THE STRONG CHROMATIC INDEX

OF HALIN GRAPHS

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Ziyu Hu

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The thesis of Ziyu Hu is approved.

Daphne Liu, Ph.D., Thesis Director

Gary Brookfield, Ph.D., Committee Member

Silvia Heubach, Ph.D., Committee Member

Grant Fraser, Ph.D., Department Chair

California State University, Los Angeles

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ABSTRACT

The Strong Chromatic Index

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A *strong edge coloring* of a graph G is an assignment of colors to the edges of G such that two distinct edges are colored differently if they have adjacent endpoints. The *strong chromatic index* of a graph G, denoted by $\chi'_{\mathrm{s}}(G)$, is the minimum number of colors needed for a strong edge coloring of G. A *Halin graph* G is a planar graph constructed by connecting all leaves of a characteristic tree T without vertices of degree two through a cycle. If a Halin graph G is different from Ne_2, Ne_4, and any wheel, then we prove $\chi'_{\mathrm{s}}(G) \leqslant 2\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. If, additionally, $\Delta(G) = 4$, we prove $\chi'_{\mathrm{s}}(G) \leqslant \chi'_{\mathrm{s}}(T) + 2$, where T is the characteristic tree of G.

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For my dear wife Jessie,

for my beloved daughter Charlotte,

and for all lovers of graph theory

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CHAPTER 1

Introduction

In this thesis, a *graph* is a representation of a set of objects where some pairs of objects are connected by links. The interconnected objects are represented by mathematical abstractions called *vertices*, and the links that connect some pairs of vertices are called *edges*. Typically, a graph is depicted in a diagrammatic form as a set of dots for the vertices, joined by lines or curves for the edges. Graphs are one of the objects of study in discrete mathematics.

The edges of a graph may be directed or undirected. In this thesis, only simple graphs, i.e., undirected graphs without loops (edges connecting vertices to themselves) and multiple edges (two or more edges connecting the same pair of vertices), are considered.

In graph theory, *graph coloring* is a special case of graph labeling; it is an assignment of labels traditionally called "*colors*" to elements of a graph (vertices or edges) subject to certain constraints. Graph coloring enjoys many practical applications as well as theoretical challenges. Besides the classical types of problems, different limitations can also be set on the graph, or on the way a color is assigned, or even on the color itself. Graph coloring has been studied as an algorithmic problem since the early 1970s: the chromatic number problem is one of Karp's 21 NP-complete problems [8] from 1972, and at approximately the same time various exponential-time algorithms were developed based on backtracking and on the deletion-contraction recurrence of Zykov (1949) [15]. One of the major applications of graph coloring, register allocation in compilers [3], was introduced in 1981. It has even reached popularity with the general public in the form of the popular number puzzle Sudoku. Graph coloring is still a very active field of research.

A *strong edge coloring* of a graph is a constrained assignment of colors to the edges of the graph such that any three consecutive edges are colored differently. The *strong chromatic index* of a graph is the minimum number of colors needed for a strong edge coloring of the graph.

A *Halin graph* is a special type of *planar graph* (a graph that can be drawn in the plane so that its edges intersect only at their endpoints). Halin graphs are named after the German mathematician Rudolf Halin, who studied them in 1971 [6], but the cubic Halin graphs (Halin graphs whose vertices have exactly three neighbors) had already been studied over a century earlier by Kirkman [9].

In this thesis, we study the strong edge coloring of Halin graphs to obtain an upper bound for the strong chromatic index under certain conditions.

CHAPTER 2

Definitions and Auxiliary Results

In this chapter, we state some basic definitions of graph theory that are involved in the thesis. Some of these definitions are illustrated with examples. All the notations we use throughout the thesis are also introduced in this chapter. Furthermore, we quote several known results which will be used in later proofs.

2.1 Basic Terminology and Examples

Definition 2.1. *A* ***graph*** $G = (V, E)$ *consists of two sets: the vertex set and the edge set, denoted by* $V(G)$ *and* $E(G)$ *(or* V *and* E *when no ambiguities arise), respectively. The elements of* V*, which are 2-element subsets of* $V(G)$*, are called* ***vertices*** *(or nodes), and the elements of* E *are called* ***edges*** *(or links). Each edge has two vertices, called* ***endpoints****, associated with it. An edge with endpoints* u *and* v *is denoted by* uv*. The cardinality of the vertex set* V *is called the* ***order*** *of* G*, denoted by* $|V|$ *(or* $|G|$*); the cardinality of the edge set* E *is called the* ***size*** *of* G*, denoted by* $|E|$ *(or* $\|G\|$*). The graph which has only one vertex and no edges is called the* ***trivial*** *graph. The graph* G *is* ***finite*** *if and only if* $|V|$ *is a finite number.*

In this thesis, we only consider finite graphs.

Definition 2.2. *An* ***undirected graph*** *is one in which edges have no orientation. A* ***loop*** *is an edge connected at both ends to the same vertex.* ***Multiple edges*** *are two* *or more edges connecting the same pair of vertices. A* ***simple graph*** *is an undirected graph that has no loops and no multiple edges.*

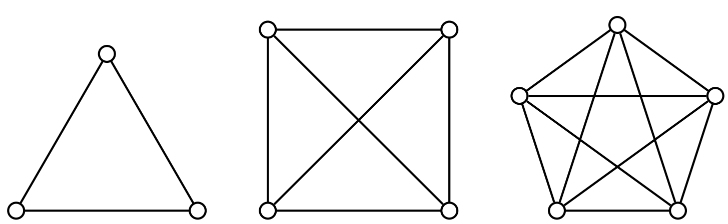
In this thesis, we only consider simple graphs.

Definition 2.3. *Two vertices u and v are said to be* ***adjacent*** *in G if $uv \in E(G)$, denoted by $u \sim v$. Adjacent vertices are* ***neighbors*** *of each other. The set of all neighbors of any vertex w is denoted by $N(w)$.*

Definition 2.4. *If vertex v is an endpoint of edge e, then we say that v is* ***incident*** *to e, and e is* ***incident*** *to v. The* ***degree*** *of a vertex v in a graph G is the number of edges incident to v, denoted by $\deg_G(v)$, or $\deg(v)$ when no ambiguities arise. The* ***maximum degree*** *of a graph G is the maximum $\deg(v)$ over all $v \in V(G)$, denoted by $\Delta(G)$, or Δ when no ambiguities arise.*

Definition 2.5. *A* ***regular graph*** *is a graph where all vertices have the same degree. A regular graph with vertices of degree k is called a* ***k-regular graph****. In particular, a 3-regular graph is called a* ***cubic graph****.*

Figure 2.1 shows examples of regular graphs. Specifically, Figure 2.1a is a 2-regular graph; Figure 2.1b is a 3-regular graph, i.e., a cubic graph; Figure 2.1c is a 4-regular graph.



(a) 2-regular graph (b) cubic graph (c) 4-regular graph

Figure 2.1: Regular graphs

Definition 2.6. *Let G be a graph. For any $u, v \in V(G)$, a u-v* ***walk*** *of G is a finite alternating sequence of vertices and edges of G which begins with vertex u and ends with vertex v, such that each edge's endpoints are the preceeding and following vertices in the sequence. A walk is said to be closed if the sequence starts and ends at the same vertex. A u-v* ***path*** *is a u-v walk with no repeated edges and vertices, denoted by $P_{u,v}$. The length of the path, denoted by $|P_{u,v}|$, is defined to be the number of edges in the sequence. A path is called an* ***empty*** *path if its length is zero, i.e., the sequence has one vertex and no edges. A* ***cycle*** *is a closed walk with no repetitions of vertices and edges allowed, other than the repetition of the starting and ending vertex. A cycle with order n is denoted by C_n, n is a positive integer and $n \geqslant 3$ for any cycle.*

Definition 2.7. *In a graph G, two vertices u and v are called* ***connected*** *if G contains a path from u to v. Otherwise, they are called* ***disconnected****. A graph is said to be* ***connected*** *if every pair of vertices in the graph is connected.*

Definition 2.8. *The* ***distance*** *between vertices u and v in graph G, denoted by $d_G(u, v)$, is the length of a shortest path between u and v. If the graph G is clear from the context, we denote the distance between vertices u and v by $d(u, v)$.*

Since only simple graphs are considered in this thesis, we use a sequence of vertices to represent a walk/path/cycle. In Figure 2.2, $P_1 = a, b, c$, $P_2 = a, b, e, d, c$, $P_3 = a, f, e, d, c$, and $P_4 = a, f, e, b, c$ are all possible a-c paths. Then $d(a, c) = \min\{|P_1|, |P_2|, |P_3|, |P_4|\} = 2$.

Definition 2.9. *A* ***complete graph*** *K_n, where n is a positive integer, is a graph of order n such that every pair of distinct vertices in K_n are connected by a unique edge. The trivial graph is K_1.*

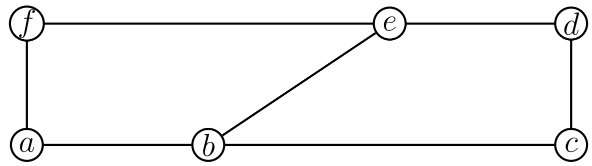


Figure 2.2: An example for path and distance

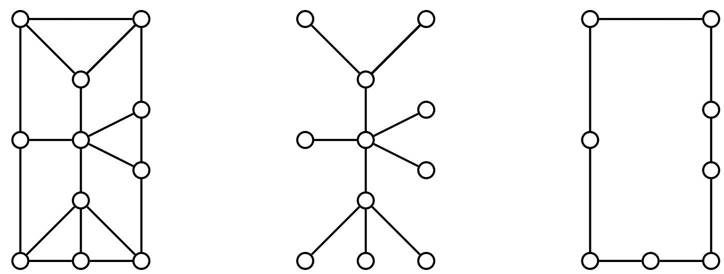
The three graphs in Figure 2.1 are also complete graphs. Figure 2.1a, Figure 2.1b, and Figure 2.1c are K_3, K_4, and K_5, respectively.

Definition 2.10. *A **tree** T is a simple connected graph that has no cycles. Any vertex in a tree with degree one is called a **leaf**.*

Definition 2.11. *A **planar graph** is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. A **plane graph** is a drawing of a planar graph on the plane such that no edges cross each other. A plane graph divides the plane into areas surrounded by edges, called **faces** (or **regions**). The area outside the plane graph is the unbounded face of the graph.*

Definition 2.12. *A **Halin graph** $G = T \cup C$ is a planar graph constructed as follows. Fix a plane drawing of a tree T that has at least four vertices and none of which have exactly two neighbors. Let C be a cycle obtained by connecting all the leaves of T in such a way that C forms the boundary of the unbounded face. The plane drawing of the tree T and the cycle C are called the **characteristic tree** and the **adjoint cycle** of G, respectively.*

An example of a Halin graph $G = T \cup C$ is given in Figure 2.3. Figure 2.3a is the Halin graph; Figure 2.3b is its characteristic tree; Figure 2.3c is its adjoint cycle.



(a) $G = T \cup C$ (b) T (c) C

Figure 2.3: A Halin graph $G = T \cup C$ and its parts

Definition 2.13. *For any integer $h \geqslant 1$, a **necklace** [12], denoted Ne_h, is a cubic Halin graph constructed as follows. Its characteristic tree T_h consists of the path $v_0, v_1, \ldots, v_h, v_{h+1}$ and leaves $v'_1, v'_2, \ldots, v'_h$ such that the unique neighbor of v'_i in T_h is v_i for $1 \leqslant i \leqslant h$ and vertices $v_0, v'_1, \ldots, v'_h, v_{h+1}$ are connected in order to form the adjoint cycle C_{h+2}.*

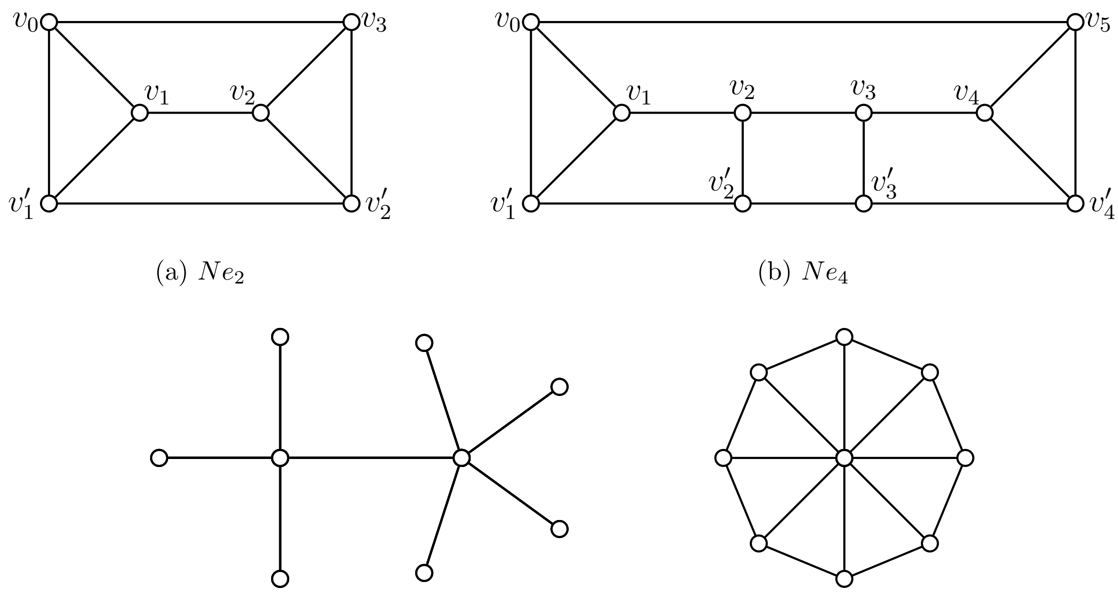
As shown in Figure 2.4, Figure 2.4a is Ne_2 and Figure 2.4b is Ne_4.

Definition 2.14. *A **star** is a tree with exactly $n-1$ leaves, where n is the order of the tree. A **doublestar** is a tree with exactly $n-2$ leaves, where n is the order of the tree. If the degrees of the two non-leaf vertices of a double star are x and y, where $x \leqslant y$, we denote such a double star by $D_{x,y}$.*

$D_{4,5}$, a double star with the degrees of the two non-leaf vertices equal to four and five is shown in Figure 2.4c.

Definition 2.15. *For any integer $n \geqslant 3$, the **wheel** W_n is a particular Halin graph with order $n+1$ and whose characteristic tree has exactly n leaves, i.e., the characteristic tree is a star. The non-leaf vertex of its characteristic tree has degree n.*

W_8, a wheel whose characteristic tree has eight leaves, is shown in Figure 2.4d.



(a) Ne_2 (b) Ne_4

(c) $D_{4,5}$ (d) W_8

Figure 2.4: Necklaces, a double star, and a wheel

Definition 2.16. *A* ***subgraph*** *of a graph $G = (V, E)$ is a graph $G' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$, denoted by $G' \leqslant G$. If G' is a subgraph of G, then G is a* ***supergraph*** *of G'. Furthermore, the subgraph G' is called an* ***induced subgraph*** *of G on V' if $uv \in E$ implies $uv \in E'$ for any $u, v \in V'$.*

Definition 2.17. *A* ***graph coloring*** *(or* ***coloring****) is an assignment of labels traditionally called "****colors****" to elements of a graph (vertices or edges) subject to certain constraints.*

Definition 2.18. *A* ***proper vertex coloring*** *(or* ***vertex coloring****) of a graph is a coloring on the vertices such that no two adjacent vertices have the same color. The minimum number of colors needed for a proper vertex coloring of a graph G is called its* ***chromatic number****, denoted by $\chi(G)$.*

As an example, $\chi(K_5) = 5$ because the vertices in K_5 are adjacent to each other as depicted in Figure 2.1c.

Definition 2.19. *A **proper edge coloring** (or **edge coloring**) of a graph is a coloring on the edges such that no two edges sharing a common endpoint have the same color. The minimum number of colors needed for a proper edge coloring of a graph G is called its **chromatic index**, denoted by $\chi'(G)$.*

Definition 2.20. *A **strong edge coloring** of a graph G is a coloring on the edges of G such that two distinct edges are colored differently if they have adjacent endpoints. The minimum number of colors needed to color all the edges of graph G is called the **strong chromatic index** of G, denoted by $\chi'_{\mathrm{s}}(G)$.*

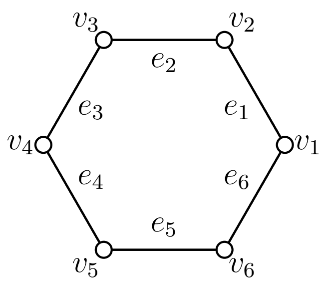
Definition 2.21. *For each edge uv of a non-trivial graph G, we define $\sigma_G(uv) = \deg_G(u) + \deg_G(v) - 1$ and it is abbreviated to $\sigma(uv)$ if no ambiguities arise. $\sigma(G)$ is the maximum of $\sigma(e)$ over all $e \in E(G)$. $\sigma(G) = 0$ if G is the trivial graph.*

$\sigma(G)$ is an important parameter, especially for trees, regarding strong edge coloring. (See Lemma 2.26.)

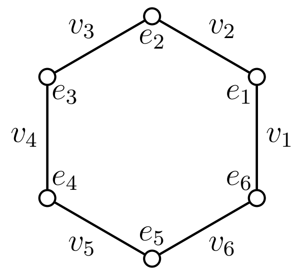
Definition 2.22. *The **line graph** $L(G)$ of G is the graph defined on the edge set $E(G)$ (each vertex in $L(G)$ represents a unique edge in G) such that two vertices in $L(G)$ are defined to be adjacent if their corresponding edges in G share a common endpoint. The **distance** between two edges in G is defined to be the distance between their corresponding vertices in $L(G)$.*

Definition 2.23. *For a graph G, the **square** of G, denoted by G^2, is the graph defined on the vertex set $V(G)$ such that two distinct vertices u and v are defined to be adjacent in G^2 if $d_G(u,v) \leqslant 2$.*

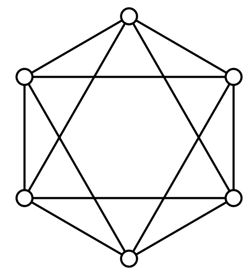
A cycle C_6, its line graph $L(C_6)$, and $L(C_6)^2$ (the square of $L(C_6)$) are shown



(a) C_6



(b) $L(C_6)$



(c) $L(C_6)^2$

Figure 2.5: A cycle, the line graph, and the square graph

in Figure 2.5. The labels on vertices and edges show how the mappings $E(C_6) \to V(L(C_6))$ and $V(C_6) \to E(L(C_6))$ work.

2.2 Auxiliary Results and Proofs

Theorem 2.24 [2]**. (Brook's Theorem)** *If G is a graph and G is not a complete graph nor an odd cycle, then $\chi(G) \leqslant \Delta(G)$.*

Theorem 2.25. *If G is a simple graph, the followings are equivalent (cf.* [14]*):*

(1) *G is connected and has no cycles, i.e., G is a tree.*

(2) *Any two vertices in G can be connected by a unique path.*

(3) *G has no cycles, and a cycle is formed if any edge is added to G.*

(4) *G is connected, but is not connected if any single edge is removed from G.*

If the order of G, n, is a finite number, the following conditions are also equivalent to the above statements (cf. [14]*):*

(5) *G is connected and has $n-1$ edges.*

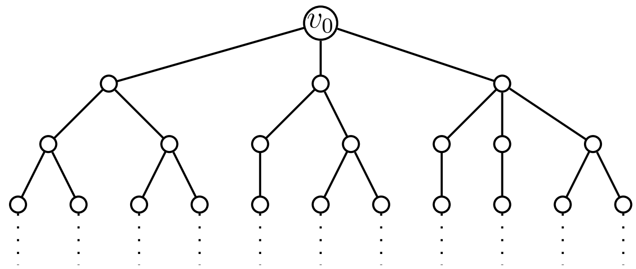
(6) *G has no cycles and has $n-1$ edges.*

Lemma 2.26 [5]**.** *If T is a tree, then $\chi'_{\mathrm{s}}(T) = \sigma(T)$.*

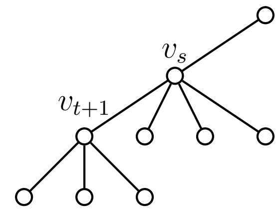
Proof. If T is trivial (it consists of a single vertex), we have $\chi'_{\mathrm{s}}(T) = 0 = \sigma(T)$. Assume that T is not trivial.

A tree is called a rooted tree if a special vertex is singled out and this vertex is called the "root". Let T be a tree rooted at vertex v_0 as indicated in Figure 2.6a.

Define the level function $l : V(T) \to \{0, 1, 2, \dots\}$ such that $l(v) = d(v, v_0)$ for any vertex $v \in V(T)$. Line up v_0 and all the non-leaf vertices of T as $v_0, v_1, v_2, \dots, v_k$ such that $l(v_i) \leqslant l(v_{i+1})$ for $i = 0, 1, 2, \dots, k$. Denote all edges incident to a vertex $v \in V(T)$ by $E(v)$.



(a) A tree rooted at v_0



(b)

Figure 2.6: Figures for the proof of Lemma 2.26

We establish a strong edge coloring of T by using at most $\sigma(T)$ colors by the following steps:

(1) We color $E(v_0)$ by $\deg(v_0) \leqslant \sigma(T)$ colors.

(2) Suppose that $E(v_i)$ have already been colored for all $0 \leqslant i \leqslant t$. Now we color $E(v_{t+1})$ as indicated in Figure 2.6b. Notice that exactly one edge, $v_s v_{t+1} \in E(v_{t+1})$, has already been colored, for some $0 \leqslant s \leqslant t$ with $l(v_s) = l(v_{t+1}) + 1$. We need to color the remaining $\deg(v_{t+1}) - 1$ edges.

Note that the only forbidden colors for $E(v_{t+1})$ are the colors assigned to $E(v_s)$. Hence, the number of available colors for $E(v_{t+1})$ is $\sigma(T) - \deg(v_s) \geqslant \sigma(v_s v_{t+1}) - \deg(v_s) = \deg(v_{t+1}) - 1$. Thus we can color the remaining edges in $E(v_{t+1})$ by legal colors.

Thus, T can be colored by using at most $\sigma(T)$ colors, i.e., $\chi'_{\mathrm{s}}(T) \leqslant \sigma(T)$.

For an edge $uv \in E(T)$ such that $\sigma(uv) = \sigma(T)$, we need $\deg(u) + \deg(v) - 1$ colors to color $E(u) \cup E(v)$. Thus $\chi'_{\mathrm{s}}(T) \geqslant \sigma(T)$. Therefore, $\chi'_{\mathrm{s}}(T) = \sigma(T)$. $\square$

Lemma 2.27 [10]**.** *For the cycle C_n,*

$$\chi'_s(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 3, \\ 5 & \text{if } n = 5, \\ 4 & \text{otherwise.} \end{cases} \tag{2.1}$$

Proof. By Definition 2.22 and Definition 2.23, it is obvious that, for a graph G, $\chi'_s(G) = \chi(L(G)^2)$. For any integer $n \geqslant 3$, $L(C_n) = C_n$. Thus $\chi'_s(C_n) = \chi(C_n^2)$. Note that $C_3^2 = C_3 = K_3$, $C_4^2 = K_4$, $C_5^2 = K_5$. For any integer $n \geqslant 6$, C_n^2 is a 4-regular graph and it is neither a complete graph nor an odd cycle, so, by Theorem 2.24, $\chi(C_n^2) \leqslant \Delta(C_n^2) = 4$.

(i) For $n \equiv 0 \pmod 3$, the vertices of C_n^2 can be colored in an alternating order of $1, 2, 3, 1, 2, 3, \ldots$ with colors $\{1, 2, 3\}$. Thus $\chi'_s(C_n) = \chi(C_n^2) = 3$.

(ii) For $n = 4$, C_4^2 is K_4. As indicated in Figure 2.1b, the four vertices of K_4 are adjacent to each other, so they have to be colored differently. Thus $\chi'_s(C_4) = \chi(C_4^2) = \chi(K_4) = 4$.

(iii) For $n = 5$, C_5^2 is K_5. As indicated in Figure 2.1c, the five vertices of K_5 are adjacent to each other, so they have to be colored differently. Thus $\chi'_s(C_5) = \chi(C_5^2) = \chi(K_5) = 5$.

(iv) Otherwise. Since $n \not\equiv 0 \pmod 3$, C_n^2 can't be colored in the same way as (i), which is the only way of a 3-coloring of any C_n^2 since every three consecutive vertices in C_n^2 on the outer cycle are adjacent to each other. Thus $\chi(C_n^2) \geqslant 4$. Hence, $\chi'_s(C_n) = \chi(C_n^2) = 4$.

Putting these four cases together proves Lemma 2.27. $\square$

Lemma 2.28 [10]**.** *For the wheel W_n,*

$$\chi'_s(W_n) = \begin{cases} n+3 & \textit{if } n \equiv 0 \pmod 3, \\ n+5 & \textit{if } n = 5, \\ n+4 & \textit{otherwise.} \end{cases} \tag{2.2}$$

Proof. Let $W_n = T \cup C_n$ be a wheel. Any edge in the characteristic tree and any edge in the adjoint cycle are within distance 2. Hence, $\chi'_s(W_n) = \|T\| + \chi'_s(C_n) = n + \chi'_s(C_n)$. $\square$

Lemma 2.29 [12]**.** *Suppose $h \geqslant 1$. For the necklace Ne_h,*

$$\chi'_s(Ne_h) = \begin{cases} 6 & \textit{if } h \textit{ is odd,} \\ 7 & \textit{if } h \textit{ is even and } h \geqslant 6, \\ 8 & \textit{if } h = 4, \\ 9 & \textit{if } h = 2. \end{cases} \tag{2.3}$$

Lemma 2.30 [10]**.** *Let $G = T \cup C$ be a Halin graph. Then $\sigma(G) = \sigma(T)$ when G is not a wheel and $\sigma(G) > \sigma(T)$ when G is a wheel.*

Proof. Let $G = T \cup C$ be a Halin graph.

(i) G is a wheel. For the non-leaf vertex vertex $u \in V(T)$, we have $\deg_G \geqslant 3$. All the vertices on the adjoint cycle C has degree three, with respect to G. It's obvious that $\sigma(G) = \sigma_G(e) = n + 2 > n = \sigma(T)$, where $e \in E(G) \cap E(T)$.

(ii) G is not a wheel. Then there are at least two non-leaves in $V(T)$. By Definition 2.12, their degrees are at least three and $\Delta(G) = \Delta(T)$. Thus $\sigma(T) =$

$\max\{\deg_T(u) + \deg_T(v) - 1 \mid u, v$ are adjacent non-leaves$\} \geqslant 3 + \Delta(T) - 1 = \Delta(T) + 2 = \Delta(G) + 2$. For each $e \in E(G)$ incident to $w \in V(C)$, $\sigma_G(e) \leqslant \deg_G(w) + \Delta(G) - 1 = \Delta(G) + 2 \leqslant \sigma(T)$. Thus $\sigma(G) = \max\{\sigma(T), \sigma_G(e) \mid e \in E(G)$ is incident to $w \in V(C)\} = \sigma(T)$.

Therefore, $\sigma(G) = \sigma(T)$ when G is not a wheel and $\sigma(G) > \sigma(T)$ when G is a wheel. $\square$

Lemma 2.31 [10]**.** *Let $G = T \cup C$ be a Halin graph and the characteristic tree T be a double star $D_{x,y}$. Suppose that $y \geqslant 4$, then*

$$\chi'_{\mathrm{s}}(G) = \begin{cases} \chi'_{\mathrm{s}}(T) + 2 & \text{if } x = 3, \\ \chi'_{\mathrm{s}}(T) + 1 & \text{if } x \geqslant 4. \end{cases} \tag{2.4}$$

CHAPTER 3

Known Results and Motivations

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. As we learned in Chapter 2, $\chi'_{\rm s}(G) = \chi(L(G)^2)$, where $L(G)$ is the line graph of G and $L(G)^2$ is the square of $L(G)$.

An *induced matching* in a graph G is a set of edges such that no two edges in this set are of distance (see Definition 2.22) at most two. A strong edge coloring can be equivalently defined as a partition of $E(G)$ into induced matchings.

The following is an outstanding conjecture concerning the strong chromatic index proposed by Faudree et al. [5].

Conjecture 3.1. *For any graph G with maximum degree $\Delta(G)$,*

$$\chi'_{\rm s}(G) \leqslant \begin{cases} \frac{5}{4}\Delta(G)^2 & \text{if } \Delta(G) \text{ is even,} \\ \frac{5}{4}\Delta(G)^2 - \frac{1}{2}\Delta(G) + \frac{1}{4} & \text{if } \Delta(G) \text{ is odd.} \end{cases} \tag{3.1}$$

Conjecture 3.1 is a refined form of the question whether $\chi'_{\rm s}(G) \leqslant \frac{5}{4}\Delta(G)^2$ asked by Erdős and Nešetřil [4]. And, as mentioned in [5], it is easy to see that $\sigma(G) \leqslant \chi'_{\rm s}(G) \leqslant 2\Delta(G)^2 - 2\Delta(G) + 1$. Hence, Conjecture 3.1 holds for $\Delta(G) \leqslant 2$. Conjecture 3.1 was proved to be true for $\Delta(G) = 3$ by Andersen [1], and independently by Horák et al. [7]. It remains open when $\Delta(G) \geqslant 4$.

Recall that for any integer $h \geqslant 1$, $Ne_h = T_h \cup C_{h+2}$ is a cubic Halin graph introduced in Definition 2.13. By Lemma 2.26, we have $\chi'_{\rm s}(T_1) = 3$ and $\chi'_{\rm s}(T_h) = 5$

for $h > 1$.

The following upper bound was first conjectured by Shiu et al. [12].

Conjecture 3.2. *If $G = T \cup C \neq W_5$ is a Halin graph, then $\chi'_s(G) \leqslant \chi'_s(T) + 4$.*

Lih et al. [10] settled this conjecture by proving the following theorem as a stronger result.

Theorem 3.3. *If a Halin graph $G = T \cup C$ is different from Ne_2 and any wheel W_n, $n \not\equiv 0 \pmod 3$, then $\chi'_s(G) \leqslant \chi'_s(T) + 3$.*

A Halin graph must be a cubic graph if its maximum degree is three. For a cubic Halin graph G that is different from all necklaces, Shiu and Tam [13] conjectured that $\chi'_s(G) \leqslant 7$. This conjecture has been confirmed by Lih and Liu [11] in the following theorem.

Theorem 3.4. *If a cubic Halin graph $G = T \cup C$ is different from Ne_2 and Ne_4, then $\chi'_s(G) \leqslant 7$.*

Theorem 3.4 lowered the upper bound suggested in Theorem 3.3 for such cubic Halin graphs since $\chi'_s(T) = 5$ for trees whose non-leaf vertices all have degree three and thus

$$\chi'_s(G) \leqslant 7 = \chi'_s(T) + 2 < \chi'_s(T) + 3. \tag{3.2}$$

We want to know whether this upper bound also works for Halin graphs with higher maximum degrees. We are interested in the following conjecture.

Conjecture 3.5. *If a Halin graph $G = T \cup C$ is different from Ne_2, Ne_4, and any wheel W_n, then $\chi'_s(G) \leqslant \chi'_s(T) + 2$.*

Recall that a Halin graph G consists two parts: the characteristic tree T and the adjoint cycle C. By Lemma 2.26, the strong chromatic index of the tree T is equal to the maximum of $\deg(u) + \deg(v) - 1$ over any $uv \in E(T)$. Since the degrees of the vertices on C are always three and the degrees of the vertices in T, except the leaves, are no less than three, then $\Delta(G) = \Delta(T)$. By Lemma 2.26, we have

$$\chi_{\mathrm{s}}'(T)+2 = \max\{\deg_T(u)+\deg_T(v) \mid uv \in E(T)\}+1 \leqslant 2\Delta(T)+1 = 2\Delta(G)+1. \quad (3.3)$$

This is an upper bound weaker than Conjecture 3.5 but stronger than Conjecture 3.2 for some cases.

In the next chapter, we are going to prove this upper bound, $2\Delta(G) + 1$, for the strong chromatic index for all Halin graphs except for a few special graphs. In addition, we extend the result of Theorem 3.4 to a maximum degree of four.

CHAPTER 4

Main Results

In this chapter, we prove the following main result of the thesis.

Theorem 4.1. *If a Halin graph $G = T \cup C$ is different from Ne_2 and Ne_4, then $\chi'_{\mathrm{s}}(G) \leqslant 2\varDelta(G) + 1$.*

Proof. Let $G = T \cup C$ be a Halin graph other than Ne_2 or Ne_4. By Theorem 3.4 and equations (3.2) and (3.3), it is easy to see that $\chi'_{\mathrm{s}}(G) \leqslant 2\varDelta(G) + 1$ is true if $\varDelta(G) = 3$. Thus we assume $\varDelta(G) \geqslant 4$.

We first consider two special families of Halin graphs.

(1) For any integer $n \geqslant 3$, consider W_n as defined in Definition 2.15 with $\varDelta(W_n) = n$. Since $\varDelta(G) \geqslant 4$, might as well assume $n \geqslant 4$ and then, by Lemma 2.28, $\chi'_{\mathrm{s}}(W_n) \leqslant n + 5 \leqslant 2n + 1 = 2\varDelta(W_n) + 1$ for all $n \geqslant 4$.

(2) Consider the Halin graph $D = D_{x,y} \cup C$ where its characteristic tree is a double star $D_{x,y}$ defined in Definition 2.14. Then, by Lemma 2.31 and (3.3), $\chi'_{\mathrm{s}}(D) \leqslant \chi'_{\mathrm{s}}(D_{x,y}) + 2 \leqslant 2\varDelta(D) + 1$.

Thus, for these two special families of Halin graphs, Theorem 4.1 is true.

Let $|E(C)| = m$. Clearly, $m \geqslant 3$. The remainder of the proof of Theorem 4.1 proceeds by induction on m.

Base cases:

- $m = 3$. W_3 is the only possible Halin graph. But $\varDelta(W_n) = 3$ and we are assuming $\Delta(G) \geqslant 4$.

- $m = 4$. W_4 and Ne_2 are the only possible Halin graphs. By Lemma 2.29, $\chi'_s(Ne_2) = 9 > 7 = 2\Delta(Ne_2) + 1$. Thus we exclude Ne_2 in Theorem 4.1.

- $m = 5$. There are three Halin graphs.

  (i) $G = W_5$. It belongs to one of the special families of Halin graphs which are discussed above.

  (ii) A Halin graph $G = T \cup C$ where $T = D_{3,4}$, i.e., the characteristic tree of G is a double star $D_{3,4}$. Such a Halin graph is also discussed above.

  (iii) $G = Ne_3$. By Lemma 2.29, $\chi'_s(G) = 6 \leqslant 2\Delta(G) + 1$.

- $m = 6$. There are three cases.

  (i) $\Delta(G) = 4$. There are four Halin graphs as shown in Figure 4.1.

  In Figure 4.1a, the characteristic tree of the Halin graph G is the double star $D_{4,4}$ and it is already done previously. If G is any of the other three graphs, as shown by the labels on the graphs with the desired bounds, $\chi'_s(G) \leqslant 8 < 2\Delta(G) + 1$.

  (ii) $\Delta(G) = 5$. The only Halin graph $G = T \cup C$ has the characteristic tree $T = D_{3,5}$, a double star. It belongs to one of the special families of Halin graphs we discussed previously.

  (iii) $\Delta(G) = 6$. The only Halin graph is W_6 and it is done previously.

Induction steps: Assume $m \geqslant 7$.

Let $P = u_0, u_1, \ldots, u_l$ be a longest path in T. Let l be the length of P and, clearly, $l = |P| \geqslant 2$.

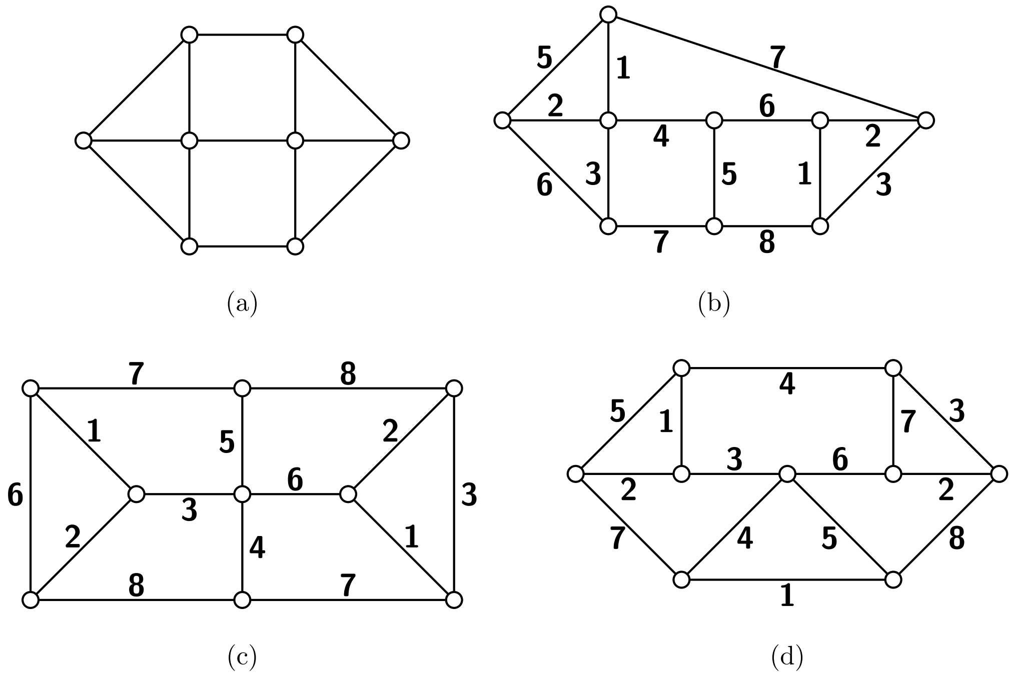


Figure 4.1: Halin graphs where $m = 6$ and $\Delta(G) = 4$

(i) For $l = 2$, G is a wheel W_n and $\chi'_{\mathrm{s}}(W_n) \leqslant 2\Delta(W_n) + 1$ for any integer $n \geqslant 3$.

(ii) For $l = 3$, the characteristic tree of G is a double star, so it is done previously.

Thus we assume $l \geqslant 4$. Without loss of generality, we also assume that $\deg_G(u_{l-1}) \geqslant \deg_G(u_1)$.

Let $u_1 = v$, $u_2 = u$, $u_3 = w$, and label the $k \geqslant 2$ neighbors of v (except u) as $v_1, v_2, \ldots, v_k$. Since P is a longest path in T, it is easy to see that $v_1, v_1, \ldots, v_k$ must be on the adjoint cycle C. Let $x_1, x_2, y_1, y_2 \in V(C)$, where $x_1 \sim v_1$, $x_1 \sim x_2$, $y_1 \sim v_k$, $y_1 \sim y_2$, and $x_3, y_3 \notin V(C)$, where $x_1 \sim x_3$, $y_1 \sim y_3$.

Claim: *There exists a path Q in T from u to x_1 or from u to y_1 with $P \cap Q = \{u\}$.*

Proof. Suppose not, i.e., for any path Q in T from u to x_1 or from u to y_1, we have $w \in P \cap Q$. Without loss of generality, assume that Q is from u to x_1. P divides the inner area of the adjoint cycle C into two parts, say top and bottom, as depicted in

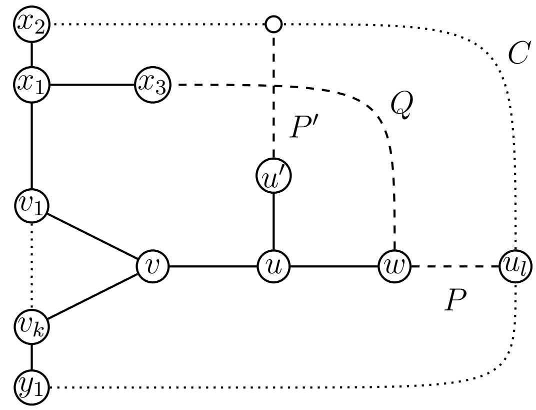


Figure 4.2: The non-existence of the path Q

Figure 4.2. Since $\deg_G(u) \geqslant 3$, there exists a neighbor of u other than v or w. Let $u' \in N(u) \setminus \{v, w\}$ be that neighbor. Without loss of generality, assume that u' is on the same side of P as x_1. Since the characteristic tree T is a plane drawing of a tree, u' must be on the unique path P' from u to a leaf on the top half of the cycle C in between x_1 and u_l. As shown in Figure 4.2, P' divides the top half area into two parts, say left and right. Since we assume that the path Q from u to x_1 is through w, Q must go across P', as indicated in Figure 4.2. Then, one of the following will occur:

(i) P' and Q share a common vertex u^* where they cross each other, i.e., $P' \cap Q = \{u, u^*\}$. Then, a cycle $u, w, \ldots, u^*, \ldots, u', u$ is created.

(ii) An edge in Q crosses an edge in P' where the two paths intersects.

By Definition 2.10 and Definition 2.11, neither of the above is allowed. Thus, such a path Q that goes through w does not exist. Then, for a path from u to x_1 or y_1, $P \cap Q = \{u\}$. $\square$

Without loss of generality, we assume that Q is from u to y_1. Since P is a longest path, it is obvious that $|Q| \leqslant 2$. Hence $u = y_3$ or $u \sim y_3$ (we will have similar result if Q is from u to x_1).

In the rest of the proof, we use notations defined as follows. For a strong edge coloring ρ of graph G, we denote the set of colors already used by edges incident to a vertex a by $C_\rho(a)$. We denote the set of forbidden colors for an edge e by $F_\rho(e)$ and denote $|e|_\rho = |F_\rho(e)|$. Assuming that the total number of colors used in ρ is fixed, we denote the set of available colors for an edge e by $A_\rho(e)$ and denote $\|e\|_\rho = |A_\rho(e)|$.

In the following, $G' = T' \cup C'$ will be a Halin graph obtained by adding some new edges to an induced subgraph of G (delete some vertices from $V(G)$) such that the length of the adjoint cycle C' is shorter than the length of the adjoint cycle C of the original graph G, i.e., $|E(C')| < |E(C)|$. We call G' a *reduction* of G. Depending on various situations, different types of G' will be used in the following proof.

Let ψ be a strong edge coloring of G' using the minimum number of colors. In general, a strong edge coloring ϕ of G is constructed as follows: we color all edges in both G and G' with the same colors used in ψ, i.e., for every $e \in E(G) \cap E(G')$, let $\phi(e) = \psi(e)$. For every $e \in E(G) \setminus E(G')$, we develop different coloring schemes for different cases.

If G' is neither Ne_2 nor Ne_4, by the induction hypothesis, we have $\chi'_s(G') \leqslant 2\Delta(G') + 1$. Otherwise, by Lemma 2.29, $\chi'_s(Ne_2) = 9$ and $\chi'_s(Ne_4) = 8$. Thus, we only need $\Delta(G') \leqslant \Delta(G)$ to be true so that it is possible for us to find the strong edge coloring ϕ using no more than $2\Delta(G) + 1 \geqslant 9$ colors.

For each of the following cases, we complete the construction of the strong edge coloring ϕ with no more than $2\Delta(G) + 1$ colors.

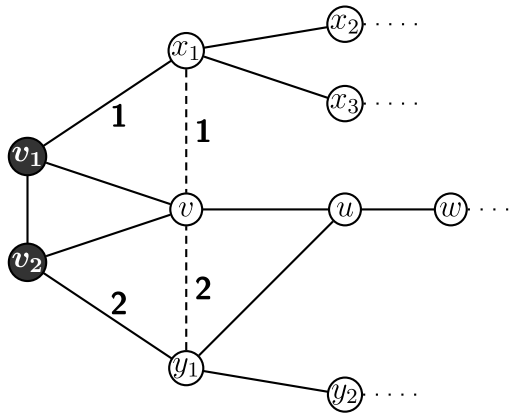


Figure 4.3: Case 1.1 in the proof of Theorem 4.1

$\boxed{\textit{Case } 1.1.}$ $k = 2$ (i.e., $\deg_G(v) = 3$) and $u = y_3$, where k is the number of neighbors of v on the adjoint cycle C.

Obtain the reduction G' of G by adding two new edges vx_1 and vy_1 to the induced subgraph of G on the vertex set $V(G) \setminus \{v_1, v_2\}$ as indicated in Figure 4.3. Clearly, $\Delta(G') = \Delta(G)$.

Without loss of generality, assume that $\psi(vx_1) = 1$ and $\psi(vy_1) = 2$. Let $\phi(v_1x_1) = \psi(vx_1)$ and $\phi(v_2y_1) = \psi(vy_1)$, i.e., $\phi(v_1x_1) = \psi(vx_1) = 1$ and $\phi(v_1x_1) = \psi(vx_1) = 2$ (we will use such abbreviation in the remaining proof). We will color the rest of edges of G, vv_1, vv_2, and v_1v_2, one after another.

(i) For $\phi(y_1y_2) = \psi(y_1y_2) \notin C_\psi(x_1)$, we let $\phi(vv_1) = \phi(y_1y_2) = 3$.

Notice that $F_\phi(vv_2) \subseteq C_\phi(u) \cup C_\phi(v_1) \cup C_\phi(y_1) = C_\phi(u) \cup \{1, 2, 3\}$.

Thus $|vv_2|_\phi \leqslant \deg_G(u) + 3$, and

$\|vv_2\|_\phi \geqslant 2\Delta(G) + 1 - (\deg_G(u) + 3) \geqslant (\Delta(G) - \deg_G(u)) + (\Delta(G) - 2) \geqslant 2.$

Thus, we color vv_2 by one of the available colors.

Next, $F_\phi(v_1v_2) \subseteq C_\phi(v) \cup C_\phi(x_1) \cup C_\phi(y_1)$

$\Rightarrow F_\phi(v_1v_2) \subseteq \{1, 2, 3, \phi(vv_2), \phi(vu), \phi(uy_1), \phi(x_1x_2), \phi(x_1x_3)\}$

$\Rightarrow |v_1v_2|_\phi \leqslant 8$

$\Rightarrow \|v_1v_2\|_\phi \geqslant 2\Delta(G) + 1 - 8 \geqslant 1.$

Thus, we color v_1v_2 by a legal color.

(ii) For $\phi(y_1y_2) = \psi(y_1y_2) \in C_\psi(x_1)$, we cannot color vv_1 by $\phi(y_1y_2) = 3$.

Notice that $F_\phi(vv_1) \subseteq C_\phi(u) \cup C_\phi(v_2) \cup C_\phi(x_1) = C_\phi(u) \cup \{1, 2, \phi(x_1x_2), \phi(x_1x_3)\}$.

Thus $|vv_1|_\phi \leqslant \deg_G(u) + 4$, and

$\|vv_1\|_\phi \geqslant 2\Delta(G) + 1 - (\deg_G(u) + 4) \geqslant 1.$

Then we assign a legal color to vv_1.

Next, $F_\phi(vv_2) \subseteq C_\phi(u) \cup C_\phi(v_1) \cup C_\phi(y_1) = C_\phi(u) \cup \{1, 2, \phi(vv_1), \phi(y_1y_2)\}$

$\Rightarrow |vv_2|_\phi \leqslant \deg_G(u) + 4$

$\Rightarrow \|vv_2\|_\phi \geqslant 2\Delta(G) + 1 - (\deg_G(u) + 4) \geqslant 1.$

Thus, we color vv_2 by an available color.

Similar to the previous condition, $|v_1v_2|_\phi \leqslant 8 \Rightarrow \|v_1v_2\|_\phi \geqslant 1$. Thus, we color v_1v_2 by a legal color.

Hence, we have colored all the edges of G with no more than $2\Delta(G)+1$ colors.

$\boxed{\textit{Case } 1.2.}$ $k = 2$ (i.e., $\deg_G(v) = 3$) and $u \sim y_3$.

Subcase 1.2.1. $\deg_G(y_3) = 3$.

Let $z_1 \sim y_2$, $z_2 \sim z_1$, and $z_3 \sim z_1$, where $z_1, z_2 \in V(C)$ and $z_3 \notin V(C)$. Obtain the reduction G' of G by adding three new edges vx_1, vy_3, and y_3z_1 to the induced subgraph of G on the vertex set $V(G) \setminus \{v_1, v_2, y_1, y_2\}$, as depicted in Figure 4.4. Clearly, $\Delta(G') = \Delta(G)$.

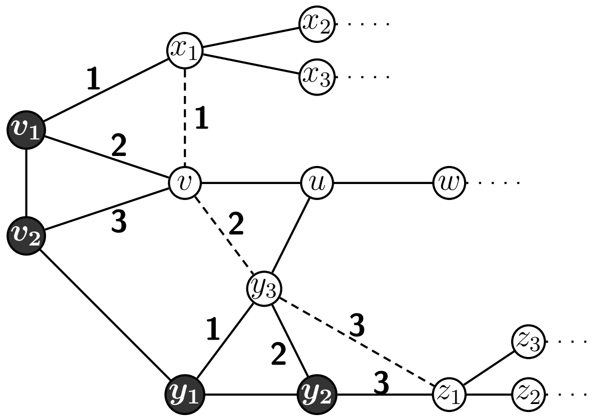


Figure 4.4: Subcase 1.2.1 in the proof of Theorem 4.1

Without loss of generality, let $\phi(v_1x_1) = \phi(y_1y_3) = \psi(vx_1) = 1$, $\phi(vv_1) = \phi(y_2y_3) = \psi(vy_3) = 2$, and $\phi(vv_2) = \phi(y_2z_1) = \psi(y_3z_1) = 3$ as indicated in Figure 4.4. We will color the remaining edges of G, v_1v_2, y_1y_2, and v_2y_1, one by one.

Note, $F_\phi(v_1v_2) \subseteq C_\phi(v) \cup C_\phi(x_1) \cup C_\phi(y_1) = \{1, 2, 3, \phi(vu), \phi(x_1x_2), \phi(x_1x_3)\}$

$\Rightarrow |v_1v_2|_\phi \leqslant 6$

$\Rightarrow \|v_1v_2\|_\phi \geqslant 2\Delta(G) + 1 - 6 \geqslant 3.$

Thus, we color v_1v_2 by one of the legal colors.

Next, $F_\phi(y_1y_2) \subseteq C_\phi(v_2) \cup C_\phi(y_3) \cup C_\phi(z_1)$

$\Rightarrow F_\phi(y_1y_2) \subseteq \{1, 2, 3, \phi(v_1v_2), \phi(uy_3), \phi(z_1z_2), \phi(z_1z_3)\}$

$\Rightarrow |y_1y_2|_\phi \leqslant 7$

$\Rightarrow \|y_1y_2\|_\phi \geqslant 2\Delta(G) + 1 - 7 \geqslant 2.$

Thus we color y_1y_2 by one of the legal colors.

Finally, $F_\phi(v_2y_1) \subseteq C_\phi(v) \cup C_\phi(v_1) \cup C_\phi(y_2) \cup C_\phi(y_3)$

$\Rightarrow F_\phi(v_2y_1) \subseteq \{1, 2, 3, \phi(v_1v_2), \phi(vu), \phi(uy_3), \phi(y_1y_2)\}$

$\Rightarrow |v_2y_1|_\phi \leqslant 7$

$\Rightarrow \|v_2y_1\|_\phi \geqslant 2\Delta(G)+1-7 \geqslant 2.$

Thus we color v_2y_1 by one of the available colors.

Hence, we have colored all the edges of G with no more than $2\Delta(G)+1$ colors.

Subcase 1.2.2. $\deg_G(y_3) \geqslant 4$ and $\Delta(G) = 4$. So $\deg_G(y_3) = 4$.

Let z be the fourth neighbor of y_3 and since P is a longest path, z must be on the adjoint cycle C. Let $z_1 \sim z$, $z_2 \sim z_1$, and $z_3 \sim z_1$, where $z_1, z_2 \in V(C)$ and $z_3 \notin V(C)$. Obtain the reduction G' of G by adding three new edges vx_1, vy_3, and y_3z_1 to the induced subgraph of G on the vertex set $V(G) \setminus \{v_1, v_2, y_1, y_2, z\}$, as shown in Figure 4.5. Clearly, $\Delta(G') \leqslant \Delta(G)$.

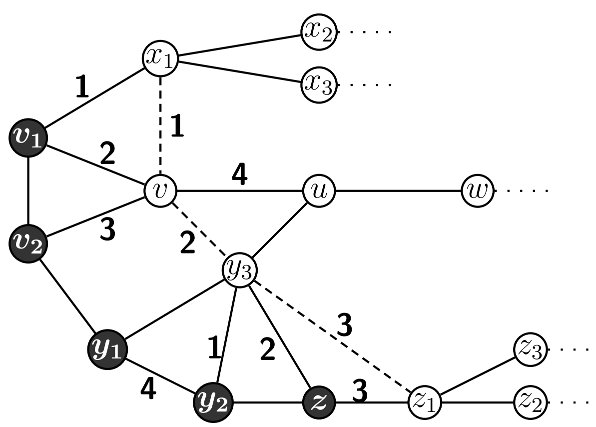


Figure 4.5: Subcase 1.2.2 in the proof of Theorem 4.1

Without loss of generality, let $\phi(v_1x_1) = \phi(y_2y_3) = \psi(vx_1) = 1$, $\phi(vv_1) = \phi(y_3z) = \psi(vy_3) = 2$, $\phi(vv_2) = \phi(zz_1) = \psi(y_3z_1) = 3$, and $\phi(y_1y_2) = \phi(vu) = 4$ as indicated in Figure 4.5. We will color the remaining edges of G, v_1v_2, v_2y_1, y_1y_3, and y_2z, one after another.

First, $F_\phi(y_1y_3) \subseteq C_\phi(u) \cup C_\phi(v_2) \cup C_\phi(y_2) \cup C_\phi(z)$

$\Rightarrow |y_1y_3|_\phi \leqslant \deg_G(u) + 3$

$\Rightarrow \|y_1y_3\|_\phi \geqslant 2\Delta(G) + 1 - (\deg_G(u) + 3) \geqslant 2.$

Thus we color y_1y_3 by one of the available colors.

Next, $F_\phi(v_1v_2) \subseteq C_\phi(v) \cup C_\phi(x_1) \cup C_\phi(y_1)$

$\Rightarrow |v_1v_2|_\phi \leqslant 7$

$\Rightarrow \|v_1v_2\|_\phi \geqslant 2\Delta(G) + 1 - 7 = 2.$

Thus we color v_1v_2 with one of the legal colors.

Observe that $F_\phi(y_2z) \subseteq C_\phi(y_1) \cup C_\phi(y_3) \cup C_\phi(z_1)$

$\Rightarrow |y_2z|_\phi \leqslant 8$

$\Rightarrow \|y_2z\|_\phi \geqslant 2\Delta(G) + 1 - 8 \geqslant 1.$

Thus we color y_2z by an available color.

Finally, $F_\phi(v_2y_1) \subseteq C_\phi(v) \cup C_\phi(v_1) \cup C_\phi(y_2) \cup C_\phi(y_3)$

$\Rightarrow |v_2y_1|_\phi \leqslant 8$

$\Rightarrow \|v_2y_1\|_\phi \geqslant 2\Delta(G) + 1 - 8 \geqslant 1.$

Thus we color v_2y_1 by an available color.

Hence, we have colored all the edges of G with no more than $2\Delta(G)+1$ colors.

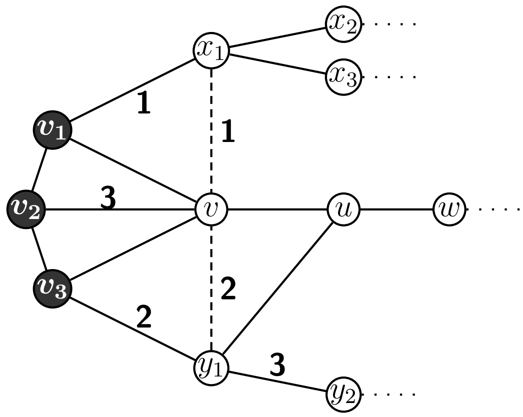
Subcase 1.2.3. $\deg_G(y_3) \geqslant 4$ and $\Delta(G) \geqslant 5$.

Take $P = y_1, y_3, u, w, u_4, \ldots, u_l$ as a longest path. Then this subcase is covered in Case 2.2.

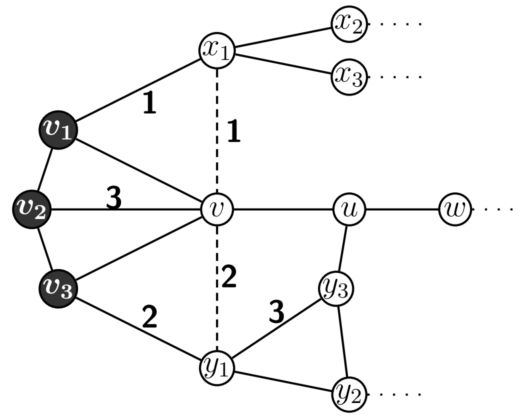
Case 2.1. $k \geqslant 3$ and $\Delta(G) = 4$. Then $\deg_G(v) = 4$ and $k = 3$.

Subcase 2.1.1. $\deg_G(u) = 3$.

Obtain the reduction G' of G by adding two new edges vx_1 and vy_1 to the induced subgraph of G on the vertex set $V(G) \setminus \{v_1, v_2, v_3\}$ as depicted in Figure 4.6.



(a) $u = y_3$



(b) $u \sim y_3$

Figure 4.6: Subcase 2.1.1 in the proof of Theorem 4.1

Since we assumed earlier that $\deg_G(u_{l-1}) \geqslant \deg_G(u_1) = \deg_G(v) = 4$, we have $\Delta(G') = \Delta(G) = 4$.

Without loss of generality, let $\phi(v_1x_1) = \psi(vx_1) = 1$, $\phi(v_3y_1) = \psi(vy_1) = 2$, and additionally:

(i) If $u = y_3$, we define $\phi(vv_2) = \phi(y_1y_2) = 3$ as indicated in Figure 4.6a;

(ii) If $u \sim y_3$, we define $\phi(vv_2) = \phi(y_1y_3) = 3$ as indicated in Figure 4.6b.

We are going to color the remaining edges, vv_1, vv_3, v_1v_2, and v_2v_3, one by one.

First, $F_\phi(vv_1) \subseteq C_\phi(u) \cup C_\phi(x_1) \cup C_\phi(v_2) \cup C_\phi(v_3)$

$\Rightarrow |vv_1|_\phi \leqslant \deg_G(u) + 5 = 8$

$\Rightarrow \|vv_1\|_\phi \geqslant 2\Delta(G) + 1 - 8 = 1.$

Thus we color vv_1 by a legal color.

Next, $F_\phi(vv_3) \subseteq C_\phi(u) \cup C_\phi(v_1) \cup C_\phi(v_2) \cup C_\phi(y_1)$

$\Rightarrow |vv_3|_\phi \leqslant \deg_G(u) + 4 = 7$

$\Rightarrow \|vv_3\|_\phi \geqslant 2\Delta(G) + 1 - 7 = 2.$

Thus we assign an available color to vv_3.

Next, $F_\phi(v_1v_2) \subseteq C_\phi(v) \cup C_\phi(x_1) \cup C_\phi(v_3)$

$\Rightarrow F_\phi(v_1v_2) \subseteq \{1, 2, 3, \phi(uv), \phi(vv_1), \phi(vv_3), \phi(x_1x_2), \phi(x_1x_3)\}$

$\Rightarrow |v_1v_2|_\phi \leqslant 8$

$\Rightarrow \|v_1v_2\|_\phi \geqslant 2\Delta(G) + 1 - 8 = 1.$

Thus we color v_1v_2 by an available color.

Finally, $F_\phi(v_2v_3) \subseteq C_\phi(v) \cup C_\phi(v_1) \cup C_\phi(y_1)$

$\Rightarrow F_\phi(v_1v_2) \subseteq \{1, 2, 3, \phi(uv), \phi(uy_1), \phi(vv_1), \phi(vv_3), \phi(v_1v_2)\}$

$\Rightarrow |v_2v_3|_\phi \leqslant 8$

$\Rightarrow \|v_2v_3\|_\phi \geqslant 2\Delta(G) + 1 - 8 = 1.$

Thus we color v_2v_3 by a legal color.

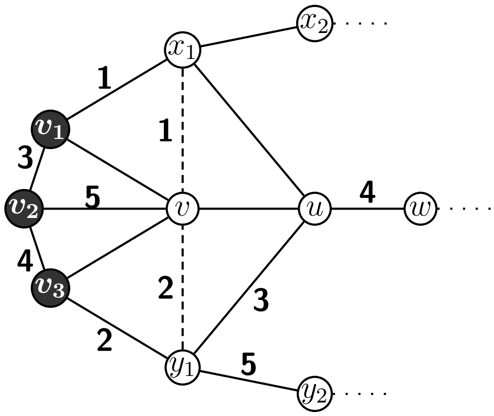
Hence, we have colored all the edges of G with no more than $2\Delta(G)+1$ colors.

Subcase 2.1.2. $\deg_G(u) = 4$.

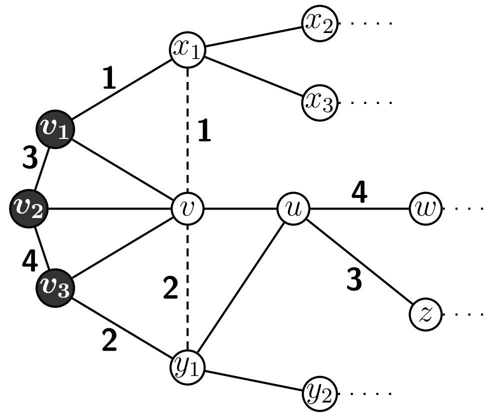
For conditions (1), (2), and (3) listed below, we obtain the reduction G' of G by adding two new edges vx_1 and vy_1 to the induced subgraph of G on the vertex set $V(G) \setminus \{v_1, v_2, v_3\}$, as shown in Figure 4.7. Clearly, $\Delta(G') = \Delta(G)$. Without loss of generality, let $\phi(v_1x_1) = \psi(vx_1) = 1$, $\phi(v_3y_1) = \psi(vy_1) = 2$. We color the remaining edges, vv_1, vv_2, vv_3, v_1v_2, and v_2v_3, as follows.

Consider the following conditions.

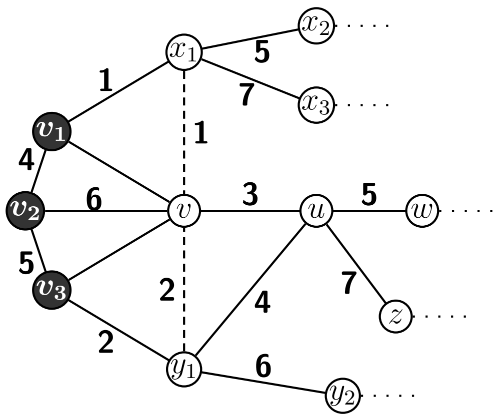
(1) $u = y_3$ and $u \sim x_1$. Let $\phi(v_1v_2) = \phi(uy_1) = 3$, $\phi(v_2v_3) = \phi(uw) = 4$, and $\phi(vv_2) = \phi(y_1y_2) = 5$ as indicated in Figure 4.7a. We color the remaining edges, vv_1 and vv_3, one by one.



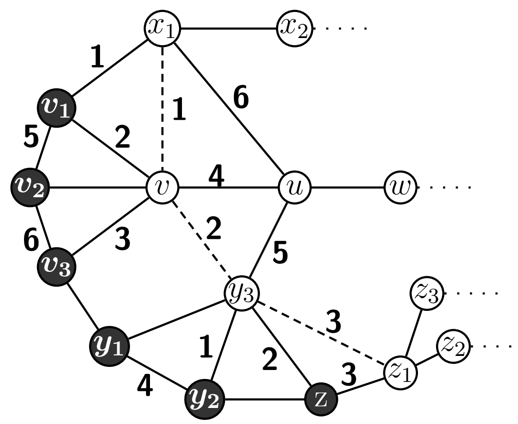
(a) Condition (1)



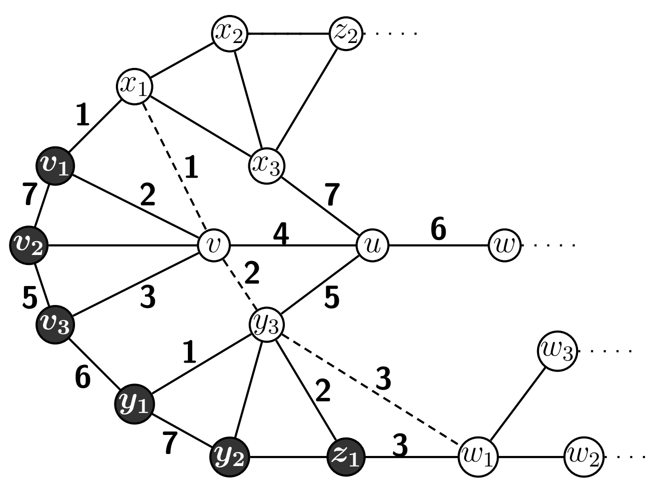
(b) Condition (2)



(c) Condition (3)



(d) Condition(5)



(e) Condition (7)

Figure 4.7: Subcase 2.1.2 in the proof of Theorem 4.1

Note, $F_\phi(vv_1) \subseteq C_\phi(u) \cup C_\phi(x_1) \cup C_\phi(v_2) \cup C_\phi(v_3)$

$\Rightarrow |vv_1|_\phi \leqslant 8$

$\Rightarrow \|vv_1\|_\phi \geqslant 2\Delta(G) + 1 - 8 = 1.$

Thus we color vv_1 with an available color.

Next, $F_\phi(vv_3) \subseteq C_\phi(u) \cup C_\phi(y_1) \cup C_\phi(v_1) \cup C_\phi(v_2)$

$\Rightarrow |vv_3|_\phi \leqslant 8$

$\Rightarrow \|vv_3\|_\phi \geqslant 2\Delta(G) + 1 - 8 = 1.$

Thus we assign a legal color to vv_3.

(2) $u = y_3$, $u \not\sim x_1$ and $|\{\phi(uw), \phi(uz)\} \cap \{\phi(x_1x_2), \phi(x_1x_3)\}| \leqslant 1$, where z is the fourth neighbor of u. Without loss of generality, assume that $\phi(uz) \notin \{\phi(x_1x_2), \phi(x_1x_3)\}$. Let $\phi(v_1v_2) = \phi(uz) = 3$ and $\phi(v_2v_3) = \phi(uw) = 4$ as indicated in Figure 4.7b. We color the remaining edges, vv_1, vv_3, and vv_2, one after another.

Note, $F_\phi(vv_1) \subseteq C_\phi(u) \cup C_\phi(x_1) \cup C_\phi(v_2) \cup C_\phi(v_3)$

$\Rightarrow |vv_1|_\phi \leqslant 8$

$\Rightarrow \|vv_1\|_\phi \geqslant 2\Delta(G) + 1 - 8 = 1.$

Thus we color vv_1 with an available color.

Next, $F_\phi(vv_3) \subseteq C_\phi(u) \cup C_\phi(y_1) \cup C_\phi(v_1) \cup C_\phi(v_2)$

$\Rightarrow |vv_3|_\phi \leqslant 8$

$\Rightarrow \|vv_3\|_\phi \geqslant 2\Delta(G) + 1 - 8 = 1.$

Thus we color vv_3 by a legal color.

Finally, $F_\phi(vv_2) \subseteq C_\phi(u) \cup C_\phi(v_1) \cup C_\phi(v_3)$

$\Rightarrow |vv_2|_\phi \leqslant 8$

$\Rightarrow \|vv_2\|_\phi \geqslant 2\mathit{\Delta}(G) + 1 - 8 = 1.$

Thus we assign an available color to vv_2.

(3) $u = y_3$, $u \nsim x_1$, and $\{\phi(uw), \phi(uz)\} = \{\phi(x_1x_2), \phi(x_1x_3)\}$ where z is the fourth neighbor of u. Without loss of generality, we assume that $\phi(x_1x_2) = \phi(uw) = 5$ and $\phi(x_1x_3) = \phi(uz) = 7$. Let $\phi(uv) = 3$, $\phi(v_1v_2) = \phi(uy_1) = 4$, $\phi(v_2v_3) = 5$, and $\phi(vv_2) = \phi(y_1y_2) = 6$ as indicated in Figure 4.7c. Clearly, vv_1 and vv_3 can be colored by any two colors other than $\{1, 2, \ldots, 7\}$.

(4) $u \sim y_3$ and $\deg_G(y_3) = 3$. Take $P = y_1, y_3, u, w, u_4, \ldots, u_l$ as a longest path and such a graph is already covered in Subcase 1.2.2.

(5) $u \sim y_3$, $\deg_G(y_3) = 4$, and $u = x_3$. Let z be the fourth neighbor of y_3, $z_1 \sim z$, $z_2 \sim z_1$, and $z_3 \sim z_1$, where $z_1, z_2 \in V(C)$ and $z_3 \notin V(C)$. Obtain the reduction of G by adding three new edges vx_1, vy_3, and y_3z_1 to the induced subgraph of G on the vertex set $V(G) \setminus \{v_1, v_2, v_3, y_1, y_2, z\}$, as depicted in Figure 4.7d. Since $\deg_G(u_{l-1}) \geqslant \deg_G(v) = 4$, obviously $\mathit{\Delta}(G') = \mathit{\Delta}(G)$. Without loss of generality, let $\phi(v_1x_1) = \phi(y_2y_3) = \psi(vx_1) = 1$, $\phi(vv_1) = \phi(y_3z) = \psi(vy_3) = 2$, $\phi(vv_3) = \phi(zz_1) = \psi(y_3z_1) = 3$, $\phi(y_1y_2) = \phi(vu) = 4$, $\phi(v_1v_2) = \phi(uy_3) = 5$, and $\phi(v_2v_3) = \phi(ux_1) = 6$ as indicated in Figure 4.7d. We color the remaining vertices, y_2z, y_1y_3, v_3y_1, and vv_2, one by one.

First, $F_\phi(y_2z) \subseteq C_\phi(y_1) \cup C_\phi(y_3) \cup C_\phi(z_1)$

$\Rightarrow |y_2z|_\phi \leqslant 7$

$\Rightarrow \|y_2z\|_\phi \geqslant 2\mathit{\Delta}(G) + 1 - 7 = 2.$

Thus we assign one of the available colors to y_2z.

Next, $F_\phi(y_1y_3) \subseteq C_\phi(u) \cup C_\phi(v_3) \cup C_\phi(y_2) \cup C_\phi(z)$

$\Rightarrow |y_1y_3|_\phi \leqslant 8$

$\Rightarrow \|y_1y_3\|_\phi \geqslant 2\Delta(G) + 1 - 8 = 1.$

Thus we color y_1y_3 using a legal color.

Next, $F_\phi(v_3y_1) \subseteq C_\phi(v) \cup C_\phi(v_2) \cup C_\phi(y_2) \cup C_\phi(y_3)$

$\Rightarrow |v_3y_1|_\phi \leqslant 8$

$\Rightarrow \|v_3y_1\|_\phi \geqslant 2\Delta(G) + 1 - 8 = 1.$

Thus we color v_3y_1 by an available color.

Finally, $F_\phi(vv_2) \subseteq C_\phi(u) \cup C_\phi(v_1) \cup C_\phi(v_3)$

$\Rightarrow |vv_2|_\phi \leqslant 8$

$\Rightarrow \|vv_2\|_\phi \geqslant 2\Delta(G) + 1 - 8 = 1.$

Thus we color vv_2 with an available color.

(6) If $u \sim y_3$, $\deg_G(y_3) = 4$, $u \sim x_3$, and $\deg_G(x_3) = 3$. Take $P = x_1, x_3, u, w, u_4, \ldots, u_l$ as a longest path and such a graph is already covered in Subcase 1.2.2.

(7) $u \sim y_3$, $\deg_G(y_3) = 4$, $u \sim x_3$, and $\deg_G(x_3) = 4$. Let $w_2 \sim w_1 \sim z_1 \sim y_2$ and $z_2 \sim x_2$ be vertices in $V(C)$, and let $w_3 \notin V(C)$ where $w_3 \sim w_1$, see Figure 4.7e. Obtain the reduction of G by adding three new edges vx_1, vy_3, and y_3w_1 to the induced subgraph on the vertex set $V(G)\backslash\{v_1, v_2, v_3, y_1, y_2, z_1\}$, as depicted in Figure 4.7e. Since $\deg_G(u_{l-1}) \geqslant \deg_G(v) = 4$, we have $\Delta(G') = \Delta(G)$. Without loss of generality, let $\phi(v_1x_1) = \phi(y_1y_3) = \psi(vx_1) = 1$, $\phi(vv_1) = \phi(y_3z_1) = \psi(vy_3) = 2$, $\phi(vv_3) = \phi(z_1w_1) = \psi(y_3w_1) = 3$, $\phi(v_2v_3) = \phi(uy_3) = 5$, $\phi(v_3y_1) = \phi(uw) = 6$,

and $\phi(v_1v_2) = \phi(y_1y_2) = \phi(ux_3) = 7$ as indicated in Figure 4.7e. We color the remaining edges, y_2z_1, y_2y_3, and vv_2, one after another.

First, $F_\phi(y_2z_1) \subseteq C_\phi(y_1) \cup C_\phi(y_3) \cup C_\phi(w_1)$

$\Rightarrow |y_2z_1|_\phi \leqslant 8$

$\Rightarrow \|y_2z_1\|_\phi \geqslant 2\mathit{\Delta}(G) + 1 - 8 = 1.$

Thus we color y_2z_1 with an available color.

Next, $F_\phi(y_2y_3) \subseteq C_\phi(u) \cup C_\phi(y_1) \cup C_\phi(z_1)$

$\Rightarrow |y_2y_3|_\phi \leqslant 8$

$\Rightarrow \|y_2y_3\|_\phi \geqslant 2\mathit{\Delta}(G) + 1 - 8 = 1.$

Thus we color y_2y_3 with a legal color.

Finally, we color vv_2 with $\phi(y_2y_3)$.

Hence, we have colored all the edges of G with no more than $2\mathit{\Delta}(G)+1$ colors.

$\boxed{\textit{Case } 2.2.}$ $k \geqslant 3$ and $\mathit{\Delta}(G) \geqslant 5$.

Obtain the reduction of G by adding two new edges vx_1 and vy_1 to the induced subgraph of G on the vertex set $V(G) \setminus \{v_1, v_2, \ldots, v_k\}$, see Figure 4.8. Since $\deg_G(u_{l-1}) \geqslant \deg_G(v)$, we have $\mathit{\Delta}(G) = \mathit{\Delta}(G')$. Without loss of generality, let $\phi(v_1x_1) = \psi(vx_1) = 1$ and $\phi(v_ky_1) = \psi(vy_1) = 2$. For $u = y_3$ or $u \sim y_3$, let $\phi(vv_2) = \phi(y_1y_2) = 3$ or $\phi(vv_2) = \phi(y_1y_3) = 3$, as indicated in Figure 4.8a or Figure 4.8b, respectively.

For $k = 3$, the coloring scheme is the same as Subcase 2.1.1. Thus we assume $k \geqslant 4$ in the following. We proceed to color the remaining edges, $vv_1, vv_3, \ldots, vv_k$ and v_jv_{j+1}, for $j = 1, 2, \ldots, k-1$.

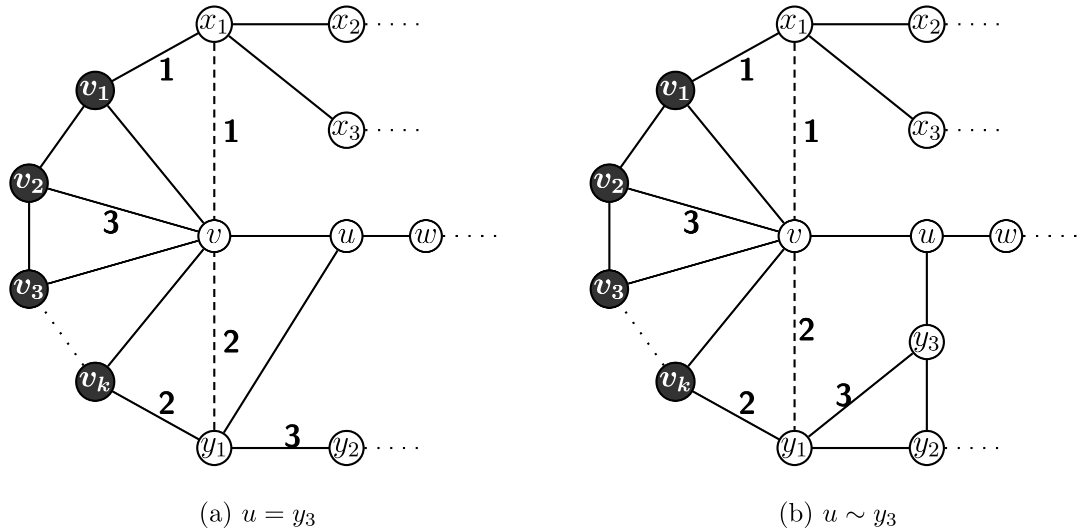


Figure 4.8: Case 2.2 in the proof of Theorem 4.1

First, $F_\phi(vv_1) \subseteq C_\phi(u) \cup C_\phi(x_1) \bigcup_{i=2}^{k} C_\phi(v_i) = C_\phi(u) \cup \{1, 2, 3, \phi(x_1x_2), \phi(x_1x_3)\}$

$\Rightarrow |vv_1|_\phi \leqslant \deg_G(u) + 5$

$\Rightarrow \|vv_1\|_\phi \geqslant 2\Delta(G) + 1 - (\deg_G(u) + 5) \geqslant \Delta(G) - 4 \geqslant 1.$

Color vv_1 by a legal color.

Secondly, $F_\phi(vv_k) \subseteq C_\phi(u) \cup C_\phi(y_1) \cup \{1, 3, \phi(vv_1)\}$

$$= \begin{cases} C_\phi(u) \cup \{1, 2, 3, \phi(vv_1)\}, & \text{for } u = y_3 \\ C_\phi(u) \cup \{1, 2, 3, \phi(vv_1), \phi(y_1y_2)\}, & \text{for } u \sim y_3 \end{cases}$$

$\Rightarrow |vv_k|_\phi \leqslant \deg_G(u) + 5$

$\Rightarrow \|vv_k\|_\phi \geqslant 2\Delta(G) + 1 - (\deg_G(u) + 5) \geqslant 1.$

Color vv_k by an available color.

Thirdly, for $i = 3, 4, \ldots, k - 1$, we have

$F_\phi(vv_i) \subseteq C_\phi(u) \cup \{1, 2, \phi(vv_1), \phi(vv_2), \ldots, \phi(vv_{i-1}), \phi(vv_k)\}$

$\Rightarrow |vv_i|_\phi \leqslant \deg_G(u) + i + 2 \leqslant \deg_G(u) + k + 1 = \deg_G(u) + \deg_G(v)$

$\Rightarrow \|vv_i\|_\phi \geqslant 2\Delta(G) + 1 - (\deg_G(u) + \deg_G(v)) \geqslant 1.$

We color vv_i with available colors one after another.

Next, $F_\phi(v_1v_2) \subseteq C_\phi(v) \cup C_\phi(x_1) \cup C_\phi(v_3) = C_\phi(v) \cup \{1, \phi(x_1x_2), \phi(x_1x_3)\}$

$\Rightarrow |v_1v_2|_\phi \leqslant \deg_G(v) + 3$

$\Rightarrow \|v_1v_2\|_\phi \geqslant 2\Delta(G) + 1 - (\deg_G(v) + 3) \geqslant 3.$

Thus we color v_1v_2 with one of the legal colors.

Next, for $j = 2, 3, \ldots, k-2$, we have

$F_\phi(v_jv_{j+1}) \subseteq C_\phi(v) \cup C_\phi(v_{j-1}) \cup C_\phi(v_{j+2})$

$= C_\phi(v) \cup \{\phi(v_{j-2}v_{j-1}), \phi(v_{j-1}v_j), \phi(v_{j+1}v_{j+2}), \phi(v_{j+2}v_{j+3})\}$

$\Rightarrow |v_jv_{j+1}|_\phi \leqslant \deg_G(v) + 4$

$\Rightarrow \|v_jv_{j+1}\|_\phi \geqslant 2\Delta(G) + 1 - (\deg_G(v) + 4) \geqslant 2.$

Thus we color v_jv_{j+1} with available colors one by one.

Finally, $F_\phi(v_{k-1}v_k) \subseteq C_\phi(v) \cup C_\phi(v_{k-2}) \cup C_\phi(y_1)$

$$= \begin{cases} C_\phi(v) \cup \{2, \phi(v_{k-3}v_{k-2}), \phi(v_{k-2}v_{k-1}), \phi(uy_1)\}, & \text{for } u = y_3 \\ \\ C_\phi(v) \cup \{2, \phi(v_{k-3}v_{k-2}), \phi(v_{k-2}v_{k-1}), \phi(y_1y_2)\}, & \text{for } u \sim y_3 \end{cases}$$

$\Rightarrow |v_{k-1}v_k|_\phi \leqslant \deg(v) + 4$

$\Rightarrow \|v_{k-1}v_k\|_\phi \geqslant 2.$

Color $v_{k-1}v_k$ with one of the legal colors.

Hence, we have colored all the edges of G with no more than $2\Delta(G)+1$ colors. Then, all possible cases are discussed and the proof is done. $\square$

Together with Theorem A.1 (see Appendix), we obtain the following result:

Corollary 4.2. *If a Halin graph $G = T \cup C$ is different from Ne_2, Ne_4, and any* *wheel W_n, and $\varDelta(G) = 4$, then $\chi'_s(G) \leqslant \chi'_s(T) + 2$.*

Proof. Let $G = T \cup C$ be a Halin graph other than Ne_2, Ne_4, or any wheel W_n, and $\varDelta(G) = 4$.

Note that for every $v \in V(T)$, $\deg_G(v)$ is either three or four. Thus, $\sigma(G) = \max\{\deg_G(u) + \deg_G(v) - 1 \mid uv \in E(G)\}$ is either six or seven.

1) $\sigma(G) = 6$. By Theorem A.1, Lemma 2.26, and Lemma 2.30, $\chi'_s(G) \leqslant 8 = \sigma(G) + 2 = \sigma(T) + 2 = \chi'_s(T) + 2$.

2) $\sigma(G) = 7$. By Theorem 4.1, Lemma 2.26, and Lemma 2.30, $\chi'_s(G) \leqslant 2\varDelta(G) + 1 = 9 = \sigma(G) + 2 = \sigma(T) + 2 = \chi'_s(T) + 2$.

Hence, $\chi'_s(G) \leqslant \chi'_s(T) + 2$ is true for all Halin graphs satisfying the requirements. $\square$

Corollary 4.2 settled Conjecture 3.5 for $\varDelta(G) = 4$, however, Conjecture 3.5 is still open for $\varDelta(G) \geqslant 5$.

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APPENDIX A

The Strong Chromatic Index of a Halin Graph G when $\sigma(G) = 6$

The following is an unpublished result of Dr. Liu. We include a sketch of the proof here.

Theorem A.1. *If a Halin graph $G = T \cup C$ with $\sigma(G) = 6$ is different from Ne_2, Ne_4, and W_4, then $\chi'_s(G) \leqslant 8$.*

Proof. Let $G = T \cup C$ be a Halin graph other than Ne_2, Ne_4, and any wheel W_n, and $\sigma(G) = 6$. Here, T and C are the characteristic tree and adjoint cycle of G, respectively. By Definition 2.21, $\sigma(G) = \max\{\sigma(uv) \mid uv \in E(G)\} = \max\{\deg_G(u) + \deg_G(v) - 1 \mid uv \in E(G)\}$. Since G is a Halin graph, the minimum degree over all vertices in G is three. Thus, $\Delta(G) = 4$ and, for any $v \in V(G)$ with $\deg_G(v) = 4$, the neighbors of v must have degree equal to three.

Let $|E(C)| = m$. By Definition 2.6, $m \geqslant 3$. The remainder of the proof proceeds by induction on m.

Base cases:

- $m = 3$. There is no Halin graph G that satisfies our assumption.
- $m = 4$. There is no Halin graph G that satisfies our assumption as G is not W_4.
- $m = 5$. There exists a Halin graph G whose characteristic tree is a double star $D_{3,4}$. By Lemma 2.26 and Lemma 2.31, $\chi'_s(G) = \chi'_s(D_{3,4}) + 2 = \sigma(D_{3,4}) + 2 = 8$.
- $m = 6$. There are three possible Halin graphs as depicted in Figure 4.1b,

Figure 4.1c, and Figure 4.1d. By the labels in the figures, eight is the upper bound for the strong chromatic index of each of the three graphs.

Induction steps: Assume $m \geqslant 7$.

Let $P = u_0, u_1, \ldots, u_l$ be a longest path in T where l is the length of P. It is clear that $l = |P| \geqslant 2$.

(i) For $l = 2$, G is W_4 and has been excluded.

(ii) For $l = 3$, the characteristic tree of G is a double star. Since $\Delta(G) = 4$ and $\deg_G(v) = 3$ for any $v \in E(C)$, one of the non-leaves in the double star must have degree four. As we have argued previously, for any $v \in V(G)$ with $\deg_G(v) = 4$, the neighbors of v must have degree equal to three. Thus, the double star must be $D_{3,4}$, and so $\chi'_s(G) = 8$.

Thus we assume $l \geqslant 4$. Without loss of generality, we also assume that $\deg_G(u_1) \leqslant \deg_G(u_{l-1})$.

In the following, we obtain a reduction G' of G depending on various situations. Assuming that ψ is a strong edge coloring of G' using at most eight colors, we try to expand ψ of G' to the original graph G by constructing a strong edge coloring ϕ of G. Similar to the proof of Theorem 4.1, for every $e \in E(G) \cap E(G')$, let $\phi(e) = \psi(e)$; for every $e \in E(G) \setminus E(G')$, we develop different coloring schemes for different cases.

If G' is different from Ne_2, Ne_4, and any wheel (W_4, to be more specific), by the induction hypothesis, we have $\chi'_s(G') \leqslant 8$. If G' is W_4 or Ne_4, we have $\chi'_s(G') = 8$ by Lemma 2.15 and Lemma 2.29. If G' is Ne_2, we discuss all possible original Halin graphs and their strong chromatic indices at the end of our proof.

For each of the following cases, we complete the construction of the strong edge coloring ϕ of G with at most eight colors.

$\boxed{\textit{Case } 1.}$ $\deg_G(u_1) = 3$ and $\deg_G(u_2) = 3$.

Subcase 1.1. In T, u_2 has exactly one neighbor that is a leaf.

Let the graph of G around the neighborhood of u_1 be as depicted in Figure A.1 and the vertices be labeled as indicated. Delete vertices u_0, u_1, v_1, and v_2 (vertices in black) from $V(G)$ and add two new edges u_2x_1 and u_2y_1 (edges in dashed lines) to the induced subgraph of G to obtain the reduction G'. Without loss of generality, let $\psi(u_2y_1) = 1$, $\psi(u_2x_1) = 2$, $\psi(u_2u_3) = 3$, $\psi(y_1y_2) = 4$, and $\psi(y_1y_3) = 5$. Complete the construction of the strong edge coloring ϕ of G by coloring edges in $E(G) \setminus E(G')$ as shown in Figure A.1. Edges u_1u_2, v_1v_2 and u_1v_1 are not yet colored.

Using the notations introduced in the proof of Theorem 4.1, we have $|u_1u_2|_\phi = |C_\phi(u_0) \cup C_\phi(u_3) \cup C_\phi(v_1) \cup C_\phi(v_2)| \leqslant \deg_G(u_3) + 2 \leqslant 6$. Thus, $\|u_1u_2\|_\phi \geqslant 2$ and we color u_1u_2 with one of the legal colors.

Notice that $F_\phi(v_1v_2) = \{1, 2, 3, 4, 5, \phi(u_1u_2)\}$, then $\|v_1v_2\|_\phi \geqslant 2$ and we color v_1v_2 by one of the available colors.

At last, $F_\phi(u_1v_1) = \{1, 2, 3, \phi(u_1u_2), \phi(v_1v_2)\}$, then $\|u_1v_1\|_\phi \geqslant 3$ and we color u_1v_1 by one of the three available colors.

Subcase 1.2. In T, none of the neighbors of u_2 is a leaf. Consider the two possibilities in Figure A.2.

For the first possibility, the graph of the neighborhood of u_1 is shown in Figure A.2a. We obtain the reduction G' by adding two new edges u_2x_1 and u_2y_1 to the subgraph of G induced on the vertex set $V(G) \setminus \{u_0, u_1, v_1, v_2, v_3, v_4\}$. Without loss

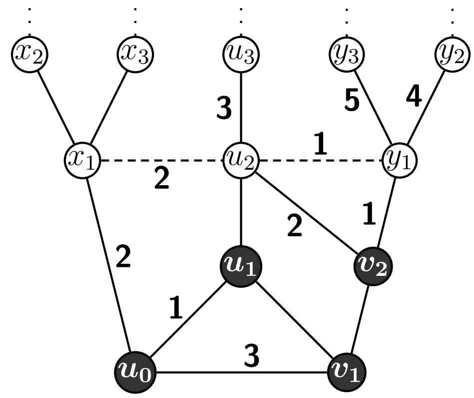


Figure A.1: Subcase 1.1 in the proof of Theorem A.1

of generality, let $\psi(u_2y_1) = 1$, $\psi(u_2x_1) = 2$, $\psi(u_2u_3) = 3$, $\psi(y_1y_2) = 4$, $\psi(y_1y_3) = 5$, $\psi(x_1x_3) = a$, and $\psi(x_1x_2) = b$. Color the edges in $E(G)\backslash E(G')$ by the colors indicated in Figure A.2a. Edges u_0u_1, v_2v_3, v_2v_4, v_3v_4, v_1v_2, and u_1v_1 are not yet colored.

Notice that $F_\phi(u_0u_1) = \{1, 2, 3, t_1, t_2\}$, then $\|u_0u_1\|_\phi \geqslant 3$ and we color u_0u_1 by one of the legal colors.

Next, observe that $F_\phi(v_2v_3) \cup F_\phi(v_2v_4) \cup F_\phi(v_3v_4) = \{1, 2, 3, 4, 5\}$. Thus we color the three edges with the three available colors.

Next, note $F_\phi(v_1v_2) = \{1, 2, 3, \phi(u_0u_1), \phi(v_2v_3), \phi(v_2v_4), \phi(v_3v_4)\}$. Thus we color v_1v_2 by a legal color.

Finally, since $F_\phi(u_1v_1) = \{1, 2, 3, \phi(u_0u_1), \phi(v_2v_3), \phi(v_2v_4), \phi(v_1v_2)\}$. We color u_1v_1 by an available color.

For the second possibility, the reduction G' is depicted and the vertices are labeled in Figure A.2b. Without loss of generality, we assume that $C_\phi(u_3) \subseteq \{3, 4, 5, 6\}$. Then we color the remaining edges in $E(G) \setminus E(G')$ with eight colors as indicated in Figure A.2b.

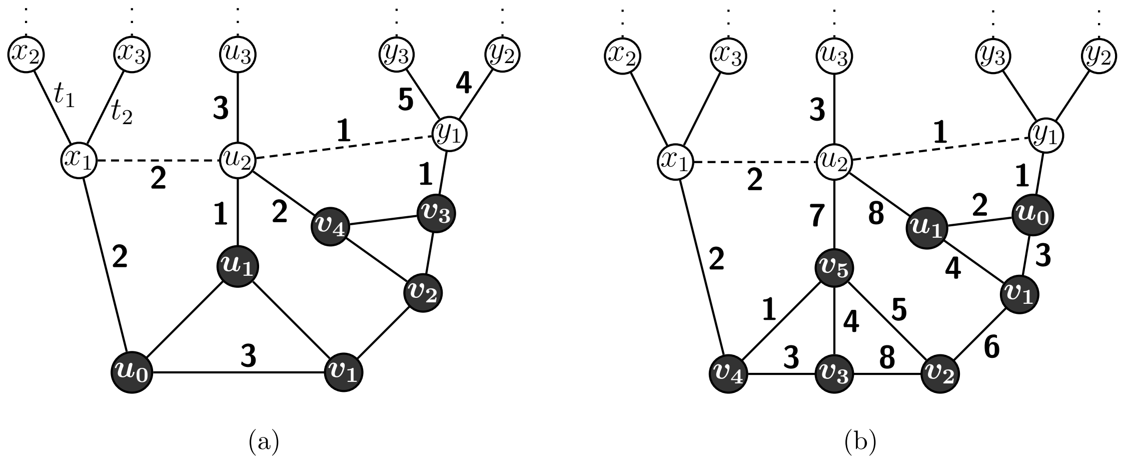


Figure A.2: Two possibilities of Subcase 1.2 in the proof of Theorem A.1

$\boxed{\textit{Case } 2.}$ $\deg_G(u_1) = 3$ and $\deg_G(u_2) = 4$. Then $\deg_G(u_3) = 3$.

<u>*Subcase* 2.1.</u> In T, u_2 has exactly two neighbors that are leaves. Consider the five possible situations depicted in Figure A.3.

In Figure A.3a, the reduction G', the labels for vertices and the coloring for some edges are shown. We need to color the remaining edges v_2v_3, u_2v_2, and u_2u_1.

Note, $F_\phi(v_2v_3) = \{1, 2, 3, 4, 5, s_1, s_2\}$, total seven forbidden colors at most. Thus we color v_2v_3 by an available color.

Next, observe that $F_\phi(u_2v_2) \cup F_\phi(u_2u_1) = \{1, 2, 3, 4, 5, \phi(v_2v_3)\}$, total six forbidden colors at most. Thus there are at least two available colors for u_2v_2 and u_2u_1 and we color both edges by two legal colors.

In Figure A.3b, the reduction G', the labels for vertices and a coloring for some edges are shown. Now we try to color edges u_2v_4, u_2v_2, and u_1u_2 in orders so that the total number of colors used in ϕ does not exceed eight.

Note, $F_\phi(u_2v_4) = \{1, 2, 3, 4, 5, s_1, s_2\}$ and there are at most seven forbidden

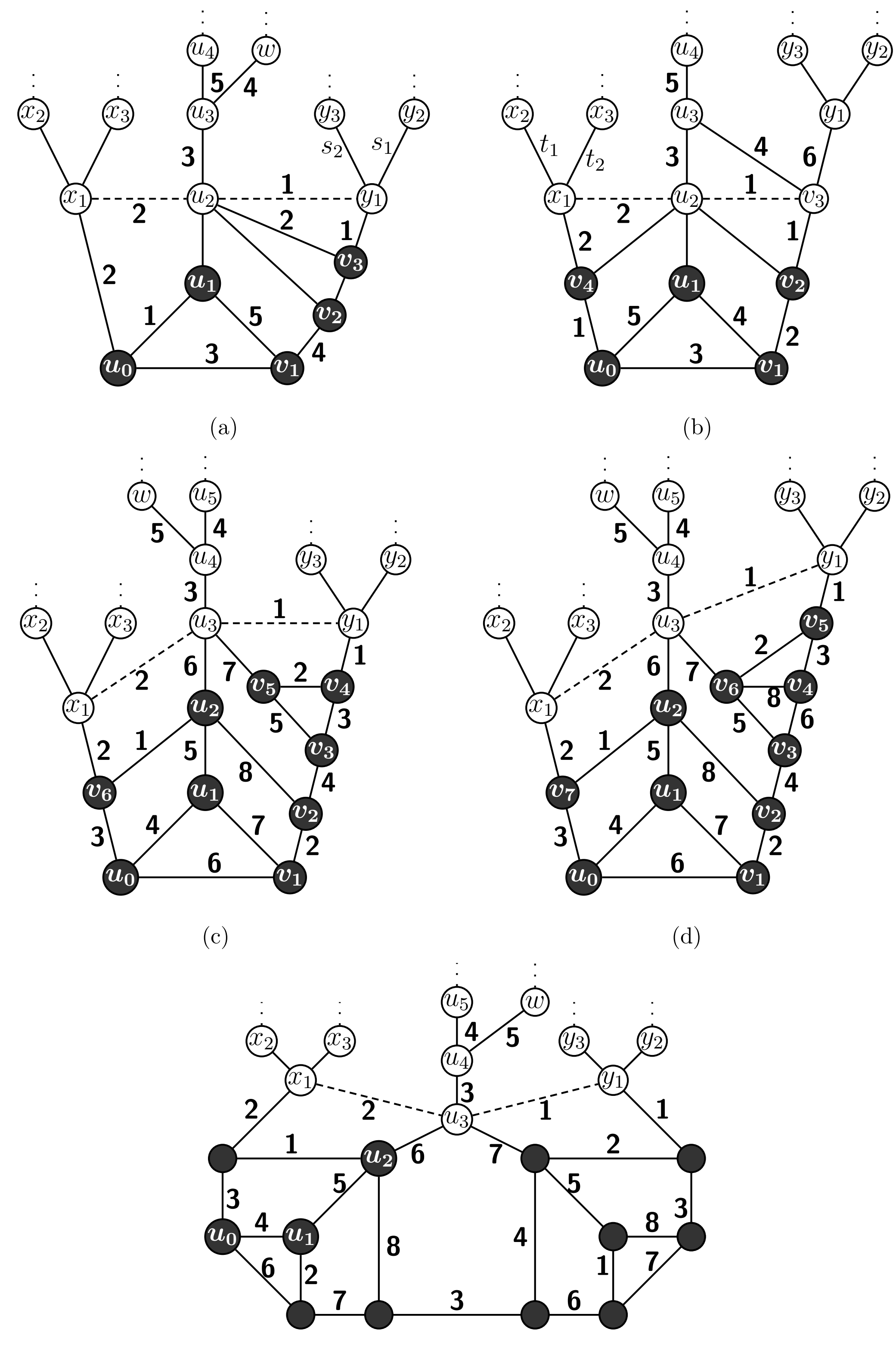


Figure A.3: Five possibilities of Subcase 2.1 in the proof of Theorem A.1

colors. Thus we color u_2v_4 with a legal color.

Next, observe that $F_\phi(u_2v_2) = \{1, 2, 3, 4, 5, 6, \phi(u_2v_4)\}$ and there are at most seven forbidden colors. Thus we color u_2v_2 with a legal color.

Finally, observe that $F_\phi(u_1u_2) = \{1, 2, 3, 4, 5, \phi(u_2v_4), \phi(u_2v_2)\}$ and there are at most seven forbidden colors. Thus we color u_1u_2 with a legal color.

In Figure A.3c to Figure A.3e, the reduction G' and the completion of ϕ using eight colors are depicted.

<u>*Subcase* 2.2.</u> In T, u_2 has exactly one neighbor that is a leaf. Consider the two possible situations as shown in Figure A.4.

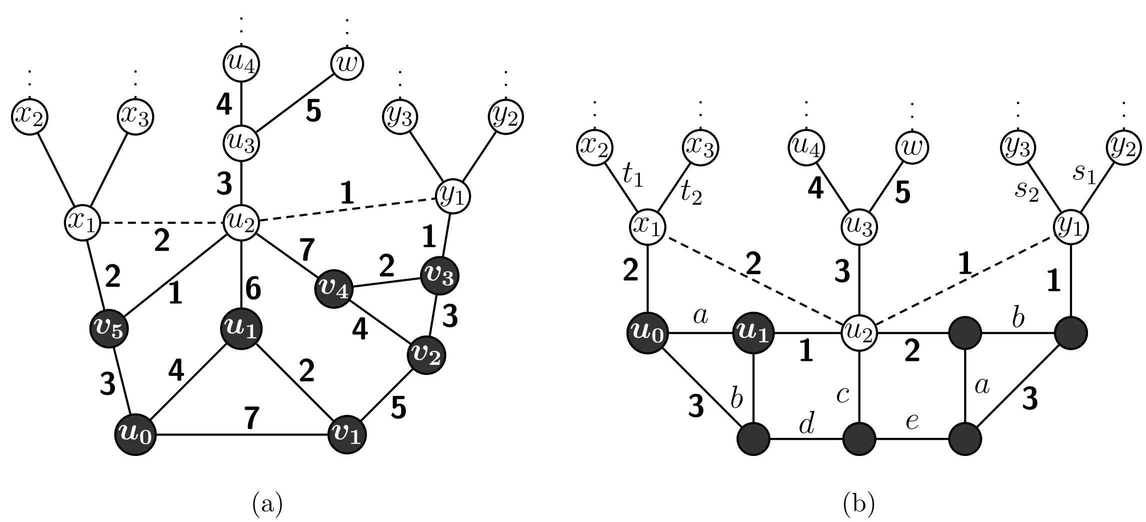


Figure A.4: Two possibilities of Subcase 2.2 in the proof of Theorem A.1

For the first situation, the reduction G', the labels for vertices and the coloring with at most eight colors are depicted in Figure A.4a.

For the second situation, the reduction G', the labels for vertices and a proposed coloring are depicted in Figure A.4b. Notice that the colors a, b, c, d, and e we assign to the edges need to be checked to make sure that they can be found one after another amongst the eight colors of ϕ.

Note, a only has to avoid all colors in $\{1, 2, 3, t_1, t_2\}$, at most five forbidden colors. Thus we can find a legal color for a from ϕ.

Next, observe that b only needs to avoid colors in $\{1, 2, 3, s_1, s_2, a\}$, at most six forbidden colors. Thus we can find an available color for b in ϕ.

Next, notice that c only has to avoid all colors in $\{1, 2, 3, 4, 5, a, b\}$, at most seven forbidden colors. Thus we can find a legal color for c in ϕ.

Finally, note that d and e only have to avoid colors in $\{1, 2, 3, a, b, c\}$, at most six forbidden colors. Thus, there are at least two available colors in ϕ for d and e.

<u>*Subcase* 2.3.</u> In T, none of the neighbors of u_2 is a leaf.

The reduction G' and the completion of ϕ using eight colors is shown in Figure A.5.

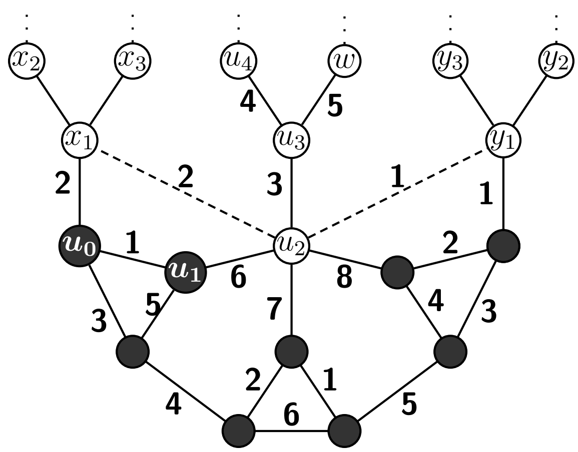


Figure A.5: Subcase 2.3 in the proof of Theorem A.1

$\boxed{\textit{Case } 3.}$ $\deg_G(u_1) = 4$. Then $\deg_G(u_2) = 3$.

<u>*Subcase* 3.1.</u> In T, u_2 has exactly one neighbor that is a leaf.

The reduction G', the labels for vertices, and a proposed coloring for some edges is depicted in Figure A.6. We need to check whether the colors a, b, c, d, and e can be found one by one in ϕ using at most eight colors.

Notice that $a = \phi(u_1u_2)$ and $F_\phi(u_1u_2) = C_\phi(u_3) \cup \{1, 2, 3\}$. Thus $\|u_1u_2\|_\phi \geqslant 6 - \deg_G(u_3) \geqslant 2$, hence, we can find a legal color for a in ϕ.

Next, observe that b only has to avoid colors in $\{1, 2, 3, 4, 5, a\}$. Therefore, we can find a legal color for b in ϕ.

Finally, observe that c, d, and e only need to avoid colors in $\{1, 2, 3, a, b\}$. Hence, we can find legal colors for c, d, and e in ϕ.

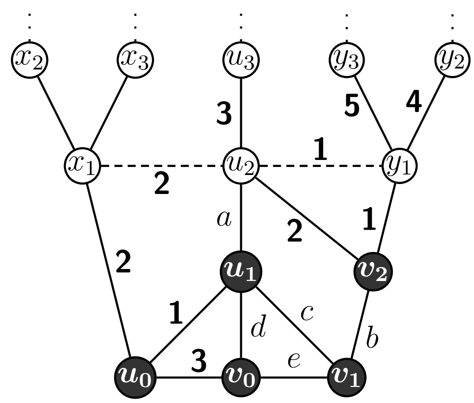


Figure A.6: Subcase 3.1 in the proof of Theorem A.1

<u>*Subcase* 3.2.</u> In T, none of the neighbors of u_2 is a leaf.

Consider the two possible situations depicted in Figure A.7.

For the first situation shown in Figure A.7a, we take $P = v_3, v_4, u_2, u_3, \ldots, u_l$ as the longest path instead and this situation is already covered in the second situation discussed in Subcase 1.2.

For the second possibility, the reduction G' is depicted in Figure A.7b. Without loss of generality, we assume that $C_\phi(u_3) \subseteq \{3, 4, 5, 6\}$. Then we complete the

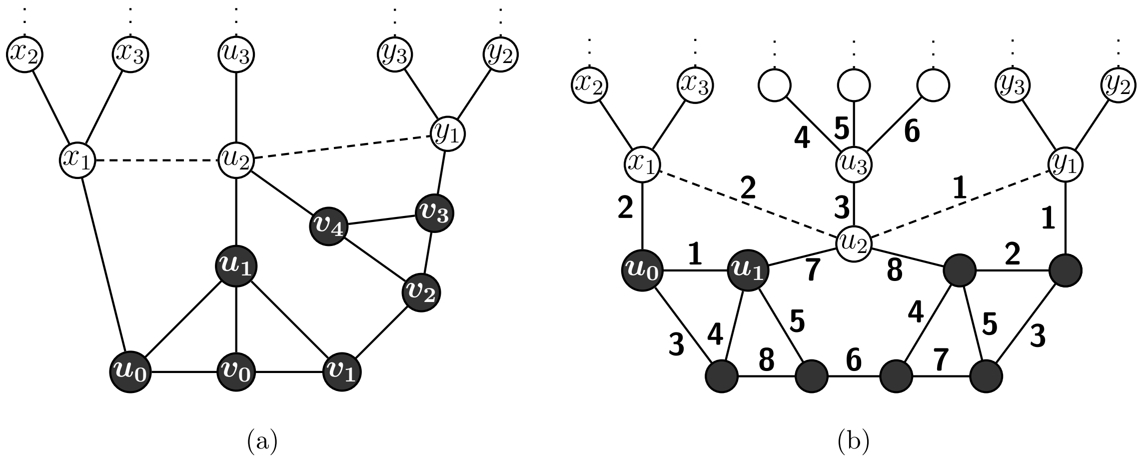


Figure A.7: Two possibilities of Subcase 3.2 in the proof of Theorem A.1

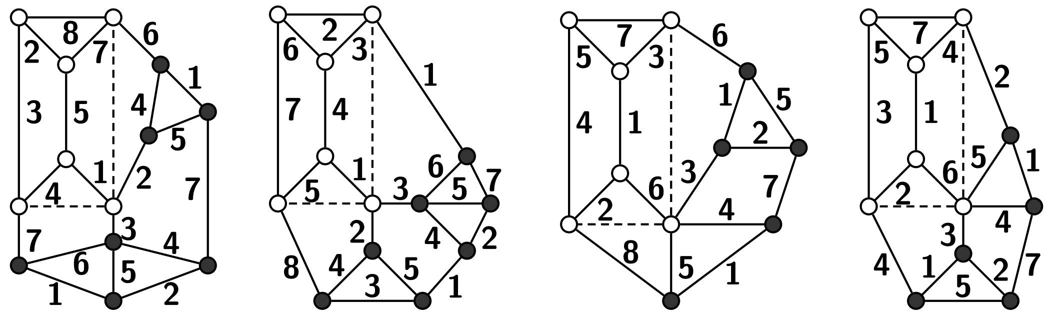
construction of ϕ by using only eight colors as indicated in Figure A.7b.

At last, as we mentioned previously, we need to check the following special graphs which will end up with a Ne_2 if we follow the chosen way of reduction. Notice that we do not need to check the reduction used in Case 3 because, by the assumption, $\deg_G(u_{l-1}) \geqslant \deg_G(u_1) = 4$.

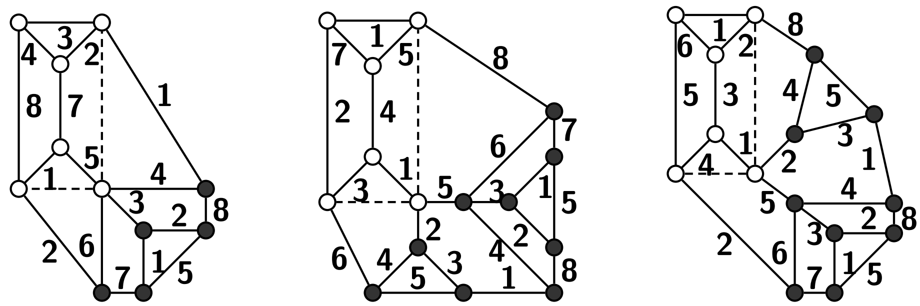
For Subcase 1.1, if $G' = Ne_2$, by the reduction we did in Figure A.1, G must be cubic. This violate the overall assumption that $\sigma(G) = 6$ which implies $\Delta(G) = 4$.

Similarly, for the first situation in Subcase 1.2, G must be cubic if $G' = Ne_2$ and it is against the assumption.

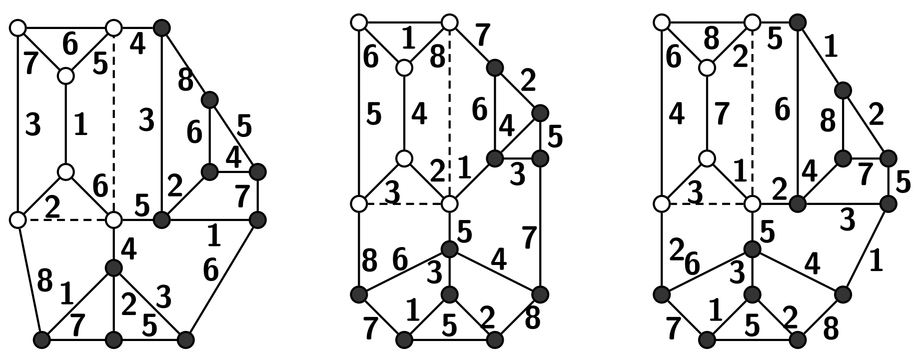
For the second situation in Subcase 1.2, all situations in Subcase 2.1, all situations in Subcase 2.2, and Subcase 2.3, fourteen special graphs for G such that the designated reduction G' is Ne_2 are depicted in Figure A.8 and a strong edge coloring using no more than eight colors is constructed for each graph. $\square$



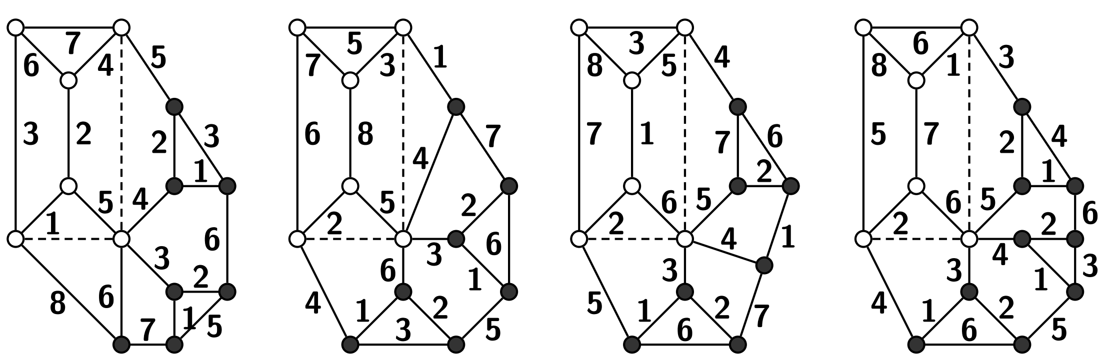
(a) Subcase 1.2-1 (b) Subcase 1.2-2 (c) Subcase 2.1-1 (d) Subcase 2.1-2



(e) Subcase 2.1-3 (f) Subcase 2.1-4 (g) Subcase 2.1-5



(h) Subcase 2.1-6 (i) Subcase 2.1-7 (j) Subcase 2.1-8



(k) Subcase 2.2-1 (l) Subcase 2.2-2 (m) Subcase 2.2-3 (n) Subcase 2.3

Figure A.8: Fourteen special graphs for the proof of Theorem A.1