_1 to colour v_α is a sink. Let t_j be any such vertex, with $t_j \neq s_j$. Since, t_j uses x_1, by Lemma 4.2.4 there is a Hamilton path in F_j from s_j to t_j.

Now, by our choice of f_0, the vertex f_0 cannot use any colour which f_{N-1} does not use. Therefore, every vertex of F_{N-1} is a sink. As such, we have enough freedom to ensure that our choice of t_{N-1} is *not* adjacent to t_0. Finally, let s_0 be the neighbour of t_{N-1} in F_0. We have $S_0 = A_{0,1}, A_{0,2}, \ldots, A_{0,t}$, with $A_{0,i} = \{x_1, x_2, \ldots, x_{l_0}, y_{i,1}\}$ for each $i \in \{1, 2, \ldots, t\}$. Then, by Lemma 4.2.3, there is a Hamilton path in F_0 from t_0 to any other vertex in F_0. In particular, there is a Hamilton path in F_0 from t_0 to s_0. Now that we have defined s_i and t_i for each $i \in \{1, 2, \ldots, N-1\}$, we may describe a Hamilton cycle in $\mathcal{C}_k(K)$.

For $i \in \{1, 2, \ldots, N-1\}$, let P_i be a Hamilton path in F_i from s_i to t_i. Then, $P_0 P_1 \cdots P_{N-1} s_0$ is a Hamilton cycle in $\mathcal{C}_k(K)$. That this cycle has Property $\mathcal{A}$ follows from Theorem 4.2.5, and the fact that $f_0, f_1, \ldots, f_{N-1}, f_0$ has Property $\mathcal{A}$. □

5.2 Upper and lower bounds for $k_0(K)$

In this section we give our main result, which uses our constructions theorems Theorem 5.1.1 and Theorem 5.1.3 to give an upper bound on $k_0(K)$. Additionally, we will show that there exist $\mathcal{C}_k(K)$ which are *not* Hamiltonian for k as large as $t + \lceil \frac{t}{2} \rceil$. The following simple lemma, which will be presented without proof, illustrates the construction used to prove our upper bound theorem.

Lemma 5.2.1. *Every non-increasing sequence of natural numbers $a_1, a_2, \ldots, a_t$ can be reduced to a sequence $b, 1$ using the following two operations:*

$$O_1(b_1, b_2, \ldots, b_s) = b_1 - 1, b_2 - 1, \ldots, b_s - 1,$$

$$O_2(b_1, b_2, \ldots, b_{s-1}, 1) = b_1, b_2, \ldots, b_{s-1}.$$

For example, consider the sequence $s_0 = 7, 5, 4, 2, 1, 1$. The series of moves which reduces s_0 to the length two sequence $3, 1$ is as follows:

- $s_1 = O_2(s_0) = 7, 5, 4, 2, 1$
- $s_2 = O_2(s_1) = 7, 5, 4, 2$
- $s_3 = O_1(s_2) = 6, 4, 3, 1$
- $s_4 = O_2(s_3) = 6, 4, 3$
- $s_5 = O_1(s_4) = 5, 3, 2$
- $s_6 = O_1(s_5) = 4, 2, 1$
- $s_7 = O_2(s_6) = 4, 2$
- $s_8 = O_2(s_7) = 3, 1$

We are now ready to prove the main result.

Theorem 5.2.2. *If $k \geq 2t$, $\mathcal{C}_k(K_{a_1, a_2, \ldots, a_t})$ is $\mathcal{A}$-Hamiltonian.*

Proof. By Lemma 5.2.1, there exists a sequence $G_0, G_1, \ldots, G_n$ of complete multipartite graphs such that:

- $G_0 = K_{b,1}$ for some b,
- $G_n = K_{a_1,a_2,\ldots,a_t}$,
- if $G_i = K_{b_1,b_2,\ldots,b_{s-1},b_s}$, then either $b_s > 1$ and $G_{i-1} = K_{b_1-1,b_2-1,\ldots,b_{s-1}-1,b_s-1}$, or $b_s = 1$ and $G_{i-1} = K_{b_1,b_2,\ldots,b_{s-1}}$, for $1 \leq i \leq n$.

Since $G_0 = K_{b,1}$ is a star, we have that $Col(G_0) = 2$. Therefore, by Theorem 2.3.4, $\mathcal{C}_l(G_0)$ is $\mathcal{A}$-Hamiltonian whenever $l \geq 4$. Since $k \geq 2t$, we must have $k-(t-2) \geq t+2 \geq 4$, and so $\mathcal{C}_{k-(t-2)}(G_0)$ is $\mathcal{A}$-Hamiltonian. Suppose that for some G_j that G_j is s-partite, and $\mathcal{C}_{k-(t-s)}(G_j)$ is $\mathcal{A}$-Hamiltonian. Since $k \geq 2t$, we have $k-(t-s) \geq t+s \geq 2s$. If G_{j+1} is obtained from G_j by adding a vertex to each part of G_j, then by Lemma 5.1.3, $\mathcal{C}_{k-(t-s)}(G_{j+1})$ is $\mathcal{A}$-Hamiltonian. If G_{j+1} is obtained from G_j by adding a single dominating vertex, then by Theorem 5.1.1, $\mathcal{C}_{k-(t-(s+1))}(G_{j+1})$ is $\mathcal{A}$-Hamiltonian. By induction, for each $j \in \{0,1,\ldots,n\}$, if G_j is s-partite, then $\mathcal{C}_{k-(t-s)}(G_j)$ is $\mathcal{A}$-Hamiltonian. In particular, $\mathcal{C}_{k-(t-t)}(G_n) = \mathcal{C}_k(K_{a_1,a_2,\ldots,a_t})$ is $\mathcal{A}$-Hamiltonian.

□

Corollary 5.2.3. *$k_0(K_{a_1,a_2,\ldots,a_t}) \leq 2t$.*

An improvement to this upper bound can be made when $K_{a_1,a_2,\ldots,a_t}$ has some number of parts of size one by applying Theorem 5.1.1.

Corollary 5.2.4. *For $K = K_{a_1,a_2,\ldots,a_t}$ with $a_1 \geq a_2 \geq \cdots \geq a_t$, if $a_i = a_{i+1} = \cdots = a_t = 1$ where $t > i \geq 3$, then $\mathcal{C}_{t+i-1}(K)$ is Hamiltonian.*

Proof. $\mathcal{C}_{2i-2}(K_{a_1,a_2,\ldots,a_{i-1}})$ is $\mathcal{A}$-Hamiltonian. The result follows by applying Theorem 5.1.1 $t-(i-1)$ times. □

We now turn our attention towards a lower bound for $k_0(K)$. It is clear that $\mathcal{C}_t(K)$ is not connected, much less Hamiltonian. Considering our upper bound, we are only left to concern ourselves with the Hamiltonicity of $\mathcal{C}_k(K)$ when $t+1 \leq k \leq 2t-1$. As will be shown in the next section, $\mathcal{C}_{t+1}(K)$ is Hamiltonian for some choices of K, and not Hamiltonian for others. As a result, it is likely the case that $k_0(K) \geq t+1$ is the best general lower bound that exists. It is however worth noting that it may be possible for $\mathcal{C}_{k'}(K)$ to be Hamiltonian, but $\mathcal{C}_k(K)$ be non-Hamiltonian for some $2t \geq k \geq k'$. The result that follows is an attempt to gain insight on why some complete multipartite graphs fail to have Hamiltonian colour graphs for a particular number of colours while others succeed. Furthermore, this result shows that for k as large as $t + \lceil \frac{t}{2} \rceil$, there are complete t-partite graphs whose k-colour graphs are *not* Hamiltonian.

For a graph G, let $\omega(G)$ denote the number of connected components of G. Before we continue, recall a well known necessary condition for Hamiltonicity of G (see [1], page 53).

Lemma 5.2.5. *If G is Hamiltonian, and $S \subseteq V(G)$, then $\omega(G - S) \leq |S|$.*

We use this condition to prove the following result.

Lemma 5.2.6. *For $K = K_{a_1,a_2,\ldots,a_t}$, if K has s parts of size two, then $\mathcal{C}_k(K)$ is not Hamiltonian for $k \leq t + \lceil \frac{s}{2} \rceil$*

Proof. Suppose $k = t+c$, where $c \leq s$. Let $T \subseteq V(\mathcal{C}_k(K))$ denote the set of colourings of K in which c parts of size two are coloured with two colours, and the remaining $t-c$ parts are coloured with a single colour. Let $S \subseteq V(\mathcal{C}_k(K))$ denote the set of colourings of K in which $c-1$ parts of size two are coloured with two colours, the remaining $t-c+1$ parts are coloured with a single colour, and one colour is unused. Notice that the subgraph induced by T contains no edges, as the only vertices which can change colour are those in the parts of size two which are coloured with two colours, and they can only be changed to the colour of the other vertex in their part. As such, vertices in T are adjacent only to vertices in S. Then, $\omega(\mathcal{C}_k(K) - S) \geq |T| + 1$. Therefore, if $|S| \leq |T|$, $\mathcal{C}_k(K)$ is not Hamiltonian by Lemma 5.2.5. It is a simple counting exercise to show that $|T| = \binom{s}{c}(t+c)!$, and $|S| = \binom{s}{c-1}(t+c)!$. So, $|S| \leq |T|$ whenever $\binom{s}{c} \geq \binom{s}{c-1}$. This inequality holds whenever $c \leq \lceil \frac{s}{2} \rceil$. □

A computer search for Hamilton cycles in colour graphs of some small complete 3-partite graphs suggests that $K_{2,2,2}$ is the *only* complete 3-partite graph whose 5-colour graph is not Hamiltonian.

5.3 Hamiltonicity of $\mathcal{C}_{t+1}(K)$

Let $V_1, V_2, \ldots, V_t$ be the t-partition of K, where $|V_i| = a_i$. It is clear that $\mathcal{C}_t(K)$ is not connected for $t \geq 2$, and of course cannot be Hamiltonian. This is not the case

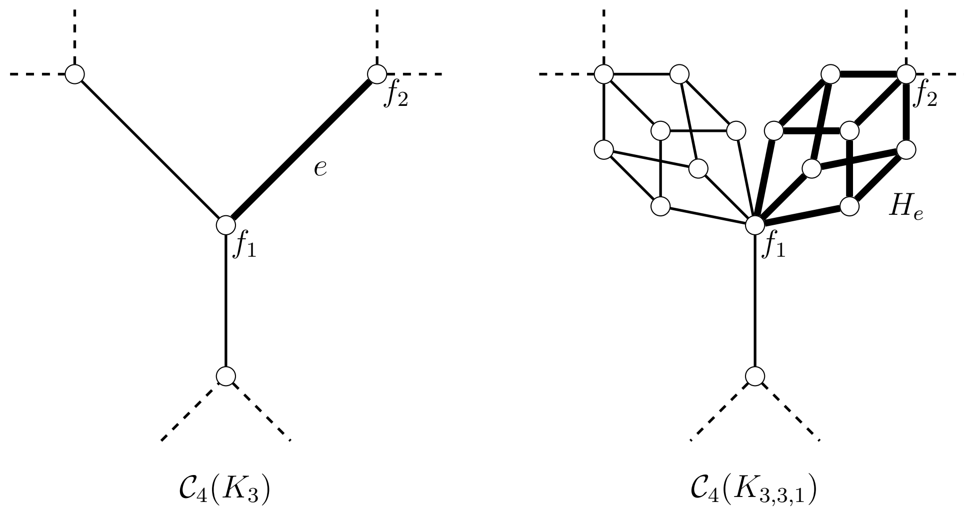

Figure 5.1: A vertex f_1 of $\mathcal{C}_4(K_3)$ and its neighbourhood, and the corresponding subgraph of $\mathcal{C}_4(K_{3,3,1})$. The thick edges represent the edge e of $\mathcal{C}_4(K_3)$, and the subgraph H_e of $\mathcal{C}_4(K_{3,3,1})$ corresponding to e. In the right hand graph, f_1 and f_2 are the two 3-colourings contained in H_e, and each other vertex of H_e represents a 4-colouring.

with $\mathcal{C}_{t+1}(K)$. In this section we characterize the complete t-partite graphs whose $(t+1)$-colour graphs are Hamiltonian. Once again our results are a generalization of the work done by Celaya et al. [3] on colour graphs of complete bipartite graphs. To begin, we will examine some basic necessary conditions.

Consider $\mathcal{C}_{t+1}(K_{1,1,\ldots,1}) = \mathcal{C}_{t+1}(K_t)$. Let $V(K_t) = \{v_1, v_2, \ldots, v_t\}$. There is an important relationship between Hamilton cycles in $\mathcal{C}_{t+1}(K_t)$ and Hamilton cycles in $\mathcal{C}_{t+1}(K)$. Let $e = (f_1, f_2) \in E(\mathcal{C}_{t+1}(K_t))$. Let H_e be the subgraph of $\mathcal{C}_{t+1}(K)$ induced by the colourings where, for each i, V_i is coloured using colours from $\{f_1(v_i), f_2(v_i)\}$. Since $f_1(v_i) = f_2(v_i)$ for all but one vertex, this subgraph simply contains colourings in which one part V_j is coloured using two colours, and each other part is monocoloured. Recall that Q_n denotes the $n-cube$, the graph whose vertex set is the set of binary strings of length n, where two strings are adjacent if they differ in exactly one position. Notice that $H_e \cong Q_{a_j}$, as there is a clear correspondence between binary strings of length a_j and colourings of V_j using two colours. Figure 5.1 gives an example of how the edges of $\mathcal{C}_4(K_3)$ correspond to subgraphs of $\mathcal{C}_4(K_{3,3,1})$.

It should be clear that every vertex in $\mathcal{C}_{t+1}(K)$ is contained in H_e for some e. More specifically, if $f \in \mathcal{C}_{t+1}(K)$ is a colouring which uses all $t+1$ colours, it will appear in exactly one subgraph H_e, and if f uses t colours, it will appear in t subgraphs H_e, corresponding to the t edges incident to the vertex in $\mathcal{C}_{t+1}(K_t)$ corresponding to f.

Consider H_e for some edge $e \in E(\mathcal{C}_{t+1}(K_t))$. Let f_1 and f_2 denote the two colourings in $V(H_e)$ which use t colours. Since no colour used on V_i can be used on V_j for any $j \neq i$ and any vertex $y \in V(H_e) - \{f_1, f_2\}$ uses all $t+1$ colours, any path from a vertex $x \in V(\mathcal{C}_{t+1}(K)) - V(H_e)$ to y must contain either f_1 or f_2. In other words, $\{f_1, f_2\}$ is a disconnecting set of $\mathcal{C}_{t+1}(K)$. As a result, a Hamilton cycle in $\mathcal{C}_{t+1}(K)$ must be composed of a sequence of Hamilton paths in the H_es which begin and end at the vertices of H_e which use t colours. Each such path corresponds to an edge $e \in \mathcal{C}_{t+1}(K_t)$, and therefore a Hamilton cycle in $\mathcal{C}_{t+1}(K)$ will correspond to a Hamilton cycle in $\mathcal{C}_{t+1}(K_t)$. Our first condition comes from the fact that we need H_e to contain a Hamilton path beginning and ending at its t colourings. Since $H_e \cong Q_{a_j}$ for some $j \in \{1, 2, \dots, t\}$, this is equivalent to finding a Hamilton path in Q_{a_j} from $00\dots0$ to $11\dots1$. The following lemma, a proof of which may be found in [9], illustrates when this can be done.

Lemma 5.3.1. *For $n \geq 1$, there is a Hamilton path in Q_n from $00\dots0$ to $11\dots1$ if and only if n is odd.*

As a result, we immediately get the following corollary.

Corollary 5.3.2. *If $\mathcal{C}_{t+1}(K_{a_1,a_2,\dots,a_t})$ is Hamiltonian, a_i is odd for each i.*

We now complete the argument for complete bipartite graphs, reaffirming the result of Celaya et al. [3]. The proof we give is similar to the proof given by Celaya et al.

Theorem 5.3.3 (Celaya at al. [3])**.** *$\mathcal{C}_3(K_{a_1,a_2})$ is Hamiltonian if and only if a_1 and a_2 are both odd.*

Proof. We know that $\mathcal{C}_3(K_2)$ is Hamiltonian, and since each vertex in $\mathcal{C}_3(K_2)$ has degree two, every edge of $\mathcal{C}_3(K_2)$ is used in this cycle. Note that $\mathcal{C}_3(K_2)$ has $3! = 6$ vertices. Let $f_0, f_1, \dots, f_5, f_0$ denote the Hamilton cycle in $\mathcal{C}_3(K_2)$, and let $e_i = (f_i, f_{i+1})$, interpreting indices modulo 6. Let f_i' and f_{i+1}' denote the t colourings in H_{e_i} which use the same colours on the same parts as f_i and f_{i+1} respectively. If a_1 and a_2 are both odd, by Lemma 5.3.1 there will always be a Hamilton path P_{e_i} in H_{e_i} from f_i' to f_{i+1}'. Then, $P_{e_0}P_{e_1}\cdots P_{e_5}$ is a Hamilton cycle in $\mathcal{C}_3(K_{a_1,a_2})$. If either a_1 or a_2 is even, then some for some edge e_i, by Lemma 5.3.1 H_{e_i} will not contain a Hamilton path from f_i' to f_{i+1}', and a Hamilton cycle in $\mathcal{C}_3(K_{a_1,a_2})$ cannot exist. $\square$

Since $\mathcal{C}_{t+1}(K_t)$ is also Hamiltonian for $t \geq 3$, one might imagine a similar idea would work to find Hamilton cycles in $\mathcal{C}_{t+1}(K_{a_1,a_2,\ldots,a_t})$ when $t \geq 3$. This however is not necessarily the case. Since the degree of a vertex in $\mathcal{C}_{t+1}(K_t)$ is t, if $t \geq 3$, there must be some edges which are not used in a Hamilton cycle C in $\mathcal{C}_{t+1}(K_t)$. Consider such an edge e, and say e changes the colour of v_i in K_t. Then, since $H_e \cong Q_{a_i}$, if $a_i \geq 2$, there will be vertices in H_e which are not visited by the cycle in $\mathcal{C}_{t+1}(K)$ which corresponds to C. Thus, C does not correspond to a Hamilton cycle in $\mathcal{C}_{t+1}(K)$. Furthermore, if $a_i > 1$, any Hamilton cycle C' in $\mathcal{C}_{t+1}(K_t)$ which *does* correspond to a Hamilton cycle in $\mathcal{C}_{t+1}(K)$ must use every edge e which changes the colour of v_i. If $a_i = 1$ then H_e contains only two vertices, the two possible t colourings, and these vertices will appear in $H_{e'}$ for some e' which *is* used by C. Every vertex in $\mathcal{C}_{t+1}(K_t)$ is incident with exactly one edge which changes the colour of v_i, and indeed the set E_i of all edges in $\mathcal{C}_{t+1}(K_t)$ which change the colour of v_i is a perfect matching.

To reiterate, each vertex v_i has a corresponding value a_i, which is the size of V_i. Additionally, if $a_i > 1$, v_i has a corresponding set of edges E_i which must be used in any Hamilton cycle in $\mathcal{C}_{t+1}(K_t)$ which corresponds with a Hamilton cycle in $\mathcal{C}_{t+1}(K) = \mathcal{C}_{t+1}(K_{a_1,a_2,\ldots,a_t})$. Note that $E(\mathcal{C}_{t+1}(K_t)) = E_1 \cup E_2 \cup \cdots \cup E_t$, and that $E_i \cap E_j = \emptyset$ whenever $i \neq j$. If there exist a_i, a_j and a_k, with $i \neq j \neq k$, $a_i \geq a_j \geq a_k \geq 2$, then there clearly cannot be a Hamilton cycle in $\mathcal{C}_{t+1}(K_t)$ which uses every edge in $E_i \cup E_j \cup E_k$. Therefore, no Hamilton cycle in $\mathcal{C}_{t+1}(K_t)$ can correspond to a Hamilton cycle in $\mathcal{C}_{t+1}(K)$. Since a Hamilton cycle in $\mathcal{C}_{t+1}(K)$ necessarily has a corresponding Hamilton cycle in $\mathcal{C}_{t+1}(K_t)$, we conclude that $\mathcal{C}_{t+1}(K)$ is not Hamiltonian. Suppose then, that only a_1 and a_2 are greater than one. A corresponding Hamilton cycle in $\mathcal{C}_{t+1}(K_t)$ must use every edge in $E_1 \cup E_2$, and since E_1 and E_2 are disjoint perfect matchings, a corresponding Hamilton cycle in $\mathcal{C}_{t+1}(K_t)$ cannot use any edge that is not in $E_1 \cup E_2$. Since these edges correspond to changing the colour of vertices v_1 and v_2 respectively, clearly we cannot have a Hamilton cycle, as the colour of v_3 is never changed using only these edges. This proves the following lemma.

Lemma 5.3.4. *If $\mathcal{C}_{t+1}(K_{a_1,a_2,\ldots,a_t})$ is Hamiltonian and $t \geq 3$, then $a_2 = a_3 = \cdots = a_t = 1$.*

At this point, we are left to consider $\mathcal{C}_{t+1}(K)$ for $K = K_{a_1,1,1,\ldots,1}$, with a_1 odd. Furthermore, a Hamilton cycle in $\mathcal{C}_{t+1}(K)$ must correspond to a Hamilton cycle in $\mathcal{C}_{t+1}(K_t)$ which uses every edge in the perfect matching E_1, the set of edges which change the colour of v_1. Notice that the edges of such a cycle must alternate between edges in E_1 and edges not in E_1, as using two edges in E_1 consecutively contradicts that E_1 is a perfect matching, and using two edges not in E_1 consecutively would result in the cycle missing some edge in E_1. We will turn our efforts towards discerning when such a cycle in $\mathcal{C}_{t+1}(K_t)$ will exist. To do so, we turn our attention to a particular Cayley graph.

Choo [9] noticed that $\mathcal{C}_{t+1}(K_t) \cong Cay(X : S_{t+1})$, where $X = \{(1, t+1), (2, t+1), \ldots, (t, t+1)\}$. We allow the permutation π of $\{1, 2, \ldots, t+1\}$ to correspond with a colouring f of $\{v_1, v_2, \ldots, v_{t+1}\}$ in the obvious manner, where $\pi(i) = f(v_i)$ for $i \in \{1, 2 \ldots, t\}$. In addition, $\pi(t+1)$ corresponds to the single colour not used by f. Thus, the transposition $(i, t+1)$ is equivalent to switching the colour of v_i to the single unused colour. A Hamilton cycle $\pi_1, \pi_2, \ldots, \pi_{(t+1)!}, \pi_1$ in $Cay(X : S_{t+1})$ can be represented by a sequence $t_1, t_2, \ldots, t_{(t+1)!}$ of transpositions in X such that $t_i \circ \pi_i = \pi_{i+1}$, interpreted modulo $(t+1)!$. Recall that we are looking for a Hamilton cycle in $\mathcal{C}_{t+1}(K_t)$ in which every other edge changes the colour of v_1. In order to find this Hamilton cycle, we can equivalently find a Hamilton cycle in $Cay(X : S_{t+1})$ which is represented by $(1, t+1), t_2, (1, t+1), t_3, \ldots, (1, t+1), t_{t+1!}$, where $t_i \in X - \{(1, t+1)\}$.

This is equivalent to finding a Hamilton cycle in the directed Cayley graph $D_{t+1} = Cay(X' : A_{t+1})$, where $X' = \{(1, t+1)(i, t+1) : 2 \leq i \leq t\} = \{(1, i, t+1) : 2 \leq i \leq t\}$, and $A_{t+1} \subset S_{t+1}$ is the set of even permutations of a set of size $t+1$. The group A_{t+1} is known as the *alternating group*. We then have the following lemma.

Lemma 5.3.5. *$\mathcal{C}_{t+1}(K_{a_1, a_2, \ldots, a_t})$, with $t \geq 3$, is Hamiltonian if and only if*

- *a_1 is odd,*
- *$a_i = 1$, for $2 \leq i \leq t$,*
- *D_{t+1} is Hamiltonian.*

Gould and Roth [15] proved the following theorem.

Theorem 5.3.6. *D_n is Hamiltonian if $n = 3$ or $n \geq 5$, and D_n is not Hamiltonian if $n = 4$.*

Therefore, we have fully characterized the complete t-partite graphs with Hamiltonian $(t+1)$-colour graphs with the following three theorems.

Theorem 5.3.7. *$\mathcal{C}_3(K_{a_1, a_2})$ is Hamiltonian if and only if a_1 and a_2 are odd.*

Theorem 5.3.8. *$\mathcal{C}_4(K_{a_1,a_2,a_3})$ is Hamiltonian if and only if $a_1 = a_2 = a_3 = 1$.*

Theorem 5.3.9. *$\mathcal{C}_{t+1}(K_{a_1,a_2,\ldots,a_t})$ is Hamiltonian if and only if a_1 is odd, and $a_i = 1$ for $2 \leq i \leq t$.*

Chapter 6

Open Problems

Our analysis leaves the following open problems regarding colouring graphs of complete multipartite graphs.

- For $k \leq t + \lceil \frac{t}{2} \rceil$, for which complete t-partite graphs is $\mathcal{C}_k(K)$ Hamiltonian?
- Given a complete t-partite graph K, is $\mathcal{C}_k(K)$ Hamiltonian whenever $k > t + \lceil \frac{t}{2} \rceil$? Our results, supported by a limited computer search, suggest that this may be the case.
- More specifically, is $K_{2,2,2}$ the only complete 3-partite graph whose 5 colour graph is non-Hamiltonian?

Concerning connectivity of colouring graphs, the following problem remains unsolved.

- Does there exist a 3-colouring graph which is connected, but not 2-connected?

On the Hamiltonicity of colouring graphs:

- Determine Gray code numbers of further classes of graphs.
- If $\mathcal{C}_k(G)$ is Hamiltonian, is $\mathcal{C}_{k+1}(G)$ always Hamiltonian?

To study the Gray code numbers of unexplored classes of graphs, the best candidates seem to be classes of highly structured graphs, such as outerplanar graphs, k-trees, and perhaps chordal graphs. With regards to whether or not Hamiltonicity of $\mathcal{C}_k(G)$ implies Hamiltonicity of $\mathcal{C}_{k+1}(G)$, a similar result by Cereceda et al. [5] suggests that this is not the case. It is shown that there exist graphs G and integers k such that $\mathcal{C}_k(G)$ is connected and $\mathcal{C}_{k+1}(G)$ is *not* connected. The graphs used for these examples are good candidates for an analagous result regarding Hamiltonicity.

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