Testing the Leggett-Garg Inequality with Solar Neutrinos

by

Mykola Murskyj

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Abstract

One of the fundamental questions in quantum physics is whether measurements reveal pre-existing values. Leggett and Garg derived an inequality that is satisfied by systems featuring both macroscopic realism and non-invasive measurability. The Leggett-Garg Inequality places a quantum-classical limit on a linear combination of correlation functions of a series of measurement outcomes. The flavor oscillations of solar neutrinos provide an interesting way to test the Leggett-Garg inequality, and on an astrophysical length scale. Beginning as electron neutrinos in the solar core, they undergo the Mikheyev-Smirnov-Wolfenstein Effect and exit the sun in either of two mass eigenstates, depending on their energy. Using the neutrino energy to predict the flavor state when it begins traversing the vacuum between the sun and the Earth, we can construct two-time correlation functions of the flavor state at creation, after the MSW effect, and upon detection here on Earth. We can then use these correlation functions to test whether neutrino flavor oscillations obey or violate the Leggett-Garg inequality.

Una delle domande fondamentali della fisica quantistica é se le misurazioni rivelano valori preesistenti. Leggett e Garg hanno derivato una disuguaglianza che viene soddisfatta da sistemi caratterizzati sia dal realismo macroscopico che dalla misurabilitá non invasiva. La disuguaglianza Leggett-Garg pone un limite quantistico-classico su una combinazione lineare di funzioni di correlazione di una serie di risultati di misura. Le oscillazioni di sapore dei neutrini solari forniscono un modo interessante per testare la disuguaglianza Leggett-Garg, e su una scala di lunghezza astrofisica. Utilizzando l'energia dei neutrini per prevedere lo stato di sapore quando comincia ad attraversare il vuoto tra il sole e la Terra, possiamo costruire funzioni di correlazione due volte dello stato di sapore al momento della creazione, dopo l'effetto MSW, e al rilevamento qui sulla Terra. Possiamo poi utilizzare queste funzioni di correlazione per verificare se oscillazioni di sapore dei neutrini obbediscono o violano la disuguaglianza Leggett-Garg.

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CONTENTS

CHAPTER 1

THE LEGGETT-GARG INEQUALITY

1.1 REALISM, LOCALITY, AND MEASUREMENT

1.1.1 THE COPENHAGEN INTERPRETATION

The predictions of quantum mechanics have been verified in a century's worth of experiments, while the theory has been subject to that century's debates over its interpretation. Quantum mechanics rejects many of the assumptions about the external world that were developed in Renaissance philosophy and enshrined in classical mechanics and electrodynamics. No longer universal are the principle of determination (that a system's later state is completely fixed in terms of its previous states), the principle of realism (that objects have fixed states even when no one looks at them), and others.

Bohr believed that quantum mechanics required these mysteries in order to reduce to classical mechanics in the $\hbar \to 0$ limit, and that no more fundamental theory could match its predictions. In his words,

> Far from being a temporary compromise in this dilemma, the recourse to essentially statistical considerations is our only conceivable means of arriving at a generation of the customary way of description sufficiently wide to account for ... the quantum postulates and reducing to classical theory

in the [classical limit].[1]

Einstein, on the other hand, described himself as a suspicious admirer of quantum mechanics, believing that there was a more fundamental theory from which the former might emerge. He, Podolsky, and Rosen presented[2] a classic thought experiment demonstrating that quantum mechanics predicted non-local effects. The EPR paradox is the apparent instantaneous movement of information between entangled particles, in contradiction with the speed limit from relativity. The authors argued that the particles had pre-determined properties that were simply hidden from quantum mechanics – using these hidden variables, one is able to avoid violating locality. Both Bohr and Heisenberg opposed this realism of Einstein's, but for different reasons. While Bohr was interested in metaphysical speculations about the underlying nature of reality, Heisenberg argued – apparently to his mentor's frustration – that what mattered were the outcomes of experiments and nothing more:

> It is possible to ask whether there is still concealed behind the statistical universe of perception a 'true' Universe in which the law of causality would be valid. but such speculation seems to us to be without value and meaningless, for physics must confine itself to the description of the relationship between perceptions.[3]

This view was codified as the Copenhagen Interpretation,[4] which asserts that quantum mechanics is all there is – there are no hidden variables, local or otherwise. The wavefunction is an exhaustive representation of what we can know about a system before a measurement is made. An important corollary of this is that once a measurement is made, the overall wavefunction instantly "collapses" in such a way that probabilities of non-observed outcomes vanish, including at spatially-separated

[1]N. Bohr, *Causality and Complementarity*, Supplementary papers edited by Jan Faye and Henry Folse as *The Philosophical Writings of Niels Bohr, Vol. IV*. Woodbridge: Ox Bow Press, 1998.

[2]A. Einstein, B. Podolsky, and N. Rosen. *Phys. Rev.* **47**, 777 (1935).

[3]W. Heisenberg, "Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik." *Zeitschrift für Physik*, **43** (1927), 172-198.

[4]According to Wikipeida, the term "Copenhagen Interpretation" was coined by Heisenberg in his 1929 lectures at the University of Chicago, as Kopenhagener Geist [spirit] der Quantentheorie.

points. This mysterious, non-local collapse is precisely what unsettled Einstein and elicited objections and skepticism from others.

In the 1960s, Bell derived a set of inequalities that would allow an experimenter to test whether there were any variables hidden from quantum mechanics. Specifically, Bell demonstrated that no locally causal theory could reproduce the predictions of quantum mechanics about spatially separated systems. That is, no theory in which the outcomes of future measurements are already determined by local measurements could match the predictions of quantum mechanics.[5] Each test of Bell's inequality has yielded a violation of the classical bound,[6] prompting us to abandon locality and other reasonable assumptions about the world: we must apparently come to terms with instantaneous wavefunction collapse.

1.1.2 MACROSCOPIC SUPERPOSITIONS?

In the spirit of Bell and EPR, Leggett and Garg ask,[7] To what extent can our classical intuitions guide our view of the microscopic world? Specifically, does our expectation to have a macroscopic object in a definite state, and not in a superposition of states, apply to microscopic systems?

The authors formalized two expectations we have about the macroscopic world: *macroscopic realism* and *noninvasive measurability*. The first says that the outcome of an observation is well-defined before the measurement is made – I'm still speeding down the highway at 75 mph even if neither I nor a police officer make a measurement of the speed. Noninvasive measurability is my expectation that the observation's outcome is not substantially affected by the measurement – the police officer's radar gun will not affect the speed of my car.

Asher Peres wrote that realism has "at least as many definitions as there are au-

[5]J.S. Bell, "On the Einstein Podolsky Rosen Paradox." *Physics* **I**, 195-200 (1964).

[6]Subject to some lingering loopholes; see, for example, J. Gallicchio, A.S. Friedman, and D.I. Kaiser, "Testing Bell's Inequality with Cosmic Photon: Closing the Setting-Independence Loophole." *Phys. Rev. Lett.* **112**, 110405 (2014). arxiv:1310:3288v2.

[7]A.J Leggett and A. Garg, "Quantum Mechanics versus Macroscopic Realism: Is the Flux There when Nobody Looks?" *Phys. Rev. Lett.* **54** 9 (857), 4 March 1985.

thors."[8] We will define realism to be *the independence of an object's properties from external measurement and observation*. On the other hand, the Copenhagen Interpretation, as expressed by what is referred to as Bohr's "indefinability thesis," says (in philosophical jargon) that the

> truth conditions of sentences ascribing a certain kinematic or dynamic value to an atomic object are dependent on the apparatus involved, in such a way that these truth conditions have to include reference to the experimental setup as well as the actual outcome of the experiment.[9]

Classical physics obeys these two assumptions, whereas quantum mechanics does not. Leggett and Garg derived an inequality that is obeyed by theories and worlds satisfying both macroscopic realism and non-invasive measurability.[10] Consider a macroscopic variable that can take on two values, say $Q = \pm 1$, and its two-time correlation function

$$C_{ij} = \langle \hat{Q}(t_i)\hat{Q}(t_j)\rangle, \tag{1.1}$$

where $\hat{Q}(t)$ is the operator corresponding to the observable Q. If macroscopic realism and noninvasive measurability are obeyed, then the Leggett-Garg inequality places classical limits on a combination of the correlations: for three measurements,

$$-3 \leq C_{12} + C_{23} - C_{13} \leq 1, \tag{1.2}$$

or more generally, for an n-measurement Leggett-Garg parameter,

$$\begin{aligned} -n \leq K_n \leq n-2 \quad & n = 3,5,7... \\ -(n-2) \leq K_n \leq n-2, \quad & n = 4,5,6... \end{aligned} \tag{1.3}$$

where $K_n = C_{12} + ... + C_{n-1\,n} - C_{n1}$.

[8] A. Peres, "Quantum Limitations on Measurement of Magnetic Flux," *Phys. Rev. Lett.* **61**, 2019 (1988).

[9] Jan Faye, "The Copenhagen Interpretation of Quantum Mechanics." *Stanford Encyclopedia of Philosophy*, 2014.

[10] This discussion follows the discussion in Emary, Lambert, and Nori, "Leggett-Garg Inequalities." 31 January 2014, arXiv:1304.5133.

To see where this comes from, at least for the case of K_3, note that the correlation function can be written as

$$C_{ij} = \sum_{Q_i, Q_j = \pm 1} Q_i Q_j P_{ij}(Q_i, Q_j), \tag{1.4}$$

where $P_{ij}(Q_i, Q_j)$ is the joint probability of measuring $Q_i = \pm 1$ at time t_i and $Q_j = \pm 1$ at time t_j.[11] The assumption of realism means that the observables Q have definite values throughout the measuring process, so the probability distribution $P_{ij}(Q_i, Q_j)$ is the marginal distribution of some other distribution that includes a third Q:

$$P_{ij}(Q_i, Q_j) = \sum_{Q_k = \pm 1,\ k \neq i,j} P_{ij}(Q_1, Q_2, Q_3). \tag{1.5}$$

So far we have said nothing about the independence of $P_{21}(Q_1, Q_2, Q_3)$, $P_{32}(Q_1, Q_2, Q_3)$, and $P_{31}(Q_1, Q_2, Q_3)$. If we include the assumption of non-invasive measurement, however, we see that each of these distributions must be the same, since the measurement will not affect the subsequent evolution of the system. Thus, the LG-string may be written

$$\begin{aligned} K_3 &= C_{21} + C_{32} - C_{31} \\ &= \sum_{Q_i, Q_j = \pm 1} \left(\sum_{Q_k,\ k \neq 1,2} P(Q_1, Q_2, Q_3) + \sum_{Q_k,\ k \neq 2,3} P(Q_1, Q_2, Q_3) - \sum_{Q_k,\ k \neq 1,3} P(Q_1, Q_2, Q_3) \right). \end{aligned} \tag{1.6}$$

Carrying through the algebra, and using the convexity of probabilities $\sum = 1$, one finds

$$K_3 = 1 - 4\big(P(+1, -1, +1) + P(-1, +1, -1)\big), \tag{1.7}$$

yielding the bounds $K_3 = -3$ for $P(+1, -1, +1) = P(-1, +1, -1) = 1$, and $K_3 = 1$ for $P(+1, -1, +1) = P(-1, +1, -1) = 1$. This yields the Leggett-Garg inequality (LGI),

$$\boxed{-3 \leq C_{12} + C_{23} - C_{13} \leq 1.} \tag{1.8}$$

[11] To see an example of this in practice, *cf.* sec. 3.1.

The LGI can also be proven in terms of hidden-variable theories, but I do not reproduce that demonstration here.[12]

It is worth noting that the correlation functions $C_{ij} = \langle Q_i Q_j \rangle$ are classical quantities, and no vagueness arises from the fact that we do not specific which of Q_i and Q_j happened first. To find the LG parameter K for a quantum system, the correlation functions are set to be equal to their symmetric combination,

$$\mathcal{C}_{ij} = \frac{1}{2}\langle \hat{Q}_i \hat{Q}_j + \hat{Q}_j \hat{Q}_i \rangle, \tag{1.9}$$

where I follow David Kaiser's notes [?] in distinguishing the symmetrized correlation function by using a script font.

Emary, Lambert, and Nori use this description to show that the maximum violation of K_3 is 1.5. They assume that each operator $\hat{Q}_i$ can be written as a linear combination of Pauli matrices.[13] Setting $\hat{Q}_i = \vec{a}_i \cdot \hat{\sigma}$, the authors find that

$$\frac{1}{2}\langle \hat{Q}_i \hat{Q}_j + \hat{Q}_j \hat{Q}_i \rangle = \vec{a}_i \cdot \vec{a}_j \langle \hat{1} \rangle = \vec{a}_i \cdot \vec{a}_j. \tag{1.10}$$

Defining θ_m to be the angle between the vectors a_m and a_{m+1}, the LG parameter K_3 can be written as

$$K_3 = \cos(\theta_1) + \cos(\theta_2) - \cos(\theta_1 + \theta_2). \tag{1.11}$$

The maximum of K_3 occurs for all angles $\theta = \pi/3$,

$$\max(K_3) = 1.5. \tag{1.12}$$

As described in Emary, Lambert, and Nori (2014) and Joe Formaggio's notes of April 27, 2015 [10], LGI violation is closely related to the commutators of the operators

[12]See, for example, Emary, Lambert, and Nori (2014) §2.2, also Dressel *et al.* (2011); J. Kofler and C. Brukner, "Condition for macroscopic realism beyond the Leggett-Garg inequalities." *Phys. Rev. A* **87**, 052115 (2013); and O.J.E. Maroney, Detectability, "Invasiveness and the Quantum Three Box Paradox." arXiv:1207.3114 (2012).

[13]One cannot exactly write weak measurements as a linear combination of Pauli matrices, which is something that ought to be addressed.

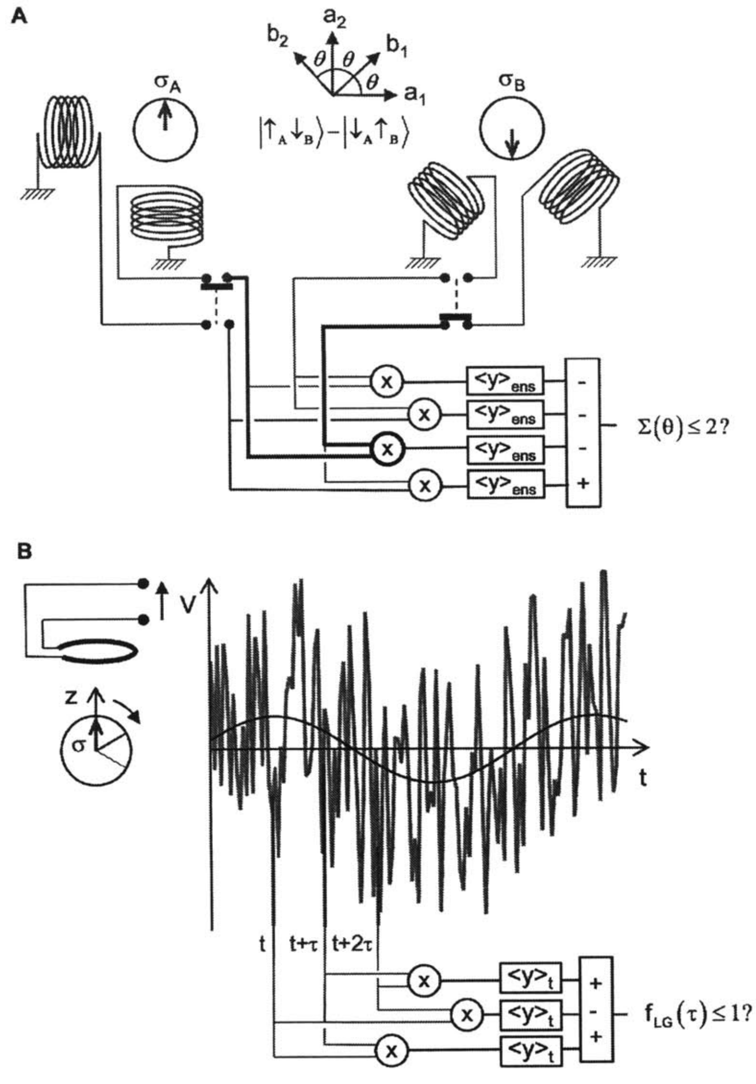

Figure 1-1: An interesting schematic comparing the setup of experiments testing the Bell (above) and Leggett-Garg (below) inequalities. While Bell experiments measure different qubits at spatial separation, Leggett-Garg tests measure the same qubit at different times. Figure taken from Palacios-Laloy *et al.*, (2010).

$\hat{Q}_i$. Setting $\hat{Q}_i = \vec{a}_i \cdot \hat{\sigma}$, one finds

$$\left[\hat{Q}_i, \hat{Q}_j\right] = 2i\hat{\sigma} \cdot (\vec{a}_i \times \vec{a}_j) \tag{1.13}$$

For more on this, see Emary, Lambert, and Nori (2014) §3.1.

Note that there is a clear analogy between the LGI and the Bell inequality. While the latter places a limit on a function of correlations of measurements taken at space-like separated points, the Leggett-Garg inequality limits a function of correlations at different points in time. Sometimes the LGIs are called the temporal Bell inequalities. Both inequalities test realism, coupled with locality in the case of the Bell inequalities, or with non-invasive measurability in the case of the Leggett-Garg inequality.

1.2 Testing the Leggett-Garg Inequality

1.2.1 Experimental Progress

The first experimental violation of a Leggett-Garg inequality was reported by Palacois-Laloy et al.,[14] who studied a superconducting qubit subjected to continuous weak measurements. Since then, violations of the LGI have been found in photon systems,[15] diamond defect centers,[16] NMR systems,[17] and more.

Each of these experimental LGI violations can be circumvented by exploiting a "clumsiness loophole": the observed violation could be explained by some unintentional invasiveness in the measurement scheme.[18] In other words, it might be possible that the excess correlations observed are due not to the absence of superpositions,

[14] A. Palacios-Laloy *et al.*, "Experimental violation of a Bell's inequality in time with weak measurement." *Nat. Phys.* **6**, 442 (2010).

[15] See, for example, Goggin *et al.*, "Violation of the Leggett-Gart inequality with weak measurements of photos." *Proc. Natl. Acad. Sci. USA* **108**, 1256 (2011); J. S. Xu *et al.*, "Experimental violation of the Leggett-Garg Inequality under decoherence." *Sci. Rep.* **1**,101 (2011); J. Dressel *et al.*, "Experimental violation of two-party Leggett-Garg inequalities with Semiweak Measurements." *Phys. Rev. Lett.* **106**. 040402 (2011).

[16] See, for example, Y. Suzuki, P. Neumann, and H.F. Hoffman, "Experimental violation of Leggett-Garg Inequalities in quantum measurements with variable resolution and back-action." *New J. Phys.* **14**, 103022 (2012).

[17] V. Athalye, S. Singha Roy, and T.S. Mahesh, "Investigation of the Leggett-Garg Inequality for Precessing Nuclear Spins." *Phys. Rev. Lett.* **107**, 130402 (2011).

[18] Emary, Lambert, and Nori (2014).

but to interactions between the system and the measuring device. It is impossible to demonstrate that a physical measurement is ideally non-invasive; the best strategy is to exploit ideal negative measurements.[19]

1.2.2 Weak Measurement Schemes

When testing the Leggett-Garg inequality in the lab, a measured violation may be attributed either to the failure of macroscopic realism or to an unwitting invasivity of the measurement. In order to actually probe macroscopic realism, the measurements leading up to the final one must interfere minimally with the system. Leggett and Garg introduced the idea of ideal negative measurements (see footnote above). However, there is some room between projective and negative measurements for *weak* ones.

Weak measurements were first proposed by Aharonov, Albert, and Vaidman in 1988.[20] Their results do not fully distinguish between possible outcomes of the measurement process, providing the experimenter with less information than if she had performed a projective measurement. An adherent of the Copenhagen interpretation would say that weak measurements cause a partial collapse of the wavefunction.

From a classical point of view, weak measurements are made using an ambiguous detector.[21] Suppose the detector has a continuous output variable q as a response to the system variable $Q = \pm 1$, with conditional probability $P(q|Q)$. Then, the expectation value of q can be constructed as

$$\langle q \rangle = \int dq \ \left[qP(q|+)P(+) + P(q|-)P(-) \right], \tag{1.14}$$

[19]An ideal negative measurement is one in which we are certain no interaction with the system took place. For example, suppose we would like to measure the charge of a particle and select the particles with zero charge to undergo further tests. We can set up an electric or magnetic field between the particle source and a screen with a hole at which the source is pointed. If we shoot a particle toward the screen and detect no hit on the screen, the particle has no charge *and it interacted with no field as it travelled,* with this absence of interaction making this process an ideal negative measurement of the particle's neutrality.

[20]Y. Aharonov, D.Z. Albert, and L. Vaidman, "How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100." *Phys. Rev. Lett.* **60**, 1351 (1988).

[21]This section follows §4.1 in Emary, Lambert, and Nori (2014).

where $P(Q)$ is the probability density function of the original variable Q. Note that it is possible for one to measure $\langle q \rangle$ outside the bounds on Q.

The quantum-mechanical analogy of this is implemented using Kraus operators. The expectation value of the variable Q is

$$\langle \hat{Q} \rangle = \text{Tr}\ \hat{\rho}\hat{Q}, \tag{1.15}$$

where $\hat{\rho}$ is the density operator. Performing a weak measurement of $\hat{Q}$ which gives outcome q can be described as causing the system to change from $\hat{\rho}$ to $\mathcal{K}(q)\hat{\rho}\mathcal{K}^\dagger(q)$, where the $\mathcal{K}$ are the Kraus operators associated to outcome q. The example given in Emary, Lambert, and Nori is

$$\mathcal{K}(q) = \left(\frac{2\lambda}{\pi}\right)^{1/4} \exp\left[-\lambda(q - \hat{Q})^2\right], \tag{1.16}$$

where λ parameterizes the strength of the measurement. For example, $\lambda \to 0$ corresponds to Kraus operators $\mathcal{K}(q) \to \mathbb{I}$, a fully weak measurement with no information gain, while $\lambda \to \infty$ corresponds to a projective measurement with a definite outcome.

Note that weak measurements are not fully non-invasive – the ambiguity of the result reflects the weakness of the measurement, while the non-invasivity comes from the fact that a process does not affect the future evolution of the system. Thus, while projective measurements are invasive, weak measurements may or may not be, and to varying degrees. An adherent of the Copenhagen interpretation might say that a weak measurement causes some negligible amount of wavefunction collapse, and so the measurement is minimally invasive. However, this explanation might not be accepted by adherents of macrorealism who do not admit any ideas of wavefunction collapse. Emary, Lambert, and Nori conclude their discussion of this confusion with the worrying statement,

> How to counterfactually assert that the measurement is non-invasive is just as much of a problem with weak measurements as it is with strong ones. This problem seems to have gone unaddressed in the literature.

CHAPTER 2

SOLAR NEUTRINOS

Neutrinos from the sun provide an interesting system in which to probe the Leggett-Garg inequality. To date, all violations of the LGI have been in terrestrial laboratories.

2.1 CREATION OF NEUTRINOS IN THE SUN

Standard Solar Models (SSMs) are founded on four basic assumptions.[1] First, to a good approximation, the Sun balances gravitational forces with outward-directed pressure from thermonuclear reactions. Second, energy is transported throughout the sun by radiation and by convection. Third, the source of most of the Sun's energy is proton fusion via the proton-proton chain (see fig. 2-1),

$$2e^- + 4p \rightarrow^4 \mathrm{He} + 2\nu_e + 26.73\ \mathrm{MeV}. \tag{2.1}$$

And finally, each SSM should give the appropriate solar mass, age, radius, and luminosity.

The different processes in the pp-chain produce neutrinos of different energies. For example, the neutrinos produced by the reaction p + p→ ^{2}H + e$^+$ + ν have a maximum energy of 0.42 MeV, while those produced in the reaction ^{8}B → ^{8}Be + e$^+$ + ν have a

[1] W. C. Haxton et al., "Solar neutrinos: Status and prospects," *Ann. Rev. Astron. Astrophys.* **51** 21 (2013). arXiv:1208.5723 [astro-ph.SR].

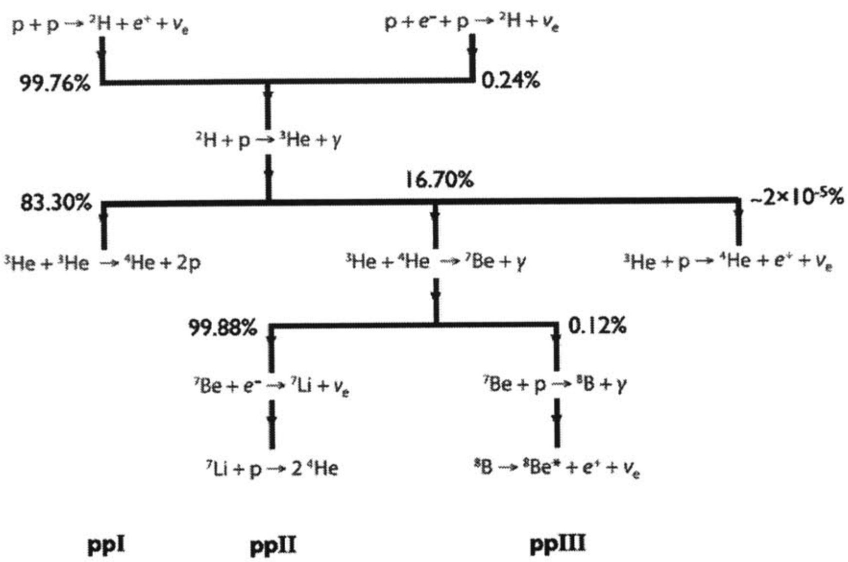

Figure 2-1: Diagram showing the main three branches of the pp-chain, denoted pp I, pp II, and pp III. Each of the three branches produces neutrinos with different energies, with a minor branch (shown furthest to the right) producing the highest energy neutrinos. Figure originally from Haxton, Robertson, and Serenelli (2013).

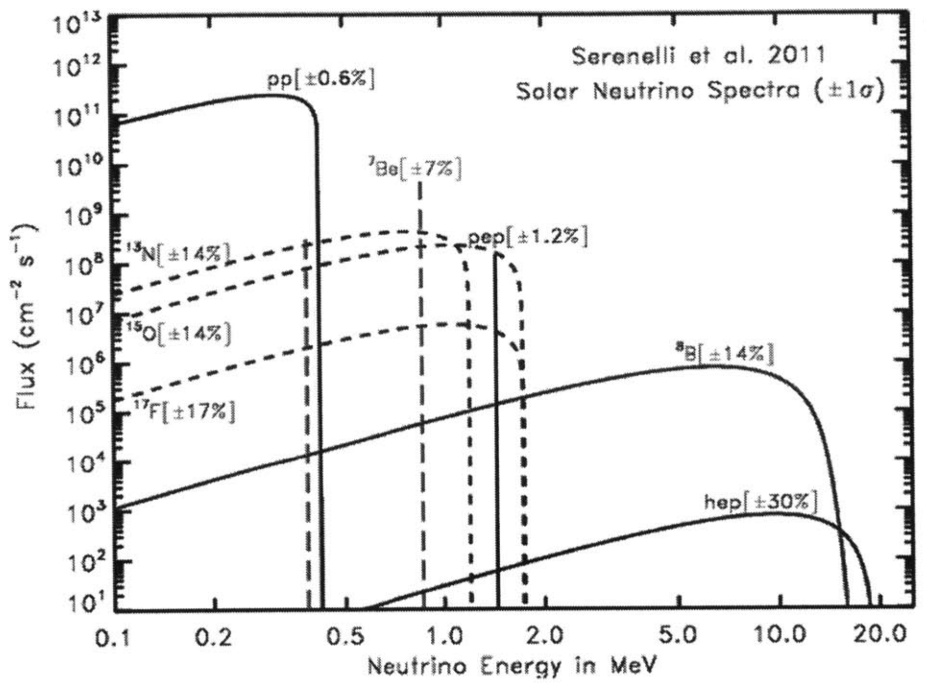

Figure 2-2: Solar neutrino spectrum. Figure originally from Serenelli, Haxton, and Pena-Garay (2011).

maximum energy of around 15±1 MeV. Figure 2-2 shows the solar neutrino spectrum together with uncertainties as of 2011.

2.2 Flavor Oscillations

Experiments have shown that neutrino mass/energy eigenstates $|\nu_1\rangle$, $|\nu_2\rangle$, $|\nu_3\rangle$ are not the same as the weak interaction states $|\nu_e\rangle$, $|\nu_\mu\rangle$, $|\nu_\tau\rangle$. The Maki-Nakagawa-Sakata (MNS) matrix gives the mixing between these two bases, $|\nu_\alpha\rangle = \sum_i U^*_{\alpha i} |\nu_i\rangle$, where α indexes flavor states e, μ, and τ.[2]

The theory of neutrino oscillations is founded on three main ideas.[3]

[2]This section is based on the discussion in Duan, Fuller, Qian, "Collective Neutrino Oscillations," Ann. Rev. Nucl. Part. Sci. 60:569 (2010) and on Mehta, "Topological Phase in Two Flavor Neutrino Oscillations," arxiv:0901.0790v2.

[3]C. Giunti and C. Kim, *Fundamentals of Neutrino Physics and Astrophysics.* Oxford: Oxford Univer-

1. Neutrinos that are produced or detected via charged-current interactions are given by flavor states $|\nu_\alpha\rangle$.

2. Each massive component of a flavor neutrino with momentum $\vec{p}$ has that same momentum $\vec{p}$ – that is, all the mass and flavor components propagate in the same direction.

3. The propagation time is equal to the distance the neutrino travels – while unjustified in the case of a plane-wave approximation of neutrino oscillations, the group velocity of the wave packet is close enough to the velocity of light.

Consider a model with a 2-dimensional neutrino flavor space, spanned by either $(|\nu_1\rangle, |\nu_2\rangle)$ or by $(|\nu_e\rangle, |\nu_\mu\rangle)$, where $|\nu_\mu\rangle$ is some combination of ν_μ and ν_τ. Assuming that general-relativistic effects are unimportant, the flavor state ψ obeys

$$i\frac{d\psi}{d\tau} = \mathbb{H}\psi, \tag{2.2}$$

where τ is the distance along the world line of the neutrino and

$$\mathbb{H} = \left(p + \frac{\delta m^2}{4p} + \frac{1}{2}V_C + V_N\right)\mathbb{I} + \frac{1}{2}\begin{pmatrix} V_C - \omega\cos 2\theta & \omega\sin 2\theta \\ \omega\sin 2\theta & -\left(V_C - \omega\cos 2\theta\right) \end{pmatrix}. \tag{2.3}$$

Here, $p \sim E$ is the momentum of the neutrino, θ is the vacuum mixing angle, δm^2 is the difference of the masses-squared, ω is $\delta m^2/2p$, V_C is the effective potential due to "coherent forward scattering of neutrinos with electrons," through charged current interaction ($V_C = \sqrt{2}G_F n_e$), V_N is the effective potential due to "coherent forward scattering of neutrinos with neutrons," through neutral current interactions ($V_N = \sqrt{2}G_F n_n$).[4]

Note that there is a value of V_C for which the diagonal elements of the mixing matrix vanish, and for which flavor oscillations occur regardless of what $\theta \neq 0$ is. This resonance gives the Mikheyev-Smirnov-Wolfenstein (MSW) Effect. On the other

sity Press, 2007, §8.1.

[4]Duan, Fuller, Qian (2010).

hand, note that if the vacuum mixing angle vanishes, no flavor mixing can occur (in vacuum or in matter). Setting the matter terms $V_C = V_N = 0$ returns the vacuum case,

$$\mathbb{H} = \left(p + \frac{\delta m2}{4p}\right)\mathbb{I} + \frac{1}{2}\begin{pmatrix} -\omega\cos 2\theta & \omega\sin 2\theta \\ \omega\sin 2\theta & \omega\cos 2\theta \end{pmatrix}. \tag{2.4}$$

2.2.1 Transition Probabilities

This section follows the discussion in Bilenky, Giunti, and Grimus (1999).[5] First we discuss the general setup for neutrino flavor transitions, and then specialize to the vacuum and matter cases.

Nuetrino oscillations, generated by the interference of different massive neutrinos, were first suggested by Pontecorvo in the 1950s.[6] Suppose there are two generations of massive neutrinos. Let α index flavor states $\nu_e,\ \nu_\mu$, and k the mass eigenstates $\nu_1,\ \nu_2$. The flavor state ν_α can be written in terms of the mass states ν_k as

$$|\nu_\alpha\rangle = \sum_{k=1}^{2} U^*_{\alpha k}\,|\nu_k\rangle\,, \tag{2.5}$$

where $U_{\alpha k}$ is a unitary mixing matrix. Since the mass eigenstates obey the Schrödinger equation,

$$|\nu_\alpha(t)\rangle = \sum_{k=1}^{2} U^*_{\alpha k}\,|\nu_k(t)\rangle = \sum_{k=1}^{2} U^*_{\alpha k}e^{-iE_k t}\,|\nu_k\rangle\,. \tag{2.6}$$

In detectors, neutrinos are measured in their flavor basis, $|\nu_\beta\rangle$, where β indexes $\nu_e,\ \nu_\mu$. That is,

$$|\nu_\alpha(t)\rangle = \sum_{\beta} \mathcal{A}_{\alpha\to\beta}(t)\,|\nu_\beta\rangle\,, \tag{2.7}$$

where

$$\mathcal{A}_{\alpha\to\beta}(t) = \sum_{k=1}^{2} U_{\beta k}e^{-iE_k t}U^*_{\alpha k}. \tag{2.8}$$

[5] S.M. Bilenky, C. Giunti, and W. Grimus, "Phenomenology of Neutrino Oscillations." *Prog. Part. Phys.* **43** (1999), 1-86. hep-ph/9812360v4.
[6] B. Pontecorvo, *Sov. Phys. JETP* **6** 429, 1957.

The transition probability is then the square of the amplitude:

$$P(\nu_a \to \nu_\beta) = \left| \sum_{k=1}^{2} U_{\beta k} e^{-iE_k t} U^*_{\alpha k} \right|^2 \tag{2.9}$$

The same relationship applies for antineutrinos but with U replaced by U^*, so

$$\mathcal{A}_{\bar{\alpha}\to\bar{\beta}}(t) = \sum_{k=1}^{2} U^*_{\beta k} e^{-iE_k t} U_{\alpha k}. \tag{2.10}$$

The unitarity of U gives, for an n-generation neutrino space,

$$P(\nu_a \to \nu_\beta) = \left| \delta_{\alpha\beta} + \sum_{k=2}^{n} U_{\beta k} U^*_{\alpha k} \left(e^{-iL\delta m^2_{k1}/2E} - 1 \right) \right|^2, \tag{2.11}$$

where $L \approx t$ in the ultrarelativistic approximation. For $n = 2$, this is

$$P(\nu_a \to \nu_\beta) = \left| \delta_{\alpha\beta} + U_{\beta 2} U^*_{\alpha 2} \left(e^{-i\omega L} - 1 \right) \right|^2, \tag{2.12}$$

where I used $\omega = \delta m^2/2p$. The first thing to note is that if the mixing matrices are the identity matrix, or if $\omega L \ll 1$, then no flavor transitions can occur (that is, $P(\alpha \to \beta) = \delta_{\alpha\beta}$). In order to get flavor transitions, we need to have non-trivial mixing matrices and also satisfy $\omega L \gtrsim 1$. Further, note that comparing the expressions for the amplitudes shows that

$$P(\alpha \to \beta) = P(\bar{\alpha} \to \bar{\beta}) \tag{2.13}$$

This is an expression of CPT symmetry / Lorentz invariance.

Vacuum Flavor Transitions

Consider the transition probability for the case of two neutrino generations,[7] given above by

$$P(\nu_a \to \nu_\beta) = \left| \delta_{\alpha\beta} + U_{\beta 2} U^*_{\alpha 2} \left(e^{-i\omega L} - 1 \right) \right|^2. \tag{2.14}$$

[7] Giunti and Kim (2007).

Let's parameterize the mixing matrix by $U_{\alpha 2} = \sin\theta$ (and so $U_{\beta 2} = \cos\theta$), giving

$$P(\nu_\alpha \to \nu_\beta) = \frac{1}{2}\sin^2(2\theta)\,(1 - \cos(\omega L)) \qquad (\alpha \neq \beta) \tag{2.15}$$

$$= \sin^2(2\theta)\sin^2(\omega L) \qquad (\alpha \neq \beta) \tag{2.16}$$

$$P(\nu_\alpha \to \nu_\alpha) = 1 - P(\nu_\alpha \to \nu_\beta) \tag{2.17}$$

$$= 1 - \frac{1}{2}\sin^2(2\theta)\,(1 - \cos(\omega L)) \tag{2.18}$$

More concretely,

$$P(\nu_\alpha \to \nu_\beta) = \frac{1}{2}\sin^2(2\theta)\left[1 - \cos\left(2.53\,\frac{\delta m^2}{\mathrm{eV}^2}\,\frac{L}{\mathrm{m}}\,\frac{\mathrm{MeV}}{E}\right)\right] \qquad (\alpha \neq \beta) \tag{2.19}$$

The amplitude of the oscillation is given by $\sin^2 2\theta = 4(U^\dagger U)^2$, and the oscillation length is

$$L^{\mathrm{osc}} = \frac{4\pi E}{\delta m^2} = 2.5 \cdot 10^{11}\ \mathrm{m} \sim \text{ 1.6 times Earth-sun distance.} \tag{2.20}$$

Neutrino oscillations can be observed if the oscillation length is not much larger than the source-detector length. In other words, the condition for observable oscillations $\delta m^2 \gtrsim E/L$ becomes $L \gtrsim L^{\mathrm{osc}}$, which is conveniently true in the case of the Earth-sun distance $1.5 \cdot 10^{11}$ m.

Experiments detecting neutrino oscillations can only detect the energy-averaged transition probability. So at large L/E the observable transition probability reduces to the constant $1 - \frac{1}{2}\sin^2 2\theta$, and the oscillatory behavior of the probability is enveloped as shown in fig. 2.2.1.

MATTER FLAVOR TRANSITIONS

Wolfenstein, as well as Mikheyev and Smirnov, pointed out that neutrino oscillations can be "significantly modified by the passage of neutrinos through matter because of the effect of coherent forward scattering."[8] Again limiting ourselves to the case of two

[8] Language from Bilenky (1998).

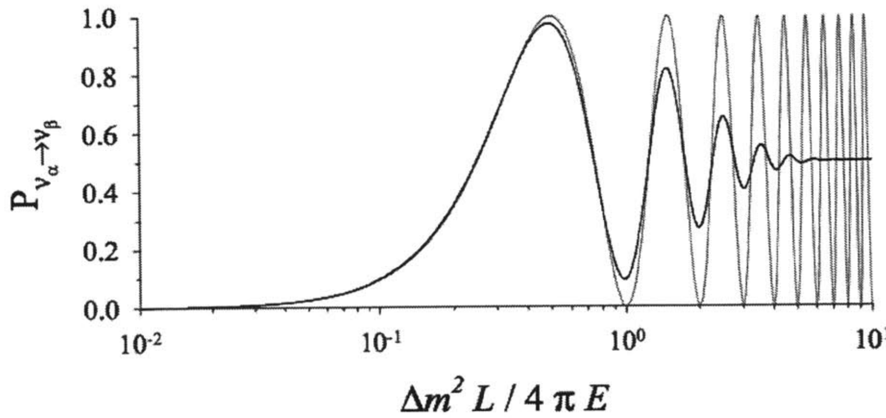

Figure 2-3: Transition probabilities for vacuum oscillations. The dark black line takes into account averaging over a Gaussian energy spectrum with mean E and standard deviation $E/10$. Figure from hep-ph/9812360.

neutrino generations, let $\psi = (a_\alpha \, a_\beta)^T$, so that the evolution equation reads

$$i\frac{d}{d\tau}\begin{pmatrix} a_\alpha \\ a_\beta \end{pmatrix} = \mathbb{H}\begin{pmatrix} a_\alpha \\ a_\beta \end{pmatrix}, \tag{2.21}$$

where $\mathbb{H}$ is given at the beginning of this section.

The sun does not have constant density, but for simplicity let's first consider a constant-density matter distribution $N_e =$ const. With the benefit of hindsight, we define the matter mixing angle to be

$$\sin 2\theta_m = \frac{\tan 2\theta}{[(1 - N_e/N_e^{\text{res}})^2 + \tan^2 2\theta]^{\frac{1}{2}}}, \tag{2.22}$$

where the "resonance density"

$$N_e^{\text{res}} \equiv \frac{\delta m^2 \cos 2\theta}{2\sqrt{2} E G_F}. \tag{2.23}$$

Then the transition probability can be shown to have the same form as the vacuum case,[9]

$$P(\nu_e \to \nu_\mu) = \sin^2 2(\theta_m) \sin^2(\omega L) = \sin^2 2(\theta_m) \sin^2(L/2L_m), \tag{2.24}$$

where L_m is the oscillation length in matter, defined by

$$L_m = 2\pi \frac{2E/\delta m^2}{\sqrt{(1 - N_e/N_e^{\text{res}})^2 \cos^2 2\theta + \sin^2 2\theta}}. \tag{2.25}$$

This is what we find for constant-density matter distributions. Now suppose that $N_e = N_e(t)$, where $t - t_0 = \tau$ is the distance travelled from the core of the sun towards the surface. In the case of the sun, the density profile is very close to exponential:

$$N_e(t) = N_e(t_0) \exp\left(-\frac{t - t_0}{r_0}\right), \tag{2.26}$$

where $r_0 = 0.1\ R_{\text{sun}}$. The change in the matter mixing angle is given by

$$\frac{d\theta_m}{d\tau} = \frac{1}{2} \frac{\sin 2\theta_m}{\delta m^2} \frac{dV_C}{d\tau}, \tag{2.27}$$

where V_C is the effective potential induced by charged current interations. Recall that the MSW resonance occurs for $V_C^{\text{res}} = \delta m^2 \cos 2\theta$ (where θ is the vacuum mixing angle). In this case, one can show[10] that in the matter basis, the evolution equation becomes

$$i\frac{d}{d\tau} \begin{pmatrix} a_1 \\ a_1 \end{pmatrix} = \frac{1}{4E} \begin{pmatrix} -\delta m^2 & -4Ei\frac{d\theta_m}{d\tau} \\ 4Ei\frac{d\theta_m}{d\tau} & \delta m^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{2.28}$$

The terms involving $d\theta_m/d\tau$ induce transitions between matter eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$. If these terms, however, are smaller than δm^2, matter eigenstates are unlikely to flip. To distinguish between these cases, we can define an adiabaticity parameter

$$\gamma(\tau) = \frac{\delta m^2}{4E|d\theta_m/d\tau|} = \frac{(\delta m^2)^2}{2E\sin(2\theta_m)|dV_C/d\tau|}. \tag{2.29}$$

The adiabatic case occurs if $\gamma(\tau) \gg 1$, and transitions between the matter eigenstates

[9] J. Beringer *et al.* (PDG), PR **D86**, 010001 (2012).
[10] Giunti and Kim (2007) §9.3

are negligible. In this case, the $|\nu_1\rangle$ and $|\nu_2\rangle$ evolve independently, and

$$\begin{aligned} a_1(\tau) &= \exp\left(i\int_0^\tau \frac{\delta m^2(\tau')}{4E}d\tau'\right) a_1(0) \\ a_2(\tau) &= \exp\left(-i\int_0^\tau \frac{\delta m^2(\tau')}{4E}d\tau'\right) a_2(0). \end{aligned} \tag{2.30}$$

Then Giunti and Kim show that the survival probability for electron neutrinos is

$$P_{ad.}(\nu_e \to \nu_e) = \frac{1}{2}+\frac{1}{2}\cos(2\theta_m^{\text{initial}})\cos(2\theta_m^{\text{final}})+\frac{1}{2}\sin(2\theta_m^{\text{initial}})\sin(2\theta_m^{\text{final}})\cos\left(\int_0^\tau \frac{\delta m^2(\tau')}{4E}d\tau'\right). \tag{2.31}$$

The energy-averaged survival probability in this case is

$$\langle P(\nu_e \to \nu_e)\rangle_E = \frac{1}{2} + \frac{1}{2}\cos(2\theta_m^{\text{initial}})\cos(2\theta). \tag{2.32}$$

Now let's consider the non-adiabatic case: the off-diagonal terms in eq. 2.28 are significant enough to cause transitions between $|\nu_1\rangle$ and $|\nu_2\rangle$. The transitions are maximized at the minimum of the adiabaticity parameter $\gamma(\tau)$; this point is called the *maximum violation of adiabaticity* or MVA. This point occurs when

$$\left[\frac{d^2\cos(2\theta_m)}{d\tau^2}\right]_{\tau=\tau_{MVA}} = 0. \tag{2.33}$$

In terms of the effective potential, this is[11]

$$\left[3\cos(2\theta_m)\sin(2\theta_m)\left(\frac{dV_C}{d\tau}\right)^2 + \delta m^2\sin(2\theta)\frac{d^2V_C}{d\tau^2}\right]_{\tau=\tau_{MVA}} = 0. \tag{2.34}$$

So in the case of solar neutrinos, we imagine electron neutrinos traveling from a high-density region through matter of decreasing density. If the neutrinos cross the resonance point τ_R non-adiabatically, then transitions between the mass eigenstates can occur. One can show that the amplitudes of the two mass eigenstates at any point

[11]Giunti and Kim (2007) §9.3.

after the non-adiabatic resonance crossing are

$$a_1(\tau) = \left[\cos(\theta_m^{\text{initial}}) e^{i\Gamma(\tau_R)} \mathcal{A}_{11}^R + \sin(\theta_m^{\text{initial}}) e^{-i\Gamma(\tau_R)} \mathcal{A}_{21}^R\right] exp\left(i \int_{\tau_R}^{\tau} \frac{\delta m^2(\tau')}{4E} d\tau'\right)$$
$$= \left[\cos(\theta_m^{\text{initial}}) e^{i\Gamma(\tau_R)} \mathcal{A}_{12}^R + \sin(\theta_m^{\text{initial}}) e^{-i\Gamma(\tau_R)} \mathcal{A}_{22}^R\right] exp\left(i \int_{\tau_R}^{\tau} \frac{\delta m^2(\tau')}{4E} d\tau'\right), \quad (2.35)$$

where we defined, for convenience,

$$\Gamma(\tau_R) = \int_0^{\tau_R} \frac{\delta m^2(\tau')}{4E} d\tau', \quad (2.36)$$

and where $\mathcal{A}_{ij}^R$ is the amplitude associated to the transition between mass eigenstates $\nu_k \to \nu_j$ at the resonance.

After doing all the calculations, the Parke formula gives the average survival probability,

$$\langle P(\nu_e \to \nu_e) \rangle_E = \frac{1}{2} + \left(\frac{1}{2} - P_c\right) \cos(2\theta_m^{\text{initial}}) \cos(2\theta), \quad (2.37)$$

where P_c is the crossing probability, which depends on the matter density profile. For an exponential distribution $N_e(\tau) \sim e^{-\tau}$,

$$P_c = \frac{\exp\left(-\pi(1 - \tan^2\theta)\,\gamma_R/2\right) - \exp\left(-\pi\gamma_R(1 - \tan^2\theta)/(2\sin^2\theta)\right)}{1 - \exp\left(-\pi\gamma_R(1 - \tan^2\theta)/(2\sin^2\theta)\right)}, \quad (2.38)$$

where $\gamma_R = \gamma(\tau_R)$ is the adiabaticity parameter at resonance crossing.

2.3 Energy-Dependent Transition Probabilities

The probability that a neutrino will emerge from the sun as an electron-flavor neutrino is a function of its energy. For $E < 1$ MeV, 35% of neutrinos change flavor while traveling from the core of the sun to its outer limb, as a result of the MSW effect. On the other hand, for $E > 10$ MeV, 70% of neutrinos change flavor. Thus, the neutrinos' energy can shed light on the flavor state of neutrinos that exit the sun. See fig. 2-4.

Note that as neutrinos travel from the sun to terrestrial detectors, the energy state and the flavor state evolve independently: the only "entanglement" between the

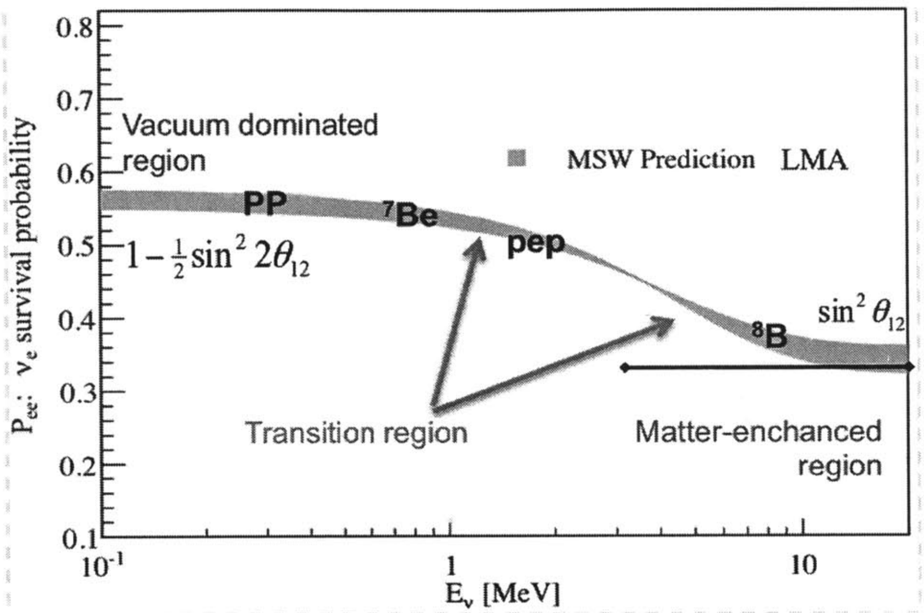

Figure 2-4: The central curve is the prediction of the MSW effect. Figure from ref. [13].

energy and flavor states occurs within the sun. In fact, the energy of the neutrino doesn't change at all as it travels to the Earth. Further, the energy of the neutrino and its flavor are measured independently by detectors. So the energy can tell us about the flavor state of the neutrino when it exited the sun.

Recall that the Leggett-Garg inequality involves the two-time correlation functions of a dichotomic variable. We define the dichotomic variable Q as corresponding to neutrino flavor, so that its eigenvalues are $\hat{Q}\,|\nu_e\rangle = |\nu_e\rangle$ and $\hat{Q}\,|\nu_\mu\rangle = -\,|\nu_\mu\rangle$. We define the set $(|\nu_e\rangle\,, |\nu_\mu\rangle)$ to be an orthonormal basis for the flavor space, so that

$$\langle\nu_e\,|\nu_e\rangle = 1 \qquad \langle\nu_\mu\,|\nu_\mu\rangle = 1 \qquad \langle\nu_e\,|\nu_\mu\rangle = 0. \tag{2.39}$$

Note that this means that a third neutrino flavor is not included in our discussions; rather, it is absorbed into the definition of $|\nu_\mu\rangle$.

The quantum circuit corresponding to this process is shown in fig. 2-5. For the sake of keeping the analogy going, let's define the energy state $E = |L\rangle$ as corresponding to $E < 1$ MeV, and $E = |H\rangle$ as corresponding to $E > 10$ MeV. Then we have a CNOT gate:

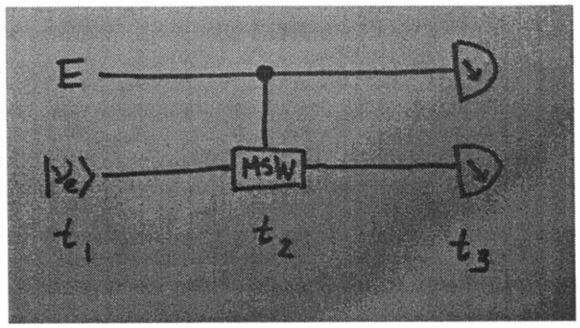

Figure 2-5: Quantum circuit illustrating how we might use the energy of neutrinos to probe their flavor state as they exit the sun. Note that this is not a weak measurement scheme: the controlled operation is reversed. Rather than controlling *on* the target system, the flavor ket is flipped controlled on the energy state. At time t_3 the flavor and energy states are independently projectively measured, and *the result of the "now" energy measurement tells us about what a flavor measurement at t_2 would have been.*

the flavor vector flips from $|\nu_e\rangle$ to $|\nu_\mu\rangle$, and vice versa, if $E = |H\rangle$, but is conserved if $E = |L\rangle$. So this version of $\hat{Q}_2$ is *not* a weak measurement, but nor is it a projective measurement. The advantages to this idea are that it's significantly simpler than a full implementation of the weak measurement formalism. Second, it's adjustable: we can change the number of cuts as necessary to maximize our test of the LGI.

CHAPTER 3

TESTING THE LGI WITHOUT WEAK MEASUREMENT

3.1 TOY MODEL

In this section we present an idealized version of the neutrino's evolution from its creation in the sun's core, through the sun and the vacuum, towards the detectors here on Earth. For now, let's assume that transition probabilities are constant in both energy bins, and that the MSW effect acts as a 'flip' of the flavor state, controlled on energy: at some fixed point along the sun-earth ray, the CNOT-like operation described in §2.2.2 takes place. In this section, we assume that the CNOT flip is ideal: *each* low-energy neutrino remains electron-flavored and *each* high-energy neutrino is flipped to muon-flavor.

3.1.1 DERIVATION OF K

Let's introduce the notation from David Kaiser's notes dated 17 April 2015 [15]. We are interested in a flavor-observable Q and define $\hat{Q}\,|\nu_e\rangle = +1$, $\hat{Q}\,|\nu_\mu\rangle = -1$. The flavor kets form an orthonormal basis, so that $\langle\nu_\alpha|\nu_\beta\rangle = \delta_{\alpha\beta}$, where $\alpha,\ \beta =$ e, μ.

The solar core radiates electron-flavor neutrinos with some energy distribution.

We can write the initial state of the neutrinos at $t = t_1$ as

$$|t_1\rangle = a\,|L\rangle\,|\nu_e\rangle + b\,|H\rangle\,|\nu_e\rangle\,, \tag{3.1}$$

where $|L, H\rangle$ gives the energy (low or high) of the neutrino, and $|a|^2+|b|^2 = 1$. Upon exiting the sun at $t = t_2$, assuming a perfect CNOT operation, the high-energy neutrinos all flip flavor to muon neutrinos:

$$|t_2\rangle = a\,|L\rangle\,|\nu_e\rangle + b\,|H\rangle\,|\nu_\mu\rangle\,. \tag{3.2}$$

Finally, the neutrinos undergo vacuum oscillations as they travel to the Earth, arriving in the state

$$|t_3\rangle = a\,|L\rangle\,(c\,|\nu_e\rangle + d\,|\nu_\mu\rangle) + b\,|H\rangle\,(e\,|\nu_e\rangle + f\,|\nu_\mu\rangle)\,. \tag{3.3}$$

The new coefficients c - f depend on the neutrino energy, vacuum mixing angle, and mass splitting.

To find the LG parameter, we must calculate the two-time correlation functions C_{ij}, which are defined in ref. [15] by

$$\begin{aligned} C_{ij} &= \sum_{\alpha,\beta} \alpha(t_i)\beta(t_j)P_{\alpha\beta}(t_i, t_j) \\ &= \sum_{\alpha,\beta} \alpha(t_i)\beta(t_j)|\langle t_i|\nu_\alpha\rangle\langle\nu_\beta|t_j\rangle|^2. \end{aligned} \tag{3.4}$$

Note that in general we should be considering $\mathcal{C}_{ij} = C_{[ij]}$, but, as we will show later, it appears as though these two quantities are the same.

For now, however, we find that

$$\begin{aligned} C_{12} &= |\langle t_1|\nu_e\rangle\langle\nu_e|t_2\rangle|^2 - |\langle t_1|\nu_\mu\rangle\langle\nu_e|t_2\rangle|^2 - |\langle t_1|\nu_e\rangle\langle\nu_\mu|t_2\rangle|^2 + |\langle t_1|\nu_\mu\rangle\langle\nu_\mu|t_2\rangle|^2 \\ &= |\langle t_1|\nu_e\rangle\,\langle\nu_e|\,t_2\rangle|^2 - |\,\langle t_1|\,\nu_e\rangle\,\langle\nu_\mu|\,t_2\rangle|^2 \qquad (3.5) \\ &= |(a^*\,\langle L| + b^*\,\langle H|)a\,|L\rangle\,|^2 - |(a^*\,\langle L| + b^*\,\langle H|)b\,|H\rangle\,|^2 \qquad (3.6) \end{aligned}$$

$$
\begin{aligned}
&= |a^*a|^2 - |b^*b|^2 \\
&= 2|a|^4 - 1
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
C_{13} &= |\langle t_1|\nu_e\rangle\langle\nu_e|t_3\rangle|^2 - |\langle t_1|\nu_\mu\rangle\langle\nu_e|t_3\rangle|^2 - |\langle t_1|\nu_e\rangle\langle\nu_\mu|t_3\rangle|^2 + |\langle t_1|\nu_\mu\rangle\langle\nu_\mu|t_3\rangle|^2 \\
&= |(a^* \langle L| + b^* \langle H|)(ac\,|L\rangle + be\,|H\rangle)|^2 - |(a^* \langle L| + b^* \langle H|)(ad\,|L\rangle + bf\,|H\rangle)|^2 \\
&= |a^*ac + b^*be|^2 - |a^*ad + b^*bf|^2 \\
&= |a|^4(|c|^2 - |d|^2) + |b|^4(|e|^2 - |f|^2) + |a|^2|b|^2(c^*e + ce^* - d^*f - df^*),
\end{aligned} \tag{3.8}
$$

where we may define an interference term $I = c^*e + ce^* - d^*f - df^*$. If each of these complex numbers is written $c = |c|e^{i\theta_c}$, then we can see that I is real:

$$
\begin{aligned}
I &= |ce| \left(e^{i\theta_e - i\theta_c} + e^{i\theta_c - i\theta_e}\right) - |df| \left(e^{i\theta_f - i\theta_d} + e^{i\theta_d - i\theta_f}\right) \\
&= 2|ce| \cos(\theta_e - \theta_c) - 2|df| \cos(\theta_d - \theta_f) \\
&\equiv 2|ce| \cos(\theta_{ec}) - 2|df| \cos(\theta_{df}).
\end{aligned} \tag{3.9}
$$

The final correlation function is

$$
\begin{aligned}
C_{23} &= |\langle t_2|\nu_e\rangle\langle\nu_e|t_3\rangle|^2 - |\langle t_2|\nu_\mu\rangle\langle\nu_e|t_3\rangle|^2 - |\langle t_2|\nu_e\rangle\langle\nu_\mu|t_3\rangle|^2 + |\langle t_2|\nu_\mu\rangle\langle\nu_\mu|t_3\rangle|^2 \\
&= |a^* \langle L|\, ac\, |L\rangle\,|^2 - |b^* \langle H|\, be\, |H\rangle\,|^2 - |a^* \langle L|\, ad\, |L\rangle\,|^2 + |b^* \langle H|\, bf\, |H\rangle\,|^2 \\
&= |a|^4(|c|^2 - |d|^2) + |b|^4(|f|^2 - |e|^2)
\end{aligned} \tag{3.10}
$$

Then the Leggett-Garg parameter K is given by

$$
K = 2|a|^2 - 1 + 2|b|^4(|f|^2 - |e|^2) - |a|^2|b|^2 I. \tag{3.11}
$$

After rearranging and using $|a|^2 + |b|^2 = 1$, the full LGI reads

$$
\boxed{-1 \leq |a|^2 + \left(1 - |a|^2\right)^2 \left(|f|^2 - |e|^2\right) - \frac{1}{2}|a|^2(1 - |a|^2)\left(2|ce| \cos(\theta_{ec}) - 2|df| \cos(\theta_{df})\right) \leq 1} \tag{3.12}
$$

3.1.2 Application to Neutrinos

So far the above has been completely general and not taken into account anything we know about neutrino physics or even two-level systems. Before going any further, let's consider what we can learn from the fact that the neutrinos' energies don't change as they travel through the sun and vacuum to the Earth.

The probability of finding a low-energy neutrino immediately after its creation in the solar core is

$$\begin{aligned} P(L,t_1) &= |\langle t_1|L\rangle\langle L|t_1\rangle|^2 \\ &= |a^* \langle \nu_e| \langle L| |L\rangle \langle L| a |L\rangle |\nu_e\rangle |^2 \\ &= |a^* \langle \nu_e| a |\nu_e\rangle |^2 = |a|^4 \end{aligned} \tag{3.13}$$

I claim that the fact that neutrinos' energies are constant implies that the probability of a *randomly chosen* neutrino to be in a given energy bin remains constant as well. Thus, $P(L,t_1)$ should be equal to $P(L,t_3)$, which is

$$\begin{aligned} P(L,t_3) &= |\langle t_3|L\rangle\langle L|t_3\rangle|^2 \\ &= |\left(a^*c^* \langle L| \langle \nu_e| + a^*d^* \langle L| \langle \nu_\mu|\right) |L\rangle \langle L| \left(ac |L\rangle |\nu_e\rangle + ad |L\rangle |\nu_\mu\rangle\right)|^2 \\ &= |a^*a\, c^*c + a^*a\, d^*d|^2 \\ &= |a|^4 \left(|c|^2 + |d|^2\right)^2 \end{aligned} \tag{3.14}$$

Requiring $P(L,t_1) = P(L,t_3)$ means that $|c|^2 + |d|^2 = 1$. Similarly, $P(H,t_1) = P(H,t_3)$ means requiring that $|e|^2 + |f|^2 = 1$.

Using $|e|^2 + |f|^2 = 1$ and $|c|^2 + |d|^2 = 1$ in eq. 3.11 gives

$$\begin{aligned} K = &4\left(a^2-1\right)a^2\left(\cos(\theta_{ce})\text{Abs}\left(\sqrt{1-d^2}\sqrt{1-f^2}\right) - |df|\cos(\theta_{df})\right) \\ &+ 2\left(a^2-1\right)^2\left(2f^2-1\right) + 2a^2 - 1 \end{aligned} \tag{3.15}$$

3.1.3 LGI Violation

In the toy model, without specifying if we're working with a neutrino or a photon or other system, and dropping the absolute value signs for simplicity, the LGI is

$$-3 \leq -1 + 2a^2 + 2(1-a^2)^2(f^2 - e^2) - a^2b^2\left(2|ce|\cos(\theta_{ec}) - 2|df|\cos(\theta_{df})\right) \leq 1 \quad (3.16)$$

From the normalization of the state at t_3, we know that

$$a^2c^2 + b^2e^2 + a^2d^2 + b^2f^2 = 1. \quad (3.17)$$

Plugging this constraint into the LG-string and maximizing numerically, we find that $K^{\max} = 1.5$, which occurs for $a = b = 1/\sqrt{2}$, $c = e = 0$, $d = f = 1$, and for interference angles $\theta_{ce} = 1.16255...$ and $\theta_{df} = 2\pi$. Using $a = b = 1/\sqrt{2}$ and $\theta_{df} = 2\pi$, the LGI reduces to

$$-3 \leq \frac{1}{2}\left(-2\cos(\theta_{ce})|ce| + 2|df| - e^2 + f^2\right) \leq 1 \quad (3.18)$$

Then using $e^2 + f^2 = c^2 + d^2 = 1$, we can have a two-parameter LG-string near the point of maximum violation (that is, near $a = b = \sqrt{2}/2$ and so on):

$$K = -\cos(\theta_{ce})\mathrm{Abs}\left(\sqrt{1-d^2}\sqrt{1-f^2}\right) + |df| + f^2 - \frac{1}{2} \quad (3.19)$$

These functions is plotted in figs. 3-1 - 3-3. As the interference angle θ_{ce} increases, a larger portion of the (d, f) parameter space is able to lead to LGI violation.

Recall that in our original description of the neutrino state at time $t = t_3$,

$$|t_3\rangle = a\,|L\rangle\,[c\,|\nu_e\rangle + d\,|\nu_\mu\rangle] + b\,|H\rangle\,[e\,|\nu_e\rangle + f\,|\nu_\mu\rangle] \quad (3.20)$$

Then according to the results visible in the figures, we need to have the complex coefficients d and f out of phase, and we are more likely to see LGI violation if the low- and high-energy states are heavily weighted to one of the neutrino flavors.

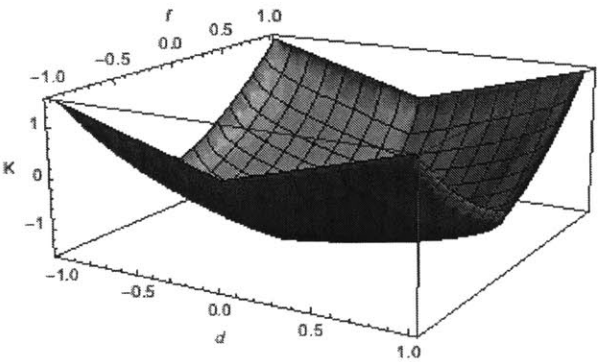

Figure 3-1: For $\theta_{ce} = 0$. The LG parameter does not exceed the classical bounds. (Recall that in this and all the figures on this page, we've set $a^2 = b^2 = 1/2$, $c^2 = 1 - d^2$, and $e^2 = 1 - f^2$).

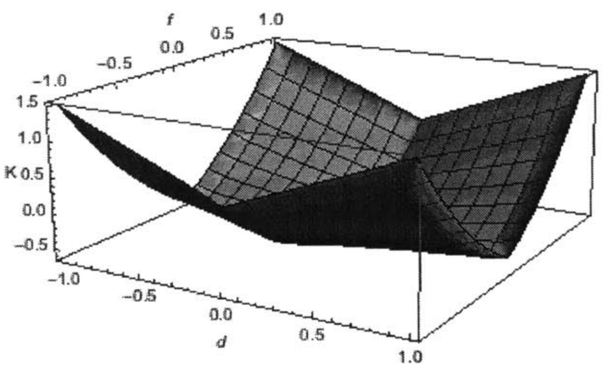

Figure 3-2: For $\theta_{ce} = \pi/2$. The classical bound of $K \leq 1$ is starting to be violated for this value of θ_{ce}, but only very extreme values of d and f can cause LGI violation.

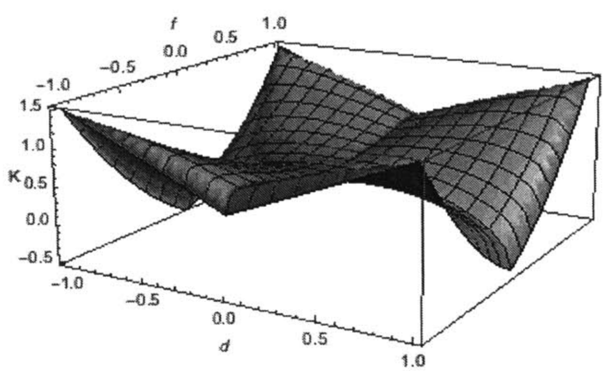

Figure 3-3: For $\theta_{ce} = \pi$. Here a larger portion of the parameter space leads to LGI violation.

3.2 From Toys to Physics

We discuss two methods of making the toy model more general. First, we expand the model in terms of the parameters of neutrino evolution, measurement, and so on. The second subsection explores what happens to the LG parameter if the CNOT flip is not ideal – that is, if only some fraction of the high-energy neutrinos become muon neutrinos, and if some fraction of low-energy neutrinos become muon-flavored.

3.2.1 Physical Parameters

General Expression for K in terms of Physical Parameters

Consider for now the evolution of our neutrino state through the vacuum. Starting as

$$|t_2\rangle = a\,|L\rangle\,|\nu_e\rangle + b\,|H\rangle\,|\nu_\mu\rangle\,, \tag{3.21}$$

it arrives on our planet in the state

$$|t_3\rangle = a\,|L\rangle\,\big(c\,|\nu_e\rangle + d\,|\nu_\mu\rangle\,\big) + b\,|H\rangle\,\big(e\,|\nu_e\rangle + f\,|\nu_\mu\rangle\,\big). \tag{3.22}$$

The Hamiltonian governing this evolution is

$$\mathbb{H}_{\text{vac}} = \big(p + \frac{\delta m^2}{4p}\big)\mathbb{I} + \frac{1}{2}\begin{pmatrix} -\omega\cos(2\theta) & \omega\sin(2\theta) \\ \omega\sin(2\theta) & \omega\cos(2\theta) \end{pmatrix}, \tag{3.23}$$

where δm^2 is the mass-squared splitting, $\omega = \delta m^2/2p$, and θ is the vacuum mixing angle.

Now, let's denote the low-energy and high-energy ωs as ω_L and ω_H, respectively. Recalling the notation for flavor vectors $\psi(e,\mu) = (a_e,\ a_\mu)^T$, let's write the low-energy part and high-energy part of $|t_2\rangle$, respectively, as

$$\psi_L(t_2) = \begin{pmatrix} a \\ 0 \end{pmatrix}, \qquad \psi_H(t_2) = \begin{pmatrix} 0 \\ b \end{pmatrix}. \tag{3.24}$$

And the low- and high-energy parts of $|t_3\rangle$, respectively, as

$$\psi_L(t_3) = \begin{pmatrix} ac \\ ad \end{pmatrix} \ , \qquad \psi_H(t_3) = \begin{pmatrix} be \\ bf \end{pmatrix} . \tag{3.25}$$

Our goal is to find a way to go from $|t_2\rangle$ to $|t_3\rangle$ using the Hamiltonian given above. Specifically, we need to figure out how to make the following viable

$$i\frac{d}{dt}\psi_L = \mathbb{H}_L\psi_L \ , \qquad i\frac{d}{dt}\psi_H = \mathbb{H}_H\psi_H \tag{3.26}$$

Considering for now only the low-energy case, we have that

$$\begin{aligned}
\psi_L(t_3) &= \exp\left(-i\int_0^{\Delta t}\mathbb{H}_L dt\right)\psi_L(t_2) \\
&= \exp\left(-i\mathbb{H}_L\Delta t\right)\psi_L(t_2) \\
&= \exp\begin{pmatrix} i\Delta t\ \omega_L\cos(2\theta)/2 & -i\Delta t\ \omega_L\sin(2\theta)/2 \\ -i\Delta t\ \omega_L\sin(2\theta)/2 & -i\Delta t\ \omega_L\cos(2\theta)/2 \end{pmatrix}\psi_L(t_2) \\
&= \begin{pmatrix} \cos(\omega_L\Delta t/2) + i\cos(2\theta)\sin(\omega_L\Delta t/2) & -i\sin(\omega_L\Delta t/2)\sin(2\theta) \\ -i\sin(\omega_L\Delta t/2)\sin(2\theta) & \cos(\omega_L\Delta t/2) - i\cos(2\theta)\sin(\omega_L\Delta t/2) \end{pmatrix}\psi_L(t_2) \\
\begin{pmatrix} ac \\ ad \end{pmatrix} &= \begin{pmatrix} \cos(\omega_L\Delta t/2) + i\cos(2\theta)\sin(\omega_L\Delta t/2) & -i\sin(\omega_L\Delta t/2)\sin(2\theta) \\ -i\sin(\omega_L\Delta t/2)\sin(2\theta) & \cos(\omega_L\Delta t/2) - i\cos(2\theta)\sin(\omega_L\Delta t/2) \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix}
\end{aligned} \tag{3.27}$$

This tells us that

$$\begin{aligned}
c &= \cos(\omega_L\Delta t/2) + i\cos(2\theta)\sin(\omega_L\Delta t/2) \\
d &= -i\sin(\omega_L\Delta t/2)\sin(2\theta)
\end{aligned} \tag{3.28}$$

Likewise, repeating this calculation for the high-energy case, we find that

$$\begin{aligned}
e &= \cos(\omega_H\Delta t/2) + i\cos(2\theta)\sin(\omega_H\Delta t/2) \\
f &= -i\sin(\omega_H\Delta t/2)\sin(2\theta)
\end{aligned} \tag{3.29}$$

Note that it is extremely convenient that given the above, we find that

$$|c|^2 + |d|^2 = 1 = |e|^2 + |f|^2. \tag{3.30}$$

Next, let's explore if our ideal CNOT is viable in terms of the physical parameters. The low- and high-energy states we begin with at $|t_1\rangle$are, respectively,

$$\psi_L(t_1) = \begin{pmatrix} a \\ 0 \end{pmatrix}, \qquad \psi_H(t_1) = \begin{pmatrix} a \\ 0 \end{pmatrix}. \tag{3.31}$$

That is, there is no muon component at all. Once the CNOT operates on the system, we get

$$\psi_L(t_2) = \begin{pmatrix} a \\ 0 \end{pmatrix}, \qquad \psi_H(t_2) = \begin{pmatrix} 0 \\ b \end{pmatrix}. \tag{3.32}$$

For now, let's assume that the Hamiltonian describing the MSW effect does not depend on how far the neutrino is along its path – that is, the CNOT happens at a specific point along the particle's trajectory. This allows us to write, for the state with energy $E = L, H$,

$$\begin{aligned} \psi_L(t_2) &= \exp\left(-i\int_0^{\Delta t} \mathbb{H}_E dt\right)\psi_L(t_1) \\ &= \exp\left(-i\mathbb{H}_E \Delta t\right)\psi_E(t_1) \\ &= \exp\begin{pmatrix} -i\left(V_C - \omega\cos(2\theta)\right)\Delta t/2 & -i\omega\sin(2\theta)\Delta t/2 \\ -i\omega\sin(2\theta)\Delta t/2 & i(V_C - \omega\cos(2\theta))\Delta t/2 \end{pmatrix}\psi_E(t_1) \end{aligned} \tag{3.33}$$

The matrix exponential is rather complicated. Defining the quantity

$$\Xi \equiv \sqrt{2\omega V_C \cos(2\theta) - V_C^2 - \omega^2} \tag{3.34}$$

for convenience, we find that for the low-energy states, two equations are true:

$$1 = i\frac{\omega_L}{\Xi_L}\cos(2\theta)\sinh\left(\frac{1}{2}\Xi_L\Delta t\right) - i\frac{V_C}{\Xi_L}\sinh\left(\frac{1}{2}\Xi_L\Delta t\right) + \cosh\left(\frac{1}{2}\Xi_L\Delta t\right) \tag{3.35}$$

and

$$0 \sim -i\frac{\omega_L}{\Xi_L}\sin(2\theta)\sinh\left(\frac{1}{2}\Delta t\Xi_L\right) \tag{3.36}$$

Likewise for the high-energy states, we find

$$0 \sim i\frac{\omega_H}{\Xi_H}\cos(2\theta)\sinh\left(\frac{1}{2}\Xi_H\Delta t\right) - i\frac{V_C}{\Xi_H}\sinh\left(\frac{1}{2}\Xi_H\Delta t\right) + \cosh\left(\frac{1}{2}\Xi_H\Delta t\right) \tag{3.37}$$

and

$$b = -ia\frac{\omega_H}{\Xi_H}\sin(2\theta)\sinh\left(\frac{1}{2}\Delta t\Xi_H\right) \tag{3.38}$$

It is these four equations (3.35 - 3.38) that define the CNOT gate in terms of physical parameters.

3.2.2 Relaxing the CNOT

Now we need to understand what happens if the CNOT operation is not perfect. The neutrinos at time t_1 are in the state

$$|t_1\rangle = a\,|L\rangle\,|\nu_e\rangle + b\,|H\rangle\,|\nu_e\rangle\,, \tag{3.39}$$

and we can model the non-ideal CNOT operation as having created the state

$$|t_2\rangle = \xi_1\,|L\rangle\,|\nu_e\rangle + \xi_2\,|L\rangle\,|\nu_\mu\rangle + \xi_3\,|H\rangle\,|\nu_e\rangle + \xi_4\,|H\rangle\,|\nu_\mu\rangle \tag{3.40}$$

where the ξ_i are complex. In order to see how this comes from eq. 3.39, we can write $|t_2\rangle$ as

$$|t_2\rangle = a'\,|L\rangle\,(\eta_L\,|\nu_e\rangle + \epsilon_L\,|\nu_\mu\rangle) + b'\,|H\rangle\,(\eta_H\,|\nu_e\rangle + \epsilon_H\,|\nu_\mu\rangle)\,, \tag{3.41}$$

where a', b' are not yet equal to a, b, and where $\epsilon_{L,H}$ represents the departure from a perfect CNOT (in which case $\epsilon_{L,H} \to 0$). With a perfect CNOT we'd expect $\eta_{L,H} \to 1$.

Note that *since the neutrinos' energy distribution does not change*, the total amplitude associated with $|L\rangle$ must be the same for $|t_1\rangle$ and $|t_2\rangle$. So $|a|^2 = |a'|^2\left(|\eta_L|^2 + |\epsilon_L|^2\right)$. Likewise for $|H\rangle$, we require that $|b|^2 = |b'|^2\left(|\eta_H|^2 + |\epsilon_H|^2\right)$. For simplicity we can choose to set $a' = a$ and $b' = b$, which leaves us with the constraints

$$|\eta_L|^2 + |\epsilon_L|^2 = 1, \qquad |\eta_H|^2 + |\epsilon_H|^2 = 1. \tag{3.42}$$

These η, ϵ parameters will depend on the mass splitting of the neutrinos, the matter and vacuum and mixing angles, and the neutrino energy.

Finally, we don't need to make any modification to our notation for the state of the neutrinos when they reach the Earth:

$$|t_3\rangle = a\,|L\rangle\,(c\,|\nu_e\rangle + d\,|\nu_\mu\rangle) + b\,|H\rangle\,(e\,|\nu_e\rangle + f\,|\nu_\mu\rangle)\,; \tag{3.43}$$

what's new is that now we have that c, d, e, and f are functions of η_L, η_H, ϵ_H, ϵ_L, in addition to depending on the vacuum mixing angle, mass splitting, and energy. For now we keep these parameters general.

Now we can proceed to calculate the correlation functions.

$$\begin{aligned}
C_{12} &= |\langle t_1|\nu_e\rangle\langle\nu_e|t_2\rangle|^2 - |\langle t_1|\nu_\mu\rangle\langle\nu_e|t_2\rangle|^2 - |\langle t_1|\nu_e\rangle\langle\nu_\mu|t_2\rangle|^2 + |\langle t_1|\nu_\mu\rangle\langle\nu_\mu|t_2\rangle|^2 \\
&= |\,(a^*\,\langle L| + b^*\,\langle H|)\,(a\eta_L\,|L\rangle + b\epsilon_H\,|H\rangle)\,|^2 - |\,(a^*\,\langle L| + b^*\,\langle H|)\,(a\epsilon_L\,|L\rangle + b\eta_H\,|H\rangle)\,|^2 \\
&= |a^*a\eta_L + b^*b\epsilon_H|^2 - |a^*a\epsilon_L + b^*b\eta_H|^2 \\
&= |a|^4(|\eta_L|^2 - |\epsilon_L|^2) + |b|^4(|\epsilon_H|^2 - |\eta_H|^2) + |a|^2|b|^2(\eta_L^*\epsilon_H + \eta_L\epsilon_H^* - \eta_H^*\epsilon_L - \eta_H\epsilon_L^*)
\end{aligned} \tag{3.44}$$

Note that in the limit that $\epsilon_{L,H} \to 0$ and $\eta_{L,H} \to 1$, we recover the ideal CNOT case, eq. 3.7. We expect to have the same C_{13} as before:

$$\begin{aligned}
C_{13} &= |\langle t_1|\nu_e\rangle\langle\nu_e|t_3\rangle|^2 - |\langle t_1|\nu_\mu\rangle\langle\nu_e|t_3\rangle|^2 - |\langle t_1|\nu_e\rangle\langle\nu_\mu|t_3\rangle|^2 + |\langle t_1|\nu_\mu\rangle\langle\nu_\mu|t_3\rangle|^2 \\
&= |(a^*\,\langle L| + b^*\,\langle H|)(ac\,|L\rangle + be\,|H\rangle)|^2 - |(a^*\,\langle L| + b^*\,\langle H|)(ad\,|L\rangle + bf\,|H\rangle)|^2 \\
&= |a^*ac + b^*be|^2 - |a^*ad + b^*bf|^2 \\
&= |a|^4(|c|^2 - |d|^2) + |b|^4(|e|^2 - |f|^2) + |a|^2|b|^2(c^*e + ce^* - d^*f - df^*) \\
&= |a|^4(|c|^2 - |d|^2) + |b|^4(|e|^2 - |f|^2) + |a|^2|b|^2\,[2|ce|\cos(\theta_e - \theta_c) - 2|df|\cos(\theta_d - \theta_f)]
\end{aligned} \tag{3.45}$$

The final correlation function is

$$\begin{aligned}
C_{23} &= |\langle t_2|\nu_e\rangle\langle\nu_e|t_3\rangle|^2 - |\langle t_2|\nu_\mu\rangle\langle\nu_e|t_3\rangle|^2 - |\langle t_2|\nu_e\rangle\langle\nu_\mu|t_3\rangle|^2 + |\langle t_2|\nu_\mu\rangle\langle\nu_\mu|t_3\rangle|^2 \\
&= |\big(a^*\eta_L^* \langle L| + b^*\epsilon_H^* \langle H|\big)\big(ac\,|L\rangle + be\,|H\rangle\big)|^2 - |\big(a^*\epsilon_L^* \langle L| + b^*\eta_H^* \langle H|\big)\big(ac\,|L\rangle + be\,|H\rangle\big)|^2 \\
&\quad - |\big(a^*\eta_L^* \langle L| + b^*\epsilon_H^* \langle H|\big)\big(ad\,|L\rangle + bf\,|H\rangle\big)|^2 + |\big(a^*\epsilon_L^* \langle L| + b^*\eta_H^* \langle H|\big)\big(ad\,|L\rangle + bf\,|H\rangle\big)|^2 \\
&= |a^*a\eta_L^*c + b^*b\epsilon_H^*e|^2 - |a^*a\epsilon_L^*c + b^*b\eta_H^*e|^2 - |a^*a\eta_L^*d + b^*b\epsilon_H^*f|^2 + |a^*a\epsilon_L^*d + b^*b\eta_H^*f|^2 \\
&= |a|^4\big(|\eta_L|^2|c|^2 + |\epsilon_L|^2|d|^2 - |\epsilon_L|^2|c|^2 - |\eta_L|^2|h|^2\big) + |b|^4\big(|\epsilon_H|^2|e|^2 + |\eta_H|^2|f|^2 - |\eta_H|^2|e|^2 - |\epsilon_H|^2|f|^2\big) + \\
&\quad + |a|^2|b|^2\big(\epsilon_H^*\eta_L(ce^* - df^*) + \epsilon_H\eta_L^*(c^*e - d^*f) - \epsilon_L^*\eta_H(ce^* - df^*) - \epsilon_L\eta_H^*(c^*e - d^*f)\big) \\
&= |a|^4(|\eta_L|^2 - |\epsilon_L|^2)(|c|^2 - |d|^2) + |b|^4(|\eta_H|^2 - |\epsilon_H|^2)(|f|^2 - |e|^2) + \\
&\quad + |a|^2|b|^2\left[(\epsilon_H^*\eta_L - \epsilon_L^*\eta_H)(ce^* - df^*) + (\epsilon_H\eta_L^* - \epsilon_L\eta_H^*)(c^*e - d^*f)\right]
\end{aligned} \tag{3.46}$$

Here the modified interference term is the sum of a complex number $(\epsilon_H^*\eta_L - \epsilon_L^*\eta_H)(ce^* - df^*)$ and its complex-conjugate $(\epsilon_H\eta_L^* - \epsilon_L\eta_H^*)(c^*e - d^*f)$, so we know the full interference term is real.

The full Leggett-Garg parameter in this case reads

$$\begin{aligned}
K &= |a|^4(|\eta_L|^2 - |\epsilon_L|^2) + |b|^4(|\epsilon_H|^2 - |\eta_H|^2) + |a|^2|b|^2\big(\eta_L^*\epsilon_H + \eta_L\epsilon_H^* - \eta_H^*\epsilon_L - \eta_H\epsilon_L^*\big) \\
&\quad + |a|^4(|\eta_L|^2 - |\epsilon_L|^2)(|c|^2 - |d|^2) + |b|^4(|\eta_H|^2 - |\epsilon_H|^2)(|f|^2 - |e|^2) \\
&\quad + |a|^2|b|^2\left[(\epsilon_H^*\eta_L - \epsilon_L^*\eta_H)(ce^* - df^*) + (\epsilon_H\eta_L^* - \epsilon_L\eta_H^*)(c^*e - d^*f)\right] \\
&\quad - \big(|a|^4(|c|^2 - |d|^2) + |b|^4(|e|^2 - |f|^2) + |a|^2|b|^2(c^*e + ce^* - d^*f - df^*)\big)
\end{aligned} \tag{3.47}$$

Dropping the absolute value signs for simplicity:

$$\begin{aligned}
K &= a^4\left[(\eta_L^2 - \epsilon_L^2)(1 + c^2 - d^2) - c^2 + d^2\right] \\
&\quad + b^4\left[(\eta_H^2 - \epsilon_H^2)(f^2 - e^2 - 1) + f^2 - e^2\right] \\
&\quad + a^2b^2\left[(\epsilon_H^*\eta_L - \epsilon_L^*\eta_H - 1)(ce^* - df^*) + (\epsilon_H\eta_L^* - \epsilon_L\eta_H^* - 1)(c^*e - d^*f)\right] \\
&\quad + a^2b^2\left[\eta_L^*\epsilon_H + \eta_L\epsilon_H^* - \eta_H^*\epsilon_L - \eta_H\epsilon_L^*\right]
\end{aligned} \tag{3.48}$$

Plugging in the normalization constraints that $a^2+b^2=1$ and $\eta_E^2+\epsilon_E^2=1$ gives

$$\begin{aligned}K &= a^4\left[(2\eta_L^2-1)(1+c^2-d^2)-c^2+d^2\right]\\&\quad+(1-a^2)^2\left[(2\eta_H^2-1)(f^2-e^2-1)+f^2-e^2\right]\\&\quad+a^2(1-a^2)\left[(\epsilon_H^*\eta_L-\epsilon_L^*\eta_H-1)(ce^*-df^*)+(\epsilon_H\eta_L^*-\epsilon_L\eta_H^*-1)(c^*e-d^*f)\right]\\&\quad+a^2(1-a^2)\left[\eta_L^*\epsilon_H+\eta_L\epsilon_H^*-\eta_H^*\epsilon_L-\eta_H\epsilon_L^*\right]\end{aligned} \tag{3.49}$$

The term proportional to $a^2(1-a^2)$ can be written

$$\begin{aligned}&=\left(\epsilon_H\bar{\eta}_L+\bar{\epsilon}_H\eta_L\right)-\left(\epsilon_L\bar{\eta}_H+\bar{\epsilon}_L\eta_H\right)-(\bar{c}e+c\bar{e})+(\bar{d}f+d\bar{f})+\left(\bar{\epsilon}_H\eta_Lc\bar{e}+\eta_H\bar{\eta}_L\bar{c}e\right)\\&\quad-\left(\bar{\epsilon}_H\eta_Ld\bar{f}+\epsilon_H\bar{\eta}_L\bar{d}f\right)-\left(\bar{\epsilon}_L\eta_Hc\bar{e}+\epsilon_L\bar{\eta}_H\bar{c}e\right)+\left(\bar{\epsilon}_L\eta_Hd\bar{f}+\epsilon_L\bar{\eta}_H\bar{d}f\right)\end{aligned} \tag{3.50}$$

Each term can be written following the pattern

$$\epsilon_H\bar{\eta}_L+\bar{\epsilon}_H\eta_L=2|\epsilon_H\eta_L|\cos(\theta_{\epsilon_H\eta_L}), \tag{3.51}$$

where $\theta_{\epsilon_H\eta_L}=\theta_{\epsilon_H}-\theta_{\eta_L}$. Doing this for each term allows us to write the LG parameter as

$$\begin{aligned}K &= a^4\big(1+4c^2(\eta_L^2-1)\big)-4(1-a^2)e^2\eta_H^2+\\&\quad+2a^2(1-a^2)\big(|ce|\cos(\theta_{ce})-|df|\cos(\theta_{df})\big)\cdot\big(1+|\epsilon_H\eta_L|\cos(\theta_{\epsilon_H\eta_L})-|\epsilon_L\eta_H|\cos(\theta_{\epsilon_L\eta_H})\big)\\&\quad+2a^2(1-a^2)\big(1+|df|\sin(\theta_{df})-|ce|\sin(\theta_{ce})\big)\cdot\big(|\epsilon_H\eta_L|\sin(\theta_{\epsilon_H\eta_L})-|\epsilon_L\eta_H|\sin(\theta_{\epsilon_L\eta_H})\big)\end{aligned} \tag{3.52}$$

3.3 Closing

While there is still a lot of work to be done, it's clear that some regions of parameter space in this CNOT model can indeed give rise to LGI violation. The next steps are to calculate what these regions are in terms of the physical neutrino parameters, and to see if solar neutrinos satisfy the restrictions.

The possibility of testing the Leggett-Garg inequality using astrophysical sources is very exciting, as is the prospect of an LGI test that does not involve weak measurements.

BIBLIOGRAPHY

[1] Y. Aharonov, D.Z. Albert, and L. Vaidman, "How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100." *Phys. Rev. Lett.* **60**, 1351 (1988).

[2] V. Athalye, S. Singha Roy, and T.S. Mahesh, "Investigation of the Leggett-Garg Inequality for Precessing Nuclear Spins." *Phys. Rev. Lett.* **107**, 130402 (2011).

[3] J.S. Bell, "On the Einstein Podolsky Rosen Paradox." *Physics* **I**, 195-200 (1964).

[4] J. Beringer *et al.* (PDG), *Phys. Rev. D* **86**, 010001 (2012).

[5] S.M. Bilenky, C. Giunti, and W. Grimus, "Phenomenology of Neutrino Oscillations." *Prog. Part. Phys.* **43** (1999), 1-86. hep-ph/9812360v4.

[6] N. Bohr, *Causality and Complementarity*, Supplementary papers edited by Jan Faye and Henry Folse as *The Philosophical Writings of Niels Bohr, Vol. IV*. Woodbridge: Ox Bow Press, 1998.

[7] H. Duan, G. Fuller, Y. Qian, "Collective Neutrino Oscillations," *Ann. Rev. Nucl. Part. Sci.* **60**, 569 (2010).

[8] A. Einstein, B. Podolsky, and N. Rosen. *Phys. Rev.* **47**, 777 (1935).

[9] J. Faye, "The Copenhagen Interpretation of Quantum Mechanics." *Stanford Encyclopedia of Philosophy*, 2014.

[10] J. Formaggio, "Notes on Correlators and Commutators." 17 April 2015.

[11] J. Gallicchio, A.S. Friedman, and D.I. Kaiser, "Testing Bell's Inequality with Cosmic Photon: Closing the Setting-Independence Loophole." *Phys. Rev. Lett.* **112**, 110405 (2013). arxiv:1310:3288v2.

[12] C. Giunti and C. Kim, *Fundamentals of Neutrino Physics and Astrophysics.* Oxford: Oxford University Press, 2007.

[13] W.C. Haxton et al., "Solar neutrinos: Status and prospects," *Ann. Rev. Astron. Astrophys.* **51**, 21 (2013). arXiv:1208.5723 [astro-ph.SR].

[14] W. Heisenberg, "Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik." *Zeitschrift fŬr Physik*, **43** (1927).

[15] D. Kaiser, "Notes on Testing the Leggett-Garg Inequality Using Solar Neutrinos," v. 2, 17 April 2015.

[16] G. Knee *et al.*, "Comment on 'A Scattering Quantum Circuit for Measuring Bell's Time Inequality,'" *New J. Phys.* (2012). arXiv:1207.2786

[17] A.J Leggett and A. Garg, "Quantum Mechanics versus Macroscopic Realism: Is the Flux There when Nobody Looks?" *Phys. Rev. Lett.* **54**, 9 (857), 4 March 1985.

[18] P. Mehta, "Topological phase in two flavor neutrino oscillations," *Phys. Rev. D* **79**, 096013 (2009). arXiv:0901.0790 [hep-ph].

[19] O. Moussa, C. Ryan, D. Cory, and R. Laflamme, "Testing Contextuality on Quantum Ensembles with One Clean Qubit." *Phys. Rev. Lett.* **104**, 160501 (2010).

[20] B. Pontecorvo, *Sov. Phys. JETP* **6** 429, 1957

[21] A. Renshaw *et al.*, "First Indication of Terrestrial Matter Effects on Solar Neutrino Oscillation." *Phys. Rev. Lett.* **112**, 091805 (2014). arxiv:1312.5176v3.

[22] J. Xu, C. Li, X. Zou, and G. Guo, "Experimental violation of the Leggett-Garg inequality under decoherence." *Sci. Rep.* **1**, 101 (2011).

Ensemble Measurement

This appendix describes how one might test the LGI using ensemble measurements, rather than observations made on the same qubit. The protocol presented in Moussa et al. and applied in the NMR PRL is an example of a weak measurement scheme.[1] The extent to which a weak measurement can be treated as one that does not disturb the target system is addressed in Knee et al., Comment on 'A Scattering Quantum Circuit for Measuring Bell's Time Inequality,' New J. Phys. (2012) arXiv:1207.2786. Assuming that the assumption of noninvasive measurement is satisfied, the LGI is testing the assumption of macroscopic realism.

The counterintuitive part of the Moussa scheme is that all the entanglement procedures do not involve any operation on the ancilla; rather, they involve only operations that are "controlled on" the ancilla. So even though the ancilla itself doesn't change, performing a standard measurement on it at the end will still give us information about the target system.

Here I follow the derivations in the LGIV-NMR paper and in Moussa et al., albeit with simplified notation. These papers provide the necessary connection between the LGI (which in its original formulation probes one system many times) and measurements on multiple systems.[2] Consider a d-dimensional target system prepared in the state ρ_T and a single probe/ancilla qubit prepared in ρ_P. The composite state is given by $\rho = \rho_P \otimes \rho_T$. Let S be a dichotomic observable with spectral decomposition

[1] Weak measurements being first proposed in Y. Aharonov, D.Z. Albert, and L. Vaidman, Phys. Rev. Lett. 60 (1988) 1351.

[2] Moussa, Ryan, Cory, and Laflamme, "Testing Contextuality on Quantum Ensembles with One Clean Qubit," PRL 104, 160501 (2010).

$S = P_+ - P_-$, where $P_+ \equiv |+\rangle\langle +|$ and $P_- \equiv |-\rangle\langle -|$ are "projection operators onto the +1 and -1 eigenspaces of S."[3] We define a unitary transformation

$$U_S = \mathbb{I}_2 \otimes P_+ - Z \otimes P_- \, , \tag{53}$$

where $\mathbb{I}_d$ is the d-dimensional identity operation[4], and Z is the Pauli Z operator $\hat{\sigma}_Z$. This transformation amounts to a controlled phase flip. To see this, note that we can write the above as

$$U_S = \frac{1}{2}(\mathbb{I}_2 + Z) \otimes \underbrace{(P_+ + P_-)}_{=\mathbb{I}_d} + \frac{1}{2}(\mathbb{I}_2 - Z) \otimes \underbrace{(P_+ - P_-)}_{=S} \tag{54}$$

$$= |0\rangle\langle 0| \otimes \mathbb{I}_d + |1\rangle\langle 1| \otimes S \, . \tag{55}$$

Thus the action of U_S is such that if the probe qubit is in the state $|0\rangle$, the system is left alone; if the probe qubit is in the state $|1\rangle$, the measurement $S = P_+ - P_-$ is applied to the system.

The important claim in these papers is that *"the ensemble measurement of the probe give correlation between successively measured commuting observables of the target."*[5] In the case we're interested in (namely, measuring two-time correlation functions), the set of observables is $\hat{\sigma}_X(t_i)_{i=1..N}$, and we will calculate the two-time correlation function of the outcomes of $\hat{\sigma}_X(t_i)$. For simplicity I'll just write $X(t_i)$ and $Z(t_i)$ to denote the Pauli X and Z operators evaluated at time t_i.

The two-time correlation function (TTCC) is $C_{ij} = \langle X(t_i)X(t_j)\rangle$, where $X(t)$ is applied to the *target* system. Our goal is to express this in terms of measurements on the *probe* qubit. Suppose that the probe is prepared in the state $|+\rangle$ so that the overall density matrix is $\rho = (|+\rangle\langle +|)_P \otimes \rho_T$. The evolution of the density matrix under $U_S(t_i, t_j) \equiv U_S(t_j)U_S(t_i)$ is $\rho' = U(t_j, t_i)\, \rho\, U^\dagger(t_i, t_j)$.

The probability of the outcome ν is given by the trace of $\rho' P_\nu$. So the probability of getting the final probe measurement to read +1 is

[3] Moussa, Ryan, Cory, and Laflamme (2010).
[4] The identity operation corresponds to doing nothing to the system.
[5] Moussa, Ryan, Cory, and Laflamme (2010).

$$p(+1) = \mathrm{tr}_{\mathrm{P,T}} \left[\rho' \left(\overbrace{|+\rangle\langle+|}^{\text{Applied to probe}} \otimes \overbrace{\mathbb{I}}^{\text{Applied to target}} \right) \right] \tag{56}$$

$$= \mathrm{tr}_{\mathrm{P,T}} \; U(t_j, t_i)\, \rho\, U^\dagger(t_i, t_j)\, (|+\rangle\langle+| \otimes \mathbb{I}) \tag{57}$$

$$= \mathrm{tr}_{\mathrm{P,T}} \; U(t_j)U(t_i)\, \rho\, U^\dagger(t_j)U^\dagger(t_i)\, (|+\rangle\langle+| \otimes \mathbb{I}) \tag{58}$$

$$= \mathrm{tr}_{\mathrm{P,T}} \; \overbrace{(\mathbb{I} \otimes P_+(t_j) + Z(t_j) \otimes P_-(t_j))}^{U(t_j)} \overbrace{(\mathbb{I} \otimes P_+(t_i) + Z(t_i) \otimes P_-(t_i))}^{U(t_i)} (|+\rangle\langle+| \otimes \rho_T) \tag{59}$$

$$\underbrace{(\mathbb{I} \otimes P_+(t_j) + Z(t_j) \otimes P_-(t_j))^\dagger}_{U^\dagger(t_j)} \underbrace{(\mathbb{I} \otimes P_+(t_i) + Z(t_i) \otimes P_-(t_j))^\dagger}_{U^\dagger(t_i)} (|+\rangle\langle+| \otimes \mathbb{I})$$

$$= \mathrm{tr}_{\mathrm{P,T}} \; (\mathbb{I} \otimes P_{+j}P_{+i} + Z_i \otimes P_{+j}P_{-i} + Z_j \otimes P_{-j}P_{+i} + Z_jZ_i \otimes P_{-j}P_{-i}) \;\; (|+\rangle\langle+| \otimes \rho_T) \tag{60}$$

$$(\mathbb{I} \otimes P_{+j}P_{+i} + Z_i \otimes P_{+j}P_{-i} + Z_j \otimes P_{-j}P_{+i} + Z_jZ_i \otimes P_{-j}P_{-i})^\dagger \;\; (|+\rangle\langle+| \otimes 1)$$

where I've really condensed notation, so that $Z_j \equiv \hat{\sigma}_Z(t_j)$ and $P_{+j} = P_+(t_j)$. Since projection operators are their own adjoints, it seems likely that the correct expansion for the third factor is

$$\mathrm{T3} = \mathrm{tr}_{\mathrm{P,T}} \; \mathbb{I} \otimes (P_{+j}P_{+i})^\dagger + Z_i \otimes (P_{+j}P_{-i})^\dagger + Z_j \otimes (P_{-j}P_{+i})^\dagger + (Z_jZ_i)^\dagger \otimes (P_{-j}P_{-i})^\dagger \tag{61}$$

$$= \mathrm{tr}_{\mathrm{P,T}} \; \mathbb{I} \otimes P_{+i}P_{+j} + Z_i \otimes P_{-i}P_{+j} + Z_j \otimes P_{+i}P_{-j} + Z_iZ_j \otimes P_{-i}P_{-j} \tag{62}$$

Plugging this into the above gives

$$p(+) = \mathrm{tr}_{\mathrm{P,T}} \; (\mathbb{I} \otimes P_{+j}P_{+i} + Z_i \otimes P_{+j}P_{-i} + Z_j \otimes P_{-j}P_{+i} + Z_jZ_i \otimes P_{-j}P_{-i}) \;\; (|+\rangle\langle+| \otimes \rho_T)$$
$$(\mathbb{I} \otimes P_{+i}P_{+j} + Z_i \otimes P_{-i}P_{+j} + Z_j \otimes P_{+i}P_{-j} + Z_iZ_j \otimes P_{-i}P_{-j}) \;\; (|+\rangle\langle+| \otimes 1) \tag{63}$$

$$= \mathrm{tr}_{\mathrm{T}} \left[(P_{+i}P_{+j} + P_{-i}P_{-j})\rho_T \right] \tag{64}$$

Analogous algebra to get $p(-1) = \mathbf{tr_T}\left[(P_{+i}P_{-j} + P_{-i}P_{+j})\rho_T\right]$.

Recall that earlier we had $\langle X \otimes 1 \rangle = p(+1) - p(-1)$. Applying the above,

$$\langle X \otimes 1 \rangle = \mathbf{tr_T}\left[(P_{+i}P_{+j} + P_{-i}P_{-j} - P_{+i}P_{-j} - P_{-i}P_{+j})\rho_T\right] \tag{65}$$

$$= \mathbf{tr_T}\left[((P_{+i} - P_{-i})P_{+j} - (P_{+i} - P_{-i})P_{-j})\rho_T\right] \tag{66}$$

$$= \mathbf{tr_T}\left[(P_{+i} - P_{-i})(P_{+j} - P_{-j})\rho_T\right] \tag{67}$$

$$= \mathbf{tr_T}\left[X_i X_j \rho_T\right] \tag{68}$$

But this is just the expectation value of X_iX_j, so

$$\langle X \otimes 1 \rangle = \langle X_i X_j \rangle \equiv C_{ij}, \tag{69}$$

exactly the two-time correlation coefficient (TTCC) we were looking for. We now have an expression for the TTCC in terms of unitary operations applied to the target system controlled on the probe qubit, "followed by an ensemble measurement of the Pauli X operator on the probe" qubit.[6]

[6]Moussa, Ryan, Cory, and Laflamme (2010).