An Asymptotic Existence Theory on Incomplete Mutually Orthogonal Latin Squares

by

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ABSTRACT

An incomplete Latin square is a $v \times v$ array with an empty $n \times n$ subarray with every row and every column containing each symbol at most once and no row or column with an empty cell containing one of the last n symbols. A set of t incomplete mutually orthogonal Latin squares of order v and hole size n, denoted $t-IMOLS(v,n)$, is a set of t incomplete Latin squares (containing the same empty subarray on the same set of symbols) with a natural extension to the condition of orthogonality. The existence of such sets have been previously explored only for small values of t. We determine an asymptotic result for the existence of $t-IMOLS(v;n)$ for general t requiring large holes, which we develop from our results on incomplete pairwise balanced designs and incomplete group divisible designs.

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DEDICATION

To Stephen Finbow,
I would not be here without your guidance.

Chapter 1

Introduction

1.1 Orthogonal Latin Squares

A *Latin square of order* v is a v by v array of cells in which each cell contains a single integer between 1 and v and every row and every column contains each integer between 1 and v exactly once. A common example of a Latin square is a completed Sudoku puzzle, which is a Latin square of order 9 with the additional requirement that the nine 3 by 3 subarrays of cells, which are denoted by thicker lines, each contain each integer between 1 and 9 exactly once. A completed Sudoku puzzle is shown in Figure 1.1.

Latin squares are easily seen to exist for all orders by constructing a group table of v elements. A *group* $\langle G, * \rangle$ is a set of elements G together with a binary operation $*$ such that $a * b \in G$ for every $a, b \in G$ (G is *closed* under $*$), $(a * b) * c = a * (b * c)$ for every $a, b, c \in G$ (G is *associative*), there exists an element $e \in G$ such that $a * e = e * a = a$ for every $a \in G$ (G has an *identity*), and for every $a \in G$ there exists an element $a' \in G$ such that $a * a' = a' * a = e$ (G contains *inverses*). As a result of these group properties, the equations $a * x = b$ and $y * a = c$, for $a, b, c \in G$,

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Figure 1.1: A Completed Sudoku

have unique solutions x, y, which is precisely what is needed to guarantee that every row and every column contains each element exactly once in the group table. Hence, $\langle \mathbb{Z}_v, + \rangle$, the integers mod v under $+$, gives a group table which forms a Latin square of order v. An example is given in Figure 1.2.

+	1	2	3	4	5	6
1	2	3	4	5	6	1
2	3	4	5	6	1	2
3	4	5	6	1	2	3
4	5	6	1	2	3	4
5	6	1	2	3	4	5
6	1	2	3	4	5	6

Figure 1.2: A Latin Square Constructed from the Group $\langle \mathbb{Z}_6, + \rangle$

Two Latin squares are said to be *orthogonal* if, by forming ordered pairs of elements of corresponding cells, each ordered pair occurs exactly once. We can use playing cards to formulate a problem whose solution is a pair of orthogonal Latin squares. The problem is to arrange the face cards and aces from a standard deck of cards in a 4 by 4 array such that each row and each column contains one card of each rank and one card of each suit. Equivalently, this problem is to construct a pair of orthogonal Latin squares, one representing the rank and the other representing the

A♠	K♡	Q♢	J♣
K♢	A♣	J♠	Q♡
Q♣	J♢	A♡	K♠
J♡	Q♠	K♣	A♢

A	K	Q	J
K	A	J	Q
Q	J	A	K
J	Q	K	A

♠	♡	♢	♣
♢	♣	♠	♡
♣	♢	♡	♠
♡	♠	♣	♢

Figure 1.3: Playing Card Problem

suit. A solution to this problem is given in Figure 1.3. A second problem is Euler's 36 Officer Problem, which requires arranging six regiments, each with six officers of different ranks, in a 6 by 6 array so that each row and each column contains one officer from each regiment and one officer from each rank. Equivalently, the problem is to construct orthogonal Latin squares of order 6. There is no solution to this problem, however; 6 is one of only two orders for which orthogonal Latin squares do not exist. This nonexistence result was proven exhaustively by Tarry [56]; a shorter and more elegant proof was later given by Stinson [52]. The nonexistence of a pair of orthogonal Latin squares of order 2 is a simple exercise. Euler determined the existence of orthogonal Latin squares whose order was odd or a multiple of four, but made the conjecture that orthogonal Latin squares of the remaining orders, that is, those orders equivalent to 2 (mod 4), did not exist. Existence for these orders was eventually shown by Bose, Shrikhande, and Parker [10].

Theorem 1.1.1. [10] *Orthogonal Latin squares of order v exist for all $v \neq 2, 6$.*

A set of Latin squares is said to be *mutually orthogonal* if every pair of Latin squares in the set is orthogonal. The maximum number of mutually orthogonal Latin squares of order v is denoted $N(v)$ and an upper bound on $N(v)$ is $v - 1$. To see this, we can assume that the first row of each Latin square in the set has its entries in ascending order, as permuting symbols within a square preserves orthogonality. Now, consider the first cell in the second row of each square. In no square can this cell contain a 1, as in each square the cell above it contains a 1. Further, each square

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

1	2	3	4	5
3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3

1	2	3	4	5
4	5	1	2	3
2	3	4	5	1
5	1	2	3	4
3	4	5	1	2

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2
2	3	4	5	1

Figure 1.4: Mutually Orthogonal Latin Squares of Orders 4 and 5

must contain a distinct entry in this cell, because between any pair of squares, the ordered pair where both entries are the same is already contained in the first row. Hence, the maximum number of mutually orthogonal Latin squares of order v is at most $v-1$. This upper bound is achieved when v is a prime power. To construct such an example, we work in $\mathbb{F}_q$, the finite field of order q, and let entry (i, j) of square s, $i, j \in \mathbb{F}_q$, $s \in \mathbb{F}_q \setminus \{0\}$, be $si + j$. Maximum sets of mutually orthogonal Latin squares of orders 4 and 5 are given in Figure 1.4. This idea also leads to the following lower bound, due to MacNeish [44].

Theorem 1.1.2. [44] *If* $v = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$*, where each* p_i *is a distinct prime, then* $N(v) \geq \min\{p_i^{e_i} - 1 : i = 1, 2, \ldots, t\}$.

Two equivalent structures to mutually orthogonal Latin squares are transversal designs and orthogonal arrays. A *transversal design* of order v and block size k, denoted $TD(k, v)$, is a triple $(V, \Pi, \mathcal{B})$ such that V is a set of vk points, Π is a partition of V into k groups of v points each, and $\mathcal{B}$ is a collection of k-subsets of V, called blocks, such that no block contains two points in the same group, and every pair of points from different groups appears in exactly one block. A $TD(k, v)$ is equivalent to $k - 2$ mutually orthogonal Latin squares of order v; the first two groups of the transversal design index the rows and the columns, and entry (i, j) in square s is the element in group $s + 2$ in the block containing i and j. Since every pair of points from different groups appears exactly once, it follows that cells have unique entries, rows and columns contain every symbol, and every pair of Latin squares contains every ordered pair of symbols. An *orthogonal array* of order v and depth k, denoted $OA(v, k)$ is an array of k rows and v^2 columns in which each cell contains a symbol from 1 to v and every pair of rows contains each ordered pair of symbols exactly once. An $OA(v, k)$ is equivalent to $k - 2$ mutually orthogonal Latin squares of order v; the first two rows of the orthogonal array index the rows and columns, and each subsequent row gives the entries of one of the Latin squares. Again, as every pair of symbols occurs exactly once in any two rows of the orthogonal array, cells of the squares have unique entries, rows and columns contain every symbol, and every pair of Latin squares contains every ordered pair of symbols. An example of an orthogonal array of order 4 and depth 5, corresponding to the mutually orthogonal Latin squares of order 4 in Figure 1.4, is given in Figure 1.5.

1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
1	2	3	4	2	1	4	3	3	4	1	2	4	3	2	1
1	2	3	4	3	4	1	2	4	3	2	1	2	1	4	3
1	2	3	4	4	3	2	1	2	1	4	3	3	4	1	2

Figure 1.5: An Orthogonal Array of Order 4 and Depth 5

Chowla, Erdős, and Straus [14] proved the asymptotic result that the maximum number of mutually orthogonal Latin squares of order v tends to infinity with v. More precisely, they showed that there exists a positive constant c such that $N(v) > v^c$ for all sufficiently large v, and obtained the specific bound $N(v) > \frac{1}{3}v^{\frac{1}{91}}$. The exponent c was improved to $\frac{1}{17}$ by Wilson [59] and further improved to $\frac{1}{14.8}$ by Beth [9]. For later convenience, we state this result (in a weak form) in terms of transversal designs.

Theorem 1.1.3. [14] *Given k, there exist $TD(k, v)$ for all sufficiently large integers v.*

On the other hand, for values of $v < 10000$, Colbourn and Dinitz [18] give a table of the best known lower bounds of the number of mutually orthogonal Latin squares of order v. Using this table, an upper bound can be stated for the minimum order v_t such that t mutually orthogonal Latin squares exist for all orders $v \geq v_t$, as presented below.

Theorem 1.1.4. [18] *A set of t mutually orthogonal Latin squares exist for all $v \geq v_t$ given by Table* 1.1.

Table 1.1: Upper Bound on v for the Existence of t Mutually Orthogonal Latin Squares of Order v

t	2	3	4	5	6	7	8	9	10	11	12
v_t	7	11	23	61	75	571	2767	3679	5805	7223	7287

1.2 Subsquares and Holes

A Latin square is said to contain a *subsquare* of order n if the subsquare itself is a Latin square, that is, it contains n symbols each occurring exactly once in each row and exactly once in each column. A pair of orthogonal Latin squares are said to have *aligned subsquares* if the subsquares consist of the same cells and symbols, and are themselves orthogonal. This concept also extends to mutually orthogonal Latin squares with aligned subsquares. An example of orthogonal Latin squares with aligned subsquares is given in Figure 1.6.

If each subsquare is removed from each Latin square in a set of mutually orthogonal Latin squares with aligned subsquares, the result is a set of *incomplete mutually orthogonal Latin squares*. A set of t incomplete mutually orthogonal Latin squares of

8	9	3	0	5	6	7	1	2	4
9	7	0	2	3	4	8	5	6	1
4	0	6	7	1	8	9	2	3	5
0	3	4	5	8	9	1	6	7	2
7	1	2	8	9	5	0	3	4	6
5	6	8	9	2	0	4	7	1	3
3	8	9	6	0	1	2	4	5	7
1	4	7	3	6	2	5	8	9	0
6	2	5	1	4	7	3	9	0	8
2	5	1	4	7	3	6	0	8	9

1	2	8	4	0	9	7	3	6	5
5	8	7	0	9	3	4	6	2	1
8	3	0	9	6	7	1	2	5	4
6	0	9	2	3	4	8	5	1	7
0	9	5	6	7	8	2	1	4	3
9	1	2	3	8	5	0	4	7	6
4	5	6	8	1	0	9	7	3	2
2	6	3	7	4	1	5	8	9	0
7	4	1	5	2	6	3	0	8	9
3	7	4	1	5	2	6	9	0	8

Figure 1.6: Orthogonal Latin Squares of Order 10 with Aligned Subsquares of Order 3

order v with hole size n, denoted $t{-}IMOLS(v,n)$ is a set of v by v arrays with a hole $N \subseteq [v]$ such that cell (i,j) is empty if $\{i,j\} \subseteq N$ and contains an integer between 1 and v otherwise, every row and every column contains each symbol at most once, symbols in N are not contained in a row or column index by N, and each ordered pair in $[v]^2 \setminus N^2$ occurs exactly once. For convenience, N is often the last n integers of $[v]$, which produces an empty subarray in the bottom-right corner of each square. Analogously, we have *incomplete transversal designs*, denoted $TD(k,v) - TD(k,n)$, and *incomplete orthogonal arrays*, denoted $IA(v,n,k)$. In an incomplete transversal design, two points index by N are not contained in any common block, and in an incomplete orthogonal array, ordered pairs in N^2 do not occur. It is possible for a set of incomplete mutually orthogonal Latin squares to exist even if the corresponding set of mutually orthogonal Latin squares with aligned subsquares does not exist. For example, two incomplete mutually orthogonal Latin squares of order 6 with hole size 2 exist, as depicted in Figure 1.7, but as there do not exist orthogonal Latin squares of order 2, the empty subarrays cannot be completed. The first example of such a design was constructed by Euler as part of his search for mutually orthogonal Latin squares.

5	6	3	4	1	2
2	1	6	5	3	4
6	5	1	2	4	3
4	3	5	6	2	1
1	4	2	3		
3	2	4	1		

1	2	5	6	3	4
6	5	1	2	4	3
4	3	6	5	1	2
5	6	4	3	2	1
2	4	3	1		
3	1	2	4		

Figure 1.7: $2{-}IMOLS(6,2)$

Horton [38] initiated the study of incomplete mutually orthogonal Latin squares and identified the following necessary condition for their existence.

Theorem 1.2.1. [38] *If* $t{-}IMOLS(v,n)$ *exist, then* $v \geq n(t+1)$.

Proof. Consider the last n integers and the top row of each square in a set of $t{-}IMOLS(v,n)$. Since there are no ordered pairs of these integers between any two squares by definition, each column contains at most one of these integers in the top row over all the squares. Further, the last n columns cannot contain these integers in any square by definition. It follows that the number of columns, and hence the order, is at least $n(t+1)$. □

While several results on the existence of t incomplete mutually orthogonal Latin squares have been determined for small values of t, there is no discussion in the literature of results for general t, with the exception of the case where the size of the hole is at most $t+2$; these results are discussed in Section 6.2. The main result of this thesis fills this hole (no pun intended). That is, we will show the existence of $t{-}IMOLS(v,n)$ for all sufficiently large v and n exceeding a ratio that is quadratic in t. This result, which we prove in Chapter 6, follows from existence results on incomplete pairwise balanced designs which we develop in Chapter 5. These designs, along with their complete counterparts, are defined in Chapter 2. Chapter 3 considers the constructions required to prove the results on incomplete pairwise balanced designs, and Chapter 4 introduces and proves the asymptotic existence of incomplete group divisible designs. Finally, Chapter 7 considers present challenges associated with improving these results and future research directions. The results of this thesis also appear in [31].

Chapter 2

Pairwise Balanced Designs

2.1 Definition and Existence

A *pairwise balanced design* on v points with block size k, denoted $PBD(v,k)$, is a pair $(V,\mathcal{B})$ such that V is a set of v points and $\mathcal{B}$ is a collection of k-subsets of V, called blocks, such that every pair of distinct points occurs together in exactly one block. In general, pairwise balanced designs also have an index λ, are denoted $PBD(v,k,\lambda)$, and each pair of distinct points occurs together in exactly λ blocks; however, we consider exclusively the case $\lambda = 1$ for this and all subsequently introduced designs, and drop the index from the notation. The case $k = 2$ is trivial: each pair forms a block in $\mathcal{B}$. At the other extreme, the case $v = k$ is trivial, and $\mathcal{B}$ consists of a single block containing all the points. The smallest nontrivial example is as follows.

Example 2.1.1. The blocks of a $PBD(7,3)$ on the point set $\{0,1,2,3,4,5,6\}$ are

$$\{0,1,3\},\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,0\},\{5,6,1\},\{6,0,2\}.$$

This design is also represented as the Fano plane as pictured in Figure 2.1, where each vertex represents a point and each line (including the circle) represents a block whose points are the three vertices on the line.

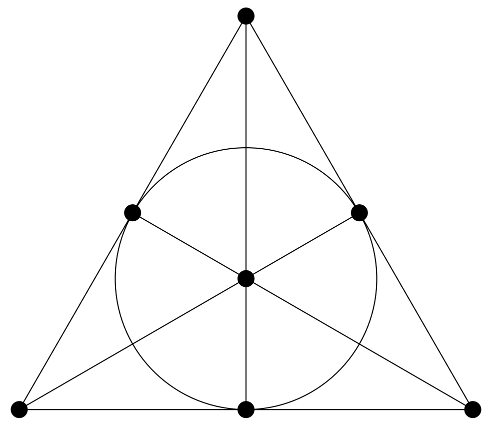


Figure 2.1: The Fano Plane

For a $PBD(v,k)$, the number of pairs of points that need to be covered by the blocks is $\binom{v}{2}$ and the number of pairs covered by each block is $\binom{k}{2}$, so the number of blocks required is $\binom{v}{2}/\binom{k}{2} = \frac{v(v-1)}{k(k-1)}$. In addition, each point is paired with $v-1$ other points, and is paired with $k-1$ other points in each block. So each point occurs in $\frac{v-1}{k-1}$ blocks; this value is called the *replication number*. Since both of these values must be integers, we obtain the following necessary conditions for pairwise balanced designs.

Proposition 2.1.2. *If a $PBD(v,k)$ exists, then*

$$v(v-1) \equiv 0 \pmod{k(k-1)}, \text{ and} \tag{2.1.1}$$

$$v-1 \equiv 0 \pmod{k-1}. \tag{2.1.2}$$

Note that as (2.1.2) implies $k-1 \mid v(v-1)$ and $\gcd(k,k-1) = 1$, we can equivalently write (2.1.1) as $v(v-1) \equiv 0 \pmod{k}$. If the above necessary conditions are satisfied by a particular set of values of v and k, those values are said to be *admissible*. These necessary conditions are not, however, sufficient, as the following example shows.

Example 2.1.3. There exists no $PBD(16, 6)$. If such a design existed, it would consist of $\frac{16\times15}{6\times5} = 8$ blocks and have replication number $\frac{15}{5} = 3$. We can assume without loss of generality that one of the blocks is $\{1, 2, 3, 4, 5, 6\}$. Then each of these six points must be contained in two other blocks, and each of those blocks must be distinct as a pair of points cannot occur together in more than one block. But then we have at least thirteen blocks, contrary to the requirement for the number of blocks.

More generally, a pairwise balanced design can have multiple block sizes. Hence, a $PBD(v, K)$, where $K \subseteq \mathbb{Z}_{\geq 2}$, the set of integers greater than or equal to 2, is a pair $(V, \mathcal{B})$ such that V is a set of v points and $\mathcal{B}$ is a collection of subsets of V, called blocks, such that the size of each block is in K and every pair of points occurs together in exactly one block. We consider the following example.

Example 2.1.4. The blocks of a $PBD(10, \{3, 4\})$ on the point set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are

$$\{1, 2, 3, 0\}, \{4, 5, 6, 0\}, \{7, 8, 9, 0\}, \{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\},$$
$$\{1, 5, 9\}, \{2, 6, 7\}, \{3, 4, 8\}, \{1, 6, 8\}, \{2, 4, 9\}, \{3, 5, 7\}.$$

Unlike the case of a single block size, there is no way to simultaneously calculate the number of blocks required in a pairwise balanced design and the replication number (which, as the previous example shows, may not be constant). In fact, the following example shows there may be multiple ways to construct a pairwise balanced design using different numbers of blocks.

Example 2.1.5. Two possible block sets of a $PBD(12, \{3, 4\})$ on the point set $V =$

$\{1,2,3,4,5,6,7,8,9,10,11,12\}$ are

1. $\{1,2,3,10\},\{4,5,6,10\},\{7,8,9,10\},\{1,4,7,11\},\{2,5,8,11\},\{3,6,9,11\},$ $\{1,5,9,12\},\{2,6,7,12\},\{3,4,8,12\},\{1,6,8\},\{2,4,9\},\{3,5,7\},\{10,11,12\}.$
2. $\{1,2,3,4\},\{5,6,7,8\},\{9,10,11,12\},\{1,5,9\},\{1,6,10\},\{1,7,11\},\{1,8,12\},$ $\{2,5,10\},\{2,6,11\},\{2,7,12\},\{2,8,9\},\{3,5,11\},\{3,6,12\},\{3,7,9\},\{3,8,10\},$ $\{4,5,12\},\{4,6,9\},\{4,7,10\},\{4,8,11\}.$

Despite being unable to simultaneously determine in advance the total number of blocks or the number of times a point occurs, certain restrictions on the number of points of a pairwise balanced design can be easily identified. The total number of pairs of points must be expressible as a (nonnegative) integer linear combination of the numbers of pairs contained in a block of each size in K, and if we choose a particular point, the remaining points must be expressible as a (nonnegative) integer linear combination of the numbers of remaining points contained in a block of each size in K. To this end, let $\beta(K) = \gcd\{k(k-1) : k \in K\}$ and $\alpha(K) = \gcd\{k-1 : k \in K\}$. Then the following necessary conditions are obtained.

Proposition 2.1.6. *If a $PBD(v, K)$ exists, then*

$$v(v-1) \equiv 0 \pmod{\beta(K)}, \text{ and} \tag{2.1.3}$$

$$v - 1 \equiv 0 \pmod{\alpha(K)}. \tag{2.1.4}$$

As with the case of a single block size, we can write an equivalent condition for (2.1.3). To this end, let $\gamma(K) = \beta(K)/\alpha(K)$. It is clear from the definitions of $\beta(K)$ and $\alpha(K)$ that $\alpha(K) \mid \beta(K)$, so $\gamma(K)$ is an integer. Further, it must be the case that $\gcd(\alpha(K), \gamma(K)) = 1$. For if not, there exists a prime p such that $p \mid \gamma(K)$ and

$p \mid \alpha(K)$. Suppose $p^e \parallel \alpha(K)$, that is p^e is the largest power of p to divide $\alpha(K)$. Then $p^{e+1} \mid \beta(K)$. As $p \mid \alpha(K)$, $p \mid k-1$ for all $k \in K$, and so $\gcd(p,k) = 1$ for all $k \in K$. Hence, as $p^{e+1} \mid \beta(K)$, $p^{e+1} \mid k-1$ for all $k \in K$, so $p^{e+1} \mid \alpha(K)$, which contradicts that $p^e \parallel \alpha(K)$. Hence, we can equivalently write (2.1.3) as $v(v-1) \equiv 0 \pmod{\gamma(K)}$.

Although these necessary conditions are not sufficient, Wilson [58] showed that the conditions are in fact asymptotically sufficient.

Theorem 2.1.7. [58] *Given any $K \subseteq \mathbb{Z}_{\geq 2}$, there exist $PBD(v, K)$ for all sufficiently large v satisfying* (2.1.3) *and* (2.1.4).

2.2 Incomplete Pairwise Balanced Designs

An *incomplete pairwise balanced design* on v points with hole size w and block size k, denoted $IPBD((v; w), k)$, is a triple $(V, W, \mathcal{B})$ such that V is a set of v points, W is a subset of V containing w points called the *hole*, and $\mathcal{B}$ is a collection of k-subsets, called *blocks*, such that no block contains two points in W, and every pair of points not both in W appears in exactly one block. The cases $w = 0$ and $w = 1$ reduce to a $PBD(v, k)$ as there are no pairs of points in W. The case $w = v$ is an empty design as every pair of points is in W. Further, the case $k = 2$ is trivial as each pair of points not both in W form a block in $\mathcal{B}$. Hence, in what follows, we consider the nontrivial cases in which $2 \leq w \leq v$ and $k \geq 3$. A small example is given below.

Example 2.2.1. The blocks of an $IPBD((11; 5), 3)$ with point set $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ and hole set $W = \{7, 8, 9, 10, 11\}$ are

$$\{1,2,7\},\quad \{1,3,8\},\quad \{1,4,9\},\quad \{1,5,10\},\quad \{1,6,11\},$$
$$\{3,6,7\},\quad \{2,4,8\},\quad \{3,5,9\},\quad \{4,6,10\},\quad \{2,5,11\},$$
$$\{4,5,7\},\quad \{5,6,8\},\quad \{2,6,9\},\quad \{2,3,10\},\quad \{3,4,11\}.$$

As with pairwise balanced designs, we can calculate the number of blocks required for an incomplete pairwise balanced design. The number of pairs of points that need to be covered by the blocks is $\binom{v}{2} - \binom{w}{2}$ and the number of pairs covered by each block is $\binom{k}{2}$, so the number of blocks required is $\frac{v(v-1)-w(w-1)}{k(k-1)}$. Replication numbers, however, need to be computed separately for points in the hole and points outside the hole. Each point outside the hole is paired with $v-1$ other points, and is paired with $k-1$ other points in each block, so its replication number is $\frac{v-1}{k-1}$. Each point in the hole is paired with $v-w$ other points, and is paired with $k-1$ other points in each block, so its replication number is $\frac{v-w}{k-1}$. Since each of these values must be an integer, and the difference between these two values $\frac{w-1}{k-1}$ must also be an integer, we obtain the following necessary conditions on incomplete pairwise balanced designs.

Proposition 2.2.2. *If an $IPBD((v;w),k)$ exists, then*

$$v(v-1) - w(w-1) \equiv 0 \pmod{k(k-1)}, \text{ and} \tag{2.2.1}$$

$$v-1 \equiv w-1 \equiv 0 \pmod{k-1}. \tag{2.2.2}$$

As (2.2.1) can also be expressed as $(v-w)(v+w-1) \pmod{k(k-1)}$, and (2.2.2) implies $v-w \equiv 0 \pmod{k-1}$, we can therefore equivalently write (2.2.1) as $v(v-1)-w(w-1) \equiv 0 \pmod{k}$. As with pairwise balanced designs, we say values for v, w, k are *admissible* if they satisfy the necessary conditions. For an incomplete pairwise balanced design to exist, however, there is also a necessary inequality that must be satisfied, as shown in the following proposition.

Proposition 2.2.3. *If an $IPBD((v;w),k)$ exists, then $v \geq (k-1)w+1$.*

Proof. The replication number of a point in the hole is $\frac{v-w}{k-1}$. Since two points in the hole cannot be in the same block, there must be at least $\frac{(v-w)w}{k-1}$ blocks. As the total number of blocks is $\frac{v(v-1)-w(w-1)}{k(k-1)}$, we have

$$\frac{v(v-1)-w(w-1)}{k(k-1)} \geq \frac{(v-w)w}{k-1},$$

which, after some algebra, reduces to

$$v \geq (k-1)w+1.$$ □

Notice that if equality holds, every block contains a point in the hole.

Analogously to the connection between incomplete mutually orthogonal Latin squares and mutually orthogonal Latin squares with aligned subsquares, we have a connection between incomplete pairwise balanced designs and pairwise balanced designs containing a subdesign. If a $PBD(v,k)$ contains the subdesign $PBD(w,k)$, the subdesign can be removed to form an $IPBD((v;w),k)$. Conversely, if both an $IPBD((v;w),k)$ and a $PBD(w,k)$ exist, the hole of the incomplete design can be filled to form a $PBD(v,k)$. Hence, the existence of a $PBD(v,k)$ implies the existence of an $IPBD((v,k),k)$, since a $PBD(k,k)$ trivially exists.

Like pairwise balanced designs, the necessary conditions for incomplete pairwise balanced designs are not sufficient. Dukes, Lamken, and Ling [30] determined the following two results approaching a result of asymptotic sufficiency.

Theorem 2.2.4. [30] *Given $w \equiv 1 \pmod{k-1}$, there exist $IPBD((v;w),k)$ for all sufficiently large v satisfying* (2.2.1) *and* (2.2.2).

Theorem 2.2.5. [30] *For any real $\epsilon > 0$, there exist $IPBD((v;w),k)$ for all suffi-*

ciently large v, w satisfying (2.2.1)*,* (2.2.2)*, and $v > (k-1+\epsilon)w$.*

As with pairwise balanced designs, we can have multiple block sizes in incomplete pairwise balanced designs. Hence, an $IPBD((v;w),K)$, where $K \subseteq \mathbb{Z}_{\geq 2}$, is a triple $(V, W, \mathcal{B})$ such that V is a set of v points, W is a subset of V containing w points called the hole, and $\mathcal{B}$ is a collection of subsets of V called blocks such that the size of each block is in K, no block contains two points in W, and every pair of points not both in W appears in exactly one block. A small example is given below.

Example 2.2.6. The blocks of an $IPBD((11;2),\{3,4\})$ with point set $V = \{1,2,3,4,5,6,7,8,9,10,11\}$ and hole set $W = \{10,11\}$ are

$$\{1,2,3,10\},\{4,5,6,10\},\{7,8,9,10\},\{1,4,7\},\{2,5,8\},\{3,6,9\},$$
$$\{1,5,9,11\},\{2,6,7,11\},\{3,4,8,11\},\{1,6,8\},\{2,4,9\},\{3,5,7\}.$$

Analogous necessary conditions are obtained as given below. As with pairwise balanced designs, the modulus in (2.2.3) can be replaced with $\gamma(K)$.

Proposition 2.2.7. *If an $IPBD((v;w),K)$ exists, then*

$$v(v-1) - w(w-1) \equiv 0 \pmod{\beta(K)}, \text{ and} \tag{2.2.3}$$

$$v - 1 \equiv w - 1 \equiv 0 \pmod{\alpha(K)}. \tag{2.2.4}$$

Proposition 2.2.8. *If an $IPBD((v;w),K)$ exists, then $v \geq (\min K - 1)w + 1$.*

Proof. A point outside the hole must appear in at least w blocks, as no two points in the hole can be in the same block, and at most $\frac{v-1}{\min K - 1}$ blocks. Hence $\frac{v-1}{\min K-1} \geq w$ and the result follows. □

As with the case of a single block size, if a $PBD(v,K)$ contains the subdesign $PBD(w,K)$, the subdesign can be removed to form an $IPBD((v;w),K)$. Conversely, if both an $IPBD((v;w),K)$ and a $PBD(w,K)$ exist, the hole of the incomplete design can be filled to form a $PBD(v,K)$. More generally, if both an $IPBD((v;w),K)$ and an $IPBD((w;x),K)$ exist, the hole of the larger incomplete pairwise balanced design can be filled with the smaller to form an $IPBD((v;x),K)$. On the other hand, if we have an $IPBD((v;w),K)$ and for each $k \in K$, there exists a $PBD(k,L)$, then we can break up the blocks of the incomplete pairwise balanced design with the smaller pairwise balanced designs to form an $IPBD((v;w),L)$ on a new block set.

The hole of an incomplete pairwise balanced design is often considered in the literature as a distinguished block, so an $IPBD((v;w),K)$ is also denoted as a $PBD(v,K \cup \{w^*\})$, where the star indicates there is at least one block of size w if $w \in K$, or there is exactly one block of size w if $w \notin K$. We will discuss existence results for particular block sets in Section 5.3, but as there are no results for incomplete pairwise balanced designs with multiple block sizes in general, proving analogues of Theorems 2.2.4 and 2.2.5 will be our main focus in the next few chapters, as they are necessary to prove our main result on incomplete mutually orthogonal Latin squares.

Chapter 3

Constructions

3.1 Group Divisible Designs

Our first set of incomplete pairwise balanced designs will be constructed using group divisible designs. A *group divisible design* of type T with block size k, denoted $GDD(T, k)$, is a triple $(V, \Pi, \mathcal{B})$ such that:

- V is a set of v points;
- $\Pi = \{V_1, V_2, \ldots, V_u\}$ is a partition of V into groups such that $T = [|V_1|, |V_2|, \ldots, |V_u|]$;
- $\mathcal{B}$ is a collection of k-subsets of V, called blocks, meeting each group in at most one point; and
- every pair of points from different groups appears in exactly one block.

Typically, T is expressed in exponential notation, where the term g^u represents u groups of size g. A transversal design is a group divisible design in which the number of groups is k, and each group contains the same number of elements, i.e. a $TD(k, n)$

is equivalent to a $GDD(n^k, k)$. At the other extreme, a $PBD(v, k)$ is equivalent to a $GDD(1^v, k)$. An example of a group divisible design is given below.

Example 3.1.1. A $GDD(6^1 4^1 2^3, 3)$ with point set $V = \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, c_1, c_2, d_1, d_2, e_1, e_2\}$ and partition $\Pi = \{\{a_1, a_2, a_3, a_4, a_5, \}, \{b_1, b_2, b_3, b_4\}, \{c_1, c_2\}, \{d_1, d_2\}, \{e_1, e_2\}\}$ consists of the following blocks:

$$\begin{array}{cccccc}
\{a_1, b_1, c_1\}, & \{a_2, b_1, d_1\}, & \{a_3, b_1, e_1\}, & \{a_4, b_1, c_2\}, & \{a_5, b_1, d_2\}, & \{a_6, b_1, e_2\}, \\
\{a_1, b_2, d_1\}, & \{a_2, b_2, e_1\}, & \{a_3, b_2, c_1\}, & \{a_4, b_2, d_2\}, & \{a_5, b_2, e_2\}, & \{a_6, b_2, c_1\}, \\
\{a_1, b_3, e_1\}, & \{a_2, b_3, c_2\}, & \{a_3, b_3, d_2\}, & \{a_4, b_3, e_2\}, & \{a_5, b_3, c_1\}, & \{a_6, b_3, d_1\}, \\
\{a_1, b_4, c_2\}, & \{a_2, b_4, d_2\}, & \{a_3, b_4, e_2\}, & \{a_4, b_4, c_1\}, & \{a_5, b_4, d_1\}, & \{a_6, b_4, e_1\}, \\
\{a_1, d_2, e_2\}, & \{a_2, e_2, c_1\}, & \{a_3, c_1, d_1\}, & \{a_4, d_1, e_1\}, & \{a_5, e_1, c_2\}, & \{a_6, c_2, d_2\}, \\
 & & \{c_1, d_2, e_1\}, & \{c_2, d_1, e_2\}. & &
\end{array}$$

Our primary focus will be *uniform* group divisible designs, which have type $T = g^u$. An example is given below.

Example 3.1.2. A $GDD(2^7, 4)$ with point set $V = \mathbb{Z}_7 \times \mathbb{Z}_2$ and partition $\Pi = \{\{i\} \times \mathbb{Z}_2 : i \in \mathbb{Z}_7\}$ consists of block set $\mathcal{B} = \{\{(0, 0), (1, 1), (4, 0), (6, 0)\}\} + (\mathbb{Z}_7 \times \mathbb{Z}_2)$, where we start with a base block and develop it additively over the group $\mathbb{Z}_7 \times \mathbb{Z}_2$.

Compared to the general case, the calculation of the number of blocks and the replication number for uniform group divisible designs is straightforward. The number of pairs of points that need to be covered by the blocks is $\frac{g^2u(u-1)}{2}$ and the number of pairs covered by each block is $\binom{k}{2}$, so the number of blocks required is $\frac{g^2u(u-1)}{k(k-1)}$. Also, each point is paired with $g(u - 1)$ other points, and is paired with $k - 1$ other points in each block, so the replication number is $\frac{g(u-1)}{k-1}$. Since both of these values must be integers, we obtain the following necessary conditions on uniform group divisible designs. As usual, we can replace the modulus in the first condition with k.

Proposition 3.1.3. *If a $GDD(g^u, k)$ exists, then*

$$g^2u(u-1) \equiv 0 \pmod{k(k-1)}, \text{ and}$$
$$g(u-1) \equiv 0 \pmod{k-1}.$$

As the following proposition shows, there is a connection between incomplete pairwise balanced designs and group divisible designs.

Proposition 3.1.4. *The following are equivalent:*

- $IPBD((v; w), k)$,
- $GDD(1^{v-w}w^1, k)$, *and*
- $GDD((k-1)^{\frac{v-w}{k-1}}(w-1)^1, k)$.

Proof. The equivalence between the first and second designs easily follows from their definitions. Going from the first to the third design, delete a point in the hole and all its incident blocks, which become groups, as does the remainder of the hole. Conversely, add a point and form new blocks from each group but the last together with the new point. □

As a result of this equivalence, the necessary conditions for uniform group divisible designs are not sufficient (even if we include the simple observation that $u \geq k$). To generalize this type of design, we can have multiple block sizes. Hence, a $GDD(T, K)$, where $K \subseteq \mathbb{Z}_{\geq 2}$, is a triple $(V, \Pi, \mathcal{B})$ such that:

- V is a set of v points;
- $\Pi = \{V_1, V_2, \ldots, V_u\}$ is a partition of V into groups such that $T = [|V_1|, |V_2|, \ldots, |V_u|]$;

- $\mathcal{B} \subseteq \bigcup_{k \in K} \binom{V}{k}$ is a collection of blocks, meeting each group in at most one point; and
- every pair of points from different groups appears in exactly one block.

The following necessary conditions are obtained in the uniform case; the first condition can also be written with the modulus $\gamma(K)$.

Proposition 3.1.5. *If a $GDD(g^u, K)$ exists, then*

$$g^2 u(u-1) \equiv 0 \pmod{\beta(K)}, \text{ and} \tag{3.1.1}$$

$$g(u-1) \equiv 0 \pmod{\alpha(K)}. \tag{3.1.2}$$

The proof of the asymptotic sufficiency of these conditions was found independently by Draganova [27] and Liu [43].

Theorem 3.1.6. [27, 43] *Given g and $K \subseteq \mathbb{Z}_{\geq 2}$, there exist $GDD(g^u, K)$ for all sufficiently large u satisfying* (3.1.1) *and* (3.1.2).

The following two constructions show the connections between group divisible designs and incomplete pairwise balanced designs in this more general case. They are relatively straightforward extensions of Proposition 3.1.4. The first construction follows by deleting a point in the hole and letting each block that contained the point be a group, as well as the remaining point in the hole. The second construction follows by filling all but one group with an incomplete pairwise balanced design, identifying each hole and adding the points of the final group to it.

Construction 3.1.7. *Suppose $(V, W, \mathcal{B})$ is an $IPBD((v; w), K)$. Choose a point $x \in W$ and let the blocks containing x have sizes $g_1, g_2, \ldots, g_r$, where r is the number of blocks containing x. Then there exists a $GDD(T, K)$ with $T = [g_1 - 1, g_2 - 1, \ldots, g_r - 1, w - 1]$.*

Construction 3.1.8. *Suppose there exists a* $GDD(T, K)$ *on* v *points and for some group size* y *in* T*, there exists an* $IPBD((g+h; h), K)$ *for each group size* g *in* $T \setminus [y]$*. Then there exists an* $IPBD((v + h; y + h), K)$*.*

Larger group divisible designs can also be constructed from smaller group divisible designs, as shown by Wilson's Fundamental Construction [60]. The output group divisible design is referred to as the 'resultant', and the smaller input designs are referred to as the 'master' and 'ingredients'.

Construction 3.1.9 (Wilson's Fundamental Construction [60])**.** *Suppose there exists a GDD* $(V, \Pi, \mathcal{B})$*, where* $\Pi = \{V_1, V_2, \ldots, V_u\}$*. Let* $\omega : V \to \mathbb{Z}_{\geq 0}$*, assigning nonnegative weights to each point in such a way that for every* $B \in \mathcal{B}$*, there exists a* $GDD([\omega(x) : x \in B], K)$*. Then there exists a* $GDD(T, K)$*, where*

$$T = \left[\sum_{x \in V_1} \omega(x), \sum_{x \in V_2} \omega(x), \ldots, \sum_{x \in V_u} \omega(x)\right].$$

We now determine the existence of a certain type of non-uniform group divisible design, in which one group has a different size from the rest. We will use this design in the construction of our first set of incomplete pairwise balanced designs. Let $u_0(g, K)$ be such that there exist $GDD(g^u, K)$ for all admissible $u \geq u_0(g, K)$; such a value exists by Theorem 3.1.6.

Lemma 3.1.10. *For any* $m \geq u_0(\alpha(K), K)$ *with* $m \equiv 0 \pmod{\gamma(K)}$*, there exists a* $GDD(s^m t^1, K)$ *for all sufficiently large integers* s *and any integer* t *satisfying* $s \equiv t \equiv 0 \pmod{\alpha(K)}$*.*

Proof. We have $m(m-1) \equiv (m+1)m \equiv 0 \pmod{\gamma(K)}$, so m and $m+1$ are both admissible for uniform group divisible designs with group size $\alpha(K)$ and block sizes in K. Hence, there exist $GDD((\alpha(K))^m, K)$ and $GDD((\alpha(K))^{m+1}, K)$ by Theorem 3.1.6. By Theorem 1.1.3, there exist $TD(m+1, \frac{s}{\alpha(K)})$ for all sufficiently large s

such that $s \equiv 0 \pmod{\alpha(K)}$. Let t be such that $0 \leq t \leq s$ and $t \equiv 0 \pmod{\alpha(K)}$. If we remove points from one of the groups of the transversal design so the group has size $\frac{t}{\alpha(K)}$, the result is a $GDD((\frac{s}{\alpha(K)})^m(\frac{t}{\alpha(K)})^1, \{m, m+1\})$. If ω assigns every point weight $\alpha(K)$, then as the group divisible designs have block sizes m and $m+1$, we require a $GDD((\alpha(K))^m, K)$ and a $GDD((\alpha(K))^{m+1}, K)$ whose existence has been previously shown. Hence, applying Wilson's Fundamental Construction results in a $GDD(s^m t^1, K)$ as required. □

Applying this result, we now construct incomplete pairwise balanced designs with v and w in the same congruence class modulo some multiple of $\beta(K)$. We let $M := m\beta(K)$ be this value, where m retains its value from the previous lemma. The small group will be used as the hole and the other groups will be filled with pairwise balanced designs.

Proposition 3.1.11. *For any $w \equiv 1 \pmod{\alpha(K)}$, there exist $IPBD((v; w), K)$ for all sufficiently large $v \equiv w \pmod{M}$.*

Proof. Let $v - w = aM = am\beta(K)$. We assume a is large enough such that there exists both a $GDD((a\beta(K))^m(w-1)^1, K)$ by Lemma 3.1.10 and a $PBD(a\beta(K)+1, K)$ by Theorem 2.1.7. By Construction 3.1.8, there exists an $IPBD((v; w), K)$. As a can be incremented, the result follows. □

3.2 Resolvable Designs

We construct our next set of incomplete pairwise balanced designs using resolvable pairwise balanced designs. A design is said to be *resolvable* if the blocks of $\mathcal{B}$ can be partitioned into *parallel classes* in such a way that each point is contained in exactly one block in each parallel class. An example of resolvable pairwise balanced designs is the solution to Kirkman's schoolgirl problem [39], which states that fifteen schoolgirls leave the schoolhouse in rows of three for each of seven days and it is required to arrange them throughout the week so that no two girls are in the same row on multiple days. The problem is solved by a resolvable $PBD(15, 3)$, where the blocks are the rows and the parallel classes represent the days. Thus, a solution to the problem is given by Figure 3.1.

{A, B, C}	{A, D, G}	{A, E, N}	{A, I, M}	{A, H, J}	{A, F, L}	{A, K, O}
{D, E, F}	{B, E, J}	{B, D, O}	{B, G, L}	{B, K, M}	{B, I, N}	{B, F, H}
{G, H, I}	{C, F, M}	{C, H, L}	{C, D, K}	{C, E, I}	{C, J, O}	{C, G, N}
{J, K, L}	{H, K, N}	{F, I, K}	{E, H, O}	{D, L, N}	{D, H, M}	{D, I, J}
{M, N, O}	{I, L, O}	{G, J, M}	{F, J, N}	{F, G, O}	{E, G, K}	{E, L, M}

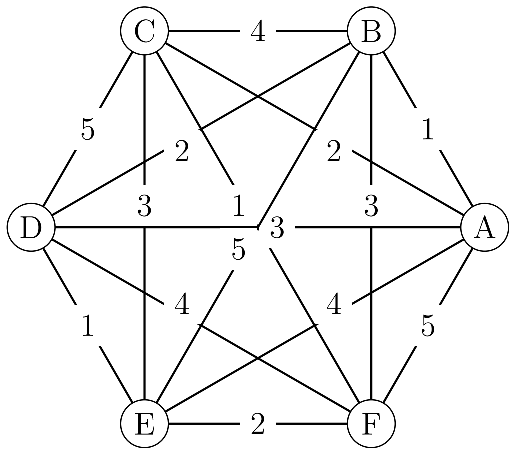
Figure 3.1: Solution to Kirkman's Schoolgirl Problem

Since a particular point is contained in exactly one block in each parallel class, it follows that the number of parallel classes is equal to the replication number, which is $\frac{v-1}{k-1}$. Furthermore, each parallel class consists of v points divided among blocks of size k, so the number of blocks in each parallel class is $\frac{v}{k}$. Since these two values must be integers, the resulting two congruences on v can be combined to obtain the following necessary condition.

Proposition 3.2.1. *If a resolvable $PBD(v, k)$ exists, then*

$$v \equiv k \pmod{k(k-1)}. \tag{3.2.1}$$

Consequently, resolvable pairwise balanced designs with block size 2 exist only if v is even. This condition is also sufficient in this case, and a resolvable $PBD(v, 2)$ is equivalent to a proper edge coloring of K_v, the complete graph on v vertices, with $v - 1$ colors. The vertices represent the points, the edges represent the blocks, and the color classes represent the parallel classes. An example for $v = 6$ is given in Figure 3.2.



{A, B}	{C, F}	{D, E}
{A, C}	{B, D}	{E, F}
{A. D}	{B, F}	{C, E}
{A, E}	{B, C}	{D, F}
{A, F}	{B, E}	{C, D}

Figure 3.2: Resolvable $PBD(6, 2)$

The question of asymptotic existence of resolvable pairwise balanced designs was settled by Ray Chaudhuri and Wilson [45].

Theorem 3.2.2. [45] *Given any integer $k \geq 2$, there exists resolvable $PBD(v, k)$ for all sufficiently large v satisfying* (3.2.1).

The following proposition demonstrates the equivalence between resolvable pairwise balanced designs and incomplete pairwise balanced designs with maximal holes, that is, designs with $v = (k-1)w + 1$.

Proposition 3.2.3. *If $v = (k-1)w + 1$, then there exists an $IPBD((v; w), k)$ if and only if there exists a resolvable $PBD(v - w, k - 1)$.*

Proof. Let $(V, W, \mathcal{B})$ be an $IPBD((v; w), k)$ with $v = (k-1)w + 1$. Since this design achieves equality in Proposition 2.2.3, then every block must contain a point in W. Removing these points results in blocks of size $k-1$, which resolve into parallel classes based on which point in W was in the block. Conversely, for each of the $\frac{v-w-1}{k-2} = w$ parallel classes of a resolvable $PBD(v - w, k - 1)$, add a new point to each of the blocks to obtain an $IPBD((v; w), k)$. $\square$

We now construct a second class of incomplete pairwise balanced designs. In this class, the parameters are such that $v \equiv 1 - w \pmod{\gamma(K)}$. Our approach is to start with an appropriate resolvable pairwise balanced design using a single block size, and then fill each of the blocks using block sizes in K.

Proposition 3.2.4. *Given K, a positive modulus $M = m\beta(K)$, and an admissible congruence class $w_0 \pmod{M}$ for incomplete pairwise balanced designs with block sizes in K, there exists an $IPBD((v; w_1), K)$ with $w_1 \equiv w_0 \pmod{M}$ and $v \equiv 1 - w_1 \pmod{\gamma(K)}$.*

Proof. Choose an integer $q \gg 0$ such that $\gcd(q, M) = 1$, $q\alpha(K)+1 \equiv 0 \pmod{\gamma(K)}$, and there exists a $PBD(q\alpha(K)+1, K)$, whose existence follows from Theorem 2.1.7. Since q and M are coprime, $q\alpha(K)$ and M have only the common factor $\alpha(K)$, and hence it follows from the Chinese remainder theorem that we can choose a $w_1 \gg 0$ (i.e. a sufficiently large value w_1) such that $w_1 \equiv w_0 \pmod{M}$ and $w_1 \equiv 1 \pmod{q\alpha(K)}$ and such that there exists a resolvable $PBD(w_1(q\alpha(K) - 1) + 1, q\alpha(K))$ by Theorem 3.2.2. By Proposition 3.2.3, there exists an $IPBD((w_1 q\alpha(K) + 1; w_1), q\alpha(K) + 1)$. Breaking up the blocks results in an $IPBD((w_1 q\alpha(K) + 1; w_1), K)$ with $v = w_1 q\alpha(K) + 1 \equiv 1 - w_1 \pmod{\gamma(K)}$ as required. □

Chapter 4

Incomplete Group Divisible Designs

4.1 Definition and Necessary Conditions

With examples of incomplete pairwise balanced designs in two distinct classes, we turn to incomplete group divisible designs to construct the remaining classes. An *incomplete group divisible design* of type T with block size k, denoted $IGDD(T, k)$, is a quadruple $(V, \Pi, \Xi, \mathcal{B})$ such that:

- V is a set of v points;
- $\Pi = \{V_1, V_2, \ldots, V_u\}$ is a partition of V into groups and $\Xi = \{W_1, W_2, \ldots, W_u\}$ is a set of holes with W_i a subset of V_i for each i and such that $T = [(|V_1|; |W_1|), (|V_2|; |W_2|), \ldots, (|V_u|; |W_u|)]$;
- $\mathcal{B}$ is a collection of k-subsets of V, called blocks, meeting each group in at most one point and containing at most one point from the hole; and
- every pair of points from different groups not both in a hole appears in exactly

one block.

As with group divisible designs, T is typically expressed in exponential notation, where the term $(g;h)^u$ represents u groups of size g, each with a hole of size h. An incomplete transversal design is an incomplete group divisible design in which the number of groups is k and each group contains the same number of points and the same size hole, i.e. a $TD(k,v) - TD(k,n)$ is equivalent to an $IGDD((v;n)^k, k)$. An example of an incomplete group divisible design is given below.

Example 4.1.1. An $IGDD((3;1)^1(2;1)^2(2;0)^1, 3)$ with point set $V = \{a_1, a_2, a_3, a_4, b_1, b_2, c_1, c_2, d_1, d_2\}$, partition $\Pi = \{\{a_1, a_2, a_3, a_4\}, \{b_1, b_2\}, \{c_1, c_2\}, \{d_1, d_2\}\}$, and hole set $\Xi = \{\{a_1\}, \{b_1\}, \{c_1\}, \{\}\}$ consists of the following blocks:

$$\{a_1, b_2, d_1\}, \{a_1, c_2, d_2\}, \{a_2, b_1, c_2\}, \{a_2, b_2, d_2\}, \{a_2, c_1, d_1\}, \{a_3, b_1, d_1\},$$
$$\{a_3, b_2, c_2\}, \{a_3, c_1, d_2\}, \{a_4, b_1, d_2\}, \{a_4, b_2, c_1\}, \{a_4, c_2, d_1\}.$$

Our focus will be on uniform incomplete group divisible designs, which have type $T = (g;h)^u$. An example is given below.

Example 4.1.2. An $IGDD((4;2)^4, 3)$ with point set $V = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4\}$, partition $\Pi = \{\{a_1, a_2, a_3, a_4\}, \{b_1, b_2, b_3, b_4\}, \{c_1, c_2, c_3, c_4\}, \{d_1, d_2, d_3, d_4\}\}$ and hole set $\Xi = \{\{a_1, a_2\}, \{b_1, b_2\}, \{c_1, c_2\}, \{d_1, d_2\}\}$ consists of the following blocks:

$$\{a_1, b_3, c_4\}, \{a_1, c_3, d_4\}, \{a_1, d_3, b_4\}, \{a_2, b_3, d_4\}, \{a_2, c_3, b_4\}, \{a_2, d_3, c_4\},$$
$$\{b_1, c_3, d_3\}, \{b_1, a_3, c_4\}, \{b_1, a_4, d_4\}, \{b_2, a_3, d_3\}, \{b_2, c_3, a_4\}, \{b_2, c_4, d_4\},$$
$$\{c_1, b_3, d_3\}, \{c_1, a_3, d_4\}, \{c_1, a_4, b_4\}, \{c_2, a_3, b_3\}, \{c_2, d_3, a_4\}, \{c_2, b_4, d_4\},$$
$$\{d_1, b_3, c_3\}, \{d_1, a_3, b_4\}, \{d_1, a_4, c_4\}, \{d_2, a_3, c_3\}, \{d_2, b_3, a_4\}, \{d_2, b_4, c_4\}.$$

In the uniform case, the calculation of the number of blocks is not difficult. The number of pairs of points that need to be covered by the blocks is $\frac{(g^2-h^2)u(u-1)}{2}$ and the number of pairs covered by each block is $\binom{k}{2}$, so the number of blocks required is $\frac{(g^2-h^2)u(u-1)}{k(k-1)}$. As with the case of incomplete pairwise balanced designs, replication numbers need to be calculated separately for points in the hole and points outside the hole. Each point outside the hole is paired with $g(u-1)$ other points, and is paired with $k-1$ other points in each block, so its replication number is $\frac{g(u-1)}{k-1}$. Each point in the hole is paired with $(g-h)(u-1)$ other points, and is paired with $k-1$ other points in each block, so its replication number is $\frac{(g-h)(u-1)}{k-1}$. Since each of these values must be integers, and the difference between the two replication numbers $\frac{h(u-1)}{k-1}$ must also be an integer, we obtain the following necessary conditions on uniform incomplete group divisible designs.

Proposition 4.1.3. *If an IGDD$((g;h)^u, k)$ exists, then*

$$(g^2 - h^2)u(u-1) \equiv 0 \pmod{k(k-1)}, \text{ and} \tag{4.1.1}$$

$$g(u-1) \equiv h(u-1) \equiv 0 \pmod{k-1}. \tag{4.1.2}$$

Additionally, there is a necessary inequality that must be satisfied in order for a uniform incomplete group divisible design to exist.

Proposition 4.1.4. *If an IGDD$((g;h)^u, k)$ exists, then $g \geq (k-1)h$.*

Proof. The replication number of a point in the hole is $\frac{(g-h)(u-1)}{k-1}$. Since two points in the hole cannot be in the same block, there must be at least $\frac{(g-h)(u-1)hu}{k-1}$ blocks. Since the total number of blocks is $\frac{(g^2-h^2)u(u-1)}{k(k-1)}$, we have

$$\frac{(g^2-h^2)u(u-1)}{k(k-1)} \geq \frac{(g-h)(u-1)hu}{k-1},$$

or equivalently,

$$g \geq (k-1)h. \qquad \square$$

As with our previous types of incomplete designs, there is a connection between incomplete group divisible designs and group divisible designs containing a subdesign. If a $GDD(T, k)$, where $T = [g_1, g_2, \ldots, g_u]$, contains the subdesign $GDD(U, k)$, where $U = [h_1, h_2, \ldots, h_u]$, the subdesign can be removed to form an $IGDD(S, k)$, where $S = [(g_1; h_1), (g_2; h_2), \ldots, (g_u; h_u)]$. Conversely, if both an $IGDD(S, k)$ and a $GDD(U, k)$ exist, the hole of the incomplete design can be filled to form a $GDD(T, k)$. There is also a connection between incomplete group divisible designs and incomplete pairwise balanced designs as described in the following proposition.

Proposition 4.1.5. *There exists an $IPBD((v; w), k)$ if and only if there exists an $IGDD((k-1; 1)^w (k-1; 0)^{\frac{v-1}{k-1}-w}, k)$.*

Proof. Starting from the IPBD, delete a point outside the hole and all its incident blocks, which become groups. The points in the hole becomes the holes of the IGDD. Conversely, add a point and form new blocks from each group together with the new point; the holes form the hole of the resulting IPBD. $\square$

Unsurprisingly, the necessary conditions for uniform incomplete group divisible designs are not sufficient. Dukes, Lamken, and Ling [30] sketched a proof of the following asymptotic existence result.

Theorem 4.1.6. [30] *Given integers g, h, k with $k \geq 2$ and $g \geq (k-1)h$, there exists an $IGDD((g; h)^u, k)$ whenever u is sufficiently large satisfying* (4.1.1) *and* (4.1.2).

As usual, we can have multiple block sizes in incomplete group divisible designs. Hence, an $IGDD(T, K)$, where $K \subseteq \mathbb{Z}_{\geq 2}$, is a quadruple $(V, \Pi, \Xi, \mathcal{B})$ such that:

- V is a set of v points;

- $\Pi = \{V_1, V_2, \ldots, V_u\}$ is a partition of V into groups and $\Xi = \{W_1, W_2, \ldots, W_u\}$ is a set of holes with W_i a subset of V_i for each i and such that $T = [(|V_1|; |W_1|), (|V_2|; |W_2|), \ldots, (|V_u|; |W_u|)]$;
- $\mathcal{B} \subseteq \bigcup_{k \in K}$ is a collection of blocks, meeting each group in at most one point and containing at most one point from the hole; and
- every pair of points from different groups not both in a hole appears in exactly one block.

The following corresponding necessary conditions are obtained.

Proposition 4.1.7. *If an $IGDD((g;h)^u, K)$ exists, then*

$$(g^2 - h^2)u(u-1) \equiv 0 \pmod{\beta(K)}, \text{ and} \tag{4.1.3}$$

$$g(u-1) \equiv h(u-1) \equiv 0 \pmod{\alpha(K)}. \tag{4.1.4}$$

Proposition 4.1.8. *If an $IGDD((g;h)^u, K)$ exists, then $g \geq (\min K - 1)h$.*

Proof. A point outside the holes must appear in at least $h(u-1)$ blocks, as no two points in the holes can be in the same block, and at most $\frac{g(u-1)}{\min K - 1}$ blocks. Hence, $\frac{g(u-1)}{\min K-1} \geq h(u-1)$ and the result follows. □

As with the case of a single block size, if a $GDD(T, K)$, where $T = [g_1, g_2, \ldots, g_u]$, contains the subdesign $GDD(U, K)$, where $U = [h_1, h_2, \ldots, h_u]$, the subdesign can be removed to form an $IGDD(S, K)$, where $S = [(g_1; h_1), (g_2; h_2), \ldots, (g_u; h_u)]$. Conversely, if both an $IGDD(S, K)$ and a $GDD(U, K)$ exist, the holes of the incomplete design can be filled to form a $GDD(T, K)$. The following two constructions show the connections between incomplete pairwise balanced designs and incomplete group divisible designs and are relatively straightforward extensions of Proposition 4.1.5.

The first construction follows by deleting a point outside the hole, letting each block that contained the point be a group, and each point in the hole be a hole of its group. The second construction follows by filling each group with an incomplete pairwise balanced design, identifying the extra points in each hole, and merging each of the holes.

Construction 4.1.9. *Suppose* $(V, W, \mathcal{B})$ *is an* $IPBD((v; w), K)$*. Choose a point* $x \in V \setminus W$ *and let the blocks containing* x *have sizes* $g_1, g_2, \ldots, g_w, g_{w+1}, \ldots, g_r$*, where* r *is the replication number of* x *and the first* w *blocks contain a point in the hole. Then there exists an* $IGDD(T, K)$ *with* $T = [(g_1; 1), (g_2; 1), \ldots, (g_w; 1), (g_{w+1}; 0), \ldots, (g_r; 0)]$*.*

Construction 4.1.10. *Suppose there exists an* $IGDD(T, K)$ *on* v *points with* w *points in the holes, and for each* $(g; h) \in T$*, there exists an* $IPBD((g+m; h+m), K)$*. Then there exists an* $IPBD((v + m; w + m), K)$*.*

In Section 4.4, we give a detailed proof of the asymptotic existence of uniform incomplete group divisible designs with multiple block sizes. The main tool of this proof is the Lamken-Wilson Theorem, which we examine in Section 4.3. However, we must consider the case of maximal holes separately; frames are introduced in the next section to prove existence in this case.

4.2 Frames

Motivated by the equivalence of resolvable pairwise balanced designs and incomplete pairwise balanced designs with maximal holes, we are led to consider an analogous object in the case of group divisible designs. A $GDD(T, k)$ with point set V and partition $\Pi = \{V_1, V_2, \ldots, V_u\}$ is said to be a *frame* if the blocks of $\mathcal{B}$ can be partitioned into partial parallel classes such that each class misses exactly one group, that is, it is a partition of $V \setminus V_i$ for some $i = 1, 2, \ldots, u$. An example of a frame is given below.

Example 4.2.1. A frame $GDD(2^4, 3)$ with point set $\{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$ and partition $\Pi = \{\{a_1, a_2\}, \{b_1, b_2\}, \{c_1, c_2\}, \{d_1, d_2\}\}$ consists of the following blocks:

$$\{a_1, b_1, c_1\}, \quad \{a_1, b_2, d_1\}, \quad \{a_1, c_2, d_2\}, \quad \{b_1, c_2, d_1\},$$
$$\{a_2, b_2, c_2\}, \quad \{a_2, b_1, d_2\}, \quad \{a_2, c_1, d_1\}, \quad \{b_2, c_1, d_2\}.$$

In addition to the necessary conditions for uniform group divisible designs, it must also be possible to form the partial parallel classes. The number of points in a partial parallel class is $g(u-1)$, and these points must be divided among blocks of size k, so the number of blocks in each parallel class is $\frac{g(u-1)}{k}$, which must be an integer. Additionally, the total number of blocks is $\frac{g^2u(u-1)}{k(k-1)}$, so the number of partial parallel classes is $\frac{gu}{k-1}$. Since the design is uniform, each group must be missed by the same number of partial parallel classes, so the number of partial parallel classes missing a particular group is $\frac{g}{k-1}$, which must also be an integer. Hence, the following necessary conditions are obtained.

Proposition 4.2.2. *If a frame $GDD(g^u, k)$ exists, then*

$$g(u-1) \equiv 0 \pmod{k}, \text{ and} \tag{4.2.1}$$

$$g \equiv 0 \pmod{k-1}. \tag{4.2.2}$$

The asymptotic existence of frames was established by Liu [43].

Theorem 4.2.3. [43] *Given $g \geq 1$ and $k \geq 2$, there exists u_0 such that there exists a frame $GDD(g^u, k)$ for all $u \geq u_0$ satisfying (4.2.1) and (4.2.2).*

The equivalence of a certain type of frame to certain incomplete transversal designs was shown by Stinson [53] with the following theorem.

Theorem 4.2.4. [53] *The existence of a* $TD(k+1, kw) - TD(k+1, w)$ *implies the existence of a frame* $GDD(((k-1)w)^{k+1}, k)$*, and conversely, the existence of a frame* $GDD(t^{k+1}, k)$ *implies the existence of a* $TD(k+1, \frac{tk}{k-1}) - TD(k+1, \frac{t}{k-1})$*.*

The following result extends the previous theorems to incomplete group divisible designs. A more general form was given by by Furino et al. [32]. We include a proof for completeness.

Proposition 4.2.5. [32] *If* $g = (k-1)h$*, then there exists an* $IGDD((g;h)^u, k)$ *if and only if there exists a frame* $GDD((g-h)^u, k-1)$*.*

Proof. Let $(V, \Pi, \Xi, \mathcal{B})$ be an $IGDD((g;h)^u, k)$, with $\Pi = \{V_1, V_2, \ldots, V_u\}$ and $\Xi = \{W_1, W_2, \ldots, W_u\}$. Let $W_i = \{w_{ij} : j = 1, 2, \ldots, h\}$, $i = 1, 2, \ldots, u$, and let $X_i = V_i \setminus W_i$, $i = 1, 2, \ldots, u$. Since $g = (k-1)h$, it follows that there is no block that does not intersect the hole, that is, for every $B \in \mathcal{B}$, $|B \cap \bigcup_{i=1}^u W_i| = 1$. For every point w_{ij}, define the partial parallel class $\mathcal{R}_{ij} = \{B \setminus \{w_{ij}\} : w_{ij} \in B \in \mathcal{B}\}$. Thus, if $V' = \bigcup_{i=1}^u X_i$, $\Pi' = \{X_1, X_2, \ldots, X_u\}$, and $\mathcal{B}' = \{B \setminus \bigcup_{i=1}^u W_i : B \in \mathcal{B}\}$, then $(V', \Pi', \mathcal{B}')$ is a frame $GDD((g-h)^u, k-1)$ with partial parallel classes $\mathcal{R}_{ij}$.

Conversely, let $(V, \Pi, \mathcal{B})$ be a frame $GDD((g-h)^u, k-1)$ with $\Pi = \{V_1, V_2, \ldots, V_u\}$ and partial parallel classes $\mathcal{R}_{ij}$, $i = 1, 2, \ldots, u$ and $j = 1, 2, \ldots, h$, as the number of partial parallel classes missing a particular group is $\frac{g-h}{k-2} = h$. For each partial parallel class, add a point w_{ij}, and let $W_i = \{w_{ij} : j = 1, 2, \ldots, h\}$, $i = 1, 2, \ldots, u$ and $X_i = V_i \cup W_i$, $i = 1, 2, \ldots, u$. Thus, if $V' = \bigcup_{i=1}^u X_i$, $\Pi' = \{X_1, X_2, \ldots, X_u\}$, $\Xi' = \{W_1, W_2, \ldots, W_u\}$, and $\mathcal{B}' = \{B \cup \{w_{ij}\} : B \in \mathcal{R}_{ij}, i = 1, 2, \ldots, u, j = 1, 2, \ldots, h\}$, then $(V', \Pi', \Xi', \mathcal{B}')$ is an $IGDD((g;h)^u, k)$. □

We verify that the preceding proposition implies the asymptotic existence of uniform incomplete group divisible designs with maximal holes.

Theorem 4.2.6. *Given h and a set $K \subseteq Z_{\geq 2}$, there exists an $IGDD((g;h)^u, K)$ with $g = (\min K - 1)h$ whenever u is sufficiently large satisfying* (4.1.3) *and* (4.1.4)*.*

Proof. As $g = (\min K - 1)h$, a point outside the hole must appear in exactly $h(u-1)$ blocks (Proposition 4.1.8). Since the number of points that must appear in a block together with a given point outside the hole is $g(u-1)$, the average size of a block containing this point is $\frac{g(u-1)}{h(u-1)} + 1 = \min K$. Since $\min K$ is the smallest block size permissible, every block containing this point must have size $\min K$, and since the design is uniform, the same can be said for every point outside the hole, so every block has size $\min K$. Hence, an $IGDD((g;h)^u, K)$ with $g = (\min K - 1)h$ exists if and only if there exists an $IGDD((g;h)^u, \min K)$, which, by Proposition 4.2.5, exists if and only if there exists a frame $GDD((g-h)^u, \min K - 1)$. As $(g-h)(u-1) \equiv 0 \pmod{\alpha(K)}$ and $\alpha(K) \mid \min K - 1$, then $(g-h)(u-1) \equiv 0 \pmod{\min K - 1}$, and as $g - h = (\min K - 2)h$, $g - h \equiv 0 \pmod{\min K - 2}$, so by Theorem 4.2.3, there exists a frame $GDD((g-h)^u, \min K - 1)$ for all sufficiently large u, and the result follows. □

4.3 The Lamken-Wilson Theorem

The Lamken-Wilson Theorem proves the asymptotic existence of decompositions of edge-colored complete (di)graphs. A *graph* is a pair (V, E) such that V is a set of points and E is a set of unordered pairs of V called *edges*. A *complete graph* on v vertices, denoted K_v, is a graph containing every possible edge. A *graph decomposition* on v points into copies of a graph G, denoted $GrD(v, G)$, is a pair $(V, \mathcal{A})$ such that V is a set of v points and $\mathcal{A}$ is a collection of copies (*blocks*) of G on points of V, such that every pair of points is an edge in exactly one copy. An example is given in Figure 4.1; note that P_3 is a path on three vertices.

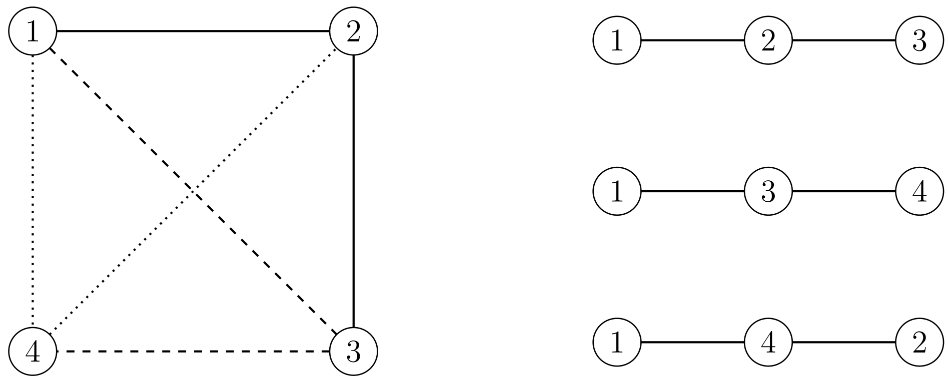


Figure 4.1: $GrD(4, P_3)$

A $GrD(v, G)$ can also be thought of as a decomposition of the complete graph K_v into copies of G. This concept can be generalized to allow decompositions of a general graph F. Hence, a graph decomposition of F into copies of graph G, denoted $GrD(F, G)$ or $F \rightsquigarrow G$, is a pair $(V(F), \mathcal{A})$ such that $V(F)$ is the vertex set of F and $\mathcal{A}$ is a collection of copies (blocks) of G on vertices of $V(F)$, such that every edge in F is an edge in exactly one copy. This generalization allows us to formulate each of the designs seen previously as a graph decomposition, as given by Table 4.1.

Table 4.1: Designs and their Equivalent Graph Decompositions

Design	Notation	Graph Decomposition
Latin Square	$LS(v)$	$\overline{3 \cdot K_v} \rightsquigarrow K_3$
MOLS	$t{-}MOLS(v)$	$\overline{(t+2) \cdot K_v} \rightsquigarrow K_{t+2}$
IMOLS	$t{-}IMOLS(v; n)$	$\overline{(t+2) \cdot K_v} - \overline{(t+2) \cdot K_n} \rightsquigarrow K_{t+2}$
PBDs	$PBD(v, k)$	$K_v \rightsquigarrow K_k$
Uniform GDDs	$GDD(g^u, k)$	$\overline{u \cdot K_g} \rightsquigarrow K_k$
IPBDs	$IPBD((v; w), k)$	$K_v - K_w \rightsquigarrow K_k$
Uniform IGGDs	$IGDD((g; h)^u, k)$	$\overline{u \cdot K_g} - \overline{u \cdot K_h} \rightsquigarrow K_k$

We can similarly determine necessary conditions for graph divisible designs based on the number of blocks and the replication number. We first consider the decompositions of v points. The number of edges that need to be covered by the blocks is

$\binom{v}{2}$, and the number of edges covered by each block is $m := |E(G)|$, so the number of blocks required is $\frac{v(v-1)}{2m}$, which must be an integer. If the graph G is not regular, that is, different points are incident with different numbers of edges, then computing the replication number of a point in advance is not the simple matter that it was with pairwise balanced designs, as a point may be incident with different numbers of edges in different copies of G. However, the $v-1$ edges required must be expressible as a (nonnegative) integer linear combination of the vertex degrees of G. To this end, let $d = \gcd\{\deg_G(x) : x \in V(G)\}$, where $\deg_G(x)$ denotes the *degree* of vertex x in graph G, the number of edges incident with x. Then the following necessary conditions are obtained.

Proposition 4.3.1. *If a $GrD(v, G)$ exists, then*

$$v(v-1) \equiv 0 \pmod{2m}, \text{ and} \tag{4.3.1}$$

$$v - 1 \equiv 0 \pmod{d}. \tag{4.3.2}$$

The necessary conditions are similarly stated for the more general case of decomposing a graph F.

Proposition 4.3.2. *If a $GrD(F, G)$ exists, then*

$$|E(F)| \equiv 0 \pmod{m}, \text{ and} \tag{4.3.3}$$

$$\deg_F(x) \equiv 0 \pmod{d} \text{ for all } x \in V(F). \tag{4.3.4}$$

Further, we can consider decompositions into families of graphs. Hence, a graph decomposition of F into $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$, denoted $GrD(F, \mathcal{G})$, is a pair $(V(F), \mathcal{A})$ such that $V(F)$ is the vertex set of F and $\mathcal{A}$ is a collection of blocks, each a copy of a graph in $\mathcal{G}$ on the vertices of F, such that every edge in F is an edge in exactly one block. The necessary conditions are similarly derived; let $\beta(\mathcal{G}) = \gcd\{|E(G)| : G \in \mathcal{G}\}$ and $\alpha(\mathcal{G}) = \gcd\{\deg_G(x) : x \in G, G \in \mathcal{G}\}$.

Proposition 4.3.3. *If a $GrD(F, \mathcal{G})$ exists, then*

$$|E(F)| \equiv 0 \pmod{\beta(\mathcal{G})}, \text{ and} \tag{4.3.5}$$

$$\deg_F(x) \equiv 0 \pmod{\alpha(\mathcal{G})} \text{ for all } x \in V(F). \tag{4.3.6}$$

We now consider decompositions of edge-r-colored complete digraphs. A *digraph* is a pair (V, A) such that V is a set of points and A is a set of **ordered** pairs of V called *arcs*. A *complete digraph* on n vertices, denoted $\overset{\leftrightarrow}{K}_n$, is a graph containing every possible arc. An *edge-r-colored complete digraph* on n vertices, denoted $K_n^{(r)}$, is a graph containing every possible arc r times, once in each of the r colors, or equivalently, is r copies of K_n defined on the same vertex set and each colored a different color. A *decomposition of* $K_n^{(r)}$ into Φ, or a Φ-*decomposition of* $K_n^{(r)}$, where Φ is a set of edge-colored digraphs, is a pair $(V, \mathcal{F})$, such that V is the set of n vertices of $K_n^{(r)}$ and $\mathcal{F}$ is a collection of blocks, each a copy of a graph in Φ, such that every colored arc of $K_n^{(r)}$ is an arc of exactly one block.

To determine the necessary conditions for such a decomposition, we must first consider whether the graphs in Φ can be used. For each $G \in \Phi$, let $\mu(G) = (m_1, m_2, \ldots, m_r)$, where m_i is the number of edges of color i in G, be the *edge vector* of G, and for each $x \in G$, let $\tau(x) = (\deg_1^-(x), \deg_1^+(x), \deg_2^-(x), \deg_2^+(x), \ldots, \deg_r^-(x), \deg_r^+(x))$, where $\deg_i^-$ is the number of arcs of color i entering x and $\deg_i^+(x)$ is the number of arcs of color i leaving x, be the *degree vector* of x. A graph $G_0 \in \Phi$ is said to be *useless* when it cannot occur in any Φ decomposition of $K_n^{(r)}$, or equivalently, every nonnegative solution to the equation $\mathbf{1} = \sum_{G \in \Phi} c_G \mu(G)$ has $c_{G_0} = 0$. We then say that Φ is *admissible* if no graph in Φ is useless, or equivalently, there exists a positive solution to the equation $\mathbf{1} = \sum_{G\in\Phi} c_G \mu(G)$. Let $\beta(\Phi)$ be the least positive integer m such that $m\mathbf{1}$ is an integral linear combination of the vectors $\mu(G)$, and let $\alpha(\Phi)$ be the least positive integer t such that $t\mathbf{1}$ is an integral linear combination of the vectors $\tau(x)$. Then the Lamken-Wilson Theorem, stated below, establishes the asymptotic existence of decompositions of edge-r-colored complete digraphs.

Theorem 4.3.4 (Lamken-Wilson [42])**.** *Let Φ be an admissible family of simple edge-r-colored digraphs. Then there exists a constant $n_0 = n_0(\Phi)$ such that Φ-decompositions of $K_n^{(r)}$ exist for all $n \geq n_0$ satisfying the congruences*

$$n(n-1) \equiv 0 \pmod{\beta(\Phi)}, \text{ and} \tag{4.3.7}$$

$$n-1 \equiv 0 \pmod{\alpha(\Phi)}. \tag{4.3.8}$$

We are now ready to prove the asymptotic existence of uniform incomplete group divisible designs.

4.4 Asymptotic Existence

Before implementing the Lamken-Wilson Theorem, we first state the following lemma useful for proving the desired result.

Lemma 4.4.1. [48] *Let M be a rational s by t matrix and $\mathbf{c}$ a rational column vector of length s. The equation $M\mathbf{x} = \mathbf{c}$ has an integral solution $\mathbf{x}$, a column vector of length t, if and only if $\mathbf{yc}$ is an integer for each rational row vector $\mathbf{y}$ such that the row vector $\mathbf{y}M$ is a vector of integers.*

We now state and prove the asymptotic existence of uniform incomplete group divisible designs.

Theorem 4.4.2. *Given integers g, h, a set $K \subseteq \mathbb{Z}_{\geq 2}$, and $g \geq (\min K - 1)h$, there exists an $IGDD((g;h)^u, K)$ whenever u is sufficiently large satisfying* (4.1.3) *and* (4.1.4).

Proof. By Theorem 4.2.6, we have the existence of uniform incomplete group divisible designs with $g = (\min K - 1)h$. Hence, we can assume $g > (\min K - 1)h$. In order to apply the Lamken-Wilson Theorem, we first establish that the existence of an $IGDD((g;h)^u, K)$ is implied by the existence of some decomposition of an edge-colored complete digraph.

To that end, we will decompose the graph $K_u^{(g^2-h^2)}$ using the set of colors $[g]^2 - [h]^2$ into the set of colored digraphs Φ defined as follows. For each $k \in K$, let F_k be the set of all possible functions f_k such that $f_k : [k] \to [g]$ and at most one element in the range belongs to $[h]$. Every function f_k induces an edge coloring $G(f_k)$ of the complete digraph $\overleftrightarrow{K}_k$ using colors in $[g]^2 - [h]^2$ where the arc (x, y), $x, y \in [k]$, is assigned the color $(f_k(x), f_k(y))$. Notice that if arc (x, y) is assigned color (a, b), then arc (y, x) must be assigned color (b, a), and further, every arc leaving x has a color of the form $(f_k(x), c)$ and every arc entering x has a color of the form $(c, f_k(x))$. Then $\Phi = \{G(f_k) : f_k \in F_k, k \in K\}$.

If a Φ-decomposition $(V', \mathcal{F})$ of $K_u^{(g^2-h^2)}$ exists, we obtain the $IGDD((g;h)^u, k)$ $(V, \Pi, \Xi, \mathcal{B})$ as follows. Let $V = V' \times [g]$, $\Pi = \{\{x\} \times [g] : x \in V'\}$, and $\Xi = \{\{x\} \times [h] : x \in V'\}$. Each block $\zeta \in \mathcal{F}$ induces a block $B_\zeta \in \mathcal{B}$ as follows. Assign each vertex x of ζ the color $c_\zeta(x)$ so that every arc (x, y) has color $(c_\zeta(x), c_\zeta(y))$. Such an assignment exists since ζ is induced by some function f_k in such a way that this property holds. Then $B_\zeta = \{(x, c_\zeta(x)) : x \in V(\zeta)\}$. As defined, the size of each $B \in \mathcal{B}$ is in K as it is induced by the vertices of a coloring of $\overleftrightarrow{K}_k$ for some $k \in K$, no block contains two points in the same group as the groups are defined by the vertices of $K_u^{(g^2-h^2)}$ or two points in the same hole as every f_k maps at most one vertex to the hole, and the block containing an arbitrary pair $\{(x,i),(y,j)\}$, $x \neq y$, $\{i,j\} \nsubseteq [h]$, corresponds to the graph block in which arc (x,y) is colored (i,j) (and hence arc (y,x) is colored (j,i)).

We must now show that the set Φ is admissible. Hence, we must find a positive solution to the equation $\mathbf{1} = \sum_{G\in\Phi} c_G \mu(G)$. Consider the average of $\mu(G(f_k))$ over all $f_k \in F_k$ and $k \in K$. As there is no distinction between any two colors outside the hole or between any two colors inside the hole, this average $\mathbf{u}$ has coordinates u_{ij} such that for some constants A and B, $u_{ij} = A$ if $(i,j) \in [g]^2 - [h]^2$ and $u_{ij} = B$ otherwise. If $A = B$, we are done. If $A < B$, consider the average of $\mu(G(f_{\min K}))$ over all $f_{\min K} \in F_{\min K}$ with no elements in the range belonging to $[h]$. Then this average $\mathbf{t}$ has coordinates t_{ij} such that for some positive constant C, $t_{ij} = C$ if $(i,j) \in [g]^2 - [h]^2$ and $t_{ij} = 0$ otherwise. Hence, some positive combination of $\mathbf{u}$ and $\mathbf{t}$ equals $\mathbf{1}$. If instead $A > B$, consider the average of $\mu(G(f_{\min K}))$ over all $f_{\min K} \in F_{\min K}$ with exactly one element in the range belonging to $[h]$. Then this average $\mathbf{s}$ has coordinates

$$s_{ij} = \begin{cases} \frac{(\min K-1)(\min K-2)}{(g-h)^2} & \text{if } (i,j) \in [g]^2 - [h]^2, \\ \frac{\min K-1}{h(g-h)} & \text{otherwise.} \end{cases}$$

As we assumed $g > (\min K - 1)h$, we have $\frac{(\min K-1)(\min K-2)}{(g-h)^2} < \frac{\min K-1}{h(g-h)}$, so some positive combination of $\mathbf{u}$ and $\mathbf{s}$ equals $\mathbf{1}$. Therefore, Φ is admissible.

Finally, we must verify that the corresponding Φ-decomposition of $K_u^{(g^2-h^2)}$ is admissible for each required value of u. To verify (4.3.7), it is sufficient to show that $u(u-1)\mathbf{1}$ is an integral linear combination of the vectors $\mu(G(f_k))$, $G(f_k) \in \Phi$. By Lemma 4.4.1, an integral linear combination exists if and only if for all rational vectors $\mathbf{x}$, $\mathbf{x} \cdot u(u-1)\mathbf{1}$ is an integer if $\mathbf{x} \cdot \mu(G(f_k))$ is an integer for every $f_k \in F_k, k \in K$.

Now, consider the structure of the vector $\mu(G(f_k))$. There exists an arc with color (i,j) for every pair of vertices (x,y) such that $f_k(x)=i$ and $f_k(y)=j$. Hence, if t_i counts the number of vertices mapped to i, then $\mu(G(f_k))_{ii}=t_i(t_i-1)$ for $i\in[g]-[h]$ and $\mu(G(f_k))_{ij}=t_it_j$ for $i\neq j$ and $(i,j)\in[g]^2-[h]^2$. Thus, for any g^2-h^2 rational numbers x_{ij}, $(i,j)\in[g]^2-[h]^2$, we must show that for all $f_k\in F_k, k\in K$,

$$\sum_{\substack{(i,j)\in[g]^2-[h]^2\\ i\neq j}} t_it_jx_{ij}+\sum_{i\in[g]-[h]} t_i(t_i-1)x_{ii}\equiv 0 \tag{4.4.1}$$

implies

$$u(u-1)\sum_{(i,j)\in[g]^2-[h]^2} x_{ij}\equiv 0,$$

where $a\equiv b$ denotes that $a-b\in\mathbb{Z}$.

Assume (4.4.1) holds. We first consider functions that have no element in their range belonging to $[h]$. For each $k\in K$ and each $(i,j)\in([g]-[h])^2$, $i\neq j$, we establish a relation between x_{ij},x_{ji} and x_{ii},x_{jj}. Consider the following three functions f_k: (1) $t_i=k$; (2) $t_i=k-1$, $t_j=1$; and (3) $t_i=k-2$, $t_j=2$. By (4.4.1) we have

$$\begin{aligned} &(1)\quad k(k-1)x_{ii}\equiv 0\\ &(2)\quad (k-1)(k-2)x_{ii}+(k-1)(x_{ij}+x_{ji})\equiv 0\\ &(3)\quad (k-2)(k-3)x_{ii}+2(k-2)(x_{ij}+x_{ji})+2x_{jj}\equiv 0. \end{aligned} \tag{4.4.2}$$

Computing $(1)-2\cdot(2)+(3)$ of (4.4.2) gives

$$2(x_{ij}+x_{ji})\equiv 2x_{ii}+2x_{jj} \tag{4.4.3}$$

which implies

$$u(u-1)(x_{ij}+x_{ji})\equiv u(u-1)x_{ii}+u(u-1)x_{jj}$$

and hence

$$u(u-1)\sum_{i,j\in[g]-[h]} x_{ij} \equiv u(u-1)(g-h)\sum_{i\in[g]-[h]} x_{ii}. \tag{4.4.4}$$

Now, for each $k \in K$ and each $i, j \in [g]-[h]$, $i \neq j$, we find a relation between x_{ii} and x_{jj}. Computing (1) – (2) of (4.4.2) gives

$$2(k-1)x_{ii} \equiv (k-1)(x_{ij}+x_{ji}) \tag{4.4.5}$$

and since this holds when we interchange i and j, we have

$$2(k-1)x_{ii} \equiv 2(k-1)x_{jj} \tag{4.4.6}$$

and as $\alpha(K) = \gcd\{k-1 : k \in K\}$, we obtain

$$2\alpha(K)x_{ii} \equiv 2\alpha(K)x_{jj}.$$

If $\alpha(K)$ is odd, then as $(g-h)(u-1) \equiv 0 \pmod{\alpha(K)}$ by (4.1.4), it follows that $(g-h)u(u-1) \equiv 0 \pmod{2\alpha(K)}$. Therefore,

$$(g-h)u(u-1)x_{ii} \equiv (g-h)u(u-1)x_{jj}. \tag{4.4.7}$$

If, however, $\alpha(K)$ is even, then each $k \in K$ must be odd, so we can multiply, for each $k \in K$, (4.4.3) by $\frac{k-1}{2}$, which together with (4.4.5) gives

$$2(k-1)x_{ii} \equiv (k-1)x_{ii} + (k-1)x_{jj}$$

which simplifies to $(k-1)x_{ii} \equiv (k-1)x_{jj}$. Hence, $\alpha(K)x_{ii} \equiv \alpha(K)x_{jj}$, and therefore (4.4.7) is again obtained.

We now consider functions that have exactly one element in their range belonging to $[h]$. For each $l \in [h]$ and each $i, j \in [g] - [h]$, $i \neq j$, we establish a relation between x_{li}, x_{il}, x_{ii} and x_{lj}, x_{jl}, x_{jj}. For each $k \in K$, consider the following two functions f_k: (1) $t_l = 1$, $t_i = k - 1$; and (2) $t_l = 1$, $t_i = k - 2$, $t_j = 1$. By (4.4.1) we have

$$\begin{aligned} &(1) \quad (k-1)(x_{li} + x_{il}) + (k-1)(k-2)x_{ii} \equiv 0 \\ &(2) \quad (k-2)(x_{li} + x_{il}) + (k-2)(k-3)x_{ii} + (x_{lj} + x_{jl}) + (k-2)(x_{ij} + x_{ji}). \end{aligned} \tag{4.4.8}$$

Computing (1) $-$ (2) of (4.4.8) gives

$$x_{li} + x_{il} + 2(k-2)x_{ii} \equiv x_{lj} + x_{jl} + (k-2)(x_{ij} + x_{ji})$$

which also holds when we interchange i and j. Combining these two forms gives

$$2(x_{li} + x_{il}) + 2(k-2)x_{ii} \equiv 2(x_{lj} + x_{jl}) + 2(k-2)x_{jj}$$

which implies

$$u(u-1)(x_{li} + x_{il}) + u(u-1)(k-2)x_{ii} \equiv u(u-1)(x_{lj} + x_{jl}) + u(u-1)(k-2)x_{jj}$$

and hence

$$\begin{aligned}
&u(u-1)\sum_{\substack{l\in[h]\\ i\in[g]-[h]}}(x_{li}+x_{il})\\
&\equiv u(u-1)\sum_{\substack{l\in[h]\\ i\in[g]-[h]}}(x_{li}+x_{il}+(k-2)x_{ii})-hu(u-1)\sum_{i\in[g]-[h]}(k-2)x_{ii}\\
&\equiv (g-h)u(u-1)\sum_{l\in[h]}(x_{l1}+x_{1l}+(k-2)x_{11})-(g-2)hu(u-1)(k-2)x_{11}\\
&\equiv (g-h)u(u-1)\sum_{l\in[h]}(x_{l1}+x_{1l}). \qquad (4.4.9)
\end{aligned}$$

Now, for each $k\in K$ and $(l,i)\in[h]\times([g]-[h])$, we find a relation between x_{li}, x_{il} and x_{ii}. Computing (4.4.2)(1) – (4.4.8)(1) gives

$$(k-1)(x_{li}+x_{il})\equiv 2(k-1)x_{ii}$$

and as $\alpha(K)=\gcd\{k-1 : k\in K\}$, we obtain

$$\alpha(K)(x_{li}+x_{il})\equiv 2\alpha(K)x_{ii}.$$

As $(g-h)(u-1)\equiv 0 \pmod{\alpha(K)}$ by (4.1.4), it follows that $(g-h)u(u-1)\equiv 0 \pmod{\alpha(K)}$. Therefore,

$$(g-h)u(u-1)(x_{li}+x_{il})\equiv 2(g-h)u(u-1)x_{ii}. \qquad (4.4.10)$$

Finally, as $\beta(K)=\gcd\{k(k-1) : k\in K\}$, it follows from (4.4.2)(1) that $\beta(K)x_{ii}\equiv 0$, and as $(g^2-h^2)u(u-1)\equiv 0 \pmod{\beta(K)}$ by (4.1.3), we obtain $(g^2-h^2)u(u-1)x_{ii}\equiv$

0. Hence, together with (4.4.4), (4.4.7), (4.4.9), and (4.4.10), it follows that

$$\begin{aligned} u(u-1)\sum_{(i,j)\in[g]^2-[h]^2} x_{ij} &\equiv u(u-1)\sum_{i,j\in[g]-[h]} x_{ij} + u(u-1)\sum_{\substack{l\in[h]\\ i\in[g]-[h]}} (x_{li}+x_{il}) \\ &\equiv (g-h)u(u-1)\sum_{i\in[g]-[h]} x_{ii} + (g-h)u(u-1)\sum_{l\in[h]}(x_{l1}+x_{1l}) \\ &\equiv (g-h)^2u(u-1)x_{11} + 2(g-h)u(u-1)x_{11} \\ &\equiv (g^2-h^2)u(u-1)x_{11} \equiv 0. \end{aligned}$$

Hence, $u(u-1) \equiv 0 \pmod{\beta(\Phi)}$.

To verify (4.3.8), it is sufficient to show that $(u-1)\mathbf{1}$ is an integral linear combination of the vectors $\tau(v)$, $v \in V(G(f_k))$, $G(f_k) \in \Phi$. By Lemma 4.4.1, an integral linear combination exists if and only if for all rational vectors $\mathbf{y}$, $\mathbf{y}\cdot(u-1)\mathbf{1}$ is an integer if $\mathbf{y}\cdot\tau(v)$ is an integer for every $v \in V(G(f_k))$, $f_k \in F_k$, $k \in K$. Now, consider the structure of the vector $\tau(v)$. If $f_k(v) = q$, then there exists an arc entering v with color (i,q) and an arc leaving v with color (q,i) for every vertex $w \in V(G(f_k))$, $w \neq v$ such that $f_k(w) = i$. Hence, if t_i counts the number of vertices mapped to i, then $\tau(v)^-_{qq} = \tau(v)^+_{qq} = t_q - 1$ for $q \in [g]-[h]$ and $\tau(v)^-_{iq} = \tau(v)^+_{qi} = t_i$ for $i \neq q$ and $(i,q) \in [g]^2-[h]^2$. Thus, for any $2(g^2-h^2)$ rational numbers y^*_{ij}, $(i,j) \in [g]^2-[h]^2$, $* \in \{-,+\}$, we must show that for all $v \in V(G(f_k))$, $f_k \in F_k$, $k \in K$,

$$\begin{cases} (t_q-1)(y^-_{qq}+y^+_{qq}) + \sum_{\substack{i\in[g]\\ i\neq q}} t_i(y^-_{iq}+y^+_{qi}) \equiv 0 & \text{if } q \in [g]-[h] \\ \sum_{i\in[g]-[h]} t_i(y^-_{iq}+y^+_{qi}) \equiv 0 & \text{if } q \in [h] \end{cases} \tag{4.4.11}$$

implies

$$(u-1)\sum_{(i,j)\in[g]^2-[h]^2} (y^-_{ij}+y^+_{ij}) \equiv 0.$$

Assume (4.4.11) holds. We first consider vertices not mapped to $[h]$. For each $k \in K$, $q \in [g]-[h]$, and $i \in [g]$, $i \neq q$, we establish a relation between y^-_{iq}, y^+_{iq} and y^-_{qq}, y^+_{qq}. Consider the following two functions f_k: (1) $t_q = k$; and (2) $t_q = k-1$, $t_i = 1$. By (4.4.11) we have

$$\begin{aligned} &(1) \quad (k-1)(y^-_{qq} + y^+_{qq}) \equiv 0 \\ &(2) \quad (k-2)(y^-_{qq} + y^+_{qq}) + (y^-_{iq} + y^+_{qi}) \equiv 0. \end{aligned} \tag{4.4.12}$$

Computing (1) – (2) of (4.4.12) gives

$$y^-_{iq} + y^+_{qi} \equiv y^-_{qq} + y^+_{qq}. \tag{4.4.13}$$

We now consider vertices mapped to $[h]$. For each $k \in K$, $q \in [h]$, and $i, j \in [g]-[h]$, $i \neq j$, we establish a relation between y^-_{iq}, y^+_{qi} and y^-_{jq}, y^+_{qj}. Consider the following two functions f_k: (1) $t_q = 1$, $t_i = k-1$; and (2) $t_q = 1$, $t_i = k-2$, $t_j = 1$. By (4.4.11) we have

$$\begin{aligned} &(1) \quad (k-1)(y^-_{iq} + y^+_{qi}) \equiv 0 \\ &(2) \quad (k-2)(y^-_{iq} + y^+_{qi}) + (y^-_{jq} + y^+_{qj}) \equiv 0. \end{aligned} \tag{4.4.14}$$

Computing (1) – (2) of (4.4.14) gives

$$y^-_{iq} + y^+_{qi} \equiv y^-_{jq} + y^+_{qj}. \tag{4.4.15}$$

Finally, as $\alpha(K) = \gcd\{k-1 : k \in K\}$, it follows from (4.4.12)(1) that $\alpha(K)(y^-_{qq} + y^+_{qq}) \equiv 0$ for $q \in [g]-[h]$ and from (4.4.14)(1) that $\alpha(K)(y^-_{iq} + y^+_{qi}) \equiv 0$ for $q \in [h]$, $i \in [g]-[h]$. As $g(u-1) \equiv 0 \pmod{\alpha(K)}$ and $(g-h)(u-1) \equiv 0 \pmod{\alpha(K)}$ by

(4.1.4), we obtain $g(u-1)(y^-_{qq}+y^+_{qq}) \equiv 0$ for $q \in [g]-[h]$ and $(g-h)(u-1)(y^-_{iq}+y^+_{qi}) \equiv 0$ for $q \in [h]$, $i \in [g]-[h]$. Hence, together with (4.4.13) and (4.4.15), it follows that

$$\begin{aligned}
&(u-1) \sum_{(i,j)\in[g]^2-[h]^2} (y^-_{ij} + y^+_{ij}) \\
&\equiv (u-1) \sum_{\substack{q\in[g]-[h] \\ i\in[g]}} (y^-_{iq} + y^+_{qi}) + (u-1) \sum_{\substack{q\in[h] \\ i\in[g]-[h]}} (y^-_{iq} + y^+_{qi}) \\
&\equiv g(u-1) \sum_{q\in[g]-[h]} (y^-_{qq} + y^+_{qq}) + (g-h)(u-1) \sum_{q\in[h]} (y^-_{1q} + y^+_{q1}) \equiv 0
\end{aligned}$$

Hence, $u - 1 \equiv 0 \pmod{\alpha(\Phi)}$, and the existence of an $IGDD((g;h)^u, K)$ whenever u is sufficiently large follows from the Lamken-Wilson Theorem. □

We are now ready to prove our results on incomplete pairwise balanced designs with multiple block sizes.

Chapter 5

Incomplete Pairwise Balanced Designs with Multiple Block Sizes

5.1 Fixed Hole Size

Having proved the asymptotic existence of uniform incomplete group divisible designs, we can now apply Construction 4.1.10 to find an example incomplete pairwise balanced design for each congruence class of v and w.

Proposition 5.1.1. *Given K, a positive modulus $M = m\beta(K)$, and admissible congruence classes $v_0, w_0 \pmod{M}$ for incomplete pairwise balanced designs with block sizes in K, there exists an $IPBD((v_2; w_2), K)$ for some $v_2 \equiv v_0$ and $w_2 \equiv w_0 \pmod{M}$.*

Proof. The incomplete pairwise balanced designs constructed in Proposition 3.2.4 can be used as ingredients in Construction 4.1.10 to produce the remaining examples outside the two classes previously considered, however, we will therefore require certain conditions on v and w. In particular, if q retains its value from Proposition 3.2.4, where it is chosen independently of w_0, then we must have $v \equiv w \equiv 1 \pmod{q}$.

Hence, we must select classes v_1 and $w_1 \pmod{Mq}$ such that $v_1 \equiv v_0 \pmod{M}$, $v_1 \equiv 1 \pmod{q}$, $w_1 \equiv w_0 \pmod{M}$, and $w_1 \equiv 1 \pmod{q}$, which can be found by the Chinese remainder theorem as $\gcd(q, M) = 1$.

Hence, let the incomplete pairwise balanced designs found in Proposition 3.2.4 be denoted as $IPBD((x; y), K)$, where $x = yq\alpha(K)+1$. If we use the uniform incomplete group divisible design $IGDD((g; h)^u, K)$, with $g-h = x-y$ and $y \geq h$, then applying Construction 4.1.10 results in an $IPBD((g(u-1)+x; h(u-1)+y), K)$. We determine an identity relating u and h to the known parameters.

$$
\begin{aligned}
v_2 - w_2 &= (g-h)(u-1) + x - y \\
v_2 - w_2 &= (x-y)u \\
v_2 - w_2 &= (yq\alpha(K) + 1 - y)u \\
v_2 - w_2 &= y(q\alpha(K) - 1)u + u \\
v_2 - w_2 &= (w_2 - h(u-1))(q\alpha(K) - 1)u + u
\end{aligned}
$$

Hence, we obtain

$$u(u-1)(q\alpha(K)-1)h = w_2(u(q\alpha(K)-1)) + w_2 + u - v_2. \tag{5.1.1}$$

Now, we wish to determine u and h such that $v_2 \equiv v_1$ and $w_2 \equiv w_1 \pmod{Mq}$. Hence, it is sufficient to determine the required congruence classes for u and h. Thus, we consider the congruence

$$u(u-1)(q\alpha(K)-1)h \equiv w_1 u(q\alpha(K)-1) + w_1 + u - v_1 \pmod{p^t}$$

for each prime power p^t such that $p^t \parallel Mq$. For convenience, we choose a congruence class for u that produces a relation independent of w_1. Hence, we choose

$$u \equiv \begin{cases} -(q\alpha(K)-1)^{-1} & \text{if } \gcd(p, q\alpha(K)-1) = 1 \\ (v_1 - w_1)(w_1(q\alpha(K)-1)+1)^{-1} & \text{otherwise.} \end{cases}$$

Since p cannot divide two consecutive values, it follows that both inverses exist when required. Then, if $\gcd(p, q\alpha(K)-1) = 1$, we obtain $(u-1)h \equiv v_1 - u$, and hence

$$h \equiv \frac{(q\alpha(K)-1)v_1 + 1}{-q\alpha(K)}$$

which is well defined since $v_1 - 1 \equiv 0 \pmod{\alpha(K)}$ as v_1 is admissible, $v_1 - 1 \equiv 0 \pmod{q}$ as a result of Proposition 3.2.4, and $\gcd(\alpha(K), q) = 1$ as $\alpha(K) \mid M$. Otherwise, $p \mid q\alpha(K) - 1$, so we obtain $u(u-1)(q\alpha(K)-1)h \equiv 0$, and hence $h \equiv 0$. Thus, we obtain a solution for u and h by the Chinese remainder theorem. We summarize our choice of parameters in Table 5.1.

Table 5.1: Choice of Parameters to Obtain a Desired Congruence Class

	$\gcd(p, q\alpha(K)-1) = 1$	$p \mid q\alpha(K) - 1$
$u \equiv$	$-(q\alpha(K)-1)^{-1}$	$(v_1 - w_1)((q\alpha(K)-1)w_1 + 1)^{-1}$
$h \equiv$	$((q\alpha(K)-1)v_1 + 1)/ - q\alpha(K)$	0

$$\begin{aligned} y &= w_1 - h(u-1) \\ x &= yq\alpha(K) + 1 \\ g &= h + x - y \end{aligned}$$

We now verify that the required incomplete group divisible design exists by Theorem 4.4.2. Checking (4.1.3), we obtain

$$\begin{aligned}(g^2 - h^2)u(u-1) &\equiv (g-h)u(g+h)(u-1) \\ &\equiv (x-y)u[g(u-1)+h(u-1)] \\ &\equiv (v_1 - w_1)(v_1 - x + w_1 - y) \\ &\equiv (v_1 - w_1)(v_1 + w_1 - y(q\alpha(K)+1) - 1) \\ &\equiv (v_1 - w_1)(v_1 + w_1 - 1) \pmod{\gamma(K)}\end{aligned}$$

which is equivalent to (2.2.3). Checking (4.1.4), we calculate

$$u - 1 \equiv -(q\alpha(K)-1)^{-1} - 1 \equiv \frac{-1 - q\alpha(K) + 1}{q\alpha K - 1} \equiv \frac{q\alpha(K)}{q\alpha(K) - 1} \equiv 0 \pmod{p^t}$$

for any $p^t \parallel \alpha(K)$, so $u - 1 \equiv 0 \pmod{\alpha(K)}$. Hence, the required $IGDD((g;h)^u, K)$ exists provided u is sufficiently large. Therefore, Construction 4.1.10 results in an $IPBD((v_2; w_2), K)$ hitting the desired congruence classes. □

We can now prove the asymptotic existence result on incomplete pairwise balanced designs for fixed hole sizes.

Theorem 5.1.2. *Given* $w \equiv 1 \pmod{\alpha(K)}$*, there exist* $IPBD((v;w), K)$ *for all sufficiently large* v *satisfying* (2.2.3) *and* (2.2.4).

Proof. Let v be sufficiently large satisfying (2.2.3) and (2.2.4). By Proposition 5.1.1, there exists an $IPBD((v_2; w_2), K)$ such that $v_2 \equiv v$ and $w_2 \equiv w \pmod{M}$. We can assume $v \gg v_2$ and $w_2 \gg w$ so that there exist both an $IPBD((v; v_2), K)$ and an $IPBD((w_2; w), K)$ by Proposition 3.1.11. Then an $IPBD((v; w), K)$ exists as a result of filling the holes of the large design with the two smaller designs. □

Using this result, we will go on to prove the existence of incomplete pairwise balanced designs for all sufficiently large v and w, but satisfying a weaker inequality than the necessary conditions suggest.

5.2 Large Hole Size

This section is devoted to extending Theorem 2.2.5 to the general case of multiple block sizes. We will first state results for two ingredient designs used by Dukes, Lamken, and Ling [30] to prove Theorem 2.2.5.

Lemma 5.2.1. [30] *For sufficiently large m with $m \equiv -1 \pmod{k}$ and $m \equiv 1 \pmod{k-2}$, there exist both $GDD((k-1)^m r^1, k)$ and $GDD((k-1)^{m+1} r^1, k)$, where $r = (k-1)(m-1)/(k-2)$.*

Lemma 5.2.2. [30] *Let s be an integer with $s \equiv 0 \pmod{k-1}$ and $s \equiv -1 \pmod{k}$. There exist both $GDD((k-1)^m s^1, k)$ and $GDD((k-1)^{m+1} s^1, k)$ for all sufficiently large $m \equiv -1 \pmod{k}$.*

Our approach for proving an existence result on incomplete pairwise balanced designs with multiple block sizes will follow the approach used by Dukes, Lamken, and Ling [30] for a single block size. To obtain a particular congruence class, we will use the construction multiple times, with a different block size at each step. We first show the modifications required at a single step; we give a full proof for completeness.

Lemma 5.2.3. *For any real $\epsilon > 0$ and a given $k \in K$, there exist $IPBD((v; w), K)$ for all sufficiently large v, w satisfying (2.2.3), (2.2.4), $v > (k-1+\epsilon)w$, and $v - w \equiv 0 \pmod{k-1}$.*

Proof. Let m be sufficiently large such that for each $x \in R := \{k-1, k^2-1, r\}$, there exist both $GDD((k-1)^m x^1, k)$ and $GDD((k-1)^{m+1} x^1, k)$. As r retains its value of $(k-1)(m-1)/(k-2)$ from Lemma 5.2.1, m is restricted as stated, and as

$k-1 \equiv k^2-1 \equiv 0 \pmod{k-1}$ and $k-1 \equiv k^2-1 \equiv -1 \pmod{k}$, m is also restricted by the existence of these two sets of group divisible designs by Lemma 5.2.2. We also choose m so it is of the order $1/\epsilon$.

Let $z = w \mod k$. By Theorem 5.1.2, there exist $IPBD((u(k-1)+z;z),K)$ for all admissible $u \geq u_0(z,K)$. As z has only k possible congruence classes, we can define $u_0(k) := \max\{u_0(z,K)\}$ to be independent of z. Let $y = w - z$; then $y \equiv 0 \pmod{k(k-1)}$.

We construct the incomplete pairwise balanced designs starting from a transversal design. By Theorem 1.1.3, there exist $TD(m+2,n)$ for all $n \geq n_0(m)$. Then, for $v-w$ sufficiently large, we can express $v-w = (k-1)(mn+p)$ such that $k \mid n$, $n \geq n_0(m)$, and both $n,p \geq u_0(K)$. We delete all but p points of one of the groups of the transversal design to obtain a $GDD(n^m p^1 n^1, \{m+1, m+2\})$, where the last group of n is separated for notational convenience. We now assign weights to the points of the group divisible design. Each point in the first $m+1$ groups receives a weight of $k-1$ and each point in the final group receives a weight in R. Hence, our ingredient group divisible designs are of form $GDD((k-1)^m x^1, k)$ and $GDD((k-1)^{m+1} x^1, k)$, whose existence was shown above, so the result of applying Wilson's Fundamental Construction is a $GDD(((k-1)n)^m((k-1)p)^1 t^1, k)$, where $t \in n * R$, the set of n-fold sums of integers taken from R. Finally, since there exists an $IPBD(((k-1)n+z;z),K)$ and an $IPBD(((k-1)p+z;z),K)$, then there exists an $IPBD(((k-1)(nm+p)+t+z;t+z),K)$ by Construction 3.1.8.

Hence, it remains to consider the values of $n * R$. We need each possible hole size, so we wish to find an arithmetic progression having difference $k(k-1)$, which is precisely the difference between the two smaller terms of R. If instead moving to the next value in the arithmetic progression requires introducing an additional r term compared to the previous sum, than some number, say c, terms of k^2-1 must be removed, and $c-1$ terms of $k-1$ must also be introduced. We calculate c below.

$$\begin{aligned} k(k-1) &= \frac{(k-1)(m-1)}{k-2} + (c-1)(k-1) - c(k^2-1) \\ c(k^2-k) &= \frac{(k-1)(m-1)}{k-2} - (k+1)(k-1) \\ c &= \frac{m-1}{k(k-2)} - \frac{k+1}{k} < \frac{n(k-2)}{m-1}, \quad \text{for all sufficiently large } n. \end{aligned}$$

If we let $t_{\max}$ be the largest value of the arithmetic progression, then we must have

$$\begin{aligned} t_{\max} &= (n-(c-1))\frac{(k-1)(m-1)}{k-2} + (c-1)(k^2-1) \\ &\geq (n-c)\frac{(k-1)(m-1)}{k-2} \\ &\geq (k-1)n\left(\frac{m-1}{k-2} - \frac{(m-1)c}{(k-2)n}\right). \\ &\geq (k-1)n\left(\frac{m-1}{k-2} - 1\right). \end{aligned}$$

It follows that we achieve $\frac{v}{w}$ ratios as small as

$$\frac{v}{w} < \frac{(k-1)n(m+1)}{t_{\max}} + 1 < (1+O(1/m))(k-2) + 1 < k-1+\epsilon$$

as required. □

We will eventually combine the individual steps to obtain our existence result, but we first must prove the following technical lemma.

Lemma 5.2.4. *Given K and admissible parameters $(v;w)$ for incomplete pairwise balanced designs with block sizes in K, then for all sufficiently large v, we can write $v-w = \sum_{k\in K_0} c_k(k-1)$, for nonnegative integers c_k and $K_0 \subseteq K$ such that $\alpha(K_0) = \alpha(K)$, in such a way that if $K_0 = \{k_1, k_2, \ldots, k_n\}$, and we let $v_i = \sum_{j=1}^{i} c_{k_j}(k_j-1)+w$, $i = 1, 2, \ldots, n$ and $v_0 = w$, then $(v_i; v_{i-1})$ is also admissible for each $i = 1, 2, \ldots, n$.*

Proof. Let $K_m = \{k_1, k_2, \ldots, k_m\}$, $m = 1, 2, \ldots, n$, and let $a_m = \gcd\{k_i - 1 : i = 1, 2, \ldots, m\}$. We assume by induction on m that if $(c{+}w; w)$ are admissible parameters for incomplete pairwise balanced designs with block sizes in K and $c \equiv 0 \pmod{a_m}$, then for all sufficiently large c, we can write $c = \sum_{i=1}^{m} c_{k_i}(k_i - 1)$ in such a way that $(v_i; v_{i-1})$ is also admissible for each $i = 1, 2, \ldots, m$.

The case $m = 1$ is trivial as $a_1 = k_1 - 1 \mid c$, so $c = (\frac{c}{k_1 - 1})(k_1 - 1)$ and $(v_1; v_0) = (c + w; w)$ is admissible by assumption. Now, assume the result holds for all $m < M$. We show the result also holds for $m = M$. Let $b = k_M - 1$. Since $a_M = \gcd(a_{M-1}, b)$ and $c \equiv 0 \pmod{a_M}$ by assumption, then $c = a_{M-1}x + by$ has integer solutions in x and y. If x_0, y_0 is a particular solution, then every solution is of the form $(x, y) = (x_0 + nb, y_0 - na_{M-1})$ for $n \in \mathbb{Z}$. It remains to find a solution x, y such that $(a_{M-1}x + w; w)$ is admissible, that is, that $a_{M-1}x(a_{M-1}x + 2w - 1) \equiv 0 \pmod{\beta(K)}$. If a_{M-1} and $\beta(K)$ have a common factor, it can be divided out. Further, if $\beta' = \frac{\beta(K)}{\gcd(a_{M-1}, \beta(K))}$, then $\gcd(a_{M-1}, \beta') = 1$. To verify this claim, we proceed by contradiction, and assume there exists a prime p such that $p \mid a_{M-1}$ and $p \mid \beta'$. Suppose $p^e \parallel a_{M-1}$. Then it must be the case that $p^{e+1} \mid \beta(K)$. As $p \mid a_{M-1}$, $p \mid k_i - 1$ for each $i = 1, 2, \ldots, M - 1$, so $\gcd(p, k_i) = 1$. Hence, as $p^{e+1} \mid \beta(K)$, $p^{e+1} \mid k_i - 1$ for each $i = 1, 2, \ldots, n - 1$, so $p^{e+1} \mid a_{M-1}$, which is the contradiction proving the claim. Hence, as $\gcd(a_{M-1}, \beta') = 1$, it remains to find a solution such that $a_{M-1}x(a_{M-1}x + 2w - 1) \equiv 0 \pmod{\beta'}$. As $a_{M-1}x = c - by$, we have

$$\begin{aligned}
a_{M-1}x(a_{M-1}x + 2w - 1) &\equiv (c - by)(c + 2w - by - 1) \\
&\equiv (c)(c + 2w - 1) - by(2c + 2w - 1 - by) \\
&\equiv -by(2c + 2w - 1 - (b + 1)y + y) \\
&\equiv b(b + 1)y^2 - by(2c + 2w - 1 + y) \\
&\equiv -by(2c + 2w - 1 + y) \pmod{\beta'}
\end{aligned}$$

and as $\gcd(a_{M-1}, \beta') = 1$, a_{M-1} is a generator of $\mathbb{Z}_{\beta'}$, so there exists a solution y in each congruence class $\pmod{\beta'}$. Therefore, choosing $y \equiv 1 - 2c - 2w \pmod{\beta'}$ ensures $(v_{M-1}; v_0) = (a_{M-1}x + w; w)$ is admissible. Then by the induction hypothesis, we can write $a_{M-1} = \sum_{i=1}^{M-1} c_{k_i}(k_i - 1)$ in such a way that $(v_i; v_{i-1})$ is also admissible for each $i = 1, 2, \ldots, M - 1$, and the result follows. □

Our asymptotic result on incomplete pairwise balanced designs with multiple block sizes now easily follows from the previous two results.

Theorem 5.2.5. *For any real $\epsilon > 0$, there exist $IPBD((v; w), K)$ for all sufficiently large v, w satisfying* (2.2.3), (2.2.4), *and $v > (\prod_{k \in K_0}(k - 1 + \epsilon))w$, where $K_0 \subseteq K$ such that $\alpha(K_0) = \alpha(K)$.*

Proof. For $v - w$ sufficiently large, we can express $v - w = \sum_{k \in K_0} c_k(k - 1)$ in a manner that satisfies the conditions of Lemma 5.2.4 and also that each $(v_i; v_{i-1})$ is sufficiently large such that, by Lemma 5.2.3, there exists an $IPBD((\sum_{j=1}^{i} c_{k_j}(k_j - 1) + w; \sum_{j=1}^{i-1} c_{k_j}(k_j - 1) + w), K)$ for $i = 1, 2, \ldots, n$. By the required inequality of Lemma 5.2.3, we must have $v_i \geq (k_i - 1 + \epsilon)v_{i-1}$ for $i = 1, 2, \ldots, n$. Combining these required inequalities gives $v_n \geq (\prod_{i=1}^{n}(k_i - 1 + \epsilon))v_0$, i.e. $v \geq (\prod_{k \in K_0}(k - 1 + \epsilon))w$. The required design exists by filling. □

For certain block sets that are highly structured, the inequality required can be improved significantly; we examine such sets in the next section.

5.3 Particular Block Sets

Based on the previous theorem, the best inequality will be obtained when $K_0 = \{\min K\}$. Hence, the block set K must have the property that $\alpha(K) = \min K - 1$. The following result identifies one such example; let $K_{1(m)} := \{x : x \equiv 1 \pmod{m}, x \neq 1\}$.

Corollary 5.3.1. *For any real $\epsilon > 0$, there exist $IPBD((v;w), K_{1(m)})$ for all sufficiently large v, w satisfying* (2.2.3), (2.2.4), *and $v > (m+\epsilon)w$.*

Proof. If $K_0 = \{m+1\}$, then $\alpha(K_0) = m = \alpha(K_{1(m)})$, so the result follows from Theorem 5.2.5. □

In fact, we obtain the same result for any subset of $K_{1(m)}$ that includes $m+1$. More explicit results have previously been shown for block sets $K_{1(m)}$ for small values of m. Colburn, Haddad, and Linek [20] showed that the necessary conditions are sufficient for $m = 2$, Wang and Shen [57] showed the necessary conditions are sufficient with a single exception for $m = 3$, and Ge et al. [33] described existence results for $m = 4$.

Theorem 5.3.2. [20] *Let v, w be odd positive integers, and $v \geq 2w + 1$. Then there exists an $IPBD((v;w), K_{1(2)})$.*

Theorem 5.3.3. [57] *There exists an $IPBD((v;w), K_{1(3)})$ if and only if $v, w \equiv 1 \pmod 3$, $v \geq 3w + 1$, and $(v, w) \neq (19, 4)$.*

Theorem 5.3.4. [33] *An $IPBD((v;w), K_{1(4)})$ with $v \geq 4w + 1$ and $v \equiv w \equiv 1 \pmod 4$ exists in the following cases:*

- $w > 1033$;
- $v \geq 5w$ *if* $w \geq 61$;
- $w > 41$ *if* $w \equiv 1 \pmod{20}$;
- $w > 45$ *if* $w \equiv 5 \pmod{20}$;
- $w > 489$ *if* $w \equiv 9 \pmod{20}$;
- $w > 717$ *if* $w \equiv 17 \pmod{20}$.

Considering the other extreme, we can construct a block set in which every element is required to be a member of K_0. For a fixed integer n, let $\Pi = p_1 p_2 \cdots p_n$, the product of the first n prime numbers, and let $K = \{\frac{\Pi}{p_i} + 1 : i = 1, 2, \ldots, n\}$. Then $\alpha(K) = 1$. Let $K_1 = K \setminus \{\frac{\Pi}{p_1} + 1\}$ and consider $\alpha(K_1)$. As each element in K_1 is congruent to 1 (mod p_1), $p_1 \mid \alpha(K_1)$. Since this is true for all p_i, K_0 must be equal to K.

Another type of block set for incomplete pairwise balanced designs that has been analyzed is $\mathbb{Z}_{\geq k}$, the set of integers at least as large as k. Applying Theorem 5.2.5, we obtain the following.

Corollary 5.3.5. *For any real $\epsilon > 0$, there exist $IPBD((v; w), \mathbb{Z}_{\geq k})$ for all sufficiently large v, w satisfying* (2.2.3), (2.2.4), *and $v \geq (k - 1 + \epsilon)(k + \epsilon)w$.*

Proof. If $K_0 = \{k, k + 1\}$, then $\alpha(K_0) = 1 = \alpha(\mathbb{Z}_{\geq k})$, so the result follows from Theorem 5.2.5. □

Previously, the block set $\mathbb{Z}_{\geq 3}$ was considered by Hartman and Heinrich [35], and their result contained possible exceptions which were later shown to exist by Heathcote [36] and Chee et al. [12], to obtain the following.

Theorem 5.3.6. [12] *An $IPBD((v; w), \mathbb{Z}_{\geq 3})$ exists if and only if $v \geq 2w + 1$ except when*

1. $v = 2w + 1$ *and* $w \equiv 0 \pmod 2$;
2. $v = 2w + 2$ *and* $w \not\equiv 4 \pmod 6$, $w > 1$;
3. $v = 2w + 3$ *and* $w \equiv 0 \pmod 2$, $w > 6$;
4. $(v, w) \in \{(7, 2), (8, 2), (9, 2), (10, 2), (11, 4), (12, 2), (13, 2), (17, 6)\}$.

Ge et al. [33] determined the following results for the block sets $\mathbb{Z}_{\geq 5}$.

Theorem 5.3.7. [33] *Let* $5 \leq w \leq 8$. *Then an* $IPBD((v; w), \mathbb{Z}_{\geq 5})$ *exists for* $v \geq 4w + 1$, *with the following definite exceptions:*

- $w = 5$ *and* $v \in \{22, 23, 24, 27, 28, 29, 31, 32, 33, 34\}$*;*
- $w = 6$ *and* $v \in \{25, 27, 28, 29, 32, 33, 34, 35\}$*;*
- $w = 7$ *and* $v \in \{29, 30, 31, 32, 33, 34\}$*;*
- $w = 8$ *and* $v \in \{33, 34, 35, 37\}$.

Theorem 5.3.8. [33] *Let* $5 \leq w$. *Then an* $IPBD((v; w), \mathbb{Z}_{\geq 5})$ *exists for* $v \geq 5w$, *with the definite exceptions of* $v \in \{27, 28, 29, 31, 32, 33, 34\}$ *when* $w = 5$, $v \in \{32, 33, 34, 35\}$ *when* $w = 6$, *and the possible exception of* $v \in \{77, 78, 79\}$ *when* $w = 15$.

Turning to the nonexistence of incomplete pairwise balanced designs, Ge et al. [33] demonstrated the following result for all block sets of the form $\mathbb{Z}_{\geq k}$ that demonstrates that the necessary conditions cannot even be asymptotically sufficient in most cases.

Theorem 5.3.9. [33] *Let* $w = (k-1)t + 1 + a$ *and* $v = (k-1)w + 1 + b$ *with* $0 \leq b < a \leq k - 1$. *Then no* $IPBD((v; w), \mathbb{Z}_{\geq k})$ *exists, unless* $b = 0$ *and* $a = k - 1$, *when the IPBD exists if and only if a resolvable* $PBD(v - w, k - 1)$ *exists.*

Notice that if there exists an $IPBD((v; w), K)$ for a general block set K, then there also exists an $IPBD((v; w), \mathbb{Z}_{\geq \min K})$. Hence, the above result can be extended to general block sets. For the block sets $K_{1(m)}$, however, we obtain no additional information, as there are no admissible values in the given range.

Chapter 6

Incomplete Mutually Orthogonal Latin Squares

6.1 Asymptotic Existence

With our asymptotic existence result for incomplete pairwise balanced designs (with multiple block sizes), we can prove our main result for incomplete mutually orthogonal Latin squares. We first need to introduce the concept of partitions in incomplete Latin squares. A *partitioned incomplete Latin square of order v and type L*, denoted $PILS(L)$, is a v by v array of cells and a partition $P = \{S_1, S_2, \ldots, S_m\}$ of $[v]$ with $L = [|S_1|, |S_2|, \ldots, |S_m|]$ such that a cell is empty if the ordered pair indexing the cell is in some $S_i \times S_i$ for $i = 1, 2, \ldots, m$, each nonempty cell contains an integer between 1 and v such that every row and every column contains each integer at most once, and if a row (respectively column) contains an empty cell in column (row) i, then that row (column) does not contain the integer i. Figure 6.1 depicts a partitioned incomplete Latin square of order 8 and type $3^1 2^1 1^3$.

A set of t partitioned incomplete Latin squares of order v and type L is said

			6	7	8	4	5
			7	8	5	6	4
			8	6	4	5	7
6	7	8			1	2	3
7	8	6			3	1	2
4	5	7	3	2		8	1
8	4	5	1	3	2		6
5	6	4	2	1	7	3	

Figure 6.1: $PILS(3^1 2^1 1^3)$

to be *mutually orthogonal*, and is denoted $t{-}OPILS(L)$, if every pair of partitioned incomplete Latin squares in the set is *orthogonal*, that is, each ordered pair of integers not in some $S_i \times S_i$ for $S_i \in P$ occurs exactly once in a common cell. These objects are also referred to as *holey mutually orthogonal Latin squares*, and denoted $HMOLS(L)$. Figure 6.2 shows two orthogonal partitioned incomplete Latin squares of order 13 and type $4^1 1^9$.

Observe that a set of t orthogonal partitioned incomplete Latin squares of type 1^v is equivalent to a set of idempotent mutually orthogonal Latin squares of order v. A set of mutually orthogonal Latin squares is said to be *idempotent* if the main diagonal of each square in the set contains each of the symbols in ascending order. That is, cell (i, i) contains the symbol i. The following proposition shows the relation between these orthogonal partitioned incomplete Latin squares and incomplete mutually orthogonal Latin squares.

Proposition 6.1.1. *If there exist* $(t+1){-}MOLS(v)$*, then there exist* $t{-}OPILS(1^v)$.

Proof. Consider one of the squares in the set of $t + 1$ mutually orthogonal Latin squares. As each cell containing a 1 is in a distinct row and a distinct column, the rows can be permuted in such a way so that each cell containing a 1 is on the main diagonal. The rows of the other Latin squares in the set must be permuted in the

				6	7	5	9	10	8	12	13	11
				8	9	10	11	13	12	5	7	6
				9	10	8	13	12	11	6	5	7
				10	8	9	12	11	13	7	6	5
7	11	12	13		1	6	2	3	4	8	9	10
5	13	11	12	7		1	4	2	3	10	8	9
6	12	13	11	1	5		3	4	2	9	10	8
10	5	7	6	11	12	13		1	9	2	4	3
8	6	5	7	12	13	11	10		1	4	3	2
9	7	6	5	13	11	12	1	8		3	2	4
13	8	10	9	2	3	4	5	7	6		1	12
11	10	9	8	3	4	2	6	5	7	13		1
12	9	8	10	4	2	3	7	6	5	1	11	

				7	5	6	10	8	9	13	11	12
				11	13	12	5	6	7	8	10	9
				12	11	13	7	5	6	10	9	8
				13	12	11	6	7	5	9	8	10
6	8	9	10		7	1	11	12	13	2	3	4
7	9	10	8	1		5	12	13	11	3	4	2
5	10	8	9	6	1		13	11	12	4	2	3
9	11	13	12	2	4	3		10	1	5	6	7
10	13	12	11	3	2	4	1		8	7	5	6
8	12	11	13	4	3	2	9	1		6	7	5
12	5	6	7	8	10	9	2	4	3		13	1
13	7	5	6	9	8	10	4	3	2	1		11
11	6	7	5	10	9	8	3	2	4	12	1	

Figure 6.2: $2-OPILS(4^1 1^9)$

same way to preserve orthogonality. Then, in each other square, the entries on the main diagonal must be unique. In each of these squares, permute the symbols so the entries on the main diagonal are in ascending order; such a permutation preserves orthogonality. Then, delete the entries on each of the main diagonals, and the result is a set of $t-OPILS(1^v)$. □

To prove our main result, we make use of the following construction of Colbourn and Dinitz [19], which produced incomplete transversal designs from incomplete pairwise balanced designs and partitioned incomplete transversal designs. For our purposes, we restate the result in terms of Latin squares.

Construction 6.1.2. [19] *Suppose there exists an $IPBD((v;n),K)$ and that for each $k \in K$, there exist $t-OPILS(1^k)$. Then there exist $t-IMOLS(v,n)$ and in fact there exist $t-OPILS(n^1 1^{v-n})$.*

Our main theorem then follows from the previous construction and our results on incomplete pairwise balanced designs.

Theorem 6.1.3. *There exist $t-IMOLS(v,n)$ for all sufficiently large v,n such that $v \geq 8(t+1)^2 n$. More precisely, if T is the smallest power of 2 greater than $t+1$, then there exist $t-IMOLS(v,n)$ for all sufficiently large v,n such that $v \geq 2T^2 n$.*

Proof. Appealing to the previous construction, we must find a set K such that $\alpha(K) = 1$, $\beta(K) = 2$, and for each $k \in K$, there exist $t-OPILS(1^k)$. By Proposition 6.1.1, $t-OPILS(1^k)$ exist if $(t+1)-MOLS(k)$ exist, whose existence can be verified for specific values of k by Theorem 1.1.2. Now, as $\alpha(K) = 1$, K must contain an even number, which by Theorem 1.1.2 means we want a power of 2 exceeding $t+1$; let T be the smallest such value. In order to apply Construction 6.1.2 with as small a ratio as possible, we seek a second value U such that $T-1$ and $U-1$ are coprime. Choosing $U = 2T$, it is clear that $T-1$ and $2T-1 = 2(T-1)+1$ are coprime.

Finally, we must complete K so that $\beta(K) = 2$, and as $K_0 = \{T, U\}$, any additional value will not affect the bound resulting from Theorem 5.2.5. We desire a value $S \equiv 3 \pmod 4$; a convenient choice is the smallest odd power of 3 larger than $t+1$. Hence, $K = \{T, 2T, S\}$, and by Theorem 5.2.5, there exist $IPBD((v; n), K)$ for all sufficiently large v, n satisfying $v > (T-1+\epsilon)(2T-1+\epsilon)n$, or more concisely, $v \geq 2T^2 n$. As each required orthogonal partitioned incomplete Latin square exists by Theorem 1.1.2, the result follows from Construction 6.1.2 and the fact $2(t+1) \geq T$. □

6.2 Improving the Required Inequality

Consider the inequality required in our main theorem. While it is necessary that K and K_0 contain a power of 2, our second value U in K_0 was chosen as a power of 2 for convenience. If instead, we choose a different value such that U is a prime power and $T-1$ and $U-1$ are coprime, we will be able to improve the required inequality. We show this improvement for small values of t in Table 6.1, dropping the ϵ terms for simplicity.

Table 6.1: Improving the Required Inequality for t–$IMOLS(v, n)$

t	1	2	3	4-5	6	7	8-14
$2T^2$	32	32	128	128	128	512	512
$\{T, U\}$	3,4	4,5	5,8	7,8	8,9	9,16	16,17
New Ratio	6	12	28	42	56	120	240

t	15-17	18-21	22-23	24-25	26-27	28-29	30
$2T^2$	2048	2048	2048	2048	2048	2048	2048
$\{T, U\}$	19,32	23,32	25,32	27,32	29,32	31,32	32,37
New Ratio	558	682	744	806	868	930	1116

Previous results have established better required inequalities for the existence of t–$IMOLS(v, n)$ for $1 \leq t \leq 6$. For the case $t = 1$, we can always fill the empty subsquare with any Latin square of order n, so the problem is equivalent to determining when a partially filled Latin square can be completed. Hence, we consider the following result of Ryser [47] concerning the completion of Latin rectangles. A *Latin rectangle* of order r by s is an r by s array of positive integers such that the integers in each row and each column are distinct.

Theorem 6.2.1. [47] *Let L be an r by s Latin rectangle based upon the integers $1, 2, \ldots, n$. Let $N(i)$ denote the number of times that the integer i occurs in L. A necessary and sufficient condition in order that L may be extended to an n by n Latin square is that for each $i = 1, 2, \ldots, n$,*

$$N(i) \geq r + s - n.$$

The previous result is used to verify that the necessary condition of $v \geq 2n$ is also sufficient.

Corollary 6.2.2. *A $1-IMOLS(v, n)$ (i.e. an incomplete Latin square) exists if and only if $v \geq 2n$.*

Proof. Let L be an n by n Latin square. Then L is an n by n Latin rectangle based upon the integers $1, 2, \ldots, v$. Hence, L can be extended to a v by v Latin square if and only if $N(i) \geq 2n - v$ for each $i = 1, 2, \ldots, v$. Since $v > n$, it follows that $N(v) = 0$. Hence, L can be extended to a v by v Latin square if and only if $v \geq 2n$. We can then remove L from the resulting Latin square and the result follows. □

For the case $t = 2$, Heinrich and Zhu [37] completed the proof that the necessary condition is sufficient with a single exception.

Theorem 6.2.3. [37] *There exist $2-IMOLS(v, n)$ if and only if $v \geq 3n$ and $(v, n) \neq (6, 1)$.*

For the case $t = 3$, Du [29] showed the the necessary condition is sufficient except for possibly a list of 109 exceptions. Constructions for almost all of these exceptions were provided by Abel and Todorov [4]; Colbourn [15, 16, 17]; Abel, Colbourn, Yin, and Zhang [2]; and Abel and Du [3], which led to the following.

Theorem 6.2.4. [3] *There exist* $3{-}IMOLS(v, n)$ *if and only if* $v \geq 4n$ *and* $(v, n) \neq (6, 1)$*, except possibly* $(v, n) = (10, 1)$*.*

For $t = 4, 5$, Zhu [63] determined results for the small hole sizes of 7, 8, and 9. Drake and Lenz [28] determined the following result for five IMOLS by considering aligned subsquares.

Theorem 6.2.5. [28] *Assume that* $N(n) \geq 7$ *and* $n \geq 93$*. Then if* $v \geq 7n + 7$*, there exist* $5{-}IMOLS(v, n)$*.*

With the improved value of v_7 given by Theorem 1.1.4, the following result is obtained.

Corollary 6.2.6. *There exist* $5{-}IMOLS(v, n)$ *if* $v \geq 7n + 7$ *and* $n \geq 571$*.*

For the case $t = 6$, the following bounds on v were obtained by Colbourn and Zhu [21] with different conditions on n.

Theorem 6.2.7. [21] *There exist* $6{-}IMOLS(v, n)$ *if* $v \geq 8n + 139$ *and* $n \geq 98$*.*

Theorem 6.2.8. [21] *For* $n \geq 781$*, there exist* $6{-}IMOLS(v, n)$ *if and only if* $v \geq 7n$*.*

Theorem 6.2.9. [21] *For* $n \geq 23$ *and* n *a prime power, there exist* $6{-}IMOLS(v, n)$ *if and only if* $v \geq 7n$*.*

This work also led to an improved result for $t = 4$.

Theorem 6.2.10. [21] *There exist* $4{-}IMOLS(v, n)$ *exist if* $v \geq 7n$ *and* $n \geq 98$*.*

Considering the problem from a different perspective, Chee et al. [13], determined the following result when the size of the hole was close to the number of squares.

Theorem 6.2.11. [13] *Let t be a positive integer and let $0 \le n \le t+2$. Then for all sufficiently large v, there exist $t-IMOLS(v,n)$.*

Finally, Table III.4.14 in the *Handbook of Combinatorial Designs* [18] gives a lower bound on the number of $IMOLS(v,n)$ for $1 \le v \le 1000$ and $0 \le n \le 50$.

On the other hand, we can also improve the required inequality if we restrict our consideration to certain congruence classes for v and n. The best required inequality for incomplete pairwise balanced designs resulted from the block sets $K_{1(m)}$, or more generally, for any subset of $K_{1(m)}$ which includes $m+1$. Using this idea, we demonstrate the following result.

Theorem 6.2.12. *If $N(m) \ge t+1$, then there exist $t-IMOLS(v,n)$ for all sufficiently large v,n such that $v \equiv n \equiv 1 \pmod{m}$ and $v \ge (m+1)n$.*

Proof. To use Construction 6.1.2, we need to find a set K such that $m+1 \in K$ and $K_0 = \{m+1\}$, and hence $\alpha(K) = m+1$ and

$$\beta(K) = \begin{cases} m & \text{if } m \text{ is even,} \\ 2m & \text{if } m \text{ is odd.} \end{cases}$$

To achieve these parameters, we can choose a value k such that $k \equiv 1 \pmod{m}$ and $k \equiv -1 \pmod{m+1}$. By the Chinese remainder theorem, such a value is of the form $cm(m+1) + 2m + 1$, where c is a positive integer. By Dirichlet's Theorem, there exists a prime in this arithmetic progression as $\gcd(m(m+1), 2m+1) = 1$. For such a prime k, $N(k) = k-1 > t+1$, so by Proposition 6.1.1, there exist $t-OPILS(1^{m+1})$ and $t-OPILS(1^k)$. Hence, $K = \{m+1, k\} \subset K_{1(m)}$, and by Theorem 5.2.5, there exist $IPBD((v;n),K)$ for all sufficiently large v,n such that $v \equiv n \equiv 1 \pmod m$ and $v \geq (m+1)n$, so the result follows from Construction 6.1.2. □

Alternatively, we can make use of product constructions to construct mutually orthogonal Latin squares with aligned subsquares, and hence incomplete mutually orthogonal Latin squares, where the order is a prescribed multiple of the size of the hole. We make use of the following construction.

Construction 6.2.13. *Suppose that there exist $t-MOLS(m)$ and $t-MOLS(n)$. Then there exist $t-IMOLS(mn,n)$.*

The subsequent construction due to Brouwer and van Rees [11] allows the additional flexibility to shift as well as multiply. It is a variation of Wilson's MOLS Construction [59], which is as follows.

Theorem 6.2.14 (Wilson's MOLS Construction [59])**.** *Suppose $t \geq 1$ and there exist $t-MOLS(m)$, $t-MOLS(m+1)$, $t-MOLS(s)$, and $(t+1)-MOLS(r)$, where $1 \leq s \leq r$. Then there exist $t-MOLS(mr+s)$.*

Construction 6.2.15. [11] *Suppose there exist $(t+1)-MOLS(r)$, $t-MOLS(m)$, and $t-MOLS(m+1)$, and that $0 \leq s \leq r$. Then there exist $t-IMOLS(mr+s,s)$. If there also exist $t-MOLS(s)$, then there exist $t-IMOLS(mr+s,r)$, $t-IMOLS(mr+s,m)$ if $s \neq r$, and $t-IMOLS(mr+s,m+1)$ if $s \neq 0$.*

Implementing the above construction would require $t-MOLS(u)$, where u is approximately $\frac{v}{n}$. It is unclear whether this construction could be modified to produce the required examples close to the lower bound. A version of this construction is used by Colbourn and Zhu [21] to prove results on six incomplete mutually orthogonal Latin squares; we examine adapting this approach in the next section.

6.3 An Alternate Approach

In this section, we examine the approach used by Colbourn and Zhu [21] to prove existence results for six incomplete mutually orthogonal Latin squares. We generalize the portion of the argument that proves Theorem 6.2.7 to determine the resulting inequality for general t and conclude with a comparison of the inequalities achieved by the two methods. The remaining work to reduce the bound to obtain the result of Theorem 6.2.8 is largely an analysis by cases; we do not explore reducing the bound in this manner.

We begin with the following lemma, stated without proof, which is a straightforward modification of a construction used by Colbourn and Zhu [21], taken as a variant of the working corollaries of Brouwer and van Rees [11]. First we introduce the following generalization of incomplete mutually orthogonal Latin squares allowing multiple holes. A set of t incomplete mutually orthogonal Latin squares of order v and hole sizes $n_1, n_2, \ldots, n_m$, denoted $t{-}IMOLS(v; n_1, n_2, \ldots, n_m)$ is a set of v by v arrays with hole set $H = \{N_1, N_2, \ldots, N_m\}$, where each $N_k \subseteq [v]$, $k = 1, 2, \ldots, m$, such that cell (i, j) is empty if $\{i, j\} \subseteq N_k$ for some $k = 1, 2, \ldots, m$ and contains an integer between 1 and v otherwise, every row and every column contains each symbol at most once, symbols in N_k, $k = 1, 2, \ldots, m$ are not contained in a row or column indexed by N_k, and each ordered pair in $[v]^2 \setminus \cup_{k=1}^{m} N_k^2$ occurs.

Lemma 6.3.1. *Suppose there exist* $(t+2){-}IMOLS(r)$ *and* $t{-}IMOLS(m + y_i + z_j; s, y_i, z_j)$ *for* $1 \le i \le r$ *and* $1 \le j \le r$. *Then there exist* $t{-}IMOLS(mr + u + v; sr, u, v)$, *where* $y = \sum_{i=1}^{r} y_i$ *and* $z = \sum_{j=1}^{r} z_j$. *Moreover, (1) if* $s = 0$ *and there exist* $t{-}MOLS(u)$, *then there exist* $t{-}IMOLS(mr + y + z, z)$; *(2) if there exist both* $t{-}MOLS(y)$ *and* $t{-}MOLS(z)$, *then there exist* $t{-}IMOLS(mr + y + z, sr)$.

Motivated by this construction, we establish a useful upper bound on a value of m_t

such that each of $t{-}IMOLS(m_t, 0)$ (equivalent to $t{-}MOLS(m_t)$), $t{-}IMOLS(m_t + 1, 1)$ (equivalent to $t{-}MOLS(m_t + 1)$) and $t{-}IMOLS(m_t + 2; 1, 1)$ (implied by the existence of $t{-}OPILS(1^{m_t+2})$). The following upper bounds are established from the MOLS table of the Handbook of Combinatorial Designs [18].

Table 6.2: Upper Bounds on m_t

t	2	3	4	5	6	7	8	9	10
m_t	3	7	7	7	7	23	47	79	208

We now prove a fundamental result regarding the density of numbers with no small prime divisors.

Proposition 6.3.2. *For any integer t, there exists a smallest integer d_t such that for any integer n, at least one of the integers $n, n+1, n+2, \ldots, n+d_t-1$ is not divisible by any prime at most t.*

Proof. It suffices to find an upper bound on d_t. Let Π be the product of all primes at most t. Then in any set of Π consecutive integers, there is a value x such that $x \equiv 1 \pmod{\Pi}$. Since each prime at most t divides Π, it cannot also divide x. Hence, $d_t \leq \Pi$. □

The following table gives d_t for some small values of t.

Table 6.3: Values of d_t

t	2	3	5	7	11	13
d_t	2	4	6	10	16	18

Using the previous results, we find a series of such integers satisfying certain additional conditions to use as ingredient orders of mutually orthogonal Latin squares.

Lemma 6.3.3. *For any integer t, there exists a series of integers $\{q_{ti}\}_{i=1,2,\ldots}$ such that $N(q_{ti}) \geq t+2$, $0 < q_{t(i+1)} - q_{ti} \leq d_{t+2}$, and $m_t q_{t(i+1)} + v_t - 1 \leq (m_t+1)q_{ti}$.*

Proof. By Proposition 6.3.2, any d_{t+2} consecutive integers contain at least one number z such that $N(z) \geq t+2$ (by Theorem 1.1.2). So, there is an infinite series $q_{t1}, q_{t2}, \ldots$ with $q_{t1} \geq m_t d_{t+2} + v_t - 1$ and such that for any $i \geq 1$, $N(q_{ti}) \geq t+2$ and $0 < q_{t(i+1)} - q_{ti} \leq d_{t+2}$. Finally, since $m_t d_{t+2} + v_t - 1 \leq q_{ti}$, $m_t d_{t+2} \leq q_{ti} - v_t + 1$ and then $q_{t(i+1)} \leq q_{ti} + d_{t+2} \leq \frac{(m_t+1)q_{ti} - v_t + 1}{m_t}$. Hence, $m_t q_{t(i+1)} + v_t - 1 \leq (m_t+1)q_{ti}$ as required. □

We now construct examples of incomplete mutually orthogonal Latin squares with small holes.

Lemma 6.3.4. *For any integer t and any integer $v \geq m_t q_{t1} + v_t + n$ and $0 \leq n \leq q_{t1}$, there exist $t{-}IMOLS(v,n)$.*

Proof. Apply Lemma 6.3.1 (1) with $r = q_{ti}$, $m = m_t$, and $y_i, z_j = 0$ or 1. Since $t{-}IMOLS(m_t + y_i + z_j; y_i, z_j)$ all exist by definition of m_t and $(t+2){-}IMOLS(q_{ti})$ exist by Lemma 6.3.3, we obtain $t{-}IMOLS(m_t q_{ti} + y + z; y, z)$ [$y = \sum_{i=1}^{r} y_i$ and $z = \sum_{j=1}^{r} z_j$]. Let $y \geq v_t$, $z = n$, and $v = m_t q_{ti} + y + z$. Since $t{-}MOLS(y)$ exist (by definition of v_t), we have $t{-}IMOLS(v,n)$ for $n \in [m_t q_{ti} + v_t + n, (m_t+1)q_{ti} + n]$, $i = 1, 2, \ldots$. Finally, since $m_t q_{ti} + v_t \leq (m_t+1)q_{ti} + 1$ by Lemma 6.3.3, we obtain $t{-}IMOLS(v,n)$ for $v \geq m_t q_{ti} + v_t + n$. □

We can now establish the main result for this section.

Theorem 6.3.5. *For any integer t and any integer $n > q_{t1}$, then whenever $v \geq (m_t+1)n + m_t(d_{t+2} - 1) + v_t$, there exist $t{-}IMOLS(v,n)$.*

Proof. For any integer $n \geq q_{t1} + 1$, let $j = \min\{i : q_{ti} \geq n\}$ and $n^* = q_{tj}$. For any $i \geq j$, we have $t{-}IMOLS(v,n)$ where $v = m_t q_{ti} + k + n$, $v_t \leq k \leq q_{ti}$, by applying Lemma 6.3.1 (1) with $r = q_{ti}$, $m = m_t$, $y_i, z_j = 0$ or 1, $y = k$, and $z = n$. This gives an interval $[m_t q_{ti} + v_t + n, (m_t + 1)q_{ti} + n]$. Lemma 6.3.3 guarantees that there is no gap between consecutive intervals $[m_t q_{ti} + v_t + n, (m_t + 1)q_{ti} + n]$ and $[m_t q_{t(i+1)} + v_t + n, (m_t + 1)q_{t(i+1)} + n]$. Hence, $t-IMOLS(v, n)$ exist for any $v \geq m_t q_{tj} + v_t + n = m_t n^* + v_t + n$. If $v \geq (m_t + 1)n + m_t(d_{t+2} - 1) + v_t$, then $v \geq m_t(n + d_{t+2} - 1) + v_t + n \geq m_t n^* + v_t + n$, and therefore $t-IMOLS(v, n)$ exist by Lemma 6.3.4. □

Table 6.4 compares the bounds due to Theorem 6.1.3, its improvement discussed in Section 6.2, and Theorem 6.3.5, giving the coefficient on the n terms. The constant term obtained from Theorem 6.3.5 is also given for the first set of entries. The best ratio is highlighted in bold. We see that the bound given by Theorem 6.3.5 is best in the case of $t \leq 10$ as well as for a couple of values near powers of two ($t = 15, 30$), but there is no improvement on Section 6.2 in the majority of cases. However, we are limited by the knowledge of the maximum number of mutually orthogonal Latin squares for most orders, and new updated entries in the MOLS table would contribute to improving the bound due to Theorem 6.3.5.

Table 6.4: Comparing the Required Inequalities for $t-IMOLS(v,n)$

t	2	3	4	5	6	7	8	9	10
T	4	8	8	8	8	16	16	16	16
U	5	5	7	7	9	9	17	17	17
$2T^2$	32	128	128	128	128	512	512	512	512
§ 6.2	12	28	42	42	56	120	240	240	240
§ 6.3	**4**	**8**	**8**	**8**	**8**	**24**	**48**	**80**	**209**
Constant	16	46	58	124	138	778	3190	4864	8925

t	11	12	13	14	15	16	17	18-21	22-23
T	16	16	16	16	32	32	32	32	32
U	17	17	17	17	19	19	19	23	25
$2T^2$	512	512	512	512	2048	2048	2048	2048	2048
§ 6.2	**240**	**240**	**240**	**240**	558	**558**	**558**	**682**	**744**
§ 6.3	272	272	272	272	**272**	607	838	838	838

t	24-25	26-27	28	29	30	31	32-39	40	41-45
T	32	32	32	32	32	64	64	64	64
U	27	29	31	31	37	41	41	47	47
$2T^2$	2048	2048	2048	2048	2048	8192	8192	8192	8192
§ 6.2	**806**	**868**	**930**	**930**	1116	**2520**	**2520**	**2898**	**2898**
§ 6.3	838	991	991	992	**992**	2592	5208	5502	6270

t	46	47-51	52-57	58-62	63	64-65	66	67-69	70
T	64	64	64	64	128	128	128	128	128
U	53	53	59	81	67	67	71	71	73
$2T^2$	8192	8192	8192	8192	32768	32768	32768	32768	32768
§ 6.2	**3276**	**3276**	**3654**	**5040**	**8382**	**8382**	**8890**	**8890**	**9144**
§ 6.3	6270	6732	9152	9152	9152	9337	9337	9342	9342

Chapter 7

Discussion

7.1 Incomplete Pairwise Balanced Designs from Resolvable Designs

In the case of incomplete pairwise balanced designs with a single block size, the existence of designs with $v = (k-1)(w+t)+1$ points has been shown for $t = 0, 1$ and large t using resolvable designs. We have previously shown this existence for $t = 0$ in Proposition 3.2.3. Dukes, Lamken, and Ling [30] used resolvable group divisible designs to show the existence for the other values of t. In dealing with multiple block sizes, however, resolvable designs do not easily extend to show existence in these cases. In fact, while the replication number for each point in a resolvable design with multiple block sizes must be the same, there are cases where multiple resolvable pairwise balanced designs can be found having different replication numbers; an example is as follows.

Example 7.1.1. Three possible block sets of a resolvable $PBD(8, \{2, 3, 4\})$ on the point set $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ are

1. $\{1, 2, 3\}$, $\{4, 7, 8\}$, $\{5, 6\}$,
$\{1, 4, 5\}$, $\{2, 6, 8\}$, $\{3, 7\}$,
$\{1, 6, 7\}$, $\{3, 5, 8\}$, $\{2, 4\}$,
$\{2, 5, 7\}$, $\{3, 4, 6\}$, $\{1, 8\}$.

2. $\{1, 2, 3, 4\}$, $\{5, 6, 7, 8\}$,
$\{1, 5\}$, $\{2, 6\}$, $\{3, 7\}$, $\{4, 8\}$,
$\{1, 6\}$, $\{2, 7\}$, $\{3, 8\}$, $\{4, 5\}$,
$\{1, 7\}$, $\{2, 8\}$, $\{3, 5\}$, $\{4, 6\}$,
$\{1, 8\}$, $\{2, 5\}$, $\{3, 6\}$, $\{4, 7\}$.

3. $\{1, 2\}$, $\{3, 8\}$, $\{4, 7\}$, $\{5, 6\}$,
$\{1, 3\}$, $\{2, 4\}$, $\{5, 8\}$, $\{6, 7\}$,
$\{1, 4\}$, $\{3, 5\}$, $\{2, 6\}$, $\{7, 8\}$,
$\{1, 5\}$, $\{4, 6\}$, $\{3, 7\}$, $\{2, 8\}$,
$\{1, 6\}$, $\{5, 7\}$, $\{4, 8\}$, $\{2, 3\}$,
$\{1, 7\}$, $\{6, 8\}$, $\{2, 5\}$, $\{3, 4\}$,
$\{1, 8\}$, $\{2, 7\}$, $\{3, 6\}$, $\{4, 5\}$.

The first block set has replication number 4, the second block set has replication number 5, and the third block set has replication number 7. If we add a point to the block set of each parallel class, we obtain an $IPBD((12; 4), \{3, 4, 5\})$, an $IPBD((13; 5), \{3, 4, 5\})$, and an $IPBD((15; 7), \{3, 4, 5\})$ respectively. Working from the other direction, an $IPBD((v; n), K)$ can be obtained from a resolvable $PBD(v - n, K - 1)$ with n resolution classes, where $K - 1 = \{k - 1 : k \in K\}$. Hence, unlike the case of resolvable pairwise balanced designs with a single block size, in which the number of resolution classes is uniquely determined by the number of points and the block size, we need to ensure a resolvable pairwise balanced design has the proper number of resolution classes to form the incomplete pairwise balanced design required.

While resolvable pairwise balanced designs with multiple block sizes have unfortunately been given little consideration in general, certain structured types of resolvable pairwise balanced designs have been considered in more detail. A *uniformly resolvable pairwise balanced design* on v points with replication number r and block set K, denoted $URD(v, r, K)$, is a $PBD(v, K)$ resolved into r parallel classes such that within each parallel class, each block is the same size. Block set 2 in Example 7.1.1 is an example of a $URD(8, 5, \{2, 3, 4\})$. Rees [46] completed the following result for the block set $\{2, 3\}$.

Theorem 7.1.2. [46] *There exists a* $URD(v, r, \{2, 3\})$ *if and only if either*

(i) $v \equiv 3 \pmod 6$ *and* $r = (v - 1)/2$*, or*

(ii) $v \equiv 0 \pmod 2$ *and* $r = v - 1$*, or*

(iii) $v \equiv 0 \pmod 6$ *and* $v/2 \leq r \leq v - 1$*,*

with the exceptions $(v, r) = (6, 3)$ *and* $(12, 6)$*.*

Uniformly resolvable pairwise balanced designs have also been studied for the block set $\{3, 4\}$ by Schuster and Ge, see [49, 50, 51]. For the block set $\{2, 4\}$, maximum uniformly resolvable pairwise balanced designs have been studied, in which the goal is the maximum number of resolution classes with blocks of size 4, see [26, 34].

A different restriction results in *class-uniformly resolvable pairwise balanced designs* on v points with replication number r and block set K, denoted $CURD(v, r, K)$, which is a $PBD(v, K)$ resolved into r parallel classes such that each parallel class contains the same number of blocks of each size. Block set 1 in Example 7.1.1 is an example of a $CURD(8, 4, \{2, 3\})$. Existence results for class-uniformly resolvable pairwise balanced designs with block set $\{2, 3\}$ were explored by Lamken, Rees, and Vanstone [41], Danziger and Stevens [22], and Dinitz and Ling [25]. Finally, Danziger and Stevens [23, 24] considered both class-uniformly resolvable group divisible designs and class-uniformly resolvable frames, in which each parallel class of these objects contains the same number of blocks of each size.

Using either type of uniformly resolvable pairwise balanced design, we can construct incomplete pairwise balanced designs as follows.

Construction 7.1.3. *If there exists a* $URD(v-w, w, K-1)$ *or a* $CURD(v-w, w, K-1)$*, then there exists an* $IPBD((v; w), K)$*.*

Proof. For each parallel class of the $URD(v-w, w, K-1)$ or $CURD(v-w, w, K-1)$, add a new point to each of the blocks. The result is an $IPBD((v; w), K)$. □

With a general result on uniformly resolvable designs or class-uniformly resolvable designs, we would likely be able to improve the required inequality for incomplete pairwise balanced designs, instead of resorting to working with one block size at a time.

7.2 Further Directions

In addition to the consideration of resolvable designs discussed in the previous section, there are several avenues that can be pursued in this area. The first is to attempt to improve the required inequality for incomplete pairwise balanced designs. While more examples of incomplete pairwise balanced designs of the form $v = (k-1)(w+t)+1$ could be used in Lemma 5.2.3 and improve the largest term of the arithmetic progression in $n * R$, we are limited to a maximum of $\frac{n(k-1)(m-1)}{k-2}$. As the groups are filled with incomplete pairwise balanced designs from Theorem 5.1.2, we are unable to achieve the maximum ratio between v and w. Hence, a different construction is necessary to improve the required inequality, which, due to Theorem 5.3.9, maybe only be possible in specific cases.

Recalling the equivalence between incomplete pairwise balanced designs and group divisible designs, we may also consider the asymptotic existence for $GDD(g^u w^1, K)$ for fixed g. Considering blocks and replication numbers, we have the following necessary conditions.

Proposition 7.2.1. *If a* $GDD(g^u w^1, K)$ *exists, then*

$$gu(g(u-1)+2w) \equiv 0 \pmod{\beta(K)}, \text{ and} \tag{7.2.1}$$

$$gu \equiv w - g \equiv 0 \pmod{\alpha(K)}. \tag{7.2.2}$$

Considering the hole, we obtain the following necessary inequality.

Proposition 7.2.2. *If a* $GDD(g^u w^1, K)$ *exists, then* $g(u-1) \geq (\min K - 2)w$.

The major challenge with this problem appears to be constructing examples with $w < g$.

The final direction for future consideration are incomplete mutually orthogonal Latin squares containing multiple holes, as introduced in Section 6.1. Lamken [40] considered the case of three orthogonal partitioned incomplete Latin squares, and proved a result for orthogonal partitioned incomplete Latin squares with equal sized holes with a number of possible exceptions. Some exceptions were eliminated by Stinson and Zhu [55], Abel, Zhang, and Zhang [6], Zhang and Zhang [62], Bennett and Zhu [8], Bennett, Colbourn, and Zhu [7], and Abel and Zhang [5] to produce the following result with eight possible exceptions remaining.

Theorem 7.2.3. [5] *If* $n \geq 5$*, then there exist* $3{-}OPILS(h^n)$*, except for* $(h,n) = (6,1)$ *and possibly for* $(h,n) \in \{(1,10),(3,6),(3,18),(3,28),(3,34),(6,18),(6,19),(6,23)\}$*.*

Bennett, Colbourn, and Zhu [7] and Abel and Zhang [5] also considered the case of one hole of size three and the rest of size two and obtained the following result.

Theorem 7.2.4. [5] *For any* $n \geq 6$*, there exist* $3{-}OPILS(2^n 3^1)$ *except possibly for* $n \in \{6,9,10,12,14,15,17,21,24,26,27\}$*.*

A more general result for the case of exactly one hole not of size two was given by Xu [61].

Theorem 7.2.5. [61] *Suppose that* $u \geq 4$ *is an integer. There exist* $3{-}OPILS(2^n u^1)$ *if* $n \geq 54$ *and* $n \geq \frac{7}{4}u + 7$.

Finally, Abel, Bennett, and Ge [1] investigated four orthogonal partitioned incomplete Latin squares with equal sized holes and determined the following result.

Theorem 7.2.6. [1] *Suppose* h, n *are integers satisfying* $h \geq 2$ *and* $n \geq 6$. *Then there exist* $4{-}OPILS(h^n)$ *except possibly in the following cases:*

1. $h = 2$ *and* $n \in \{28, 30, 32, 33, 34, 35, 38, 39, 40, 45\}$.
2. $h = 3$ *and* $n \in \{6, 12, 18, 24, 28, 46, 54, 62\}$.
3. $h = 4$ *and* $n \in \{20, 22, 24, 28, 30, 32, 33, 34, 35, 38, 39, 40\}$.
4. $h = 5$ *and* $n \in \{18, 22, 26, 30\}$.
5. $h = 6$ *and* $n \in \{18, 22, 24, 26\}$.
6. $h = 9$ *and* $n \in \{10, 18, 22\}$.
7. $h = 10$ *and* $n \in \{32, 33, 35, 38\}$.
8. $h = 11$ *and* $n \in \{10, 15\}$.
9. $h = 14$ *and* $n \in \{34\}$.
10. $h = 17$ *and* $n \in \{10, 18, 22\}$.
11. $h = 22$ *and* $n \in \{33, 34, 35, 39, 40\}$.
12. $n = 6$ *and* h *is not of the form* $m \times b$ *where* $4{-}MOLS(m)$ *exist,* $2 \leq b \leq 13$, *and* $b \neq 3$.

13. $n = 15$ *and* h *is not of the form* $m \times b$ *where* $4-MOLS(m)$ *exist,* $2 \leq b \leq 12$, *and* $b \neq 11$.

To adapt our approach for the study of orthogonal partitioned incomplete Latin squares, we would likely require a more general type of incomplete pairwise balanced design which itself contains multiple holes. One such example, introduced by Stinson [54], are $\Diamond$-incomplete pairwise balanced designs. A *$\Diamond$-incomplete pairwise balanced design* on v points with hole sizes w_1, w_2, intersection w_3, and block set K, denoted $\Diamond-IPBD((v; w_1, w_2; w_3), K)$, is a quadruple $(V, W_1, W_2, \mathcal{B})$ such that V is a set of v points, W_1 and W_2 are subsets of V containing w_1 and w_2 points respectively called holes which intersect in w_3 points, and $\mathcal{B}$ is a collection of subsets of V called blocks such that the size of each block is in K, no block contains two points in the same hole, and every pair of points not both in the same hole appears in exactly one block. An example is given below.

Example 7.2.7. [54] The blocks of a $\Diamond-IPBD((15; 7, 7; 3), \{3\})$ on the point set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ and hole sets $\{5, 6, 7, 8, 13, 14, 15\}$ and $\{9, 10, 11, 12, 13, 14, 15\}$ are

$$\{1,5,9\},\{1,6,10\},\{1,7,11\},\{1,8,12\},\{2,5,10\},\{2,6,11\},\{2,7,12\},\{2,8,9\},$$
$$\{3,5,11\},\{3,6,12\},\{3,7,9\},\{3,8,10\},\{4,5,12\},\{4,6,9\},\{4,7,10\},\{4,8,11\},$$
$$\{1,2,13\},\{3,4,13\},\{1,3,14\},\{2,4,14\},\{1,4,15\},\{2,3,15\}.$$

In general, however, more than two holes will be needed for dealing with orthogonal partitioned incomplete Latin squares.

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