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Portfolio Optimization Using Stochastic Programming with Market Trend Forecast

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Portfolio Optimization Using Stochastic Programming with Market Trend Forecast

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Dedication

To my dear parents, Shourong Yang and Mian Lei

and

my beloved wife, Ying Chen.

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Portfolio Optimization Using Stochastic Programming with Market Trend Forecast

by

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The University of Texas at Austin, 2014

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This report discusses a multi-stage stochastic programming model that maximizes expected ending time profit assuming investors can forecast a bull or bear market trend. If an investor can always predict the market trend correctly and pick the optimal stochastic strategy that matches the real market trend, intuitively his return will beat the market performance. For investors with different levels of prediction accuracy, our analytical results support their decision of selecting the highest return strategy. Real stock prices of 154 stocks on 73 trading days are collected. The computational results verify that accurate prediction helps to exceed market return while portfolio profit drops if investors partially predict or forecast incorrectly part of the time. A sensitivity analysis shows how risk control requirements affect the investor's decision on selecting stochastic strategies under the same prediction accuracy.

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1. Introduction

This project discusses how to maximize portfolio expected return assuming investor can forecast market trend, which is represented by rise or fall of S&P 500 Stock Index. Note that investors only anticipate if the market index rises or falls in each coming period, but cannot forecast how much it changes.

An ordinary multi-stage stochastic programming model is based upon one scenario tree of random assets returns. A straightforward way of generating the scenario tree is to draw several random vectors (assets returns) from a multivariate normal distribution in each node, and then sequentially build the tree. Historical data indicates constant means and variances of the random assets returns in each node. Since stock returns in this scenario tree only represent the market in average, this optimal strategy is optimal only for a market without clear trend.

One question that has been asked often in modern portfolio optimization model is whether means and variances of historical returns can represent future returns or not. Since means and variances of assets returns are estimated from historical data, excellent investors should adjust them according to their own judgments on future market, which means making prediction.

However, forecasting individual asset return is too hard to be right. We take a step back and we think predicting market trends will be relative easier. If an investor predicts that market will rise tomorrow, he will pick an optimal stochastic strategy corresponding to a scenario tree in which means and variances of assets returns are adjusted for a bull market. Details about generating the scenario tree will be presented in Chapter 5. Thus, our stochastic programming approach generates multiple scenario trees of assets returns. In each of them, means and variances may vary in stages and depend on the corresponding market trend in that stage. Finally, we generate optimal stochastic strategies for scenario trees that represent all possible combinations of market trends in a multistage problem.

If an investor can always pick the optimal stochastic strategy that matches with the real market trend, which means he made predictions and made them right, intuitively his return exceeds the market index return. It is also interesting to see how portfolio profit drops if investors partially predict or forecast wrong sometimes. For investors with different level of prediction accuracy, our analytical results help them select the highest return stochastic strategy. To test our algorithm, real stock prices of 154 stocks in 73 trading days are collected. We also show that risk control requirements affect investor's decision on selecting stochastic strategies under the same prediction accuracy.

This report is organized as follows. Chapter 2 reviews previous studies on stochastic programming, risk measure conditional value at risk and market prediction. Chapter 3 introduces the portfolio optimization problem in this report. Chapter 4 presents the equivalent deterministic linear model of stochastic model. In Chapter 5, we show the procedure of generating scenario trees of market returns and scenario trees of stock returns. In Chapter 6, we analyze how to utilize the optimal stochastic strategies for investors with imperfect market prediction. Computational results and sensitivity analysis are represented in Chapter 7. Finally, we make conclusion and discuss directions for further research in Chapter 8.

2. Literature Review

Markowitz (1959) proposes a mean-variance approach in which portfolio optimization integrates the trade-off between expected returns and market volatility. Following this groundbreaking work in portfolio optimization, modern portfolio theory has emerged and it has been studied extensively.

Compared with Markowitz's mean-variance model that focuses on single-period, Stochastic Programming (SP) have been proposed to handle multi-period optimization problem. Since late 1950s SP models have been well-studied by Dantzig (1964), Charnes and Cooper (1959) and others. Birge (1997) gives a survey of general SP approach in terms of its computation and applications. Yu et al. (2003) provides a survey of SP models developed for financial optimization. They also summaries typical scenario generation methods and computational difficulty of solving large-scale problems. While transaction costs are not considered in Markowitz's model, SP model can handle it. Gülpınar et al. (2003) considers a multi-stage problem consists of four risky asset classes, a set of liabilities and risk-free assets. Their goal is to minimize risk with specified return level while transaction cost is included. Yu et al. (2004) applies SP model in bond markets. In addition to maximizing expected ending time return, they minimize the weighted sum of shortfall costs along planning horizon. The multistage portfolio optimization model in Pınar (2007) uses an objective including expected return and downside deviation. When he generates scenarios, he develops a simulated market model and randomly create scenarios in order to approximate the market stochasticity.

Due to the computational difficulty of solving large-scale stochastic problem, several decomposition methods are proposed. Birge (1985) develops two algorithms for multistage stochastic linear problem. The first one is a nested bender's decomposition method that extends two-stage L-shaped method. The second one is a piecewise linear partitioning method. Mulvey and Ruszczyński (1995) provides a new scenario decomposition method that temporarily relaxes non-anticipativity constraints as Lagrangian relaxation. And then they approximate subproblem as diagonal quadratic problem and solve them using interior point algorithm. Mulvey and Shetty (2004) describes a framework for modeling large-scale financial planning problems and solves it by the similar approach.

The portfolio optimization model in this report is based on the risk measure named conditional value at risk (CVaR). Rockafellar and Uryasev (2000) firstly introduces this risk measure to portfolio optimization problem in conjunction with a linear formulation. This combination is particularly attractive as it allows an easy implementation based on standard linear programming software. Consequently, CVaR gains increasing acceptance as a risk measure that goes beyond the classical mean-variance model. In the literature, rich empirical tests and analyses (Andersson et al. (2001); Topaloglou, Vladimirou and Zenios (2002); Rockafellar and Uryasev (2002)) have been conducted and confirm its applicability to a wide range of financial optimization problems.

Market prediction is one of the hottest fields of financial research lately because of its significant commercial benefit. In the earlier stage, most studies about market prediction were mainly focused on applying artificial neural network (ANN) models for stock market prediction. Kohara et al. (1997) uses prior knowledge and neural networks to predict the stock price. Trippi et al. (1992) predicts the price of the S&P 500 Stock Index futures using an ANN model. Newly research tends to embed artificial intelligence (AI) technique to improve the market prediction performance. Tsaih, Hsu, and Lai (1998) combines ANN and several rule-based techniques to forecast if the S&P 500 Stock Index futures rises or falls on a daily basis. Wang (2002) predicts stock price with a hybrid AI system and then selects investment stocks according to a set of decision-making rules and the predicting results.

In this paper, we are not providing a method that predicts stock market, but assume that investors are capable to predict the trend of stock market with certain level of accuracy. The problem we proposed is how to maximum portfolio return via stochastic programming when investors are able to predict market trends with mistakes. To our best knowledge, there is no such a paper that integrates stochastic programming and market trend prediction in portfolio optimization.

3. Problem Ingredients

The timing of decisions in a multi-stage stochastic programming problem is important. We present it in Section 3.1.

This portfolio includes one risk-free asset, such as cash or Treasury bill, and a number of risky assets, such as stocks. The objective is to maximize the expected ending time returns including all assets. When calculating the final return in the ending period, value of stocks are estimated by its market price in the last period. The risk measure in this problem is conditional value at risk (CVaR), which is shown in Section 3.2.

Instead of long term planning, this project mainly focus on relative short planning horizon (e.g. two days). As a consequence, we omit the interest rate of the risk-free asset in the model. Transaction fee are included while buying or selling stocks, except the initial purchase.

3.1 Timing of decisions

In this problem, investor buys stocks at time $t = 0$ without observing anything. At next time point $t = 1$, investor observes stock returns before time $t = 1$ and then decide to sell or purchase stocks at time point $t = 1$. The Figure 1 illustrates the timing of decisions in this problem.

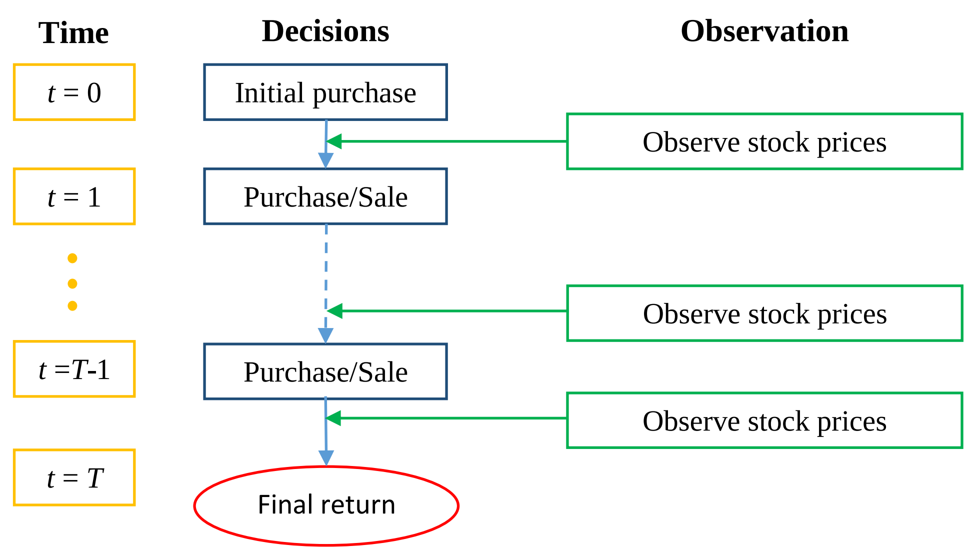

Figure 1. Timing of decisions

3.2 Conditional value at risk

Since the notion of Conditional Value at Risk (CVaR) is an extension of Value at Risk (VaR), we start by presenting the definition of VaR. Let *Y* be a random variable with densify function ϕ. In this section, we assume *Y* is the value of loss of a portfolio at some future time. Assume $\Phi(y) = P(Y \leq y)$ increases strictly when $0 < \Phi(y) < 1$. Let $0 < \alpha < 1$ be the confidence level (e.g. $\alpha = 0.97$). Then $\mathrm{VaR}_\alpha(Y) = \Phi^{-1}(\alpha)$ is the α-level quantile of *Y*.

We define CVaR as $\mathrm{CVaR}_\alpha(Y) = \mathrm{E}(Y \mid Y > \mathrm{VaR}_\alpha(Y))$. If *Y* represents a loss, then the $\mathrm{CVaR}_\alpha(Y)$ represents the conditional expectation of that loss, conditional on the loss exceeding the α-level quantile of *Y*.

According to Rockafellar and Uryasev (2000), we also define VaR and CVaR in an alternative way which is equivalent with the definition before.

$$\mathrm{VaR}_\alpha(Y) = \arg\min_u (u + \frac{1}{1-\alpha}\mathrm{E}(Y-u)^+)$$

$$\mathrm{CVaR}_\alpha(Y) = \min_u (u + \frac{1}{1-\alpha}\mathrm{E}(Y-u)^+)$$

It is clear that VaR only represents the total probability that is covered by the left tail of portfolio profit's pdf (i.e. the right tail of the pdf of loss). In the contrary, CVaR extends VaR by measuring the mean of the part of distribution that is beyond VaR. This is not a complete description of that piece of distribution, but it is an improvement compared with VaR.

In our problem, we maximize the expected ending time profit while satisfying risk control requirements, in which the CVaR at each stage should be less than a given bound. Following the above definition, let *Y* represents a loss and $\overline{CVaR}$ to be the upper bound. The CVaR risk control requirement is formulated as follows. In the formulation, *a* and *Z* are non-negative slack variables while α and $\overline{CVaR}$ are parameters.

$$a + \frac{1}{1-\alpha}\mathrm{E}(Z) \leq \overline{CVaR}, \text{ where } Z \geq Y - a \text{ and } Z \geq 0$$

Since every stage needs risk control, our model applies this constraint to every stage with the universal upper bound $\overline{CVaR}$.

4. Modeling

This is an equivalent deterministic linear model for the stochastic portfolio optimization model. The goal is to maximize the expected ending time return while satisfying CVaR requirements in each stage.

Indices:

T = planning horizon

t = index of time

I = set of risky assets

i = index of asset

Ω = set of nodes in a scenario tree

w_t = index of node at stage *t* in a scenario tree

Parameters

$a(w_t)$ = the ancestor of scenario w_t

u_0 = investor's initial wealth

θ_i = transaction cost for purchases and sales of asset *i* (a percentage of the transaction value)

m_t = maximum percentage of the portfolio value that can be allocated into a single stock at time *t*

α_t = the probability level in VaR at time *t*

$\overline{CVaR}$ = represents the maximum CVaR allowed at time *t*.

Random variables:

r_{it} = return for asset *i* between time *t*-1 and time *t*

Realization of the random variable r_{it} :

$r_{it}^{w_t}$ = return for asset *i* between time *t*-1 and time t in scenario w_t

Decision variables:

a_t = slack variable for CVaR constraint at time *t*

$v_t^{w_t}$ = the total value of the portfolio at time *t* in scenario w_t

$X_{it}^{w_t}$ = amount allocated to asset i at time t in scenario w_t

$A_{it}^{w_t}$ = amount of asset i bought at time t in scenario w_t

$D_{it}^{w_t}$ = amount of asset i sold at time t in scenario w_t

$C_t^{w_t}$ = amount allocated to riskless asset at time t in scenario w_t

$Y_t^{w_t}$ = represents loss exceeding VaR in scenario w_t

Objective function:

$$\max \sum_{w_T \in \Omega} p^{w_T} v_T^{w_T} \tag{1}$$

Constraints:

The initial portfolio value is

$$C_0 + \sum_{i \in I} X_{i0} = u_0 \tag{2}$$

Constraint (3) is the conservation constraint for stocks. The value of stock at time t equals to the current value of stock obtained at time t-1 plus the additional amount that purchased or sold at time t.

$$X_{it}^{w_t} = r_{it}^{w_t} X_{it-1}^{a(w_t)} + A_{it}^{w_t} - D_{it}^{w_t} \quad \text{for } i \in I, t \in \{1,2,..,T-1\}, w_t \in \Omega \tag{3}$$

Constraint (4) keeps tracking the cash flow at time t. Note that transaction costs are included when purchasing or selling stocks.

$$C_t^{w_t} = C_{t-1}^{a(w_t)} - \sum_{i \in I} (1+\theta_i) A_{it}^{w_t} + \sum_{i \in I} (1-\theta_i) D_{it}^{w_t} \quad \text{for } t \in \{1,2,..,T-1\}, w_t \in \Omega \tag{4}$$

The value of the portfolio at time t includes cash and current value of stocks.

$$v_t^{w_t} = C_{t-1}^{a(w_t)} + \sum_{i \in I} r_{it}^{w_t} X_{it-1}^{a(w_t)} \quad \text{for } t \in \{1,2,..,T\}, w_t \in \Omega \tag{5}$$

Forcing constraints for the non-negative slack variable Y^{w_t}. Note that $u_0 - v_t^{w_t}$ in constraint (6) represents total loss at time t.

$$Y_t^{w_t} \geq u_0 - v_t^{w_t} - a_t \quad \text{for } t \in \{1,2,..,T\}, w_t \in \Omega \tag{6}$$

Constraint (7) is the CVaR risk control constraint at time t.

$$a_t + \frac{1}{1-\alpha_t} \sum_{w_t \in \Omega} p^{w_t} Y_t^{w_t} \leq \overline{CVaR} \quad \text{for } t \in \{1,2,..,T\} \tag{7}$$

Empirical experiences show that investment diversion is helpful in portfolio management in order to limit loss. Thus, constraint (8) enforces a universal upper bound on the percentage of any risky asset in the portfolio, in which m_t is chosen by the investor (e.g. 10%).

$$0 \le X_{it}^{w_t} \le m_t (C_t^{w_t} + \sum_{i \in I} X_{it}^{w_t}) \qquad \text{for } i \in I, t \in \{0,1,2,..,T\}, w_t \in \Omega \tag{8}$$

$$X_{it}^{w_t}, A_{it}^{w_t}, D_{it}^{w_t}, C_t^{w_t}, Y_t^{w_t}, a_t \ge 0 \qquad \text{for } i \in I, t \in T, w_t \in \Omega \tag{9}$$

5. Generate Scenario Trees of Random Assets Returns

In our approach, we first list all possible combinations of market trend predictions in multi-stages and then generate scenario trees of random market index returns in which the mean and variance depends on the corresponding market trend prediction.

In each stage, an investor has three possible attitudes about the market: unclear (no prediction), rise or fall. If the investor does not make prediction, the mean of random market index returns is zero and variance is relatively large. If he predicts bull or bear market, the mean of random market index returns becomes positive or negative respectively and the corresponding variance shrinks.

In the rest of this report, we restrict the number of stages as two for simplicity. Thus, there are in total nine different combinations of market trend prediction during two stages. For each combination, we build one scenario tree of market index returns by draw random variables as the pre-specified mean and variance.

Given nine scenario trees of index returns, we transform them into scenario trees of random assets returns by the *Single Index Model*, in which coefficients are estimated from historical data. Thus, means and variances of the random assets returns may vary in stages and depend on the corresponding type of trend prediction.

5.1 Scenario trees of Market trends

Instead of considering the market in average, I classify the prediction of two days market trends into nine different types and develop strategies separately. If an investor prefers not to make a prediction, the prediction type is defined as *unclear*; otherwise his prediction is either *rise* or *fall*. The Table 1 illustrates all possible types. Each type of prediction corresponds to stochastic problem and the solution is the optimal strategy for that type of trend prediction.

Note that the real market only is either bull or bear, so that only market trend type 5, 6, 8, and 9 will be realized.

Table 1. All possible types of prediction for two days market trend

	Type 1	Type 2	Type 3	Type 4	Type 5	Type 6	Type 7	Type 8	Type 9
Day1	*unclear*	*unclear*	*unclear*	*rise*	*rise*	*rise*	*fall*	*fall*	*fall*
Day2	*unclear*	*rise*	*fall*	*unclear*	*rise*	*fall*	*unclear*	*rise*	*fall*

For each type of market trend prediction, we generate a scenario tree of index returns by drawing random variables from a normal distribution in each node. The mean and variance of the normal distribution in each node depends on the anticipated market trend in that day. If the market rises, the mean of index return is 1%; if it falls, the mean of index return is -1%. If investors cannot judge, the variance of random index returns reduces. Table 2 summarizes means and variances of the normal random index return under each type of trend prediction.

Table 2. Means and Variances of random index returns

Market trend	Mean of index returns	Variance of index returns
unclear	0%	0.02
rise	1%	0.01
fall	-1%	0.01

Clearly, we use random sampling to generate scenario tree of index returns. The random sampling procedure for generating event tree with multi-periods may leads to unstable investment strategies. An obvious way to deal with this problem is to increase the number of nodes in event tree, such that the approximation error decreases. When we generate scenario tree of stock index returns, we draw 20 random variables in each node. And 20 is relative large number for this problem. Since we only try to solve a two stages problem, the stochastic program is still computationally tractable.

In Table 3, we illustrate an example of scenario tree of index returns under prediction Type 6. In the first stage, we draw ten random values and add ten nodes to the tree. For each node in the first stage, we add ten sub-nodes so that there are in total 100 nodes in the second stage.

Table 3. Example of Scenario tree of index returns for prediction Type 6

Daily market index return rate r_{mt}											
Day1	-0.01%	1.15%	0.13%	1.63%	0.53%	1.46%	3.01%	1.74%	1.20%		
Day2	-1.97%	-0.09%	-2.82%	-1.44%	-1.45%	-1.40%	0.39%	-1.09%	-1.53%	-0.39%	…(100)

5.2 Single-index model

Single index model is a simple model to measure systemic risk and firm-specific return of a stock in finance industry. It shows how a stock return is influenced by the market (β_i) and also indicates stock return has a firm specific expected value (α_i) and un-expected component ε_{it}.

Define:

r_{it} = return to stock *i* in period *t*

r_f = the risk free rate (i.e. constant interest rate for cash)

r_{mt} = the return to the market portfolio in period *t* (e.g. S&P 500 Stock Index)

α_i = the stock's alpha, or abnormal return

β_i = the stocks' beta, or responsiveness to the market return

ε_{it} = the residual (random) returns, which are assumed independent normally distributed with mean zero and standard deviation σ_i.

$$r_{it} - r_f = \alpha_i + \beta_i(r_{mt} - r_f) + \varepsilon_{it}$$
$$\varepsilon_{it} \sim N(0, \sigma_i)$$

In this report, we consider investment on a daily base so that the risk free rate r_f is omitted.

By taking a linear regression on historical data, we estimate the parameters in the Single-Index model for each stock. Table 4 provides an example of the linear regression on an Excel table. In the table, the second row indicates the S&P 500 Stock Index daily return rate from Jan 3rd to Feb. 26th 2014, which are only the first half of real data (37 days) that we collected. The following rows represent stock daily returns. We use the

LINEST function in Excel to run a linear regression of S&P 500 daily returns and stock daily returns.

Table 4. Estimate coefficients of Single-Index Model

	20140103	…	20140225	20140226		β_i	α_i	σ_i^2
SP500	-0.00033		-0.00135	0.00002				
AA	0.00381		-0.01274	0.03701		1.09422	0.00377	0.00053
AAPL	-0.02197		-0.01041	-0.00902		0.35288	-0.00100	0.00029
…								
GPC	0.00492		-0.00474	0.00511		1.47937	0.00133	0.00006
GPS	0.02062		0.02085	0.00229		1.29564	0.00290	0.00016

Single-Index model builds the relationship between individual stock return rates with market return rate in a statistic model. Now given a scenario tree of market index returns and this Single-index model, we create a scenario tree of random assets returns.

5.3 Scenario tree of assets returns

Given a node in the tree of market index returns, we create a node that contains returns for all stocks, which are represented by $\vec{r}_t$, according to the Single-Index model. Let r_{mt} represent a realization of the market index return at time *t*. The corresponding return of stock *i* and time *t* is defined as r_{it}. Therefore $r_{it} = \alpha_t + \beta_i r_{mt} + N(0, \sigma_i)$ for every stock *i* and time *t*, and we sequentially build a scenario tree of assets returns. The following Figure 3 illustrates the transaction.

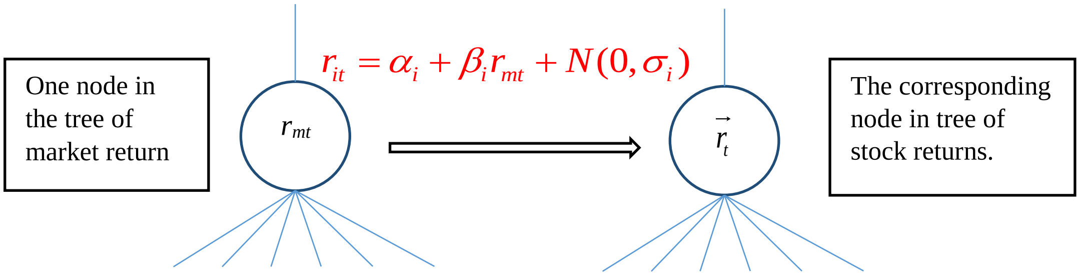

Figure 2. Construct scenario tree of stock returns

In total, for the two-stage problem, we build nine scenario trees of assets returns. Each scenario tree of assets returns represents a stochastic problem that we will solve in Section 6.

6. Utilize Optimal Stochastic Strategies

If an investor can forecast market trends with 100% accuracy, then of course he should always select the stochastic strategy that matches with his prediction. In reality, such investor does not exist and a wrong prediction may loss serious amount of money, so every investor should carefully decide how to utilize those optimal stochastic strategies for the best return.

In this problem, investors have the following five options and need to choose one of them based on his own prediction accuracy. If an investor predict market trend in both days and choose the corresponding strategy, there are four possible outcomes. If investor uses the strategy that predict market trend in only one day, then two outcomes may be realized. Intuitively, the risk increases as stochastic strategies forecast more trends. But investors with high prediction accuracy intend to choose risky strategies to obtain more profit.

The Table 5 lists those five options and the corresponding possible outcomes. We define the average returns for two days in each situation in the Table 5. And we will give the formulation for each return in Table 6.

Table 5. Investor's Options

Investor's decision	predict result	Average return
Buy index portfolio		r_0
Apply strategy that does not make prediction		r_1
Predict market trend for two days and choose from Strategy 5, 6, 8 or 9	Only right in day 1	r_2
	Only right in day 2	r_3
	both right	r_4
	both wrong	r_5
Predict only in day 1 and choose either Strategy 4 or 7	right	r_6
	wrong	r_7
Predict only in day 2 and choose either Strategy 2 or 3	right	r_8
	wrong	r_9

The return of buying index portfolio r_0 is obtained from historical data. The return r_1 equals to the average return of applying the strategy for the Type 1 prediction (i.e. strategy that does not predict in both days) to every testing point.

To calculate the rest of returns, we define probabilities that type 5, 6, 8 and 9 market trends are realized as u_5, u_6, u_8 and u_9 respectively. Let $R(i, j)$ denote the average return of applying strategy i to type j market trend, where $i \in \{1,2,...,9\}$, $j \in \{5,6,8,9\}$. The Table 6 summarizes the equation of calculating returns of cases in Table 5.

Table 6. Equations of calculating average returns

Average return	Formulation
r_2	$r_2 = u_5 R(6,5) + u_6 R(5,6) + u_8 R(9,8) + u_9 R(8,9)$
r_3	$r_3 = u_5 R(8,5) + u_6 R(9,6) + u_8 R(5,8) + u_9 R(6,9)$
r_4	$r_4 = u_5 R(5,5) + u_6 R(6,6) + u_8 R(8,8) + u_9 R(9,9)$
r_5	$r_5 = u_5 R(9,5) + u_6 R(8,6) + u_8 R(6,8) + u_9 R(5,9)$
r_6	$r_6 = u_5 R(2,5) + u_6 R(3,6) + u_8 R(2,8) + u_9 R(3,9)$
r_7	$r_7 = u_5 R(3,5) + u_6 R(2,6) + u_8 R(3,8) + u_9 R(2,9)$
r_8	$r_8 = u_5 R(4,5) + u_6 R(4,6) + u_8 R(7,8) + u_9 R(7,9)$
r_9	$r_9 = u_5 R(7,5) + u_6 R(7,6) + u_8 R(4,8) + u_9 R(4,9)$

Suppose an investor knows his success probability of predicting day 1 (p_1) and day 2 (p_2) market trend respectively. We can show his expected return of each option in the Table 7. According to this table, investors can pick the strategy with the maximum expected return rate.

Table 7. Expected returns

Investor's decision	Expected return
Buy index portfolio	r_0
Apply strategy that does not make prediction	r_1
Predict market trend for two days and choose from Strategy 5, 6, 8 or 9	$p_1(1-p_2)r_2 + (1-p_1)p_2 r_3 + p_1 p_2 r_4 + (1-p_1)(1-p_2)r_5$
Predict only in day 1 and choose either Strategy 4 or 7	$p_1 r_6 + (1-p_1)r_7$
Predict only in day 2 and choose either Strategy 2 or 3	$p_2 r_8 + (1-p_2)r_9$

7. Computational Results

In the first Section 7.1, we solve the stochastic model for each scenario tree of stock returns. As a consequence, we have nine optimal stochastic strategies for nine types of market trend predictions respectively.

In Section 7.2, we collect real prices of 154 stocks and S&P 500 Stock Index in 73 trading days and classify the 72 pieces of real two-day market into 4 types: 1) rise and rise; 2) rise and fall; 3) fall and rise; 4) fall and fall. Interesting statistic results about the real market data are presented. In the next step Section 7.3, we apply our nine stochastic strategies to the real market data.

Assuming a perfect investor always makes right prediction and applies the right optimal stochastic strategy to real stock prices. In Section 7.4, we compare his performances with the real market returns and the stochastic strategy that does not predict market trend. Beside the prefect investor, we assume there are other investors who don't predict or forecast incorrectly sometimes and we compare them with the prefect investor. More importantly, we test when an investor should take strategies that make prediction given the success probabilities of his predictions in this section.

The optimal return rate and investor's decision vary when important input feature changes. In the section 7.5, we will vary the risk control requirement and check how it affects return rates and invest decisions.

In 7.6, we finally summarize all computation experiments we have conducted.

We implement models with Java programming language and solved them using ILOG CPLEX 12.5 solver. All computational tests were conducted on a 3.33 GHz processor with 2 GB RAM.

7.1 Generate optimal stochastic strategies

As shown in Section 5, we develop nine stochastic problems that represent all possible types of trend predictions for two days. In each stochastic problem, the mean and variance of random asset returns depends on the market trend prediction. Thus, the resulting optimal stochastic solution provides the optimal strategy if an investor picks the corresponding type of prediction.

The model size exponentially increases as the number of sub-nodes in scenario tree. Since we draw 20 sub-nodes for each node in the scenario tree, the number of constraints grows up to 68777, in which 94% are constraint (8). Even though solving times are not the major concern, they are all within ten minutes.

For all nine stochastic problems, we use the same parameters in the CVaR risk control constraints. We set $\alpha_t = 0.8$ and $\overline{CVaR} = 0.005$ which means conditional expectation of the loss is no greater than 0.005, conditional on the loss exceeding the 80%-level qunatile of random total loss. Clearly, this set of risk control constraints is quite restrictive. We solve the nine stochastic problems and obtain the following optimal return rates in Table 8.

We would like to give an example of an optimal solution to the stochastic problem in which investor predicts rise in the first day and fall in the second day (Type 6 prediction). In this stochastic problem, market return scenarios in day 1 have positive means and market return scenarios in day 2 have negative means. Therefore, we can expect that the optimal solution will initially purchase stocks that are positively correlated with market returns and then sell them out at the end of day 1.

Table 8. Optimal return rates for nine stochastic problems

Stochastic problem for each type of two-day prediction	Optimal return rate
1 (unclear, unclear)	1.00454
2 (unclear, rise)	1.02087
3 (unclear, fall)	1.00314
4 (rise, unclear)	1.02586
5 (rise, rise)	1.04058
6 (rise, fall)	1.02017
7 (fall, unclear)	1.00258
8 (fall, rise)	1.01910
9 (fall, fall)	1.00055

At stage 0, no information is revealed. Suppose the initial wealth is \$1. Since first day is expected to be bull market, the optimal solution uses all cash to buy stock that are positively correlated with the market index ($\beta > 0$). Table 9 shows the amount of value allocated to asset i in stage 0. Note that constraint (8) enforces that the amount of value allocated to a single asset cannot exceed 10% of total value.

Table 9. Initial purchase at stage 0

Stock ID	Amount of value invested	α	β
6	0.1	0.00357	1.40994
10	0.1	0.00290	1.45019
29	0.1	0.00652	1.13084
31	0.1	-0.00639	2.67334
33	0.1	0.00162	1.68942
49	0.1	0.00577	2.00780
69	0.1	0.00173	1.80162
107	0.1	0.00116	1.56558
125	0.1	0.00327	1.50518
141	0.1	0.01539	1.43013

At stage 1, the asset returns between stage 0 and stage 1 are revealed. Therefore, investor's decision also depends on the realization of index return and assets returns at that node of the scenario tree.

In one scenario node, the index return between stage 0 and stage 1 is 1.0112 which means about 1.1% increase in a day. In this case, the actions of the optimal solution are shown in Table 10. At the end of stage 1, the optimal solution would invest $0.1021 to stock 141 and buy $0.1021 stock 92. Notice that stock 141 is not strongly correlated with Index return and its α value is relative large. The reason of picking stock 92 is that the Shape Ratio of stock 92 is high.

In summary, the optimal solution takes the advantage of trend prediction. Initially, it purchased as much as possible stocks that are positively correlated with index return. At the end of the first stage, it sold them out and bought a small amount of stocks that either have large abnormal return or have high Sharp Ratio. This observation meets our expectation.

Table 10. Actions at the end of stage 1

	Stock ID	Amount of value	α	β
Sold at stage 1	6	0.10194	0.00357	1.40994
	10	0.10192	0.00290	1.45019
	29	0.10192	0.00652	1.13084
	31	0.10236	-0.00639	2.67334
	33	0.10206	0.00162	1.68942
	49	0.10283	0.00577	2.00780
	69	0.10220	0.00173	1.80162
	107	0.10187	0.00116	1.56558
	125	0.10202	0.00327	1.50518
Purchased at stage 1	92	0.10212	0.00253	0.11663
Investment at end of stage 1	92	0.10212	0.00253	0.11663
	141	0.10212	0.01539	1.43014

7.2 Classify real market data

We collect real prices of 154 stocks and S&P 500 Stock Index in 73 trading days. The Figure 3 shows the movement of S&P 500 Stock Index in the time horizon. In the Figure 4, the daily return rates of S&P 500 Stock Index are shown. The total return rate from Jan. 2nd to April 17th is 1.018 which is obtained by dividing the ending date index price by the starting date index price. According to Figure 4, the average index return for two days is 1.0005.

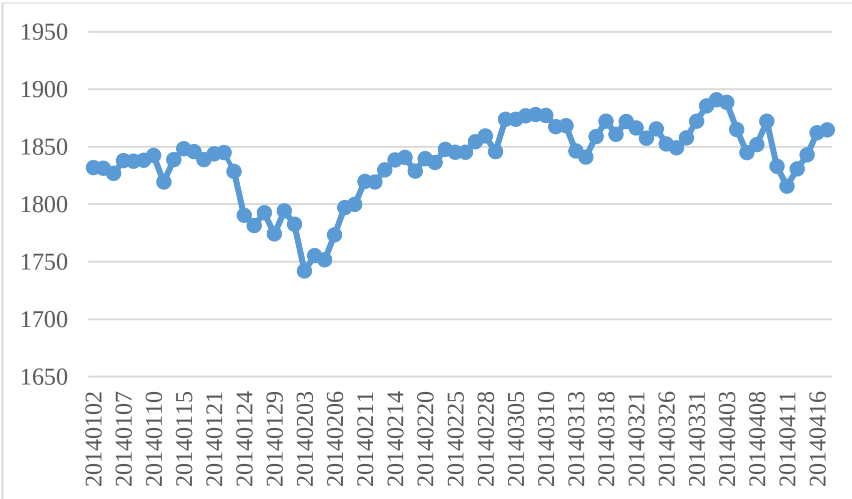

Figure 3. S&P 500 Stock Index

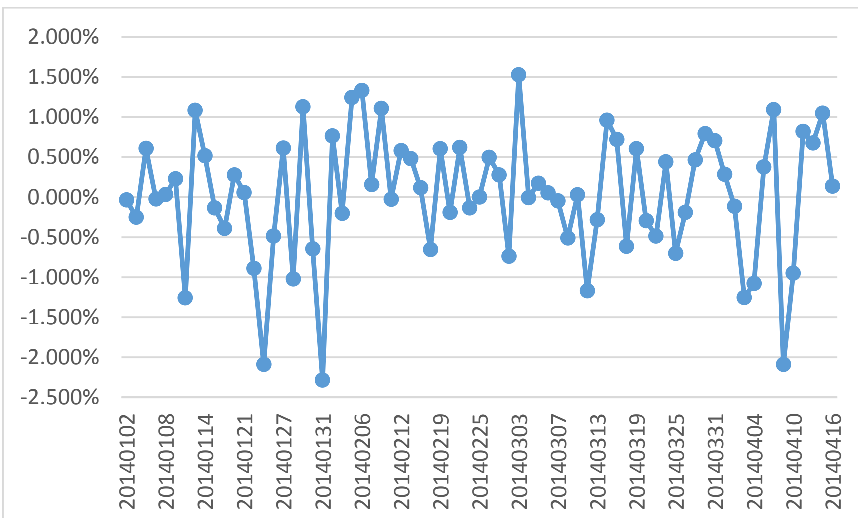

Figure 4. S&P 500 Stock Index daily return rate

If the S&P 500 Stock Index rises in one day, no matter how much it rises, we treat that day as a bull market trend. If the following day is also a bull market trend, then this piece of two days market belongs to Type 5. Therefore, for a two days problem, there are in total four types of market trends: 1) rise, rise; 2) rise, fall; 3) fall, rise; 4) fall, fall. They correspond to prediction Type 4, 5, 8 and 9 respectively.

The Table 11 summarizes the number of real two days market trends in each type. When we calculate average index return for Type 5 trend, we first calculate daily index return rate for each piece of two days market that is in Type 5 and then take the average.

Note that we need to separate the 73 days data into two parts. We used the first part (37 days) real market data to estimate stock's parameters in the Single-Index model. In the contrast, the rest of real market data is totally out of sample test for the nine stochastic strategies.

Table 11. Summary of real market data

Trend Type	Number of two day market in each trend type	Average index return
5	19	1.0116
6	20	1.0002
8	21	1.0005
9	12	0.9837

7.3 Apply stochastic strategies

When we apply a stochastic strategy to a piece of two days market data, we measure how close in absolute value the actual S&P 500 index returns is to the scenario value and pick the closest one.

We apply all stochastic strategies to every piece of two days market data. Some of them are lower than the index return rate while others that match with the real market trends outperform the index return. Appendix A shows all return rate results for 72 pieces of two-day market data. To get an insight from those return rates, Table 12 summarizes Appendix A and represents the average return rate of each strategy under each type of real two-day market data. Note that both index return and strategy return in all following tables are two days return rate.

The bolded numbers in Table 12 represent the average return rate of applying stochastic strategy correctly. For instance, compared with the average index return for Type 5 trends, stochastic Strategy 5 can obtain one percent extra return, which is also the best strategy among all strategies.

Table 12 provides the values of $R(i, j)$ defined in Section 6.

Table 12. Average return of each strategy

Trend type	Index return	Average return rate of each stochastic strategy								
		1	2	3	4	5	6	7	8	9
5	1.0116	1.0044	1.0089	1.0037	1.0153	**1.0196**	1.0116	1.0048	1.0087	1.0011
6	1.0002	1.0016	0.9931	1.0034	1.0085	1.0027	**1.0099**	1.0006	0.9923	1.0010
8	1.0005	1.0024	1.0105	1.0024	0.9981	1.0051	0.9942	1.0029	**1.0100**	1.0016
9	0.9837	1.0002	0.9855	0.9994	0.9804	0.9716	0.9823	0.9995	0.9856	**1.0014**

Suppose there is an investor who always makes right prediction and applies the right optimal stochastic strategy, then the expected return rate for this perfect investor would be (19*1.0196+20*1.0099+21*1.01+12*1.0014)/72 = 1.011. Compared with the average index return 1.0005, the perfect investor can gain more than 1% extra return per two days.

Remember that the parameters in the Single-Index model are estimated according to the first half of the data. We want to compare the in sample test result and out of sample test results. Table 13 shows the average return of each strategy while applying to the first half of the data (in sample test results). In the contrast, Table 14 summary the average return when we test the second half of data (out of sample test).

Table 13. Average return rate (in sample test)

Trend type	# of each trend type	Index return	Average return rate in the first half of data								
			1	2	3	4	5	6	7	8	9
5	9	1.0108	1.0080	1.0114	1.0067	1.0185	**1.0224**	1.0141	1.0086	1.0111	1.0023
6	11	1.0011	1.0058	1.0000	1.0065	1.0179	1.0124	**1.0154**	1.0044	0.9988	1.0012
8	12	1.0005	1.0031	1.0128	1.0031	1.0008	1.0084	0.9962	1.0031	**1.0121**	1.0014
9	5	0.9815	0.9995	0.9812	0.9989	0.9827	0.9688	0.9844	0.9988	0.9807	**1.0002**

Table 14. Average return rate (out of sample test)

Trend type	# of each trend type	Index return	Average return rate in the first half of data								
			1	2	3	4	5	6	7	8	9
5	10	1.0123	1.0012	1.0066	1.0010	1.0125	**1.0171**	1.0093	1.0013	1.0065	1.0001
6	9	0.9991	0.9964	0.9846	0.9997	0.9969	0.9909	**1.0033**	0.9961	0.9843	1.0007
8	9	1.0006	1.0015	1.0074	1.0014	0.9946	1.0007	0.9915	1.0026	**1.0072**	1.0019
9	7	0.9852	1.0007	0.9886	0.9997	0.9789	0.9737	0.9808	1.0000	0.9890	**1.0023**

The average index returns in the first half days is 1.0006 and it is 1.0005 in the second half days.

Compared with Table 13, a large proportion of return rates in Table 14 drop. For example, in the first half data, an investor who makes perfect prediction obtains return rate (9*1.0224 + 11*1.0154 + 12*1.0121 + 5*1.002)/37 = 1.014, which exceeds average index return by 1.34%. However, in the second half days (out of sample test), the perfect prediction only gains 1.008 return rate, which only exceeds average index returns by 0.75%. The reason performance of stochastic strategies goes off in the second half data is that parameters in Single-Index model bias for the out of sample test.

In the following discussion, we analyze first and second half of data as a whole.

7.4 Investor's decision of choosing strategies

In reality, no one can predict market trend with 100 percent accuracy. Therefore, investors may pick stochastic strategies that only predict one day or would rather apply the stochastic strategy that does not make prediction. Table 15 shows their average return rates under each case and it is a realization of the Table 5.

The probabilities that type 5, 6, 8 and 9 market trends are realized as u_5, u_6, u_8 and u_9 respectively. We assume that u_5 equals to the number of Type 2 trends divided by 72. Similarly, u_6 equals 20/72; u_8 equals 21/72; u_9 equals 12/72.

Assuming an investor knows his success probability of predicting day 1 (p_1) and day 2 (p_2) market trend respectively, how the investor picks strategies is an interesting question. In general, investor can take one of the five options in Table 15. And the optimal decision gives him the maximum expected return rate. It is clear that applying strategy that does not forecast outperforms buying index portfolio, so we can ignore the first option.

Table 15. Investor's decision and return rates

Investor's decision	predict result	Average return
Buy index portfolio		1.0005
Do not predict trend and take Strategy 1		1.0023
Predict market trend for two days and choose from Strategy 5, 6, 8 or 9	Only right in day 1	1.0019
	Only right in day 2	1.0011
	both right	1.0111
	both wrong	0.9917
Predict only in day 1 and choose either Strategy 4 or 7	right	1.0062
	wrong	0.9973
Predict only in day 2 and choose either Strategy 2 or 3	right	1.0072
	wrong	0.9976

Table 16 shows the average return rate if investor only predicts trend in day 1 and takes the corresponding stochastic strategy when p_1 changes from zero to 100 percent. In the contrary, Table 17 shows the average return rate if investor predicts market trend in the second day when p_2 varies. When we compare Table 16 and 17, we observe that making trend prediction on day 1 incurs higher risk and profit than forecasting market trend on day 2.

Table 16. Expected return rate of forecasting trend only in day 1

p_1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Average return	0.9986	0.9995	1.0005	1.0014	1.0024	1.0033	1.0043	1.0053	1.0062	1.0072

Table 17. Expected return rate of forecasting trend only in day 2

p2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Average return	0.9982	0.9991	1.0000	1.0009	1.0018	1.0027	1.0036	1.0045	1.0054	1.0062

If investor uses strategies that predict in both days, his average return rate depends on p_1 and p_2. Table 18 shows the average returns with a discrete set of p_1 and p_2 pairs.

For each pair value of p_1 and p_2, investor should take the strategy with maximum expected return rate. The maximum return equals to the maximum of return rate of strategy that does not predict, strategy that only predicts day 1 trend with p_1, strategy that only predicts day 2 trend with p_2, and strategy that predicts two day trends with p_1 and p_2. Table 19 presents the results for a set of p_1 and p_2. According to Table 19, an investor with estimated success probabilities p_1 and p_2 would know which strategy he should choose.

Table 18. Expected return rate of forecasting trends in both days

		p_2									
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
p_1	0.1	0.9945	0.9955	0.9966	0.9976	0.9986	0.9997	1.0007	1.0017	1.0028	1.0038
	0.2	0.9954	0.9964	0.9974	0.9984	0.9995	1.0005	1.0015	1.0025	1.0036	1.0046
	0.3	0.9962	0.9972	0.9983	0.9993	1.0003	1.0013	1.0023	1.0033	1.0044	1.0054
	0.4	0.9971	0.9981	0.9991	1.0001	1.0011	1.0021	1.0031	1.0041	1.0052	1.0062
	0.5	0.9979	0.9989	0.9999	1.0009	1.0019	1.0029	1.0039	1.0049	1.0059	1.0070
	0.6	0.9988	0.9998	1.0008	1.0018	1.0028	1.0038	1.0048	1.0058	1.0067	1.0077
	0.7	0.9997	1.0006	1.0016	1.0026	1.0036	1.0046	1.0056	1.0066	1.0075	1.0085
	0.8	1.0005	1.0015	1.0025	1.0035	1.0044	1.0054	1.0064	1.0074	1.0083	1.0093
	0.9	1.0014	1.0024	1.0033	1.0043	1.0053	1.0062	1.0072	1.0082	1.0091	1.0101
	1	1.0022	1.0032	1.0042	1.0051	1.0061	1.0070	1.0080	1.0090	1.0099	1.0109

Table 19. Investor's maximum average return rate

		p_2									
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
p_1	0.1	1.0023	1.0023	1.0023	1.0023	1.0023	1.0027	1.0036	1.0045	1.0054	1.0062
	0.2	1.0023	1.0023	1.0023	1.0023	1.0023	1.0027	1.0036	1.0045	1.0054	1.0062
	0.3	1.0023	1.0023	1.0023	1.0023	1.0023	1.0027	1.0036	1.0045	1.0054	1.0062
	0.4	1.0023	1.0023	1.0023	1.0023	1.0023	1.0027	1.0036	1.0045	1.0054	1.0062
	0.5	1.0024	1.0024	1.0024	1.0024	1.0024	1.0027	1.0036	1.0045	1.0054	1.0062
	0.6	1.0033	1.0033	1.0033	1.0033	1.0033	1.0034	1.0043	1.0052	1.0062	1.0071
	0.7	1.0043	1.0043	1.0043	1.0043	1.0043	1.0044	1.0053	1.0062	1.0072	1.0081
	0.8	1.0053	1.0053	1.0053	1.0053	1.0053	1.0054	1.0063	1.0072	1.0082	1.0091
	0.9	1.0062	1.0062	1.0062	1.0062	1.0062	1.0064	1.0073	1.0082	1.0092	1.0101
	1	1.0072	1.0072	1.0072	1.0072	1.0072	1.0074	1.0083	1.0093	1.0102	1.0111

Do not predict trend and take Strategy 1
Predict trend only in day 1 and choose either Strategy 4 or 7
Predict trend only in day 2 and choose either Strategy 2 or 3
Predict market trend for two days and choose from Strategy 5, 6, 8 or 9

7.5 Risk constraints sensitivity analysis

Recall that the risk control requirements are quite restrictive in the above computational tests. We want to see the effect of relaxing the risk control requirements one by one. In order to make the difference significant large, we remove CVaR constraints and repeat the above tests, in Section 7.5.1. Besides the CVaR constraints, we also want to study the effect of constraints (8) that enforces an upper bound on the portion of total value that can be invested in one asset. In Section 7.5.2, we only remove constraints (8) and repeat the above tests.

7.5.1 Relax CVaR constraints

We re-generate nine optimal stochastic strategies that are the optimal solutions to the nine stochastic problems respectively. But this time, the stochastic model does not require to control risk via CVaR constraints. The following Table 20 shows the optimal return rates for each stochastic problem.

Table 20. Optimal return rates (without CVaR)

Stochastic problem for each prediction type	Optimal return rate
1 (unclear, unclear)	1.01900
2 (unclear, rise)	1.02432
3 (unclear, fall)	1.00354
4 (rise, unclear)	1.02503
5 (rise, rise)	1.03961
6 (rise, fall)	1.01856
7 (fall, unclear)	1.00573
8 (fall, rise)	1.02072
9 (fall, fall)	1.00055

Then we apply those stochastic strategies to each piece of two-day market data. We summarize the results in Table 21. The return rate for an investor who always makes right prediction is (19*1.0196+20*1.0101+21*1.0104+12*1.0013)/72 = 1.0112.

Table 21. Average return of each strategy (without CVaR)

Trend type	Index return	Average return rate of each stochastic strategy								
		1	2	3	4	5	6	7	8	9
5	1.0116	1.0179	1.0173	1.0078	1.0148	**1.0196**	1.0114	1.0067	1.0107	1.0011
6	1.0002	1.0048	1.0005	1.0060	1.0078	1.0027	**1.0101**	1.0015	0.9942	1.0010
8	1.0005	1.0052	1.0068	1.0006	1.0028	1.0051	0.9942	1.0070	**1.0104**	1.0016
9	0.9837	0.9763	0.9774	0.9960	0.9753	0.9716	0.9823	0.9948	0.9851	**1.0013**

Given this set of high risky stochastic strategies, investors also face the problem of utilizing them. Table 22 shows the expected return rates when investors only predict one day or does not make prediction and apply the corresponding strategy.

In Table 22, we also compare return rates from the risky strategies with those from the low risk strategies (with CVaR constraints). The strategy that does not make prediction gains higher profit when CVaR constraints are removed.

Table 22. Investor's decision and return rates (without CVaR)

Investor's decision	predict result	Average return without CVaR	Average return with CVaR
Buy index portfolio		1.0005	1.0005
Apply strategy that does not make prediction		1.0036	1.0023
Predict market trend for two days and choose from Strategy 5, 6, 8 or 9	Only right in day 1	1.0017	1.0019
	Only right in day 2	1.0016	1.0011
	both right	1.0112	1.0111
	both wrong	0.9923	0.9917
Predict only in day 1 and choose either Strategy 4 or 7	right	1.0075	1.0062
	wrong	0.9986	0.9973
Predict only in day 2 and choose either Strategy 2 or 3	right	1.0072	1.0072
	wrong	0.9989	0.9976

Table 23 indicates the return rate if an investor only predicts in day 1 and takes the corresponding the high risk stochastic strategy, while Table 24 shows the return rate if an investor only predicts in the second day and takes the corresponding the high risk stochastic strategy. If investor uses risky strategies that predict in both days, his average return rate depends on p_1 and p_2. Table 25 shows the average returns with a discrete set of p_1 and p_2.

Table 23. Expected return rate of forecasting only in day 1 (without CVaR)

p_1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Average return	0.9997	1.0005	1.0014	1.0022	1.0030	1.0039	1.0047	1.0055	1.0064	1.0072

Table 24. Expected return rate of forecasting only in day 2 (without CVaR)

p_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Average return	0.9995	1.0004	1.0013	1.0022	1.0031	1.0040	1.0048	1.0057	1.0066	1.0075

Table 25. Expected return rate of forecasting in both days (without CVaR)

		p_2									
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
p_1	0.1	0.9941	0.9951	0.9960	0.9970	0.9979	0.9988	0.9998	1.0007	1.0017	1.0026
	0.2	0.9951	0.9960	0.9970	0.9979	0.9989	0.9998	1.0007	1.0017	1.0026	1.0036
	0.3	0.9960	0.9970	0.9979	0.9989	0.9998	1.0007	1.0017	1.0026	1.0036	1.0045
	0.4	0.9970	0.9979	0.9989	0.9998	1.0008	1.0017	1.0026	1.0036	1.0045	1.0055
	0.5	0.9979	0.9989	0.9998	1.0008	1.0017	1.0027	1.0036	1.0045	1.0055	1.0064
	0.6	0.9989	0.9998	1.0008	1.0017	1.0027	1.0036	1.0046	1.0055	1.0064	1.0074
	0.7	0.9998	1.0008	1.0017	1.0027	1.0036	1.0046	1.0055	1.0065	1.0074	1.0084
	0.8	1.0008	1.0017	1.0027	1.0036	1.0046	1.0055	1.0065	1.0074	1.0084	1.0093
	0.9	1.0017	1.0027	1.0036	1.0046	1.0055	1.0065	1.0074	1.0084	1.0093	1.0103
	1	1.0027	1.0036	1.0046	1.0055	1.0065	1.0074	1.0084	1.0093	1.0103	1.0112

Investors always would like to take the strategy with maximum average return rate. Table 26 presents the maximum return rate strategy and its return rate for each pair of p_1 and p_2. Therefore, Table 26 supports investor's decision in choosing the best strategy when he knows his estimated success probabilities p_1 and p_2.

Compared with Table 19, Table 26 suggests that the high risk strategies that make predictions (e.g. Strategy 5, 6, 8 and 9) need higher the prediction accuracy (e.g. larger than 50 percent for one day prediction). This observation makes senses since the higher risk strategies intend to loss more when the trend prediction is wrong.

Table 26. Investor's maximum average return rate (without CVaR)

		p_2									
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
p_1	0.1	1.0036	1.0036	1.0036	1.0036	1.0036	1.0040	1.0048	1.0057	1.0066	1.0075
	0.2	1.0036	1.0036	1.0036	1.0036	1.0036	1.0040	1.0048	1.0057	1.0066	1.0075
	0.3	1.0036	1.0036	1.0036	1.0036	1.0036	1.0040	1.0048	1.0057	1.0066	1.0075
	0.4	1.0036	1.0036	1.0036	1.0036	1.0036	1.0040	1.0048	1.0057	1.0066	1.0075
	0.5	1.0036	1.0036	1.0036	1.0036	1.0036	1.0040	1.0048	1.0057	1.0066	1.0075
	0.6	1.0039	1.0039	1.0039	1.0039	1.0039	1.0040	1.0048	1.0057	1.0066	1.0075
	0.7	1.0047	1.0047	1.0047	1.0047	1.0047	1.0047	1.0055	1.0065	1.0074	1.0084
	0.8	1.0055	1.0055	1.0055	1.0055	1.0055	1.0055	1.0065	1.0074	1.0084	1.0093
	0.9	1.0064	1.0064	1.0064	1.0064	1.0064	1.0065	1.0074	1.0084	1.0093	1.0103
	1	1.0072	1.0072	1.0072	1.0072	1.0072	1.0074	1.0084	1.0093	1.0103	1.0112

Do not predict trend and take Strategy 1
Predict trend only in day 1 and choose either Strategy 4 or 7
Predict trend only in day 2 and choose either Strategy 2 or 3
Predict market trend for two days and choose from Strategy 5, 6, 8 or 9

7.5.2 Relax maximum investment percentage restriction

In this section, we study the effect of relaxing constraints (8) that enforces an upper bound on the percentage of total value that can be invested in one asset (stock).

Once again, we re-generate nine optimal stochastic strategies that are the optimal solutions to the nine stochastic problems respectively. But this time, the stochastic model does not include constraint (8). The following Table 27 shows the optimal return rates for each stochastic problem.

Table 27. Optimal return rates (without constraint (8))

Stochastic problem for each prediction type	Optimal return rate
1 (unclear, unclear)	1.01165
2 (unclear, rise)	1.03021
3 (unclear, fall)	1.00499
4 (rise, unclear)	1.04613
5 (rise, rise)	1.06086
6 (rise, fall)	1.03055
7 (fall, unclear)	1.00629
8 (fall, rise)	1.02917
9 (fall, fall)	1.00276

Then we apply those stochastic strategies to each piece of two-day market data and record the average return rate for each strategy in Table 28. According to Table 28, the average return rate for an investor who makes perfect prediction jumps up to (19*1.0256 + 20*1.0236 + 21*1.0054 + 12*1.0038)/72 = 1.0155.

Table 28. Average return of each strategy (without constraint (8))

Trend type	Index return	Average return rate of each stochastic strategy								
		1	2	3	4	5	6	7	8	9
5	1.0116	1.0190	1.0214	1.0074	1.0256	**1.0256**	1.0263	1.0119	1.0216	1.0064
6	1.0002	1.0053	1.0026	1.0043	1.0239	1.0239	**1.0236**	1.0028	1.0034	1.0031
8	1.0005	1.0038	1.0051	1.0039	1.0127	1.0127	1.0064	1.0034	**1.0054**	1.0026
9	0.9837	0.9981	0.9930	1.0019	0.9764	0.9764	0.9830	0.9979	0.9924	**1.0038**

In Table 29, we show the expected return rates for each option and each outcome and we also compare return rates from the risky strategies with those from the low risk strategies with constraint (8). From Table 29, we can see that relaxing constraint (8) boosts up return rates for all options and all outcomes.

Table 29. Investor's decision and return rates (without constraint (8))

Investor's decision	predict result	Average return without constraint 8	Average return with constraint 8
Buy index portfolio		1.0005	1.0005
Apply strategy that does not make prediction		1.0073	1.0023
Predict market trend for two days and choose from Strategy 5, 6, 8 or 9	Only right in day 1	1.0131	1.0019
	Only right in day 2	1.0074	1.0011
	both right	1.0155	1.0111
	both wrong	1.0006	0.9917
Predict only in day 1 and choose either Strategy 4 or 7	right	1.0086	1.0062
	wrong	1.0026	0.9973
Predict only in day 2 and choose either Strategy 2 or 3	right	1.0140	1.0072
	wrong	1.0037	0.9976

Investors always would like to take the strategy with maximum average return rate. Table 30 presents the maximum possible average return rate strategy and the corresponding return rate for each pair of p_1 and p_2. Therefore, it helps investors decide how to choose the best strategy given success probabilities p_1 and p_2.

Table 30. Investor's maximum average return rate (without constraint (8))

		p_2									
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
p_1	0.1	1.0073	1.0073	1.0073	1.0073	1.0073	1.0073	1.0073	1.0074	1.0080	1.0086
	0.2	1.0073	1.0073	1.0073	1.0073	1.0073	1.0073	1.0073	1.0079	1.0085	1.0091
	0.3	1.0073	1.0073	1.0073	1.0073	1.0073	1.0076	1.0082	1.0088	1.0093	1.0099
	0.4	1.0078	1.0078	1.0078	1.0078	1.0081	1.0086	1.0091	1.0097	1.0102	1.0107
	0.5	1.0089	1.0089	1.0089	1.0089	1.0091	1.0096	1.0101	1.0105	1.0110	1.0115
	0.6	1.0099	1.0099	1.0099	1.0099	1.0102	1.0106	1.0110	1.0114	1.0119	1.0123
	0.7	1.0109	1.0109	1.0109	1.0109	1.0112	1.0116	1.0120	1.0123	1.0127	1.0131
	0.8	1.0120	1.0120	1.0120	1.0120	1.0122	1.0126	1.0129	1.0132	1.0136	1.0139
	0.9	1.0130	1.0130	1.0130	1.0130	1.0133	1.0136	1.0138	1.0141	1.0144	1.0147
	1	1.0140	1.0140	1.0140	1.0140	1.0143	1.0145	1.0148	1.0150	1.0153	1.0155

Do not predict trend and take Strategy 1
Predict trend only in day 1 and choose either Strategy 4 or 7
Predict trend only in day 2 and choose either Strategy 2 or 3
Predict market trend for two days and choose from Strategy 5, 6, 8 or 9

7.6 Computational results summary

As the Chapter 6 generates scenario tree of asset returns for each trend type prediction, the beginning of Chapter 7 first solves the nine stochastic problems and obtains nine optimal stochastic strategies. Section 7.2 describes the input data and then we apply every stochastic strategy to each piece of two-day market data in section 7.3. Based on results in Section 7.3, we discuss how to utilize the optimal stochastic strategies for an investor with estimated prediction accuracy in Section 7.4.

Since the risk control requirements (CVaR constraints and maximal proportion constraints) in Section 7.4 is tight, we relax them in Section 7.5 as a sensitivity analysis. While comparing Table 15 and Table 22, it is clear that relaxing CVaR constraints increases the return rate difference between a successful prediction and a wrong prediction. If we compare Table 19 and Table 26, we can conclude that making prediction and taking the corresponding high risk strategies that ignore CVaR constraints need higher prediction accuracy than forecasting trend and taking the strategies with CVaR constraints.

In Section 7.5.2, we relax the constraint (8) that enforces an upper bound on the percentage of the portfolio that we can invest into a single asset/stock. The results shows that removing this constraints can increase returns rates for all action options and all outcomes. One reasonable explanation is that the upper bound we proposed in computation experiment (e.g. m_t = 10%) is too tight. Or the sample size is not large enough to represent the case that one stock drops dramatically due to its firm specific reason while the market remains stable.

8. Conclusion

In this report, we discuss how to maximize portfolio return via multi-stage stochastic programming assuming investors can forecast market trend. In each stage, investors forecast the market trend as unclear, rise or fall. So there are nine possible types of prediction about market trends in a two-stage problem.

Instead of generating a single scenario tree of asset returns with fixed means and covariance and solving the corresponding stochastic problem, we generate nine scenario trees of asset returns for a two-stage problem and solve them separately. Those nine scenario trees represent nine possible types of prediction about market trends in two stages. When we generate those nine scenario trees of asset returns, the means and variances of random return rates depend on market trend on each stage.

After solving the nine stochastic problems, we discuss how to utilize the resulting optimal stochastic strategies. Compared with strategies that do not depend on trend forecasting, strategies that need trend prediction gain higher return if the prediction is right; otherwise they loss more money. If an investor cannot forecast market trend with 100 percent accuracy, he needs to decide which strategy he choose according to his prediction accuracy on day 1 and day 2. We address this problem by providing a table in which investor can select the best strategy with specific prediction accuracies.

In this report, risk is measure by CVaR. We firstly solve all stochastic problems with CVaR constraints. But then we relax the CVaR constraints and repeat the procedure in order to conduct a sensitivity analysis. While comparing the results of low risk stochastic model (i.e. CVaR constraints are included) and high risk stochastic model, we find that investors intend to forecast market trend and apply low risk stochastic model under the same prediction accuracy.

For future research, the risk aversion constraints in stochastic model should coordinate with investor's prediction accuracy. If investor can forecast with high accuracy, the stochastic model should take advantage of that and allow more risk in strategies.

To extend this problem, the prediction accuracy can go beyond a universal point estimator. For example, investors may predict bull market with 80 percent accuracy but predict bear market with 90 percent accuracy. Moreover, the probability of a successful prediction can be a range rather than a point estimator.

In the real portfolio management, investors usually apply multi-stage stochastic strategy dynamically. For example, when trend prediction in day 1 goes wrong, how to adjust strategy is also an interesting question.

Appendix A

Appendix A shows return rate results for 72 pieces of two-day market data applying each stochastic strategy which considers risk control requirements.

Start Date	Trend Type	Index Return	Stochastic Strategy Return								
			1	2	3	4	5	6	7	8	9
0	9	0.9972	1.0011	0.9952	1.0014	0.9994	0.993	0.9951	1.0011	0.9944	1.0033
1	8	1.0036	1.0055	1.0108	1.0046	1.0039	1.0174	1.0016	1.0048	1.0167	1.0043
2	6	1.0059	1.0038	1.0308	1.0015	1.0173	1.0478	1.0103	1.0153	1.0298	1.0011
3	8	1.0001	1.0002	0.9973	1	1.0136	1.0266	1.0291	1.0004	0.9978	1.0016
4	5	1.0027	1.005	1.0054	1.0038	0.9968	0.9986	1	1.0042	1.0055	1.0035
5	6	0.9897	1.0001	0.9803	1.0019	0.9912	0.9853	1.0021	0.9998	0.9789	1.0014
6	8	0.9981	1.0014	1.0174	0.9988	0.9901	1.0027	0.9805	1.0011	1.0179	0.9987
7	5	1.0160	0.9996	1.005	1.0007	1.0149	1.0292	1.0205	0.9991	1.0037	0.9992
8	6	1.0038	1.0037	1.0011	1.0031	1.0096	1.0084	1.0074	1.0031	0.9967	1.0027
9	9	0.9948	1.003	0.9977	1.0033	1.0098	1.0005	1.003	1.0028	0.9967	1.0021
10	8	0.9989	1.0042	1.0092	1.003	1.0068	1.0187	1.0006	1.0066	1.0091	1.0021
11	5	1.0033	1.0039	1.003	1.0053	1.0242	1.0191	1.0216	1.0034	1.0021	1.0047
12	6	0.9917	1.0029	0.9933	1.0034	1.0017	0.9954	1.004	1.0035	0.9936	1.0038
13	9	0.9704	0.9987	0.9645	1.0005	0.9947	0.9565	0.993	0.9975	0.9643	1.001
14	9	0.9743	0.9983	0.9952	0.9978	0.9834	0.9616	0.9679	0.9988	0.9903	0.999
15	8	1.0012	1.0003	1.0116	0.9997	0.9998	1.0068	0.9925	1.0012	1.0113	0.9998
16	6	0.9959	1.0007	0.9873	1.0016	1.0051	1.0057	1.0135	0.9991	0.9875	1.001
17	8	1.0009	1.0084	1.0248	1.0062	1.0043	1.0127	0.9905	1.0092	1.0164	1.0031
18	6	1.0047	1.006	0.9816	1.0052	1.0067	0.9898	1.0165	1.0048	0.9781	1.0021
19	9	0.9708	0.999	0.9823	0.9988	0.9901	0.9386	0.9793	0.9986	0.9591	0.9994
20	8	0.9846	0.9986	1.012	0.9993	0.9892	0.9805	0.9695	0.9988	1.011	1.0002
21	6	1.0056	0.9987	0.9957	0.999	1.0025	1.0075	1.016	0.9999	0.9963	0.9994
22	8	1.0104	1.0037	1.0107	1.0031	1.0018	0.9985	0.9927	1.0044	1.0111	1.0011
23	5	1.0259	1.0057	1.03	1.0052	1.0198	1.0361	1.0095	1.0079	1.0281	1.0031
24	5	1.0149	1.0026	0.9973	1.0019	1.0061	1.0197	1.0234	1.0022	0.9973	1.0022
25	5	1.0126	1.0037	1.006	1.0018	1.0084	1.0014	0.9938	1.0033	1.0053	1.002
26	6	1.0108	1.0017	1.0077	1.0017	1.01	1.0164	1.0083	1.0014	1.008	1
27	8	1.0055	1.0049	1.0166	1.0035	1.0057	1.0196	1.0003	1.0045	1.0151	1.0004
28	5	1.0106	1.0065	1.0203	1.0055	1.0145	1.0262	1.0104	1.0069	1.0187	1.0036
29	5	1.0060	1.0123	1.0443	1.0042	1.0331	1.0513	1.0054	1.0277	1.0437	1.0033
30	6	0.9946	1.0007	0.9872	1.0013	1.0441	1.0329	1.0411	1.003	0.9857	1.0011
31	8	0.9995	1.0024	1.0085	1.0015	1.0064	1.0051	0.9933	1.0027	1.0127	1.0015
32	6	1.0041	1.0045	0.9987	1.0024	1.0062	1.0075	1.0113	1.0018	0.998	1.0021

Start Date	Trend Type	Index Return	Stochastic Strategy Return								
			1	2	3	4	5	6	7	8	9
34	6	1.0048	0.9978	0.9998	0.9986	1.006	1.0144	1.0092	0.9989	1.0006	0.999
35	8	0.9987	1.0001	0.9955	0.9988	1.0011	1.0041	1.0015	1.0005	0.996	0.9991
36	5	1.0050	0.9981	0.9989	0.9989	1.0013	0.9989	0.9991	0.9984	0.9986	0.9989
37	5	1.0077	1.0001	1.0002	1.0003	1.0026	0.9931	1	0.999	0.9926	1.0006
38	6	0.9954	0.9983	0.9888	1	0.9975	0.991	0.9996	0.9981	0.988	1.0006
39	8	1.0078	1.0019	1.0096	1.0007	0.9958	1.012	0.9919	1.0011	1.0195	1.0017
40	6	1.0152	0.999	1.0087	1.0008	1.0134	1.0274	1.017	0.9986	1.0082	1.001
41	8	1.0017	0.9939	1.0083	0.9954	0.9942	1.0132	1.0034	0.9947	1.0096	0.9976
42	5	1.0023	0.9975	0.9928	0.9975	0.9952	1.013	1.0105	0.9977	0.9945	0.9986
43	6	1.0001	0.998	0.9908	0.9993	0.9961	0.9889	0.9973	0.9971	0.9908	0.9994
44	9	0.9945	1.0005	0.9941	1.0009	0.9961	0.9938	0.9964	1.0003	0.9944	1.0015
45	8	0.9952	1.0052	0.9963	1.0036	1.0003	0.9946	0.9938	1.0056	1.001	1.0028
46	6	0.9886	1.0064	0.9854	1.0061	1.0039	0.9788	0.9954	1.005	0.9839	1.002
47	9	0.9855	1.0059	1.0095	1.0049	1.0004	0.9951	0.9885	1.0056	1.0064	1.0024
48	8	1.0068	1.0056	1.0119	1.0053	1.009	1.019	1.0063	1.0056	1.0139	1.0039
49	5	1.0169	1.0034	1.0106	1.003	1.0095	1.0218	1.0117	1.0043	1.0093	1.0021
50	6	1.0010	0.9969	0.984	0.9994	0.9941	0.9927	1.0077	0.9969	0.9838	1.0002
51	8	0.9999	0.9999	1.0026	0.9998	0.9921	0.9918	0.9885	1.0007	1.0025	1.0008
52	6	1.0031	1.0083	0.9747	1.0059	1.0053	0.9801	1.0065	1.0051	0.975	1.0035
53	9	0.9922	1.0081	1.0038	1.0088	1.0101	0.9805	0.9853	1.0083	0.9982	1.0054
54	8	0.9995	1.0031	1.0155	1.0033	1.0031	1.0113	0.9977	1.004	1.013	1.0029
55	6	0.9974	0.9968	0.9884	0.9985	0.9968	0.9966	1.0082	0.9963	0.9805	0.999
56	9	0.9911	1.0007	1.0071	0.9997	0.9956	0.9953	0.9916	1.0008	0.9969	0.9993
57	8	1.0027	1.0059	0.9943	1.0047	1.017	1	1.009	1.0069	0.9939	1.0031
58	5	1.0126	1.0061	1.0184	1.0041	1.0133	1.0196	0.9988	1.006	1.0213	1.0027
59	5	1.0150	1.0027	1.012	1.0035	1.0145	1.0292	1.015	1.0034	1.0117	1.002
60	5	1.0099	0.9996	1.0044	1.0015	1.0057	1.0142	1.0097	1	1.0034	1.0011
61	6	1.0017	1.0024	0.9864	1.0018	1.0144	0.9958	1.0025	1.0025	0.9871	1.0007
62	9	0.9863	1.004	0.9793	1.006	1.0199	0.9913	1.0132	1.0044	0.9777	1.0028
63	9	0.9768	1.0027	0.9835	1.003	1.0027	0.9714	0.9898	1.0028	0.9827	1.0019
64	8	0.9930	1.0032	1.0027	1.0029	0.9914	0.9816	0.9808	1.0029	1.0028	1.0014
65	5	1.0147	1.0069	1.0334	1.0031	1.0108	1.0327	1.0012	1.0068	1.0321	1.0012
66	6	0.9898	0.9985	0.9584	0.9999	0.999	0.9881	1.0236	0.9961	0.9583	0.9999
67	9	0.9698	1.0006	0.9944	1.0003	0.9882	0.9541	0.9717	0.9998	0.9794	1.0003
68	8	0.9987	1.0034	1.0135	1.0022	0.9968	1.0025	0.9853	1.0026	1.0098	1.0012
69	5	1.0150	1.0007	1.0088	0.999	1.0114	1.0257	1.0178	1.0007	1.0076	0.9966
70	5	1.0173	0.9984	1.0138	0.9976	1.0109	1.0287	1.0092	0.999	1.0199	0.996
71	5	1.0119	0.9975	1.0083	0.9992	1.0127	1.0255	1.0157	0.9975	1.0084	1.0002

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